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TESIS DOCTORAL

Espacios de preórdenes izquierdos de grupos.
Spaces of left-preorders on groups

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Memoria para optar al grado de doctor presentada por

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A mis padres y mi hermana

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Resumen

Un preorden izquierdo en un grupo es una relación binaria, reflexiva, transitiva, total e invariante al multiplicar elementos por la izquierda. El interés principal de la tesis es estudiar algunos espacios de preórdenes izquierdos, que son espacios topológicos que aparecen al inducir una topología natural en el conjunto de preórdenes izquierdos. Estos espacios generalizan los espacios de órdenes izquierdos, los cuales han sido considerados en sucesivas ocasiones en la teoría de grupos ordenables.

En el Capítulo 1 proporcionamos las definiciones y propiedades básicas relacionadas con los preórdenes izquierdos. Estudiamos los conos positivos como concepto central para comprender los preórdenes izquierdos. Desarrollamos una definición de preorden izquierdo conradiano. Definimos el espacio de preórdenes izquierdos y el espacio de preórdenes izquierdos conradianos, estudiando algunas propiedades topológicas que presentan. De igual modo, para un subgrupo dado, definimos el espacio de preórdenes izquierdos relativos al subgrupo y el espacio de preórdenes izquierdos conradianos relativos al subgrupo.

En el Capítulo 2 realizamos una caracterización para la finitud del espacio de preórdenes izquierdos conradianos. Esta caracterización involucra la topología de dicho espacio, y relaciona tal finitud con una condición algebraica en términos de series racionales. También proporcionamos resultados análogos para el espacio de preórdenes izquierdos conradianos relativos a un subgrupo. Estos hechos son empleados para demostrar que el espacio de preórdenes izquierdos conradianos es finito o no numerable, y análogamente para el espacio de preórdenes izquierdos conradianos relativos a un subgrupo.

En el Capítulo 3 fijamos alguna notación sobre teoría de grafos y proporcionamos algunos resultados técnicos empleando la teoría de Bass-Serre de grupos actuando en árboles. Esta parte sobre teoría de Bass-Serre es algo técnica, y se realiza para obtener resultados que serán necesarios en conclusiones clave de capítulos posteriores. Estos resultados versan sobre propiedades de subgrupos de rango de Kurosh finito en productos libres de grupos.

En el Capítulo 4 estudiamos algunos hechos de los conos positivos de los preórdenes izquierdos en productos libres de grupos. Demostramos que el cono positivo de todo preorden izquierdo en un producto libre relativo a un subgrupo con rango de Kurosh finito no es conexo a gran escala en el grafo de Schreier. Probamos que el cono positivo de todo preorden izquierdo en un producto libre de grupos no puede ser descrito en términos de lenguajes regulares. También proporcionamos una relación entre la conexión a gran escala y la propiedad de ser descriptible por lenguajes regulares.

En el Capítulo 5 consideramos realizaciones dinámicas como un modo de estudiar preórdenes izquierdos. Probamos que el espacio de preórdenes izquierdos de un producto libre no tiene elementos aislados. Para lograrlo, demostramos tal hecho para los espacios de preórdenes izquierdos relativos a ciertos subgrupos. Esto se prueba empleando realizaciones dinámicas. Como consecuencia, al considerar el producto libre de grupos finitamente generados admitiendo un preorden izquierdo, el espacio de preórdenes izquierdos resulta un conjunto de Cantor. También demostramos que, para productos libres de grupos, el espacio de preórdenes izquierdos es finito o no numerable.

Abstract

A left-preorder on a group is a left-invariant total transitive reflexive binary relation. The aim of the thesis is studying some spaces of left-preorders, which are topological spaces that arise by endowing a natural topology on the set of left-preorders on a given group. These spaces generalize spaces of left-orders, which has been considered many times on the theory of orderable groups.

In Chapter 1, we provide the basic definitions and the basic properties related to left-preorders. We study positive cones as a central notion to understand left-preorders. We give a definition for conradian left-preorders. We define the space of left-preorders and the space of conradian left-preorders, and we study some topological properties. For a given subgroup, we also define the space of left-preorders relative to that subgroup and the space of conradian left-preorders relative to that subgroup.

In Chapter 2 we provide a characterization for the space of conradian left-preorders being finite. This characterization involves the topology of that space. It also relates the finiteness of the space with an algebraic condition in terms of rational series. We also provide an analogous result for the space of conradian left-preorders relative to a subgroup. These facts are used to prove that the space of conradian left-preorders is either finite or uncountable, and analogously for the space of conradian left-preorders relative to a subgroup.

In Chapter 3 we fix some notation about graph theory and we provide some technical results using Bass-Serre theory of groups acting on trees. This Bass-Serre theory part is quite technical, and it is considered in order to obtain some results that will be needed on some central results of next chapters. These results are about some properties of subgroups of free products of groups having finite Kurosh rank.

In Chapter 4, we study some properties of positive cones of left-preorders on free products of groups. We prove that the positive cone of each left-preorder on a free product relative to a subgroup of finite Kurosh rank is not coarsely connected on the Schreier graph. We show that the positive cone of each left-preorder on a free product cannot be described in terms of regular languages. We also provide a relation between being coarsely connected and being describable using regular languages.

In Chapter 5, we consider dynamical realizations as a way to study left-preorders. We show that the space of left-preorders on a free product does not have isolated elements. In order to do that, we prove that the same fact holds for spaces of left-preorders relative to certain subgroups. This fact is shown by using dynamical realizations. As a consequence, when considering the free product of finitely generated groups admitting a left-preorder, the space of left-preorders is a Cantor set. We also prove that, for free products of groups, the space of left-preorders is either finite or uncountable.

Chapter 0

Introduction

The main objective of the thesis is studying spaces of left-preorders for some groups.

Left-preorders are a generalization of left-orders and this can be readily seen from the definition. An *order* is a total, transitive, reflexive, antisymmetric binary relation $\preceq$ on a set. For a *preorder* we drop the antisymmetry requirement. By a *left-order* (resp. *left-preorder*) we mean an order (resp. preorders) $\preceq$ on a group G that is invariant under the left-multiplication action on itself. As we will see, left-preorders arise naturally on the study of left-orders and this thesis will focus on the study of the topology on the set of left-preorders on a given group.

The theory of left-orderable groups (groups admitting a left-order) has been developed since the beginnings of the XX century, and its starting point goes back to works of Dedekind, Hölder and Hilbert. Many relevant groups in Geometric Group Theory are left-orderable, including, for example, braid groups (see [15]) and right-angled Artin groups (see [20]). The theory of left-orderable groups and 3-manifolds are connected through the L-space Conjecture (see [8, 33]). Left-orderability is useful when working with the group ring, and it is well-known that left-orderable groups satisfies Kaplansky's Conjecture about zero-divisors. The solution of the long-standing H. Neumann's Conjecture (see [39]) used in a key way the left-orderability of free groups. Recently, this theory has attracted a lot of interest because of the strong relation with the theory of group actions on 1-manifolds (see Navas' survey [43] in the proceedings of ICM in 2018, where he was an invited speaker). Through this connection, the first examples of finitely generated or finitely presented simple left-orderable groups have been constructed (see [29, 37] for finitely generated and [28] for finitely presented).

There is a natural way to endow the set of left-preoders with a topology. The idea goes back to Sikora [53]. In this paper Sikora proved that for a finitely generated group, the space of left-orders is compact and totally disconnected topological space. Moreover, Sikora described the space of left-orders for free abelian groups, showing that when the group is non-cyclic, this topological space is homeomorphic to the Cantor set. Other cases where the space of left-orders is a Cantor set include non-abelian free groups, proved by McClearly in [38]; free products of two non-trivial left-orderable groups, proved by Rivas [49]; fundamental groups of hyperbolic surfaces, proved by Alonso, Brun and Rivas [2]. It is also remarkable that in [41] Muliarchyk showed that the subspace of bi-orders (left-orders that are also right-invariant) of a free product of two non-trivial bi-orderable groups is a Cantor set. One of the main results of this thesis is showing that the space of left-preorders on non-abelian free groups and a free product of left-orderable groups are Cantor sets.

A left-preorder $\preceq$ on G is *conradian* if for every pair of positive elements $g, h \in G$ (i.e. $1 \prec g$ and $1 \prec h$), one has that $g \prec hg^n$ for some $n \in \mathbb{N}$. The conradian left-orders forms an important subclass of left-orders. The groups admitting a conradian left-order are precisely the ones that are *locally indicable*, that is, every non-trivial finitely generated subgroup maps onto the integers. In [44], Navas proved that the set of conradian left-orders on a group is closed on the space of left-orders on that group. Later, [50], Rivas characterizes when the space of conradian left-orders is finite. He also shows that this space is either finite or uncountable. One of the contributions of this thesis is showing that the definition of conradian

left-preorder coincides with the definition of left-preorder with no crossings of [4]. Moreover, the thesis also studies the topology of the space of conradian left-preorders understanding when it is a closed subspace and characterizing when the space is finite.

For further discussing the result of the thesis, we need the concept of the positive cone of a left-preoder. Given a left-preorder $\preceq$ in G, the equivalence class of 1, the set of elements g such that $1 \preceq g$ and $g \preceq 1$, forms a subgroup C of G. In this case, we say that $\preceq$ it is a left-preorder on G *relative to* C. Moreover, we have a unique decomposition $G = P \sqcup P^{-1} \sqcup C$ where P is the positive cone of $\preceq$. The *positive cone* of $\preceq$ is the set of elements g such that $1 \prec g$. The positive cone is a subsemigroup of G and completely determines $\preceq$.

It follows from the result of McClearly mentioned above, that no left-order on a non-abelian free group is isolated in the space of left-orders, which in turn implies that no positive cone is finitely generated as semigroup. Hermiller and Šunić [23], were able to strengthen this last result showing that no positive cone in a free product is described by a regular language. Some of the applications of the methods of the thesis go towards proving that non-abelian free groups and free products do not have left-preoders with regular positive cones.

Denote by $\mathcal{PO}(G)$, $\mathcal{PO}_C(G)$, $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ the set of left-preorders on G, left-preorders on G relative to C, conradinan left-preorders, and conradian left-preorders relative to C, respectively. These are the central objects of this thesis and we are going to describe now their topologies. Given a left-preorder $\preceq$ on G, consider the unique decomposition $G = P \sqcup P^{-1} \sqcup C$ where P is the positive cone of $\preceq$. We can associate to $\preceq$ an element of $\{\pm 1, 0\}^G$ with the map

$$\begin{aligned} \phi_{\preceq} : G &\longrightarrow \{\pm 1, 0\} \\ x &\mapsto \begin{cases} 1 & \text{if} \quad x \in P \\ -1 & \text{if} \quad x \in P^{-1} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The previous map allows us to see $\mathcal{PO}(G)$, $\mathcal{PO}_C(G)$, $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ as subsets of $\{\pm 1, 0\}^G$. Endowing $\{\pm 1, 0\}^G$ with the product topology, this induces the topologies that we consider in the thesis. Notice that for $\mathcal{PO}_{\{1\}}(G)$ this is exactly the topology that Sikora introduced and $\mathcal{CO}_{\{1\}}(G)$ is the space of conradian left-orders studied by Navas and Rivas.

For convenience to the reader, we describe the space of left-preorders on some examples. The space $\mathcal{PO}(\mathbb{Z})$ is discrete and has 2 elements (Example 1.6.19). The space $\mathcal{PO}(\mathbb{Z}^2)$ is the union of a Cantor set and a countable discrete set (Example 1.6.19). For K the Klein-bottle group (the one which is generated by $\{a, b\}$ with the relation $aba^{-1} = b^{-1}$), we have that $\mathcal{PO}(K)$ is discrete and has 6 elements (Example 2.2.16). We note that for these groups all left-preorders are conradian, so we have also provided descriptions of $\mathcal{CO}(G)$ in these three cases.

The spaces $\mathcal{PO}(G)$, $\mathcal{PO}_C(G)$, $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ are totally disconnected and Hausdorff, and, when G is countable (for example, finitely generated), they are all metrizable. The spaces $\mathcal{PO}_C(G)$ and $\mathcal{CO}_C(G)$ are always compact, and the spaces $\mathcal{PO}(G)$ and $\mathcal{CO}(G)$ are compact when G is finitely generated.

The thesis aims to further expand the knowledge of the topological spaces of left-preoders that we have mention. We now will give a detail account of the contributions of the thesis to the area in the next two sections. The next section of the introduction will be devoted to the work on the space of conradian left-preorders. The second section of the introduction deals with the structure of $\mathcal{PO}(G * H)$. Before this, let us mention the few other places of the literature that we are aware that $\mathcal{PO}(G)$ has been considered. In [4] an analague of Morris theorem [40] is proved for left-preoders using the compactness of $\mathcal{PO}(G)$. Decaup and Rond in [14] computed $\mathcal{PO}(G)$ for G an abelian group. The space of left-preorders also plays an important role in [9], where it is shown that a certain natural quotient on a product of the space of left-preorders it is homeomorphic to a space of normalized harmonic actions. It is remarkable that space of normalized harmonic actions used in other important results, as in [16], where it is shown that an irreducible lattice in a real semi-simple Lie group of real rank at least two and finite center is not left-orderable.

0.1 Conradian left-preorders

Conradian condition for left-orders was firstly considered by Conrad in [12]. Conrad was interested on these kind of left-orders as they satisfy a "local" version of the conclusion provided by Hölder for Archimedean left-orders (meaning that for any positive $g, h \in G$, we have $h \preceq g^n$ for some $n \in \mathbb{N}$). The mentioned Hölder's result says that every *Archimedean* left-order on a group is isomorphic to a subgroup of the additive group of the real numbers, and such isomorphism is order preserving. As an example of the importance of these left-orders, in [36] Linnell showed that the space of left-orders is either finite or uncountable using some properties of conradian left-orders.

There are some characterizations of conradian left-orders in the literature. We have mentioned Conrad's characterization as a left-order satifying the conclusion of Hölder's theorem but locally. In [31], it is shown that, in the definition of being conradian, one can take $n = 2$. In [45], there is a dynamical characterization of conradian left-orders as a left-order having no crossings. We provide three characterizations of conradian left-preorders generalizing these facts. We state it without defining what a crossing is, the definition will be given in Section 5.2.

Theorem 0.1.1 (Theorem 1.4.13 & Theorem 5.2.2)**.** *Let $\preceq$ be a left-preorder on G relative to C. The following conditions are equivalent:*

a) For every (K, L) $\preceq$-convex jump (that is, a pair of distinct convex subgroups with no other convex subgroup between them), we have that K is normal in L and the group L/K is isomorphic to a subgroup of the additive group of the real numbers, and such isomorphism is order preserving.

b) For every pair of positive elements $g, h \in G$ we have $g \prec hg^2$.

c) The action of G on G/C by left-multiplication admits no crossing.

d) $\preceq$ is conradian (for every pair of positive elements $g, h \in G$, one has that $g \prec hg^n$ for some $n \in \mathbb{N}$).

We may ask for conditions on G (resp. C) determining when is the space $\mathcal{CO}(G)$ (resp. $\mathcal{CO}_C(G)$) finite. For the $\mathcal{PO}_{\{1\}}(G)$, this was done by a well known result of Tararin. For $\mathcal{CO}_{\{1\}}(G)$, this was solved by Rivas in [50]. We provide a characterization for the finiteness of $\mathcal{CO}(G)$(resp. $\mathcal{CO}_C(G)$). We need to introduce some further notation.

Definition 0.1.2. Let G be a group. A rational series starting on G is a finite series of the form

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \tag{1}$$

satisfying that each G_i is normal on G_{i+1} and the group G_{i+1}/G_i is torsion-free abelian of abelian rank 1 for all $0 \leq i < n$. Such a series is said to be normal if each G_i is normal on G. An abelian jump on this series is a subseries the form

$$G_i \trianglelefteq G_{i+1} \trianglelefteq G_{i+2}$$

for $i \in \{0, \ldots, n-2\}$ such that G_i is normal on G_{i+2} and the group G_{i+2}/G_i is abelian. We say that such a series is of maximal length if for any other

$$H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G$$

rational series starting on G, we have $k \leq n$.

Theorem 0.1.3. *[Theorem 2.2.8] Let G be a group. Assume that $\mathcal{CO}(G)$ is non-empty. Let $C_0 = \bigcap_{H \in \mathcal{H}} H$ for $\mathcal{H} = \{H$ subgroup with $\mathcal{CO}_H(G) \neq \emptyset\}$. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}(G)$ are isolated.

b) There exists a finite rational series starting on G as in Equation (1) of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$.

c) $\mathcal{CO}_{C_0}(G)$ is finite.

d) $\mathcal{CO}(G)$ is finite.

In condition b) of the previous theorem, one can also prove that such series is unique. In fact, in that series, we have that $G_0 = C_0$. We can use the length n of the series to compute the cardinality, since $\mathrm{Card}\,(\mathcal{CO}(G)) = 2^{n+1} - 2$. Notice that the uniqueness of the series implies that the series is normal.

For $\mathcal{CO}_C(G)$ we obtain a similar description (Theorem 2.1.14), but the corresponding series satisfies that $G_0 = C$. For $\mathcal{CO}_C(G)$, the series is not normal in general, as C does not need to be a normal subgroup. The fact that the series is not normal will give an extra layer of technicality in our proofs. In this case, one can also add as equivalent condition that some element on $\mathcal{CO}_C(G)$ is isolated. As a consequence, by topological arguments, if G is finitely generated, then $\mathcal{CO}_C(G)$ is either finite or a Cantor set. The characterization of $\mathcal{CO}_C(G)$ being finite generalizes the result of Rivas for $\mathcal{CO}_{\{1\}}(G)$.

These characterizations of the finiteness of $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ can be used to obtain more information about their cardinalities, as we state now.

Theorem 0.1.4 (Corollary 2.2.11)**.** *Let G be a group. Then $\mathcal{CO}(G)$ (resp. $\mathcal{CO}_C(G)$) is either finite or uncountable.*

0.2 Spaces of left-preorders on free products

Free products has been considered so many times on the theory of orderable groups. The fact that non-abelian free groups admit left-orders was known since the 1940's, appearing on published or unpublished works of some mathematicians as Neumann, Tarski, Birkhoff, Shimbireva and Iwasawa (see [46] and [47]). In parallel, Vinogradov proved that free products of bi-orderable groups are bi-orderable (see [59]), and Passman explained that such argument also proves that free products of left-orderable groups are left-orderable (see [48]). The proof used groups of positive units of ordered rings. Johnson simplified Vinogradov's arguments (see [32]) and Holland and Scrimger made explicit that Johnson's arguments are valid for the left-orderable case (see [26]). In [7], Bergman developed a more simplified argument for Vinogradov result. It is also remarkable that, in [10], Burns and Hale provedided a proof of Passman's statement by considering the natural map from a free product to a direct product, whose kernel is free. Dicks and Šunić provided a short proof by constructing total orders on the set of vertices of oriented trees (see [19]).

Recall that, as we have discussed before, McClearly proved that the space of left-orders on any non-abelian free group of finite rank has no isolated elements, and Rivas proved this fact for free products of two non-trivial left-orderable groups. On the thesis, we study the space of left-preorders on free products, providing a similar answer.

Theorem 0.2.1. *[Theorem 5.3.8 & Corollary 5.3.10] Let G and H be two non-trivial groups. Then, $\mathcal{PO}(G * H)$ has no isolated elements.*

As a corollary, we deduce that $\mathcal{PO}(G * H)$ is either empty or uncountable. Also, when G and H are finitely generated, one can prove that $\mathcal{PO}(G * H)$ is a Cantor set by using topological arguments. Notice that the conclusion of the previous theorem does not depend on the structure of the space of left-preorders on each factor on the free product.

We now briefly describe the strategy used on the proof of the theorem. Say that a proper subgroup C of G is *left-relatively convex* on G if $\mathcal{PO}_C(G)$ is non-empty. Then, the space $\mathcal{PO}(G)$ can be decomposed as a union of $\mathcal{PO}_C(G)$ where C ranges over all left-relatively convex proper subgroups on G. In order to derive the theorem, we show the following theorem about the structure of $\mathcal{PO}_C(G)$.

Theorem 0.2.2. *[Theorem 5.3.1 & Corollary 5.3.6] Let G and H be two non-trivial groups. Let C be a left-relatively convex proper subgroup. If C is finitely generated, then $\mathcal{PO}_C(G*H)$ has no isolated elements.*

The proof uses the same strategy done by Rivas. We recover Rivas' result when taking $C = \{1\}$. Actually, we prove a more general result: we can change the condition of C being finitely generated by having finite Kurosh rank on the free product $G * H$, which is a weaker condition. The *Kurosh rank* of a subgroup C of $G * H$ is defined as the sum of the number of vertices of $C\backslash\mathtt{T}$ with associated non-trivial

stabilizer plus the number of edges of $C\backslash \mathtt{T}$ outside of a maximal subtree, for $\mathtt{T}$ the Bass-serre universal covering tree relative to the free product $G * H$.

The proof of the theorem needs a result providing a generating set for an intersection of subgroups of a free product. This fact, as far as the author knows, has not been considered in the literature, so we state it as a result that could be interesting by itself.

Theorem 0.2.3 (Theorem 3.3.5)**.** *Let C be a proper subgroup of $G * H$. If C has finite Kurosh rank, then there exists $\Omega, \{\zeta_i\}_{i\in I}, \{\alpha_j\}_{j\in J}, \{\beta_k\}_{k\in K} \subseteq G * H$ finite sets satisfying the following property:*

For every G_0 subgroup of G and every H_0 subgroup of H satisfying $\Omega \subseteq \langle G_0, H_0\rangle$ there exists G_j subgroup of G_0 for each $j \in J$ and there exists G_k subgroup of H_0 for each $k \in K$ such that

$$C \cap \langle G_0, H_0\rangle = \langle \{\zeta_i \ / \ i \in I\} \cup \left(\bigcup_{j\in J} \alpha_j G_j \alpha_j^{-1}\right) \cup \left(\bigcup_{k\in K} \beta_k H_k \beta_k^{-1}\right)\rangle.$$

It is remarkable that, for such C and for such G_0 and H_0 satisfying $\Omega \subseteq \langle G_0, H_0\rangle$, we have that the Kurosh rank of $C \cap \langle G_0, H_0\rangle$ is bounded by the Kurosh rank of C.

We also mention a relevant consequence of the theorem about $\mathcal{PO}_C(G*H)$. Recall that the positive cone of a left-preorder is always a semigroup. The fact that $\mathcal{PO}_C(G * H)$ has no isolated elements implies that the positive cone of any left-preorder relative to C is not finitely generated as a semigroup. In particular, the mentioned theorem implies the following.

Corollary 0.2.4. *[Corollary 5.3.12] Let G and H be non-trivial finitely generated groups. Let C be a left-relatively convex proper subgroup. If C is finitely generated, then no left-preorder on $G * H$ relative to C has a positive cone P that is finitely generated as a subsemigroup.*

On the previous corollary, one can change the condition of C being finitely generated by the condition of C having finite Kurosh rank.

Finitely generated subsemigroups are examples of regular sets from the formal languages point of view. Then, a natural question arises from the last result: is it possible to be regular under the hypotheses of last corollary? This question is related with the study of the computational complexity of the positive cone, which has been studied in other papers for other groups (see for example [5], [13], [22],[51] and [54]). A left-preorder is considered to be regular if the positive cone can be described just using a regular language or, equivalently, if there is a finite state automaton whose accepting language represents all elements on the positive cone. In [23] it is shown that it is impossible to find a regular positive cone of a left-order on a free product of non-trivial groups. We obtain the same conslusion for left-preorders.

Theorem 0.2.5. *[Theorem 4.2.25] Let G and H be non-trivial finitely generated groups. Let C be a left-relatively convex proper subgroup on $G * H$. Then, no left-preorder on $G * H$ relative to C has a regular positive cone.*

As a consequence, we deduce that in the previous corollary we can drop the requirement of C being finitely generated (or the requirement of C having finite Kurosh rank). In other words, no left-preorder on a free product $G*H$ of non-trivial groups has a positive cone P that is finitely generated as a subsemigroup.

In [1] it is shown that, on a group, the regularity of a subset implies that it is coarsely connected on the Cayley graph. Then, we can ask if positive cones of left-preorders on a free product relative to C are coarsely connected on the Schreier graph associated to the subgroup C. We state the answer simplifying the notation and without all the technical details in order to be more clear on the exposition.

Theorem 0.2.6. *[Theorem 4.1.10] Let G and H be non-trivial finitely generated groups. Let C be a left-relatively convex proper subgroup. If C is finitely generated (resp. has finite Kurosh rank), then no left-preorder on $G * H$ relative to C has a coarsely connected positive cone (on the Schreier graph).*

We discuss briefly the structure of the thesis. The first chapter is devoted to basic definitions and properties of left-preorders. In the second chapter we study finiteness conditions for the spaces of conradian left-preorders. In the third chapter we use Bass-serre theory to derive results about subgroups having finite kurosh rank on free products. The fourth chapter is devoted to study some properties of left-preorders from the point of view of geometry and formal languages. In the fifth chapter we consider dynamical realizations and we prove that the space of left-preorders on a free product does not have isolated elements.

Chapter 1

Left-preorders

In this chapter, we present the central notions of the thesis and their basic properties.

In the first section, we define left-preorders and we study some first properties and examples. In the second section, we focus on positive cones. The third section is devoted to define convexity, to study some properties related to convexity and to define a useful bijection between set of left-preorders involving the notion of convexity. In the fourth section we define and study conradian left-preorders. The fifth section is focused on checking that the bijection considered at the third section preserves the property of being conradian. Finally, in the sixth one it is defined some different spaces of left-preorders endowed with a topology following the construction of Sikora.

The dynamical tools, although they are part of the basic constructions on left-preorders, are not considered in this chapter. The basic dynamical properties are studied in Sections 5.1 and 5.2. However, we have determined not including them here in order to make it easy for the reader, as these tools are not used until chapter 5.

1.1 Defining left-preorders

The notion of left-preorder can be also found in other works. For example, it is present in [14] and [9] with the name of *left-invariant preorder*. This notion will be equivalent to the notion of *relative order* in [4]. We emphasize that in this thesis we will impose in the definition that left-preoders are non-trivial. In this section we detail some definitions and basic properties related to left-preorders.

Definition 1.1.1. Let X be a set. A preorder $\preceq$ on X is a transitive reflexive binary relation that is total (for all $x_1, x_2 \in X$ we have that $x_1 \preceq x_2$ or that $x_2 \preceq x_1$). Denote by $[x]_\preceq$ the set of $y \in X$ such that $x \preceq y$ and $y \preceq x$. A preorder $\preceq$ on X is trivial if $x_1 \preceq x_2$ for all $x_1, x_2 \in X$. An order on X is a preorder that is antisymmetric.

Definition 1.1.2. Let G be a group. A left-preorder $\preceq$ on G is a preorder on G that is non-trivial and that is invariant under the action of G on itself by left multiplication:

$$g_1 \preceq g_2 \text{ implies } gg_1 \preceq gg_2 \text{ for all } g, g_1, g_2 \in G.$$

We define $\mathcal{PO}(G)$ the set of left-preorders on G. A left-order on G is an antisymmetric left-preorder on G.

Definition 1.1.3. Let G be a group. A left-order $\preceq$ on G is a left-preorder on G that is antisymmetric. We say that G is left-orderable if it admits some left-order.

If a group is left-orderable, then it is torsion-free. We now provide some examples of left-orderable groups. All these examples are well known in the literature (for references, see [11] and [17]). The following groups are left-orderable:

- Torsion-free abelian groups.

- Direct products of left-orderable groups.
- Extensions of left-orderable groups.
- Free products of left-orderable groups (in particular non-abelian free groups).
- Locally left-orderable groups.
- Torsion-free nilpotent groups.
- Locally indicable groups (in particular, one-relator torsion-free groups).
- Braid groups of any number of strands.
- The group of orientation preserving homeomorphisms of the real line.
- Thompson's group F.
- Right angled Artin groups.
- Surface groups.
- Limit groups.

We can equivalently describe left-preorders as left-invariant orders on the set of left cosets of G with respect to the subgroup $[1]_\preceq$. With this, we obtain the formulation from [4]. In order to do that, we prove that $[1]_\preceq$ is a proper subgroup and then we present this equivalent vision.

Lemma 1.1.4. *Let G be a group and $\preceq$ a left-preorder on G. Then $[1]_\preceq$ is a proper subgroup of G.*

Proof. The fact that $[1]_\preceq$ is proper follows by the fact that $\preceq$ is non-trivial. We now prove that $[1]_\preceq$ is a subgroup. Consider $g, h \in [1]_\preceq$, so that $g \preceq 1$, $h \preceq 1$, $1 \preceq g$ and $1 \preceq h$. On the one hand, from $h \preceq 1$ and $1 \preceq g$ we deduce $h \preceq g$, so left-multiplying by g^{-1} we obtain $g^{-1}h \preceq 1$. On the other hand, from $g \preceq 1$ and $1 \preceq h$ we deduce $g \preceq h$, so left-multiplying by g^{-1} we obtain $1 \preceq g^{-1}h$. Therefore, $g^{-1}h \in [1]_\preceq$. □

Remark 1.1.5. A left-preorder $\preceq$ on G is a left-order on G if and only if $[1]_\preceq = \{1\}$. Next lemma is trivial for left-orders as one can trivially identify G with $G/\{1\}$.

Lemma 1.1.6. *Let G be a group. Given $\preceq$ a left-preorder on G, the relation $\preceq_\bullet$ on $G/[1]_\preceq$ defined by*

$$g[1]_\preceq \preceq_\bullet h[1]_\preceq \Leftrightarrow g \preceq h$$

is an order invariant under left multiplication. Reciprocally, given C a proper subgroup of G and $\preceq_\bullet$ an order on G/C invariant under left multiplication, the relation $\preceq$ on G defined by

$$g \preceq h \Leftrightarrow gC \preceq_\bullet hC$$

is a left-preorder on G satisfying $C = [1]_\preceq$.

Proof. We start by proving the first statement. Consider $\preceq$ a left-preorder on G and define $\preceq_\bullet$ on $G/[1]_\preceq$ by

$$g[1]_\preceq \preceq_\bullet h[1]_\preceq \Leftrightarrow g \preceq h.$$

We claim that $\preceq_\bullet$ is a well defined relation on $G/[1]_\preceq$. Indeed, take $g', h' \in G$ such that $g[1]_\preceq = g'[1]_\preceq$ and $h[1]_\preceq = h'[1]_\preceq$. Then, $g^{-1}g', h^{-1}h' \in [1]_\preceq$. In particular, $g^{-1}g' \preceq 1$ and $1 \preceq h^{-1}h'$, so then $g' \preceq g$ and $h \preceq h'$. Using that $g \preceq h$ we obtain $g' \preceq h'$. Therefore, $\preceq_\bullet$ is a well defined relation on $G/[1]_\preceq$, as we claimed.

Clearly $\preceq_\bullet$ is transitive and reflexive as $\preceq$ satisfies these properties. We now prove that $\preceq_\bullet$ is antisymmetric. Indeed, assume that $g[1]_\preceq \preceq_\bullet h[1]_\preceq$ and $h[1]_\preceq \preceq_\bullet g[1]_\preceq$. In particular, from the definition we have $g \preceq h$ and $h \preceq g$. Then, applying the definition of $[1]_\preceq$, it follows that $g[1]_\preceq = h[1]_\preceq$. This shows that $\preceq_\bullet$ is antisymmetric, as we wanted. The fact that $\preceq_\bullet$ is total and invariant under left multiplication follows directly by the fact that $\preceq$ also has these properties. Therefore, $\preceq_\bullet$ is an order invariant under left multiplication,

so we finish the proof of the first statement.

We now prove the second statement. Consider C a proper subgroup of G and $\preceq_\bullet$ an order on G/C invariant under left multiplication. Define

$$g \preceq h \Leftrightarrow gC \preceq_\bullet hC,$$

which give us a relation $\preceq$ on G. Notice that $\preceq$ is transitive, reflexive, total and invariant under left-multiplication since $\preceq_\bullet$ is. Hence, $\preceq$ is a left-preorder on G, so we only need to prove that $[1]_\preceq = C$. It is clear that $C \subseteq [1]_\preceq$ since $fC = 1C$ for all $f \in C$ and since $\preceq$ is reflexive. For the other inclusion, consider $g \in [1]_\preceq$. We have $1 \preceq g$ and $g \preceq 1$, so we deduce $C \preceq_\bullet gC$ and $gC \preceq_\bullet C$. Then, using the antisymmetry of $\preceq_\bullet$ we obtain $gC = C$, then $g \in C$. This proves that that $[1]_\preceq = C$. Notice that $\preceq$ is non-trivial since C is proper. This proves the second statement. □

Notice that, if the subgroup $[1]_\preceq$ is normal, then the induced order on the group $G/[1]_\preceq$ is a left-order.

Notation 1. The last lemma says that left-preorders can be identified with an order on $G/[1]_\preceq$ which is invariant under left multiplication (left-invariant to abbreviate) and vice versa. Hence, from now on, we will identify each left-preorder on G with a left-invariant order on a set of cosets over a subgroup of G. Given a left-preorder $\preceq$ on G, the expression $g[1]_\preceq \prec h[1]_\preceq$ will mean that $g[1]_\preceq \preceq h[1]_\preceq$ and $g[1]_\preceq \neq h[1]_\preceq$.

Definition 1.1.7. Let G be a group and let C be a proper subgroup of G. A left-preorder on G relative to C is a left-preorder $\preceq$ satisfying that $C = [1]_\preceq$. We denote by $\mathcal{PO}_C(G)$ the set of left-preorders on G relative to C.

We can introduce also the notion of left-relatively convex subgroup, which is classical in the context of left orders (for example, see the start of Section 2.2 of [11], where it is presented as the notion of relatively-convex subgroup). We follow the formulation given in [6] using left-preorders. Notice that in the next definition we are implicitly assuming that C is a proper subgroup as we are not allowing trivial left-preorders.

Definition 1.1.8. Let G be a group and let C be a subgroup of G. We say that C is left-relatively convex on G if there is a left-preorder on G relative to C.

We can obtain left-preorders from left-orders.

Remark 1.1.9. Let G be a group. The trivial subgroup $\{1\}$ will be left-relatively convex on G if and only if G is left-orderable and, if that is the case, can see each left-order on G as a left-preorder on G relative to $\{1\}$ and vice versa. More generally, if C is a normal subgroup of G, C will be left-relatively convex on G if and only if G/C is left-orderable and, in this case, each left-order on G/C is identified with a left-preorder on G relative to C and vice versa. This is a direct consequence of Lemma 1.1.6.

We can adapt this to the abelian case. Recall that an abelian group is left-orderable if and only if it is torsion-free.

Example 1.1.10. *Consider G an abelian group that is not torsion-free. This group is not left-orderable. However, if we consider C the torsion subgroup, then G/C is an abelian torsion-free group, so it is left-orderable. In particular, C is left-relatively convex on G and each left-preorder on G relative to C corresponds to a left-order on G/C.*

We can consider some examples of left-relatively convex subgroups.

Example 1.1.11. *In [6] we can find many other examples of left-relatively convex subgroups. We list a few below.*

- *If C is a proper normal subgroup of G and G/C is left-orderable, then C is left-relatively convex on G (as remarked before).*
- *If G is a non-abelian free group, each maximal cyclic subgroup of G is left-relatively convex on G (see Corollary 9 on [6]).*

- *If G and H are non-abelian free groups and C is a maximal cyclic group in both G and H, then G and H are left-relatively convex on $G *_C H$ (see Example 22 on [6]).*
- *If G is a subgroup of a right-angled Artin group and $n \in \mathbb{N}$, then every maximal n-generated abelian subgroup of G is left-relatively convex on G (see Corollary 28 on [6]).*
- *If $(G_i)_{i\in I}$ is a family of non-trivial left-orderable groups, then G_{i_0} is a left-relatively convex subgroup on $*_{i\in I}G_i$ for all $i_0 \in I$ (see Theorem 17 on [6]).*

1.2 Positive cones

This section is devoted to present and study positive cones as a central notion on left-preorders. Our discussion is inspired in some classical ideas about left-orders that works for left-preorders (see Lemma 2.1 in [4] and Definition 4 of [6] for left-preorders, and Section 1.1.1 of [17] for left-orders).

Definition 1.2.1. Let G be a group and C be left-relatively convex subgroup. Let $\preceq$ be a left-preorder on G relative to C. We define the positive cone of $\preceq$ to be the set $\{g \in G \,/\, C \prec gC\}$.

We are now going to prove that describing the positive cone is equivalent to describing the left-preorder. We split part of the proof by considering two previous lemmas to be more clear in the exposition. These proofs are routine and elementary, but we provide all the details on the proofs to familiarize the reader that is not familiar with left-(pre)orders on groups.

Lemma 1.2.2. *Let G be a group, let C be a subgroup of G and let P is a subsemigroup of G. Suppose that $G = P \cup P^{-1} \cup C$ and that P and C are disjoint. Then, we have*

$$CP \subseteq P \quad \text{and} \quad PC \subseteq P.$$

Proof. We only prove $CP \subseteq P$ as the proof of $PC \subseteq P$ is analogous. We do it by contradiction. Take $g \in C$ and $h \in P$ such that $gh \notin P$. Applying that $G = P \cup P^{-1} \cup C$ by hypothesis, we deduce $gh \in P^{-1} \cup C$. If we are in the case $gh \in C$, since $g \in C$, we deduce $h \in C$, and this contradicts that P and C are disjoint. Hence, it must be the case $gh \in P^{-1}$. Then $h^{-1}g^{-1} \in P$. Using that $h \in P$ and P is a subsemigroup by hypothesis, we obtain $g^{-1} \in P$. This is a contradiction with P and C being disjoint by hypothesis. □

Lemma 1.2.3. *Let G be a group, let C be a subgroup of G and let P is a subsemigroup of G. Suppose that $G = P \cup P^{-1} \cup C$. Then, P and C are disjoint if and only if P, P^{-1} and C are pairwise disjoint.*

Proof. Suppose that P and C are disjoint. If we assume that P and P^{-1} are not disjoint, we can take $g \in P \cap P^{-1}$, so $1 = gg^{-1} \in P$, and using the fact that C is subgroup, we obtain $1 \in P \cap C$, a contradiction. If we assume that P^{-1} and C are not disjoint, we can consider $g \in P^{-1} \cap C$, and then $g^{-1} \in P \cap C$, a contradiction. Therefore, P, P^{-1} and C are pairwise disjoint. The reciprocal is clear. □

We are now ready to present the announced fact.

Proposition 1.2.4. *Given $\preceq$ a left-preorder on G relative to C, the positive cone P of $\preceq$ satisfies both*

a) P is a subsemigroup of G.

b) $G = P \cup P^{-1} \cup C$.

c) P, P^{-1} and C are pairwise disjoint.

and we call P the positive cone of $\prec$. Reciprocally, if C is a proper subgroup of G and $P \subseteq G$ is a subset in such a way the last three conditions are satisfied, defining

$$gC \preceq hC \Leftrightarrow g^{-1}h \in P \cup C$$

give us a left-preorder on G relative to C.

Proof. We start by proving the first affirmation. Assume $\preceq$ is a left-preorder on G relative to C. We prove the Condition a). Consider $g, h \in P$. Since $C \prec hC$, we have $gC \prec ghC$, so then $C \prec gC \prec ghC$. This shows $gh \in P$. Hence, the Condition a) holds.

Secondly, Condition b) holds. Indeed, consider $g \in G$, so by totality $C \preceq gC$ or $gC \preceq C$. Then, $g \in P \cup P^{-1} \cup C$, and hence $G = P \cup P^{-1} \cup C$.

Thirdly, in order to prove Condition c), by Lemma 1.2.3 it is enough to show that $P \cap C = \emptyset$. This is clear since $g \in P \cap C$ implies $C \prec gC = C$, a contradiction. Hence, we have shown the first affirmation.

We now prove the second statement. Assume that P satisfies the conditions a), b) and c). We claim that

$$(P \cup C)(P \cup C) \subseteq P \cup C. \tag{1.1}$$

Notice that $(P \cup C)(P \cup C)$ is contained in $PP \cup PC \cup CP \cup CC$. Then, applying Lemma 1.2.2 and that P is a subsemigroup and C is a subgroup, we deduce Equation (1.1), as we claimed.

Consider $\preceq$ defined by

$$gC \preceq hC \Leftrightarrow g^{-1}h \in P \cup C.$$

We claim that $\preceq$ is well defined. Indeed, assume that $gC \preceq hC$ and $f, t \in C$. From $gC \preceq hC$ we deduce that $g^{-1}h \in P \cup C$, but since $f, t^{-1} \in C$, applying Equation (1.1) two times, $f^{-1}g^{-1}ht \in P \cup C$. Hence, $gfC \preceq htC$. This shows that $\preceq$ is well defined.

The second statement follows if we prove that $\preceq$ is an order on G/C. We do it by checking that it is transitive, antisymmetric, reflexive, left-invariant and total.

We prove that $\preceq$ is transitive. Indeed, assume

$$gC \preceq hC \text{ and } hC \preceq fC,$$

so then $g^{-1}h, h^{-1}f \in P \cup C$. Using Equation (1.1), we obtain $g^{-1}f \in P \cup C$. Hence

$$gC \preceq fC,$$

so $\preceq$ is transitive, as we claimed.

We claim that $\preceq$ is antisymmetric. Suppose that $gC \preceq hC$ and $hC \preceq gC$. By definition, we have $g^{-1}h, h^{-1}g \in P \cup C$. If $gC \neq hC$, we obtain $g^{-1}h, h^{-1}g \in P$, so Condition a) impies $1 \in P$. This contradicts Condition c), so we have proven that $\preceq$ is antisymmetric, as we claimed.

We show that $\preceq$ is reflexive. Indeed, take $gC \in G/C$. Clearly $g^{-1}g = 1$ is in $P \cup C$, so then $gC \preceq gC$. This proves that $\preceq$ is reflexive, as we claimed.

We prove that $\preceq$ is left-invariant. Indeed, take $f \in G$ and assume that

$$gC \preceq hC.$$

By definition, we have $g^{-1}h \in P \cup C$, so then $g^{-1}f^{-1}fh \in P \cup C$. By definition,

$$fgC \preceq fhC.$$

Hence, $\preceq$ is left-invariant as we wanted.

We show that $\preceq$ is total. Indeed, take $gC, hC \in G/C$. By Condition b), we have that $g^{-1}h \in P \cup P^{-1} \cup C$. Then, $g^{-1}h \in P \cup C$ or $g^{-1}h \in P^{-1} \cup C$. By definition of $\preceq$, $gC \preceq hC$ or $hC \preceq gC$. This shows that $\preceq$ is total. □

We state an elementary consequence that we will apply later many times.

Corollary 1.2.5. *Let $\preceq$ be a left-preorder on G relative to C with positive cone P. Then, we have*

$$CP \subseteq P \quad \text{and} \quad PC \subseteq P.$$

Proof. Follows by Proposition 1.2.4 and Lemma 1.2.2. □

Also, as a consequence of the previous proposition, we have the following.

Remark 1.2.6. Assume that $\preceq$ is a left-preorder on G relative to C having positive cone P. We have that P generates G. Indeed, take $x \in G$. Since P is the positive cone of $\preceq$, we have $x \in P$, $x \in P^{-1}$ or $x \in C$ (by the previous proposition). If we are in the case $x \in P$ or in the case $x \in P^{-1}$, then we have $x \in \langle P \rangle$. If we are in the case $x \in C$, by taking an auxiliary element $y \in P$, we have $xy \in CP$. In particular, $x \in \langle P \rangle$. This proves the claim.

1.3 Convexity and constructions of left-preorders

We now define convexity, which will be necessary to define the concept of conradian left-preorder. Also, in this section we use convexity to establish canonical bijections between certain sets of left preorders that we will need later. The notion of convexity is quite classical on the context of (pre)ordered sets. However, we provide a definition that is adapted to the context of left-preorders. For a discussion about convexity for left-orders see Section 2.2 of [11]. In that reference, it is exhibited the relation between convexity and the notion of left-relatively convex subgroup (here it is called relatively-convex subgroup), which justifies the nomenclature.

Notation 2. During all the section, G will denote a group and C will denote a proper subgroup

Definition 1.3.1. Let $\preceq$ be a left-preorder on G relative to C. A subgroup H of G is $\preceq$-convex if for each $g, h, f \in G$ such that $gC \preceq hC \preceq fC$ and such that $g, f \in H$, then $h \in H$. In particular, $C \subseteq H$.

Proposition 1.3.2. *Let $\preceq$ be a left-preorder on G relative to C. Then, the set of $\preceq$-convex subgroups is totally ordered by the inclusion.*

Proof. We are going to prove it by contradiction, so assume that the set of $\preceq$-convex subgroups is not totally ordered by the inclusion. Then, there are H and K two $\preceq$-convex subgroups such that $K \not\subseteq H$ and $H \not\subseteq K$. Then, we can take an element $g \in H$ such that $g \notin K$ and an element $h \in H$ such that $h \notin K$. By possibly taking inverses, we can assume that $C \preceq gC$ and $C \preceq hC$. Since $\preceq$ is total, we have $gC \preceq hC$ or $hC \preceq gC$. Assume that we are in the case $gC \preceq hC$, as the other case is analogous. Then, we have

$$C \preceq gC \preceq hC.$$

Then, applying that $1, h \in K$ and that K is a $\preceq$-convex subgroup, we deduce that $g \in K$. This contradicts the fact that $g \notin K$. □

We define a bijection that involves left-preorders using convexity, which will be useful later. This bijection $\mu_{G,C,H}$ will be defined between the set $\mathcal{PO}_C(H) \times \mathcal{PO}_H(G)$ and the set $\{\preceq \in \mathcal{PO}_C(G) \ / \ H \text{ is } \preceq -\text{convex in } G\}$, where G is a group and H, C are subgroups satisfying that $C \subsetneq H \subsetneq G$. Moreover, we will see that this bijection preserves convexity in some sense. Its corresponding inverse will be $\rho_{G,C,H}$. We start by defining $\mu_{G,C,H}$. This is an adaption to left-preorders of known facts on left-orders (see Remark 2.1.5 of [17]).

Notation 3. Let A_1, A_2, B be three sets. We write $B = A_1 \sqcup A_2$ if $B = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$.

Next objective is to prove the following result, that will be proved later. We state here in order to keep in mind the objective.

Theorem (Theorem 1.3.11)**.** *Let H be a proper subgroup of G such that $C \subsetneq H$. Consider the following maps described, identifying each left-preorder with its positive cone, by*

$$\begin{array}{rccc} \mu_{G,C,H} \colon & \mathcal{PO}_C(H) \times \mathcal{PO}_H(G) & \longrightarrow & \{\preceq \in \mathcal{PO}_C(G) \ / \ H \text{ is } \preceq -\text{convex in } G\} \\ & (P_1, P_2) & \mapsto & P_1 \cup P_2 \end{array}$$

$$\begin{array}{rcl}\rho_{G,C,H}\colon \{\preceq \in \mathcal{PO}_C(G) \;/\; H \text{ is } \preceq\text{-convex in } G\} & \longrightarrow & \mathcal{PO}_C(H)\times\mathcal{PO}_H(G)\\ P & \mapsto & (P\cap H, P-H)\end{array}$$

Then, these two maps $\mu_{G,C,H}$ and $\rho_{G,C,H}$ are bijective and $\mu_{G,C,H}=(\rho_{G,C,H})^{-1}$.

Before proving that, we first need to show that the maps are well defined. We prove it by splitting it in few results.

Proposition 1.3.3. *Let H be a proper subgroup of G such that $C\subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Then, the set $P_1\cup P_2$ defines a positive cone of a left-preorder on G relative to C, which we denote by $\mu_{G,C,H}(\preceq_1,\preceq_2)$.*

Proof. By Proposition 1.2.4, in order to prove that $P_1\cup P_2$ defines a positive cone of a left-preorder on G relative to C, we need to prove that $P_1\cup P_2$ is a subsemigroup of G and that $G=(P_1\sqcup P_2)\sqcup C\sqcup(P_1\sqcup P_2)^{-1}$. We are going to prove that each condition hold one by one.

Firstly, we prove that $P_1\cup P_2$ is a subsemigroup of G. Since $\preceq_2$ is a left-preorder on G relative to H with positive cone P_2, Corollary 1.2.5 says that HP_2 and P_2H are both contained in P_2. In particular, since $P_1\subseteq H$ (P_1 is the positive cone of a left-preorder on H), we deduce that P_1P_2 and P_2P_1 are both contained in P_2. Since P_1 and P_2 are semigroups (Corollary 1.2.5), we also deduce that P_1P_1 is contained in P_1 and P_2P_2 is contained in P_2. In particular, we have that P_1P_2, P_2P_1, P_1P_1 and P_2P_2 are all contained in $P_1\cup P_2$. Hence,

$$(P_1\cup P_2)(P_1\cup P_2)\subseteq P_1P_1\cup P_2P_2\cup P_1P_2\cup P_2P_1\subseteq P_1\cup P_2,$$

so then $P_1\cup P_2$ is a subsemigroup of G, as we wanted.

Secondly, we show that $G=(P_1\sqcup P_2)\sqcup C\sqcup(P_1\sqcup P_2)^{-1}$. By the definition of P_1 and P_2 as positive cones, we have

$$H=P_1\sqcup C\sqcup P_1^{-1} \text{ and } G=P_2\sqcup H\sqcup P_2^{-1}.$$

Hence, we have

$$G=P_2\sqcup H\sqcup P_2^{-1}=P_2\sqcup P_1\sqcup C\sqcup P_1^{-1}\sqcup P_2^{-1}=(P_1\sqcup P_2)\sqcup C\sqcup(P_1\sqcup P_2)^{-1}.$$

□

Definition 1.3.4. Let H be a proper subgroup of G such that $C\subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. We define $\mu_{G,C,H}(\preceq_1,\preceq_2)$ to be the left-preorder on G relative to C with positive cone $P_1\cup P_2$.

We now study the relation between $\mu_{G,C,H}$ and convexity, which will be split in few results.

Proposition 1.3.5. *Let H be a proper subgroup of G such that $C\subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq=\mu_{G,C,H}(\preceq_1,\preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1\cup P_2$. Let K be a subgroup of G such that $H\subseteq K$. Then, K is $\preceq$-convex if and only if K is $\preceq_2$-convex.*

Proof. Suppose that K is $\preceq$-convex. We are going to prove that K is $\preceq_2$-convex. Indeed, consider $g,f\in K$ and $h\in G$ such that

$$gH\preceq_2 hH\preceq_2 fH.$$

Then we have three cases: $gH=hH$, $hH=fH$ or $gH\prec_2 hH\prec_2 fH$. We are going to prove that, in all this cases, we have $h\in K$.

If we are in the case $gH=hH$, then $g^{-1}h$ is in $H\subseteq K$, and since $g\in K$, we obtain $h\in K$. The case $hH=fH$ is analogous, so we also obtain $h\in K$.

Assume now we are in the case $gH \prec_2 hH \prec_2 fH$. As P_2 defines the positive cone of $\preceq_2$, we deduce that $g^{-1}h$ and $h^{-1}f$ are both in P_2. Since $P_1 \cup P_2$ is the positive cone of $\preceq$, we deduce

$$gC \prec hC \prec fC.$$

Using that K is $\preceq$-convex by hypothesis, we deduce that $h \in K$. Therefore, we have $h \in K$ in the three cases, so we conclude that K is $\preceq_2$-convex.

Reciprocally, assume that K is $\preceq_2$-convex. We are going to prove that that K is $\preceq$-convex. Indeed, take $g, f \in K$ and $h \in G$ such that

$$gC \preceq hC \preceq fC.$$

From last equation, since $P_1 \cup P_2$ defines the positive cone of $\preceq$, we obtain $g^{-1}h, h^{-1}f \in P_1 \cup P_2 \cup C$. Then, noticing that $P_1 \cup C \subseteq H$, we deduce that $g^{-1}h, h^{-1}f \in P_2 \cup H$. Since P_2 is the positive cone of the left-preorder $\preceq_2$ relative to H, we deduce that

$$gH \preceq_2 hH \preceq_2 fH.$$

Using that K is $\preceq_2$-convex by hypothesis and that $g, f \in K$, we obtain that $h \in H$. We conclude that K is a $\preceq$-convex subgroup, as we wanted. □

Corollary 1.3.6. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Then, H is $\preceq$-convex.*

Proof. Follows directly by Proposition 1.3.5 and by the fact that H is clearly $\preceq_2$-convex. □

Notation 4. Let $\preceq$ be a left-preorder on G relative to C. Let H be a subgroup of G. We denote by $\preceq_{|H}$ the restriction of $\preceq$ to H. The relation $\preceq_{|H}$ on H is a left-preorder on G relative to $C \cap H$. Equivalently, as in Lemma 1.1.6, when seeing $\preceq$ as a relation on G/C, $\preceq_{|H}$ corresponds to the natural order induced by $\preceq$ on $H/(C \cap H)$.

Lemma 1.3.7. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Then, $\preceq_1$ is equal to the restriction $\preceq_{|H}$.*

Proof. We only need to prove that, for every $g, h \in H$, we have $gC \preceq_1 hC$ if and only if $gC \preceq hC$. We prove it with a chain of equivalences. Take $g, h \in H$. The fact $gC \preceq_1 hC$ is equivalent to $g^{-1}h \in P_1 \cup C$. Since $H \cap P_2 = \emptyset$, the previous fact is equivalent to $g^{-1}h \in P_1 \cup P_2 \cup C$, and this is the same as saying $gC \preceq hC$. □

Proposition 1.3.8. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Let L be a subgroup of G such that $L \subseteq H$. Then, L is $\preceq$-convex if and only if L is $\preceq_1$-convex.*

Proof. The fact that being $\preceq$-convex implies being $\preceq_1$-convex is a direct consequence of the fact that, by Lemma 1.3.7, $\preceq_1$ is equal to the restriction $\preceq_{|H}$. Reciprocally, suppose that L is $\preceq_1$-convex. We are going to prove that that L is $\preceq$-convex. In order to do this, take $g, f \in L$ and $h \in G$ such that

$$gC \preceq hC \preceq fC.$$

By Corollary 1.3.6, H is $\preceq$-convex, so g, f in $L \subseteq H$ implies $h \in H$. Hence, we have

$$gC \preceq_{|H} hC \preceq_{|H} fC.$$

By Lemma 1.3.7 we have that $\preceq_1$ is equal to $\preceq_{|H}$, so applying that L is $\preceq_1$-convex by hypothesis, we obtain $h \in L$. We conclude that L is $\preceq$-convex. □

We define $\rho_{G,C,H}$.

Proposition 1.3.9. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq$ be a left-preorder on G relative to C with positive cone P. We have the following:*

- *The set $P \cap H$ defines a positive cone of a left-preorder $\preceq_1$ on H relative to C.*
- *If H is $\preceq$-convex, then the set $P - H$ defines a positive cone of a left-preorder $\preceq_2$ on G relative to H.*

When H is $\preceq$-convex, we denote $\rho_{G,C,H}(\preceq) = (\preceq_1, \preceq_2)$.

Proof. We are going to prove each of the two conditions of the statement separately. We start by proving the first condition, that is, we want to show that $P \cap H$ defines a positive cone of a left-preorder $\preceq_1$ on H relative to C. By Proposition 1.2.4, we only need to show that $P \cap H$ is a subsemigroup of G and that $H = (P \cap H) \sqcup C \sqcup (P \cap H)^{-1}$. This is direct by the fact that H is a subgroup containing C and by the fact that, by applying Proposition 1.2.4 to $\preceq$ with positive cone P, we know that P is a subsemigroup of G and that $G = P \sqcup C \sqcup P^{-1}$. Therefore, $P \cap H$ defines a positive cone of a left-preorder $\preceq_1$ on H relative to C, as we wanted to prove.

We continue by proving the second condition, that is, we want to prove that, if H is $\preceq$-convex, then $P - H$ defines a positive cone of a left-preorder $\preceq_2$ on G relative to H. Assume that H is $\preceq$-convex. In order to achieve this, by Proposition 1.2.4, we only need to show that $P - H$ is a subsemigroup of G and that $G = (P - H) \sqcup H \sqcup (P - H)^{-1}$. The fact that $P - H$, $(P - H)^{-1}$ and H are pairwise disjoint is direct. The fact

$$G = (P - H) \cup H \cup (P - H)^{-1}$$

follows by

$$G = P \cup C \cup P^{-1} \subseteq P \cup H \cup P^{-1} = (P - H) \cup H \cup (P - H)^{-1},$$

as the other content is trivial. The last condition we need to check is that $P - H$ is a subsemigroup. We are going to prove it. Take $g, h \in P - H$. We want to prove that $gh \in P - H$. In order to do that, we are going to show that $gh \in P$ but $gh \notin H$. The fact that

$$gh \in P$$

follows directly by the fact that P is a semigroup. Hence, we only need to prove that $gh \notin H$. Since P is the positive cone of $\preceq$ and since $g, h \in P - H$, we deduce

$$C \prec gC \quad \text{and} \quad C \prec hC.$$

Then, we obtain

$$C \prec gC \prec ghC.$$

Applying the fact that H is $\preceq$-convex by hypothesis, $1 \in H$ and $g \notin H$, we obtain $gh \in H$. Therefore, we have checked that $P - H$ is a subsemigroup. Hence, we have proved all the needed conditions to show that $P - H$ defines a positive cone of a left-preorder $\preceq_2$ on G relative to H, as we wanted. □

Definition 1.3.10. Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq$ be a left-preorder on H relative to C with positive cone P. Suppose that H is $\preceq$-convex. We define $\rho_{G,C,H}(\preceq) = (\preceq_1, \preceq_2)$ where $\preceq_1$ is the left-preorder on H relative to C with positive cone $P \cap H$ and $\preceq_2$ is the left-preorder on G relative to H with positive cone $P - H$.

We are now ready to state and prove the announced result.

Theorem 1.3.11. *Let H be a proper subgroup of G such that $C \subsetneq H$. Consider the following maps described, identifying each left-preorder with its positive cone, by*

$$\begin{array}{rccc} \mu_{G,C,H} \colon & \mathcal{PO}_C(H) \times \mathcal{PO}_H(G) & \longrightarrow & \{\preceq \in \mathcal{PO}_C(G) \ / \ H \text{ is } \preceq\text{-convex in } G\} \\ & (P_1, P_2) & \mapsto & P_1 \cup P_2 \end{array}$$

$$\begin{array}{rccc} \rho_{G,C,H} \colon & \{\preceq \in \mathcal{PO}_C(G) \ / \ H \text{ is } \preceq\text{-convex in } G\} & \longrightarrow & \mathcal{PO}_C(H) \times \mathcal{PO}_H(G) \\ & P & \mapsto & (P \cap H, P - H) \end{array}$$

Then, these two maps $\mu_{G,C,H}$ and $\rho_{G,C,H}$ are bijective and $\mu_{G,C,H} = (\rho_{G,C,H})^{-1}$.

Proof. The maps $\mu_{G,C,H}$ is well defined by Proposition 1.3.3, and $\rho_{G,C,H}$ is well defined by Proposition 1.3.9. In order to prove that $\mu_{G,C,H}$ and $\rho_{G,C,H}$ are bijective and $\mu_{G,C,H} = (\rho_{G,C,H})^{-1}$, we only need to show that $\mu_{G,C,H} \circ \rho_{G,C,H}$ and $\rho_{G,C,H} \circ \mu_{G,C,H}$ are identity maps on the corresponding sets. We are going to prove it.

We first prove that $\mu_{G,C,H} \circ \rho_{G,C,H}$ is the identity map. Indeed, given P the positive cone of an element in $\mathcal{PO}_C(G)$ having H as a convex subgroup, we have

$$(\mu_{G,C,H} \circ \rho_{G,C,H})(P) = \mu_{G,C,H}(P \cap H, P - H) = (P \cap H) \cup (P - H) = P.$$

Therefore, $\mu_{G,C,H} \circ \rho_{G,C,H}$ is the identity map.

We lastly prove that $\rho_{G,C,H} \circ \mu_{G,C,H}$ is the identity map. Indeed, take P_1 the positive cone of an element in $\mathcal{PO}_C(H)$ and P_2 the positive cone of an element in $\mathcal{PO}_H(G)$. Notice that $P_1 \subseteq H$ and $P_2 \cap H = \emptyset$. Using this and following the definitions we have

$$(\rho_{G,C,H} \circ \mu_{G,C,H})(P_1, P_2) = \rho_{G,C,H}(P_1 \cup P_2) = ((P_1 \cup P_2) \cap H, (P_1 \cup P_2) - H) = (P_1, P_2).$$

Therefore, $\rho_{G,C,H} \circ \mu_{G,C,H}$ is the identity map. □

We can apply these bijections to consider extensions of left-preorders. This generalizes to left-preorders context what it is called the convex extension procedure in Section 2.1.1 of [17].

Proposition 1.3.12. *Let $\hat{C}$ be a proper subgroup of G such that $C \subseteq \hat{C}$. Let $\preceq$ be a left-preorder on G relative to $\hat{C}$. If C is left-relatively convex on $\hat{C}$, there exists $\preceq'$ a left-preorder on G relative to C extending $\preceq$, meaning that $g_1\hat{C} \prec g_2\hat{C}$ implies $g_1C \prec' g_2C$.*

Proof. Assume that C is left-relatively convex on $\hat{C}$. Consider P the positive cone of $\preceq$. Since C is left-relatively convex on $\hat{C}$ by hypothesis, there is $\preceq^*$ a left-preorder on $\hat{C}$ relative to C, and denote by Q its positive cone. By Proposition 1.3.3, we can consider $\preceq' = \mu_{G,C,\hat{C}}(\preceq^*, \preceq)$, which is a left-preorder on G relative to C whose positive cone is $Q \cup P$. We finish the proof if we show that $g_1\hat{C} \prec g_2\hat{C}$ implies that $g_1C \prec' g_2C$ for all $g_1, g_2 \in G$. Indeed, consider $g_1, g_2 \in G$ such that $g_1\hat{C} \prec g_2\hat{C}$. Since P is the positive cone of $\preceq$, we deduce that $g_1^{-1}g_2 \in P$. Then, $g_1^{-1}g_2 \in Q \cup P$. Hence, using the fact that $Q \cup P$ is the positive cone of $\preceq'$, we deduce that $g_1C \prec' g_2C$, as we wanted. □

1.4 Conradian left-preorders

We now present the definition of conradian left-preorder. Notice that, for $C = \{1\}$, we recover the classical definition of conradian left-order. For a discussion of conradian left-orders, see Chapter 9 of [11].

Definition 1.4.1. A left-preorder $\preceq$ on G relative to C is conradian if for all $g, h \in G$ such that $C \prec gC$ and $C \prec hC$ there is $n \in \mathbb{N}$ satisfying $gC \prec hg^nC$. We define $\mathcal{CO}_C(G)$ to be the set of conradian left-preorders on G relative to C and $\mathcal{CO}(G)$ to be the set of conradian left-preorders on G.

There are some classical definitions of conradian left-order where it is assumed that $n \neq 0$. In our definition of conradian left-preorder we allow to be $n = 0$ as both definitions are equivalent, as one can easily notice.

Given G a group, if G is abelian, then every left-preorder on G is conradian. Also, the restriction of a conradian left-preorder to a non-trivial subgroup is again a conradian left-preorder.

Remark 1.4.2. Let G be a group and C be a proper normal subgroup. By Remark 1.1.9, we can identify left-orders on G/C with left-preorders on G relative to C. In fact, by definition of being conradian, we can identify conradian left-orders on G/C with conradian left-preorders on G relative to C.

We now want an equivalent definition of conradian left-preorder using convex jumps. We state the theorem with the equivalence, we need some auxiliary results.

Definition 1.4.3. Let $\preceq$ be a left-preorder on G relative to C. A $\preceq$-convex jump is a pair (K, L) of $\preceq$-convex subgroups of G such that $K \subsetneq L$ and such that there is no $\preceq$-convex subgroup strictly between them.

Proposition 1.4.4. *Let $\preceq$ be a left-preorder on G relative to C. Consider $g \in G$. Define*

$$K_g = \bigcup_{g \notin K \text{ convex}} K \quad \text{and} \quad L_g = \bigcap_{g \in L \text{ convex}} L.$$

Then, K_g and L_g are $\preceq$-convex subgroups and (K_g, L_g) is a $\preceq$-convex jump.

Proof. We start by proving that K_g is a $\preceq$-convex subgroup. It is a subgroup since it is defined as an union of a linearly ordered family of subgroups, as a consequence of Proposition 1.3.2. We now check that it is $\preceq$-convex. Indeed, take $h_1, h_2 \in K_g$ and $f \in G$ satisfying that

$$h_1 C \preceq fC \preceq h_2 C. \tag{1.2}$$

Since $h_1, h_2 \in K_g$, using the definition and Proposition 1.3.2 we deduce that there is a $\preceq$-convex subgroup K such that $g \notin K$ and satisfying that $h_1, h_2 \in K$. Applying Equation (1.2) knowing that K is $\preceq$-convex and that $h_1, h_2 \in K$, we deduce that $f \in K$. Then, we have $f \in K_g$. This shows that K_g is a $\preceq$-convex subgroup, as we wanted.

We now prove that L_g is a $\preceq$-convex subgroup. It is a subgroup since it is defined as an intersection of a (non empty) family of subgroups. We now show that it is $\preceq$-convex. In order to do that, take $h_1, h_2 \in L_g$ and $f \in G$ satisfying that

$$h_1 C \preceq fC \preceq h_2 C. \tag{1.3}$$

Consider L an arbitrary $\preceq$-convex subgroup such that $g \in L$. Since $h_1, h_2 \in L_g$, we deduce that $h_1, h_2 \in L$ by definition of L_g. Combining Equation (1.2) with the fact that L is $\preceq$-convex and that $h_1, h_2 \in L$, we deduce that $f \in L$. Hence, we have shown that $f \in L$ for L an arbitrary $\preceq$-convex subgroup satisfying that $g \in L$, so we conclude that $f \in L_g$. This proves that L_g is a $\preceq$-convex subgroup, as we wanted.

We finally prove that (K_g, L_g) is a $\preceq$-convex jump. Since we know that K_g and L_g are $\preceq$-convex subgroups and since $g \notin K_g$ and $g \in L_g$, applying that the set of $\preceq$-convex subgroups is totally ordered by the inclusion (done in Proposition 1.3.2), we obtain that

$$K_g \subsetneq L_g.$$

Hence, in order to show that (K_g, L_g) is a $\preceq$-convex jump we only need to show that there is no $\preceq$-convex subgroup H strictly between them. However, this is clear since, by definition, we have $H \subseteq K_g$ if $g \notin H$ and $L_g \subseteq H$ if $g \in H$. We conclude that (K_g, L_g) is a $\preceq$-convex jump. □

As a corollary, for finitely generated groups, we can always consider a maximal proper $\preceq$-subgroup for each $\preceq$ conradian.

Corollary 1.4.5. *Let $\preceq$ be a conradian left-preorder on G relative to C. If G is finitely generated, then there exists a maximal proper $\preceq$-convex subgroup.*

Proof. Assume that G is finitely generated, say $G = \langle g_1, \dots, g_k \rangle$ for $k \in \mathbb{N}_{>0}$. By possibly taking inverses, we can assume that $C \preceq g_i C$ for all $i = 1, \dots, k$. Moreover, by possibly changing the indexes, we can assume that

$$C \preceq g_1 C \preceq \dots \preceq g_k C. \tag{1.4}$$

Define

$$K = \bigcup_{g_k \notin K \text{ convex}} K \quad \text{and} \quad L = \bigcap_{g_k \in L \text{ convex}} L.$$

By Proposition 1.4.4 we know that (K, L) is a $\preceq$-convex jump. This means that K and L are $\preceq$-convex subgroups such that $K \subsetneq L$ and such that there is no $\preceq$-convex subgroups strictly between them. In particular, if we prove that $L = G$, we have that a maximal proper $\preceq$-convex subgroup, as we wanted. Hence, we only need to show that $L = G$. We prove this by proving that $g_i \in L$ for $i = 1, \dots, k$.
Indeed, take $i \in \{1, \dots, k\}$. Notice that $g_k \in L$ by definition of L. From Equation (1.4) we have

$$C \preceq g_i C \preceq g_k C.$$

This, combined with the fact that $g_k \in L$ and that L is $\preceq$-convex, implies that $g_i \in L$, as we wanted. □

The condition of being finitely generated in the previous corollary cannot be avoided, as we see in next example.

Example 1.4.6. *Consider the group $G = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$. We can write $G = \bigoplus_{i\in\mathbb{N}} \langle x_i\rangle$ where $\langle x_i\rangle$ is isomorphic to $\mathbb{Z}$. Define*

$$(x_i^{j_i}) \preceq (x_i^{k_i}) \quad \Longleftrightarrow \quad j_{i_0} < k_{i_0} \text{ for } i_0 \text{ the minimum such that } j_{i_0} \neq k_{i_0}.$$

It is routine to check that $\preceq$ is a left-order on G. In other words, $\preceq$ is a left-preorder on G relative to $\{1\}$. Since G is abelian, we have that $\preceq$ is conradian. One can check that the set of $\preceq$ convex subgroups is

$$\{\langle x_1, \dots, x_k\rangle \ / \ k \in \mathbb{N}\} \cup G.$$

Clearly, there is no maximal proper $\preceq$-convex subgroup.

From now, we are going to adapt some classical ideas from the theory of orderable groups (we follow Chapter 9 of [11] as a reference). The following lemma follows the discussion on Lemma 9.7 from [11].

Lemma 1.4.7. *Let $\preceq$ be a left-preorder on G relative to C. Suppose that there exists H a maximal $\preceq$-convex proper subgroup of G. If $\preceq$ is conradian, then for every $g, h \in G - H$ such that $C \prec gC$ and $C \prec hC$ there exists $n \in \mathbb{N}$ such that $hC \prec g^n C$.*

Proof. Assume that $\preceq$ is conradian. Consider $g \in G - H$ such that $C \prec gC$. Define

$$S_0 = \{t \in G \ / \ \exists n \in \mathbb{N} \text{ such that } tC \prec g^n C\}.$$

Notice that the proof will conclude if we show that every $h \in G - H$ such that $C \prec hC$ satisfies $h \in S_0$. In order to do that, we are going to consider the action of G on the power set of G by left-multiplication and we are going to show that the stabilizer of S_0 is $\preceq$-convex properly containing H. Notice that, by definition of S_0, we have

$$gS_0 = S_0. \tag{1.5}$$

In other words, g stabilizes S_0 under the left-multiplication action.

We start by showing that

$$S_0 \subseteq yS_0 \text{ for all } y \in G \text{ such that } C \preceq yC, \tag{1.6}$$

as we will use this fact many times at the proof. Indeed, consider $y \in G$ such that $C \preceq yC$. Notice that $C \preceq g^m C$ for all $m \in \mathbb{N}$. Since $\preceq$ is conradian, for all $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that $g^m C \prec y(g^m)^{n_m} C$, and hence $y^{-1}g^m C \prec (g^m)^{n_m} C$. Now, if we take $x \in S_0$, by definition of S_0, there is $m \in \mathbb{N}$ such that $xC \prec g^m C$, so then $y^{-1}xC \prec y^{-1}g^m C$. Now, since we know that $y^{-1}g^m C \prec (g^m)^{n_m} C$, we deduce that $y^{-1}xC \prec (g^m)^{n_m} C$, and hence $y^{-1}x \in S_0$. Therefore, we have shown that $y^{-1}x \in S_0$ for all $x \in S_0$, so then $x = y(y^{-1}x) \in yS_0$. This proves Equation (1.6), as we wanted.

We consider $\mathrm{Stab}_G(S_0)$ the stabilizer of S_0 by the action on the power set of G by left multiplication. We are now going to prove that $\mathrm{Stab}_G(S_0)$ is a $\preceq$-convex subgroup. Indeed, consider $x, y, z \in G$ such that $xC \preceq yC \preceq zC$ and such that $x, z \in \mathrm{Stab}_G(S_0)$. From $xC \preceq yC$ we deduce that $C \preceq x^{-1}yC$. Then, using Equation (1.6) we deduce that $S_0 \subseteq (x^{-1}y)S_0$. From this we deduce $xS_0 \subseteq yS_0$, but $xS_0 = S_0$ since $x \in \mathrm{Stab}_G(S_0)$, so then $S_0 \subseteq yS_0$. Analogously, we can deduce that $yS_0 \subseteq S_0$ from the facts $yC \preceq zC$ and $z \in \mathrm{Stab}_G(S_0)$. Therefore, we obtain $yS_0 = S_0$, and hence $y \in \mathrm{Stab}_G(S_0)$. This shows that $\mathrm{Stab}_G(S_0)$ is a $\preceq$-convex subgroup, as we wanted.

We are now going to prove that $H \subseteq \mathrm{Stab}_G(S_0)$. Indeed, let $f \in H$. Recall that, at the start of the proof, we considered an element $g \in G - H$ such that $C \prec gC$. Since $f \in H$ and H is $\preceq$-convex, we deduce $C \preceq f^{\pm 1}gC$. Then, applying Equation (1.6), we obtain $S_0 \subseteq f^{\pm 1}gS_0$. Then, $f^{\pm 1}S_0 \subseteq gS_0$, and combining this with Equation (1.5), we deduce $f^{\pm 1}S_0 \subseteq S_0$. We have shown that $fS_0 \subseteq S_0$ and $f^{-1}S_0 \subseteq S_0$, or equivalently $fS_0 \subseteq S_0$ and $S_0 \subseteq fS_0$, so we deduce $S_0 = fS_0$. This shows that $f \in \mathrm{Stab}_G(S_0)$. Hence, we conclude that $H \subseteq \mathrm{Stab}_G(S_0)$, as we wanted.

We now focus on showing that $G = \mathrm{Stab}_G(S_0)$. At this point, we have shown that $\mathrm{Stab}_G(S_0)$ is a $\preceq$-convex subgroup and that $H \subseteq \mathrm{Stab}_G(S_0)$. Since H be a maximal $\preceq$-convex proper subgroup of G by hypothesis, if we show that $H \neq \mathrm{Stab}_G(S_0)$, we can conclude that $G = \mathrm{Stab}_G(S_0)$. Indeed, we know that $g \notin H$, but $g \in \mathrm{Stab}_G(S_0)$ as a consequence of Equation (1.5), so this shows that $H \neq \mathrm{Stab}_G(S_0)$. We conclude that $G = \mathrm{Stab}_G(S_0)$, as we wanted.

Finally, we are now able to complete the proof from the proven fact $G = \mathrm{Stab}_G(S_0)$. Take $h \in G - H$ such that $C \prec hC$. Since $G = \mathrm{Stab}_G(S_0)$, we deduce $h \in \mathrm{Stab}_G(S_0)$, so then $hS_0 = S_0$. Since $1C = C \prec hC$, we obtain $1 \in S_0$, so $h \in hS_0$, and hence $h \in S_0$. Therefore, by definition of S_0, we conclude that there exists $n \in \mathbb{N}$ such that $hC \prec g^nC$. Since the element g was fixed at the beginning as an arbitrary $g \in G - H$ such that $C \prec gC$, we have shown that for every $g, h \in G - H$ such that $C \prec gC$ and $C \prec hC$ there exists $n \in \mathbb{N}$ such that $hC \prec g^nC$. □

Corollary 1.4.8. *Let $\preceq$ be a left-preorder on G relative to C and let (K, L) be a $\preceq$-convex jump. If $\preceq$ is conradian, then for every $g, h \in L - K$ such that $C \prec gC$ and $C \prec hC$ there exists $n \in \mathbb{N}$ such that $hC \prec g^nC$.*

The following lemma is based on Lemma 9.8 from [11].

Lemma 1.4.9. *Let $\preceq$ be a left-preorder on G relative to C. If $\preceq$ is conradian and (K, L) is a $\preceq$-convex jump, then K is normal in L.*

Proof. We are going to show that $fKf^{-1} = K$ for all $f \in L - K$ such that $C \prec fC$, as for $fC \prec C$ follows by taking the inverse and for $f \in K$ is trivial.

We start by proving that $fKf^{-1} \subseteq K$ by contradiction. Indeed, suppose that there is $g \in K$ such that $fgf^{-1} \notin K$. In particular, $fgf^{-1} \notin C$ as $C \subseteq K$, so we deduce that $C \prec fgf^{-1}C$ or $fgf^{-1}C \prec C$. If we are in the case $C \prec fgf^{-1}C$, by Corollary 1.4.8 knowing that $C \prec fC$, we deduce that there exists $n \in \mathbb{N}$ such that $fC \prec (fgf^{-1})^nC$, so then $fC \prec fg^nf^{-1}C$, and hence $g^{-n}C \prec f^{-1}C$, but since $C \prec fC$, we obtain that

$$g^{-n}C \prec f^{-1}C \prec C,$$

but $g^{-n}, 1 \in K$ and K is $\preceq$-convex, so $f^{-1} \in K$, and then $f \in K$, a contradiction. If we are in the case $fgf^{-1}C \prec C$, then applying the previous case for $(fgf^{-1})^{-1} = fg^{-1}f^{-1}$ we also conclude $f \in K$, a contradiction. Therefore, we have shown that $fKf^{-1} \subseteq K$, as we wanted.

To finish the proof, we only need to show that $K \subseteq fKf^{-1}$. In order to do that, we claim that $f^{-1}tfC \prec fC$ for all $t \in K$. Consider $t \in K$. As we know that $f \in L - K$ and $C \prec fC$, applying that K is $\preceq$-convex, $K \subseteq L$ and $t \in K$, we deduce that $tC \prec fC$, so then $C \prec t^{-1}fC$. Applying Corollary 1.4.8 noticing that $t^{-1}f \in L - K$, we deduce that there is $n \in \mathbb{N}$ such that $fC \prec (t^{-1}f)^nC$. Then, $C \prec f^{-1}(t^{-1}f)^nC$, but we know that $C \prec fC$, so multiplying this two $\preceq$-positive elements we obtain $C \prec f^{-1}(t^{-1}f)^nfC$, that is, $C \prec (f^{-1}(t^{-1}f)f)^nC$. As $\preceq$-positivity is preserved by positive roots (note that $n > 0$), we obtain $C \prec f^{-1}(t^{-1}f)fC$. Left-multiplying by $(f^{-1}t^{-1}f)^{-1} = f^{-1}tf$, we obtain $f^{-1}tfC \prec fC$, as we claimed.

We are now able to prove that $K \subseteq fKf^{-1}$. Indeed, take $t \in K$. Then, we have three cases: $C \prec f^{-1}tfC$, $f^{-1}tfC \prec C$ or $f^{-1}tfC = C$. Assume that we are in the case $C \prec f^{-1}tfC$. As $t \in K$ implies $t^l \in K$ for all $l \in \mathbb{N}_{>0}$, and applying the conclusion of the previous paragraph we deduce that $f^{-1}t^lfC \prec fC$ for all $l \in \mathbb{N}_{>0}$. This is equivalent to $(f^{-1}tf)^lC \prec fC$ for all $l \in \mathbb{N}_{>0}$. Then, by $C \prec f^{-1}tfC$ and by Corollary 1.4.8, we deduce that $f^{-1}tf$ cannot be in $L - K$, so then $f^{-1}tf \in K$. Then $t \in fKf^{-1}$, as we wanted. If we are in the case $f^{-1}tfC \prec C$, we can just apply the previous case for $(f^{-1}tf)^{-1} = f^{-1}t^{-1}f$, so we obtain $t^{-1} \in fKf^{-1}$, and hence $t \in fKf^{-1}$. Finally, if we are in the case $f^{-1}tfC = C$, as $C \subseteq K$, we have $f^{-1}tf \in K$, and hence $t \in fKf^{-1}$. Therefore, we have shown that $K \subseteq fKf^{-1}$, as we needed. □

Notation 5. Let G be a group. Let $\preceq$ be an order on G. An injective map $\tau : G \longrightarrow \mathbb{R}$ is said to be $\preceq$-preserving if for all $g, h \in G$ we have

$$g \preceq h \iff \tau(g) \leq \tau(h).$$

As a consequence of total orders' properties and injectivity, this is equivalent to say that for all $g, h \in G$ we have

$$g \prec h \iff \tau(g) < \tau(h).$$

Notice that implicitly we are considering in $\mathbb{R}$ the usual order.

Also, we previously need the following well known theorem (for a reference see Theorem 2.6 of [11]).

Theorem 1.4.10 (Hölder's theorem)**.** *Let G be a group. Let $\preceq$ be a left-order on G. Assume that $\preceq$ is archimedean, meaning that for all $g, h \in G - \{1\}$ such that $1 \prec g$ and $1 \prec h$ there exists $n \in \mathbb{N}$ such that $h \prec g^n$. Then, there exists a $\preceq$-preserving injective group morphism $\tau : G \longrightarrow \mathbb{R}$.*

Before characterizing conradian left-preorders we need a definition and a remark.

Definition 1.4.11. Let $\preceq$ be a left-preorder on G relative to C. Let K and L be $\preceq$-convex subgroups of G such that $K \subsetneq L$. We define the left-preorder induced by $\preceq$ on L relative to K to be the left-preorder on L relative to K having positive cone $(P \cap L) - K$.

Remark 1.4.12. With the notation of the previous definition, notice that the notion of left-preorder induced by $\preceq$ on L relative to K is well defined. Indeed, applying statement 1 from Proposition 1.3.9, we deduce that $P \cap L$ defines the positive cone of a left-preorder on L relative to C, and applying statement 2 from Proposition 1.3.9, we conclude that $(P \cap L) - K$ defines the positive cone of a left-preorder induced by $\preceq$ on L relative to K.

In general, if we denote by $\hat{\preceq}$ left-preorder induced by $\preceq$ on L relative to K, then for each $g, h \in K$ we have

$$gK \hat{\prec} hK \implies gC \prec hC.$$

This is a direct consequence of the fact that $(P \cap L) - K \subseteq P$. In a equivalent way, using the properties of total orders, for each $g, h \in K$ we have

$$gC \preceq hC \implies gK \hat{\preceq} hK.$$

We can now state the theorem with the equivalent version of being a conradian left-preorder using convex jumps. We also prove that, in the definition of conradian left-preorder, one can take $n = 2$. This last fact was proven for conradian left-orders on [31] but using other kind of arguments. The theorem is proved using the ideas on Theorem 9.5 from [11]. Apart from these two characterizations of being conradian, we also provide another one in Theorem 5.2.2.

Theorem 1.4.13. *Let $\preceq$ be a left-preorder relative to C. Then, the following conditions are equivalent:*

a) For every (K, L) $\preceq$-convex jump, K is normal in L and there exists a $\hat{\preceq}$-preserving injective morphism $\tau_{K,L} : L/K \longrightarrow \mathbb{R}$ for $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K.

b) For all $g, h \in G$ such that $C \prec gC$ and $C \prec hC$ we have $gC \prec hg^2C$.

c) $\preceq$ is conradian.

Proof. We are going to prove that each condition implies the next one. We start by proving that a) implies b). Assume that for every (K, L) $\preceq$-convex jump, K is normal in L and there exists a $\hat{\preceq}$-preserving injective morphism $\tau_{K,L} : L/K \longrightarrow \mathbb{R}$ for $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K. Let $g, h \in G$ be such that $C \prec gC$ and $C \prec hC$. Define the subgroups

$$K_g = \bigcup_{g \notin K \text{ convex}} K \quad \text{and} \quad L_g = \bigcap_{g \in L \text{ convex}} L.$$

By Proposition 1.4.4, K_g and L_g are $\preceq$-convex subgroups and (K_g, L_g) is a $\preceq$-convex jump. One can analogously define K_h and L_h, so we also have that K_h and L_h are convex and (K_h, L_h) is a $\preceq$-convex jump. Using the hypothesis, K_g is normal in L_g, K_h is normal in L_h and we can consider the order preserving morphisms

$$\tau_g = \tau_{K_g, L_g} : L_g/K_g \longrightarrow \mathbb{R} \qquad \tau_h = \tau_{K_h, L_h} : L_h/K_h \longrightarrow \mathbb{R}.$$

Applying Proposition 1.3.2, we have that $K_g \subsetneq L_g \subseteq K_h \subsetneq L_h$, or that $K_h \subsetneq L_h \subseteq K_g \subsetneq L_g$ or that $K_g = K_h \subsetneq L_g = L_h$. Hence, we can consider (K, L) the $\preceq$-convex jump that is bigger or equal than the other, and $\tau_{K,L} : L/K \longrightarrow \mathbb{R}$ the corresponding $\hat{\preceq}$-preserving injective morphism given by the hypothesis, where $\hat{\preceq}$ is the left-preorder induced by $\preceq$ on L relative to K. Notice that $g, h \in L$. Since $C \prec gC$ and $C \prec hC$, by Remark 1.4.12, we deduce that $K \hat{\preceq} gK$ and $K \hat{\preceq} hK$. Then, as τ is order preserving, we deduce that $\tau(gK) \geq 0$ and $\tau(hK) \geq 0$. By definition, it is clear that $g \notin K_g$ and $h \notin K_h$, so then $g \notin K$ or $h \notin K$, and hence $\tau(gK) \neq 0$ or $\tau(hK) \neq 0$, but we know that $\tau(gK) \geq 0$ and $\tau(hK) \geq 0$, so then $\tau(gK) + \tau(hK) > 0$. Hence, we have

$$\tau(g^{-1}hg^2K) = -\tau(gK) + \tau(hK) + 2\tau(gK) = \tau(gK) + \tau(hK) > 0.$$

Then, using that τ is $\hat{\preceq}$-preserving, we obtain that

$$K \hat{\prec} g^{-1}hg^2K.$$

By Remark 1.4.12, we conclude that

$$C \prec g^{-1}hg^2C.$$

Therefore, we have

$$gC \prec hg^2C.$$

This shows that a) implies b).

Clearly b) implies c). We now prove that c) implies a). Assume that $\preceq$ is conradian. Let (K, L) be a $\preceq$-convex jump. By Lemma 1.4.9, we know that K is normal in L. Hence, we can consider the group L/K. We can consider $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K, which is in fact a left-order on the group L/K. We claim that $\hat{\preceq}$ is an archimedean left-order. Indeed, take $g, h \in L - K$ such that $K \hat{\prec} gK$ and $K \hat{\prec} hK$. By Remark 1.4.12, we have $C \prec gC$ and $C \prec hC$. Then, applying Corollary 1.4.8, we deduce that there exists $n \in \mathbb{N}$ such that $hC \prec g^nC$. Then, by Remark 1.4.12, we deduce that there exists $n \in \mathbb{N}$ such that $hK \hat{\preceq} g^nK$. From this by changing n by $n+1$, since we know that $K \hat{\prec} gK$, we deduce that there exists $n \in \mathbb{N}$ such that $hK \hat{\prec} g^nK$. This proves that $\hat{\preceq}$ is an archimedean left-order, as we claimed. Applying Theorem 1.4.10, there exists a $\hat{\preceq}$-preserving injective morphism $\tau_{K,L} : L/K \longrightarrow \mathbb{R}$, as we wanted. This proves that c) implies a). □

Apart from the previous characterization of being conradian, we will introduce a dynamical characterization in Section 5.2.

1.5 Constructions of left-preorders that preserves being conradian

In this section, we show that the functions $\mu_{G,C,H}$ and $\rho_{G,C,H}$ preserves being conradian using the characterization of being conradian in Theorem 1.4.13. In particular, we can restrict that functions so we obtain a bijection between certain sets of conradian left-preorders. We start proving that they preserve convex jumps.

Notation 6. During all the section, G will denote a group and C will denote a proper subgroup

Lemma 1.5.1. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let K, L be two subgroups of G. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Then, (K, L) is a $\preceq_i$-convex jump for some $i = 1, 2$ if and only if (K, L) is a $\preceq$-convex jump.*

Proof. We prove each implication. Firstly, assume that (K, L) a $\preceq_i$-convex jump for some $i = 1, 2$, and we are going to prove that (K, L) a $\preceq$-convex jump. Indeed, the fact that (K, L) is a $\preceq_i$-convex jump implies that K and L are $\preceq_i$-convex. Then, by Proposition 1.3.8 or Proposition 1.3.5, we deduce that K and L are $\preceq$-convex. Notice that there is not a $\preceq$-convex subgroup strictly between them K and L, as Proposition 1.3.8 or Proposition 1.3.5 would imply that there is a $\preceq_i$-convex subgroup strictly between K and L, a contradiction with the fact that (K, L) a $\preceq_i$-convex jump. Hence, we have shown that (K, L) is a $\preceq$-convex jump.

Secondly, suppose that (K, L) is a $\preceq$-convex jump, and we are going to show that (K, L) is a $\preceq_i$-convex jump for some $i = 1, 2$. Since (K, L) is a $\preceq$-convex jump, we know that there is no $\preceq$-convex subgroup strictly between K and L. Then, as H is $\preceq$-convex by Corollary 1.3.6, we deduce that

$$H \subseteq K \subsetneq L \quad \text{or} \quad K \subsetneq L \subseteq H.$$

If we are in the case

$$H \subseteq K \subsetneq L,$$

then Proposition 1.3.8 implies that K and L are $\preceq_1$-convex. Moreover, in this case (K, L) is a $\preceq_1$-convex jump as, if there is a $\preceq_1$-convex subgroup strictly between them, by Proposition 1.3.8 we would obtain a $\preceq$-convex subgroup strictly between them, which is impossible. If we are in the case

$$K \subsetneq L \subseteq H,$$

we can do an analogous argumentation as in the other case but using Proposition 1.3.5. Therefore, we have shown that (K, L) is a $\preceq_i$-convex jump for some $i = 1, 2$, as we wanted. □

We now state a technical lemma.

Lemma 1.5.2. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Let K and L be $\preceq$-convex subgroups of G such that $K \subsetneq L$.*

1. *If $K \subsetneq L \subseteq H$, then the left-preorder induced by $\preceq$ on L relative to K and the left-preorder induced by $\preceq_1$ on L relative to K are equal.*
2. *If $H \subseteq K \subsetneq L$, then the left-preorder induced by $\preceq$ on L relative to K and the left-preorder induced by $\preceq_2$ on L relative to K are equal.*

Proof. We are going to prove each affirmation. We start by proving the first one, so assume that $K \subseteq L \subseteq H$. We are going to prove that the left-preorder induced by $\preceq$ on L/K and the left-preorder induced by $\preceq_1$ on L/K are equal by showing that both left-preorders have the same positive cone. The left-preorder induced by $\preceq$ on L/K has positive cone

$$((P_1 \cup P_2) \cap L) - K.$$

The left-preorder induced by $\preceq_1$ on L/K has positive cone

$$(P_1 \cap L) - K.$$

Since $L \subseteq H$ and P_2 is the positive cone of a left-preorder on G relative to H, we deduce that $L \cap P_2 = \emptyset$. Then, this implies

$$((P_1 \cup P_2) \cap L) - K = (P_1 \cap L) - K.$$

Therefore, we have shown that the left-preorder induced by $\preceq$ on L/K and the left-preorder induced by $\preceq_1$ on L/K have the same positive cone, so then they are equal, as we wanted to prove.

We continue by proving the second affirmation, so suppose that $H \subseteq K \subseteq L$. We are going to show that the left-preorder induced by $\preceq$ on L/K and the left-preorder induced by $\preceq_2$ on L/K are equal by proving that they have the same positive cone. The positive cone of the left-preorder induced by $\preceq$ on L/K is given by

$$((P_1 \cup P_2) \cap L) - K.$$

The positive cone of the left-preorder induced by $\preceq_2$ on L/K is given by

$$(P_2 \cap L) - K.$$

Since $H \subseteq K$ and P_1 is the positive cone of a left-preorder on H relative to C, we deduce that $P_1 \subseteq K$. Then, this implies

$$((P_1 \cup P_2) \cap L) - K = ((P_1 \cup P_2) - K) \cap L = (P_2 - K) \cap L = (P_2 \cap L) - K.$$

Therefore, we have proven that the positive cone of the left-preorder induced by $\preceq$ on L/K and the positive cone of the left-preorder induced by $\preceq_2$ on L/K are equal, so then this two left-preorders are equal, as we needed to prove. □

We can prove that the bijection preserves being conradian.

Proposition 1.5.3. *Let H be a proper subgroup of G such that $C \subsetneq H$. Let $\preceq_1$ be a left-preorder on H relative to C with positive cone P_1 and let $\preceq_2$ be a left-preorder on G relative to H with positive cone P_2. Let $\preceq = \mu_{G,C,H}(\preceq_1, \preceq_2)$, which is a left-preorder on G relative to C with positive cone $P_1 \cup P_2$. Then, $\preceq_1$ and $\preceq_2$ are conradian if and only if $\preceq$ is conradian.*

Proof. We are going to prove each implication. Assume that $\preceq_1$ and $\preceq_2$ are conradian. We are going to prove that $\preceq$ is conradian. In order to do that, by Theorem 1.4.13, we want to prove that for every (K, L) $\preceq$-convex jump, K is normal in L and there exists a $\hat{\preceq}$-preserving injective morphism $\tau_{K,L} : L/K \longrightarrow \mathbb{R}$ where $\hat{\preceq}$ is the left-preorder induced by $\preceq$ on L relative to K. Take (K, L) a $\preceq$-convex jump. By Lemma 1.5.1, (K, L) is a $\preceq_i$-convex jump for $i = 1$ or $i = 2$. Since $\preceq_i$ is conradian by hypothesis and since we know that (K, L) is a $\preceq_i$-convex jump, using Theorem 1.4.13, we deduce that K is normal in L and there exists an $\hat{\preceq}_i$-preserving injective group morphism $\tau_{H,K} : L/K \longrightarrow \mathbb{R}$ where is $\hat{\preceq}_i$ is the left-preorder induced by $\preceq_i$ on L relative to K. Applying Lemma 1.5.2, we deduce that K is normal in L and there exists an $\hat{\preceq}$-preserving injective group morphism $\tau_{H,K} : L/K \longrightarrow \mathbb{R}$ for $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K, as we wanted. This proves that $\preceq$ is conradian.

Conversely, we now prove the other implication. Suppose that $\preceq$ is conradian. We are going to prove that $\preceq_1$ and $\preceq_2$ are conradian. In other words, if we take an arbitrary $i \in \{1, 2\}$, we are going to show that $\preceq_i$ is conradian. In order to deduce that, by Theorem 1.4.13, we just need to prove that for every (K, L) $\preceq_i$-convex jump, K is normal in L and there exists an $\hat{\preceq}_i$-preserving injective group morphism $\tau_{H,K} : L/K \longrightarrow \mathbb{R}$ where $\hat{\preceq}_i$ is the left-preorder induced by $\preceq_i$ on L relative to K. Indeed, take (K, L) a $\preceq_i$-convex jump. From Lemma 1.5.1, we deduce that (K, L) is a $\preceq$-convex jump. As $\preceq$ is conradian by hypothesis and (K, L) is a $\preceq$-convex jump, by Theorem 1.4.13, we deduce that K is normal in L and there exists an $\hat{\preceq}$-preserving injective group morphism $\tau_{H,K} : L/K \longrightarrow \mathbb{R}$ for $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K. Using Lemma 1.5.2, we deduce that, in fact, $\tau_{H,K} : L/K \longrightarrow \mathbb{R}$ is a $\hat{\preceq}_i$-preserving injective group morphism where is $\hat{\preceq}_i$ is the left-preorder induced by $\preceq_i$ on L relative to K. With this, we can conclude that $\preceq_i$ is conradian, as we wanted. □

We can now prove that $\mu_{G,C,H}$ and $\rho_{G,C,H}$ are bijective functions defined on sets consisting of conradian left-preorders.

Proposition 1.5.4. *Let H be a proper subgroup of G such that $C \subsetneq H$. Consider the following maps described, identifying each left-preorder with its positive cone, by*

$$\begin{array}{rccl} \mu_{G,C,H}\colon & \mathcal{CO}_C(H) \times \mathcal{CO}_H(G) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \ / \ H \text{ is } \preceq\text{-convex in } G\} \\ & (P_1, P_2) & \mapsto & P_1 \cup P_2 \end{array}$$

$$\begin{array}{rccl} \rho_{G,C,H}\colon & \{\preceq \in \mathcal{CO}_C(G) \ / \ H \text{ is } \preceq\text{-convex in } G\} & \longrightarrow & \mathcal{CO}_C(H) \times \mathcal{CO}_H(G) \\ & P & \mapsto & (P \cap H, P - H) \end{array}$$

Then, these two maps $\mu_{G,C,H}$ and $\rho_{G,C,H}$ are bijective and $\mu_{G,C,H} = (\rho_{G,C,H})^{-1}$.

Proof. Follows from Theorem 1.3.11 and Proposition 1.5.3. □

We can generalize the last proposition inductively. In fact, one can also generalize Theorem 1.3.11 in an analogous way, but we do not state it as we will not need it.

Proposition 1.5.5. *Let*

$$C = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_n = G$$

be a finite chain of proper subgroups of G for $n \in \mathbb{N}_{>0}$. Consider the following maps, described by identifying each left-preorder with its positive cone, which are defined by

$$\begin{array}{rccl} \mu\colon & \prod_{j=1}^{n} \mathcal{CO}_{G_{j-1}}(G_j) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \ / \ G_i \text{ are } \preceq\text{-convex for all } i\} \\ & (P_1, \ldots, P_n) & \mapsto & \bigcup_{j=1}^{n} P_j \end{array}$$

$$\begin{array}{rccl} \rho\colon & \{\preceq \in \mathcal{CO}_C(G) \ / \ G_i \text{ is } \preceq\text{-convex for all } i\} & \longrightarrow & \prod_{j=0}^{n-1} \mathcal{CO}_{G_{j-1}}(G_j) \\ & P & \mapsto & ((P \cap G_i) - G_{i-1})_{i=1}^{n} \end{array}$$

Then, these two maps μ and ρ are bijective and $\mu = \rho^{-1}$.

Proof. We are going to prove the statement by induction on n. For $n = 1$ is trivial, and for $n = 2$ is just Proposition 1.5.4. We now take $n > 2$ and we are going to prove the statement for n assuming that it is true for $n-1$ as inductive hypothesis. Applying the inductive hypothesis to the finite chain

$$C = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{n-1}$$

we obtain two bijective and mutually inverse maps

$$\begin{array}{rccl} \hat{\mu}_0 \colon & \prod_{j=1}^{n-1} \mathcal{CO}_{G_{j-1}}(G_j) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ are } \preceq\text{-convex for all } i < n\} \\ & (P_1, \dots, P_{n-1}) & \mapsto & \bigcup_{j=1}^{n-1} P_j \end{array}$$

$$\begin{array}{rccl} \hat{\rho}_0 \colon & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ is } \preceq\text{-convex for all } i < n\} & \longrightarrow & \prod_{j=0}^{n-1} \mathcal{CO}_{G_{j-1}}(G_j) \\ & P & \mapsto & ((P \cap G_i) - G_{i-1})_{i=1}^{n-1}. \end{array}$$

We can consider the Cartesian product of these maps with the identity on $\mathcal{CO}_{G_{n-1}}(G_n)$, so this give us two bijective and mutually inverse maps

$$\begin{array}{rccl} \hat{\mu} \colon & \prod_{j=1}^{n} \mathcal{CO}_{G_{j-1}}(G_j) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ are } \preceq\text{-convex for all } i < n\} \times \mathcal{CO}_{G_{n-1}}(G_n) \\ & (P_1, \dots, P_{n-1}, P_n) & \mapsto & (\bigcup_{j=1}^{n-1} P_j, P_n) \end{array}$$

$$\begin{array}{rccl} \hat{\rho} \colon & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ is } \preceq\text{-convex for all } i < n\} \times \mathcal{CO}_{G_{n-1}}(G_n) & \longrightarrow & \prod_{j=0}^{n} \mathcal{CO}_{G_{j-1}}(G_j) \\ & (P, Q) & \mapsto & \left(((P \cap G_i) - G_{i-1})_{i=1}^{n-1}, Q\right). \end{array}$$

By applying Proposition 1.5.4, we have two mutually inverse bijections

$$\begin{array}{rccl} \overline{\mu}_0 \colon & \mathcal{CO}_C(G_{n-1}) \times \mathcal{CO}_{G_{n-1}}(G_n) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G_n) \,/\, G_{n-1} \text{ is } \preceq\text{-convex in } G_n\} \\ & (\overline{P}, P_n) & \mapsto & \overline{P} \cup P_n \end{array}$$

$$\begin{array}{rccl} \overline{\rho}_0 \colon & \{\preceq \in \mathcal{CO}_C(G_n) \,/\, G_{n-1} \text{ is } \preceq\text{-convex in } G_n\} & \longrightarrow & \mathcal{CO}_C(G_{n-1}) \times \mathcal{CO}_{G_{n-1}}(G_n) \\ & P & \mapsto & (P \cap G_{n-1}, P - G_{n-1}). \end{array}$$

By Corollary 1.3.6, the following restrictions of the last mutually inverse bijections are well defined

$$\begin{array}{rccl} \overline{\mu} \colon & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ are } \preceq\text{-convex for all } i < n\} \times \mathcal{CO}_{G_{n-1}}(G_n) & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ is } \preceq\text{-convex for all } i\} \\ & (\overline{P}, P_n) & \mapsto & \overline{P} \cup P_n \end{array}$$

$$\begin{array}{rccl} \overline{\rho} \colon & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ is } \preceq\text{-convex for all } i\} & \longrightarrow & \{\preceq \in \mathcal{CO}_C(G) \,/\, G_i \text{ are } \preceq\text{-convex for all } i < n\} \times \mathcal{CO}_{G_{n-1}}(G_n) \\ & P & \mapsto & (P \cap G_{n-1}, P - G_{n-1}). \end{array}$$

Notice that the fact that $\overline{\mu}_0$ and $\overline{\rho}_0$ are mutually inverse bijections implies that $\overline{\mu}$ and $\overline{\rho}$ are mutually inverse bijections.

By following the definitions, we have

$$\mu = \overline{\mu} \circ \hat{\mu} \quad \text{and} \quad \rho = \hat{\rho} \circ \overline{\rho}.$$

Recall that we needed to prove that μ and ρ are bijective and $\mu = \rho^{-1}$. The fact that μ and ρ are bijective is direct from the fact that we have expressed them as a composition of two bijective functions. The fact that $\mu = \rho^{-1}$ follows by

$$\rho^{-1} = (\hat{\rho} \circ \overline{\rho})^{-1} = (\overline{\rho})^{-1} \circ (\hat{\rho})^{-1} = \overline{\mu} \circ \hat{\mu} = \mu.$$

Therefore, we have shown that the statement is true for n, and this finishes the induction. □

1.6 Spaces of left-preorders

In this section we introduce different subspaces of the space of left-preorders and we study some topological properties of them. The space of left-preorders was defined in [4] using the name *space of relative orders*. This space generalizes the space of left-orders introduced in [53]. In [50] it is defined the space of conradian left-orders as a subspace of the space of left-orders, so we do an analogous version to define the space of conradian left-preorders as a subspace of the space of left-preorders. We use some common ideas and techniques from [4], [44] and [53].

Notation 7. Given G a group and C a left-relatively convex subgroup, Lemma 1.2.3 provides us a way to identify left-preorders with elements of $\{\pm 1, 0\}^G$. Specifically, given $\preceq$ a left-preorder on G relative to C and denoting by P its positive cone, we can define a $\phi_\preceq \in \{\pm 1, 0\}^G$ by:

$$\begin{array}{rccl} \phi_\preceq \colon & G & \longrightarrow & \{\pm 1, 0\} \\ & g & \mapsto & \begin{cases} 1 & \text{if} \quad g \in P \\ -1 & \text{if} \quad g \in P^{-1} \\ 0 & \text{if} \quad g \in C \end{cases} . \end{array}$$

In particular, we can identify the set $\mathcal{PO}(G)$ of left-preorders on G with a subset of $\{\pm 1, 0\}^G$. The space of left-preorders on G is the set $\mathcal{PO}(G)$ endowed with the subspace topology of $\{\pm 1, 0\}^G$ with the product topology. The space of left-preorders on G relative to C is the set $\mathcal{PO}_C(G)$ endowed with the subspace topology of $\mathcal{PO}(G)$. In the same way, we can consider $\mathcal{CO}(G)$ the space of conradian left-preorders on G, and $\mathcal{CO}_C(G)$ the space of conradian left-preorders on G relative to C.

Remark 1.6.1. Given $\mathcal{A} \subseteq \mathcal{PO}(G)$ endowed with the subspace topology, then a basis of open neighborhoods of $\in \mathcal{A}$ on $\mathcal{A}$ is given by the sets

$$U_{g_1,\dots,g_n}(\preceq_0) = \{\preceq \in \mathcal{A} \ / \ \phi_\preceq(g_i) = \phi_{\preceq_0}(g_i) \ \text{ for all } \ i = 1, \dots, n\}$$

for all $g_1, \dots, g_n \in G$ and all $n \in \mathbb{N}_{>0}$. Also, the sets

$$\overline{U}_{g_1,\dots,g_n}(\preceq_0) = \{\preceq \mathcal{A} \ / \ \phi_\preceq(g_i) \neq \phi_{\preceq_0}(g_i) \ \text{ for all } \ i = 1, \dots, n\}$$

are also open on $\mathcal{A}$. In particular, these two facts hold by taking $\mathcal{A} = \mathcal{PO}_C(G)$, $\mathcal{A} = \mathcal{CO}(G)$ or $\mathcal{A} = \mathcal{CO}_C(G)$ (for C a left-relatively convex subgroup of G).

We now study some different topological properties of these spaces. In some properties, we will sometimes consider also the space $\mathcal{PO}(G) \cup \{\preceq_T\}$ for $\preceq_T$ the trivial preorder. We do it only because it will be helpful to prove Corollary 5.3.9. Essentially, it is sometimes convenient to consider $\mathcal{PO}(G) \cup \{\preceq_T\}$ instead of $\mathcal{PO}(G)$ to guarantee compactness without any extra condition in G.

Proposition 1.6.2. *Let G be a group. Consider $\mathcal{A} \subseteq \mathcal{PO}(G)$ endowed with the subspace topology. Then, $\mathcal{A}$ is totally disconnected and Hausdorff.*

Proof. Notice that $\mathcal{A}$ is endowed with the subspace topology of $\{\pm 1, 0\}^G$. Recall that the properties of being totally disconnected and being Hausdorff are hereditary for subspaces. Since $\{\pm 1, 0\}^G$ is totally disconnected and Hausdorff, we conclude that $\mathcal{A}$ is totally disconnected and Hausdorff. □

Corollary 1.6.3. *Let G be a group. The spaces $\mathcal{PO}(G)$, $\mathcal{PO}(G) \cup \{\preceq_T\}$, $\mathcal{PO}_C(G)$, $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ are totally disconnected and Hausdorff for $\preceq_T$ the trivial preorder and for C any left-relatively convex subgroup.*

Proposition 1.6.4. *Let G be a group. Consider $\mathcal{A} \subseteq \mathcal{PO}(G)$ endowed with the subspace topology. If G is countable, then $\mathcal{A}$ is metrizable.*

Proof. Assume that G is countable. Notice that $\mathcal{A}$ is endowed with the subspace topology of $\{\pm 1, 0\}^G$. It is clear that being metrizable is hereditary for subspaces. Since G is countable, we have that $\{\pm 1, 0\}^G$ is is metrizable, so we conclude that $\mathcal{A}$ is metrizable. □

Corollary 1.6.5. *Let G be a group. If G is countable, then $\mathcal{PO}(G)$, $\mathcal{PO}_C(G)$, $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ are metrizable spaces.*

Notation 8. Let G be a group. Define the following subsets of $\{\pm 1, 0\}^G$ as follows:

$$V_g^r = \{\psi \in \{\pm 1, 0\}^G \ / \ \psi(g) = r\} \quad \text{for each } g \in G \text{ and each } \ r \in \{\pm 1, 0\},$$

$$\overline{V}_g^r = \{\psi \in \{\pm 1, 0\}^G \ / \ \psi(g) \neq r\} = \bigcup_{k \in \{\pm 1, 0\} - \{r\}} V_g^k \quad \text{for each } g \in G \text{ and each } \ r \in \{\pm 1, 0\}.$$

Notice that they are clopen (simultaneously open and closed) sets on $\{\pm 1, 0\}^G$.

We prove now that $\mathcal{PO}_C(G)$ is compact by showing that it is closed on the compact space $\{\pm 1, 0\}^G$.

Proposition 1.6.6. *Let G be a group and C a left-relatively convex subgroup of G. Then, the set $\mathcal{PO}_C(G)$ is closed on $\{\pm 1, 0\}^G$.*

Proof. We prove that $\mathcal{PO}_C(G)$ is closed by describing it as a complement of an open set made of intersections and finite unions of open sets. Consider the open sets V_g^r and $\overline{V}_g^r$ of $\{\pm 1, 0\}^G$ as above. From Proposition 1.2.4 we deduce that having an element of $\mathcal{PO}_C(G)$ is equivalent to give a decomposition of the form $G = P \sqcup P^{-1} \sqcup C$ where P is a subsemigroup of G. This is equivalent to give an element ϕ of $\{\pm 1, 0\}^G$ satisfying the following conditions:

1. $\left(\phi^{-1}(\{1\})\right)^{-1} = \phi^{-1}(\{-1\})$.
2. $\phi^{-1}(\{0\}) = C$.
3. $\phi^{-1}(\{1\})$ is a subsemigroup of G.

This give us the description

$$\{\pm 1, 0\}^G - \mathcal{PO}_C(G) = \left((\bigcup_{g \in G} V^1_{g^{-1}} \cap \overline{V}^{-1}_g)) \cup (\bigcup_{g \in G} \overline{V}^1_{g^{-1}} \cap V^{-1}_g)) \right) \cup \left((\bigcup_{g \in C} \overline{V}^0_g)) \cup (\bigcup_{g \in C - G} V^0_g)) \right) \cup$$
$$\cup \left(\bigcup_{g,h \in G} V^1_g \cap V^1_h \cap \overline{V}^1_{gh} \right)$$

where each of the four sets between parentheses is an open set that corresponds to the respective negation of each condition below. Therefore, we have shown that $\mathcal{PO}_C(G)$ is a closed subset. □

Corollary 1.6.7. *Let G be a group and C a left-relatively convex subgroup of G. Then, the space $\mathcal{PO}_C(G)$ is compact.*

For the compactness of $\mathcal{PO}(G)$ we need to assume that G is finitely generated.

Proposition 1.6.8. *Let G be a group. Let $\preceq_T$ be the trivial preorder. Then, the set $\mathcal{PO}(G) \cup \{\preceq_T\}$ is closed on $\{\pm 1, 0\}^G$.*

Proof. The proof is similar to the one done in Proposition 1.6.6. Consider the open sets V_g^r and $\overline{V}_g^r$ of $\{\pm 1, 0\}^G$ as above.

By Proposition 1.2.4, having a element of $\mathcal{PO}(G) \cup \{\preceq_T\}$ is equivalent to give a decomposition of the form $G = P \sqcup P^{-1} \sqcup C$ where P is a subsemigroup of G satisfying that C is a subgroup (for $C = G$ it give us $\preceq_T$). This is equivalent to give an element $\phi \in \{\pm 1, 0\}^G$ satisfying the following:

1. $\left(\phi^{-1}(\{1\})\right)^{-1} = \phi^{-1}(\{-1\})$.
2. $\phi^{-1}(\{0\})$ is a subgroup.
3. $\phi^{-1}(\{1\})$ is a subsemigroup of G.

This give us the description

$$\{\pm 1, 0\}^G - (\mathcal{PO}(G) \cup \{\preceq_T\}) = \left((\bigcup_{g \in G} V^1_{g^{-1}} \cap \overline{V}^{-1}_g)) \cup (\bigcup_{g \in G} \overline{V}^1_{g^{-1}} \cap V^{-1}_g)) \right) \cup \left(\bigcup_{g,h \in G} V^0_g \cap V^0_h \cap \overline{V}^0_{gh^{-1}} \right) \cup$$
$$\cup \left(\bigcup_{g,h \in G} V^1_g \cap V^1_h \cap \overline{V}^1_{gh} \right)$$

Therefore, $\mathcal{PO}(G) \cup \{\preceq_T\}$ is closed on $\{\pm 1, 0\}^G$. □

Corollary 1.6.9. *Let G be a group. Let $\preceq_T$ be the trivial preorder. Then, the space $\mathcal{PO}(G) \cup \{\preceq_T\}$ is compact.*

Proposition 1.6.10. *Let G be a group. If G is finitely generated, then the set $\mathcal{PO}(G)$ is closed on $\{\pm 1, 0\}^G$.*

Proof. Since G is finitely generated, we can take $G = \langle g_1, \dots, g_n \rangle$. Consider the open sets V_g^r as before. Let $\preceq_T$ be the trivial preorder. By Proposition 1.2.4, having a element of $\mathcal{PO}(G) \cup \{\preceq_T\}$ is equivalent to give a decomposition of the form $G = P \sqcup P^{-1} \sqcup C$ where P is a subsemigroup of G satisfying that C is a subgroup (for $C = G$ it give us $\preceq_T$). In particular, since $G = \langle g_1, \dots, g_n \rangle$, we have

$$(\mathcal{PO}(G) \cup \{\preceq_T\}) \cap \left(\bigcap_{i=1,\dots,n} V_{g_i}^0 \right) = \{\preceq_T\}.$$

Then, since $V_{g_i}^0$ is open on $\{\pm 1, 0\}^G$, we deduce that $\{\preceq_T\}$ is open on $\mathcal{PO}(G) \cup \{\preceq_T\}$. In particular, $\mathcal{PO}(G)$ is closed on $\mathcal{PO}(G) \cup \{\preceq_T\}$. Since $\mathcal{PO}(G) \cup \{\preceq_T\}$ is closed on $\{\pm 1, 0\}^G$ by Proposition 1.6.8, we conclude that $\mathcal{PO}(G)$ is closed on $\{\pm 1, 0\}^G$. □

Corollary 1.6.11. *Let G be a group. If G is finitely generated, then the space $\mathcal{PO}(G)$ is compact.*

We can do a similar discussion for $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$. Here it plays a fundamental role the characterization of being conradian that allows us to take $n = 2$ in the definition of conradian (see Theorem 1.4.13).

Proposition 1.6.12. *Let G be a group and C a left-relatively convex subgroup of G. Then, the set $\mathcal{CO}_C(G)$ is closed on $\{\pm 1, 0\}^G$.*

Proof. As before, consider the open sets V_g^r and $\overline{V}_g^r$ of $\{\pm 1, 0\}^G$ as above. By the characterization of being conradian (Theorem 1.4.13), $\mathcal{CO}_C(G)$ is equal to the set of elements of $\mathcal{PO}_C(G)$ such that its positive cone P satisfies that $g, h \in P$ implies $g^{-1}hg^2 \in P$. Then, the elements on $\mathcal{CO}_C(G)$ are given by the elements $\phi \in \{\pm 1, 0\}^G$ such that $\phi \in \mathcal{PO}_C(G)$ and satisfying that for all $g, h \in G$, we have that $g, h \in \phi^{-1}(\{1\})$ implies $g^{-1}hg^2 \in \phi^{-1}(\{1\})$. Hence, we obtain the description

$$\mathcal{CO}_C(G) = \mathcal{PO}_C(G) \cap \left(\{\pm 1, 0\}^G - \left(\bigcup_{g,h \in G} V_g^1 \cap V_h^1 \cap \overline{V}_{g^{-1}hg^2}^1 \right) \right)$$

where $\mathcal{PO}_C(G)$ is closed by Proposition 1.6.6 and we know that $\bigcup_{g,h \in G} V_g^1 \cap V_h^1 \cap \overline{V}_{g^{-1}hg^2}^1$ is open. Therefore, $\mathcal{CO}_C(G)$ is closed on $\{\pm 1, 0\}^G$. □

Corollary 1.6.13. *Let G be a group and C a left-relatively convex subgroup of G. Then, the space $\mathcal{CO}_C(G)$ is compact.*

Proposition 1.6.14. *Let G be a group. If G is finitely generated, then the set $\mathcal{CO}(G)$ is closed on $\{\pm 1, 0\}^G$.*

Proof. Same proof as in Proposition 1.6.12 but considering $\mathcal{CO}(G)$ instead of $\mathcal{CO}_C(G)$ and $\mathcal{PO}(G)$ instead of $\mathcal{PO}_C(G)$, and applying espacio de ordenes es cerrado instead of Proposition 1.6.6. □

Corollary 1.6.15. *Let G be a group. If G is finitely generated, then the space $\mathcal{CO}(G)$ is compact.*

We need the condition of G being finitely generated to guarantee that $\mathcal{PO}(G)$ and $\mathcal{CO}(G)$ are compact, as we will see in the next example from [4]. Notice that we are considering the same group we have considered in Example 1.4.6.

Example 1.6.16. *Consider the group $G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$. We can write $G = \bigoplus_{i \in \mathbb{N}} \langle x_i \rangle$ where $\langle x_i \rangle$ is isomorphic to $\mathbb{Z}$. Fix the notation $\overline{V}_g^0 = \{\psi \in \{\pm 1, 0\}^G \ / \ \psi(g) \neq 0\}$ for each $g \in G$. Notice that*

$$\mathcal{PO}(G) = \bigcup_{i \in \mathbb{N}} \left(\mathcal{PO}(G) \cap \overline{V}_{x_i}^0 \right),$$

since every element $\preceq \in \mathcal{PO}(G)$ satisfies that $\phi_{\preceq}(x_i) \neq 0$ for some $i \in \mathbb{N}$, as otherwise $\preceq$ would be trivial. The sets $\mathcal{PO}(G) \cap \overline{V}_{x_i}^0$ are open in $\mathcal{PO}(G)$ since the sets $\overline{V}_{x_i}^0$ are open in $\{\pm 1, 0\}^G$. Hence, the collection of

sets $\mathcal{PO}(G) \cap \overline{V}^0_{x_i}$ for $i \in \mathbb{N}$ is an open cover of $\mathcal{PO}(G)$. However, for every finite $i_1, \dots, i_k \in \mathbb{N}$ for $k \in \mathbb{N}$, we have that

$$\mathcal{PO}(G) \neq \bigcup_{j=1}^{k} \left(\mathcal{PO}(G) \cap \overline{V}^0_{x_{i_j}} \right).$$

Indeed, if one takes $l \in \mathbb{N}$ such that $i_j < l$ for all $j = 1, \dots, l$, since $G/\langle x_l \rangle$ is isomorphic to the left-orderable group $G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$, one can take $\preceq_l \in \mathcal{PO}(G)$ such that $\phi_{\preceq}(x_i) = 0$ if and only if $i = l$. Therefore, this shows that $\mathcal{PO}(G)$ is not compact, where notice that G is not finitely generated. The same conclusion holds for $\mathcal{CO}(G)$, since $\mathcal{PO}(G) = \mathcal{CO}(G)$ as G is abelian.

In some situations these spaces will become a Cantor set. In this sense, it is very useful the following topological characterization since the spaces we consider are compact, Hausdorff, totally disconnected, and non-empty metric space (depending on the case, we may need to require that G is finitely generated or countable, as discussed in the previous results).

Proposition 1.6.17. *Every compact, Hausdorff, totally disconnected, non-empty metric space having no isolated points is homeomorphic to the Cantor set.*

Proof. See Corollary 2-98 from [24]. □

We now state a proposition that provide us some conditions to guarantee that $\mathcal{PO}_C(G)$ or $\mathcal{PO}(G)$ have an isolated element. The same statement holds by assuming the left-preorder to be conradian and by interchanging $\mathcal{PO}(G)$ with $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ with $\mathcal{CO}_C(G)$. This fact is a consequence of the fact that an element being isolated is preserved when considering subspaces. However, we do not state this other version as we will not need it.

Notation 9. Given G a group and $S \subset G$, $\langle S \rangle^+$ denotes the subsemigroup generated by S.

Proposition 1.6.18. *Let G be a group and C a left-relatively convex subgroup of G. Let $\preceq$ be a left-preorder on G relative to C having positive cone P. If P is finitely generated as a subsemigroup, then $\preceq$ is isolated in $\mathcal{PO}_C(G)$. Moreover, if we also assume that C is a finitely generated subgroup, then $\preceq$ is also isolated in $\mathcal{PO}(G)$.*

Proof. Consider $P = \langle g_1, \dots, g_n \rangle^+$ the positive cone of $\preceq$, so that $G = P \sqcup P^{-1} \sqcup C$. In order to prove that $\preceq$ is isolated in $\mathcal{PO}_C(G)$ we need to prove that $\{\preceq\}$ is an open set. For this purpose, we are going to prove that

$$\{\preceq' \in \mathcal{PO}_C(G) \ / \ \phi_{\preceq'}(g_1) = 1, \dots, \phi_{\preceq'}(g_n) = 1\} = \{\preceq\},$$

so the fact that $\{\preceq\}$ is an open set will follow directly. Indeed, take $\preceq'$ an element belonging to the set in the left term of the equation. Then, $\preceq'$ is a left-preorder on G relative to C such that $C \preceq' g_i C$ for $i = 1, \dots, n$. Denote by P' its positive cone. Using the fact that $P = \langle g_1, \dots, g_n \rangle^+$, we obtain that

$$P \subseteq P' \text{ and } P^{-1} \subseteq (P')^{-1}.$$

Notice also that $G = P' \sqcup (P')^{-1} \sqcup C$ by Proposition 1.2.4. Hence, $P = P'$. We deduce that $\preceq' = \preceq$, as we wanted to prove.

Assume now that C is a finitely generated subgroup. Considering $x_1, \dots, x_k$ a finite set of generators, following similar arguments as before we can conclude that

$$\{\preceq' \in \mathcal{PO}(G) \ / \ \phi_{\preceq'}(g_1) = 1, \dots, \phi_{\preceq'}(g_n) = 1, \phi_{\preceq'}(x_1) = 0, \dots, \phi_{\preceq'}(x_k) = 0\} = \{\preceq\}.$$

Therefore, $\preceq$ is isolated in $\mathcal{PO}(G)$. □

We now briefly study abelian groups as an example and we give a brief description of some properties of the space of left-preorders for $\mathbb{Z}^m$. We will not provide the details, as this description is done in [14]. In this paper, the notion of Patch topology on the set of left-preorders corresponds to the topology we consider on the space of left-preorders. Notice that when working with abelian groups, the space of left-preorders is equal to the space of conradian left-preorders since all left-preorders on an abelian group are conradian.

Example 1.6.19. *Let G be a group and C be a proper normal subgroup such that G/C is torsion-free abelian. One can prove that $\mathcal{PO}_C(G) = \mathcal{CO}_C(G)$ is naturally homeomorphic to the space of left-orders on the group G/C as topological spaces. By [53], we know that the space of left-orders on a non-trivial torsion-free abelian group is a discrete space with two elements if the abelian rank is 1 and it is a Cantor set (having no isolated elements) if the abelian rank is greater than 1. Hence, we conclude the following:*

1. *If G/C is a torsion-free abelian group of abelian rank 1, then $\mathcal{PO}_C(G) = \mathcal{CO}_C(G)$ is a discrete space with two elements.*
2. *If G/C is a torsion-free abelian group of abelian rank greater than 1, then $\mathcal{PO}_C(G) = \mathcal{CO}_C(G)$ has no isolated orders and it is a Cantor set.*

Now, in order to study $\mathcal{PO}(G)$, we consider the group $G = \mathbb{Z}^m$ with $m \in \mathbb{N}_{>0}$ for simplicity. For $m = 1$, $\mathcal{PO}(\mathbb{Z})$ is a two point discrete space. Assume $m \geq 2$. We will show that $\mathcal{PO}(\mathbb{Z}^m)$ consists of a numerable discrete space and a Cantor set. Consider C a left-relatively convex subgroup of abelian co-rank 1, so we can consider $x_1, \dots, x_{m-1}$ a set of generators of C and gC a generator of the cyclic group $\mathbb{Z}^m/C$. Since $\mathbb{Z}^m/C$ is cyclic, there are only 2 left-preorders on $\mathbb{Z}^m$ relative to C, say $\preceq_C^+$ and $\preceq_C^-$. Then, we have

$$\{\preceq' \in \mathcal{PO}(\mathbb{Z}) \ / \ \phi_{\preceq}(x_1) = 0, \dots, \phi_{\preceq}(x_{m-1}) = 0, \phi_{\preceq}(g) = 1\} = \{\preceq_C^+\} \quad \textit{and}$$

$$\{\preceq' \in \mathcal{PO}(\mathbb{Z}) \ / \ \phi_{\preceq}(x_1) = 0, \dots, \phi_{\preceq}(x_{m-1}) = 0, \phi_{\preceq}(g) = -1\} = \{\preceq_C^-\}.$$

Hence, both left-preorders are isolated.

On the other hand, assume C is a left-relatively convex subgroup of abelian co-rank greater than 1. Take $\preceq$ a left-preorder on $\mathbb{Z}^m/C$. Then $\preceq$ is not isolated on $\mathcal{PO}_C(\mathbb{Z}^m)$ as it is a Cantor set. In particular, $\preceq$ is not isolated on $\mathcal{PO}(\mathbb{Z}^m)$. Hence, the set of left-preorders relative to a subgroup of abelian co-rank greater than 1 has no isolated elements. In fact, it is a Cantor set, as it is a closed subset of $\mathcal{PO}(\mathbb{Z}^m)$.

We conclude that $\mathcal{PO}(\mathbb{Z}^m)$ consists of a numerable discrete space and a Cantor set. The numerable discrete space is the set of left-preorders relative to a subgroup of abelian co-rank 1 and the Cantor set is the set of left-preorders relative to a subgroup of abelian co-rank greater than 1.

From now until the end of the section, we focus on obtaining some interesting properties that are consequence of the compactness of $\{\pm 1, 0\}^G$. These properties will be useful in other chapters. We do this by firstly proving a technical proposition that allows us to obtain the other conclusions without repeating some similar arguments. We use topological arguments inspired in the ideas from Lemma 2.2 of [49] and Proposition 1.4 of [44].

Proposition 1.6.20. *Let G be a group and C be a proper subgroup of G. Let $\mathcal{A}$ be a closed subset of $\{\pm 1, 0\}^G$ (endowed with the product topology). Let $X \subseteq G - C$ be a finite set. Assume that for every finite family $f_1, \dots, f_k$ of elements of $G - C$ there exists $\eta_1 \dots, \eta_k \in \{\pm 1\}$ and there exists $\chi \in \mathcal{A}$ such that*

$$C \subseteq \chi^{-1}(\{0\}) \ , \quad \langle X \cup \{f_1^{\eta_1}, \dots, f_k^{\eta_k}\}\rangle^+ \subseteq \chi^{-1}(\{1\}) \ \textit{and} \quad (\langle X \cup \{f_1^{\eta_1}, \dots, f_k^{\eta_k}\}\rangle^+)^{-1} \subseteq \chi^{-1}(\{-1\}).$$

Then, there exists $\preceq \in \mathcal{PO}_C(G)$ such that[1] *$\phi_{\preceq} \in \mathcal{A}$ and such that $C \preceq xC$ for all $x \in X$.*

Proof. Consider the subsets V_g^r and $\overline{V}_g^r$ of $\{\pm 1, 0\}^G$ we have defined before in this section, which are clopen. Define

$$\begin{aligned} A_0 &= \bigcap_{g \in G} \left((V_g^1 \cap V_{g^{-1}}^{-1}) \cup (V_g^{-1} \cap V_{g^{-1}}^1) \cup (V_g^0 \cap V_{g^{-1}}^0) \right), \\ A_1 &= \{\pm 1, 0\}^G - \bigcup_{g,h \in G} \left(V_g^1 \cap V_h^1 \cap \overline{V}_{gh}^1 \right), \\ A_2 &= \bigcap_{g \in C} V_g^0, \\ A_3 &= \bigcup_{x \in X} V_x^1 \end{aligned}$$

[1] Recall that $\phi_{\preceq}$ denotes the element of associated to $\{\pm 1, 0\}^G$.

Notice that, by following the definitions, we have

$$\begin{array}{lll} \chi \in A_0 & \iff & \left(\chi^{-1}(\{1\})\right)^{-1} = \chi^{-1}(\{-1\}), \\ \chi \in A_1 & \iff & \chi^{-1}(\{1\}) \text{ is a subsemigroup of } G, \\ \chi \in A_2 & \iff & C \subseteq \chi^{-1}(\{0\}), \\ \chi \in A_3 & \iff & X \subseteq \chi^{-1}(\{1\}) \end{array} \tag{1.7}$$

Define the subsets

$$\tilde{A}_{\{f_1,\dots,f_l\}} = (V^1_{f_1} \cup V^1_{f_1^{-1}}) \cap \cdots \cap (V^1_{f_n} \cup V^1_{f_n^{-1}}) \quad \text{for each } f_1,\dots,f_l \in G - C \text{ and } l \in \mathbb{N}_{>0}. \tag{1.8}$$

By definition, for every $f_1,\dots,f_l \in G - C$ we have

$$\chi \in \tilde{A}_{\{f_1,\dots,f_l\}} \iff f_1^{\epsilon_1},\dots,f_l^{\epsilon_l} \in \chi^{-1}(\{1\}) \text{ for some } \epsilon_1,\dots,\epsilon_l \in \{\pm 1\}.$$

We also define

$$B_{\{f_1,\dots,f_l\}} = \mathcal{A} \cap A_0 \cap A_1 \cap A_2 \cap A_3 \cap \tilde{A}_{\{f_1,\dots,f_l\}} \quad \text{for each } f_1,\dots,f_l \in G - C \text{ and } l \in \mathbb{N}_{>0}. \tag{1.9}$$

We now show that

$$B_{\{f_1,\dots,f_l\}} \neq \emptyset \text{ for all } f_1,\dots,f_l \in G - C \text{ where } l \in \mathbb{N}_{>0}. \tag{1.10}$$

Indeed, consider $f_1,\dots,f_l \in G - C$ where $l \in \mathbb{N}_{>0}$. By hypothesis, there exists $\eta_1 \dots,\eta_k \in \{\pm 1\}$ and there exists $\chi \in \mathcal{A}$ such that

$$C \subseteq \chi^{-1}(\{0\}) \ , \quad S \subseteq \chi^{-1}(\{1\}) \text{ and } \quad S^{-1} \subseteq \chi^{-1}(\{-1\}) \quad \text{for} \quad S = \langle X \cup \{f_1^{\eta_1},\dots,f_k^{\eta_k}\}\rangle^+.$$

We are going to prove that

$$\chi \in B_{\{f_1,\dots,f_l\}}.$$

By Equation (1.9), we only need to prove that

$$\chi \in A_0 \ , \ \chi \in A_1 \ , \ \chi \in A_2 \ , \ \chi \in A_3 \text{ and } \chi \in \tilde{A}_{\{f_1,\dots,f_l\}}$$

It is clear that $\chi \in \mathcal{A}$, $\chi \in A_0$ and $\chi \in A_2$. Also $\chi \in A_1$ since S is a subsemigroup of G. By definition of S we have $X \subseteq \chi^{-1}(\{1\})$, so $\chi \in A_3$. Also by definition of S it is clear that $f_1^{\eta_1},\dots,f_l^{\eta_l} \in \chi^{-1}(\{1\})$, so $\chi \in \tilde{A}_{\{f_1,\dots,f_l\}}$. Hence, we have shown that

$$\chi \in B_{\{f_1,\dots,f_l\}},$$

and this proves Equation (1.10), as we wanted.

We consider the family Ω of non-empty closed subsets of $\{\pm 1, 0\}^G$ given by the sets $B_{\{f_1,\dots,f_l\}}$ for $f_1,\dots,f_l \in G - C$ and $l \in \mathbb{N}_{>0}$. We claim that

$$\bigcap_{B \in \Omega} B \neq \emptyset. \tag{1.11}$$

Since all the elements in Ω are closed sets of the compact space $\{\pm 1, 0\}^G$, Equation (1.11) will be proven if we show that Ω has the finite intersection property. We prove it. Take $B_{\{f_{1,1},\dots,f_{1,l_1}\}},\dots,B_{\{f_{t,1},\dots,f_{t,l_t}\}} \in \Omega$. By following the definitions, we have

$$B_{\{f_{1,1},\dots,f_{1,l_1},\dots\dots,f_{t,1},\dots,f_{t,l_t}\}} \subseteq B_{\{f_{1,1},\dots,f_{1,l_1}\}} \cap \cdots \cap B_{\{f_{t,1},\dots,f_{t,l_t}\}},$$

but $B_{\{f_{1,1},\dots,f_{1,l_1},\dots\dots,f_{t,1},\dots,f_{t,l_t}\}}$ is non-empty by Equation (1.10), so the intersection above is non-empty. Then, we have shown that Ω has the finite intersection property, so Equation (1.11) holds.

We are now going to prove that there exists a $\chi_* \in \{\pm 1, 0\}^G$ that will provide us a left-preorder with the needed properties. By Equation (1.10), we can consider

$$\chi_* \in \bigcap_{B \in \Omega} B.$$

By definition of Ω, we have $\chi_* \in B_{\{f_1,\dots,f_l\}}$ for all $f_1,\dots,f_l \in G-C$ and all $l \in \mathbb{N}_{>0}$. By Equation (1.7), we have that $\chi_* \in \mathcal{A}$, that $\chi_* \in A_i$ for $i=0,1,2,3$ and that $\chi_* \in \tilde{A}_{\{f_1,\dots,f_l\}}$ for all $f_1,\dots,f_l \in G-C$ and all $l \in \mathbb{N}_{>0}$. We now prove that

$$C = \chi_*^{-1}(\{0\}). \tag{1.12}$$

Since $\chi_* \in A_2$, we deduce that $C \subseteq \chi_*^{-1}(\{0\})$. For the other inclusion, given $f \in G-C$, we know that $\chi_* \in \tilde{A}_{\{f\}}$, so $f \notin \chi_*^{-1}(\{0\})$. Hence, we have shown Equation (1.12).

We are now ready use χ_* to obtain the left-preorder, where we know that $\chi_* \in \mathcal{A}$ and $\chi_* \in A_i$ for $i=0,1,2,3$. We can decompose G as

$$G = \chi_*^{-1}(\{1\}) \sqcup \chi_*^{-1}(\{-1\}) \sqcup \chi_*^{-1}(\{0\}).$$

Define $P = \chi_*^{-1}(\{1\})$, and notice that P is a subsemigroup of G since $\chi_* \in A_1$. Notice also that $P^{-1} = \chi_*^{-1}(\{-1\})$ as a consequence of $\chi_* \in A_0$ and that $X \subseteq P$ as a consequence of $\chi_* \in A_3$. Then, we can write the the decomposition below as

$$G = P \sqcup P^{-1} \sqcup \chi_*^{-1}(\{0\})$$

where P is a subsemigroup of G such that $X \subseteq P$. Therefore, using Equation (1.12), the decomposition below is in fact

$$G = P \sqcup P^{-1} \sqcup C$$

where P is a subsemigroup of G. Hence, by Proposition 1.2.4, there is a left-preorder $\preceq$ on G relative to C with positive cone P satisfying that $X \subseteq P$, and notice that $\chi_* = \phi_{\preceq}$ by construction, so $\phi_{\preceq} \in \mathcal{A}$. Therefore, we conclude that there exists $\preceq \in \mathcal{PO}_C(G)$ such that $\phi_{\preceq} \in \mathcal{A}$ and such that $C \preceq xC$ for all $x \in X$. □

As a first consequence, we obtain a way to left-preorder a whole group by just left-preordering all finitely generated subgroups. We do it in the next proposition. This will help us to prove Theorem 5.3.1 in such a way we will be able to restrict to the finitely generated case.

Definition 1.6.21. Let G be a group and C be a proper subgroup of G. Let $X \subseteq G-C$ be a finite set. We say that X has property (E_C) if and only if for every finite family $f_1,\dots,f_k$ of elements of $G-C$ for $k \in \mathbb{N}$ (possibly empty for $k=0$) there exists $\eta_1 \dots,\eta_k \in \{\pm 1\}$ such that $C \cap \langle X \cup \{f_1^{\eta_1},\dots,f_k^{\eta_k}\}\rangle^+ = \emptyset$.

Proposition 1.6.22. *Let G be a group and C be a proper subgroup of G. Let $X \subseteq G-C$ be a finite set. Then, G admits a left-preorder $\preceq$ relative to C such that $C \prec xC$ for all $x \in X$ if and only if X has property (E_C).*

Proof. The necessity of the equivalence is clear. The other implication, notice that, given a finite family $f_1,\dots,f_k$ of elements of $G-C$ for $k \in \mathbb{N}$ (possibly empty for $k=0$), and given $\eta_1 \dots,\eta_k \in \{\pm 1\}$ such that $C \cap \langle X \cup \{f_1^{\eta_1},\dots,f_k^{\eta_k}\}\rangle^+ = \emptyset$, then we have $C \cap (\langle X \cup \{f_1^{\eta_1},\dots,f_k^{\eta_k}\}\rangle^+)^{-1} = \emptyset$, as it follows by taking inverses. Then, the implication we want to prove follows by applying Proposition 1.6.20 for $\mathcal{A} = \{\pm 1, 0\}^G$ as a closed subset of $\{\pm 1, 0\}^G$ and for $X \subseteq G-C$ as a finite set. □

As a second consequence, we prove that the intersection of "conradian" left-relatively convex sugbroups is again "conradian" left-relatively convex. We do this by previously proving a lemma for the finite case. After that, we state an analogous version for left-relatively convex sugbroups instead of "conradian" ones. This version is proven with less details as it is completely analogous and we will not use it later.

Lemma 1.6.23. *Let G be a group and $C_1,\dots,C_k$ be proper subgroups of G for $k \in \mathbb{N}_{>0}$. If $\mathcal{CO}_{C_i}(G) \neq \emptyset$ for all $i=1,\dots,k$, then $\mathcal{CO}_{C_1\cap\cdots\cap C_k}(G) \neq \emptyset$.*

Proof. We prove the statement by induction on $k \in \mathbb{N}_{>0}$. For $k=1$ it is trivial. We now do the inductive case. Take $k \in \mathbb{N}_{>0}$ such that $k>1$ and assume that the statement is true for $k-1$. We want to prove it for k, so consider $C_1,\dots,C_k$ proper subgroups of G such that $\mathcal{CO}_{C_i}(G) \neq \emptyset$ for all $i=1,\dots,k$. By inductive hypothesis, we know that $\mathcal{CO}_{C_1\cap\cdots\cap C_{k-1}}(G) \neq \emptyset$. Then, we can take $\preceq_1$ a conradian left-preorder on G relative to $C_1 \cap \cdots \cap C_{k-1}$. Since we have $\mathcal{CO}_{C_k}(G) \neq \emptyset$, we can also take $\preceq_2$ a conradian left-preorder on G relative to C_k. Consider $\preceq_1'$ the restriction of $\preceq_1$ to C_k, so that $\preceq_1'$ is a conradian left-preorder on C_k relative to $C_1 \cap \cdots \cap C_k$. By Proposition 1.5.4, we have that $\mu_{G,C_1\cap\cdots\cap C_k,C_k}(\preceq_1',\preceq_2)$ is a conradian left-preorder on G relative to $C_1 \cap \cdots \cap C_k$. Hence, $\mathcal{CO}_{C_1\cap\cdots\cap C_k}(G) \neq \emptyset$. This finishes the proof for the inductive case. □

Proposition 1.6.24. *Let G be a group and $(C_i)_{i\in I}$ be a family of proper subgroups of G. If $\mathcal{CO}_{C_i}(G) \neq \emptyset$ for all $i \in I$, then*

$$\mathcal{CO}_{\bigcap_{i\in I} C_i}(G) \neq \emptyset.$$

Proof. Assume that $\mathcal{CO}_{C_i}(G) \neq \emptyset$ for all $i \in I$. Consider the subsets V_g^r and $\overline{V}_g^r$ of $\{\pm 1, 0\}^G$ we have defined before in this section, which are clopen. Define

$$\mathcal{A} = \{\pm 1, 0\}^G - \bigcup_{g,h\in G} \left(V_g^1 \cap V_h^1 \cap \overline{V}_{g^{-1}hg^2}^1 \right),$$

so then $\mathcal{A}$ is a closed subset of $\{\pm 1, 0\}^G$. Notice that, by following the definitions, for every $\preceq \in \mathcal{PO}(G)$ we have[2]

$$\phi_\preceq \in \mathcal{A} \iff \preceq \text{ is conradian.} \tag{1.13}$$

Now, we are going to prove that we are under the hypotheses of Proposition 1.6.20 in order to apply it. Consider a finite family $f_1, \dots, f_k$ of elements of $G - (\bigcap_{i\in I} C_i)$. Then, for all $r = 1, \dots, k$ we have that $f_r \notin C_{i_r}$ for some $i_r \in I$. Then, we deduce that

$$f_1, \dots, f_k \in G - (\bigcap_{r=1}^{k} C_{i_r}). \tag{1.14}$$

By hypothesis we know that $\mathcal{CO}_{C_i}(G) \neq \emptyset$ for all $i \in I$ so, by Lemma 1.6.23 we deduce that $\mathcal{CO}_{C_{i_1}\cap\cdots\cap C_{i_k}}(G)$ is non-empty. Then, take $\preceq \in \mathcal{CO}_{C_{i_1}\cap\cdots\cap C_{i_k}}(G)$. Then, from Equation (1.14), we deduce that there exists $\eta_1 \dots, \eta_k \in \{\pm 1\}$ such that $f_r^{\eta_r}$ is on the positive cone of $\preceq$. Then, $\langle f_1^{\eta_1}, \dots, f_k^{\eta_k}\rangle^+$ is contained on the positive cone of $\preceq$. Hence, by definition of $\phi_\preceq$, we deduce that

$$\langle f_1^{\eta_1}, \dots, f_k^{\eta_k}\rangle^+ \subseteq \phi_\preceq^{-1}(\{1\}) \text{ and } \quad (\langle f_1^{\eta_1}, \dots, f_k^{\eta_k}\rangle^+)^{-1} \subseteq \phi_\preceq^{-1}(\{-1\}).$$

Also by definition of $\phi_\preceq$, we have

$$\phi_\preceq^{-1}(\{0\}) = C_{i_1} \cap \cdots \cap C_{i_k} \supseteq \bigcap_{i\in I} C_i,$$

and by Equation (1.13) we have $\phi_\preceq \in \mathcal{A}$ since $\preceq$ is conradian.

This shows that we are under the hypotheses of Proposition 1.6.20 for $C = \bigcap_{i\in I} C_i$, for $\mathcal{A}$ as a closed subset of $\{\pm 1, 0\}^G$ and for $X = \emptyset$ as a finite set, so we deduce that there exists $\preceq_* \in \mathcal{PO}_{\bigcap_{i\in I} C_i}(G)$ such that $\phi_\preceq \in \mathcal{A}$. By Equation (1.13), we deduce that $\preceq_*$ is conradian, so $\preceq_* \in \mathcal{CO}_{\bigcap_{i\in I} C_i}(G)$. We conclude that $\mathcal{CO}_{\bigcap_{i\in I} C_i}(G) \neq \emptyset$, as we wanted. □

Lemma 1.6.25. *Let G be a group and $C_1, \dots, C_k$ be proper subgroups of G for $k \in \mathbb{N}_{>0}$. If $\mathcal{PO}_{C_i}(G) \neq \emptyset$ for all $i = 1, \dots, k$, then $\mathcal{PO}_{C_1\cap\cdots\cap C_k}(G) \neq \emptyset$.*

Proof. Analogous to the proof done in Lemma 1.6.23 but applying Proposition 1.3.3 instead of Proposition 1.5.4. □

Proposition 1.6.26. *Let G be a group and $(C_i)_{i\in I}$ be a family of proper subgroups of G. If $\mathcal{PO}_{C_i}(G) \neq \emptyset$ for all $i \in I$, then*

$$\mathcal{PO}_{\bigcap_{i\in I} C_i}(G) \neq \emptyset.$$

Proof. Analogous to the proof done in Proposition 1.6.24 but applying Lemma 1.6.25 instead of Lemma 1.6.23 and considering $\mathcal{A} = \{\pm 1, 0\}^G$ instead of the subspace considered on that proof. □

[2] As before, $\phi_\preceq$ denotes the element of associated to $\{\pm 1, 0\}^G$.

Chapter 2

Characterizing finiteness of spaces of $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$

The main objective of this chapter is proving Theorem 2.2.8, which provides us equivalent conditions for $\mathcal{CO}(G)$ being finite. This is done in the second section of this chapter. We also prove Theorem 2.1.14, which give us equivalent conditions for $\mathcal{CO}_C(G)$ being finite, and is used to show the other result we have mentioned. This is done in the first section of the chapter. These results are a generalization to the conradian left-preorder context of the result by Rivas (see theorems A and B on [50]).

In this chapter we also prove that the spaces $\mathcal{CO}(G)$ and $\mathcal{CO}_C(G)$ are finite or uncountable. This is done in the second and the first sections respectively.

2.1 Characterizing finiteness of spaces of $\mathcal{CO}_C(G)$

In this section we prove Theorem 2.1.14. We state the main result here in order to keep in mind the objective. Before we need to define some notions.

Definition 2.1.1. Let G be a group. A rational series starting on G is a finite series of the form

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

satisfying that each G_i is normal on G_{i+1} and the group G_{i+1}/G_i is torsion-free abelian of abelian rank 1 for all $0 \leq i < n$. Wwe say that such a series starts in G_0, ends in G_n and has length n. Such a series is said to be normal if each G_i is normal on G. An abelian jump on this series is a subseries the form

$$G_i \trianglelefteq G_{i+1} \trianglelefteq G_{i+2}$$

for $i \in \{0, \ldots, n-2\}$ such that G_i is normal on G_{i+2} and the group G_{i+2}/G_i is abelian.

Theorem (Theorem 2.1.14)**.** *Let G be a group and C be a proper subgroup. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}_C(G)$ are isolated and $\mathcal{CO}_C(G)$ is non-empty.

b) There exists an element on $\mathcal{CO}_C(G)$ that is isolated.

c) There exists a finite rational series starting on G, ending on C and without abelian jumps.

d) There exists a unique finite rational series starting on G and ending on C, and this series does not have abelian jumps.

e) $\mathcal{CO}_C(G)$ is finite and non-empty.

Moreover, under these these equivalent conditions, we have that $\mathrm{Card}\,(\mathcal{CO}_C(G)) = 2^n$ *for n the length of any series as in c) and d).*

The proof of that result it follows the ideas in [50]. In order to prove it, we previously show a partial version of the theorem (in Theorem 2.1.11) without the uniqueness. We do it by splitting the proof in some propositions and some auxiliary lemmas.

Lemma 2.1.2. *Let G be a group and C be a proper subgroup. Let $\preceq$ be a conradian left-preorder on G relative to C. Let H_1, H_2 be proper subgroups of G such that*

$$C \subseteq H_1 \subseteq H_2 \subseteq G.$$

If H_1 is $\preceq_{|H_2}$-convex and H_2 is $\preceq$-convex, then H_1 is $\preceq$-convex.

Proof. Assume that H_1 is $\preceq_{|H_2}$-convex and H_2 is $\preceq$-convex. Consider g, f in $H_1 \subseteq H_2$ and h in G such that

$$gC \preceq hC \preceq fC.$$

By $\preceq$-convexity of H_2 we deduce that $h \in H_2$. Then, since H_1 is $\preceq_{|H_2}$-convex, we deduce that $h \in H_1$. □

Lemma 2.1.3. *Let G be a group. Take*

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

a rational series on G without abelian jumps. Then, for each $0 \leq i < n-1$ there exists $g \in G_{i+1}$ and $h \in G_{i+2} - G_{i+1}$ such that

$$h^{-1}ghG_i \neq gG_i.$$

Proof. Take $0 \leq i < n-1$. We are going to prove it by contradiction, so assume that

$$h^{-1}ghG_i = gG_i \text{ for all } g \in G_{i+1} \text{ and all } h \in G_{i+2}. \tag{2.1}$$

Notice that this equation has sense on the group G_{i+1}/G_i because $h^{-1}gh \in G_{i+1}$ as G_{i+1} is normal on G_{i+2}. We are going to prove that

$$G_i \trianglelefteq G_{i+1} \trianglelefteq G_{i+2}$$

is an abelian jump. In order to do that, we are going to prove that G_i is normal on G_{i+2} and that the group G_{i+2}/G_i is abelian.

We first prove that G_i is normal on G_{i+2}. Take $f \in G_i$ and $h \in G_{i+2}$. Since $G_i \subseteq G_{i+1}$, we have $f \in G_{i+1}$, so by Equation (2.1), we deduce that $h^{-1}fhG_i = fG_i$. Notice that $fG_i = G_i$ as $f \in G_i$, so then $h^{-1}fhG_i = G_i$. Hence, $h^{-1}fh \in G_i$. Therefore, we have shown that G_i is normal on G_{i+2}.

We now prove that the group G_{i+2}/G_i is abelian. We are going to prove it by showing that for all $h_1, h_2 \in G_{i+2}$ we have $h_1h_2G_i = h_2h_1G_i$. Consider $h_1, h_2 \in G_{i+2}$. If $h_1 \in G_{i+1}$ or $h_2 \in G_{i+1}$ then $h_1h_2G_i = h_2h_1G_i$ follows by Equation (2.1) when only one of the conditions holds, or by the fact that G_{i+1}/G_i is abelian when both holds. Hence, we can assume that $h_1, h_2 \in G_{i+2} - G_{i+1}$. As G_{i+2}/G_{i+1} is torsion-free abelian of rank 1, we know that there exists $h_0 \in G_{i+2} - G_{i+1}$ and $r, s \in \mathbb{Z} - \{0\}$ such that $h_0^rG_{i+1} = h_1G_{i+1}$ and $h_0^sG_{i+1} = h_2G_{i+1}$. Then, there exists $g_1, g_2 \in G_{i+1}$ such that $h_0^rg_1 = h_1$ and $h_0^sg_2 = h_2$. By Equation (2.1), we deduce that h_0G_i commutes with g_1G_i and with g_2G_i in the group G_{i+2}/G_i. Also notice that g_1G_i and g_2G_i commute as they are elements of the abelian group G_{i+1}/G_i. Hence, we have

$$h_1h_2G_i = h_0^rg_1h_0^sg_2G_i = h_0^{r+s}g_1g_2G_i = h_0^{r+s}g_2g_1G_i = h_0^sg_2h_0^rg_1G_i = h_2h_1G_i,$$

as we wanted. Therefore, G_{i+2}/G_i is an abelian group. □

Assuming the notation of the previous lemma, if $g \in G_i$, then $h^{-1}gh$ is an element in $G_{i+1} - G_i$ satisfying the same properties. In particular, by changing g by $h^{-1}gh$ if necessary, we deduce the following version of the previous lemma.

Lemma 2.1.4. *Let G be a group. Take*

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n$$

a rational series on G without abelian jumps. Then, for each $0 \leq i < n-1$ there exists $g \in G_{i+1} - G_i$ and $h \in G_{i+2} - G_{i+1}$ such that

$$h^{-1}ghG_i \neq gG_i.$$

Proposition 2.1.5. *Let G be a group and C be a proper subgroup. Assume that G admits a rational series*

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

on G starting on G, ending on C and without abelian jumps. Then, G_j is $\preceq$-convex for all $\preceq$ conradian left-preorder on G relative to C and for all $0 \leq j \leq n$.

Proof. Suppose that G admits a rational series

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

without abelian jumps. Let $\preceq$ be a conradian left-preorder on G relative to C. We want to prove that G_j is $\preceq$-convex for all $0 \leq j \leq n$. If we prove that G_j is $\preceq_{|G_{j+1}}$-convex for all $0 \leq j < n$, then the fact that G_j is $\preceq$-convex for all $0 \leq j \leq n$ follows directly by applying finitely many times Lemma 2.1.2. Hence, we are going to prove that G_j is $\preceq_{|G_{j+1}}$-convex for all $0 \leq j < n$ by induction on j in order to prove the statement.

For $j = 0$ it is trivial, as $G_0 = C$ is always convex on every left-preorder relative to C. Given $0 < j < n$ and assuming that G_{j-1} is $\preceq_{|G_j}$-convex as inductive hypothesis, we are going to prove that G_j is $\preceq_{|G_{j+1}}$-convex. We do it by contradiction, so suppose that G_j is not $\preceq_{|G_{j+1}}$-convex. Hence, there exists $g_1, g_2 \in G_j$ and $h_0 \in G_{j+1} - G_j$ such that

$$g_1 C \preceq h_0 C \preceq g_2 C \tag{2.2}$$

By Lemma 2.1.4, we know that there exists $\tilde{g} \in G_j - G_{j-1}$ and $\tilde{h} \in G_{j+1} - G_j$ such that

$$h^{-1}\tilde{g}\tilde{h}G_{j-1} \neq \tilde{g}G_{j-1}.$$

Then, using that $\tilde{h}^{-1}\tilde{g}\tilde{h}, \tilde{g} \in G_j$ and that G_j/G_{j-1} is a torsion-free abelian group of rank 1, we deduce that

$$\tilde{h}^{-1}\tilde{g}^r\tilde{h}G_{j-1} = \tilde{g}^s G_{j-1} \text{ for some } s, r \in \mathbb{Z} \text{ such that } s \neq r \tag{2.3}$$

Notice that, in the previous equation, the fact that $s \neq r$ is a consequence of the fact that G_j/G_{j-1} has unique roots.

Consider the subgroup $L = \langle g_1, g_2, \tilde{g}, h_0, \tilde{h} \rangle$. Notice that $\preceq$, when restricted to L, is a conradian left-preorder on L. By Corollary 1.4.5, since L is finitely generated, we can consider K a maximal proper $\preceq$-convex subgroup of L. In other words, we have that (K, L) is a $\preceq$-convex jump. Then, by Theorem 1.4.13, we know that K is normal in L and there exists a $\hat{\preceq}$-preserving injective morphism $\tau : L/K \longrightarrow \mathbb{R}$ for $\hat{\preceq}$ the left-preorder induced by $\preceq$ on L relative to K. Notice that

$$f_1 C \preceq f_2 C \quad \Longrightarrow \quad \tau(f_1 K) \leq \tau(f_2 K) \quad \text{for all } f_1 f_2 \in L \tag{2.4}$$

as a consequence of Remark 1.4.12 and τ being $\hat{\preceq}$-preserving.

Firstly, we claim that

$$\tau(fK) = 0 \text{ for all } f \in G_{j-1} \cap L. \tag{2.5}$$

Indeed, take $f \in G_{j-1} \cap L$. Define $|\tilde{g}|$ to be $\tilde{g}$ if $C \prec \tilde{g}C$ and $\tilde{g}^{-1}$ if $\tilde{g}C \prec C$. By inductive hypothesis, G_{j-1} is $\preceq_{|G_j}$-convex, so using the fact that $f^m \in G_{j-1}$ for all $m \in \mathbb{Z}$ and that $|\tilde{g}|, |\tilde{g}|^{-1} \in G_j - G_{j-1}$, we deduce that

$$|\tilde{g}|^{-1} C \preceq f^m C \preceq |\tilde{g}| C \text{ for all } m \in \mathbb{Z}.$$

Then, applying Equation (2.4) for $|\tilde{g}|, f \in L$, we obtain

$$-\tau(|\tilde{g}|K) \leq m\tau(fK) \leq \tau(|\tilde{g}|K) \text{ for all } m \in \mathbb{Z}.$$

It follows that $\tau(fK) = 0$, as we claimed.

Secondly, we claim that

$$\tau(\tilde{g}K) = 0. \tag{2.6}$$

By Equation (2.3), we deduce that $\tilde{h}^{-1}\tilde{g}^r\tilde{h} = \tilde{g}^s\tilde{f}$ for some $\tilde{f} \in G_{j-1}$ where $r \neq s$. Notice that $\tilde{f} \in G_{j-1} \cap L$, so by Equation (2.5) we obtain $\tau(\tilde{f}K) = 0$. Therefore, we deduce

$$r\tau(\tilde{g}K) = -\tau(\tilde{h}K) + r\tau(\tilde{g}K) + \tau(\tilde{h}K) = \tau(\tilde{h}^{-1}\tilde{g}^r\tilde{h}K) = \tau(\tilde{g}^s\tilde{f}K) = s\tau(\tilde{g}K) + \tau(\tilde{f}K) = s\tau(\tilde{g}K),$$

but $r \neq s$, so $\tau(\tilde{g}K) = 0$. This proves Equation (2.6).

Thirdly, we claim

$$\tau(gK) = 0 \text{ for all } g \in G_j \cap L. \tag{2.7}$$

We prove it. Consider $g \in G_j \cap L$. Since $\tilde{g} \in G_j - G_{j-1}$ and since G_j/G_{j-1} is a torsion-free abelian group of rank 1, we deduce that there exists $l, m \in \mathbb{Z}$ with $l \neq 0$ such that

$$g^l G_{j-1} = \tilde{g}^m G_{j-1}.$$

Hence, there exists $f_0 \in G_{j-1}$ such that $g^l = \tilde{g}^m f_0$. Notice that $f_0 \in G_{j-1} \cap L$, so applying Equation (2.5) we deduce $\tau(f_0K) = 0$. From Equation (2.6), from $\tau(f_0K) = 0$ and from $g^l = \tilde{g}^m f_0$ we deduce that

$$l\tau(gK) = \tau(g^lK) = \tau(\tilde{g}^m f_0 K) = m\tau(\tilde{g}K) + \tau(f_0K) = 0.$$

Since $l \neq 0$, we obtain $\tau(gK) = 0$, as we claimed.

From Equation (2.2) and from Equation (2.4), we deduce that

$$\tau(g_1K) \leq \tau(h_0K) \leq \tau(g_2K).$$

Notice that $\tau(g_1K) = 0$ and $\tau(g_2K) = 0$ follows from Equation (2.7) since $g_1, g_2 \in G_j \cap L$. Therefore, we deduce that

$$\tau(h_0K) = 0. \tag{2.8}$$

Since $h_0, \tilde{h} \in G_{j+1}$ and G_{j+1}/G_j is a torsion-free abelian group of rank 1, then there exists $l, m \in \mathbb{Z}$ with $l, m \neq 0$ such that

$$\tilde{h}^l G_j = h_0^m G_j.$$

Then, there is $t \in G_j$ such that $\tilde{h}^l = h_0^m t$. In particular, we deduce that $t \in G_j \cap L$ and that

$$l\tau(\tilde{h}K) = \tau(\tilde{h}^lK) = \tau(h_0^m tK) = m\tau(h_0K) + \tau(tK).$$

Since $\tau(h_0K) = 0$ from Equation (2.8) and since $\tau(tK) = 0$ from Equation (2.7) and $t \in G_j \cap L$, we obtain $l\tau(\tilde{h}K) = 0$. As $l \neq 0$, we obtain

$$\tau(\tilde{h}K) = 0. \tag{2.9}$$

Finally, recall that $L = \langle g_1, g_2, \tilde{g}, h_0, \tilde{h}\rangle$. From Equation (2.6), Equation (2.7), Equation (2.8) and Equation (2.9), we deduce that the image of the generators of L is 0. Hence, as τ is injective, we obtain $K = L$. This is a contradiction with the fact that $K \neq L$ as (K, L) is a $\preceq$-convex jump. □

Proposition 2.1.6. *Let G be a group and C be a proper subgroup. Assume that G admits a rational series*

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

on G starting on G, ending on C, having length n and without abelian jumps. Then,

$$\text{Card}\left(\mathcal{CO}_{G_l}(G_k)\right) = 2^{k-l} \quad \textit{for all } \ 0 \leq l < k \leq n.$$

In particular,

$$\text{Card}\left(\mathcal{CO}_C(G)\right) = 2^n.$$

Proof. Fix $l, k \in \mathbb{N}$ such that $0 \leq l < k \leq n$. As a direct consequence of Proposition 1.5.5, we obtain

$$\text{Card}\left(\{\preceq \in \mathcal{CO}_{G_l}(G_k) \;/\; G_i \text{ are } \preceq\text{-convex for all } i = l+1, \dots, k-1\}\right) = \prod_{j=l+1}^{k} \text{Card}\left(\mathcal{CO}_{G_{j-1}}(G_j)\right).$$

Notice that $\text{Card}\left(\mathcal{CO}_{G_{j-1}}(G_j)\right) = 2$ follows by Example 1.6.19 as G_{j+1}/G_j is torsion-free abelian of rank 1 since the series considered above is rational. Hence, we obtain

$$\text{Card}\left(\{\preceq \in \mathcal{CO}_{G_l}(G_k) \;/\; G_i \text{ are } \preceq\text{-convex for all } i = l+1, \dots, k-1\}\right) = 2^{k-l}.$$

By Proposition 2.1.5, we know that, for all $\preceq \in \text{Card}\left(\mathcal{CO}_{G_l}(G_k)\right)$, we have G_j is $\preceq$-convex for all $j = l+1, \dots, k-1$. Then, we obtain

$$\text{Card}\left(\mathcal{CO}_{G_l}(G_k)\right) = \text{Card}\left(\{\preceq \in \mathcal{CO}_{G_l}(G_k) \;/\; G_i \text{ are } \preceq\text{-convex for all } i = l+1, \dots, k-1\}\right) = 2^{k-l}.$$

□

The following proposition is also true for $\mathcal{PO}_C(G)$, but we do not state it as we will not need it.

Proposition 2.1.7. *Let G be a group and C be a proper subgroup. Let $\preceq$ be a conradian left-preorder on G relative to C. If $\preceq$ is isolated on $\mathcal{CO}_C(G)$ then there are finitely many $\preceq$-convex subgroups.*

Proof. We prove this by contradiction. In this way, assume that $\preceq$ is isolated on $\mathcal{CO}_C(G)$ but there are infinitely many $\preceq$-convex subgroups. Since $\preceq$ is isolated on $\mathcal{CO}_C(G)$, there are $g_1, \dots, g_k \in G$ for $k \in \mathbb{N}_{>0}$ such that

$$\{\preceq_* \in \mathcal{CO}_C(G) \;/\; C \prec_* g_i C \;\text{ for } 0 < i \leq k\} = \{\preceq\}. \tag{2.10}$$

We know that there are infinitely many $\preceq$-convex subgroup and we know that the set of $\preceq$-convex subgroups is linearly ordered by Proposition 1.3.2, so we can consider

$$H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_{k+1} \subsetneq H_{k+2}$$

$k+2$ different $\preceq$-convex subgroups. Then, we obtain $k+1$ disjoint sets $H_{j+1} - H_j$ for $j = 1, \dots, k+1$. Then, since we have that $g_1, \dots, g_k$ are k different elements, by cardinality reasons, we deduce that there is a $j_0 \in \{1, \dots, k+1\}$ such that no element in $\{g_1, \dots, g_k\}$ belongs to $H_{j_0+1} - H_{j_0}$. For simplicity on the notation, assume that $j_0 = 1$, so that

$$\{g_1, \dots, g_k\} \cap (H_2 - H_1) = \emptyset.$$

We now consider the mutually inverse bijections μ and ρ from Proposition 1.5.5 for the finite chain of subgroups

$$C \subsetneq H_1 \subsetneq H_2 \subsetneq G.$$

As we know that H_1 and H_2 are $\preceq$-convex subgroups, we can consider

$$\rho(\preceq) = (\preceq_1, \preceq_2, \preceq_3) \;\text{ where }\; \preceq_1 \in \mathcal{CO}_C(H_1)\,, \;\; \preceq_2 \in \mathcal{CO}_{H_1}(H_2) \;\text{ and }\; \preceq_3 \in \mathcal{CO}_{H_2}(G). \tag{2.11}$$

Define P the positive cone of $\preceq$, P_1 the positive cone of $\preceq_1$, P_2 the positive cone of $\preceq_2$ and P_3 the positive cone of $\preceq_3$. Then, by Equation (2.10), we have

$$g_1, \dots, g_k \in P.$$

By the definition of the mutually inverse bijections μ and ρ (see Proposition 1.5.5), we deduce that

$$g_1, \dots, g_k \in P_1 \cup P_2 \cup P_3 \quad \text{and} \quad P_2 \subseteq H_2 - H_1.$$

Since $\{g_1, \dots, g_k\} \cap (H_2 - H_1) = \emptyset$, we deduce that

$$g_1, \dots, g_k \in P_1 \cup P_3. \tag{2.12}$$

Consider $\preceq_2^-$ the element of $\mathcal{CO}_{H_1}(G_2)$ whose positive cone is P_2^{-1}, so that $\preceq_2^-$ is not equal to $\preceq_2$. Define

$$\preceq_* = \mu(\preceq_1, \preceq_2^-, \preceq_3), \tag{2.13}$$

so we have that $\preceq_*$ is not equal to $\preceq$, as μ is a bijection. We are going to reach a contradiction with the fact that $\preceq_*$ is not equal to $\preceq$. By Proposition 1.5.5, the positive cone P_* of $\preceq_*$ is given by

$$P_* = P_1 \cup P_2^{-1} \cup P_3.$$

From this and from Equation (2.12) we obtain

$$g_1, \dots, g_k \in P_*,$$

which is equivalent to say that

$$C \prec_* g_i C \quad \text{for } 0 < i \leq k.$$

Applying Equation (2.10), we deduce that $\preceq$ is equal to $\preceq_*$, which is a contradiction, as we wanted. □

Definition 2.1.8. Let $\preceq$ be a conradian left-preorder on G relative to C. By Proposition 1.3.2, we know that the set of $\preceq$-convex subgroups is linearly ordered, so we can consider the subgroups series given by all $\preceq$-convex subgroups. We call it the series of $\preceq$-convex subgroups.

Lemma 2.1.9. *Let $\preceq$ be a conradian-left preorder on G relative to C. Let G_1, G_2 be proper subgroups of G such that*

$$C \subsetneq G_1 \subsetneq G_2 \subsetneq G.$$

If G_1 and G_2 are $\preceq$-convex subgroups and $\mathcal{CO}_{G_1}(G_2)$ has no isolated elements, then $\preceq$ is not isolated on $\mathcal{CO}_C(G)$.

Proof. Assume that G_1 and G_2 are $\preceq$-convex subgroups and that $\mathcal{CO}_{G_1}(G_2)$ has no isolated elements. Denote by P the positive cone of $\preceq$. We now show that $\preceq$ is not isolated on $\mathcal{CO}_C(G)$, that is, we now prove that for every finite collection of elements belonging to P, there is a left-preorder $\bar{\preceq}$ on G relative to C such that all this collection is on the positive cone of $\bar{\preceq}$ but $\bar{\preceq} \neq \preceq$. We prove now this fact.

Take $g_1, \dots, g_k \in P$ a finite collection. We are going to consider the mutually inverse bijections μ and ρ from Proposition 1.5.5 for the finite chain of subgroups

$$C \subsetneq G_1 \subsetneq G_2 \subsetneq G.$$

Since G_1 and G_2 are $\preceq$-convex subgroups, we can consider

$$\rho(\preceq) = (\preceq_1, \preceq_2, \preceq_3) \quad \text{where} \quad \preceq_1 \in \mathcal{CO}_C(H_1)\,, \quad \preceq_2 \in \mathcal{CO}_{H_1}(G_2) \quad \text{and} \quad \preceq_3 \in \mathcal{CO}_{G_2}(G).$$

Consider P_1 the positive cone of $\preceq_1$, P_2 the positive cone of $\preceq_2$ and P_3 the positive cone of $\preceq_3$. By the definition the mutually inverse bijections μ and ρ given in Proposition 1.5.5, we have

$$P = P_1 \cup P_2 \cup P_3.$$

By possibly reordering the elements $g_1, \dots, g_k \in P$, we can assume

$$g_1, \dots, g_m \in P_2 \quad \text{and} \quad g_{m+1}, \dots, g_k \in P_1 \cup P_3. \tag{2.14}$$

Since $g_1, \dots, g_m$ is a finite (possibly empty) collection of $\preceq_2$-positive elements where $\preceq_2 \in \mathcal{CO}_{G_1}(G_2)$, and since $\mathcal{CO}_{G_1}(G_2)$ has no isolated elements by hypothesis, we can take $\bar{\preceq}_2 \in \mathcal{CO}_{G_1}(G_2)$ with positive cone $\bar{P}_2$ and satisfying that

$$g_1, \dots, g_m \in \bar{P}_2 \tag{2.15}$$

and $\bar{\preceq}_2$ is not equal to $\preceq_2$. Define $\bar{\preceq} = \mu(\preceq_1, \bar{\preceq}_2, \preceq_3)$. As μ and ρ mutually inverse bijections by Proposition 1.5.5, we obtain

$$\rho(\bar{\preceq}) = (\preceq_1, \bar{\preceq}_2, \preceq_3) \tag{2.16}$$

By the definition the mutually inverse bijections μ and ρ given in Proposition 1.5.5, we know that

$$\bar{P} = P_1 \cup \bar{P}_2 \cup P_3.$$

Then, from Equation (2.14) and Equation (2.15) we obtain

$$g_1, \dots, g_k \in \bar{P}. \tag{2.17}$$

The map ρ is bijective by Proposition 1.5.5, and we know that $\bar{\preceq}_2$ is not equal to $\preceq_2$, so from Equation (2.15) and Equation (2.15), we deduce that $\bar{\preceq}$ is not equal to $\preceq$. Therefore, $\bar{\preceq}$ is a conradian left-preorder on G relative to C such that $\bar{\preceq}$ is not equal to $\preceq$ and, by Equation (2.17), the elements $g_1, \dots, g_k$ are on the positive cone of $\bar{\preceq}$, as we wanted to prove. Hence, $\preceq$ is not isolated on $\mathcal{CO}_C(G)$, as we wanted. □

Proposition 2.1.10. *Let G be a group and C be a proper subgroup. Let $\preceq$ be a conradian left-preorder on G relative to C. If $\preceq$ is isolated on $\mathcal{CO}_C(G)$, then the series of $\preceq$-convex subgroups is a finite rational series starting on G, ending on C and without abelian jumps.*

Proof. Assume that $\preceq$ is isolated on $\mathcal{CO}_C(G)$. By Proposition 2.1.7, the set of $\preceq$-convex subgroups is finite, so the series of $\preceq$-convex subgroups is a finite. Then, we can write the series of $\preceq$-convex subgroups as

$$C = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

where $\{G_0 \dots G_n\}$ is the set of $\preceq$-convex subgroups. In particular, (G_i, G_{i+1}) is a $\preceq$-convex jump for all $0 \leq i < n$ since there are not any other $\preceq$-convex subgroups strictly in between. Using the fact that $\preceq$ is conradian, by Theorem 1.4.13, we deduce that G_i is normal on G_{i+1} and the group G_{i+1}/G_i is torsion-free abelian and non-trivial. We claim that G_{i+1}/G_i has abelian rank equal to 1. Indeed, if G_{i+1}/G_i has abelian rank greater than 1, then, by Example 1.6.19, $\mathcal{CO}_{G_i}(G_{i+1})$ has no isolated orders. Then, by Lemma 2.1.9 and since G_i and G_{i+1} are $\preceq$-convex, we deduce that $\preceq$ is not isolated on $\mathcal{CO}_C(G)$, a contradiction. Hence, we have shown that the series of convex subgroups given by

$$C = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

satisfies that G_i is normal on G_{i+1} and the group G_{i+1}/G_i is torsion-free abelian of abelian rank one for all $0 \leq i < n$. Hence, the series of convex subgroups given by

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

is a finite rational series. We claim that this series has no abelian jump. We prove it by contradiction. Assume that

$$G_i \trianglelefteq G_{i+1} \trianglelefteq G_{i+2}$$

is an abelian jump. Then, G_i is normal on G_{i+2} and the group G_{i+2}/G_i is torsion-free abelian, and it must have abelian rank 2 since we know that G_{i+1}/G_i and G_{i+2}/G_{i+1} both have abelian rank equal to 1. Then, by Example 1.6.19, $\mathcal{CO}_{G_i}(G_{i+2})$ has no isolated orders. Then, by Lemma 2.1.9 and since G_i and G_{i+2} are $\preceq$-convex, we deduce that $\preceq$ is not isolated on $\mathcal{CO}_C(G)$, a contradiction. Hence, we have shown that

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

is a finite rational series without abelian jumps. □

We can now prove the partial version of the main theorem of the section.

Theorem 2.1.11. *Let G be a group and C be a proper subgroup. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}_C(G)$ are isolated and $\mathcal{CO}_C(G)$ is non-empty.

b) There exists an element on $\mathcal{CO}_C(G)$ that is isolated.

c) There exists a finite rational series starting on G, ending on C and without abelian jumps.

d) $\mathcal{CO}_C(G)$ is finite and non-empty.

Proof. We are going to prove that each condition implies the next one. The fact that condition a) implies condition b) is direct. The fact that condition b) implies condition c) follows from Proposition 2.1.10. The fact that condition c) implies condition d) is a direct consequence of Proposition 2.1.6. We finally prove that condition d) implies condition a). By Corollary 1.6.3, $\mathcal{CO}_C(G)$ is Hausdorff, so $\mathcal{CO}_C(G)$ being finite implies that $\mathcal{CO}_C(G)$ is discrete, so then all the elements on $\mathcal{CO}_C(G)$ are isolated. □

In order to prove the main theorem of the section, we need two previous results.

Lemma 2.1.12. *Let G be a group and let C be a proper subgroup. Let H be a subgroup of G such that $C \subseteq H$. If $\mathcal{CO}_C(G) \neq \emptyset$, then*

$$\mathrm{Card}\left(\mathcal{CO}_H(G)\right) \leq \mathrm{Card}\left(\mathcal{CO}_C(G)\right).$$

Proof. Assume that $\mathcal{CO}_C(G) \neq \emptyset$. Then, we can consider $\preceq$ a conradian left-preorder on G relative to C. Take $\preceq_0$ the restriction of $\preceq$ to H. The map

$$\begin{array}{ccc} \mathcal{CO}_H(G) & \longrightarrow & \mathcal{CO}_C(G) \\ \preceq^* & \mapsto & \mu_{G,H,C}(\preceq_0, \preceq^*) \end{array}$$

from the set $\mathcal{CO}_H(G)$ to the set $\mathcal{CO}_C(G)$ is injective by Proposition 1.3.3. We deduce that $\mathrm{Card}\left(\mathcal{CO}_H(G)\right) \leq \mathrm{Card}\left(\mathcal{CO}_C(G)\right)$. □

Proposition 2.1.13. *Let G be a group and C be a proper subgroup. Let $\preceq$ be a conradian left-preorder on G relative to C. If $\mathcal{CO}_C(G)$ is finite, then the series of $\preceq$-convex subgroups is the unique finite rational series starting on G and ending on C.*

Proof. Assume $\mathcal{CO}_C(G)$ is finite. By Theorem 2.1.11, we deduce that $\preceq$ is isolated on $\mathcal{CO}_C(G)$. Hence, by Proposition 2.1.10, we obtain that the series of $\preceq$-convex subgroups is a finite rational series starting on G and ending on C. Therefore, it only remains to check the uniqueness. In order to do that, consider

$$C = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

the series of $\preceq$-convex subgroups. We are going to prove the uniqueness by induction in the length $n \in \mathbb{N}_{>0}$ of the previous series (the case $n = 0$ is impossible as C is proper). For $n = 1$ it is trivial.

We now do the inductive case. We assume as inductive hypothesis that it is true for all length smaller than n, and we are going to prove it for $n > 0$. Let

$$C = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = G$$

be another finite rational series starting on G and ending on C. Denote $K = G_{n-1}$ and $H = H_{m-1}$. If we assume that

$$K = H, \tag{2.18}$$

then applying inductive hypothesis to the series

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{n-1} = K \quad \text{and} \quad C = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{m-1} = H = K$$

we conclude that $n = m$ and the series

$$C = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \quad \text{and} \quad C = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = G$$

are equal, as we wanted. Therefore, we only need to prove Equation (2.18).

Before proving it, we claim that $G/(K \cap H)$ is a torsion-free abelian group of abelian rank 1. We now prove this. By definition of being rational, we have that G/K and G/H are torsion-free abelian groups. Then, the group $G/K \times G/H$ is torsion-free abelian. When we consider the group morphism

$$\begin{array}{ccc} G & \longrightarrow & G/K \times G/H \\ g & \mapsto & (gK, gH) \end{array}$$

it has kernel $K \cap H$, so the first isomorphism theorem give us an injective morphism from $G/(K \cap H)$ to $G/K \times G/H$, then $G/(K \cap H)$ is a torsion-free abelian group. By Lemma 2.1.12, since $\preceq \in \mathcal{CO}_C(G)$ and since $\mathrm{Card}\,(\mathcal{CO}_C(G))$ is finite, we have that $\mathcal{CO}_{K\cap H}(G)$ is finite. Hence, we have that $G/(K \cap H)$ is a torsion-free abelian group such that $\mathcal{CO}_{K\cap H}(G)$ is finite. This implies, by Example 1.6.19, that $G/(K \cap H)$ is a torsion-free abelian group of abelian rank 1 or 0. Moreover, $G/(K \cap H)$ is a torsion-free abelian group of abelian rank 1 since $K \cap H \neq G$. This proves the claim, as we wanted.

We are now able to prove Equation (2.18). We prove this by showing that $g \notin K$ implies $g \notin H$ for each $g \in G$ and that $g \notin H$ implies $g \notin K$ for each $g \in G$. We just prove the first implication as the second one is analogous. Consider $g \in G$ such that $g \notin K$, we want to prove that $g \notin H$. As $H \neq G$, we can take $h \in G$ such that

$$hH \neq H. \tag{2.19}$$

Then we have that $h \notin H$ and $g \notin H$, so then

$$h \notin K \cap H \quad \text{and} \quad g \notin K \cap H.$$

Then, using the fact that $G/(K \cap H)$ is a torsion-free abelian group of abelian rank 1, we deduce that there exists $r, s \in \mathbb{Z}$ non-zero numbers such that

$$g^r(K \cap H) = h^s(K \cap H).$$

Then, we have

$$g^{-r}h^s \in K \cap H \subseteq H.$$

Then, we deduce

$$g^r H = h^s H.$$

Applying Equation (2.19) and the fact that r and s are non-zero, we deduce

$$gH \neq H.$$

Hence, $g \notin H$. This finishes the proof of Equation (2.18). □

We are now able to prove the main theorem of the section.

Theorem 2.1.14. *Let G be a group and C be a proper subgroup. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}_C(G)$ are isolated and $\mathcal{CO}_C(G)$ is non-empty.

b) There exists an element on $\mathcal{CO}_C(G)$ that is isolated.

c) There exists a finite rational series starting on G, ending on C and without abelian jumps.

d) There exists a unique finite rational series starting on G and ending on C, and this series does not have abelian jumps.

e) $\mathcal{CO}_C(G)$ is finite and non-empty.

Moreover, under these these equivalent conditions, we have that $\mathrm{Card}\,(\mathcal{CO}_C(G)) = 2^n$ *for n the length of any series as in c) and d).*

Proof. The conditions a), b), c) and e) are equivalent by Theorem 2.1.11. The condition d) obviously implies the condition c). The condition c) implies the condition d) by Proposition 2.1.13 since $\mathcal{CO}_C(G)$ is finite by Theorem 2.1.11. By Proposition 2.1.6, we have that $\mathrm{Card}\,(\mathcal{CO}_C(G)) = 2^n$ for n the length of any series as in c) and d). □

Remark 2.1.15. Following the notation of the condition d) from Theorem 2.2.8, we have that C is not normal in general. However, when C is normal on G, then the unique finite rational series starting on G and ending on C satisfies that all the subgroups appearing on this series are normal on G. This is a direct from the uniqueness and the fact that conjugating a finite rational series starting on G and ending on C provide us another finite rational series starting on G and ending on C, since C is normal on G.

As a consequence, we can study the cardinality of Card $(\mathcal{CO}_C(G))$.

Corollary 2.1.16. *Let G be a group and C be a proper subgroup. If $\mathcal{CO}_C(G)$ is finite and non-empty, then* Card $(\mathcal{CO}_C(G)) = 2^n$ *for some $n \in \mathbb{N}$ such that $n > 1$.*

Proof. Apply Theorem 2.1.14. □

Corollary 2.1.17. *Let G be a group and C be a proper subgroup. Then $\mathcal{CO}_C(G)$ is either finite or uncountable.*

Proof. If $\mathcal{CO}_C(G)$ is not finite, by Theorem 2.1.14, we deduce that $\mathcal{CO}_C(G)$ has no isolated points. Then, by Corollary 1.6.13 and Corollary 1.6.3, we deduce that $\mathcal{CO}_C(G)$ is a non-empty compact Hausdorff space without isolated points. Then, we deduce that $\mathcal{CO}_C(G)$ is uncountable (see Theorem 27.7 from [42] for example). □

Remark 2.1.18. From the topological properties of $\mathcal{CO}_C(G)$ discussed in Section 1.6, by applying the characterization of Cantor sets (Proposition 1.6.17), when $\mathcal{CO}_C(G)$ is not finite and G is countable, we have that $\mathcal{CO}_C(G)$ is a Cantor set.

2.2 Characterizing finiteness of spaces of $\mathcal{CO}(G)$

In this section show Theorem 2.2.8. We will prove it using Theorem 2.1.14. As in the previous section, we state the main result here in order to keep in mind the objective.

Definition 2.2.1. Let G be a group. A rational series starting on G of the form

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

is of maximal length if for any other

$$H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G$$

rational series starting on G, we have $k \leq n$.

Theorem (Theorem 2.2.8)**.** *Let G be a group. Assume that $\mathcal{CO}(G)$ is non-empty. Let $C_0 = \bigcap_{H \in \mathcal{H}} H$ for $\mathcal{H} = \{H$ subgroup with $\mathcal{CO}_H(G) \neq \emptyset\}$. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}(G)$ are isolated.

b) There exists a finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end.

c) There exists a unique finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end.

d) There exists a finite rational series starting on G and ending on C_0, and this series has no abelian jumps.

e) There exists a unique finite rational series starting on G and ending on C_0, and this series has no abelian jumps.

f) $\mathcal{CO}_{C_0}(G)$ is finite.

g) $\mathcal{CO}(G)$ is finite.

Moreover, under these these equivalent conditions, we have that Card $(\mathcal{CO}(G)) = 2^{n+1} - 2$ *and* Card $(\mathcal{CO}_{C_0}(G)) = 2^n$ *for n the length of any series as in b), c), d) and e).*

We prove a partial version of the theorem (Theorem 2.2.5). Before, we need some propostions and some auxiliary lemmas.

Lemma 2.2.2. *Let G be a group. Let*

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \quad \text{and} \quad H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G$$

be two finite rational series on G both starting on G. Then, there exists

$$G_0 \cap H_0 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_l = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G$$

a finite rational series starting on G and ending on $G_0 \cap H_0$ of length $k+l$ for some $l \leq n$.

Proof. Notice that for every $i = 1, \ldots, n$ we have

$$G_{i-1} \cap H_0 \trianglelefteq G_i \cap H_0$$

since $G_{i-1} \trianglelefteq G_i$. First, we claim that for every $i = 1, \ldots, n$ we have

$$G_{i-1} \cap H_0 = G_i \cap H_0 \quad \text{or} \quad (G_i \cap H_0)/(G_{i-1} \cap H_0) \text{ is torsion-free abelian of abelian rank 1.} \tag{2.20}$$

We now prove it. Consider the morphism

$$\begin{array}{rccc} \phi\colon & G_i \cap H_0 & \longrightarrow & G_i/G_{i-1} \\ & g & \mapsto & gG_{i-1} \end{array}$$

and notice that its kernel is

$$\{g \in G_i \cap H_0 \ / \ gG_{i-1} = G_{i-1}\} = G_{i-1} \cap G_i \cap H_0 = G_{i-1} \cap H_0.$$

Then, by the isomorphism theorem, we can identify $(G_i \cap H_0)/(G_{i-1} \cap H_0)$ with a subgroup of G_i/G_{i-1}. Then, since G_i/G_{i-1} is a torsion-free abelian group of abelian rank 1 as the corresponding series is rational, we deduce that $(G_i \cap H_0)/(G_{i-1} \cap H_0)$ is a torsion-free abelian group of abelian rank 1 or 0. If $(G_i \cap H_0)/(G_{i-1} \cap H_0)$ has abelian rank 1 we are done, and if it has abelian rank 0, then the group is trivial, so we have $G_{i-1} \cap H_0 = G_i \cap H_0$. Hence, we have proven Equation (2.20).

Now, consider the finite series given by

$$H_0 \cap G_0 \trianglelefteq H_0 \cap G_1 \trianglelefteq \ldots \trianglelefteq H_0 \cap G_n = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G.$$

When removing the repeated subgroups, using Equation (2.20) and the fact that

$$H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k$$

is rational, we obtain

$$G_0 \cap H_0 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_l = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_k = G$$

a finite rational series starting on G and ending on $G_0 \cap H_0$ of length $k+l$ for some $l \leq n$. □

Proposition 2.2.3. *Let G be a group. Assume that $\mathcal{CO}(G) \neq \emptyset$. Let $\mathcal{H} = \{H \text{ subgroup with } \mathcal{CO}_H(G) \neq \emptyset\}$ and $C_0 = \bigcap_{H \in \mathcal{H}} H$. If $\mathrm{Card}\,(\mathcal{CO}_{C_0}(G)) < \infty$, then $\mathrm{Card}\,(\mathcal{CO}(G)) < \infty$. Moreover, if $n \in \mathbb{N}$ is such that $\mathrm{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$, then*

$$\mathrm{Card}\,(\mathcal{CO}(G)) = 2^{n+1} - 2.$$

Proof. Assume that $\mathrm{Card}\,(\mathcal{CO}_{C_0}(G)) < \infty$. Notice that $\mathcal{CO}_{C_0}(G) \neq \emptyset$ by applying Proposition 1.6.24 and looking at the definition of C_0. Take $n \in \mathbb{N}$ such that $\mathrm{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$, which is possible by Corollary 2.1.16. Then, by Theorem 2.1.11, we can take

$$C_0 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

a finite rational series starting on G, ending on C_0 and without abelian jumps.

We claim that for each H subgroup of G we have

$$\mathcal{CO}_H(G) \neq \emptyset \quad \Rightarrow \quad \text{Card}\left(\mathcal{CO}_H(G)\right) < \infty. \tag{2.21}$$

We now prove it. Consider H a subgroup of G such that $\mathcal{CO}_H(G) \neq \emptyset$. Since $\mathcal{CO}_{C_0}(G) \neq \emptyset$ and since $\text{Card}\left(\mathcal{CO}_{C_0}(G)\right) < \infty$, by applying Lemma 2.1.12, we deduce that $\text{Card}\left(\mathcal{CO}_H(G)\right) < \infty$. This proves Equation (2.21), as we claimed.

We claim that for each H subgroup of G we have

$$\mathcal{CO}_H(G) \neq \emptyset \quad \Rightarrow \quad H = G_i \text{ for some } i \in \{0, \dots, n-1\}. \tag{2.22}$$

We now prove it. Consider H subgroup of G such that $\mathcal{CO}_H(G) \neq \emptyset$. By Equation (2.21), we know that $\text{Card}\left(\mathcal{CO}_H(G)\right) < \infty$. Then, applying Theorem 2.1.11, we deduce that we can take

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_k = G$$

a finite rational series starting on G, ending on H and without abelian jumps. Then, applying Lemma 2.2.2 to

$$C_0 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G \quad \text{and} \quad H = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_k = G,$$

we deduce that there exists

$$C_0 \cap H = L_0 \trianglelefteq L_1 \trianglelefteq \dots \trianglelefteq H = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_k = G$$

a finite rational series starting on G and ending on $C_0 \cap H$. Notice that $C_0 \cap H = C_0$ by definition of C_0 and since $\mathcal{CO}_H(G) \neq \emptyset$. Hence, there exists

$$C_0 = L_0 \trianglelefteq L_1 \trianglelefteq \dots \trianglelefteq H = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_k = G$$

a finite rational series starting on G and ending on C_0, but we know that $\text{Card}\left(\mathcal{CO}_{C_0}(G)\right) < \infty$ and that

$$C_0 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

is another finite rational series starting on G and ending on C_0, so then the uniqueness given in Proposition 2.1.13 implies that $H = G_i$ for some $i \in \{0, \dots, n-1\}$. This proves Equation (2.22).
By definition of $\mathcal{CO}(G)$, we have

$$\mathcal{CO}(G) = \bigcup_{L \text{ subgroup of } G} \mathcal{CO}_L(G).$$

Applying Equation (2.22), we obtain

$$\mathcal{CO}(G) = \bigcup_{i=0}^{n-1} \mathcal{CO}_{G_i}(G).$$

We deduce that

$$\text{Card}\left(\mathcal{CO}(G)\right) = \sum_{i=0}^{n-1} \text{Card}\left(\mathcal{CO}_{G_i}(G)\right).$$

Then, applying Proposition 2.1.6, we obtain

$$\text{Card}\left(\mathcal{CO}(G)\right) = \sum_{i=0}^{n-1} 2^{n-i} = \sum_{j=1}^{n} 2^j = \left(\sum_{j=0}^{n} 2^j\right) - 1 = \left(2^{n+1} - 1\right) - 1 = 2^{n+1} - 2.$$

Therefore, $\text{Card}\left(\mathcal{CO}(G)\right) = 2^{n+1} - 2$ and, in particular, $\text{Card}\left(\mathcal{CO}(G)\right) < \infty$. □

Lemma 2.2.4. *Let G be a group. Assume that $\mathcal{CO}(G)$ is non-empty. Let $C_0 = \bigcap_{H \in \mathcal{H}} H$ for $\mathcal{H} = \{H \text{ subgroup with } \mathcal{CO}_H(G) \neq \emptyset\}$. Suppose that there exists a finite rational series starting on G of maximal length, and this series ends on G_0 satisfying $\mathcal{CO}(G_0) = \emptyset$ and has no abelian jump. Then, $G_0 = C_0$.*

Proof. The inclusion $C_0 \subseteq G_0$ is clear by definition of C_0 and since $\mathcal{CO}_{G_0}(G) \neq \emptyset$ as a consequence of Theorem 2.1.11. We prove the other inclusion $G_0 \subseteq C_0$ by contradiction, so assume that $C_0 \subsetneq G_0$. By Proposition 1.6.24 and by the definition of C_0, we deduce that $\mathcal{CO}_{C_0}(G) \neq \emptyset$. Hence, we can take $\preceq$ a conradian left-preorder on G relative to C_0. Since $C_0 \subsetneq G_0$, the restriction of $\preceq$ on G_0 give us a left-preorder on G relative to C_0. Hence, we have obtained a left-preorder on G relative to C_0, so we have an element on $\mathcal{CO}(G_0)$, a contradiction with the fact that $\mathcal{CO}(G_0) = \emptyset$. This shows that $G_0 \subseteq C_0$. Therefore, we conclude that $G_0 = C_0$. □

We state the partial version of the main theorem of the section.

Theorem 2.2.5. *Let G be a group. Assume that $\mathcal{CO}(G)$ is non-empty. Let $C_0 = \bigcap_{H \in \mathcal{H}} H$ for $\mathcal{H} = \{H$ subgroup with $\mathcal{CO}_H(G) \neq \emptyset\}$. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}(G)$ are isolated.

b) There exists a finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end.

c) There exists a finite rational series starting on G and ending on C_0, and this series has no abelian jumps.

d) $\mathcal{CO}_{C_0}(G)$ is finite.

e) $\mathcal{CO}(G)$ is finite.

Proof. We are going to prove that a) implies d), b) implies c), c) implies b), c) implies d), d) implies e), e) implies a) and e) implies c).

We prove that a) implies d). Assume that all the elements on $\mathcal{CO}(G)$ are isolated. Since all the elements on $\mathcal{CO}(G)$ are isolated and $\mathcal{CO}_{C_0}(G)$ is a subspace of $\mathcal{CO}(G)$, we deduce that all the elements on $\mathcal{CO}_{C_0}(G)$ are isolated. Then, by Theorem 2.1.11, we deduce that $\mathcal{CO}_{C_0}(G)$ is finite. This proves that a) implies d).

We show that b) implies c). Assume that there exists a finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end. By Lemma 2.2.4, we have $G_0 = C_0$. Then, the previous series is a finite rational series starting on G and ending on C_0 without abelian jumps, as we need. This shows that that b) implies c).

We prove that c) implies b). Suppose that there exists a finite rational series starting on G and ending on C_0, and this series has no abelian jumps. If we show that the previous series has maximal length and $\mathcal{CO}(C_0) = \emptyset$, we obtain b). We are going to prove each fact independently. Firstly, we prove that $\mathcal{CO}(C_0) = \emptyset$ by contradiction. If $\mathcal{CO}(C_0)$ is non-empty, then there exists a subgroup $H \subsetneq C_0$ such that $\mathcal{CO}_H(C_0)$ is non-empty. Then, as $\mathcal{CO}_{C_0}(G)$ is non-empty by Theorem 2.1.11, applying Lemma 2.1.12 we obtain that $\mathcal{CO}_H(G)$ is non-empty. This contradicts the fact that $H \subsetneq C_0$ and the definition of C_0. Secondly, we prove that the finite rational series starting on G and ending on C_0 has maximal length. We do this by contradiction. Denote by n the length of this series, and assume that there exists a finite rational series starting on G and ending on some subgroup K and having length k for $k > n$. Then, applying Lemma 2.2.2 to these two series, we obtain a finite rational series starting on G and ending on $K \cap C_0$ and having length r for $r \geq k > n$. Notice that $K \cap C_0 = C_0$ by definition of C_0 and since $\mathcal{CO}_K(G) \neq \emptyset$ by Theorem 2.1.11. Hence, we have a finite rational series starting on G and ending on C_0 and having length r for $r > n$, but we also have a finite rational series starting on G and ending on C_0 having length n, so we have a contradiction with the uniqueness provided by Proposition 2.1.13 (we know that $\mathcal{CO}_{C_0}(G)$ is finite by Theorem 2.1.11). This finishes the proof of c) implies b).

Now, c) implies d) is direct from Theorem 2.1.11. The fact that d) implies e) is clear by Proposition 2.2.3. The fact that e) implies a) follows since $\mathcal{CO}(G)$ is Hausdorff by Corollary 1.6.3 and every finite Hausdorff space is discrete, so all its elements are isolated. The fact that e) implies c) follows by noticing that $\mathcal{CO}_{C_0}(G)$ is s subset of $\mathcal{CO}(G)$. □

Remark 2.2.6. Following the notation of Theorem 2.2.5, if we consider

$$C_0 = \bigcap_{H \in \mathcal{H}} H \quad \text{for} \quad \mathcal{H} = \{H \text{ subgroup with } \mathcal{CO}_H(G) \neq \emptyset\},$$

then C_0 is a normal subgroup of G. This is direct from the fact that $\mathcal{CO}_H(G) \neq \emptyset$ implies that $\mathcal{CO}_{gHg^{-1}}(G) \neq \emptyset$ for all $g \in G$ by just conjugating the positive cone. Then, one can consider the group G/C_0. One can build a map from $\mathcal{CO}(G)$ to $\mathcal{CO}(G/C_0)$ induced by the projection and, using the definition of C_0, one can show that it is in fact an homeomorphism. We do not prove this and we left it as an exercise since this is not necessary for our work, but we find it illustrative. In this way, knowing this fact, it is easy to prove that $\mathcal{CO}(G)$ being finite is equivalent to say that there is C a normal subgroup in G satisfying that $\mathcal{CO}(C)$ is empty and $\mathcal{CO}_{\{1\}}(G/C)$ is finite.

Previous to the proof of the main theorem we need a proposition.

Proposition 2.2.7. *Let G be a group. If there exists a finite rational series starting on G of maximal length, then it is unique.*

Proof. Take

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \quad \text{and} \quad H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

two finite rational series starting on G of maximal length n. We want to prove that both series are equal. If $n = 0$ is trivial, so assume $n > 0$. In particular, we have that G_0 and H_0 are proper subgroups of G. By applying Lemma 2.2.2, we deduce that there exists

$$G_0 \cap H_0 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_l = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

a finite rational series starting on G and ending on $G_0 \cap H_0$ of length $n + l$ for some $l \leq n$. Since n is the maximal length of such series, we deduce that $l = 0$ and, in particular, by looking at the last series we have $G_0 \cap H_0 = H_0$. Hence, we have shown that $H_0 \subseteq G_0$. Analogously and interchanging the two series of the start of the proof, we can also show that $G_0 \subseteq H_0$. Therefore, we conclude that $G_0 = H_0$. Hence, the series

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \quad \text{and} \quad G_0 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G$$

are two finite rational series starting on G and ending on $G_0 = H_0$. Also notice that $\mathcal{CO}_{G_0}(G)$ is finite by Theorem 2.1.11. Then, by the uniqueness from Proposition 2.1.13, we deduce that these two series are equal. □

We now present the main theorem.

Theorem 2.2.8. *Let G be a group. Assume that $\mathcal{CO}(G)$ is non-empty. Let $C_0 = \bigcap_{H \in \mathcal{H}} H$ for $\mathcal{H} = \{H$ subgroup with $\mathcal{CO}_H(G) \neq \emptyset\}$. Then, the following conditions are equivalent:*

a) All the elements on $\mathcal{CO}(G)$ are isolated.

b) There exists a finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end.

c) There exists a unique finite rational series starting on G of maximal length, and this series has no abelian jumps and satisfies that $\mathcal{CO}(G_0) = \emptyset$ for G_0 the subgroup on which the series end.

d) There exists a finite rational series starting on G and ending on C_0, and this series has no abelian jumps.

e) There exists a unique finite rational series starting on G and ending on C_0, and this series has no abelian jumps.

f) $\mathcal{CO}_{C_0}(G)$ is finite.

g) $\mathcal{CO}(G)$ is finite.

Moreover, under these these equivalent conditions, we have that $\mathrm{Card}\,(\mathcal{CO}(G)) = 2^{n+1} - 2$ *and* $\mathrm{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$ *for n the length of any series as in b), c), d) and e).*

Proof. The fact that a), b), d), f) and g) are equivalent between them is by Theorem 2.2.5. The fact that b) and c) are equivalent follows by Proposition 2.2.7, and the fact that d) and e) are equivalent follows by Proposition 2.1.13 since $\mathcal{CO}_{C_0}(G)$ is finite as a consequence of Theorem 2.1.11. Finally, we prove that under these these equivalent conditions, we have that $\text{Card}\,(\mathcal{CO}(G)) = 2^{n+1} - 2$ and $\text{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$ for n the length of any series as in b), c), d) and e). Assume that we are under all the equivalent conditions of the statement. Since b) holds, Lemma 2.2.4 implies that the rational series appearing on conditions b) and c) they end on C_0. Then, by the uniqueness given by Proposition 2.1.13, we deduce that there is a unique rational series satisfying some of the equivalent conditions b), c), d) and e). Hence, the all have the same length, say n. By Proposition 2.1.6, we have $\text{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$. Then, by Proposition 2.2.3, we also deduce that $\text{Card}\,(\mathcal{CO}(G)) = 2^{n+1}-2$. This shows that $\text{Card}\,(\mathcal{CO}(G)) = 2^{n+1}-2$ and $\text{Card}\,(\mathcal{CO}_{C_0}(G)) = 2^n$ for n the length of any series as in b), c), d) and e). □

Remark 2.2.9. Following the notation of the condition c) from Theorem 2.2.8, the unique finite rational series starting on G of maximal length satisfies that all the subgroups appearing on the series are normal on G. This is a consequence of the uniqueness and the fact that conjugating a finite rational series starting on G of maximal length give us another finite rational series starting on G of maximal length.

If one tries to establish an analogy between the two main theorems of each section (Theorem 2.2.8 and Theorem 2.1.14), one may think that it is reasonable to say that in the list of equivalent conditions of the main theorem of this section (Theorem 2.2.8) is missing the following condition: there is an isolated element on $\mathcal{CO}(G)$. However, it is false that it is equivalent to the other conditions. Clearly, the condition a) implies that there is an isolated element on $\mathcal{CO}(G)$, but the reciprocal is not true. We show this with an example of a group having some isolated element and some non-isolated element.

Example 2.2.10. *Consider the group $\mathbb{Z}^2$. Since it is abelian, all the subgroups are normal and all the left-preorders are conradian. Consider $C = \langle (0,1) \rangle$. Since $\mathbb{Z}^2/C$ is cyclic, there are only 2 conradian left-preorders on $\mathbb{Z}^2$ relative to C, say $\preceq_C^+$ and $\preceq_C^-$. Then, we have*

$$\{\preceq' \in \mathcal{CO}(\mathbb{Z}) \,/\, \phi_{\preceq}(0,1) = 0,\ \phi_{\preceq}(1,0) = 1\} = \{\preceq_C^+\} \quad \textit{and}$$

$$\{\preceq' \in \mathcal{CO}(\mathbb{Z}) \,/\, \phi_{\preceq}(0,1) = 0,\ \phi_{\preceq}(1,0) = -1\} = \{\preceq_C^-\}.$$

Hence, both conradian left-preorders are isolated on $\mathcal{CO}(\mathbb{Z}^2)$. However, by Example 1.6.19 we know that $\mathcal{CO}_{\{(0,0)\}}(\mathbb{Z}^2)$ has no isolated elements, so all the elements in $\mathcal{CO}_{\{(0,0)\}}(\mathbb{Z}^2)$ are not isolated on $\mathcal{CO}(\mathbb{Z}^2)$.

We can use the previous theorem to study the cardinality of $\text{Card}\,(\mathcal{CO}_C(G))$.

Corollary 2.2.11. *Let G be a group. Then $\mathcal{CO}(G)$ is either finite or uncountable.*

Proof. Assume that $\mathcal{CO}(G)$ is not finite. By Theorem 2.2.8, we deduce that $\mathcal{CO}_C(G)$ is not finite for a certain C subgroup. Then, by Corollary 2.1.17, we deduce that $\mathcal{CO}_C(G)$ is uncountable. Then, since $\mathcal{CO}_C(G)$ is a subset of $\mathcal{CO}(G)$, we deduce that $\mathcal{CO}(G)$ is uncountable. □

We now present a proposition that allows us to skip the requirement of $\mathcal{CO}(G_0) = \emptyset$ on conditions b) and c) of Theorem 2.2.8 when we know that G_0 is finitely generated (for example, for groups whose subgroups are all finitely generated).

Proposition 2.2.12. *Let G be a group. Assume that there exists a finite rational series starting on G of maximal length. Denote by G_0 the subgroup where this series ends. If G_0 is finitely generated, then $\mathcal{CO}(G_0) = \emptyset$.*

Proof. Suppose that exists

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

a finite rational series starting on G of maximal length n, so that this series ends on the finitely generated subgroup G_0. We do this by contradiction, so assume $\mathcal{CO}(G_0) \neq \emptyset$. Then, we can consider $\preceq$ a conradian left-preorder on G_0 relative to C for C a proper subgroup of G_0. By Corollary 1.4.5, there exists a maximal proper $\preceq$-convex subgroup H. This implies that (H, G_0) is a $\preceq$-convex jump. Then, since $\preceq$ is conradian, by Theorem 1.4.13, we deduce that H is normal on G_0 and the group G_0/H is torsion-free abelian and non-trivial. In fact G_0/H is finitely generated as G_0 is by hypothesis, so H is normal on G_0 and the

group G_0/H is finitely-generated torsion-free abelian and non-trivial. Then, we can take $\phi : G_0/H \longrightarrow \mathbb{Z}$ a surjective group morphism, and we can also take the surjective group morphism $\pi : G_0 \longrightarrow G_0/H$ given by the canonical projection. Hence, the group morphism $\phi \circ \pi : G_0 \longrightarrow \mathbb{Z}$ is surjective. Then, if we consider L the kernel of $\phi \circ \pi$, we deduce that L is normal on G_0 and the group G_0/L is isomorphic to $\mathbb{Z}$. In particular, L is normal on G_0 and G_0/L is a torsion-free abelian group of abelian rank 1. Hence,

$$L \trianglelefteq G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

is a finite rational series starting on G of length $n+1$. This contradicts that the maximal length is n. □

In the previous proposition it is necessary to put the condition of G_0 being finitely generated, as we can see in the following example inspired on the collection of groups $(T_n)_{n\in\mathbb{N}_{>0}}$ from Section 4.2 of [50] and from Example 2.2.14 of [17].

Example 2.2.13. *Consider T to be the group generated by the set of generators $\{a_i \ / \ i \in \mathbb{N}_{>0}\}$ and by the relations*

$$a_{i+1}a_ia_{i+1}^{-1} = a_i^{-1} \quad \text{for all } \ i \in \mathbb{N}_{>0} \qquad \text{and} \qquad a_ia_j = a_ja_i \quad \text{for all } \ |i-j| \geq 2.$$

For each $n \in \mathbb{N}_{>0}$ we consider the subgroup

$$T_n = \langle a_1, \ldots, a_n \rangle,$$

and this T_n corresponds to the one given in [50], and in this reference it is shown that T_n admits a conradian left-order (in fact, it has 2^n conradian left-orders). Notice that we have

$$\{1\} \trianglelefteq T_1 \trianglelefteq T_2 \trianglelefteq \ldots \trianglelefteq T_n \trianglelefteq \ldots \subseteq T \qquad \text{and} \qquad T = \bigcup_{n\in\mathbb{N}_{>0}} T_n,$$

where each $T_i \trianglelefteq T_{i+1}$ is rational. Clearly, each finitely generated subgroup of T is contained in some T_n, which we know admits a conradian left-order. Then, all finitely generated subgroup of T admits a conradian left-order so, by Theorem 9.15 from [11], we deduce that T admits a conradian left-order. Hence, $\mathcal{CO}(T) \neq \emptyset$.

Notice that T has trivial abelianization, so there is no normal proper subgroup on T whose quotient is abelian. Hence, the unique finite rational series on T starting on T of maximal length is the one of length 0, that is, the series with just T and no other subgroups. As we have shown before $\mathcal{CO}(T) \neq \emptyset$. This example shows us that in Proposition 2.2.12 the condition of G_0 being finitely generated cannot be avoided.

Also, as a consequence of the theorem, we obtain a result that implies theorems A and B of [50] when realizing that the uniqueness of the series below implies the normality.

Corollary 2.2.14. *Let G be a group. Assume that G admits a conradian left-order. Then, the following conditions are equivalent:*

a) There is an isolated element on $\mathcal{CO}_{\{1\}}(G)$.

b) All the elements on $\mathcal{CO}_{\{1\}}(G)$ are isolated.

c) All the elements on $\mathcal{CO}(G)$ are isolated.

d) There exists a finite rational series starting on G and ending on $\{1\}$, and this series has no abelian jumps.

e) There exists a unique finite rational series starting on G and ending on $\{1\}$, and this series has no abelian jumps.

f) $\mathcal{CO}_{\{1\}}(G)$ is finite.

g) $\mathcal{CO}(G)$ is finite.

Moreover, under these these equivalent conditions, we have that $\text{Card}\left(\mathcal{CO}(G)\right) = 2^{n+1}-2$ *and* $\text{Card}\left(\mathcal{CO}_{\{1\}}(G)\right) = 2^n$ *for n the length of any series as in d) and e).*

Proof. The set of conradian left-orders on G is equal to $\mathcal{CO}_{\{1\}}(G)$. Then, the fact that G admits a conradian left-order implies that $\mathcal{CO}_{\{1\}}(G)$ is non-empty. This implies that $\mathcal{CO}(G)$ is non-empty and that

$$\bigcap_{H \text{ subgroup with } \mathcal{CO}_H(G)\neq\emptyset} H = \{1\}.$$

Knowing this, we deduce from Theorem 2.2.8 that the conditions c), d), e), f) and g) are equivalent and that, under these conditions, we have that $\text{Card}\left(\mathcal{CO}(G)\right) = 2^{n+1} - 2$ and $\text{Card}\left(\mathcal{CO}_{\{1\}}(G)\right) = 2^n$ for n the length of any series as in d) and e). From Theorem 2.1.11 we deduce that the conditions a), b) and d) are equivalent. □

We use all this machinery to compute the number of left-preorders and the number of conradian left-preorders in the collection of groups $(T_n)_{n\in\mathbb{N}_{>0}}$ from Section 4.2 of [50] and from Example 2.2.14 of [17], that has also appeared in Example 2.2.13.

Example 2.2.15. *For each $n \in \mathbb{N}_{>0}$, define T_n as the group generated by $\{a_1, \dots, a_n\}$ with the relations*

$$a_{i+1}a_i a_{i+1}^{-1} = a_i^{-1} \quad \text{for all } i = 1, \dots, n-1 \qquad \text{and} \qquad a_i a_j = a_j a_i \quad \text{for all } |i-j| \geq 2.$$

Fix $n \in \mathbb{N}_{>0}$. One can prove that, if $\preceq$ is a left-preorder on T_n relative to C, then $C = \langle a_1, \dots, a_k\rangle$ for some $k = 0, \dots, n-1$ (follows by some easy computations using the relations and basic properties of left-preorders). Then, since T_n is left-orderable, by Proposition 1.3.12, we can extend it to a left-order. As in Example 2.2.14 of [17], one can check that $\preceq$ is determined by the sign (which is 1 if it is on the positive cone or -1 if its inverse is on the positive cone) of the elements $a_{k+1}, \dots, a_n$, so there are at most 2^{n-k} left-preorders of this form. When considering all possible values of k, we deduce that there are at most

$$\sum_{k=0}^{n-1} 2^{n-k} = 2^{n+1} - 2$$

left-preorders on T_n. As in Example 2.2.14, notice that

$$\{1\} \trianglelefteq \langle a_1\rangle \trianglelefteq \langle a_1, a_2\rangle \trianglelefteq \dots \trianglelefteq \langle a_1, \dots, a_n\rangle = T_n$$

is a rational series, and it is easy to see that it has no abelian jumps. By Corollary 2.2.14, T_n has $2^{n+1} - 2$ conradian left-preorders. Since we know that T_n has at most $2^{n+1} - 2$ left-preorders, we conclude that the space of left preorders on T_n and the space of conradian left preorders on T_n are equal and consists on $2^{n+1} - 2$ left-preorders.

The previous example can be used to determine the set of left-preorder on the Klein-bottle group.

Example 2.2.16. *Consider K the Klein-bottle group, which is generated by $\{a, b\}$ with the relation $aba^{-1} = b^{-1}$. As it is done in Example 2.2.15, we deduce that it has 6 left-preorders. In fact, from Example 2.2.15 one can deduce that the possible left-preorders are given by the following different positive cones: $\langle a, b\rangle^+$, $\langle a^{-1}, b\rangle^+$, $\langle a, b^{-1}\rangle^+$, $\langle a^{-1}, b^{-1}\rangle^+$, $\langle a\rangle^+$ and $\langle a^{-1}\rangle^+$.*

In the last two examples we have that the space of left-preorders and the space of conradian left-preorders are both finite. However, this is not always the case, as we see in the following example.

Example 2.2.17. *In Section 4.2 of [50], Rivas considers a collection of groups $(C_n)_{n\in\mathbb{N}_{>0}}$ such that C_n has 2^n conradian left-orders but infinitely many left-orders. Since all left-orders are left-preorders, we deduce that C_n has infinitely many left-preorders for all $n \in \mathbb{N}_{>0}$. Applying Corollary 2.2.14, we deduce that this collection $(C_n)_{n\in\mathbb{N}_{>0}}$ satisfies that C_n has $2^{n+1} - 2$ conradian left-preorders. This, for each $n \in \mathbb{N}_{>0}$, provides us an example of a group having $2^{n+1} - 2$ conradian left-preorders but infinitely many left-preorders*

Chapter 3

Auxiliary results on subgroups of finite Kurosh rank

In this chapter we use Bass-Serre theory to obtain some results about subgroups of finite Kurosh rank. We have three sections. The first one is about basic definitions and notations on graph theory. In the second section we build the Bass-Serre universal covering tree relative to a free product and we consider its quotient by a subgroup having finite Kurosh rank. In the third chapter we apply Bass-Serre theory to prove some results about subgroups of finite Kurosh rank on a free product.

3.1 Basic notions on graph theory

In this section we define some basic concepts and we fix some notation from graph theory that we will need in next section and in next chapter.

Definition 3.1.1. A graph Γ consists of a non-empty set of vertices $\mathtt{V}\Gamma$, a set of edges $\mathtt{E}\Gamma$, an inverse-edge map $^{-1} : \mathtt{E}\Gamma \longrightarrow \mathtt{E}\Gamma$ which is an involution without fixed-points, a initial-vertex map $\mathtt{o} : \mathtt{E}\Gamma \longrightarrow \mathtt{V}\Gamma$ and a terminal-vertex map $\mathtt{t} : \mathtt{E}\Gamma \longrightarrow \mathtt{V}\Gamma$ satisfying $\mathtt{o}(\mathtt{e}) = \mathtt{t}(\mathtt{e}^{-1})$ for all $\mathtt{e} \in \mathtt{E}\Gamma$. We call $\mathtt{o}(\mathtt{e})$ the initial vertex of $\mathtt{e}$ and $\mathtt{t}(\mathtt{e})$ the terminal vertex of $\mathtt{e}$, and we say that the initial vertex and the terminal vertex of $\mathtt{e}$ are the end-points of $\mathtt{e}$.

Definition 3.1.2. Let Γ be a graph. A path in Γ of length $n \in \mathbb{N}$ is a sequence $\mathtt{p} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_n, \mathtt{v}_n$ for $n \in \mathbb{N}$ satisfying that $\mathtt{e}_1, \ldots, \mathtt{e}_n \in \mathtt{E}\Gamma$, that $\mathtt{v}_0, \ldots, \mathtt{v}_n \in \mathtt{V}\Gamma$, and that $\mathtt{t}(\mathtt{e}_i) = \mathtt{v}_i$ and $\mathtt{o}(\mathtt{e}_i) = \mathtt{v}_{i-1}$ for all $i = 1, \ldots, n$, and in this case we say that $\mathtt{p}$ starts on $\mathtt{v}_0$ and ends on $\mathtt{v}_n$. We say that such a path $\mathtt{p}$ is reduced if $\mathtt{e}_i \neq (\mathtt{e}_{i+1})^{-1}$ for all $i = 1, \ldots, n-1$. Given a path of the form $\mathtt{p} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_n, \mathtt{v}_n$, we can define its inverse path as the path $\mathtt{p}^{-1} = \mathtt{v}_n, \mathtt{e}_n^{-1}, \mathtt{v}_{n-1}, \ldots, \mathtt{e}_1^{-1}, \mathtt{v}_0$. Given two paths of the form $\mathtt{p}_1 = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_n, \mathtt{v}_n$ and $\mathtt{p}_2 = \mathtt{v}_n, \mathtt{e}_{n+1}, \mathtt{v}_{n+1}, \ldots, \mathtt{e}_{n+m}, \mathtt{v}_{n+m}$ (the second ends on the same vertex where the first starts), we can define its concatenation as the path $\mathtt{p}_1\mathtt{p}_2 = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_n, \mathtt{v}_n, \mathtt{e}_{n+1}, \mathtt{v}_{n+1}, \ldots, \mathtt{e}_{n+m}, \mathtt{v}_{n+m}$.

Definition 3.1.3. Let Γ be a graph. A subgraph of Γ is a graph such that its set of vertices is a subset of $\mathtt{V}\Gamma$, its set of edges is a subset of $\mathtt{E}\Gamma$, its inverse-edge map is the restriction to its set of vertices of the inverse-edge map of Γ, its initial-vertex map is the restriction to its set of vertices of the initial-vertex map of Γ and its terminal-vertex map is the restriction to its set of vertices of the terminal-vertex map of Γ.

Definition 3.1.4. Let Γ be a graph. We say that Γ is connected if for every $\mathtt{v}, \mathtt{u} \in \mathtt{V}\Gamma$ there exists a path that starts on $\mathtt{v}$ and ends on $\mathtt{u}$.

Definition 3.1.5. A tree is a graph $\mathtt{T}$ satisfying that for every $\mathtt{v}, \mathtt{u} \in \mathtt{VT}$ there exists a unique reduced path that starts on $\mathtt{v}$ and ends on $\mathtt{u}$, and such a path is called the $\mathtt{T}$-geodesic from $\mathtt{v}$ to $\mathtt{u}$, and it is denoted by $[\mathtt{v}, \mathtt{u}]_{\mathtt{T}}$. In particular, every tree is connected as a graph.

Definition 3.1.6. Let G be a group and let Γ be a graph. We say that G acts on the graph Γ if G acts on the sets $\mathtt{V}\Gamma$ and $\mathtt{E}\Gamma$ in such a way the maps $^{-1} : \mathtt{E}\Gamma \longrightarrow \mathtt{E}\Gamma$, $\mathtt{o} : \mathtt{E}\Gamma \longrightarrow \mathtt{V}\Gamma$ and $\mathtt{t} : \mathtt{E}\Gamma \longrightarrow \mathtt{V}\Gamma$ are G-invariant

for the action, that is, we have $(g\mathtt{e})^{-1} = g(\mathtt{e}^{-1})$, $\mathtt{o}(g\mathtt{e}) = g\,(\mathtt{o}(\mathtt{e}))$ and $\mathtt{t}(g\mathtt{e}) = g\,(\mathtt{t}(\mathtt{e}))$ for all $g \in G$ and all $\mathtt{e} \in \mathtt{E}\Gamma$.

Notation 10. When we say that G acts on a graph Γ, we will also assume that this action has no inversions, meaning that there are not $g \in G$ and $\mathtt{e} \in \mathtt{E}\Gamma$ such that $g\mathtt{e} = \mathtt{e}^{-1}$.

Notation 11. Let G be a group acting on a set A. Given $a \in A$, we define the orbit of a under G as the set $Ga = \{ga \ / \ g \in G\}$. The set of orbits determines a partition on A. Given $a \in A$, we denote by $\mathrm{Stab}_G(a) = \{g \in G \ / \ ga = a\}$ the stabilizer of a.

Definition 3.1.7. Let G be a group acting on a graph Γ. We define $G\backslash\Gamma$ to be the graph whose set of vertices and edges is given by

$$\mathtt{V}(G\backslash\Gamma) = \{G\mathtt{v} \ / \ \mathtt{v} \in \mathtt{V}\Gamma\} \qquad \mathtt{E}(G\backslash\Gamma) = \{G\mathtt{e} \ / \ \mathtt{e} \in \mathtt{E}\Gamma\},$$

whose inverse-edge map $^{-1} : \mathtt{E}(G\backslash\Gamma) \longrightarrow \mathtt{E}(G\backslash\Gamma)$ is defined by $(G\mathtt{e})^{-1} = G\left(\mathtt{e}^{-1}\right)$, whose initial-vertex map $\mathtt{o} : \mathtt{E}(G\backslash\Gamma) \longrightarrow \mathtt{V}(G\backslash\Gamma)$ is defined by $\mathtt{o}(G\mathtt{e}) = G\,(\mathtt{o}(\mathtt{e}))$ and whose terminal-vertex map $\mathtt{t} : \mathtt{E}(G\backslash\Gamma) \longrightarrow \mathtt{V}(G\backslash\Gamma)$ is defined by $\mathtt{t}(G\mathtt{e}) = G\,(\mathtt{t}(\mathtt{e}))$.

Definition 3.1.8. Let G be a group acting on a graph Γ. Let Ω be a subgraph of Γ. We define the projection of Ω on $G\backslash\Gamma$ to be the subgraph of $G\backslash\Gamma$ whose set of vertices is $\{G\mathtt{v} \ / \ \mathtt{v} \in \mathtt{V}\Omega\}$ and whose set of edges is $\{G\mathtt{e} \ / \ \mathtt{e} \in \mathtt{E}\Omega\}$. Moreover, for $\mathtt{v} \in \mathtt{V}\Omega$, we say that $G\mathtt{v}$ is the projection of $\mathtt{v}$ on $G\backslash\Gamma$, and for $\mathtt{e} \in \mathtt{V}\Omega$, we say that $G\mathtt{e}$ is the projection of $\mathtt{e}$ on $G\backslash\Gamma$.

3.2 The Bass-Serre tree of a free-product

In this section we use Bass-Serre theory to prove some technical properties that we will need for next sections. The section is a bit technical. The only notion that is defined in this section and that is used in next chapters is the notion of Kurosh rank with respect to $G * H$ (for G and H non-trivial groups), which is detailed in Definition 3.2.5. We are just interested on that Kurosh rank being finite or infinite. Every finitely generated subgroup has finite Kurosh rank. Apart from that, the only properties we will use in other chapters are Remark 3.2.8, Corollary 3.3.2, Corollary 3.3.4 and Theorem 3.3.5. For a non-deep lecture, one can skip this section by just assuming the previous facts and by just considering the Kurosh rank as a black box that satisfies that properties. However, it is remarkable that Theorem 3.3.5 could be an interesting result by itself. As far as the author knows, that result has not been considered in the literature, and it provides a description for certain intersections on free products and, as a consequence, one obtain a bound for the Kurosh rank of that intersection.

Our basic reference for Bass-Serre theory is [52]. We start the section by fixing some notations and by proving some properties that we will be used in next results.

We use some notions from [52] that are standard in Bass-Serre theory. We are not writing down the concrete definitions as we will not need them. Instead of that, we provide the needed conclusions with the corresponding references. For the definition of graph of groups see Definition 8 from [52]. For different equivalent definitions of the fundamental group of a graph of groups see Section 5.1 from [52]. For the definition of the universal covering relative to a graph of groups see Section 5.3 from [52]. Notice that, on that reference, on the definition of the universal covering relative to a graph of groups, there is an implicit choice of a maximal tree of the underlying graph of the graph of groups, but can skip that since we are going to consider a tree as such underlying graph, so the only maximal tree is itself.

Notation 12. Through all the section, G and H will denote two non-trivial groups.

Notation 13. Consider the graph of groups (Φ, Γ) whose underlying graph is

$$\Gamma = \mathtt{v}_G \xrightarrow{\mathtt{e}_G} \mathtt{v}_0 \xleftarrow{\mathtt{e}_H} \mathtt{v}_H$$

(we don't write the corresponding $\mathtt{e}_G^{-1}$ and $\mathtt{e}_H^{-1}$) and that is given by $\Phi_{\mathtt{v}_G} = G$, $\Phi_{\mathtt{v}_0} = \{1\}$, $\Phi_{\mathtt{v}_H} = H$, $\Phi_{\mathtt{e}_G} = \{1\}$ and $\Phi_{\mathtt{e}_H} = \{1\}$. In Section 5.1 of [52], at the definition b) of the fundamental group, it is

provided a definition of the fundamental group using generations and relations. From this presentation, it is direct that the fundamental group of the graph of groups (Φ, Γ) is $G * H$. Consider $\mathtt{T}$ the universal covering relative to the graph of groups (Φ, Γ). By looking at the definition of the graph given in Section 5.3 of [52] and by changing the notation in order to be more clear, we can describe $\mathtt{T}$ in terms of the way $G * H$ acts as follows: the set of edges and the set of vertices are given by

$$\mathtt{VT} = (G*H)\mathtt{v}_G \sqcup (G*H)\mathtt{v}_0 \sqcup (G*H)\mathtt{v}_H \quad \text{and} \quad \mathtt{ET} = \mathtt{ET}^+ \sqcup \mathtt{ET}^-$$
$$\text{where } \mathtt{ET}^+ = (G*H)\mathtt{e}_G \sqcup (G*H)\mathtt{e}_H \text{ and } \mathtt{ET}^- = (G*H)\mathtt{e}_G^{-1} \sqcup (G*H)\mathtt{e}_H^{-1}$$

and the structure of the graph is determined by

$$x_1\mathtt{v}_G = x_2\mathtt{v}_G \iff x_1G = x_2G; \quad x_1\mathtt{v}_0 = x_2\mathtt{v}_0 \iff x_1 = x_2; \quad x_1\mathtt{v}_H = x_2\mathtt{v}_H \iff x_1H = x_2H;$$
$$x_1\mathtt{e}_G = x_2\mathtt{e}_G \iff x_1 = x_2; \quad x_1\mathtt{e}_H = x_2\mathtt{e}_H \iff x_1 = x_2;$$
$$(x\mathtt{e}_G)^{-1} = x\mathtt{e}_G^{-1}; \quad (x\mathtt{e}_H)^{-1} = x\mathtt{e}_H^{-1};$$
$$\mathtt{o}(x\mathtt{e}_G) = x\mathtt{v}_G; \quad \mathtt{t}(x\mathtt{e}_G) = x\mathtt{v}_0; \quad \mathtt{o}(x\mathtt{e}_H) = x\mathtt{v}_H; \quad \mathtt{t}(x\mathtt{e}_H) = x\mathtt{v}_0.$$

The graph $\mathtt{T}$ is a tree as a consequence of Theorem 12 from [52].

We can identify elements of $G * H$ with paths in $\mathtt{T}$.

Remark 3.2.1. On the tree $\mathtt{T}$, notice that the action of $G*H$ on the orbit of $1\mathtt{v}_0$ is free. In particular, given $x \in G * H$, we can identify x with a path in $\mathtt{T}$ by considering $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$, which is the unique reduced path in $\mathtt{T}$ starting on $1\mathtt{v}_0$ and ending on $x\mathtt{v}_0$. We now describe $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$. Write $x = x_1 \dots x_n$ as a reduced word in $G * H$ (so that each x_i is a non trivial element of G or H in such a way they alternate when increasing i by one). The path $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is of the form

$$[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \dots, \mathtt{e}_{2n}, \mathtt{v}_{2n}.$$

We have $\mathtt{v}_0 = 1\mathtt{v}_0$ and $\mathtt{v}_{2i} = (x_1 \dots x_i)\mathtt{v}_0$ for all $i = 1, \dots, n$. Also, for each $i = 1, \dots, n$, either $\mathtt{e}_{2i-1} = (x_1 \dots x_{i-1})\mathtt{e}_G^{-1}$, $\mathtt{v}_{2i-1} = (x_1 \dots x_{i-1})\mathtt{v}_G$ and $\mathtt{e}_{2i-1} = (x_1 \dots x_i)\mathtt{e}_G$ (this happens when $x_i \in G$), or $\mathtt{e}_{2i-1} = (x_1 \dots x_{i-1})\mathtt{e}_H^{-1}$, $\mathtt{v}_{2i-1} = (x_1 \dots x_{i-1})\mathtt{v}_H$ and $\mathtt{e}_{2i-1} = (x_1 \dots x_i)\mathtt{e}_H$ (this happens when $x_i \in H$).

Remark 3.2.2. Consider $x \in G * H$. From the description of $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ given in Remark 3.2.1 we deduce that the following fact holds: given G_0 subgroup of G and given H_0 subgroup of H then

$$x \in \langle G_0, H_0 \rangle \text{ and } [1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \dots, \mathtt{e}_m, \mathtt{v}_m \quad \text{implies}$$
$$\mathtt{e}_i \in \langle G_0, H_0 \rangle \mathtt{e}_G \cup \langle G_0, H_0 \rangle \mathtt{e}_G^{-1} \cup \langle G_0, H_0 \rangle \mathtt{e}_H \cup \langle G_0, H_0 \rangle \mathtt{e}_H^{-1} \text{ for all } i = 1, \dots, m.$$

We now consider the graph $\overline{\mathtt{T}}_C = C\backslash\mathtt{T}$ for a subgroup C of $G * H$.

Notation 14. Let C be a subgroup of $G * H$. Since $G * H$ acts on $\mathtt{T}$, we can consider the induced action of C on $\mathtt{T}$. Then, we can consider the graph $\overline{\mathtt{T}}_C = C\backslash\mathtt{T}$. Its set of vertices and edges are described by

$$\mathtt{V}\overline{\mathtt{T}}_C = C\backslash(G*H)\mathtt{v}_G \sqcup C\backslash(G*H)\mathtt{v}_0 \sqcup C\backslash(G*H)\mathtt{v}_H \quad \text{and}$$
$$\mathtt{E}\overline{\mathtt{T}}_C = C\backslash(G*H)\mathtt{e}_G \sqcup C\backslash(G*H)\mathtt{e}_G^{-1} \sqcup C\backslash(G*H)\mathtt{e}_H \sqcup C\backslash(G*H)\mathtt{e}_H^{-1},$$

where the inverse edge and the two endpoints of a given edge are the induced by the ones from $\mathtt{T}$.

Remark 3.2.3. Take $x \in G * H$. Notice that

$$\mathrm{Stab}_C(x\mathtt{v}_G) = \{y \in C \,/\, yx\mathtt{v}_G = x\mathtt{v}_G\} = \{y \in C \,/\, x^{-1}yx\mathtt{v}_G = \mathtt{v}_G\} = \{y \in C \,/\, x^{-1}yx \in G\} = C \cap (xGx^{-1}).$$

We can do analogous computations for the other vertices and edges so we obtain

$\mathrm{Stab}_C(x\mathtt{v}_G) = C \cap (xGx^{-1})$, $\mathrm{Stab}_C(x\mathtt{v}_0) = \{1\}$ and $\mathrm{Stab}_C(x\mathtt{v}_H) = C \cap (xHx^{-1})$ on vertices, and
$\mathrm{Stab}_C(x\mathtt{e}_G) = \{1\}$ and $\mathrm{Stab}_C(x\mathtt{e}_H) = \{1\}$ on edges.

Remark 3.2.4. Take $x \in G * H$. We have that

$$\mathrm{Stab}_C(x\mathtt{v}_G) = \{1\} \quad \text{if and only if} \quad Cxg \neq Cx \ \text{ for all } g \in G - \{1\}$$

and

$$\mathrm{Stab}_C(x\mathtt{v}_H) = \{1\} \quad \text{if and only if} \quad Cxh \neq Cx \ \text{ for all } h \in H - \{1\}.$$

It follwos from $\mathrm{Stab}_C(x\mathtt{v}_G) = C \cap (xGx^{-1})$ and $\mathrm{Stab}_C(x\mathtt{v}_G) = C \cap (xGx^{-1})$ (done in the previous remark).

We now introduce the definition of the Kurosh rank of a free product. For us, the Kurosh rank of a subgroup will mean, in terms of Definition 2.1 from [3], the Kurosh rank of that subgroup with respect to $\mathtt{T}$.

Definition 3.2.5. Let C be a subgroup of $G * H$. Let $c(C)$ be the sum of the number of vertices of $\overline{\mathtt{T}}_C$ of the form $Cx\mathtt{v}_G$ such that $\mathrm{Stab}_C(x\mathtt{v}_G) \neq \{1\}$ plus the number of vertices of $\overline{\mathtt{T}}_C$ of the form $Cx\mathtt{v}_H$ such that $\mathrm{Stab}_C(x\mathtt{v}_H) \neq \{1\}$. Notice that this number is well defined as it does not depend on the chosen representatives by Remark 3.2.4. Let $rank(\overline{\mathtt{T}}_C)$ be the number of edges of $\overline{\mathtt{T}}_C$ outside of a maximal subtree (this is the same as the free rank of the fundamental group of the graph $\overline{\mathtt{T}}_C$, which it is known that is a free group). We define the Kurosh rank of C with respect to $G * H$ as the number $c(C) + rank(\overline{\mathtt{T}}_C)$.

We now state the Kurosh's theorem about the structure of subgroups of a free product (for a reference see Theorem 14 of [52]).

Theorem 3.2.6 (Kurosh's theorem)**.** *Let C be a subgroup of $G * H$. Then,*

$$C = \mathbb{F} * \left(*C \cap xGx^{-1}\right) * \left(*C \cap yHy^{-1}\right)$$

where $\mathbb{F}$ is a free subgroup of $G * H$ and where x ranges over a set of representatives of the double coset $C\backslash(G * H)/G$ and y ranges over a set of representatives of the double coset $C\backslash(G * H)/H$.

Remark 3.2.7. Let C be a subgroup of $G * H$. Notice that having finite Kurosh rank is equivalent to say that, in the formulation of Kurosh's theorem, we have that $\mathbb{F}$ has finite free rank and that x and y range over finite sets (all this is a consequence of the proof of Kurosh's theorem, see the cited Theorem 14 of [52]). In particular, if C has infinite Kurosh rank with respect to $G * H$, then for each finite subset of C there exists a proper free factor C_1 of C containing that subset.

Notation 15. Let C be a subgroup of $G * H$. Define $\mathtt{u}_* = C1\mathtt{v}_0$. Define the fundamental core of $\overline{\mathtt{T}}_C$ to be the smallest connected subgraph of $\overline{\mathtt{T}}_C$ containing all reduced paths on $\overline{\mathtt{T}}_C$ simultaneously starting and ending at $\mathtt{u}_*$ and containing all vertices of $\overline{\mathtt{T}}_C$ of the form $Cx\mathtt{v}_G$ such that $\mathrm{Stab}_C(x\mathtt{v}_G) \neq \{1\}$ and all vertices of $\overline{\mathtt{T}}_C$ of the form $Cx\mathtt{v}_H$ such that $\mathrm{Stab}_C(x\mathtt{v}_H) \neq \{1\}$.

Remark 3.2.8. Let C be a subgroup of $G * H$. If C has finite Kurosh rank with respect to $G * H$, then the fundamental core of $\overline{\mathtt{T}}_C$ is a finite graph. This is direct from the definitions.

Notation 16. Let C be a subgroup of $G * H$. Recall that, in Remark 3.2.1, we provide a description of the path $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ for each $x \in G * H$, which is a path in $\overline{\mathtt{T}}_C$ which is canonically associated to x. For each $x \in G * H$ we define the path $\mathfrak{p}(x)$ in $\overline{\mathtt{T}}_C$ as the projection of $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ on $\overline{\mathtt{T}}_C$, that is: if $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_{2n}, \mathtt{v}_{2n}$ then $\mathfrak{p}(x) = C\mathtt{v}_0, C\mathtt{e}_1, C\mathtt{v}_1, \ldots, C\mathtt{e}_{2n}, C\mathtt{v}_{2n}$. Notice that $\mathfrak{p}(x)$ is a path having $\mathtt{u}_* = C1\mathtt{v}_0$ as initial vertex and having $Cx\mathtt{v}_0$ as terminal vertex. Notice that $\mathfrak{p}(x)$ always starts on $\mathtt{u}_*$, and that $\mathfrak{p}(x)$ ends on $\mathtt{u}_*$ if and only if $x \in C$.

Lemma 3.2.9. *Let C be a subgroup of $G * H$. If $x \in C$, then $\mathfrak{p}(x)$ is a path on the fundamental core of $\overline{\mathtt{T}}_C$. In particular, if $x \in C$, then every edge and every vertex appearing on the path $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$, when projected on $\overline{\mathtt{T}}_C$, give us an edge or a vertex that belongs to the fundamental core of $\overline{\mathtt{T}}_C$.*

Proof. Assume that $x \in C$. In particular, $\mathfrak{p}(x)$ is a closed path simultaneously starting and ending at $\mathtt{u}_*$. If $\mathfrak{p}(x)$ is reduced, then it is on the fundamental core of $\overline{\mathtt{T}}_C$ by definition. If it is not reduced, we can take the minimum i such that $C\mathtt{e}_i, C\mathtt{v}_i, C\mathtt{e}_i^{-1}$, and this condition implies that $\mathrm{Stab}_C(\mathtt{v}_i) \neq \{1\}$. Then, by definition, the subpath of $\mathfrak{p}(x)$ from the startng point until $C\mathtt{v}_i$. By repeating the argument with next vertices where $\mathfrak{p}(x)$ is not reduced, we prove the lemma. □

3.3 The auxiliary results

In this section we prove the auxiliary results on subgroups of finite Kurosh rank. Here we are going to assume the notation and the conclusions from last section. In particular, we fix consider the definition of $\mathtt{T}$ and $\mathtt{T}_C$ as it is done in that section.

In next proposition, we need to state it considering fixed generating sets of G and H as we will need that later.

Proposition 3.3.1. *Let G and H two non-trivial groups. Let C be a subgroup of $G*H$ having finite Kurosh rank with respect to $G*H$. Let $\mathcal{X}_G$ be a generating set of G and $\mathcal{X}_H$ be a generating set of H. Define $\Xi \subseteq C\backslash(G*H)$ to be the set of cosets $C\lambda$ such that there exists $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ satisfying $C\lambda g \neq C\lambda$, $C\lambda h \neq C\lambda$ and $C\lambda g^{\varepsilon_G} \neq C\lambda h^{\varepsilon_H}$ for every $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$. Then, $(C\backslash(G*H)) - \Xi$ is a finite set.*

Proof. By Remark 3.2.8, since C has finite Kurosh rank with respect to $G*H$, we deduce that the fundamental core of $\overline{\mathtt{T}}_C$ is finite. If we show that all vertices of the form $C\lambda\mathtt{v}_0$ for $C\lambda \in (C\backslash(G*H)) - \Xi$ are vertices of the fundamental core of $\overline{\mathtt{T}}_C$, we can conclude that $(C\backslash(G*H)) - \Xi$ is finite.

Indeed, assume that $C\lambda\mathtt{v}_0$ for $C\lambda \in C\backslash(G*H)$ is not on the fundamental core of $\overline{\mathtt{T}}_C$. We want to prove that $C\lambda$ belongs to Ξ. Consider the edges $C\lambda\mathtt{e}_G$ and $C\lambda\mathtt{e}_H$ of $\overline{\mathtt{T}}_C$ and the vertices $C\lambda\mathtt{v}_G$ and $C\lambda\mathtt{v}_H$ of $\overline{\mathtt{T}}_C$ that satisfies $\mathtt{o}(C\lambda\mathtt{e}_G) = C\lambda\mathtt{v}_G$, $\mathtt{o}(C\lambda\mathtt{e}_H) = C\lambda\mathtt{v}_H$ and $\mathtt{t}(C\lambda\mathtt{e}_G) = \mathtt{t}(C\lambda\mathtt{e}_H) = C\lambda\mathtt{v}_0$. In other words, we have

$$C\lambda\mathtt{v}_G \xrightarrow{C\lambda\mathtt{e}_G} C\lambda\mathtt{v}_0 \xleftarrow{C\lambda\mathtt{e}_H} C\lambda\mathtt{v}_H.$$

We remark that either $C\lambda\mathtt{v}_G$ or $C\lambda\mathtt{v}_H$ is not on the fundamental core of $\overline{\mathtt{T}}_C$. Indeed, if both $C\lambda\mathtt{v}_G$ and $C\lambda\mathtt{v}_H$ are on the fundamental core of $\overline{\mathtt{T}}_C$, since the path $C\lambda\mathtt{v}_G, C\lambda\mathtt{e}_H, C\lambda\mathtt{v}_0, (C\lambda\mathtt{e}_H)^{-1}, C\lambda\mathtt{v}_H$ is a reduced path starting on $C\lambda\mathtt{v}_G$ and ending on $C\lambda\mathtt{v}_H$, it would imply that $C\lambda\mathtt{v}_0$ is on the fundamental core of $\overline{\mathtt{T}}_C$, a contradiction with the fact that it is not on the fundamental core of $\overline{\mathtt{T}}_C$. Therefore, we know that either $C\lambda\mathtt{v}_G$ or $C\lambda\mathtt{v}_H$ is not on the fundamental core of $\overline{\mathtt{T}}_C$. We consider the case where $C\lambda\mathtt{v}_G$ is not on the fundamental core of $\overline{\mathtt{T}}_C$, as the case where $C\lambda\mathtt{v}_H$ is not on the fundamental core of $\overline{\mathtt{T}}_C$ is analogous.

Assume that $C\lambda\mathtt{v}_G$ is not on the fundamental core of $\overline{\mathtt{T}}_C$. Fix a nontrivial $g \in \mathcal{X}_G$. Since $C\lambda\mathtt{v}_G$ is not on the fundamental core of $\overline{\mathtt{T}}_C$, we have $\mathrm{Stab}_C(\lambda\mathtt{v}_H) = \{1\}$. Then, by Remark 3.2.4, we deduce that

$$C\lambda g \neq C\lambda \text{ for all } g \in \mathcal{X}_G - \{1\}. \tag{3.1}$$

We now prove that

$$\text{There exists } h \in \mathcal{X}_H \text{ such that } C\lambda h \neq C\lambda. \tag{3.2}$$

We do it by contradiction, so assume that $C\lambda h = C\lambda$ for all $h \in \mathcal{X}_H$. Since $\mathcal{X}_H$ is a generating set of H, we deduce that

$$C\lambda h = C\lambda \text{ for all } h \in H. \tag{3.3}$$

This implies, by the structure of $\overline{\mathtt{T}}_C$, that $C\lambda\mathtt{e}_H$ is the unique edge of $\overline{\mathtt{T}}_C$ having $C\lambda\mathtt{v}_H$ as initial vertex. Equation (3.3) implies that $C\lambda\mathtt{v}_H$ is on the fundamental core of $\overline{\mathtt{T}}_C$. In particular, any path starting on $C\lambda\mathtt{v}_H$ and ending on $\mathtt{u}_*$ pass through the vertex $C\lambda\mathtt{v}_0$. Hence, $C\lambda\mathtt{v}_0$ is on the fundamental core of $\overline{\mathtt{T}}_C$, a contradiction. Therefore, we have shown Equation (3.2).

Fix $g \in \mathcal{X}_G - \{1\}$ satisfying that $C\lambda g^{\varepsilon_G} \neq C\lambda$ for all $\varepsilon_G \in \{\pm 1\}$, which is possible by Equation (3.1) since G is non-trivial. Fix $h \in \mathcal{X}_H$ satisfying $C\lambda h^{\varepsilon_H} \neq C\lambda$ for all $\varepsilon_H \in \{\pm 1\}$, which is possible by Equation (3.2). Fix $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$. In particular, we have that $C\lambda g^{\varepsilon_G} \neq C\lambda$ and $C\lambda h^{\varepsilon_H} \neq C\lambda$. We can finish the proof if we show that

$$C\lambda g^{\varepsilon_G} \neq C\lambda h^{\varepsilon_H}. \tag{3.4}$$

We do it by contradiction, so assume that $C\lambda g^{\varepsilon_G} = C\lambda h^{\varepsilon_H}$. In particular, $C\lambda g^{\varepsilon_G}\mathtt{v}_0 = C\lambda h^{\varepsilon_H}\mathtt{v}_0$. Then, we can consider the path

$$\mathtt{p} = C\lambda g^{\varepsilon_G}\mathtt{v}_0, C\lambda g^{\varepsilon_G}\mathtt{e}_G^{-1}, C\lambda\mathtt{v}_G, C\lambda\mathtt{e}_G, C\lambda\mathtt{v}_0, C\lambda\mathtt{e}_H^{-1}, C\lambda\mathtt{v}_H, C\lambda h^{\varepsilon_H}\mathtt{e}_H, C\lambda h^{\varepsilon_H}\mathtt{v}_0,$$

which is a path starting and ending on $C\lambda g^{\varepsilon_G}\mathtt{v}_0 = C\lambda h^{\varepsilon_H}\mathtt{v}_0$, and notice that it is reduced since we know that $C\lambda g^{\varepsilon_G} \neq C\lambda$ and $C\lambda h^{\varepsilon_H} \neq C\lambda$. In particular, and since $C\lambda\mathtt{v}_0$ is a vertex appearing on $\mathtt{p}$, we can use this fact to build a reduced path having $C\lambda\mathtt{v}_0$ as a vertex and starting and ending on $\mathtt{u}_*$. Hence, by definition of the fundamental core of $\overline{\mathtt{T}}_C$, we deduce that $C\lambda\mathtt{v}_0$ is on the fundamental core of $\overline{\mathtt{T}}_C$. This contradicts the fact that $C\lambda\mathtt{v}_G$ is not on the fundamental core of $\overline{\mathtt{T}}_C$ by hypothesis. Therefore, we have shown Equation (3.4), as we wanted. □

Corollary 3.3.2. *Let G and H two non-trivial groups. Let C be a subgroup of $G*H$ having finite Kurosh rank with respect to $G*H$. Let $\mathcal{X}_G$ be a generating set of G and $\mathcal{X}_H$ be a generating set of H. Let $(\lambda_l)_{n\in\mathbb{N}}$ be a sequence of elements in $G*H$ such that $\lambda_l C \neq \lambda_m C$ for all $l \neq m$. Then, there exists $n \in \mathbb{N}$ such that for all $k \geq n$ there are $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ satisfying $g\lambda_k C \neq \lambda_k C$, $h\lambda_k C \neq \lambda_k C$ and $g^{\varepsilon_G}\lambda_k C \neq h^{\varepsilon_H}\lambda_k C$ for all $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$.*

Proof. Using the fact that C is a subgroup, we have

$$C\lambda_l^{-1} = C^{-1}\lambda_l^{-1} = (\lambda_l C)^{-1} \neq (\lambda_m C)^{-1} = C^{-1}\lambda_m^{-1} = C\lambda_m^{-1}$$

for all $l \neq m$. Applying Proposition 3.3.1 to the elements $C\lambda_l^{-1}$ for $l \in \mathbb{N}$ and to the generating sets $\mathcal{X}_G^{-1}$ and $\mathcal{X}_H^{-1}$, we deduce that there exists $n \in \mathbb{N}$ such that for all $k \geq n$ there are $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ satisfying $C\lambda_k^{-1}g^{-1} \neq C\lambda_k^{-1}$, $C\lambda_k^{-1}h^{-1} \neq C\lambda_k^{-1}$ and $C\lambda_k^{-1}g^{-\varepsilon_G} \neq C\lambda_k^{-1}h^{-\varepsilon_H}$ for all $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$. Now fix $k \geq n$, fix $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ as before. We have $g\lambda_k C = \left(C\lambda_k^{-1}g^{-1}\right)^{-1} \neq \left(C\lambda_k^{-1}\right)^{-1} = \lambda_k C$ so $g\lambda_k C \neq \lambda_k C$, and similarly $h\lambda_k C \neq \lambda_k C$ and $g^{\varepsilon_G}\lambda_k C \neq h^{\varepsilon_H}\lambda_k C$ for all $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$. □

The next result is stated in geometrical terms. After that, we provide an algebraic reformulation in order to be applied later in the thesis.

Proposition 3.3.3. *Let G and H two non-trivial groups. Let C be a subgroup of $G*H$ having finite Kurosh rank with respect to $G*H$. Let $\Lambda \subseteq G*H - \{1\}$ be an infinite subset satisfying that $C\lambda_1 \neq C\lambda_2$ for all $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$. Let Λ_* be the set of elements $\lambda \in \Lambda$ such that there exists $y_\lambda \in G \cup H$ satisfying $\lambda y_\lambda \lambda^{-1} \notin C$ and such that any path in $\overline{\mathtt{T}}_C$ starting on $\mathtt{u}_*$ and ending on $C\lambda y_\lambda^{\varepsilon}\lambda^{-1}\mathtt{v}_0$ for some $\varepsilon \in \{\pm 1\}$ it passes through $C\lambda\mathtt{v}_0$. Then, the set $\Lambda - \Lambda_*$ is finite.*

Proof. Since C has finite Kurosh rank with respect to $G*H$ by hypothesis, applying Remark 3.2.8, we deduce that the fundamental core of $\overline{\mathtt{T}}_C$ is finite. By hypothesis, we know that $C\lambda_1 \neq C\lambda_2$ for all $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$. Hence, in order to prove that $\Lambda - \Lambda_*$ is finite we only need to show that every $\lambda \in \Lambda$ such that $C\lambda\mathtt{v}_0$ is not a vertex of the fundamental core of $\overline{\mathtt{T}}_C$, then $\lambda \in \Lambda_*$. We are going to prove that.

Consider $\lambda \in \Lambda$ such that $C\lambda\mathtt{v}_0$ is not a vertex of the fundamental core of $\overline{\mathtt{T}}_C$. Notice that, since $C\lambda\mathtt{v}_0$ is not on the fundamental core of $\overline{\mathtt{T}}_C$, we deduce that the edges having $C\lambda\mathtt{v}_0$ as initial or terminal vertex (which, by the structure of $\mathtt{T}$, they are $C\lambda\mathtt{e}_G$, $C\lambda\mathtt{e}_G^{-1}$, $C\lambda\mathtt{e}_H$ and $C\lambda\mathtt{e}_H^{-1}$) are not on the fundamental core of $\overline{\mathtt{T}}_C$. Then, by the definition of the fundamental core of $\overline{\mathtt{T}}_C$, we deduce that, when removing on $\overline{\mathtt{T}}_C$ the vertex $C\lambda\mathtt{v}_0$ and the edges having $C\lambda\mathtt{v}_0$ as initial or terminal vertex, we obtain a graph that is not connected and that consists on the union of two disjoint connected graphs $\overline{\mathtt{T}}_0$ and $\overline{\mathtt{T}}_1$ satisfying that $\overline{\mathtt{T}}_0$ contains $\mathtt{u}_*$ and the fundamental core of $\overline{\mathtt{T}}_C$.

The fact that $\overline{\mathtt{T}}_0$ contains $\mathtt{u}_* = C1\mathtt{v}_0$ and that every vertex of $\overline{\mathtt{T}}_1$ is not on the fundamental core of $\overline{\mathtt{T}}_C$ implies, by applying Remark 3.2.4 recursively, the following: given $x \in G*H$ non-trivial such that the product λx presents no reductions as words on G and H (meaning that, as reduced words on G and H, the rightmost nontrivial element of λ and the leftmost nontrivial element of x are not both in G or are not both in H), then $C\lambda x\mathtt{v}_0$ is a vertex of $\overline{\mathtt{T}}_1$. Consider $y_\lambda \in G \cup H$ in such a way λy_λ presents no reductions as words on G and H. In particular, we have that $C\lambda y_\lambda^{\varepsilon}\lambda^{-1}\mathtt{v}_0$ is a vertex of $\overline{\mathtt{T}}_1$ for all $\varepsilon \in \{\pm 1\}$. Since $C\lambda y_\lambda^{\varepsilon}\lambda^{-1}\mathtt{v}_0$ is a vertex of $\overline{\mathtt{T}}_1$ and $\mathtt{u}_*$ is a vertex of $\overline{\mathtt{T}}_0$, we deduce that any path staring on $\mathtt{u}_*$ and ending on $C\lambda y_\lambda^{\varepsilon}\lambda^{-1}\mathtt{v}_0$ must go through $C\lambda\mathtt{v}_0$, as $C\lambda\mathtt{v}_0$ is a vertex needed to connect $\overline{\mathtt{T}}_1$ and $\overline{\mathtt{T}}_0$. This proves that $\lambda \in \Lambda_*$. □

We reformulate the previous proposition in an algebraic way in order to adapt it to the precise formulation we will need.

Corollary 3.3.4. *Let G and H two non-trivial groups. Let C be a subgroup of $G*H$ having finite Kurosh rank with respect to $G*H$. Let $(\lambda_l)_{n\in\mathbb{N}}$ be a sequence of elements in $G*H$ such that $\lambda_l C \neq \lambda_m C$ for all $l \neq m$. Then, there exists $n \in \mathbb{N}$ such that for all $k \geq n$ there exists $y_{\lambda_k} \in G \cup H$ satisfying $\lambda_k^{-1} y_{\lambda_k} \lambda_k \notin C$ and satisfying the following property: for every finite collection $x_1, \dots, x_m \in G*H$ with $m \in \mathbb{N}_{>0}$ satisfying that $x_1 = 1$, that $x_m C = \lambda_k^{-1} y_{\lambda_k}^{\varepsilon} \lambda_k C$ for some $\varepsilon \in \{\pm 1\}$ and that $x_i = f_i x_{i+1}$ for some $f_i \in G \cup H$ where $i = 1, \dots, m-1$, then it holds that $x_i C = \lambda_k C$ for some $i \in \{1, \dots, m\}$.*

Proof. Notice that, for $x \in G*H$, we have $(xC)^{-1} = Cx^{-1}$. Then, for all $x, \tilde{x} \in G*H$, we have

$$xC = \tilde{x}C \quad \text{ if and only if } \quad Cx^{-1} = C\tilde{x}^{-1}.$$

The result follows by applying Proposition 3.3.3 the set of elements $C\lambda_l^{-1}$ for $l \in \mathbb{N}$. Notice that, for the elements x_i as in the statement of the result we are proving, by possibly considering a subsequence, we have that the sequence Cx_i for each i induces a path as the one in Proposition 3.3.3. □

We conclude with the main result of the section. As we said at the beginning of the section, as far as the author knows, this conclusion has not been considered in the literature. In [3] it si proven that the reduced Kurosh rank of the intersection of two subgroups of a free product of left-orderable groups is bounded above by the product of the reduced Kurosh ranks of each subgroup. We provide the same conclusion for a certain case where one of the intersecting subgroups has Kurosh rank 2 but dropping left-orderable requirement. Notice that this property is a generalization of strengthened Hanna Neumann Conjecture. Related to this, it is remarkable that Tardos proved strengthened Hanna Neumann Conjecture on free groups for the case where one of the subgroups has free rank 2, and this was published on Inventiones Mathematicae ([58]). We now state the theorem.

Theorem 3.3.5. *Let G and H two non-trivial groups. Let C be a proper subgroup of $G*H$ such that C has finite Kurosh rank with respect to $G*H$. Then, there exists finite sets $\Omega, \{\zeta_i\}_{i\in I}, \{\alpha_j\}_{j\in J}, \{\beta_k\}_{k\in K} \subseteq G*H$ satisfying the following property:*
For every subgroup G_0 of G and every subgroup H_0 of H satisfying $\Omega \subseteq \langle G_0, H_0\rangle$ there exists G_j subgroup of G_0 for each $j \in J$ and there exists G_k subgroup of H_0 for each $k \in K$ such that

$$C \cap \langle G_0, H_0\rangle = \langle \{\zeta_i \ / \ i \in I\} \cup \left(\bigcup_{j\in J} \alpha_j G_j \alpha_j^{-1} \right) \cup \left(\bigcup_{k\in K} \beta_k H_k \beta_k^{-1} \right) \rangle. \tag{3.5}$$

Proof. Since C has finite Kurosh rank with respect to $G*H$, applying Remark 3.2.8, we deduce that the fundamental core of $\overline{\mathtt{T}}_C$ is a finite subgraph of $\overline{\mathtt{T}}_C$. Then, we can take $\overline{\mathtt{T}_0}$ a maximal tree of the fundamental core of $\overline{\mathtt{T}}_C$, which is finite and satisfies that every vertex of the fundamental core of $\overline{\mathtt{T}}_C$ is a vertex of $\overline{\mathtt{T}_0}$. Since $\overline{\mathtt{T}_0}$ is finite, we can recursively build $\mathtt{T}_0$ a subgraph of $\mathtt{T}$ that is a finite tree containing $1\mathtt{v}_0$ in such a way the projection of $\mathtt{T}_0$ on $\overline{\mathtt{T}}_C$ is $\overline{\mathtt{T}_0}$ and every vertex and every edge of $\overline{\mathtt{T}_0}$ is the projection of a unique vertex or edge of $\mathtt{T}_0$. Notice that every vertex of the fundamental core of $\overline{\mathtt{T}}_C$ is the projection of a unique vertex of $\mathtt{T}_0$. We now construct a fundamental tranvsersal $\mathtt{Y}$ (see [18] for a definition). We give all the details to make the exposition sefl-contained. Since $\overline{\mathtt{T}_0}$ is finite, we can choose $\mathtt{Y}$ a finite subset of $\mathtt{ET}^+ - \mathtt{ET}_0$ such that every $\mathtt{f} \in \mathtt{Y}$ satisfies that $\mathtt{o}(\mathtt{f}) \in \mathtt{VT}_0$ and such that every edge of the fundamental core of $\overline{\mathtt{T}}_C$ not in $\overline{\mathtt{T}_0}$ is the projection of a unique element in $\mathtt{Y} \sqcup \mathtt{Y}^{-1}$. Define $\mathtt{Y}^{\pm} = \mathtt{Y} \sqcup \mathtt{Y}^{-1}$. Hence, every vertex of the fundamental core of $\overline{\mathtt{T}}_C$ is the projection of a unique element of $\mathtt{VT}$ and every edge of the fundamental core of $\overline{\mathtt{T}}_C$ is the projection of a unique element of $\mathtt{ET} \sqcup \mathtt{Y}^{\pm}$.

We now provide some definitions:

- Define $I = \mathtt{Y}$, which is a finite set.
- Define J to be the set of all $x\mathtt{v}_G$ for $x \in G*H$ satisfying that $x\mathtt{v}_G \in \mathtt{VT}_0$ and that $\mathrm{Stab}_C(x\mathtt{v}_G) \neq \{1\}$. The fact that $\mathtt{T}_0$ is finite implies that J is finite.
- Define K to be the set of all $x\mathtt{v}_H$ for $x \in G*H$ satisfying that $x\mathtt{v}_H \in \mathtt{VT}_0$ and that $\mathrm{Stab}_C(x\mathtt{v}_G) \neq \{1\}$. The fact that $\mathtt{T}_0$ is finite implies that K is finite.
- For each $i \in I$, there exists an element of $\mathtt{VT}_0$ whose projection is $C\mathtt{t}(i)$, and this implies that $\zeta_i \mathtt{t}(i) \in \mathtt{VT}_0$ for some $\zeta_i \in C$. Hence, for each $i \in I$, we define $\zeta_i \in C$ such that $\zeta_i \mathtt{t}(i) \in \mathtt{VT}_0$.

- For each $j \in J$ define $\alpha_j \in G*H$ to be such that $\mathtt{o}(\alpha_j \mathtt{e}_G) = j$ and such that $\alpha_j \mathtt{e}_G$ is an edge appearing on the the $\mathtt{T}$-geodesic from $1\mathtt{v}_0$ to j.
- For each $k \in K$ define $\beta_k \in G*H$ to be such that $\mathtt{o}(\beta_k \mathtt{e}_H) = k$ and such that $\beta_k \mathtt{e}_H$ is an edge appearing on the the $\mathtt{T}$-geodesic from $1\mathtt{v}_0$ to k.

Note that, in the previous definitions, the elements ζ_i, α_j and β_k are the unique elements having the corresponding properties. This is a consequence of the fact that the action of $G*H$ on $\mathtt{T}$ is free on edges and on vertices in the orbit of $\mathtt{v}_0$.

Define the finite set

$$\Omega = \{\zeta_i \ / \ i \in I\} \cup \{\alpha_j \ / \ j \in J\} \cup \{\beta_k \ / \ k \in K\} \cup \{x \in G*H \ / \ x\mathtt{e}_G \in \mathtt{ET}_0 \cup \mathtt{Y}^{\pm} \ \text{ or } \ x\mathtt{e}_H \in \mathtt{ET}_0 \cup \mathtt{Y}^{\pm}\}$$

We are going to prove that the theorem holds for these subsets we have defined. Consider G_0 a subgroup of G and H_0 a subgroup of H satisfying that

$$\Omega \subseteq \langle G_0, H_0 \rangle.$$

Define

$$G_j = \left(\alpha_j^{-1}\mathrm{Stab}_C(\alpha_j \mathtt{e}_G)\alpha_j\right) \cap G_0 \ \text{ for } \ j \in J \quad \text{ and } \quad H_k = \left(\beta_k^{-1}\mathrm{Stab}_C(\beta_k \mathtt{e}_H)\beta_k\right) \cap H_0 \ \text{ for } \ k \in K.$$

This, by Remark 3.2.3, is equivalent to

$$G_j = \left(\alpha_j^{-1} C \alpha_j\right) \cap G_0 \ \text{ for } \ j \in J \quad \text{ and } \quad H_k = \left(\beta_k^{-1} C \beta_k\right) \cap H_0 \ \text{ for } \ k \in K. \tag{3.6}$$

Define

$$S = \{\zeta_i \ / \ i \in I\} \cup \left(\bigcup_{j \in J} \alpha_j G_j \alpha_j^{-1}\right) \cup \left(\bigcup_{k \in K} \beta_k H_k \beta_k^{-1}\right).$$

We want to prove is that $C \cap \langle G_0, H_0 \rangle = \langle S \rangle$. We first prove that

$$\langle S \rangle \subseteq C \cap \langle G_0, H_0 \rangle. \tag{3.7}$$

In order to do that, we check that $S \subseteq C \cap \langle G_0, H_0 \rangle$. The fact that $\zeta_i \in C \cap \langle G_0, H_0 \rangle$ for all $i \in I$ follows by the fact that $\zeta_i \in C$ for all $i \in I$ by definition and that $\zeta_i \in \Omega \subseteq \langle G_0, H_0 \rangle$ for all $i \in I$. The fact that, for each $j \in J$, we have $\alpha_j G_j \alpha_j^{-1} \subseteq C \cap \langle G_0, H_0 \rangle$, follows from $G_j = \left(\alpha_j^{-1} C \alpha_j\right) \cap G_0$ by Equation (3.6) and from $\alpha_j \in \Omega \subseteq \langle G_0, H_0 \rangle$. The fact that $\beta_k H_k \beta_k^{-1} \subseteq C \cap \langle G_0, H_0 \rangle$ for each $k \in K$ is analogous. That completes the proof of Equation (3.7).

Recall that all we wanted to prove is $C \cap \langle G_0, H_0 \rangle = \langle S \rangle$. By Equation (3.7), we only need to show that $C \cap \langle G_0, H_0 \rangle \subseteq \langle S \rangle$.

We now introduce some auxiliary definitions. A vertex $\mathtt{v}$ of $\mathtt{T}$ is said to be critical if $\mathtt{v} \notin \mathtt{VT}_0$. An edge $\mathtt{e}$ of $\mathtt{T}$ is said to be critical if $\mathtt{e} \notin \mathtt{ET}_0 \cup \mathtt{Y}^{\pm}$. Define the critical number of a path $\mathtt{p}$ to be the number of critical vertices appearing on $\mathtt{p}$ plus the number of critical edges appearing on $\mathtt{p}$. Notice that, for any $\mathtt{v}_1, \mathtt{v}_2 \in \mathtt{VT}$, the critical number of $[\mathtt{v}_1, \mathtt{v}_2]_{\mathtt{T}}$ is smaller or equal than the critical number of any path starting on $\mathtt{v}_1$ and ending on $\mathtt{v}_2$. For each $x \in C \cap \langle G_0, H_0 \rangle$, define the critical number of x to be the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$. We are going to prove $C \cap \langle G_0, H_0 \rangle \subseteq \langle S \rangle$ by induction on the critical number.

Now, we are ready to do the induction. Assume that $x \in C \cap \langle G_0, H_0 \rangle$ has critical number 0. In particular, $x\mathtt{v}_0$ is a vertex of $\mathtt{T}_0$. As $x \in C$, we deduce that $1\mathtt{v}_0$ and $x\mathtt{v}_0$ both project to the same vertex of $\overline{\mathtt{T}_0}$. We deduce that $1\mathtt{v}_0 = x\mathtt{v}_0$. Hence $x = 1$, so then $x \in \langle S \rangle$.

Before doing the inductive step, we prove a claim that we will use few times later on the inductive proof. We claim that, for each $x \in C \cap \langle G_0, H_0 \rangle$, if

$$[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}} = 1\mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \ldots, \mathtt{e}_i, \mathtt{v}_i, \mathtt{e}_{i+1}, \ldots, \mathtt{e}_n, x\mathtt{v}_0 \quad \text{ where } \mathtt{v}_1, \ldots, \mathtt{v}_{n-1} \in \mathtt{VT}, \ \mathtt{e}_1, \ldots, \mathtt{e}_{n-1} \in \mathtt{ET}, \ n \in \mathbb{N}_{>0}, \tag{3.8}$$

if $\mathtt{v}_i$ (respectively $\mathtt{e}_i$) is critical for some i, and if $y \in C$ is such that $y\mathtt{v}_i \in \mathtt{VT}_0$ (resp. $y\mathtt{e}_i \in \mathtt{ET}_0$), then yx has critical number strictly smaller than the critical number of x.

We prove the claim. Consider $x \in C \cap \langle G_0, H_0\rangle$. Assume that $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is of the form as in Equation (3.8), that $\mathtt{v}_i$ is critical for some i, and that $y \in C$ is such that $y\mathtt{v}_i \in \mathtt{VT}_0$ (the case for edges is analogous). Recall that $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is a reduced path on the tree $\mathtt{T}$ where $1\mathtt{v}_0$ is a vertex of the tree $\mathtt{T}_0$. From this fact and from the properties of $\mathtt{Y}$, we deduce that the path $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is a concatenation of a sequence of non critical elements and then a sequence of critical elements. Then, by erasing on that path all the elements appearing before $\mathtt{v}_i$ and left-multiplying by y, we obtain a path starting on $y\mathtt{v}_i$, ending on $yx\mathtt{v}_0$ and having critical number strictly smaller than the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ (that is, the critical number of x). By concatenating $[1\mathtt{v}_0, y\mathtt{v}_i]_\mathtt{T}$ with that path, since all elements of $[1\mathtt{v}_0, y\mathtt{v}_i]_\mathtt{T}$ are not critical as it is a path on $\mathtt{T}_0$, we obtain a path starting on $1\mathtt{v}_0$, ending on $yx\mathtt{v}_0$ and having critical number strictly smaller than the critical number of x. Then, $[1\mathtt{v}_0, yx\mathtt{v}_0]_\mathtt{T}$ has critical number strictly smaller than the critical number of x. This proves the claim.

We are now ready to study the inductive case. Consider $x \in C \cap \langle G_0, H_0\rangle$ having critical number $s > 0$. Then, $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ has some critical vertex or some critical edge. Recall that $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is a reduced path on the tree $\mathtt{T}$ where $1\mathtt{v}_0$ is a vertex of the tree $\mathtt{T}_0$. From this and the properties of $\mathtt{Y}$, we deduce that the path $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is either a sequence of elements of $\mathtt{T}_0$ (ending with a vertex) and then critical elements, or a sequence of elements of $\mathtt{T}_0$ (ending with a vertex), followed by an edge of $\mathtt{Y}$ and then critical elements. Knowing this, we have some distinct cases. In all the cases, we are going to use the fact that all edges and vertices appearing in $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$, when projected to $\overline{\mathtt{T}}_C$, they are all elements of the fundamental core of $\overline{\mathtt{T}}_C$, since $x \in C$ and as a consequence of Lemma 3.2.9.

Case I Suppose that $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is a sequence of elements of $\mathtt{T}_0$ (ending with a vertex), followed by an edge of $\mathtt{Y}$ and then critical elements. In other words and more formally, we can write

$$\begin{aligned}&[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T} = 1\mathtt{v}_0, \mathtt{e}'_1, \mathtt{v}'_1, \dots, \mathtt{e}'_{n-1}, \mathtt{v}'_{n-1}, \mathtt{e}'_n, \mathtt{v}_1, \mathtt{e}_1, \dots, \mathtt{v}_r, \mathtt{e}_r, x\mathtt{v}_0\\ &\text{ where } \mathtt{v}'_1, \dots, \mathtt{v}'_{n-1} \in \mathtt{VT}_0,\ \mathtt{e}'_1, \dots, \mathtt{e}'_{n-1} \in \mathtt{ET}_0,\ \mathtt{e}'_n \in \mathtt{Y},\\ &\mathtt{v}_1, \dots, \mathtt{v}_r \in \mathtt{VT} - \mathtt{VT}_0,\ \mathtt{e}_1, \dots, \mathtt{e}_r \in \mathtt{ET} - (\mathtt{ET}_0 \cup \mathtt{Y}^{\pm}),\ n \in \mathbb{N}_{>0},\ r \in \mathbb{N},\end{aligned}$$

where the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is $s = 2r + 1$.

Since $\mathtt{e}'_n \in \mathtt{Y}$ and $\mathtt{Y} = i$, we deduce that $\zeta_{\mathtt{e}'_n}\mathtt{t}(\mathtt{e}'_n) \in \mathtt{VT}_0$, so that $\zeta_{\mathtt{e}'_n}\mathtt{v}_1 \in \mathtt{VT}_0$. Then, by the claim, we deduce that $\zeta_{\mathtt{e}'_n}x$ has critical number strictly smaller than the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$, which is s.

We have $\zeta_{\mathtt{e}'_n} \in \Omega \subseteq \langle G_0, H_0\rangle$. As we know that $\zeta_{\mathtt{e}'_n} \in C$ by definition of $\zeta_{\mathtt{e}'_n}$, we obtain $\zeta_{\mathtt{e}'_n} \in C \cap \langle G_0, H_0\rangle$. Applying that $x \in C \cap \langle G_0, H_0\rangle$ by hypothesis, we obtain that $\zeta_{\mathtt{e}'_n}x \in C \cap \langle G_0, H_0\rangle$. Then, since $\zeta_{\mathtt{e}'_n}x$ has critical number strictly smaller than s, by induction hypothesis, we obtain that $\zeta_{\mathtt{e}'_n}x \in \langle S\rangle$. As we know that $\zeta_{\mathtt{e}'_n} \in \langle S\rangle$ from the definition of S, we conclude that $x \in \langle S\rangle$. This finishes the Case I.

Case II Suppose that the first critical element is a critical edge, and suppose that the previous vertex is an element of the form $y\mathtt{v}_0$ for some $y \in G * H$. Formally, this means that we can write

$$\begin{aligned}&[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T} = 1\mathtt{v}_0, \mathtt{e}'_1, \mathtt{v}'_1, \dots, \mathtt{e}'_n, \mathtt{v}'_n, \mathtt{e}_1, \mathtt{v}_1, \dots, \mathtt{e}_r, x\mathtt{v}_0 \ \text{ where } \mathtt{v}'_1, \dots, \mathtt{v}'_n \in \mathtt{VT}_0,\ \mathtt{e}'_1, \dots, \mathtt{e}'_n \in \mathtt{ET}_0\\ &\mathtt{v}_1, \dots, \mathtt{v}_{r-1} \in \mathtt{VT} - \mathtt{VT}_0,\ \mathtt{e}_1, \dots, \mathtt{e}_r \in \mathtt{ET} - (\mathtt{ET}_0 \cup \mathtt{Y}^{\pm}),\ \mathtt{v}'_n = y\mathtt{v}_0,\ y \in G * H,\ n \in \mathbb{N},\ r \in \mathbb{N}_{>0},\end{aligned}$$

where the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_\mathtt{T}$ is $s = 2r$.

Notice that $\mathtt{e}_1$ projects on $\overline{\mathtt{T}}_C$ to an edge of $\overline{\mathtt{T}}_C$ that is not an edge of $\overline{\mathtt{T}_0}$ or of $\mathtt{Y}$. Then, $\mathtt{e}_1$ projects on $\overline{\mathtt{T}}_C$ to an edge of $\mathtt{Y}^{-1}$. We deduce that there exists an element $\hat{\mathtt{e}} \in \mathtt{Y}$ such that both $\hat{\mathtt{e}}$ and $\mathtt{e}_1^{-1}$ projects on $\overline{\mathtt{T}}_C$ to the same edge. Then, by looking at their final vertices and using that $\zeta_{\hat{\mathtt{e}}} \in C$, we deduce that $\mathtt{t}(\mathtt{e}_1^{-1}) = \mathtt{v}'_n$ and $\zeta_{\hat{\mathtt{e}}}\mathtt{t}(\hat{\mathtt{e}}) \in \mathtt{VT}_0$ projects on $\overline{\mathtt{T}}_C$ to the same vertex, so then $\mathtt{v}'_n = \zeta_{\hat{\mathtt{e}}}\mathtt{t}(\hat{\mathtt{e}})$. In particular, since the action of C is free on the orbits of $\mathtt{v}'_n$ (the same orbit as $\mathtt{t}(\hat{\mathtt{e}})$) and $\mathtt{e}_1^{-1}$ (the same orbit as $\hat{\mathtt{e}}$), we deduce that $\mathtt{e}_1^{-1} = \zeta_{\hat{\mathtt{e}}}\hat{\mathtt{e}}$. Therefore, by looking at their initial vertices, we obtain

$\zeta_{\hat{\mathtt{e}}}^{-1}\mathtt{v}_1 \in \mathtt{VT}_0$. Then, by the claim, we obtain that $\zeta_{\hat{\mathtt{e}}}^{-1}x$ has critical number strictly smaller than the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$, which is s.

We have $\zeta_{\hat{\mathtt{e}}} \in \Omega \subseteq \langle G_0, H_0\rangle$. As $\zeta_{\hat{\mathtt{e}}} \in C$ by definition, we deduce $\zeta_{\hat{\mathtt{e}}} \in C \cap \langle G_0, H_0\rangle$. Applying that $x \in C \cap \langle G_0, H_0\rangle$ by hypothesis, we obtain that $\zeta_{\hat{\mathtt{e}}}^{-1}x \in C \cap \langle G_0, H_0\rangle$. Then, as $\zeta_{\hat{\mathtt{e}}}^{-1}x$ has critical number strictly smaller than s, by induction hypothesis, we obtain that $\zeta_{\hat{\mathtt{e}}}^{-1}x \in \langle S\rangle$. As we know that $\zeta_{\hat{\mathtt{e}}} \in \langle S\rangle$ from the definition of S, we obtain that $x \in \langle S\rangle$. This concludes the Case II.

Case III Suppose that the first apparition of a critical element in $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ is a critical edge, and suppose that the previous vertex is an element of the form $y\mathtt{v}_G$ for some $y \in G * H$. That is, we can write

$$[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}} = 1\mathtt{v}_0, \mathtt{e}'_1, \mathtt{v}'_1, \dots, \mathtt{e}'_n, \mathtt{v}'_n, \mathtt{e}_1, \mathtt{v}_1, \dots, \mathtt{e}_r, x\mathtt{v}_0 \quad \text{where } \mathtt{v}'_1, \dots, \mathtt{v}'_n \in \mathtt{VT}_0,\ \mathtt{e}'_1, \dots, \mathtt{e}'_n \in \mathtt{ET}_0,$$
$$\mathtt{v}_1, \dots, \mathtt{v}_{r-1} \in \mathtt{VT} - \mathtt{VT}_0,\ \mathtt{e}_1, \dots, \mathtt{e}_r \in \mathtt{ET} - (\mathtt{ET}_0 \cup \mathtt{Y}^{\pm}),\ \mathtt{v}'_n = y\mathtt{v}_G,\ y \in G * H,\ n \in \mathbb{N},\ r \in \mathbb{N}_{>0},$$

where the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ is s. Since $\mathtt{v}'_n = y\mathtt{v}_G$ and $\mathtt{v}'_n \in \mathtt{VT}_0$, from the definition of J, we deduce that $\mathtt{v}'_n \in J$. In particular, we deduce that

$$\mathtt{e}'_n = \alpha_{\mathtt{v}'_n}\mathtt{e}_G^{-1}\ ,\ \mathtt{v}'_n = \alpha_{\mathtt{v}'_n}\mathtt{v}_G \ \text{ and } \ \mathtt{e}_1 = \alpha_{\mathtt{v}'_n}g\mathtt{e}_G \quad \text{for some } \ g \in G. \tag{3.9}$$

By definition of $\mathtt{T}_0$ and $\mathtt{Y}$, we deduce that there is $\hat{\mathtt{e}} \in \mathtt{ET}_0 \cup \mathtt{Y}$ such that $\hat{\mathtt{e}}$ and $\mathtt{e}_1$ projects on $\overline{\mathtt{T}}_C$ to the same edge. Then, since the initial vertex of $\mathtt{e}_1$ is $\mathtt{v}'_n \in \mathtt{VT}_0$, we deduce that the initial vertex of $\mathtt{e}_1$ and the initial vertex of $\hat{\mathtt{e}}$ are both equal to $\mathtt{v}'_n$. Then, from Equation (3.9) and from the definition of $\mathtt{T}$, obtain

$$\mathtt{e}'_n = \alpha_{\mathtt{v}'_n}\mathtt{e}_G^{-1}\ ,\ \mathtt{v}'_n = \alpha_{\mathtt{v}'_n}\mathtt{v}_G\ ,\ \mathtt{e}_1 = \alpha_{\mathtt{v}'_n}g\mathtt{e}_G \ \text{ and } \ \hat{\mathtt{e}} = \alpha_{\mathtt{v}'_n}\hat{g}\mathtt{e}_G \quad \text{for some } \ g, \hat{g} \in G \ \text{ and some } \ \hat{\mathtt{e}} \in \mathtt{ET}_0 \cup \mathtt{Y}. \tag{3.10}$$

Notice that $\alpha_{\mathtt{v}'_n} \in \langle G_0, H_0\rangle$ and that $\alpha_{\mathtt{v}'_n}\hat{g} \in \langle G_0, H_0\rangle$ since $\hat{\mathtt{e}} = \hat{g}\alpha_{\mathtt{v}'_n}\mathtt{e}_G$ (these facts are consequence of these elements belonging to Ω, which is contained in $\langle G_0, H_0\rangle$). Also notice that $\alpha_{\mathtt{v}'_n}g \in \langle G_0, H_0\rangle$ by applying Remark 3.2.2 (which says that every vertex and edge on $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ is in $\langle G_0, H_0\rangle$ if we assume that $x \in \langle G_0, H_0\rangle$). From these facts we deduce that $\hat{g}g^{-1}, \alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle G_0, H_0\rangle$. Moreover, by the properties of reduced word in $G * H$, since G_0 is a subgroup of G and H_0 is a subgroup of H and since $\hat{g}g^{-1} \in G$, we have obtained

$$\hat{g}g^{-1} \in G_0 \quad \text{ and } \quad \alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle G_0, H_0\rangle. \tag{3.11}$$

Recall that, by definition of $\hat{\mathtt{e}}$, we have that $\hat{\mathtt{e}}$ and $\mathtt{e}_1$ projects on $\overline{\mathtt{T}}_C$ to the same edge. Then, knowing that $\hat{\mathtt{e}} = \alpha_{\mathtt{v}'_n}\hat{g}\mathtt{e}_G$ and $\mathtt{e}_1 = \alpha_{\mathtt{v}'_n}g\mathtt{e}_G$ from Equation (3.10), we deduce that $C\alpha_{\mathtt{v}'_n}\hat{g}\mathtt{e}_G = C\alpha_{\mathtt{v}'_n}g\mathtt{e}_G$ as edges of $\mathtt{T}$. By definition of the edges of $\mathtt{T}$, we obtain $C\alpha_{\mathtt{v}'_n}\hat{g} = C\alpha_{\mathtt{v}'_n}g$. Hence, $\hat{g}g^{-1} \in \alpha_{\mathtt{v}'_n}^{-1}C\alpha_{\mathtt{v}'_n}$. Then, since $\hat{g}g^{-1} \in G_0$ by Equation (3.11), we obtain $\hat{g}g^{-1} \in \left(\alpha_{\mathtt{v}'_n}^{-1}C\alpha_{\mathtt{v}'_n}\right) \cap G_0$. In other words, by definition of $G_{\mathtt{v}'_n}$ (see Equation (3.6)), we have shown that $\hat{g}g^{-1} \in G_{\mathtt{v}'_n}$. In particular, we obtain

$$\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle S\rangle, \tag{3.12}$$

when applying the definition of S.

Recall that $\mathtt{e}_1 = \alpha_{\mathtt{v}'_n}g\mathtt{e}_G$ and $\hat{\mathtt{e}} = \alpha_{\mathtt{v}'_n}\hat{g}\mathtt{e}_G$ by Equation (3.10), so we deduce that

$$\left(\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}\right)\mathtt{e}_1 = \hat{\mathtt{e}},$$

where $\hat{\mathtt{e}} \in \mathtt{VT}_0$. Then, $\left(\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}\right)\mathtt{e}_1$ is an edge of $\mathtt{T}_0$. Then, by the claim, we deduce that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}x$ has critical number strictly smaller than the critical number of $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$, which is s.

From Equation (3.11) we have $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle G_0, H_0\rangle$. Notice that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle S\rangle \subseteq C$ where the first containment is Equation (3.12) and the second is consequence of Equation (3.7). We deduce that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in C \cap \langle G_0, H_0\rangle$. Applying that $x \in C \cap \langle G_0, H_0\rangle$ by hypothesis, we deduce that

$\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}x \in C \cap \langle G_0, H_0\rangle$. Then, as we know that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}x$ has critical number strictly smaller than s, by induction hypothesis, we obtain that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1}x \in \langle S\rangle$. As we know that $\alpha_{\mathtt{v}'_n}\hat{g}g^{-1}\alpha_{\mathtt{v}'_n}^{-1} \in \langle S\rangle$ from Equation (3.12), we deduce that $x \in \langle S\rangle$. This concludes the Case III.

Case IV Suppose that the first apparition of a critical element in $[1\mathtt{v}_0, x\mathtt{v}_0]_{\mathtt{T}}$ is a critical edge, and suppose that the previous vertex is an element of the form $y\mathtt{v}_H$ for some $y \in G * H$. This case is analogous to Case III.

□

Remark 3.3.6. Notice, on the proof of the previous result, for such C and for such G_0 and H_0 satisfying $\Omega \subseteq \langle G_0, H_0\rangle$, we have that the Kurosh rank of $C \cap \langle G_0, H_0\rangle$ is bounded by the Kurosh rank of C. Also notice that, on that proof, one can consider

$$\Omega = \{\zeta_i \,/\, i \in I\} \cup \{\alpha_j \,/\, j \in J\} \cup \{\beta_k \,/\, k \in K\}$$

instead of

$$\Omega = \{\zeta_i \,/\, i \in I\} \cup \{\alpha_j \,/\, j \in J\} \cup \{\beta_k \,/\, k \in K\} \cup \{x \in G * H \,/\, x\mathtt{e}_G \in \mathtt{ET}_0 \cup \mathtt{Y}^{\pm} \text{ or } x\mathtt{e}_H \in \mathtt{ET}_0 \cup \mathtt{Y}^{\pm}\},$$

as one can prove that any $G_0 \subseteq G$ and $H_0 \subseteq H$ such that $\langle G_0, H_0\rangle$ contains the smaller finite set then it contains the bigger one.

Chapter 4

Geometry and language complexity of positive cones

The chapter is divided in two sections. The first section studies some geometrical properties of positive cones on free products of groups when seen in the corresponding Schreier graph. The second one studies the properties of positive cones on free products of groups from the language complexity point of view. Both approaches are related on Proposition 4.2.16.

4.1 Coarsely connexion

We start by giving some definitions and fixing some notation that we need to study geometrical properties of Scherier graphs.

Definition 4.1.1. Let Γ be a graph. Let $\mathtt{v},\mathtt{u} \in \mathtt{V}\Gamma$. We define the distance of $\mathtt{v}$ and $\mathtt{u}$ in Γ to be the minimum of the set of lengths of paths starting on $\mathtt{v}$ and ending on $\mathtt{u}$ if there is such path, and to be ∞ otherwise. We denote the distance of $\mathtt{v}$ and $\mathtt{u}$ in Γ by $d_\Gamma(\mathtt{v},\mathtt{u})$.

Definition 4.1.2. Let Γ be a graph. Let $\mathtt{W}$ be a subset of $\mathtt{V}\Gamma$. Let $R \in \mathbb{N}_{>0}$. We define the R-neighborhood of $\mathtt{W}$ in Γ to be the set

$$\mathbb{B}_\Gamma(\mathtt{W}, R) = \{\mathtt{v} \in \mathtt{V}\Gamma \ / \ d_\Gamma(\mathtt{v},\mathtt{u}) \leq R \ \text{ for some } \ \mathtt{u} \in \mathtt{W}\}.$$

Definition 4.1.3. Let Γ be a graph. Let $\mathtt{W}$ be a subset of $\mathtt{V}\Gamma$. Let $\mathtt{p} = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \dots, \mathtt{e}_n, \mathtt{v}_n$ be a path in Γ. We say that $\mathtt{p}$ is said to be supported on $\mathtt{W}$ if $\mathtt{v}_i \in \mathtt{W}$ for all $i = 0, \dots, n$.

Definition 4.1.4. Let Γ be a graph. A subset $\mathtt{W}$ of $\mathtt{V}\Gamma$ is coarsely connected on Γ if there exists $R \in \mathbb{N}_{>0}$ such that for all $\mathtt{v},\mathtt{u} \in \mathtt{W}$ there is path in Γ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on $\mathtt{v}$ and ends on $\mathtt{u}$.

Definition 4.1.5. Let G be a group and $\mathcal{X}$ a generating set satisfying $\mathcal{X} = \mathcal{X}^{-1}$. Let C be a subgroup of G. We define the (left) Schreier graph of G and C with respect to $\mathcal{X}$ to be the graph $Sch(G, C; \mathcal{X})$ whose set of vertices are G/C and whose set of edges are $(G/C) \times \mathcal{X} \times \{\pm 1\}$ in such a way the edge $(gC, x, 1)$ starts on gC and ends on $(xg)C$, and $(gC, x, 1)^{-1} = (gC, x, -1)$.

Notice that, in the previous definition, there is an edge having g_1C as initial vertex and having g_2C as terminal vertex if and only if $g_1C = xg_2C$ for some $x \in \mathcal{X}$.

Definition 4.1.6. Let G be a group and $\mathcal{X}$ a generating set satisfying $\mathcal{X} = \mathcal{X}^{-1}$. Let C be a left-relatively convex subgroup of G and let $\preceq$ be a left-preorder on G relative to C. We define the positive region of $Sch(G, C; \mathcal{X})$ with respect to $\preceq$ as the set of vertices gC of $Sch(G, C; \mathcal{X})$ satisfying that $C \prec gC$. We will just call it positive region when $Sch(G, C; \mathcal{X})$ and $\preceq$ are clear by the context.

Notice that, in next definitions, the set $\mathcal{X}_G$ could possibly not be a generating set of the whole G.

Definition 4.1.7. Let G be a group and $\mathcal{X}_G$ a finite subset such that $\mathcal{X}_G^{-1} = \mathcal{X}_G$. Given $x \in \langle \mathcal{X}_G \rangle$, we denote by $|x|_{\mathcal{X}_G}$ the word length of x with respect to $\mathcal{X}_G$, with is the minimum number of elements of $\mathcal{X}_G$ whose product is x. For $n \in \mathbb{N}$, we define the ball of radius n with respect to $\mathcal{X}_G$ to be the set

$$\mathbb{B}_{\mathcal{X}_G}(n) = \{x \in \langle \mathcal{X}_G \rangle \ / \ |x|_{\mathcal{X}_G} \leq n\}.$$

The notation used in last definition is justified by the fact that $\mathbb{B}_{\mathcal{X}_G}(n)$ corresponds to the n-neighborhood of $\{1\}$ in the Cayley graph of G with respect to $\mathcal{X}_G$ (which can be identified with the Schreier graph of G and $\{1\}$ with respect to $\mathcal{X}_G$).

We introduce the notion of $\preceq$-maximal (respectively minimal) element on a ball. This notion is very important also for Chapter 5, as these elements plays a central role on the proofs.

Definition 4.1.8. Let G be a group and $\mathcal{X}_G$ a finite subset such that $\mathcal{X}_G^{-1} = \mathcal{X}_G$. Let C be a left-relatively convex subgroup of G and let $\preceq$ be a left-preorder on G relative to C. We say that an element $g \in G$ is $\preceq$-maximal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$ if

$$g \in \mathbb{B}_{\mathcal{X}_G}(n) \text{ and } gC = \max_{\preceq}\{wC \,/\, w \in \mathbb{B}_{\mathcal{X}_G}(n)\}.$$

We say that an element $g \in G$ is $\preceq$-minimal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$ if

$$g \in \mathbb{B}_{\mathcal{X}_G}(n) \text{ and } gC = \min_{\preceq}\{wC \,/\, w \in \mathbb{B}_{\mathcal{X}_G}(n)\}.$$

Assuming the notation of the previous definition, we have that there always exists a $\preceq$-maximal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$ and a $\preceq$-minimal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$. This is a direct consequence of the fact that $\mathbb{B}_{\mathcal{X}_G}(n)$ is finite since $\mathcal{X}_G$ is finite. Also notice that these elements might not be unique.

Lemma 4.1.9. *Let G be a group and $\mathcal{X}_G$ a finite subset such that $\mathcal{X}_G^{-1} = \mathcal{X}_G$. Let $\preceq$ be a left-preorder on G relative to C. Assume that $C \cap \langle \mathcal{X}_G\rangle \neq \langle \mathcal{X}_G\rangle$. For each $n \in \mathbb{N}$, consider λ_n^+ (resp. λ_n^-) a $\preceq$-maximal (resp. $\preceq$-minimal) element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$. Then, we have $\lambda_n^+ C \prec \lambda_{n+1}^+ C$ and $\lambda_{n+1}^- C \prec \lambda_n^- C$ for all $n \in \mathbb{N}$. In particular, $\lambda_n^+ C \neq \lambda_k^+ C$ and $\lambda_n^- C \neq \lambda_k^- C$ for all $n, k \in \mathbb{N}$ such that $n \neq k$.*

Proof. We will prove the first equation $\lambda_n^+ C \prec \lambda_{n+1}^+ C$ for all $n \in \mathbb{N}$, as the proof of $\lambda_{n+1}^- C \prec \lambda_n^- C$ for all $n \in \mathbb{N}$ is analogous. Fix $n \in \mathbb{N}$. Since $C \cap \langle \mathcal{X}_G\rangle \neq \langle \mathcal{X}_G\rangle$ and since $\mathcal{X}_G^{-1} = \mathcal{X}_G$, we can take $y \in \mathcal{X}_G$ such that $C \prec yC$. In particular,

$$\lambda_n^+ C \prec \lambda_n^+ yC.$$

Notice that $\lambda_n^+ y \in \mathbb{B}_{\mathcal{X}_G}(n+1)$, so then $\lambda_n^+ yC \preceq \lambda_{n+1}^+ C$. Therefore

$$\lambda_n^+ C \prec \lambda_n^+ yC \preceq \lambda_{n+1}^+ C,$$

so we are done. □

We present the main theorem of the section. We generalize Corollary 1.2 from [1] by considering free products instead of non-abelian free groups and left-preorders instead of left-orders. The proof is similar to the proof of the fact that any finitely generated group with infinitely many ends is not a group of fractions of any finitely generated proper subsemigroup in [34].

Theorem 4.1.10. *Let G and H be two non-trivial groups having $\mathcal{X}_G$ and $\mathcal{X}_H$ as finite generating sets respectively. Assume that $\mathcal{X}_G^{-1} = \mathcal{X}_G$ and $\mathcal{X}_H^{-1} = \mathcal{X}_H$. Let $\mathcal{X} = \mathcal{X}_G \cup \mathcal{X}_H$. Let $\preceq$ be a left-preorder on $G * H$ relative to C. If C has finite Kurosh rank with respect to the free product $G * H$, then the positive region of $Sch(G * H, C; \mathcal{X})$ with respect to $\preceq$ is not coarsely connected on $Sch(G * H, C; \mathcal{X})$.*

Proof. Consider $\mathtt{W}$ the positive region of $Sch(G * H, C; \mathcal{X})$ with respect to $\preceq$. We prove it by contradiction, so we consider $R \in \mathbb{N}_{>0}$ such that for all $\mathtt{v}, \mathtt{u}$ vertices on the positive region there is path in $Sch(G*H, C; \mathcal{X})$ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on $\mathtt{v}$ and ends on $\mathtt{u}$. For each $s \in \mathbb{N}$, consider λ_s^- a $\preceq$-minimal element in the ball $\mathbb{B}_{\mathcal{X}}(s)$. By definition, we have

$$\lambda_s^- C \preceq gC \quad \text{for all} \quad s \in \mathbb{N} \quad \text{and for all} \quad g \in \mathbb{B}_{\mathcal{X}}(s). \tag{4.1}$$

Also, from Lemma 4.1.9, we deduce that $\lambda_s^- C \neq \lambda_t^- C$ for all $s, t \in \mathbb{N}$ such that $s \neq t$.

For each $s \in \mathbb{N}$, we are going to say that the pair $(\lambda_s^-, y_{\lambda_s^-})$ satisfies the property (P) if the following holds: for every finite collection $x_1, \dots, x_m \in G * H$ with $m \in \mathbb{N}_{>0}$ satisfying that $x_1 = 1$, that $x_m C = (\lambda_s^-)^{-1} y_{\lambda_s^-}^{\varepsilon} \lambda_s^- C$ for some $\varepsilon \in \{\pm 1\}$ and that $x_i = f_i x_{i+1}$ for some $f_i \in G \cup H$ where $i = 1, \dots, m-1$, then

it holds that $x_iC = \lambda_s^- C$ for some $i \in \{1, \dots, m\}$. Since we know that $\lambda_s^- C \neq \lambda_t^- C$ for all $s, t \in \mathbb{N}$ such that $s \neq t$, applying Corollary 3.3.4, we deduce that we can fix a $n_0 \in \mathbb{N}$ in such a way for every $s \in \mathbb{N}$ satisfying $n_0 \leq s$, there exists $y_{\lambda_s^-} \in G \cup H$ such that $(\lambda_s^-)^{-1} y_{\lambda_s^-}^{\varepsilon} \lambda_s^- \notin C$ and such that the pair $(\lambda_s^-, y_{\lambda_s^-})$ satisfies the property (P).

Take $n \in \mathbb{N}$ such that $R \leq n$ and $n_0 \leq n$. Since $n_0 \leq n$, by the definition of n_0, we have that there exists $y_{\lambda_n^-} \in G \cup H$ such that $(\lambda_n^-)^{-1} y_{\lambda_n^-} \lambda_n^- \notin C$ and such that the pair $(\lambda_n^-, y_{\lambda_n^-})$ satisfies the property (P). Since $(\lambda_n^-)^{-1} y_{\lambda_n^-} \lambda_n^- \notin C$ and since $\left((\lambda_n^-)^{-1} y_{\lambda_n^-} \lambda_n^-\right)^{-1} = (\lambda_n^-)^{-1} y_{\lambda_n^-}^{-1} \lambda_n^-$, we can fix $\varepsilon \in \{\pm 1\}$ such that $C \prec (\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$. In particular, $(\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$ is a vertex on the positive region. Fix any $x \in \mathcal{X}$ such that $C \prec xC$, which is possible since $\mathcal{X}$ generates $G * H$. Therefore, $(\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$ and xC are vertices on the positive region. Then, by the hypothesis we have assumed at the start of the proof, we deduce that there is path $\mathtt{p}$ in $Sch(G * H, C; \mathcal{X})$ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on xC and ends on $(\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$. In other words, we can write the path $\mathtt{p}$ as

$$\mathtt{p} = \mathtt{v}_1, \mathtt{e}_2, \mathtt{v}_1, \dots, \mathtt{e}_l, \mathtt{v}_l$$

where $\mathtt{v}_1 = xC$, where $\mathtt{v}_l = (\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$ and where we know that

$$\mathtt{v}_i \in \mathbb{B}_\Gamma(\mathtt{W}, R) \quad \text{for all} \quad i = 1, \dots, l. \tag{4.2}$$

Notice that, since $x \in \mathcal{X}$, by definition of the graph $Sch(G * H, C; \mathcal{X})$, we can extend the previous path, so we can consider the path

$$\mathtt{p}_0 = \mathtt{v}_0, \mathtt{e}_1, \mathtt{v}_1, \mathtt{e}_2, \mathtt{v}_2, \dots, \mathtt{e}_l, \mathtt{v}_l$$

in $Sch(G * H, C; \mathcal{X})$ for $\mathtt{v}_0 = C$. Notice that, by the structure of $Sch(G * H, C; \mathcal{X})$, the description of $\mathtt{p}_0$ implies that there are $x_1, \dots, x_l \in G * H$ satisfying that $\mathtt{v}_i = x_i C$ for all $i = 0, \dots, l$, that $x_1 = 1$, that $x_l C = (\lambda_n^-)^{-1} y_{\lambda_n^-}^{\varepsilon} \lambda_n^- C$ and that $x_i = f_i x_{i+1}$ for some $f_i \in \mathcal{X}$ where $i = 1, \dots, l-1$. Hence, applying that $(\lambda_n^-, y_{\lambda_n^-})$ satisfies the property (P), we deduce that $x_i C = \lambda_n^- C$ for some $i \in \{1, \dots, l\}$. Then, applying Equation (4.2) and the fact that $\mathtt{v}_i = x_i C$ for all $i = 0, \dots, l$, we deduce that

$$\lambda_n^- C \in \mathbb{B}_{Sch(G*H, C; \mathcal{X})}(\mathtt{W}, R).$$

In particular, by definition of $Sch(G * H, C; \mathcal{X})$, we deduce that there is $y \in \mathbb{B}_{\mathcal{X}}(R)$ such that $y \lambda_n^- C$ is on the positive region. Hence, $y^{-1} C \prec \lambda_n^- C$ where $y^{-1} \in \mathbb{B}_{\mathcal{X}}(R)$. Then, as we know that $R \leq n$, we get a contradiction with Equation (4.1). □

4.2 Positive cones of free products are not regular

We now focus on studying the complexity of the positive cone of left-preorders on free products. We will prove that there is no left-relative order on free products whose positive cone can be described by a regular language. We use the ideas from [23], where it is shown that the positive cone of a left-order on a free product cannot be described by a regular language. There is an alternative proof of this fact in [1]. The definitions are also inspired in [23] but generalized to the left-preorder context. We start introducing the main concepts involved and fixing some notation. For a reference about the theory of regular languages and automata, see [27].

Notation 17. Let $\mathbf{X}$ be a set. We denote by $\mathbf{X}^*$ the set of finite strings of elements in $\mathbf{X}$. This set is a monoid with the concatenation operation. The neutral element of $\mathbf{X}^*$ is the empty string.

Definition 4.2.1. Let $\mathbf{X}$ be a finite set. A language over $\mathbf{X}$ is a subset $\mathbf{L}$ of $\mathbf{X}^*$.

Definition 4.2.2. Let $\mathbf{X}$ be a finite set. Let $\mathbf{w}, \mathbf{v} \in \mathbf{X}^*$. We say that $\mathbf{w}$ is a prefix of $\mathbf{v}$ if $\mathbf{v} = \mathbf{wu}$ for some $\mathbf{u} \in \mathbf{X}^*$. We say that $\mathbf{w}$ is a suffix of $\mathbf{v}$ if $\mathbf{v} = \mathbf{uw}$ for some $\mathbf{u} \in \mathbf{X}^*$.

Definition 4.2.3. Let $\mathbf{X}$ be a finite set. Let $\mathbf{L}$ be language over $\mathbf{X}$. We define the set of prefixes of $\mathbf{L}$ as the set $Pref(\mathbf{L})$ of all elements in $\mathbf{X}^*$ that are the prefix of some element in $\mathbf{L}$. We define the set of suffixes of $\mathbf{L}$ as the set $Suf(\mathbf{L})$ of all elements in $\mathbf{X}^*$ that are the prefix of some element in $\mathbf{L}$.

Definition 4.2.4. Let G be a finitely generated group. Let S be a subset of G. Let $\mathbf{X}$ be a finite set. Let $\phi : \mathbf{X}^* \longrightarrow G$ be a surjective homomorphism. Let $\mathbf{L}$ be language over $\mathbf{X}$. We say that $\mathbf{L}$ represents S via ϕ if we have $\phi(\mathbf{L}) = S$.

Definition 4.2.5. Let $\mathbf{X}$ be a finite set. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be a languages over $\mathbf{X}$. We define

$$\mathbf{L}_1\mathbf{L}_2 = \{\mathbf{u}_1\mathbf{u}_2 \ / \ \mathbf{u}_1 \in \mathbf{L}_1, \ \mathbf{u}_2 \in \mathbf{L}_2\}.$$

Definition 4.2.6. Let $\mathbf{X}$ be a finite set. Let $\mathbf{L}$ be a language over $\mathbf{X}$. We define the Klenee star of $\mathbf{L}$ as the set

$$\mathbf{L}^* = \{\mathbf{u}_1 \cdots \mathbf{u}_n \ / \ \mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbf{L} \ \text{ for some } \ n \in \mathbb{N}\}.$$

Notice that for $n = 0$ we mean the empty string.

We introduce the central notion of a regular language.

Definition 4.2.7. Let $\mathbf{X}$ be a finite set. We say that a language over $\mathbf{X}$ is regular over $\mathbf{X}$ if it belongs to the smallest set of languages over $\mathbf{X}$ satisfying the following:

- The empty set $\emptyset$ is regular over $\mathbf{X}$.
- The singleton $\{\epsilon\}$ is regular over $\mathbf{X}$ for ϵ the empty string.
- The singleton $\{\mathbf{x}\}$ is regular over $\mathbf{X}$ for all $\mathbf{x} \in \mathbf{X}$.
- If $\mathbf{L}$ is regular over $\mathbf{X}$, then $\mathbf{L}^*$ is regular over $\mathbf{X}$.
- If $\mathbf{L}_1$ and $\mathbf{L}_2$ are regular over $\mathbf{X}$, then $\mathbf{L}_1 \cup \mathbf{L}_2$ is regular over $\mathbf{X}$.
- If $\mathbf{L}_1$ and $\mathbf{L}_2$ are regular over $\mathbf{X}$, then $\mathbf{L}_1\mathbf{L}_2$ is regular over $\mathbf{X}$.

Remark 4.2.8. Let $\mathbf{X}$ be a finite set. If $\mathbf{L}$ is a finite language over $\mathbf{X}$, then it is regular over $\mathbf{X}$. This follows by the definition of being regular over $\mathbf{X}$, since we know that the empty set is regular over $\mathbf{X}$, since every singleton is regular over $\mathbf{X}$ and since being regular over $\mathbf{X}$ is closed under finite unions.

Remark 4.2.9. Let $\mathbf{X}$ and $\mathbf{Y}$ be finite sets satisfying $\mathbf{X} \subseteq \mathbf{Y}$. By definition, If $\mathbf{L}$ is a regular language over $\mathbf{X}$, then it is regular over $\mathbf{Y}$.

We define finite state automata. We can use them to provide an equivalent definition of regular language that will be helpful.

Definition 4.2.10. A finite state automaton is a tuple $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$, where $\mathcal{S}$ is a finite set, $\mathcal{A}$ is a subset of $\mathcal{S}$, s_0 is an element of $\mathcal{S}$, $\mathbf{X}$ is a finite set and τ is a function of the form $\tau : \mathcal{S} \times \mathbf{X} \longrightarrow \mathcal{S}$. For a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$, the elements of $\mathcal{S}$ are called accepting states, the elements of $\mathcal{A}$ are called states, we call s_0 the initial state, we call $\mathbf{X}$ the input alphabet and we call τ the transition function. For a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$, we can identify τ with its extension $\tau : \mathcal{S} \times \mathbf{X}^* \longrightarrow \mathcal{S}$, which is recursively defined by $\tau(s, \mathbf{ux}) = \tau(\tau(s, \mathbf{u}), \mathbf{x})$ for each $s \in \mathcal{S}$, $\mathbf{u} \in \mathbf{X}^*$ and $\mathbf{x} \in \mathbf{X}$.

Remark 4.2.11. Following the notation from the previous definition, from the fact that $\tau(s, \mathbf{ux}) = \tau(\tau(s, \mathbf{u}), \mathbf{x})$ for each $s \in \mathcal{S}$, $\mathbf{u} \in \mathbf{X}^*$ and $\mathbf{x} \in \mathbf{X}$, we can recursively deduce that $\tau(s, \mathbf{uw}) = \tau(\tau(s, \mathbf{u}), \mathbf{w})$ for each $s \in \mathcal{S}$ and each $\mathbf{u}, \mathbf{w} \in \mathbf{X}^*$.

Proposition 4.2.12. *Let $\mathbf{X}$ be a finite set. A language $\mathbf{L}$ over $\mathbf{X}$ is regular over $\mathbf{X}$ if and only if there is a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$ that accepts $\mathbf{L}$, that is,*

$$\mathbf{L} = \{\mathbf{u} \in \mathbf{X}^* \ / \ \tau(s_0, \mathbf{u}) \in \mathcal{A}\}.$$

Proof. See Section 3.2 from [27]. □

We introduce some properties of regular languages that we will need later.

Proposition 4.2.13. *Let $\mathbf{X}$ and $\mathbf{Y}$ be finite sets. Let $\alpha : \mathbf{X}^* \longrightarrow \mathbf{Y}^*$ be a monoid homomorphism. Let $\mathbf{L}$ be a language over $\mathbf{X}$. If $\mathbf{L}$ is regular, then $\alpha(\mathbf{L})$ is regular.*

Proof. See Theorem 4.14 from [27]. □

We now state and prove two technical lemmas. These lemmas are direct consequences of the Pumping Lemma for regular languages (see Theorem 4.1 from [27]). However, we have decided to write the proof of the first statement as it is not difficult and may be helpful to understand the ideas behind it. The second lemma can be directly deduced from the first one by using the fact that regular languages are closed under taking reversals (see Theorem 4.11 from [27]). However, the second lemma can be proved also in a similar way as in the first lemma.

Lemma 4.2.14. *Let* $\mathbf{X}$ *be a finite set. Let* $\mathbf{L}$ *be a regular language over* $\mathbf{X}$ *and let* $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$ *be a finite state automaton that accepts* $\mathbf{L}$*. Let* $k = \operatorname{Card}(\mathcal{S})$*. If* $\mathbf{w} \in \mathit{Pref}(\mathbf{L})$*, then* $\mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n \in \mathbf{L}$ *for some* $n \in \mathbb{N}$ *such that* $n \leq k-1$ *and some* $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$*.*

Proof. Consider $\mathbf{w} \in \mathit{Pref}(\mathbf{L})$. Then, we have that $\mathbf{w}\mathbf{u} \in \mathbf{L}$ for some $\mathbf{u} \in \mathbf{X}^*$. Equivalently, we have that $\mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n \in \mathbf{L}$ for some $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$ and some $n \in \mathbb{N}$ (possibly 0 when $\mathbf{u}$ is the empty string). We are going to prove that if $k-1 < n$ then there is $m < n$ and there are $\mathbf{x}'_1, \ldots, \mathbf{x}'_m \in \mathbf{X}$ such that $\mathbf{w}\mathbf{x}'_1 \cdots \mathbf{x}'_n \in \mathbf{L}$.

We have that $\tau(s_0, \mathbf{w}), \tau(s_0, \mathbf{w}\mathbf{x}_1), \ldots, \tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n)$ is a sequence of $n+1$ elements of $\mathcal{S}$. Then, since $k-1 < n$ and $k = \operatorname{Card}(\mathcal{S})$, two elements of the sequence must be equal. That is,

$$\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_i) = \tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_j)$$

for some $i, j \in \mathbb{N}$ such that $0 \leq i < j \leq n$. From this and from Remark 4.2.11, we deduce that

$$\begin{aligned} &\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_i\mathbf{x}_{j+1} \cdots \mathbf{x}_n) = \tau(\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_i), \mathbf{x}_{j+1} \cdots \mathbf{x}_n)) = \\ &= \tau(\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_j), \mathbf{x}_{j+1} \cdots \mathbf{x}_n)) = \tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n). \end{aligned}$$

Now, notice that $\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n) \in \mathcal{A}$ since $\mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n \in \mathbf{L}$. Hence, we deduce $\tau(s_0, \mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_i\mathbf{x}_{j+1} \cdots \mathbf{x}_n) \in \mathcal{A}$, so then $\mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_i\mathbf{x}_{j+1} \cdots \mathbf{x}_n \in \mathbf{L}$, as we wanted. □

Lemma 4.2.15. *Let* $\mathbf{X}$ *be a finite set. Let* $\mathbf{L}$ *be a regular language over* $\mathbf{X}$ *and let* $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$ *be a finite state automaton that accepts* $\mathbf{L}$*. Let* $k = \operatorname{Card}(\mathcal{S})$*. If* $\mathbf{w} \in \mathit{Suf}(\mathbf{L})$*, then* $\mathbf{x}_1 \cdots \mathbf{x}_n\mathbf{w} \in \mathbf{L}$ *for some* $n \in \mathbb{N}$ *such that* $n \leq k-1$ *and some* $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$*.*

Proof. The proof is similar to the proof of Lemma 4.2.14, □

We now establish a relation between regular languages and coarsely connected sets on the Schreier graph. This generalizes Proposition 7.2 from [1], and the proof follows the ideas there. However, notice that next proposition can be deduced from the mentioned reference since the Schreier graph is a quotient of the Cayley graph.

Proposition 4.2.16. *Let* G *be a finitely generated group. Let* C *be a subgroup of* G*. Let* S *be a subset of* G*. Let* $\mathbf{X}$ *be a finite set. Let* $\phi : \mathbf{X}^* \longrightarrow G$ *be a surjective homomorphism. Let* $\mathbf{L}$ *be a language over* $\mathbf{X}$ *that represents* S *via* ϕ*. If* $\mathbf{L}$ *is regular, then the set* $\{xC \;/\; x \in S\}$ *is coarsely connected on the Schreier graph of* G *and* C *with respect to* $\phi(\mathbf{X})$*.*

Proof. Define $\mathtt{W} = \{xC \;/\; x \in S\}$. Consider Γ the Schreier graph of G and C with respect to $\phi(\mathbf{X})$. Suppose that $\mathbf{L}$ is regular over $\mathbf{X}$ so, by Proposition 4.2.12, there is a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$ accepting $\mathbf{L}$. Consider $R = \operatorname{Card}(\mathcal{S}) - 1$.

We claim that

$$\mathbf{w} \in \mathit{Suf}(\mathbf{L}) \quad \text{implies} \quad \phi(\mathbf{w})C \in \mathbb{B}_\Gamma(\mathtt{W}, R). \tag{4.3}$$

We prove the claim. Take $\mathbf{w} \in \mathit{Suf}(\mathbf{L})$. By Lemma 4.2.14 and since $R = \operatorname{Card}(\mathcal{S}) - 1$, we deduce that $\mathbf{x}_1 \cdots \mathbf{x}_n\mathbf{w} \in \mathbf{L}$ for some $n \in \mathbb{N}$ such that $n \leq R$ and some $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$. Notice that

$$d_\Gamma(\phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n)\phi(\mathbf{w})C, \phi(\mathbf{w})C) \leq n \leq R$$

since $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$, and that

$$\phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n)\phi(\mathbf{w}) \in S$$

since $\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\phi(\mathbf{w}) = \phi(\mathbf{x}_1\cdots\mathbf{x}_n\mathbf{w})$ is an element of $\phi(\mathbf{L}) = S$. This shows Equation (4.3), as claimed.

We are going to prove that the set $\mathtt{W} = \{xC \,/\, x \in S\}$ is coarsely connected on Γ. Take $x_1, x_2 \in S$. Since $x_1 \in S$ and since $\phi(\mathbf{L}) = S$, we deduce that there exists $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbf{X}$ for some $n \in \mathbb{N}$ such that $\mathbf{y}_1 \dots \mathbf{y}_n \in \mathbf{L}$ and $\phi(\mathbf{y}_1 \dots \mathbf{y}_n) = x_1$. Notice that $\mathbf{y}_{n-j+1} \dots \mathbf{y}_n \in \mathit{Suf}(\mathbf{L})$ for all $j = 0, \dots, n$, where for $j = 0$ it is the empty string. Then, by Equation (4.3), we deduce that

$$\phi(\mathbf{y}_{n-j+1} \dots \mathbf{y}_n) \in \mathbb{B}_\Gamma(\mathtt{W}, R) \quad \text{for all } \; j = 0, \dots, n.$$

In particular, every element on the sequence $\{\phi(\mathbf{y}_{n-i+1}) \dots \phi(\mathbf{y}_n)\}_{i=0}^n$ belongs to $\mathbb{B}_\Gamma(\mathtt{W}, R)$. Then, we deduce that this sequence induces a path $\mathtt{p}_1$ in Γ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on 1 and ends on $x_1 C$. Similarly for x_2, we obtain a path $\mathtt{p}_2$ in Γ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on 1 and ends on $x_2 C$. Hence, the path $\mathtt{p}_1^{-1}\mathtt{p}_2$ is a path in Γ supported on $\mathbb{B}_\Gamma(\mathtt{W}, R)$ that starts on $x_1 C$ and ends on $x_2 C$. This proves that $\mathtt{W} = \{xC \,/\, x \in S\}$ is coarsely connected on Γ, as we wanted. □

We are going to define a notion of regularity for left-preorders. In order to do that, we previously need to show that the regularity does not depend on the chosen set $\mathbf{X}$ and the chosen surjective monoid homomorphism ϕ. This proposition is standard.

Proposition 4.2.17. *Let G be a finitely generated group and S a subset. Let $\mathbf{X}, \mathbf{Y}$ be finite sets and $\phi : \mathbf{X}^* \longrightarrow G, \psi : \mathbf{Y}^* \longrightarrow G$ be surjective homomorphisms. If there is a regular language $\mathbf{L}$ over $\mathbf{X}$ that represents S via ϕ, then there exists a regular language $\mathbf{L}'$ over $\mathbf{Y}$ that represents S via ψ.*

Proof. Since ψ is surjective, for all $\mathbf{x} \in \mathbf{X}$ we can choose a $\mathbf{w}_\mathbf{x} \in \mathbf{Y}^*$ such that $\psi(\mathbf{w}_\mathbf{x}) = \phi(\mathbf{x})$. By the universal property of the free monoid, we have a monoid homomorphism $\alpha : \mathbf{X}^* \longrightarrow \mathbf{Y}^*$ such that $\alpha(\mathbf{x}) = \mathbf{w}_\mathbf{x}$ for all $\mathbf{x} \in \mathbf{X}$. Then,

$$\psi \circ \alpha(\mathbf{x}) = \psi(\mathbf{w}_\mathbf{x}) = \phi(\mathbf{x}) \;\text{ for all } \mathbf{x} \in \mathbf{X}.$$

In particular, since $\mathbf{X}$ generates $\mathbf{X}^*$ as a monoid, we deduce that

$$\psi \circ \alpha = \phi.$$

By taking $\mathbf{L}' = \alpha(\mathbf{L})$, since $\mathbf{L}$ is regular, when applying Proposition 4.2.13 we deduce that $\mathbf{L}'$ is regular over $\mathbf{Y}$. Then, since

$$\psi(\mathbf{L}') = (\psi \circ \alpha)(\mathbf{L}) = \phi(\mathbf{L}) = S,$$

we have that $\mathbf{L}' \subseteq \mathbf{Y}^*$ is a regular language over $\mathbf{Y}$ that represents S via ψ. □

Definition 4.2.18. Let G be a finitely generated group and $\preceq$ a left-preorder on G relative to C with positive cone P. Let $\mathbf{X}$ be a finite set and let $\phi : \mathbf{X}^* \longrightarrow G$ be a surjective homomorphism. We say that $\preceq$ is regular if there exists a regular language $\mathbf{L}$ over $\mathbf{X}$ such that $\mathbf{L}$ represents P via ϕ.

Remark 4.2.19. By Proposition 4.2.17, we can assume that in last definition $\mathbf{X} = \mathbf{Y} \sqcup \mathbf{Y}^{-1}$ for $\mathbf{Y}$ finite and $\phi : \mathbf{X}^* \longrightarrow G$ such that $\phi(\mathbf{x}^{-1}) = \phi(\mathbf{x})^{-1}$ for all $\mathbf{x} \in \mathbf{X}$, where we are denoting $(\mathbf{y}^{-1})^{-1} = \mathbf{y}$ for $\mathbf{y} \in \mathbf{Y}$ as formal letters.

We can establish, by using a result from the previous section, a relation between being regular for positive cones and not being coarsely connected for positive regions.

Proposition 4.2.20. *Let G, H be non-trivial finitely generated groups and $\preceq$ a left-preorder on $G * H$ relative to C. If C has finite Kurosh rank with respect to the free product $G * H$, then $\preceq$ is not regular.*

Proof. We prove it by contradiction, so assume that C has finite Kurosh rank with respect to the free product $G * H$ and that $\preceq$ is regular. As G, H are finitely generated, we can consider $\mathbf{X} = \mathbf{Y}_G \sqcup \mathbf{Y}_H$ a finite set and $\phi : \mathbf{X}^* \longrightarrow G * H$ a surjective homomorphism such that $\phi(\mathbf{Y}_G)$ generates G and $\phi(\mathbf{Y}_G) = \phi(\mathbf{Y}_G)^{-1}$ and such that $\phi(\mathbf{Y}_H)$ generates H and $\phi(\mathbf{Y}_H) = \phi(\mathbf{Y}_H)^{-1}$. Since $\preceq$ is regular, we deduce that there is a regular language $\mathbf{L}$ over $\mathbf{X}$ that represents P via ϕ, for P the positive cone of $\preceq$. Then, by Proposition 4.2.16, the set $\{xC \,/\, x \in P\}$ is coarsely connected on the Schreier graph of $G * H$ and C with respect to $\phi(\mathbf{X})$. In other words, we have shown that the positive region is coarsely connected on the Schreier graph of $G * H$ and C with respect to $\phi(\mathbf{X})$. This contradicts Theorem 4.1.10 since C has finite Kurosh rank with respect to the free product $G * H$. □

We now prove few technical results in order to prove Theorem 4.2.25. We apply the ideas on [23].

Proposition 4.2.21. *Let G be a finitely generated group and let $\preceq$ be a left-preorder on G relative to C with positive cone P. Let $\mathbf{X}$ be a finite set, $\phi : \mathbf{X}^* \longrightarrow G$ be a surjective homomorphism and $\mathbf{L}$ be a language over $\mathbf{X}$ representing P via ϕ. Suppose that there exists a subset P_0 of P such that P_0 is not upper bounded, that is, for all $x \in G$ there exists $y \in P_0$ such that $xC \preceq yC$. If*

$$P_0^{-1} \subseteq \phi(Pref(\mathbf{L}))$$

then $\mathbf{L}$ is not regular over $\mathbf{X}$.

Proof. We will proceed by contradiction. Suppose that $\mathbf{L}$ is a regular language over $\mathbf{X}$ representing P via ϕ and that $P_0 \subseteq P$ is not upper bounded with P_0^{-1} contained in $\phi(Pref(\mathbf{L}))$. Since $\mathbf{L}$ is regular over $\mathbf{X}$, there is a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, \mathbf{X}, \tau)$ accepting $\mathbf{L}$. Consider $k = \text{Card}\,(\mathcal{S}) - 1$. Consider $\mathbb{B}_{\phi(\mathbf{X})}(k)$ the ball of radius k with respect to $\phi(\mathbf{X})$. Since $\mathbb{B}_{\phi(\mathbf{X})}(k)$ is finite and P_0 is not upper bounded, there exists $y \in P_0$ such that

$$xC \preceq yC \quad \text{for all } x \in \mathbb{B}_{\phi(\mathbf{X})}(k).$$

In this way, we have that $y^{-1} \in P_0^{-1}$, but $P_0^{-1} \subseteq \phi(Pref(\mathbf{L}))$ by hypothesis, so there exists $\mathbf{w} \in Pref(\mathbf{L})$ such that

$$\phi(\mathbf{w}) = y^{-1}.$$

Since $\mathbf{w} \in Pref(\mathbf{L})$ and since $k = \text{Card}\,(\mathcal{S}) - 1$, by Lemma 4.2.14 we deduce that $\mathbf{w}\mathbf{x}_1 \cdots \mathbf{x}_n \in \mathbf{L}$ for some $n \in \mathbb{N}$ such that $n \leq k$ and some $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{X}$. In particular, by defining $\mathbf{u} = \mathbf{x}_1 \cdots \mathbf{x}_n$, we have

$$\phi(\mathbf{u}) \in \mathbb{B}_{\phi(\mathbf{X})}(k) \quad \text{and} \quad \mathbf{wu} \in \mathbf{L}.$$

From $\mathbf{wu} \in \mathbf{L}$, since $\mathbf{X}$ represents P via ϕ, we obtain $\phi(\mathbf{wu}) \in P$, so then $\phi(\mathbf{w})\phi(\hat{\mathbf{u}}) \in P$.

Therefore, we have that $C \prec \phi(\mathbf{w})\phi(\mathbf{u})C$. From this and from $\phi(\mathbf{w}) = y^{-1}$, we obtain $C \prec y^{-1}\phi(\mathbf{u})C$. Then, $yC \prec \phi(\mathbf{u})C$. Recall that $\phi(\mathbf{u}) \in \mathbb{B}_{\phi(\mathbf{X})}(k)$. Hence, we have obtained a contradiction with the fact that $xC \preceq yC$ for all $x \in \mathbb{B}_{\phi(\mathbf{X})}(k)$. □

Corollary 4.2.22. *Let G be a finitely generated group and let $\preceq$ be a left-preorder on G with positive cone P. Let $\mathbf{X}$ be a finite set, $\phi : \mathbf{X}^* \longrightarrow G$ be a surjective homomorphism and $\mathbf{L}$ be a language over $\mathbf{X}$ representing P via ϕ. If*

$$G - \{1\} \subseteq \phi(Pref(\mathbf{L}))$$

then $\mathbf{L}$ is not regular over $\mathbf{X}$.

Proof. Since $P^{-1} \subseteq G - \{1\}$, using the hypothesis we deduce that $P^{-1} \subseteq \phi(Pref(\mathbf{L}))$. The proof is direct by taking $P_0 = P$ on the previous theorem and noticing that obviously it is not upper bounded. □

Lemma 4.2.23. *Let G, H be non-trivial finitely generated groups. Let $\mathbf{X} = \mathbf{Y}_G \sqcup \mathbf{Y}_H \sqcup \mathbf{Y}_G^{-1} \sqcup \mathbf{Y}_H^{-1}$ be a finite set and $\phi : \mathbf{X}^* \longrightarrow G * H$ be a surjective homomorphism such that $\phi(\mathbf{x}^{-1}) = \phi(\mathbf{x})^{-1}$ for all $\mathbf{x} \in \mathbf{X}$ and such that $\phi(\mathbf{Y}_G)$ generates G and $\phi(\mathbf{Y}_H)$ generates H. If $\mathbf{w} \in \mathbf{X}^*$ and $\phi(\mathbf{w}) \in G * H$ has a reduced factorization $\phi(\mathbf{w}) = g_1 h_1 \ldots g_m h_m$ as words on G and H, then for all $i \in \{1, \ldots, m\}$ the elements $g_1 h_1 \ldots g_i h_i$ and $g_1 h_1 \ldots g_i$ of $G * H$ are represented by a prefix of the word $\mathbf{w}$.*

Proof. See Lemma 8 in [23]. □

Lemma 4.2.24. *Let G, H be non-trivial groups and let C be a proper subgroup of $G * H$. Let $x \in G * H$ be such that $x \neq 1$. Then, there exists $y \in G * H$ such that $xyx^{-1} \notin C$ and such that xyx^{-1} has no cancellations as a word in G and H.*

Proof. We know that $x \neq 1$ and $x \in G * H$. Then, x has a non-trivial reduced factorization $x = g_1 h_1 \ldots g_m h_m$ as words on G and H where $g_1, \ldots, g_m \in G$ and $h_1, \ldots, h_m \in H$. Notice that if $h_m = 1$ then the reduced word of x ends in a element of G so, by interchanging G and H if necessary, we can assume that the reduced word of x ends in a non-trivial element h_m of H.

Define $\hat{C} = x^{-1}Cx$. We claim that there exists $y \in (G * H) - \hat{C}$ whose reduced word ends and starts in a element of G. In order to do that, notice that $\hat{C} \neq G * H$ since $C \neq G * H$. We are now going to prove the claim by distinguishing two cases:

Case I Suppose that $G \subseteq \hat{C}$. Then, we notice that $H \not\subseteq \hat{C}$. Indeed, if this is not the case, we would have $G, H \subseteq \hat{C}$, so then $G * H = \hat{C}$, and this contradicts the fact that $\hat{C} \neq G * H$. Since $H \not\subseteq \hat{C}$, there exists $h \in H$ such that $h \notin \hat{C}$ and, in particular h is non-trivial. Since G is non-trivial, we can take $g \in G$ non-trivial. We have that $g \in \hat{C}$ since $G \subseteq \hat{C}$. Now take $y = ghg^{-1}$, and notice that its reduced word ends and starts in a element of G. Also, notice that $y = ghg^{-1} \notin \hat{C}$ since $g \in \hat{C}$, $h \notin \hat{C}$ and $\hat{C}$ is a subgroup. Hence, $y \in (G * H) - \hat{C}$ has a reduced word that ends and starts in a element of G.

Case II Suppose that $G \not\subseteq \hat{C}$. Then, there is $g \in G$ such that $g \notin \hat{C}$, and in particular it is non-trivial. Taking $y = g$, it is clear that its reduced word ends and starts in a element of G and $y \in (G * H) - \hat{C}$.

By the claim, there exists $y \in (G * H) - \hat{C}$ whose reduced word ends and starts in a element of G. As $y \in (G * H) - \hat{C}$ and $\hat{C} = x^{-1}Cx$, we deduce that $xyx^{-1} \notin C$. Notice that the reduced word of y ends and starts in a element of G, that the reduced word of x ends in a element of H and the reduced word of x^{-1} starts in a element of H. Therefore, we can conclude that $xyx^{-1} \notin C$ and xyx^{-1} has no cancellations as a word in G and H. □

We now prove the theorem.

Theorem 4.2.25. *Let G, H be non-trivial finitely generated groups and $\preceq$ a left-preorder on $G * H$. Then $\preceq$ is not regular.*

Proof. As G, H are finitely generated, we can consider $\mathbf{X} = \mathbf{Y}_G \sqcup \mathbf{Y}_H \sqcup \mathbf{Y}_G^{-1} \sqcup \mathbf{Y}_H^{-1}$ a finite set and $\phi : \mathbf{X}^* \longrightarrow G * H$ a surjective homomorphism such that $\phi(\mathbf{x}^{-1}) = \phi(\mathbf{x})^{-1}$ for all $\mathbf{x} \in \mathbf{X}$ and such that $\phi(\mathbf{Y}_G)$ generates G and $\phi(\mathbf{Y}_H)$ generates H. In light of Proposition 4.2.17, we only need to show that there is no regular language $\mathbf{L}$ over $\mathbf{X}$ that represents the positive cone P via ϕ. In order to do that, we are going to prove that, given $\mathbf{L} \subseteq \mathbf{X}^*$ such that $\phi(\mathbf{L}) = P$, then $\mathbf{L}$ is not a regular language over $\mathbf{X}$. By Corollary 4.2.22, if we show that

$$(G * H) - \{1\} \subseteq \phi(Pref(\mathbf{L}))$$

we are done. We are going to prove it.

Let $x \in (G * H) - \{1\}$. By Lemma 4.2.24, there exists $y \in G * H$ such that $xyx^{-1} \notin C$ and such that xyx^{-1} has no cancellations as a word in G and H. Since $xyx^{-1} \notin C$, we have $xyx^{-1} \in P$ or $xy^{-1}x^{-1} = (xyx^{-1})^{-1} \in P$. Without loss of generality, we can assume that $xyx^{-1} \in P$ since the same argument works for $xy^{-1}x^{-1}$. We have $xyx^{-1} \in P = \phi(\mathbf{L})$, so there is $\mathbf{v} \in \mathbf{L}$ such that $\phi(\mathbf{v}) = xyx^{-1}$. We know that xyx^{-1} has no cancellations as a word in G and H and that $\phi(\mathbf{v}) = xyx^{-1}$, so by Lemma 4.2.23, we deduce that there is $\mathbf{v}' \in \mathbf{X}^*$ a prefix of $\mathbf{v}$ such that $\phi(\mathbf{v}') = x$. Therefore, $x = \phi(\mathbf{v}') \in \phi(Pref(\mathbf{L}))$, as we needed. □

Corollary 4.2.26. *Let G, H be non-trivial groups, C a left-relatively convex subgroup on $G * H$ and $\preceq$ a left-preorder on $G * H$ relative to C with positive cone P. Then P is not finitely generated as a subsemigroup of $G * H$.*

Proof. We do it by contradiction, so assume P is finitely generated as a subsemigroup of $G * H$. In Remark 1.2.6 it is shown that P generates $G * H$ as a cosequence of P being the positive cone of a left-preorder on $G * H$. Since $G * H$ is finitely generated, applying Grushko's theorem, we deduce that G and H are finitely generated groups. Since P is finitely generated as a subsemigroup of $G * H$, then P can be represented by a language of the form $\mathbf{Y}^*$ for $\mathbf{Y}$ finite. Notice that $\mathbf{Y}$ being finite implies that it is regular over $\mathbf{X}$ by Remark 4.2.8, so then $\mathbf{Y}^*$ is regular over $\mathbf{X}$ by definition of being regular over $\mathbf{X}$. Hence, we have shown that $\preceq$ is regular. This contradicts Theorem 4.2.25. □

The complexity of the positive cone depends strongly on the ambient group G and not only on the structure of the set of cosets G/C when considering the left-preorder. We will see it in the following example.

Example 4.2.27. *Consider the free product $\mathbb{Z} * \mathbb{Z} = \langle x \rangle * \langle y \rangle$. Now consider the homomorphism defined by*

$$\begin{array}{rccc} \varphi : \langle x \rangle * \langle y \rangle & \longrightarrow & \langle x \rangle \\ x & \mapsto & x \\ y & \mapsto & 1 \end{array}$$

Clearly, φ is surjective, so $(\mathbb{Z}*\mathbb{Z})/\ker(\varphi) \approx \langle x\rangle = \mathbb{Z}$. The group $\mathbb{Z}$ is left-orderable, so $\ker(\varphi)$ is left-relatively convex on $\mathbb{Z}*\mathbb{Z}$. By Theorem 4.2.25, any left-preorder on $\mathbb{Z}*\mathbb{Z}$ relative to $\ker(\varphi)$ is not regular, but $(\mathbb{Z}*\mathbb{Z})/\ker(\varphi) \approx \mathbb{Z}$ where the two possible orders on $\mathbb{Z}$ are always regular. This is because regularity is defined in terms of the positive cone of the ambient group.

This example motivates the following new definition of regularity avoiding the dependence on the ambient group. As before, by Proposition 4.2.17, this definition does not depend on the chosen generators of the group.

Definition 4.2.28. Let G be a finitely generated group and $\preceq$ a left-preorder on G relative to C with positive cone P. We say that $\preceq$ is relatively regular if there exists $\mathbf{L}$ regular language over $\mathbf{X}$ and a set P' such that $P = P'C$ and such that $\mathbf{L}$ represents P' via ϕ for $\phi : \mathbf{X}^* \longrightarrow G$ a surjective homomorphism and $\mathbf{X}$ a finite set.

When we say that a left-order is regular we mean that it is regular seen as a left-preorder with respect to the trivial subgroup. Notice that the definition of being relatively regular and regular are equivalent for left-orders. Both definitions, when considered for left-orders, they coincide with the one given in [23].

The notion of relatively regular is more general than the notion of regular, as we prove in next proposition.

Proposition 4.2.29. *Let G be a finitely generated group and $\preceq$ a left-preorder on G relative to C with positive cone P. If $\preceq$ is regular then it is relatively regular.*

Proof. By hypothesis, there is a regular language $\mathbf{L}$ over $\mathbf{X}$ that represents P via $\phi : \mathbf{X}^* \longrightarrow G$, so taking $P' = P$ we deduce that $\mathbf{L}$ represents P' via ϕ. The proposition follows since $PC = P$ (Corollary 1.2.5). □

We have a reciprocal for the case when C is finitely generated.

Proposition 4.2.30. *Let G be a finitely generated group and $\preceq$ a left-preorder on G relative to C with positive cone P. If C is finitely generated, then $\preceq$ is relatively regular if and only if it is regular.*

Proof. Assume that $\preceq$ is relatively regular, that is, there exists $\mathbf{L}$ regular language over $\mathbf{X}$ and a set P' such that $P = P'C$ and such that $\mathbf{L}$ represents P' via a surjective morphism $\phi : \mathbf{X}^* \longrightarrow G$ with $\mathbf{X}$ a finite set. In particular, $\phi(\mathbf{L}) = P'$. As C is finitely generated, we have $C = \langle h_1, \dots, h_n\rangle$. Then, we take

$$\tilde{\mathbf{Y}} = \{\mathbf{y}_1, \dots, \mathbf{y}_n, \mathbf{y}_1^{-1}, \dots, \mathbf{y}_n^{-1}\} \quad \text{and} \quad \mathbf{Y} = \mathbf{X} \sqcup \tilde{\mathbf{Y}},$$

and we define $\psi : \mathbf{Y}^* \longrightarrow G$ given by

$$\psi(\mathbf{x}) = \phi(\mathbf{x}) \ \text{ for } \mathbf{x} \in \mathbf{X} \quad \text{and} \quad \psi(\mathbf{y}_k^{\pm 1}) = h_k^{\pm 1} \ \text{ for } k = 1 \dots, n.$$

Clearly, we have $\psi_{|\mathbf{X}^*} = \phi$. Also, it is clear by the definition that $\psi(\tilde{\mathbf{Y}}^*) = C$. Now, take $\tilde{\mathbf{L}} = \mathbf{L}\tilde{\mathbf{Y}}^* \subseteq \mathbf{Y}^*$. Combining all these facts and the definitions, we have

$$\psi(\tilde{\mathbf{L}}) = \psi(\mathbf{L}\tilde{\mathbf{Y}}^*) = \psi(\mathbf{L})\psi(\tilde{\mathbf{Y}}^*) = \phi(\mathbf{L})\psi(\tilde{\mathbf{Y}}^*) = P'C = P.$$

Hence, we have shown that $\tilde{\mathbf{L}}$ represents P via ψ. Notice that $\mathbf{L}$ is regular over $\mathbf{Y}$ applying Remark 4.2.9 since it is regular over $\mathbf{X}$. Notice also that $\tilde{\mathbf{Y}}$ is regular over $\mathbf{Y}$ by Remark 4.2.8 as it is finite. Then, from the definition of being regular, we deduce that $\tilde{\mathbf{L}}$ is regular over $\mathbf{Y}$. Therefore, we have proven that $\tilde{\mathbf{L}}$ is a regular language over $\mathbf{Y}$ that represents P via ψ. Finally, so we conclude that $\preceq$ is regular.

The converse follows directly by Proposition 4.2.29. □

We can see that this new definition behaves as expected with quotient groups, as in the example before.

Proposition 4.2.31. *Let G be a finitely generated group and $\preceq$ a left-preorder on G relative to C with positive cone P. Assume that C is normal in G. Then, $\preceq$ is regular seen as a left-order of the group G/C if and only if $\preceq$ is relatively regular seen as a left-preorder on G relative to C.*

Proof. By definition, the positive cone Q of $\preceq$ seen as a left-order of the group G/C is given by

$$Q = \{gC \in G/C \ / \ C \prec gC\}$$

and the positive cone P of $\preceq$ seen as a left-preorder on G relative to C is given by

$$P = \{g \in G \ / \ C \prec gC\}.$$

Consider $\psi_1 : \mathbf{X}^* \longrightarrow G$ a surjective homomorphism. Take $\pi : G \longrightarrow G/C$ the natural quotient homorphism. Consider $\psi_2 = \pi \circ \psi_1$, so that $\psi_2 : \mathbf{X}^* \longrightarrow G/C$ is a surjective homomorphism. We now prove each implication of the statement.

Assume that $\preceq$ is relatively regular seen as a left-preorder on G relative to C. This means that there is a regular language $\mathbf{L}_1 \subseteq \mathbf{X}^*$ over $\mathbf{X}$ such that $\psi_1(\mathbf{L}_1) = P$. Then, we deduce that $\psi_2(\mathbf{L}_1) = Q$. Therefore, $\preceq$ is regular seen as a left-order of the group G/C.

Reciprocally, suppose that $\preceq$ is regular seen as a left-order of the group G/C. By definition, there is a regular language $\mathbf{L}_2 \subseteq \mathbf{X}^*$ over $\mathbf{X}$ such that $\psi_2(\mathbf{L}_2) = Q$. Denote $P' = \psi_1(\mathbf{L}_2)$. In order to conclude that $\preceq$ is relatively regular seen as a left-preorder on G relative to C, we prove $P'C = P$. By definition $P' \subseteq P$. Then, we deduce that $P'C \subseteq P$ (recall that $PC \subseteq P$ as it si done in Corollary 1.2.5). For the other content, take $g \in P$. Then, there exists $\mathbf{u} \in \mathbf{L}_2$ such that $\psi_2(\mathbf{u}) = \pi(g)$. Then, $g = \psi_1(\mathbf{u})h$ for some $h \in C$. Hence $P \subseteq P'C$, as we wanted. □

Then, we can give a version of Proposition 4.2.20 with the new definition.

Theorem 4.2.32. *Let G, H be non-trivial finitely generated groups and $\preceq$ a left-preorder on $G * H$ relative to C. If C has finite Kurosh rank with respect to the free product $G * H$, then $\preceq$ is not relatively regular.*

Proof. If $\preceq$ is not relatively regular this means that there exists a regular language representing a set P' such that $P = P'C$ for P the positive cone of $\preceq$. The results follows by applying the proof given in Proposition 4.2.20 but for P' instead of P. □

Chapter 5

Left-preorders and dynamics

The chapter is devoted to use dynamical techniques to prove some facts about spaces of left-preorders. For a wide reference about the study of spaces of left-orders from a dynamical point of view, see [17].

5.1 Left-preorders from a dynamical point of view

We now describe left-preorders from a dynamical point of view. We will follow the ideas from Section 1 of [49] adapting the result and the discussion from the left-orders context. We start with a proposition that can be found in [4] as Proposition 2.2, which is inspired on Theorem 6.8 from [21]. The notation below and the notion of dynamical realization-like is based on [49] but adapted to the left-preorders case.

We will denote by $\mathrm{Homeo}_+(\mathbb{R})$ the set of orientation preserving homeomorphisms of the real line. Every orientation preserving homeomorphism of the real line preserves the usual order on $\mathbb{R}$. In particular, $\mathrm{Homeo}_+(\mathbb{R})$ is a subgroup of the group of order preserving bijections of the real line endowed with the usual order.

Notation 18. Let G be a group. Each homomorphism $\rho : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ induces an action of G on $\mathbb{R}$ by orientation preserving homeomorphisms and vice versa. Given an homomorphism $\rho : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ and $p \in \mathbb{R}$, we will denote by $\mathrm{Stab}_\rho(p)$ the stabilizer of p with respect to the action induced by the homomorphism ρ. In other words,

$$\mathrm{Stab}_\rho(p) = \{g \in G \ / \ \rho(g)(p) = p\}.$$

Proposition 5.1.1. *Let G be a group and C a proper subgroup. If there is a homomorphism $\rho : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ and $p \in \mathbb{R}$ such that $\mathrm{Stab}_\rho(p) = C$, then defining*

$$g_1C \preceq_\rho g_2C \iff \rho(g_1)(p) \leq \rho(g_2)(p)$$

give us a left-preorder $\preceq_\rho$ on G relative to C. Reciprocally, if G/C is countable and $\preceq$ is a left-preorder on G relative to C, then there is a homomorphism $\rho_\preceq : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ and an element $p \in \mathbb{R}$ such that $\mathrm{Stab}_{\rho_\preceq}(p) = C$ and

$$g_1C \preceq g_2C \iff \rho_\preceq(g_1)(p) \leq \rho_\preceq(g_2)(p).$$

We say that $\rho_\preceq$ is a dynamical realization-like homomorphism for $\prec$ with reference point p. Moreover, we can build $\rho_\preceq$ in such a way $p = 0$.

Proof. We start by proving the first statement. Assume that $\rho : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ and $p \in \mathbb{R}$ such that $\mathrm{Stab}_\rho(p) = C$. We are going to check that $\preceq_\rho$ is a left-preorder on G relative to C. The relation $\preceq_\rho$ is well defined because if we take $f_1, f_2 \in C = \mathrm{Stab}_\rho(p)$ and we take $g_1, g_2 \in G$ such that $\rho(g_1)(p) \leq \rho(g_2)(p)$, then we have

$$\rho(g_1f_1)(p) = (\rho(g_1) \circ \rho(f_1))(p) = \rho(g_1)(p) \leq \rho(g_2)(p) = (\rho(g_2) \circ \rho(f_2))(p) = \rho(g_2f_2)(p),$$

so then $\rho(g_1f_1)(p) \leq \rho(g_2f_2)(p)$.

We prove that the relation $\preceq_\rho$ is antisymmetric. Indeed, take $g_1C \preceq_\rho g_2C$ and $g_2C \preceq_\rho g_1C$. Then, $\rho(g_1)(p) \leq \rho(g_2)(p)$ and $\rho(g_2)(p) \leq \rho(g_1)(p)$. We deduce that $\rho(g_1)(p) = \rho(g_2)(p)$. Then, using that $\mathrm{Stab}_\rho(p) = C$, we obtain $g_1C = g_2C$. This proves that $\preceq_\rho$ is antisymmetric.

We now show that the relation $\preceq_\rho$ is transitive. Indeed, take $g_1C \preceq_\rho g_2C$ and $g_2C \preceq_\rho g_3C$. Then, we have $\rho(g_1)(p) \leq \rho(g_2)(p)$ and $\rho(g_2)(p) \leq \rho(g_3)(p)$. We deduce $\rho(g_1)(p) \leq \rho(g_3)(p)$, and then $g_1C \preceq_\rho g_3C$. This shows that $\preceq_\rho$ is transitive.

Notice that the relation $\preceq_\rho$ is reflexive by definition.

The fact that $\preceq_\rho$ is total follows directly by the definition and by the fact that the usual order on $\mathbb{R}$ is total.

We prove that $\preceq_\rho$ is left invariant. Indeed, take $g \in G$ and assume that $g_1C \preceq_\rho g_2C$. Then we have $\rho(g_1)(p) \leq \rho(g_2)(p)$. From this and the fact that $\rho(g)$ is order preserving, we have

$$\rho(gg_1)(p) = (\rho(g) \circ \rho(g_1))(p) \leq (\rho(g) \circ \rho(g_2))(p) = \rho(gg_2)(p).$$

Then, $gg_1C \preceq_\rho gg_2C$, and this proves that $\preceq_\rho$ is left invariant.

Hence, we have shown that $\preceq_\rho$ is a well defined relation on G/C that is antisymmetric, transitive, reflexive, total and left-invariant, so it is a left-preorder. Therefore, we conclude the proof of the first statement.

Reciprocally, we are going to show the second statement. Suppose that $\preceq$ is a left-preorder on G relative to C. We are going to build a homomorphism $\rho_\preceq : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ and an element $p \in \mathbb{R}$ such that $\mathrm{Stab}_{\rho_\preceq}(p) = C$ and

$$g_1C \preceq g_2C \iff \rho_\preceq(g_1)(p) \leq \rho_\preceq(g_2)(p).$$

We start by building the needed action that induces the needed homomorphism $\rho_\preceq$. Since G/C is countable we can fix an injective enumeration $(g_iC)_{i\in\mathbb{N}}$ of G/C such that $g_0 = 1$. We now define a function $t : G/C \longrightarrow \mathbb{R}$ inductively. We define $t(g_0C) = t(C) = 0$. Now, assume that $t(g_0C), \dots, t(g_iC)$ are defined and $t(g_{i+1}C)$ is not. We have three cases. Firstly, if $g_lC \prec g_{i+1}C$ for all $l = 0, \dots, i$, then define

$$t(g_{i+1}C) = \max\{t(g_0C), \dots, t(g_iC)\} + 1.$$

Secondly, if $g_{i+1}C \prec g_lC$ for all $l = 0, \dots, i$, then define

$$t(g_{i+1}C) = \min\{t(g_0C), \dots, t(g_iC)\} - 1.$$

Thirdly, if we are not in the other two cases, we know that there are $l, m \in \{0, \dots, i\}$ such that $g_lC \prec g_{i+1}C \prec g_mC$ and it does not exist $j \in \{0, \dots, i\} - \{l, m\}$ such that $g_lC \prec g_jC \prec g_mC$. In this case we set

$$t(g_{i+1}C) = \frac{t(g_lC) + t(g_mC)}{2},$$

so we have

$$t(g_lC) < t(g_{i+1}C) < t(g_mC).$$

This give us an inductive definition of the function $t : G/C \longrightarrow \mathbb{R}$.

By construction, it is clear that t is order preserving. We can now use this function to build the action we wanted. Note that G acts on $t(G)$ by setting

$$g(t(\tilde{g}C)) = t(g\tilde{g}C) \tag{5.1}$$

for all $g, \tilde{g} \in G$. This action can be extended continuously to an action of G by homeomorphisms of the line, as in [21]. Denote by $\rho_\preceq : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ the homomorphism corresponding to this action.

At this point, we have defined the homomorphism $\rho_{\preceq}$. Take $p = 0$, so $p = t(g_0C) = t(C)$. We are now going to show that the homomorphism $\rho_{\preceq}$ satisfies

$$\mathrm{Stab}_{\rho_{\preceq}}(p) = C \quad \text{and} \quad g_1C \preceq g_2C \Longleftrightarrow \rho_{\preceq}(g_1)(p) \leq \rho_{\preceq}(g_2)(p).$$

On the one hand, using the fact that $\rho_{\preceq}$ is induced by an extension of the action on $t(G)$ described in Equation (5.1) and since $p \in t(G)$, we have

$$\mathrm{Stab}_{\rho_{\preceq}}(p) = \{g \in G \,/\, g(p) = p\} = \{g \in G \,/\, g(t(C)) = t(C)\} = \{g \in G \,/\, t(gC) = t(C)\} = \{g \in G \,/\, gC = C\} = C.$$

On the other hand, for each $\tilde{g}_1, \tilde{g}_2 \in G$, using that t is order preserving and that $\rho_{\preceq}$ is induced by an extension of the action on $t(G)$ described in Equation (5.1), we have

$$\tilde{g}_1C \preceq \tilde{g}_2C \Leftrightarrow t(\tilde{g}_1C) \leq t(\tilde{g}_2C) \Leftrightarrow \tilde{g}_1(t(C)) \leq \tilde{g}_2(t(C)) \Leftrightarrow \tilde{g}_1(p) \leq \tilde{g}_2(p) \Leftrightarrow \rho_{\preceq}(\tilde{g}_1)(p) \leq \rho_{\preceq}(\tilde{g}_2)(p)$$

as we wanted. □

Remark 5.1.2. Let $D : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ be a dynamical realization-like homomorphism for $\preceq$ with reference point p and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be an order preserving homeomorphism. We can define the conjugation $D_\phi : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ given by $D_\phi(g) = \phi \circ D(g) \circ \phi^{-1}$ for each $g \in G$. Then, D_ϕ is a dynamical realization-like homomorphism for $\preceq$ with reference point $\phi(p)$.

5.2 Conradian left-preorders from a dynamical point of view

In the previous section, we used dynamical tools to characterize left-preorders. Specifically, each left-preorder corresponds to an action of the group on $\mathbb{R}$, and we use the fact that $\mathbb{R}$ is a totally ordered set with the usual order. Hence, the dynamical version uses a totally ordered space where the group acts to describe a left-preorder on the group. One can consider another totally ordered space instead of $\mathbb{R}$. For example, when G is a group and C is a left-relatively convex subgroup, then a left-preorder is a total order on G/C, and the action of G on G/C by left-multiplication is order-preserving. In this context, one may ask, is there a way to describe the conradian condition in a dynamical language. This was provided in Theorem 1.4 from [45] for left-orders. We now generalize it using the tools of the proof of that theorem, so we obtain a dynamical characterization of conradian left-preorders. Notice that the actions we consider also appears on [4].

Recall that $\preceq$ is conradian if for every pair of positive elements $g, h \in G$, one has that $g \prec hg^n$ for some $n \in \mathbb{N}$.

Definition 5.2.1. Let G be a group acting on a totally ordered set $\mathsf{X}, <$ by order-preserving bijections. A crossing for this action consists on $(g, h; \mathsf{u}, \mathsf{v}, \mathsf{w})$, where $g, h \in G$ and $\mathsf{u}, \mathsf{v}, \mathsf{w} \in \mathsf{X}$, and satisfying the following conditions:

1. $\mathsf{u} < \mathsf{w} < \mathsf{v}$.
2. $h^n(\mathsf{u}) < \mathsf{v}$ and $\mathsf{u} < g^n(\mathsf{v})$ for all $n \in \mathbb{N}$.
3. There exists $N, M \in \mathbb{N}$ such that $g^N(\mathsf{v}) < \mathsf{w} < h^M(\mathsf{u})$.

Theorem 5.2.2. *Let G be a group and C be a proper subgroup. Let $\preceq$ be a left-preorder relative to C. Then, $\preceq$ is conradian if and only if the action of G on $(G/C, \preceq)$ by left multiplication admits no crossing.*

Proof. We start by proving that $(G/C, \preceq)$ admits no crossing implies that $\preceq$ is conradian. We do it by contraposition. Assume that $\preceq$ is not conradian, so then

$$gh^nC \prec hC \quad \text{for all} \quad n \in \mathbb{N}. \tag{5.2}$$

Define $\mathsf{u} = C$, $\mathsf{v} = g^{-1}hC$ and $\mathsf{w} = h^2C$. We prove that $(g, h; \mathsf{u}, \mathsf{v}, \mathsf{w})$ is a crossing for the action by checking each condition from the definition of a crossing.

We start by checking the first condition. Using Equation (5.2), we have $gh^2C \prec hC$. We deduce $h^2C \prec g^{-1}hC$. Then, using the fact that $C \prec h^2C$ since $C \prec hC$, we obtain $C \prec h^2C \prec g^{-1}hC$. Hence, we have proven $\mathsf{u} < \mathsf{w} < \mathsf{v}$, so first condition of the definition of crossing holds.

We now check the second condition. Fix an arbitrary $n \in \mathbb{N}$. On the one hand, Equation (5.2) implies $h^nC \prec g^{-1}hC$, which is equivalent to $h^n(\mathsf{u}) < \mathsf{v}$. On the other hand, Equation (5.2) implies $gC \prec hC$, so then $C \prec g^{-1}hC$. Then, since $C \prec gC$ and since the product of $\preceq$-positive elements is $\preceq$-positive, we deduce that $C \prec g^ng^{-1}hC$, which is the same as saying $\mathsf{u} < g^n(\mathsf{v})$. Hence, the second condition of the definition holds.

We check the third condition. Firstly, since $C \prec hC$, we have $hC \prec h^2C$. Then, since $gg^{-1}hC = hC$, we have $gg^{-1}hC \prec h^2C$. This is the same as saying $g^N(\mathsf{v}) < \mathsf{w}$ for $N = 1$. Secondly, we know $C \prec hC$, so then $h^2C \prec h^3C$. This is equivalent to $\mathsf{w} < h^M(\mathsf{u})$ for $M = 3$. Therefore, the third condition of the definition holds. We conclude that $(g, h; \mathsf{u}, \mathsf{v}, \mathsf{w})$ is a crossing for the corresponding action, as we wanted.

Conversely, we want to prove that $\preceq$ is conradian implies that $(G/C, \preceq)$ admits no crossing, and we are going to do it by contraposition. Assume that $(g, h; \mathsf{u}, \mathsf{v}, \mathsf{w})$ is a crossing for $(G/C, \preceq)$, so we are going to prove that $\preceq$ is not conradian. Take $\mathsf{u} = u_0C$, $\mathsf{v} = v_0C$ and $\mathsf{w} = w_0C$. By definition of crossing, we have that

$$u_0C \prec w_0C \prec v_0C \tag{5.3}$$

$$h^nu_0C \prec v_0C \text{ and } u_0C \prec g^nv_0C \text{ for all } n \in \mathbb{N} \tag{5.4}$$

$$\text{There exists } N, M \in \mathbb{N} \text{ such that } g^Nv_0C \prec w_0C \prec h^Mu_0C \tag{5.5}$$

holds. We are going to show that $\preceq$ is not conradian by showing that the elements $w_0^{-1}h^Mg^Nw_0$ and $w_0^{-1}h^Mw_0$ satisfies that $C \prec w_0^{-1}h^Mg^Nw_0C$, that $C \prec w_0^{-1}h^Mw_0C$ and that $(w_0^{-1}h^Mg^Nw_0)(w_0^{-1}h^Mw_0)^nC \prec w_0^{-1}h^Mw_0C$ for all $n \in \mathbb{N}$.

Firstly, we are going to prove that $C \prec w_0^{-1}h^Mg^Nw_0C$. On the one hand, from Equation (5.5) we obtain $g^Nv_0C \prec w_0C$, so then $h^Mg^{2N}v_0C \prec h^Mg^Nw_0C$. On the other hand, from Equation (5.4) we have $u_0C \prec g^{2N}v_0C$, so then $h^Mu_0C \prec h^Mg^{2N}v_0C$. Combining these two conclusions, we have

$$h^Mu_0C \prec h^Mg^{2N}v_0C \prec h^Mg^Nw_0C,$$

but $w_0C \prec h^Mu_0C$ by Equation (5.5), so then $w_0C \prec h^Mg^Nw_0C$, and hence $C \prec w_0^{-1}h^Mg^Nw_0C$, as we wanted.

Secondly, we are going to show that $C \prec w_0^{-1}h^Mw_0C$. We notice that $w_0C \prec hw_0C$ because, if not, we would have $hw_0C \preceq w_0C$, and it would imply

$$h^Mw_0C \preceq h^{M-1}w_0C \preceq \ldots \preceq hw_0C \preceq w_0C,$$

but using $h^Mu_0C \preceq h^Mw_0C$ which is direct from Equation (5.3), we deduce $h^Mu_0C \preceq w_0C$, but by Equation (5.5) we have $w_0C \prec h^Mu_0C$, hence $w_0C \prec w_0C$, which is a contradiction. Hence, we have $w_0C \prec hw_0C$, so then

$$w_0C \prec hw_0C \prec \ldots \prec h^{M-1}w_0C \prec h^Mw_0C,$$

and by left multiplying by w_0^{-1}, we obtain $C \prec w_0^{-1}h^Mw_0C$, as we needed.

Thirdly, it only remains to prove $(w_0^{-1}h^Mg^Nw_0)(w_0^{-1}h^Mw_0)^nC \prec w_0^{-1}h^Mw_0C$ for all $n \in \mathbb{N}$. Consider an arbitrary $n \in \mathbb{N}$. From Equation (5.5) we have $w_0C \prec h^Mu_0C$, so then $h^{Mn}w_0C \prec h^{Mn+M}u_0C$, and using $h^{Mn+M}u_0C \prec v_0C$ which is direct from Equation (5.4), we deduce $h^{Mn}w_0C \prec v_0C$, and hence $(h^M)^nw_0C \prec v_0C$. Then, we obtain

$$(h^Mg^N)(h^M)^nw_0C \prec h^Mg^Nv_0C,$$

but we have $h^Mg^Nv_0C \prec h^Mw_0C$ from Equation (5.5) and left multiplying by h^M, so then

$$(h^Mg^N)(h^M)^nw_0C \prec h^Mw_0C,$$

and hence

$$w_0^{-1}(h^M g^N)(h^M)^n w_0 C \prec w_0^{-1} h^M w_0 C.$$

Notice that

$$w_0^{-1}(h^M g^N)(h^M)^n w_0 = (w_0^{-1} h^M g^N w_0)(w_0^{-1} h^M w_0)^n C \prec w_0^{-1} h^M w_0 C,$$

so we conclude

$$(w_0^{-1} h^M g^N w_0)(w_0^{-1} h^M w_0)^n C \prec w_0^{-1} h^M w_0 C$$

as we wanted. □

5.3 The space of left-preorders on free products

In this section, we use dynamical properties to prove Theorem 5.3.1. We also use that result to prove Theorem 5.3.8. These main results provide us, as direct consequences, some relevant properties about the spaces $\mathcal{PO}(G * H)$ and $\mathcal{PO}_C(G * H)$.

We start by stating a theorem.

Theorem 5.3.1. *Let G and H be two non-trivial groups. Let C be a left-relatively convex subgroup on $G * H$. If C has finite Kurosh rank with respect to $G * H$, then $\mathcal{PO}_C(G * H)$ has no isolated elements.*

The theorem will follow from the following result, which allows us to reduce to the case where G and H are finitely generated.

Theorem 5.3.2. *Let G and H be two non-trivial groups. Let $\preceq$ be a left-preorder on $G * H$ relative to a subgroup C of finite Kurosh rank with respect to $G * H$. Let $\mathscr{T}$ be a finite subset of $G * H$ such that $C \prec tC$ for all $t \in \mathscr{T}$. Then, there exists finite sets $\mathcal{X}_G \subseteq G$ and $\mathcal{X}_H \subseteq H$ and an element $\gamma_* \in \langle \mathcal{X}_G, \mathcal{X}_H \rangle$ (they depend on the set $\mathscr{T}$) such that $C \prec \gamma_* C$ and satisfying the following property:*

For all finitely generated subgroups $G_1 \subseteq G$ and $H_1 \subseteq H$ containing $\mathcal{X}_G$ and $\mathcal{X}_H$ respectively, the group $\Gamma = G_1 * H_1$ admits a left-preorder $\preceq^*$ relative to $C_\Gamma = \Gamma \cap C$ such that $\gamma_* C_\Gamma \prec^* C_\Gamma$ and $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$.

We now prove Theorem 5.3.1 assuming Theorem 5.3.2.

Proof of Theorem 5.3.1. Consider $\preceq$ a left-preorder on $G * H$ relative to C with C of finite Kurosh rank with respect to $G * H$. We need to show that $\preceq$ is not isolated on $\mathcal{PO}_C(G * H)$. In order to do that, we fix $\mathscr{T}$ a finite set of $G * H$, and we want to build another left-preorder on $G * H$ relative to C that agrees on $\mathscr{T}$ with $\preceq$ but that is not equal to $\preceq$. Since all left-preorder on $G * H$ relative to C agrees on C with $\preceq$, we can assume that $\mathscr{T}$ has no elements in C. Then, by possibly changing some elements of $\mathscr{T}$ by its inverses, we can assume that all the elements t in $\mathscr{T}$ satisfies that $C \prec tC$ or, in other words, we can assume that all the elements in $\mathscr{T}$ are $\preceq$-positive. Consider $\mathcal{X}_G \subseteq G$, $\mathcal{X}_H \subseteq H$ and $\gamma_* \in \langle \mathcal{X}_G, \mathcal{X}_H \rangle$ the sets and the element provided by Theorem 5.3.2 for the finite set $\mathscr{T}$. In order to build the needed new left-preorder we are going to use Proposition 1.6.22 with the set $\mathscr{T} \cup \{\gamma_*^{-1}\}$, which is a finite set contained in $(G * H) - C$.

To be able to use Proposition 1.6.22, we are going to prove that $\mathscr{T} \cup \{\gamma_*^{-1}\}$ satisfies property (E_C). Take $\mathscr{R} = \{r_1, \dots, r_k\}$ a finite subset of $(G * H) - C$. We need to prove that there exists $\eta_1, \dots, \eta_k \in \{\pm 1\}$ such that

$$\langle \mathscr{T} \cup \{\gamma_*^{-1}\} \cup \{r_1^{\eta_1}, \dots, r_k^{\eta_k}\} \rangle^+ \cap C = \emptyset. \tag{5.6}$$

In order to do that, consider $G_1 \subseteq G$ and $H_1 \subseteq H$ two finitely generated subgroups such that

$$\mathscr{T} \cup \{\gamma_*^{-1}\} \cup \mathcal{X}_G \cup \mathcal{X}_H \cup \mathscr{R} \subset \Gamma = G_1 * H_1.$$

By Theorem 5.3.2, we know that there is a left-preorder $\preceq^*$ on Γ relative to $C_\Gamma = \Gamma \cap C$ such that $\gamma_* C_\Gamma \prec^* C_\Gamma$ and $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$. Since $\mathscr{R} \cap C$ is empty, for each $i = 1, \dots, k$ we can take an $\eta_i \in \{\pm 1\}$ such that $C_\Gamma \prec^* r_i^{\eta_i} C_\Gamma$. Hence, we know that $\gamma_* C_\Gamma \prec^* C_\Gamma$, that $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$ and that $C_\Gamma \prec^* r_i^{\eta_i} C_\Gamma$

for all $i = 1, \dots, k$. Then, we deduce that the subsemigroup $\langle \mathscr{T} \cup \{\gamma_*^{-1}\} \cup \{r_1^{\eta_1}, \dots, r_k^{\eta_k}\}\rangle^+$ is contained in the positive cone of $\preceq^*$. By Proposition 1.2.4, we deduce that

$$\langle \mathscr{T} \cup \{\gamma_*^{-1}\} \cup \{r_1^{\eta_1}, \dots, r_k^{\eta_k}\}\rangle^+ \cap (\Gamma \cap C) = \emptyset.$$

Since $\langle \mathscr{T} \cup \{\gamma_*^{-1}\} \cup \{r_1^{\eta_1}, \dots, r_k^{\eta_k}\}\rangle^+$ is contained in $\Gamma = G_1 * H_1$ by definition of G_1 and H_1, we obtain Equation (5.6), as we wanted. Therefore, we have proven that $\mathscr{T} \cup \{\gamma_*^{-1}\}$ satisfies property (E_C).

As we have shown that $\mathscr{T} \cup \{\gamma_*^{-1}\}$ satisfies property (E_C), by Proposition 1.6.22, there is a left-preorder $\hat{\preceq}$ on $G * H$ relative to C such that $C \prec \gamma_*^{-1} C$ and $C \prec tC$ for all $t \in \mathscr{T}$. In particular, $\hat{\preceq}$ coincides with $\preceq$ on $\mathscr{T}$ but they are not equal since they not coincide on γ_*. Hence, we have built another left-preorder on $G * H$ relative to C that agrees on $\mathscr{T}$ with $\preceq$ but is not equal to $\preceq$. We conclude that $\preceq$ is not isolated on $\mathcal{PO}_C(G * H)$, as we wanted. □

The proof of Theorem 5.3.2 uses dynamical properties following the ideas on [49]. We introduce some notation and some technical conclusions in order to prove that theorem. Notice that we will make use of some of the definitions and facts of Section 4.1.

Lemma 5.3.3. *Let G be a countable group and $\mathcal{X}_G$ a finite subset such that $\mathcal{X}_G^{-1} = \mathcal{X}_G$. Consider C a left-relatively convex subgroup on G and $\preceq$ a left-preorder on G relative to C. Let $D : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ be a dynamical realization-like homomorphism for $\preceq$ with reference point p and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider λ_n^+ a $\preceq$-maximal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$ and consider λ_n^- a $\preceq$-minimal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$. Then,*

1. *For each $g \in \mathbb{B}_{\mathcal{X}_G}(n)$, we have that $D(g)(p)$ lies in the interval $[D(\lambda_n^-)(p), D(\lambda_n^+)(p)]$.*
2. *Suppose that $\tilde{D} : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ is a homomorphism. If for all $g \in \mathcal{X}_G$ we have that $D(g)$ and $\tilde{D}(g)$ are equal as maps when restricted to $[D(\lambda_n^-)(p), D(\lambda_n^+)(p)]$, then $D(g)(p) = \tilde{D}(g)(p)$ for all $g \in \mathbb{B}_{\mathcal{X}_G}(n+1)$.*

Proof. We prove each statement. We start by proving the first one. Take $g \in \mathbb{B}_{\mathcal{X}_G}(n)$. By definition, we have

$$\lambda_n^- C \preceq gC \preceq \lambda_n^+ C.$$

Then, applying D is a dynamical realization-like homomorphism for $\preceq$, we deduce

$$D(\lambda_n^-)(p) \leq D(g)(p) \leq D(\lambda_n^+)(p),$$

as we wanted. This finishes the proof of the first statement.

We now show the second statement. Consider $g \in \mathbb{B}_{\mathcal{X}_G}(n+1)$. This means that we can write

$$g = \alpha_1 \cdots \alpha_j \text{ for some } \alpha_1, \dots, \alpha_j \in \mathcal{X}_G \text{ and where } j \leq n+1.$$

Notice that $\alpha_i \cdots \alpha_1 \in \mathbb{B}_{\mathcal{X}_G}(n)$ for all $i < j$, so we deduce that

$$\lambda_n^- C \preceq \alpha_i \cdots \alpha_1 C \preceq \lambda_n^+ C \text{ for all } i < j.$$

Then, applying D, we obtain that the elements

$$x_0 = p,\ x_1 = D(\alpha_1)(p),\ \dots,\ x_{j-1} = D(\alpha_{j-1} \cdots \alpha_1)(p)$$

they all belong to $[D(\lambda_n^-)(p), D(\lambda_n^+)(p)]$. Hence, applying the hypothesis, we obtain recursively that

$$\tilde{D}(\alpha_k \alpha_{k-1} \cdots \alpha_1)(p) = \tilde{D}(\alpha_k)\left(\tilde{D}(\alpha_{k-1} \cdots \alpha_1)(p)\right) = \tilde{D}(\alpha_k)(x_{k-1}) =$$
$$= D(\alpha_k)(x_{k-1}) = D(\alpha_k)\left(D(\alpha_{k-1} \cdots \alpha_1)(p)\right) = D(\alpha_k \alpha_{k-1} \cdots \alpha_1)(p) = x_k$$

for all $k = 1, \dots, j$. □

We can extend left-preorders induced from a dynamical realization as follows.

Lemma 5.3.4. *Let G be a group and C a left-relatively convex subgroup on G. Let $D : G \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ be a homomorphism and let $p \in \mathbb{R}$ be such that $C \subseteq \mathrm{Stab}_D(p)$ and $\mathrm{Stab}_D(p) \neq G$. Consider $\preceq_p$ the relation on G/C defined by $C \preceq_p gC \Leftrightarrow p \leq D(g)(p)$. Then $\preceq_p$ can be extended to a left-preorder on G relative to C, say $\preceq$, in the sense that $g_1C \prec_p g_2C$ implies that $g_1C \prec g_2C$ for all $g_1, g_2 \in G$.*

Proof. When $C = \mathrm{Stab}_D(p)$, the result follows directly by taking $\prec_p = \prec$, so assume that $C \subsetneq \mathrm{Stab}_D(p)$. Since C a left-relatively convex subgroup on G, there is some left-preorder on G relative to C. Then, when applying Proposition 1.3.9 with respect to the subgroup $\mathrm{Stab}_D(p)$ where $C \subseteq \mathrm{Stab}_D(p)$, we deduce that there is $\preceq^*$ a left-preorder on $\mathrm{Stab}_D(p)$ relative to C. In particular, we have shown that C is relatively convex on $\mathrm{Stab}_D(p)$. Notice also that $\preceq_p$ is a left-preorder on G relative to $\mathrm{Stab}_D(p)$ by applying Proposition 5.1.1. Hence, the result follows by applying Proposition 1.3.12. □

We are ready to prove Theorem 5.3.2.

Proof of Theorem 5.3.2. Let $\preceq$ be a left-preorder on $G * H$ relative to a subgroup C of finite Kurosh rank with respect to $G * H$. Let $\mathscr{T}$ be a finite subset of $G * H$ such that $C \prec tC$ for all $t \in \mathscr{T}$.

Since C has finite Kurosh rank with respect to $G*H$, applying Theorem 3.3.5, we deduce that there exists $\Omega, \{\zeta_i\}_{i=1}^r, \{\alpha_j\}_{j=1}^s, \{\beta_t\}_{t=1}^l \subseteq G * H$ finite sets satisfying the following property: for every G_0 subgroup of G and every H_0 subgroup of H satisfying $\Omega \subseteq \langle G_0, H_0\rangle$ there exists A_j subgroup of G_0 for each $j = 1, \dots, s$ and there exists B_t subgroup of H_0 for each $t = 1, \dots, l$ such that

$$C \cap \langle G_0, H_0\rangle = \langle \{\zeta_i \,/\, i = 1, \dots, r\} \cup \left(\bigcup_{j=1}^{s} \alpha_j A_j \alpha_j^{-1}\right) \cup \left(\bigcup_{l=1}^{t} \beta_l B_l \beta_l^{-1}\right)\rangle.$$

Consider $\mathcal{X}_G \subseteq G$ and $\mathcal{X}_H \subseteq H$ two finite subsets, both having some non-trivial element, satisfying that $\mathcal{X}_G^{-1} = \mathcal{X}_G$ and $\mathcal{X}_H^{-1} = \mathcal{X}_H$ and satisfying that

$$\mathscr{T} \cup \Omega \subseteq \langle \mathcal{X}_G\rangle * \langle \mathcal{X}_H\rangle.$$

There are two facts that we remark here as we are going to use them later in the proof. Fact I is consequence of the property that have $\Omega, \{\zeta_i\}_{i=1}^r, \{\alpha_j\}_{j=1}^s, \{\beta_t\}_{t=1}^l$, as we have discussed. Fact II is a consequence of Corollary 3.3.2. These two facts are the following:

Fact I: If $G_1 \subseteq G$ and $H_1 \subseteq H$ are subgroups such that $\mathcal{X}_G \subseteq G_1$ and $\mathcal{X}_H \subseteq H_1$, and if we consider $\Gamma = G_1 * H_1$ and $C_\Gamma = C \cap \Gamma$, then we have that

$$C_\Gamma = \langle \{\zeta_i \,/\, i = 1, \dots, r\} \cup \left(\bigcup_{j=1}^{s} \alpha_j A_j \alpha_j^{-1}\right) \cup \left(\bigcup_{l=1}^{t} \beta_l B_l \beta_l^{-1}\right)\rangle$$

where A_j is subgroup of G_0 for $j = 1, \dots, s$ and where B_t is a subgroup of H_0 for $t = 1, \dots, l$.

Fact II: Let $(\lambda_l)_{l\in\mathbb{N}}$ be a sequence of elements in $G * H$ such that $\lambda_l C \neq \lambda_m C$ for all $l \neq m$. Then, there exists $n \in \mathbb{N}$ such that for all $k \geq n$ there are $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ satisfying $g^{\varepsilon_G}\lambda_k C \neq \lambda_k C$, $h^{\varepsilon_H}\lambda_k C \neq \lambda_k C$ and $g^{\varepsilon_G}\lambda_k C \neq h^{\varepsilon_H}\lambda_k C$ for every $\varepsilon_G, \varepsilon_H \in \{\pm 1\}$.

We now continue the proof.

For each $n \in \mathbb{N}$, consider λ_n^+ a $\preceq$-maximal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$ and consider λ_n^- a $\preceq$-minimal element in the ball $\mathbb{B}_{\mathcal{X}_G}(n)$. By Lemma 4.1.9, we have $\lambda_n^+ C \neq \lambda_k^+ C$ for all $n, k \in \mathbb{N}$. This implies that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfies the hypothesis of Fact II. Using the finiteness of $\mathscr{T}$ and applying Fact II, we can consider a big enough $n \in \mathbb{N}$ and we can take $g \in \mathcal{X}_G$ and $h \in \mathcal{X}_H$ satisfying the following:

$$\mathscr{T} \subseteq \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n), \tag{5.7}$$

$$\{\zeta_1, \dots, \zeta_r, \alpha_1, \dots \alpha_s, \beta_1, \dots, \beta_t\} \subseteq \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n), \tag{5.8}$$

$$g\lambda_{n+1}C \neq \lambda_{n+1}C \;\;,\;\; h\lambda_{n+1}C \neq \lambda_{n+1}C \;\text{ and }\; g^{\varepsilon_G}\lambda_{n+1}C \neq h^{\varepsilon_H}\lambda_{n+1}C \;\text{ for every }\; \varepsilon_G, \varepsilon_H \in \{\pm 1\}.$$

From this last fact, and by changing g or h (or both) by its inverse if necessary, we can assume that $\lambda_{n+1}^+ C \prec g\lambda_{n+1}^+ C$, that $\lambda_{n+1}^+ C \prec h\lambda_{n+1}^+ C$ and that $g\lambda_{n+1} C \neq h\lambda_{n+1} C$. Notice that, by interchanging the groups G and H by symmetry if necessary, we can also suppose that $h\lambda_{n+1}^+ C \prec g\lambda_{n+1}^+ C$. Hence, we have

$$\lambda_{n+1}^+ C \prec h\lambda_{n+1}^+ \prec g\lambda_{n+1}^+ C \tag{5.9}$$

We now define

$$\gamma_* = (h\lambda_{n+1}^+)^{-1} g\lambda_{n+1}^+,$$

so we have

$$C \prec \gamma_* C.$$

Notice that $g, h, \lambda_{n+1}^+ \in \langle \mathcal{X}_G, \mathcal{X}_H \rangle$, so we also have $\gamma_* \in \langle \mathcal{X}_G, \mathcal{X}_H \rangle$.

At this point, we have built two finite sets $\mathcal{X}_G \subseteq G$ and $\mathcal{X}_H \subseteq H$ and an element $\gamma_* \in \langle \mathcal{X}_G, \mathcal{X}_H \rangle$ such that $C \prec \gamma_* C$. In order to conclude the proof, we are going to prove that the triple $(\mathcal{X}_G, \mathcal{X}_H, \gamma_*)$ satisfies the property claimed in the statement. In other words, we need to prove that for all finitely generated subgroups $G_1 \subseteq G$ and $H_1 \subseteq H$ containing $\mathcal{X}_G$ and $\mathcal{X}_H$ respectively, the group $\Gamma = G_1 * H_1$ admits a left-preorder $\preceq^*$ relative to $C_\Gamma = \Gamma \cap C$ such that $\gamma_* C_\Gamma \prec^* C_\Gamma$ and $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$.

Consider $G_1 \subseteq G$ and $H_1 \subseteq H$ two finitely generated subgroups containing $\mathcal{X}_G$ and $\mathcal{X}_H$ respectively. Consider $\Gamma = G_1 * H_1$, which is finitely generated. We are going to build the left-preorder $\preceq^*$ by using the point of view of dynamical realizations and then applying Lemma 5.3.4 to obtain the needed left-preorder.

The restriction of $\preceq$ to Γ is a left-preorder on Γ relative to $C_\Gamma = C \cap \Gamma$, which we will denote by $\preceq$ as an abuse of notation. Notice that this left-preorder is given by $\gamma_1 C \preceq \gamma_2 C$ if and only if $\gamma_1 C_\Gamma \preceq \gamma_2 C_\Gamma$ for all $\gamma_1, \gamma_2 \in \Gamma$. Since Γ is finitely generated, by Proposition 5.1.1, we can consider $D : \Gamma \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ a dynamical realization-like homomorphism for $\preceq$ as left-preorder on Γ relative to C_Γ with reference point 0. Then, we have

$$\gamma_1 C_\Gamma \preceq \gamma_2 C_\Gamma \Leftrightarrow D(\gamma_1)(0) \leq D(\gamma_2)(0) \text{ for all } \gamma_1, \gamma_2 \in \Gamma. \tag{5.10}$$

From Equation (5.9), we deduce that

$$D(\lambda_{n+1}^+)(0) < D(h\lambda_{n+1}^+)(0) < D(g\lambda_{n+1}^+)(0).$$

By density of $\mathbb{R}$, we can consider $\mathsf{x}_0, \mathsf{x}_1, \mathsf{y}_0, \mathsf{y}_1 \in \mathbb{R}$ such that

$$D(\lambda_{n+1}^+)(0) < \mathsf{x}_0 < \mathsf{x}_1 < D(h\lambda_{n+1}^+)(0) < D(g\lambda_{n+1}^+)(0) < \mathsf{y}_1 < \mathsf{y}_0. \tag{5.11}$$

The support of a map $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is defined to be the set $\{\mathsf{x} \in \mathbb{R} \ / \ \mu(\mathsf{x}) \neq \mathsf{x}\}$. We take $\phi \in \mathrm{Homeo}_+(\mathbb{R})$ such that its support is equal to $(\mathsf{x}_0, \mathsf{y}_0)$ and satisfying that $\phi(\mathsf{x}_1) > \mathsf{y}_1$. We claim that

$$\phi(0) = 0 \text{ and } \phi^{-1}(0) = 0. \tag{5.12}$$

Indeed, since $C_\Gamma \prec \lambda_{n+1}^+ C_\Gamma$, then $0 < D(\lambda_{n+1}^+)(0)$, and it implies that $0 \notin (\mathsf{x}_0, \mathsf{y}_0)$. In particular, 0 is not on the support of ϕ, so this proves Equation (5.12).

Recall that $\Gamma = G_1 * H_1$. By the universal property of the free product, we can build $D_\phi : \Gamma \longrightarrow \mathrm{Homeo}_+(\mathbb{R})$ a group homomorphism defined by $D_\phi(\overline{g}) = D(\overline{g})$ for all $\overline{g} \in G_1$ and $D_\phi(\overline{h}) = \phi \circ D(\overline{h}) \circ \phi^{-1}$ for all $\overline{h} \in H_1$.

To make the proof easier to follow, we are going to conclude the proof by assuming three claims, and we are going to prove them when we are done with the main proof. The three equations we are going to assume are

$$D_\phi(w)(0) = D(w)(0) \text{ for all } w \in \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n+1), \tag{5.13}$$

$$D_\phi(g\lambda_{n+1}^+)(0) < D_\phi(h\lambda_{n+1}^+)(0) \text{ and} \tag{5.14}$$

$$C_\Gamma \subseteq \mathrm{Stab}_{D_\phi}(0). \tag{5.15}$$

We are able to build the needed left-preorder using D_ϕ. Combining Equation (5.15) and Lemma 5.3.4, we obtain a left-preorder $\preceq^*$ on Γ relative to C_Γ such that

$$D_\phi(\gamma_1)(0) < D_\phi(\gamma_2)(0) \implies \gamma_1 C_\Gamma \prec^* \gamma_2 C_\Gamma \quad \text{for all} \gamma_1, \gamma_2 \in \Gamma. \tag{5.16}$$

To conclude the proof, we only need to prove that $\gamma_* C_\Gamma \prec^* C_\Gamma$ and $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$. We check each condition. From Equation (5.14) and Equation (5.16) we deduce that

$$g\lambda_{n+1}^+ C_\Gamma \prec^* h\lambda_{n+1}^+ C_\Gamma.$$

Then, we deduce

$$C_\Gamma \prec^* (g\lambda_{n+1}^+)^{-1} h\lambda_{n+1}^+ C_\Gamma = (\gamma_*)^{-1} C_\Gamma.$$

This proves the condition $\gamma_* C_\Gamma \prec^* C_\Gamma$. Now, given $t \in \mathscr{T}$, by hypothesis we know that $C \prec tC$. Since D is the dynamical realization-like homomorphism for the restriction of $\preceq$ with reference point 0, we deduce that

$$0 < D(t)(0).$$

Combining this fact with Equation (5.7) and Equation (5.13), we deduce

$$0 < D_\phi(t)(0).$$

Applying Equation (5.16), we obtain

$$C_\Gamma \prec^* tC_\Gamma.$$

This proves the condition $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$. Therefore, we have shown $\gamma_* C_\Gamma \prec^* C_\Gamma$ and $C_\Gamma \prec^* tC_\Gamma$ for all $t \in \mathscr{T}$. This finishes the main proof of the theorem.

Recall that, for the proof, we have assumed that Equation (5.13), Equation (5.14) and Equation (5.15) hold. Now, we are going to prove each statement.

We start by proving Equation (5.13). We claim that

$$D_\phi(\overline{h})(\mathsf{x}) = D(\overline{h})(\mathsf{x}) \ \text{ for all } \overline{h} \in \mathcal{X}_H \ \text{ and all } \mathsf{x} \leq D(\lambda_n^+)(0). \tag{5.17}$$

Indeed, consider $\overline{h} \in \mathcal{X}_H$ and $\mathsf{x} \leq D(\lambda_n^+)(0)$. Notice that have $\overline{h}\lambda_n^+ \in \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n+1)$. Then, by definition of λ_n^+, we have

$$\overline{h}\lambda_n^+ C_\Gamma \preceq \lambda_{n+1}^+ C_\Gamma.$$

Then, using Equation (5.10), we obtain

$$D(\overline{h}\lambda_n^+)(0) \leq D(\lambda_{n+1}^+)(0).$$

Combining this with $\mathsf{x} \leq D(\lambda_n^+)(0)$, with Equation (5.11) and with the fact that $D(\overline{h})$ is order preserving, we obtain

$$D(\overline{h})(\mathsf{x}) \leq D(\overline{h})\left(D(\lambda_n^+)(0)\right) = D(\overline{h}\lambda_n^+)(0) \leq D(\lambda_{n+1}^+)(0) < \mathsf{x}_0.$$

Hence, $D(\overline{h})(\mathsf{x}) < \mathsf{x}_0$. Notice also that $\mathsf{x} < \mathsf{x}_0$ as a consequence of $\mathsf{x} \leq D(\lambda_n^+)(0)$ and Equation (5.11). We have shown that $D(\overline{h})(\mathsf{x})$ and x are both smaller than x_0, so they are not at the support of ϕ. Using this and the definition of D_ϕ, we have $D_\phi(\overline{h})(\mathsf{x}) = D(\overline{h})(\mathsf{x})$. This proves Equation (5.17), as we claimed.

By definition of D_ϕ, we have

$$D_\phi(\overline{g})(\mathsf{x}) = D(\overline{g})(\mathsf{x}) \ \text{ for all } \overline{g} \in \mathcal{X}_G \ \text{ and all } \mathsf{x} \leq D(\lambda_n^+)(0).$$

Combining this with Equation (5.17), we deduce that, for all $\gamma \in \mathcal{X}_G \cup \mathcal{X}_H$, we have that $D(\gamma)$ and $D_\phi(\gamma)$ are equal as maps when restricted to $[D(\lambda_n^-)(0), D(\lambda_n^+)(0)]$. Then, from this fact and from Lemma 5.3.3, we deduce Equation (5.13), as we wanted.

We are now going to prove Equation (5.14). Recall that g and h are the fixed non-trivial elements of $\mathcal{X}_G$ and $\mathcal{X}_H$. From Equation (5.11) and Equation (5.12), we obtain

$$\left(D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0) = D(h\lambda_{n+1}^+)\left(\phi^{-1}(0)\right) = D(h\lambda_{n+1}^+)(0) > \mathsf{x}_1.$$

Applying that ϕ is order preserving and $\phi(\mathsf{x}_1) > \mathsf{y}_1$ by definition, we deduce

$$\left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0) > \phi(\mathsf{x}_1) > \mathsf{y}_1.$$

Combining this with Equation (5.11), we deduce

$$D(g\lambda_{n+1}^+)(0) < \mathsf{y}_1 < \left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0).$$

Hence, we have shown that

$$D(g\lambda_{n+1}^+)(0) < \left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0). \tag{5.18}$$

Now since $\lambda_{n+1}^+ \in \mathbb{B}_{\mathcal{X}_G\cup\mathcal{X}_H}(n+1)$, applying Equation (5.13), we obtain

$$D_\phi(\lambda_{n+1}^+)(0) = D(\lambda_{n+1}^+)(0). \tag{5.19}$$

By definition of D_ϕ knowing that $g \in G_1$, we obtain $D_\phi(g) = D(g)$. From this and from Equation (5.19) we deduce

$$D_\phi(g\lambda_{n+1}^+)(0) = D_\phi(g)\left(D_\phi(\lambda_{n+1}^+)(0)\right) = D(g)\left(D(\lambda_{n+1}^+)(0)\right) = D(g\lambda_{n+1}^+)(0).$$

Then, we have shown that

$$D_\phi(g\lambda_{n+1}^+)(0) = D(g\lambda_{n+1}^+)(0). \tag{5.20}$$

From Equation (5.11) we notice that $D(\lambda_{n+1}^+)(0)$ is not in the support of ϕ, so then

$$\phi^{-1}\left(D(\lambda_{n+1}^+)(0)\right) = D(\lambda_{n+1}^+)(0). \tag{5.21}$$

By definition of D_ϕ knowing that $h \in H_1$, we obtain

$$D_\phi(h) = (\phi\circ D\circ\phi^{-1})(h).$$

From this, Equation (5.12), Equation (5.21) and Equation (5.19) we deduce

$$D_\phi(h\lambda_{n+1}^+)(0) = D_\phi(h)\left(D_\phi(\lambda_{n+1}^+)(0)\right) = D_\phi(h)\left(D(\lambda_{n+1}^+)(0)\right) = \left(\phi\circ D(h)\circ\phi^{-1}\right)\left(D(\lambda_{n+1}^+)(0)\right) =$$
$$= \left(\phi\circ D(h)\right)\left(D(\lambda_{n+1}^+)(0)\right) = \left(\phi\circ D(h\lambda_{n+1}^+)\right)(0) = \left(\phi\circ D(h\lambda_{n+1}^+)\right)(\phi^{-1}(0)) = \left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0).$$

Then, we have proven that

$$D_\phi(h\lambda_{n+1}^+)(0) = \left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0). \tag{5.22}$$

We can combine Equation (5.20), Equation (5.18) and Equation (5.22), so we obtain

$$D_\phi(g\lambda_{n+1}^+)(0) = D(g\lambda_{n+1}^+)(0) < \left(\phi\circ D(h\lambda_{n+1}^+)\circ\phi^{-1}\right)(0) = D_\phi(h\lambda_{n+1}^+)(0).$$

This proves Equation (5.14), as we wanted.

We finally prove Equation (5.15). In order to prove it, consider the following description of C_Γ provide by Fact I:

$$C_\Gamma = \langle \{\zeta_i \ / \ i = 1,\dots,r\} \cup \left(\bigcup_{j=1}^{s} \alpha_j A_j \alpha_j^{-1}\right) \cup \left(\bigcup_{l=1}^{t} \beta_l B_l \beta_l^{-1}\right)\rangle$$

where A_j is subgroup of G_0 for $j = 1,\dots,s$ and where B_t is a subgroup of H_0 for $t = 1,\dots,l$. From this description of C_Γ, in order to prove Equation (5.15), we only need to prove the following three equations involving the generators of C_Γ:

$$\zeta_i \in \mathrm{Stab}_{D_\phi}(0) \ \text{ for all } i = 1,\dots,r\ , \tag{5.23}$$

$$\alpha_j a \alpha_j^{-1} \in \mathrm{Stab}_{D_\phi}(0) \ \text{ for all } j = 1,\dots,s \ \text{ and all } a \in A_j\ , \tag{5.24}$$

$$\beta_l b \beta_l^{-1} \in \mathrm{Stab}_{D_\phi}(0) \text{ for all } l = 1, \dots, t \text{ and all } b \in B_l. \tag{5.25}$$

We are going to prove each equation separately. Before doing that, recall that

$$C_\Gamma = \mathrm{Stab}_D(0)$$

since D is a dynamical realization-like homomorphism for a left-preorder relative to C_Γ with reference point 0.

We are going to prove Equation (5.23). Take $i \in \{1, \dots, r\}$. Since $C_\Gamma = \mathrm{Stab}_D(0)$ and $\zeta_i \in C_\Gamma$, we obtain $D(\zeta_i)(0) = 0$. The previous equation combined with Equation (5.8) and Equation (5.13) give us that $D_\phi(\zeta_i)(0) = D(\zeta_i)(0)$ is equal to 0. This shows Equation (5.23), as we wanted.

We are going to show Equation (5.24). Take $j \in \{1, \dots, s\}$ and $a \in A_j$. In order to prove Equation (5.24) we are going to show

$$D_\phi(\alpha_j^{-1})(0) = D_\phi(a\alpha_j^{-1})(0),$$

as they are equivalent statements because D_ϕ is an homomorphism. On the one hand, Equation (5.8) implies

$$\alpha_j^{-1} \in \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n).$$

Combining this with Equation (5.13), we deduce

$$D_\phi(\alpha_j^{-1})(0) = D(\alpha_j^{-1})(0). \tag{5.26}$$

On the other hand, as $\alpha_j a \alpha_j^{-1} \in C_\Gamma$ and $C_\Gamma = \mathrm{Stab}_D(0)$, using that D is an homomorphism we deduce

$$D(\alpha_j^{-1})(0) = D(a\alpha_j^{-1})(0). \tag{5.27}$$

Also, as $a \in G_1$, from the definition of D_ϕ we obtain

$$D_\phi(a) = D(a). \tag{5.28}$$

Combining Equation (5.26), Equation (5.27) and Equation (5.28), and using that D and D_ϕ are homomorphisms, we deduce that

$$D_\phi(a\alpha_j^{-1})(0) = D_\phi(a)\left(D_\phi(\alpha_j^{-1})(0)\right) = D(a)\left(D(\alpha_j^{-1})(0)\right) = D(a\alpha_j^{-1})(0) = D(\alpha_j^{-1})(0) = D_\phi(\alpha_j^{-1})(0)$$

Hence, we have proven $D_\phi(\alpha_j^{-1})(0) = D_\phi(a\alpha_j^{-1})(0)$, as we wanted. Therefore, we have shown Equation (5.24).

We are going to prove Equation (5.25). Consider $l \in \{1, \dots, t\}$ and $b \in B_l$. Analogously as before, in order to prove Equation (5.25) we are going to prove

$$D_\phi(\beta_l^{-1})(0) = D_\phi(b\beta_l^{-1})(0),$$

as both are equivalent statements as D_ϕ is an homomorphism. Previously, we are going to derive some useful equations that, when combined, will let us obtain the expression we needed. From Equation (5.8), we obtain

$$\beta_l^{-1} \in \mathbb{B}_{\mathcal{X}_G \cup \mathcal{X}_H}(n). \tag{5.29}$$

This implies, by Equation (5.13), that

$$D_\phi(\beta_l^{-1})(0) = D(\beta_l^{-1})(0). \tag{5.30}$$

Now, from Equation (5.29) and the definition of λ_{n+1}^+ we obtain

$$\beta_l^{-1} C_\Gamma \prec \lambda_{n+1}^+ C_\Gamma.$$

Combining this with Equation (5.10) and Equation (5.11), we obtain

$$D(\beta_l^{-1})(0) < D(\lambda_{n+1}^+)(0) < \mathsf{x}_0.$$

In particular, $D(\beta_l^{-1})(0)$ is not in the support of ϕ, so then

$$\phi^{-1}\left(D(\beta_l^{-1})(0)\right) = D(\beta_l^{-1})(0) \text{ and } \phi\left(D(\beta_l^{-1})(0)\right) = D(\beta_l^{-1})(0). \tag{5.31}$$

On the other hand, as we know that $\beta_l b \beta_l^{-1} \in C_\Gamma$ and $C_\Gamma = \mathrm{Stab}_D(0)$, using that D is an homomorphism we obtain

$$D(\beta_l^{-1})(0) = D(b\beta_l^{-1})(0). \tag{5.32}$$

Also, as $b \in H_1$, from the definition of D_ϕ we obtain

$$D_\phi(b) = \phi \circ D(b) \circ \phi^{-1}. \tag{5.33}$$

Combining Equation (5.30), Equation (5.31), Equation (5.32) and Equation (5.33), and using that D and D_ϕ are homomorphisms, we obtain the chain of equalities

$$\begin{aligned} D_\phi(b\beta_l^{-1})(0) = D_\phi(b)\left(D_\phi(\beta_l^{-1})(0)\right) = \left(\phi \circ D(b)\phi^{-1}\right)\left(D(\beta_l^{-1})(0)\right) = (\phi \circ D(b))\left(\phi^{-1}\left(D(\beta_l^{-1})(0)\right)\right) = \\ = (\phi \circ D(b))\left(D(\beta_l^{-1})(0)\right) = \phi\left(D(b\beta_l^{-1})(0)\right) = \phi\left(D(\beta_l^{-1})(0)\right) = D(\beta_l^{-1})(0) = D_\phi(\beta_l^{-1})(0) \end{aligned}$$

This proves that $D_\phi(\beta_l^{-1})(0) = D_\phi(b\beta_l^{-1})(0)$, as we wanted. Therefore, we have shown Equation (5.25). This concludes the proof of Equation (5.15), so we can finish the proof. □

We have some consequences of the theorem.

Corollary 5.3.5. *Let G and H be two non-trivial groups. Let C be a proper subgroup of $G * H$ having finite Kurosh rank. If C is left-relatively convex in $G * H$, then $\mathcal{PO}_C(G * H)$ is uncountable.*

Proof. Assume that C is left-relatively convex in $G * H$, so then $\mathcal{PO}_C(G * H)$ is non-empty. Follows by applying Theorem 27.7 of [42], knowing that $\mathcal{PO}_C(G * H)$ is compact, Hausdorff and has no isolated points by Corollary 1.6.7, Corollary 1.6.3 and Theorem 5.3.1. □

Corollary 5.3.6. *Let G and H be two finitely generated non-trivial groups. Let C be a proper subgroup of $G * H$ having finite Kurosh rank. If C is left-relatively convex in $G * H$, then $\mathcal{PO}_C(G * H)$ is homeomorphic to the Cantor set.*

Proof. Assume that C is left-relatively convex in $G * H$, so we deduce that $\mathcal{PO}_C(G * H)$ is non-empty. The groups G and H are finitely generated so $G * H$ is countable. By Theorem 5.3.1 $\mathcal{PO}_C(G * H)$ has no isolated points. Hence, the result follows by applying the characterization of Cantor sets (Proposition 1.6.17) to the topological properties we know about $\mathcal{PO}_C(G * H)$ (Corollary 1.6.7, Corollary 1.6.3 and Corollary 1.6.5). □

Remark 5.3.7. As one can see on its proof, last corollary is also true if we change the condition of G and H being finitely generated by the condition of G and H being countable. In particular, it is also true if we change that condition by G and H being countable.

We can state now the theorem about $\mathcal{PO}(G * H)$.

Theorem 5.3.8. *Let G and H be two non-trivial groups. Then, $\mathcal{PO}(G * H)$ has no isolated elements.*

Proof. We want to show that, if $\preceq \in \mathcal{PO}(G * H)$, then $\preceq$ is not isolated in $\mathcal{PO}(G * H)$. Indeed, consider $\preceq$ an element of $\mathcal{PO}(G * H)$. Then, $\preceq$ is a left-preorder on $G * H$ relative to C for some proper subgroup C of $G * H$, that is, $\preceq$ is an element of the subspace $\mathcal{PO}_C(G * H)$. We are going to prove that $\preceq$ is not isolated $\mathcal{PO}(G * H)$ by considering two cases: when the Kurosh rank of C with respect to $G * H$ is finite or infinite. If C has finite Kurosh rank with respect to $G * H$, by Theorem 5.3.1, we know that $\preceq$ is not isolated on $\mathcal{PO}_C(G * H)$, so $\preceq$ is not isolated on $\mathcal{PO}(G * H)$. Hence, we can assume that we are in the case where C has infinite Kurosh rank with respect to $G * H$.

Consider $\mathscr{R} \subseteq G * H$ a finite subset. The fact that $\preceq$ is not isolated on $\mathcal{PO}(G * H)$ will follow if we build a left-preorder $\preceq'$ that is non equal to $\preceq$ but both coinciding on the set $\mathscr{R}$. Consider P the positive cone of $\preceq$. The set $\mathscr{R}$ is finite and C has infinite Kurosh rank with respect to $G * H$, so, as it is remarked in Remark 3.2.7, there is a proper free factor C_1 of C such that $\mathscr{R} \cap C$ is contained in C_1. Then, we can write

$$C = C_1 * C_2.$$

Since the subgroup C_1 is left-relatively convex on C (as it is remarked in Example 1.1.11), we deduce from Proposition 1.3.12 that there is $\preceq'$ a left-preorder on $G*H$ relative to C_1 such that $x_1C \prec x_2C$ implies $x_1C_1 \prec' x_2C_1$. In particular, $\preceq$ and $\preceq'$ coincide on $\mathscr{R} - C$. Notice that $\preceq$ and $\preceq'$ coincide on $\mathscr{R} \cap C$ since $\preceq'$ is a left-preorder on $G*H$ relative to C_1 and $\mathscr{R} \cap C$ is contained in C_1. Therefore, $\preceq$ and $\preceq'$ coincide on $\mathscr{R}$. The fact that they are not equal is direct since C_1 and C are not equal and since $\preceq$ is a left-preorder on $G*H$ relative to C and $\preceq'$ is a left-preorder on $G*H$ relative to C_1. □

We have some consequences of the theorem.

Corollary 5.3.9. *Let G and H be two non-trivial groups. If $\mathcal{PO}(G*H)$ is non-empty, then it is uncountable.*

Proof. Assume that $\mathcal{PO}(G*H)$ is non-empty. Let $\preceq_T$ be the trivial preorder. We are going to consider two cases depending on if $\preceq_T$ is isolated on the set $\mathcal{PO}(G*H) \cup \{\preceq_T\}$ or not.

Assume that we are on the case where $\preceq_T$ is not isolated on the set $\mathcal{PO}(G*H) \cup \{\preceq_T\}$. Then, by Theorem 5.3.8, we deduce that $\mathcal{PO}(G*H) \cup \{\preceq_T\}$ has no isolated elements. Hence, applying Corollary 1.6.9 and Corollary 1.6.3, we deduce that $\mathcal{PO}(G*H) \cup \{\preceq_T\}$ is compact, Hausdorff and has no isolated points. Applying Theorem 27.7 of [42], we deduce that $\mathcal{PO}(G*H) \cup \{\preceq_T\}$ is uncountable. Therefore, $\mathcal{PO}(G*H)$ is uncountable.

Assume that we are on the case where $\preceq_T$ is isolated on the set $\mathcal{PO}(G*H) \cup \{\preceq_T\}$. This means that $\mathcal{PO}(G*H)$ is closed on $\mathcal{PO}(G*H) \cup \{\preceq_T\}$, but $\mathcal{PO}(G*H) \cup \{\preceq_T\}$ is compact by Corollary 1.6.9, so then $\mathcal{PO}(G*H)$ is compact. Then, applying Corollary 1.6.3 and Theorem 5.3.8, we deduce that $\mathcal{PO}(G*H)$ is compact, Hausdorff and has no isolated points. Then, by Theorem 27.7 of [42], we deduce that $\mathcal{PO}(G*H)$ is uncountable. □

Corollary 5.3.10. *Let G and H be two non-trivial finitely generated left-orderable groups. Then, $\mathcal{PO}(G*H)$ is homeomorphic to the Cantor set.*

Proof. Notice that, by Theorem 5.3.8 $\mathcal{PO}(G*H)$ has no isolated points. Also, since the free product of left-orderable groups is left-orderable by Corollary 6.1.3 of [35], we deduce that $\mathcal{PO}(G*H)$ is non-empty since G and H are left-orderable. Hence, the result follows by applying the characterization of Cantor sets (Proposition 1.6.17) to the topological properties we know about $\mathcal{PO}(G*H)$ (Corollary 1.6.11, Corollary 1.6.3 and Corollary 1.6.5). □

We obtain a description of this spaces for non-abelian free groups. Recall that, for a subgroup, being finitely generated implies having a finite Kurosh rank.

Corollary 5.3.11. *Let $\mathbb{F}$ be a non-abelian free group. Then,*

1. *$\mathcal{PO}_C(\mathbb{F})$ has no isolated elements for each C finitely generated left-relatively convex subgroup on $\mathbb{F}$. In particular, if $\mathbb{F}$ has at most countable free rank, then $\mathcal{PO}_C(\mathbb{F})$ is a Cantor set.*
2. *$\mathcal{PO}(\mathbb{F})$ has no isolated elements. In particular, if $\mathbb{F}$ has finite free rank, then $\mathcal{PO}(\mathbb{F})$ is a Cantor set.*

As a corollary of Theorem 5.3.1, we can give another proof of Corollary 4.2.26 for the case where the left-relatively convex subgroup has finite Kurosh rank.

Corollary 5.3.12. *Let G and H be two non-trivial groups. Let C be a left-relatively convex subgroup on $G*H$ having finite Kurosh rank with respect to $G*H$. Let $\preceq$ be a left-preorder on $G*H$ relative to C. Then, the positive cone of $\preceq$ is not finitely generated as a subsemigroup of $G*H$.*

Proof. Apply Theorem 5.3.1 and Proposition 1.6.18. □

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