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Problems in Elliptic and Parabolic PDEs coming from Probability, Geometry and Game theory

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Mientras la ciencia a descubrir no alcance
las fuentes de la vida,
y en el mar o en el cielo haya un abismo
que al cálculo resista,
mientras la humanidad siempre avanzando
no sepa a dó camina,
mientras haya un misterio para el hombre,
¡habrá poesía!

Rima IV. Gustavo Adolfo Bécquer.

A mi madre.

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Introducción y resumen

El objetivo de esta tesis doctoral es recopilar los cinco artículos que la autora ha desarrollado durante los últimos cuatro años. El marco de estos trabajos es el de las Ecuaciones en Derivadas Parciales y sus conexiones con la Probabilidad, la Geometría y la Teoría de Juegos. En particular, en esta tesis presentamos los trabajos recogidos en:

- [48], M.M. González, I. Gonzálvez and F. Quirós. "Traveling wave behavior for Fisher-KPP equations on the hyperbolic space". Pronto disponible en arXiv.

- [49], I. Gonzálvez, A. Miranda and J.D Rossi. "Monotone iterations of two obstacle problem with different operators". J. Elliptic Parabol Equations. 10.1 2024, pp 679-709.

- [50], I. Gonzálvez, A. Miranda and J.D Rossi. "The two membranes problem in a regular tree". Proceedings of the Royal Society of Edinburgh: Section A Mathematics. 2025, pp 1-30.

- [51], I. Gonzálvez, A. Miranda, J.D Rossi and, J. Ruiz-Cases. "Finding the convex hull of a set using the flow by minimal curvature with an obstacle. A game theoretical approach". 2025. Aceptado en Communications in Analysis and Geometry.

- [52], I. Gonzálvez, F. Quirós, F. Soria and Z. Vondraček. "On the nonlocal heat equation for certain Lévy operators and the uniqueness of positive solutions". 2025. arXiv:2504.04246

0.0.1 Ecuaciones en Derivadas Parciales: un campo transversal.

Una ecuación en derivadas parciales (EDP) es una ecuación que involucra una función desconocida y sus derivadas parciales de distintos órdenes. El origen de las EDP está profundamente relacionado con la observación de la naturaleza. Al estudiar fenómenos físicos y biológicos, uno descubre que existen relaciones intrínsecas entre ciertas magnitudes y sus variaciones en el tiempo y en el espacio. Por ejemplo, mediante las EDPs se pueden describir como se propaga el calor, como una mutación beneficiosa se extiende en una población, como evoluciona una burbuja de jabón en el tiempo o como se deforma una membrana elástica en presencia de un obstáculo rígido.

En el caso de la modelización de la difusión del calor, en 1822, Fourier propone usar la siguiente ecuación

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \tag{0.0.1}$$

La ecuación anterior, denominada *ecuación del calor*, describe la evolución en el tiempo de u, que representa la densidad de temperatura. Dicha ecuación nos indica que u varía en el tiempo a la misma velocidad que la suma de sus segundas derivadas espaciales.

La ecuación (0.0.1) también es conocida como la *ecuación de difusión*. De manera intuitiva, la difusión es un fenómeno en el cual una cantidad (como una población, el calor, el agua o el dinero) se desplaza desde una región de alta concentración hacia otra de baja concentración, con el objetivo de alcanzar una distribución más uniforme. Así pues, la ecuación de difusión es una expresión matemática que modela como ciertos procesos tienden a evolucionar hacia un estado de mayor desorden con el tiempo. Además, en 1905, Einstein descubrió la relación entre esta ecuación y los paseos aleatorios (movimiento browniano).

Por otro lado, en 1937, de manera simultánea pero independiente, Fisher y el grupo conformado por Kolmogorov, Petrovskii y Piskunov (KPP) modelaron la propagación de un gen ventajoso en una población [41, 64]. Considerando $0 \leq u(x,t) \leq 1$ la proporción de la población que, en el tiempo t y en el punto x, posee la mutación beneficiosa. Si se tiene en cuenta únicamente la selección natural, es decir, la reacción, u satisface la ecuación diferencial ordinaria introducida por Verhulst en 1845:

$$u_t = u(1-u) \quad \text{in } \mathbb{R}^d \times (0, \infty)$$

que solo involucra una derivada temporal. Sin embargo, Fisher y el grupo KPP contemplaron la posibilidad de que la descendencia de un individuo ubicado en el punto x no permaneciera en el mismo lugar, sino que pudiera establecerse en un entorno alrededor de x. Esto introdujo en el problema el fenómeno de difusión. Como resultado, u se rige por la ecuación en derivadas parciales de reacción-difusión

$$u_t - \Delta u = u(1-u) \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{0.0.2}$$

Respecto a la modelización de la evolución de la burbuja de jabón, se puede observar que su forma debido a la tensión superficial evoluciona de tal manera que busca disminuir su área de la forma más eficiente. Como consecuencia, la pompa de jabón evoluciona por curvatura media. Si u es una función cuyo conjunto de nivel cero representa la burbuja de jabón, entonces u satisface

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{0.0.3}$$

La ecuación anterior fue estudiada por primera vez mediante soluciones viscosas por Evans y Spruck en su artículo [40] de 1991.

Por último, con el siguiente problema asociado a una ecuación elíptica finalizamos esta exposición sobre unos ejemplos de EDPs clásicas. Dado un dominio acotado $\Omega \subset \mathbb{R}^d$, el problema de obstáculo

$$\begin{cases} \min\{-\Delta u, u - \psi\} = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{0.0.4}$$

describe como se comporta un membrana u en equilibrio sujeta por el borde a un bastidor f cuando se encuentra con un objeto voluminoso y rígido llamado obstáculo ψ. El obstáculo obliga a la membrana a adoptar su forma en el conjunto de coincidencia, definido como el espacio o región de contacto entre ambas superficies. En el conjunto de no coincidencia la membrana se mantine en equilibrio por encima del obstáculo.

Este problema se cataloga en los problemas de frontera libre, ya que borde entre los conjuntos de contacto y no contacto es a priori desconocida. De ahí que se denomine "libre"a dicha frontera. El estudio de la regularidad de la frontera libre en el problema del obstáculo ha atraído el interés de muchos matemáticos como Lewy, Stampacchia, Nirenberg y Caffarelli, entre otros.

Además, como resumió Freidman en [42], el problema de obstáculo tiene aplicaciones en muchos otros contextos, como en matemáticas financieras y control óptimo.

Las ecuaciones y problemas anteriores son ejemplos clásicos de como modelar fenómenos naturales usando EDPs. Una vez elegido qué problema se quiere estudiar, el siguiente paso es abordar (al menos una de) las siguientes preguntas fundamentales:

- Existencia y unicidad: Dada una EDP, la primera pregunta que se debe responder es si existe una función que la satisfaga (una solución). La segunda es saber si esta solución es única bajo las condiciones dadas. Si la existencia o la unicidad no se cumplen, el problema se considera *mal planteado* (en inglés *an ill-posed problem*).

  Un ejemplo de un problema mal planteado es el problema de Cauchy para la ecuación del calor (0.0.1). Cuando la traza inicial es idénticamente cero, existen infinitas soluciones, como por ejemplo la solución trivial (la función nula) y cualquier múltiplo del contraejemplo de Tychonov, [109].

  Para abordar este problema y obtener unicidad, es necesario imponer condiciones adicionales. Por ejemplo, condiciones sobre el crecimiento de la solución, [109], o imponer que la solución sea no negativa, [5].

  En esta tesis, se aborda la cuestión de existencia y unicidad para una generalización de la ecuación del calor en el Capítulo 2.

- Resultados cualitativos: Una vez que se ha demostrado que el problema asociado a una EDP admite una solución y esta es única (es decir, es un problema *bien planteado*), el siguiente objetivo es describir la solución. Los dos tipos de resultados cualitativos más comunes son los relacionados, por un lado, con la regularidad, y por el otro, con el comportamiento asintótico.

  - La teoría de regularidad se ocupa del estudio de la integrabilidad y la diferenciabilidad de la solución.
  - El comportamiento asintótico se refiere a si la solución presenta algún ratio de decaimiento en el infinito, ya sea en el espacio o en el tiempo, o si converge a un perfil específico.

En el Capítulo 3 estudiamos el comportamiento a largo plazo de la solución del problema de Cauchy asociado a una ecuación de reacción-difusión en el espacio hiperbólico para una familia de datos iniciales apropiados. Obtenemos que la solución converge a una onda viajera específica, *traveling wave*, en coordenadas móviles (en el tiempo). En el Capítulo 3 también se estudia el comportamiento asintótico de la solución al problema del obstáculo para un operador elíptico y degenerado que modela la evolución por curvatura mínima. Se obtiene que supernivel cero converge a la envoltura convexa del soporte del obstáculo.

- Cómo obtener una solución: Otro objetivo en el estudio de las EDP es obtener la solución de un problema asociado a una ecuación dada. A veces, es posible construir un método algorítmico para obtener una solución (como hacemos en el Capítulo 5), mientras que en otros casos, se puede utilizar un enfoque de teoría de juegos para obtener la solución (como se muestra en el Capítulo 4 y en la última subsección del Capítulo 5).

  Por otro lado, en problemas lineales con existencia y unicidad a menudo es posible obtener una fórmula de representación (como se hace en el Capítulo 2).

  Además, el hecho de proporcionar una fórmula o un método para encontrar una solución a menudo conlleva poder obtener resultados cualitativos para dicha solución.

Por otro lado, la teoría de las EDPs está profundamente conectada con otras áreas de las matemáticas, de las cuales toma herramientas y técnicas para abordar estas cuestiones fundamentales. A su vez, las EDPs proporcionan a estos campos nuevas perspectivas y resultados que contribuyen a su desarrollo.

En particular, las EDPs estudiadas en esta tesis están estrechamente relacionadas con los campos de la Probabilidad, la Geometría y la Teoría de Juegos:

Probabilidad. Una de las conexiones más estrechas entre la Probabilidad y las EDPs es el *movimiento browniano* y la ecuación del calor, (1.0.1).

El movimiento browniano describe como una partícula se mueve de forma aleatoria en un fluido. Matemáticamente, el movimiento browniano es un proceso estocástico. Es decir, dado un espacio de probabilidad $(\Omega, \mathcal{A}, \mathbb{P})$, el movimiento browniano es una familia de variables aletorias definidas en dicho espacio de probabilidad, indexadas en tiempo t y tomando valores en $\mathbb{R}^d$ que satisfacen las siguientes propiedades: X tiene incrementos independientes y es casi seguramente continua (*almost surely continuous*), $X_0 = 0$ con probabilidad 1 y $X_t - X_s$ sigue una distribución normal con media cero y matriz de covarianza $2(t-s)I$ para todo $t > s \geq 0$.

Además, el moviento browniano X tiene función característica

$$\mathbb{E}[e^{i\xi X_t}] = e^{-t|\xi|^2}, \quad \text{para todo } \xi \in \mathbb{R}^d,\ t \geq 0$$

y $P : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ núcleo del calor asociado

$$P(x,t) = \mathcal{F}\big(e^{-t|\cdot|^2}\big)(x), \quad x \in \mathbb{R}^d,\ \ t > 0.$$

donde $\mathcal{F}$ denota la transformada de Fourier.

La relación entre el movimiento browniano X y la ecuación del calor se establece a través del generador infinitesimal de X: el laplaciano Δ. Este se obtiene como

$$\Delta f = \lim_{t\to 0} \frac{P(\cdot,t) * f - f}{t} \quad \text{para toda } f \in C_c^\infty(\mathbb{R}^d).$$

Como consecuencia de lo anterior, $u = P(\cdot,t) * f$ es una solución de la ecuación del calor (0.0.1) para toda f función suave.

Por otro lado, los procesos de Lévy generalizan el movimiento browniano al permitir discontinuidades en sus trayectorias. A diferencia del movimiento browniano, que tiene trayectorias continuas, los procesos de Lévy pueden realizar saltos finitos.

Los ejemplos carasterísticos de este tipo de procesos son los procesos α-estables y simétricos $Y^\alpha = \big(Y_t^\alpha\big)_{t\geq 0}$ con $\alpha \in (0,2)$. Para un valor fijo de alfa α, la función carasterística y el núcleo del calor asociado a Y^α viene dado por

$$\mathbb{E}[e^{i\xi Y_t^\alpha}] = e^{-t|\xi|^\alpha}, \quad P^\alpha(x,t) = \mathcal{F}\big(e^{-t|\cdot|^\alpha}\big)(x), \quad \xi, x \in \mathbb{R}^d, \ \ t > 0.$$

Por tanto, el generador infinitesimal de Y^α, denotado como $-(-\Delta)^{\alpha/2}$, es dado por

$$-(-\Delta)^{\alpha/2} f := \lim_{t\to 0} \frac{P^\alpha(\cdot,t) * f - f}{t} \quad \text{para toda } f \in C_c^\infty(\mathbb{R}^d). \tag{0.0.5}$$

El operador $(-\Delta)^{\alpha/2}$ es denominado como el laplaciano $\alpha/2$-fraccionario. La notación y el nombre provienen del comportamiento de las derivadas bajo la transformada de Fourier y del hecho de que

$$\mathcal{F}\big(-\Delta f\big)(\xi) = |\xi|^2\big(\mathcal{F}f\big)(\xi), \quad \mathcal{F}\big((-\Delta)^{\alpha/2} f\big)(\xi) = |\xi|^\alpha\big(\mathcal{F}f\big)(\xi), \quad \xi \in \mathbb{R}^d.$$

Una de las definiciones equivalentes del laplaciano $\alpha/2$-fraccionario (en todo el espacio $\mathbb{R}^d$) es la siguiente, expresada en términos de sus segundas diferencias:

$$(-\Delta)^{\alpha/2} f(x) = C_{d,\alpha} \iint\!\!\!\!\!\!\diagup_{\mathbb{R}^d} \big(f(x) - \frac{1}{2}\big(f(x-y) - f(x+y)\big)\big) \frac{\mathrm{d}y}{|y|^{d+\alpha}}, \tag{0.0.6}$$

para $x \in \mathbb{R}^d$ y f suave donde $C_{d,\alpha}$ es una constante positiva que depende de la dimension d y del parámetro α.

La fórmula anterior resalta la naturaleza no local del laplaciano fraccionario, que surge de los saltos de su proceso de Lévy asociado. La expresión (0.0.6) muestra que el comportamiento de la función f lejos del punto x juega un papel significativo, en contraste con el laplaciano clásico.

Además, gracias a (0.0.5), para f suave, la función $u = P^\alpha(\cdot,t) * f$ resuelve la *ecuación fraccionaria del calor*

$$u_t + (-\Delta)^{\alpha/2} u = 0, \quad \mathbb{R}^d \times (0,\infty).$$

Debido al comportamiento no local del laplaciano fraccionario, la ecuación del calor fraccionaria y sus análogos para otros procesos de Lévy sirven como modelos simplificados

para fenómenos que exhiben difusión anómala o interacciones de largo alcance. Por ejemplo, dos casos de comportamientos aleatorios discontinuos en la naturaleza son la movilidad de cargas en semiconductores amorfos [22] y los patrones de búsqueda de alimento de los monos araña en libertad en los bosques del Yucatán [96].

En el Capítulo 2, aprovechamos la caracterización del núcleo del calor en términos de un proceso de Lévy para estudiar la existencia, unicidad y traza inicial de soluciones no negativas para ecuaciones del calor no locales.

Geometría. La relación entre la Geometría y las EDPs se remonta a los primeros desarrollos en este segundo campo: estudio de problemas con un importante carácter geométrico, como el problema isoperimétrico o la curva tautócrona, son algunos de los precursores del desarrollo de las EDPs dentro del marco del Cálculo de Variaciones.

A su vez, el nacimiento de la teoría de las EDPs proporcionó a la Geometría nuevas perspectivas y herramientas. Por ejemplo, Euler, utilizando la geometría diferencial (muy relacionada con las EDPs), obtuvo la primera ecuación analítica de las geodésicas, y Lagrange la primera ecuación diferencial para describir una superficie mínima (en términos de las ecuaciones de Euler-Lagrange).

A finales del siglo XX, la relación entre la Geometría y las EDPs se hizo aún más evidente con el surgimiento del Análisis Geométrico de la mano de Karen Uhlenbeck, ganadora de un Premio Abel. Esta subdisciplina emplea tanto métodos geométricos en el estudio de las EDPs como aplica la teoría de las EDPs a la Geometría.

Por otro lado, Caffarelli, también galardonado con el Premio Abel, aplicó la teoría de las EDPs para caracterizar las singularidades que aparecen en problemas de frontera libre. Entre muchas otras contribuciones, estudió desde una perspectiva analítica la geometría de la frontera libre en problemas como el problema de obstáculo (ver (0.0.4)) y el problema de Stefan (que describe la evolución en el tiempo del borde de un hielo sumergido en agua).

Además de estas importantes líneas de investigación abiertas por estos dos galardonados con el Premio Abel, existe un interés reciente en revisar los problemas clásicos de las EDPs cambiando el espacio euclídeo por variedades, [14, 35, 75, 76, 94, 95, 113]. Estos problemas, más allá de su interés matemático intrínseco, nos ayudan a entender como las soluciones de las EDPs dependen del entorno circundante. Esta dependencia a menudo no se observa cuando se elige el espacio euclídeo como el espacio ambiente.

Por ejemplo, consideremos la ecuación análoga a (1.0.1) en el espacio hiperbólico $\mathbb{H}^d$

$$u_t - \Delta_{\mathbb{H}^d} u = 0 \quad \text{in } \mathbb{H}^d \times (0,\infty) \tag{0.0.7}$$

donde $\Delta_{\mathbb{H}^d}$ denota el operador de Laplace-Beltrami en dicho espacio. Tal y como Vázquez recalca en [113], la solución fundamental de (0.0.7) decae en tiempo exponencialmente más rápido que su homólogo en el espacio euclídeo. Esto se debe al *desplazamiento desde el infinito*, el comportamiento repelente de la frontera ideal del espacio hiperbólico. Como consecuencia de este fenómeno, como muestra [75], la propagación de un gen beneficioso en el espacio hiperbólico modelado por la ecuación

$$u_t - \Delta_{\mathbb{H}^d} u = u(1-u) \quad \text{in } \mathbb{H}^d \times (0,\infty) \tag{0.0.8}$$

no está siempre garantizada, a diferencia del caso euclídeo donde la solución de (0.0.2) siempre presenta propagación. Esto ilustra como el comportamiento en el infinito de una variedad es otro factor importante a tener en cuenta en el estudio de las ecuaciones de reacción-difusión.

En el Capítulo 3 presentamos nuevos resultados que exploran las implicaciones de la geometría del espacio hiperbólico en la ecuación de reacción-difusión (1.0.9) y para ecuaciones con lado derecho más general dado por funciones tipo KPP. En el Capítulo 5, regresamos al problema de obstáculo para construir una solución al problema de las dos membranas mediante iteraciones.

Teoría de Juegos. Para mostrar la profunda relación entre la teoría de juegos y las EDPs, introducimos a continuación un juego teórico para ilustrarlo. En [63], los autores proporcionan una interpretación del movimiento por curvatura media (la evolución de una burbuja de jabón) a través de un juego determinista de dos personas y suma cero.

Una versión de las reglas del juego son:

i) El tablero: sea Ω un dominio acotado de $\mathbb{R}^d$ y el soporte de una función u_0 dada. Aquí u_0 respresenta una función de pagos.

ii) Los jugadores: hay dos jugadores: Paul y Carol. El objetivo de Paul es ganar tanto dinero como pueda, es decir, quiere maximizar u_0. Por otro lado, Carol debe pagar a Paul lo que gane. Por lo tanto, el propósito de Carol es reducir al mínimo las ganancias de Paul. Para ello, su meta es bien expulsar a Paul del tablero Ω_0, porque en el exterior u_0 toma valores negativos, o bien, si esto no es posible, llevar a Paul al punto de Ω donde u_0 alcanza su mínimo.

iii) Movimientos permitidos: sea $x = x_0$ la posición inicial, $t = t_0 > 0$ el tiempo inicial y $\varepsilon > 0$ fijo. En cada i-ésima ronda, Paul elige $d-1$ vectores unitarios ortogonales $\{v_i^1, ..., v_i^{d-1}\}$ y Carol selecciona un colección de signos $b_i^j = \pm 1$ con $j = 1, ..., d-1$ después de la elección de Paul. Así pues, la siguiente posición en el tablero viene dada por

$$x_i = x_{i-1} + \frac{\varepsilon}{d-1} \sum_{j=1}^{d-1} b_i^j v_i^j$$

y el tiempo decrece acorde a $t_i = t_{i-1} - \frac{1}{2}\varepsilon^2$. Después de $n = \left\lceil \frac{2t_0}{\varepsilon^2} \right\rceil$ turnos, el juego acaba y Carol paga a Paul el cómputo $u_0(x_n)$.

Para una representación visual del juego, consúltese la figura 1.

Heurísticamente, la *función valor* del juego, es decir, el coste final optimizado por ambos jugadores viene dada por

$$u^\varepsilon(x,t) = \text{ máximo valor de } u_0(x_n) \text{ empezando en } x \in \mathbb{R}^d \text{ en tiempo } t > 0.$$

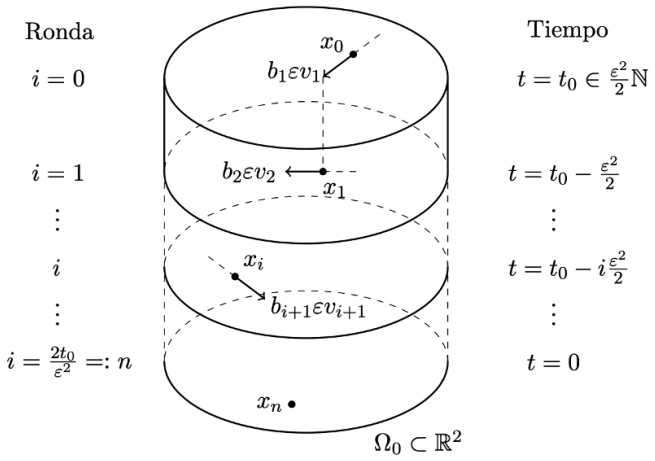


Figure 1: Juego asociado al movimiento por curvatura media en $\mathbb{R}^2$.

La función valor satisface una ecuación que, por analogía con el análisis armónico, desempeña el papel de una fórmula del valor medio. Esta ecuación, conocida como el *principio de programación dinámica* (DPP por sus siglas en inglés), está dada por

$$u^\varepsilon(x,t) = \sup_{\left\{\{v^1,\dots,v^{d-1}\}\subset\mathbb{R}^d:\, \langle v^i,v^j\rangle=\delta_{ij}\right.} \min_{\left\{\{b^1,\dots,b^{d-1}\}\subset\{\pm 1\}\right.} u^\varepsilon\Big(x + \tfrac{\varepsilon}{d-1}\sum_{j=1}^{d-1} b^j v^j, t - \tfrac{1}{2}\varepsilon^2\Big).$$

El punto clave es que, usando el desarroyo de Taylor y la teoría de soluciones viscosas, se puede establecer una conexión entre la Teoría de Juegos y las EDPs. Específicamente, cuando $\varepsilon \to 0$, la familia de funciones $\{u^\varepsilon\}_{\varepsilon>0}$ converge a la solución viscosa de (0.0.3).

Una vez establecido el puente entre ambos campos, es posible combinar resultados de los dos. Por ejemplo, se obtiene que el conjunto de nivel cero de la función límite evoluciona según la curvatura media. Esto, a su vez, implica que si ε es lo suficientemente pequeño y el juego se desarrolla durante un tiempo suficientemente largo, Carol gana. La razón es que la solución de (0.0.3) se extingue en tiempo finito. Además, gracias a la convergencia, se pueden implementar estrategias para ambos jugadores que proporcionan una comprensión intuitiva de como evoluciona la burbuja de jabón (y formas más complejas) a lo largo del tiempo.

En el Capítulo 4, en línea con la discusión anterior, definiremos un juego determinista de suma cero entre dos jugadores para obtener la envolvente convexa de un conjunto vía el flujo por curvatura mínima.

Además, en el Capítulo 5, se proporcionará una interpretación del problema de las dos membranas a través de un juego teórico y probabilístico.

0.0.2 Resultados principales

Teoría de Widder para la ecuación del calor no local para ciertos operadores de Lévy

En el Capítulo 2 presentamos los resultados del artículo [52],

On the nonlocal heat equation for certain Lévy operators and the uniqueness of positive solutions,

escrito por Fernando Quirós, Fernando Soria, Zoran Vondraček (UZ) y la autora de ésta memoria de tesis.

Como explicamos anteriormente en la Introducción de ésta tesis, el problema de Cauchy para la ecuación de calor clásica (1.0.1) no está bien planteado en general. Para establecer unicidad, Widder propuso en 1944, [116], un enfoque innovador. Sugirió estudiar este problema dentro de la clase de soluciones *no negativas*, basándose en el hecho físico de que la temperatura siempre permanece por encima del cero absoluto. En 1971, Aronson [5] completó la teoría de unicidad para soluciones no negativas en el marco clásico (local).

En este capítulo, motivados por los trabajos [12, 19], en los cuales los autores estudiaron la unicidad de soluciones no negativas para la ecuación de calor fraccionaria (1.0.7), desarrollamos una teoría de tipo Widder para ecuaciones de calor no locales de la forma

$$\partial_t u + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{0.0.9}$$

dónde $\mathcal{L}$ es un operador de Lévy tan general como sea posible, $d \geq 1$ y $T \in (0,+\infty)$. De este modo, consideramos soluciones no negativas y buscamos condiciones sobre el operador que garanticen: (i) la unicidad de soluciones no negativas, tanto de soluciones clásicas como de soluciones muy débiles, con una traza inicial dada; (ii) la existencia de una traza inicial perteneciente a una cierta clase admisible; y (iii) la existencia de una solución, expresada mediante una fórmula de representación, para cualquier traza inicial admisible.

Estos resultados se obtienen inicialmente para operadores de Lévy puramente no locales de la forma

$$\mathcal{L}_K u(x) = \iint\!\!\!\!\!\!\diagup_{\mathbb{R}^d} \Big(u(x) - \frac{u(x+y)+u(x-y)}{2}\Big) K(y)\,\mathrm{d}y, \tag{0.0.10}$$

donde $K : \mathbb{R}^d \to \mathbb{R}$ es un núcleo de Lévy positivo y simétrico, comparable con funciones radiales de crecimiento polinomial mixto. Después, extendemos estos resultados a operadores más generales, incluyendo operadores anisotrópicos de la forma $\mathcal{L}_\nu = \sum_{j=1}^{\ell} \mathcal{L}_{K_j}$, $\ell \in \{1,\dots,d\}$, donde cada $\mathcal{L}_{K_j}$es de la forma (0.0.10) y actúa únicamente en subespacios $\mathbb{R}^{d_j}$, con $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$. Asímismo, consideramos operadores que combinan una parte local y una no local, es decir, operadores como $\mathcal{L}_{\Delta,K} := -\Delta + \mathcal{L}_K$, con $\mathcal{L}_K$ dado por (0.0.10).

El esquema que seguimos para obtener una respuesta afirmativa a las tres cuestiones planteadas es el siguiente: comenzamos estableciendo propiedades fundamentales del núcleo del calor asociado a $\mathcal{L}$, las cuales nos permiten demostrar nuestros resultados principales.

En particular, utilizando los resultados de [10], obtenemos que el núcleo del calor en dos tiempos distintos es comparable y que el núcleo de Lévy K y el núcleo del calor tienen colas comparables

A continuación, dividimos nuestra estrategia en dos partes. En primer lugar, estudiamos soluciones muy débiles, abordando la unicidad, la existencia de una traza inicial y la existencia de soluciones. Nuestro objetivo es obtener una fórmula de representación para soluciones muy débiles que sean no negativas o que satisfagan una condición integral adecuada, expresada en términos del núcleo del calor.

En segundo lugar, investigamos las soluciones clásicas, centrándonos en la unicidad de las soluciones clásicas no negativas de (0.0.9). La idea clave en esta parte es demostrar que cualquier solución clásica no negativa es también una solución muy débil (no negativa). Dado que previamente hemos establecido una fórmula de representación para soluciones muy débiles no negativas, esto nos permite obtener directamente nuestro resultado principal: una fórmula de representación para soluciones clásicas no negativas de (0.0.9). Queremos descar que, a diferencia del caso local, no es inmediato que una solución clásica sea también una solución muy débil. Esta dificultad surge debido a la naturaleza no local del núcleo de Lévy.

También demostramos la existencia de soluciones clásicas cuando los operadores $\mathcal{L}_K$, $\mathcal{L}_\nu$ y $\mathcal{L}_{\Delta,K}$ están relacionados con el laplaciano fraccionario, empleando la teoría de funciones de Bernstein.

El problema KPP en el espacio hiperbólico

El Capítulo 3 esta dedicado a los resultados de [48],

Traveling wave behavior for Fisher-KPP equations on the hyperbolic space,

un artículo coescrito por María del Mar González (UAM), Fernando Quirós y la autora de la tesis.

Estudiamos el comportamiento asintótico preciso de la solución al problema de KPP

$$\begin{cases} u_t = \Delta_{\mathbb{H}^d} u + f(u), & \mathbb{H}^d \times (0,\infty), \\ u(\cdot,0) = u_0, & \mathbb{H}^d \end{cases}$$

en el espacio hiperbólico $\mathbb{H}^d$ con $d \geq 2$. Aquí, $\Delta_{\mathbb{H}^d}$ representa el operador de Laplace-Beltrami en $\mathbb{H}^d$ y f es una función tipo KPP, como por ejemplo $f(s) := s(1-s)$, $s \in [0,1]$.

Nuestra motivación es comprender como la geometría hiperbólica afecta la evolución del problema anterior, así como analizar sus diferencias y similitudes con el problema análogo en el contexto euclídeo.

Por un lado, una de las diferencias radica en la cuestión de la *propagación* frente a la *extinción*. En [7], los autores demuestran que, para un dato inicial admisible, la solución del problema de KPP en el espacio euclídeo siempre alcanza el nivel máximo de saturación en conjuntos compactos, es decir, la propagación siempre ocurre. Por otro lado, en el pionero artículo [75], los autores muestran que, en el espacio hiperbólico, existe una competencia

entre la fuente f y λ_1, (aquí λ_1 es el ínfimo del espectro de $-\Delta_{\mathbb{H}^d}$). Establecen que si $f'(0) > \lambda_1$ ocurre la propagación, mientras que si $f'(0) < \lambda_1$ la solución se extingue (extinción). En el Capítulo 4 abordamos el caso crítico en el que $f'(0) = \lambda_1$, demostrando que se produce extinción para una familia de datos iniciales.

Por otro lado, la principal similitud es que, siempre que ocurre la propagación, la solución del problema de KPP en ambos espacios converge hacia una onda viajera de velocidad mínima Φ_{c_0} en un sistema de coordenadas móviles. Para el caso euclídeo, véase [56, 84, 100, 101].

Sin embargo, en este capítulo demostramos que el sistema de coordenadas móviles en el espacio hiperbólico presenta diferencias importantes con su análogo euclídeo.

Más precisamente, demostramos que el tipo de simetrías del dato inicial determina la elección del sistema de coordenadas móviles en el cual la solución converge a Φ_{c_0}. Exis-ten tres tipos de simetrías, que están estrechamente relacionados con los tres tipos de isometrías de $\mathbb{H}^d$: elípticas, parabólicas e hiperbólicas.

Además, probamos que los términos dependientes del tiempo en las coordenadas móviles también están dictados por la geometría del espacio hiperbólico. El desplazamiento desde el infinito en $\mathbb{H}^d$ afecta directamente al término lineal del sistema de coordenadas en movimiento, el cual depende únicamente de la dimensión de $\mathbb{H}^d$. Asimismo, es necesario introducir una corrección logarítmica que se mantiene constante en los tres tipos de invariancia.

Ambos aspectos representan una diferencia significativa respecto al problema análogo en el espacio euclídeo. En $\mathbb{R}^d$, existen únicamente dos tipos de invariancia que determinan la elección del sistema de coordenadas en movimiento: rotaciones y traslaciones (correspondientes a los dos tipos de isometrías: elípticas y parabólicas). Además, el término lineal del sistema de coordenadas móviles en el espacio euclídeo no depende de la dimensión, mientras que la corrección logarítmica sí depende del tipo de invariancia del dato inicial.

Aproximación teórica basada en un juego para recuperar la envoltura convexa de un conjunto vía el flujo por curvatura mínima

Los resultados presentados en el Capítulo 4 están publicados en el artículo [51] escrito Alfredo Miranda (UBA), Julio D. Rossi (UBA), Jorge Ruiz-Cases (UAM) y la autora de la tesis bajo el nombre de

Finding the convex hull of a set using the flow by minimal curvature with an obstacle. A game theoretical approach.

El objetivo de este capítulo es estudiar el comportamiento asintótico de la solución del problema del obstáculo parabólico para una ecuación completamente no lineal degenerada, que modela el movimiento por curvatura mínima de una hipersuperficie.

Demostramos que, bajo ciertas condiciones sobre la condición inicial y el obstáculo, el conjunto de positividad de esta solución converge a la envolvente convexa del conjunto donde el obstáculo es positivo cuando t tiende a infinito.

Más precisamente, utilixamos el enfoque de conjuntos de nivel (*level set approach*) para describir la evolución geométrica de una familia de $\mathbb{R}^d$-hipersuperfices $\{\Omega_t\}_{t\geq 0}$ hacia el envolvente convexo de un conjunto acotado $K \subset \mathbb{R}^d$, $co(K)$.

Suponemos que existe una función $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ cuyos superniveles de nivel cero son Ω_t para cada $t \geq 0$, y que satisface la siguiente propiedad:

$$\Omega_t \to co(K) \quad \text{as } t \to \infty. \tag{0.0.11}$$

Nosotros demostramos que u puede ser descrita como la solucion del problema

$$\begin{cases} \max\Big\{ -\partial_t u(x,t) - \mathscr{L}(u(x,t)), \psi(x) - u(x,t)\Big\} = 0, & (x,t) \in \mathbb{R}^d \times \{t>0\}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{0.0.12}$$

donde $\mathscr{L}$ es el operador elíptico y degenerado dado por

$$\mathscr{L}(u(x)) := \inf_{\substack{|v|=1 \\ v \perp \nabla f(x)}} \Big(- \langle D^2 u(x) v, v\rangle\Big).$$

que modela la evolución por curvatura mínima y donde ψ es una función suave cuyo supernivel zero es el conjunto K.

La idea de considerar el flujo por curvatura mínima radica en el hecho de que la envolvente convexa de un conjunto dado K se caracteriza por la propiedad de que su frontera puede ser tocada por hiperplanos que dejan a un lado de ellos todos los puntos de K. Además, nótese que cualquier punto que pertenezca a $co(K) \setminus K$ también pertenece a uno de estos hiperplanos. En dicho punto, la curvatura principal de la frontera debe ser cero, mientras que las demás curvaturas deben ser no negativas para garantizar la convexidad de $co(K)$.

Por lo tanto, tenemos que $co(K)$ es una solución a la curvatura mínima igual a cero en los puntos de la frontera que no pertenecen a $\overline{K}$ (es decir, donde la superficie no toca el obstáculo). Dado que nuestro objetivo, $co(K)$, es una solución estacionaria de nuestra ecuación de evolución, resulta natural esperar que recuperemos la envolvente convexa cuando el tiempo tienda a infinito para las soluciones del problema (0.0.12)

En este capítulo, nuestro primer paso es establecer la existencia y unicidad de soluciones para el problema del obstáculo (1.0.13), utilizando un principio de comparación y el método de Perron. Para ello, nos basaremos en la teoría previamente desarrollada para soluciones de viscosidad en flujos geométricos; véase [36, 45, 78, 79].

Una vez establecida la existencia y unicidad, centramos nuestra atención en el comportamiento asintótico de las soluciones. Para obtener nuestro resultado principal (0.0.11), utilizamos un juego determinista de suma cero para dos jugadores, inspirado en resultados previos de [63] y [82].

En este juego, los jugadores son Paul, cuyo objetivo es maximizar el resultado esperado, y Carol, que busca minimizarlo. Para un $\varepsilon > 0$ fijo, en cada turno del juego, los jugadores realizan una serie de elecciones que les permiten moverse un paso de tamaño ε a la siguiente

posición. El objetivo principal de Paul es alcanzar el obstáculo K, ya que, al hacerlo, gana automáticamente. Si Carol impide este objetivo, el segundo propósito de Paul es permanecer dentro de $co(K)$. Por otro lado, el primer objetivo de Carol es expulsar a Paul hacia Ω_0, ya que allí el coste final es negativo. El coste final optimizado por ambos jugadores, comenzando en x y en el tiempo t, está dado por el valor del ε-juego, denotado por $u^\varepsilon(x,t)$.

Los puntos clave para obtener (0.0.11) son, por un lado, que las funciones valor u^ε convergen a la solución u del problema parabólico del obstáculo (1.0.13) cuando $\varepsilon \to 0$. Por otro lado, al introducir el concepto de ε-convexidad y considerar estrategias que permiten a los jugadores alcanzar sus objetivos, demostramos que el supernivel cero de u^ε, para ε suficientemente pequeño, aproxima (bajo ciertas hipótesis sobre K y u_0) a la envolvente convexa de K cuando $t \to \infty$.

El problema de las dos membranas

El Capítulo 5 está dedicado a presentar los resultados recogidos en

- [49], *Monotone iterations of two obstacle problem with different operators*,
- [50], *The two membranes problem in a regular tree*,

dos trabajos coescritos por Alfredo Miranda (UBA), Julio D. Rossi (UBA) y la autora de ésta memoria para optar al título de doctor.

Sea Ω un dominio Lipschitz y acotado y $\mathbb{T}$ un árbol regular. Consideramos L_1 y L_2 ó bien dos operadores diferenciales definidos sobre funciones $u : \Omega \to \mathbb{R}$ ó bien dos operadores tipo media que actúan en funciones $u : \mathbb{T} \mapsto \mathbb{R}$. De manera similar, sean (f, g) un par de funciones ordenadas en $\partial\Omega$ o en $\partial\mathbb{T}$.

Una solución al problema de las dos membranas en Ω o en $\mathbb{T}$ con operadores L_1 y L_2 y dato de frontera (f, g) es un par de funciones (u, v) que satisface

$$\begin{cases} \min\Big\{L_1(u)(x), (u-v)(x)\Big\} = 0, & x \in \Omega \ \text{ ó } x \in \mathbb{T}, \\ \max\Big\{L_2(v)(x), (v-u)(x)\Big\} = 0, & x \in \Omega \ \text{ ó } x \in \mathbb{T}, \\ u(x) = f(x), \qquad v(x) = g(x), & x \in \partial\Omega \ \text{ ó } x \in \partial\mathbb{T}. \end{cases}$$

En otras palabras, u es la solución del obtáculo por abajo con operador L_1, obstáculo v y condición de Dirichlet g mientras que v es la solución del problema del obstáculo por arriba con operador L_2, obstáculo u y dato de frontera f.

El problema de las dos membranas para laplaciano fue considerado por primera vez en [111] mediante argumentos variacionales. Una vez que se ha obtenido la existencia de una solución (en un sentido apropiado), surgen numerosas cuestiones de interés, como la unicidad, la regularidad de las funciones involucradas, la descripción del conjunto de contacto, la regularidad de dicho conjunto, entre otras. Véanse [27, 26, 104], la tesis doctoral [112] y las referencias allí citadas. Además, versiones más generales de este problema

involucrando más de dos membranas han sido estudiada por diversos autores (véanse, por ejemplo, [9, 31, 34]).

En este Capítulo 5, por un lado, establecemos la existencia de soluciones para el problema de las dos membranas en un árbol bajo condiciones adecuadas sobre los datos en la frontera y los términos fuente involucrados. Obsérvese que la existencia de solución para el problema de las dos membranas en un dominio está siempre garantizada por una aplicación clásica del método de Perron cuando L_1 y L_2 son operadores elípticos generales.

Por otro lado, analizamos iteraciones del problema del obstáculo en ambos espacios ambiente. Resolvemos de manera iterativa el problema del obstáculo desde arriba o desde abajo para dos operadores distintos, L_1 y L_2, con obstáculos dados por las funciones obtenidas en el paso terativo previo.

Asímismo, cuando el espacio ambiente es Ω, implementamos este método iterativo en dos enfoques distintos: el marco variacional, cuando los operadores L_1 y L_2 admiten la forma en divergencia, y el enfoque basado en soluciones de viscosidad, para operadores más generales (siempre que sean elípticos). Comenzamos el método iterativo con una supersolución o una subsolución de uno de los operadores. El procedimiento descrito en este capítulo genera dos sucesiones monótonas de funciones las cuales convergen a una solución del problema de las dos membranas.

Cuando es espacio ambiente es un árbol $\mathbb{T}$, con un método iterativo análogo obtenemos una solución al problema de las dos membranas en un árbol. Además, gracias a este método, cuando los datos de frontera f y g están estrictamente separados, demostramos que el conjunto de coincidencia está alejado de la frontera y, en consecuencia, contiene solo un número finito de nodos. Finalmente, proporcionamos una interpretación del problema de las dos membranas en un árbol desde la teoría de juegos.

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Chapter 1

Introduction and Main Results

The aim of this PhD dissertation is to encompass the five papers that the author has carried out over the past four years. The framework of these works is Partial Differential Equations and their connections with Probability, Geometry and Game theory. Specifically, in this thesis, we present the works compiled in:

- [48], M.M. González, I. Gonzálvez and F. Quirós. "Traveling wave behavior for Fisher-KPP equations on the hyperbolic space". Coming soon on arXiv.
- [49], I. Gonzálvez, A. Miranda and J.D Rossi. "Monotone iterations of two obstacle problem with different operators". J. Elliptic Parabol Equations. 10.1 2024, pp 679-709.
- [50], I. Gonzálvez, A. Miranda and J.D Rossi. "The two membranes problem in a regular tree". Proceedings of the Royal Society of Edinburgh: Section A Mathematics. 2025, pp 1-30.
- [51], I. Gonzálvez, A. Miranda, J.D Rossi and, J. Ruiz-Cases. "Finding the convex hull of a set using the flow by minimal curvature with an obstacle. A game theoretical approach". 2025. Accepted in Communications in Analysis and Geometry.
- [52], I. Gonzálvez, F. Quirós, F. Soria and Z. Vondraček. "On the nonlocal heat equation for certain Lévy operators and the uniqueness of positive solutions". 2025. arXiv:2504.04246

1.0.1 Partial Differential Equations: a transversal field

A partial differential equation (PDE) is an equation involving an unknown function and its partial derivaties of different orders. The origin of PDEs is deeply related with the observation of nature. When studying physical and biological phenomena, one finds that there are intrinsic relationships between quantities and their variations over time and space. For example, PDEs can describe how heat propagates, how a beneficial mutation propagates in

a population, how a soap bubble evolves in time or how an elastic membrane is deformed in the presence of a rigid obstacle.

In the case of modelling the heat propagation, in 1822, Fourier proposed using the following equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \tag{1.0.1}$$

The above, called the *heat equation*, describes the evolution in time of u, the density of the temperature. That equation tell us that u varies in time at the same rate that the sum of its second spatial derivaties.

The equation (1.0.1) is also known as the *diffusion equation*. Roughly speaking, diffusion is a phenomenon in which a quantity or substance (such as a population, heat, water, money...) moves from a region of high concentration to one of low concentration, aiming to achieve a more uniform distribution. Then, the diffusion equation is a mathematical expression that modeled how processes tend to move toward greater disorder over time. Moreover, in 1905, Einstein discovered the relationship between that equation and the random walks (Brownian motions).

On the other hand, in 1937, simultaneously but independently, Fisher and the group formed by Kolmogorov, Petrovskii, and Piskunov, (KPP), modeled the propagation of an advantageous gene in a population, [41, 64]. Let $0 \leq u(x,t) \leq 1$ be the proportion of the population that, at time t and at point x, possess the beneficial mutation. Then, considering only the natural selecction, that is, the reaction, u satisfies the ordinary partial equation

$$u_t = u(1-u) \quad \text{in } \mathbb{R}^d \times (0, \infty)$$

introducted by Verhulst in 1845 that only involves a time derivate. However, Fisher and the KPP group contemplated the possibility that the offspring of an invidual located at point x need not remain in the same place. It could be established in an environment around that point x. This introduced the phenomenon of diffusion. As a result, they established that u is governed by the reaction-diffusion PDE

$$u_t - \Delta u = u(1-u) \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1.0.2}$$

Regarding soap bubble, one can observe that it evolves in such a way to decrease its area most efficiently due to surface tension. That is, it evolves by mean curvature. If u is a function whose zero level set represents the soap bubble, then u satisfies

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1.0.3}$$

The above PDE has been introduced in terms of viscosity solutions in 1991 by Evans and Spruck, [40].

To finish this exposition of a sample of classical PDEs, we introduce a problem asociated to an elliptic equation. Given a bounded domain $\Omega \subset \mathbb{R}^d$, the obstacle problem

$$\begin{cases} \min\{-\Delta u, u - \psi\} = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.0.4}$$

describes how a membrane u, which is balanced and fastened to a fixed boundary f, behaves when it encounters a rigid and voluminous object: the obstacle ψ. The obstacle constrains the membrane, forcing it to conform to its shape within the obstacle set, i.e., the region where the membrane and the obstacle are in contact. In the non-contact set, the membrane remains above the obstacle and in equilibrium.

This problem falls within the category of free boundary problems because the interface between the contact and non-contact sets (the free boundary) is a priori unknown. The study of the regularity of the free boundary has attracted the interest of many mathematicians, such as Lewy, Stampacchia, Nirenberg and Caffarelli, among others. Moreover, as Friedman summarized in [42], the obtacle problem has aplications to may other contexts such as financial math and optimal control.

The aforementioned equations are classical examples of modeling natural phenomena using PDEs. Once a specific equation is chosen for study, one must address (at least one of) the following fundamental questions:

- Existence and uniqueness: Given a PDE, the first question to answer is whether there exists a function that satisfies it (a solution). The second is know if this solution is unique under the given conditions. If existence or uniqueness fails, the problem is called *ill-posed*.

  An example of such an ill-posed problem is the Cauchy problem for the heat equation (1.0.1). When the initial trace is identically zero, there exist infinitely many solutions, such the trivial solution (the null function) and any multiple of Tychonov's counterexample [109].

  To address this issue and obtain uniqueness, one have to impose additional conditions. For example, conditions over the growth of the solution, [109], or to impose the solution to be nonnegative, [5].

  The question of existence and uniqueness for a generalization of the heat equation is adressed in Chapter 2.

- Qualitative results: Once that it has been proved that the problem asociated to some PDE admits an unique solution (it is a *well-posed* problem), the next goal is to describe the solution. The two most common types of qualitative results are those related, on the one hand, to regularity properties, and on the other hand, to the asymptotic behavior of solutions.

  - Regularity theory is concerned with the study of the integrability and differentiability of the solution.
  - Asymptotic behavior refers to whether the solution exhibits some decay rate at infinity in space or time or if it converges to a specific profile.

    In Chapter 3 we study the long-time bahaviour of the solution of the Cauchy problem associated to a reaction-diffusion equation in the hyperbolic space for a family of appropiate initial data. We obtain that solution converges to a

specific trawelingwave in a moving coordinates. In Chapter 3 we also study the asymptotic behavior of the solution of the parabolic obstacle problem for a fully nonlinear and elliptic operator that models the evolution by minimal curvature flow. We obtain that the zero superlevel set of such solution converges to the convex hull of the support of the obstacle. c

- Construction of solutions: Another aim in the study of PDEs is to obtain an explicit solution of a problem associated with a given equation. Sometimes, it is possible to construct an algorithmic method to obtain a solution (as we do in Chapter 5), while in other cases, a game-theoretical approach can be used to derive the solution (as shown in Chapter 4 and in the last subsection of Chapter 5). In linear problems where existence and uniqueness hold, it is often possible to obtain a representation formula (as it is done in Chapter 2).

  Moreover, being able to give a formula or a method to obtain a solution often infers qualitative results to that solution.

Notably, the theory of PDEs is deeply connected with other areas of mathematics, from which it draws tools and techniques to explore these fundamental questions. Conversely, PDEs provide these fields with new perspectives and results that contribute to their advancement.

In particular, the PDEs studied in this dissertation are closely related to the fields of Probability, Geometry and Game theory:

Probability. One of the strongest conections between Probability and PDEs is the *Brownian motion* and the heat equation, (1.0.1).

The Browian motion describes how particles move randomly in a fluid. Mathematically, the Brownian motion is a stochastic process. To be precise, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probabilty space. The Brownian motion $X := \big(X_t\big)_{t\geq 0}$ is a familly of radon variables defined in that probability space indexed by time t and taking values in $\mathbb{R}^d$ that satisfies the following: X has independent increments and it is almost surely continuous, $X_0 = 0$ with probability 1 and $X_t - X_s$ is normally distributed with zero mean and coviariance matrix $2(t-s)I$ for all $t > s \geq 0$. Moreover, the Brownian motion X has the following characteristic function

$$\mathbb{E}[e^{i\xi X_t}] = e^{-t|\xi|^2}, \quad \text{for all } \xi \in \mathbb{R}^d,\, t \geq 0,$$

and an associated heat kernel $P : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ given by

$$P(x,t) = \mathcal{F}\big(e^{-t|\cdot|^2}\big)(x), \quad x \in \mathbb{R}^d, \;\; t > 0,$$

where $\mathcal{F}$ denotes the Fourier transform.

The conection between the Brownian motion X and the heat equation is obtained through the infinitesimal generator of X: the Laplacian Δ. It is obtained as

$$\Delta f = \lim_{t\to 0} \frac{P(\cdot,t) * f - f}{t} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

As a consequence of the above, $u = P(\cdot,t) * f$ is a solution to the heat equation (1.0.1) for all smooth function f.

On the other hand, Lévy processes generalize the Brownian motion by allowing for discontinuities in their trajectories. Unlike the Brownian motion, which exhibits continuous paths, Lévy processes can make finite jumps.

The main characteristic examples of this type of processes are the symmetric α-stable processes $Y^\alpha = \big(Y_t^\alpha\big)_{t\geq 0}$ with $\alpha \in (0,2)$. For a fixed α, the characteristic function and the heat kernel P^α associated to Y^α are given by

$$\mathbb{E}[e^{i\xi Y_t^\alpha}] = e^{-t|\xi|^\alpha}, \quad P^\alpha(x,t) = \mathcal{F}\big(e^{-t|\cdot|^\alpha}\big)(x), \quad \xi, x \in \mathbb{R}^d, \ \ t>0.$$

Then, the associated infinitesimal generator of Y^α, denoted by $-(-\Delta)^{\alpha/2}$ is obtained as

$$-(-\Delta)^{\alpha/2} f := \lim_{t\to 0} \frac{P^\alpha(\cdot,t) * f - f}{t} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \tag{1.0.5}$$

The operator $(-\Delta)^{\alpha/2}$ is called the $\alpha/2$-fractional Laplacian. Its notation and name arise from the behavior of derivatives under the Fourier transform and the fact that

$$\mathcal{F}\big(-\Delta f\big)(\xi) = |\xi|^2\big(\mathcal{F}f\big)(\xi), \quad \mathcal{F}\big((-\Delta)^{\alpha/2} f\big)(\xi) = |\xi|^\alpha\big(\mathcal{F}f\big)(\xi), \quad \xi \in \mathbb{R}^d.$$

One of the equivalent characterizations of the $\alpha/2$-fractional Laplacian of a smooth function f in $\mathbb{R}^d$ is the following, expressed in terms of its second differences:

$$(-\Delta)^{\alpha/2} f(x) = C_{d,\alpha} \iint\!\!\!\!\!\!\diagup_{\mathbb{R}^d} \big(f(x) - \frac{1}{2}\big(f(x-y) - f(x+y)\big)\big) \frac{\mathrm{d}y}{|y|^{d+\alpha}} \tag{1.0.6}$$

for $x \in \mathbb{R}^d$ where $C_{d,\alpha}$ is a positive constant depended on d and α.

The above formula highlights the nonlocal nature of the fractional Laplacian, which araises from the jumps of its associated Lévy process. The expression (1.0.6) shows that the behavior of the function f far from the point x plays a significant role, in contrast with the clasical Laplacian.

Moreover, thanks to (1.0.5), for a f smooth, $u = P^\alpha(\cdot,t) * f$ solves the *fractional heat equation*

$$u_t + (-\Delta)^{\alpha/2} u = 0, \quad \mathbb{R}^d \times (0,\infty). \tag{1.0.7}$$

Due to the nonlocal behavior of the fractional Laplacian, the fractional heat equation and their analogous for others Lévy processes serves as toy models for phenomena exhibiting anomalous diffusion or long-range interactions. For instance, two examples of discontinuous random behaviors in nature are mobility of charges in amorphous semiconductors, [22], and the foraging patterns of freeranging spider monkeys in the forest of Yucatan, [96].

In Chapter 2, we take advantage of the characterization of the heat kernel in terms of a Lévy process to study the existence, uniqueness and initial trace of nonnegative solutions for generalized nonlocal heat equations.

Geometry. The relationship between Geometry and PDEs dates back to the earliest developments in this second field. The study of problems with important geometrical character,

such as the isoperimetric problem or the tautochrone curve, are some of the precursors to the development of PDEs through the framework of the Calculus of Variations.

In turn, the birth of the theory of PDEs provided Geometry with new perspectives and tools. For example, Euler, using Differential Geometry (very related with PDEs), derivated the first analytical geodesic equation and Lagrage obtained the first differential equation to describe a minimal surface in terms of Euler-Lagrange equations.

At the end of the 20^{th} century, the relationship between Geometry and PDEs became even more evident with the emergence of Geometric Analysis, pioneered by the Abel Prize laureate Karen Uhlenbeck. This sub-discipline uses both geometric methods in the study of PDEs and applies PDEs theory to Geometry.

On the other hand, the also Abel Prize honered Caffarelli applied the theory of PDEs to characterize singularities that appear in free-boundary problems. Among many other contributions, he studied from an analytic perspective the geometry of the free boundary in problems such as the obtacle problem (see (1.0.4)) and Stefan problem (which describes the evolution over time of interface of submerged in water).

Futhermore, apart from these remarkable lines of research opened by theses two Abel Prize winners, there is a recent interest in revisiting the classical PDE problem by changing the ambient Euclidean space to manifolds, [14, 35, 75, 76, 94, 95, 113]. These problems, beyond their intrinsic mathematical interest, helps us to understand how the solutions of PDEs depend on the surrounding environment. This dependence is often not observed when the PDE is posed on the Euclidean space.

For example, let us consider the analogous equation of (1.0.1) in the hyperbolic space $\mathbb{H}^d$

$$u_t - \Delta_{\mathbb{H}^d} u = 0 \quad \text{in } \mathbb{H}^d \times (0,\infty), \tag{1.0.8}$$

where $\Delta_{\mathbb{H}^d}$ denotes Laplace-Beltrami operator in that space. As Vázquez remaks in [113], the fundamental solution of (1.0.8) decays in time exponentially faster than its counterpart in Euclidean space. This is due to the *drift from infinity*, the repellent behavior of the ideal boundary of the hyperbolic space. As a consequence of this phenomenon, as [75] shows, the propagation of a beneficial gene in the hyperbolic space modeled by the equation

$$u_t - \Delta_{\mathbb{H}^d} u = u(1-u) \quad \text{in } \mathbb{H}^d \times (0,\infty), \tag{1.0.9}$$

is not always guaranteed, unlike in the Euclidean setting, where the solution to (1.0.2) has always propagation. This illustrates how the behavior at infinity in a manifold is another important factor to consider in the study of reaction-diffusion equations.

In Chapter 3, we present new results that continue exporing the implications of the geometry of the hyperbolic space for generalizations of the reaction-diffusion equation (1.0.9). In Chapter 5 we go back to the obstacle problem to construct a solution of the two membranes problem via iterations.

Game theory. To show the deep relation between Game theory and PDEs, let us introduce a game to ilustrate it. In [63], the authors provide an interpretation of the motion by

mean curvature (the evolution of a soap bubble) via a deterministic two-person, zero-sum game.

A version of the the rules of that game are the following:

i) The board: let Ω_0 be a bounded domain of $\mathbb{R}^d$. It is the support a given function u_0, the (initial) cost function.

ii) The players: there are two players, Paul and Carol. Paul aims to earn as much money as possible. That is, his goal is to maxime the u_0. On the other hand, Carol must pay Paul whatever he earns. Consequently, Carol's goal is to minimize Paul's earnings. Therefore, to achieve this, she will attempt to either push Paul out of the board Ω_0, where u_0 takes negative values, or, if that is not possible, to guide Paul towards the location within Ω_0, where u_0 attains its minimum.

iii) Allowed moves: let $x = x_0$ be the initial position, $t = t_0 > 0$ the initial time and fix $\varepsilon > 0$. At the $i-$th round, Paul chooses of $d-1$ othogonall unitary vectors $\{v_i^1, ..., v_i^{d-1}\}$ and Carol selects a collection of sign $b_i^j = \pm 1$ for $j = 1, ..., d-1$ after Paul's choice. Then, the next position of the game is given by

$$x_i = x_{i-1} + \frac{\varepsilon}{d-1} \sum_{j=1}^{d-1} b_i^j v_i^j,$$

and time decreases acording to $t_i = t_{i-1} - \frac{1}{2}\varepsilon^2$. After $n = \left\lceil \frac{2t_0}{\varepsilon^2} \right\rceil$ rounds, the game ends and Carol pays the terminal cost $u_0(x_n)$ to Paul.

For a visual representation of the game, see Figure 1.1.

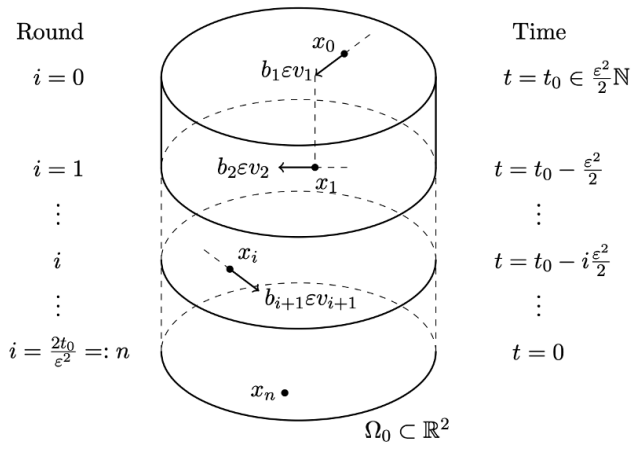


Figure 1.1: Game associated to motion by mean curvature in $\mathbb{R}^2$.

Heuristically, the *value function* of the game, that is, the final cost optimized by both players is given by

$$u^\varepsilon(x,t) = \text{ maximal value of } u_0(x_n) \text{ starting from } x \in \mathbb{R}^d \text{ at time } t > 0.$$

This satisfies an equation that, by analogy with harmonic analysis, plays the role of a mean value formula. This equation, known as the *dynamic programming principle* (DPP), is given by

$$u^\varepsilon(x,t) = \sup_{\left\{\{v^1,\dots,v^{d-1}\}\subset\mathbb{R}^d:\,\langle v^i,v^j\rangle=\delta_{ij}\right.} \min_{\left\{\{b^1,\dots,b^{d-1}\}\subset\{\pm 1\}\right.} u^\varepsilon\Big(x+\tfrac{\varepsilon}{d-1}\sum_{j=1}^{d-1} b^j v^j, t-\tfrac{1}{2}\varepsilon^2\Big).$$

The key point is that, by applying a Taylor expansion and the theory of viscosity solutions, one can establish a connection between Game Theory and PDEs. Specifically, as $\varepsilon \to 0$, the family of functions $\{u^\varepsilon\}_{\varepsilon>0}$ converges to the viscosity solution of (1.0.3).

Once the bridge between both fields is established, we can combining results from both. For example, we obtain that the zero level set of the limiting function evolves by mean curvature. This, in turn, implies that if ε is sufficiently small and the game is played for a long enough time, Carol wins. The reason is that the solution to (1.0.3) has an extinction time. Moreover, thanks to the convergence, we can implement strategies for both players that provide an intuitive understanding of how the soap bubble (and more complex forms) evolves over time.

In Chapter 4, in the spirit of the above discussion, we will define a deterministic two-person, zero-sum game to find the convex envelope of a set via the minimal curvature flow. Moreover, in Chapter 5 we provide an interpretation of the two membranes problem via theoretical and probabilistic game.

1.0.2 Main results

A Widder's type theory for the nonlocal heat equation for certain Lévy operators

In Chapter 2, we present the results in [52] by Fernando Quirós, Fernando Soria, Zoran Vondraček (UZ) and the author of this dissertation memoir under the name

On the nonlocal heat equation for certain Lévy operators and the uniqueness of positive solutions.

As we explained before in the Introduction of this thesis, the Cauchy problem for the classical heat equation (1.0.1) is not well-posed in general. To establish uniqueness, Widder, in 1944, [116], proposed a groundbreaking approach. He suggested studying this problem within the class of *nonnegative* solutions, based on the fact that temperatures always remain above absolute zero. In 1971, Aronson, [5], completed the theory of uniqueness for nonnegative solutions in the classical (local) setting.

In this chapter, motivated by the papers [12, 19], where the authors studied the uniqueness of nonnegative solutions to the fractional heat equation (1.0.7), we develop a Widder-type theory for nonlocal heat equations of the form

$$\partial_t u + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{1.0.10}$$

where $\mathcal{L}$ is a Lévy-type operator as general as possible, $d \geq 1$ and $T \in (0, +\infty)$. Thus, we consider *nonnegative* solutions and look for conditions on the operator that ensure: (i) uniqueness of nonnegative classical and very weak solutions with a given initial trace; (ii) the existence of an initial trace, belonging to certain admissibility class; and (iii) the existence of a solution, given by a representation formula, for any admissible initial trace.

Such results are obtained first for purely nonlocal Lévy operators

$$\mathcal{L}_K u(x) = \iint_{\mathbb{R}^d} \Big(u(x) - \frac{u(x+y) + u(x-y)}{2} \Big) K(y)\, \mathrm{d}y, \tag{1.0.11}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a positive symmetric Lévy kernel, comparable to radial functions with mixed polynomial growth. Latter we extended to more general operators, including anisotropic ones such as $\mathcal{L}_\nu = \sum_{j=1}^{\ell} \mathcal{L}_{K_j}$, $\ell \in \{1, \dots, d\}$, where each $\mathcal{L}_{K_j}$ is of the form (1.0.11) and acts only on the subspace $\mathbb{R}^{d_j}$, with $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$ and operators that have both a local and a nonlocal part, namely operators of the form $\mathcal{L}_{\Delta,K} := -\Delta + \mathcal{L}_K$, with $\mathcal{L}_K$ as in (1.0.11).

The scheme we follow to obtain positive answerto the three questions is as follows. We begin by establishing fundamental properties of the heat kernel associated with $\mathcal{L}$ which allow us to prove our main results. In particular, using the results of [10], we obtain that the heat kernel at two different times is comparable and that the Lévy kernel K and the heat kernel have comparable tails.

Next, we split our strategy into two parts. First, we study very weak solutions, addressing uniqueness, the existence of an initial trace, and existence of solutions. Our goal is to obtain a representation formula for very weak solutions that are either nonnegative or satisfy a suitable integral condition, expressed in terms of the heat kernel.

Second, we investigate classical solutions, focusing on uniqueness for nonnegative classical solutions to (1.0.10). The key idea in this part is to prove that any nonnegative classical solution is also a (nonnegative) very weak solution. Since we have already established a representation formula for nonnegative very weak solutions, this directly yields our main result: a representation formula for nonnegative classical solutions to (1.0.10). We emphasize that, unlike the local case, it is not straightforeard that a classical solution is also a very weak solution. This complexity arises due to the non local nature of the Lévy kernel.

We also prove existence of classical solution when $\mathcal{L}_K$, $\mathcal{L}_\nu$ and $\mathcal{L}_{\Delta,K}$ are related to the fractional Laplacian, using the theory of Bernstein functions.

The KPP problem on the hyperbolic space

Chapter 3 is devoted to presenting the results in [48],

Traveling wave behavior for Fisher-KPP equations on the hyperbolic space,

a joint work with María del Mar González (UAM) and Fernando Quirós.

We study the sharp long-time behavior of solution to KPP problem

$$\begin{cases} u_t = \Delta_{\mathbb{H}^d} u + f(u), & \mathbb{H}^d \times (0, \infty), \\ u(\cdot, 0) = u_0, & \mathbb{H}^d \end{cases}$$

in the hyperbolic space $\mathbb{H}^d$ with $d \geq 2$. Here, $\Delta_{\mathbb{H}^d}$ is the Laplace-Beltrami operator in $\mathbb{H}^d$ and f is a Fisher-KPP function, for example $f(s) := s(1-s)$, $s \in [0,1]$.

Our motivation is to understand how hyperbolic geometry affects the evolution of the above problem and their differences and similarities with the analogous problem in the Euclidean setting.

On the one hand, one of the differences lies in the question of *propagation vs. extinction*. In [7], the authors prove that for, an admissible initial datum, the solution of the Euclidean KPP problem always reaches the maximum saturation level in compact sets, i.e, propagation always occurs. While, in the pioneering article [75], the authors show that, in the hyperbolic space, there is a competition between the source f and λ_1, the infimum of the spectrum of $-\Delta_{\mathbb{H}^d}$; (spectral gap). They establish that if $f'(0) > \lambda_1$ propagation occurs and if $f'(0) < \lambda_1$ the solution vanishes (*extinction*). In chapter 4 we adress the critical case where $f'(0) = \lambda_1$ proving that extinction ocurrs for a family of initial data.

On the other hand, the major similarity is that, whenever propagation occurs, the solution to the KPP problem in both spaces converges to a trawelingwave of minimal speed Φ_{c_0} in a moving coordinate frame. For the Euclidean case, we refere to [56, 84, 100, 101]. However, in this work, we show that the moving frame in hyperbolic space exhibits significant differences compared to its Euclidean counterpart.

More precisely, we prove that the type of invariances of the initial datum determine the choice of the moving coordinate system in which the solution converges to Φ_{c_0}. There are three types of invariance, which are closely related to the three types of isometries of $\mathbb{H}^d$: elliptic, parabolic, and hyperbolic.

In addition, we prove that the time-dependent terms in the moving coordinates are also dictated by the geometry of hyperbolic space. The drift from infinity in $\mathbb{H}^d$ directly affects the linear term of the moving frame, which depends only on the dimension of $\mathbb{H}^d$. We also need to introduce a logarithmic correction, which remains the same across all three types of invariance.

Both aspects represent a significant difference from the analogous KPP problem in Euclidean space. In the Euclidean setting, there are only two types of invariance that determine the choice of the moving frame: rotations and translations (corresponding to the two types of isometries: elliptic and parabolic). Furthermore, the linear term of the moving frame in the Euclidean space does not depend on the dimension, whereas the logarithmic correction does depend on the type of invariance of the initial datum.

A game theoretical approach to the convex hull of a set via minimal curvature flow

The results we present in Chapter 4 are published in [51] by Alfredo Miranda (UBA), Julio D. Rossi (UBA), Jorge Ruiz-Cases (UAM) and the author of this thesis under the name

Finding the convex hull of a set using the flow by minimal curvature with an obstacle. A game theoretical approach.

The aim of this chapter is to study the asymptotic behavior of the solution to the parabolic obstacle problem for a degenerate fully nonlinear equation that models the motion by minimal curvature of a hypersurface. We prove (under certain conditions on the initial condition and the obstacle) that the positivity set of this solution converges to the convex hull of the set where the obstacle is positive as t goes to infinity.

More precisely, to describe the geometric evolution to a family of $\mathbb{R}^d$-hypersufaces $\{\Omega_t\}_{t\geq 0}$ to the convex hull of a bounded set $K \subset \mathbb{R}^d$, $co(K)$, we use a level set approach. Assume that there is a function $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ which its zero superlevel sets are Ω_t for each $t \geq 0$ satifying that

$$\Omega_t \to co(K) \quad \text{as } t \to \infty. \tag{1.0.12}$$

Then, we prove that u can be modell as the solution to

$$\begin{cases} \max\Big\{ -\partial_t u(x,t) - \mathscr{L}(u(x,t)), \psi(x) - u(x,t)\Big\} = 0, & (x,t) \in \mathbb{R}^d \times \{t>0\}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.0.13}$$

where $\mathscr{L}$ is the elliptic and degenerate operator given by

$$\mathscr{L}(u(x)) := \inf_{\substack{|v|=1 \\ v \perp \nabla f(x)}} \Big(-\langle D^2 u(x) v, v\rangle \Big).$$

that encode the envolution by minimal curvature and ψ is assume to be a smooth function which zero superlevel set is K.

The idea of considering the minimal curvature flow is in that the convex hull of a given set K is characterized by the property that its boundary can be touched by supporting hyperplanes. Note that a point that belongs to $co(K) \setminus K$ also belongs to a one of these supporting hyperplanes. Then, in that point, the main curvature of the boundary has to be zero, while all the other curvatures need to be nonnegative in order to ensure the convexity of $co(K)$.

Therefore, we have that $co(K)$ is a solution to the minimal curvature equal to zero at boundary points that are not in $\overline{K}$ (where the surface does not touch the obstacle). Since our target ($co(K)$) is a stationary solution to our evolution equation it seems natural to expect that we find the convex hull in the long time behavior of the solutions to our parabolic problem (1.0.13).

In this chapter, our first step is to prove existence and uniqueness of solutions of the obstacle problem (1.0.13) via a comparison principle and Perron's method. We will rely on previous theory developed for viscosity solutions to geometric flows, see [36, 45, 78, 79].

Once that we have obtain existence and uniqueness, we turn our attention to their asymptotic behavior. To obtain our main result (1.0.12) we use a deterministic two-person zero-sum game inspired by previous results by [63] and [82].

In this game, the players are Paul who wants to maxize the expect outcome and Carol who aims to minizime it. For a fixed $\varepsilon > 0$, in each round of the game the players perform a series of chooices that allow them to move a ε-step to the next position. The main goal of Paul is to achieve the obstacle K because there he automatically wins. If Carol does not

allow it, the second Paul's objetive is to remain in the $co(K)$. While, the Carol's first aim is to expel Paul to Ω_0 since there the terminal cost is negative. The final cost opimized by both players starting at x and time t is given by the value of the ε- game $u^\varepsilon(x,t)$.

The key points to obtain (1.0.12) are the following: the value functions u^ε converge to the solution u of the parabolic obstacle problem (1.0.13) as $\varepsilon \to 0$. On the other hand, introducing the concept of ε-convexity and considering strategies that enable the players to achieve their objectives, we prove that the zero superlevel set of u^ε, for ε small enough, approximates (under certain hypotheses on K and u_0) the convex hull of K as $t \to \infty$.

The two membranes problem

Chapter 5 is devoted to presenting the results published in:

- [49], *Monotone iterations of two obstacle problem with different operators*,
- [50], *The two membranes problem in a regular tree*,

two joint works with Alfredo Miranda (UBA) and Julio D. Rossi (UBA).

Let Ω be a bounded Lipschitz domain and $\mathbb{T}$ be a regular tree. Consider L_1 and L_2 be two differential operators defined in functions $u : \Omega \to \mathbb{R}$ or two operators L_1 and L_2 given by mean value formulas acting on functions $u : \mathbb{T} \mapsto \mathbb{R}$. Similary, let (f, g) be a pair of ordered functions in $\partial\Omega$ or in $\partial\mathbb{T}$.

A solution to the two membranes problem in Ω or in $\mathbb{T}$ with operators L_1 and L_2 and boundary datum (f, g) is a pair of funtions (u, v) satisfying

$$\begin{cases} \min\Big\{L_1(u)(x), (u-v)(x)\Big\} = 0, & x \in \Omega \text{ or } x \in \mathbb{T}, \\ \max\Big\{L_2(v)(x), (v-u)(x)\Big\} = 0, & x \in \Omega \text{ or } x \in \mathbb{T}, \\ u(x) = f(x), \qquad v(x) = g(x), & x \in \partial\Omega \text{ or } x \in \partial\mathbb{T}. \end{cases}$$

That is, u is the solution of the obstacle problem from below with operator L_1, obstacle v and Dirichlet condition g; while v is the solution of the obstacle problem from above with operator L_2, obstacle u and boundary datum f.

The two membranes problem for the Laplacian was first considered in [111] using variational arguments. Once existence of a solution (in an appropriate sense) is obtained, a lot of interesting questions arise, such us uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [26, 27, 104], the dissertation [112] and references therein. Futhermore, a more general version of this problem involving more than two membranes was considered by several authors (see for example [9, 31, 34]).

In this Chapter 5, first, we establish the existence of solutions of the two membranes problem in a tree under adequate conditions on the boundary data and the involved source terms. Observe that the existence of a solution to the two membranes problem in a domain is guaranteed by a classical application of Perron's method, provided that L_1 and L_2 are general elliptic operators.

Second, we analyze iterations of the obstacle problem in both ambient spaces. We solve iteratively the obstacle problem from above or below for two different operators L_1 and L_2 with obstacles given by the previous functions in the iterative process.

Moreover, when the ambient space is Ω, we implement this iterative method in two different frameworks: the variational setting, where L_1 and L_2 admit a divergence form, and via viscosity solutions, where we consider more general elliptic operators. We initiate the iterations with a supersolution or a subsolution of one of the operators. The procedure described in this chapter generates two monotone sequences of functions that converge to a solution to the two membranes problem for the two different operators.

Similarly, we apply an analogous iterative method to obtain a solution to the two membranes problem in a tree. Futhermore, thanks to this method, when the boundary data f and g are strictly separated, we prove that the coincidence set is separated from the boundary and, consequently, it contains only a finite number of nodes. Finally, we provide a game theoretical interpretation of the two membranes problem in a tree.

Chapter 2

A Widder's type theory for the nonlocal heat equation for certain Lévy operators

In this chapter we address the results presented in [52] named *On the nonlocal heat equation for certain Lévy operators and the uniqueness of positive solutions* by Fernando Quirós, Fernando Soria, Zoran Vondraček (UZ) and the author of this dissertation memoir.

2.1 Introduction

Goal. Our aim is to develop a Widder-type theory for nonlocal heat equations of the form

$$\partial_t u + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{NLHE}$$

where $\mathcal{L}$ is a Lévy-type operator as general as possible and $T \in (0,+\infty)$. Thus, we consider *nonnegative* solutions to the problem and look for conditions on $\mathcal{L}$ that guarantee: (i) uniqueness of nonnegative classical and very weak solutions with a given initial trace; (ii) the existence of an initial trace, belonging to certain admissibility class; and (iii) the existence of a solution, given by a representation formula, for any admissible initial trace.

We start by considering the case of purely nonlocal Lévy operators of the form

$$\mathcal{L}_K u(x) = \fint_{\mathbb{R}^d} \Big(u(x) - \frac{u(x+y)+u(x-y)}{2} \Big) K(y)\,\mathrm{d}y, \tag{2.1.1}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a positive symmetric Lévy kernel, that is, a measurable function such that

$$K(x) > 0, \quad K(x) = K(-x) \quad \text{for a.e. } x \in \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} (1 \wedge |y|^2) K(y)\,\mathrm{d}y < \infty, \tag{A0}$$

with $a \wedge b := \min\{a,b\}$. We will give additional growth conditions on K allowing us to answer in the positive to all three questions: uniqueness, existence of a trace, and existence of a solution with a given (admissible) initial trace.

A careful analysis of the proofs will allow us to extend the results to some more general purely nonlocal operators of the form

$$\mathcal{L}_\nu u(x) = \fint_{\mathbb{R}^d} \left(u(x) - \frac{u(x+y)+u(x-y)}{2} \right) \mathrm{d}\nu(y), \tag{2.1.2}$$

where ν is a symmetric Lévy measure, that is, a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\nu(B) = \nu(-B) \quad \text{for all } B \in \mathbb{B}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} (1 \wedge |y|^2)\, \mathrm{d}\nu(y) < \infty, \tag{A_ν}$$

with ν not necessarily absolutely continuous with respect to the Lebesgue measure. Here $\mathbb{B}(\mathbb{R}^d)$ denotes the σ-algebra of Borel sets in $\mathbb{R}^d$. Our results cover, for instance, the family of anisotropic operators $\mathcal{L}_\nu = \sum_{j=1}^{\ell} \mathcal{L}_{K_j}$, $\ell \in \{1, \dots, d\}$, where each $\mathcal{L}_{K_j}$ is of the form (2.1.1) and acts only on the subspace $\mathbb{R}^{d_j}$, with $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell}$.

We will finally consider Lévy operators having both a local and a nonlocal part, namely operators of the form $\mathcal{L}_{\Delta,K} := -\Delta + \mathcal{L}_K$, with $\mathcal{L}_K$ as in (2.1.1). We will also obtain positive answers to the three questions if the kernel K satisfies the same growth conditions as in the purely nonlocal case.

Widder's theory for the local heat equation. The Cauchy problem for the local heat equation

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^d \times (0, T) \tag{2.1.3}$$

is not well-posed in general: it has infinitely many classical solutions with zero initial datum, for instance, any multiple of Tychonov's counterexample [109]. In order to get uniqueness, one has to restrict the class of functions within which one looks for solutions. One possibility, that may not seem natural from a physical point of view, is to consider only solutions that do not grow too rapidly as $|x| \to \infty$; see [109]. A second common option is to require some integrability in space for almost every time, regarding solutions as curves in some L^p space, as is done in the Theory of Semigroups. This approach matches well with the conservation of energy when $p = 1$.

D. V. Widder considered a third approach: since the Third Principle of Thermodynamics asserts that temperatures remain always above zero Kelvin degrees, it is sensible to restrict the analysis to *nonnegative* classical solutions. In his celebrated paper [116] he proved two important results in this framework, both in the one-dimensional case $d = 1$. The first one is a uniqueness result for nonnegative classical solutions with trivial initial data. Unfortunately, this does not give uniqueness for other initial data, since the difference of nonnegative solutions is not necessarily nonnegative. The second one is a representation theorem: if u is a nonnegative classical solution of (2.1.3), there exists a nonnegative Borel measure μ such that $u(\cdot, t) = G_t * \mu$ for all $t > 0$, where

$$G_t(x) = G(x,t) := (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d,\ t > 0, \tag{2.1.4}$$

is the heat kernel (or fundamental solution) of the heat equation. Besides, μ cannot have too much mass near infinity,

$$\iint\!\!\!\!\!\!\diagup_{\mathbb{R}^d} e^{-|y|^2/(4t)}\,\mathrm{d}\mu(y) < \infty, \quad t \in (0,T). \tag{2.1.5}$$

Some years later, Krzyżański extended Widder's representation theorem to higher dimensions [65]; see also [4], where the uniqueness of the representing measure μ is proved. Note that (2.1.5) implies that μ is locally finite, hence a Radon measure, since $\mathbb{R}^d$ is a locally-compact Hausdorff space.

The converse of the representation theorem is also true: if μ is a Radon measure satisfying (2.1.5), then $u(\cdot,t) = G_t * \mu$, $t > 0$, is a solution to (2.1.3); see [116] for $d = 1$ and [5] for higher dimensions.

Widder's representation theorem did not settle the question of uniqueness for the Cauchy problem. For this we have to wait until Aronson's paper [5], where he proves that if μ is the measure representing a nonnegative solution u to the heat equation (2.1.3), then

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} \psi(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\mu \quad \text{for all } \psi \in C_{\mathrm{c}}(\mathbb{R}^d); \tag{2.1.6}$$

that is, $u(\cdot,t) \to \mu$ as $t \to 0^+$ as Radon measures in the weak* topology. This gives a positive answer to the uniqueness question for nonnegative solutions of the Cauchy problem having a Radon measure as initial trace, if we understand that u has μ as initial datum when (2.1.6) holds. In this chapter the initial data is meant to be taken in this sense.

Remark 2.1.1. Let $d = 1$. If μ is the measure representing a nonnegative solution u to (2.1.3), then

$$\lim_{t\to 0^+} \iint\!\!\!\!\!\!\diagup_a^b u(x,t)\,\mathrm{d}x = \mu(a,b) + \frac{1}{2}\mu(a) + \frac{1}{2}\mu(b) \quad \text{for all } a,b \in \mathbb{R} \text{ with } a < b;$$

see [117]. Hence, if we understand that u has μ as initial trace when

$$\lim_{t\to 0^+} \iint\!\!\!\!\!\!\diagup_a^b u(x,t)\,\mathrm{d}x = \mu(a,b) \quad \text{for all } a < b \text{ such that } \mu\{a\} = \mu\{b\} = 0,$$

we would have uniqueness for nonnegative solutions of the Cauchy problem; see [118]. A similar result in higher dimensions is given in [6].

Precedents for nonlocal operators. As already announced, our purpose is to extend Widder's theory to the nonlocal heat equation (NLHE) for the widest possible class of nonlocal operators $\mathcal{L}$. We have two direct nonlocal precedents that have been a source of inspiration for us, [12] and [19], both of them dealing with the special case in which $\mathcal{L}$ is the fractional Laplacian $(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$, an operator of the form (2.1.1) with $K_\alpha(y) = C_{d,\alpha}|y|^{-(d+\alpha)}$ for some normalization constant $C_{d,\alpha}$; see also [86, Chapter 12]. In contrast with the

local heat equation, the definition of very weak solution for this problem requires some integrability, at least $u \in L^1_{\rm loc}((0,T);L_{\alpha/2})$, where

$$L_{\alpha/2}(\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ measurable} : \fint_{\mathbb{R}^d} \frac{|u(x)|}{1+|x|^{d+\alpha}}\,\mathrm{d}x < \infty \right\}.$$

We will have an analogous situation when dealing with more general operators, with integrability spaces adapted to $\mathcal{L}$.

The first of these papers, [12], gives a representation formula in terms of the initial datum for nonnegative classical solutions to the equation that are continuous down to $t = 0$. This implies uniqueness. The strategy is to prove uniqueness for very weak solutions within $L^1_{\rm loc}([0,T);L_{\alpha/2})$, and then show that nonnegative classical solutions are very weak solutions in this class. This last step is not as straightforward as in the local case, due to the nonlocal character of the fractional Laplacian. This approach can be adapted easily to cope with more general initial data. The second paper, [19], develops a complete Widder-type theory, though only for nonnegative very weak solutions, including the existence of an initial trace and considering the possibility of having (admissible) nonnegative Radon measures as initial data. The existence of a trace had already been proved in the previous paper [20], where the authors analyze a nonlinear variant of the equation.

Let us remark that the notions of very weak solution used in [12] and [19] do not coincide. We will make use of both of them (adapted to general operators), and of their equivalence in certain cases.

As for Widder-type results for more general linear nonlocal operators, we are only aware of the work [85], where the authors prove uniqueness of very weak solutions to (NLHE) when $\mathcal{L}$ is an α-stable diffusion operators given by an anisotropic symmetric Lévy measure. Such operators have motivated us to develop the general theory in Section 2.4.

Operators of the form $\mathcal{L}_{\Delta,K}$ have attracted some attention in recent years; see, for instance, [33], [**Zoran2**] and [39]. However, we are not aware of any previous work dealing with Widder's theory for them.

Concepts of solution and initial trace. As we have just mentioned, there are in the literature different notions of solution to (NLHE). Hence, before going further, let us make precise what we will mean by a very weak solution and a classical solution to this equation.

Definition 2.1.2. A *very weak solution* of (NLHE) is a function $u \in L^1_{\rm loc}(\mathbb{R}^d \times (0,T))$ such that:

$$\fint_0^T \fint_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\theta(x,t)|\,\mathrm{d}x\,\mathrm{d}t < \infty \quad \text{for all } \theta \in C_c^\infty(\mathbb{R}^d \times (0,T)); \tag{2.1.7}$$

$$\fint_0^T \fint_{\mathbb{R}^d} u(x,t)(\partial_t\theta(x,t) - \mathcal{L}\theta(x,t))\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{for all } \theta \in C_c^\infty(\mathbb{R}^d \times (0,T)). \tag{2.1.8}$$

We will give, and use, an alternative definition of very weak solution in Section 2.3; see (2.3.3)–(2.3.4).

Definition 2.1.3. A *classical solution* of (NLHE) is a function $u \in C(\mathbb{R}^d \times (0,T))$ such that $\partial_t u$ and $\mathcal{L}u$ belong to $C(\mathbb{R}^d \times (0,T))$ if $\mathcal{L}$ is purely nonlocal, and moreover $u \in C^{2,1}_{x,t}(\mathbb{R}^d \times (0,T))$ if $\mathcal{L}$ has a local part, and

$$(\partial_t u + \mathcal{L}u)(x,t) = 0 \quad \text{for all } (x,t) \in \mathbb{R}^d \times (0,T). \tag{2.1.9}$$

Since we are dealing with the Cauchy problem, it is also convenient to make precise what we mean by the "initial datum" of the problem. Let us recall that the space of Radon measures on $\mathbb{R}^d$ is the dual space of $C_c(\mathbb{R}^d)$. We will ask the solution to converge to the initial datum as $t \to 0^+$ as Radon measures in the weak* topology (also known as vague topology).

Definition 2.1.4. Let μ_0 a Radon measure. We say that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ has (initial) trace or initial datum μ_0 if

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^d} \psi(x) u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi \,\mathrm{d}\mu_0 \quad \text{for all } \psi \in C_c(\mathbb{R}^d). \tag{2.1.10}$$

Mixed polynomial growth. Our main goal is to show that, under certain conditions, very weak and classical solutions to (NLHE) with initial datum μ_0 are given by the representation formula

$$u(x,t) = \int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}\mu_0(y) \quad \text{in } \mathbb{R}^d \times (0,T). \tag{RF}$$

Here, $P_t(x)$ is the heat kernel of (NLHE), given by $P_t = \mathcal{F}(e^{-tm})$, $t>0$, where $\mathcal{F}$ denotes Fourier transform, and m is the Fourier symbol of $\mathcal{L}$. In order to accomplish this task when $\mathcal{L} = (-\Delta)^{\alpha/2}$, the papers [12] and [19] use in an essential way that the heat kernel associated to that operator satisfies

$$P_t(x) \asymp \frac{t}{\left(t^{1/s} + |x|^2\right)^{(d+2s)/2}} \quad \text{for } (x,t) \in \mathbb{R}^d \times (0,\infty).$$

(Here and below we denote $f \asymp g$ if the quotient f/g remains bounded between two positive constants.) In particular, P_t is comparable to the Lévy kernel K_α of the fractional Laplacian away from the origin for every $t>0$. This comparability between the heat and the Lévy kernel will also be essential for our arguments in Section 2.3, when dealing with more general operators $\mathcal{L}_K$. Fortunately, there is a wide class of Lévy kernels K for which it holds: those comparable to radial functions with *mixed polynomial growth*; see [10]. Thus, besides (A0), we will also assume that

$$\frac{C^{-1}}{|x|^d \tilde{\varphi}(|x|)} \le K(x) \le \frac{C}{|x|^d \tilde{\varphi}(|x|)} \quad \text{for all } x \neq 0 \tag{A1}$$

for some constant $C>1$ and some nondecreasing function $\tilde{\varphi}:(0,+\infty)\to(0,+\infty)$, known as the *scale function*, satisfying the global (lower and upper) *weak scaling conditions*

$$\lambda_1 \Big(\frac{R}{r}\Big)^{\beta_1} \le \frac{\tilde{\varphi}(R)}{\tilde{\varphi}(r)} \le \lambda_2 \Big(\frac{R}{r}\Big)^{\beta_2} \quad \text{for all } 0<r\le R<\infty \tag{A2}$$

for constants λ_1, $\lambda_2 > 0$ and exponents $\beta_2 \ge \beta_1 > 0$.

Remark 2.1.5. Since the Lévy kernel K is assumed to satisfy (A0), then necessarily $\beta_2 \leq 2$. Note, however, that we are admitting the borderline case $\beta_2 = 2$, which is not common in the literature.

For some results we will also require the scale function $\tilde{\varphi}$ to satisfy

$$\tilde{\varphi} \in C^2((1,\infty)), \qquad |\tilde{\varphi}'/\tilde{\varphi}| \leq c, \quad |\tilde{\varphi}''/\tilde{\varphi}| \leq c \quad \text{in } [2,\infty) \quad \text{for some constant } c > 0. \tag{A3}$$

Here is a table of examples satisfying assumptions (A0)–(A3), with its lower and upper exponents.

Examples of operators $\mathcal{L}$ or Lévy kernels K		β_1	β_2
$\mathcal{L} = (-\Delta)^{\alpha/2}$,	$0 < \alpha < 2$	α	α
$\mathcal{L} = (-\Delta)^{\alpha_1/2} + (-\Delta)^{\alpha_2/2}$,	$0 < \alpha_1 < \alpha_2 < 1$	α_1	α_2
$K(x) = \dfrac{1}{\lvert x\rvert^{d+\alpha}(2+\cos\langle\theta, x\rangle)}$,	$\theta \in \mathbb{S}^{d-1}$, $0 < \alpha < 2$	α	α
$K(x) = \dfrac{\log^{\varepsilon}(1+\lvert x\rvert)}{\lvert x\rvert^{d+2}}$,	$0 < \varepsilon < 2$	$2-\varepsilon$	2
$K(x) = \dfrac{1}{\lvert x\rvert^{d+2}(1+\log_+^{\alpha}(\frac{1}{\lvert x\rvert}))}$,	$1 < \alpha < 2$	$2-\varepsilon$ for all $\varepsilon \in (0,2)$	2

Main results and organization of the chapter. In Section 2.2, we first review some known background about the relationship between Lévy processes and heat kernels. We then obtain some fundamental properties of the heat kernel when the Lévy operator is of the form $\mathcal{L}_K$, with K satisfying assumptions (A0)–(A2). These properties will be useful in proving our main results in Section 2.3. Finally, by means of Bernstein's theorem for completely monotone functions, we establish additional properties for the heat kernel when $\mathcal{L} = (-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$, that will be used to prove the existence of classical solutions in this case. For a complete description of Bernstein's theorem see the monograph [103].

The core of the chapter is Section 2.3, where we analyze operators of the form (2.1.1), assuming that K satisfies (A0)–(A2) (plus (A3) for some results). Let P_t be the associated heat kernel. Under these assumptions, $P_{t_1} \asymp P_{t_2}$ for every $t_1, t_2 > 0$, an important fact in our proofs. This is a significant difference with respect to the local heat equation, since G_{t_1} is never comparable to G_{t_2} if $t_1 \neq t_2$.

In the first part of the section we focus on very weak solutions. It turns out that the integrability condition (2.1.7) can be substituted by an integrability condition involving

the heat kernel; see Proposition 5.2.5. Thus, a natural space for very weak solutions is $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, where

$$L^1_{P_1} := \big\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |v(x)| P_1(x)\,\mathrm{d}x < \infty \quad .$$

There is nothing special about P_1, since all the functions P_t, $t>0$, are comparable. We could have picked any other reference time instead of $t=1$.

We start by proving uniqueness for very weak solutions to the Cauchy problem in the sense of the alternative definition (2.3.3)–(2.3.4); this is Theorem 2.3.5. But very weak solutions in the sense of Definition 2.1.2 having μ_0 as initial trace satisfy (2.3.3)–(2.3.4) if they belong to $L^\infty((0,T);L^1_{P_1})$, see Proposition 2.3.3, and uniqueness for very weak solutions in that class follows; see Corollary 2.3.6.

We next prove that nonnegative very weak solutions belong to $L^\infty((0,T);L^1_{P_1})$. As a consequence we obtain a uniqueness result for them; see Corollary 2.3.10.

The third step, Theorem 2.3.15, is to show that every nonnegative very weak solution has an initial trace which is a Radon measure satisfying the integral growth condition

$$\iint_{\mathbb{R}^d} P_1 \,\mathrm{d}\mu_0 < \infty. \tag{GC}$$

The final step is to prove that if μ_0 is a (signed) Radon measure satisfying

$$\iint_{\mathbb{R}^d} P_1 \,\mathrm{d}|\mu_0| < \infty, \tag{GCS}$$

then, the function

$$U(x,t) := \int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}\mu_0(y) \quad \text{for all } (x,t)\in\mathbb{R}^d\times(0,\infty) \tag{S_{μ_0}}$$

is a very weak solution to (NLHE) in $L^\infty((0,T);L^1_{P_1})$ *for every* $T>0$, having μ_0 as initial trace; see Theorem 5.2.9. Here comes a second important difference with the local heat equation (intimately related to the first one): the integrability of the initial trace against the heat kernel at any time gives existence *for all times*. In contrast, in the local framework, if the initial datum satisfies (2.1.5), we can only guarantee existence for $t\in(0,T)$.

Remark 2.1.6. Our existence result admits Radon measures with sign changes as initial data, though this is not necessary for Widder's theory.

The four steps that we have just described imply in particular that a very weak solution to (NLHE) in $L^\infty((0,T);L^1_{P_1})$ with initial trace μ_0 is given by the representation formula (RF). Since nonnegative very weak solutions belong to $L^\infty((0,T);L^1_{P_1})$, we have completed Widder's theory for this kind of solutions; see Corollary 2.3.17.

In the second part of Section 2.3 we consider classical solutions to (NLHE), still for operators of the form (2.1.1). We give two conditions that together are sufficient for a

classical solution to be a very weak solution; see Proposition 2.3.18. The first one, that concerns the regularity of the function, is

$$\iint\!\!\!\!\!\!\diagup_{\{|y|\le 1\}} |\Lambda u(x,y,t)|K(y)\,\mathrm{d}y \in L^1_{\mathrm{loc}}(\mathbb{R}^d\times(0,T)), \tag{H}$$

where $\Lambda u(x,y,t) := u(x,t) - \frac{1}{2}\big(u(x+y,t)+u(x-y,t)\big)$. This would correspond to the regularity requirement $u \in C^{2,1}_{x,t}$ in Widder's theory for the classical heat equation, and is used to ensure that $\mathcal{L}u$ is integrable against any test function. The second one, $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, is required by the definition of very weak solution for a nonlocal operator. Nonnegative classical solutions always satisfy the second condition; see Corollary 2.3.23.

Thanks to Proposition 2.3.18, we can take profit of the results for very weak solutions to prove results for classical solutions, always under the assumption (H): uniqueness and a representation formula in the class $L^\infty((0,T);L^1_{P_1})$, Theorem 2.3.19; and uniqueness, the existence of an initial trace and a representation formula for nonnegative solutions, Theorem 2.3.24.

What about the existence of classical solutions with a given Radon measure μ_0 as initial trace? If μ_0 satisfies the integral growth condition (GCS), we have a natural candidate: the function U given by (S_{μ_0}). However, proving that this very weak solution is a classical solution is not a trivial task, and requires some assumptions, that we write in terms of the heat kernel associated to the operator; see Theorem 2.3.25. These assumptions are satisfied, for instance, when the operator is the fractional Laplacian.

An inspection of the proofs in Section 2.3 shows that they do not use assumptions (A0)–(A3) directly, but their implications for the heat kernel, that could be satisfied by more general operators. In Section 2.4 we rewrite the theorems in Section 2.3 in terms of such conditions for the heat kernel, so that they may encompass a wider class of purely nonlocal operators $\mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). To show the usefulness of these results, we check that they may be applied to the already mentioned family of anisotropic operators $\mathcal{L}_\nu = \sum_{j=1}^{\ell}\mathcal{L}_{K_j}$, $\ell\in\{1,\dots,d\}$, where each $\mathcal{L}_{K_j}$ is of the form (2.1.1) and acts only on the subspace $\mathbb{R}^{d_j}$, with $\mathbb{R}^d = \mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_\ell}$. The key here, besides the special form of the operator, is that the heat kernel associated to $\mathcal{L}_\nu$ is the product of the heat kernels associated to each of the operators $\mathcal{L}_{K_j}$.

Finally, in Section 2.5, we consider operators having both a local and a nonlocal part, namely $\mathcal{L}_{\Delta,K} := -\Delta + \mathcal{L}_K$, with $\mathcal{L}_K$ as in (2.1.1), and K satisfying (A0)–(A2) (and (A3) in some cases). The results, and the proofs, go in parallel to those in Section 2.3. The clue to deal with this kind of operators is that the heat kernel is the convolution of the heat kernels of $-\Delta$ and $\mathcal{L}_K$, which helps in several ways, and even allows to remove some of the restrictions that we had in Section 2.3.

Some notations. Throughout the chapterwe will often use the following notations:

- Fourier transform: $\mathcal{F}(f)(\xi) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}\xi$.
- Inverse Fourier transform: $\mathcal{F}^{-1}(f)(x) = \displaystyle\iint\!\!\!\!\!\!\diagup_{\mathbb{R}^d} f(\xi)e^{i\langle x,\xi\rangle}\,\mathrm{d}\xi$.

- Second order incremental quotients:

$$\Lambda v(x,y) := v(x) - \frac{1}{2}(v(x+y) + v(x-y)). \tag{2.1.11}$$

- Spatial cutoff functions: $\{\psi_R\}_{R\geq R_0}$, $R_0 > 0$, where

$$\psi_R(x) = \psi(x/R), \quad x \in \mathbb{R}^d, \quad \text{for some } \psi \in C_c^\infty(\mathbb{R}^d) \text{ such that } \mathcal{X}_{B_1(0)} \leq \psi \leq \mathcal{X}_{B_2(0)}. \tag{2.1.12}$$

- Set of Lebesgue points of $h \in L^1_{\mathrm{loc}}((0,T))$ with respect to a summability kernel $\{\eta_k\}_{k\in\mathbb{N}}$: $L(h;\{\eta_k\}_{k\in\mathbb{N}}) := \{t \in (0,T) : |h(t)| < \infty, \lim_{k\to\infty}(\eta_k * h)(t) = h(t)\}$.

2.2 The heat kernel

After reviewing some known background concerning the relation between Lévy processes and heat kernels in subsection 2.2.1, we obtain in subsection 2.2.2 several fundamental properties of the heat kernel associated to an operator $\mathcal{L}_K$ with K satisfying (A0)–(A2) that will be later used to prove our main results. When the operator is the fractional Laplacian we obtain further properties in subsection 2.2.3, using the theory of completely monotone functions.

2.2.1 Lévy processes and heat kernels

In this paragraph we explain the relation between Lévy processes and heat kernels, that allows us to use well-known results for the former field to deal with the latter. Our main references for this preliminary material are [3, 62].

Lévy process. A stochastic process on $\mathbb{R}^d$ is said to be a Lévy process if it starts at zero (i.e., $X_0 = 0$ a.s.), has stationary and independent increments, and is continuous in probability.

A Lévy process $(X_t)_{t\geq 0}$ has a characteristic function, that determines its distribution, given by

$$\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)} \quad \text{for all } \xi \in \mathbb{R}^d,\ t \geq 0, \tag{2.2.1}$$

where $\psi : \mathbb{R}^d \to \mathbb{C}$ is the so-called characteristic exponent, [62, Corollary 2.5]. By Lévy-Khintchine's theorem [62, Theorem 6.8], ψ is a continuous function given by Lévy-Khintchine's formula

$$\psi(\xi) = \frac{1}{2}\langle \xi, Q\xi\rangle - \langle b, \xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle \mathcal{X}_{B_1(0)}(y)\right) \mathrm{d}\nu(y). \tag{2.2.2}$$

Here Q is a positive semi-definite $d\times d$ matrix with real coefficients, $b \in \mathbb{R}^d$, and ν is a Lévy measure, that is, a nonnegative Radon (that is, Borel regular and locally finite) measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\fint_{\mathbb{R}^d} (1 \wedge |y|^2)\, \mathrm{d}\nu(y) < \infty.$$

A triplet (Q, b, ν) that satisfies the above properties is known as a Lévy triplet.

Conversely, by Lévy-Khintchine's theorem [62, Theorem 7.5], a Lévy triplet (Q, b, ν) has univocally associated a Lévy process whose characteristic exponent is given by (2.2.2).

Markov transition density. By [62, Lemma 4.4], if $\big(X_t\big)_{t\geq 0}$ is a Lévy process, then

$$p_t(x, B) = \int_B p_t(x, \mathrm{d}y) := \mathbb{P}(x + X_t \in B), \quad t \geq 0,\ x \in \mathbb{R}^d,\ B \in \mathbb{B}(\mathbb{R}^d), \tag{2.2.3}$$

is a Markov transition density: (i) $B \mapsto p_t(x, B)$ is a probability for every $t \geq 0$ and $x \in \mathbb{R}^d$; (ii) $(x, t) \mapsto p_t(x, B)$ is a Borel measurable function for every $B \in \mathbb{B}(\mathbb{R}^d)$; and (iii) the Chapman-Kolmogorov equations hold,

$$p_{t+s}(x, B) = \int_{\mathbb{R}^d} p_t(y, B) p_s(x, \mathrm{d}y) \quad \text{for all } t, s \geq 0,\ x \in \mathbb{R}^d,\ B \in \mathbb{B}(\mathbb{R}^d).$$

Besides, by definition, this Markov transition density is invariant by translations,

$$p_t(x, B) = p_t(0, B - x) \quad \text{for all } t \geq 0, \quad x \in \mathbb{R}^d \quad \text{and } B \in \mathbb{B}(\mathbb{R}^d).$$

In slight abuse of notation, we write $p_t(B - x) := p_t(0, B - x)$.

Feller semigroup. Associated with the above Markov transition density, we define a family of linear operators indexed by $t \geq 0$ acting on the set of bounded Borel functions $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$,

$$\mathcal{P}_t : \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \to \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d), \quad \mathcal{P}_t f(x) := \mathbb{E}[f(x+X_t)] = \int_{\mathbb{R}^d} f(y) p_t(x, \mathrm{d}y), \quad f \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{2.2.4}$$

Since p_t is invariant under translations, it follows that $\mathcal{P}_t$ is a convolution operator for all $t \geq 0$,

$$\mathcal{P}_t f(x) = \int_{\mathbb{R}^d} f(x + y) p_t(\mathrm{d}y) = f * P_t^K(x), \quad f \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d),\ x \in \mathbb{R}^d, \tag{2.2.5}$$

where $P_t^K(B) := p_t(-B)$ for all Borel sets B and $t \geq 0$. Moreover, the family of operators $\big(\mathcal{P}_t\big)_{t\geq 0}$ is a Feller semigroup; that is, $\big(\mathcal{P}_t\big)_{t\geq 0}$:

- is a semigroup: $\mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s$ for all $t, s \geq 0$ and $\mathcal{P}_0 = \mathrm{Id}$;
- is sub-Markovian: if $0 \leq f \leq 1$ then $0 \leq \mathcal{P}_t f \leq 1$;
- is contractive: $\|\mathcal{P}_t f\|_\infty \leq \|f\|_\infty$;
- has the Feller property: $\mathcal{P}_t(C_0(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$;
- is strongly continuous on $C_0(\mathbb{R}^d)$: $\lim_{t\to 0^+} \|\mathcal{P}_t f - f\|_\infty = 0$ for all $C_0(\mathbb{R}^d)$.

The strong continuity on $C_0(\mathbb{R}^d)$ implies in particular that

$$p_t \to \delta_0 \quad \text{as } t \to 0^+ \quad \text{in the weak* topology in } (C_0(\mathbb{R}^d))'. \tag{2.2.6}$$

Let us recall that $(C_0(\mathbb{R}^d))'$ is the set of finite Radon measures.

Infinitesimal generator. Since $\big(\mathcal{P}_t\big)_{t\geq 0}$ is a Feller semigroup, it has an associated infinitesimal generator $\mathcal{A}$, given by

$$\mathcal{A}f := \lim_{t\to 0^+} \frac{\mathcal{P}_t f - f}{t} \quad \text{(uniform limit) for all } f \text{ in the domain of } \mathcal{A},$$
$$\mathcal{D}(\mathcal{A}) := \Big\{ f \in C_0(\mathbb{R}^d) : \exists\, g \in C_0(\mathbb{R}^d) \text{ such that } \lim_{t\to 0^+} \frac{\mathcal{P}_t f - f}{t} - g \Big\|_\infty = 0 \Big\}.$$

It holds that $\mathbb{P}_t(\mathcal{D}(\mathcal{A})) \subset \mathcal{D}(\mathcal{A})$ for all $t \geq 0$, and

$$\frac{d}{dt}\mathbb{P}_t f = \mathcal{A}(\mathbb{P}_t f) \quad \text{for all } f \in \mathcal{D}(\mathcal{A}), \tag{2.2.7}$$

so that $u(t) = \mathbb{P}_t f$ solves the initial value problem $u' = \mathcal{A}u$, $u(0) = f$ in the Banach space $C_0(\mathbb{R}^d)$. In addition,

$$\mathbb{P}_t f - f = \mathcal{A} \fint_0^t \mathbb{P}_s f \, \mathrm{d}s \quad \text{if } f \in C_0(\mathbb{R}^d), \qquad \mathbb{P}_t f - f = \fint_0^t \mathbb{P}_s \mathcal{A} f \, \mathrm{d}s \quad \text{if } f \in \mathcal{D}(\mathcal{A}).$$

As a consequence of the above properties, $\mathcal{D}(\mathcal{A})$ is dense in $C_0(\mathbb{R}^d)$, $\mathcal{A}$ is a closed operator and $\mathcal{A}$ determines $(\mathbb{P}_t)_{t\geq 0}$ uniquely; see [62, Lemma 5.4, Remark 5.5]. Moreover, since $(\mathbb{P}_t)_{t\geq 0}$ comes from a Lévy process, $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class in $\mathbb{R}^d$.

Fourier symbol. By [3, Theorem 3.3.3], the infinitesimal generator $\mathcal{A}$ of a Feller semigroup $(\mathbb{P}_t)_{t\geq 0}$ associated with a Lévy process $(X_t)_{t\geq 0}$ can be expressed in terms of Fourier transform as

$$\mathcal{A}f(x) = -\fint_{\mathbb{R}^d} \mathcal{F}(f)(\xi)\psi(\xi)e^{i\langle \xi, y\rangle} \, \mathrm{d}\xi, \quad f \in \mathcal{S}(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where ψ is the characteristic exponent of $(X_t)_{t\geq 0}$.

The operators in the Feller semigroup $(\mathbb{P}_t)_{t>0}$ can also be rewritten via Fourier transform. Indeed, $\mathbb{P}_t(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$ and, due to (2.2.4) and (2.2.1),

$$\begin{aligned}\mathbb{P}_t f &= \mathbb{P}_t \int_{\mathbb{R}^d} \mathcal{F}(f)(\xi)e^{i\langle \xi,\cdot\rangle} \, \mathrm{d}\xi = \fint_{\mathbb{R}^d} \mathcal{F}(f)(\xi)\mathbb{P}_t e^{i\langle \xi,\cdot\rangle} \, \mathrm{d}\xi = \int_{\mathbb{R}^d} \mathcal{F}(f)(\xi)e^{i\langle \xi,\cdot\rangle}e^{-t\psi(\xi)} \, \mathrm{d}\xi \\ &= \langle \mathcal{F}^{-1}(f(\cdot - \xi)), e^{-t\psi(\xi)} \, \mathrm{d}\xi\rangle_{\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d),\ t \geq 0.\end{aligned}$$

As a consequence, using also (2.2.5), we have that

$$p_t(\mathrm{d}y) = \mathcal{F}^{-1}(e^{-t\psi(-y)}\mathrm{d}y) \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{2.2.8}$$

Heat kernel. A Feller semigroup $(\mathbb{P}_t)_{t\geq 0}$ is a strong Feller semigroup if $\mathbb{P}_t(B_{\rm b}(\mathbb{R}^d)) \subset C_{\rm b}(\mathbb{R}^d)$. We have the following important result due to Hawkes, [62, Lemma 4.9]. The semigroup $(\mathbb{P}_t)_{t\geq 0}$ defined by (2.2.4) associated to a Lévy process $(X_t)_{t\geq 0}$ is a strong Feller semigroup if and only if its Markov transition density p_t, $t \geq 0$ (defined in (2.2.3)) is absolutely continuous with respect to the Lebesgue measure for all $t > 0$; that is, if

$$p_t(\mathrm{d}y) = P_t(y)\,\mathrm{d}y \quad \text{for all } t > 0$$

for some $P_t \in L^1(\mathbb{R}^d)$, known as the heat kernel of $(X_t)_{t\geq 0}$. Note that $\|P_t\|_{L^1(\mathbb{R}^d)} = 1$, since p_t is a probability measure. In what follows we will use several times the alternative notation

$$P(y,t) = P_t(y), \quad (y,t) \in \mathbb{R}^d \times (0,\infty).$$

If the characteristic exponent of a Lévy process $(X_t)_{t\geq 0}$ satisfies $e^{-t\psi} \in L^1(\mathbb{R}^d)$ for all $t > 0$, then, by Riemann-Lebesgue's Theorem, $\mathcal{F}^{-1}(e^{-t\psi}) \in C_0(\mathbb{R}^d)$. Moreover, if ψ is symmetric, by (2.2.8), $(X_t)_{t\geq 0}$ has a heat kernel given by

$$P_t = \mathcal{F}^{-1}(e^{-t\psi}), \quad t > 0.$$

Notice that for each $t > 0$ we have $|P_t(x)| \leq \fint_{\mathbb{R}^d} e^{-t\psi(\xi)}\,\mathrm{d}\xi = P_t(0)$.

Thanks to (2.2.6) we know that the heat kernel has an initial trace:

$$P_t \to \delta_0 \quad \text{as } t \to 0^+ \text{ as finite Radon measures in the weak* topology.}$$

Besides, by Fourier transform, $(P_t)_{t\geq 0}$ satisfies the semigroup property

$$P_t * P_s = P_s * P_t = P_{t+s} \quad \text{for all } t, s \geq 0. \tag{2.2.9}$$

2.2.2 Heat kernel properties for operators $\mathcal{L}_K$

Our next goal is to obtain further properties of the heat kernel when the Lévy operator $\mathcal{L}_K$ has the form (2.1.1) with K satisfying (A0)–(A2).

Fourier symbol of the operator. By Fourier transform,

$$\mathcal{L}_K f(x) = \fint_{\mathbb{R}^d} m(\xi)\mathcal{F}(f)(\xi)e^{i\langle x,\xi\rangle}\,\mathrm{d}\xi, \quad f \in C_{\rm c}^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where m is the Fourier symbol of $\mathcal{L}_K$, given by

$$m(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, y\rangle) K(y)\,\mathrm{d}y, \quad \xi \in \mathbb{R}^d. \tag{2.2.10}$$

We know from the properties of the cosine function that there is a constant $C > 0$ such that

$$0 \leq 1 - \cos\langle \xi, y\rangle \leq C(1 \wedge |\xi|^2|y|^2) \leq C(1 + |\xi|^2)(1 \wedge |y|^2). \tag{2.2.11}$$

Then, since K is a Lévy kernel,

$$0 \le m(\xi) \le C(1+|\xi|^2), \quad \xi \in \mathbb{R}^d; \tag{2.2.12}$$

see (A0). Moreover, because K is symmetric and $\langle \xi, \cdot \rangle \mathcal{X}_{B_1(0)}$ is an odd function,

$$m(\xi) = \text{P.V.} \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, y\rangle}) K(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle \mathcal{X}_{B_1(0)}(y)) K(y)\, \mathrm{d}y.$$

Since $|1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle \mathcal{X}_{B_1(0)}(y)| \le C(1+|\xi|^2)(1 \wedge |y|^2)$, the last integral is finite, thanks to the Lévy condition (A0). Therefore, m is the characteristic exponent of the Lévy process $(Y_t)_{t\ge 0}$ that has $(0, 0, K(y)\,\mathrm{d}y)$ as Lévy triplet; see Lévy-Khintchine's formula (2.2.2). As a consequence,

$$-\mathcal{L}_K \, _{\mathcal{S}(\mathbb{R}^d)} = \mathcal{A} \, _{\mathcal{S}(\mathbb{R}^d)}, \tag{2.2.13}$$

where $\mathcal{A}$ is the infinitesimal generator associated to the process.

We now prove the continuity of the Fourier symbol and estimate its size.

Proposition 2.2.1. *Let m be the Fourier symbol of $\mathcal{L}_K$, with K satisfying assumptions (A0)–(A2). Then, $m \in C(\mathbb{R}^d)$ and there exist constants $0 < C_1 \le C_2$ such that*

$$C_1|\xi|^{\beta_1} \le m(\xi) \le C_2|\xi|^{\beta_2} \quad \textit{for all } |\xi| \ge 1, \tag{2.2.14}$$

where $0 < \beta_1 \le \beta_2$ are the exponents in the global weak scaling conditions (A2).

Proof. Thanks to (2.2.11), we can apply the Dominated Convergence Theorem, (DCT from now on), to obtain the desired continuity of m,

$$\lim_{\xi_1 \to \xi_2} (m(\xi_1) - m(\xi_2)) = \int_{\mathbb{R}^d} \lim_{\xi_1 \to \xi_2} (\cos\langle y, \xi_2\rangle - \cos\langle y, \xi_1\rangle) K(y)\, \mathrm{d}y = 0.$$

As for the lower bound for m when $|\xi| \ge 1$, performing the change of variables $y \mapsto y/|\xi|$ in (2.2.10),

$$m(\xi) = \frac{1}{|\xi|^d} \int_{\mathbb{R}^d} (1 - \cos\langle \xi/|\xi|, y\rangle) K(y/|\xi|)\, \mathrm{d}y.$$

Using hypothesis (A1) and the global lower weak scaling condition in (A2),

$$K(y/|\xi|) \ge \frac{|\xi|^d}{C|y|^d \tilde{\varphi}(y/|\xi|)} = \frac{|\xi|^d}{C|y|^d \tilde{\varphi}(y)} \frac{\tilde{\varphi}(y)}{\tilde{\varphi}(y/|\xi|)} \ge \frac{\lambda}{C} \frac{|\xi|^{d+\beta_1}}{|y|^d \tilde{\varphi}(y)} \ge \frac{\lambda}{C^2} |\xi|^{d+\beta_1} K(y),$$

where C is the same constant as in (A1) and λ and β_1 come from (A2). Therefore,

$$m(\xi) \ge \frac{\lambda}{C^2} |\xi|^{\beta_1} \int_{\mathbb{R}^d} (1 - \cos\langle \xi/|\xi|, y\rangle) K(y)\, \mathrm{d}y = \frac{\lambda}{C^2} |\xi|^{\beta_1} m(\xi/|\xi|).$$

Since K is assumed to be positive, then $m(\xi) \ge 0$, with equality if and only if $\xi = 0$; see (2.2.10). Therefore, the continuity of m implies that

$$m(\xi) \ge C_1|\xi|^{\beta_1}, \quad \text{where } C_1 = \frac{\lambda}{C^2} \min_{\theta \in \mathbb{S}^n} m(\theta) > 0.$$

The proof of the upper bound is completely analogous. □

The heat kernel and its regularity. Let $(Y_t)_{t\geq 0}$ be the Lévy process with Lévy triplet $(0,0,K(y)\,\mathrm{d}y)$. We have just seen that its characteristic exponent is m. The lower bound for m in (2.2.14) implies that $e^{-tm}\in L^1(\mathbb{R}^d)$ for all $t>0$, whence, since m is symmetric, $(Y_t)_{t\geq 0}$ has a heat kernel given by

$$P_t=\mathcal{F}^{-1}(e^{-tm}),\quad t>0; \tag{2.2.15}$$

see (2.2.8). The lower bound also allows us to prove that the heat kernel is smooth.

Corollary 2.2.2. *Let K satisfying (A0)–(A2) and let P be the heat kernel associated to $\mathcal{L}_K$. Then:*

(i) *$\partial_t^k P(\cdot,t), D^\alpha P_t\in C_0(\mathbb{R}^d)$ for all $t>0$, $k\in\mathbb{N}\cup\{0\}$, and $\alpha=(\alpha_1,\dots,\alpha_d)\in(\mathbb{N}\cup\{0\})^d$;*

(ii) *$\mathcal{L}_K P_t\in C_0(\mathbb{R}^d)$ for all $t>0$;*

(iii) *$\partial_t^k P\in C(\mathbb{R}^d\times(0,\infty))$ for all $k\in\mathbb{N}\cup\{0\}$.*

Proof. (i) By the definition (2.2.15) of the heat kernel via Fourier transform,

$$\begin{aligned}\partial_t^k P(\cdot,t)&=\partial_t^k\mathcal{F}^{-1}(e^{-tm})=\mathcal{F}^{-1}\big(\partial_t^k e^{-tm}\big)=\mathcal{F}^{-1}\big((-1)^k m^k e^{-tm}\big),\\ D^\alpha P_t&=D^\alpha\mathcal{F}^{-1}(e^{-tm})=\mathcal{F}^{-1}\big(pe^{-tm}\big)\quad\text{for some }\mathbb{C}\text{-polynomial }p.\end{aligned}$$

Hence, the result will follow from Riemann-Lebesgue's Lemma if we prove that

$$m^k e^{-tm},\ pe^{-tm}\in L^1(\mathbb{R}^d)\quad\text{for all } t>0 \tag{2.2.16}$$

for any $k\in\mathbb{N}\cup\{0\}$ and any $\mathbb{C}$-polynomial $p=p(\xi_1,\dots,\xi_d)$.

By the lower bound for m in (2.2.14), for any $k\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}\int_{\mathbb{R}^d}(1+|\xi|^k)e^{-tm(\xi)}\mathrm{d}\xi&=\iint_{\{|\xi|\leq 1\}}(1+|\xi|^k)e^{-tm(\xi)}\mathrm{d}\xi+\int_{\{|\xi|\geq 1\}}(1+|\xi|^k)e^{-tm(\xi)}\mathrm{d}\xi\\&\leq 2|B_1(0)|+2\iint_{\{|\xi|\geq 1\}}|\xi|^k e^{-tC_1|\xi|^{\beta_1}}\mathrm{d}\xi<\infty\quad\text{for all } t>0,\end{aligned} \tag{2.2.17}$$

from where (2.2.16) follows, since $0\leq m^k(\xi)\leq C(1+|\xi|^{2k})$ for some constant $C>0$ for all $\xi\in\mathbb{R}^d$, by (2.2.12), and $|p(\xi_1,\dots,\xi_d)|\leq C(1+|\xi|^k)$ for all $\xi\in\mathbb{R}^d$ for some constants $C>0$ and $k\in\mathbb{N}\cup\{0\}$.

(ii) We have already proved in (i) that $P_t\in C_0^2(\mathbb{R}^d)$. Hence, by Taylor's expansion,

$$\begin{aligned}|\Lambda P_t(x,y)|&\leq\frac{1}{2}\|D^2P_t\|_\infty|y|^2\mathcal{X}_{B_1(0)}(y)+2\|P_t\|_\infty\mathcal{X}_{B_1^c(0)}(y)\\&\leq(\frac{1}{2}\|D^2P_t\|_\infty+2\|P_t\|_\infty)(1\wedge|y|^2)\end{aligned} \tag{2.2.18}$$

where Λ denotes the second order incremental quotients (see (2.1.11)). Hence, since K satisfies the Lévy property (A0), we can apply the DCT to obtain

$$|\mathcal{L}P_t(x_1)-\mathcal{L}P_t(x_2)|\to 0\quad\text{as } |x_1-x_2|\to 0,\quad |\mathcal{L}P_t(x)|\leq\int_{\mathbb{R}^d}|\Lambda P_t(x,y)|K(y)\,\mathrm{d}y\to 0\quad\text{as } |x|\to\infty.$$

(iii) It is enough to check that $\partial_t^k P(x,\cdot) \in C((0,\infty))$ for all $x \in \mathbb{R}^d$, since we already now that $\partial_t^k P(\cdot,t) \in C_0(\mathbb{R}^d)$ for all $t > 0$. Consider $x \in \mathbb{R}^d$ and $t_1, t_2 > 0$. By (2.2.17), using the DCT, we have

$$|\partial_t^k P(x,t_1) - \partial_t^k P(x,t_2)| \le \fint_{\mathbb{R}^d} m^k(\xi)\ e^{-t_1 m(\xi)} - e^{-t_2 m(\xi)}\ \mathrm{d}\xi \to 0 \quad \text{as } |t_1 - t_2| \to 0. \qquad \square$$

Size estimates. If K satisfies (A0)–(A2), [10, Theorem 1.2] implies that the tails of the heat kernel and the Lévy kernel are comparable uniformly in bounded times away from zero: for each $\varepsilon > 0$ and $T > \varepsilon$ there exists a constant $C_{\varepsilon,T} \ge 1$ depending only on ε and T such that

$$C_{\varepsilon,T}^{-1} K(x) \le P_t(x) \le C_{\varepsilon,T} K(x) \quad \text{for all } |x| \ge 1 \text{ and } t \in [\varepsilon, T]. \tag{C_{PK}}$$

Remember that $P_t \in C_0(\mathbb{R}^d)$ for all $t > 0$. In particular, P_t is bounded. On the other hand, $P_t > 0$ for all $t > 0$; see [10, Theorem 1.2]. Combining these two facts with (C_{PK}), we have that there is some constant $C_{\varepsilon,T} > 1$ such that

$$C_{\varepsilon,T}^{-1} P_t(x) \le P_s(x) \le C_{\varepsilon,T} P_t(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t, s \in [\varepsilon, T]. \tag{C_{P_t}}$$

Besides, since $P_1 > 0$ and continuous, for each $\tilde\theta \in C_c(\mathbb{R}^d \times [0,T])$ there is $C_{\tilde\theta} > 0$ such that

$$|\tilde\theta(x,t)| \le C_{\tilde\theta} P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0,T], \tag{2.2.19}$$

which together with the semigroup property (2.2.9) and (C_{P_t}) yields that there is $C_{\tilde\theta,T} > 0$ such that

$$\big|\big(P_t * \tilde\theta(\cdot,t)\big)(x)\big| \le C_{\tilde\theta} P_t * P_1(x) \le C_{\tilde\theta,T} P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0,T]. \tag{2.2.20}$$

Thanks to the positivity and continuity of P_1, we can control the image of test functions under the action of $\mathcal{L}_K$.

Lemma 2.2.3. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2) and let P be the corresponding heat kernel. Let $0 \le \delta < \tau < \infty$. For each $\tilde\theta \in C_c^2(\mathbb{R}^d \times [\delta,\tau])$ there is a constant $C_{\tilde\theta} > 0$ such that*

$$|\mathcal{L}_K \tilde\theta(x,t)| \le \int_{\mathbb{R}^d} |\Lambda\tilde\theta(x,y,t)| K(y)\,\mathrm{d}y \le C_{\tilde\theta} P_1(x) \quad \textit{for all } (x,t) \in \mathbb{R}^d \times [\delta,\tau].$$

Proof. The first inequality is immediate. As for the second, we split the integration domain in two sets, to take profit of the different properties of K in different sets,

$$\int_{\mathbb{R}^d} |\Lambda\tilde\theta(x,y,t)| K(y)\,\mathrm{d}y = \underbrace{\fint_{\{|y|\le 1\}} |\Lambda\tilde\theta(x,y,t)| K(y)\,\mathrm{d}y}_{\mathrm{I}(x,t)} + \underbrace{\int_{\{|y|\ge 1\}} |\Lambda\tilde\theta(x,y,t)| K(y)\,\mathrm{d}y}_{\mathrm{II}(x,t)}.$$

By Taylor's expansion, for each $x, y \in \mathbb{R}^d$ and $t \in [0, \tau]$ there is some λ, $|\lambda| \leq 1$, such that

$$|\Lambda\tilde{\theta}(x, y, t)| \leq \frac{|y|^2}{2} \sum_{j,k=1}^{d} |\partial_j \partial_k \tilde{\theta}(x + \lambda y, t)|.$$

Let $R_{\tilde{\theta}}$ be such that $\operatorname{supp}(\tilde{\theta}(\cdot, t)) \subset B_{R_{\tilde{\theta}}}(0)$ for all $t \in [0, \tau]$. If $|\lambda|, |y| \leq 1$, then

$$\operatorname{supp}(\partial_j \partial_k \tilde{\theta}(\cdot + \lambda y, t)) \subset B_{1+R_{\tilde{\theta}}}(0), \quad j, k \in \{1, \dots, d\},$$

whence, since P_1 is continuous and positive, there exists a constant $C_{\tilde{\theta}}$ such that

$$\sum_{j,k=1}^{d} |\partial_j \partial_k \tilde{\theta}(x + \lambda y, t)| \leq C_{\tilde{\theta}} P_1(x).$$

The Lévy condition (A0) implies then that $\mathrm{I}(x, t) \leq C'_{\tilde{\theta}} P_1(x)$ for some constant $C'_{\tilde{\theta}} > 0$.

As for II, since K is symmetric,

$$\mathrm{II}(x, t) \leq |\tilde{\theta}(x, t)| \int_{\{|y| \geq 1\}} K(y)\, \mathrm{d}y + \int_{\{|y| \geq 1\}} |\tilde{\theta}(x - y, t)| K(y)\, \mathrm{d}y.$$

By (2.2.19), there is a constant $C_{\tilde{\theta}} > 0$ such that $|\tilde{\theta}(\cdot, t)| \leq C_{\tilde{\theta}} P_1$ for all $t \in [0, \tau]$. Then,

$$\mathrm{II}(x, t) \leq C_{\tilde{\theta}} P_1(x) \int_{\{|y| \geq 1\}} K(y)\, \mathrm{d}y + C_{\tilde{\theta}} \int_{\{|y| \geq 1\}} P_1(x - y) K(y)\, \mathrm{d}y.$$

The integral in the first term of the right-hand side is finite, thanks to the Lévy condition (A0). To handle the second term we use that we can control the tail of K by the tail of P_1, see (C_{PK}). Thus,

$$\mathrm{II}(x, t) \leq C_{\tilde{\theta}} \Big(P_1(x) + \int_{\{|y| \geq 1\}} P_1(x - y) P_1(y)\, \mathrm{d}y \Big) \leq C_{\tilde{\theta}} \Big(P_1(x) + \int_{\mathbb{R}^d} P_1(x - y) P_1(y)\, \mathrm{d}y \Big).$$

The integral on the rightmost term is equal to $P_2(x)$, due to the semigroup property (2.2.9). Moreover, thanks to (C_{P_t}), there exists $C > 0$ such that $P_2 \leq C P_1$. We conclude that

$$\mathrm{II}(x, t) \leq C_{\tilde{\theta}} (P_1(x) + P_2(x)) \leq C_{\tilde{\theta}} P_1(x). \qquad \square$$

Our next result shows that we can compare P_1 with its translations.

Lemma 2.2.4. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2) and let P be the corresponding heat kernel. Given $R > 0$, there exists a constant $C_R > 0$ such that*

$$P_1(x - y) \leq C_R P_1(y) \quad \text{for all } y \in \mathbb{R}^d \text{ and } x \in B_R(0). \tag{2.2.21}$$

Proof. Let $x \in B_R(0)$ and $y \in A_x := \{y : |y| \le 2\} \cup \{y : |x-y| \le |y|/2,\ |y| \ge 2\}$. Thus we have $A_x \subset \{y : |y| \le 2(R+1)\}$. Since $P_1 \in C(\mathbb{R}^d)$ and $0 < P_1 \le P_1(0)$, then $P_1(x-y) \le P_1(0) \le C_R P_1(y)$, where $C_R = P_1(0)/\tilde{C}_R$ with $\tilde{C}_R = \min\{P_1(y) : |y| \le 2(R+1)\} > 0$.

On the other hand, if $y \in (A_x)^c = \{y : |x-y| \ge |y|/2 \ge 1\}$, using the estimates for P_1 following from (C_{PK}) and (A1), and the fact that $\tilde{\varphi}$ is a nondecreasing function that satisfies the upper weak scaling condition in (A2), we have, for constants not depending on x,

$$P_1(x-y) \le \frac{C}{|x-y|^d \tilde{\varphi}(|x-y|)} \le \frac{C}{|y|^d \tilde{\varphi}(|y|/2)} \le \frac{C}{|y|^d \tilde{\varphi}(|y|)} \le C P_1(y). \qquad \square$$

Rate of change. Our next aim is to prove that P_1 is essentially constant on dyadic annuli, that is, on sets of the form $\{x : r \le |x| \le 2r\}$, with a rate of change that is independent of r. To be more precise, we formulate the following.

Definition 2.2.5. We will say that a function $f : \mathbb{R}^d \to (0,\infty)$ has a *bounded rate of change* (or simply, that f is *changing slowly*) if for all $0 < \rho_1 \le \rho_2$ there exists a constant $C_{\rho_1,\rho_2} \ge 1$ such that

$$C_{\rho_1,\rho_2}^{-1} \le \frac{f(x)}{f(y)} \le C_{\rho_1,\rho_2} \quad \text{if } \rho_1 \le \frac{|y|}{|x|} \le \rho_2.$$

A function with this property is one that does not exhibit rapid oscillations nor a rapid decay or growth. A typical example is that where f is monotone decreasing with a certain polynomial decay. On the contrary, the Gaussian represents a function which does not satisfy the condition. As we will see, the above property will be essential throughout this work.

The slow change of P_1 near zero stems from its continuity and positivity. Away from the origin it is inherited directly from K, which has this property thanks to assumptions (A1) and (A2). In fact, we know that the tails of P_1 and K are comparable. We are now ready to prove the following.

Proposition 2.2.6. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2) and let P be the corresponding heat kernel. Then, P_1 is a slowly changing function.*

Proof. Let $0 < \rho_1 \le \rho_2$. We consider two cases, depending on the size of $|x|$.

Let $|x| \le 1 \vee \rho_1^{-1}$, where $a \vee b := \max\{a,b\}$. If $\rho_1 \le |y|/|x| \le \rho_2$, then $|y| \le \rho_2(1 \vee \rho_1^{-1})$. Since P_1 is positive and continuous,

$$C^{-1} \le \frac{P_1(x)}{P_1(y)} \le C \quad \text{for all } |x| \le 1 \vee \rho_1^{-1} \text{ and } |y| \le \rho_2(1 \vee \rho_1^{-1}),$$

for some constant $C \ge 1$ depending only on ρ_1 and ρ_2, and we are done.

Suppose now that $|x| \ge 1 \vee \rho_1^{-1}$. Then $|x|, |y| \ge 1$ whenever $\rho_1 \le |y|/|x| \le \rho_2$. By ($C_{PK}$), K and P_1 are comparable outside the ball of radius 1, whence, by hypothesis (A1),

$$\frac{C^{-1}}{|x|^d \tilde{\varphi}(|x|)} \le P_1(x) \le \frac{C}{|x|^d \tilde{\varphi}(|x|)} \quad \text{for all } |x| \ge 1$$

for some $C > 1$. As a consequence,

$$C^{-2}\rho_1^d \frac{\tilde{\varphi}(|y|)}{\tilde{\varphi}(|x|)} \leq \frac{P_1(x)}{P_1(y)} \leq C^2\rho_2^d \frac{\tilde{\varphi}(|y|)}{\tilde{\varphi}(|x|)} \quad \text{whenever } \rho_1 \leq \frac{|y|}{|x|} \leq \rho_2 \text{ and } |x| \geq 1 \vee \rho_1^{-1},$$

from where the result will follow if there is a constant $C \geq 1$ depending only on ρ_1 and ρ_2 such that

$$C^{-1} \leq \frac{\tilde{\varphi}(|y|)}{\tilde{\varphi}(|x|)} \leq C \quad \text{whenever } \rho_1 \leq \frac{|y|}{|x|} \leq \rho_2. \tag{2.2.22}$$

Assume that $\rho_1 \leq |y|/|x| \leq \rho_2$. To prove (2.2.22) we consider two cases, depending on the size of $|y|$ compared to $|x|$. If $|y| \geq |x|$, then $(1 \vee \rho_1) \leq |y|/|x| \leq \rho_2$. Hence, by (A2),

$$\lambda_1(1 \vee \rho_1)^{\beta_1} \leq \frac{\tilde{\varphi}(|y|)}{\tilde{\varphi}(|x|)} \leq \lambda_2\rho_2^{\beta_2}.$$

On the other hand, if $|y| \leq |x|$, then $\rho_1| \leq |y|/|x| \leq (1 \wedge \rho_2)$. Therefore, again by (A2),

$$\lambda_2^{-1}\rho_1^{\beta_2} \leq \frac{\tilde{\varphi}(|y|)}{\tilde{\varphi}(|x|)} \leq \lambda_1^{-1}(1 \wedge \rho_2)^{\beta_1}. \qquad \square$$

As a corollary, we get the following result that will show to be useful in next section.

Lemma 2.2.7. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2) and let P be the corresponding heat kernel. There is a constant $C > 0$ such that*

$$P_1(z) \leq CP_1(x) \quad \textit{for all } x \in \mathbb{R}^d \textit{ and } z \in B_1(x). \tag{2.2.23}$$

Proof. If $|x| \geq 2$, then $|x|/2 \leq |z| \leq 3/2|x|$, whence, since P_1 is slowly changing, $P_1(z) \leq CP_1(x)$ for some constant $C > 0$ independent of x and z. Besides, since $P_1 \in C(\mathbb{R}^d)$ and $0 < P_1 \leq P_1(0)$, then

$$P_1(z) \leq \frac{P_1(0)}{\min_{B_2(0)} P_1} P_1(x) \quad \text{for all } |x| \leq 2. \qquad \square$$

The function P_1 is also *almost decreasing*.

Proposition 2.2.8. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2) and let P be the corresponding heat kernel. Then, P_1 is almost decreasing: there exists a constant $C > 0$ such that*

$$P_1(z) \leq CP_1(x) \quad \textit{for all } x, z \in \mathbb{R}^d \textit{ such that } |x| \leq |z|.$$

Proof. If $1 \leq |x| \leq |z|$ the result follows easily, since in this region P_1 is comparable to K, see (C_{PK}), and K is almost decreasing, see (A1). Besides, since $P_1 \in C(\mathbb{R}^d)$ and $0 < P_1 \leq P_1(0)$,

$$P_1(z) \leq \frac{P_1(0)}{\min_{B_1(0)} P_1} P_1(x) \quad \text{if } |x| \leq |z| \leq 1.$$

Finally, if $|x| \leq 1 \leq |z|$, we consider ξ such that $|\xi| = 1$, and, using the two previous cases we conclude that $P_1(z) \leq CP_1(\xi) \leq CP_1(x)$. $\square$

Remark 2.2.9. Since, by (C_{P_t}), P_t is comparable to P_1 if $t > 0$, and the latter is slowly changing and almost decreasing, so is the former.

The heat kernel solves the equation. By assumption (A0), K is symmetric, whence the associated Markov transition density is also symmetric. Then, by (2.2.5),

$$\mathcal{P}_t f = P_t * f \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d) \text{ and } t > 0. \tag{2.2.24}$$

Since $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$ and $\mathbb{P}_t(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$, using (2.2.7), (2.2.13) and (2.2.24) we get that for every $f \in \mathcal{S}(\mathbb{R}^d)$ the function $u(t) = P_t * f$ is a (classical) solution of $\partial_t u + \mathcal{L}_K u = 0$ with initial data f.

The operators $\mathcal{L}_K$ and $-\mathcal{A}$ in principle only coincide in $\mathcal{S}(\mathbb{R}^d)$, see (2.2.13), and the heat kernel at time t, P_t, does not belong in general to this class (see for instance the case of the fractional Laplacian). However, we are able to prove that P does not only solve the heat equation associated to $-\mathcal{A}$, but also the one associated to $\mathcal{L}_K$.

Proposition 2.2.10. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2). The corresponding heat kernel P is a classical solution of equation (NLHE) for every $T > 0$.*

Proof. By Corollary 2.2.2, we already know that $P, \partial_t P, \mathcal{L}P_t \in C(\mathbb{R}^d \times (0,\infty))$. Hence it is enough to check that P_t satisfies (2.1.9) for all $T > 0$.

Recall that $P_t \in C_0^2(\mathbb{R}^d)$; see Corollary 2.2.2. Hence, there is a sequence $\{f_n\} \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to P_t$ in $W_0^{2,\infty}(\mathbb{R}^d)$ as $n \to \infty$. Then, since K satisfies the Lévy condition (A0),

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\big\{|\mathcal{L}f_n(x) - \mathcal{L}P_t(x)| \quad &\le \sup_{x\in\mathbb{R}^d}\Big\{\int_{\mathbb{R}^d} |\Lambda(f_n - P_t)(x,y)|K(y)\,dy \\ &\le \frac{1}{2}\|D^2(f_n - P_t)\|_\infty \int_{\{|y|\le 1\}} |y|^2 K(y)\,dy + 2\|f_n - P_t\|_\infty \int_{\{|y|\ge 1\}} K(y)\,dy \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

That is, $\mathcal{L}f_n \to \mathcal{L}P_t$ uniformly as $n \to \infty$. But, since $f_n \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, then $\mathcal{L}f_n = -\mathcal{A}f_n$ for all $n \in \mathbb{N}$; see (2.2.13). Thus, we have proved that $-\mathcal{A}f_n \to \mathcal{L}P_t$ uniformly as $n \to \infty$ converges uniformly to $\mathcal{L}P_t$ as $n \to \infty$. Since $\mathcal{A}$ is a closed operator (see [62, Remark 5.5]), we conclude that $P_t \in \mathcal{D}(\mathcal{A})$ and $-\mathcal{A}P_t = \mathcal{L}P_t$.

By (2.2.7), since $P_h \in \mathcal{D}(\mathcal{A})$ for all $h > 0$, then $\frac{d}{dt}\mathbb{P}_t P_h = \mathcal{A}\mathbb{P}_t P_h$ for all $h > 0$. But, thanks to (2.2.24) and the semigroup property (2.2.9), we have that $\mathbb{P}_t P_h = P_{t+h}$ for all $t \ge 0$ and $h > 0$, whence we conclude that P_t satisfies (2.1.9). □

Corollary 2.2.11. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying assumptions (A0)–(A2). The corresponding heat kernel P is a very weak solution to equation (NLHE) for every $T > 0$.*

Proof. By Proposition 2.2.2, $P \in C(\mathbb{R}^d \times (0,\infty))$, whence $P \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$.

Let us check that P satisfies (2.1.7). Given $\theta \in C_c^\infty(\mathbb{R}^d \times (0,T))$, let $0 < \varepsilon < \tau < T$ such that $\operatorname{supp}(\theta) \subset \mathbb{R}^d \times (\varepsilon, \tau)$. Using Lemma 2.2.3, the symmetry of P_t, the semigroup property (2.2.9), and (C_{P_t}),

$$\begin{aligned}\iint_0^T \int_{\mathbb{R}^d} P_t(x)|\mathcal{L}\theta(x,t)|\,dx\,dt &\le C_\theta \iint_0^T \int_{\mathbb{R}^d} P_t(x)P_1(x)\,dx\,dt \\ &\le C_\theta \iint_0^T P_{1+t}(0)\,dt \le C_{\theta,\tau}\tau P_1(0) < \infty.\end{aligned} \tag{2.2.25}$$

Let us prove now that P satisfies (2.1.8). By (2.2.25), using first Fubini-Tonelli's Theorem and then changing variables, we obtain that

$$\begin{aligned}\fint_0^T\int_{\mathbb{R}^d} P(x,t)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \fint_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P(x,t)\Lambda\theta(x,y,t)K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &= \fint_0^T\int_{\mathbb{R}^d} K(y)\int_{\mathbb{R}^d} P(x,t)\Lambda\theta(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &= \fint_0^T\int_{\mathbb{R}^d} K(y)\int_{\mathbb{R}^d} \Lambda P(x,y,t)\theta(x,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned} \tag{2.2.26}$$

On the other hand, since $0 < P_t(x) \le P_t(0)$ for all $x \in \mathbb{R}^d$, $t > 0$, and $P \in C(\mathbb{R}^d \times (0,\infty))$, see Corollary 2.2.2 (iii), then $\|P_t\|_\infty \le \max_{t\in[\varepsilon,\tau]} P_t(0) = c_{\varepsilon,\tau} < \infty$ if $t \in [\varepsilon,\tau]$. Besides, using Fourier transform and the estimate from below for the symbol m in (2.2.14),

$$|\partial_i\partial_j P_t(x)| \le \int_{\mathbb{R}^d} |\xi_i||\xi_j| e^{-tm(\xi)}\,\mathrm{d}\xi \le \int_{\mathbb{R}^d} |\xi|^2 e^{-\varepsilon|\xi|^{\beta_1}}\,\mathrm{d}\xi = c_\varepsilon < \infty, \quad x \in \mathbb{R}^d,\ t \in [\varepsilon,\tau],$$

for all $i,j \in \{1,\dots,d\}$. These estimates together with (2.2.18) yield $|\Lambda P(x,y,t)| \le C_{\varepsilon,\tau}(1\wedge|y|^2)$ for some constant $C_{\varepsilon,\tau} > 0$ depending only on ε and τ. Therefore,

$$\fint_0^T\int_{\mathbb{R}^d}\fint_{\mathbb{R}^d}|\theta(x,t)||\Lambda P(x,y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \le C_{\varepsilon,\tau}\fint_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\theta(x,t)|(1\wedge|y|^2)K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

since θ is a test function. As a consequence, using Fubini-Tonelli's Theorem again, we have that

$$\begin{aligned}\fint_0^T\int_{\mathbb{R}^d} \theta(x,t)\mathcal{L}P(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \fint_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \theta(x,t)\Lambda P(x,y,t)K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_\varepsilon^\tau\int_{\mathbb{R}^d} K(y)\fint_{\mathbb{R}^d}\theta(x,t)\Lambda P(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned} \tag{2.2.27}$$

Combining (2.2.26) and (2.2.27), we get that

$$\fint_0^T\int_{\mathbb{R}^d} P(x,t)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t = \fint_0^T\int_{\mathbb{R}^d} \theta(x,t)\mathcal{L}P(x,t)t\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, since P is a classical solution to (NLHE), (see Proposition 2.2.10), and integrating by parts in time, we conclude that

$$\fint_0^T\int_{\mathbb{R}^d} P(x,t)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\fint_0^T\int_{\mathbb{R}^d} \theta(x,t)\partial_t P(x,t)\,\mathrm{d}x\,\mathrm{d}t = \fint_0^T\int_{\mathbb{R}^d} P(x,t)\partial_t\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad \square$$

2.2.3 Completely monotone functions and fractional heat kernels

We now devote our attention to the fractional Laplacian of order α, $\alpha \in (0,2)$, denoted by $(-\Delta)^{\alpha/2}$. It is defined as the operator $\mathcal{L}_{K_\alpha}$ with Lévy kernel, satisfying hypotheses (A0)–(A3),

$$K_\alpha(y) = \frac{C_{d,\alpha}}{|y|^{d+\alpha}} \quad \text{and} \quad C_{d,\alpha} = \left(\fint_{\mathbb{R}^d}\frac{1-\cos(\xi_1)}{|\xi|^{d+\alpha}}\,\mathrm{d}\xi\right)^{-1}.$$

The Fourier multiplier of this operator is $m(\xi) = |\xi|^\alpha$, whence the corresponding heat kernel, that we will denote by $P_t^\alpha(x)$ or $P^\alpha(x,t)$, is given by $\mathcal{F}(P_t^\alpha)(\xi) = e^{-t|\xi|^\alpha}$ for all $t > 0$. Since the multiplier is homogeneous, the heat kernel has a self-similar structure,

$$P^\alpha(x,t) = t^{-d/\alpha} P^\alpha(xt^{-\alpha}, 1). \tag{2.2.28}$$

We will obtain estimates for P^α by means of Bernstein's Theorem for *completely monotone* functions; for a reference see, for instance, [3, Theorem 1.3.23] and [103, Theorem 1.4].

Definition 2.2.12. A function $g \in C^\infty([0,\infty))$ is *completely monotone* if $(-1)^n g^{(n)} \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Example. The function $g_\alpha(y) = e^{-y^{\alpha/2}}$, $y \in [0,\infty)$, is completely monotone for any $\alpha \in (0,2]$.

Theorem 2.2.13 (Bernstein's Theorem)**.** *If g is a completely monotone function, it is the Laplace transform of some nonnegative Borel measure μ,*

$$g(y) = \iint_0^\infty e^{-sy}\, \mathrm{d}\mu(s), \quad y \in [0,\infty).$$

Remark 2.2.14. (i) The Borel measure $\mathrm{d}\mu_\alpha$ associated to the function g_α in the above example is a probability measure since, by Bernstein's Theorem,

$$g_\alpha(y) = e^{-y^{\alpha/2}} = \iint_0^\infty e^{-sy}\, \mathrm{d}\mu_\alpha(s), \quad y \in [0,\infty),$$

which implies that $1 = g_\alpha(0) = \iint_0^\infty \mathrm{d}\mu_\alpha(s)$.

(ii) If $\alpha = 2$, then $\mu_2 = \delta_1$, and if $\alpha = 1$, $\mathrm{d}\mu_1(s) = (4\pi)^{-1/2} s^{-3/2} e^{-1/4s}\, \mathrm{d}s$.

Corollary 2.2.15. *Let P^α be the heat kernel associated to the operator $(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. Then,*

$$P^\alpha(x,1) = \iint_0^\infty G(x,s)\, \mathrm{d}\mu_\alpha(s), \quad x \in \mathbb{R}^d, \tag{2.2.29}$$

where μ_α is the Borel measure given by Bernstein's Theorem and G is the classical heat kernel associated to $-\Delta$, given by (2.1.4).

Proof. By Bernstein's Theorem, $e^{-|\xi|^\alpha} = g_\alpha(|\xi|^2) = \iint_0^\infty e^{-s|\xi|^2}\, \mathrm{d}\mu_\alpha(s)$, whence

$$P^\alpha(x,1) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} e^{-i\langle x,\xi\rangle}\, \mathrm{d}\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \iint_0^\infty e^{-s|\xi|^2} e^{-i\langle x,\xi\rangle}\, \mathrm{d}\mu_\alpha(s)\, \mathrm{d}\xi. \tag{2.2.30}$$

By Tonelli's Theorem

$$\iint_0^\infty G(0,s)\, \mathrm{d}\mu_\alpha(s) = \iint_0^\infty \int_{\mathbb{R}^d} \frac{e^{-s|\xi|^2}}{(2\pi)^d}\, \mathrm{d}\xi\, \mathrm{d}\mu_\alpha(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \iint_0^\infty e^{-s|\xi|^2}\, \mathrm{d}\mu_\alpha(s)\, \mathrm{d}\xi = P^\alpha(0,1) < \infty.$$

Therefore, since $G(x,s) \le G(0,s)$ for all $x \in \mathbb{R}^d$, $s > 0$,

$$\iint_0^\infty G(x,s)\,\mathrm{d}\mu_\alpha(s) < \infty \quad \text{for all } x \in \mathbb{R}^d.$$

As a consequence, by Fubini's Theorem, we can interchange the integrals in (2.2.30) to get

$$P^\alpha(x,1) = \iint_0^\infty \int_{\mathbb{R}^d} \frac{e^{-s|\xi|^2}}{(2\pi)^d} e^{-i\langle x,\xi\rangle}\,\mathrm{d}\xi\,\mathrm{d}\mu_\alpha(s) = \iint_0^\infty G(x,s)\,\mathrm{d}\mu_\alpha(s) \quad \text{for all } x \in \mathbb{R}^d. \qquad \square$$

The representation (2.2.29) combined with the self-similar structure (2.2.28) of the kernel P^α allows us to obtain estimates for its derivatives. Let us state some of them explicitly.

Proposition 2.2.16. *Let P^α be the heat kernel associated to the operator $(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. There exists a positive constant $C = C(\alpha,d)$ such that for all $x \in \mathbb{R}^d$:*

(i) $|\partial_j P_1^\alpha(x)| \le C\Big(1 \wedge \dfrac{1}{|x|}\Big) P_1^\alpha(x)$ *for all* $j = 1,\dots,d$;

(ii) $|\partial_i\partial_j P_1^\alpha(x)| \le C\Big(1 \wedge \dfrac{1}{|x|^2}\Big) P_1^\alpha(x)$ *for all* $i,j = 1,\dots,d$;

(iii) $|\langle x, \nabla P_1^\alpha(x)\rangle| \le C(|x| \wedge 1) P_1^\alpha(x)$;

(iv) $|\partial_t P^\alpha(x,t)| = |\mathcal{L} P_t^\alpha(x)| \le \dfrac{C}{t} P_t^\alpha(x)$ *for all* $t \in (0,\infty)$;

(v) $|\partial_j P_t^\alpha(x)| \le C\Big(\dfrac{1}{t^\alpha} \wedge \dfrac{1}{|x|}\Big) P_1^\alpha(x)$ *for all* $j = 1,\dots,d$ *and* $t \in (0,\infty)$;

(vi) $|\partial_i\partial_j P_t^\alpha(x)| \le C\Big(\dfrac{1}{t^{2\alpha}} \wedge \dfrac{1}{|x|}\Big) P_1^\alpha(x)$ *for all* $i,j = 1,\dots,d$ *and* $t \in (0,\infty)$.

Proof. (i)–(ii) These properties are clearly satisfied for any bounded domain, since $P_1^\alpha > 0$ and $P_1^\alpha \in C^\infty(\mathbb{R}^d)$; see Corollary 2.2.2. Therefore, it is enough to prove them for $|x| \ge 2$.

A direct computation of the spatial derivatives of the classical heat kernel G yields

$$\partial_j G(x,s) = -\frac{x_j}{2s} G(x,s), \quad \partial_i\partial_j G(x,s) = \Big(\frac{-\delta_{ij}}{2s} + \frac{x_i x_j}{8s^2}\Big) G(x,s),$$
$$|\partial_j G(x,s)| \le \frac{|x|}{2s} G(x,s), \quad |\partial_i\partial_j G(x,s)| \le \Big(\frac{1}{2s} + \frac{|x|^2}{8s^2}\Big) G(x,s)$$

for all $(x,s) \in \mathbb{R}^d \times (0,\infty)$. Then, since $a^k \le k! e^a$ for all $k \in \mathbb{N}$ and $a \ge 0$, we have that

$$|x||\partial_j G(x,s)| \le 4 e^{\frac{|x|^2}{8s}} G(x,s) = 4G(x/2,s), \tag{2.2.31}$$
$$|x|^2|\partial_i\partial_j G(x,s)| \le 20 e^{\frac{|x|^2}{8s}} G(x,s) = 20G(x/2,s). \tag{2.2.32}$$

Since $G(x,s) \le G(0,s)$ for all $(x,s) \in \mathbb{R}^d \times (0,\infty)$, using the MVT and these estimates we get

$$\frac{G(x+he_j,s)-G(x,s)}{h} = |\partial_j G(x'(x,h,s),s)| \le \frac{4}{|x'|}G(0,s) \le \frac{C}{|x|}G(0,s),$$
$$\frac{\partial_j G(x+he_i,s)-\partial_j G(x,s)}{h} = |\partial_i\partial_j G(x''(x,h,s),s)| \le \frac{20}{|x''|^2}G(0,s) \le \frac{C}{|x|^2}G(0,s),$$

if $|h| \le 1$, because $x', x'' \in B_1(x)$ for all $(x,s) \in \mathbb{R}^d \times (0,\infty)$.

Since $G(0,\cdot) \in L^1((0,\infty), \mathrm{d}\mu_\alpha)$, see (2.2.29), we are allowed to interchange the derivatives and the integral in the representation formula (2.2.29) for the heat kernel via the DCT and, by (2.2.31) and (2.2.32), we obtain

$$|\partial_j P_1^\alpha(x)| = \left| \fint_0^\infty \partial_j G(x,s)\,\mathrm{d}\mu_\alpha(s) \right| \le \frac{C}{|x|} \fint_0^\infty G(x/2,s)\,\mathrm{d}\mu_\alpha(s) = \frac{C}{|x|}P_1^\alpha(x/2),$$
$$|\partial_i\partial_j P_1^\alpha(x)| = \left| \fint_0^\infty \partial_i\partial_j G(x,s)\,\mathrm{d}\mu_\alpha(s) \right| \le \frac{C}{|x|^2} \fint_0^\infty G(x/2,s)\,\mathrm{d}\mu_\alpha(s) = \frac{C}{|x|^2}P_1^\alpha(x/2).$$

Finally, since P^α is slowly changing, see Proposition 2.2.6, there is $C > 0$ such that $P_1^\alpha(x/2) \le C P_1^\alpha(x)$ for all $x \in \mathbb{R}^d$, and the result follows.

(iii) It is a direct consequence of (i).

(iv) Using the self-similar form (2.2.28) of P^α and (iii), we get

$$|\partial_t P^\alpha(x,t)| = t^{-(1+\frac{d}{\alpha})} \left| \frac{d}{\alpha}P_1^\alpha(xt^{-\alpha}) + \frac{1}{\alpha}\langle xt^{-\alpha}, \nabla P_1^\alpha(xt^{-\alpha})\rangle \right| \le \frac{C}{t}P_1^\alpha(x)$$

for all $t > 0$. The estimate for $|\mathcal{L}P_t^\alpha|$ follows immediately, since P^α solves the fractional heat equation; see Proposition 2.2.10.

(v) Using again the self-similarity of P^α, now in combination with (i), we obtain

$$|\partial_j P_t^\alpha(x)| = t^{-(\alpha+\frac{d}{\alpha})}|\partial_j P_1^\alpha(xt^{-\alpha})| \le C\Big(\frac{1}{t^\alpha} \wedge \frac{1}{|x|}\Big)P_1^\alpha(x).$$

(vi) Using once more the self-similarity of P^α, now together with (ii),

$$|\partial_i\partial_j P_t^\alpha(x)| = t^{-(2\alpha+\frac{d}{\alpha})}|\partial_i\partial_j P_1^\alpha(xt^{-\alpha})| \le C\Big(\frac{1}{t^{2\alpha}} \wedge \frac{1}{|x|}\Big)P_1^\alpha(x). \qquad \square$$

2.3 Operators $\mathcal{L}_K$

This section constitutes the main body of the chapter. It is devoted to develop a complete Widder-type theory for (NLHE) when $\mathcal{L} = \mathcal{L}_K$, with K satisfying (A0)–(A2), plus (A3) for some results. It is divided into two parts. In the first one we focus on very weak solutions, addressing three key topics: the existence of an initial trace when the solutions are nonnegative, uniqueness under various conditions, and existence. In the second part, we turn our attention to classical solutions. We begin by exploring uniqueness. Later, we give conditions on the heat kernel guaranteing the existence of classical solutions. Such conditions are satisfied by the heat kernel of the fractional Laplacian.

2.3.1 Very weak solutions

We now give conditions ensuring uniqueness for very weak solutions. In addition, we prove, on the one hand, that nonnegative very weak solutions have an initial trace which is a Radon measure satisfying certain integral growth condition and, on the other hand, the existence of a solution for any initial data satisfying a growth condition that coincides with the above mentioned one when the data are nonnegative. As a corollary, we obtain a representation formula. As a first step, we discuss a useful alternative definition of very weak solution.

An alternative definition of very weak solution

Our notion of very weak solution requires u to be integrable against $\mathcal{L}\theta$ for any test function θ. We will see now that this is equivalent to requiring u to be integrable against the heat kernel at any positive time, say $t = 1$. This equivalence relies on the Lévy kernel K and P_1 being comparable far from the origin (see (C_{PK})) and on P_1 being slowly changing by Proposition 2.2.6.

Proposition 2.3.1. *Let $\mathcal{L} = \mathcal{L}_K$, with K satisfying (A0)–(A2).*
(i) *Let $0 \le \delta < \tau \le T$. Then $u \in L^1((\delta,\tau); L^1_{P_1})$ if and only if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [\delta,\tau])$ and*

$$\fint_\delta^\tau \int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\theta(x,t)|\,\mathrm{d}x\,\mathrm{d}t < \infty \quad \text{for all } \theta \in C_{\mathrm{c}}^\infty(\mathbb{R}^d \times [\delta,\tau]). \tag{2.3.1}$$

(ii) *$u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$ if and only if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ and (2.1.7) holds.*

Proof. (i) $\Rightarrow$) Since $u \in L^1((\delta,\tau); L^1_{P_1})$, the local integrability in $\mathbb{R}^d \times [\delta,\tau]$ follows immediately from the positivity of P_1. On the other hand, by Lemma 2.2.3, for a given $\theta \in C_{\mathrm{c}}^\infty(\mathbb{R}^d \times [\delta,\tau])$,

$$\fint_\delta^\tau \int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\theta(x,t)|\,\mathrm{d}x \le C_\theta \fint_\delta^\tau \int_{\mathbb{R}^d} |u(x,t)|P_1(x)\,\mathrm{d}x\,\mathrm{d}t < \infty.$$

$\Leftarrow$) Consider $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ such that $\mathcal{X}_{B_1(0)} \le \psi \le \mathcal{X}_{B_2(0)}$. Since $\psi \ge 0$, K is symmetric and $\mathrm{supp}(\psi) \subset B_2(0)$, if $|x| \ge 4$, then $|\mathcal{L}\psi(x)| = \int_{B_2(x)} \psi(x-y)K(y)\,\mathrm{d}y$. Notice that we are avoiding the singularity of K, since $|y| \ge 2$ in the domain of integration. On the other hand, if $|x| \ge 4$ and $|y-x| \le 2$, then $|y| \le 3|x|/2$, whence

$$K(y) \ge \frac{C}{|y|^d \tilde{\varphi}(|y|)} \ge \frac{C}{\left(\frac{3^d|x|^d}{2^n}\right)\tilde{\varphi}\left(\frac{3|x|}{2}\right)} \ge CK\left(\frac{3x}{2}\right) \ge CP_1\left(\frac{3x}{2}\right) \ge CP_1(x) \quad \text{whenever } y \in B_2(x),$$

due to (A1), the fact that $\tilde{\varphi}$ is nondecreasing, (C_{PK}), and the fact that P_1 is slowly changing, see Proposition 2.2.6. Then,

$$|\mathcal{L}\psi(x)| = \int_{B_2(x)} \psi(x-y)K(y)\,\mathrm{d}y \ge CP_1(x) \int_{B_2(0)} \psi(y)\,\mathrm{d}y = CP_1(x) \quad \text{for all } |x| \ge 4. \tag{2.3.2}$$

As a consequence, using also that $P_1 \leq P_1(0)$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [\delta, \tau])$ and (2.3.1),

$$\fint_0^T \int_{\mathbb{R}^d} |u(x,t)| P_1(x)\,\mathrm{d}x\,\mathrm{d}t \leq P_1(0) \fint_0^T \int_{B_4(0)} |u(x,t)|\,\mathrm{d}x\,\mathrm{d}t + C \fint_0^T \int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\psi|(x)\,\mathrm{d}x\,\mathrm{d}t < \infty.$$

(ii) $\Rightarrow$) This follows directly from (i) $\Rightarrow$).
$\Leftarrow$) Let $0 < \varepsilon < T' < T$. We consider ψ as in the proof of (i) $\Leftarrow$) and $\varphi : [0,T] \to \mathbb{R}$ such that $\mathcal{X}_{[\varepsilon,T']} \leq \varphi \leq \mathcal{X}_{[\varepsilon/2,T'+(T-T')/2]}$. Then, $\varphi\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d \times (0,T))$. Hence, since $P_1 \leq P_1(0)$, by (2.1.7) and (2.3.2), with $\varphi\psi$ replacing ψ, we have

$$\begin{aligned}\fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)| P_1(x)\,\mathrm{d}x\,\mathrm{d}t &\leq P_1(0) \fint_0^{T'} \int_{B_4(0)} |u(x,t)|\,\mathrm{d}x\,\mathrm{d}t + C \fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\psi|(x)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq P_1(0) \fint_0^{T'} \int_{B_4(0)} |u(x,t)|\,\mathrm{d}x\,\mathrm{d}t + C \fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}(\varphi\psi)|(x,t)\,\mathrm{d}x\,\mathrm{d}t < \infty. \qquad \square\end{aligned}$$

Remark 2.3.2. Thanks to Proposition 5.2.5 (ii), if u is a very weak solution of equation (NLHE), then $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$.

When dealing with very weak solutions to the Cauchy Problem for (NLHE), a common choice is asking u to be a very weak solution of the equation in the sense of Definition 2.1.2 and to take the data (in general a Radon measure μ_0) in the sense of Definition 2.1.4; see for instance [19]. This is our choice in the present chapter. Another option is asking u to satisfy

$$u \in L^1_{\mathrm{loc}}([0,T); L^1_{P_1}); \tag{2.3.3}$$

$$\fint_0^T \int_{\mathbb{R}^d} u(x,t)(\partial_t \tilde{\theta}(x,t) - \mathcal{L}\tilde{\theta}(x,t))\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^d} \tilde{\theta}(x,0)\,\mathrm{d}\mu_0(x) \quad \text{for all } \tilde{\theta} \in C_{\mathrm{c}}^\infty(\mathbb{R}^d \times [0,T)); \tag{2.3.4}$$

see for instance [12] for a definition in this spirit. Both choices have advantages and disadvantages. Formulation (2.3.3)–(2.3.4) facilitates obtaining uniqueness without any requirement of positivity or integrability, excepting (2.3.3), although it requires such conditions to establish existence. On the contrary, proving existence is more straightforward with our choice, but you pay the price of needing additional assumptions to achieve uniqueness.

If $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2) and $u \in L^\infty((0,T); L^1_{P_1})$, both notions are equivalent. We will take profit of this equivalence later.

Proposition 2.3.3. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure and $u \in L^\infty((0,T); L^1_{P_1})$.*
(i) *If u satisfies (2.3.4), then it is a very weak solution of (NLHE) with initial trace μ_0.*
(ii) *If u is a very weak solution to (NLHE) with initial trace μ_0, then it satisfies (2.3.3)–(2.3.4).*

Proof. (i) Condition (2.1.7) follows from Proposition 5.2.5, since $L^\infty((0,T); L^1_{P_1}) \subset L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$. Moreover, since $C_{\mathrm{c}}^\infty(\mathbb{R}^d \times (0,T)) \subset C_{\mathrm{c}}^\infty(\mathbb{R}^d \times [0,T))$, (2.3.4) implies (2.1.8).

It remains to prove that μ_0 is the initial trace of u. To this aim, given $\psi \in C_c(\mathbb{R}^d)$, we would like to take $H(\tau - \cdot)\psi$, $\tau \in (0,T)$, where H is the Heaviside function, as test function in (2.3.4), and then pass to the limit $\tau \to 0^+$. However, this test is not admissible, since it does not have the required smoothness; hence, we have to proceed through an approximation argument.

Approximating ψ is easy: we take a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ converging to ψ uniformly in $\mathbb{R}^d$ such that $|\psi_j| \le M\mathcal{X}_{B_R(0)}$ for some constants $M > 0$, $R > 0$.

In order to approximate the time factor, we consider $\varphi_k(t) = \displaystyle\int_{2^k t}^\infty \eta(s)\,\mathrm{d}s$, $k \in \mathbb{N}$, $t \in \mathbb{R}$, where

$$\eta(s) := \begin{cases} C_1 \mathrm{e}^{-\frac{1}{1-s^2}} & \text{if } |s| < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad C_1^{-1} := \fint_{-1}^{1} \mathrm{e}^{-\frac{1}{1-s^2}}\,\mathrm{d}s. \tag{2.3.5}$$

Then $\{(\varphi_k)'\}_{k\in\mathbb{N}} = \{-\eta_k\}_{k\in\mathbb{N}}$, with $\eta_k(s) := 2^k\eta(2^k s)$, $s \in \mathbb{R}$. Notice that $\{\eta_k\}_{k\in\mathbb{N}}$ is a summability kernel. The desired approximating sequence is then $\{\varphi_k^\tau\}_{k\in\mathbb{N}}$, where $\varphi_k^\tau(t) = \varphi_k(t-\tau)$ for $\tau \in (0,T)$, $k \ge k_0 > -\ln(T-\tau)/\ln 2$ and $t \in \mathbb{R}$.

Let $g_j(t) = \displaystyle\int_{\mathbb{R}^d} \psi_j(x)u(x,t)\,\mathrm{d}x$. Take $\varphi_k^\tau\psi_j \in C_c^\infty(\mathbb{R}^d \times [0,T))$ as test function in (2.3.4). Then,

$$\fint_0^T (\varphi_k^\tau)'(t)g_j(t)\,\mathrm{d}t - \fint_0^T \int_{\mathbb{R}^d} \varphi_k^\tau(t)u(x,t)\mathcal{L}\psi_j(x)\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^d} \varphi_k^\tau(0)\psi_j\,\mathrm{d}\mu_0. \tag{2.3.6}$$

Since $|\psi_j| \le M\mathcal{X}_{B_R(0)}$ for all $j \in \mathbb{N}$ and $P_1 > 0$, then $|\psi_j| \le CP_1$ for some $C > 0$. Hence, $g_j \in L^1((0,T))$, as $u \in L^\infty((0,T);L^1_{P_1}) \subset L^1((0,T);L^1_{P_1})$. Therefore, $(\varphi_k)' * g_j(\tau) \to -g_j(\tau)$ if $\tau \in (0,T) \cap L(g_j;\{\eta_k\}_{k\in\mathbb{N}})$, whence

$$\lim_{k\to\infty} \fint_0^T (\varphi_k^\tau)'(t)g_j(t)\,\mathrm{d}t = -g_j(\tau) \quad \text{for all } \tau \in (0,T) \cap L(g_j;\{\eta_k\}_{k\in\mathbb{N}}).$$

Note that, by Lebesgue's Differentiation Theorem, $|(0,T) \setminus L(g_j;\{\eta_k\}_{k\in\mathbb{N}})| = 0$.

Since $\mathcal{X}_{(-\infty,-2^{-k}]} \le \varphi_k \le \mathcal{X}_{(-\infty,2^{-k})}$, then $0 \le \varphi_k^\tau \le 1$. Besides, $|\mathcal{L}\psi_j| \le C_j P_1$ (see Lemma 2.2.3). Hence,

$$\begin{aligned} \limsup_{k\to\infty} \fint_0^T \int_{\mathbb{R}^d} \varphi_k^\tau(t)u(x,t)\mathcal{L}\psi_j(x)\,\mathrm{d}x\,\mathrm{d}t &\le C_j \lim_{k\to\infty} \fint_0^{\tau+2^{-k}} \int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= C_j \fint_0^T \int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

On the other hand, $\varphi_k^\tau(0) = \varphi_k(-\tau) = 1$ if $\tau \in (0,T)$ and $k \ge -\log\tau/\log 2$, whence

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \varphi_k^\tau(0)\psi_j\,\mathrm{d}\mu_0 = \int_{\mathbb{R}^d} \psi_j\,\mathrm{d}\mu_0 \quad \text{for all } \tau \in (0,T) \cap L(g_j;\{\eta_k\}_{k\in\mathbb{N}}).$$

Thus, letting $k \to \infty$ in (2.3.6), we obtain

$$\fint_{\mathbb{R}^d} \psi_j(x)u(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}^d} \psi_j\,\mathrm{d}\mu_0 \le C_j \fint_0^T \int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \tau \in (0,T) \cap L(g_j;\{\eta_k\}_{k\in\mathbb{N}}),$$

whence

$$\operatorname*{ess\,lim}_{\tau\to 0^+} \int_{\mathbb{R}^d} \psi_j(x)u(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}^d} \psi_j\,\mathrm{d}\mu_0 \ = 0. \tag{2.3.7}$$

This will be enough to complete the approximation argument. Indeed, on the one hand, since $|\psi_j - \psi| = |\psi_j - \psi|\mathcal{X}_{B_R(0)}$, $j \in \mathbb{N}$, then

$$\iint_{\mathbb{R}^d} |\psi_j - \psi|\,\mathrm{d}|\mu_0| \le \|\psi - \psi_j\|_{L^\infty(\mathbb{R}^d)}|\mu_0|(B_R(0)).$$

On the other hand, as $P_1 > 0$, there is a constant $C > 0$ such that $\mathcal{X}_{B_R(0)} \le CP_1$, so that

$$\begin{aligned}\int_{\mathbb{R}^d} |\psi - \psi_j|(x)u(x,\tau)\,\mathrm{d}x &\le C\|\psi - \psi_j\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} u(x,\tau)P_1(x)\,\mathrm{d}x \\ &\le C\|\psi - \psi_j\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^\infty((0,T);L^1_{P_1})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(x)u(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}^d} \psi\,\mathrm{d}\mu_0 \ \le \ &\iint_{\mathbb{R}^d} \psi_j(x)u(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}^d} \psi_j\,\mathrm{d}\mu_0 \\ &+ \|\psi - \psi_j\|_{L^\infty(\mathbb{R}^d)}\big(|\mu_0|(B_R(0)) + C\|u\|_{L^\infty((0,T);L^1_{P_1})}\big).\end{aligned}$$

Since $|(0,T)\setminus \cap_{j=1}^\infty L(g_j;\{\eta_k\}_{k\in\mathbb{N}})| = 0$, and thanks to (2.3.7), letting first $\tau \to 0^+$ and then $j \to \infty$, we arrive at (2.1.10), as desired.

(ii) Condition (2.3.3) follows directly from $u \in L^\infty((0,T);L^1_{P_1})$.

Let us check that u satisfies (2.3.4). We define

$$\tilde{\varphi}_k(t) = \iint_{-\infty}^{2^k t} \eta(s)\,\mathrm{d}s, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

It satisfies that $\mathcal{X}_{[\frac{1}{2^k},\infty)} \le \tilde{\varphi}_k \le \mathcal{X}_{[-\frac{1}{2^k},\infty)}$ for all $k \in \mathbb{N}$ and $(\tilde{\varphi}_k)' = \eta_k$. For fixed $\tau \in (0,T)$, let $\tilde{\varphi}_k^\tau(t) = \tilde{\varphi}_k(t-\tau)$. Given $\tilde{\theta} \in C_c^\infty(\mathbb{R}^d \times [0,T))$, the function $\tilde{\varphi}_k^\tau\tilde{\theta} \in C_c^\infty(\mathbb{R}^d \times (0,T))$ for all $k > -\ln\tau/\ln 2$. Hence, because u is a very weak solution,

$$\underbrace{\iint_0^T (\tilde{\varphi}_k^\tau)'(t)\int_{\mathbb{R}^d} u(x,t)\tilde{\theta}(x,t)\,\mathrm{d}x\,\mathrm{d}t}_{\mathrm{I}_k(\tau)} + \underbrace{\iint_0^T \tilde{\varphi}_k^\tau(t)\int_{\mathbb{R}^d} u(x,t)(\partial_t\tilde{\theta}(x,t) - \mathcal{L}\tilde{\theta}(x,t))\,\mathrm{d}x\,\mathrm{d}t}_{\mathrm{II}_k(\tau)} = 0. \tag{2.3.8}$$

Let us analyze the double limit, first a $k \to \infty$ and then as $\tau \to 0^+$, of $\mathrm{I}_k(\tau)$ and $\mathrm{II}_k(\tau)$.

As P_1 is strictly positive, we can make the decomposition $\mathrm{I}_k(\tau) = \mathrm{I}_{1,k}(\tau) + \mathrm{I}_{2,k}(\tau)$, where

$$\begin{aligned}\mathrm{I}_{1,k}(\tau) &:= \iint_0^T (\tilde{\varphi}_k^\tau)'(t)\int_{\mathbb{R}^d} u(x,t)\tilde{\theta}(x,0)\,\mathrm{d}x\,\mathrm{d}t, \\ \mathrm{I}_{2,k}(\tau) &:= \iint_0^T (\tilde{\varphi}_k^\tau)'(t)\iint_{\mathbb{R}^d} u(x,t)\frac{\tilde{\theta}(x,t) - \tilde{\theta}(x,0)}{P_1(x)}P_1(x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $|\tilde\theta(\cdot,0)| \le CP_1$ and $u \in L^\infty((0,T);L^1_{P_1})$, then $\int_{\mathbb{R}^d} u(x,\cdot)\tilde\theta(x,0)\,\mathrm{d}x \in L^1((0,T))$. Therefore, because $\{\eta_k\}_{k\in\mathbb{N}}$ is a summability kernel,

$$\lim_{k\to\infty} \mathrm{I}_{1,k}(\tau) = \int_{\mathbb{R}^d} u(x,\tau)\tilde\theta(x,0)\,\mathrm{d}x \quad \text{for a.e. } \tau\in(0,T),$$

whence, as μ_0 is the initial trace of u,

$$\operatorname*{ess\,lim}_{\tau\to0^+}\lim_{k\to\infty} \mathrm{I}_{1,k}(\tau) = \operatorname*{ess\,lim}_{\tau\to0^+}\int_{\mathbb{R}^d} u(x,\tau)\tilde\theta(x,0)\,\mathrm{d}x = \int_{\mathbb{R}^d}\tilde\theta(x,0)\,\mathrm{d}\mu_0(x).$$

On the other hand, since P_1 is strictly positive and $\tilde\theta\in C_{\mathrm{c}}(\mathbb{R}^d\times[0,T))$, by the MVT,

$$\frac{|\tilde\theta(x,s)-\tilde\theta(x,0)|}{P_1(x)} \le \frac{\|\partial_t\theta\|_{L^\infty(\mathbb{R}^d\times(0,T))}}{\min\limits_{(y,s)\in\operatorname{supp}(\tilde\theta)}\{P_1(y)\}}t = C_{\tilde\theta,P_1}t \quad \text{for all } (x,t)\times\mathbb{R}^d\times[0,T).$$

Besides, $(\tilde\varphi_k^\tau)' = \eta_k(\cdot-\tau)\ge0$ and $\operatorname{supp}(\eta_k(\cdot-\tau))\subset[\tau-2^{-k},\tau+2^{-k}]$, so that

$$\begin{aligned}|\mathrm{I}_{2,k}(\tau)| &\le C_{\tilde\theta,P_1}\fint_{-2^{-k}}^{\tau+2^{-k}} t\eta_k(t-\tau)\int_{\mathbb{R}^d}|u(x,t)|P_1(x)\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_{\tilde\theta,P_1}\|u\|_{L^\infty((0,T);L^1_{P_1})}\fint_{-2^{-k}}^{\tau+2^{-k}} t\eta_k(t-\tau)\,\mathrm{d}t.\end{aligned}$$

Since $\varphi_k^\tau(\tau-2^{-k})=0$ and $\varphi_k^\tau(\tau+2^{-k})=1$, integrating by parts we get

$$0\le\fint_{-2^{-k}}^{\tau+2^{-k}} t\eta_k(t-\tau)\,\mathrm{d}t = \tau+2^{-k}-\fint_{-2^{-k}}^{\tau+2^{-k}}\varphi_k^\tau(t)\,\mathrm{d}t \le \tau+2^{1-k},$$

because $\varphi_k^\tau\ge0$. Thus,

$$\operatorname*{ess\,lim}_{\tau\to0^+}\limsup_{k\to\infty}|\mathrm{I}_{2,k}(\tau)| \le C_{\tilde\theta,P_1}\|u\|_{L^\infty((0,T);L^1_{P_1})}\operatorname*{ess\,lim}_{\tau\to0^+}\limsup_{k\to\infty}(\tau+2^{1-k}) = 0.$$

Concerning $\mathrm{II}_k(\tau)$, since $L^\infty((0,T);L^1_{P_1})\subset L^1((0,T);L^1_{P_1})$, Proposition 5.2.5 (i) implies that

$$\fint_0^T\int_{\mathbb{R}^d}|u(x,t)|(|\partial_t\tilde\theta(x,t)|+|\mathcal{L}\tilde\theta(x,t)|)\,\mathrm{d}x\,\mathrm{d}t<\infty.$$

As a consequence, since $|\tilde\varphi_k^\tau|\le1$, we may apply the DCT twice to obtain

$$\begin{aligned}\operatorname*{ess\,lim}_{\tau\to0^+}\lim_{k\to\infty}\mathrm{II}_k(\tau) &= \operatorname*{ess\,lim}_{\tau\to0^+}\fint_0^T\int_{\mathbb{R}^d}u(x,t)(\partial_t\tilde\theta(x,t)-\mathcal{L}\tilde\theta(x,t))\,\mathrm{d}x\,\mathrm{d}t\\ &= \fint_0^T\int_{\mathbb{R}^d}u(x,t)(\partial_t\tilde\theta(x,t)-\mathcal{L}\tilde\theta(x,t))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Summarizing: by letting first $k\to\infty$ and then $\tau\to0^+$ in (2.3.8) we obtain (2.3.4). □

Remark 2.3.4. An equivalence result similar to Proposition 2.3.3 is given in [108] for a family of nonlocal operators wider than the one considered here, although within a class of solutions more restrictive than $L^\infty((0,T);L^1_{P_1})$.

Uniqueness of very weak solutions

We will prove uniqueness of very weak solutions within two different classes: $L^\infty((0,T);L^1_{P_1})$ and the class of the ones that are nonnegative.

The strategy is as follows. We first prove uniqueness for solutions in the sense of the alternative definition (2.3.3)–(2.3.4), whence, thanks to Proposition 2.3.3, we obtain uniqueness for very weak solutions within $L^\infty((0,T);L^1_{P_1})$. Next, we prove a smoothing result for nonnegative very weak solutions, Lemma 2.3.9 below, showing that they belong to $L^\infty((0,T);L^1_{P_1})$, from where uniqueness follows.

We start with the uniqueness result for solutions in the sense of (2.3.3)–(2.3.4).

Theorem 2.3.5. *Let $\mathcal{L}=\mathcal{L}_K$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure. Let u_1 and u_2 satisfy (2.3.3)–(2.3.4). Then, $u_1=u_2$ a.e. in $\mathbb{R}^d\times(0,T)$.*

Proof. Let $u=u_1-u_2$. Then $u\in L^1_{\mathrm{loc}}([0,T);L^1_{P_1})$ and

$$\fint_0^T\int\!\!\!\!\!\!\int_{\mathbb{R}^d} u(x,t)(\partial_t\tilde\theta(x,t)-\mathcal{L}\tilde\theta(x,t))\,\mathrm{d}x\,\mathrm{d}t=0\quad\text{for all }\tilde\theta\in C_{\mathrm{c}}^\infty(\mathbb{R}^d\times[0,T)).\tag{2.3.9}$$

We have to prove that $u=0$ for a.e. in $\mathbb{R}^d\times(0,T)$. For that purpose, it is enough to show that

$$\fint_0^T\int\!\!\!\!\!\!\int_{\mathbb{R}^d} u(x,t)\tilde\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t=0\quad\text{for all }\tilde\theta\in C_{\mathrm{c}}^\infty(\mathbb{R}^d\times[0,T)).\tag{2.3.10}$$

To this aim we will use Hilbert's duality method. We consider a solution φ for the backward problem

$$\partial_t\varphi-\mathcal{L}\varphi=\tilde\theta\quad\text{in }\mathbb{R}^d\times(0,T).\tag{2.3.11}$$

If φ belonged to $C_{\mathrm{c}}^\infty(\mathbb{R}^d\times[0,T))$, we could use it as a test function in (2.3.9) to obtain

$$0=\fint_0^T\int_{\mathbb{R}^d} u(x,t)(\partial_t\varphi(x,t)-\mathcal{L}\varphi(x,t))\,\mathrm{d}x\,\mathrm{d}t=\fint_0^T\int_{\mathbb{R}^d} u(x,t)\tilde\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

and we would be done. Let $0<t_0<T$ and $R_0>0$ be such that $\operatorname{supp}(\tilde\theta)\subset B_{R_0}\times[0,t_0]$. By Duhamel's formula,

$$\varphi(x,t)=\begin{cases}\displaystyle\int_0^{t_0-t}\int_{\mathbb{R}^d} P_{t_0-t-s}(x-y)\tilde\theta(y,s)\,\mathrm{d}y\,\mathrm{d}s, & (x,t)\in\mathbb{R}^d\times[0,t_0),\\ \varphi=0, & (x,t)\in\mathbb{R}^d\times[t_0,T)\end{cases}\tag{2.3.12}$$

is a solution to (2.3.11). However, though it has the required smoothness, it is not compactly supported in space. Despite this fact, we will show that it is anyway an admissible test function for (2.3.9).

Let $\{\psi_R\}_{R\geq R_0}$ be the family of cutoff functions given by (2.1.12). Since $\psi_R\varphi\in C_{\mathrm{c}}^\infty(\mathbb{R}^d\times[0,T))$, we can use it as a test function in (2.3.9), and we have

$$\fint_0^T\int\!\!\!\!\!\!\int_{\mathbb{R}^d} u(x,t)(\psi_R\partial_t\varphi-\mathcal{L}(\psi_R\varphi))(x,t)\,\mathrm{d}x\,\mathrm{d}t=0.\tag{2.3.13}$$

The operator $\mathcal{L}$ satisfies the nonlocal Leibniz's formula

$$\mathcal{L}(uv) = u\mathcal{L}v + v\mathcal{L}u - B(u,v), \tag{2.3.14}$$

where $B(u,v)$ is the bilinear form given by

$$B(u,v)(x) = \int_{\mathbb{R}^d} (u(x) - u(x-y))(v(x) - v(x-y))K(y)\,\mathrm{d}y. \tag{2.3.15}$$

Then, because of (2.3.11), if $R \geq R_0$ we have that

$$\psi_R \partial_t \varphi - \mathcal{L}(\psi_R \varphi) = \tilde{\theta} - \varphi \mathcal{L}\psi_R + B(\psi_R, \varphi),$$

since $\psi_R \tilde{\theta} = \tilde{\theta}$ for all $R \geq R_0$, whence we may rewrite (2.3.13) as

$$\begin{aligned} 0 = {} & \fint_0^T \int_{\mathbb{R}^d} u(x,t)\tilde{\theta}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \fint_0^T \int_{\mathbb{R}^d} u(x,t)\varphi(x,t)\mathcal{L}\psi_R(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ & + \fint_0^T \int_{\mathbb{R}^d} u(x,t)B(\phi_R,\varphi)(x,t)\,\mathrm{d}t. \end{aligned} \tag{2.3.16}$$

If the integrals

$$\begin{aligned} I_R^1 = \fint_0^{t_0} \int_{\mathbb{R}^d} |u||\varphi\mathcal{L}\psi_R|, \quad I_R^2 = \fint_0^{t_0} \int_{\mathbb{R}^d} |u|B_R^1, \quad I_R^3 = \fint_0^{t_0} \int_{\mathbb{R}^d} |u|B_R^2, \quad \text{where} \\ B_R^1(x,t) := \int_{\{|y|\leq 1\}} |\psi_R(x) - \psi_R(x-y)||\varphi(x,t) - \varphi(x-y,t)|K(y)\,\mathrm{d}y, \\ B_R^2(x,t) := \int_{\{|y|\geq 1\}} |\psi_R(x) - \psi_R(x-y)||\varphi(x,t) - \varphi(x-y,t)|K(y)\,\mathrm{d}y, \end{aligned}$$

go to zero as $R \to \infty$, the desired result (2.3.10) will follow immediately from (2.3.16).

As for I_R^1, since $\psi_R(x) = \psi(x/R)$, then $\Lambda\psi_R(x,y) \to 0$ for all $x, y \in \mathbb{R}^d$ as $R \to \infty$. Moreover, there exists $C > 0$ depending on the uniform norm of ψ and $D^2\psi$, but not on R, such that

$$|\Lambda\psi_R(x,\cdot)|K \leq C(1 \wedge |\cdot|^2)K \in L^1(\mathbb{R}^d) \quad \text{for all } x \in \mathbb{R}^d,\ R \geq R_0, \tag{2.3.17}$$

since K is a Lévy kernel. Then, by the DCT, $\mathcal{L}\psi_R \to 0$ as $R \to 0$. On the other hand, by (2.2.19), the semigroup property (2.2.9) and (C_{P_t}), we have

$$\begin{aligned} |\varphi(x,t)| &\leq C_{\tilde{\theta}} \fint_0^{t_0-t} \int_{\mathbb{R}^d} P_{t_0-t-s}(x-y)P_1(y)\,\mathrm{d}y\mathrm{d}s \\ &= C_{\tilde{\theta}} \int_0^{t_0-t} P_{1+t_0-t-s}(x)\,\mathrm{d}x \leq C_{\tilde{\theta},T} t_0 P_1(x). \end{aligned} \tag{2.3.18}$$

Hence, since $u \in L^1_{\mathrm{loc}}([0,T); L^1_{P_1})$, we may use the DCT to conclude that $I_R^1 \to 0$ as $R \to \infty$.

Let us proceed with the analysis of I_R^2. By the MVT,

$$|\psi_R(x)-\psi_R(x-y)|\le\frac{1}{R}\|\nabla\psi\|_\infty|y|,\quad |\varphi(x,t)-\varphi(x-y,t)|\le|\nabla\varphi(z,t)||y|,$$

where $z\in B_1(x)$. Since

$$\partial_i\varphi(\cdot,t)=\fint_0^{t_0-t}\int_{\mathbb{R}^d}P_{t_0-t-s}(\cdot-y)\partial_i\theta(y,t)\,\mathrm{d}y\,\mathrm{d}s\quad\text{for all } i=1,\dots,d,$$

arguing as when we obtained (2.3.18), we get $|\partial_i\varphi|\le C_{\partial_i\tilde\theta,T}P_1$. Hence, $|\nabla\varphi(z,t)|\le C_{\tilde\theta,T}P_1(z)$ for some $z\in B_1(x)$. Using now (2.2.23) we conclude that

$$|\varphi(x,t)-\varphi(x-y,t)|\le C_{\tilde\theta,T}|y|P_1(x).$$

Thus, since K is a Lévy kernel,

$$B_R^1(x,t)\le\frac{1}{R}CP_1(x)\|\nabla\psi\|_\infty\int_{\{|y|\le1\}}|y|^2K(y)\,\mathrm{d}y\le\frac{1}{R}CP_1(x)\quad\text{for all }(x,t)\in\mathbb{R}^d\times[0,t_0)$$

for some constant $C>0$. Therefore, since $u\in L^1([0,T);L^1_{P_1})$, by the DCT, $I_R^2\to0$ as $R\to\infty$.

Finally, we deal with I_R^3. Since $\|\psi_R\|_\infty=1$, $\|\varphi\|_\infty\le\|\tilde\theta\|_\infty$ and K is a Lévy kernel, then

$$|\psi_R(x)-\psi_R(x-y)||\varphi(x,t)-\varphi(x-y,t)|K(y)\le4\|\tilde\theta\|_\infty K(y)\in L^1(\mathbb{R}^d\setminus B_1(0)).$$

Besides, $|\psi_R(x)-\psi_R(x-y)|\to0$ a.e. as $R\to\infty$, whence, by the DCT, $B_R^2(x,t)\to0$ for all $x\in\mathbb{R}^d$ and $0\le t\le t_0$ as $R\to\infty$. On the other hand, $\|\psi_R\|_\infty=1$,

$$B_R^2(x,t)\le2|\varphi(x,t)|\int_{\{|y|\ge1\}}K(y)\,\mathrm{d}y+2\int_{\{|y|\ge1\}}|\varphi(x-y,t)|K(y)\,\mathrm{d}y.$$

The first term on the right-hand side is easily seen to be less or equal than $CP_1(x)$, using again that K is a Lévy kernel and (2.3.18). Besides, by (2.3.18), (C_{PK}), the semigroup property (2.2.9), and (C_{P_t}),

$$\begin{aligned}\fint_{\{|y|\ge1\}}|\varphi(x-y,t)|K(y)\,\mathrm{d}y&\le C_{\tilde\theta}\int_{\{|y|\ge1\}}P_1(x-y)K(y)\,\mathrm{d}y\le C_{\tilde\theta}\int_{\{|y|\ge1\}}P_1(x-y)P_1(y)\,\mathrm{d}y\\&\le C_{\tilde\theta}\int_{\mathbb{R}^d}P_1(x-y)P_1(y)\,\mathrm{d}y=C_{\tilde\theta}P_2(x)\le C_{\tilde\theta,T}P_1(x).\end{aligned}$$

Thus, $B_R^2(x,t)\le C_{\tilde\theta,T}P_1(x)$. Therefore, since $u\in L^1_{\mathrm{loc}}([0,T);L^1_{P_1})$, we may use the DCT and we get $I_R^3\to0$ as $R\to\infty$. □

As an immediate consequence of Theorem 2.3.5 and Proposition 2.3.3, we obtain uniqueness for very weak solutions within the class $L^\infty((0,T);L^1_{P_1})$.

Corollary 2.3.6 (Uniqueness for very weak solutions within $L^\infty((0,T);L^1_{P_1})$)**.** *Let $\mathcal{L}=\mathcal{L}_K$ with K satisfying (A0)–(A2). There is at most one very weak solution of (NLHE) belonging to $L^\infty((0,T);L^1_{P_1})$ with a given initial trace.*

Now we will obtain uniqueness in the class of nonnegative very weak solutions. If u is a very weak solution, then $u\in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$; see Remark 5.2.16. Hence, if we are able to prove that nonnegative solutions in that class belong to $u\in L^\infty((0,T);L^1_{P_1})$, uniqueness will be inherited from Corollary 2.3.6. In order to prove this smoothing result we follow ideas from [19].

Let us start with a heuristic proof. Let u be a nonnegative very weak solution to (NLHE) and $f(t):=\int_{\mathbb{R}^d}u(x,t)P_1(x)\,\mathrm{d}x$. Then, formally ($f$ is not necessarily smooth; it may even be infinite for some times, so we will have to take some extra care when performing the rigorous proof),

$$|f'(t)|=\int_{\mathbb{R}^d}\partial_t u(x,t)P_1(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\mathcal{L}u(x,t)P_1(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}u(x,t)\mathcal{L}P_1(x)\,\mathrm{d}x\,.$$

Therefore, if $|\mathcal{L}P_1|\le cP_1$ for some constant $c>0$, then $|f'(t)|\le cf(t)$, since u is nonnegative, which yields, upon integration,

$$e^{-c|t-\tau|}f(t)\le f(\tau)\le e^{c|t-\tau|}f(t)\quad\text{for almost every } 0<t,\tau<T. \tag{2.3.19}$$

By Remark 5.2.16, $u\in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$; hence $f(t_0)<\infty$ for some time $t_0\in(0,T)$. Therefore, by (2.3.19), $u\in L^\infty((0,T);L^1_{P_1})$.

Besides the already mentioned regularity issues, we find a second difficulty when trying to follow the above programme: up to now, the inequality $|\mathcal{L}P_1|\le cP_1$ is only known to hold when $\mathcal{L}$ is the fractional Laplacian; see Proposition 2.2.16 (iv). To overcome this hitch, the strategy is to construct a positive $\phi\in C^2(\mathbb{R}^d)$ comparable to P_1 such that $|\mathcal{L}\phi|\le C\phi$ for some $C>0$ and use it instead of P_1, multiplied by a convenient smooth cutoff function of time, as a test function in the definition of very weak solution. This will yield an integrated version of the ODE inequality $|g'|\le Cg$, with $g(t):=\int_{\mathbb{R}^d}u(x,t)\phi(x)\,\mathrm{d}x$, that will imply that

$$e^{-C|t-\tau|}\|u(\cdot,\tau)\|_{L^1(\mathbb{R}^d,\,\phi\,\mathrm{d}x)}\le\|u(\cdot,t)\|_{L^1(\mathbb{R}^d,\,\phi\,\mathrm{d}x)}\le e^{C|t-\tau|}\|u(\cdot,\tau)\|_{L^1(\mathbb{R}^d,\,\phi\,\mathrm{d}x)} \tag{2.3.20}$$

for almost every $0<t,\tau<T$. Since P_1 and ϕ are comparable, this is equivalent to (2.3.19).

The first step is then to construct the desired function ϕ. It is here that we need assumption (A3).

Lemma 2.3.7. *Let $\mathcal{L}=\mathcal{L}_K$ with K satisfying (A0)–(A3), and B the bilinear form defined in (2.3.15). There exist $\phi\in C^2(\mathbb{R}^d)$ and a constant $c>1$ such that*

$$c^{-1}P_1\le\phi\le cP_1, \tag{2.3.21}$$

$$|\mathcal{L}\phi|\le c\phi,\qquad |B(\psi,\phi)|\le c\max\{\|\psi\|_\infty,\|\nabla\psi\|_\infty\}\phi\quad\text{for all }\psi\in C^\infty_{\mathrm{c}}(\mathbb{R}^d). \tag{2.3.22}$$

Proof. Let $\tilde{\varphi}$ as in (A1)–(A3). Let $\phi \in C^2(\mathbb{R}^d)$ such that $\phi > 0$ and

$$\phi(x) = \begin{cases} 1, & |x| \le 1, \\ \frac{1}{|x|^d \tilde{\varphi}(|x|)}, & |x| \ge 2. \end{cases}$$

Note that ϕ satisfies (2.3.21), due to the way in which it is defined.

Let us start by proving that ϕ satisfies

$$\int_{\{|y|\le 1\}} |\Lambda\phi(x,y)|K(y)\,dy \le c\phi(x), \quad \int_{\{|y|\le 1\}} |y||\phi(x)-\phi(x-y)|K(y)\,dy \le c\phi(x), \quad x \in \mathbb{R}^d. \tag{2.3.23}$$

A direct calculation shows that there is a constant $C > 0$ such that

$$\begin{aligned} &|\nabla\phi(x)| \le C\phi(x)\big(1 + \frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(x)}\big), \\ &|D^2\phi(x)| \le C\phi(x)\big(1 + \frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(x)} + \big(\frac{\tilde{\varphi}'(x)}{\tilde{\varphi}(x)}\big)^2 + \frac{\tilde{\varphi}''(x)}{\tilde{\varphi}(x)}\big), \end{aligned} \qquad \text{for all } |x| \ge 3,$$

whence, thanks to (A3), there is a constant $C > 0$ such that

$$|\nabla\phi(x)| \le C\phi(x), \quad |D^2\phi(x)| \le C\phi(x) \quad \text{for all } |x| \ge 3. \tag{2.3.24}$$

Then, applying the MVT once or twice as appropriate, if $|x| \ge 3$ and $|y| \le 1$,

$$\begin{aligned} |\Lambda\phi(x,y)|K(y) &\le |D^2\phi(z)||y|^2K(y) \le C\phi(z)|y|^2K(y), \\ |y||\phi(x)-\phi(x-y)|K(y) &\le |\nabla\phi(z)||y|^2K(y) \le C\phi(z)|y|^2K(y), \end{aligned}$$

where $z = z(x,y) \in B_1(x)$. Moreover, since P_1 is slowly changing (see Proposition 2.2.6) and ϕ satisfies (2.3.21), we have that ϕ is also slowly changing. Then, since $z \in B_1(x)$,

$$\begin{aligned} &|\Lambda\phi(x,y)|K(y) \le C\phi(x)|y|^2K(y), \\ &|y||\phi(x)-\phi(x-y)|K(y) \le C\phi(x)|y|^2K(y) \end{aligned} \qquad \text{if } |x| \ge 3,\ |y| \le 1, \tag{2.3.25}$$

and (2.3.23) follows for $|x| \ge 3$, since K is a Lévy kernel. Besides, since $\phi > 0$, applying the MVT,

$$\begin{aligned} &|\Lambda\phi(x,y)|K(y) \le \|D^2\phi\|_{L^\infty(B_4(0))}|y|^2K(y) \le C\phi(x)|y|^2K(y), \\ &|y||\phi(x)-\phi(x-y)|K(y) \le \|\nabla\phi\|_{L^\infty(B_4(0))}|y|^2K(y) \le C\phi(x)|y|^2K(y), \end{aligned} \qquad \text{if } |x| \le 3, \tag{2.3.26}$$

whence (2.3.23) also holds for all $|x| \le 3$, because of K being a Lévy kernel.

On the other hand, since ϕ is positive, K is a symmetric Lévy kernel, (C_{PK}), the semigroup property (2.2.9), (C_{P_t}) and (2.3.21), there is $C > 1$ such that

$$\begin{aligned} &\max\Big\{\int_{\{|y|\ge 1\}} |\Lambda\phi(x,y)|K(y)\,dy, \iint_{\{|y|\ge 1\}} |\phi(x)-\phi(x-y)|K(y)\,dy\Big\} \\ &\quad\le \phi(x)\int_{\{|y|\ge 1\}} K(y)\,dy + \int_{\{|y|\ge 1\}} \phi(x-y)K(y)\,dy \\ &\quad\le C\Big(\phi(x) + \int_{\{|y|\ge 1\}} P_1(x-y)P_1(y)\,dy\Big) \le C(\phi(x)+P_2(x)) \le C(\phi(x)+P_1(x)) \le C\phi(x), \end{aligned}$$

for all $x \in \mathbb{R}^d$. Then,

$$\int_{\{|y|\geq 1\}} |\Lambda\phi(x,y)|K(y)\,\mathrm{d}y \leq c\phi(x), \quad \int_{\{|y|\geq 1\}} |\phi(x)-\phi(x-y)|K(y)\,\mathrm{d}y \leq c\phi(x), \quad x \in \mathbb{R}^d. \tag{2.3.27}$$

Thus, combining (2.3.23) and (2.3.27), we obtain $|\mathcal{L}\phi| \leq c\phi$, and by the MVT, for a given $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} |B(\psi,\phi)(x)| &\leq \|\nabla\psi\|_\infty \int_{\{|y|\leq 1\}} |y||\phi(x)-\phi(x-y)|K(y)\,\mathrm{d}y \\ &\quad + 2\|\psi\|_\infty \int_{\{|y|\geq 1\}} |\phi(x)-\phi(x-y)|K(y)\,\mathrm{d}y \\ &\leq c\max\{\|\psi\|_\infty, \|\nabla\psi\|_\infty\}\phi(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

□

Remark 2.3.8. The key points to obtain (2.3.22) are inequalities (2.3.23) and (2.3.27). The latter is guaranteed if ϕ has a tail comparable to P_1. The former may be understood as a weak requirement of boundedness for the Hessian and the gradient of ϕ respectively, in relationship with the Lévy kernel K. It follows from (A3). We could have used the existence of a function ϕ satisfying (2.3.23)–(2.3.27) as an assumption instead of (A3). However, we prefer (A3), since it is more easily verified.

We can proceed now with a rigorous proof.

Lemma 2.3.9. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A3).*
(i) *If u is a nonnegative very weak solution to (NLHE), there is a constant $c > 0$ such that*

$$e^{-c|t-\tau|}\|u(\cdot,\tau)\|_{L^1_{P_1}} \leq \|u(\cdot,t)\|_{L^1_{P_1}} \leq e^{c|t-\tau|}\|u(\cdot,\tau)\|_{L^1_{P_1}} \quad \text{for a.e. } t,\tau \in (0,T). \tag{2.3.28}$$

(ii) *If u is a nonnegative very weak solution to (NLHE), then $u \in L^\infty((0,T);L^1_{P_1})$.*

Proof. (i) As we have just explained, it is enough to prove (2.3.20).

Since u is a very weak solution, if $g(t) := \int_{\mathbb{R}^d} u(x,t)\phi(x)\,\mathrm{d}x$ were well defined for all $t \in (0,T)$ and $\phi\mathcal{X}_{[t_0,t_1]}$ were an admissible test function, we would have

$$\begin{aligned} \int_{\mathbb{R}^d} u(x,t_0)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x,t_1)\phi(x)\,\mathrm{d}x &= \fint_0^T \int_{\mathbb{R}^d} u(x,t)\partial_t(\phi\mathcal{X}_{[t_0,t_1]})(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ = \fint_0^T \int_{\mathbb{R}^d} u(x,t)\mathcal{L}(\phi\mathcal{X}_{[t_0,t_1]})(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \fint_{t_0}^{t_1} \int_{\mathbb{R}^d} u(x,t)\mathcal{L}\phi(x)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $0 < t_0 < t_1 < T$. Then, by the first inequality in (2.3.22), since u is nonnegative, there would be $C' > 0$ such that

$$\int_{\mathbb{R}^d} u(x,t_0)\phi(x)\,\mathrm{d}x - \fint_{\mathbb{R}^d} u(x,t_1)\phi(x)\,\mathrm{d}x \leq C' \fint_{t_0}^{t_1} \int_{\mathbb{R}^d} u(x,t)\phi(x)\,\mathrm{d}x\,\mathrm{d}t, \quad 0 < t_0 < t_1 < T, \tag{2.3.29}$$

whence, using a Gronwall type argument, $e^{-C'|t_0-t_1|}g(t_0) \le g(t_1) \le e^{C'|t_0-t_1|}g(t_0)$, $t_0, t_1 \in (0,T)$.

However, $\phi\mathcal{X}_{[t_0,t_1]}$ is not an admissible test function, and we do not know if g is well defined for all $t \in (0,T)$, only a.e. in $(0,T)$, since $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ (see Remark 5.2.16), and P_1 and ϕ are comparable; see (2.3.21). Therefore, we will need to construct suitable approximations of ϕ and $\mathcal{X}_{[t_0,t_1]}$ such that the product is a suitable test function that in the limit yields (2.3.29) for a.e. $t_0, t_1 \in (0,T)$.

We approximate ϕ by $\phi\psi_n$, $n \in \mathbb{N}$ with ψ_n a cutoff function of the form (2.1.12) with $R = n$.

As for $\mathcal{X}_{[t_0,t_1]}$, let $\eta : \mathbb{R} \to \mathbb{R}$ as in (2.3.5). The family $\{\eta_k\}_{k\in\mathbb{N}}$ given by $\eta_k(t) := 2^k\eta(2^kt)$ is a summability kernel. For $0 < t_0 < t_1 < T$ and $k_0 > -\max\{\ln(t_1-t_0), \ln t_0, \ln(T-t_1)\}/\ln 2$, let

$$\varphi_k^{t_0,t_1}(t) = \fint_{2^k(t_0-t)}^{2^k(t_1-t)} \eta(s)\,\mathrm{d}s, \quad k \ge k_0,\ t > 0.$$

Then, since η is symmetric,

$$\mathcal{X}_{[t_0+\frac{1}{2^k},t_1-\frac{1}{2^k}]} \le \varphi_k^{t_0,t_1} \le \mathcal{X}_{[t_0-\frac{1}{2^k},t_1+\frac{1}{2^k}]}, \quad (\varphi_k^{t_0,t_1})'(t) = \eta_k(t-t_0) - \eta_k(t-t_1)$$

for all $k \ge k_0$ and $t > 0$. In particular, $(\varphi_k^{t_0,t_1})' \to \delta_{t_0} - \delta_{t_1}$ as $k \to \infty$ in the sense of distributions.

Let $g_n(t) := \int_{\mathbb{R}^d} u(x,t)\psi_n(x)\phi(x)\,\mathrm{d}x$, $n \in \mathbb{N}$. For notational convenience, let $g_\infty(t) = g(t)$ for $t > 0$. Since u is a very weak solution, then $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ (see Remark 5.2.16). On the other hand, as ϕ and P_1 are comparable, see (2.3.21), then $g_\infty \in L^1_{\mathrm{loc}}((0,T))$ and $g_n \in L^1_{\mathrm{loc}}((0,T))$ for all $n \in \mathbb{N}$, whence, by Lebesgue's Differentiation Theorem, $|(0,T) \setminus L(g_\infty;\{\eta_k\}_{k\in\mathbb{N}})| = 0$ and $|(0,T) \setminus L(g_n;\{\eta_k\}_{k\in\mathbb{N}})| = 0$ for all $n \in \mathbb{N}$. As a straightforward consequence,

$$|(0,T) \setminus \bigcap_{n=1}^{\infty} L(g_n;\{\eta_k\}_{k\in\mathbb{N}})| = 0. \tag{2.3.30}$$

Finally, consider two times $0 < \tau_0 < \tau_1 < T$ such that $\tau_0, \tau_1 \in \cap_{n=1}^\infty L(g_n;\{\eta_k\}_{k\in\mathbb{N}})$.

Since u is a very weak solution and $\varphi_k^{\tau_0,\tau_1}\psi_n\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d \times (0,T))$ for all $k > k_0$ and $n \in \mathbb{N}$,

$$\fint_0^T (\varphi_k^{\tau_0,\tau_1})'(t) \int_{\mathbb{R}^d} \psi_n(x)\phi(x)u(x,t)\,\mathrm{d}x\,\mathrm{d}t = \fint_0^T \varphi_k^{\tau_0,\tau_1}(t) \int_{\mathbb{R}^d} u(x,t)\mathcal{L}(\psi_n\phi)(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.3.31}$$

Besides, thanks to (2.3.17), there is $C > 0$ depending on $\|\psi\|_\infty$ and $\|D^2\psi\|_\infty$ such that $|\mathcal{L}\psi_n| \le C$ for all $n \in \mathbb{N}$. Moreover, by (2.3.22) there exists $C >$ such that $|\mathcal{L}\phi| \le C\phi$ and $|B(\psi_n,\phi)| \le C\phi$ for all $n \in \mathbb{N}$. Therefore, by Leibniz's formula (2.3.14), there is $C_2 > 0$ such that

$$|\mathcal{L}(\psi_n\phi)(x)| \le C_2\phi(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } n \in \mathbb{N}.$$

Then, since u and $\varphi_k^{\tau_0,\tau_1}$ are nonnegative, (2.3.31) yields

$$\fint_0^T (\varphi_k^{\tau_0,\tau_1})'(t) g_n(t)\,\mathrm{d}t \ \le C_2 \fint_0^T \varphi_k^{\tau_0,\tau_1}(t) g_\infty(t)\,\mathrm{d}t \quad \text{for all } k > k_0 \text{ and } n \in \mathbb{N}.$$

Then, since $\tau_0, \tau_1 \in \cap_{n=1}^\infty L(g_n; \{\eta_k\}_{k\in\mathbb{N}})$, taking $k \to \infty$,

$$|g_n(\tau_1) - g_n(\tau_0)| \le C_2 \fint_{\tau_0}^{\tau_1} g_\infty(t)\,\mathrm{d}t \quad \text{for all } n \in \mathbb{N}.$$

Since ψ_n converges monotonically to 1 as $n \to \infty$ and u is nonnegative, the Monotone Convergence Theorem (MCT from now on) yields $g_n(\tau_i) \to g_\infty(\tau_i)$ as $n \to \infty$ for $i = 0, 1$. Therefore,

$$|g_\infty(\tau_1) - g_\infty(\tau_0)| \le C_2 \fint_{\tau_0}^{\tau_1} g_\infty(t)\,\mathrm{d}t. \tag{2.3.32}$$

Since C_2 does not depend on τ_0 and τ_1, we have

$$g_\infty(t) \le g_\infty(\tau_1) + C_2 \fint_t^{\tau_1} g_\infty(s)\,\mathrm{d}s \quad \text{for all } t \in [\tau_0, \tau_1) \cap \bigcap_{n=1}^\infty L(g_n; \{\eta_k\}_{k\in\mathbb{N}}). \tag{2.3.33}$$

We will prove now, using an induction argument, a Gronwall-type lemma: if g_∞ satisfies the integral estimate (2.3.33), it satisfies a pointwise estimate.

For $k = 1$, by (2.3.33) and (2.3.30), we have

$$g_\infty(t) \le g_\infty(\tau_1) + C_2 \fint_t^{\tau_1} \Big(g_\infty(\tau_1) + C_2 \fint_s^{\tau_1} g_\infty(r)\,\mathrm{d}r \Big)\,\mathrm{d}s.$$

Therefore, since

$$\fint_t^{\tau_1} \fint_s^{\tau_1} g_\infty(r)\,\mathrm{d}r\,\mathrm{d}s = \fint_t^{\tau_1} \fint_t^{r} g_\infty(r)\,\mathrm{d}s\,\mathrm{d}r = \fint_t^{\tau_1} g_\infty(r)(r-t)\,\mathrm{d}r,$$

we have that

$$g_\infty(t) \le g_\infty(\tau_1)(1 + C_2(\tau_1 - t)) + C_2^2 \fint_t^{\tau_1} g_\infty(s)(s-t)\,\mathrm{d}s$$

for all $t \in \cap_{n=1}^\infty L(g_n)$ such that $\tau_0 \le t < \tau_1$. Iterating, we obtain that

$$g_\infty(t) \le g_\infty(\tau_1)\Big(\sum_{j=0}^{k} \frac{C_2^j (\tau_1 - t)^j}{j!} \Big) + C_2^{k+1} \fint_t^{\tau_1} g_\infty(s) \frac{(s-t)^k}{k!}\,\mathrm{d}s$$

for all $t \in [\tau_0, \tau_1) \cap \bigcap_{n=1}^\infty L(g_n)$, $k \in \mathbb{N}$. Note that

$$\fint_t^{\tau_1} g_\infty(s) \frac{(s-t)^k}{k!}\,\mathrm{d}s \le \frac{(\tau_1 - t)^k}{k!} \fint_t^{\tau_1} g_\infty(s)\,\mathrm{d}s \le c\frac{T^k}{k!} \fint_{\tau_0}^{\tau_1} \int_{\mathbb{R}^d} u(x,t) P_1(x)\,\mathrm{d}x$$

where c is the constant in (2.3.21). Therefore,

$$g_\infty(t) \le g_\infty(\tau_1)\Big(\sum_{j=0}^{k} \frac{C_2^j(\tau_1 - t)^j}{j!}\Big) + c\frac{C_2^{k+1}T^k}{k!} \fint_{\tau_0}^{\tau_1} \int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x,$$

for all $t \in \cap_{n=1}^{\infty} L(g_n)$ such that $t \in [\tau_0, \tau_1)$ and $k \in \mathbb{N}$. By Remark 5.2.16, $u \in L^1((\tau_0,\tau_1); L^1_{P_1})$ and therefore taking $k \to \infty$ in the above inequality, we obtain

$$g_\infty(t) \le g_\infty(\tau_1)e^{C_2(\tau_1 - t)} \quad \text{a.e. } t \in (\tau_0, \tau_1)$$

due to (2.3.30). Since C_2 does not depend on τ_0 or τ_1, the result can be extrapolated and we obtain

$$g_\infty(t) \le g_\infty(\tau)e^{C_2(\tau - t)} \quad \text{a.e. } t, \tau \in (0,T),\ t \le \tau. \tag{2.3.34}$$

Repeating the induction argument, starting now from

$$g_\infty(\tau_1) \le g_\infty(t) + C_2 \fint_t^{\tau_1} g_\infty(s)\,\mathrm{d}s \quad \text{for all } t \in [\tau_0,\tau_1) \cap \bigcap_{n=1}^{\infty} L(g_n; \{\eta_k\}_{k\in\mathbb{N}}),$$

which is also implied by (2.3.32), we arrive at

$$g_\infty(\tau) \le g_\infty(t)e^{C_2(\tau - t)} \quad \text{a.e. } t, \tau \in (0,T),\ t \le \tau,$$

which combined with (2.3.34) gives (2.3.20).
(ii) Since u is a very weak solution, $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$; see Remark 5.2.16. Hence, $\|u(\cdot,\tau)\|_{L^1_{P_1}} < \infty$ for some $\tau \in (0,T)$. The result now follows from the second inequality in (2.3.28). □

The combination of Corollary 2.3.6 and Lemma 2.3.9 (ii) yields the desired uniqueness result for nonnegative very weak solutions.

Corollary 2.3.10 (Uniqueness for nonnegative very weak solutions)**.** *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A3). There is at most one nonnegative very weak solution of (NLHE) with a given initial trace.*

We finish this paragraph proving that nonnegative solutions have some regularity in time.

Proposition 2.3.11. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A3), and let u be a nonnegative very weak solution to (NLHE).*
(i) *Let $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22). There is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^d} u(x,\tau_1)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x,\tau_2)\phi(x)\,\mathrm{d}x \ \le C\|u\|_{L^\infty((0,T);L^1_{P_1})}|\tau_1 - \tau_2| \tag{2.3.35}$$

for a.e. $\tau_1, \tau_2 \in (0,T)$.

(ii) *Given $\psi \in C_c^2(\mathbb{R}^d)$, there is a constant $C_\psi > 0$ such that*

$$\int_{\mathbb{R}^d} u(x,\tau_1)\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x,\tau_2)\psi(x)\,\mathrm{d}x \ \le C_\psi \|u\|_{L^\infty((0,T);L^1_{P_1})}|\tau_1-\tau_2| \tag{2.3.36}$$

for a.e. $\tau_1,\tau_2 \in (0,T)$.

Proof. (i) We have seen in the proof of Lemma 2.3.9 that there is a constant $C>0$ such that

$$\int_{\mathbb{R}^d} u(x,\tau_1)\phi(x)\,\mathrm{d}x - \fint_{\mathbb{R}^d} u(x,\tau_2)\phi(x)\,\mathrm{d}x \ \le C \fint_{\tau_1\wedge\tau_2}^{\tau_1\vee\tau_2} u(x,t)P_1(x)\,\mathrm{d}x \quad \text{for a.e. } \tau_1,\tau_2 \in (0,T);$$

see (2.3.32) and (2.3.21). The result now follows from Lemma 2.3.9 (ii).
(ii) By Lemma 2.2.3, there is a constant $C_\psi > 0$ such that $|\mathcal{L}\psi| \le C_\psi P_1$. Hence, we can proceed as in the proof of Lemma 2.3.9, with ϕ replaced by ψ, to obtain that

$$\int_{\mathbb{R}^d} u(x,\tau_1)\psi(x)\,\mathrm{d}x - \fint_{\mathbb{R}^d} u(x,\tau_2)\psi(x)\,\mathrm{d}x \ \le C_\psi \fint_{\tau_1\wedge\tau_2}^{\tau_1\vee\tau_2} u(x,t)P_1(x)\,\mathrm{d}x \quad \text{for a.e. } \tau_1,\tau_2 \in (0,T),$$

whence the result, thanks again to Lemma 2.3.9 (ii). □

If $\mathcal{L} = (-\Delta)^{\alpha/2}$, the smooth function $\phi = P_1^\alpha$, where P_1^α denotes the heat kernel associated to $(-\Delta)^{\alpha/2}$ (see Subsection 2.2.3), satisfies (trivially) (2.3.21). Next lemma implies that it also satisfies (2.3.22). Therefore, nonnegative very weak solutions are Lipschitz as functions in time with values in $L^1_{P_1}$.

Remark 2.3.12. We already know by Proposition 2.2.16 (iv) that the first inequality in (2.3.22) holds. However, we give an alternative proof that might be useful to deal with other operators.

Lemma 2.3.13. *Let $\mathcal{L} = (-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$, and $0 < \varepsilon < T$. Then, there is $C > 0$ such that*

$$|(-\Delta)^{\alpha/2}P_t^\alpha| \le \int_{\mathbb{R}^d} |\Lambda P^\alpha(\cdot,y,t)|K(y)\,\mathrm{d}y \le CP_1^\alpha, \quad |B(\psi,P_t^\alpha)| \le C\max\{\|\psi\|_\infty, \|\nabla\psi\|_\infty\}P_1^\alpha \tag{2.3.37}$$

for all $t \in (\varepsilon,T)$ where B is the bilinear form given by (2.3.15) with $K = K_\alpha$.

Proof. The key to get (2.3.37) is to prove that there is $c > 1$ such that

$$\int_{\{|y|\le 1\}} |\Lambda P_t^\alpha(x,y)|K(y)\,\mathrm{d}y \le cP_1^\alpha(x), \quad \int_{\{|y|\le 1\}} |y||P_t^\alpha(x) - P_t^\alpha(x-y)|K(y)\,\mathrm{d}y \le cP_1^\alpha(x), \tag{2.3.38}$$

$$\int_{\{|y|\ge 1\}} |\Lambda P_t^\alpha(x,y)|K(y)\,\mathrm{d}y \le cP_1^\alpha(x), \quad \int_{\{|y|\ge 1\}} |P_t^\alpha(x) - P_t^\alpha(x-y)|K(y)\,\mathrm{d}y \le cP_1^\alpha(x). \tag{2.3.39}$$

for all $x \in \mathbb{R}^d$; see the proof of Lemma 2.3.7. The estimates (2.3.39) are obtained easily, following closely the proof of Lemma 2.3.7, using that K_α is a symmetric Lévy kernel, K_α and P_1^α have comparable tails (see (C_{PK})), the semigroup property (2.2.9) and (C_{P_t}).

On the other hand, by Proposition 2.2.16 (vi), using Taylor's expansion, for $x \in \mathbb{R}^d$ and $|y| \le 1$,

$$|\Lambda P^\alpha(x,y,t)| \le C|D^2 P_t^\alpha(z)||y|^2 \le C P_1^\alpha(z)|y|^2,$$
$$|y||P_t^\alpha(x) - P_t^\alpha(x-y)| \le |\nabla P_t^\alpha(z')||y|^2 \le C P_1^\alpha(z')|y|^2$$

where z and z' depends on x and y and $z, z' \in B_1(x)$. Since $P_1^\alpha(\cdot)$ is slowly changing, Proposition 2.2.6,

$$|\Lambda P^\alpha(x,y,t)| \le C P_1^\alpha(x)|y|^2, \quad |y||P_1^\alpha(x) - P_1^\alpha(x-y)| \le C P_1^\alpha(x)|y|^2.$$

Therefore, since K_α is a Lévy kernel, we obtain (2.3.38). □

Corollary 2.3.14. *Let $\mathcal{L} = (-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$ and let u be a nonnegative very weak solution to (NLHE). There is a constant $C > 0$ such that*

$$\|u(\cdot,\tau_1)\|_{L^1_{P_1}} - \|u(\cdot,\tau_2)\|_{L^1_{P_1}} \le C\|u\|_{L^\infty((0,T);L^1_{P_1})}|\tau_1 - \tau_2| \quad \text{for a.e. } \tau_1, \tau_2 \in T.$$

Existence of an initial trace

We now prove that every nonnegative solution has an initial trace. We also have to assume (A3) for this purpose.

Theorem 2.3.15 (Existence of initial trace integrable against P_1)**.** *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A3). Let u be a nonnegative very weak solution of (NLHE). There is a nonnegative Radon measure μ_0 such that (2.1.10) holds; that is, μ_0 is the initial trace of u. Moreover, μ_0 satisfies the integral growth condition (GC).*

Proof. Let u be a nonnegative very weak solution of (NLHE). By Remark 5.2.16, $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, whence there is some $\tilde{T} \in (0,T)$ such that $\int_{\mathbb{R}^d} u(x,\tilde{T})P_1(x)\,\mathrm{d}x < \infty$. Besides, by Lemma 2.3.9, there is some exceptional set $E \subset (0,T)$ with $|E| = 0$ such that

$$\int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x \le e^{cT} \fint_{\mathbb{R}^d} u(x,\tilde{T})P_1(x)\,\mathrm{d}x < \infty \quad \text{for all } t \in (0,\tilde{T}) \setminus E.$$

On the other hand, for each $R > 0$, let $c_R := 1/\min\{P_1(x) : |x| \le R\}$. This quantity is finite for each R, since P_1 is continuous and $P_1 > 0$. Hence

$$\sup_{t\in(0,T)\setminus E} \int_{B_R(0)} u(x,t)\,\mathrm{d}x \le c_R \fint_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x \le c_R e^{cT} \fint_{\mathbb{R}^d} u(x,\tilde{T})P_1(x)\,\mathrm{d}x = C_R.$$

This implies weak* compactness in the sense of Radon measures: there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}} \subset (0,T)\setminus E$ with $t_n \to 0^+$ as $n \to \infty$ and a nonnegative Radon measure μ_0 such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \psi(x)u(x,t_n)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\mu_0 \quad \text{for all } \psi \in C_{\mathrm{c}}(\mathbb{R}^d).$$

For a reference, see for instance [105, Theorem 4.4]. Moreover, for any $R > 0$, $\mu_0(B_R(0)) \leq C_R$.

However, we are not done yet, since the limit might depend on the sequence. Let us discard this possibility. Suppose that for some sequence $\{\tilde{t}_n\}_{n\in\mathbb{N}} \subset (0,T)\setminus E$ with $\tilde{t}_n \to 0^+$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \psi(x) u(x,\tilde{t}_n)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\tilde{\mu}_0(x) \quad \text{for all } \psi \in C_{\mathrm{c}}(\mathbb{R}^d).$$

If moreover $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$, taking $\tau_1 = t_n$ and $\tau_2 = \tilde{t}_k$ in (2.3.36), we obtain

$$\int_{\mathbb{R}^d} u(x,t_n)\psi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} u(x,\tilde{t}_k)\psi(x)\,\mathrm{d}x \leq C_\psi \|u\|_{L^\infty((0,T);L^1_{P_1})}|t_n - t_k| \quad \text{for all } n,k \in \mathbb{N}.$$

Letting first $n \to \infty$ and then $k \to \infty$, we conclude that

$$\int \psi\,\mathrm{d}\mu_0 = \int_{\mathbb{R}^d} \psi\,\mathrm{d}\tilde{\mu}_0(x) \quad \text{for all } \psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d). \tag{2.3.40}$$

By a density argument, (2.3.40) is also true for $\psi \in C_{\mathrm{c}}(\mathbb{R}^d)$, hence $\mu_0 \equiv \tilde{\mu}_0$ as Radon measures on $\mathbb{R}^d$.

To conclude, we have to prove that μ_0 satisfies (GC). To this aim, we consider a family of cut-off functions $\{\psi_R\}_{R>0}$ with ψ_R as in (2.1.12). By Lemma 2.3.9, for all $t \in (0,T)\setminus E$ we have

$$\fint_{\mathbb{R}^d} u(x,t)\psi_R(x)P_1(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} u(x,t)P_1(x)\,\mathrm{d}x \leq e^{cT}\fint_{\mathbb{R}^d} u(x,\tilde{T})P_1(x)\,\mathrm{d}x = C_{\tilde{T},T}$$

Thus, letting $t \to 0^+$, we get $\int_{\mathbb{R}^d} \psi_R P_1\,\mathrm{d}\mu_0 \leq C_{\tilde{T},T}$, whence, by the MCT, $\int_{\mathbb{R}^d} P_1\,\mathrm{d}\mu_0 \leq C_{\tilde{T},T}$. □

Existence of very weak solutions

We have just seen that nonnegative weak solutions to equation (NLHE) have an initial trace μ_0 that is a Radon measure satisfying (GC). We now investigate the reciprocal. Is it true that for any nonnegative Radon measure μ_0 satisfying (GC) there is a very weak solution to (NLHE) having μ_0 as initial trace? We will answer to this question positively. In fact, we will be able to construct also solutions with sign changes, if the initial trace satisfies a suitable growth condition closely related to (GC). These solutions belong to the class $L^\infty((0,T);L^1_{P_1})$ for which there is uniqueness. As a corollary we will obtain a representation formula.

Theorem 2.3.16. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure with the admissible growth at infinity (GCS). Then, for each $T > 0$ the function U given in (S_{μ_0}) is a very weak solution of (NLHE) in $C(\mathbb{R}^d \times (0,T)) \cap L^\infty((0,T);L^1_{P_1})$ with initial trace μ_0.*

Proof. We first check that U is well defined. By (C_{P_t}), $P_t \le C_t P_1$ for some $C_t > 0$ depending on t. Then, using Lemma 2.2.4 and (GCS), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}|\mu_0|(y) &\le C_t \int_{\mathbb{R}^d} P_1(x-y)\,\mathrm{d}|\mu_0|(y)\\ &\le C_{x,t} \iint_{\mathbb{R}^d} P_1(y)\,\mathrm{d}|\mu_0|(y) < \infty, \quad (x,t)\in\mathbb{R}^d\times(0,\infty).\end{aligned} \tag{2.3.41}$$

Now, we will prove that $U \in L^\infty((0,T);L^1_{P_1})$. By Tonelli's theorem, the fact that P_t is symmetric, the semigroup property (2.2.9), (C_{P_t}) and (GCS),

$$\begin{aligned}\iint_{\mathbb{R}^d} |U(x,t)|P_1(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_t(y-x)P_1(y)\,\mathrm{d}x\,\mathrm{d}|\mu_0|(y)\\ &= \int_{\mathbb{R}^d} P_{1+t}(y)\mathrm{d}|\mu_0|(y) \le C_T \int_{\mathbb{R}^d} P_1(y)\,\mathrm{d}|\mu_0|(y) < \infty \quad \text{for all } t\in(0,T).\end{aligned} \tag{2.3.42}$$

Let us prove next that μ_0 is the initial trace of U. Let $\psi \in C_c(\mathbb{R}^d)$. By (2.2.20) and (GCS),

$$\int_{\mathbb{R}^d}\iint_{\mathbb{R}^d} P_t(x-y)|\psi(x)|\,\mathrm{d}x\,\mathrm{d}|\mu_0|(y) \le C_{T,\psi}\int_{\mathbb{R}^d} P_1(y)\,\mathrm{d}|\mu_0|(y) < \infty.$$

Then, by Fubini-Tonelli's theorem,

$$\iint_{\mathbb{R}^d} U(x,t)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\iint_{\mathbb{R}^d} P_t(x-y)\psi(x)\,\mathrm{d}\mu_0(y)\,\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_t(x-y)\psi(x)\,\mathrm{d}x\,\mathrm{d}\mu_0(y).$$

Since P_t is symmetric, we may use the DCT in $L^1(\mathbb{R}^d,\mathrm{d}|\mu_0|)$ to get

$$\lim_{t\to 0^+}\int_{\mathbb{R}^d} U(x,t)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\lim_{t\to 0^+}\iint_{\mathbb{R}^d} P_t(y-x)\psi(x)\,\mathrm{d}x\,\mathrm{d}\mu_0(y) = \int_{\mathbb{R}^d}\psi(y)\,\mathrm{d}\mu_0(y),$$

because $P_t(\cdot - y)$ has the Dirac delta centered at y as initial trace; see Proposition 2.2.10.

Besides, since $U \in L^\infty((0,T);L^1_{P_1}) \subset L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, by Proposition 5.2.5, U satisfies (2.1.7).

Finally, let us prove that U satisfies (2.1.8). Let $\theta \in C_c^\infty(\mathbb{R}^d\times(0,T))$. Due to Lemma 2.2.3, (2.3.42), the symmetry of P_t, the semigroup property (2.2.9) and (C_{P_t}),

$$\begin{aligned}\iint_0^T \int_{\mathbb{R}^d}\iint_{\mathbb{R}^d} P_t(x-y)|\mathcal{L}\theta(x,t)|\,\mathrm{d}x\,\mathrm{d}|\mu_0|(y)\,\mathrm{d}t &\le C_\theta \iint_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_t(x-y)P_1(x)\,\mathrm{d}x\,\mathrm{d}|\mu_0|(y)\,\mathrm{d}t\\ &\le C_\theta C_T T\int_{\mathbb{R}^d} P_1(y)\,\mathrm{d}|\mu_0|(y) < \infty.\end{aligned}$$

Since P is a very weak solution, see Corollary 2.2.11, so is any spatial translation of it, due to the translation invariance of the equation. Therefore, using also Fubini-Tonelli's Theorem,

$$\begin{aligned}\fint_0^T \fint_{\mathbb{R}^d} U(x,t)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathbb{R}^d} \fint_0^T \int_{\mathbb{R}^d} P_t(x-y)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu_0(y)\\ &= \int_{\mathbb{R}^d}\int_0^T \fint_{\mathbb{R}^d} P_t(x-y)\partial_t\theta(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mu_0(y).\end{aligned} \tag{2.3.43}$$

On the other hand, using (2.2.20) and (GCS),

$$\int_{\mathbb{R}^d} \fint_0^T \fint_{\mathbb{R}^d} P_t(x-y)|\partial_t\theta(x,t)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}|\mu_0|(y) < \infty.$$

Then, we can use Fubini-Tonelli's Theorem to change the order of integration for the last integral in (2.3.43), and we finally obtain that U satisfies (2.1.8).

Now, we will prove that $U \in C(\mathbb{R}^d \times (0,T))$. This comes from the fact that U is the limit, uniform in compact sets, of continuous functions.

Let us define the approximating functions. For each $r > 0$ we define the finite Radon measure

$$\mu^r(B) := \mu_0(B \cap B_r(0)) \quad \text{for all Borel set } B,$$

and we set

$$U^r(x,t) := \int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}\mu^r(y) = \int_{\{|y|\le r\}} P_t(x-y)\,\mathrm{d}\mu_0(y).$$

Let us check that $U^r \in C(\mathbb{R}^d \times (0,T))$. Let $(x_1,t_1) \in \mathbb{R}^d \times (0,T)$. Then

$$|U^r(x_1,t_1) - U^r(x_2,t_2)| \le \int_{\{|y|\le r\}} |P_{t_1}(x_1-y) - P_{t_2}(x_2-y)|\,\mathrm{d}|\mu_0|(y).$$

Since P is uniformly continuous in $\mathbb{R}^d \times (\varepsilon, T')$ if $0 < \varepsilon < T' < T$ (Corollary 2.2.2), given $\widetilde{\varepsilon} > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$|P_{t_1}(x_1-y) - P_{t_2}(x_2-y)| \le \widetilde{\varepsilon}$$

for each $|x_1 - x_2| \le \delta_1$ and $|t_1 - t_2| < \delta_2$, so that we have the desired continuity,

$$|U^r(x_1,t_1) - U^r(x_2,t_2)| \le \widetilde{\varepsilon}|\mu_0|(B_r(0)).$$

Let us now prove that U^r converges uniformly to U in compact sets of $\mathbb{R}^d \times (0,T)$. Let $R > 0$ and $0 < \varepsilon < T' < T$. We define

$$I_r(R) := \sup_{\varepsilon\le t\le T'} \sup_{|x|\le R} |U(x,t) - U^r(x,t)| \le \sup_{\varepsilon\le t\le T'} \sup_{|x|\le R} \int_{\{|y|>r\}} P_t(x-y)\,\mathrm{d}|\mu_0|(y).$$

By (C_{P_t}), there is a positive constant C only depending on ε and T, such that $P_t(x) \leq CP_1(x)$ for all $(x,t) \in \mathbb{R}^d \times (\varepsilon, T')$. Thus, using also (2.2.21) and the growth condition (GCS) for the initial datum,

$$I_r(R) \leq C \int_{\{|y|>r\}} P_1(y)\, \mathrm{d}|\mu_0|(y) \to 0 \quad \text{as } r \to +\infty,$$

which shows the desired uniform convergence. □

As a corollary to our existence and uniqueness results, Theorem 5.2.9, Corollary 2.3.6 and Corollary 2.3.10, there is a representation formula for very weak solutions belonging to $L^\infty((0,T); L^1_{P_1})$ and for nonnegative very weak solutions whenever the initial datum μ_0 is admissible.

Corollary 2.3.17 (Representation formula)**.** *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying* (A0)–(A2) *and let μ_0 be a Radon measure satisfying* (GCS)*. Let u be a very weak solution to* (NLHE) *with trace μ_0. If either $u \in L^\infty((0,T); L^1_{P_1})$, or u is nonnegative and K satisfies in addition* (A3)*, then u is given by representation formula* (RF)*.*

2.3.2 Classical solutions

We consider now classical solutions. If we are able to prove that classical solutions are very weak solutions, our uniqueness theorems for the latter class, Corollary 2.3.6 and Corollary 2.3.10, will yield uniqueness for the former. In the case of nonnegative solutions, this will also guarantee the existence of an initial trace, thanks to Theorem 2.3.15, and a representation formula, because of Theorem 5.2.9. However, proving this is not as straightforward as in the case of the classical heat equation, due to the nonlocal character of the diffusion operator $\mathcal{L}$. This will be our first goal.

Next, we will give sufficient conditions, in terms of the heat kernel, ensuring the existence of a classical solution with initial trace any Radon measure μ_0 satisfying the integral growth condition (5.2.9). Such conditions are satisfied, for instance, by the heat kernel of the fractional heat equation.

Uniqueness of classical solutions

The following proposition presents two conditions that together are sufficient for a classical solution, even one with sign changes, to be a very weak solution. The first condition, denoted by (H), concerns the regularity of the function. We require a classical solution u to possess this regularity to ensure that $\mathcal{L}u$ is integrable against any test function. This would correspond to the regularity requirement $u \in C^{2,1}_{x,t}$ in the classical result by Widder. The second one, the integrability condition $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$, arises from the nonlocal character of the operator $\mathcal{L}$, and is required in the definition of very weak solution; see Proposition 5.2.5. However, we will prove later that it is satisfied by all nonnegative classical solutions.

Proposition 2.3.18. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2) and u a classical solution to (NLHE). If $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$ and satisfies (H), then it is a very weak solution to the same equation.*

Proof. Since $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$, u satisfies (2.1.7); see Proposition 5.2.5.

It remains to prove that u satisfies (2.1.8). Since u is a classical solution,

$$0 = \fint_0^T \int_{\mathbb{R}^d} (\partial_t u(x,t) + \mathcal{L}u(x,t))\theta(x,t)\,\mathrm{d}x\,\mathrm{d}y = \fint_0^T \int_{\mathbb{R}^d} (-u(x,t)\partial_t\theta(x,t) + \mathcal{L}u(x,t)\theta(x,t))\,\mathrm{d}x\,\mathrm{d}y$$

for all $\theta \in C_c^\infty(\mathbb{R}^d \times (0,T))$. If

$$\mathcal{C} := \fint_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\theta(x,t)||\Lambda u(x,y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.3.44}$$

we would have, via Fubini's theorem, that

$$\fint_0^T \int_{\mathbb{R}^d} \mathcal{L}u(x,t)\theta(x,t)\,\mathrm{d}x\,\mathrm{d}y = \fint_0^T \int_{\mathbb{R}^d} u(x,t)\mathcal{L}\theta(x,t)\,\mathrm{d}x\,\mathrm{d}y,$$

and we would be done. However, proving (2.3.44) is not straightforward, because of the singularity of Lévy kernel K at the origin. It is here that we need hypothesis (H).

To prove (2.3.44), we make the decomposition $\mathcal{C} = \mathrm{I} + \mathrm{II}$, where

$$\begin{aligned}
\mathrm{I} &:= \fint_0^T \int_{\mathbb{R}^d}\int_{\{|y|\le 1\}} |\theta(x,t)|\,|\Lambda u(x,y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{II} &:= \fint_0^T \int_{\mathbb{R}^d}\int_{\{|y|\ge 1\}} |\theta(x,t)|\,|\Lambda u(x,y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The term I is finite, due to (H). As for II, since K is symmetric, then $\mathrm{II} \le \mathrm{II}_1 + \mathrm{II}_2$, where

$$\begin{aligned}
\mathrm{II}_1 &:= \fint_0^T \int_{\mathbb{R}^d} |\theta(x,t)||u(x,t)|\,\mathrm{d}x\,\mathrm{d}t \int_{\{|y|\ge 1\}} K(y)\,\mathrm{d}y,\\
\mathrm{II}_2 &:= \fint_0^T \int_{\mathbb{R}^d}\int_{\{|y|\ge 1\}} |\theta(x,t)|\,|u(x+y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&= \fint_0^T \int_{\mathbb{R}^d} |u(x,t)| \int_{\{|y|\ge 1\}} |\theta(x-y,t)|K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As $u \in C(\mathbb{R}^d \times (0,T))$ and K is a Lévy kernel, then

$$\mathrm{II}_1 \le \|\theta u\|_{L^1(\mathbb{R}^d\times(0,T))} \int_{\{|y|\ge 1\}} K(y)\,\mathrm{d}y < \infty.$$

Let now $0 < \varepsilon < T' < T$ such that $\operatorname{supp}(\theta) \subset \mathbb{R}^d \times (\varepsilon, T')$. Using ($C_{PK}$), (2.2.19), (2.2.9) and (C_{P_t}),

$$\begin{aligned}\mathrm{II}_2 &\le C_{\theta,T} \fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)| \fint_{\mathbb{R}^d} P_1(x-y)P_1(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = C_{\theta,T} \fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)| P_2(x)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C_{\theta,T} \fint_0^{T'} \int_{\mathbb{R}^d} |u(x,t)| P_1(x)\,\mathrm{d}x\,\mathrm{d}t < \infty,\end{aligned}$$

because $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$. □

Corollary 2.3.19. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2).*
(i) *There is at most one classical solution $u \in L^\infty((0,T); L^1_{P_1})$ to (NLHE) satisfying (H) with initial trace a given Radon measure.*
(ii) *Let K satisfy also (A3). If there exists a classical solution $u \in L^\infty((0,T); L^1_{P_1})$ to (NLHE) satisfying (H) with initial trace a measure μ_0 verifying (GCS), then u is given by the representation formula (RF).*

Proof. (i) Suppose that $u \in L^\infty((0,T); L^1_{P_1})$ is a classical solution to (NLHE) that satisfies (H). Since $L^\infty((0,T); L^1_{P_1}) \subset L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$, then u is a very weak solution of (NLHE); see Proposition 2.3.18. Uniqueness now follows from Corollary 2.3.6, since $u \in L^\infty((0,T); L^1_{P_1})$.
(ii) It follows immediately from Corollary 2.3.17, since u is a very weak solution of (NLHE). □

We move on now to the class of nonnegative classical solutions u to (NLHE) satisfying (H). We will prove that $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$, whence, by Proposition 2.3.18, it will be a very weak nonnegative solution, from where uniqueness and a representation formula will follow. The aforementioned integrability will be a consequence of the inequality

$$U(x,t) := \int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}\mu_0(y) \le u(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (0,T), \tag{2.3.45}$$

satisfied by nonnegative classical solutions, even if (H) does not hold. The first step to prove (2.3.45) is to obtain a maximum principle, which we do next. This only requires (A0).

Lemma 2.3.20 (A maximum principle)**.** *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0). Let $\Omega \subset \mathbb{R}^d$ be non-empty, open, bounded and smooth, and $v \in C(\overline{\Omega} \times [0,T))$ such that $\partial_t v \in C(\Omega \times (0,T))$ and $\mathcal{L}v(x,t)$ is defined for all $(x,t) \in \Omega \times (0,T)$. If*

$$\begin{cases} (\partial_t v + \mathcal{L}v)(x,t) \le 0 & \text{for all } (x,t) \in \Omega \times (0,T), \\ v(x,t) \le 0 & \text{for all } (x,t) \in \big(\mathbb{R}^d \times [0,T)\big) \setminus \big(\Omega \times (0,T)\big), \end{cases} \tag{2.3.46}$$

then $v \le 0$ for all $x \in \mathbb{R}^d \times [0,T)$.

Proof. Choose any $T' \in (0,T)$. Let $(x_0,t_0) \in \bar{\Omega} \times [0,T']$ such that

$$v(x_0,t_0) := \max_{\bar{\Omega}\times[0,T']} v(x,t).$$

Assume for contradiction that $v(x_0,t_0) > 0$. Thanks to the "boundary conditions", necessarily $(x_0,t_0) \in \Omega \times (0,T']$. Observe that $\partial_t v(x_0,t_0) \geq 0$ and $\Lambda v(x_0,y,t_0) \geq 0$ for all $y \in \mathbb{R}^d$, because v reaches it maximum at time t_0 at $x = x_0$. Hence, for any $R_0 > 0$,

$$0 \geq -\partial_t v(x_0,t_0) \geq \mathcal{L}v(x_0,t_0) \geq \int_{B^c_{2R_0}(0)} \Lambda v(x_0,y,t_0) K(y)\,\mathrm{d}y.$$

Let $R_0 > 0$ be such that $\Omega \subset B_{R_0}(0)$, and $|y| > 2R_0$. Then, $|x_0 \pm y| > R_0$, whence $x_0 \pm y \notin \Omega$ and therefore $v(x_0 \pm y, t_0) \leq 0$. Thus,

$$0 \geq \int_{B^c_{2R_0}(0)} \Lambda v(x_0,y,t_0) K(y)\,\mathrm{d}y \geq v(x_0,t_0) \int_{B^c_{2R_0}(0)} K(y)\,\mathrm{d}y > 0,$$

because K is positive, see (A0), and we arrive at a contradiction. □

We now proceed to prove (2.3.45).

Lemma 2.3.21. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2). If u is a nonnegative classical solution of the nonlocal heat equation (NLHE) with initial data a nonnegative Radon measure μ_0, then (2.3.45) holds.*

Proof. Let $\delta \in (0,T)$. Since u is a classical solution, $u(\cdot,\delta) \in C(\mathbb{R}^d)$. We consider the family of cutoff functions $\{\psi_R\}_{R>0}$ with ψ_R as in (2.1.12). Let

$$v^\delta_R(x,t) = \int_{\mathbb{R}^d} P_t(x-y)\psi_R(y)u(y,\delta)\,\mathrm{d}y.$$

Since the heat kernel $P \in C^\infty(\mathbb{R}^d \times (0,\infty))$, see Corollary 2.2.2, and $\big(\mathcal{P}_t\big)_{t\geq 0}$ is a strongly continuous semigroup in $C_0(\mathbb{R}^d)$, then $v^\delta_R \in C^\infty(\mathbb{R}^d \times [0,T))$, and solves

$$\begin{cases} \partial_t v^\delta_R + \mathcal{L}v^\delta_R = 0 & \text{everywhere in } \mathbb{R}^d \times (0,T), \\ v^\delta_R \geq 0 & \text{everywhere in } \mathbb{R}^d \times [0,T), \\ v^\delta_R(\cdot,0) = \psi_R(\cdot)u(\cdot,\delta) & \text{everywhere in } \mathbb{R}^d. \end{cases}$$

By (2.2.20),

$$0 \leq v^\delta_R(x,t) \leq \|u(\cdot,\delta)\|_{L^\infty(B_{2R}(0))}(P_t * \psi_R)(x) \leq C_{R,T}\|u(\cdot,\delta)\|_{L^\infty(B_{2R}(0))}P_1(x).$$

On the other hand, $P_1(x) \to 0$ as $|x| \to \infty$, since P_1 is comparable to K away from the origin, (C_{PK}), and $K(x) \to 0$ as $|x| \to \infty$, see (A1) and (A2). Therefore, given $\varepsilon > 0$ we

can choose $\rho > 0$ large such that $C_{R,T}\|u(\cdot,\delta)\|_{L^\infty(B_{2R}(0))}P_1(x) \le \varepsilon$ for all $|x| \ge \rho$. Hence, since u is nonnegative,

$$\begin{aligned} &0 \le v_R^\delta(x,t) \le \varepsilon \le u(x,t+\delta) + \varepsilon && \text{for any } |x| \ge \rho,\ t \in (0, T-\delta) \\ &v_R^\delta(x,0) = \psi_R(x)u(x,\delta) \le u(x,\delta) \le u(x,\delta) + \varepsilon && \text{for any } x \in \mathbb{R}^d. \end{aligned}$$

Thus, applying the Maximum Principle, Lemma 2.3.20, with $\Omega = B_\rho(0)$ to $w := v_R^\delta - u^\delta - \varepsilon$, we get

$$v_R^\delta(x,t) \le u(x,t+\delta) + \varepsilon \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0, T-\delta).$$

Since ε is arbitrary,

$$v_R^\delta(x,t) = \int_{\mathbb{R}^d} P_t(x-y)\psi_R(y)u(y,\delta)\,\mathrm{d}y \le u(x,t+\delta) \quad \text{for all } (x,t) \in \mathbb{R}^d \times [0, T-\delta).$$

Using that the initial trace of u is μ_0, for all $(x,t) \in \mathbb{R}^d \times (0,T)$ we have

$$\int_{\mathbb{R}^d} P_t(x-y)\psi_R(y)\,\mathrm{d}\mu_0(y) = \operatorname*{ess\,lim}_{\delta\to 0^+} \int_{\mathbb{R}^d} P_t(x-y)\psi_R(y)u(y,\delta)\,\mathrm{d}y \le \lim_{\delta\to 0^+} u(x,t+\delta) = u(x,t),$$

since $u \in C(\mathbb{R}^d \times (0,T))$. Letting $R \to \infty$, using the MCT, we arrive at (2.3.45). □

Remark 2.3.22. Thanks to inequality (2.3.45) and the fact that μ_0 is a nonnegative measure, the integral $\int_{\mathbb{R}^d} P_t(x-y)\,\mathrm{d}\mu_0(y)$ is well defined as long as the nonnegative classical solution u exists. Moreover, $\int_{\mathbb{R}^d} P_1\,\mathrm{d}\mu_0 \le u(0,1) < \infty$, because P_1 is symmetric. Hence, we do not need to assume the integrability condition (GC) in advance in order for U to be well defined.

We can prove now the desired integrability, and even a bit more than required.

Corollary 2.3.23. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2). If u is a nonnegative classical solution of the nonlocal heat equation (NLHE) with initial trace a nonnegative Radon measure μ_0, then $u \in L^1_{\mathrm{loc}}([0,T); L^1_{P_1})$.*

Proof. Since u is nonnegative, it is enough to prove that for all $T' \in (0,T)$ we have

$$\fint_0^{T'} \int_{\mathbb{R}^d} P_1(y)u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau < \infty.$$

Let $\tau \in (0,T')$ and $u^\tau(x,t) := u(x,t+\tau)$, $(x,t) \in \mathbb{R}^d \times [0, T-\tau)$. Since u is a classical solution to (NLHE), so it is u^τ, for $t \in (0, T-\tau)$. The initial trace of u^τ is $u(\cdot,\tau)$. By Lemma 2.3.21,

$$\fint_{\mathbb{R}^d} P_t(x-y)u(y,\tau)\,\mathrm{d}y \le u(x,t+\tau) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (0, T-\tau)$$

which implies that

$$\fint_0^{T'} \int_{\mathbb{R}^d} P_t(x-y)u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \le \fint_t^{t+T'} u(x,s)\,\mathrm{d}s. \tag{2.3.47}$$

Let $T'' \in (T', T)$. By (C_{P_t}) we know that $P_1(y) \le C_{T',T''} P_{T''-T'}(y)$ for all $y \in \mathbb{R}^d$. Then, since P_t is symmetric for all times, using the estimate (2.3.47) with $x = 0$, we get

$$\begin{aligned} \fint_0^{T'} \int_{\mathbb{R}^d} P_1(y) u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau &\le C_{T',T''} \fint_0^{T'} \int_{\mathbb{R}^d} P_{T''-T'}(-y) u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\le C_{T''-T'} \fint_{T''-T'}^{T''} u(0,\tau)\,\mathrm{d}\tau < \infty, \end{aligned}$$

due to the fact that $u(0,\cdot) \in C([T''-T', T''])$. □

At this point we have gathered all the necessary ingredients to prove what may be considered as one of the main outcomes of the paper: a Widder-type representation formula (implying uniqueness) for nonnegative classical solutions (assuming that the "energy assumption" (H) holds).

Theorem 2.3.24. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A3). Let u be a nonnegative classical solution of the nonlocal heat equation (NLHE) with initial trace a nonnegative Radon measure μ_0. If (H) holds, then u is given by the representation formula (RF).*

Proof. Thanks to its nonnegativity, $u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$; see Corollary 2.3.23. Therefore, since u satisfies (H), it is a nonnegative very weak solution of (NLHE); see Proposition 2.3.18. Finally, the integral growth condition (GC) holds; see Remark 4.4.15. Thus, u satisfies the hypotheses of Corollary 2.3.17, from where the result follows immediately. □

Existence of classical solutions

We would like to prove now that given a Radon measure μ_0 satisfying the integral growth condition (GCS), there is a classical solution to (NLHE) having it as initial trace. We have a natural candidate: the function $U(x,t) = P_t * \mu_0$, which is a very weak solution of the equation and has μ_0 as initial trace; see Theorem 5.2.9. Moreover, it belongs to the class $L^\infty((0,T); L^1_{P_1})$ in which there is uniqueness. However, proving that U is a classical solution is not a trivial task, and will require some assumptions, that we will write in terms of the heat kernel associated to the operator. These assumptions are satisfied, for instance, when the operator is the fractional Laplacian.

Theorem 2.3.25. *Let $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2), μ_0 a Radon measure satisfying the integral growth condition (GCS), and U the very weak solution to (NLHE) given in (S_{μ_0}). Suppose that for each $T > 0$ and $\varepsilon \in (0,T)$ there is a constant $C > 0$ such that*

$$|\partial_t P_t(x)| \le C P_1(x), \qquad \int_{\mathbb{R}^d} |\Lambda P(x,y,t)| K(y)\,\mathrm{d}y \le C P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T). \tag{2.3.48}$$

(i) *Then, U is a classical solution to (NLHE) within $L^\infty((0,T); L^1_{P_1})$ and with initial trace μ_0.*

(ii) *If in addition for each* $T > 0$ *and* $\varepsilon \in (0, T)$ *there is a constant* $C > 0$ *such that*

$$|D^2 P_t(x)| \le C P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T), \tag{2.3.49}$$

then U *is the unique classical solution to* (NLHE) *with initial trace* μ_0 *that satisfies* (H) *and belongs to* $L^\infty((0,T); L^1_{P_1})$.
(iii) *If, in addition to all the above,* K *satisfies* (A3) *and* μ_0 *is nonnegative, then* U *is the unique nonnegative classical solution to* (NLHE) *with initial trace* μ_0 *that satisfies* (H).

Remark 2.3.26. Since K satisfies the Lévy condition (A0), it is easily checked that condition (2.3.49) implies the second condition in (2.3.48).

Remark 2.3.27. Let $\mathcal{L} = (-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. Thanks to Proposition 2.2.16 (iv) and (vi),

$$|\partial_t P_t(x)| \le C\varepsilon^{-1} P_1(x), \quad |D^2 P^\alpha_t(x)| \le C\varepsilon^{-2\alpha} P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T),$$

whence Theorem 2.3.25 (ii)–(iii) apply to this case.

Thanks to Theorem 5.2.9 we already know that the function U has initial datum μ_0 and belongs to $C(\mathbb{R}^d \times (0,T)) \cap L^\infty((0,T); L^1_{P_1})$, which implies that $U \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$. Thus, it only remains to check that $\partial_t U$ belongs to $C(\mathbb{R}^d \times (0,T))$ and that U satisfies (2.1.9). This is what we do in the next two lemmas.

Lemma 2.3.28. *Under the general assumptions of Theorem 2.3.25,* $\partial_t U(x,t) \in C(\mathbb{R}^d \times (0,T))$.

Proof. The first step is to prove that

$$\partial_t U(x,t) = \int_{\mathbb{R}^d} \partial_t P(x-y,t)\, \mathrm{d}\mu_0(y). \tag{2.3.50}$$

Let $0 < \varepsilon < T' < T$ and $(x,t) \in \mathbb{R}^d \times (\varepsilon, T')$. By definition,

$$\partial_t U(x,t) = \lim_{h \to 0} \int_{\mathbb{R}^d} \frac{P(x-y,t+h) - P(x-y,t)}{h}\, \mathrm{d}\mu_0(y).$$

The MVT and the first hypothesis in (2.3.48) imply that

$$\frac{P(x-y,t+h) - P(x-y,t)}{h} \le |\partial_t P(x-y,\tilde{t})| \le C P_1(x-y)$$

for all $x, y \in \mathbb{R}^d$, $t \in (\varepsilon, T')$ and $2|h| \le \varepsilon \wedge (T - T')$, where $\tilde{t}$ depends on x, y, t, and h and $|\tilde{t} - t| \le |h|$. Moreover, C only depends on ε and T'. Since μ_0 satisfies (GCS), we already know that $\int_{\mathbb{R}^d} P(x-y,1)\, \mathrm{d}|\mu_0|(y) < \infty$ for each $(x,t) \in \mathbb{R}^d \times (\varepsilon, T')$; see (2.3.41). Therefore, applying the DCT we obtain that $\partial_t U$ is well defined and

$$\partial_t U(x,t) = \lim_{h \to 0} \int_{\mathbb{R}^d} \frac{P(x-y,t+h) - P(x-y,t)}{h}\, \mathrm{d}\mu_0(y) = \int_{\mathbb{R}^d} \partial_t P(x-y,t)\, \mathrm{d}\mu_0(y).$$

The continuity of $\partial_t U$ follows from the fact that it is the limit as $r \to +\infty$, uniform in compact sets, of the approximating functions

$$(\partial_t U)^r(x,t) = \int_{\mathbb{R}^d} \partial_t P(x-y,t)\,\mathrm{d}\mu^r(y),$$

arguing as in the proof of continuity of U performed in Theorem 5.2.9; we omit the details. □

Lemma 2.3.29. *Under the general assumptions of Theorem 2.3.25, U satisfies (2.1.9).*

Proof. We start from (2.3.50). Since P is a classical solution to the nonlocal heat equation (NLHE),

$$\partial_t U(x,t) = -\fint_{\mathbb{R}^d} \mathcal{L}P(x-y,t)\,\mathrm{d}\mu_0(y) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Lambda P(x-y,z,t)K(z)\,\mathrm{d}z\,\mathrm{d}\mu_0(y). \quad (2.3.51)$$

At this point, we would like to change the order of integration in the last integral to conclude that $\partial_t U + \mathcal{L}U = 0$. By the second hypothesis in (2.3.48),

$$\int_{\mathbb{R}^d} \fint_{\mathbb{R}^d} |\Lambda P(x-y,z,t)|K(z)\,\mathrm{d}z\,\mathrm{d}|\mu_0|(y) \le C\int_{\mathbb{R}^d} P_1(x-y)\,\mathrm{d}|\mu_0|(y) < \infty,$$

because the last integral is finite for all $x \in \mathbb{R}^d$, as we proved in (2.3.41). Since the last integral in (2.3.51) converges absolutely, using Fubini-Tonelli's Theorem

$$\partial_t U(x,t) = -\int_{\mathbb{R}^d} \fint_{\mathbb{R}^d} \Lambda P(x-y,z,t)K(z)\,\mathrm{d}\mu_0(y)\,\mathrm{d}z = -\int_{\mathbb{R}^d} \Lambda U(x,z,t)K(z)\,\mathrm{d}z = -\mathcal{L}U(x,t)$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$. □

Proof of Theorem 2.3.25. (i) By Theorem 5.2.9, Lemma 2.3.28 and Lemma 2.3.29, U is a classical solution of (NLHE) with initial trace μ_0 and $U \in L^\infty((0,T);L^1_{P_1})$.
(ii) It only remains to prove that U satisfies (H), since then uniqueness will follow from Corollary 2.3.19. Given $\theta \in C_c^\infty(\mathbb{R}^d \times (0,T))$, let $0 < \varepsilon < T' < T$ be such that $\operatorname{supp}(\theta) \subset \mathbb{R}^n \times (\varepsilon, T')$. Using the definition of U and applying Taylor's expansion,

$$\begin{aligned}
&\fint_0^T \int_{\mathbb{R}^d} \theta(x,t) \int_{\{|y|\le 1\}} \Lambda U(x,y,t)K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\le \fint_0^{T'} \int_{\mathbb{R}^d} |\theta(x,t)| \fint_{\{|y|\le 1\}} K(y) \int_{\mathbb{R}^d} |\Lambda P(x-z,y,t)|\,\mathrm{d}|\mu_0|(z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\fint_0^{T'} \int_{\mathbb{R}^d} |\theta(x,t)| \fint_{\{|y|\le 1\}} |y|^2 K(y) \int_{\mathbb{R}^d} |D^2 P(x-z+w(x,z,y,t),t)|\,\mathrm{d}|\mu_0|(z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $w(x,z,y,t) \in B_1(0)$ because $|y| \le 1$. By assumption (2.3.49),

$$|D^2 P(x-z+w(x,z,y,t),t)| \le C P_1(x-z+w(x,z,y,t)),$$

where C only depends on ε and T'. Since P_1 is slowly changing, see Proposition 2.2.6, and $|w| \le 1$, and it is also symmetric we have $|P_1(x-z+w(x,z,y,t))| \le CP_1(z-x)$. Therefore, using, one after another, Tonelli's Theorem, assumption (A0), (2.2.20) and (GCS), we arrive at

$$\begin{aligned}\fint_0^T \int_{\mathbb{R}^d} |\theta(x,t)| \int_{\{|y|\le 1\}} |\Lambda U(x,y,t)| K(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\le C \int_{\{|y|\le 1\}} |y|^2 K(y)\,\mathrm{d}y \fint_0^{T'} \int_{\mathbb{R}^d} (P_1 * |\theta(\cdot,t)|)(z)\,\mathrm{d}|\mu_0|(z)\,\mathrm{d}t \\ &\le C(T'-\varepsilon) \int_{\mathbb{R}^d} P_1(z)\,\mathrm{d}|\mu_0|(z) < \infty.\end{aligned}$$

(iii) If μ_0 is nonnegative, then U is nonnegative. Uniqueness in the class of nonnegative solutions satisfying (H) is then granted by Theorem 2.3.24. □

2.4 General purely nonlocal operators

We have developed in Section 2.3 a quite complete Widder-type theory for operators of the form $\mathcal{L} = \mathcal{L}_K$ with K satisfying (A0)–(A2), plus (A3) for some results. A careful inspection of the proofs shows that they do not use these assumptions directly, but their implications for the heat kernel associated to the operator, that could be satisfied by more general operators. Our aim in this section is to rewrite the theorems in terms of this more general conditions, so that they may include a wider class of purely nonlocal operators $\mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). In order to check that this is not a useless generalization, we will show that these new versions of the theorems encompass an important class of anisotropic operators that were not covered by the theory in Section 2.3.

In addition to the Lévy condition (A_ν), we will always assume that the Lévy process associated to ν has a heat kernel P satisfying

$$P_t \in C(\mathbb{R}^d), \quad P_t > 0 \quad \text{for all } t > 0. \tag{2.4.1}$$

Thus, we are leaving out, for instance, the case in which $\mathrm{d}\nu = J\,\mathrm{d}x$ with $J \in L^1(\mathbb{R}^d)$, for which there is a transition density but not a heat kernel, or the operator $\log(1-\Delta)$, with Fourier multiplier $\log(1+|\xi|^2)$, that has a positive heat kernel, though not with the desired continuity until some time has passed.

Observe that, since we are assuming that ν is symmetric, so it is P_t for all $t > 0$.

2.4.1 Very weak solutions

We will be able to recover all the important results for very weak solutions, with less restrictive assumptions than in Section 2.3.

Alternative definitions of very weak solution

A quick inspection of the proof of Lemma 2.2.3 shows that it can be easily adapted to show the following more general result, that will be helpful to establish the equivalence of our two definitions of very weak solution if $u \in L^\infty((0,T);L^1_{P_1})$.

Lemma 2.4.1. *Let $\mathcal{L}=\mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P satisfying (2.4.1) and*

$$\fint\!\!\!\int_{\{|y|\ge 1\}} P_1(x-y)\,\mathrm{d}\nu(y) \le CP_1(x) \quad \text{for all } x\in\mathbb{R}^d \tag{2.4.2}$$

for some constant C. If $\theta\in C_c^\infty(\mathbb{R}^d\times[\delta,\tau])$, $0\le\delta<\tau\le T$, there is a constant C_θ such that

$$|\mathcal{L}_\nu\theta(x,t)| \le \int_{\mathbb{R}^d} |\Lambda\theta(x,y,t)|\,\mathrm{d}\nu(y) \le C_\theta P_1(x), \quad (x,t)\in\mathbb{R}^d\times[\delta,\tau]. \tag{2.4.3}$$

Remark 2.4.2. We proved in Section 2.3, under certain assumptions on K, that, given $\theta\in C_c^\infty(\mathbb{R}^d\times[\delta,\tau])$, $0\le\delta<\tau\le T$, there are constants $C,R_0>0$ depending on θ such that $|\mathcal{L}_K\theta(x)|\ge CP_1(x)$ if $|x|\ge R_0>0$. This was helpful to prove that very weak solutions belong to $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$. Unfortunately, hypotheses (A_ν), (2.4.1) and (2.4.2) are not enough to ensure the validity of such an inequality. As a counterexample (in spatial dimension $d=2$) we consider the operator $\mathcal{L}_\nu=(-\partial^2_{x_1x_1})^{1/2}+(-\partial^2_{x_2x_2})^{1/2}$, which satisfies the three hypotheses, and the function $\theta(x)=\theta_1(x_1)\theta_2(x_2)$, with $\theta_1,\theta_2\in C_c^\infty(\mathbb{R})$, so that $\theta\in C_c^\infty(\mathbb{R}^2)$. If $\mathrm{supp}(\theta)\subset B_R(0)$ for some $R>0$, then

$$\mathcal{L}_\nu\theta(x)=\theta_2(x_2)(-\partial^2_{x_1x_1})^{1/2}\theta_1(x_1)+\theta_1(x_1)(-\partial^2_{x_2x_2})^{1/2}\theta_2(x_2)=0 \quad \text{if } |x_1|>R \text{ and } |x_2|>R.$$

Lemma 2.4.1 is enough to show that the integrability condition (2.1.7) is satisfied if $u\in L^1([\delta,\tau];L^1_{P_1})$.

Proposition 2.4.3. *Let $\mathcal{L}=\mathcal{L}_\nu$ and P as in Lemma 2.4.1. If $u\in L^1([\delta,\tau];L^1_{P_1})$, $0\le\delta<\tau\le T$, then*

$$\fint\!\!\!\int_\delta^\tau \fint\!\!\!\int_{\mathbb{R}^d} |u(x,t)||\mathcal{L}\theta(x,t)|\,\mathrm{d}x\,\mathrm{d}t<\infty \quad \text{for all } \theta\in C_c^\infty(\mathbb{R}^d\times[\delta,\tau]).$$

As a consequence, if $u\in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, u satisfies (2.1.7).

With this integrability condition at hand, we can now prove the equivalence of both definitions of very weak solution if $u\in L^\infty((0,T);L^1_{P_1})$, just following step by step the proof of Proposition 2.3.3.

Proposition 2.4.4. *Let $\mathcal{L}=\mathcal{L}_\nu$ and P as in Lemma 2.4.1, μ_0 a Radon measure and $u\in L^\infty((0,T);L^1_{P_1})$.*
(i) *If u satisfies (2.3.4), then it is a very weak solution of (NLHE) with initial trace μ_0.*
(ii) *If u is a very weak solution to (NLHE) with initial trace μ_0, then it satisfies (2.3.3)–(2.3.4).*

Uniqueness for very weak solutions

We have now all the tools to prove uniqueness for very weak solutions in the sense of (2.3.3)–(2.3.4) under two new assumptions on the kernel, that have already appeared in Section 2.3. This is done by following the steps in the proof of Theorem 2.3.5, where now φ is a solution of the backward problem $\partial_t\varphi - \mathcal{L}_\nu\varphi = \tilde{\theta}$ and B is the bilinear form

$$B_\nu(u,v)(x) = \iint_{\mathbb{R}^d} \big(u(x)-u(x-y)\big)\big(v(x)-v(x-y)\big)\,\mathrm{d}\nu(y), \tag{2.4.4}$$

taking profit of the nonlocal Leibniz's formula

$$B_\nu(uv) = u\mathcal{L}_\nu v + v\mathcal{L}_\nu u - B_\nu(u,v).$$

Theorem 2.4.5. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P satisfying (2.4.1), (2.4.2), (C_{P_t}) and (2.2.23). Let μ_0 be a Radon measure. If u_1 and u_2 satisfy (2.3.3)–(2.3.4), then, $u_1 = u_2$ a.e. in $\mathbb{R}^d \times (0,T)$.*

Since by Proposition 2.4.4 a very weak solution $u \in L^\infty((0,T);L^1_{P_1})$ with initial datum a Radon measure μ_0 satisfies (2.3.4), we obtain the following corollary.

Corollary 2.4.6 (Uniqueness for very weak solutions within $L^\infty((0,T);L^1_{P_1})$)**.** *Let $\mathcal{L} = \mathcal{L}_\nu$ and P as in Theorem 2.4.5. There is at most one nonnegative very weak solution of (NLHE) belonging to $L^\infty((0,T);L^1_{P_1})$ with a given initial trace.*

We consider now the class of nonnegative very weak solutions that belong to $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$. As in Section 2.3, the strategy to prove uniqueness in this class is to show that very weak solutions within it belong to $L^\infty((0,T);L^1_{P_1})$. Uniqueness will then be inherited from Corollary 2.4.6.

The required boundedness is obtained following the proof of Lemma 2.3.9, that depends only on the existence of a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22), with $\mathcal{L} = \mathcal{L}_\nu$ and $B = B_\nu$, remembering that (2.4.3) holds. Thus we have the following result.

Lemma 2.4.7. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν) and $B = B_\nu$ the bilinear form defined in (2.4.4). Assume that $\mathcal{L}_\nu$ has associated a heat kernel P satisfying (2.4.1) and that there is a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22).*
(i) *If u is a nonnegative very weak solution to (NLHE) in $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, there is a constant $c > 0$ such that (2.3.28) holds.*
(ii) *If $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ is a nonnegative very weak solution of (NLHE), then $u \in L^\infty((0,T);L^1_{P_1})$.*

Remark 2.4.8. Lemma 2.3.9 does not have $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ among its hypotheses, though this fact is used in its proof. The reason is that when we consider $\mathcal{L}_K$ instead of $\mathcal{L}_\nu$ any very weak solution has this integrability, as noted in Remark 5.2.16. But this is not guaranteed *a priori* in the case of more general operators.

Mimicking the proof of Proposition 2.3.11 we get also some regularity in time for general operators.

Proposition 2.4.9. *Let $\mathcal{L} = \mathcal{L}_\nu$ and P as in Lemma 2.4.7, and let u be a nonnegative very weak solution to (NLHE).*
(i) *Let $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22). There is a constant $C > 0$ such that (2.3.35) holds.*
(ii) *Given $\psi \in C_c^2(\mathbb{R}^d)$, there is a constant $C_\psi > 0$ such that (2.3.36) holds.*

The combination of Lemma 2.4.7 (ii) and Corollary 2.4.6 yields the desired uniqueness result for nonnegative very weak solutions.

Corollary 2.4.10 (Uniqueness for nonnegative very weak solutions)**.** *Let $\mathcal{L} = \mathcal{L}_\nu$ and P as in Theorem 2.4.5, and $B = B_\nu$ the bilinear form defined in (2.4.4). If there is a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22), then there is at most one nonnegative very weak solution of (NLHE) belonging to $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ with a given initial trace.*

Existence of an initial trace

The existence of an initial trace satisfying the integral growth condition (GC) for nonnegative very weak solutions is based on the estimates (2.3.28); see the proof of Theorem 2.3.15. Following that proof we obtain easily our next result.

Theorem 2.4.11 (Existence of initial trace integrable against P_1)**.** *Assume the same hypotheses as in Lemma 2.4.7. If $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ is a nonnegative very weak solution to (NLHE), then there is a nonnegative Radon measure μ_0 such that (2.1.10) holds; that is, μ_0 is the initial trace of u. Moreover, μ_0 satisfies the integral growth condition(GC).*

Existence of very weak solutions

The existence of very weak solutions, for suitable initial data, is obtained by mimicking the proof of Theorem 5.2.9, with two new hypotheses.

Theorem 2.4.12. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P that is a very weak solution of (NLHE) and satisfies (2.4.1), (C_{P_t}) and (2.2.21). Let μ_0 be a Radon measure with the admissible growth at infinity (GCS). Then, for each $T > 0$, the function U given in (S_{μ_0}) is a very weak solution of (NLHE) in $C(\mathbb{R}^d \times (0,T)) \cap L^\infty((0,T);L^1_{P_1})$ with initial trace μ_0.*

As a corollary to Theorem 2.4.12 and the uniqueness results Corollary 2.4.6 and Corollary 2.4.10, solutions are given by a representation formula.

Corollary 2.4.13 (Representation formula for very weak solutions)**.** *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P satisfying (2.4.1), (2.4.2), (C_{P_t}), (2.2.21) and (2.2.23), and let μ_0 be a Radon measure satisfying (GCS). Let u be a very weak solution to (NLHE) with trace μ_0. If either $u \in L^\infty((0,T);L^1_{P_1})$, or u is nonnegative and in addition there is a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22), then u is given by representation formula (RF).*

2.4.2 Classical solutions

It is turn now to analyze classical solutions for general operators. We have to tailor the energy condition (H) on the solutions to the general situation. Thus, we will look for solutions satisfying

$$\iint_{\{|y|\le 1\}} |\Lambda u(x,y,t)|\,\mathrm{d}\nu(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^d\times(0,T)). \tag{H'}$$

Uniqueness of classical solutions.

The clue for uniqueness is to prove that classical solutions satisfying the energy condition (H') are very weak solutions, so that we can apply to them the results that we have already obtained for the latter class. This is done easily following the proof of Proposition 2.3.18, adding assumption (2.4.2), so that we have the following result.

Proposition 2.4.14. *Let $\mathcal{L}=\mathcal{L}_\nu$ and P as in Lemma 2.4.1, and u a classical solution to (NLHE). If $u\in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ and satisfies (H'), then it is a very weak solution to the same equation.*

As a consequence of Proposition 2.4.14, Corollary 2.4.6 and Corollary 2.4.13, we obtain uniqueness for classical solutions in $L^\infty((0,T);L^1_{P_1})$, and a representation formula if there is existence in this class.

Corollary 2.4.15. *Let $\mathcal{L}=\mathcal{L}_\nu$ and P as in Theorem 2.4.5.*
(i) *There is at most one classical solution $u\in L^\infty((0,T);L^1_{P_1})$ to (NLHE) satisfying (H') with initial trace a given Radon measure.*
(ii) *If there exists a classical solution $u\in L^\infty((0,T);L^1_{P_1})$ to (NLHE) satisfying (H') with initial trace a measure μ_0 verifying (GCS), then u is given by the representation formula (RF).*

We move on now to the class of nonnegative classical solutions u to (NLHE) satisfying (H'). Following the strategy of Section 2.3, we will prove that they satisfy inequality (2.3.45). This will imply that they belong to $L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$, so that they are nonnegative very weak solutions because of Proposition 2.4.14. Uniqueness and a representation formula will then follow from the results for this class of solutions.

Inequality (2.3.45) follows easily if we have a comparison principle, $P\in C^{2,1}_{x,t}(\mathbb{R}^d\times(0,\infty))$ and $P_1\in C_0(\mathbb{R}^d)$. An inspection of the proof of Lemma 2.3.20 shows that such a comparison principle will be available for $\mathcal{L}_\nu$ if the Lévy measure ν satisfies that

$$\nu\big((B_R(0))^{\mathrm{c}}\big)>0 \quad \text{for all } R\ge R_0 \text{ for some } R_0>0. \tag{2.4.5}$$

Lemma 2.4.16 (A maximum principle)**.** *Let $\mathcal{L}=\mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν) and (2.4.5). Let $\Omega\subset\mathbb{R}^d$ be a non-empty, open, bounded and smooth set, and $v\in C(\Omega\times[0,T))$ such that $\partial_t v\in C(\bar\Omega\times(0,T))$ and $\mathcal{L}v(x,t)$ is defined for all $(x,t)\in\mathbb{R}^d\times(0,T)$. If (2.3.46) holds, then $v\le 0$ for all $x\in\mathbb{R}^d\times[0,T)$.*

We can now show the validity of (2.3.45), following the proof of Lemma 2.3.21.

Lemma 2.4.17. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν) and (2.4.5). Assume that it has associated a heat kernel P satisfying (2.4.1) and (C_{P_t}), and such that $P \in C^{2,1}_{x,t}(\mathbb{R}^d \times (0,\infty))$ and $P_1 \in C_0(\mathbb{R}^d)$. If u is a nonnegative classical solution of the nonlocal heat equation (NLHE) with initial data a nonnegative Radon measure μ_0, then (2.3.45) holds.*

Remark 2.4.18. Since μ_0 is a nonnegative measure, $U(x,t)$ is well defined as long as the nonnegative classical solution u exists, thanks to inequality (2.3.45). Moreover, thanks to the symmetry of P_1, (GC) holds, since $\int_{\mathbb{R}^d} P_1 \, \mathrm{d}\mu_0 \le u(0,1) < \infty$.

The desired integrability is now obtained easily, proceeding as in the proof of Corollary 2.3.23.

Corollary 2.4.19. *Let $\mathcal{L} = \mathcal{L}_\nu$ and P as in Lemma 2.4.17. If u is a nonnegative classical solution of the nonlocal heat equation (NLHE) with initial trace a nonnegative Radon measure μ_0, then $u \in L^1_{\mathrm{loc}}([0,T); L^1_{P_1})$.*

We have now all the elements to obtain a representation formula (implying uniqueness) for nonnegative classical solutions, following the proof of Theorem 2.3.24.

Theorem 2.4.20. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν) and (2.4.5), and $B = B_\nu$ the bilinear form defined in (2.4.4). Assume that $\mathcal{L}_\nu$ has associated a heat kernel P satisfying (2.4.1), (2.4.2), (C_{P_t}), (2.2.21) and (2.2.23), and such that $P \in C^{2,1}_{x,t}(\mathbb{R}^d \times (0,\infty))$ and $P_1 \in C_0(\mathbb{R}^d)$, and that there is a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22). Let u be a nonnegative classical solution to the nonlocal heat equation (NLHE) with initial trace a nonnegative Radon measure μ_0 satisfying (GCS). If (H') holds, then u is given by the representation formula (RF).*

Existence of classical solutions

Let μ_0 a Radon measure (that may have sign changes) satisfying the integral growth condition (GCS). We have already obtained conditions ensuring the function $U(\cdot,t) = P_t * \mu_0$ to be a very weak solution to (NLHE) having μ_0 as initial trace; see Theorem 2.4.12. Mimicking the proof of Theorem 2.3.25 we get conditions showing that it is also a classical solution. Let us note that U belongs to the class $L^\infty((0,T); L^1_{P_1})$ in which there is uniqueness.

Before proceeding, let us make a short technical detour. The starting point of our desired existence result is Theorem 2.4.12, that requires P to be a very weak solution to (NLHE). But our proof needs P to be a classical solution also. Do we need both assumptions? Fortunately not. As can be easily seen following the proof Lemma 2.2.11, under some additional mild hypothesis that we need anyway, if P is a classical solution, it is also a very weak solution.

Lemma 2.4.21. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P that is a classical solution to (NLHE) and satisfies (2.4.1) and (C_{P_t}). If*

$$\iint_{\mathbb{R}^d} |\Lambda P(x,y,t)|\,\mathrm{d}\nu(y) \le C P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T),$$

then P is also a very weak solution to (NLHE).

We can now state the existence theorem.

Theorem 2.4.22. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν satisfying (A_ν). Assume that it has associated a heat kernel P that is a classical solution to (NLHE) and satisfies (2.4.1), (C_{P_t}) and (2.2.21). Let μ_0 a Radon measure satisfying the growth condition (GCS), and U the very weak solution to (NLHE) given in (S_{μ_0}). Suppose that for each $T > 0$ and $\varepsilon \in (0,T)$ there is a constant $C > 0$ such that*

$$|\partial_t P_t(x)| \le C P_1(x), \qquad \int_{\mathbb{R}^d} |\Lambda P(x,y,t)|\,\mathrm{d}\nu(y) \le C P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T).$$

(i) *Then, U is a classical solution to (NLHE) within $L^\infty((0,T); L^1_{P_1})$ and with initial trace μ_0.*
(ii) *If, in addition, for each $T > 0$ and $\varepsilon \in (0,T)$ there is a constant $C > 0$ such that*

$$|D^2 P_t(x)| \le C P_1(x) \quad \text{for all } (x,t) \in \mathbb{R}^d \times (\varepsilon, T),$$

then U is the unique classical solution to (NLHE) with initial trace μ_0 that satisfies (H') and belongs to $L^\infty((0,T); L^1_{P_1})$.
(iii) *If, in addition to all the above, ν satisfies (2.4.5), P satisfies (2.4.2), (2.2.23), $P \in C^{2,1}_{x,t}(\mathbb{R}^d \times (0,\infty))$ and $P_1 \in C_0(\mathbb{R}^d)$, and there is a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22), where $B = B_\nu$ is the bilinear form defined in (2.4.4), and μ_0 is nonnegative, then U is the unique nonnegative classical solution to (NLHE) with initial trace μ_0 that satisfies (H').*

2.4.3 A family of anisotropic operators to which the theory applies

Our goal now is to show that the general theory we have just presented includes many interesting examples that were not covered by the results of Section 2.3. To this aim, we will introduce a family of anisotropic operators $\mathcal{L}_\nu$ that satisfy all the assumptions in the above results.

The Lévy measure and the operator

We consider a family of Lévy kernels $\{K_j\}_{j=1}^{\ell}$, with $\ell \in \{1,\dots,d\}$, where each $K_j : \mathbb{R}^{d_j} \to (0,\infty)$ satisfies assumptions (A0)–(A2) (and (A3) for some results) with $d = d_j$, and $\sum_{j=1}^{\ell} d_j = d$. Given $y \in \mathbb{R}^d$, we write it as $y = (y_1,\dots,y_\ell)$, with $y_j \in \mathbb{R}^{d_j}$, $j \in \{1,\dots,\ell\}$. The (in general anisotropic) measure defining our operator $\mathcal{L}_\nu$ is

$$\mathrm{d}\nu(y) := \sum_{j=1}^{\ell} K_j(y_j)\,\mathrm{d}y_j \prod_{k=1,k\neq j}^{\ell} \delta_0^{d_k}(y_k), \quad y \in \mathbb{R}^d, \tag{2.4.6}$$

where $\delta_0^{d_k}$ is the Dirac delta in $\mathbb{R}^{d_k}$. It is easily checked that this is a Lévy measure.

Proposition 2.4.23. *If K_j, $j \in \{1,\dots,\ell\}$, satisfy (A0) with $d = d_j$, then ν in (2.4.6) satisfies (A_ν).*

Proof. Since K_j is a Lévy kernel in $\mathbb{R}^{d_j}$ for all $j \in \{1,\dots,\ell\}$,

$$\fint_{\mathbb{R}^d} (1\wedge|y|^2)\,\mathrm{d}\nu(y) = \sum_{j=1}^{\ell} \int_{\mathbb{R}^{d_j}} (1\wedge|y_j|^2) K_j(y_j)\,\mathrm{d}y_j < \infty. \qquad \square$$

The Lévy operator $\mathcal{L}_\nu$ defined by ν can be written as $\mathcal{L}_\nu = \sum_{j=1}^{\ell} \mathcal{L}_{K_j}$, where each of the operators $\mathcal{L}_{K_j}$ is of the form (2.1.1) and acts only on the subspace $\mathbb{R}^{d_j}$; to be specific,

$$\mathcal{L}_{K_j} u(x) = \fint_{\mathbb{R}^{d_j}} \Big(u(x) - \frac{u(x_1,\dots,x_j+y_j,\dots,x_\ell) + u(x_1,\dots,x_j-y_j,\dots,x_\ell)}{2}\Big) K_j(y_j)\,\mathrm{d}y_j.$$

In what follows we write for short $\mathcal{L}^j$ to denote $\mathcal{L}_{K_j}$.

The measure ν in (2.4.6) satisfies the condition needed to apply the comparison principle.

Proposition 2.4.24. *Let ν as in Proposition 2.4.23. Then ν satisfies (2.4.5).*

Proof. Since $K_j > 0$ for all $j \in \{1,\dots,\ell\}$, then $\nu\big((B_R(0))^c\big) = \sum_{j=1}^{\ell} \int_{\{|y_j|>R\}} K_j(y_j)\,\mathrm{d}y_j > 0$. $\square$

Just for the record, $-\mathcal{L}_\nu$ is the infinitesimal generator of the Lévy process $X_t = (Y_t^1,\dots,Y_t^\ell)$, $t \geq 0$, where each $\big(Y_t^j\big)_{t\geq 0}$, $j \in \{1,\dots,\ell\}$, is a Lévy process with infinitesimal generator $-\mathcal{L}^j$.

The heat kernel

As explained in Section 2, the Fourier symbol of the operator $\mathcal{L}^j$ is

$$m^j(\xi_j) = \int_{\mathbb{R}^{d_j}} (1 - \cos\langle \xi_j, y_j\rangle)\, \mathrm{d}y_j, \quad \xi_j \in \mathbb{R}^{d_j}, \quad j \in \{1, \dots, \ell\}.$$

and the corresponding heat kernel $P^j(x_j, t) = P^j_t(x_j)$, $(x_j, t) \in \mathbb{R}^{d_j} \times (0, \infty)$, is given by

$$\mathcal{F}(P^j_t)(\xi_j) = e^{-tm^j(\xi_j)}, \quad (\xi_j, t) \in \mathbb{R}^{d_j} \times (0, \infty), \quad j \in \{1, \dots, \ell\}.$$

Using Lévy-Khintchine's formula we obtain the Lévy-Khintchine exponent m for the operator $\mathcal{L}_\nu$,

$$m(\xi) = \fint_{\mathbb{R}^d} (1 - \cos\langle \xi, y\rangle)\, \mathrm{d}\nu(y) = \sum_{j=1}^{\ell} \fint_{\mathbb{R}^{d_j}} (1 - \cos\langle \xi_j, y_j\rangle)\, \mathrm{d}y_j = \sum_{j=1}^{\ell} m^j(\xi_j), \quad \xi = (\xi_1, \dots, \xi_\ell) \in \mathbb{R}^d,$$

with $\xi_j \in \mathbb{R}^{d_j}$ for $j \in \{1, \dots, \ell\}$. Therefore, the heat kernel P_t associated to the nonlocal heat equation (NLHE) for this operator is given by

$$\begin{aligned} P_t(x) &= \int_{\mathbb{R}^d} e^{-tm(\xi)} e^{\langle \xi, x\rangle}\, \mathrm{d}\xi = \fint_{\mathbb{R}^d} e^{-t\sum_{j=1}^{\ell} m^j(\xi_j)} e^{i\sum_{j=1}^{\ell}\langle \xi_j, x_j\rangle}\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_\ell \\ &= \prod_{j=1}^{\ell} \fint_{\mathbb{R}^{d_j}} e^{-tm^j(\xi_j)} e^{i\langle \xi_j, x_j\rangle}\, \mathrm{d}\xi_j = \prod_{j=1}^{\ell} P^j_t(x_j), \quad (x, t) \in \mathbb{R}^d \times (0, \infty). \end{aligned} \tag{AHK}$$

This special structure in separate variables, combined with the special form of the Lévy measure, is the clue to prove all the desired properties for the kernel, that will be inherited in most of the cases from the corresponding ones for the factors P^j, $j \in \{1, \dots, \ell\}$.

Proposition 2.4.25. *Let $\mathcal{L}_\nu$ of the form (2.1.2) with ν given by (2.4.6) and K_j, $j \in \{1, \dots, \ell\}$, satisfying (A0)–(A2) with $d = d_j$. The corresponding heat kernel P belongs to $C^{2,1}_{x,t}(\mathbb{R}^d \times (0, \infty)) \cap C_0(\mathbb{R}^d)$, satisfies ($\mathrm{C}_{P_t}$), (2.2.21), (2.2.23), (2.4.1) and (2.4.2), and solves the equation (NLHE) in the classical and in the very weak sense.*

Proof. All the properties, except (2.4.2), have already been proved for the factors P^j. The corresponding properties for P can be easily recovered from the results for the factors thanks to the special form of the Lévy measure. We omit the details.

The proof of (2.4.2) also uses the special form (2.4.6) of the Lévy measure ν and the product structure (AHK) of P_1. Indeed,

$$\fint_{\{|y|\geq 1\}} P_1(x - y)\, \mathrm{d}\nu(y) = \sum_{j=1}^{\ell} \underbrace{\fint_{\{|y_j|\geq 1\}} P^j_1(x_j - y_j) |K^j(y_j)\, \mathrm{d}y_j}_{\mathrm{I}_j(x_j)} \prod_{k=1, k\neq j}^{\ell} P^k_1(x_k).$$

But we can control the tail of K^j by the tail of P_1^j; see (C_{PK}). Thus, using also the semigroup property (2.2.9), for some constant $C_j > 0$,

$$\mathrm{I}_j(x_j) \le C_j \int_{\{|y_j|\ge 1\}} P_1^j(x_j - y_j) P_1^j(y_j)\,\mathrm{d}y_j \le C_j \int_{\mathbb{R}^{d_j}} P_1^j(x_j - y_j) P_1^j(y_j)\,\mathrm{d}y_j = C_j P_2^j(x_j).$$

On the other hand, because of (C_{P_t}), there exists $C_j > 0$ such that $P_2^j \le C_j P_1^j$, and we conclude that $\mathrm{I}_j \le C_j P_1^j$ for some constant $C_j > 0$ for each $j \in \{1,\dots,\ell\}$. Therefore,

$$\iint\!\!\!\!\!\!\int_{\{|y|\ge 1\}} P_1(x-y)\,\mathrm{d}\nu(y) \le \sum_{j=1}^{\ell} C_j P_1^j(x_j) \prod_{k=1,k\neq j}^{\ell} P_1^k(x_k) = \sum_{j=1}^{\ell} C_j P_1(x).$$

To check that P is a classical solution to (NLHE), we use that $\partial_t P^j + \mathcal{L}^j P^j = 0$, $j \in \{1,\dots,\ell\}$. Hence, for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$,

$$-\partial_t P(x,t) = -\sum_{j=1}^{\ell} \partial_t P^j(x_j,t) \prod_{k=1,k\neq j}^{\ell} P^k(x_k,t) = \sum_{j=1}^{\ell} \mathcal{L}^j P^j(x_j,t) \prod_{k=1,k\neq j}^{\ell} P^k(x_k,t) = \mathcal{L}_\nu P(x,t).$$

Once we know that P is a classical solution to the equation we can show that it is also a very weak solution mimicking the proof of Lemma 2.2.11. □

The function ϕ

The last thing that we have to check is the existence of a function $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22). We have already proved that such function ϕ^j exists for each of the operators $\mathcal{L}^j$, $j \in \{1,\dots,\ell\}$ (and the corresponding bilinear form B_j) if the kernels K_j satisfy (A0)–(A3). Our function ϕ will then be given by

$$\phi(x) := \prod_{j=1}^{\ell} \phi^j(x_j) \quad \text{for all } x = (x_1,\dots,x_\ell) \in \mathbb{R}^d. \tag{2.4.7}$$

Lemma 2.4.26. *Let $\mathcal{L} = \mathcal{L}_\nu$ of the form (2.1.2) with ν given by (2.4.6) with functions K_j, $j = 1,\dots,\ell$ satisfying (A0)–(A3). Let P be the associated heat kernel, given by (AHK), and $B = B_\nu$ the bilinear form defined in (2.4.4). The function ϕ defined in (2.4.7) belongs to $C^2(\mathbb{R}^d)$ and satisfies (2.3.21)–(2.3.22).*

Proof. Since P_1^j is comparable to ϕ^j for each $j \in \{1,\dots,\ell\}$, (2.3.21) follows easily thanks to the structure in separate variables of both ϕ and P.

On the other hand, since $|\mathcal{L}^j \phi^j| \le c_j \phi^j$ for some c_j for all $j \in \{1,\dots,\ell\}$, using again the structure in separate variables of ϕ and the special form (2.4.6) of the Lévy measure we arrive at

$$|\mathcal{L}_\nu \phi(x)| = \sum_{j=1}^{\ell} \mathcal{L}^j \phi^j(x_j) \prod_{k=1,k\neq j}^{\ell} \phi^k(x_k) \le \sum_{j=1}^{\ell} c_j \phi(x) = c\phi(x) \quad \text{for all } x \in \mathbb{R}^d.$$

As for the second inequality in (2.3.22), we have $|B_\nu(\psi,\phi)(x)| \le \|\nabla\psi\|_\infty \mathrm{I}(x) + 2\|\psi\|_\infty \mathrm{II}(x)$, where

$$\mathrm{I}(x) := \int_{\{|y|\le 1\}} |y||\phi(x)-\phi(x-y)|\,\mathrm{d}\nu(y),$$
$$\mathrm{II}(x) := \underbrace{\phi(x)\int_{\{|y|\ge 1\}} \mathrm{d}\nu(y)}_{\mathrm{II}_1(x)} + \underbrace{\int_{\{|y|\ge 1\}} \phi(x-y)\,\mathrm{d}\nu(y)}_{\mathrm{II}_2(x)}.$$

Because of the special form (2.4.6) of the Lévy measure ν and the product structure (2.4.7) of ϕ, since K_j and ϕ^j satisfy (2.3.25)–(2.3.26) and the kernels K^j satisfy the Lévy condition (A0), then

$$\begin{aligned}\mathrm{I}(x) &= \sum_{j=1}^{\ell} \fint_{\{|y_j|\le 1\}} |y_j||\phi^j(x_j)-\phi^j(x_j-y_j)|K_j(y_j)\,\mathrm{d}y_j \prod_{k=1,k\neq j}^{\ell}\phi^k(x_k)\\ &\le \sum_{j=1}^{\ell} C_j \fint_{\{|y_j|\le 1\}} |y_j|^2 K_j(y_j)\,\mathrm{d}y_j \prod_{k=1}^{\ell}\phi^k(x_k) = C\phi(x).\end{aligned}$$

On the other hand, since ν is a Lévy measure, then $\mathrm{II}_1(x)\le C\phi(x)$. Finally, using again the special form of ν and ϕ, and then, for all $j\in\{1,\dots,\ell\}$, the comparability (C_{PK}) of K^j and P_1^j outside the origin, the semigroup property (2.2.9), (C_{P_t}) (for P^j) and the comparability of P_1^j and ϕ^j,

$$\begin{aligned}\mathrm{II}_2(x) &= \sum_{j=1}^{\ell} \fint_{\{|y_j|\ge 1\}} \phi^j(x_j-y_j)K^j(y_j)\,\mathrm{d}y_j \prod_{k=1,k\neq j}^{\ell}\phi^k(x_k)\\ &\le \sum_{j=1}^{\ell}\Big(\prod_{k=1,k\neq j}^{\ell}\phi^k(x_k)\Big)\fint_{\mathbb{R}^{d_j}} C_j P_1^j(x_j-y_j)P_1^j(y_j)\,\mathrm{d}y_j \le \sum_{j=1}^{\ell}\Big(\prod_{k=1,k\neq j}^{\ell}\phi^k(x_k)\Big)C_j P_2^j(x_j)\\ &\le \sum_{j=1}^{\ell}\Big(\prod_{k=1,k\neq j}^{\ell}\phi^j(x_j)\Big)C_j P_1^j(x_j) \le \sum_{j=1}^{\ell}\Big(\prod_{k=1,k\neq j}^{\ell}\phi^j(x_j)\Big)C_j\phi^j(x_j) = C\phi(x).\end{aligned}$$

□

Existence in a special case

As in the case of operators of the form $\mathcal{L}_K$, we will only be able to prove existence for a special case, namely $\mathcal{L}_\nu = \sum_{j=1}^{\ell}(-\Delta_{d_j})^{\alpha/2}$, where $\alpha\in(0,2)$ and $(-\Delta_{d_j})^{\alpha/2}$ is the d_j-dimensional α-fractional Laplacian.

We denote the heat kernels associated to $(-\Delta_{d_j})^{\alpha/2}$, $j=1,\dots,\ell$, and $\sum_{j=1}^{\ell}(-\Delta_{d_j})^{\alpha/2}$ respectively by $P^{j,\alpha}$, $j=1,\dots,\ell$ and P^α. We recall that

$$P^\alpha(x,t) = \prod_{j=1}^{\ell} P^{j,\alpha}(x_j,t), \quad x=(x_1,\dots,x_\ell)\in\mathbb{R}^d,\ t>0. \tag{2.4.8}$$

Then the heat kernel P^α inherits the self-similar structure from the kernels P_j^α, $j = 1, \dots, \ell$,

$$P^\alpha(x,t) = \prod_{j=1}^{\ell} \left(t^{-d_j/\alpha} P^{j,\alpha}(x_j t^{-\alpha}, 1)\right) = t^{-d/\alpha} \prod_{j=1}^{d} P^{j,\alpha}(x_j t^{-\alpha}, 1) = t^{-d/\alpha} P^\alpha(x t^{-\alpha}, 1). \tag{2.4.9}$$

This is the clue to prove the necessary conditions for the existence result.

Proposition 2.4.27. *For all $\varepsilon \in (0,T)$ there is a constant $C > 0$ such that*

$$\text{(i) } |\partial_t P^\alpha(x,t)| \le C P_1^\alpha(x), \quad \text{(ii) } |D^2 P_t(x)| \le C P_1(x), \qquad x \in \mathbb{R}^d,\ t \in (\varepsilon, T).$$

Proof. (i) The selfsimilar structure (2.4.9) of P^α and its product structure (2.4.8) imply that

$$\partial_t P^\alpha(x,t) = -\frac{d}{\alpha t} t^{-d/\alpha} P_1^\alpha(x t^{-\alpha}) - \frac{t^{-d/\alpha}}{\alpha t} \sum_{j=1}^{\ell} \langle x_j t^{-\alpha}, \nabla_{x_j} P_1^{j,\alpha}(x_j t^{-\alpha})\rangle \prod_{k=1, k\neq j}^{\ell} P_1^{k,\alpha}(x_k t^{-\alpha}).$$

On the other hand, by Proposition 2.2.16 (iii), for each $j \in \{1, \dots, \ell\}$ there is a constant C_j such that $|\langle x_j t^{-\alpha}, \nabla P_1^{j,\alpha}(x_j t^{-\alpha})\rangle| \le C_j P_1^{j,\alpha}(x_j t^{-\alpha})$, and the result follows using (2.4.8) and (C_{P_t}).
(ii) There are two possibilities. If the two variables, x_i, x_l, $i, l \in \{1, \dots, d\}$, with respect to which we differentiate fall within the same subspace $\mathbb{R}^{d_j}$, $j \in \{1, \dots, \ell\}$, then, thanks to (2.4.8),

$$\partial_i \partial_l P_t^\alpha(x) = \partial_i \partial_l P_t^{j,\alpha}(x_j) \prod_{k=1, k\neq j}^{\ell} P_t^{k,\alpha}(x_k),$$

and the result follows using Proposition 2.2.16 (vi). If, instead, each of them falls in a different subspace, say x_i in $\mathbb{R}^{d_{j_1}}$ and x_l in $\mathbb{R}^{d_{j_2}}$, $j_1, j_2 \in \{1, \dots, \ell\}$, $j_1 \neq j_2$, then,

$$\partial_i \partial_l P_t^\alpha(x) = \partial_i P_t^{j_1,\alpha}(x_{j_1}) \partial_l P_t^{j_2,\alpha}(x_{j_2}) \prod_{k=1, k\neq j_1, j_2}^{\ell} P_t^{k,\alpha}(x_k),$$

and now the result follows using Proposition 2.2.16 (v). □

2.5 Operators having a local and a nonlocal part

We consider now operators that combine both a local and a nonlocal part, more precisely those of the form $\mathcal{L} = \mathcal{L}_{\Delta,K} := -\Delta + \mathcal{L}_K$, with $\mathcal{L}_K$ as in (2.1.1) and K satisfying (A0)–(A2) (and (A3) in some cases). Estimates of the associated heat kernels have been the subject of other papers, such as [9, 18] and the more recent [12]. The latter addresses the qualitative properties of solutions to a class of fractional Schrödinger equations. However, in all three papers only the case $K = K_\alpha$ related to the fractional Laplacian $(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$ is considered.

2.5.1 The heat kernel

When $\mathcal{L} = \mathcal{L}_{\Delta,K}$, with K satisfying (A0)–(A2), the Lévy process associated to equation (NLHE) is the sum of two independent Lévy processes, each one with its heat kernel: the Lévy process associated to $\partial_t u + \mathcal{L}_K u = 0$, with heat kernel P^K, and the Brownian process associated to the classical heat equation, with the well-known heat kernel G, given in (2.1.4).

Let m_K be the Lévy-Khintchine exponent of $\mathcal{L}_K$, given by (2.2.10). Since the Fourier symbol of $-\Delta$ is $|\xi|^2$, the heat kernel P associated to $\mathcal{L}_{\Delta,K}$ is then given in terms of Fourier transform by

$$\mathcal{F}(P_t)(\xi) = e^{-t(m_K(\xi)+|\xi|^2)}, \quad \xi \in \mathbb{R}^d,\ t > 0.$$

Therefore, using the properties of the Fourier transform,

$$P_t(x) = P_t^K * G_t(x), \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.5.1}$$

Since G_t decays as the inverse of a quadratic exponential and $\tilde{P}_t$ as the inverse of some polynomial (see (C_{PK}), (A1) and (A2)), there is a constant $C_t > 0$ such that

$$G_t(x) \le C_t P_t^K(x) \quad \text{for all } x \in \mathbb{R}^d. \tag{2.5.2}$$

From this and (2.5.1) we deduce now that P_t is comparable with P_t^K, a key fact for what follows.

Proposition 2.5.1. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2) and let P be corresponding heat kernel. For each $t > 0$ there is a constant $C_t > 1$ such that $C_t^{-1} P_t^K \le P_t \le C_t P_t^K$.*

The result when $K = K_\alpha$ can be seen in [**Zoran2**]. Our proof for more general kernels K is, somehow, shorter.

Proof. As for the first inequality, since $P_t^K, G_t > 0$, then $P_t > 0$. Therefore, there is a constant $C_t > 0$ such that $P_t^K(x) \le C_t P_t(x)$ for all $|x| \le 2$. Consider now $|x| \ge 2$. If $|y| < 1$, then $|x - y| < |x| + 1 \le 3|x|/2$. Hence, since P_t^K is almost decreasing and slowly changing, see Remark 2.2.9,

$$P_t(x) \ge \int_{\{|y|<1\}} \tilde{P}_t(x-y) G_t(y)\,\mathrm{d}y \ge C_t P_t^K(3x/2) \int_{\{|y|<1\}} G_t(y)\,\mathrm{d}y \ge C_t P_t^K(x).$$

To prove the second inequality we make use of (2.5.2), the semigroup structure and (C_{P_t}) to deduce

$$P_t(x) = P_t^K * G_t(x) \le C_t P_t^K * P_t^K(x) = C_t P_{2t}^K(x) \le \tilde{C}_t P_t^K(x).$$

□

As a corollary of Proposition 2.5.1, since P^K is comparable in two different times, see (C_{P_t}), so is P too.

Corollary 2.5.2. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2) and let P be corresponding heat kernel. For all $t_1, t_2 > 0$ there exists $C > 1$ such that*

$$C^{-1} P_{t_1} \leq P_{t_2} \leq C P_{t_1}.$$

Moreover, since P_t^K is slowly changing and almost decreasing for all $t > 0$ and P_1^K and P_1 are comparable, the heat kernel associated to $\mathcal{L}_{\Delta,K}$ has also these properties.

Corollary 2.5.3. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2) and let P be corresponding heat kernel. Then, P_t is slowly changing and almost decreasing for all $t > 0$.*

2.5.2 Very weak solutions

Thanks to the comparability between P_t^K and P_t, we can develop a complete Widder-type theory for very weak solutions for the operator $\mathcal{L}_{\Delta,K}$.

Alternative definitions of very weak solution

We can adapt easily the proof of Proposition 5.2.5 to check that integrability against $\mathcal{L}_{\Delta,K}\theta$ with θ a test function is equivalent to integrability against P_1. The only delicate point is to check that if $\psi \in C_c^\infty(\mathbb{R}^d)$ is such that $\mathcal{X}_{B_1(0)} \leq \psi \leq \mathcal{X}_{B_2(0)}$, then there is a constant $C > 0$ such that $|\mathcal{L}_{\Delta,K}\psi(x)| \geq C P_1(x)$ for all $|x| \geq 4$. But this follows easily from (2.3.2), since $-\Delta$ is a local operator and $\mathrm{supp}(\psi) \subset B_2(0)$. Thus we have the following result.

Proposition 2.5.4. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$, with K satisfying (A0)–(A2).*
(i) *Let $0 \leq \delta < \tau \leq T$. Then $u \in L^1((\delta,\tau); L^1_{P_1})$ if and only if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times [\delta,\tau])$ and (2.3.1) holds.*
(ii) *$u \in L^1_{\mathrm{loc}}((0,T); L^1_{P_1})$ if and only if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ and (2.1.7) holds.*

It is now easy to mimic the proof of Proposition 2.3.3 (that essentially only has to deal with the way the initial datum is taken) to show the equivalence in the class $L^\infty((0,T); L^1_{P_1})$ between the two definitions of very weak solution that we are handling.

Proposition 2.5.5. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure and $u \in L^\infty((0,T); L^1_{P_1})$.*
(i) *If u satisfies (2.3.4), then it is a very weak solution to (NLHE) with initial trace μ_0.*
(ii) *If u is a very weak solution to (NLHE) with initial trace μ_0, then it satisfies (2.3.3)–(2.3.4).*

Uniqueness for very weak solutions

The proof of uniqueness for very weak solutions is similar to that of Theorem 2.3.5, taking care of an extra term in the bilinear form associated to the operator.

Theorem 2.5.6. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure. Let u_1 and u_2 satisfy (2.3.3)–(2.3.4). Then, $u_1 = u_2$ a.e. in $\mathbb{R}^d \times (0,T)$.*

Proof. The main difference with respect to the proof of Theorem 2.3.5 is that the bilinear form has an extra term with respect to the one appearing there, so that now it is given by

$$B_{\Delta,K}(u,v) = B(u,v) + 2\langle \nabla u, \nabla v\rangle,$$

with B as in (2.3.15). Hence, using $\psi_R\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ as test function in (2.3.9), where $\{\psi_R\}_{R\geq R_0}$ is the family of cutoff functions given by (2.1.12) and φ is the solution to the backward problem $\partial_t\varphi - \mathcal{L}_{\Delta,K}\varphi = \tilde{\theta}$ in $\mathbb{R}^d \times (0,T)$ given by (2.3.12), we arrive at

$$\begin{aligned} 0 = &\fint_0^T \int_{\mathbb{R}^d} u(x,t)\tilde{\theta}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \fint_0^T \int_{\mathbb{R}^d} u(x,t)\varphi(x,t)\mathcal{L}_{\Delta,K}\psi_R(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \fint_0^T \int_{\mathbb{R}^d} u(x,t)B(\phi_R,\varphi)(x,t)\,\mathrm{d}t + 2\fint_0^T \int_{\mathbb{R}^d} u(x,t)\langle\nabla\phi_R,\nabla\varphi\rangle(x,t)\,\mathrm{d}t. \end{aligned}$$

If we check that the three last terms on the right-hand side go to 0 as $R\to\infty$ the proof will be complete. The two first of them are handled essentially as in the proof of Theorem 2.3.5, using that P_1 is comparable to the kernel K far from the origin, since P_1 is comparable to P_1^K, and observing also that $\|\Delta\psi_R\|_\infty = R^{-2}\|\Delta\psi\|_\infty$.

As for the last term, it is also handled easily, since $|\nabla\varphi| \leq CP_1$ (see once more the proof of Theorem 2.3.5), $\|\nabla\psi_R\|_\infty = R^{-1}\|\nabla\psi\|_\infty$ and $u \in L^1((0,t_0);L^1_{P_1})$. Therefore, we may apply the DCT to arrive at

$$\lim_{R\to\infty} \fint_0^T \int_{\mathbb{R}^d} |u(x,t)||\langle\nabla\phi_R(x),\nabla\varphi(x,t)\rangle|\,\mathrm{d}x\,\mathrm{d}t = 0. \qquad \square$$

As a consequence, since by Proposition 2.5.5 a very weak solution $u \in L^\infty((0,T);L^1_{P_1})$ with initial datum a Radon measure μ_0 satisfies (2.3.4), we obtain the following corollary.

Corollary 2.5.7 (Uniqueness for very weak solutions within $L^\infty((0,T);L^1_{P_1})$)**.** *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2). There is at most one very weak solution to equation (NLHE) belonging to $L^\infty((0,T);L^1_{P_1})$ with a given initial trace.*

Now, we move on to the class of nonnegative very weak solutions, following the strategy of Section 2.3.

Let $\phi \in C^2(\mathbb{R}^d)$ be the function given by Lemma 2.3.7. By Proposition 2.5.1, P_1 and P_1^K are comparable, and by (2.3.21), $\tilde{P}_1$ and ϕ are comparable. Hence, (2.3.21) holds: P_1 and ϕ are comparable. Besides, by (2.3.24) and the positivity and continuity of $\phi > 0$, there is some $C > 0$ such that

$$|\Delta\phi| \leq C\phi, \qquad |\langle\nabla\psi,\nabla\phi\rangle| \leq \|\nabla\psi\|_\infty|\nabla\phi| \leq C\|\nabla\psi\|_\infty\phi \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^d).$$

Combining this with (2.3.22) (applied to $\mathcal{L}_K$), we arrive at (2.3.22) with $\mathcal{L} = \mathcal{L}_{\Delta,K}$ and $B = B_{\Delta,K}$. Therefore, arguing as in the proof of Lemma 2.3.9, using also Proposition 2.5.4, we have the desired smoothing effect.

Lemma 2.5.8. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A3).*
(i) *If u is a nonnegative very weak solution to (NLHE), there is $c > 0$ such that (2.3.28) holds.*
(ii) *If u is a nonnegative very weak solution to (NLHE), then $u \in L^\infty((0,T); L^1_{P_1})$.*

Imitating the proof of Proposition 2.3.11 we get some regularity in time for general operators.

Proposition 2.5.9. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A3), and let u be a nonnegative very weak solution to (NLHE).*
(i) *Let $\phi \in C^2(\mathbb{R}^d)$ satisfying (2.3.21)–(2.3.22). There is a constant $C > 0$ such that (2.3.35) holds.*
(ii) *Given $\psi \in C^2_c(\mathbb{R}^d)$, there is a constant $C_\psi > 0$ such that (2.3.36) holds.*

The desired uniqueness result for nonnegative very weak solutions now follows from the combination of Lemma 2.5.8 (ii) and Corollary 2.5.7.

Corollary 2.5.10 (Uniqueness for nonnegative very weak solutions)**.** *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A3). There is at most one nonnegative very weak solution of (NLHE) with a given initial trace.*

Existence of initial trace

The proof of Theorem 2.3.15 giving the existence of an initial trace satisfying the integral growth condition (GC) for nonnegative very weak solutions is based on the estimates (2.3.28), and may therefore be easily adapted to cope with operators $\mathcal{L} = \mathcal{L}_{\Delta,K}$.

Theorem 2.5.11 (Existence of initial trace integrable against P_1)**.** *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A3). Let u be a nonnegative very weak solution of (NLHE) in $\mathbb{R}^d \times (0,T)$. There is a nonnegative Radon measure μ_0 such that (2.1.10) holds; that is, μ_0 is the initial trace of u. Moreover, μ_0 satisfies the integral growth condition (GC).*

Existence of very weak solutions

The existence of very weak solutions, for suitable initial data μ_0, is obtained by imitating the proof of Theorem 5.2.9. They will be given by $U(\cdot,t) = P_t * \mu_0$. The news with respect to that theorem are that now $U \in C^\infty(\mathbb{R}^d \times (0,\infty))$. Indeed, because of (2.5.1), $U(\cdot,t) = G_t * (P^K_t * \mu_0)$ for all $t > 0$, whence, since $(P^K_t * \mu_0)$ is continuous, the regularity of U coincides with that of G.

Theorem 2.5.12. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2). Let μ_0 be a Radon measure with the admissible growth at infinity (GCS). Then, for each $T > 0$, the function U given in (S_{μ_0}) is a very weak solution of (NLHE) in $C^\infty(\mathbb{R}^d \times (0,T)) \cap L^\infty((0,T); L^1_{P_1})$ with initial trace μ_0.*

As a consequence, thanks to the uniqueness results, Corollary 2.5.7 and Corollary 2.5.10, we have a representation formula.

Corollary 2.5.13 (Representation formula)**.** *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying* (A0)–(A2) *and let μ_0 be a Radon measure satisfying* (GCS)*. Let u be a very weak solution to* (NLHE) *with trace μ_0. If either $u \in L^\infty((0,T);L^1_{P_1})$, or u is nonnegative and K satisfies in addition* (A3)*, then u is given by representation formula* (RF)*.*

2.5.3 Classical solutions

The main novelty in the analysis of classical solutions is that for operators $\mathcal{L}_{\Delta,K}$ we don't need to impose the energy assumption (H), since in this case classical solutions belong by definition to $C^2(\mathbb{R}^d \times (0,T))$ and, therefore, are already known to satisfy it.

Uniqueness of classical solutions.

We can now imitate the proof of Proposition 2.3.18 to prove that classical solutions are very weak solutions, the key step towards uniqueness in $L^\infty((0,T);L^1_{P_1})$.

Proposition 2.5.14. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying* (A0)–(A2)*. If $u \in L^1_{\mathrm{loc}}((0,T);L^1_{P_1})$ is a classical solution to* (NLHE)*, then it is a very weak solution to the same equation.*

As a byproduct of Proposition 2.5.14, Corollary 2.5.7 and Corollary 2.5.13, we obtain uniqueness for classical solutions in $L^\infty((0,T);L^1_{P_1})$, and a representation formula if there is existence in this class.

Corollary 2.5.15. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying* (A0)–(A2)*.*
(i) *There is at most one classical solution $u \in L^\infty((0,T);L^1_{P_1})$ to* (NLHE) *with initial trace a given Radon measure.*
(ii) *If K satisfies also* (A3)*, if there exists a classical solution $u \in L^\infty((0,T);L^1_{P_1})$ to* (NLHE) *with initial trace a measure μ_0 verifying* (GCS)*, then u is given by the representation formula* (RF)*.*

To prove uniqueness of nonnegative classical solutions, we first obtain a comparison principle. This can be done following the proof of Lemma 2.3.20, observing that at a point (x_0,t_0) where v achieves a maximum we have $\Delta v(x_0,t_0) \le 0$ and $\mathcal{L}_K v(x_0,t_0) > 0$, so that $\mathcal{L}_{\Delta,K} v(x_0,t_0) > 0$.

Lemma 2.5.16 (A maximum principle)**.** *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying* (A0)*. Let $\Omega \subset \mathbb{R}^d$ be non-empty, open, bounded and smooth, and $v \in C(\mathbb{R}^d \times [0,T))$ such that $v \in C^{2,1}_{x,t}(\Omega \times (0,T))$ and $\mathcal{L}_K v(x,t)$ is defined for all $(x,t) \in \Omega \times (0,T)$. If* (2.3.46) *holds, then $v \le 0$ for all $x \in \mathbb{R}^d \times [0,T)$.*

We can now show the validity of (2.3.45), following the proof of Lemma 2.3.21.

Lemma 2.5.17. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying* (A0)–(A2)*, and u a nonnegative classical solution to* (NLHE) *with initial data a nonnegative Radon measure μ_0. Then,* (2.3.45) *holds.*

As a consequence, proceeding as in the proof of Corollary 2.3.23, we obtain that $u \in L^1_{\mathrm{loc}}([0,T);L^1_{P_1})$.

Corollary 2.5.18. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2), and u a nonnegative classical solution to (NLHE) with initial data a nonnegative Radon measure μ_0. Then, $u \in L^1_{\mathrm{loc}}([0,T);L^1_{P_1})$.*

We have now all the ingredients to get a representation formula (implying uniqueness) for nonnegative classical solutions, mimicking the proof of Theorem 2.3.24.

Theorem 2.5.19. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A3). Let u be a nonnegative classical solution to (NLHE) with initial trace a nonnegative Radon measure μ_0. Then u is given by the representation formula (RF).*

Existence of classical solutions

Our goal now is to prove that, if $\mathcal{L} = \mathcal{L}_{\Delta,K}$, then for any Radon measure μ_0 satisfying the integral growth condition (GCS) there is a classical solution to (NLHE) having μ_0 as initial trace. The natural candidate is the function $U(x,t) = P_t * \mu_0$, which is a very weak solution of the equation and has μ_0 as initial trace, and moreover belongs to the class $L^\infty((0,T);L^1_{P_1})$ in which there is uniqueness. In the case of operators of the form $\mathcal{L}_K$ we had to impose certain conditions on the heat kernel, namely (2.3.48), in order for such U to be a classical solution. When $\mathcal{L} = \mathcal{L}_{\Delta,K}$, the local part of the diffusion, the Laplacian, helps us, and these conditions are always satisfied, as we see next. Let us remark that the energy condition (H) is not required now for uniqueness; nevertheless, P also satisfies (2.3.49). Besides, the local part of the operator does not require any extra hypothesis.

Lemma 2.5.20. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2), and let P be the corresponding heat kernel. For each $T > 0$ and $\varepsilon \in (0,T)$ there is a constant $C > 0$ such that (2.3.48)–(2.3.49) hold.*

Proof. We start by proving that for all $0 < \varepsilon < T$ there is $C_{\varepsilon,T} > 0$ such that $|\partial_t P_t(x)| \le C_{\varepsilon,T} P_1$ for all $t \in (\varepsilon, T)$. On the one hand, a direct computation shows that

$$\begin{aligned}|\partial_i\partial_j G_t(x)| &\le \Big(\frac{1}{4t} + \frac{|x|^2}{(4t)^2}\Big) G_t(x) \le \Big(\frac{1}{4\varepsilon} + \frac{|x|^2}{(4\varepsilon)^2}\Big)\frac{1}{(4\pi\varepsilon)^{d/2}} e^{-|x|^2/(4T)} \\ &\le C_\varepsilon(1+|x|^2)e^{-|x|^2/(4T)} \le C_{\varepsilon,T} G_T(x) \quad \text{for all } x \in \mathbb{R}^d,\ t \in (\varepsilon,T).\end{aligned}$$

Moreover, thanks to (2.5.2) and the fact that P^K satisfies (C_{P_t}), we have $G_T \le C_T P_1^K$, whence

$$\max_{t\in[\varepsilon,T]} \max_{i,j\in\{1,\dots,d\}} |\partial_i\partial_j G_t(x)| \le C_{\varepsilon,T} P_1^K(x) \quad \text{for all } x \in \mathbb{R}^d. \tag{2.5.3}$$

On the other hand, from the properties of P^K and the MVT, we obtain that

$$\begin{aligned}|\mathcal{L}_K G_t(x)| &\le G_t(x)\int_{\{|y|\ge 1\}} K(y)\,\mathrm{d}y + \fint_{\{|y|\ge 1\}} G_t(x-y)K(y)\,\mathrm{d}y + \int_{\{|y|\le 1\}} |\Lambda G_t(x,y)|K(y)\,\mathrm{d}y\\ &\le C_{\varepsilon,T}\Big(P_1^K(x) + \int_{\{|y|\ge 1\}} P_1^K(x-y)P_1^K(y)\,\mathrm{d}y\Big) + \int_{\{|y|\le 1\}} |\Lambda G_t(x,y)|K(y)\,\mathrm{d}y\\ &\le C_{\varepsilon,T}\Big(P_1^K(x) + P_2^K(x)\Big) + \int_{\{|y|\le 1\}} |D^2 G_t(z)||y|^2 K(y)\,\mathrm{d}y,\end{aligned}$$

for some $z = z(x,y) \in B_1(x)$. Now, using (2.5.3) we conclude that

$$|\mathcal{L}_K G_t(x)| \le C_{\varepsilon,T}\Big(P_1^K(x) + \int_{\{|y|\le 1\}} P_1^K(z)|y|^2 K(y)\,\mathrm{d}y\Big) \le C_{\varepsilon,T} P_1^K(x), \tag{2.5.4}$$

where, for the last inequality, we have used that $P_1^K(z) \le C P_1^K(x)$, due to Lemma 2.2.4, and the fact that $|x-z| \le 1$. As a consequence of (2.5.4), since $P_t = P_t^K * G_t$ is the fundamental solution to (NLHE) with $\mathcal{L} = \mathcal{L}_{\Delta,K}$, using (2.5.3), the semigroup property of P^K and Proposition 2.5.1, we have

$$\begin{aligned}|\partial_t P_t(x)| = |\mathcal{L}_{\Delta,K} P_t(x)| = \ & P_t^K * \mathcal{L}_{\Delta,K} G_t(x) \ \le P_t^K * |\mathcal{L}_K G_t|\,(x) + P_t^K * |\Delta G_t|\,(x)\\ &\le C_{\varepsilon,T} P_t^K * P_1^K(x) \le C_{\varepsilon,T} P_{1+t}^K(x) \le C_{\varepsilon,T} P_1(x) \quad \text{for all } x \in \mathbb{R}^d,\end{aligned}$$

which is the first inequality in (2.3.48).

Second, by (2.5.3), the semigroup property of P_t^K and Proposition 2.5.1, we get

$$|D^2 P_t| = \max_{i,j\in\{1,\dots,d\}} |\partial_i\partial_j P_t| \le P_t^K * \max_{i,j\in\{1,\dots,d\}} |\partial_i\partial_j G_t| \le C_{\varepsilon,T} P_t^K * P_1^K = C_{\varepsilon,T} P_{1+t}^K \le C_{\varepsilon,T} P_1 \tag{2.5.5}$$

for all $t \in (\varepsilon, T)$, which is estimate (2.3.49).

Finally, we will obtain the second inequality in (2.3.48). Procceding as before, using the MVT and (2.5.5), we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\Lambda P_t(x,t)|K(y)\,\mathrm{d}y &\le \fint_{\{|y|\ge 1\}} \big(P_t(x) + P_t(x-y)\big)K(y)\,\mathrm{d}y + \int_{\{|y|\le 1\}} |D^2 P_t(z)|y|^2 K(y)\,\mathrm{d}y\\ &\le C P_t(x) + \int_{\{|y|\ge 1\}} P_t(x-y)K(y)\,\mathrm{d}y + C_{\varepsilon,T}\int_{\{|y|\le 1\}} P_1(z)|y|^2 K(y)\,\mathrm{d}y,\end{aligned}$$

where $z = z(x,y,t) \in B_1(x)$. We estimate the second term as follows

$$\int_{\{|y|\ge 1\}} P_t(x-y)K(y)\,\mathrm{d}y \le C\int_{\{|y|\ge 1\}} P_t(x-y)P_1(y)\,\mathrm{d}y \le C P_{1+t}(x) \le C_T P_1(x),$$

whereas for the third term we use that P_1 is slowly changing (see Corollary 2.5.3), that $z \in B_1(x)$ and that K is a Lévy kernel, to obtain

$$\int_{\{|y|\le 1\}} P_1(z)|y|^2 K(y)\,\mathrm{d}y \le C P_1(x)\int_{\{|y|\le 1\}} |y|^2 K(y)\,\mathrm{d}y = C P_1(x).$$

Putting all together we have that

$$\iint_{\mathbb{R}^d} |\Lambda P_t(x,t)| K(y)\,\mathrm{d}y \leq C P_1(x) \quad \text{for all } x \in \mathbb{R}^d. \qquad \square$$

We can now reproduce the proof of Theorem 2.3.25 to arrive at our last result.

Theorem 2.5.21. *Let $\mathcal{L} = \mathcal{L}_{\Delta,K}$ with K satisfying (A0)–(A2), μ_0 a Radon measure satisfying the integral growth condition (GCS), and U the very weak solution to (NLHE) given in (S_{μ_0}). Then, for each $T > 0$, U is the unique classical solution to (NLHE) within $L^\infty((0,T); L^1_{P_1})$ with initial datum μ_0. If in addition K satisfies (A3) and μ_0 is nonnegative, then U is the unique nonnegative classical solution to (NLHE) with initial trace μ_0.*

Chapter 3

The KPP problem on the hyperbolic space

This chapter is devoted to the results of [48], named *Travelling wave behaviour for Fisher-KPP equations in the hyperbolic space.* This paper is authored by María del Mar González (UAM), Fernando Quirós and the author of this thesis.

3.1 Introduction and statement of results

We study the sharp long-time behavior of solutions to the Cauchy problem for the semilinear parabolic Fisher-KPP equation posed in the hyperbolic space $\mathbb{H}^d$, $d \geq 2$,

$$u_t = \Delta_{\mathbb{H}^d} u + f(u) \quad \text{in } \mathbb{H}^d \times \mathbb{R}_+, \qquad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{H}^d. \tag{KPP}$$

Here $\Delta_{\mathbb{H}^d}$ is the Laplace-Beltrami operator in $\mathbb{H}^d$ and $\mathbb{R}_+ := (0, \infty)$. Solutions to (KPP) are always meant in the classical sense. Throughout the paper we always assume that $0 \leq u_0 \leq 1$ and that f is a KPP *function*,

$$\begin{gathered} f \in C^2([0,1]), \quad f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0), \\ f(s) > 0 \text{ and } f(s) \leq f'(0)s \quad \text{for all } s \in (0,1). \end{gathered} \tag{H_f}$$

The main example of a KPP function is $f(s) := s(1-s)$, $s \in [0,1]$. Our motivation is to understand how the underlying hyperbolic geometry affects the evolution of solutions u to (KPP), in comparison with the Euclidean case.

Under assumption (H_f) on the nonlinearity f, problem (KPP) has two stationary solutions, one stable, $u \equiv 1$, and the other unstable, $u \equiv 0$. One might think that all nontrivial solutions will converge to the stable stationary solution $u \equiv 1$. But there is a competition between the repulsion of the level 0 (measured in terms of $f'(0)$) and diffusion (measured in terms of the infimum of the spectrum of the Laplace-Beltrami operator), so that in principle solutions might also approach 0 as t goes to infinity. The first case is known as *propagation* (or spreading) and the second one as *extinction* (or vanishing). When the

problem is posed in the Euclidean space, diffusion is not strong enough and, if u_0 is non-trivial, there is always propagation. However, in the hyperbolic space diffusion is stronger than in the Euclidean case, and, while for large values of $f'(0)$ (strong repulsion) there is propagation, if $f'(0)$ is small, diffusion dominates and solutions become extinct; this had already been noticed in the pioneering work of Matano, Punzo and Tesei [75]. Here, as a first step, we explore further the dichotomy between propagation and extinction, covering in particular the case in which $f'(0)$ is critical, which was left open in [75].

In the case of propagation, we exploit the invariancies of the hyperbolic space in order to describe the long-time behavior of the solution to (KPP). More precisely, if the initial datum has some symmetry (in terms of cohomogeneity one subgroups of isometries of the hyperbolic space), then the solution itself is also invariant, which reduces the problem to a unidimensional PDE. We will give a moving coordinate system, that takes into account the symmetry, in which the solution to (KPP) converges to a specific traveling wave of minimal speed. Let us remark that, differently from the Euclidean case, where isometries consist of rotations and translations only, the hyperbolic space has a more complex geometry with three types of such isometries: elliptic, parabolic, and hyperbolic.

Another significant difference is that propagation in $\mathbb{H}^d$ is slower than in $\mathbb{R}^d$, due to the stronger diffusion mentioned above. This effect is described as the *drift from infinity* in the work [75], as it directly affects the velocity of the moving frame. We also need to introduce a logarithmic correction which, contrary to the Euclidean case, stays the same across all three types of invariance.

To explore this in greater detail, we first provide an in-depth introduction to the well-established theory of the KPP problem in the Euclidean space. Later, we present the known theory of that equation posed in the hyperbolic space. Finally, we introduce and discuss our main results, together with the main ideas in the proof.

The KPP problem on $\mathbb{R}^d$. We summarize here some landmark results from the vast literature on the KPP problem in the Euclidean setting. Without being exhaustive, we refer to [8, 7, 24, 25, 41, 56, 64, 84, 100, 101] and the references therein.

In 1937, Fisher, on the one side, and Kolmogorov, Petrovsky and Piskunov on the other, published simultaneously, but independently, two papers, [41, 64] respectively, in which they describe the evolution of the spatial distribution of a dominant gene within a population. Fisher modelled it by the reaction-diffusion spatial one-dimensional equation

$$u_t = u_{xx} + u(1-u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

where the term $u(1-u)$ models natural selection as a saturation process, while the diffusion term u_{xx} accounts for the dispersal of the descendants. On the other hand, Kolmogorov, Petrovsky, and Piskunov proposed a more general formulation:

$$u_t = u_{xx} + f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \tag{3.1.1}$$

where f is a reaction term satisfying condition (H_f). As a consequence, the Cauchy problem

$$u_t = \Delta u + f(u) \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \qquad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{3.1.2}$$

where $u_0 : \mathbb{R}^d \to [0,1]$ is a given initial condition, is known as the Fisher-KPP problem. For short, we refer to it as the KPP problem in the Euclidean space. This is the Euclidean counterpart of (KPP).

Being f a KPP function (see (H_f)), there exists a unique solution u to (3.1.2) and $0 \leq u \leq 1$ in $\mathbb{R}^d \times \mathbb{R}_+$. Moreover, since for the ODE $q' = f(q)$, $q = 0$ is an unstable equilibrium and $q = 1$ is a stable one, Aronson and Weinberger proved in [7] that the state $u \equiv 1$ invades the state $u \equiv 0$ whenever u_0 is not the null function. That is, for all K compact subsets of $\mathbb{R}^d$,

$$\lim_{t\to\infty} \inf_{x\in K} u(x,t) = 1. \tag{3.1.3}$$

Furthermore, as proven in [110] (see also [8, 7, 64]) the equation (3.1.1) admits a family of traveling wave solutions $\{u_c\}_{c\geq c_0}$ (see Figure 3.1), that is, of the form $u_c(x,t) := U_c(x-ct)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ where

$$c_0 := 2\lambda_0, \quad \lambda_0 := \sqrt{f'(0)} > 0, \tag{3.1.4}$$

and the traveling wave profile $U_c : \mathbb{R} \to [0,1]$ is the unique solution (up to translations) to the ODE

$$U'' + cU' + f(U) = 0 \quad \text{in } \mathbb{R}, \tag{3.1.5}$$

that satisfies

$$U(-\infty) = \lim_{s\to-\infty} U(s) = 1, \quad U' < 0, \quad U(+\infty) = \lim_{s\to+\infty} U(s) = 0. \tag{3.1.6}$$

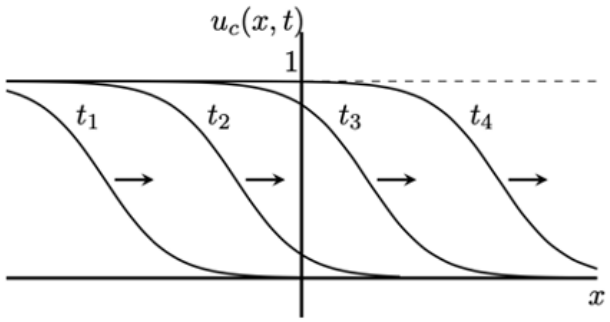


Figure 3.1: Representation of $u_c(x,t) = U_c(x-ct)$ for some $c \geq c_0$ at times $t_j > t_i > 0$, $j > i$ with $j, i \in \{1,2,3,4\}$.

Among all the translates that solve (3.1.5)–(3.1.6), we define the *normalized* traveling wave profile of minimal speed Φ_{c_0} as the unique one that satisfies $\Phi_{c_0}(0) = 1/2$. This profile Φ_{c_0} behaves as

$$\Phi_{c_0}(s) = (s+k)e^{-\lambda_0 s} + O\big(e^{-(\lambda_0+\delta)s}\big), \quad s \to \infty \tag{3.1.7}$$

with two constants $\delta > 0$ and $k \in \mathbb{R}$.

We now explain the illustrative case of the unidimensional KPP problem. In [64], Kolmogorov, Petrovski and Piskunov proved that the solution u to (3.1.1) posed on $\mathbb{R}$ with initial datum a step function $u_0 = \mathcal{X}_{(-\infty,0]}$ converges to the traveling wave profile Φ_{c_0} in the following sense: let $h : \mathbb{R}_+ \to \mathbb{R}$, the centering function, given by $u(h(t), t) = 1/2$. Then,

$$u(x + h(t)) \to \Phi_{c_0}(x) \text{ uniformly in } x \in \mathbb{R}, \quad h'(t) \to c_0 \quad \text{as } t \to \infty.$$

This motivates the study of the convergence of the solution u to Φ_{c_0} in a coordinate system that translates in time and moves with linear velocity c_0; that is, moving coordinates $x' = x + c_0 t + o(t)$. Several years later, in 1975, Aronson and Weinberger proved in [8] that, in this setting, with nontrivial and right compactly supported initial datum u_0 (that is, $u_0 \not\equiv 0$ and $u_0(x) = 0$ for all $x \geq A$ for some $A > 0$), we have

$$\lim_{t\to\infty} \sup_{\{x>ct\}} u(x,t) = 0 \quad \text{for all } c > c_0, \qquad \lim_{t\to\infty} \inf_{\{0\leq x<ct\}} u(x,t) = 1 \quad \text{for all } 0 < c < c_0. \tag{3.1.8}$$

Such c_0 is called the *spreading speed.*

Moreover, this result may be refined, since the speed of the moving is not exactly c_0 but there is a asymptotic logarithmic delay in the convergence of u to the traveling wave profile Φ_{c_0}. In this regard, Bramson established in [25, 24], using probabilistic methods, that for an initial datum $u_0 = \mathcal{X}_{(-\infty,0]}$, there is a constant β_0 that depends on u_0 such that

$$u\left(x + c_0 t - \tfrac{3}{2}\ln t + \beta_0 + o(1)\right) \to \Phi_{c_0}(x) \quad \text{uniformly in } x > 0 \text{ as } t \to \infty. \tag{3.1.9}$$

Using purely PDE arguments, in [56] the authors found a logarithmic correction in the convergence whenever the initial datum is compactly supported from the right. As a continuation of this work, in [84], Nolen, Roquejoffre, and Ryzhik extended these results by reproving Bramson's result (3.1.9). They achieved this through a comparison principle and compactness arguments, for continuous initial data trapped between two step functions.

Consider now the multidimensional equation

$$u_t = \Delta u + f(u), \quad \mathbb{R}^d \times \mathbb{R}_+ \tag{3.1.10}$$

with $d > 1$. Aronson and Weinberger proved in their famous paper [7] that the solution u has also spreading spead c_0 whenever the intial datum is compactly supported, that is,

$$\lim_{t\to\infty} \sup_{\{|x|>ct\}} u(x,t) = 0 \quad \text{for all } c > c_0, \qquad \lim_{t\to\infty} \inf_{\{|x|<ct\}} u(x,t) = 1 \quad \text{for all } 0 < c < c_0.$$

Moreover, the moving frame in which the solution of the multimensional equation (3.1.10) converges to the profile Φ_{c_0} also needs a logarithmic correction.

Several years later, for more general initial data, Roquejoffre, Roussier-Michon and Rossi showed in a couple of papers that the choice of the coordinate system and the constant of the logarithmic correction depend strongly on the geometry of the support of the initial datum (see Figure 3.2). More precisely:

(i) Ball case. In [100] the authors proved that if the initial datum u_0 is compactly supported and lies between

$$\mathcal{X}_{B_{R_1}(0)} \leq u_0 \leq \mathcal{X}_{B_{R_2}(0)}, \quad \text{for some } 0 < R_1 < R_2,$$

then there is a bounded Lipschitz function $s_1^\infty : \mathbb{S}^{d-1} \to \mathbb{R}$ such that

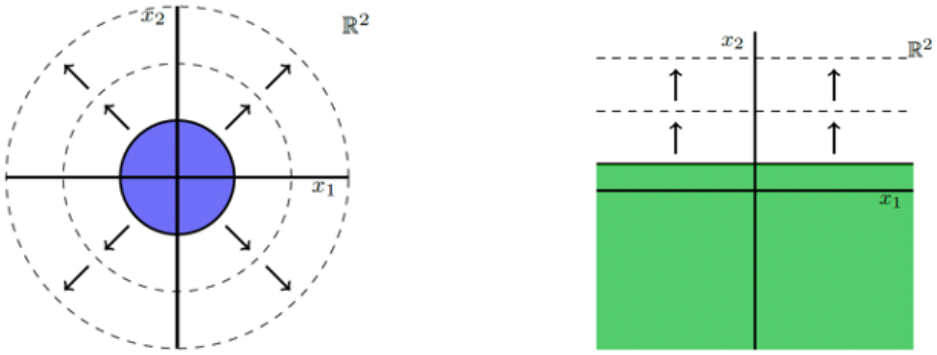
$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d\setminus\{0\}} u(x,t) - \Phi_{c_0}\left(|x| - c_0 t + \tfrac{d+2}{c_0}\ln t + s_1^\infty\left(\frac{x}{|x|}\right)\right) = 0. \tag{3.1.11}$$

(ii) Half-plane case. By [101], if the initial datum u_0 is trapped between two translates (in the same direction) of the step function $\mathcal{X}_{(-\infty,0)}$, say, for simplicity,

$$\mathcal{X}_{(-\infty,1)}(x_d) \leq u_0(x) \leq \mathcal{X}_{(-\infty,2)}(x_d), \quad \text{for } x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

then there exists a bounded Lipschitz function $s_2^\infty : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{R}^d :\, x_d\geq 0\}} u(x,t) - \Phi_{c_0}\left(x_d - c_0 t + \tfrac{3}{c_0}\ln t + s_2^\infty(x_1, ..., x_{d-1})\right) = 0. \tag{3.1.12}$$



(a) u_0 is radial invariant and compactly supported.

(b) u_0 is translation invariant and suported in a haf-space.

Figure 3.2: Propagation of the support of u.

The KPP problem in $\mathbb{H}^d$. The preceding exposition of the KPP problem in the Euclidean space naturally leads to the following questions regarding the same problem in the hyperbolic space: Is propagation always guaranteed? If so, does the solution converge to a traveling wave? In which coordinate system does this occur? Does this moving frame have a spreading speed, and does it require a logarithmic correction? In this article, we aim to address these questions, focusing on how the geometry of the hyperbolic space affects these issues.

The first difference with the Euclidean setting is encoded λ_1, the infimum of the L^2-spectrum of the operator $-\Delta_{\mathbb{H}^d}$ (spectral gap), which is given by

$$\lambda_1 := \frac{(d-1)^2}{4}. \tag{3.1.13}$$

It is well known (see, for instance, Section 3.3 and the paper [113] for the necessary background) that this spectral gap due to geometry creates a stronger diffusion (sometimes known as ballistic behavior). Non-linear variations can be found in [13, 115], for the porous medium equation.

This effect is also present in the KPP problem on hyperbolic space. Here, Matano, Punzo y Tesei are the pioneers in the study of (KPP) with their paper [75]. They proved that there is a competition between the source f and λ_1. If $f'(0) > \lambda_1$, propagation takes place, and if $f'(0) < \lambda_1$ and the initial datum is compact there is extinction, that is, the state $u \equiv 0$ invades the state $u \equiv 1$. Specifically, they establish the following result:

Theorem 3.1.1 (Theorem 3.2 in [75]). *Let u be the solution to (KPP) with $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$.*

(i) *If $f'(0) < \lambda_1$ and u_0 has compact support, then $\lim_{t\to\infty} \sup_{x\in\mathbb{H}^d} u(x,t) = 0$.*

(ii) *If $f'(0) > \lambda_1$, then, for any compact subset K of $\mathbb{H}^d$, $\lim_{t\to\infty} \inf_{x\in K} u(x,t) = 1$.*

The above result shows the first significant difference with the KPP problem in the Euclidean space, where propagation always occurs; recall (3.1.3).

Observe also that the dichotomy between $f'(0)$ and λ_1 can be rephrased in terms of c_0 by setting

$$c_* := c_0 - (d-1). \tag{3.1.14}$$

By (3.1.4) and (3.1.13), we have propagation if and only if $c_* > 0$. This can be explained by saying that the extra diffusion due to the geometry is slowing down the propagation. Indeed, if the initial datum $u_0 \not\equiv 0$ is compactly supported and propagation occurs, the solution of (KPP) has spreading speed c_*, as the following theorem shows:

Theorem 3.1.2 (Theorem 3.6 in [75]). *Let $f'(0) > \lambda_1$ and $u_0 \not\equiv 0$ compactly supported. Then, the solution u to (KPP) satisfies*

(i) $\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d : \rho_{\mathfrak{e}}(x) > ct\}} u(x,t) = 0$ *for all $c > c_*$;*

(ii) $\lim_{t\to\infty} \inf_{\{x\in\mathbb{H}^d : 0 \leq \rho_{\mathfrak{e}}(x) < ct\}} u(x,t) = 1$ *for all $0 < c < c_*$*

where $\rho_{\mathfrak{e}} : \mathbb{H}^d \to [0,\infty)$ is the hyperbolic geodesic distance to the origin.

The extra term $d-1$ in the expression of c_*, described as a drift from infinity in [75], can be explained also from the structure of the Laplacian in $\mathbb{H}^d$ (see, for instance, formula (3.1.18) in radial coordinates). The coefficient of the drift term in radial coordinates, differently from Euclidean space, does not vanish at infinity, but is asymptotically constant. As a consequence, this drift from infinity must be overcome in order to have propagation.

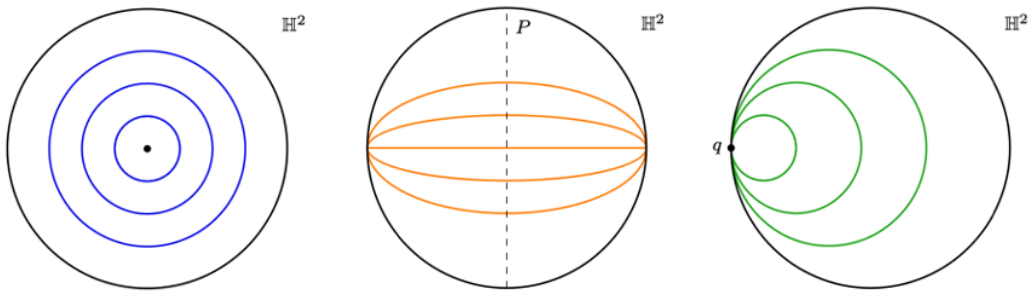
The first goal of this paper is to address the question of propagation *versus* extinction, extending Theorem 3.1.1 to include more general initial data and the critical case $f'(0) =$

λ_1. This is done in Section 3.4. For further discussion on propagation versus extinction in the context of the KPP problem on other complete, non-compact Riemannian manifolds, we refer to [95].

Our second goal is to address, in the case of propagation, the convergence of the solution of (KPP) to a traveling wave in a moving coordinate system, written in terms of the isometries of hyperbolic space.

As the Euclidean results (3.1.11) and (3.1.12) highlight, there is a relation between the initial datum, the coordinate system in which the solution converges to the traveling wave and the chosen isometry. Indeed, for (3.1.12), in [101] the authors require the initial datum to be trapped between two functions invariant under translations in one direction, that is, invariant under the same Euclidean parabolic isometry (translations). On the other hand, to obtain (3.1.11), one of the hypotheses in [100] is that the initial datum must be trapped between two radial functions, i.e., between two functions that are invariant under the same elliptic Euclidean isometry (rotations). The underlying idea is that these are two settings where the problem can be essentially characterized in terms of solutions to an unidimensional PDE.

In the hyperbolic space there are also subgroups of isometries that generate orbits of co-dimension one, that is. In this setting, solutions of a PDE with symmetric initial datum are also invariant and depend on one variable only. These are the cohomogeneity one subgroups of isometries of $\mathbb{H}^d$. Here we find a sriking difference with Euclidean space since there are three types such isometries: elliptic ($\mathfrak{e}$), parabolic ($\mathfrak{p}$) and hyperbolic ($\mathfrak{h}$). In Section 2, we provide some geometric background on $\mathbb{H}^d$ and its isometries in detail, for convenience of the non expert reader. For a representation of the invariant sets under each isometry group of the Poincaré disk $\mathbb{D}^2$, a model of $\mathbb{H}^2$, see Figure 3.3.



(a) Elliptic orbits (circles centered at origin)

(b) Hyperbolic orbits respect hyperplane P.

(c) Parabolic orbits (horospheres centered at q)

Figure 3.3: Orbits in the Poincaré disk $\mathbb{D}^2$ in each isometry group.

This rich structure has been explored in several PDEs, for instance [14, 76] for Allen-Cahn, [35] for Swift-Hohenberg, and the aforementioned [75] for (KPP).

Analytically, each of the three isometry subgroups generates a coordinate system of the form

$$x = (\rho_i, \theta_i) \in \mathbb{H}^d, \quad i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}, \tag{3.1.15}$$

where ρ_i is unidimensional. This yields a splitting in the expression of the Laplace-Beltrami operator on $\mathbb{H}^d$ (see (3.2.2)). We will consider initial data that are symmetric, that is, that only depend on the variable ρ_i.

Now we are able present our main result in a simplified version. For the precise statement, we refer to Theorem 3.5.1.

Main result. *Let $f'(0) > \lambda_1$. Consider $u_0 : \mathbb{H}^d \to [0,1]$ that can be expressed as a unidimensional ρ_i-function for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$ according to the i-coordinate system (3.1.15). Moreover, for such i, u_0 is trapped between two unidimensional ρ_i-step functions. Then, the solution u to (KPP) with initial datum u_0 converges to a traveling wave of minimal speed in the variable ρ_i. More precisely,*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\ \rho_i(x)\geq 0\}} u\big(x,t\big) - \Phi_{c_0}\Big(\rho_i(x) - \big[c_* t - \tfrac{3}{c_0}\log t + \beta\big]\Big) = 0 \tag{3.1.16}$$

for some $\beta \in \mathbb{R}$.

Notice that the limit profile Φ_{c_0} in moving coordinates is the same as in the Euclidean case, since it only depends on the solution of an ODE. Nevertheless, we find here two important differences with respect to the Euclidean case: while the linear term of the moving frame in $\mathbb{H}^d$ varies with the dimension, the constant in the logarithmic correction remains fixed and is independent of the chosen isometries. This contrasts with (3.1.11) and (3.1.12), where the linear term is independent of the dimension while the logarithmic correction depends on the type of invariance.

We further conjecture that the hypothesis of the initial datum being invariant (that is, only depending on the ρ_i variable for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$) can be replaced by u_0 trapped between two unidimensional ρ_i-step functions, as the Euclidean results (3.1.11) and (3.1.12) suggest. More precisely, for $f'(0) > \lambda_1$, we aim to study the long-time behavior of the solution u to the (KPP) problem with an initial datum $u_0 : \mathbb{H}^d \to [0,1]$ satisfying

$$\mathcal{X}_{[0,\rho_1]}(\rho_i(x)) \leq u_0(x) \leq \mathcal{X}_{[0,\rho_2]}(\rho_i(x)) \quad \text{for all } x \in \mathbb{H}^d :\ \rho_i(x) \geq 0. \tag{3.1.17}$$

for some $0 < \rho_1 < \rho_2$ and some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. In this situation we expect that for some $\tilde{\beta} : \mathcal{M}_i \to \mathbb{R}$ bounded, Lipschitz function

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\ \rho_i(x)\geq 0\}} u(x,t) - \Phi_{c_0}\left(\rho_i(x) - \left[c_* t - \tfrac{3}{c_0}\log t + \tilde{\beta}(\theta_i(x))\right]\right) = 0.$$

Our Lemma 3.5.10 is a first step in ths direction.

A second goal for the future would be to give the intermediate asymptotics for solutions when they vanish; that is, to determine their rate of decay to 0, and their possible convergence towards a profile once the solution is rescaled to annihilate the decay. One expects vanishing solutions to behave like a multiple of the heat kernel. One main difficulty would be to determine the right multiple, which should be given by the asymptotic mass of the solution. We hope to return to this problem elsewhere.

Discussion: why convergence to a traveling wave in a moving coordinate system? Here we present a heuristic outline of the argument in the proof of our main result (3.1.16). Although the statement addresses more general initial data, we focus here on a simplified scenario ($i = \mathfrak{e}$, that is, elliptic isometries) to highlight the core ideas. For this, suppose that the initial datum $0 \leq u_0 \leq 1$ is compactly supported and radially symmetric. That is, there exists $\tilde{u}_0 : [0, \infty) \to [0, 1]$ such that $u_0(x) = \tilde{u}_0(\rho(x))$ for all $x \in \mathbb{H}^d$ where $\rho(x)$ is the geodesic distance between x and a fixed pole $o \in \mathbb{H}^d$.

We choose the polar representation of the unit disc (see Section 3.2 for the geometric background). Since the Laplace-Beltrami operator in radial coordinates is given by

$$\Delta_{\mathbb{H}^d} u = u_{\rho\rho} + (d-1)\coth\rho\, u_\rho + \frac{1}{\sinh^2\rho}\Delta_{\mathbb{S}^{d-1}} u, \quad \text{where } \rho = \rho(x) \geq 0, \tag{3.1.18}$$

the comparison principle holds (see Theorem 3.3.1) and the initial datum is invariant under rotations, the solution u to (KPP) is also radial and one reduces to working with one spatial dimension only. That is, $u(x,t) = \tilde{u}(\rho(x), t)$ for all (x,t) belonging to $\mathbb{H}^d \times \mathbb{R}_+$ where the function $\tilde{u} : [0,\infty) \times \mathbb{R}_+ \to [0,1]$ satisfies

$$\begin{cases} \tilde{u}_t = \tilde{u}_{\rho\rho} + (d-1)\coth\rho\,\tilde{u}_\rho + f(\tilde{u}), & \rho \geq 0, \quad t > 0, \\ \tilde{u}_\rho(0,t) = 0, & t > 0, \\ \tilde{u}(\rho,0) = \tilde{u}_0(\rho), & \rho \geq 0. \end{cases}$$

Moreover, since

$$\lim_{\rho\to\infty} \coth\rho = 1,$$

we expect $\hat{u}(\rho,t) := \tilde{u}(\rho - (d-1)t, t)$ to be governed for ρ large by the one-dimensional Euclidean KPP equation

$$v_t = v_{\rho\rho} + f(v) \quad \text{in } \mathbb{R}\times\mathbb{R}_+. \tag{3.1.19}$$

Let $\tilde{v}$ be the solution of (3.1.19) with a compactly supported initial datum. By (3.1.8), $\tilde{v}$ spreads with speed c_0. Therefore, we predict that $\hat{u}$ spreads also with velocity c_0 which implies that the asymptotic spreading speed for $\tilde{u}$ should be c_* as given in see (3.1.14). In fact, in [75], the authors show that u spreads in radial coordinates with velocity c_*, i.e.,

$$(i)\ \lim_{t\to\infty}\sup_{\{\rho>ct\}} \tilde{u}(\rho,t) = 0 \quad \text{for all } c > c_*, \qquad (ii)\ \lim_{t\to\infty}\inf_{\{0\leq\rho<ct\}} \tilde{u}(\rho,t) = 1 \quad \text{for all } 0 < c < c_*. \tag{3.1.20}$$

Note that the correction $(d-1)$ that appears in c_*, that comes from the transport term, is slowing the propagation, as we have mentioned above.

Furthermore, we will show in Proposition 3.4.6 that, even at the position $\rho = c_* t$, $\tilde{u}$ vanishes as $t \to \infty$. This suggests that, as in the KPP problem in the Euclidean space, we may need a logarithmic correction to capture the precise behavior of u.

Moreover, by (3.1.20) and Proposition 3.4.6, we anticipate that u will be near zero in $\{\rho \geq c_* t\}$ as t goes to infinity. In this region, $f(\tilde{u})$ behaves asymptotically as $f'(0)\tilde{u}$. For

that reason, we expect that $\tilde{u}$ will be close to solution w of the linearized problem

$$\begin{cases} w_t = w_{\rho\rho} + (d-1)\coth\rho\, w_\rho + f'(0)w, & \rho \geq c_* t, \ \ t > 0, \\ w(c_* t, t) = 0, & t > 0. \end{cases}$$

Consider $\hat{w}(\rho, t) := w(\rho + c_* t, t)$, then, by (3.1.14), $\hat{w}$ satisfies (in moving coordinates)

$$\begin{cases} \hat{w}_t = \hat{w}_{\rho\rho} + [c_0 - (d-1)\big(\coth(\rho + c_* t) - 1\big)]\hat{w}_\rho + f'(0)\hat{w}, & \rho \geq 0, \ \ t > 0, \\ \hat{w}(0, t) = 0, & t > 0. \end{cases} \tag{3.1.21}$$

The key idea here is that, since

$$\lim_{t\to\infty} \sup_{\rho\geq 0} |\coth(\rho + c_* t) - 1| = 0,$$

the drift term in the first equation in (3.1.21) should be negligible. Hence, we expect $\check{u}(\rho, t) := \tilde{u}(\rho + c_* t, t)$ to be comparable for t large enough to the solution $\check{w}$ of

$$\check{w}_t = \check{w}_{\rho\rho} + c_0 \check{w}_\rho + f'(0)\check{w}, \quad \rho \geq 0, \ t > 0, \qquad \check{w}(0, t) = 0, \quad t > 0.$$

At this point, we realize that the theory developed for the KPP problem on the Euclidean space in the papers [56, 84, 100] can also be used on the hyperbolic space with the necessary modifications. A fundamental idea in these articles is that the behavior of $\check{w}$ in the region $\{-t^\delta \leq \rho \leq t^\delta\}$, with small positive δ, determines the behavior of $\check{w}$ in all $\mathbb{R}$. Or, in terms of u, we expect to obtain (3.1.16) to capture the behavior $\tilde{u}$ in the region $\{\rho = c_* t + o(\sqrt{t})\}$. For simplicity, we only argue here for the points on the right of $c_* t$.

Following this idea, consider

$$p(\rho, t) := e^{-\lambda_0 \rho}\check{w}, \quad \rho \geq 0, \ t > 0.$$

Recalling (3.1.4), p solves

$$p_t = p_{\rho\rho}, \quad \rho \geq 0, \ t > 0, \qquad p(0, t) = 0, \quad t > 0.$$

Thus the long-time behavior of p is given by a multiple of the dipole solutions to the heat equation, that is,

$$p(\rho, t) \asymp \alpha_\infty \frac{\rho}{t^{3/2}} e^{-\frac{\rho^2}{4t}}, \quad \rho \geq 0,$$

where α_∞ is a positive constant that depends of the first moment of the initial datum $\tilde{u}_0$. This suggests that

$$\tilde{u}(\rho + c_* t, t) \asymp \alpha_\infty \frac{\rho}{t^{3/2}} e^{-\frac{\rho^2}{4t}} e^{-\lambda_0 \rho}, \quad \rho \geq 0$$

and t large enough. Combining this with the fact that $c_0 = 2\lambda_0$, we obtain

$$\tilde{u}(\rho + c_* t, t) \asymp \alpha_\infty \rho e^{-\lambda_0(\rho + \frac{3}{c_0}\ln t)}, \quad 0 \leq \rho \leq o(\sqrt{t}) \tag{3.1.22}$$

for t large enough. This result establishes a connection with the behavior of the tail of the traveling wave Φ_{c_0} (see (3.1.7)), justifies the need of a logarithmic correction and establishes the value of β in (3.1.16).

On the other hand, by (3.1.7), there is $B > 0$ such that

$$\begin{aligned}\Phi_{c_0}\big(\rho + \tfrac{3}{c_0}\ln t\big) &\asymp B\big(\rho + \tfrac{3}{c_0}\ln t\big)e^{-\lambda_0(\tilde{\rho}+\frac{3}{c_0}\ln t)} \\ &\asymp B\rho e^{-\lambda_0(\rho+\frac{3}{c_0}\ln t)} \asymp \alpha_\infty \rho e^{-\lambda_0(\rho+\frac{3}{c_0}\ln t-\beta)}, \quad \text{as } \rho\to\infty,\end{aligned} \tag{3.1.23}$$

for some constant $\beta \in \mathbb{R}$ depending on α_∞. Then, comparing the behaviors (3.1.22) and (3.1.23), we obtain that

$$\tilde{u}(\rho,t) \asymp \Phi_{c_0}\big(\rho - (c_* t - \tfrac{3}{c_0}\ln t + \beta)\big) \quad \text{whenever } \rho \asymp c_* t + o(\sqrt{t}).$$

Finally, once that we have captured the behavior at $\rho - c_* t \asymp \sqrt{t}$, we will be able to prove that $\tilde{u}(\rho,t)$ converges to $\Phi_{c_0}\big(\rho - (c_* t - \frac{3}{c_0}\ln t + \beta)\big)$ as t goes to infinity, uniformly for $\rho \geq 0$. This shows (3.1.16) and completes the proof of our main result.

Organization of the chapter. In Subsection 2, we present the necessary background on $\mathbb{H}^d$, its isometries and the three coordinate systems that we will chose in our analysis. Subsection 3 gathers known preliminary results concerning the heat kernel and the existence and uniqueness of solutions to (KPP). In Subsection 4, we extend the discussion of propagation versus extinction. Finally, in Subsection 5, we prove our main theorem, establishing the convergence of the solution in a specific moving coordinate system to the traveling wave of minimal speed.

3.2 Geometric background

For the convenience of the reader, this section collects some background on the hyperbolic space which we believe might be helpful for the PDE community. A classical reference is [97], while a quick summary is found in [14].

The hyperbolic space $\big(\mathbb{H}^d, g\big)$ is the unique simply connected, complete d-dimensional Riemannian manifold with sectional curvature identically equal to -1.

Let $d_{\mathbb{H}^d} : \mathbb{H}^d \times \mathbb{H}^d \to [0,\infty)$ be the hyperbolic geodesic distance. For any two different points in $\mathbb{H}^d$ there is exactly one geodesic passing through them. Moreover, every geodesic is defined on the whole real line. A subset $A \subset \mathbb{H}^d$ is a totally geodesic hyperbolic subspace if, given any two different points in A, the entire geodesic passing through them is contained in A. A hyperplane P is a totally geodesic subset of $\mathbb{H}^d$ of co-dimension one; in particular, P is isomorphic to $\mathbb{H}^{d-1}$.

Let us recall the definition of the reflection across a hyperplane $P \subset \mathbb{H}^d$. Consider $x \in \mathbb{H}^d \setminus P$, then there is a unique $y \in P$ such that $d_{\mathbb{H}^d}(x,y) = \min_{z\in P} d_{\mathbb{H}^d}(x,z)$. Let $\gamma : \mathbb{R} \to \mathbb{H}^d$ be the unique geodesic passing through x and y parametrized such that

$\gamma(0) = y$ and $\gamma(1) = x$. Then, the reflection with respect to P is the function $\mathcal{R}_P$ such that

$$\mathcal{R}_P : \mathbb{H}^d \to \mathbb{H}^d, \quad \mathcal{R}_P(x) := \gamma(-1).$$

The geodesic compactification $\overline{\mathbb{H}^d} = \mathbb{H}^d \cup \partial\mathbb{H}^d$ is obtained by fixing a point $x \in \mathbb{H}^d$ and adding a point at both ends of each geodesic passing through x. The points of $\partial\mathbb{H}^d$ can be regarded as the points at infinity of $\mathbb{H}^d$. A horosphere centered at $q \in \partial\mathbb{H}^d$ is a closed hypersurface of $\mathbb{H}^d$ orthogonal to all the geodesics with endpoint q. Each horosphere is isomorphic to $\mathbb{R}^{d-1}$.

Moreover, the hyperbolic space $(\mathbb{H}^d, g)$ as a Riemannian manifold, has an associated Laplace-Beltrami operator. In terms of the metric it is given by

$$\Delta_{\mathbb{H}^d} = \frac{1}{\sqrt{\det g}} \sum_{j=1}^{d} \sum_{l=1}^{d} \partial_j \big(\sqrt{\det g} \, (g^{-1})_{jl} \, \partial_l \big).$$

The infimum of the L^2-spectrum of the operator $-\Delta_{\mathbb{H}^d}$ in $\mathbb{H}^d$ is λ_1, given by (3.1.13).

There are several isomorphic models to describe the $\mathbb{H}^d$. In this paper, we consider two of them: the Poincaré disk and Poincaré half-space model.

The Poincaré disk model $\mathbb{D}^d$ is the unit ball $B_1(0) := \{x \in \mathbb{R}^d : |x| < 1\}$, where $|\cdot|$ is the Euclidean metric, with the Riemannian metric g given by

$$g_{jl} = \frac{4}{(1 - |x|^2)^2} \delta_{jl}, \quad x \in B_1(0), \quad j, l \in \{1, ..., d\}$$

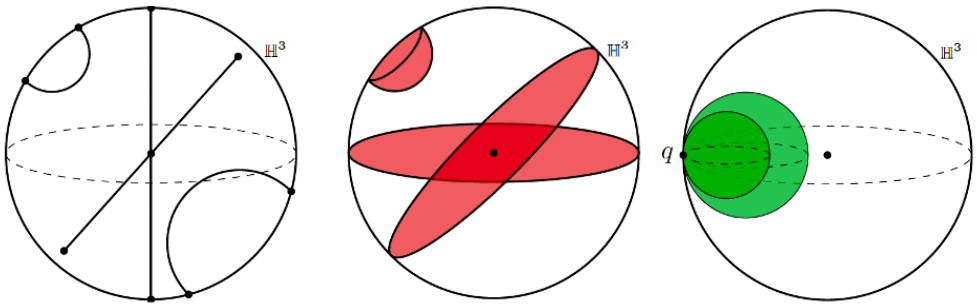
where $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$.

In this model:

- geodesics are either semicircles orthogonal to $\partial B_1(0)$, or diameters of $B_1(0)$.
- a subset $P \subset \mathbb{H}^d$ is a hyperplane if and only if P is the intersection of $B_1(0)$ with a $(d-1)$-dimensional Euclidean sphere orthogonal to $\partial B_1(0)$ or it is the intersection of $B_1(0)$ with a Euclidean hyperplane $\pi \subset \mathbb{R}^d$ such that $0 \in \pi$.
- a horosphere with center $q \in \partial B_1(0)$ is a $(d-1)$-dimensional Euclidean sphere tangent to $\partial B_1(0)$ in q and contained in $B_1(0)$.
- The Laplace-Beltrami operator is

$$\Delta_{\mathbb{H}^d} u = \frac{1}{4}(1 - |x|^2)^2 \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} + \frac{d-2}{2}(1 - |x|^2) \sum_{j=1}^{d} x_j \frac{\partial u}{\partial x_j}.$$

For a visual representation of some geodesics, hyperplanes and horospheres in $\mathbb{D}^3$, see Figure 3.4.



(a) Geodesics. (b) Hyperplanes. (c) Horoespheres centered at q.

Figure 3.4: Poincaré disk model $\mathbb{D}^3$.

The Poincaré half-space model $\mathbb{U}^d$ is the upper half-space $\{x \in \mathbb{R}^d : x_d > 0\}$ endowed with the Riemannian metric

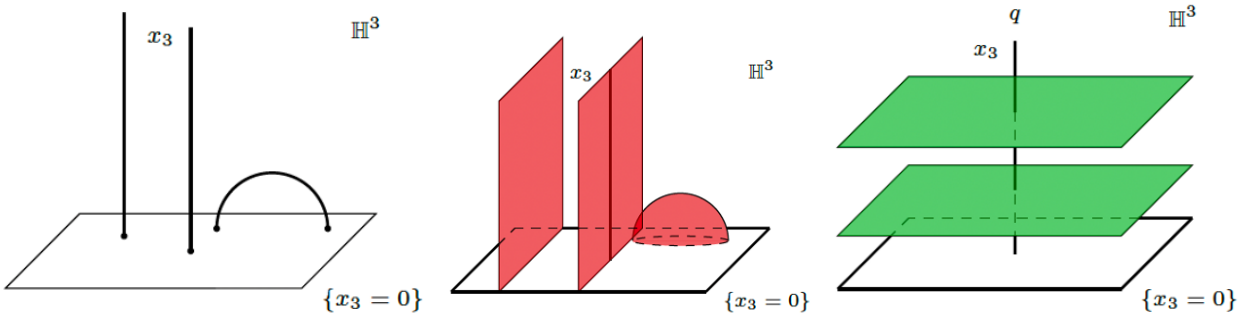
$$g_{jl} = \frac{1}{x_d^2}\delta_{jl}, \quad x \in \mathbb{R}^d : x_d > 0, \quad j, l \in \{1, ..., d\}.$$

Here:

- geodesics are either upper half-lines or upper semicircles orthogonal to $\{x_d = 0\}$,
- a subset $P \subset \mathbb{H}^d$ is a hyperplane if and only if P is the intersection of $\{x \in \mathbb{R}^d : x_d > 0\}$ with a Euclidean hyperplane orthogonal to $\{x_d = 0\}$ or it is the intersection of $\{x \in \mathbb{R}^d : x_d > 0\}$ with a d-dimensional Euclidean sphere with center in $\{x_d = 0\}$,
- for a given $q \in \partial\mathbb{H}^d$, we can identify q with the point $-\infty$. Then, in $\mathbb{U}^d$ any horosphere with center q is given by $\{x_d = \delta\}$ with $\delta > 0$.
- The Laplace-Beltrami operator is

$$\Delta_{\mathbb{H}^d} u = x_d^2 \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} + (2 - d) x_d \frac{\partial u}{\partial x_d}.$$

For a visual representation of some geodesics, hyperplanes and horospheres in $\mathbb{U}^3$, see Figure 3.5.



(a) Geodesics. (b) Hyperplanes. (c) Horoespheres centered at q.

Figure 3.5: Poincaré half-space model $\mathbb{U}^3$.

Isometries of the hyperbolic space. There are three subgroups of $\mathrm{Isom}(\mathbb{H}^d)$ which have orbits in $\mathbb{H}^d$ of codimension one: $H_{\mathfrak{e}}$, $H_{\mathfrak{h}}$ and $H_{\mathfrak{p}}$. The subscripts "$\mathfrak{e}$", "$\mathfrak{h}$"and "$\mathfrak{p}$"stand for elliptic, parabolic and hyperbolic, respectively.

$H_{\mathfrak{e}}$, elliptic isometries. The action of $H_{\mathfrak{e}}$ is by rotations around a fixed point $p \in \mathbb{H}^d$ that we identify with the origin or by reflections across a totally geodesic hyperplane P such that $p \in P$. Then, the invariant sets are hyperbolic spheres centered at the origin.

$H_{\mathfrak{h}}$, hyperbolic isometries. The inclusion of $H_{\mathfrak{h}}$ in $\mathrm{Isom}(\mathbb{H}^d)$ relies on the selection of a totally geodesic hyperplane $P \subset \mathbb{H}^d$. Since P is isomorphic to $\mathbb{H}^{d-1}$, any isometry h in P can be uniquely extended to an isometry of the full space $\mathbb{H}^d$, up to a reflection across P. The subgroup $H_{\mathfrak{h}}$ is composed of all isometries constructed in this manner. The *half-rugby ball* sets (defined precisely in (3.2.1)) remain invariant under the action of hyperbolic isometries.

$H_{\mathfrak{p}}$, parabolic isometries. The parabolic isometries are Euclidean motions on each horosphere centered at a point $q \in \partial\mathbb{H}^d$. Here, the invariant sets are the horospheres. Without loss of generality we can identify q with $-\infty$.

We notice here that each particular isometry (elliptic, hyperbolic, parabolic) involves a choice (the origin point, the hyperplane P, the point q, respectively), which will be implicitly assumed in our discussion.

Associated with those subgroups of $\mathrm{Isom}(\mathbb{H}^d)$, we introduce three different coordinate systems:

• $\mathfrak{e}$-coordinates (radial coordinates, Figure 3.6): Let $x \in \mathbb{D}^d = \{y = (y_1, ..., y_d) \in \mathbb{R}^d, |y| := \left(\sum_{j=1}^d y_j^2\right)^{1/2} \leq 1\}$,

$$\rho_{\mathfrak{e}}(x) := \log\left(\frac{1+|x|}{1-|x|}\right), \qquad \theta_{\mathfrak{e}}(x) := \frac{x}{|x|}.$$

Then $(\rho_{\mathfrak{e}}, \theta_{\mathfrak{e}}) \in [0, \infty) \times \mathbb{S}^{d-1}$ and the metric has a simple expression given by $g = \mathrm{d}\rho_{\mathfrak{e}}^2 + \sinh^2(\rho_{\mathfrak{e}})\,|\mathrm{d}\theta_{\mathfrak{e}}|^2$.

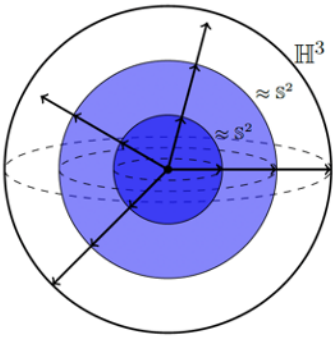


Figure 3.6: Elliptic coordinates in the Poincaré disk model $\mathbb{D}^3$.
Arrowed lines represent the geodesics parameterised by $\rho \geq 0$.
We color somes hyperbolic spheres centered at the origin in blue.

• $\mathfrak{h}$-coordinates (Figure 3.7): Consider $x \in \mathbb{D}^d = \{y = (y_1, ..., y_d) \in \mathbb{R}^d, |y| \leq 1\}$,

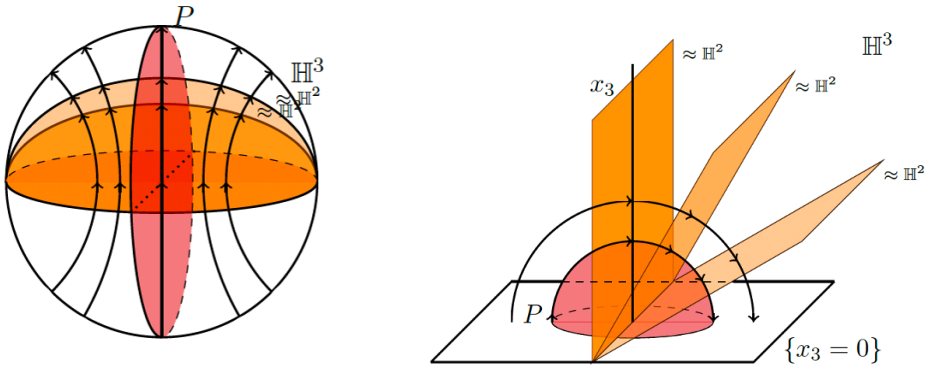
$$x_d = \sinh(\rho_{\mathfrak{h}}), \qquad x_j = \cosh(\rho_{\mathfrak{h}})\theta_{\mathfrak{h}_j} \quad \text{for } j \in \{1, ..., d-1\}.$$

Then $(\rho_{\mathfrak{h}}, \theta_{\mathfrak{h}}) \in (-\infty, \infty) \times \mathbb{H}^{d-1}$ and $g = \mathrm{d}\rho_{\mathfrak{h}}^2 + \cosh^2(\rho_{\mathfrak{h}})\,|\mathrm{d}\theta_{\mathfrak{h}}|^2$.

We define the *half-rugby ball* set of level s as

$$\mathcal{R}_{\mathfrak{h}}(s) := \big\{x \in \mathbb{H}^d : \rho_{\mathfrak{h}}(x) = s \qquad \text{with } s \in (-\infty, \infty). \tag{3.2.1}$$

The name for $\mathcal{R}_{\mathfrak{h}}$ is obviously inspired by its shape in the three dimensional Poincaré disk model $\mathbb{D}^3$.



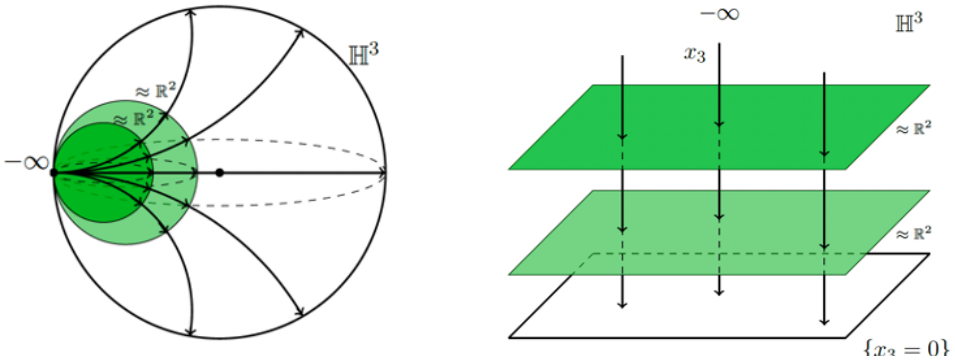
(a) Poincaré disk model $\mathbb{D}^3$. (b) Poincaré half-space model $\mathbb{U}^3$.

Figure 3.7: Hyperbolic coordinates. Arrowed curves represent geodesics parameterised by $\rho \in (-\infty, \infty)$. We color in orange some half-rugby balls. In red, the hyperplane P with respect to which we consider the hyperbolic isometries.

• $\mathfrak{p}$-coordinates (Figure 3.8): For $x \in \mathbb{U}^d = \{x \in \mathbb{R}^d : x_d > 0\}$, we write

$$\rho_{\mathfrak{p}} = -\log x_d, \qquad \theta_{\mathfrak{p}_j} = x_j \quad \text{for } j \in 1, ..., d-1.$$

Then $(\rho_{\mathfrak{p}}, \theta_{\mathfrak{p}}) \in (-\infty, \infty) \times \mathbb{R}^{d-1}$ and $g = \mathrm{d}\rho_{\mathfrak{p}}^2 + e^{2\rho_{\mathfrak{p}}} |\mathrm{d}\theta_{\mathfrak{p}}|^2$.



(a) Poincaré disk model $\mathbb{D}^3$. (b) Poincaré half-space model $\mathbb{U}^3$.

Figure 3.8: Parabolic coordinates. Arrowed lines and curves represent geodesics parameterised by $\rho \in (-\infty, \infty)$. We color in green some horospheres.

Now, in the former coordinates $(\rho, \theta) = (\rho_i, \theta_i)$ with $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$, the operator $\Delta_{\mathbb{H}^d}$ is splits as

$$\Delta_{\mathbb{H}^d} u = u_{\rho\rho} + (d-1) g_1^i u_\rho + g_2^i \Delta_{\mathcal{M}_i} u, \quad (\rho, \theta) \in I_i \times \mathcal{M}_i, \tag{3.2.2}$$

where

Type	i	$g_1^i(\rho)$	$g_2^i(\rho)$	I_i	$\mathcal{M}_i$
Elliptic	$\mathfrak{e}$	$\coth \rho$	$\sinh^{-2} \rho$	$[0, +\infty)$	$\mathbb{S}^{d-1}$
Hyperbolic	$\mathfrak{h}$	$\tanh \rho$	$\cosh^{-2} \rho$	$(-\infty, +\infty)$	$\mathbb{H}^{d-1}$
Parabolic	$\mathfrak{p}$	1	$e^{-2\rho}$	$(-\infty, +\infty)$	$\mathbb{R}^{d-1}$

Observe that

$$\lim_{\rho \to \infty} g_1^i(\rho) = 1 \quad \text{for all } i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}.$$

Differently from the Euclidean Laplacian in polar coordinates, here there is a drift of constant velocity $d-1$. As we have mentioned in the introduction, this is known as the *drift from infinity*, that is slowing down the propagation.

3.3 Analytic preliminaries.

Heat kernel. Let $G : \mathbb{H}^d \times \mathbb{H}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ be the fundamental solution of the hyperbolic heat equation

$$v_t = \Delta_{\mathbb{H}^d} v, \quad \text{in } \mathbb{H}^d \times \mathbb{R}_+. \tag{3.3.1}$$

Fixed a bounded function $v_0 : \mathbb{H}^d \to \mathbb{R}$, the unique bounded solution to (3.3.1) with initial datum v_0 is given by

$$v(x,t) = \int_{\mathbb{H}^d} G(x,y,t) v_0(y) \, \mathrm{d}\mu(y), \tag{3.3.2}$$

where $\mathrm{d}\mu$ denotes the volume element of $\mathbb{H}^d$. The heat kernel satisfies

$$\|G(x,\cdot,t)\|_{L^1(\mathbb{H}^d)} = 1 \quad \text{for all } x \in \mathbb{H}^d, \ t > 0, \tag{3.3.3}$$

and is an approximation of the identity as $t \to 0^+$. Moreover, there is a family of decreasing functions $\{P_t\}_{\{t>0\}}$ (the heat kernel profiles) such that

$$G(x,y,t) = P_t(d_{\mathbb{H}^d}(x,y)) \quad x, y \in \mathbb{H}^d, \ t > 0. \tag{3.3.4}$$

We have an explicit formula for this heat kernel profile. Indeed, for all $r \geq 0$,

$$P_t(r) = \begin{cases} \dfrac{(-1)^m}{2^n \pi^n (4\pi t)^{1/2}} \left(\dfrac{1}{\sinh r} \dfrac{\partial}{\partial r} \right)^n e^{-n^2 t - \frac{r^2}{4t}} & \text{if } d = 2n+1, \\ \dfrac{(-1)^n}{2^{n+5/2} \pi^{n+3/2} t^{3/2}} e^{-\frac{(2n+1)^2 t}{4}} \left(\dfrac{1}{\sinh r} \dfrac{\partial}{\partial r} \right)^n \displaystyle\int_r^\infty \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh r)^{1/2}} \, \mathrm{d}s & \text{if } d = 2n, \end{cases}$$

for $n \in \mathbb{N}$. For a complete reference, see [54]. Moreover, in [38, Theorem 5.7.2], Davies proved that there exists a positive constant $c_d > 1$ such that

$$c_d^{-1} h_t(r) \leq P_t(r) \leq c_d h_t(r), \quad r \geq 0, \ t > 0, \tag{3.3.5}$$

where

$$h_t(r) = \frac{(1+r)}{(4\pi t)^{d/2}} (1+r+t)^{\frac{d-3}{2}} e^{-\lambda_1 t - \frac{d-1}{2} r - \frac{r^2}{4t}}, \quad r \geq 0, \ t > 0. \tag{3.3.6}$$

By (3.1.13), h_t can be rewritten as

$$h_t(r) = \frac{1}{(4\pi)^{d/2}} e^{\frac{-(r+(d-1)t)^2}{4t}} H_t(r), \quad \text{where } H_t(r) = \frac{(1+r+t)^{\frac{d-3}{2}}(1+r)}{t^{d/2}}, \quad r \geq 0, \ t > 0. \tag{3.3.7}$$

Existence and uniqueness of solutions to (KPP).

Theorem 3.3.1 (Comparison Principle in $\mathbb{H}^d$ for bounded functions)**.** *Consider $T > 0$ and $g : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function. Let $\underline{u}$ and $\overline{u}$ be two bounded $C^2(\mathbb{H}^d \times (0,T))$-functions such that*

$$\underline{u}_t - \Delta_{\mathbb{H}^d} \underline{u} - g(\underline{u}) \leq 0 \quad \text{and} \quad \overline{u}_t - \Delta_{\mathbb{H}^d} \overline{u} - g(\overline{u}) \geq 0 \quad \text{in } \mathbb{H}^d \times (0,T).$$

If $\underline{u}(x,0) \leq \overline{u}(x,0)$ for all $x \in \mathbb{H}^d$, then $\underline{u} \leq \overline{u}$ in $\mathbb{H}^d \times (0,T)$.

The above comparison result is contained in the proof [94, Theorem 2.6].

Proposition 3.3.2. *Let $u_0 \in C(\mathbb{H}^d)$ with values in $[0,1]$. There exists a unique solution u to (KPP) with initial datum u_0. Moreover, $u \in C^\infty(\mathbb{H}^d \times \mathbb{R}_+)$ and $0 \le u \le 1$ in $\mathbb{H}^d \times \mathbb{R}_+$.*

Proof. If a solution u of (KPP) problem with initial datum u_0 exists, it must be a solution to the integral equation

$$u(x,t) = \int_{\mathbb{H}^d} G(x,y,t)u_0(y)\,\mathrm{d}\mu(y) + \fint_0^t \int_{\mathbb{H}^d} G(x,y,t-s)f(u(y,s))\,\mathrm{d}\mu(y)\,\mathrm{d}s. \tag{3.3.8}$$

Since f has been extended as a globally Lipschitz function C^2 in all $\mathbb{R}$ with $f(x) < 0$ when $x \notin [0,1]$, the heat kernel $G : \mathbb{H}^d \times \mathbb{H}^d \times \mathbb{R}_+ \to \mathbb{R}$ is the fundamental solution of (3.3.1) and satisfies (3.3.3) and u_0 is a continuous and bounded function, we can repeat the proof of [64, Theorem 1]. Then we obtain, by a standard fixed point argument, that exists a unique $u \in L^\infty(\mathbb{H}^d \times [0,T])$ solution of (3.3.8) with T only depending of the size of u_0 (which is less than 1). Moreover, since $G \in C^\infty(\mathbb{H}^d \times \mathbb{H}^d \times \mathbb{R}_+)$ and $f \in C^2(\mathbb{R})$, $u \in C^2(\mathbb{H}^d \times (0,T))$ and solves

$$u_t = \Delta_{\mathbb{H}^d} u + f(u) \quad \text{in } \mathbb{H}^d \times (0,T], \qquad u(\cdot,0) = u_0 \quad \text{on } \mathbb{H}^d$$

in the classical sense.

As a consequence, by the the comparison principle Theorem 3.3.1, since the constant functions $u_1 \equiv 0$ and $u_2 \equiv 1$ solve the first equation in the above problem due to $f(0) = f(1) = 0$ and $0 \le u_0 \le 1$, we also have that $0 < u < 1$. Then, since T only depends on the size of the initial datum and we already know that $0 \le u(\cdot,T) \le 1$, we can repeat the fixed point argument as many times as necessary to reach any time $t > 0$. Therefore, there exists a unique u solution to (KPP), it satisfies $0 \le u \le 1$, and solves (3.3.8) for all $t > 0$. Note that this solution is unique by comparison principle Theorem 3.3.1. □

Since the Laplace-Beltrami operator $\Delta_{\mathbb{H}^d}$ is invariant under the action of isometries, thanks to the comparison principle we have the following result:

Corollary 3.3.3 (Invariant by isometries)**.** *Let $u_0 \in C(\mathbb{H}^d)$ with values in [0,1] invariant under the action of a group of isometries of the hyperbolic space. The solution u to (KPP) with initial datum u_0 is invariant under the action of the same group of isometries.*

3.4 Propagation vs. extinction

In this section we obtain new propagation and extinction results, not contained in [75, Theorem 3.2]. More precisely, we address these issues for more general data and we also study the critical case $f'(0) = \lambda_1$ with λ_1 given by (3.1.13).

Remark 3.4.1 (Linealization of equation (KPP)). A useful tool that we employ in this section is the following linearization of equation (3.3.1). Let u be a solution of (KPP) with initial datum $u_0 : \mathbb{H}^d \to [0,1]$. Since f is a KPP function, see (H_f), we have that $f(s) \leq f'(0)s$ for all $s \in (0,1)$. Then, by the comparison principle Theorem 3.3.1, the solution $\tilde{u}$ of the lineal equation

$$\tilde{u}_t = \Delta_{\mathbb{H}^d}\tilde{u} + f'(0)\tilde{u} \quad \text{in } \mathbb{H}^d \times \mathbb{R}_+$$

with $\tilde{u}(\cdot,0) = u_0$ satisfies that $u \leq \tilde{u}$ in $\mathbb{H}^d \times \mathbb{R}_+$. Moreover, since u_0 is bounded, $\tilde{u}$ is given by

$$\tilde{u}(x,t) = e^{f'(0)t}\int_{\mathbb{H}^d} G_t(x,y)u_0(y)\,\mathrm{d}\mu(y) \quad \text{for all } (x,t) \in \mathbb{H}^d \times \mathbb{R}_+$$

due to the fact that the function $v := e^{-f'(0)t}\tilde{u}$ satisfies the heat equation (3.3.1) with initial datum u_0.

Summarising, thanks to the above linealization we know that

$$u(x,t) \leq e^{f'(0)t}\int_{\mathbb{H}^d} G_t(x,y)u_0(y)\,\mathrm{d}\mu(y) \quad \text{for all } (x,t) \in \mathbb{H}^d \times \mathbb{R}_+. \tag{3.4.1}$$

Extinction. Before we give the statement, let us start with some preliminary bounds:

Proposition 3.4.2. *Let $u_0 : \mathbb{H}^d \to [0,1]$ be in $L^p(\mathbb{H}^d)$ with $p \in [1,\infty)$. Then the solution u to (KPP) satisfies that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{H}^d)} = O\big(t^{-\frac{3}{2p}}e^{(f'(0)-\frac{\lambda_1}{p})t}\big) \quad \text{as } t \to \infty. \tag{3.4.2}$$

Proof. Using (3.4.1), (3.3.2) and (3.3.4) we have that

$$u(x,t) \leq e^{f'(0)t}\int_{\mathbb{H}^d} P_t(d_{\mathbb{H}^d}(x,y))u_0(y)\,\mathrm{d}\mu(y) \leq e^{f'(0)t}\|u_0\|_{L^p(\mathbb{H}^d)}\|P_t(d_{\mathbb{H}^d}(x,\cdot))\|_{L^{p'}(\mathbb{H}^d)} \tag{3.4.3}$$

with $1/p + 1/p' = 1$.

On the one hand, if $p = 1$, since P_t is a decreasing function of the geodesic distance for all $t > 0$ and using (3.3.5) and (3.3.6), there is $C > 0$ such that

$$\|P_t(d_{\mathbb{H}^d}(x,\cdot))\|_{L^\infty(\mathbb{H}^d)} \leq P_t(0) \leq Ct^{-3/2}e^{-\lambda_1 t} \quad \text{for all } x \in \mathbb{H}^d,\ t > 0. \tag{3.4.4}$$

On the other hand, if $p \in (1,\infty)$,

$$\|P_t(d_{\mathbb{H}^d}(x,\cdot))\|_{L^{p'}(\mathbb{H}^d)} \leq \|P_t(d_{\mathbb{H}^d}(x,\cdot))\|^{\frac{1}{p'}}_{L^1(\mathbb{H}^d)}\|P_t(d_{\mathbb{H}^d}(x,\cdot))\|^{\frac{p'}{1-p'}}_{L^\infty(\mathbb{H}^d)} \leq Ct^{-\frac{3}{2p}}e^{-\frac{\lambda_1}{p}t} \tag{3.4.5}$$

for all $x \in \mathbb{H}^d$ and $t > 0$ due to (3.4.4) and the fact that $\|P_t(d_{\mathbb{H}^d}(x,\cdot))\|_{L^1(\mathbb{H}^d)} = 1$ for all $x \in \mathbb{H}^d$ and $t > 0$. Therefore, combining (3.4.3) with (3.4.4) or with (3.4.5), we obtain (3.4.2). □

As a consequence of the previous Proposition 3.4.2, we obtain the following extinction result that covers the critical case when $f'(0) = \lambda_1$.

Corollary 3.4.3. *Let $f'(0) \leq \lambda_1$. If $u_0 \in L^p(\mathbb{H}^d)$, $0 \leq u_0 \leq 1$, and $1 \leq p \leq \lambda_1/f'(0)$, then the solution u to (KPP) vanishes, that is, $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{H}^d)} = 0$.*

Extinction may also occurs for initial data that do not belong to any L^p space except L^∞. For example, consider an initial datum whose support in parabolic coordinates is located in the half-space $\{x \in \mathbb{H}^d : \rho_{\mathfrak{p}}(x) < R_0\}$ for some constant R_0, as shown below in Proposition 3.4.4. Unfortunately, the analogous result in the hyperbolic coordinates setting cannot be proved with the current method due to the behavior of the associated ODE.

Proposition 3.4.4. *Let $f'(0) < \lambda_1$ and u the solution to (KPP) with $u_0(x) \leq \mathcal{X}_{(-\infty,R_0)}(\rho_{\mathfrak{p}}(x))$ for some $R_0 \geq 0$. Then, for all $\delta \in \mathbb{R}$,*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\, \rho_{\mathfrak{p}}(x)\geq\delta\}} u(x,t) = 0. \tag{3.4.6}$$

Proof. Since $f'(0) < \lambda_1$ implies that $c_0 < (d-1)$, see (3.1.13) and (3.1.14), we have, due to [7, Proposition 4.1], that there exists $q : [0,\infty) \times \mathbb{R}_+ \to [0,1]$ such that

$$q'' + (d-1)q' + f(q) = 0 \quad \text{in } \mathbb{R}_+, \qquad q(0) = 1, \quad \lim_{s\to\infty} q(s) = 0.$$

Let v be the solution to

$$v_t = v_{\rho\rho} + (d-1)v_\rho + f(v) \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \qquad v(\rho,0) = \varphi(\rho - R_0), \quad \rho \in \mathbb{R},$$

where

$$\varphi(\rho) = 1, \quad \text{if } \rho \leq 0 \quad \text{and } \varphi(\rho) = q(\rho) \text{ if } \rho \geq 0.$$

Then, since $f(0) = 0$ and $q > 0$, by the comparison principle, $v \geq 0$. Moreover, by the proof of [7, Theorem 5.1], it holds that $v_\rho(\cdot,t) \leq 0$ for all $t > 0$ and for all $\delta \in \mathbb{R}$

$$\lim_{t\to\infty} \sup_{\{\rho\geq\delta\}} v(\rho,t) = 0.$$

As a consequence, by (3.2.2) and the comparison principle (Theorem 3.3.1), we have that the solution u to (KPP) problem with initial datum $u_0(x) \leq \mathcal{X}_{(-\infty,R_0)}(\rho_{\mathfrak{p}}(x))$ with $x \in \mathbb{H}^d$, satisfies that

$$0 \leq u(x,t) \leq v(\rho_{\mathfrak{p}}(x),t) \quad \text{for all } (x,t) \in \mathbb{H}^d \times \mathbb{R}_+,$$

and this yields (3.4.6). □

Propagation. Theorem 3.1.1, which ensures propagation, and the fact that the Laplace-Beltrami operator $\Delta_{\mathbb{H}^d}$ is invariant under isometries yield the following result.

The following proposition is is a consequence of the comparison principle, which yields a stronger result than Theorem 3.1.2 (Theorem 3.6 in [75]) for some initial data, since it shows uniform convergence in sets that may not be compact.

Proposition 3.4.5. *Let $f'(0) > \lambda_1$ and $u_0 : \mathbb{H}^d \to [0,1]$ satisfying that for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$ and some $v_0 : I_i \to [0,1]$, $v_0 \not\equiv 0$,*

$$u_0(x) \geq v_0(\rho_i(x)) \quad \text{for all } x \in \mathbb{H}^d.$$

Then the solution u to the (KPP) problem satisfies that, for all $\delta > 0$,

$$\lim_{t\to\infty} \inf_{\{x\in\mathbb{H}^d : |\rho_i(x)|\leq\delta\}} u(x,t) = 1. \tag{3.4.7}$$

Proof. Let $\tilde{u}$ be the solution to (KPP) with initial datum $\tilde{u}(x,0) = v_0(\rho_i(x))$ for all $x \in \mathbb{R}$. Then, by Theorem 3.1.1 and the comparison principle (Theorem 3.3.1), we have that $\|u(\cdot,t)\|_{L^\infty(K)} \to 1$ as $t \to \infty$ if K is a compact subset of $\mathbb{H}^d$. Moreover, by Corollary 3.3.3, $\tilde{u}(x,t) = v(\rho_i(x),t)$. Then,

$$\lim_{t\to\infty} \inf_{\{x\in\mathbb{H}^d : |\rho_i(x)|\leq\delta\}} \tilde{u}(x,t) = 1 \quad \text{for all } \delta > 0.$$

Finally, by the comparison principle (Theorem 3.3.1) again, we obtain (3.4.7). □

In view of Theorem 3.1.2, the solution u with nontrivial, compactly supported initial datum spreads with velocity $c = c_*$. However, the following result shows that even at velocity $c = c_*$ we have that the solution vanishes. Moreover, it proves that to capture the behavior of u we need a logarithmic correction in the velocity.

Proposition 3.4.6. *Let $f'(0) > \lambda_1$ and let u be a solution to (KPP) with a compactly supported initial datum $u_0 : \mathbb{H}^d \to [0,1]$. Then,*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d :\, \rho_{\mathfrak{e}}(x) > c_* t - C\log t\}} u(x,t) = 0 \quad \text{for all } 0 \leq C < 1/c_0. \tag{3.4.8}$$

Proof. Since u_0 is compactly supported there exists $R_0 > 0$ such that $u_0(x) \leq \mathcal{X}_{[0,R_0]}(\rho_{\mathfrak{e}}(x))$ for all $x \in \mathbb{H}^d$. Then, using inequeality (3.4.1), we have that

$$\begin{aligned} u(x,t) &\leq e^{f'(0)t} \int_{\mathbb{H}^d} G_t(x,y) X_{[0,R_0]}(\rho_{\mathfrak{e}}(y))\, \mathrm{d}\mu(y) \\ &= \omega_{d-1} e^{f'(0)t} \iint_0^{R_0} P_t\big(d_{\mathbb{H}^d}(\rho_{\mathfrak{e}}(x), \theta_{\mathfrak{e}}(x)), (r,\xi)\big) \sinh^{d-1} r\, dr \end{aligned}$$

for all $(x,t) \in \mathbb{H}^d \times \mathbb{R}_+$ where ω_{d-1} is the volume of $\mathbb{S}^{d-1}$ due to the fact that $G_t(x,y) = P_t(d_{\mathbb{H}^d}(x,y))$ for all $x, y \in \mathbb{H}^d$.

On the other hand, let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a $o(t^{1/2})$ that will be determined at the end of the proof. If $r \geq c_* t - g(t)$, then $d_{\mathbb{H}^d}((\rho,\theta),(r,\xi)) \geq c_* t - g(t) - R_0 > 0$ for all $r \in [0, R_0]$ and $\theta, \xi \in \mathbb{S}^{d-1}$. Therefore, since the heat kernel profile P_t is decreasing in the geodesic distance variable, we have that

$$\begin{aligned} \sup_{\{x\in\mathbb{H}^d :\, \rho_{\mathfrak{e}}(x) > c_* t - C\log t\}} u(x,t) &\leq \omega_{d-1} e^{f'(0)t} P_t(c_* t - g(t) - R_0) \iint_0^{R_0} \sinh^{d-1} r\, dr \\ &\leq C_{d,R_0} t^{-1/2} \exp\big(\lambda_0 g(t)\big) \quad \text{for } t > 0 \text{ large enough,} \end{aligned}$$

due to (3.3.5), (3.3.7), (3.1.14) and (3.1.4). Taking $g(t) := C\log(t)$ with $0 \leq C < 1/c_0$, we obtain (3.4.8). □

3.5 Propagation: convergence to a traveling wave

This section is devoted to proving our main theorem, which concerns the long-time asymptotic behavior of the solution u to the (KPP) problem with initial datum that satisfies condition (A2) below for some i belonging to $\{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. More precisely, we will show that u converges to the traveling wave profile of minimal speed Φ_{c_0} in a moving i-coordinate frame with linear velocity c_*, logarithmic correction with coefficient $3/c_0$ and with a specific shift $\beta \in \mathbb{R}$.

We consider initial data satisfying $u_0 : \mathbb{H}^d \to [0,1]$ given by a unidimensional ρ_i-function for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$ and, for such i, trapped between two unidimensional ρ_i-step functions. That is, u_0 satisfies

$$\begin{cases} u_0 : \mathbb{H}^d \to [0,1] \text{ for some } i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}, \quad u_0 = \tilde{u}_0(\rho_i(x)) \quad \text{for all } x \in \mathbb{H}^d; \\ \text{there exist } 0 < \rho_1 < \rho_2 \text{ such that } \mathcal{X}_{[0,\rho_1]}(\rho) \le \tilde{u}_0(\rho) \le \mathcal{X}_{[0,\rho_2]}(\rho) \quad \text{for all } \rho \ge 0. \end{cases} \tag{A2}$$

Theorem 3.5.1. *Let $f'(0) > \lambda_1$ and u_0 satisfying (A2) for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. Then, the solution u to (KPP) satisfies that*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\ \rho_i(x)\ge 0\}} u(x,t) - \Phi_{c_0}\left(\rho_i(x) - (c_* t - \tfrac{3}{c_0}\log t + \beta)\right) = 0 \tag{3.5.1}$$

for some $\beta \in \mathbb{R}$.

Strategy. Reduction to the Euclidean case and the choice of parameters. Our aim here is to introduce several transformations to the equation in (KPP) in order to capture the dynamics of u. In particular, we will give a precise description of the behavior of u at positions $\rho_i(x) \asymp c_* t + o(\sqrt{t})$, known as the *diffusive region*. This region is crucial as it determines the overall behavior of u in the remaining regions (for $\rho_i(x) > 0$). The transformations we apply are strongly inspired by those introduced by Roquejoffre, Rossi and Roussier-Michon in [100, Section 2] to address the KPP problem in the d-dimensional Euclidean space (3.1.10). We have already introduced some of these changes in the Introduction; we now formalize and describe the full picture.

Fix $f'(0) > \lambda_1$. Let u be the solution to (KPP) with initial datum u_0 satisfying (A2) for some $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. Let us explain the main steps of our approach.

1. i-invariance. Since u_0 satisfies (A2), by Corollary 3.3.3, $u(x,t) = \tilde{u}(\rho_i(x), t)$ for all points (x,t) in $\mathbb{H}^d \times \mathbb{R}_+$, where the function $\tilde{u} : I_i \times \mathbb{R}_+ \to [0,1]$ satisfies

$$\begin{cases} \widetilde{u}_t(\rho,t) = \widetilde{u}_{\rho\rho}(\rho,t) + (d-1)g_1^i(\rho)\widetilde{u}_\rho(\rho,t) + f(\widetilde{u}(\rho,t)), & \rho \in I_i, \quad t > 0, \\ \widetilde{u}(\rho,0) = \widetilde{u}_0(\rho), & \rho \in I_i. \end{cases}$$

If $i = \mathfrak{e}$, we need to add the extra condition $\tilde{u}_\rho(0,t) = 0$ for all $t > 0$.

2. Moving coordinates. We expect (see the discussion in the Introduction and Proposition 3.4.6) the transition zone where u is not 0 nor 1 to be located, for t big enough, around

$$R(t) := c_* t - k \ln t + \rho_0, \tag{3.5.2}$$

where c_* is given by (3.1.14), and k and ρ_0 are positive constants that we will choose later. As a consequence, we study $\tilde{u}$ in the moving coordinates $\rho' = \rho - R(t)$, that is,

$$\hat{u}(\rho, t) := \tilde{u}(\rho + R(t), t - 1), \tag{3.5.3}$$

which is a solution to

$$\begin{cases} \hat{u}_t = \hat{u}_{\rho\rho} + \big[(d-1)\big(g_1^i(\rho + R(t)) - 1\big) + c_0 - \frac{k}{t}\big]\hat{u}_\rho + f(\hat{u}), & \rho \geq -R(t), \quad t > 1, \\ \hat{u}(\rho, 1) = \widetilde{u}_0(\rho + R(1)), & \rho \geq -R(1). \end{cases}$$

The translation in time is just to avoid difficulties with the logarithm at the initial time. Observe that we only consider the points with $\rho \geq -R(t)$ due to the fact that Theorem 3.5.1 is only valid for $\rho \geq 0$ in the original reference system.

3. Matching the asymptotics of Φ_{c_0}. To compensate for the decay of Φ_{c_0} at infinity, given in (3.1.7), we consider

$$v(\rho, t) := e^{\lambda_0 \rho} u(\rho, t). \tag{3.5.4}$$

Thus, v solves

$$\begin{cases} v_t = v_{\rho\rho} + \big(h^i - \frac{k}{t}\big)(v_\rho - \lambda_0 v) + e^{\lambda_0\rho} f(e^{-\lambda_0\rho} v) - f'(0)v, & \rho \geq -R(t), \quad t > 1, \\ v(\rho, 1) = e^{\lambda_0\rho}\widetilde{u}_0(\rho + R(1)), & \rho \geq -R(1), \end{cases}$$

where

$$h^i(\rho, t) := (d-1)\big(g_1^i(\rho + R(t)) - 1\big), \quad i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}. \tag{3.5.5}$$

Observe that taking $\rho_0 > 0$ large enough, we can assume $R(t) \geq t^{1/2}$ for all $t \geq 1$. Therefore, $h^i(\rho, t)$ is bounded for all $\rho \in I_i$, $t \geq 1$ and $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$, thus overcoming the problem introduced by the asymptote of $g_1^{\mathfrak{e}}(\rho) = \coth \rho$ at $\rho = 0$.

4. Self-similar coordinates. We use self-similar coordinates for the Euclidean heat equation in order to study the behavior of v in the region $\rho \asymp \sqrt{t}$ for $t \geq 1$. Let

$$\eta := \frac{\rho}{\sqrt{t}}, \qquad \tau := \ln t \quad \text{and} \quad \hat{w}(\eta, \tau) := e^{-\tau/2} v(\rho, t). \tag{3.5.6}$$

Here, $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t + \mathcal{L}\hat{w} = e^{-\tau/2}\hat{h}^i \hat{w}_\tau - (\lambda_0 \hat{h}^i + \frac{3}{2})\hat{w} + \hat{F}(\eta, \tau, \hat{w}), & \eta e^{\tau/2} \geq -c_* e^\tau + k\tau - \rho_0, \quad \tau > 0, \\ \hat{w}(\eta, 0) = e^{\lambda_0\eta}\widetilde{u}_0(\eta + R(1)), & \eta \geq -R(1), \end{cases} \tag{3.5.7}$$

where $\mathcal{L}$ is the Fokker-Planck operator $\mathcal{L}z := -z_{\eta\eta} - \dfrac{\eta}{2} z_\eta - z$ and

$$\begin{aligned} \hat{h}^i(\eta, \tau) :&= e^\tau h^i(\eta e^{\tau/2}, e^\tau) - k \qquad (3.5.8) \\ &= e^\tau (d-1)\big(g_1^i(\eta e^{\tau/2} + c_* e^\tau - k\tau + \rho_0) - 1\big) - k, \quad i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}, \\ \hat{F}(\eta, \tau, z) :&= e^{\tau/2 + \lambda_0 \eta e^{\tau/2}} f\big(e^{\tau/2 - \lambda_0\eta e^{\tau/2}} z\big) - e^\tau f'(0) z. \qquad (3.5.9) \end{aligned}$$

Our aim is to prove that

$$\hat{w}(\eta,\tau)\to\alpha_\infty\phi_0(\eta)\quad\text{as }\tau\to\infty\ \text{ in the region }\{\eta\geq e^{-(\frac{1}{2}-\delta)\tau}\}\tag{3.5.10}$$

for some $\alpha_\infty>0$, δ a positive and small enough constant (smaller that $1/4$) and

$$\phi_0(\eta):=\eta e^{-\frac{\eta^2}{4}}.\tag{3.5.11}$$

Note that ϕ_0 is a solution of $\mathcal{L}\phi_0=0$ for $\eta>0$ such that $\phi_0(0)=\phi_0(\infty)=0$.

To obtain (3.5.10), we will follow the same strategy as in [100]. The reason behind is that problem (3.5.7) is very close to the one given by [100, equation (12)], with the error terms being small, provided that k is chosen properly (as we will see in the upcoming point 6). The precise scheme is to prove the compactness of the trajectories $\{\hat{w}(\cdot,\cdot+T)\}_{T>0}$ as $T\to\infty$ in a weighted L^∞ norm. The limit trajectory will satisfy the Euclidean heat equation in self-similar coordinates. Now, in order to obtain this compactness, we need to construct sub and supersolutions to $\tilde{w}$.

5. Sub and supersolutions by translations. Let

$$\delta\in(0,\tfrac{1}{4}),\quad \eta_\delta\in\{\eta_\delta^-,0,\eta_\delta^+\}\quad\text{with }\eta_\delta^\pm=\eta_\delta^\pm(\tau):=\pm e^{-(\frac{1}{2}-\delta)\tau}.\tag{3.5.12}$$

Observe that $\eta_\delta^\pm(\tau)$ vanishes exponentially fast as $\tau\to\infty$.

Our goal is to construct a subsolution in the region $\{\eta\geq\eta_\delta^+(\tau)\}$, i.e, to built a lower barrier of $\tilde{w}$ where ρ is far ahead of the front of the traveling wave, and to construct a supersolution in the region $\{\eta\geq\eta_\delta^-(\tau)\}$, that is, where ρ is far at the right of the front of Φ_{c_0}.

Although the precise construction will be given in Proposition 3.5.4 below, tbe key idea here would be to consider the translation $\tilde{w}(\eta,\tau):=\hat{w}(\eta+\eta_\delta,\tau)$. It is a solution to

$$\begin{cases}\tilde{w}_t+\mathcal{L}\tilde{w}=\big((\delta-\frac{1}{2})\eta_\delta+e^{-\tau/2}\hat{h}^i(\eta+\eta_\delta,\tau)\big)\tilde{w}_\tau\\ \qquad\qquad -(\lambda_0\hat{h}^i(\eta+\eta_\delta,\tau)+\frac{3}{2})\tilde{w}+\hat{F}(\eta+\eta_\delta,\tau,\tilde{w}), & \eta e^{\tau/2}+\eta_\delta\geq -c_*e^\tau+k\tau-\rho_0,\quad\tau>0,\\ \hat{w}(\eta,0)=e^{\lambda_0(\eta+\eta_\delta)}\widetilde{u}_0(\eta+\eta_\delta(0)+R(1)), & \eta\geq-\eta_\delta+R(1).\end{cases}\tag{3.5.13}$$

6. Turning $\mathcal{L}$ into a self-adjoint operator, and choice of k. Let us describe the general structure at this stage. For this, let $\mathcal{M}$ be the operator given by

$$\mathcal{M}z:=-z_{\eta\eta}+\big(\tfrac{\eta^2}{16}-\tfrac{3}{4}\big)z.$$

Here $\mathcal{M}$ is self-adjoint and has an associated quadratic form in $H_0^1(\mathbb{R}_+,\eta\,\mathrm{d}\eta)$. It is well known (see, for instance, the proof of [56, Lemma 2.2]) that a function that solves

$$\begin{cases}z_t+\mathcal{M}z=-\varepsilon e^{-\frac{\tau}{2}}(z_\eta-\frac{\eta}{4}z),\quad \eta\geq0,\quad\tau>0,\\ z(0,\tau)=0,\quad\tau>0.\end{cases}\tag{3.5.14}$$

with $\varepsilon>0$ small enough and a fast decaying initial datum is a perturbation of $e^{\eta^2/8}\phi_0$.

Going back go our problem, consider the last transformation

$$w := e^{\frac{\eta^2}{8}}\tilde{w}.$$

Then w solves

$$\begin{cases} w_t + \mathcal{M}w = l_1^i(\eta,\tau)w_\eta + l_2^i(\eta,\tau)w + F(\eta,\eta_\delta,\tau,w), & \eta e^{\tau/2} + \eta_\delta \geq -c_* e^\tau + k\tau - \rho_0,\ \tau > 0, \\ w(\eta,0) = e^{\lambda_0(\eta+\eta_\delta)+\frac{\eta^2}{8}}\widetilde{u}_0(\eta+\eta_\delta(0)+R(1)), & \eta \geq -\eta_\delta + R(1), \end{cases}$$

where

$$\begin{aligned} F(\eta,\eta_\delta,\tau,z) &:= e^{\tau/2+\frac{\eta^2}{8}+\lambda_0(\eta+\eta_\delta)e^{\tau/2}} f\big(e^{\tau/2-\frac{\eta^2}{8}-\lambda_0(\eta+\eta_\delta)e^{\tau/2}}z\big) - e^\tau f'(0)z, \\ l_1^i(\eta,\eta_\delta,\tau) &:= (\delta-\tfrac{1}{2})\eta_\delta(\tau) + e^{\tau/2}(d-1)\big[g_1^i\big((\eta+\eta_\delta)e^{-\tau/2} + c_* e^\tau - k\tau + \rho_0\big) - 1\big] - ke^{-\tau/2}, \\ l_2^i(\eta,\eta_\delta,\tau) &:= -\lambda_0(d-1)e^\tau\big[g_1^i\big((\eta+\eta_\delta)e^{-\tau/2} + c_* e^\tau - k\tau + \rho_0\big) - 1\big] - \frac{\eta}{4}l_1^i(\eta,\tau) + \lambda_0 k - \tfrac{3}{2}. \end{aligned}$$

Our aim is to prove that w is a perturbation of some multiple of $e^{\eta^2/8}\phi_0$. However, w does not satisfy (3.5.14). Nevertheless, in [100], motivated by the difficulties that arise in the study of the KPP problem in a d-dimensional Euclidean space in polar coordinates, the authors develop an extensive theory of convergence in a broader setting (more precisely, for solutions of some version of (3.5.14) with more general error terms in the right hand side), that we will adapt for our problem in the hyperbolic space. For this purpose, the key point is to show that l_1^i and l_2^i satisfy the same bounds [100, eq. (19)] for all $i \in \{\mathfrak{e},\mathfrak{h},\mathfrak{p}\}$ as their counterparts in the euclidean case, l_1 and l_2, defined in [100, transformation 7, section 2]. Thus we have to prove that l_j^i, $j = 1,2$, vanish as τ goes to infinity. It is precisely here where we choose k satisfying

$$\lambda_0 k = \frac{3}{2}. \tag{3.5.15}$$

Therefore,

$$l_2^i(\eta,\eta_\delta,\tau) = -\lambda_0 e^\tau(d-1)\big[g_1^i\big((\eta+\eta_\delta)e^{-\tau/2} + c_* e^\tau - k\tau + \rho_0\big) - 1\big] - \frac{\eta}{4}l_1^i(\eta,\tau).$$

On the other hand, since

$$|g_1^i(s) - 1| = \begin{cases} \frac{2}{e^{2s}-1}, & i = \mathfrak{e}, \\ \frac{2}{e^{2s}+1}, & i = \mathfrak{h}, \\ 0, & i = \mathfrak{p}, \end{cases} \tag{3.5.16}$$

for $s > 0$, taking $\rho_0 = \rho_0(\lambda_0)$ big enough such that

$$s(\eta,\eta_\delta,\tau) := (\eta+\eta_\delta)e^{-\tau/2} + c_* e^\tau - k\tau + \rho_0 \geq -e^{\tau/4} + c_* e^\tau - k\tau + \rho_0 \geq 1$$

for all $\eta \geq 0$ and $\tau > 0$, we have $|g_1^i(s(\eta,\eta_\delta,\tau)) - 1| \leq \frac{2}{e^2-1}$ and

$$e^{3\tau/2} \sup_{\delta\in(0,\frac{1}{4})} \|g_1^i(s(\cdot,\eta_\delta(\tau),\tau)) - 1\|_{L^\infty(0,\infty)} \to 0 \quad \text{as } \tau\to\infty \quad \text{for all } i \in \{\mathfrak{e},\mathfrak{h},\mathfrak{p}\}. \tag{3.5.17}$$

As a consequence, there exists $C > 0$ depending only on d, δ and $f'(0)$ such that

$$|l_1^i(\eta, \eta_\delta, \tau)| \leq Ce^{-(\frac{1}{2}-\delta)\tau}, \quad |l_2^i(\eta, \eta_\delta, \tau)| \leq C(1+\eta)e^{-(\frac{1}{2}-\delta)\tau} \tag{3.5.18}$$

for all $\eta \geq 0$, $\tau > 0$, $\eta_\delta \in \{\eta_\delta^-, 0, \eta_\delta^+\}$ with $\delta \in (0, \frac{1}{4})$ and $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. Note that the bounds in (3.5.18) improve the ones in [100, eq. (19)].

Remark 3.5.2. The parameter k does not depend on the dimension d; recall (3.5.15). This is an important difference with the KPP problem in the multidimensional Euclidean space (3.1.2), for compactly supported, symmetric initial datum. Indeed, in [100], they need to take $k = \frac{d+2}{c_0}$, (recall $c_0 = 2\lambda_0$), due to the fact that the equivalent w in [100, Transformation 7, section 2], solves

$$\begin{aligned}
\partial_\tau z + \mathcal{M}z &= l_1(\eta, \tau)z_\eta + l_2(\eta, \tau)z + \hat{F}(\eta + \eta_\delta, \tau, z), \quad \text{where} \\
l_1(\eta, \eta_\delta, \tau) &= \delta\eta_\delta + \left(\lambda_0 e^\tau \frac{d-1}{(\eta+\eta_\delta)e^{\tau/2} + c_0 e^\tau - k\tau} + \lambda_0 k - \frac{3}{2}\right) e^{-\tau/2}, \\
l_2(\eta, \eta_\delta, \tau) &= -\lambda_0 e^\tau \frac{d-1}{(\eta+\eta_\delta)e^{\tau/2} + c_0 e^\tau - k\tau} + \lambda_0 k - \frac{3}{2} - \frac{\eta}{4} l_1(\eta, \tau)
\end{aligned}$$

where the following limit

$$e^\tau \sup_{\delta \in (0, \frac{1}{4})} \frac{d-1}{(\eta+\eta_\delta)e^{\tau/2} + c_0 e^\tau - k\tau} \Big\|_{L^\infty(0,\infty)} \to \frac{d-1}{c_0} \quad \text{as } \tau \to \infty$$

that not converges to zero, in contrast to (3.5.17).

On the other hand, in the Euclidean space $\mathbb{R}^d$ (with $d > 1$), if the initial datum u_0 satisfies

$$\mathcal{X}_{(-\infty,a]}(x_d) \leq u_0(x_1, ..., x_d) \leq \mathcal{X}_{(-\infty,b]}(x_d)$$

with $b < a$, then $k = 3/c_0$ (see [101]). That is, k does not depend of the dimension (and it is exactly the same as in $\mathbb{R}^1$).

Therefore, in $\mathbb{R}^d$, the parameter k depends on the geometry of the support of the initial datum in terms of the isometries of $\mathbb{R}^d$. Specifically, k changes depending on whether the initial datum can be bounded between two radial functions or two functions invariant under some translation. In this case, k reflects the type of isometry; elliptic (radial) or parabolic (translations) and, as a consequence, it may depend on the dimension d. In $\mathbb{H}^d$, by contrast, k is independent of the type of isometry and the dimension.

Once we have understood the challenges introduced by the hyperbolic space, we are able to adapt the results from [100].

Preliminary estimates. First, we will construct sup and sub-solutions for

$$z_t + \mathcal{M}z = l_1^i(\eta, \tau)z_\eta + l_2^i(\eta, \tau)z + F(\eta, \eta_\delta, \tau, z), \tag{3.5.19}$$

under some conditions. Specifically, we will obtain two functions $\overline{w}$ and $\underline{w}$ such that

$$\begin{aligned}
\overline{w}_t + \mathcal{M}\overline{w} &\geq l_1^i(\eta, \tau)\overline{w}_\eta + l_2^i(\eta, \tau)\overline{w} + F(\eta, \eta_\delta, \tau, \overline{w}), \\
\underline{w}_t + \mathcal{M}\underline{w} &\leq l_1^i(\eta, \tau)\underline{w}_\eta + l_2^i(\eta, \tau)\underline{w} + F(\eta, \eta_\delta, \tau, \underline{w}),
\end{aligned}$$

in $\{\eta \geq \eta_\delta(\tau)\}$ (chosen $\eta_\delta(\tau)$ in (3.5.12) as we need: we will chose $\eta_\delta = \eta_\delta^-$ for $\overline{w}$ and $\eta_\delta = \eta_\delta^+$ for $\underline{w}$) such that $\overline{w}(0,\tau)$ and $\underline{w}(0,\tau)$ vanish as $\tau \to \infty$.

The following proposition tells us that $\overline{w}$ and $\underline{w}$ are comparable to $e^{\eta^2/8}\phi_0$ as $\tau \to \infty$. This is because f is a KPP function and l_1^i, l_2^i satisfy (3.5.18) for $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$.

Proposition 3.5.3. *[100, Proposition 1]* (i) *We may give six functions $q_1^\pm(\tau)$, $q_2^\pm(\tau)$, $\xi^\pm(\tau)$ that satisfy:*

- *there exists $C > 1$ such that*

$$C^{-1}e^{-\tau}q_j^+(\tau) \leq Ce^{-(\frac{1}{2}-\delta)\tau}, \quad 0 \leq q_j^-(\tau) \leq Ce^{-(\frac{1}{2}-\delta)\tau} \quad \text{for all } \tau > 0 \text{ and } j \in \{1,2\};$$

- *there are $\overline{\xi}_0 \geq \underline{\xi}_0 > 0$ such that*

$$\xi^\pm(\tau) \in [\underline{\xi}_0, \overline{\xi}_0], \quad \xi^+(\tau) > 0, \quad \xi^-(\tau) < 0 \text{ for all } \tau > 0.$$

(ii) *Let $a \in (0,1)$, $\gamma_1(\tau)$ a nonnegative smooth function, identically 1 if $\eta \leq a/2$ and zero if $\eta \geq 2a/3$, and $\gamma_2(\tau)$ be a nonnegative function, identically zero if $\tau \geq 1$ and equal to 1 if $\eta \geq 2$. Then,*

$$\overline{w}(\eta,\tau) := \xi^+(\tau)e^{\eta^2/8}\phi_0(\eta) + q_1^+(\tau)\gamma_1(\eta)\cos(\tfrac{\pi}{2a}\eta) + q_2^+(\tau)\gamma_2(\eta)e^{-\frac{\eta^2}{16}}$$

is a supersolution to (3.5.19) with $\eta_\delta = \eta_\delta^-$ in the range $\tau > 0$, $\eta > 0$, whereas the function

$$\underline{w}(\eta,\tau) := \xi^-(\tau)e^{\eta^2/8}\phi_0(\eta) - q_1^-(\tau)\gamma_1(\eta)\cos(\tfrac{\pi}{2a}\eta) - q_2^-(\tau)\gamma_2(\eta)e^{-\frac{\eta^2}{16}}$$

is a subsolution to (3.5.19) with $\eta_\delta = \eta_\delta^+$ in the range $\tau > \tau_1$, $\eta > 0$ for some $\tau_1 > 0$.

Moreover, since $g_1^i(\cdot + \rho_0)$ is in $L^\infty([-1,\infty))$ for $\rho_0 \geq 2$, for all $i \in \{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$ (here we can see the use of the constant ρ_0 in order to obtain that $\coth\big((\eta+\eta_\delta)e^{-\tau/2} + c_*e^\tau - k\tau + \rho_0\big)$ is uniformly bounded for all $\eta \geq 0$ and $\tau > 0$) we have the standard comparison principle for (3.5.19). In addition, as $w_0(\eta) = e^{\lambda_0(\eta+\eta_\delta)+\frac{\eta^2}{8}}\widetilde{u}_0(\eta+\eta_\delta(0)+R(1))$ is compactly supported from the right and $0 \leq u \leq 1$, we can compare $\overline{w}$ and $\underline{w}$ with w for $\eta \geq \eta_\delta$ (where the choice of η_δ depends of $\overline{w}$ and $\underline{w}$). Repeating the proof of [100, Proposition 5 and Corollary 1] we obtain the following proposition.

Proposition 3.5.4. *[100, Proposition 5 and Corollary 1] Let $f'(0) > \lambda_1$ and u_0 satisfying (A2). Let ϕ_0 and $\hat{w}$ be the functions given by (3.5.11) and (3.5.6), respectively. Set $q_1^\pm(\tau)$, $q_2^\pm(\tau)$, $\xi^\pm(\tau)$ as in Proposition 3.5.3.*
(i) ***Control of $\hat{w}$ from above and from the rear of the front.*** *There is a pair of positive functions $(q_+(\tau), \xi_+(\tau))$ such that $\xi_+(\tau)$ is bounded and bounded away from 0 by constants that depend only on the initial datum and the constants appearing in the equation, whereas $\delta \in (0, 1/4)$ such that $q_+(\tau) = O(e^{-(\frac{1}{2}-\delta)\tau})$ as $\tau \to \infty$. Moreover, we have that*

$$\hat{w}(\eta + \eta_\delta^-, \tau) \leq \xi_+(\tau)\phi_0(\eta) + q_+(\tau)\big(\mathcal{X}_{[0,a_0/2]}(\eta)\cos(\tfrac{\pi}{2a_0}\eta) + e^{-\frac{\eta^2}{16}}\big)e^{-\frac{\eta^2}{8}}, \quad \eta \geq 0, \ \tau > 0. \tag{3.5.20}$$

(ii) ***Control of $\hat{w}$ from below and from the head of the front.*** *There is a pair of positive functions $(q_-(\tau), \xi_-(\tau))$ satisfying the same estimates as $(q_+(\tau), \xi_+(\tau))$ in* (i) *and such that, for some $\tau_1 > 0$,*

$$\hat{w}(\eta + \eta_\delta^-, \tau) \geq \xi_-(\tau)\phi_0(\eta) - q_-(\tau)\big(\mathcal{X}_{[0,a_0/2]}(\eta)\cos(\tfrac{\pi}{2a_0}\eta) + e^{-\frac{\eta^2}{16}}\big)e^{-\frac{\eta^2}{8}}, \quad \eta \geq 0,\ \tau \geq \tau_1. \tag{3.5.21}$$

Moreover, there exists $C > 0$ such that

$$|\hat{w}_\tau(\eta,\tau)| + |\hat{w}_\eta(\eta,\tau)| \leq Ce^{-\frac{3\eta^2}{16}} \quad \text{for all } \eta \geq 0,\ \tau > 1. \tag{3.5.22}$$

Summarizing, Proposition 3.5.3 states that there is a constant $C > 1$ such that

$$C^{-1}\phi_0(\eta) \leq \hat{w}(\eta,\tau) \leq C\phi_0(\eta) \quad \text{for } \tau > 0 \text{ big enough and } \eta \geq 0 \tag{3.5.23}$$

As explained in [100], this is the best one can do with barrier arguments, because $\phi_0 \in \mathrm{Ker}(\mathcal{L})$. This is far from a convergence result, since it does not even determine the right multiple of ϕ_0 that one should obtain in the limit. However, Proposition 3.5.4 is a first step towards the desired result, since it will give the compactness of the trajectories $\{\hat{w}(\cdot, \cdot + T)\}_{T>0}$ in a spatial weighted L^∞-norm.

Proposition 3.5.5. *(Adaptation of [100, Theorem 2.1] and [84, Lemma 5.1]) Consider the same assumptions of Proposition 3.5.4. Then, there exists a positive constant α_∞ such that*

$$\lim_{\tau\to\infty} \sup_{\{\eta>0\}} e^{\frac{\eta^2}{16}}\ \hat{w}(\eta,\tau) - \alpha_\infty\phi_0(\eta)\ = 0. \tag{3.5.24}$$

Proof. We will follow the ideas of the proof of [84, Lemma 5.1]. Consider a sequence of times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to \infty$ as $n \to \infty$. By the barriers (3.5.20) and (3.5.21), the bounds for the derivatives (3.5.22) and using standard compactness argument, there exists a subsequence $\{\tau_{n'}\}_{n'\in\mathbb{N}} \subset \{\tau_n\}_{n\in\mathbb{N}}$ and two functions $\hat{w}_\infty : [0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ and $\widehat{W}_\infty : [0,\infty) \to \mathbb{R}$ such that

$$\lim_{n'\to\infty} \sup_{\eta\geq 0,\,\tau\in[0,T]} e^{\frac{\eta^2}{16}}\ \widehat{w}(\eta,\tau+\tau_{n'}) - \hat{w}_\infty(\eta,\tau)\ = 0, \quad \lim_{n'\to\infty} \sup_{\eta\geq 0} e^{\frac{\eta^2}{16}}\ \hat{w}(\eta,\tau_{n'}) - \widehat{W}_\infty(\eta)\ = 0 \tag{3.5.25}$$

for all $T > 0$. Moreover, $\hat{w}_\infty$ solves

$$\begin{cases} (\hat{w}_\infty)_\tau + \mathcal{L}\hat{w}_\infty = 0, & \eta \geq 0,\ \tau > 0, \\ \hat{w}_\infty(0,\tau) = 0, & \tau > 0, \\ \hat{w}_\infty(\eta,0) = \widehat{W}_\infty(\eta), & \eta \geq 0. \end{cases}$$

Then,

$$\lim_{\tau\to\infty} \sup_{\eta\geq 0} e^{\frac{\eta^2}{16}}\ \hat{w}_\infty(\eta,\tau) - \alpha_\infty\phi_0(\eta)\ = 0, \tag{3.5.26}$$

where $\alpha_\infty := \int_0^\infty \eta \widehat{W}_\infty(\eta)\,\mathrm{d}\eta$, (we refer to [114] for the full description of solutions for the Fokker-Planck equation). In addition, by (3.5.23), we have that $\alpha_\infty > 0$ and by (3.5.20), α_∞ is finite.

To prove (3.5.24) we need to show that the convergence does not depend on the choice of subsequences.

By (3.5.25), for a given $\varepsilon > 0$, there is T_ε big enough such that

$$\hat{w}_\infty(\eta, T_\varepsilon) - \alpha_\infty \phi_0(\eta) \ \leq \varepsilon e^{-\frac{\eta^2}{16}} \quad \text{for all } \eta \geq 0$$

and, by the first limit in (3.5.26), there is $n'_\varepsilon \in \mathbb{N}$ (that depends also in T_ε) such that

$$\widehat{w}(\eta, T_\varepsilon + \tau_{n'_\varepsilon}) - \hat{w}_\infty(\eta, T_\varepsilon) \ \leq \varepsilon e^{-\frac{\eta^2}{16}} \quad \text{for all } \eta \geq 0.$$

Therefore, putting all together,

$$\widehat{w}(\eta, T_\varepsilon + \tau_{n'_\varepsilon}) - \alpha_\infty \phi_0(\eta) \ \leq 2\varepsilon e^{-\frac{\eta^2}{16}} \quad \text{for all } \eta \geq 0. \tag{3.5.27}$$

Now, we define

$$z(\eta, \tau) := \hat{w}(\eta + \eta_\delta^+, \tau + T_\varepsilon + \tau_{n'_\varepsilon}) - \hat{w}(\eta_\delta^-, \tau + T_\varepsilon + \tau_{n'_\varepsilon})\varphi(\eta), \tag{3.5.28}$$

where φ is a smooth function such that $\mathcal{X}_{[0,1]} \leq \varphi \leq \mathcal{X}_{[0,2]}$.

Let $\omega \in (0, \frac{1}{2} - \delta)$. Since by Proposition 3.5.4, $\hat{w}$ is bounded and since f is a KPP function, by Taylor expansion, there exists a constant $C > 0$ such that $\hat{F}$ given by (3.5.9) satisfies

$$|\hat{F}(\eta + \eta_\delta^+, \tau, \hat{w})| \leq C \exp\left(\tfrac{3}{2}\tau - \lambda_0(\eta + \eta_\delta^+)e^{\frac{\tau}{2}}\right)\hat{w}^2 \leq C \exp\left(\tfrac{3}{2}\tau - \lambda_0 e^{\delta\tau}\right)\hat{w}^2 \leq Ce^{-\omega\tau}|\hat{w}| \tag{3.5.29}$$

for all $\eta \geq 0$ and $\tau > 0$. Furthermore, since $\hat{h}^i$ is given by (3.5.8), $\lambda_0 k = 3/2$ (recall (3.5.15)) and (3.5.16), we have that

$$e^{-\tau/2}|\hat{h}^i(\eta, \tau)| + |\lambda_0 \hat{h}^i(\eta, \tau) + \tfrac{3}{2}| \leq Ce^{-\omega\tau} \quad \text{for all } \eta \geq 0,\ \tau > 0. \tag{3.5.30}$$

Recall we have defined before $\tilde{w}(\eta, \tau) = \hat{w}(\eta + \eta_\delta, \tau)$ (here $\eta_\delta = \eta_\delta^+(\tau)$). Since $\tilde{w}$ solves (3.5.13), by (3.5.20), $\hat{w}(\eta_\delta^+, \eta) \leq Ce^{-(\frac{1}{2}-\delta)\tau}$ for some $C > 0$ and using (3.5.29) and (3.5.30), we have that z satisfies

$$\begin{cases} z_t + \mathcal{L}z \ \leq \varepsilon e^{-\omega\tau}\big(|z_\eta| + |z| + G(\eta, \tau)\big), & \eta \geq \tau > 0, \\ z(0, \tau) = 0 & \tau > 0, \end{cases}$$

for some $G(\eta, \tau) : [0, \infty) \times \mathbb{R}_+ \to \mathbb{R}$ smooth, bounded and compactly supported in η and where $\tilde{\varepsilon} = o(T_\varepsilon)$ as $T_\varepsilon \to \infty$. Moreover, by (3.5.27), $z(\tau, 0) \in L^2(\mathbb{R}_+, e^{\frac{\eta^2}{6}}\mathrm{d}\eta)$. Then, by [84, Lemma 5.2], there exists $Z_\infty : [0, \infty) \to \mathbb{R}$ such that

$$(\alpha_\infty - C\tilde{\varepsilon})\phi_0 \leq Z_\infty \leq (\alpha_\infty + C\tilde{\varepsilon})\phi_0, \qquad \lim_{\tau\to\infty} \sup_{\eta \geq 0} e^{\frac{\eta^2}{16}}|z(\eta, \tau) - Z_\infty(\eta)| = 0.$$

As a consequence, since $\tilde{\varepsilon}$ is arbitrary small and since $\eta_\delta^+(\tau) \to 0$ and by (3.5.20), $\hat{w}(\eta_\delta^-, \tau + T_\varepsilon + \tau_{n'_\varepsilon}) \to 0$ as $\tau \to \infty$ using the definition of z from (3.5.28), the above implies (3.5.24), as desired. □

As a corollary, following the proof of [100, Proposition 7] we obtain sharp bounds for $\hat{w}$.

Proposition 3.5.6. *[100, Proposition 7] Consider the assumptions of Proposition 3.5.4. Let $\alpha_\infty > 0$ be the constant given by Proposition 3.5.5. Then, for every $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ (possibly depending also on δ) and $\tilde{\eta}_\varepsilon > 1$ such that*

$$(\alpha_\infty - C\varepsilon)(\eta - Ce^{-(\frac{1}{2}-\delta)(\tau-\tau_\varepsilon)}) \leq \hat{w}(\eta,\tau) \leq (\alpha_\infty + C\varepsilon)(\eta + Ce^{-(\frac{1}{2}-\delta)(\tau-\tau_\varepsilon)}), \quad \tau \geq \tau_\varepsilon,\ \eta \in [\eta_\delta^-, \tilde{\eta}_\varepsilon].$$

We are now prepared to capture the dynamics of u for all $\rho \geq 0$. In particular, we will consider three different regions: the diffusive region (where $\rho \asymp R(t) + o(\sqrt{t})$), the subdiffusive ($0 \leq \rho \leq R(t) + o(\sqrt{t})$) and superdiffusive ($\rho \geq R(t) + o(\sqrt{t})$). We will prove that u converges uniformly to 1 in the subdiffusive region and to 0 in the superdiffusive region (as well as $\Phi_{c_0}(\cdot - R(t))$).

Furthermore, we will establish that u converges to $\Phi_{c_0}(\cdot - R(t) + \beta))$, where $\beta \in \mathbb{R}$ is related with the constant $\alpha_\infty > 0$ provided in Proposition 3.5.5. We will employ the techniques developed in [56, 84, 100].

The following result captures the behavior of u in the superdiffusive region.

Theorem 3.5.7. *Consider $f'(0) > \lambda_1$ and u_0 satisfying (A2) with the corresponding i. Then, there is $C > 0$ for which the solution u to (KPP) with initial datum u_0 satisfies*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\ \rho_i(x) > c_* t - \frac{2}{c_0}\log t + g(t)\}} u(x,t) = 0 \tag{3.5.31}$$

for all $g : [0,\infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} g(t) = \infty$.

Proof. By point *1)* in Proposition 3.5.4 and recalling that the solution u to (KPP) is rewritten as $u(\rho + R(t), t) = e^{-\lambda_0\rho} t^{1/2} \hat{w}(\rho t^{-1/2}, \ln t)$, we have that, for each $\delta \in (0, 1/4)$ there is $C > 0$ such that

$$u(\rho + R(t), t) \leq Ce^{-\lambda_0\rho}\Big((\rho + t^\delta)e^{-\frac{(\rho+t^\delta)^2}{4t}} + t^\delta\big(1 + e^{-\frac{\rho^2}{16t}}\big)e^{-\frac{\rho^2}{8t}}\Big) \quad \text{for all } \rho \geq 0 \text{ and } t > 0.$$

Then, since $\delta \in (0, 1/4)$ and $0 \leq se^{-s^2} \leq 1$ for all $s \geq 0$, we have that for all $t_1 > 0$ there is $C > 0$ such that

$$u(\rho + R(t), t) \leq Ct^{1/2}e^{-\lambda_0\rho} \quad \text{for all } \rho \geq 0 \text{ and } t \geq t_1.$$

Therefore, since $c_0 = 2\lambda_0$, for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \sup_{\{\rho \geq (\frac{1}{c_0} + \varepsilon)\ln t\}} u(\rho + R(t), t) = 0.$$

Using that $R(t) = c_* t - k \ln t + \rho_0$ with $k = 3/c_0$ (see (3.5.15)), we conclude (3.5.31). □

Now, we will prove that u converges to 1 in the subdiffusive region.

Theorem 3.5.8. *Consider $f'(0) > \lambda_1$ and u_0 satisfying (A2) with the corresponding i. Then, the solution u to the (KPP) problem with initial datum u_0 satisfies that for all $g : [0,\infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} g(t) = \infty$, we have*

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d,\ 0\le\rho_i(x)\le c_* t - \frac{3}{c_0}\ln t - g(t)\}} u(x,t) = 1. \tag{3.5.32}$$

Proof. Step 1. Fix $m \in (0,1)$. Let $f_1 : [0,1] \to \mathbb{R}$ be a C^1 function such that

$$f_1(0) = f_1(m) = 0, \quad f_1'(0) = f'(0), \qquad f_1 \le f \quad \text{in } [0,m], \quad f_1 > 0 \quad \text{in } (0,m). \tag{3.5.33}$$

Then, following arguments of [7, Theorem 4.1], there exists $\tilde\Phi_{c_0} : \mathbb{R} \to (0,m)$ such that

$$\begin{gathered} \tilde\Phi_{c_0}'' + c_0\tilde\Phi_{c_0}' + f_1(\tilde\Phi_{c_0}) = 0 \quad \text{in } \mathbb{R}, \\ \lim_{s\to-\infty} \tilde\Phi_{c_0}(s) = m, \quad \tilde\Phi_{c_0}' < 0, \quad \lim_{s\to+\infty} \tilde\Phi_{c_0}(s) = 0, \end{gathered} \tag{3.5.34}$$

and for some $B > 1$ and $\rho_1 > 0$

$$B^{-1} s e^{-\lambda_0 s} \le \tilde\Phi_{c_0}(s) \le B s e^{-\lambda_0 s} \quad \text{for all } s \ge \rho_1. \tag{3.5.35}$$

Our aim is to prove that

$$\underline{u}(\rho,t) = \tilde\Phi_{c_0}(\rho - R(t) + \rho_3)$$

is a subsolution for u in $\{(\rho,t) :\ \rho \in A_i,\ t \ge T\}$ where $\rho_3, T > 0$ will be chosen later and

$$A_i(t) = \begin{cases} [0, c_* t + t^{1/2}], & \text{if } i \in \{\mathfrak{h},\mathfrak{p}\}, \\ [\frac{c_*}{2} t, c_* t + t^{1/2}], & \text{if } i = \mathfrak{e}. \end{cases}$$

On the one hand, since $\tilde\Phi_{c_0}$ satisfies (3.5.34), then, using (3.5.33) and the fact that $0 \le \underline{u} \le m$,

$$\begin{aligned} \underline{u}_t - \underline{u}_{\rho\rho} - (d-1)g_1^i \underline{u}_\rho - f(\underline{u}) &= f_1(\underline{u}) - f(\underline{u}) + \big(\frac{k}{t} + (d-1)(1 - g_1^i(\rho))\big)\underline{u}_\rho \\ &\le \big(\frac{k}{t} + (d-1)(1-g_1^i)\big)\underline{u}_\rho. \end{aligned}$$

Moreover, $1 - g_1^i \ge 0$ for $i \in \{\mathfrak{h},\mathfrak{p}\}$ and since $\tilde\Phi_{c_0}' < 0$ implies $u_\rho < 0$, we have that

$$\underline{u}_t - \underline{u}_{\rho\rho} - (d-1)g_1^i(\rho)\underline{u}_\rho - f(\underline{u}) \le 0 \quad \text{in } \{(\rho,t) : \rho \in A_i,\ t > 0\} \tag{3.5.36}$$

for $i \in \{\mathfrak{h},\mathfrak{p}\}$. For $i = \mathfrak{e}$, we have $g_{\mathfrak{e}}^i(\rho) = \coth\rho$. Then,

$$1 - \coth\rho = -\frac{2}{e^{2\rho} - 1}, \quad \rho > 0,$$

which implies that there exists $t_1 > 0$ such that

$$\inf_{\rho\in A_{\mathfrak{e}}(t)}\left(\frac{k}{t}+(d-1)(1-g_1^i)\right)=\frac{k}{t}-\frac{2(d-1)}{e^{c_*t}-1}>0 \quad \text{for all } t\geq t_1.$$

Thus,

$$\underline{u}_t-\underline{u}_{\rho\rho}-(d-1)\coth\rho\,\underline{u}_\rho-f(\underline{u})\leq 0 \quad \text{in } \{(\rho,t):\rho\in A_{\mathfrak{e}},\ t\geq t_1\}. \tag{3.5.37}$$

On the other hand, by [75, Theorem 3.2 ii)] we know that there exists $t_2 \geq t_1$ such that

$$u(0,t)\geq m \quad \text{for all } t\geq t_2 \ \text{ for } i\in\{\mathfrak{h},\mathfrak{p}\},$$

and by [75, Theorem 3.6, ii)] it holds that

$$\inf_{\{0\leq\rho\leq\frac{c_*}{2}t\}} u(\rho,t)\geq m \quad \text{for all } t\geq t_2 \ \text{ for } i=\mathfrak{e}.$$

Then, since $\tilde{\Phi}_{c_0}\leq m$, for all $t\geq t_2$, $u(0,t)\geq\underline{u}(0,t)$ if $i\in\{\mathfrak{h},\mathfrak{p}\}$ and $u(\frac{c_*}{2}t,t)\geq\underline{u}(\frac{c_*}{2}t,t)$ if $i=\mathfrak{e}$.

In addition, since $u(\rho+R(t),t)=e^{-\lambda_0\rho}t^{1/2}\hat{w}(\rho t^{-1/2},\ln t)$, by Proposition 3.5.6, we have that, for $\delta\in(0,1/4)$, there exists $t_3\geq t_2$ such that

$$u(\rho+R(t),t)\geq\frac{\alpha_\infty}{2}(\rho-Ct^\delta)e^{-\lambda_0\rho} \quad \text{for all } \rho\in[-t^\delta,at^{1/2}],\ \ t\geq t_3, \tag{3.5.38}$$

for some $a>1$ and where α_∞ and C denotes the same positive constants as in Proposition 3.5.6. Then, since there is $T\geq t_3$ such that $\rho=t^{1/2}+\frac{3}{c_0}\ln t-\rho_0\in[-t^\delta,at^{1/2}]$ for all $t\geq T$, using (3.5.38), we have that

$$u(c_*t+t^{1/2},t)\geq\frac{\alpha_\infty e^{\lambda_0\rho_0}e^{-\lambda_0t^{1/2}}}{2t^{-3/2}}\left(t^{1/2}+k\ln t-\rho_0-Ct^\delta\right)\geq\frac{\alpha_\infty e^{\lambda_0\rho_0}e^{-\lambda_0t^{1/2}}}{4t^{-1/2}}$$

for all $t\geq T$. This implies, together with (3.5.35) and (3.5.15), that for $\rho_3>0$ big enough,

$$u(c_*t+t^{1/2},t)\geq\underline{u}(c_*t+t^{1/2},t)=\tilde{\Phi}_{c_0}(t^{1/2}-k\ln t+\rho_3) \quad \text{for all } t\geq T.$$

Finally, since the solution of the heat equation (3.3.1) with initial datum u_0 is strictly positive and $f\geq 0$, by the comparison principle, Theorem 3.3.1, we have that

$$\min_{\{0\leq\rho\leq c_*T+T^{1/2}\}} u(\rho,T)>0.$$

Moreover, since $\lim_{s\to\infty}\tilde{\Phi}_{c_0}(s)=0$, it holds that, for $\rho_3>0$ big enough,

$$u(\rho,t)\geq\min_{\{0\leq\rho\leq c_*T+T^{1/2}\}} u(\rho,T)\geq\tilde{\Phi}_{c_0}(-R(T)+\rho_3)\geq\tilde{\Phi}_{c_0}(\rho-R(T)+\rho_3)=\underline{u}(\rho,t)$$

for all $\rho\in[0,c_*T+T^{1/2}]$.

Summarizing, $\underline{u} \leq u$ in the boundary $\partial\{(\rho,t) : \rho \in A_i, t \geq T\}$. Therefore, taking into account (3.5.36) and (3.5.37), by the comparison principle, $\underline{u} \leq u$ in the full region $\{(\rho,t) : \rho \in A_i, t \geq T\}$.

STEP 2. Now we prove (3.5.32). Let g be such that $\lim_{s\to\infty} g(t) = \infty$. If $i \in \{\mathfrak{h}, \mathfrak{p}\}$, by Step 1,

$$\lim_{t\to\infty} \inf_{\{0\leq\rho\leq R(t)-g(t)\}} u(\rho,t) \geq \lim_{t\to\infty} \inf_{\{0\leq\rho\leq R(t)-g(t)\}} \tilde{\Phi}_{c_0}(\rho - R(t) + \rho_3) = \lim_{t\to\infty} \tilde{\Phi}_{c_0}(-g(t)+\rho_3) = m.$$

Since $u \in [0,1]$ and $m \in (0,1)$ arbitrary, we obtain (3.5.32) for $i \in \{\mathfrak{h}, \mathfrak{p}\}$.

As for $i = \mathfrak{e}$, by [75, Theorem 3.6, ii)], we know that

$$\lim_{t\to\infty} \inf_{\{0\leq\rho\leq \frac{c_*}{2}t\}} u(\rho,t) = 1.$$

Then, since $u \in [0,1]$, using Step 1 and proceeding as before,

$$\lim_{t\to\infty} \inf_{\{0\leq\rho\leq R(t)-g(t)\}} u(\rho,t) = \lim_{t\to\infty} \inf_{\{\frac{c_*}{2}t\leq\rho\leq R(t)-g(t)\}} u(\rho,t) \geq \lim_{t\to\infty} \tilde{\Phi}_{c_0}(-g(t)+\rho_3) = m.$$

Once more, since the above holds for all $m \in (0,1)$, we get (3.5.32). □

The study of the diffusive region will be a crucial step in the proof of our main Theorem 3.5.1. Note that this region is where $|\rho| \leq o(\sqrt{t})$ in moving coordinates.

Before we do that, let us give a technical lemma to show that the error between a shift of the traveling wave of minimal speed and the solution vanishes as t goes to infinity.

Lemma 3.5.9. *Let $\gamma \in (0,1/2)$ and $s : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ be a locally bounded solution to*

$$\begin{cases} s_t - s_{\rho\rho} - \left(h^i - \frac{k}{t}\right)(s_\rho - \lambda_0 s) = O(t^{-(1-2\gamma)}), & |\rho| \leq t^\gamma,\ t > 0, \\ s(-t^\gamma, t) = O(e^{-\lambda_0 t^\gamma}), \quad s(t^\gamma, t) = 0, & t > 0. \end{cases}$$

Then, $\lim_{t\to\infty} \sup_{|\rho|\leq t^\gamma} |s(\rho,t)| = 0.$

Proof. Following ideas from the proof of [84, Proposition 4], we consider a supersolution to (3.5.9) of the form

$$\bar{s}(\rho,t) = t^{-\lambda}\cos\left(\frac{\rho}{t^\gamma}\right) > 0, \quad |\rho| \leq t^\gamma,\ t > 0, \tag{3.5.39}$$

where $\lambda > 0$. It holds that

$$-\bar{s}_{\rho\rho}(\rho,t) = t^{-2\gamma}\bar{s}(\rho,t), \quad \bar{s}_t(\rho,t) \geq -(\lambda+1)t^{-1}\bar{s}(\rho,t) \quad \text{for all } |\rho| \leq t^\gamma,\ t \geq 1,$$

and, using (3.5.5), (3.5.2) and (3.5.39), there exists $C > 0$ such that

$$(h^i(\rho,t) - \tfrac{k}{t})(\bar{s}_\rho - \lambda_0\bar{s}) \ \leq Ct^{-1}\bar{s} \quad \text{for all } |\rho| \leq t^\gamma,\ t \geq 1.$$

Therefore, if $\gamma \in (0, 1/2)$, there exists $t_0 > 1$ such that

$$\bar{s}_t - \bar{s}_{\rho\rho} - \big(h^i - \tfrac{k}{t}\big)(\bar{s}_\rho - \lambda_0 \bar{s}) \geq \big(t^{-2\gamma} - (\lambda + 1 + C)t^{-1}\big)\bar{s} \geq Ct^{-1}\bar{s} \quad \text{for all } |\rho| \leq t^\gamma,\ t \geq t_0.$$

That is,

$$\begin{cases} \bar{s}_t - \bar{s}_{\rho\rho} - \big(h^i - \frac{k}{t}\big)(\bar{s}_\rho - \lambda_0 \bar{s}) \geq Ct^{-(1+\lambda)}, & |\rho| \leq t^\gamma,\ t > t_0, \\ \bar{s}(-t^\gamma, t) = \bar{s}(t^\gamma, t) = t^{-\lambda}\cos(1), & t > t_0, \\ \bar{s}(\rho, t_0) \geq t_0^{-\lambda}\cos(1), & |\rho| \leq t_0^\gamma. \end{cases}$$

Since s is a locally bounded function, there exists $C > 0$ such that $|s(\rho, t_0)| \leq C\bar{s}(\rho, t_0)$ for all $|\rho| \leq t_0^\gamma$. Then, by the linearity of the equation, the comparison principle yields

$$-C\bar{s}(\rho, t) \leq s(\rho, t) \leq C\bar{s}(\rho, t) \quad \text{for all } |\rho| \leq t^\gamma,\ t > t_0.$$

Therefore, $\lim\limits_{t\to\infty} \sup\limits_{|\rho|\leq t^\gamma} |s(\rho, t)| \leq \lim\limits_{t\to\infty} \sup\limits_{|\rho|\leq t^\gamma} C\bar{s}(\rho, t) = 0$, as desired. □

At this point, we have all the ingredients to prove our main theorem.

Proof of Theorem 3.5.1. STEP 1. Here we capture the behavior of u in the diffusive area. For this, we estimate the difference between u and an appropriate shift to $\Phi_{c_0}(\cdot - R(t))$ in moving coordinates. The key idea here is to prepare u and Φ_{c_0} in order to use Lemma 3.5.9.

Consider the family of functions

$$\psi^\pm_{\varepsilon,T,\delta}(\rho) = (\alpha_\infty \pm C\varepsilon)(\rho \pm T^{\frac{1-\delta}{2}}\sqrt{\rho}), \quad \rho \in \mathbb{R}.$$

Fix $\delta_0 > 0$ small enough, for example $0 < \delta_0 < 1/100$. By Proposition 3.5.6, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there is $t_\varepsilon > 0$ such that the function v given by (3.5.4) satisfies

$$0 \leq \psi^-_{\varepsilon,t_\varepsilon,\delta_0}(t^\delta) \leq v(t^{\delta_0}, t) \leq \psi^+_{\varepsilon,t_\varepsilon,\delta_0}(t^{\delta_0}) \quad \text{for all } t \geq t_\varepsilon.$$

On the one hand, let $\xi^\pm_\varepsilon$ two functions such that

$$\Phi_{c_0}\big(t^\delta + \xi^\pm_{\varepsilon,\delta_0}(t)\big) = e^{-\lambda_0 t^{\delta_0}}\psi^\pm_{\varepsilon,t_\varepsilon,\delta_0}(t^\delta) \quad \text{for all } t \geq t_\varepsilon.$$

Then, by the implicit function theorem, we have that

$$\xi^\pm_{\varepsilon,\delta_0}(t) = -\ln(\lambda_\infty \pm C\varepsilon) + O(t^{-\delta_0}), \quad (\xi^\pm_\varepsilon)'(t) = O(t^{-(1+\delta_0)}). \tag{3.5.40}$$

On the other hand, consider $v^\pm_{\varepsilon,\delta_0}$ the solution to

$$\begin{cases} z_t = z_{\rho\rho} + \big(h^i - \frac{k}{t}\big)(z_\rho - \lambda_0 z) + e^{\lambda_0\rho} f(e^{-\lambda_0\rho} z) - f'(0)z, & \rho \in (-t^{\delta_0}, t^{\delta_0}),\ t > t_\varepsilon, \\ z(\rho, t_\varepsilon) = v(\rho, t_\varepsilon), & \rho \in (-t_\varepsilon^{\delta_0}, t_\varepsilon^{\delta_0}), \end{cases}$$

with Dirichlet conditions

$$v^\pm_\varepsilon(t, t^{\delta_0}) = \psi^\pm_{\varepsilon,t_\varepsilon,\delta_0}(t^{\delta_0}), \quad v^+_{\varepsilon,\delta_0}(t, -t^{\delta_0}) = e^{-\lambda_0 t^{\delta_0}}, \quad v^-_{\varepsilon,\delta_0}(t, -t^{\delta_0}) = 0 \quad \text{for all } t \geq t_\varepsilon.$$

Since $0 \le u \le 1$ and $v(\rho,t) = e^{\lambda_0\rho}u(\rho + R(t),t)$, we have, by the comparison principle,

$$v^-_{\varepsilon,\delta_0}(\rho,t) \le v(\rho,t) \le v^+_{\varepsilon,\delta_0}(\rho,t) \quad \text{for all } t \ge t_\varepsilon \text{ and } \rho \in (-t^\delta, t^\delta). \tag{3.5.41}$$

We claim that, to be proven later,

$$\lim_{t\to\infty} \sup_{|\rho|\le t^{\delta_0}} |e^{-\lambda_0\rho}v^\pm_{\varepsilon,\delta_0}(\rho,t) - \Phi_{c_0}\big(\rho + \xi^\pm_{\varepsilon,\delta_0}(t)\big)| = 0. \tag{3.5.42}$$

Therefore, as a consequence of the above, since Φ_{c_0} is continuous and ε is arbitrary small, using (3.5.40) and (3.5.41), we obtain that

$$\lim_{t\to\infty} \sup_{|\rho|\le t^{\delta_0}} |e^{-\lambda_0\rho}v(\rho,t) - \Phi_{c_0}(\rho - \ln\lambda_\infty)| = 0. \tag{3.5.43}$$

Thus, by the definition to v, (see (3.5.3) and (3.5.4)), we have

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\, R(t)-t^{\delta_0}\le\rho_i(x)\le R(t)+t^{\delta_0}\}} u\left(\rho_i(x),t\right) - \Phi_{c_0}(\rho - [c_* t - \tfrac{3}{c_0}\ln t + \beta]\Big) \;= 0, \tag{3.5.44}$$

where $\beta := \rho_0 + \ln\alpha_\infty$.

Step 2. We now prove claim (3.5.43). Let

$$V^\pm_{\varepsilon,\delta_0}(\rho,t) = v^\pm_{\varepsilon,\delta}(\rho,t) - e^{\lambda_0\rho}\Phi(\rho + \xi^\pm_{\varepsilon,\delta_0}(t)).$$

Then $V^\pm_{\varepsilon,\delta_0}(\rho,t)$ is a solution to

$$\begin{aligned} z_t - z_{\rho\rho} - \big(h^i - \tfrac{k}{t}\big)(z_\rho - \lambda_0 z) &= e^{\lambda_0\rho}f(e^{-\lambda_0\rho}v^\pm_{\varepsilon,\delta_0}) - f'(0)v^\pm_{\varepsilon,\delta}(\rho,t) \\ &\quad + e^{\lambda_0\rho}\big(f(\Phi_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t))) - f'(0)\Phi_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t))\big) \\ &\quad + \big((\xi^\pm_{\varepsilon,\delta_0})' - h^i + \tfrac{k}{t}\big)e^{\lambda_0\rho}\Phi'_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t)) \end{aligned}$$

for all $\rho \in (-t^{\delta_0}, t^{\delta_0})$ and $t > t_\varepsilon$, with Dirichlet conditions $V^\pm_{\varepsilon,\delta_0}(t^{\delta_0},t) = 0$,

$$\begin{aligned} &V^+_{\varepsilon,\delta_0}(-t^{\delta_0},t) = e^{-\lambda_0 t^{\delta_0}}(1 - \Phi_{c_0}(-t^{\delta_0} + \xi^+_{\varepsilon,\delta_0}(-t^{\delta_0}))) && \\ &V^-_{\varepsilon,\delta_0}(-t^{\delta_0},t) = e^{-\lambda_0 t^{\delta_0}}\Phi_{c_0}(-t^{\delta_0} + \xi^-_{\varepsilon,\delta_0}(-t^{\delta_0})) && \text{for all } t \ge t_\varepsilon, \\ &V^\pm_{\varepsilon,\delta_0}(\rho,t_\varepsilon) = v^\pm_{\varepsilon,\delta}(\rho,t_\varepsilon) - e^{\lambda_0\rho}\Phi_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t_\varepsilon)) && \text{for all } |\rho| \le t^{\delta_0}. \end{aligned}$$

Observe that

$$e^{\lambda_0\rho}f(e^{-\lambda_0\rho}v^\pm_{\varepsilon,\delta_0}) - f'(0)v^\pm_{\varepsilon,\delta}(\rho,t) + e^{\lambda_0\rho}\big(f(\Phi_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t))) - f'(0)\Phi_{c_0}(\rho + \xi^\pm_{\varepsilon,\delta_0}(t))\big) \le 0, \tag{3.5.45}$$

since f is a KPP function, see (H_f). Furthermore,

$$(\xi^\pm_{\varepsilon,\delta_0})' - h^i + \frac{k}{t} = O\Big(\frac{1}{t}\Big). \tag{3.5.46}$$

Moreover, we claim that there exists $C > 0$ with

$$|e^{\lambda_0\rho}\Phi_{c_0}'(\rho)| \leq C(1+\rho^2\mathcal{X}_{\{\rho>0\}}(\rho)) \quad \text{for all } \rho \in \mathbb{R}. \tag{3.5.47}$$

To see this, let w be the function given by $w(\rho) = e^{\lambda_0\rho}\Phi_{c_0}'(\rho)$, $\rho \in \mathbb{R}$. Then, w satisfies that

$$w_\rho + \lambda_0 w = -e^{\lambda_0\rho} f(\Phi_{c_0}(\rho)), \quad \rho \in \mathbb{R}.$$

Therefore, since Φ_{c_0} is decreasing, for a constant $C > 0$ we have

$$w(\rho) = -Ce^{-\lambda_0\rho} - \fint_0^\rho e^{\lambda_0(2s-\rho)} f(\Phi_{c_0}(s))\,\mathrm{d}s.$$

Thus, using again that f is a KPP function and the fact that $0 \leq \Phi_{c_0} \leq 1$ has the decay (3.1.7), we get (3.5.47).

Then, the bound (3.5.47), together with (3.5.40) and (3.5.46), implies that

$$\sup_{|\rho|\leq t^{\delta_0}} \left((\xi^\pm_{\varepsilon,\delta_0})' - h^i + \frac{k}{t} \right) e^{\lambda_0\rho}\Phi_{c_0}'(\rho + \xi^\pm_{\varepsilon,\delta_0}(t)) \Big) = O(t^{-(1-2\delta_0)}). \tag{3.5.48}$$

As a consequence, using Kato's inequality $|w|_{\rho\rho} \leq \mathrm{sign}(w) w_{\rho\rho}$, by (3.5.45) and (3.5.48), there is $C > 0$ such that $|V^\pm_{\varepsilon,\delta_0}|$ is a subsolution of

$$\begin{cases} z_t - z_{\rho\rho} - \left(h^i - \frac{k}{t}\right)(z_\rho - \lambda_0 z) = Ct^{-(1-2\delta_0)}, & |\rho| \leq t^{\delta_0},\ t > 0, \\ s(-t^{\delta_0}, t) = O(e^{-\lambda_0 t^\gamma}), \quad s(t^{\delta_0}, t) = 0, & t > 0. \end{cases}$$

Thus, taking $\delta_0 \in (0, 1/2)$, since $V^\pm_{\varepsilon,\delta_0}$ is locally bounded, by Lemma 3.5.9, and the comparison principle we obtain (3.5.42).

Step 3. In this final step, we will prove that u converges to $\Phi_{c_0}(\cdot - [c_* t - \frac{3}{c_0}\ln t + \beta])$ in the sub and superdiffusive areas.

Since Φ_{c_0} is a decreasing continuous function, $\lim_{s\to-\infty} \Phi_{c_0}(s) = 1$ and $\lim_{s\to\infty} \Phi_{c_0}(s) = 0$, we get

$$\lim_{t\to\infty} \left\| 1 - \Phi_{c_0}(\cdot - [c_* t - \tfrac{3}{c_0}\ln t + \beta]) \right\|_{L^\infty(0, R(t)-t^{\delta_0})} = \lim_{t\to\infty} \left(1 - \Phi_{c_0}(-t^{\delta_0} - \ln\alpha_\infty)\right) = 0,$$
$$\lim_{t\to\infty} \left\| \Phi_{c_0}(\cdot - [c_* t - \tfrac{3}{c_0}\ln t + \beta]) \right\|_{L^\infty(R(t)+t^{\delta_0}, \infty)} = \lim_{t\to\infty} \Phi_{c_0}(t^{\delta_0} - \ln\alpha_\infty) = 0.$$

Moreover, by Theorems 3.5.7 and 3.5.8, we have that

$$\lim_{t\to\infty} \inf_{\{x\in\mathbb{H}^d:\, 0\leq\rho_i(x)\leq R(t)-t^{\delta_0}\}} u(x,t) = 1, \quad \lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\, \rho_i(x)\geq R(t)+t^{\delta_0}\}} u(x,t) = 0.$$

As a consequence,

$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\, 0\leq\rho_i(x)\leq R(t)-t^{\delta_0}\}} u(\rho_i(x),t) - \Phi_{c_0}\left(\rho - [c_* t - \tfrac{3}{c_0}\ln t + \beta]\right) = 0,$$
$$\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d:\, \rho_i(x)\geq R(t)+t^{\delta_0}\}} u\left(\rho_i(x),t\right) - \Phi_{c_0}(\rho - [c_* t - \tfrac{3}{c_0}\ln t + \beta]) = 0.$$

Combining this with (3.5.44) we obtain (3.5.1), and this completes the proof of our main theorem. □

Future directions. As we have mentioned in the Introduction, we would like to allow more general initial data, not necessarily unidimensional. The following lemma is a first step toward achieving this goal.

Lemma 3.5.10. *Let $f'(0) > \lambda_1$. Consider $u_0 : \mathbb{H}^d \to [0,1]$ satisfying (3.1.17) for some $0 < \rho_1 < \rho_2$ and some i belonging to $\{\mathfrak{e}, \mathfrak{h}, \mathfrak{p}\}$. Then, there exist $\beta_1 \leq \beta_2$ such that for all $\varepsilon > 0$ one may find $t_\varepsilon > 0$ such that*

$$-\varepsilon + \Phi_{c_0}\big(\rho_i(x) - [c_* t - \tfrac{3}{c_0}\log t + \beta_1]\big) \leq u(x,t) \leq \Phi_{c_0}\big(\rho_i(x) - [c_* t - \tfrac{3}{c_0}\log t + \beta_2]\big) + \varepsilon \quad (3.5.49)$$

for all $x \in \mathbb{H}^d$ such that $\rho_i(x) \geq 0$ and $t \geq t_\varepsilon$.

Proof. Consider $\underline{u}, \overline{u} : \mathbb{H}^d \to [0,1]$ and $\underline{\tilde{u}}, \tilde{\overline{u}} : I_i \to [0,1]$ such that

$$\begin{gathered}
\underline{u}_0(x) \leq u_0(x) \leq \overline{u}_0(x), \quad x \in \mathbb{H}^d, \\
\underline{u}_0(x) = \underline{\tilde{u}}_0(\rho_i(x)), \quad \overline{u}_0(x) = \tilde{\overline{u}}_0(\rho_i(x)), \\
\mathcal{X}_{[0,\rho_1/2]}(\rho) \leq \underline{\tilde{u}}_0(\rho) \leq \mathcal{X}_{[0,\rho_1]}(\rho), \quad \mathcal{X}_{[0,\rho_2]}(\rho) \leq \tilde{\overline{u}}_0(\rho) \leq \mathcal{X}_{[0,2\rho_2]}(\rho) \quad \text{for all } \rho \geq 0.
\end{gathered}$$

Let $\underline{u}$, u and $\overline{u}$ the solution of the (KPP) problem with initial datum $\underline{u}_0$, u_0 or $\overline{u}_0$ respectively. Then, by the comparison principle (Theorem 3.3.1), we have that

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) \quad \text{for all } (x,t) \in \mathbb{H}^d \times \mathbb{R}_+. \quad (3.5.50)$$

On the other had, by Theorem 3.5.1, there exists $\beta_1 \leq \beta_2$ such that

$$\begin{aligned}
\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d : \rho(x)\geq 0\}} \underline{u}\Big(\rho(x),t\Big) - \Phi_{c_0}\big(\rho(x) - [(c_0 - d + 1)t - \tfrac{3}{c_0}\log t + \beta_1]\big) &= 0, \\
\lim_{t\to\infty} \sup_{\{x\in\mathbb{H}^d : \rho(x)\geq 0\}} \overline{u}\Big(\rho(x),t\Big) - \Phi_{c_0}\big(\rho(x) - [(c_0 - d + 1)t - \tfrac{3}{c_0}\log t + \beta_2]\big) &= 0.
\end{aligned}$$

Combining this with (5.3.4), we obtain (3.5.49). □

Chapter 4

A game theoretical approach to the convex hull of a set via minimal curvature flow

This chapter is dedicated to the results of [51], named *Finding the convex hull of a set using the flow by minimal curvature with an obstacle. A game theoretical approach.* This paper is authored by Alfredo Miranda (UBA), Julio D. Rossi (UBA), Jorge Ruiz-Cases (UAM) and the author of this thesis.

4.1 Introduction

4.1.1 Description of the main goal

The aim of this chapter is to study the asymptotic behavior of the solution to the obstacle problem for a degenerate fully nonlinear parabolic equation that models the motion by minimal curvature of a hypersurface. We prove (under certain conditions on the initial condition and the obstacle) that the positivity set of this solution converges to the convex hull of the set where the obstacle is positive as t goes to infinity.

To obtain our results we use a deterministic two-person zero-sum game (inspired by previous results by [63] and [82]) and the theory of viscosity solutions (we will rely on previous theory developed for viscosity solutions to geometric flows, see [36] and [45]).

Now, let us describe the main ingredients that we need to state and prove our results. First we introduce a parabolic nonlinear partial differential equation (PDE) related to the geometric motion by minimal curvature with an obstacle and next we describe an associated game whose value functions approximate solutions to the PDE.

The movement by minimal curvature with an obstacle and its associated parabolic equation

Our aim is to describe how a hypersurface that is the boundary of a connected domain, $S = \partial\Omega_0 \subset \mathbb{R}^d$, $d \geq 2$, evolves in time according to the minimal curvature flow. We will use a level set approach to describe this geometric evolution. Assume that there is a real valued function, $u(x,t)$, defined for $(x,t) \in \mathbb{R}^d \times [0,\infty)$, and consider the zero superlevel sets of $u(x,t)$,

$$\Omega_t = \{x : u(x,t) > 0\}.$$

In what follows we denote by $\nabla u(x,t)$ and by $D^2u(x,t)$ the gradient and the Hessian of u with respect to the spatial variable, x. Assume that $\partial\Omega_t$ is smooth. At a point $(x,t) \in \partial\Omega_t$ we have $\nabla u(x,t) \perp \partial\Omega_t$ (also assume that (x,t) is a regular point and that we have $|\nabla u|(x,t) = 1$) and for a unitary vector $v \perp \nabla u(x,t)$ (notice that v is tangential to the hypersurface $\partial\Omega_t$) the quantity $-\langle D^2u(x,t)v, v\rangle$ gives the curvature of $\partial\Omega_t$ in the direction of v. Therefore, under these conditions, the minimum of the curvatures of $\partial\Omega_t$ at a regular point is given by

$$\kappa_{min} = \min_{\substack{|v|=1 \\ v \text{ tangent to } \partial\Omega_t}} \Big\{\kappa(v)\Big\} = \inf_{\substack{|v|=1 \\ v\perp\nabla u(x,t)}} \Big(- \langle D^2u(x,t)v, v\rangle\Big).$$

Then, we consider the geometric evolution of the hypersurface $\partial\Omega_t$ moving its points in the direction of the normal vector (pointing inside the set Ω_t) with speed given by the minimal curvature,

$$V = -\kappa_{min} \qquad \text{on } \partial\Omega_t,$$

and we obtain, as the associated level set formulation for this geometric evolution, the parabolic equation

$$\partial_t u(x,t) + \mathcal{L}(u(x,t)) = 0, \tag{4.1.1}$$

associated with the elliptic and degenerate operator $\mathcal{L}$ given by

$$\mathcal{L}(f(x)) := \inf_{\substack{|v|=1 \\ v\perp\nabla f(x)}} \Big(- \langle D^2f(x)v, v\rangle\Big). \tag{4.1.2}$$

In this chapter we add to this geometric evolution the presence of an open and bounded obstacle K (the hypersurface is stopped when it touches K). We assume that our geometric evolution starts with the prescribed set inside, that is, we have $K \subset\subset \Omega_0$. To include the obstacle in the level set formulation, we consider a continuous function ψ such that

$$K = \{x : \psi(x) > 0\}$$

and we assume that the initial condition for our evolution satisfies

$$u_0(x) = u(x,0) \geq \psi(x)$$

in the whole $\mathbb{R}^d$. Notice that under this assumption we have

$$K = \{x : \psi(x) > 0\} \subset \{x : u_0(x) > 0\} = \Omega_0,$$

that is, the initial hypersurface $\partial\Omega_0$ surrounds the obstacle K. Therefore, we get that the parabolic problem that corresponds to the level set formulation for the movement by minimum curvature with an obstacle is the following:

$$\begin{cases} u(x,t) \geq \psi(x), & (x,t) \in \mathbb{R}^d \times \{t > 0\}, \\ \partial_t u(x,t) + \mathcal{L}(u(x,t)) \geq 0, & (x,t) \in \mathbb{R}^d \times \{t > 0\}, \\ \partial_t u(x,t) + \mathcal{L}(u(x,t)) = 0, & (x,t) \in \{(x,t) : u(x,t) > \psi(x)\}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{P}$$

This problem can also be written as

$$\begin{cases} \max\Big\{ -\partial_t u(x,t) - \mathcal{L}(u(x,t)), \psi(x) - u(x,t) \Big\} = 0, & (x,t) \in \mathbb{R}^d \times \{t > 0\}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

For this obstacle problem we will prove that there exists a unique viscosity solution. To this end, we first show that a comparison principle holds for (P), let $\overline{u}$ be a supersolution and $\underline{u}$ be a subsolution of (P), then $\overline{u}(x,t) \geq \underline{u}(x,t)$ for every (x,t) belonging to $\mathbb{R}^d \times \{t > 0\}$. Next, we will obtain the existence and uniqueness of the solution via Perron's method, that is, we obtain the unique solution of the obstacle problem as the infimum of supersolutions to $u_t + \mathcal{L}u = 0$ in $\mathbb{R}^d \times (0, \infty)$ that are greater or equal than the obstacle ψ and are also greater or equal than u_0 at time $t = 0$.

Once we have existence and uniqueness of solutions we turn our attention to their asymptotic behavior. Now, we will assume that $\overline{K} \subset \Omega_0$ (for instance, this happens when $\psi(x) < u_0(x)$). Let us consider again the positivity sets of $u(x,t)$, $\Omega_t = \{x : u(x,t) > 0\}$. According to our previous discussion, these sets encode the evolution by minimal curvature of the hypersurface $\partial\Omega_0 = \partial\{x : u_0(x) > 0\}$ with the obstacle $K = \{x : \psi(x) > 0\}$.

Our main goal is to prove that the sets $\Omega_t = \{x : u(x,t) > 0\}$ approximate the convex hull of K as $t \to \infty$. To get some insight on why this result holds let us look at a simple example. In $\mathbb{R}^3$ let K be the union of two disjoint balls,

$$K = B_1(2,0,0) \cup B_1(-2,0,0).$$

For this set K its convex hull, that we denote by $\mathrm{co}(K)$, is given by a tube that joins the two balls, see Figure 4.1. Now, take a set Ω_0 that is convex and contains $\mathrm{co}(K)$. When the set evolves by minimal curvature it shrinks until it touches the obstacle and then converges to the convex hull. In fact, remark that at points on the boundary of the convex hull that are not in $\overline{K}$ we have that one the principal curvatures is positive but the one that corresponds to the horizontal direction is zero. Therefore, the flow by minimal curvature with K as obstacle has this convex hull as a stationary solution, see Figure 4.1 below.

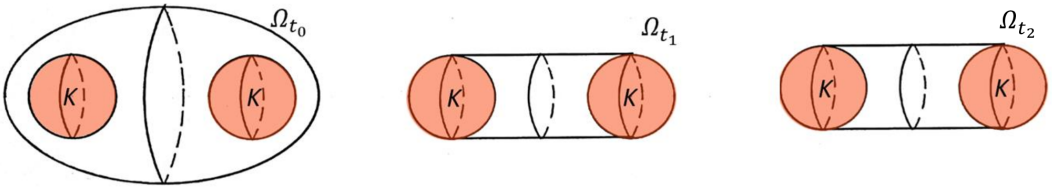


Figure 4.1: Motion by minimal curvature flow.

Notice that this does not happens for the usual mean curvature flow. In fact, the mean curvature of a point on the boundary of the convex hull that is not in K is strictly positive (one of the main curvatures is zero but the other one is strictly positive), and therefore we have that the evolution by mean curvature flow with obstacle K continues shrinking the set when it reaches the convex hull of K, see Figure 4.2 below.

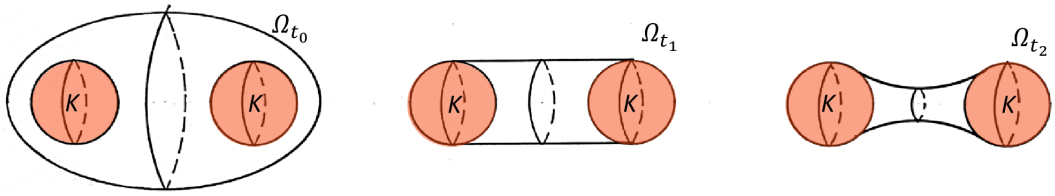


Figure 4.2: Motion by mean curvature flow.

The motion by mean curvature is nowadays a classical subject, we refer to [23, 40, 32, 44, 55, 68, 69] and references therein. Since there is only one curvature for a curve in dimension two our evolution problem coincides with the mean curvature flow with an obstacle studied in [47, 78, 79, 82]. However, for higher dimensions, $d \geq 3$, the minimal curvature flow studied here is quite different from the mean curvature flow.

In the special case that the initial domain is a ball $\Omega_0 = B_R(0) \subset \mathbb{R}^d$, since all the curvatures are equal, the evolution according to minimal curvature coincides with the evolution by mean curvature (and is given by $B_{R(t)}(0)$ with $R'(t) = 1/R(t)$) until the first time when the surface touches the obstacle, from that time on, the two evolutions are different. Then, when there is no obstacle, $K = \emptyset$, and the initial domain is a ball $B_R(0) \subset \mathbb{R}^d$ we get that the surface shrinks to a point in finite time. Since there is a comparison principle for the motion by minimal curvature we have that when $K = \emptyset$ the hypersurface shrinks and disappears in finite time.

A game approximation for the parabolic problem

Next, let us describe a game whose value function approximates the solution to (P).

The game is a deterministic two-person zero-sum game. As in [63], we use Paul for the name of the first player and Carol for the second player. Take $\varepsilon > 0$ (a parameter that controls the size of the possible movements in the game), $\Omega_0 \subset \mathbb{R}^d$ an open, bounded set and $K_\varepsilon \subset \Omega_0$ an open set (it will be convenient that the obstacle for the game also depends

on the parameter ε). We also have two functions, $u_0 : \mathbb{R}^d \mapsto \mathbb{R}$ and $\psi_\varepsilon : \mathbb{R}^d \mapsto \mathbb{R}$ that will give us the final payoff of the game. We will assume that the obstacle satisfies

$$K_\varepsilon = \{x : \psi_\varepsilon(x) > 0\}.$$

On the other hand, the function $u_0(x)$ will give the initial condition for the limit equation and then we assume that $u_0(x) \geq \psi_\varepsilon(x)$. The initial condition u_0 may also depend on ε but, to simplify, we drop this dependence here.

Let $x = x_0$ be the initial position of the game and $t = t_0 > 0$ the initial time. The game is played as follows: at the $i-$th round, Paul chooses a unitary vector v_i and Carol chooses a sign $b_i = \pm 1$ after Paul's choice. Then, the next position of the game is given by

$$x_i = x_{i-1} + b_i v_i \varepsilon,$$

while time decreases as

$$t_i = t_{i-1} - \frac{1}{2}\varepsilon^2.$$

As a remark we point out that one can also choose to play with $x_i = x_{i-1} + \sqrt{2} b_i v_i \varepsilon$ and $t_i = t_{i-1} - \varepsilon^2$ as in [63, 82]. We prefer to keep the spatial steps of size ε in order to simplify some computations when analyzing the game.

After $n = \left\lceil \frac{2t_0}{\varepsilon^2} \right\rceil$ rounds, the game ends and Carol pays the terminal cost $u_0(x_n)$ to Paul. We also add an extra stopping rule. At any round Paul has the right to quit the game and Carol pays him the value given by the obstacle, $\psi_\varepsilon(x_i)$. We define $u^\varepsilon(x,t)$ as the value of this game starting at x with time t (the value of the game is just the final cost optimized by both players, Paul wants to maximize the expected outcome, while Carol aims to minimize it). The value of the game is given by the following formula,

$$u^\varepsilon(x_0,t_0) = \max\left\{ \psi_\varepsilon(x_0),\ \sup_{|v_1|=1} \min_{b_1=\pm 1} \left\{ \cdots \max\left\{ \psi(x_{n-1}),\ \sup_{|v_n|=1} \min_{b_n=\pm 1} u_0(x_{n-1} + b_n v_n \varepsilon) \right\} \right\} \right\}$$

for $n = \left\lceil \frac{2t_0}{\varepsilon^2} \right\rceil$. Notice that the rule that allows Paul to stop the game at any time and obtain ψ_ε ensures that the value of the game satisfies

$$u^\varepsilon(x,t) \geq \psi_\varepsilon(x).$$

After only one round of the game starting from (x,t) we have that the outcome is given by

$$\begin{cases} u^\varepsilon(x,t) = \max\left\{ \psi_\varepsilon(x),\ \sup_{|v|=1} \min_{b=\pm 1} u^\varepsilon\left(x + bv\varepsilon, t - \frac{1}{2}\varepsilon^2\right) \right\}, & x \in \mathbb{R}^d,\ t > 0, \\ u^\varepsilon(x,t) = u_0(x), & x \in \mathbb{R}^d,\ t \leq 0. \end{cases} \tag{P_ε}$$

The equation that appears in (P_ε) is known as the Dynamic Programming Principle (DPP) in the game literature, see [71], and reflects the rules of the game. In fact, Paul

wants to maximize the outcome and he can choose between getting $\psi_\varepsilon(x_0)$ or the value after making one move (that is given by the supremum among directions (Paul's choice) of the minimum among a sign (Carol's choice) of the value at the new position at time $t_0 - \frac{1}{2}\varepsilon^2$). The successive positions of the game are illustrated in Figure 4.3 below.

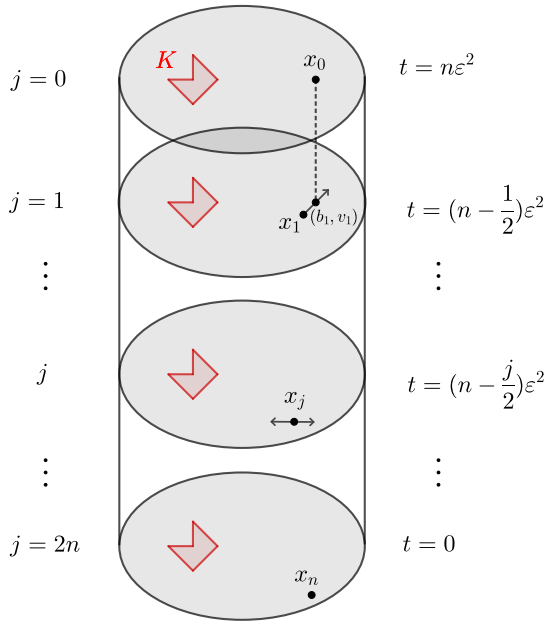


Figure 4.3: Positions playing the game.

We observe that Paul plays aiming to stay inside Ω_0 and also trying to reach the obstacle K_ε. Notice that the continuous and bounded terminal payoff, u_0, is such that $u_0(x) > 0$ for $x \in \Omega_0$ and $u_0(x) < 0$ for $x \in \mathbb{R}^d \setminus \overline{\Omega_0}$. Hence, Paul has a strong motivation to remain inside Ω_0 when the game ends. On the other hand, under this setting, Carol will try to stay outside Ω_0 and plays her best in order to evade the set K_ε at every move.

Now, to look for a partial differential equation related to this game, we argue formally (one of the main goals of this chapter is to make rigorous what follows). Let us find the asymptotic behavior as $\varepsilon \approx 0$ of the DPP, (P_ε), evaluated at a smooth function with an obstacle ψ that is independent of ε (we will have that $\psi_\varepsilon \to \psi$ uniformly as $\varepsilon \to 0$, hence we can assume here that ψ does not depend on ε). Assume that for a $C^{2,1}$ function ϕ we have

$$\phi(x,t) \approx \max\left\{\psi(x), \sup_{|v|=1}\min_{b=\pm 1}\phi(x + bv\varepsilon, t - \frac{1}{2}\varepsilon^2)\right\},$$

that is,

$$0 \approx \max\left\{\psi(x) - \phi(x,t), \sup_{|v|=1}\min_{b=\pm 1}\frac{\phi(x + bv\varepsilon, t - \frac{1}{2}\varepsilon^2) - \phi(x, t - \frac{1}{2}\varepsilon^2)}{\varepsilon^2} - \frac{\phi(x,t) - \phi(x, t - \frac{1}{2}\varepsilon^2)}{\varepsilon^2}\right\}.$$

Using that $\phi \in C^{2,1}$ and neglecting higher order terms, after multiplying by 2 the second term, we arrive to

$$0 \approx \max\left\{\psi(x) - \phi(x,t), \sup_{|v|=1} \min_{b=\pm 1} \frac{2}{\varepsilon} b\langle \nabla\phi(x,t), v\rangle + \langle D^2\phi(x,t)v, v\rangle - \frac{\partial \phi}{\partial t}(x,t)\right\}.$$

Now, assume that $\phi(x,t) > \psi(x)$. In this case, we notice that when computing the supremum we want to use a vector v orthogonal to $\nabla\phi(x,t)$ (otherwise, we can select the right sign when we compute the minimum in order to obtain $-2\varepsilon^{-1}|\langle \nabla\phi(x,t), v\rangle|$, that is negative and large for $\varepsilon \approx 0$). Then, assuming that the vector v is orthogonal to $\nabla\phi(x,t)$ we obtain

$$0 \approx \max\left\{\psi(x) - \phi(x,t), \sup_{\substack{|v|=1,\\ v\perp\nabla\phi(x,t)}} \langle D^2\phi(x,t)v, v\rangle - \frac{\partial \phi}{\partial t}(x,t)\right\},$$

that is, we find the obstacle problem for the equation

$$\frac{\partial \phi}{\partial t}(x,t) = \sup_{\substack{|v|=1,\\ v\perp\nabla\phi(x,t)}} \langle D^2\phi(x,t)v, v\rangle \tag{4.1.3}$$

with ψ as obstacle. Notice that the right hand side of our equation (4.1.3) is given by the minimum of the curvatures of $\partial\Omega_t$,

$$\sup_{\substack{|v|=1,\\ v\perp\nabla\phi(x,t)}} \langle D^2\phi(x,t)v, v\rangle = -\inf_{\substack{|v|=1,\\ v\perp\nabla\phi(x,t)}} \Big(-\langle D^2\phi(x,t)v, v\rangle\Big) = -\min_{\substack{|v|=1,\\ v \text{ tangent to } \partial\Omega_t}} \{\kappa(v)\}.$$

Then, we have that (4.1.3) is just the level set formulation of the geometric evolution of a hypersurface by the minimum curvature with an obstacle. Hence, we conclude that the DPP associated to the game, (P_ε), is related to the obstacle problem (with obstacle ψ) for the geometric flow of a hypersurface driven by the minimum curvature. That is, the limit as $\varepsilon \to 0$ for the value of the game, gets us back to the parabolic problem (P). In fact, we prove that the value function of our game converges locally uniformly to the viscosity solution to (P) as $\varepsilon \to 0$. This is an alternative proof of existence of viscosity solutions for problem (P) that avoids the use of Perron's method.

For similar results for the classical mean curvature flow we refer to [63, 82]. Again, we point out that, in dimension two, our game coincides with the one studied in [63] (without the obstacle) and in [82] (with an obstacle). However, in dimensions bigger than three the game presented here and the game for the mean curvature flow are quite different.

Since Paul aims to maximize the payoff and Carol wants to minimize it, we introduce the sets

$$\Omega_t^\varepsilon = \{x : u^\varepsilon(x,t) > 0\},$$

that is, the set of positions in $\mathbb{R}^d$ where starting at time t Paul expects to win (and Carol to loose) a positive payoff after $\left\lceil \frac{2t}{\varepsilon^2} \right\rceil$ rounds. Notice that

$$K_\varepsilon = \{x : \psi_\varepsilon(x) > 0\} \subset \Omega_t^\varepsilon$$

for every t (Paul can end the game at any round and obtain a positive payoff inside the set $\{x : \psi_\varepsilon(x) > 0\}$).

As a consequence of the previous locally uniform convergence of u^ε to u, we have that for each $t > 0$, the set where the value function of the ε-game is positive at time t, $\Omega_t^\varepsilon = \{x : u^\varepsilon(x,t) > 0\}$ approximates as $\varepsilon \to 0$ the set where the solution of the obstacle problem is positive, $\Omega_t = \{x : u(x,t) > 0\}$.

Finally, for ε fixed, we also study the shape of the sets Ω_t^ε for t large. We find that the limit is related to the convex hull of the set K_ε with respect to a notion of convexity that takes into account that we have a fixed particular length, ε, for the movements of the game. Taking the limit $\varepsilon \to 0$ yields the results for the asymptotic limit of Ω_t when $t \to \infty$ and in this way we show our main result, that Ω_t approaches the convex hull of K as $t \to \infty$.

4.1.2 Statements of the main results

Let us state rigorously the main theorems that are included in this chapter.

In Subsection 4.2 we analyze the parabolic problem (P) for which we will prove the following results: First, a comparison principle holds for viscosity sub and supersolutions to (P) (for the precise definition of being a sub or a supersolution we refer to Subsection 4.2).

Theorem 4.1.1 (Comparison Principle)**.** *Assume that the obstacle ψ is bounded and has a uniform modulus of continuity ω, that is,*

$$|\psi(x) - \psi(y)| \leq \omega(|x-y|), \qquad \text{for all } x, y \in \mathbb{R}^d$$

for an increasing continuous function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$. Also assume that the initial datum $u_0 \in C(\mathbb{R}^d)$ is such that

$$\lim_{|x|\to\infty} u_0(x) = \mu \in \mathbb{R}_{<0}$$

and

$$u_0(x) \geq \psi(x)$$

for $x \in \mathbb{R}^d$. If $\underline{u}$ is a viscosity subsolution and $\overline{u}$ a supersolution to problem (P), then

$$\underline{u}(x,t) \leq \overline{u}(x,t) \qquad \text{in } \mathbb{R}^d \times [0,\infty).$$

From now on and along the whole chapter, we will assume that u_0 and ψ satisfy the hypotheses in Theorem 4.1.1. We will omit the explicit mention to these hypotheses in the following statements.

With this comparison result we can show existence and uniqueness of solutions to (P).

Theorem 4.1.2 (Existence and uniqueness for the PDE)**.** *There exists a unique solution u to the Obstacle Problem (P) with obstacle ψ and initial datum u_0. Moreover, u is a continuous function in $\mathbb{R}^d \times [0,\infty)$ that is given by*

$$u(x,t) = \inf\{w(x,t) \ : \ \ w \in \mathcal{A}$$

where

$$\mathcal{A} := \Big\{ w(x,t) \ : \ \ w \ \text{ is a viscosity supersolution to } (P), \text{ that is,}$$
$$w_t + \mathcal{L}w \geq 0 \ \text{ in } \ \mathbb{R}^d \times (0,\infty), \ w(\cdot,t) \geq \psi \ \text{ for all } \ t>0 \ \text{ and } \ w(\cdot,0) \geq u_0 \Big\}.$$

Now, let us describe our results for the game. To this end, we first need to be precise in the way that we enlarge the obstacle. Let us consider $K_\varepsilon = K + B_{\varepsilon d}(0)$. Notice that here we are using the dimension N to enlarge the obstacle by adding a ball or radius εd. That the added ball has exactly radius εd has a purpose later on, but let us mention that adding any ball $B_{r(\varepsilon)}(0)$ with $r(\varepsilon) \geq \varepsilon d$ and $r(\varepsilon) \to 0$ as $\varepsilon \to 0$ is enough for our arguments. We assume that

$$K_\varepsilon = K + B_{\varepsilon d}(0) \subset \Omega_0.$$

This holds for ε small when $\overline{K} \subset \Omega_0$. Recall that ψ is a continuous function such that

$$K = \{x \in \mathbb{R}^d : \psi(x) > 0\}.$$

We need to characterize the set $K_\varepsilon = K + B_{\varepsilon d}(0)$ in a similar way, with a function ψ_ε. We want to choose a function ψ_ε that is positive when x is an interior point of K_ε, that is,

$$K_\varepsilon = \{x \in \mathbb{R}^d : \psi_\varepsilon(x) > 0\},$$

and such that, in addition, $\psi_\varepsilon \to \psi$ uniformly when $\varepsilon \to 0$. We use this function ψ_ε as the obstacle for our game. To see that there is a function ψ_ε that fulfills our requirements we argue as follows: We assumed that ψ is bounded and has a uniform modulus of continuity ω, that is,

$$|\psi(x) - \psi(y)| \leq \omega(|x-y|), \qquad \text{for all } x, y \in \mathbb{R}^d.$$

Now, assuming that ω is continuous, let h_ε be given by

$$h_\varepsilon(x) = \begin{cases} \min\limits_{y \in \partial K_\varepsilon} (\psi(y) + 2\omega(|x-y|)) & \text{if } x \in K_\varepsilon, \\ \max\limits_{y \in \partial K_\varepsilon} (\psi(y) - 2\omega(|x-y|)) & \text{if } x \in \mathbb{R}^d \setminus K_\varepsilon. \end{cases}$$

Notice that h_ε is uniformly continuous, $h_\varepsilon(x) = \psi(x)$ for $x \in \partial K_\varepsilon$, $\psi(x) \leq h_\varepsilon(x)$ for $x \in K_\varepsilon$ and $h_\varepsilon(x) \leq \psi(x)$ for $x \in \mathbb{R}^d \setminus K_\varepsilon$. Also notice that there is a constant $C_\varepsilon > 0$ that goes to zero as $\varepsilon \to 0$ such that $0 < h_\varepsilon(x) - \psi(x) < C_\varepsilon$ for $x \in K_\varepsilon \setminus K$ and $0 > h_\varepsilon(x) - \psi(x) > -C_\varepsilon$ for $x \in K_{2\varepsilon} \setminus K_\varepsilon$. Now, we choose

$$\psi_\varepsilon(x) := \begin{cases} \max\Big\{\psi(x), \min\{h_\varepsilon(x) - \psi(x), C_\varepsilon\}\Big\} & \text{if } x \in K_\varepsilon, \\ \min\Big\{\psi(x), \max\{h_\varepsilon(x) - \psi(x), -C_\varepsilon\}\Big\} & \text{if } x \in \mathbb{R}^d \setminus K_\varepsilon. \end{cases}$$

In Subection 4.3 we include the following results:

Theorem 4.1.3. *Let ψ_ε be as above, and u_0 such that $u_0 \geq \psi_\varepsilon$ for every ε small. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of value functions of the game with obstacle ψ_ε and final payoff u_0, that is, for each $\varepsilon > 0$, u^ε is a solution to (P_ε).*

Then, u^ε converges locally uniformly as $\varepsilon \to 0$ to u, the unique solution to (P).

As a consequence, we can obtain the behavior of the positivity sets of u^ε as $\varepsilon \to 0$.

Corollary 4.1.4. *Consider*

$$\Omega_t^\varepsilon := \{x \, : \, u^\varepsilon(x,t) > 0\} \text{ and } \Omega_t := \{x \, : \, u(x,t) > 0\}. \tag{4.1.4}$$

Then, for each $t > 0$, we have that

$$\Omega_t \subset \liminf_{\varepsilon\to 0} \Omega_t^\varepsilon \subset \limsup_{\varepsilon\to 0} \Omega_t^\varepsilon \subset \overline{\Omega}_t.$$

Next, we drive our attention to the behavior of the positivity sets of u^ε as $t \to \infty$. We need to introduce a graph $G = (\mathcal{V}, \mathcal{E})$ associated with the geometric configuration of K and Ω_0. The vertices of the graph G are the connected components of K and two vertices are connected by an edge if there is a segment in $\mathbb{R}^d$ with endpoints in two connected components of K that correspond to the two vertices, that is contained in Ω_0. The main hypothesis in the next results is going to be the following:

$$\text{The graph } G \text{ is connected.} \tag{G}$$

Notice that (G) holds when we have the stronger condition $\text{co}(K) \subset \Omega_0$.

In the next statement recall that we denoted by $\text{co}(A)$ the convex hull of a set A. In Subsection 4.4 we prove:

Theorem 4.1.5. *Assume that $K \subset \mathbb{R}^d$ and $\Omega_0 \subset \mathbb{R}^d$ satisfy condition (G). If $co(K) \subset \Omega_0$ or $d = 2$, then there exists $\varepsilon_0 > 0$ such that*

$$co(K) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon \subset \limsup_{t\to\infty} \Omega_t^\varepsilon \subset co(K_\varepsilon) \subset co(K) + B_{\varepsilon d}(0)$$

for every $0 < \varepsilon \leq \varepsilon_0$.

In the proof of the previous result is where we use that we enlarged the obstacle and considered the set $K_\varepsilon = K + B_{\varepsilon d}(0)$ and its associated function ψ_ε as the obstacle for our game. Notice that we have $\text{co}(K) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon$ and $\limsup_{t\to\infty} \Omega_t^\varepsilon \subset \text{co}(K_\varepsilon)$ when we play with $K + B_{\varepsilon d}(0)$ and its associated function ψ_ε as obstacle, but, in general, these inclusions are not true if we play with K and ψ as the obstacle. We can give a slightly better result when we assume that $\text{co}(K) \subset \Omega_0$, see Subsection 4.4.

Finally, we use our previous results and some extra arguments to recover exactly the convex hull of K when we take the limit as $t \to \infty$ of the geometric evolution by minimal curvature. The following result is valid for any dimension.

Theorem 4.1.6. *Assume the K and Ω_0 satisfy condition (G). The family of positivity sets of the unique viscosity solution $u(x,t)$ to the obstacle problem (P), $(\Omega_t)_{t>0}$ defined in (4.1.4), verifies*

$$co(K) = \lim_{t\to\infty} \Omega_t.$$

We want to emphasize that, for the result above, inclusions $\limsup_{t\to\infty} \Omega_t^\varepsilon \subset co(K_\varepsilon)$ and $co(K) \subset \liminf_{t\to\infty} \Omega_t$ are both fundamental and original of this chapter. The main novelty of the first one, $\limsup_{t\to\infty} \Omega_t^\varepsilon \subset co(K_\varepsilon)$, is that we avoid the need to use $\overline{co(K_\varepsilon)}$. In fact it is also true for $\varepsilon = 0$, although this is not deduced from taking the limit $\varepsilon \to 0$. The second one, $co(K) \subset \liminf_{t\to\infty} \Omega_t$, is completely new for $N \geq 3$. The absence of ε in the previous inclusion is important, since we found several difficulties when trying to prove it for $\varepsilon > 0$. All of this will be further explained in Subsection 4.4.

4.1.3 An alternative game

Let us describe an alternative game. As before, we have two players, Paul (who wants to maximize the final payoff) and Carol (who wants to minimize it). Again as before, the final payoff is given by u_0 and an obstacle ψ_ε.

Now, the rules of the game change. At each turn, Carol chooses a subspace S_i of dimension $N-1$, then Paul chooses a unit vector in that subspace, $v_i \in S$, $|v_i| = 1$, and the new position of the game is

$$x_i = x_{i-1} + \varepsilon v_i$$

and time decreases as before,

$$t_i = t_{i-1} - \frac{1}{2}\varepsilon^2.$$

Again, we have the stopping rule and the terminal rule. Paul has the right to end the game at any turn and he gets the value given by the obstacle, $\psi_\varepsilon(x_i)$ and if time is consumed (after $n = \left\lceil \frac{2t_0}{\varepsilon^2} \right\rceil$ rounds) Paul gets the final payoff $u_0(x_n)$.

Let $u^\varepsilon(x,t)$ be the value of this game starting at x with time t. For this game the corresponding Dynamic Programming Principle (DPP) is given by

$$\begin{cases} u^\varepsilon(x,t) = \max\left\{\psi_\varepsilon(x), \inf\limits_{\substack{S\subset\mathbb{R}^d,\\ dim(S)=d-1}} \sup\limits_{\substack{v\in S,\\ |v|=1}} u^\varepsilon\big(x+v\varepsilon, t-\frac{1}{2}\varepsilon^2\big)\right\}, & x\in\mathbb{R}^d,\ t>0, \\ u^\varepsilon(x,t) = u_0(x), & x\in\mathbb{R}^d,\ t\leq 0. \end{cases} \tag{P_ε}$$

The partial differential equation related to this game is exactly the level set formulation of the geometric evolution of a hypersurface by the minimum curvature with an obstacle. To see this, formally, we find the asymptotic behavior as $\varepsilon \approx 0$ of the DPP, (P_ε), evaluated at a smooth function ϕ assuming again that the obstacle ψ is independent of ε. We have

$$\phi(x,t) \approx \max\left\{\psi_\varepsilon(x), \inf_{\substack{S\subset\mathbb{R}^d,\\ dim(S)=d-1}} \sup_{\substack{v\in S,\\ |v|=1}} \phi\big(x+v\varepsilon, t-\frac{1}{2}\varepsilon^2\big)\right\}.$$

Using similar Taylor expansions as we did before we get

$$0 \approx \max\Big\{\psi(x)-\phi(x,t),\ \inf_{\substack{S\subset\mathbb{R}^d,\\ dim(S)=d-1}}\ \sup_{\substack{v\in S,\\ |v|=1}} \frac{2}{\varepsilon}\langle\nabla\phi(x,t),v\rangle + \langle D^2\phi(x,t)v,v\rangle - \frac{\partial\phi}{\partial t}(x,t)\Big\}.$$

Now, in the second term, we notice that when computing the infimum we use a subspace S orthogonal to $\nabla\phi(x,t)$ (otherwise, we can select a unit vector $v\in S$ such that $2\varepsilon^{-1}\langle\nabla\phi(x,t),v\rangle$ is positive and large for $\varepsilon\approx 0$). Then, assuming that the subspace S is orthogonal to $\nabla\phi(x,t)$ we arrive to

$$0 \approx \max\Big\{\psi(x)-\phi(x,t),\ \sup_{\substack{|v|=1,\\ v\perp\nabla\phi(x,t)}} \langle D^2\phi(x,t)v,v\rangle - \frac{\partial\phi}{\partial t}(x,t)\Big\},$$

that is, we find the obstacle problem for the level set formulation of the geometric evolution by the minimum curvature with an obstacle.

For this game, one can show (using similar computations to the ones gathered in Subsection 4.3) that the limit as $\varepsilon\to 0$ for the value of the game, gives the solution to the parabolic problem (P).

We have chosen to include the details of the proofs of our results concerning the asymptotic behavior of the positivity set for the value function for our previous game and leave to the reader the analysis of this alternative game.

4.1.4 Comments on the ideas involved in the proofs and on our hypotheses

The idea of considering the flow by minimal curvature with an obstacle in order to recover the convex hull of the obstacle is natural since the boundary of the convex hull of a set can be touched by hyperplanes and when these hyperplanes touch the convex hull at points that are not in the original set we obtain that one of the main curvatures of the boundary has to be zero at those points, while all the other curvatures need to be nonnegative in order for the convex hull to be a convex set. Therefore, we have that the convex hull of a set is a solution to the minimal curvature equal to zero at boundary points that are not in the closure of the original set (where the surface does not touch the obstacle). Since our target (the convex hull of a set) is a stationary solution to our evolution equation it seems natural to expect that we find the convex hull in the long time behavior of the solutions to our parabolic problem.

We use viscosity theory to deal with the parabolic problem (P) using ideas from [78, 79]. Here we assume that the obstacle ψ has a uniform modulus of continuity and that the initial condition is continuous and bounded with a limit as $|x|\to\infty$. These assumptions allow us to use a doubling of variables technique in order to show the comparison principle for our problem. Once we have a comparison principle, existence and uniqueness follow from Perron's method.

The analysis of the game relies on ideas from [63, 82]. However, here we enlarge the obstacle set and we introduce the idea of ε-convexity in order to study the long time behavior of the solutions. These modifications of the arguments in [63, 82] prove to be quite useful for the analysis of the asymptotic behavior of the positivity sets, but introduce new difficulties in the analysis.

The proof of the convergence of the values of the game u^ε to the unique viscosity solution to (P) is based on the half-relaxed limits. Some care needs to be taken here since taking these limits we want to obtain sub and supersolutions to the degenerate parabolic equation (4.1.1).

Concerning the study of the asymptotic behavior of the solutions to (P) and the values of the game we borrow some geometric ideas from [82] and use that the positivity set of the initial condition u_0 contains some segments that connect the connected components of the obstacle K (condition (G)). Without this assumption the general asymptotic result may be false. For example, in the case where there are different connected components of K contained in different connected components of Ω_0 the positivity set of a solution to the evolution problem may remain disconnected and we can only recover in the limit the union of the convex hulls of the components that are connected. The study of the long time behavior of the positivity set of the value function of the game relies on the construction of clever strategies for Paul and Carol that allow us to obtain lower an upper bounds on the sets where Paul wins (that is, for the set where the value of the game is strictly positive).

We want to highlight that our hypothesis that the obstacle set K is open can be relaxed at some points, in particular when we have that $\mathrm{co}(K) \subset \Omega_0$. We will make further comments on this along the text when appropriate.

To see the utility of enlarging the obstacle considering $K_\varepsilon = K + B_{\varepsilon d}(0)$ and that we can deal with sets K that are not necessarily open, let us describe a concrete example. Assume that we want to recover the convex hull of only two points, that is, we take $K = \{x, y\}$. In this simple case the convex hull of K is just the segment that connects x and y. Assume that we are in the most favorable case, the positivity set of the initial condition contains the segment and also assume that the initial position x_0 is inside the segment. In this situation, since Paul wins when the position of the game arrives to the obstacle or remains in the segment, starting at a point inside the segment Paul will choose that direction to play. However, if ε is such that the starting point x_0 is inside the segment but is such that $|x - x_0|/\varepsilon$ or $|y - x_0|/\varepsilon$ is not a natural number, then Carol may choose the signs in such a way that the position of the game for large times exits any large ball without touching the obstacle (and then Paul loses). See Figure 4.4. Therefore, we get that $\liminf_{t\to\infty} \Omega_t^\varepsilon = K = \{x, y\}$ and in this limit we do not recover the convex hull of K (the segment).

Nevertheless, when we enlarge the set K, taking $K_\varepsilon = B_{\varepsilon d}(x) \cup B_{\varepsilon d}(y)$, it is simple to observe that when x_0 is inside the segment that connects x and y and Paul chooses that direction, then no matter what Carol does the position will reach the obstacle or remain in the segment and hence Paul wins. See Figure 4.4. In this case we have that the segment $\mathrm{co}(K)$ is included in $\liminf_{t\to\infty} \Omega_t^\varepsilon$ as we wanted.

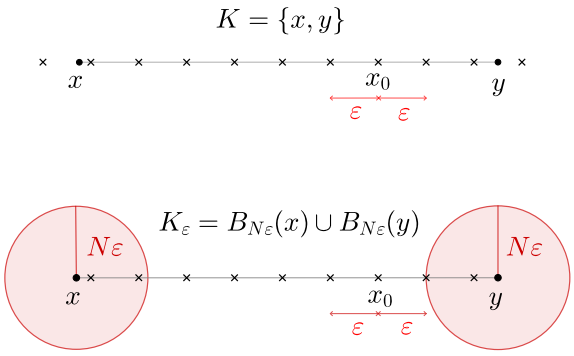


Figure 4.4: Positions of the game with or without enlarging the obstacle.

4.1.5 Precedents

Let us end this introduction with a brief description of previous results in the literature.

Our immediate precedent and source of inspiration is [82] where the author studies the mean curvature flow with an obstacle. In two space dimensions the mean curvature flow is the well known curve shortening problem and the curve evolution was analyzed in [43, 53] and references therein. Without obstacles, it is known that the curve becomes convex at some time and then it converges to a single point in finite time, provided the initial curve is closed and smooth ([43, 53]). Since there is only one curvature for a curve in dimension two our evolution problem coincides with the one studied in [82]. However, for dimensions bigger than two the two problems are different in nature and, as we explained before, the evolution by minimal curvature studied here recovers the convex hull of the obstacle in the limit as $t \to \infty$; while the mean curvature flow does not.

The level set method for surface evolution equations was first rigorously analyzed in [32, 40]. The level set method is applied to various surface evolution equations including the mean curvature flow equation (with or without an obstacle), see [1, 32, 40]. For general references on level sets formulation of geometric flows we refer to [23, 46, 55, 47, 68, 69, 79] and the book [44].

On the other hand, the relation between game theory and nonlinear PDEs is quite rich and has attracted considerable attention in recent years. We quote [17, 63, 99, 70, 72, 73, 67, 81, 82, 87, 88, 102] and the books [16, 66]. As we have mentioned, in [63] a game for the motion by mean curvature was studied and in [82] the same game with an obstacle was tackled. We use some ideas for the game in dimension two that were obtained in [82]. In particular, we extend the geometric condition on the geometric configuration of the obstacle K and the initial set Ω_0 from [82] to several space dimensions. We refer to [74, 67] for games for obstacle problems and to [81] for a game for the two membranes problem.

4.2 The level set formulation for the movement of a hypersurface by minimal curvature with an obstacle

This subsection is devoted to the proof of Theorem 4.1.1 (comparison principle) and Theorem 4.1.2 (existence and uniqueness). Although both results are standard in the theory of viscosity solutions, we provide detailed proofs in order to make the chapter self-contained.

First, we start with the definition of what we understand by a sub and supersolution. We recall the definition of semicontinuous functions, since they are useful when working in the viscosity framework.

Definition 4.2.1. We say that $f : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R}$ is a *lower semicontinuous function*, l.s.c., at $(x,t) \in \mathbb{R}^d \times (0,\infty)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x,t) \leq f(y,s) + \varepsilon \text{ for all } (y,s) \in B_\delta(x,t) \setminus \{(x,t)\},$$

or, equivalently,

$$f(x,t) \leq \liminf_{(y,s)\to(x,t)} f(y,s).$$

If a function f is l.s.c. at every $(x,t) \in \mathbb{R}^d \times (0,\infty)$, we say that f is l.s.c. in $\mathbb{R}^d \times (0,\infty)$.

On the other hand, we say that $g : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R}$ is an *upper semicontinuous function*, u.s.c, at $(x,t) \in \mathbb{R}^d \times (0,\infty)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$g(y,s) \leq g(x,t) + \varepsilon \text{ for all } (y,s) \in B_\delta(x,t) \setminus \{(x,t)\},$$

or, equivalently,

$$\limsup_{(y,s)\to(x,t)} f(y,s) \leq f(x,t).$$

If a functions g is u.s.c. at every $(x,t) \in \mathbb{R}^d \times (0,\infty)$, we say that it is u.s.c. in $\mathbb{R}^d \times (0,\infty)$.

We will also require the concepts of lower and upper semicontinuous envelopes of a function.

Definition 4.2.2. The *lower semicontinuous envelope* of f is defined as

$$f_*(x,t) = \liminf_{(y,s)\to(x,t)} f(y,s).$$

It verifies that is the largest lower semicontinuous function smaller or equal than f, i.e,

$$f_* = \sup\Big\{h : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R} \text{ l.s.c. } : \ h(x,t) \leq f(x,t) \text{ for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\Big\}.$$

The *upper semicontinuous envelope* of f is

$$f^*(x,t) = \limsup_{(y,s)\to(x,t)} f(y,s).$$

It verifies that is the smallest upper semicontinuous function larger or equal than f, i.e,

$$g^* = \inf\Big\{h : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R} \text{ u.s.c. } : \ h(x,t) \geq f(x,t) \text{ for all } (x,t) \in \mathbb{R}^d \times (0,\infty)\Big\}.$$

Now, to use viscosity theory for our operator

$$\mathcal{L}(f(x)) := \inf_{\substack{|v|=1 \\ v\perp\nabla f(x)}} \Big(- \langle D^2 f(x)v, v\rangle \Big).$$

we look at it using the function $\mathcal{L} : \mathbb{R}^d \setminus \{0\} \times \mathbb{S}^{d\times d} \mapsto \mathbb{R}$ (here $\mathbb{S}^{d\times d}$ denotes the set of real symmetric matrices) given by

$$\mathcal{L}(p, X) := \inf_{\substack{|v|=1 \\ v\perp p}} \Big(- \langle Xv, v\rangle \Big).$$

Observe that, with an abuse of notation, we use $\mathcal{L}$ to denote both the operator and the function associated with it. When $p = 0$, we take the infimum over all the unitary vectors. The function $\mathcal{L}$ is discontinuous at $p = 0$. Therefore, when we define sub and supersolutions we will need to use $\mathcal{L}_*$, the lower semicontinuous envelope and $\mathcal{L}^*$, the upper semicontinuous envelope of $\mathcal{L}$.

Now, let us state the definition of being a viscosity subsolution and a viscosity supersolution (we refer to [36] for more details on viscosity theory).

Definition 4.2.3. Let $\mathcal{D} \subset \mathbb{R}^d \times (0,\infty)$ be an open set. A function $u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ is called a *viscosity subsolution* of the parabolic equation (4.1.1) in $\mathcal{D}$ if its upper semicontinuous envelope, u^*, satisfies that for every $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ such that ϕ touches u^* at $(x,t) \in \mathcal{D}$ strictly from above, that is, $\phi - u^*$ has a strict minimum at (x,t), we have,

$$\partial_t\phi(x,t) + \mathcal{L}_*(\nabla\phi(x,t), D^2\phi(x,t)) \le 0.$$

Conversely, $u : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ is called a *viscosity supersolution* of the parabolic equation (4.1.1) in $\mathcal{D}$ if its lower semicontinuous envelope function, u_*, satisfies that for every $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ such that ϕ touches u_* at $(x,t) \in \mathcal{D}$ strictly from below, that is, $u_* - \phi$ has a strict minimum at (x,t), we have

$$\partial_t\phi(x,t) + \mathcal{L}^*(\nabla\phi(x,t), D^2\phi(x,t)) \ge 0.$$

If u is both a sub and supersolution of (4.1.1), we say that u is a viscosity solution to (4.1.1).

To motivate the following equivalent notion of viscosity solution, suppose that u is a viscosity subsolution to the equation (4.1.1). Assume that $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ is such that $\phi - u^*$ has a strict minimum at (x,t). Then,

$$\phi(x,t) - u^*(x,t) < \phi(y,s) - u^*(y,s) \ \text{ for all } \ (y,s) \in \mathbb{R}^d \times (0,\infty).$$

With this in mind, using the Taylor's expansion of $\phi(y,s)$ on (x,t) we get

$$u^*(y,s) \le u^*(x,t) + a(s-t) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|t-s| + |y-x|^2) \quad (4.2.1)$$

as $(y,s) \to (x,s)$ where

$$a = \partial_t \phi(x,t), \quad p = \nabla \phi(x,t) \text{ and } X = D^2\phi(x,t).$$

Moreover, if $u \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ and (4.2.1) holds for some $(p,X) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}$, then

$$a = \partial_t u(x,t), \quad p = \nabla u(x,t) \text{ and } X = D^2 u(x,t).$$

Definition 4.2.4. Consider a function $v : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R}$. The parabolic super 2-jet of v at (x,t), called $\mathcal{P}^{2,+}v(x,t)$, is the set of all $(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^{d\times d}$ such that

$$v(y,s) \leq v(x,t) + a(s-t) + \langle p, y-z\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t|+|y-x|^2) \text{ as } (y,s) \to (x,t).$$

Similarly, the parabolic lower super 2-jet of v at (x,t), that we call $\mathcal{P}^{2,-}v(x,t)$, is given by

$$\mathcal{P}^{2,-}v(x,t) = -\mathcal{P}^{2,+}(-v)(x,t).$$

Furthermore, as explained in [36], if $w : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R}$ is an upper semicontinuous function, there is a correspondence between the tuples (a,p,X) belonging to $\mathcal{P}^{2,-}w(x,t)$ and the test functions that touch w at (x,t) strictly from above: $(a,p,X) \in \mathcal{P}^{2,-}w(x,t)$ if and only if there exists a $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty)$ such that $(a,p,X) = (\partial_t\phi(x,t), \nabla\phi(x,t), D^2\phi(x,t))$ and ϕ touches w at (x,t) strictly form above. That is, the parabolic upper super 2-jet of the upper semicontinuous function w at (x,t) can be expressed as

$$\mathcal{P}^{2,+}w(x,t) = \left\{ (\partial_t\phi(x,t), \nabla\phi(x,t), D^2\phi(x,t)) \; : \; \begin{array}{l} \phi \in C^2(\mathbb{R}^d \times (0,\infty)) \text{ and} \\ \phi - w \text{ has an strict minimun at } (x,t) \end{array} \right\}.$$

With the aim of applying [36, Theorem 8.2] in the proof of the comparison principle, we reformulate the definition of viscosity super and subsolutions in terms of inequalities that involves elements of parabolic super and lower 2-jets of a function.

Definition 4.2.5. Let $\mathcal{D}$ be non-empty subset of $\mathbb{R}^d \times (0,\infty)$. Let $\mathcal{L}$ be defined as before. Consider the equation

$$w_t + \mathcal{L}(\nabla w, D^2 w) = 0 \text{ in } \mathcal{D}. \tag{4.2.2}$$

The function $u : \mathbb{R}^d \times (0,\infty) \mapsto \mathbb{R}$ is called a viscosity subsolution of the above equation (4.2.2) in $\mathcal{D}$ if its upper semicontinuous envelope, u^*, satisfies that

$$a + \mathcal{L}_*(p,X) \leq 0 \text{ for all } (a,p,X) \in \mathcal{P}^{2,+}u^*(x,t).$$

Reciprocally, we say that u is a viscosity supersolution of (4.2.2) in $\mathcal{D}$ if its lower semicontinuous envelope, u_*, satisfies that

$$a + \mathcal{L}^*(p,X) \geq 0 \text{ for all } (a,p,X) \in \mathcal{P}^{2,-}u_*(x,t).$$

Moreover, if u is both a sub and supersolution to (4.2.2), we say that u is a viscosity solution to (4.2.2) in $\mathcal{D}$.

Next we will prove some important properties of $\mathcal{L}$. Subsequently, as a consequence, we will establish the equivalence between Definition 4.2.3 and Definition 4.2.5.

Proposition 4.2.6. *The function $\mathcal{L}$ satisifies the following properties:*

i) *$\mathcal{L}$ is continuous in $\left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$,*

ii) *$\lambda(-X) \leq \mathcal{L}_*(p,X) \leq \mathcal{L}^*(p,X) \leq \Lambda(-X)$ for all $(p,X) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}$ where*

$$\lambda(Y) = \min\left\{\mu \in \mathbb{R}\,;\ \ \mu \ \text{ eigenvalue of } \ Y\right\} = \min_{|v|=1} \langle Yv, v\rangle \tag{4.2.3}$$

and

$$\Lambda(Y) = \max\left\{\mu \in \mathbb{R}\,;\ \ \mu \ \text{ eigenvalue of } \ Y\right\} = \max_{|v|=1} \langle Yv, v\rangle \tag{4.2.4}$$

for $Y \in \mathbb{S}^{d\times d}$.

iii) *$\mathcal{L}_*(0,O) = \mathcal{L}^*(0,O) = 0$ where O is the null matrix in $\mathbb{S}^{d\times d}$.*

iv) *$\mathcal{L}$ is degenerate elliptic, i.e.,*

$$\mathcal{L}(p,Y) \leq \mathcal{L}(p,X)$$

whenever $X \leq Y$ and $p \neq 0$.

v) *$\mathcal{L}$ is geometric, i.e, for all $\alpha > 0$ and $\sigma \in \mathbb{R}^d$,*

$$\mathcal{L}(\alpha p, \alpha X + \sigma p \otimes p) = \alpha \mathcal{L}(p,X)$$

for all $(p,X) \in \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$.

Proof. We begin with the proof of *i)*. Let $p_1, p_2 \in \mathbb{R}^d \setminus \{0\}$ and $X_1, X_2 \in \mathbb{S}^{d\times d}$. Notice that

$$|\mathcal{L}(p_1,X_1) - \mathcal{L}(p_2,X_2)| \leq |\mathcal{L}(p_1,X_1) - \mathcal{L}(p_1,X_2)| + |\mathcal{L}(p_1,X_2) - \mathcal{L}(p_2,X_2)|$$

and

$$|\mathcal{L}(p_1,X_1) - \mathcal{L}(p_1,X_2)| \leq \sup_{\substack{|v|=1\\ v\perp p_1}} |\langle (X_1 - X_2)v, v\rangle| \leq \|X_1 - X_2\|.$$

Then, to prove that $\mathcal{L}$ is continuous in $\left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$, it remains to prove that $\mathcal{L}(\cdot, X)$ is continuous in $\mathbb{R}^d \setminus \{0\}$ for every $X \in \mathbb{S}^{d\times d}$. We fix $(p,X) \in \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$. We have to show that

$$\mathcal{L}(p,X) = \lim_{q\to p} \mathcal{L}(q,X).$$

Let us prove first that $\mathcal{L}$ is lower semicontinuous in $(p,X) \in \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$ on the first variable, that is,

$$\liminf_{q\to p} \mathcal{L}(q,X) \geq \mathcal{L}(p,X). \tag{4.2.5}$$

We argue by contradiction. Suppose that there is a $\delta > 0$ such that

$$\liminf_{q \to p} \mathcal{L}(q, X) + \delta \leq \mathcal{L}(p, X).$$

Note that, due to the definition of $\mathcal{L}$, $\mathcal{L}(q, X) \geq \lambda(-X)$ for all $q \neq 0$ with λ given by (4.2.3). Therefore, there is a sequence of vectors $\{q_n\}_{n \in \mathbb{N}}$ with $|q_n| = 1$ for every $n \in \mathbb{N}$ such that $q_n \to p$ and

$$\mathcal{L}(q_n, X) \to \liminf_{q \to p} \mathcal{L}(q, X)$$

as $n \to \infty$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{L}(q_n, X) - \frac{\delta}{2} \leq \liminf_{q \to p} \mathcal{L}(q, X) \qquad \text{for all } \ n \geq n_0.$$

Putting all together, and using the definition of $\mathcal{L}$, we have

$$\sup_{\substack{|v|=1 \\ v \perp p}} \langle Xv, v \rangle + \frac{\delta}{2} \leq \sup_{\substack{|v|=1 \\ v \perp q_n}} \langle Xv, v \rangle \qquad \text{for all } \ n \geq n_0. \tag{4.2.6}$$

Moreover, since

$$\sup_{\{|v|=1 \ \ v \perp q_n\}} \langle Xv, v \rangle \leq \Lambda(X)$$

and the set $\{v \in \mathbb{S}^d : \ v \perp q_n\}$ is a compact set for all $n \in \mathbb{N}$, there exists a sequence of vectors in $\{w_n\} \subset \mathbb{S}^{d \times d}$ with $w_n \perp q_n$ such that the maximums are attained,

$$\langle Xw_n, w_n \rangle = \sup_{\substack{|v|=1 \\ v \perp q_n}} \langle Xv, v \rangle \qquad \text{for all } \ n \in \mathbb{N}. \tag{4.2.7}$$

In addition, there is a subsequence $\{w_{n_k}\} \subset \{w_n\}$ so that $w_{n_k} \to w_0$ for some $w_0 \in \mathbb{S}^{d \times d}$ as $k \to \infty$. Observe, since $w_{n_k} \perp q_{n_k}$ and $|w_{n_k}| = 1$, we get

$$|\langle w_{n_k}, p \rangle| = |\langle w_{n_k}, p - q_{n_k} \rangle| + |\langle w_{n_k}, q_{n_k} \rangle| \leq |p - q_{n_k}|$$

and this implies that the limit must satisfy

$$w_0 \perp p$$

due to the fact that $q_{n_k} \to p$ as k goes to infinity. As a consequence, using (4.2.6) and (4.2.7),

$$\langle Xw_0, w_0 \rangle + \frac{\delta}{2} \leq \sup_{\substack{|v|=1 \\ v \perp p}} \langle Xv, v \rangle + \frac{\delta}{2} \leq \langle Xw_{k_n}, w_{k_n} \rangle \qquad \text{for all } \ n_k \geq n_0$$

and this is a contradiction, because $w_{n_k} \to w_0$ as k goes to infinity.

We now need to prove that $\mathcal{L}$ is upper semicontinuous in the first variable for $p \neq 0$, that is

$$\limsup_{q\to p} \mathcal{L}(q,X) \leq \mathcal{L}(p,X). \tag{4.2.8}$$

To see this, we proceed by contradiction, arguing in a similar way to the one used before. Suppose that there is a $\delta > 0$ such that

$$\mathcal{L}(p,X) + \delta \leq \limsup_{q\to p} \mathcal{L}(q,X).$$

Observe that $\mathcal{L}(q,X) \leq \Lambda(-X)$ for all $q \neq 0$ with Λ given by (4.2.4). For that reason, there is a family of vectors $\{q_n\}_{n\in\mathbb{N}}$ such that $q_n \to p$,

$$\mathcal{L}(q_n,X) \longrightarrow \limsup_{q\to p} \mathcal{L}(q,X)$$

as n goes to infinity and $n_0 \in \mathbb{N}$ for which

$$\sup_{\substack{|v|=1\\ v\perp q_n}} \langle Xv,v\rangle + \frac{\delta}{2} \leq \sup_{\substack{|v|=1\\ v\perp p}} \langle Xv,v\rangle \qquad \text{for all } \ n \geq n_0.$$

Moreover, there exists a vector w_0 with $w_0 \perp p$ and $|w_0| = 1$ such that

$$\langle Xw_0,w_0\rangle = \sup_{\substack{|v|=1\\ v\perp p}} \langle Xv,v\rangle.$$

Therefore,

$$\langle Xw_n,w_n\rangle + \frac{\delta}{2} \leq \sup_{\substack{|v|=1\\ v\perp q_n}} \langle Xv,v\rangle + \frac{\delta}{2} \leq \langle Xw_0,w_0\rangle \ \text{ for all } \ n \geq n_0 \tag{4.2.9}$$

with

$$w_n = \frac{w_0 - \frac{\langle w_0,q_n\rangle}{|q_n|^2} q_n}{\left|w_0 - \frac{\langle w_0,q_n\rangle}{|q_n|^2} q_n\right|},$$

due to the fact that $w_n \perp q_n$ and $|w_n| = 1$. Notice that , since $q_n \to p$ as $n \to \infty$ and $w_0 \perp p$, we get that $w_n \to w_0$ as n goes to infinity. Therefore, (4.2.9) gives a contradiction. Thus, from (4.2.5) and (4.2.8) we have that

$$\mathcal{L}(p,X) = \lim_{q\to p} \mathcal{L}(q,X) \ \text{ for all } \ (p,X) \in \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d},$$

and, as a consequence, we conclude that $\mathcal{L}$ is continuous in $\left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$.

Next, we prove *ii)*. Since, $p \neq 0$,

$$\mathcal{L}_*(p,X) = \lim_{\varepsilon\to 0^+} \inf \Big\{ \mathcal{L}(q,Y) \ : \ \ 0 < |p-q| \leq \varepsilon, \ \ 0 < |X-Y| \leq \varepsilon \Big\},$$
$$\mathcal{L}^*(p,X) = \lim_{\varepsilon\to 0^+} \sup \Big\{ \mathcal{L}(q,Y) \ : \ \ 0 < |p-q| \leq \varepsilon, \ \ 0 < |X-Y| \leq \varepsilon \Big\}$$

and

$$\lambda(-Y) \leq \mathcal{L}(q,Y) \leq \Lambda(-Y) \text{ for all } (q,Y) \in \big(\mathbb{R}^d \setminus \{0\}\big) \times \mathbb{S}^{d\times d},$$

we have that,

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} \Big\{\lambda(-Y) \,:\, 0<|X-Y|\leq \varepsilon\Big\} &\leq \mathcal{L}_*(p,X)\\ &\leq \mathcal{L}^*(p,X) \leq \limsup_{\varepsilon\to 0^+} \Big\{\Lambda(-Y) \,:\, 0<|X-Y|\leq \varepsilon\Big\}\end{aligned}$$

for all $(p,X) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}$. Therefore, we have

$$\lambda(-X) \leq \mathcal{L}_*(p,X) \leq \mathcal{L}^*(p,X) \leq \Lambda(-X) \qquad \text{for all } (p,X) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}.$$

Moreover, the previous arguments imply that $\mathcal{L}_*(0,O) = \mathcal{L}^*(0,O)$. Thus, we have also proved *iii)*.

Now, we continue the proof with *iv)*. If $X, Y \in \mathbb{S}^{d\times d}$ are such that $0 \leq Y - X$, then $0 \leq \langle (Y-X)v, v\rangle$ for all $v \in \mathbb{R}^d$. Therefore,

$$\mathcal{L}(p,Y) = \inf_{\substack{|v|=1\\ v\perp p}} \Big\{ -\langle Yv, v\rangle\Big\} \leq \inf_{\substack{|v|=1\\ v\perp p}} \Big\{ -\langle Xv, v\rangle\Big\} = \mathcal{L}(p,X)$$

for all $p \neq 0$. So, $\mathcal{L}$ is degenerate elliptic.

Finally, we prove *v)*. Consider $p \in \mathbb{R}^d \setminus \{0\}$. For $v \perp p$ it holds that

$$(p\otimes p)v = 0, \tag{4.2.10}$$

and hence we have that

$$\langle Xv, v\rangle = \langle Xv, \big(I - \frac{p\otimes p}{|p|^2}\big)v\rangle = \langle \big(I - \frac{p\otimes p}{|p|^2}\big)Xv, v\rangle$$

for every $v \perp p$ and every $X \in \mathbb{S}^{d\times d}$, due to the fact that $\big(I - \frac{p\otimes p}{|p|^2}\big)$ is a symmetric matrix. Then, $\mathcal{L}$ can be also to be expressed as

$$\mathcal{L}(p,X) = \inf_{\substack{|v|=1\\ v\perp p}} \Big\{ -\langle \big(I - \frac{p\otimes p}{|p|^2}\big)Xv, v\rangle\Big\} \qquad (p,X) \in \big(\mathbb{R}^d \setminus \{0\}\big) \times \mathbb{S}^{d\times d}.$$

Now, since $\alpha p \otimes \alpha p = \alpha^2 (p\otimes p)$ and (4.2.10) holds, we have that

$$\langle \big(I - \frac{\alpha p\otimes \alpha p}{|\alpha p|^2}\big)(\alpha X + \sigma p\otimes p)v, v\rangle = \alpha \langle \big(I - \frac{p\otimes p}{|p|^2}\big)Xv, v\rangle$$

and, therefore, we conclude that $\mathcal{L}$ is geometric. □

In the following result, we aim to provide an intuition for the scenario when the gradient is zero. In this case, as we will see in the proof, we cannot define $\mathcal{L}(0,X)$ properly as the first eigenvalue of $-X$. This is due to the fact that $\mathcal{L}_*(0,X) = \lambda(-X)$ but in general $\mathcal{L}^*(0,X) \neq \lambda(-X)$.

Corollary 4.2.7. *Let $X \in \mathbb{S}^{d\times d} \setminus \{O\}$. Then $\mathcal{L}_*(0,X) = \lambda(-X)$. However, there exists $X \neq O$ such that, $\mathcal{L}^*(0,X) \neq \lambda(-X)$.*

Proof. Let $X \in \mathbb{S}^{d\times d} \setminus \{O\}$. By *i)* and *ii)* of the previous Proposition 4.2.6, we have

$$\begin{aligned}\lambda(-X) \leq \mathcal{L}_*(0,X) &= \lim_{\varepsilon\to 0^+} \inf\{\mathcal{L}(q,Y) \ : \ 0<|q|\leq\varepsilon, \ \ 0<|X-Y|\leq\varepsilon\} \\ &= \lim_{\varepsilon\to 0^+} \inf\{\mathcal{L}(q,X) \ : \ 0<|q|\leq\varepsilon\} \leq \lim_{\varepsilon\to 0^+} \mathcal{L}(q_\varepsilon,X)\end{aligned}$$

for every family of vectors $\{q_\varepsilon\}_{\varepsilon>0}$ such that $q_\varepsilon \neq 0$ and $|q_\varepsilon| \leq \varepsilon$. Since $X \in \mathbb{S}^{d\times d}$, there is a unit vector v_1 that satisfies $\langle -Xv_1, v_1\rangle = \lambda(-X)$. We take a unit vector v_2 such that $v_2 \perp v_1$ and define $q_\varepsilon = \varepsilon v_2$. Therefore, we obtain

$$\lambda(-X) \leq \mathcal{L}_*(0,X) \leq \lim_{\varepsilon\to 0^+} \inf_{\substack{|v|=1 \\ v\perp\varepsilon v_2}} \Big\{ -\langle Xv,v\rangle \quad = \min_{\substack{|v|=1 \\ v\perp v_2}} \Big\{ -\langle Xv,v\rangle \quad = \lambda(-X)$$

due to the fact that $v_1 \perp v_2$.

Now, we will show that $\mathcal{L}^*(0,X) \neq \lambda(-X)$ in general. Let $X \in \mathbb{S}^{d\times d}$ such that the eigenvalue $\lambda(-X)$ has multiplicity one. By definition of the upper semicontinuous envelope and *i)* of Proposition 4.2.6, it holds that

$$\mathcal{L}^*(0,X) = \lim_{\varepsilon\to 0^+} \sup\Big\{\mathcal{L}(q,Y) \ : \ 0<|q|\leq\varepsilon, \ \ 0<|X-Y|\leq\varepsilon\Big\} \geq \lim_{\varepsilon\to 0^+} \mathcal{L}(p_\varepsilon,X)$$

for every family of vectors $\{p_\varepsilon\}_{\varepsilon>0}$ such that $p_\varepsilon \neq 0$ and $|p_\varepsilon| \leq \varepsilon$. Let v_1 be the unit eigenvector associated to the eigenvalue $\lambda(-X)$ as before. We take $p_\varepsilon = \varepsilon v_1$. Since the eigenvalue $\lambda(-X)$ has multiplicity one,

$$\mathcal{L}^*(p,X) \geq \lim_{\varepsilon\to 0^+} \mathcal{L}(\varepsilon v_1,X) = \min_{\substack{|v|=1 \\ v\perp v_1}} \Big\{ -\langle Xv,v\rangle \Big\} > \lambda(-X).$$

This ends the proof. □

Now, we can state that the two definitions of being a viscosity solution to our evolution problem coincide when the test function has a nonvanishing gradient, that is, at regular points of $\mathcal{L}$. The proof follows from the fact that the function $\mathcal{L}$ is continuous in $\big(\mathbb{R}^d \setminus \{0\}\big) \times \mathbb{S}^{d\times d}$.

Theorem 4.2.8. *Since $\mathcal{L}$ is continuous in $\big(\mathbb{R}^d \setminus \{0\}\big) \times \mathbb{S}^{d\times d}$, the definitions 4.2.3 and 4.2.5 are equivalent for regular points of $\mathcal{L}$. That is, if u is viscosity subsolution (resp. supersolution) to (4.1.1) and $\phi \in C^{2,1}(\mathbb{R}^d \times (0,T))$ touches u^* (u_*) strictly from above (from below) and $\nabla\phi(x,t) \neq 0$, then*

$$\partial_t\phi(x,t) + \mathcal{L}(\phi(x,t)) \leq 0 \ \ (\geq 0)$$

if and only if

$$\partial_t\phi(x,t) + \mathcal{L}(\nabla\phi(x,t), D^2\phi(x,t)) \leq 0 \ \ (\geq 0).$$

Finally, we introduce the notion of viscosity sub and supersolution of the obstacle problem (P).

Definition 4.2.9. We say that $u : \mathbb{R}^d \times [0,\infty) \mapsto \mathbb{R}$ is a viscosity subsolution to (P) if its upper semicontinuous envelope, u^*, satisfies that $u^*(x,0) \leq u_0(x)$ and for every $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ such that ϕ touches u^* at $(x,t) \in \mathbb{R}^d \times (0,\infty)$ strictly from above, that is, $\psi - u^*$ has a strict minimum at (x,t), we have that,

$$\max\Big\{ -\partial_t\phi(x,t) - \mathcal{L}_*(\phi(x,t)), \psi(x) - u(x,t)\Big\} \geq 0.$$

That means, $u^*(x,0) \leq u_0(x)$ and, if $\psi(x) < u(x,t)$, then,

$$\partial_t\phi(x,t) + \mathcal{L}_*(\phi(x,t)) \leq 0.$$

Equivalently, in terms of the jets $\mathcal{P}^{2,+}u^*(x,t)$, u is a viscosity subsolution to the problem (P) if $u_*(x,0) \leq u_0(x)$ and

$$\max\Big\{ -a - \mathcal{L}_*(p,X), \psi(x) - u(x,t)\Big\} \geq 0 \qquad \text{for all } (a,p,X) \in \mathcal{P}^{2,+}u^*(x,t).$$

Conversely, u is called a viscosity supersolution to (P) if its lower semicontinuous envelope, u_*, satisfies the following: $u_*(x,0) \geq u_0(x)$ and for every $\phi \in C^{2,1}(\mathbb{R}^d \times (0,\infty))$ such that ϕ touches u_* at $(x,t) \in \mathbb{R}^d \times (0,\infty)$ strictly from below, that is, $u_* - \phi$ has a strict minimum at (x,t), we have that,

$$\max\Big\{ -\partial_t\phi(x,t) - \mathcal{L}^*(\phi(x,t)), \psi(x) - u(x,t)\Big\} \leq 0.$$

That is,

$$u(x,t) \geq \psi(x,t), \;\; \partial_t\phi(x_0,t_0) + \mathcal{L}^*(\phi(x_0,t_0)) \geq 0 \;\text{ and }\; u_*(x,0) \geq u_0(x).$$

Equivalently, in terms of the jets $\mathcal{P}^{2,-}u(x,t)$, u is a viscosity supersolution to (P) if $u_*(x,0) \geq u_0(x)$ and

$$\max\Big\{ -a - \mathcal{L}^*(p,X), \psi(x) - u(x,t)\Big\} \leq 0 \qquad \text{for all } (a,p,X) \in \mathcal{P}^{2,-}u_*(x,t).$$

Finally, u is a viscosity solution to (P) if it is both a viscosity supersolution and a viscosity subsolution.

That our problem (P) has a comparison principle is due to the fact that the problem without the obstacle also has a comparison principle. In [78] the author adapted the proof of Theorem 8.2 in [36] incorporating an obstacle to the problem. For completeness, we introduce and prove the version stated in Theorem 4.1.1 of the comparison principle for our problem.

Proof of the Comparison Principle, Theorem 4.1.1. We prove this result arguing by contradiction, assuming that we have a viscosity supersolution $\overline{u}$ and a viscosity subsolution $\underline{u}$ such that $\overline{u} \not\geq \underline{u}$. The contradiction that we are pursuing comes from the application of [36, Theorem 8.3] combined with the properties of $\mathcal{L}$.

We split the proof into two steps. In the first step of the proof we include the necessary conditions in order to apply [36, Theorem 8.2] to the function

$$w(x,y,t) = \underline{u}_\eta(x,t) - \overline{u}(x,t) - \widetilde{\phi}_{\varepsilon,\alpha}(x,y,t),$$

with $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,T)$. Here $\underline{u}_\eta$ is a modification of $\underline{u}$ and $\widetilde{\phi}_{\varepsilon,\alpha}$ is an admisible test function with doubled variables, both will be defined later. We will look at $w(x,y,t)$ at a family of points

$$\{(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha})\} \subset \mathbb{R}^d \times \mathbb{R}^d \times (0,T).$$

We will also prove that, at these points, $\underline{u}_\eta$ is larger than the obstacle as well as useful bounds for $\hat{x}_{\varepsilon,\alpha}$ and $\hat{y}_{\varepsilon,\alpha}$. Finally, in the second step, we apply [36, Theorem 8.2] and, using the properties of $\mathcal{L}$ stated in Proposition 4.2.6, we obtain the desired contradiction.

Step 1. For each $\eta > 0$, we take

$$\underline{u}_\eta(x,t) := \max\left\{\underline{u}(x,t) - \frac{\eta}{(T-t)}, \psi(x)\right\}.$$

Since $\underline{u}$ is a viscosity subsolution to (P), $\underline{u}_\eta$ is a viscosity subsolution to

$$\max\left\{ -\partial_t w(x,t) - \mathcal{L}(\nabla w(x,t), D^2 w(x,t)) - \frac{\eta}{T^2}, \psi - w\right\} = 0$$

in $\mathbb{R}^d \times (0,T)$.

We will prove that $\underline{u}_\eta \leq \overline{u}$. Once we get this result, taking the limit $\eta \to 0^+$ we obtain the desired conclusion, $\underline{u} \leq \overline{u}$. To show this, we argue by contradiction. Suppose that there exists a point $(\bar{x}, \bar{t}) \in \mathbb{R}^d \times (0,T)$ such that

$$\underline{u}_\eta(\bar{x},\bar{t}) - \overline{u}(\bar{x},\bar{t}) \geq 2\delta > 0.$$

We consider the function $\phi_{\alpha,\varepsilon} : \mathbb{R}^d \times \mathbb{R}^d \times (0,T) \mapsto \mathbb{R}$, given by

$$\phi_{\alpha,\varepsilon}(x,y,t) := \underline{u}_\eta(x,t) - \overline{u}(y,t) - \frac{\alpha}{4}|x-y|^4 - \frac{\varepsilon}{2}\Big(|x|^2 + |y|^2\Big).$$

Since $\phi_{\varepsilon,\alpha}$ is an upper semicontinuous function, it attains maximums in compact sets. In addition, due to the penalization term $-\frac{\varepsilon}{2}(|x|^2+|y|^2)$ and the definition of $\underline{u}_\eta$, we get that, for each $\varepsilon > 0$ and $\eta > 0$, there exist $R_\varepsilon > 0$ and $0 < T_\eta < T$ such that

$$\phi_{\alpha,\varepsilon}(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha}) = \sup_{\mathbb{R}^d \times \mathbb{R}^d \times (0,T)} \{\phi_{\alpha,\varepsilon}(\cdot,\cdot,\cdot)\}$$

for some $(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha}) \in \bar{B}_{R_\varepsilon}(0) \times \bar{B}_{R_\varepsilon}(0) \times [0, T_\eta]$. Observe that $\phi_{\varepsilon,\alpha}$ and $(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha})$ also depend on η but, to simplify the notation, we are not making this dependence explicit.

The distance between $\hat{x}_{\varepsilon,\alpha}$ and $\hat{y}_{\varepsilon,\alpha}$ remains uniformly bounded in ε for each positive $\eta > 0$ and $\alpha > 0$: if ε is small enough

$$\phi_{\alpha,\varepsilon}(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha}) \geq \phi_{\varepsilon,\alpha}(\bar{x}, \bar{x}, \bar{t}) \geq \delta \tag{4.2.11}$$

and then

$$\|\underline{u}\|_\infty + \|\overline{u}\|_\infty \geq \underline{u}_\eta(\hat{x}_{\alpha,\varepsilon}, \hat{t}_{\alpha,\varepsilon}) - \overline{u}(\hat{y}_{\alpha,\varepsilon}, \hat{t}_{\alpha,\varepsilon}) \geq \delta + \frac{\alpha}{4}|\hat{x}_{\alpha,\varepsilon} - \hat{y}_{\alpha,\varepsilon}|^4 + \frac{\varepsilon}{2}(|\hat{x}_{\alpha,\varepsilon}|^2 + |\hat{y}_{\alpha,\varepsilon}|^2), \tag{4.2.12}$$

which implies that there is a positive constant C depending only on the infinity norm of $\underline{u}$ and $\overline{u}$ such that

$$|\hat{x}_{\alpha,\varepsilon} - \hat{y}_{\alpha,\varepsilon}| \leq C\alpha^{-1/4}. \tag{4.2.13}$$

Also, by (4.2.12), the sequences $\{\varepsilon|\hat{x}_{\alpha,\varepsilon}|^2\}$ and $\{\varepsilon|\hat{y}_{\alpha,\varepsilon}|^2\}$ are bounded (with respect to both ε and α). Then, for each $\alpha > 0$, we have

$$\varepsilon\hat{x}_{\alpha,\varepsilon},\ \varepsilon\hat{y}_{\alpha,\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0^+. \tag{4.2.14}$$

Moreover, for every $\alpha > 0$ large enough and for every subsequence of times $\left(\hat{t}_{\alpha,\tilde{\varepsilon}}\right)_{\tilde{\varepsilon}>0} \subset \left(\hat{t}_{\alpha,\varepsilon}\right)_{\varepsilon>0}$, $\hat{t}_{\alpha,\tilde{\varepsilon}} \not\to 0$ as $\tilde{\varepsilon} \to 0$. This property comes from the uniform continuity of the initial datum u_0. Let us prove this by contradiction. Suppose that for some $\alpha_0 > 0$, $\hat{t}_{\alpha_0,\tilde{\varepsilon}} \to 0$ as $\tilde{\varepsilon} \to 0$. Then, by (4.2.11),

$$0 < \delta \leq \underline{u}(\hat{x}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}}) - \overline{u}(\hat{y}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}}).$$

Suppose that there exists $C' > 0$ such that $|\hat{x}_{\alpha_0,\tilde{\varepsilon}}| \leq C'$ for every $\tilde{\varepsilon} > 0$. Then, by (5.2.7), there exist $x_{\alpha_0}, y_{\alpha_0} \in \mathbb{R}^d$ such that $\hat{x}_{\alpha_0,\tilde{\varepsilon}} \to x_{\alpha_0}$ (up to a subsequence) and $\hat{y}_{\alpha_0,\tilde{\varepsilon}} \to y_{\alpha_0}$ and they satisfy

$$|\hat{x}_{\alpha_0} - \hat{y}_{\alpha_0}| \leq C\alpha_0^{-1/4}. \tag{4.2.15}$$

Here C is the same constant that appears in (5.2.7). Hence, since $\underline{u} - \overline{u}$ is upper semicontinuous, we get

$$0 < \delta \leq \limsup_{\tilde{\varepsilon}\to 0^+} \left(\underline{u}(\hat{x}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}}) - \overline{u}(\hat{y}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}})\right) \leq \underline{u}(\hat{x}_{\alpha_0}, 0) - \overline{u}(\hat{y}_{\alpha_0}, 0) \leq u_0(\hat{x}_{\alpha_0}) - u_0(\hat{y}_{\alpha_0})$$

since $\underline{u}(\cdot, 0) \leq u_0(\cdot) \leq \overline{u}(\cdot, 0)$. Therefore, since u_0 is uniformly continuous, if α_0 is large enough, we obtain from (4.2.15) the following contradiction,

$$0 < \delta \leq u_0(\hat{x}_{\alpha_0}) - u_0(\hat{y}_{\alpha_0}) \leq \frac{\delta}{2}.$$

On the other hand, if $|\hat{x}_{\alpha_0,\tilde{\varepsilon}}| \to \infty$ as $\tilde{\varepsilon} \to 0^+$, by (5.2.7), $|\hat{y}_{\alpha_0,\tilde{\varepsilon}}| \to \infty$. Then, since

$$\underline{u}(\cdot, 0) \leq u_0(\cdot) \leq \overline{u}(\cdot, 0)$$

and

$$\lim_{|x|\to\infty} u_0(x) = \mu \in \mathbb{R},$$

we have that

$$0 < \delta \leq \limsup_{\tilde{\varepsilon}\to 0^+} \left(\underline{u}(\hat{x}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}}) - \overline{u}(\hat{y}_{\alpha_0,\tilde{\varepsilon}}, \hat{t}_{\alpha_0,\tilde{\varepsilon}})\right) \leq \mu - \mu = 0,$$

which is a contradiction. Thus, if $\alpha > 0$ is large enough, we have that $\liminf_{\varepsilon\to 0} \hat{t}_{\alpha,\varepsilon} > 0$. In what follows, let us write $(\hat{x}, \hat{y}, \hat{t})$ insted of $(\hat{x}_{\varepsilon,\alpha}, \hat{y}_{\varepsilon,\alpha}, \hat{t}_{\varepsilon,\alpha})$.

Our next goal is to prove that $\underline{u}_\eta$ is bigger that the obstacle at the point $(\hat{x}, \hat{t})$, i.e, we aim to show that

$$\underline{u}_\eta(\hat{x}, \hat{t}) > \psi(\hat{x}).$$

Let us establish this inequality arguing by contradiction. Suppose that $\underline{u}_\eta(\hat{x}, \hat{t}) = \psi(\hat{x})$. Since $\psi \leq \overline{u}$ in $\mathbb{R}^d \times (0, T)$, we get,

$$\begin{aligned} 0 <& \delta \leq \phi_{\varepsilon,\alpha}(\hat{x}, \hat{y}, \hat{t}) \leq \underline{u}_\eta(\hat{x}, \hat{y}) - \overline{u}(\hat{y}, \hat{t}) = \psi(\hat{x}) - \overline{u}(\hat{y}, \hat{t}) \\ &\leq \psi(\hat{x}) - \psi(\hat{y}) + \psi(\hat{y}) - \overline{u}(\hat{y}, \hat{t}) \leq \psi(\hat{x}) - \psi(\hat{y}) \leq \omega(|x - y|) \leq \omega(C\alpha^{-1/4}), \end{aligned}$$

where C is the constant in (5.2.7). Then, if α is large enough, we obtain

$$0 < \delta \leq \phi_{\varepsilon,\alpha}(\hat{x}, \hat{y}, \hat{t}) \leq \frac{\delta}{2},$$

and we get a contradiction. Then, we conclude that

$$\underline{u}_\eta(\hat{x}, \hat{t}) > \psi(\hat{x}),$$

which implies that for every test function φ such that $\varphi - \underline{u}_\eta \geq 0$ and attains its maximum at $(\hat{x}, \hat{t})$, the following inequality holds,

$$\varphi_t(\hat{x}, \hat{t}) + \mathcal{L}(\nabla\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) + \frac{\eta}{T^2} \leq 0$$

since we have that $\hat{t} \neq 0$.

Now, let us consider

$$w(x, y, t) = \underline{u}_\eta(x, t) - \overline{u}(y, t) - \widetilde{\phi}_{\varepsilon,\alpha}(x, y, t),$$

where

$$\widetilde{\phi}_{\varepsilon,\alpha}(x, y, t) = \phi_{\varepsilon,\alpha}(\hat{x}, \hat{y}, \hat{t}) + \frac{\alpha}{4}|x - y|^4 + \frac{\varepsilon}{2}\Big(|x|^2 + |y|^2\Big).$$

Notice that $w(x, y, t) \leq 0$ with equality only if $(x, y, t) = (\hat{x}, \hat{y}, \hat{t})$.

Step 2. Now, we are ready to apply [36, Theorem 8.2] to w, leading to the following result: for every $\mu > 0$ and for every pair of tuples $(a, X), (b, Y) \in \mathbb{R}^d \times \mathbb{S}^{d\times d}$ such that

$$(a, \nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x}, \hat{y}, \hat{t}), X) \in \mathcal{P}^{2,+}\underline{u}_\eta(\hat{x}, \hat{t}) \quad \text{and} \quad (b, \nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x}, \hat{y}, \hat{t}), Y) \in \mathcal{P}^{2,-}\overline{u}(\hat{y}, \hat{t})$$

we get that

$$a - b = 0$$

and

$$-\big(\frac{1}{\mu}+||A||\big)I_{2d}\leq\begin{pmatrix}X&0\\0&-Y\end{pmatrix}\leq A+\mu A^2, \tag{4.2.16}$$

where

$$A:=D^2_{x,y}\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t})=\varepsilon I_{2d}+\begin{pmatrix}P&-P\\-P&P\end{pmatrix}\quad\text{and}\quad P:=\frac{\alpha}{2}(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})+\frac{\alpha}{4}|\hat{x}-\hat{y}|^2I_d.$$

We remark that X and Y depend on ε and α. In particular, the inequalities in (4.2.16) give us

$$\|X\|+\|Y\|\leq C_\mu(\alpha|\hat{x}-\hat{y}|^2+\varepsilon) \tag{4.2.17}$$

and

$$X-Y\leq 2\varepsilon(1+\mu\varepsilon)I_d \tag{4.2.18}$$

for some constant C_μ that only depends on μ. Furthermore, since $\underline{u}_\eta$ is a viscosity subsolution to

$$\partial_t w(x,t)+\mathcal{L}(\nabla w(x,t),D^2w(x,t))+\frac{\eta}{T^2}=0$$

at $(\hat{x},\hat{t})$ and $\overline{u}$ is a viscosity supersolution to

$$\partial_t w(x,t)+\mathcal{L}(\nabla w(x,t),D^2w(x,t))=0$$

at $(\hat{y},\hat{t})$, we obtain

$$a+\mathcal{L}_*(\nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),X)+\frac{\eta}{T^2}\leq 0,$$

and

$$b+\mathcal{L}^*(-\nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),Y)\geq 0.$$

Since $a=b$, using the above inequalities we obtain

$$0<\frac{\eta}{T}\leq\mathcal{L}^*(-\nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),Y)-\mathcal{L}_*(\nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),X).$$

In addition, by (4.2.18) and the fact that $\mathcal{L}$ is degenerate elliptic (point *iv)* in Proposition 4.2.6), we have

$$0<\frac{\eta}{T}\leq\mathcal{L}^*(-\nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),X-2\varepsilon(1+\mu\varepsilon)I_n)-\mathcal{L}_*(\nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t}),X).$$

On the other hand, from a direct computation, we get

$$\nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t})=\hat{p}+\varepsilon\hat{x}\quad\text{and}\quad-\nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t})=\hat{p}-\varepsilon\hat{y}\quad\text{where}\quad\hat{p}:=\alpha|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y}).$$

Thanks to (5.2.7) and (4.2.14), $\nabla_x\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t})$ and $-\nabla_y\widetilde{\phi}_{\varepsilon,\alpha}(\hat{x},\hat{y},\hat{t})$ converge to the same vector $p_0\in\mathbb{R}^d$ as ε goes to zero (up to extracting a subsequence if necessary). Moreover, by (5.2.7) and (4.2.17), the matrices X are uniformly bounded with respect ε and therefore

X and $(X - 2\varepsilon(1+\mu\varepsilon)I_d)$ converge to some $X_0 \in \mathbb{S}^{d\times d}$ as ε goes to zero (again, up to a subsequence). Therefore, since $\mathcal{L}^*$ and $-\mathcal{L}_*$ are upper semicontinuous functions, we have

$$0 < \frac{\eta}{T^2} \leq \limsup_{\varepsilon\to 0^+} \left(\mathcal{L}^*(\hat{p}+\varepsilon\hat{y}, X - 2\varepsilon(1+\mu\varepsilon)I_n) - \mathcal{L}_*(\hat{p}+\varepsilon\hat{x}, X)\right) = \left(\mathcal{L}^* - \mathcal{L}_*\right)(p_0, X_0),$$

and the contradiction is straightforward. If $p_0 \neq 0$, then $\mathcal{L}^*(p_0, X_0) = \mathcal{L}_*(p_0, X_0)$ since $\mathcal{L}$ is continuous in $\left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{S}^{d\times d}$ as we proved in point *i)* of Proposition 4.2.6. If $p_0 = 0$, we have that $|\hat{x} - \hat{y}| \to 0$ as $\varepsilon \to 0^+$ which implies, by (4.2.17), $X_0 = 0$, and, by point iii) in Proposition 4.2.6, we obtain $\mathcal{L}_*(0, O) = 0 = \mathcal{L}^*(0, O)$. In both cases, we conclude that $\left(\mathcal{L}^* - \mathcal{L}_*\right)(p_0, X_0) = 0$ and we get the contradiction $0 < \eta/T^2 \leq 0$.

Therefore, we have proved that $\underline{u}_\eta \leq \overline{u}$ in $\mathbb{R}^d \times (0, T)$ for every $\eta > 0$. Thus, taking the limit $\eta \to 0$, we obtain the desired result, it holds that

$$\underline{u} \leq \overline{u}$$

in $\mathbb{R}^d \times (0, T)$. □

Once we know that our evolution problem (P) has a comparison principle (Theorem 4.1.1), we can prove that this problem has a unique solution. The proof of the existence is based on Perron's method as described in [78]. This proof also works for more general operators $\mathcal{L}$ that have a comparison principle and satisfy items *i)-v)* in Proposition 4.2.6.

Now, we will prove that there is a unique solution to (P), that is, Theorem (4.1.2). Let us consider W given by

$$W(x,t) = \inf\left\{w(x,t) \ : \quad w \in \mathcal{A}\right. \tag{4.2.19}$$

with

$$\mathcal{A} := \Big\{w(x,t) \ : \quad w \text{ is a viscosity supersolution} \tag{4.2.20}$$
$$w_t + \mathcal{L}w \geq 0 \text{ in } \mathbb{R}^d \times (0,\infty),\ w_*(\cdot,t) \geq \psi \text{ for all } t > 0 \text{ and } w_*(\cdot,0) \geq u_0\Big\}.$$

Our aim is to show that W is also a supersolution of the Obstacle Problem (P) and, in addition, it is a subsolution to

$$u_t + \mathcal{L}u = 0$$

inside the set $\{(x,t) : W(x,t) > \psi(x)\}$.

Before doing that, we need to establish that W is well defined and that it is the smallest supersolution of the Obstacle Problem (P) using the following lemma.

Lemma 4.2.10. *Let $\mathcal{F}$ be a family of viscosity supersolutions to $u_t + \mathcal{L}u = 0$ in $\mathbb{R}^d \times (0,\infty)$ and assume that the family is bounded below. Assume*

$$U(x,t) = \inf\left\{u(x,t) \ : \quad u \in \mathcal{F}\right\}$$

is a well defined function with $U(x,t) > -\infty$ for every $(x,t) \in \mathbb{R}^d \times (0,\infty)$. Then, U is also a supersolution to $u_t + \mathcal{L}u = 0$ in $\mathbb{R}^d \times (0,\infty)$.

Proof. Fix $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$. Suppose that $(a, p, X) \in \mathcal{P}^{2,-}U_*(x_0, t_0)$. We want to see that

$$a + F^*(p, X) \geq 0.$$

Thanks to the way in which U is defined, we can take a sequence $\{(x_n, t_n, u_n)\}_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{F}$ and $(x_n, t_n, u_n(x_n, t_n)) \to (x_0, t_0, U_*(x_0, t_0))$. By a parabolic version of [36, Proposition 4.3], there exists a sequence $\{(\hat{x}_n, \hat{t}_n, a_n, p_n, X_n)\}_{n\in\mathbb{N}}$ belonging to $\mathbb{R}^d \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^{d\times d}$ such that $(a_n, p_n, X_n) \in \mathcal{P}^{2,-}u_{n*}(\hat{x}_n, \hat{t}_n)$ for all $n \in \mathbb{N}$ and

$$\big(a_n, p_n, X_n\big) \to (a, p, X) \ \text{ as } \ n \to \infty.$$

Therefore, we have

$$a_n + \mathcal{L}^*(p_n, X_n) \geq 0$$

due to the fact that u_n is a viscosity supersolution in $\mathbb{R}^d \times (0, \infty)$ for every $n \in \mathbb{N}$. Since $\mathcal{L}^*$ is upper semicontinuous, we obtain

$$a + \mathcal{L}^*(p, X) \geq 0$$

as we wanted to show. □

In the following proposition, we show that W, defined in (4.2.19), is the smallest supersolution of (4.1.1) that is larger than the obstacle and with initial datum larger than u_0. As a consequence, W is a supersolution to the Obstacle Problem (P). To prove this result, we strongly use Lemma 4.2.10 and the fact that $\mathcal{L}$ is a geometric function (see point *v)* in Proposition 4.2.6).

Proposition 4.2.11. *The function $W : \mathbb{R}^d \times (0, \infty) \mapsto \mathbb{R}$ given by*

$$W(x, t) = \inf\big\{w(x, t) \ : \ \ w \in \mathcal{A}$$

with $\mathcal{A}$ as in (4.2.20) is well defined and belongs to $\mathcal{A}$. That is, W a supersolution to (4.1.1) in $\mathbb{R}^d \times (0, \infty)$ such that $W(\cdot, 0) \geq u_0$ and $W \geq \psi$ or, equivalently, W is a supersolution to (P). Furthermore, $W(x, 0) = u_0(x)$ for all $x \in \mathbb{R}^d$ and W is a lower semicontinuous function in $\mathbb{R}^d \times (0, \infty)$.

Proof. We divide the proof in several steps.

First step. We begin by constructing an upper barrier w^+. We look for a function $w^+ \in \mathcal{A}$ such that $w^+(\cdot, 0) = u_0$. Consider two constants α and β such that $\beta > 2\alpha > 0$. Fix $z \in \mathbb{R}^d$. We define the function

$$h_z(x, t) := \alpha|x - z|^2 + \beta t$$

for $(x, t) \in \mathbb{R}^d \times (0, \infty)$. Observe that, since $D^2 h_z = -2\alpha I_d$, we have that $\mathcal{L}_*(\nabla h_z, D^2 h_z) = \mathcal{L}^*(\nabla h_z, D^2 h_z) = -2\alpha$ in $\mathbb{R}^d \times (0, \infty)$ due to the item *ii)* in Proposition 4.2.6. Then, since

$$\partial_t h_z(x, t) + \mathcal{L}(\nabla h_z(x, t), D^2 h_z(x, t)) = \beta - 2\alpha > 0$$

for every $(x,t) \in \mathbb{R}^d \times (0,\infty)$, we have that h_z is a supersolution to (4.1.1) in $\mathbb{R}^d \times (0,\infty)$ for every $z \in \mathbb{R}^d$. Now, our goal is to construct a supersolution to (4.1.1) based on h_z that is ordered with the initial datum at $t = 0$. Since u_0 is bounded and continuous, we consider

$$H_z(s) := \max\Big\{u_0(y) \ : \ \ y \in \mathbb{R}^d \ \text{ satisfies that } \ \alpha|y-z|^2 \leq s\Big\}, \qquad s \in \mathbb{R}.$$

Since $\mathcal{L}$ is degenerate elliptic and geometric (see Proposition 4.2.6) and H_z is a nondecreasing continuous function and h_z is a supersolution to (4.1.1) for all $z \in \mathbb{R}^d$, then, by [32, Theorem 5.2.], the function

$$H_z \circ h_z(x,t) = \max\Big\{u_0(y) : \ \ \alpha(|y-z|^2 - |x-z|^2) \leq \beta t\Big\}, \quad (x,t) \in \mathbb{R}^d \times (0,\infty)$$

is, endeed, a supersolution to (4.1.1) for all $z \in \mathbb{R}^d$. Moreover, since $H_z \circ h_z \geq u_0$ and $u_0 \geq \psi$, we have that $H_z \circ h_z \geq \psi$ for all $z \in \mathbb{R}^d$. Therefore, $H_z \circ h_z \in \mathcal{A}$ for every $z \in \mathbb{R}^d$, so $\mathcal{A}$ is not empty.

Now, to construct a function in $\mathcal{A}$ that takes the initial datum u_0 at $t = 0$, we define

$$w^+(x,t) := \inf_{z\in\mathbb{R}^d}\Big\{H_z \circ h_z(x,t)\Big\},$$

for $(x,t) \in \mathbb{R}^d \times [0,\infty)$. Observe that, by construction, $\|H_z \circ h_z\|_\infty \leq \|u_0\|_\infty$ for every $z \in \mathbb{R}^d$. Moreover, since $H_z \circ h_z$ is a supersolution to (4.1.1) in $\mathbb{R}^d \times (0,\infty)$ for every $z \in \mathbb{R}^d$, by Lemma 4.2.10, we get that w^+ is also a supersolution of the parabolic equation (4.1.1) in $\mathbb{R}^d \times (0,\infty)$.

Finally, since that $H_z \circ h_z \geq \psi$ and $H_z \circ h_z(\cdot,0) \geq u_0(\cdot)$ for every $z \in \mathbb{R}^d$, we obtain that w^+ satisfies

$$w^+(x,t) \geq \psi(x) \ \text{ and } \ w^+(\cdot,0) \geq u_0(\cdot).$$

In addition, $w^+(\cdot,0) = u_0(\cdot)$ due to the fact that

$$u_0(x) \leq w^+(x,0) \leq H_x \circ h_x(x,0) = \max\{u_0(y) \ : \ \ \alpha|y-x|^2 \leq 0\} = u_0(x)$$

for every $x \in \mathbb{R}^d$. Hence, we conclude that $w^+ \in \mathcal{A}$ and $w^+(\cdot,0) = u_0$.

Second step. We will prove that W is well defined, $W \in \mathcal{A}$ and $W(\cdot,0) = u_0(\cdot)$.

First, observe that since $w^+ \in \mathcal{A}$, $w^+ \leq \|u_0\|_\infty$ and $w \geq \psi \geq -\|\psi\|_\infty$ for every $w \in \mathcal{A}$, the infimum in $\mathcal{A}$, W, is well defined and satisfies $-\|\psi\|_\infty \leq W \leq \|u_0\|_\infty$. Furthermore, by Lemma 4.2.10, W is also a supersolution to (4.1.1) in $\mathbb{R}^d \times (0,\infty)$.

On the other hand, since $w \geq \psi$ and $w(\cdot,0) \geq u_0(\cdot)$ for every $w \in \mathcal{A}$, we have that $W \geq \psi$ and $W(\cdot,0) \geq u_0(\cdot)$. In addition, it holds that

$$u_0(x) \leq W(x,0) \leq w^+(x,0) = u_0(x)$$

for every $x \in \mathbb{R}^d$. □

Now, we show that W is also a subsolution to our obstacle problem.

Proposition 4.2.12. *The function defined by (4.2.19), W, is a subsolution to the obstacle problem (P) in $\mathbb{R}^d \times (0,\infty)$ with $W(\cdot,0) = u_0(\cdot)$.*

Proof. Let $(x_0,t_0) \in \mathbb{R}^d \times (0,\infty)$ such that $W(x_0,t_0) - \psi(x_0) \geq \gamma > 0$. We want to prove that, for every $(a,p,X) \in \mathcal{P}^{2,+}W^*(x_0,t_0)$, it holds that

$$a + \mathcal{L}_*(p,X) \leq 0.$$

We will argue by contradiction. Suppose that there exists $(a_0,p_0,X_0) \in \mathcal{P}^{2,+}W^*(x_0,t_0)$ such that

$$a_0 + \mathcal{L}_*(p_0,X_0) \geq \gamma > 0. \tag{4.2.21}$$

Our goal is to construct a supersolution U to (4.1.1) that is strictly smaller than W at (x_0,t_0). Notice that, by the definition of W as the infimum of supersolutions, this leads to the desired contradiction.

First, we define

$$\begin{aligned} u_{\mu,\sigma}(x,t) := W(x_0,t_0) - \frac{\gamma}{3} + a_0(t-t_0) + \langle p_0, x-x_0\rangle \\ + \frac{1}{2}\langle X_0(x-x_0), x-x_0\rangle + \sigma(|t-t_0| + |x-x_0|^2). \end{aligned}$$

where $\mu,\sigma > 0$ and they will be chosen later. Observe that, since

$$W(x_0,t_0) > \psi(x_0) + \gamma,$$

we have that there is a r_0 depending on γ, $|a_0|$, $|p_0|$, $|X_0|$ and ψ such that

$$W(x_0,t_0) - \frac{\mu}{3} + a_0(t-t_0) + \langle p_0, x-x_0\rangle + \frac{1}{2}\langle X_0(x-x_0), x-x_0\rangle \geq \psi(x) + \frac{\gamma}{3}$$

for every $(x,t) \in B_{r_0}(x_0,t_0)$ (here we use the assumption that ψ is continuous). Moreover, thanks to the fact that $\sigma > 0$, the above implies that

$$u_{\mu,\sigma}(x,t) \geq \psi(x) + \frac{\gamma}{3} \quad \text{for all} \quad (x,t) \in B_{r_0}(x_0,t_0).$$

In addition, by (4.2.21), we have

$$a_0 + \mathcal{L}^*(p_0,X_0) \geq a_0 + \mathcal{L}_*(p_0,X_0) \geq \gamma > 0.$$

Then, there exists $0 < r_1 \leq r_0$ where r_1 does not depend on σ such that $u_{\mu,\sigma}$ is supersolution to $u_t + \mathcal{L}u = 0$ in $B_{r_1}(x_0,t_0)$.

On the other hand, we claim that we can choose $\sigma > 0$ large enough such that

$$W(x,t) \leq u_{\mu,\sigma}(x,t) \qquad \text{for all} \quad \frac{r_0}{2} \leq |(x,t) - (x_0,t_0)| \leq r_0.$$

Considering

$$\tau := \max_{B_{r_0}(x_0,t_0)} W^* - \min_{B_{r_0}(x_0,t_0)} W \geq 0,$$

we have that $W(x,t) \leq W(x_0,t_0) + \tau$ and, as a consequence,

$$W(x,t) \leq u_{\mu,\sigma}(x,t) + \frac{\gamma}{3} + \tau - a_0(t-t_0) - \langle p_0, x-x_0\rangle \\ -\frac{1}{2}\langle X_0(x-x_0), x-x_0\rangle - \sigma(|t-t_0| + |x-x_0|^2)$$

for every $(x,t) \in B_{r_0}(x_0,t_0)$. Now, suppose that $r_0/2 \leq |(x,t)-(x_0,t_0)| \leq r_0$. Then, taking

$$\sigma = \frac{2(\gamma/3 + \tau + |a_0|r_0 + |p_0|r_0 + |X_0|r_0^2/2)}{r_0 + r_0^2}$$

it holds that

$$W(x,t) \leq u_{\mu,\sigma}(x,t) + \frac{\gamma}{3} + \tau + |a_0|r_0 + |p_0|r_0 + \frac{|X_0|}{2}r_0^2 - \frac{\sigma}{2}(r_0 + r_0^2) = u_{\gamma,\sigma}(x,t).$$

Then the claim is proven.

Now, we define

$$U(x,t) := \begin{cases} \min\{W(x,t), u_{\delta,\sigma}(x,t)\} & \text{if } |(x,t)-(x_0,t_0)| \leq r_1, \\ W(x,t) & \text{otherwise.} \end{cases}$$

Observe that, by Lemma 4.2.10 and the previous steps, U is a supersolution to $u_t + \mathcal{L}u = 0$ in $\mathbb{R}^d \times (0,\infty)$ such that $U \geq \psi$ and $U(\cdot,0) = u_0(\cdot)$ (taking r_1 small enough such that $U(\cdot,0) = W(\cdot,0)$). That is, $U \in \mathcal{A}$ with $\mathcal{A}$ defined in (4.2.20). As a consequence, by the definition of W in (4.2.19) and using that W is lower semicontinuous (see Proposition 4.2.11), we get

$$W(x_0,t_0) \leq U_*(x_0,t_0).$$

However,

$$U_*(x_0,t_0) = \min\left\{W(x_0,t_0), u_{\gamma,\sigma}(x_0,t_0)\right\}_* \\ = \min\left\{W(x_0,t_0), W(x_0,t_0) - \frac{\gamma}{3}\right\} = W(x_0,t_0) - \frac{\gamma}{3} < W(x_0,t_0)$$

which is a contradiction with the previous inequality. Then, we conclude that W is a subsolution to the obstacle problem (P). □

Now we are ready to conclude the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. By Proposition 4.2.11, the function W is a viscosity supersolution of (P) such that $W(x,0) = u_0$ for all $x \in \mathbb{R}^d$. On the other hand, by Proposition 4.2.12, W is a subsolution to (P). Therefore, thanks to the Comparison Principle 4.1.1, $W^* \leq W_*$ in $\mathbb{R}^d \times [0,\infty)$. Hence, W is a continuous function in $\mathbb{R}^d \times [0,\infty)$ and W is the unique solution to the obstacle problem (P). □

4.3 The limit of the value functions for the game is the solution to the parabolic equation

In this subsection, we will obtain that the unique solution u to the obstacle problem (P) is the locally uniform limit of the value functions u^ε of the ε-game as ε goes to zero, see Theorem 4.1.3. Our strategy is to prove that the functions

$$\overline{u}(x,t) = \limsup_{\substack{\varepsilon\to 0^+\\(y,s)\to(x,t)}} u^\varepsilon(y,s), \quad \underline{u}(x,t) = \liminf_{\substack{\varepsilon\to 0^+\\(y,s)\to(x,t)}} u^\varepsilon(y,s), \tag{4.3.1}$$

are, respectively, a viscosity subsolution and a viscosity supersolution to the problem (4.1.1) with $\overline{u}(x,0) = \underline{u}(x,0) = u_0(x)$. Therefore, by the Comparison Principle, Theorem 4.1.1, we get that $\overline{u} \le \underline{u}$. Since by definition $\underline{u} \le \overline{u}$, we then obtain that $\overline{u} = \underline{u}$. Hence, the limit of the family u^ε exists and it is the unique solution u to (4.1.1). Finally, as a consequence of the convergence $u^\varepsilon \to u$, for each $t > 0$ we will be able to approximate the set where $u(\cdot,t)$ is positive by the set where $u^\varepsilon(\cdot,t)$ is positive for ε small enough, see Corollary 4.1.4.

Notice that the final payoff is uniformly bounded,

$$\sup_{x\in\mathbb{R}^d} \max\Big\{|u_0(x)|, |\psi_\varepsilon(x)|\Big\} \le C$$

for a constant C independent of ε. Therefore, we conclude that the value function of the game, u^ε is uniformly bounded.

Lemma 4.3.1. *There exists a constant C, independent of ε, such that*

$$\sup_{(x,t)\in\mathbb{R}^d\times[0,\infty)} |u^\varepsilon(x,t)| \le C.$$

Next, we prove that the half-relaxed limits of the family u^ε, $\overline{u}$ and $\underline{u}$, given by (4.3.1), attain the initial condition.

Proposition 4.3.2. *Let $\overline{u}$ and $\underline{u}$ be defined as in (4.3.1). Then*

$$\overline{u}(x,0) = \underline{u}(x,0) = u_0(x), \qquad \text{for every } x \in \mathbb{R}^d.$$

Proof. We give strategies for both Paul and Carol that will yield the two inequalities needed for the proof.

Paul's strategy. In this strategy Paul will never stop playing and collect the amount given by the obstacle. Fix $z \in \mathbb{R}^d$. Paul chooses vectors v_k perpendicular to $x_{k-1} - z$ for $k = 1, \cdots n$. We let Carol choose the sign freely. This implies that, independently of Carol's choice,

$$|x_n - x_0|^2 = |x_0 - z|^2 + n\varepsilon^2.$$

Let $t = n\varepsilon^2$. Then, we have that

$$|x_n - x_0| = \sqrt{|x_0 - z|^2 + t}.$$

Since Paul has been choosing the vectors, we get a bound from below

$$u_0(x_n) \leq u^\varepsilon(x_0, t), \qquad \text{with } |x_n - x_0| = \sqrt{|x_0 - z|^2 + t}.$$

Carol's strategy. Now we let Paul choose freely the possibility to stop and obtain the payoff given by the obstacle and the direction he wishes to point to, but Carol chooses the sign b_k for $k = 1, \cdots, n$ so that each x_k stays as near as possible to x_0. The furthest that each position $\widetilde{x}_k$ can be from x_0 is, following the previous computation, $\sqrt{|x_0 - z|^2 + t}$. So we get

$$|\widetilde{x}_n - x_0| \leq \sqrt{|x_0 - z|^2 + t}.$$

Notice that Paul could choose to stop at the obstacle at any moment. Since we selected a particular strategy for Carol, what we get in this case is that

$$u^\varepsilon(x_0, t) \leq \max\left\{\psi(x_0), \max\left\{\psi(\widetilde{x}_1), \cdots \max\left\{\psi(\widetilde{x}_{n-1}), u_0(\widetilde{x}_n) \right\}\right\}\right\}.$$

Attaining the initial condition. From our previous computations we have that

$$u_0(x_n) \leq u^\varepsilon(x_0, t) \leq \max\left\{\psi(x_0), \max\left\{\psi(\widetilde{x}_1), \cdots \max\left\{\psi(\widetilde{x}_{n-1}), u_0(\widetilde{x}_n) \right\}\right\}\right\}$$

with

$$|x_n - z| = \sqrt{|x_0 - z|^2 + t}, \qquad |\widetilde{x}_k - z| \leq \sqrt{|x_0 - z|^2 + t}.$$

for $k = 1, \cdots n$. Now, taking the lim sup and lim inf when $\varepsilon \to 0$, $x_0 \to z$ and $t \to 0$, using the continuity of u_0 and ψ the we get

$$u_0(z) \leq \underline{u}(z, 0) \leq \overline{u}(z, 0) \leq \max\left\{\psi(z), u_0(z)\right.$$

To conclude, observe that $u_0(z) \geq \psi(z)$ for every $z \in \mathbb{R}^d$. So, we have obtained that

$$\underline{u}(z, 0) = \overline{u}(z, 0) = u_0(z)$$

for an arbitrary $z \in \mathbb{R}^d$, that is what we wanted to prove. □

Remark 4.3.3. When u_0 is a Lipschitz function, from the previous estimates in the proof of Proposition 4.3.2, we get that

$$u_0(x_n) \leq u^\varepsilon(x_0, t) \leq \max\left\{u_0(x_0), u_0(\widetilde{x}_1), \cdots u_0(\widetilde{x}_{n-1}), u_0(\widetilde{x}_n)\right\},$$

with

$$|x_n - z| = \sqrt{|x_0 - z|^2 + t}, \qquad |\widetilde{x}_k - z| \leq \sqrt{|x_0 - z|^2 + t}.$$

for $k = 1, \cdots n$. Hence, using that u_0 is Lipschitz we obtain an explicit bound of the form

$$|u_0(z) - u^\varepsilon(x_0, t)| \leq C(|x_0 - z| + \sqrt{t}).$$

Now, let us show that the half-relaxed limits are sub and supersolutions to our parabolic obstacle problem. Recall that ψ is a continuous and bounded function such that

$$K = \{x \in \mathbb{R}^d : \psi(x) > 0\},$$

and that we have characterized the enlarged set $K_\varepsilon = K + B_{N\varepsilon}(0)$ in a similar way, that is, there exists a function ψ_ε, such that we have

$$K_\varepsilon = \{x \in \mathbb{R}^d : \psi_\varepsilon(x) > 0\}$$

and $\psi_\varepsilon \to \psi$ uniformly when $\varepsilon \to 0$.

Theorem 4.3.4. *The functions $\overline{u}$ and $\underline{u}$ defined in (4.3.1) are respectively viscosity subsolution and supersolutions to the obstacle problem (P).*

Proof. First, remark that $\overline{u}$ is upper semicontinuous and $\underline{u}$ is lower semicontinuous. We know that $u^\varepsilon(x,t) \geq \psi(x)$ thanks to the DPP. This implies that both $\overline{u}$ and $\underline{u}$ satisfy the same inequality.

Let us write the Dynamic Programming Principle of our parabolic game as

$$0 = \max\left\{\psi_\varepsilon(x) - u^\varepsilon(x,t), \sup_{|v|=1}\min_{b=\pm 1}\left\{u^\varepsilon(x+bv\varepsilon, t-\frac{1}{2}\varepsilon^2) - u^\varepsilon(x,t)\right\}\right\}.$$

The use of this equation is the key to pass to the limit in the viscosity sense.

Let us show that $\overline{u}$ is a viscosity subsolution. To this end, let $\phi \in C^{2,1}(\mathbb{R}^d \times \mathbb{R})$ be a function such as $\phi - \overline{u}$ achieves a strict minimum at (x_0,t_0) in $\overline{B}_{R_0}(x_0,t_0)$. We want to prove that

$$\min\left\{\partial_t\phi(x_0,t_0) + \mathcal{L}(\phi(x_0,t_0)), \overline{u}(x_0,t_0) - \psi(x_0)\right\} \leq 0.$$

For every $\varepsilon > 0$, we can choose $(x_\varepsilon, t_\varepsilon) \in \overline{B}_{R_0}(x_0,t_0)$ such that

$$\phi(x_\varepsilon,t_\varepsilon) - u^\varepsilon(x_\varepsilon,t_\varepsilon) - \varepsilon^3 \leq \inf_{\overline{B}_{R_0}(x_0,t_0)}\{\phi(x,t) - u^\varepsilon(x,t)\} \leq \phi(x,t) - u^\varepsilon(x,t),$$

for every $(x,t) \in \overline{B}_{R_0}(x_0,t_0)$. As $\overline{B}_{R_0}(x_0,t_0)$ is a compact set, $(x_\varepsilon,t_\varepsilon)$ converge, up to a subsequence, to some $(\overline{x},\overline{t})$ as $\varepsilon \to 0^+$. We have that $(\overline{x},\overline{t}) = (x_0,t_0)$, since (x_0,t_0) is the only maximum of $(\overline{u} - \phi)$ in $B_{R_0}(x_0,t_0)$ by construction. Rearranging the previous expression, we get

$$u^\varepsilon(y,s) - u^\varepsilon(x_\varepsilon,t_\varepsilon) \leq \phi(y,s) - \phi(x_\varepsilon,t_\varepsilon) + \varepsilon^3,$$

for every $(y,s) \in \overline{B}_{R_0}(x_0,t_0)$.

Evaluating the DPP expression at $(x,t) = (x_\varepsilon,t_\varepsilon)$ we obtain

$$\begin{aligned} 0 &= \max\left\{\psi_\varepsilon(x_\varepsilon) - u^\varepsilon(x_\varepsilon,t_\varepsilon), \sup_{|v|=1}\min_{b=\pm 1}\left\{u^\varepsilon(x_\varepsilon+bv\varepsilon, t_\varepsilon-\frac{1}{2}\varepsilon^2) - u^\varepsilon(x_\varepsilon,t_\varepsilon)\right\}\right\} \\ &\leq \max\left\{\psi_\varepsilon(x_\varepsilon) - u^\varepsilon(x_\varepsilon,t_\varepsilon), \sup_{|v|=1}\min_{b=\pm 1}\left\{\phi(x_\varepsilon+bv\varepsilon, t_\varepsilon-\frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon,t_\varepsilon)\right\} + \varepsilon^3\right\}. \end{aligned}$$

This says that either

$$\psi_\varepsilon(x_\varepsilon) \geq u^\varepsilon(x_\varepsilon, t_\varepsilon) \quad \text{or} \quad \sup_{|v|=1} \min_{b=\pm 1} \left\{ \phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon) \right\} + \varepsilon^3 \geq 0.$$

Since $u^\varepsilon(x,t) \geq \psi_\varepsilon(x)$, we can rewrite this as

$$\psi_\varepsilon(x_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon) \quad \text{or} \quad \sup_{|v|=1} \min_{b=\pm 1} \left\{ \phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon) \right\} + \varepsilon^3 \geq 0,$$

which, dividing by ε^2 the inequality, is equivalent to

$$\psi_\varepsilon(x_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon), \quad \text{or} \quad \sup_{|v|=1} \min_{b=\pm 1} \left\{ \frac{\phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2} \right\} + \varepsilon \geq 0,$$

that is,

$$0 \leq \max \left\{ \psi_\varepsilon(x_\varepsilon) - u^\varepsilon(x_\varepsilon, t_\varepsilon), \sup_{|v|=1} \min_{b=\pm 1} \left\{ \frac{\phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2} \right\} + \varepsilon \right\}.$$

Now, we divide the proof into two cases. First, assume that there is a sequence $(x_\varepsilon, t_\varepsilon)$ that satisfies that $\psi_\varepsilon(x_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon)$. In this case, we have that $\psi_\varepsilon(x_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon)$ for a sequence $\varepsilon \to 0$. Thus, passing to the limit in

$$u^\varepsilon(y,s) \leq \psi(x_\varepsilon) + \phi(y,s) - \phi(x_\varepsilon, t_\varepsilon) + \varepsilon^3,$$

that holds for every $(y,s) \in \overline{B}_{R_0}(x_0, t_0)$, we obtain $\psi(x_0) = \bar{u}(x_0, t_0)$.

For the second case, we can assume that $\psi_\varepsilon(x_\varepsilon, t_\varepsilon) < u^\varepsilon(x_\varepsilon, t_\varepsilon)$ and, therefore, we have

$$\sup_{|v|=1} \min_{b=\pm 1} \left\{ \frac{\phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2} \right\} + \varepsilon \geq 0. \tag{4.3.2}$$

Let us analyze this expression. We have

$$\begin{aligned} &\phi\Big(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big) - \phi(x_\varepsilon, t_\varepsilon) \\ &= \phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big) - \phi(x_\varepsilon, t_\varepsilon) + \phi\Big(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big) - \phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big). \end{aligned}$$

The first couple of terms can be estimated with a first order Taylor expansion

$$\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big) - \phi(x_\varepsilon, t_\varepsilon) = -\frac{1}{2}\varepsilon^2 \partial_t \phi(x_\varepsilon, t_\varepsilon) + o_\varepsilon(\varepsilon^2),$$

while the second couple requires a second order Taylor expansion

$$\begin{aligned} &\phi\Big(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big) - \phi(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) \\ &= b\varepsilon\Big\langle \nabla\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big), v\Big\rangle + \frac{1}{2}\varepsilon^2 \langle D^2\phi(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2)v, v\rangle + o_\varepsilon(\varepsilon^2). \end{aligned}$$

Choose $v_\varepsilon \in \mathbb{S}^{N-1}$ such that

$$\min_{b=\pm 1}\left\{\frac{\phi(x_\varepsilon + bv_\varepsilon\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2}\right\} = \sup_{|v|=1}\min_{b=\pm 1}\left\{\frac{\phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2}\right\},$$

such a $v_\varepsilon \in \mathbb{S}^{N-1}$ exists due to the continuity of ϕ. Then,

$$\begin{aligned}&\sup_{|v|=1}\min_{b=\pm 1}\left\{\frac{\phi(x_\varepsilon + bv\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2) - \phi(x_\varepsilon, t_\varepsilon)}{\varepsilon^2}\right\}\\ &= -\frac{1}{2}\partial_t\phi(x_\varepsilon, t_\varepsilon) + \frac{1}{2}\langle D^2\phi(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2)v_\varepsilon, v_\varepsilon\rangle + \min_{b=\pm 1}\left\{\frac{b}{\varepsilon}\left\langle\nabla\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big), v_\varepsilon\right\rangle\right\} + o_\varepsilon(1).\end{aligned}$$

Since $|v_\varepsilon| = 1$, up to a subsequence, we can suppose that $v_\varepsilon \to v_0$ for some $v_0 \in \mathbb{S}^{N-1}$ when $\varepsilon \to 0^+$. This limit direction v_0 is going to satisfy that $v_0 \perp \nabla\phi(x_0, t_0)$. If it did not, for ε small enough we have that

$$c\langle\nabla\phi(x_0, t_0), v_0\rangle \leq \left\langle\nabla\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big), v_\varepsilon\right\rangle \leq C\langle\nabla\phi(x_0, t_0), v_0\rangle$$

for two positive constants $c, C > 0$ independent of ε. But if this was the case and, for instance, $\langle\nabla\phi(x_0, t_0), v_0\rangle > 0$, we would have that

$$\lim_{\varepsilon\to 0^+}\min_{b=\pm 1}\left\{\frac{b}{\varepsilon}\left\langle\nabla\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big), v_\varepsilon\right\rangle\right\} \leq \lim_{\varepsilon\to 0^+} -c\frac{1}{\varepsilon}\langle\nabla\phi(x_0, t_0), v_0\rangle = -\infty.$$

Assuming that $\langle\nabla\phi(x_0, t_0), v_0\rangle < 0$ we arrive to the same conclusion (since we can choose the opposite sign, $b = +1$). This contradicts (4.3.2).

With the perpendicularity condition in mind, $\langle\nabla\phi(x_0, t_0), v_0\rangle = 0$, and taking into account that

$$\min_{b=\pm 1}\left\{\frac{b}{\varepsilon}\left\langle\nabla\phi\Big(x_\varepsilon, t_\varepsilon - \frac{1}{2}\varepsilon^2\Big), v_\varepsilon\right\rangle\right\} \leq 0,$$

we can take limits and obtain

$$0 \leq -\frac{1}{2}\partial_t\phi(x_0, t_0) + \frac{1}{2}\langle D^2\phi(x_0, t_0)v_0, v_0\rangle \leq -\frac{1}{2}\partial_t\phi(x_0, t_0) + \sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_0,t_0)}}\frac{1}{2}\langle D^2\phi(x_0, t_0)v, v\rangle.$$

Hence, since $\overline{u}(\cdot, 0) = u_0(\cdot)$ (see Proposition 4.3.2), we have that $\overline{u}$ is a viscosity subsolution for the parabolic obstacle problem, (P).

To show that $\underline{u}$ is a supersolution, the structure of the proof remains basically the same (with the appropriate changes in the inequalities), but the last step in this case is simpler,

since we arrive to

$$\begin{aligned}0 &\geq \sup_{|v|=1}\min_{b=\pm 1}\left\{\frac{\phi(x_\varepsilon+bv\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)-\phi(x_\varepsilon,t_\varepsilon)}{\varepsilon^2}\right\}\\ &\geq \sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)}}\min_{b=\pm 1}\left\{\frac{\phi(x_\varepsilon+bv\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)-\phi(x_\varepsilon,t_\varepsilon)}{\varepsilon^2}\right\}\\ &= -\frac{1}{2}\partial_t\phi(x_\varepsilon,t_\varepsilon)+\sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)}}\left\{\frac{1}{2}\langle D^2\phi(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)v,v\rangle+o_\varepsilon(1)\right\}.\end{aligned}$$

Choose v_0 such that

$$\langle D^2\phi(x_0,t_0)v_0,v_0\rangle=\sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_0,t_0)}}\langle D^2\phi(x_0,t_0)v,v\rangle.$$

Define the vector w_ε as

$$w_\varepsilon=\left\langle v_0,\nabla\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)\right\rangle\frac{\nabla\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)}{\nabla\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)^2}$$

if $\nabla\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)\neq 0$, and $w_\varepsilon=0$ otherwise. Observe that

$$(v_0-w_\varepsilon)\perp\nabla\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right).$$

With this in mind, we get

$$\begin{aligned}&\sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2)}}\left\{\frac{1}{2}\left\langle D^2\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)v,v\right\rangle+o_\varepsilon(1)\right\}\\ &\quad\geq\left\langle\frac{1}{2}D^2\phi\left(x_\varepsilon,t_\varepsilon-\frac{1}{2}\varepsilon^2\right)\frac{v_0-w_\varepsilon}{\|v_0-w_\varepsilon\|},\frac{v_0-w_\varepsilon}{\|v_0-w_\varepsilon\|}\right\rangle+o_\varepsilon(1)\\ &\quad\to\frac{1}{2}\langle D^2\phi(x_0,t_0)v_0,v_0\rangle=\sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_0,t_0)}}\frac{1}{2}\langle D^2\phi(x_0,t_0)v,v\rangle,\end{aligned}$$

since $w_\varepsilon\to 0$. Therefore, we found what we wanted, that is,

$$0\geq-\frac{1}{2}\partial_t\phi(x_0,t_0)+\sup_{\substack{|v|=1\\ v\perp\nabla\phi(x_0,t_0)}}\left\{\frac{1}{2}\langle D^2\phi(x_0,t_0)v,v\rangle\right\}.$$

Thus, since $\underline{u}(\cdot,0)=u_0(\cdot)$ (see Proposition 4.3.2) and that we have $\underline{u}\geq\psi$, we obtain that $\underline{u}$ is a viscosity supersolution to (P). □

Our aim is to establish the convergence of the functions u^ε given by the deterministic game to the solution to the parabolic problem (P). Having proved that the half relaxed limits satisfy that $\overline{u}$ is a viscosity subsolution and $\underline{u}$ is a viscosity supersolution to (P), our next goal is to prove that $\overline{u}$ and $\underline{u}$ coincide. We will achieve this using the Comparison Principle, Theorem 4.1.1.

Proof of Theorem 4.1.3 . Consider the upper semicontinuous function $\overline{u}$ and lower semicontinuous function $\underline{u}$ defined by (4.3.1).

On the one hand, by definition, we have that $\underline{u}(x,t) \leq \overline{u}(x,t)$ for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$.

By Proposition 4.3.4, $\overline{u}$ is subsolution and $\underline{u}$ is a supersolution of problem (P). Then, $\underline{u} \geq \overline{u}$ in $\mathbb{R}^d \times [0,\infty)$ due to the Comparison Principle, Theorem 4.1.1. Therefore, we conclude that

$$\overline{u} \equiv \underline{u}$$

which implies that $u := \overline{u} = \underline{u}$ is a continuous function in $\mathbb{R}^d \times [0,\infty)$. Moreover, u is the unique solution to the obstacle problem (P) and u^ε converges to u pointwise in $\mathbb{R}^d \times (0,\infty)$. This convergence is locally uniform. Let us show this by a contradiction argument. Assume there is a compact set $A \subset \mathbb{R}^d \times [0,\infty)$ such that the uniform convergence is not satisfied there. Then, there exist a positive number $\mu > 0$, and sequences $\{\varepsilon_j\}_{j\in\mathbb{N}}$, $\{(x_j,t_j)\}_{j\in\mathbb{N}} \subset A$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$|u^{\varepsilon_j}(x_j,t_j) - u(x_j,t_j)| \geq \mu \quad \text{for all } j \in \mathbb{N}.$$

Then, taking a subsequence if necessary, there exists $(x_\infty,t_\infty) \in A$ so that (x_j,t_j) converges to (x_∞,t_∞) as $j \to \infty$. Since u is continuous at (x_∞,t_∞), and we know by the previous part of this result that

$$\lim_{\substack{\varepsilon\to 0^+ \\ (y,s)\to(x_0,t_0)}} u^\varepsilon(y,s) = u(x_0,t_0),$$

we obtain that

$$\mu \leq |u^{\varepsilon_j}(x_j,t_j) - u(x_j,t_j)| \leq |u^{\varepsilon_j}(x_j,t_j) - u(x_\infty,t_\infty)| + |u(x_j,t_j) - u(x_\infty,t_\infty)| \to 0$$

when $j \to \infty$, which is a contradiction. □

Now, recall that we we want to study the positivity sets of u^ε and u,

$$\Omega_t^\varepsilon := \{x \,:\, u^\varepsilon(x,t) > 0\} \text{ and } \Omega_t := \{x \,:\, u(x,t) > 0\}.$$

Next, as a consequence of the convergence of u^ε to u, we will prove that for each fixed $t > 0$,

$$\Omega_t \subset \liminf_{\varepsilon\to 0} \Omega_t^\varepsilon \subset \limsup_{\varepsilon\to 0} \Omega_t^\varepsilon \subset \overline{\Omega}_t.$$

Proof of Corollary 4.1.4. Fix $t > 0$. We start proving that

$$\Omega_t \subset \liminf_{\varepsilon\to 0} \Omega_t^\varepsilon.$$

Suppose that $x \in \Omega_t$. Since

$$\liminf_{\varepsilon\to 0} \Omega_t^\varepsilon = \bigcup_{\varepsilon_0>0} \bigcap_{0<\varepsilon\leq\varepsilon_0} \Omega_t^\varepsilon,$$

we want to prove that there exists $\varepsilon_0 > 0$ such that $x \in \Omega_t^\varepsilon$ for all $0 < \varepsilon \leq \varepsilon_0$.

Using that $x \in \Omega_t$, we get that there is $\mu > 0$ such that $u(x,t) \geq \mu$. By Theorem 4.1.3 , $u^\varepsilon \to u$. Then, there is $\varepsilon_0 > 0$ such that $u^\varepsilon(x,t) \geq \mu/2$ for each $0 < \varepsilon \leq \varepsilon_0$. Thus, $x \in \Omega_t^\varepsilon$ for all $0 < \varepsilon \leq \varepsilon_0$.

On the other hand, suppose that $x \notin \overline{\Omega_t}$. Our aim is to prove that

$$x \notin \limsup_{\varepsilon\to 0} \Omega_t^\varepsilon.$$

Since

$$\limsup_{\varepsilon\to 0} \Omega_t^\varepsilon = \bigcap_{\varepsilon_0>0} \bigcup_{0<\varepsilon\leq\varepsilon_0} \Omega_t^\varepsilon,$$

the above is equivalent to prove that there exists $\varepsilon_0 > 0$ for which $x_0 \notin \Omega_t^\varepsilon$ for all $0 < \varepsilon \leq \varepsilon_0$. Due to the fact that $x \notin \overline{\Omega_t}$, we get that there exists $\mu > 0$ such that $u(x,t) \leq -\mu$. Using again the convergence of u^ε to u, we obtain that there exists $\varepsilon_0 > 0$ such that $u^\varepsilon(x,t) \leq -\mu/2$ for every $0 < \varepsilon \leq \varepsilon_0$, that is, for each $0 < \varepsilon \leq \varepsilon_0$, $x \notin \Omega_t^\varepsilon$. □

4.4 Long time behavior of the value function of the game and of the solution to the parabolic problem

The main goal of this subsection is to prove Theorem 4.1.6, that is, to prove that in the asymptotic limit as $t \to +\infty$ we recover the convex hull of the obstacle as the limit of the positivity set of the solution to the parabolic problem, that is,

$$\mathrm{co}(K) = \lim_{t\to\infty} \Omega_t.$$

In other words, we expect that the set where the solution stays positive is close to the convex hull of K for large times. This holds for any open set K as long as the condition on Ω_0 and K that we called (G) is satisfied. At this point we recall that we assumed that K is an open set. However, we can also work with a more general K, not necessarily open (but always assuming (G)), and in this case the best that we can conclude is

$$\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega_t \subset \limsup_{t\to\infty} \Omega_t \subset \overline{\mathrm{co}(K)}.$$

The game is going to be our main tool in this subsection, so we will also study the value of the game $u^\varepsilon(x,t)$ for large times before taking the limit in ε. To this end, we need to introduce a different notion of convexity that takes into account that there is a fixed length for the steps that one can make in the game. With this notion of convexity (that we will call ε-convexity), we define a better suited concept of "convex hull" that is adapted to the discrete nature of the game: the ε-convex hull of a set.

4.4.1 Defintion of ε-convexity and preliminaries

Let us first present some previous definitions to understand ε-convexity and see how they reproduce the expected properties of convexity. We will write $\ell_{x,y}$ as the standard segment between x and y (including both x and y),

$$\ell_{x,y} := \Big\{ z = \theta x + (1-\theta)y \ : \ 0 \leq \theta \leq 1 \Big\}.$$

Definition 4.4.1. For $0 < \varepsilon < 1$, we define the ε-segment between x and y as

$$\ell^{\varepsilon}_{x,y} := \begin{cases} \Big\{ z = \dfrac{k}{M}x + (1-\dfrac{k}{M})y \ : \ j = 0, 1, ..., M \Big\} & \text{if } \dfrac{|x-y|}{\varepsilon} = M \in \mathbb{N}, \\ \Big\{ x, y \Big\} & \text{if } \dfrac{|x-y|}{\varepsilon} \notin \mathbb{N}. \end{cases}$$

Notice that when $|x-y|/\varepsilon \notin \mathbb{N}$ the ε-segment contains just the two points x and y.
With this definition of ε-segment, we can define the concept of ε-convexity.

Definition 4.4.2. $A \subset \mathbb{R}^d$ is a *ε-convex* set if for any $x, y \in A$, $\ell^{\varepsilon}_{x,y} \subset A$.

Remark 4.4.3. The arbitrary intersection of ε-convex sets is ε-convex.

The reason behind using ε-convexity and not usual convexity is that due to the discrete nature of the game, Carol could force Paul to exit the convex hull of K, but not the ε-convex hull. Observe, for example, that if the obstacle was $K = \{a, b\}$ (without enlarging it) and $\ell^{\varepsilon}_{a,b} \subset \Omega_0$, Paul would win if the game starts at $x \in \ell^{\varepsilon}_{a,b}$ no matter how many plays the game last, but would loose if $x \in \ell_{a,b} \setminus \ell^{\varepsilon}_{a,b}$ if they play a large number of times.

Thanks to the fact that the intersection of ε-convex sets is ε-convex we can introduce the ε-convex hull of a set.

Definition 4.4.4. We define the ε-convex hull of a set A, $co_{\varepsilon}(A)$, as the smallest ε-convex set that contains A, that is,

$$\mathrm{co}_{\varepsilon}(A) := \bigcap_{\{C \ \varepsilon\text{-convex}: \ A \subset C\}} C.$$

Note that, thanks to the previous remark, the set $\mathrm{co}_{\varepsilon}(A)$ is ε-convex.
With these definitions at hand, what we aim to prove is Theorem 4.1.5, that is,

$$\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega^{\varepsilon}_t \subset \limsup_{t\to\infty} \Omega^{\varepsilon}_t \subset \mathrm{co}(K_{\varepsilon}) \subset \mathrm{co}(K) + B_{\varepsilon d}(0). \tag{4.4.1}$$

On the one hand, the ideas used in the proof of the inclusion $\limsup_{t\to\infty} \Omega^{\varepsilon}_t \subset \mathrm{co}(K_{\varepsilon})$ are a major novelty of this chapter. This result is derived using the characterization of of Ω^{ε}_t in Proposition 4.4.8 along with the strategies outlined in Definition 4.4.9, which each player can implement to achieve their goals. The key point here is that we avoid the need to use $\overline{\mathrm{co}(K_{\varepsilon})}$. We also emphasize that the inclusion holds true for $\varepsilon = 0$, although this is not deduced directly from taking the limit as $\varepsilon \to 0$.

On the other hand, the inclusion $\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega_t$ under the hypothesis (G) is one of the most relevant results of this work. Specifically, for the case $d \geq 3$, this inclusion constitutes the most essential contribution to the obstacle problem (P). The proof relies on an original argument based on polygonal constructions.

We will deduce Theorem 4.1.6 from these inclusions.

Under the hypothesis that $\mathrm{co}(K) \subset \Omega_0$ we will be able to prove a stronger statement, in fact, under this condition, we have

$$\mathrm{co}(K) \subset \mathrm{co}_\varepsilon(K_\varepsilon) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon \subset \limsup_{t\to\infty} \Omega_t^\varepsilon \subset \mathrm{co}(K_\varepsilon) \subset \mathrm{co}(K) + B_{\varepsilon d}(0). \tag{4.4.2}$$

Now, let us state a result concerning classical convex hulls that proves the last inclusion in the above chain.

Lemma 4.4.5. *It holds that*

$$co(K_\varepsilon) \subset co(K) + B_{\varepsilon d}(0).$$

Proof. Let us call $U := co(K) + B_{\varepsilon d}(0)$. We will prove that U is convex and $K_\varepsilon \subset U$. Then $\mathrm{co}(K_\varepsilon) \subset U$ because $\mathrm{co}(K_\varepsilon)$ is the intersection of all convex sets that contains K_ε.

On the one hand, U is convex. By definition

$$U = \{y = x + z : x \in \mathrm{co}(K) \text{ and } z \in B_{\varepsilon d}(0)\}.$$

Given $y_1, y_2 \in U$, $y_1 = x_1 + z_1$ and $y_2 = x_2 + z_2$ with $x_1, x_2 \in \mathrm{co}(K)$ and $z_1, z_2 \in B_{\varepsilon d}(0)$. Given $0 \leq \theta \leq 1$ let us consider

$$\theta y_1 + (1-\theta)y_2 = \big(\theta x_1 + (1-\theta)x_2\big) + \big(\theta z_1 + (1-\theta)z_2\big).$$

Using that $\mathrm{co}(K)$ and $B_{\varepsilon d}(0)$ are convex we get that $(\theta y_1 + (1-\theta)y_2) \in co(K) + B_{\varepsilon d}(0)$.

On the other hand, it holds that $K_\varepsilon \subset U$ because

$$K_\varepsilon = \{y = x + z : x \in K \text{ and } z \in B_{\varepsilon d}(0)\}$$

and $K \subset \mathrm{co}(K)$. □

Next, we want to characterize the ε-convex hull of a set by means of the ε-segments that we introduced. We need the following auxiliary definitions.

Definition 4.4.6. Let $A \subset \mathbb{R}^d$. Let us consider $\ell^0(A) := A$ and $\ell^{0,\varepsilon}(A) := A$. We define $\ell^j(A)$ and $\ell^{j,\varepsilon}(A)$ for $j \in \mathbb{N}$ as

$$\ell^j(A) := \Big\{x \in \mathbb{R}^d : \ \exists\, a, b \in \ell^{j-1}(A) \ \text{ such that } \ x \in \ell_{a,b}\Big\},$$
$$\ell^{j,\varepsilon}(A) := \Big\{x \in \mathbb{R}^d : \ \exists\, a, b \in \ell^{j-1,\varepsilon}(A) \ \text{ such that } \ x \in \ell^\varepsilon_{a,b}\Big\}.$$

Remark 4.4.7. Observe that the sequences $\{\ell^j(A)\}_{j=1}^\infty$ and $\{\ell^{j,\varepsilon}(A)\}_{j=1}^\infty$ are increasing sequences. We also have that $\ell^j(A) = \ell^1\left(\ell^{j-1}(A)\right)$ and $\ell^{j,\varepsilon}(A) = \ell^{1,\varepsilon}\left(\ell^{j-1,\varepsilon}(A)\right)$.

Notice that what the previous definitions do is add to a set the segments whose endpoints lie in the set. A naive first impression may suggest that $\mathrm{co}(A) = \ell^1(A)$, but if the dimension of the ambient space is bigger that one, this is not true in general. Next, we are going to characterize both $\mathrm{co}(A)$ and $\mathrm{co}_\varepsilon(A)$ of a set $A \subset \mathbb{R}^d$ with the families $\{\ell^j(A)\}$ and $\{\ell^{j,\varepsilon}(A)\}$, $j \in \mathbb{N}$.

Proposition 4.4.8. *Let $A \subset \mathbb{R}^d$. Then*

$$co_\varepsilon(A) = \bigcup_{j=0}^{\infty} \ell^{j,\varepsilon}(A), \qquad co(A) = \ell^d(A).$$

Proof. We begin proving the result for the ε-convex hull. First, note that the family $\{\ell^{j,\varepsilon}(A)\}_{j=1}^{\infty}$ is an increasing sequence of sets as we mentioned in Remark 4.4.7. Then, if $x, y \in \bigcup_{j=0}^{\infty} \ell^{j,\varepsilon}(A)$ are such that $|x-y|/\varepsilon \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $x, y \in \ell^{k,\varepsilon}(A)$ which implies $\ell^{\varepsilon}_{x,y} \in \ell^{k+1,\varepsilon}(A)$ and therefore, $\bigcup_{j=0}^{\infty} \ell^{j,\varepsilon}(A)$ is ε-convex. Since $A = \ell^{0,\varepsilon}(A)$, we obtain

$$\mathrm{co}_\varepsilon(A) \subset \bigcup_{j=0}^{\infty} \ell^{j,\varepsilon}(A).$$

Next, we want prove that $\ell^{n,\varepsilon}(A) \subset \mathrm{co}_\varepsilon(A)$ for every $n \in \mathbb{N}$. For $n = 1$, if $x \in \ell^{1,\varepsilon}(A)$ there exist $a, b \in A$ such that $x \in \ell^{\varepsilon}_{a,b}$. This implies that $x \in \mathrm{co}_\varepsilon(A)$ because $\mathrm{co}_\varepsilon(A)$ is ε-convex. Now, arguing inductively, suppose that $\ell^{n-1,\varepsilon}(A) \subset \mathrm{co}_\varepsilon(A)$. Let $x \in \ell^{n,\varepsilon}(A) \setminus \ell^{n-1,\varepsilon}(A)$. Therefore, there exists $a, b \in \ell^{n-1,\varepsilon}(A)$ such that $x \in \ell^{\varepsilon}_{a,b}$ and, applying the inductive hypothesis, we obtain that $x \in \mathrm{co}_\varepsilon(A)$ because $a, b \in \mathrm{co}_\varepsilon(A)$ and $\mathrm{co}_\varepsilon(A)$ is ε-convex. Thus,

$$\mathrm{co}_\varepsilon(A) \supset \bigcup_{j=0}^{\infty} \ell^{j,\varepsilon}(A).$$

The last part of the proposition, namely, the fact that $\mathrm{co}(A) = \ell^d(A)$ (notice that the dimension of the ambient space plays a role here), is just a simple consequence of Caratheodory's Theorem. This is a standard result for the convex hull of a set, which states that for any element $x \in \mathrm{co}(A)$, x is the result of a convex combination of at most $d+1$ points in A. □

Now, under our hypothesis on the relationship between $\mathrm{co}(K)$ and Ω_0 we will be able to prove the chain of inclusions (4.4.1), and, with further hypotheses, (4.4.2).

First, we introduce two important strategies (a version for each player) that will help us to understand the asymptotic behavior of the value of the game.

Definition 4.4.9. Let

$$\{(v_n, b_n)\}_{n\in\mathbb{N}} \subset \mathbb{S}^{d-1} \times \{1, -1\}$$

be a family of choices for Paul and Carol.

We say that the choice $\{(v_n, b_n)\}_{n\in\mathbb{N}}$ follows a concentric strategy at x for the ε-game if for some $z \in \mathbb{R}^d$ the sequence

$$x_1 := x + b_1 v_1, \quad x_n = x_{n-1} + b_n v_n, \quad n \in \mathbb{N}$$

satisfies that

$$|x_n - z|^2 \geq |x - z|^2 + n\varepsilon^2, \quad \text{for every } n \in \mathbb{N}. \tag{4.4.3}$$

For each starting point x and centre z, each player can implement a concentric strategy regardless the choices of the other player (if Paul plays without stopping at the obstacle).

- *Concentric strategy for Paul:* Here, at each round of the game, Paul chooses to continue playing and a vector v_n such that $v_n \perp x_{n-1} - z$ for $n \geq 2$ and $v_1 \perp x - z$. With these choices, no matter which election of sign Carol makes, we have that

$$|x_1 - z|^2 = |x - z|^2 + \varepsilon^2, \quad |x_n - z|^2 = |x_{n-1} - z|^2 + \varepsilon^2 \quad n \geq 2.$$

  Then, (5.3.10) holds for the sequence of positions of the game $\{x_n\}_{n\in\mathbb{N}}$.

- *Concentric strategy for Carol:* Assume that Paul chooses to play at every round. Let v_1 be the choice of Paul at the first round. Carol chooses $b_1 = \pm$ such that $x_1 \notin B_{R_1}(z)$ with $R_1 = |x - z|$. Then, $|x_1 - z|^2 \geq |x - z|^2 + \varepsilon^2$ with equality if and only if $v_1 \perp x - z$. For the n-th round, given Paul's selection of a vector, v_n, Carol chooses again the sign $b_n = \pm$ that satisfies that $x_n \notin B_{R_n}(z)$ with $R_n = |x_{n-1} - z|$. By an iterative argument, one can see that the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies (5.3.10).

Although Paul and Carol can use a concentric strategy for each x, this approach is not always optimal. However, in some cases, merely having the option to execute this strategy determines the winner of the game. The final outcome heavily depends on the position of the starting point x relative to $\mathrm{co}(K)$ and the time t at which the game starts (this determines the maximum number of plays). For an example, see for instance the proof of the following result.

Lemma 4.4.10. *For every $\varepsilon > 0$, it holds that*

$$\limsup_{t\to\infty} \Omega_t^{\tilde{\varepsilon}} \subset \overline{co(K_\varepsilon)} \qquad \textit{for all } \ 0 < \tilde{\varepsilon} \leq \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and choose $x \notin \overline{\mathrm{co}(K_\varepsilon)}$. We will prove that

$$x \notin \limsup_{t\to\infty} \Omega_t^{\tilde{\varepsilon}} = \bigcap_{t\geq 0} \bigcup_{s\geq t} \Omega_s^{\tilde{\varepsilon}} \text{ for every } \ 0 < \tilde{\varepsilon} \leq \varepsilon. \tag{4.4.4}$$

That is, we want to prove that there exists $t \geq 0$ such that $x \notin \Omega_s^{\tilde{\varepsilon}}$ for all $s \geq t$ and all $0 < \tilde{\varepsilon} \leq \varepsilon$.

We claim that there exists a time $\tau_x \geq 0$, that may depend on x, such that

$$u^{\tilde{\varepsilon}}(x,t) \leq -\mu/2 \text{ for all } t \geq \tau_x \text{ and } 0 < \tilde{\varepsilon} \leq \varepsilon, \tag{4.4.5}$$

where $\mu > 0$ is such that $u_0(x) \to -\mu$ as $|x| \to \infty$.

Using the claim, since $\Omega_s^{\tilde{\varepsilon}} = \{x \in \mathbb{R}^d : u^{\tilde{\varepsilon}}(x,s) > 0\}$, we have that $x \notin \Omega_s^{\tilde{\varepsilon}}$ for all $s \geq \tau_x$ and $0 < \tilde{\varepsilon} < \varepsilon$. Then, we have just proved (4.4.4).

Now, let us prove the claim. Since u_0 is continuous function and $u_0(x) \to -\mu$ as $|x| \to \infty$ for some $\mu > 0$, there exists $R_1 > 0$ such that

$$\{x \in \mathbb{R}^d : u_0(x) \geq -\mu/2\} \subset B_{R_1}(0).$$

On the other hand, since $x \notin \overline{\text{co}(K_\varepsilon)}$, there exist $z \in \mathbb{R}^d$ and $R_2 > 0$ such that $\text{co}(K_\varepsilon) \subset B_{R_2}(z)$ and $x \in \partial B_{R_2}(z)$ due to the fact that $\overline{\text{co}(K_\varepsilon)}$ is a bounded, closed and convex set. In this situation, using the concentric strategy, playing the $\tilde{\varepsilon}$-game, (with $0 < \tilde{\varepsilon} < \varepsilon$), Carol wins. No matter which choice of $v \in \mathbb{S}^{N-1}$ Paul makes, Carol can choose the next position $x_1 = x \pm v\tilde{\varepsilon}$ such that $|x_1 - z|^2 \geq R_2^2 + \tilde{\varepsilon}^2$ with equality if and only if $x - z \perp v$. For the next position, x_2, for any vector that Paul could choose, Carol choose the sign that increases the distance between the position and z, that is, she chooses the sign that ensures that $|x_2 - z|^2 \geq |x_1 - z|^2 + \tilde{\varepsilon}^2$. Then, $|x_2 - z|^2 \geq R_2^2 + 2\tilde{\varepsilon}^2$. Iterating this strategy, after n rounds, we have that $|x_n - z|^2 \geq R_2^2 + n\tilde{\varepsilon}^2$ and then we get

$$|x_n|^2 \geq R_2^2 + n\tilde{\varepsilon}^2 - |z|^2.$$

Taking, τ_x such that $\tau_x \geq R_1^2 + |z|^2$, we have

$$|x_n|^2 \geq R_2^2 + n\tilde{\varepsilon}^2 - |z|^2 \geq n\tilde{\varepsilon}^2 - |z|^2 \geq R_1^2 \text{ for all } n\tilde{\varepsilon}^2 \geq \tau_x.$$

Therefore, $x_n \notin B_{R_1}(0)$ for all $n\tilde{\varepsilon}^2 \geq \tau_x$ which implies, by the choice of R_1, that

$$u^{\tilde{\varepsilon}}(x,t) \leq \sup\{u_0(y) : y \notin B_1(0) \leq -\mu/2 \text{ for all } t \geq \tau_x \text{ and } 0 < \tilde{\varepsilon} \leq \varepsilon.$$

This ends the proof. □

Let us improve the previous result and show that if $x \notin \text{co}(K_\varepsilon)$, then Carol has a strategy that allows her to leave the domain Ω_0 if they play a large number of times (and hence Paul looses).

Lemma 4.4.11. *It holds that*

$$\limsup_{t\to\infty} \Omega_t^\varepsilon \subset co(K_\varepsilon).$$

Proof. We are going to describe a strategy for Carol so that if x is in $\partial\text{co}(K)$, then it forces Paul to eventually move to one position that is in $\mathbb{R}^d \setminus \overline{\text{co}(K)}$. Then, we can apply our previous result to conclude.

Since $x \notin \text{co}(K)$, there exists a hyperplane that separates x from $\text{co}(K)$. If Paul chooses a vector v not included in the hyperplane, Carol may choose the direction that will make

x_1 go to a positive distance to $\mathrm{co}(K)$. Then, from that point, Carol may apply a concentric strategy just like in the previous result and exit the domain Ω_0.

Now, if we assume that Paul always chooses vectors included in the hyperplane H, Carol may perform a concentric strategy inside the hyperplane so that she can also exit the domain Ω_0. In this case, Carol treats $H \cap \Omega_0$ as the new domain and $H \cap \overline{co(K)}$ as the new obstacle. Carol just needs to apply the concentric strategy to move out of $H \cap \Omega_0$. □

Remark 4.4.12. In this proof we only used that K_ε was an open set, but we are not using that it is an enlarged version of K. Recall that we are assuming K to be open. Then, if we consider the same game but using K instead of the enlarged version of K, this result would hold just the same, that is, we can deal with $x \in \partial\mathrm{co}(K)$ playing the game without enlarging the obstacle. This fact will be used to obtain the inclusion

$$\limsup_{t\to\infty} \Omega_t \subset \mathrm{co}(K)$$

that we prove next.

Notice that this inclusion cannot be deduced taking the limit as $\varepsilon \to 0$ in the previous result, $\limsup_{t\to\infty} \Omega_t^\varepsilon \subset \mathrm{co}(K_\varepsilon)$.

Theorem 4.4.13. *Recall that K is an open set. We have that*

$$\limsup_{t\to\infty} \Omega_t \subset co(K).$$

Proof. Let $x \notin \overline{\mathrm{co}(K)}$. Then, there exists $\varepsilon_0 > 0$, depending on x, such that $x \notin \overline{\mathrm{co}(K_{\varepsilon_0})}$. By the claim (4.4.5) in the proof of Lemma 4.4.10, there is a time $\tau_x > 0$ such that $u^\varepsilon(x,t) < -\mu/2$ for all $t \geq \tau_x$ and $0 < \varepsilon < \varepsilon_0$ where $\mu > 0$ is the constant such that $u_0(x) \to -\mu$ as $|x| \to \infty$. Since u^ε converges locally uniformly to u (see Theorem 4.1.3), we have that $u(x,t) \leq -\mu/2$ for all $t \geq \tau_x$ and every ε small enough and therefore, $x \notin \limsup_{t\to\infty} \Omega_t$.

If $x \in \partial\mathrm{co}(K)$ then the result follows from Remark 4.4.12 and the previous computations. □

Now we describe some strategies that Paul may use to win. As we already mentioned, it is convenient to divide our arguments into different situations. We present our ideas from the most to the least favorable situation for Paul.

4.4.2 Convex hull of K included in Ω_0

For this whole subsection we assume that

$$\mathrm{co}(K) \subset \Omega_0 \qquad \text{and} \qquad K_\varepsilon \subset \Omega_0 \text{ (for } \varepsilon \text{ small)}.$$

Notice that $K_\varepsilon \subset \Omega_0$ for ε small follows from $\overline{K} \subset \Omega_0$. Observe that when Ω_0 is bigger Paul has more opportunities to win. We are going to prove (4.4.2).

We need the next result for the first part of the chain of inclusions (4.4.2). In the next lemma we use our choice of the enlarged obstacle as $K_\varepsilon = K + B_{\varepsilon d}(0)$.

Lemma 4.4.14. *Let $K \subset \mathbb{R}^d$ be an open set and let $K_\varepsilon = K + B_{\varepsilon d}(0)$. Then, it holds that*

$$co(K) \subset co_\varepsilon(K_\varepsilon).$$

Proof. First, we show that $\ell^1(K) \subset \ell^{1,\varepsilon}(K_\varepsilon)$. Choose $a, b \in K$. Let us assume that $|a-b|/\varepsilon \notin \mathbb{N}$, since the proof in the case that $|a-b|/\varepsilon \in \mathbb{N}$ is simpler and follows exactly the same ideas.

If $|a-b|$ is not a natural multiple of ε, there exists $\hat{b}$ in the line formed by a, b such that $|a-\hat{b}|$ is a multiple of ε and the distance between b and $\hat{b}$ is at most $\varepsilon/2$ (if it was larger than $\varepsilon/2$ forward, it would be less than $\varepsilon/2$ backwards and vice versa). See Figure 4.5. Notice that if $|a-b|$ was a natural multiple of ε, the only necessary change to the proof would be choosing $\hat{b} = b$.

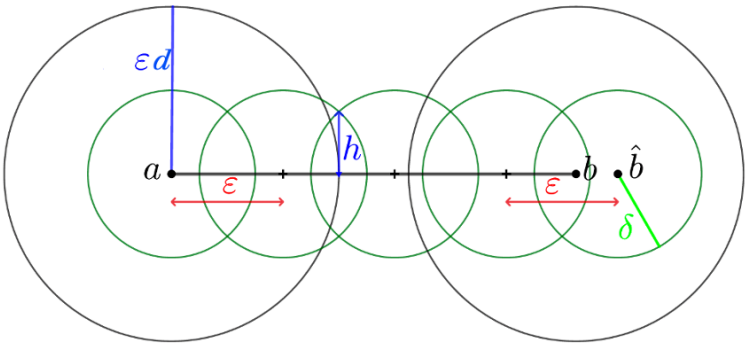


Figure 4.5

Since $a, b \in K$, then $B_{\varepsilon d}(a), B_{\varepsilon d}(b) \subset K_\varepsilon$. We know that the maximum distance between b and $\hat{b}$ is $\varepsilon/2$, so choose $\delta = \varepsilon d - \varepsilon/2 < \varepsilon d$ and we get that $B_\delta(\hat{b}) \subset B_{\varepsilon d}(b)$. From the construction of $\ell^{1,\varepsilon}(K_\varepsilon)$, we know that it contains copies of the set B_δ from a to $\hat{b}$ centered at each natural multiple of ε between them. Then, we have that these B_δ balls overlaps. Let us compute this overlap.

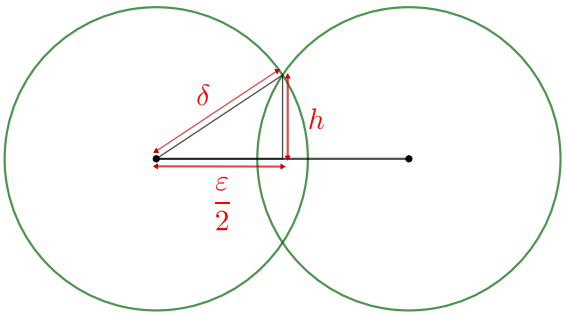


Figure 4.6

Define h as the radius of the overlap (see Figure 4.6). Then, using Pythagoras Theorem we get that

$$h^2 = \delta^2 - \left(\frac{\varepsilon}{2}\right)^2 = \left(\varepsilon d - \frac{\varepsilon}{2}\right)^2 - \left(\frac{\varepsilon}{2}\right)^2 = d(d-1)\varepsilon^2 > 0.$$

Then, we have that the segment $[a, \hat{b}]$ is included in $\ell^{1,\varepsilon}(K_\varepsilon)$, and in particular the segment $[a, b]$ we were interested in is inside $\ell^{1,\varepsilon}(K_\varepsilon)$. This proves $\ell^1(K) \subset \ell^{1,\varepsilon}(K_\varepsilon)$.

To prove that $\mathrm{co}(K) \subset \mathrm{co}_\varepsilon(K_\varepsilon)$, we are going to iterate the previous argument to obtain that $\ell^k(K) \subset \ell^{k,\varepsilon}(K_\varepsilon)$. Since by Proposition 4.4.8 we know that $\mathrm{co}(K) = \ell^d(K)$, there is a finite amount of iterations and our choice of K_ε as the set $K + B_{\varepsilon d}(0)$ is enough (notice that in $\mathbb{R}^d$ we need $d-1$ iterations). □

Let us treat first the situation in which Ω_0 is ε-convex. This particular case is interesting because the set Ω_0 preserves the ε-convexity through its evolution via the game, that is, the positivity set of the value of the game after k rounds of the game is still ε-convex. To simplify the notation let us call Ω_k^ε (instead of $\Omega^\varepsilon_{t=k\varepsilon^2/2}$) the positivity set of the value of the game, u^ε, after k plays.

Remark 4.4.15. Note that, when $\overline{K} \subset \Omega_0$, it holds that $K_\varepsilon \subset \Omega_k^\varepsilon$ for all $\varepsilon > 0$ small enough and all $k \in \mathbb{N}$. It is due to the fact that, if $x \in K_\varepsilon$, Paul can always choose to stop the game and receives the positive payment $\psi_\varepsilon(x)$.

Lemma 4.4.16 (Preservation of ε-convexity)**.** *Let Ω_0 be ε-convex. Then, Ω_k^ε is ε-convex for any $k \in \mathbb{N}$ and moreover,*

$$K_\varepsilon \subset co_\varepsilon(K_\varepsilon) \subset \cdots \subset \Omega_k^\varepsilon \subset \Omega^\varepsilon_{(k-1)} \subset \cdots \subset \Omega_0.$$

Proof. Assume Ω^ε_{k-1} is ε-convex. Let us show Ω_k^ε is ε-convex and $\Omega_k^\varepsilon \subset \Omega^\varepsilon_{k-1}$. Let us start with the latter. If $x \in \Omega_k^\varepsilon$, we have that $u(x, k\varepsilon^2/2) > 0$. According to the DPP, this implies that either

$$0 < \sup_{|v|=1} \min_{\pm} u(x \pm \varepsilon v, (k-1)\varepsilon^2/2)$$

or

$$0 < \psi_\varepsilon(x).$$

This second case directly implies that $x \in K_\varepsilon \subset \Omega_k^\varepsilon$ due to Remark 4.4.15. Then, let us assume that we are in the first case. If that supremum is bigger than zero, we have that there is some vector v_0 such that

$$0 < \min_{\pm} u(x \pm \varepsilon v_0, (k-1)\varepsilon^2/2).$$

According to this expression, both $x + v\varepsilon, x - \varepsilon v$ belong to Ω^ε_{k-1}. Since Ω^ε_{k-1} is ε-convex by assumption, we also have that $x \in \Omega^\varepsilon_{k-1}$. Thus, $\Omega_k^\varepsilon \subset \Omega^\varepsilon_{k-1}$.

Now let us check that Ω_k^ε is also ε-convex. Choose $x, y \in \Omega_k^\varepsilon$, and $z \in \ell^\varepsilon_{x,y} \setminus (\{x\} \cup \{y\})$ (if this set is empty, the conclusion is trivial). We want to check that $z \in \Omega_k^\varepsilon$. Since

$\Omega_k^\varepsilon \subset \Omega_{k-1}^\varepsilon$, both $x, y \in \Omega_{k-1}^\varepsilon$. Using the ε-convexity of this set, we get that $\ell_{x,y}^\varepsilon \subset \Omega_{k-1}^\varepsilon$. Choosing $v_0 = \frac{x-y}{|x-y|}$, this implies that

$$u(z, k\varepsilon^2) \geq \max_{|v|=1} \min_{\pm} u(z \pm \varepsilon v, (k-1)\varepsilon^2/2) \geq \min_{\pm} u(z \pm \varepsilon v_0, (k-1)\varepsilon^2/2) > 0$$

Hence, we conclude that $z \in \Omega_k^\varepsilon$, making the set Ω_k^ε ε-convex. □

Now we prove the main result of this subsection.

Lemma 4.4.17. *Suppose that $co_\varepsilon(K_\varepsilon) \subset \Omega_0$. Then,*

$$co_\varepsilon(K_\varepsilon) \subset \Omega_k^\varepsilon \text{ for every } k \in \mathbb{N}.$$

Proof. To prove that $\mathrm{co}_\varepsilon(K_\varepsilon) \subset \Omega_k^\varepsilon$ for each $k \in \mathbb{N}$ is equivalent to show that for every time $t = k\varepsilon^2/2$, there exists a strategy for Paul such that he receives a positive payment regardless of Carol's choices for $x \in \mathrm{co}_\varepsilon(K_\varepsilon)$, as a consequence of the existence of this strategy, playing with the optimal strategy Paul also wins which implies that $u^\varepsilon(x, k\varepsilon^2/2) > 0$.

Suppose $x \in \mathrm{co}_\varepsilon(K_\varepsilon) \setminus K_\varepsilon$. Otherwise, if $x \in K_\varepsilon$, Paul automatically wins (he can stop the game at this point) and receives the payment of the obstacle. Then, if $x \in \mathrm{co}_\varepsilon(K_\varepsilon) \setminus K_\varepsilon$, by characterization in Proposition 4.4.8 of the ε-convex hull, there exists an integer $n > 0$ such that $x \in \ell^{n,\varepsilon}(K_\varepsilon)$. Then, there are $a, b \in \ell^{n-1,\varepsilon}(K_\varepsilon) \subset \mathrm{co}_\varepsilon(K_\varepsilon)$ such that $x \in \ell_{a,b}^\varepsilon$. Based on this information, Paul selects the vector $v = (b-a)/|b-a|$ because that choice ensures the next point $x_1 \in \ell_{a,b}^\varepsilon$ no matter what Carol does. Then, if Paul and Carol only play once, $k = 1$, Paul will receive a positive payoff because $x_1 \in \mathrm{co}_\varepsilon(K_\varepsilon)$. That implies Paul will either get money from the obstacle if $x_1 \in K_\varepsilon$ or from the initial data, since $\mathrm{co}_\varepsilon(K_\varepsilon) \subset \Omega_0$.

If $k \neq 1$, Paul repeats the same strategy for x_1 choosing $v = (b-a)/|b-a|$. Suppose that Carol always chooses the sign for the next the movement such that $x_j \in \ell_{a,b}^\varepsilon \setminus \{a, b\}$ for each $j \in \{1, ..., k\}$. Consequently, at the end of the game Paul will receive a positive payment since $x_j \in \ell_{a,b}^n \subset \Omega_0$. On the other hand, suppose that for some $j \in \{1, ..., k-1\}$, $x_j = a$ (or $x_j = b$), then $x_j \in \ell^{n-1,\varepsilon}(K_\varepsilon)$. Thus, in the next turn, Paul's choice will be to select the vector $v = (b_j - a_j)/|b_j - a_j|$ where $a_j, b_j \in \ell^{n-2,\varepsilon}(K_\varepsilon$ and $x_j \in \ell_{a_j,b_j}^\varepsilon \subset \mathrm{co}_\varepsilon(K_\varepsilon)$. Once again, if the game concludes at this stage, Paul will obtain a positive reward since $x_{j+1} \in \ell_{a_j,b_j}^\varepsilon \subset \mathrm{co}_\varepsilon(K_\varepsilon)$. However, if the game continues and $x_{j+1} \notin K_\varepsilon$ either Paul can keep the same strategy ensuring he wins when the time expires or Paul reaches K_ε and he also wins in this case. □

As a consequence we get the following corollary.

Corollary 4.4.18. *Assume that $co_\varepsilon(K_\varepsilon) \subset \Omega_0$, then*

$$co_\varepsilon(K_\varepsilon) \subset \liminf_{k\to\infty} \Omega_k^\varepsilon.$$

Moreover, when Ω_0 is ε-convex, it holds that

$$co_\varepsilon(K_\varepsilon) \subset \lim_{k\to\infty} \Omega_k^\varepsilon.$$

Proof. Since we have $\mathrm{co}_\varepsilon(K_\varepsilon) \subset \Omega_0$, thanks to Lemma 4.4.17, we get

$$\mathrm{co}_\varepsilon(K_\varepsilon) \subset \liminf_{k\to\infty} \Omega_k^\varepsilon.$$

Moreover, if Ω_0 is ε-convex, by Lemma 4.4.16, using that $\{\Omega_k^\varepsilon\}_{k\in\mathbb{N}}$ is a monotone sequence of sets, we obtain that

$$\liminf_{k\to\infty} \Omega_k^\varepsilon = \limsup_{k\to\infty} \Omega_k^\varepsilon.$$

□

Observe that we are not able to prove that $\limsup_{k\to\infty} \Omega_k^\varepsilon \subset \mathrm{co}_\varepsilon(K_\varepsilon)$ with a fixed ε. However, in the next subsection, we will be able to prove that $\limsup_{t\to\infty} \Omega_t \subset \mathrm{co}(K)$ and conclude $\lim_{t\to\infty} \Omega_t = \mathrm{co}(K)$.

4.4.3 Convex hull of K not included in Ω_0

For an open obstacle K and an initial open set Ω_0, we defined in the introduction of this chapter the graph $G = (\mathcal{V}, \mathcal{E})$ as follows: the set of vertices is given by

$$\mathcal{V} := \Big\{ V \subset \mathbb{R}^d : V \text{ is a connected component of } K \Big\},$$

and the set of edges by

$$\mathcal{E} := \Big\{ \langle V_1, V_2 \rangle : V_1, V_2 \in \mathcal{V} \text{ and } \ell_{x,y} \subset \Omega_0 \text{ for some } x \in V_1 \text{ and some } y \in V_2 \Big\}.$$

Let us also define the open set

$$L = \Big\{ x \in \ell_{a,b} : a, b \in K, \ell_{a,b} \subset \Omega_0 \Big\},$$

which is a set analogue of the graph G. It is clear that $\mathrm{co}(K) = \mathrm{co}(L)$, since we have that $K \subset L \subset \ell^1(K)$. Observe that if G is connected, then L is connected. That the graph G is connected is going to be the main hypothesis of this subsection.

We start by showing the following elementary result.

Lemma 4.4.19. *Let $K \subset \mathbb{R}^d$ be an open set. Then, K can only have at most a countable amount of connected components.*

Proof. For each connected component of K, which are open, choose an element of $\mathbb{Q}^d$ belonging to the connected component. Since every one of these rational elements can only belong to one connected component, there can be at most a countable amount of connected components. □

The proof of the result in [82] requires to choose a sufficiently small ε, depending on the starting position x: the reason for doing that is that if the jumps are large enough, it may happen that you jump over the set you want to land on and then the proof would not work anymore. This fact (that one has to choose ε depending on the starting position x) prevents the proofs in [82] to shed some light on the asymptotic behavior for large times of the sets Ω_t^ε. At this crucial point we have to take into account that we enlarged the obstacle set, since this will help Paul to win.

What we are going to show now is that

$$\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon. \tag{4.4.6}$$

What we would actually like to prove is that

$$\mathrm{co}_\varepsilon(K_\varepsilon) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon,$$

but there are several difficulties in order to show this. We have proved (4.4.6) in the previous subsection under the additional assumption that $\mathrm{co}(K) \subset \Omega_0$. When this hypothesis is not in force, we want an argument that takes into the account both the concentric strategy used in [82] and the ε-convex hull of K_ε, something that is missing for the moment.

Remark 4.4.20. Once we prove that Paul has a winning strategy in a certain region, we can may assume without loss of generality that that region is part of the obstacle. For example, we may assume the connected components of the obstacle K to be convex, or that L is the obstacle, instead of K, and thus we can consider the whole obstacle as a connected set.

From this point onwards, we have to differentiate between the proof for $d = 2$ and the proof for $d \geq 3$. This is due to the fact that the proof for $d = 2$ is more natural for this game, while the proof for $d \geq 3$ is a generalization of the two-dimensional one. In fact, when $d = 2$ we will be able to say more than in higher dimensions, since in this case we will be able to choose a fixed ε, uniform for every $x \in \mathrm{co}(K)$. The two-dimensional proof is a slight refinement of what is done in [82], while the proof in higher dimensions is completely new.

Let us start with the case of $d = 2$. We state a well-known property of the convex hull of a path-connected set in two dimensions.

Lemma 4.4.21. *If $L \subset \mathbb{R}^2$ is a path-connected set, then $co(L) = \ell^1(L)$.*

This next auxiliary result will be needed in the proof.

Lemma 4.4.22. *Let $K_1, K_2 \subset \mathbb{R}^2$ be two connected sets, and let $(a, b) \in K_1 \times K_2$. If r is a line that intersects $\ell_{a,b} \setminus (K_1 \cup K_2)$, then for any pair $(x, y) \in K_1 \times K_2$ the line r intersects the segment $\ell_{x,y}$.*

Proof. Assume without loss of generality that the line r is the x-axis, that is, $\{(x, y) \subset \mathbb{R}^2 : y = 0\}$.

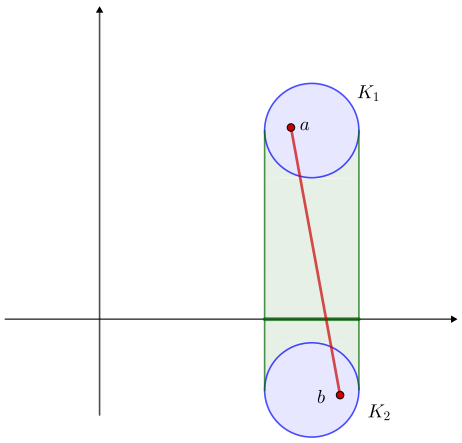


Figure 4.7: Scheme of the proof.

Then, if K_1 and K_2 do not touch the x-axis, they must lie either inside the set $H^+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ or inside $H^-\{(x, y) \in \mathbb{R}^2 : y < 0\}$. If both of them are in the same set, it is impossible that any segment would intersect the x-axis. So we can assume that $K_1 \subset H^+$ and $K_2 \subset H^-$. Therefore, due to Bolzano's Theorem, for any couple of points $(a, b) \in K_1 \times K_2$, the segment $\ell_{a,b}$ joining them must intersect the x-axis. □

Let us now write the main result of this section for two space dimensions, $d = 2$.

Proposition 4.4.23. *Assume we are in $\mathbb{R}^2$ and condition (G) be satisfied, that is, let the graph G be connected. Then,*

$$co(K) \subset \liminf_{t\to\infty} \Omega_t^\varepsilon,$$

for every ε small enough.

Proof. The proof follows the ideas in [82], but with a few changes to take into account that the enlarged obstacle set, K_ε, is going to help us. The idea is that we want to prove that the asymptotic limit $t \to \infty$ can be taken for a fixed ε small enough, independently of the starting position $x \in \mathrm{co}(K)$.

Case $x \in K$. In this case it is clear that

$$u^\varepsilon(x, t) \geq \min_{x\in K} \psi_\varepsilon(x) > 0 \quad \text{for } t > 0.$$

Case $x \in L \setminus K$. In this case, by the definition of L, there exists a, b in two different connected components of K, such that $x \in \ell_{a,b} \subset \Omega_0$. With this in mind, an clear strategy for Paul is to move along the segment $\ell_{a,b}$. The only consideration needed is to observe that, since $\varepsilon < 2\varepsilon$, Paul will not jump over the enlarged set K_ε and can always stop inside. We get that

$$u^\varepsilon(x, t) \geq \min\left\{\min_{y\in K+B_\varepsilon} \psi_\varepsilon(x), \min_{y\in\ell_{a,b}} u_0(y)\right\} > 0 \quad \text{for } t > 0.$$

So far it has not been necessary to take ε to be small.

Case $x \in \mathrm{co}(K) \setminus L$. In this the case we have to slightly alter the arguments from the ones used in [82]. Recall that $\mathrm{co}(K) = \mathrm{co}(L) = \ell^1(L)$, see Lemma 4.4.21.

Let us give a sketch of the argument. Since $x \in \ell^1(L)$, there exists $a, b \in L$ such that $x \in \ell_{a,b}$. Now, the essential part of the argument lies in the fact that you can build a polygonal path in L between a and b and, this polygonal, together with the segment $\ell_{a,b}$, form a closed curve. We now perform a concentric argument for Paul so that he moves away from the segment $\ell_{a,b}$ and towards the polygonal. When the position lies close enough to the polygonal, Paul wins, since the problem reduces to one of the previous cases.

This approach has one disadvantage, which is that for every x the polygonal chosen may be different. And if this polygonal lies too close to the boundary of Ω_0, we have to take the step ε appropiately small enough so that Paul does not jump over the set. Therefore, the step ε depends on the initial position x. If we chose a polygonal path that is valid for every x, then we avoid this problem and the ε step, if chosen small enough, can be valid for every x.

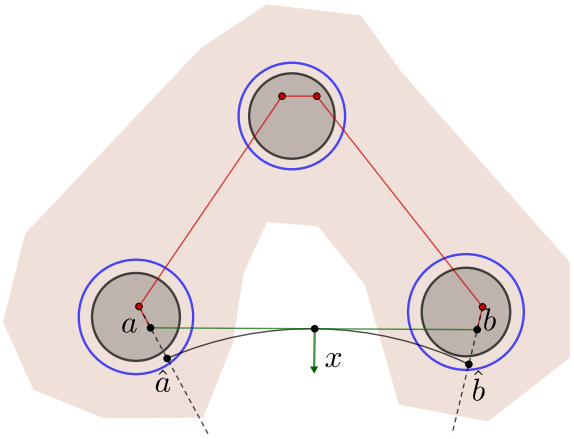


Figure 4.8: Scheme of the proof.

To build this polygonal, recall that K only has a countable amount of connected components K_j with $j \in \mathbb{N}$ (see Lemma 4.4.19). Then for each pair of connected components K_i, K_j with $i \neq j$ such that there is a segment $\ell_{y_i,y_j} \subset \Omega_0$ with $y_i \in K_i$ and $y_j \in K_j$, take this segment ℓ_{y_i,y_j} and consider the set Γ_L of all the points in these segments, that is, $\Gamma_L = \cup \ell_{y_i,y_j}$.

This Γ_L connects all of the connected components K_j, but inside of each K_j it is not necessarily connected. We add the required polygonal paths inside of each K_j, which we call Γ_K, to make $\Gamma := \Gamma_L \cup \Gamma_K$ a connected polygonal path. The polygonal Γ is a compact subset of L, so there exists $\delta > 0$ such that $B_{3\delta}(\Gamma) \subset L$. Now we have to choose $\varepsilon < \delta$ to make sure Paul cannot jump over the set. Let us also assume that $\delta < 2\varepsilon$. Notice that δ is choosed depending on ε and that such a choice is possible for every ε small enough.

As we mentioned before, for any $x \in \mathrm{co}(K) \setminus L$ there exists $a, b \in L$ such that $x \in \ell_{a,b}$ due to $\mathrm{co}(K) = \ell^1(L)$. We are going to show that a, b can be chosen either in Γ or in

K, that is, $x \in \ell^1(\Gamma \cup K)$. Assume, a is in $L \setminus K$. Then a belongs to a segment joining two connected components without intersecting them. Hence, thanks to Lemma 4.4.22, we can change a to be in Γ, and we still have $x \in \ell_{a,b}$. We perform the same change for b if necessary. If either a or b lie in K, we add the necessary segments inside of K to the polygonal Γ so that it connects with them, and discard unnecessary segments so that the polygonal path starts in a and ends in b.

Now we have the polygonal path joining a, b and lying inside of L ready. Notice that, without loss of generality, we may assume that Γ and $\ell_{a,b}$ only intersect at a and b. If we modify Γ extend the segment joining Γ with a and b with a length δ, we obtain two new corresponding endpoints, $\hat{a}$ and $\hat{b}$. We define $\hat{\Gamma}$ as the modified polygonal and C as the arc that passes through the points $\hat{a}$, x and $\hat{b}$. Bear in mind that if a or b were in K, $\hat{a}$ or $\hat{b}$ may lie in K_ε. Observe that $\hat{\Gamma} \cup C$ form a closed curve. Therefore, there is a Jordan closed curve $\gamma \subset \hat{\Gamma} \cup C$, such that $x \in \gamma$. Then, thanks to the Jordan curve theorem, there exists a bounded open set D_γ enclosed by the curve γ. Note that the exterior normal vector of C points towards the set D_γ.

From this point the proof follows using a concentric argument for Paul. The arc C has a center z which we take as the center for the concentric strategy. Since D_γ is a bounded set, it takes at most finite time τ such that $D_\gamma \subset B_{R_\tau}(z)$, for

$$R_\tau = \sqrt{|x - z|^2 + 2\tau}$$

Before this time τ, we can be sure that Paul has arrived at a neighbourhood of Γ and thus lies inside of L. Therefore, we are now in one of the previous cases and have concluded the proof. □

Now, let us try to generalize the previous proof for $d = 2$ to any dimension $d \geq 3$. The fundamental idea lies in the fact that Paul can always play in a two-dimensional plane as if we were in $\mathbb{R}^2$ by choosing vectors contained in that plane. In order to use this, we develop a different approach to the convex hull of a set that allows us to extend the proof from $\mathbb{R}^2$ to $\mathbb{R}^d$.

Let us try to give a sketch of the proof in $\mathbb{R}^3$. The simplest geometrical example in $\mathbb{R}^2$ where we can see how the previous proof works is a triangle. If we lie inside a triangle without one of the sides, then Paul will be able to move from the interior of the triangle to one of the other sides. Hence, starting from the convex hull of a triangle without one of the sides, Paul can find a strategy to win. In $\mathbb{R}^3$, if we think of a tetrahedron, the idea is to move from the interior to one of the faces, and then reduce to two dimensions and move to the sides. Observe that thinking about the previous argument in terms of triangles greatly simplifies the geometry of the problem. The intuition behind our adaptation relies on this idea.

We mention again one important difference between the results of $d = 2$ and $d \geq 3$: in two dimensions we can give the result for a fixed $\varepsilon > 0$, but for higher dimensions we cannot. This is the main difference between Theorems 4.1.5 and 4.1.6, and it occurs because in the proof of Proposition 4.4.23 we can choose the same polygonal path for every x. This type of argument clearly does not work for $d \geq 3$.

Let us define, for a set Γ, the new set

$$\mathcal{T}^1(\Gamma) := \Big\{ z \in \mathbb{R}^d : z \in \ell_{x,y}, \text{ with } x \in \ell_{a,b} \subset \Gamma \text{ and } y \in \ell_{a,c} \subset \Gamma, \text{ for } a \neq b \text{ and } a \neq c \Big\}.$$

This definition is relevant when the set Γ is polygonally connected. It is easy to observe from this construction that $\mathcal{T}(\Gamma)$ is open whenever Γ is open.

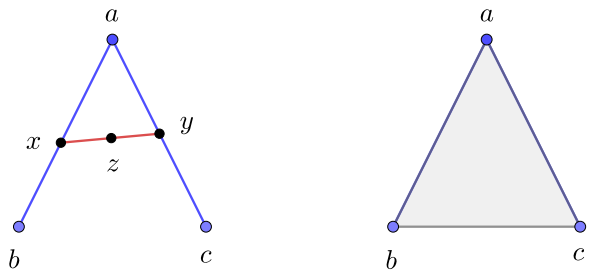


Figure 4.9: Construction of $\mathcal{T}^1(\Gamma)$.

Let us define $\mathcal{T}^2(\Gamma) = \mathcal{T}^1(\mathcal{T}^1(\Gamma))$. Also define $\mathcal{T}^0(\Gamma) = \Gamma$. Let

$$\mathcal{T}(\Gamma) = \bigcup_{n=0}^{\infty} \mathcal{T}^d(\Gamma).$$

We will show that if Paul starts in $\text{co}(K)$ plays with a strategy that bears in mind the iterations $\mathcal{T}^d(K)$, he will be able to win the game. It is immediate to observe that $\mathcal{T}^d(\Gamma) \subset \text{co}(\Gamma)$, since this is a more restrictive construction than the one of the convex hull, so $\mathcal{T}(\Gamma) \subset \text{co}(\Gamma)$. What we need to prove now is that $\text{co}(\Gamma) \subset \mathcal{T}(\Gamma)$.

First, let us show that these sets preserve polygonal connectedness.

Lemma 4.4.24. *Let Γ be a polygonally connected set. Then, $\mathcal{T}^1(\Gamma)$ is poligonally connected.*

Proof. Let $x, y \in \mathcal{T}^1(\Gamma)$. This implies that there is a segment joining x with a point $b_x \in \Gamma$ and a segment joining y with a point b_y in Γ. By hypothesis, there is a polygonal joining both b_x and b_y, so we can extend this polygonal to join x and y. □

Now let us prove an auxiliary lemma that will be useful.

Lemma 4.4.25. *Let Γ be a polygonally connected set. Then,*

$$\ell^1(\Gamma) \subset \mathcal{T}(\Gamma).$$

Proof. Let $x, y \in \Gamma$. Since Γ is polygonally connected, there is a polygonal joining x and y. Let k be the number of segments in the polygonal joining both points. We can see by an inductive argument that the segment joining x and y is necessarily in $\mathcal{T}^{k-1}(\Gamma) \subset \mathcal{T}(\Gamma)$. Thus, $\ell^1(\Gamma) \subset \mathcal{T}(\Gamma)$. □

Lemma 4.4.26. *Let Γ a polygonally connected set. Then,*

$$\mathcal{T}(\Gamma) = \mathcal{T}(\mathcal{T}(\Gamma)).$$

Proof. That $\mathcal{T}(\Gamma) \subset \mathcal{T}(\mathcal{T}(\Gamma))$ is obvious. We want to prove the reciprocal. It suffices to prove that $\mathcal{T}^1(\mathcal{T}(\Gamma)) = \mathcal{T}(\Gamma)$. If $x \in \mathcal{T}^1(\mathcal{T}(\Gamma))$, then this implies there exist $y, z \in \mathcal{T}(\Gamma)$ such that x is in $\ell_{y,z}$, and there are $a, b, c \in \mathcal{T}(\Gamma)$, $b \neq a \neq c$, such that $y \in \ell_{a,b} \subset \mathcal{T}(\Gamma)$ and $z \in \ell_{a,c} \subset \mathcal{T}(\Gamma)$. But if $a, b \in \mathcal{T}(\Gamma)$, then there exist $k_1, k_2 \in \mathbb{N}$ such that $a, b \in \mathcal{T}^{k_1}(\Gamma)$ and $a, c \in \mathcal{T}^{k_2}(\Gamma)$. Recall that Lemma 4.4.24 implies $\mathcal{T}^k(\Gamma)$ is polygonally connected for any $k \in \mathbb{N}$. Then, just like in the proof of Lemma 4.4.25, there is a polygonal in $\mathcal{T}^{k_1}(\Gamma)$ joining a and b and another joining a and c in $\mathcal{T}^{k_2}(\Gamma)$, so we know that for $m \in \mathbb{N}$ such that $m \geq \max\{k_1, k_2\}$ both $\ell_{a,b}$ and $\ell_{a,c}$ lie inside $\mathcal{T}^m(\Gamma)$. Hence, $x \in \mathcal{T}^1\left(\mathcal{T}^m(\Gamma)\right) = \mathcal{T}^{m+1}(\Gamma) \subset \mathcal{T}(\Gamma)$. □

With these two lemmas we are ready to show that $\mathrm{co}(\Gamma) \subset \mathcal{T}(\Gamma)$.

Corollary 4.4.27. *Let Γ be a polygonally connected set. Then,*

$$co(\Gamma) \subset \mathcal{T}(\Gamma).$$

Proof. We know from Proposition 4.4.8 that $\mathrm{co}(\Gamma) = \ell^d(\Gamma)$. Recall $\ell^2(\Gamma) = \ell^1(\ell^1(\Gamma))$. Notice that, thanks to Lemma 4.4.25 and Lemma 4.4.26 we get $\mathcal{T}(\Gamma) = \mathcal{T}(\ell^1(\Gamma))$, using that $\mathcal{T}(\ell^1(\Gamma)) \subset \mathcal{T}\left(\mathcal{T}(\Gamma)\right)$. Then, thanks to Lemma 4.4.25 again, we get that $\ell^2(\Gamma) \subset \mathcal{T}(\Gamma)$. Iterating this argument we obtain that $\ell^k(\Gamma) \subset \mathcal{T}(\Gamma)$, so $\mathrm{co}(\Gamma) \subset \mathcal{T}(\Gamma)$. □

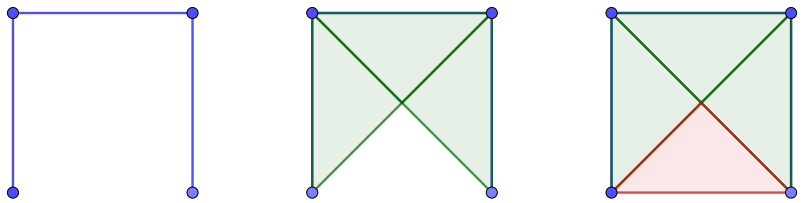


Figure 4.10: Recovering $\mathrm{co}(\Gamma)$ through iterations of $\mathcal{T}^k(\Gamma)$.

We now have the tools to prove the main result concerning the asymptotic behavior of our problem, that is, $\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega_t$, under condition (G). Recall that in this case we cannot fix a small ε for every starting point x, since the argument in Proposition 4.4.23 relies on choosing a polygonal valid for every $x \in \mathrm{co}(K)$, but this can only be done in $\mathbb{R}^2$.

Theorem 4.4.28. *Let condition (G) be satisfied, that is, let the graph G be connected. Then*

$$co(K) \subset \liminf_{t\to\infty} \Omega_t.$$

Proof. First, recall that G being connected implies that L is connected. Since L is also open, it follows that it is polygonally connected. Let $x \in \mathrm{co}(K) = \mathrm{co}(L)$. This implies, due to Corollary 4.4.27 that $x \in \mathcal{T}(L)$, which means $x \in \mathcal{T}^k(L)$ for some large enough $k \in \mathbb{N}$. Let us show that if $x \in \mathcal{T}^k(L)$ for $k \in \mathbb{N}$, then the Proposition 4.4.23 shows that starting from x Paul can win. We just need to prove it for $\mathcal{T}^1(L)$, since $\mathcal{T}^k(L) = \mathcal{T}^1\left(\mathcal{T}^{k-1}(L)\right)$. If $x \in \mathcal{T}^1(L)$, x is inside a segment with extreme points y and z, where $y \in \ell_{a,b}$ and $z \in \ell_{a,c}$, with $a, b, c \in L$. Then the arguments done for $\mathbb{R}^2$ imply that from x Paul can move a finite number of steps and end up as close as needed to either $\ell_{a,b}$ or $\ell_{a,c}$. Recall that L is open, so we can end up in L after a finite amount of steps choosing ε sufficiently small. Since $L \subset \Omega_0$, we just proved Paul has a strategy to win starting from x, provided that the time is large enough. □

Proof of Theorem 4.1.6. Since $\limsup_{t\to\infty} \Omega_t \subset \mathrm{co}(K)$ was already proved in Theorem 4.4.13, the previous Theorem 4.4.28 concludes the whole proof of Theorem 4.1.6. Concerning the asymptotic behavior of the minimal curvature flow with an obstacle, under the hypotheses of the Comparison Principle Theorem 4.1.1 and condition (G), we have that

$$\mathrm{co}(K) \subset \liminf_{t\to\infty} \Omega_t \subset \limsup_{t\to\infty} \Omega_t \subset \mathrm{co}(K),$$

and we conclude that the limit exists,

$$\lim_{t\to\infty} \Omega_t = \mathrm{co}(K),$$

as we wanted to show. □

Chapter 5

Two membranes problem

This chapter is devoted to presenting the results published in:

- [49], *Monotone iterations of two obstacle problem with different operators*,
- [50], *The two membranes problem in a regular tree*,

two joint works with Alfredo Miranda (UBA) and Julio D. Rossi (UBA).

5.1 Introduction

In this chapter, we analyze iterations of the two membranes problem where the ambient space is a bounded Lipschitz domain Ω and where it is a regular tree $\mathbb{T}$.

Let us first describe the obstacle problem (in a domain, from above or below) and the then two membranes problem (as described in [26]).

The obstacle problem is one of the main problems in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is fixed, and which is constrained to lie above a given obstacle. In mathematical terms, given an operator L (notice that here we can consider fully nonlinear problems of the form $Lu = F(D^2u, Du, u, x)$) that describes the elastic configuration of the membrane, a bounded Lipschitz domain Ω and a boundary datum f, the obstacle problem from below (here solutions are assumed to be above the obstacle) reads as

$$\begin{cases} u \geq \phi & \text{in } \Omega, \\ Lu \geq 0 & \text{in } \Omega, \\ Lu = 0 & \text{in } \{u > \phi\}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

or identically

$$\begin{cases} \min\{Lu, u - \phi\} = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

When the operator L is in divergence form and is associated to an energy functional $E(u)$ the problem becomes a variational problem and a solution can be obtained minimizing E in the set of functions in the appropriate Sobolev space that are above the obstacle and take the boundary datum. We refer to Section 5.2.1 for details.

The obstacle problem can be also stated as follows: we look for the smallest supersolution of L (with boundary datum f) that is above the obstacle. This formulation is quite convenient when dealing with fully nonlinear problems using viscosity solutions. We will assume here that the problem (5.1.1) has a unique continuous viscosity solution (for general theory of viscosity solutions we refer to [30] and [36]). This is guaranteed if L has a comparison principle and one can construct barriers close to the boundary so that the boundary datum f is taken continuously. See Section 5.2.2.

We can also consider the obstacle problem from above (here solutions are assumed to be below the obstacle)

$$\begin{cases} v \leq \varphi & \text{in } \Omega, \\ Lv \leq 0 & \text{in } \Omega, \\ Lv = 0 & \text{in } \{u < \varphi\}, \\ v = g & \text{on } \partial\Omega, \end{cases} \tag{5.1.2}$$

or equivalently

$$\begin{cases} \max\{Lv, v - \varphi\} = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

On the other hand, the obstacle problem can be also define when the ambient space Ω is replaced by an infinite graph with a regular structure (a regular tree) $\mathbb{T}$ and the involved operator L is given by mean value formula with a source term.

In this case, formally, the obstacle problem from below in a tree with obstacle ϕ and bounday datum f, (where ϕ and f are functions defined in $\mathbb{T}$ and $\partial\mathbb{T}$ respectively), is given by

$$\begin{cases} u \geq \phi & \text{in } \mathbb{T}, \\ Lu \geq 0 & \text{in } \mathbb{T}, \\ Lu = 0 & \text{in } \{u > \phi\}, \\ u = f & \text{on } \partial\mathbb{T}, \end{cases} \tag{5.1.3}$$

or identically

$$\begin{cases} \min\{Lu, u - \phi\} = 0 & \text{in } \mathbb{T}, \\ u = f & \text{on } \partial\mathbb{T}. \end{cases}$$

Conversely, the obstacle problem from above with obstacle φ and boundary datum g

(two functions defined in $\mathbb{T}$ and $\partial\mathbb{T}$ repect.) satisfies

$$\begin{cases} v \leq \varphi & \text{in } \mathbb{T}, \\ Lv \leq 0 & \text{in } \mathbb{T}, \\ Lv = 0 & \text{in } \{u < \varphi\}, \\ v = g & \text{on } \partial\mathbb{T}, \end{cases} \tag{5.1.4}$$

or equivalently

$$\begin{cases} \max\{Lv, v - \varphi\} = 0 & \text{in } \mathbb{T}, \\ v = g & \text{on } \partial\mathbb{T}. \end{cases}$$

We refer to Section 5.3 for proper definitions and details.

In both case, where the ambient space is a domain Ω or in a tree $\mathbb{T}$, the obstacle problem from below can be viewed as follows:

- Obstacle problem from below: we look for the minimal supersolution of L (with boundary datum f) that is above the obstacle ϕ. We called

  $$v = \underline{O}(L, \varphi, g)$$

  the solution to the obstacle problem in a domain, (5.1.1), or in a tree, (5.1.3).

- Obstacle problem from above: it is the largest subsolution of L (with boundary datum g) that is below the obstacle φ. We will denote by

  $$v = \overline{\mathrm{O}}(L, \varphi, g)$$

  the solution to the obstacle problem in a domain, (5.1.2), or in a tree, (5.1.4).

For general references on the obstacle problem in domains (including regularity of solutions, that in some cases are proved to be optimally $C^{1,1}$) we just mention [29], [28], [37], [89], the survey [83] and references therein.

On the other hand, when we deal with the obstacle problem in a tree $\mathbb{T}$, the first point is to verify that the the single equation

$$Lu = 0 \quad \text{in } \mathbb{T}, \qquad u = g \quad \text{in } \partial\mathbb{T}$$

admits a solution. In [90] it was studied the solvability for the above equation equation where L has null source term. In section 5.2.1 we discuss the existence of solutions for mean value fomulas Ls that admit nontrivial sources. Concerning previous results in the literature for mean value formulas on trees we refer to [2, 15, 90, 93, 91, 92, 60, 61, 106, 107] and references therein.

The two membranes problem. Closely related to the obstacle problem, one of the systems that attracted the attention of the PDE community is the two membranes problem.

This problem models the behaviour of two elastic membranes that are clamped at the boundary of a prescribed domain, they are assumed to be ordered, one membrane is above the other, and they are subject to different external forces (the membrane that is on top is pushed down and the one that is below is pushed up). The main assumption here is that the two membranes do not penetrate each other (they are assumed to be ordered in the whole domain). This situation can be modeled by two obstacle problems; the lower membrane acts as an obstacle from below for the free elastic equation that describes the location of the upper membrane, while, conversely, the upper membrane is an obstacle from above for the equation for the lower membrane.

With our previous notation, a solution of the two membranes problem can be written as

$$u = \underline{\mathrm{O}}(L_1, v, f) \qquad \text{and} \qquad v = \overline{\mathrm{O}}(L_2, u, g).$$

When the equations that obey the two membranes in a domain have a variational structure this problem can be tackled using calculus of variations (one aims to minimize the sum of the two energies subject to the constraint that the functions that describe the position of the membranes are always ordered inside the domain, one is bigger or equal than the other), see [111]. When the involved equations in a domain are not variational the analysis relies on monotonicity arguments (using the maximum principle). Once existence of a solution (in an appropriate sense) is obtained a lot of interesting questions arise, like uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [27, 26, 104], the dissertation [112] and references therein.

On the other hand, when the ambient space is a tree, we obtain in this chapter the existence of a solution to the two membrane problem via an iterative method (under the conditions for the existence of solutions to the single equations for, on one side, L_1 and, on the other side f, and L_2 and g respectively)

For another systems of mean value formulas on trees we quote [80] where two coupled equations (but not of obstacle type) were considered (a probabilistic interpretation of the equations is also provided there).

Description of the main results Our main goal is to obtain solutions to the two membranes problems (in a domain and in a tree) as limits of iterations of the obstacle problem. For iterations of the obstacle problem in a different context we refer to [18].

Let us consider two different operators L_1 and L_2 (with boundary data f and g respectively, we assume that $f > g$) and generate two sequences iterating the obstacle problems from above and below. Given an initial function v_0, take u_0 as the solution to the obstacle problem from below for L_1 with boundary datum f and obstacle v_0, that is,

$$u_0 = \underline{\mathrm{O}}(L_1, v_0, f).$$

Now, we use this u_0 as the obstacle from above for L_2 with datum g and obtain

$$v_1 = \overline{\mathrm{O}}(L_2, u_0, g).$$

We can iterate this procedure (solving the obstacle problem for L_1 or L_2 with boundary data f or g and obstacle the previous u or v) to obtain two sequences $\{u_n\}_n$, $\{v_n\}_n$, given by

$$u_n = \underline{O}(L_1, v_n, f), \qquad v_n = \overline{O}(L_2, u_{n-1}, g).$$

Our main goal here is to show that, when the initial function v_0 is a subsolution for L_2 with boundary datum g, then both sequences of functions $\{u_n\}_n$, $\{v_n\}_n$ are nondecreasing sequences that converge to a limit pair that gives a solution of the two membranes problem, i.e., there exists a pair of functions (u_∞, v_∞) such that

$$u_n \to u_\infty \qquad \text{and} \qquad v_n \to v_\infty$$

and the limit functions satisfy

$$u_\infty = \underline{O}(L_1, v_\infty, f) \qquad \text{and} \qquad v_\infty = \overline{O}(L_2, u_\infty, g).$$

An analogous result can be obtained when we start the iteration with u_0 a supersolution to L_1 with boundary datum f and consider $\{u_n\}_n$, $\{v_n\}_n$, given by

$$u_n = \underline{O}(L_1, v_{n-1}, f), \qquad v_n = \overline{O}(L_2, u_n, g).$$

In this case the sequences are non-increasing sequences and also converge to a solution to the two membranes problem.

Moreover, where the ambient space is a tree and f and g are strictly separated, we will prove coincidence set of u_∞ and v_∞ is separeted from $\partial\mathbb{T}$ and thus it contains only a finite number of nodes. Finaly, in Section 5.3.4, we provied a probabilistic interpretation of the two membranes problem in a tree.

We will study this iterative scheme using three very different frameworks for the solutions. On the one hand, in Section (5.2), we deal with variational methods and understand solutions in the weak sense (this imposes that L_1 and L_2 must be in divergence form) and next we deal with viscosity solutions (allowing L_1 and L_2 to be general elliptic operators) for domains. On the other hand, in Section (5.3), we introduce and analyze the two membranes problem and their terms involved.

5.2 The two membranes problem in a domain

Given two differential operators $L_1(D^2u, Du, u, x)$ and $L_2(D^2v, Dv, v, x)$ the mathematical formulation the two membranes problem (with Dirichlet boundary conditions) is the following:

Definition 5.2.1. A pair (u, v) is called a solution to the two membranes problem if it solves

$$\begin{cases} \min\Big\{L_1(u)(x), (u-v)(x)\Big\} = 0, & x \in \Omega, \\ \max\Big\{L_2(v)(x), (v-u)(x)\Big\} = 0, & x \in \Omega, \\ u(x) = f(x), \qquad v(x) = g(x), & x \in \partial\Omega. \end{cases}$$

That is, with our previous notation introduced in the introduction, this system can be written as

$$u = \underline{O}(L_1, v, f) \qquad \text{and} \qquad v = \overline{O}(L_2, u, g).$$

Remark that, in general, the two membranes problem as stated in Definition 5.2.1 does not have uniqueness. To see this, just take a solution to $L_1(D^2u) = 0$ with $u|_{\partial\Omega} = f$ and v the solution to the obstacle problem for $L_2(D^2v)$ with u as obstacle from above, that is, $v = \overline{O}(L_2, u, g)$. One can easily check that the pair (u, v) is a solution to the general formulation of the two membranes problem stated Definition 5.2.1. Analogously, one can take a solution to $L_2(D^2v) = 0$ with $v|_{\partial\Omega} = g$ and u the solution to the corresponding obstacle problem, $u = \underline{O}(L_1, v, f)$, to obtain another solution to the two membranes problem. In general, these two pairs do not coincide. For example, consider Ω as the interval $(0, 1)$, $f \equiv 1$, $g \equiv 0$ on $\partial\Omega$ and the operators $L_1(u'') = u'' - 10$ and $L_2(v'') = v'' + 2$. The solution to $L_1(u'') = 0$ with $u(0) = u(1) = 1$ is $\widehat{u}(x) = -5x(1 - x) + 1$ and the solution to $L_2(v'') = 0$ with $v(0) = v(1) = 0$ is $\tilde{v}(x) = x(1 - x)$ for $x \in (0, 1)$. Let $\widehat{v} = \overline{O}(L_2, \widehat{u}, g)$ and $\tilde{u} = \underline{O}(L_1, \tilde{v}, f)$. Both pairs $(\widehat{u}, \widehat{v})$ and $(\tilde{u}, \tilde{v})$ are solutions to the two membranes problem but they are different. Since $\widehat{u}$ is a solution and $\tilde{u}$ is a supersolution to L_1 with the same boundary datum, by the comparison principle, we get $\widehat{u} \leq \tilde{u}$. On the other hand, since $\tilde{u}$ is the solution to the obstacle problem for $L_1(u'')$ with $\tilde{v}$ as obstacle from below, $\tilde{u} \geq \tilde{v}$. Since $\widehat{u}$ and $\tilde{v}$ are not ordered, we obtain that $\widehat{u} \not\equiv \tilde{u}$.

The two membranes problem for the Laplacian with a right hand side, that is, for $L_1(D^2u) = -\Delta u + h_1$ and $L_2(D^2v) = -\Delta v - h_2$, was considered in [111] using variational arguments. Latter, in [27] the authors solve the two membranes problem for two different fractional Laplacians of different order (two linear non-local operators defined by two different kernels). Notice that in this case the problem is still variational. In these cases an extra condition appears, namely, the sum of the two operators vanishes, $L_1(D^2u) + L_2(D^2v) = 0$, inside Ω. Moreover, this extra condition together with the variational structure is used to prove a $C^{1,\gamma}$ regularity result for the solution.

The two membranes problem for a fully nonlinear operator was studied in [27, 26, 104]. In particular, in [26] the authors consider a version of the two membranes problem for two different fully nonlinear operators, $L_1(D^2u)$ and $L_2(D^2v)$. Assuming that L_1 is convex and that $L_2(X) = -L_1(-X)$, they prove that solutions are $C^{1,1}$ smooth.

We also mention that a more general version of the two membranes problem involving more than two membranes was considered by several authors (see for example [9, 31, 34]).

Variational operators. To simplify the notation and the arguments, when we deal with the problem using variational methods, we will concentrate in the particular choice of L_1 and L_2 as two different $p-$Laplacians; that is, we let

$$\mathcal{L}_p(w) = -\Delta_p w + h_p \qquad \text{and} \qquad \mathcal{L}_q(w) = -\Delta_q w + h_q,$$

where $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$, $\Delta_q w = \operatorname{div}(|\nabla w|^{q-2}\nabla w)$ are two different $p-$Laplacians and h_p, h_q are two given functions. These operators are associated to the energies

$$E_p(w) = \frac{1}{p}\int_\Omega |\nabla w|^p + \iint_\Omega h_p w \qquad \text{and} \qquad E_q(w) = \frac{1}{q}\int_\Omega |\nabla w|^q + \int_\Omega h_q w.$$

These energies are naturally well-posed in the Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively.

In this variational framework the obstacle problem can be solved minimizing the energy in the appropriate set of functions. In fact, the solution to the obstacle problem from below, $u = \underline{O}(\mathcal{L}_p, \varphi, f)$, can be obtained as the minimizer of the energy E_p among functions are constrained to be above the obstacle, that is,

$$E_p(u) = \min_{w \in \underline{\Lambda}^p_{f,\varphi}} \Big\{ E_p(w) \Big\}, \tag{5.2.1}$$

with

$$\underline{\Lambda}^p_{f,\varphi} = \Big\{ w \in W^{1,p}(\Omega) \ : \ \ w - f \in W^{1,p}_0(\Omega), \ \ w \geq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

The minimizer in (5.2.1) exists and is unique provided the set $\underline{\Lambda}^p_{f,\varphi}$ is not empty. We refer to Section 5.2.1 for details and precise definitions.

Due to the fact that the problem satisfied by the limit of the sequences constructed iterating the obstacle problem from above and below does not have uniqueness (see the comments after Definition 5.2.1) we need to rely on monotonicity of the sequences to obtain convergence (rather than use compactness arguments that provide only convergence along subsequences).

We notice at this point that, in this variational setting, one can solve the two membranes problem just minimizing the total energy $E(u,v) = E_p(u) + E_q(v)$ in the subset of $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ given by $\{(u,v) : u - f \in W^{1,p}_0(\Omega), v - g \in W^{1,q}_0(\Omega), u \geq v \text{ a.e } \Omega\}$. The fact that there is a unique pair that minimizes $E(u,v)$ follows from the strict convexity of the functional using the direct method of calculus of variations. One can check that the minimizing pair is in fact a solution to the two membranes problem given in Definition 5.2.1. However, in general, there are other solutions to the two membranes problem in the sense of Definition 5.2.1 that are not minimizers of $E(u,v)$. Since solutions according to Definition 5.2.1 are, in general, not unique, we observe that the limit that we prove to exist for the sequences that we construct iterating the obstacle problem from above and below does not necessarily converge to the unique minimizer to $E(u,v)$.

Fully nonlinear operators. On the other hand, when we deal with viscosity solutions to the obstacle problems we use the general framework described in [36] and we understand sub and supersolutions applying the operator to a smooth test function that touches the graph of the sub/supersolution from above or below at some point in the domain. Remark that here we can consider fully nonlinear elliptic operators that need not be in divergence form.

When we develop the viscosity theory of the iterations in this general viscosity setting it is more difficult to obtain estimates in order to pass to the limit in the sequences $\{u_n\}_n$, $\{v_n\}_n$ and hence we have to rely again on monotonicity (that is obtained using comparison arguments).

In the viscosity framework there is no gain in considering two particular operators. Therefore, we will state and prove the results for two general second order elliptic operators

that satisfy the comparison principle. However, if one looks for a model problem one can think on two normalized $p-$Laplacians, that is, let

$$L_1(w) = -\beta_1 \Delta w - \alpha_1 \Delta_\infty w + h_p \quad \text{and} \quad L_2(w) = -\beta_2 \Delta w - \alpha_2 \Delta_\infty w + h_q.$$

Here $\Delta w = trace(D^2 w)$ (the usual Laplacian), $\Delta_\infty w = \langle D^2 w \frac{\nabla w}{|\nabla w|}, \frac{\nabla w}{|\nabla w|} \rangle$ (the normalized infinity Laplacian), α_i, β_i are nonnegative coefficients and h_p, h_q are continuous functions. This operator, the normalized $p-$Laplacian, appears naturally when one considers game theory to solve nonlinear PDEs, we refer to [73, 87, 88] and the recent books [16, 66].

Let us finish the introduction with a short paragraph concerning the equivalence between weak and viscosity solutions for divergence form operators. For the Dirichlet problem $\Delta u = 0$ the notions of weak and viscosity solutions coincide (and in fact the Dirichlet problem has a unique classical solution), see [58] and [98]. Moreover, the equivalence between weak and viscosity solutions include quasi-linear equations, [59, 77], and some non-local equations, [11, 21].

5.2.1 Variational solutions

In this subsection we will obtain two sequences of solutions to variational obstacle problems whose limits constitute a pair of functions that is a variational solution to the two membranes problem.

Let us start by introducing some notations and definitions. We follow [57]. For short, we write

$$\mathcal{L}_p(w) = -\Delta_p w + h_p, \qquad E_p(w) = \frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega h_p w,$$
$$\mathcal{L}_q(w) = -\Delta_q w + h_q, \qquad E_q(w) = \frac{1}{q}\int_\Omega |\nabla w|^q + \int_\Omega h_q w,$$

for the operators and their associated energies. In this variational context, we now introduce the definition of weak super and subsolutions.

Definition 5.2.2. Given $p \in (1, \infty)$ and $h_p \in W^{-1,p}(\Omega)$ we say that $u \in W^{1,p}(\Omega)$ is a weak supersolution (resp. subsolution) for $\mathcal{L}_p$ if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w + \int_\Omega h_p w \geq 0 \quad (\text{resp. } \leq 0)$$

for all non-negative $w \in W_0^{1,p}(\Omega)$ and in this case we writte $\mathcal{L}_p u \geq 0$ (resp. $\mathcal{L}_p u \leq 0$) weakly in Ω. We say that $u \in W^{1,p}(\Omega)$ is a weak solution for $\mathcal{L}_p$ if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w + \int_\Omega h_p w = 0$$

for all $w \in W_0^{1,p}(\Omega)$ and we write $\mathcal{L}_p u = 0$ weakly in Ω.

Definition 5.2.3. Given p, $q \in (1,\infty)$, let $f \in W^{1,p}(\Omega)$, $\varphi \in L^1(\Omega)$, $h_p \in W^{-1,p}$ and

$$\underline{\Lambda}^p_{f,\varphi} = \Big\{ w \in W^{1,p}(\Omega) \,:\, \ w - f \in W^{1,p}_0(\Omega),\ \ w \geq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

When $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$, we say $u \in \underline{\Lambda}^p_{f,\varphi}$ is the solution of the p-Laplacian upper obstacle problem in a varational sense with obstacle φ and boundary datum f if

$$E_p(u) = \min_{w \in \underline{\Lambda}^p_{f,\varphi}} \Big\{ E_p(w) \Big\},$$

i.e, u minimizes the energy associated with the operator $\mathcal{L}_p := -\Delta_p + h_p$ in the set of functions that are above the obstacle and take the boundary datum, $\underline{\Lambda}^p_{f,\varphi}$. In this case, we denote $u = \underline{O}(\mathcal{L}_p, \varphi, f)$. In Proposition 5.2.4 we show that this solution exists and is unique provided the set $\underline{\Lambda}^p_{f,\varphi}$ is not empty.

Analogously, for $g \in W^{1,q}(\Omega)$, $\varphi \in L^1(\Omega)$, $h_q \in W^{-1,q}$, we define

$$\overline{\Lambda}^q_{g,\varphi} = \Big\{ w \in W^{1,q}(\Omega) \,:\, \ w - g \in W^{1,q}_0(\Omega),\ \ w \leq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

When $\overline{\Lambda}^q_{g,\varphi} \neq \emptyset$, we say that $v \in \overline{\Lambda}^q_{g,\varphi}$ is the solution of the q-Laplacian lower obstacle problem in a varational sense with obstacle φ and boundary g data if

$$E_q(v) = \min_{w \in \overline{\Lambda}^q_{g,\varphi}} \Big\{ E_q(w) \Big\}.$$

For short, we denote v as $v = \overline{O}(\mathcal{L}_q, \varphi, g)$.

Proposition 5.2.4. *Let be $h_p \in W^{-1,p}(\Omega)$ and $f \in W^{1,p}(\Omega)$. If $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$ there exists a unique $u \in \underline{\Lambda}^p_{f,\varphi}$ such that $E_p(u) \leq E_p(w)$ for all $w \in \underline{\Lambda}^p_{f,\varphi}$, i.e, there exists a unique $u \in \underline{\Lambda}^p_{f,\varphi}$ that minimizes the energy E_p in $\underline{\Lambda}^p_{f,\varphi}$. That is, $u = \underline{O}(\mathcal{L}_p, \varphi, f)$. This function satisfies*

$$\begin{cases} u \geq \varphi & \text{a.e. in } \Omega, \\ \mathcal{L}_p u \geq 0 & \text{weakly in } \Omega, \\ \mathcal{L}_p u = 0 & \text{weakly in } \{u > \varphi\} \text{ (for } \varphi \text{ continuous)}, \\ u = f & \text{a.e. in } \partial\Omega. \end{cases}$$

Analogously, let $h_q \in W^{-1,q}(\Omega)$ and $g \in W^{1,q}(\Omega)$, if $\overline{\Lambda}^q_{g,\varphi} \neq \emptyset$, there exists a unique $v \in \overline{\Lambda}^q_{g,\varphi}$ such that $E_q(v) \leq E_q(w)$ for all $w \in \overline{\Lambda}^q_{g,\varphi}$. That is, $v = \overline{O}(\mathcal{L}_q, \varphi, g)$. This function satisfies

$$\begin{cases} v \leq \varphi & \text{a.e. in } \Omega, \\ \mathcal{L}_q v \leq 0 & \text{weakly in } \Omega, \\ \mathcal{L}_q v = 0 & \text{weakly in } \{v < \varphi\} \text{ (for } \varphi \text{ continuous)}, \\ v = g & \text{a.e. in } \partial\Omega. \end{cases}$$

Proof. Let us prove existence and uniqueness of the solution to the obstacle problem from below. The proof is contained in Theorem 3.21 in [57] but we reproduce some details here for completeness. Suppose that $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$. Since E_p is a coercive and strictly convex functional in $W^{1,p}(\Omega)$, we obtain that all minimizing sequences $u_j \in \underline{\Lambda}^p_{f,\varphi}$ converge weakly to the same limit $u \in W^{1,p}(\Omega)$ (that we want to show that is the unique minimizer of E_p in $\underline{\Lambda}^p_{f,\varphi}$). Let us show that $u \in \underline{\Lambda}^p_{f,\varphi}$. Rellich-Kondrachov's Theorem implies that, taking a subsequence, $u_j \to u$ strongly in $L^p(\Omega)$. Then, taking a subsequence, we have that $u_j \to u$ a.e. in Ω. Thus, since $u_j \geq \varphi$ a.e. Ω and $u_j - f \in W^{1,p}_0(\Omega)$ for all $j \geq 1$, we get $u \geq \varphi$ a.e. Ω and $u - f \in W^{1,p}_0(\Omega)$.

Let us prove that u verifies the properties stated in the proposition. Let us start with the fact that $\mathcal{L}_p u \geq 0$ weakly in Ω. Since $\underline{\Lambda}^p_{f,\varphi}$ is a convex set, $u + t(v-u) \in \underline{\Lambda}^p_{f,\varphi}$ for all $v \in \underline{\Lambda}^p_{f,\varphi}$ and $t \in [0,1]$. Thanks to the fact that E_p reaches its minimum in the set $\underline{\Lambda}^p_{f,\varphi}$ at u, we have that

$$i(t) := E_p[u + t(v-u)]$$

satisfies $i(t) \geq i(0)$ and therefore $i'(0) \geq 0$. Let us compute

$$i'(t) = \int_\Omega [|\nabla u + t\nabla(v-u)|^{p-2}(\nabla u + t\nabla(v-u)) \cdot \nabla(v-u) - h_p(v-u)]dx.$$

Taking $t = 0$, we get

$$i'(0) = \int_\Omega [|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) - h_p(v-u)]dx \geq 0. \tag{5.2.2}$$

If we consider $w \in W^{1,p}_0(\Omega)$ $w \geq 0$ a.e. Ω, and $v = u + tw \in \underline{\Lambda}^p_{f,\varphi}$ for $t \geq 0$ small enough, if we come back to (5.2.2), we obtain

$$t\int_\Omega [|\nabla u|^{p-2}\nabla u \cdot \nabla w - h_p w]dx \geq 0$$

with $t \geq 0$. This implies that

$$\mathcal{L}_p u \geq 0$$

weakly in Ω.

Finally, we show that

$$\mathcal{L}_p u = 0$$

weakly in $\{u(x) > \varphi(x)\}$ when φ is continuous. From Section 3.26 in [57] we have that the solution to the obstacle problem with continuous obstacle becomes also continuous (after a redefinition in a set of measure zero). Therefore, for a continuous obstacle the set $\{x : u(x) > \varphi(x)\}$ is an open set. Take a ball

$$\overline{B}_r \subset \{x : u(x) > \varphi(x)\},$$

and consider a nonnegative $w \in W^{1,p}_0(B_r) \cap C_0(\overline{B_r})$. Using that $\varphi < u$ in ∂B_r, we get $v = u + tw \in \underline{\Lambda}^p_{f,\varphi}$ for $|t|$ small enough. Then, let us come back to (5.2.2) and obtain

$$t\int_{B_r} [|\nabla u|^{p-2}\nabla u \cdot \nabla w - h_p w]dx \geq 0.$$

In this case t can be positive or negative. Thus, we get

$$\fint_{B_r}[|\nabla u|^{p-2}\nabla u\nabla w - h_p w]dx = 0,$$

wich implies

$$\mathcal{L}_p u = 0$$

weakly in B_r and therefore in $\{u > \varphi\}$.

The other case (an obstacle from above) is analogous. □

This minimizer of the energy is in fact the infimum of weak supersolutions are above/below the obstacle.

Proposition 5.2.5. *Let $u = \underline{O}(\mathcal{L}_p, \varphi, f)$. This function satisfies*

$$u = \inf\Big\{w \in \underline{\Lambda}^p_{f,\varphi} \ : \ \mathcal{L}_p w \geq 0 \ \text{ weakly in } \ \Omega\Big\}, \tag{5.2.3}$$

Analogously, let $v = \overline{O}(\mathcal{L}_q, \varphi, g)$, then

$$v = \sup\Big\{w \in \overline{\Lambda}^q_{g,\varphi} \ : \ \mathcal{L}_q w \leq 0 \ \text{ weakly in } \ \Omega\Big\}. \tag{5.2.4}$$

Proof. Let us consider

$$\overline{u} = \inf\Big\{w \in \underline{\Lambda}^p_{f,\varphi} \ : \ \mathcal{L}_p w \geq 0 \ \text{ weakly in } \ \Omega\Big\}.$$

We will prove that $\overline{u} = u$ in Ω.

Let us start proving that $u \geq \overline{u}$. By Proposition 5.2.4, we have

$$u \in \Big\{w \in \underline{\Lambda}^p_{f,\varphi} \ : \ \mathcal{L}_p w \geq 0 \ \text{ weakly in } \ \Omega\Big\},$$

and then we get

$$u \geq \overline{u} = \inf\Big\{w \in \underline{\Lambda}^p_{f,\varphi} \ : \ \mathcal{L}_p w \geq 0 \ \text{ weakly in } \ \Omega\Big\}.$$

Now, let us prove that $u \leq \overline{u}$. Given $w \in \underline{\Lambda}^p_{f,\varphi}$ such that $\mathcal{L}_p w \geq 0$ weakly in Ω, we have

$$\fint_{\Omega}[|\nabla w|^{p-2}\nabla w \cdot \nabla v - h_p v]dx \geq 0 \quad \text{for all } v \in W^{1,p}_0(\Omega)$$

and then, using (5.2.2), we get

$$\fint_{\Omega}[|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) - h_p(v-u)]dx \geq 0 \quad \text{for all } v \in \underline{\Lambda}^p_{f,\varphi}(\Omega). \tag{5.2.5}$$

Let us consider $z = \min\{w, u\} \in \underline{\Lambda}^p_{f,\varphi}$, then $u - z \in W^{1,p}_0(\Omega)$, $u - z \geq 0$. Then

$$0 \leq \int_\Omega [|\nabla w|^{p-2}\nabla w \cdot \nabla(u - z) - h_p(u - z)]dx$$

and, using (5.2.5), we obtain

$$0 \geq \int_\Omega [|\nabla u|^{p-2}\nabla u \cdot \nabla(u - z) - h_p(u - z)]dx.$$

Substracting these inequalities we conclude that

$$0 \leq \int_\Omega [(|\nabla w|^{p-2}\nabla w - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - z)]dx,$$

which implies

$$0 \leq \int_{\{u>z\}} [(|\nabla w|^{p-2}\nabla w - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - w)]dx$$

using that $(|b|^{p-2}b - |a|^{p-2}a)(a - b) \leq 0$ (with a strict inequality for $a \neq b$) we obtain $|\{u > z\}| = 0$. Then, $u = z \leq w$. Thus, we have obtained that $u \leq \overline{u}$.

The other case is analogous. □

Remark 5.2.6. Notice that Proposition 5.2.5 implies that the solution to the obstacle problem is monotone with respect to the obstacle in the sense that, for $\varphi_1 \geq \varphi_2$, we have $u_1 = \underline{O}(\mathcal{L}_p, \varphi_1, f) \geq u_2 = \underline{O}(\mathcal{L}_p, \varphi_2, f)$.

Now, we are ready to introduce the definition of a weak solution to the two membranes problem.

Definition 5.2.7. Let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces. Take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. We say that the pair (u, v) with $u \in \underline{\Lambda}^p_{f,v}$, $v \in \overline{\Lambda}^q_{g,u}$ is a solution of the two membranes problem if

$$(u, v) \text{ satisfies } \begin{cases} E_p(u) = \min\limits_{w \in \underline{\Lambda}^p_{f,v}} \Big\{E_p(w)\Big\}, \\ E_q(v) = \min\limits_{w \in \overline{\Lambda}^q_{g,u}} \Big\{E_q(w)\Big\}, \end{cases}$$

i.e.,

$$u = \underline{O}(\mathcal{L}_p, v, f) \text{ and } v = \overline{O}(\mathcal{L}_q, u, g),$$

Remark 5.2.8. In general, given a pair (f, g), the solution to the two membranes problem is not unique.

Iterative method.

The main result of this subsection reads as follows.

Theorem 5.2.9. *For $p \geq q$ let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces and take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. Let us consider v_0 a weak subsolution of $\mathcal{L}_q v = 0$ in Ω such that $v_0 - g \in W_0^{1,q}(\Omega)$ and then define inductively the sequences*

$$u_n = \underline{O}(\mathcal{L}_p, v_n, f), \qquad v_n = \overline{O}(\mathcal{L}_q, u_{n-1}, g).$$

Both sequences of functions $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega)$, $\{v_n\}_{n=0}^{\infty} \subset W^{1,q}(\Omega)$ converge strongly in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Moreover, the limits of the sequences are a solution of the two membranes problem. That is, there exists a pair of functions $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ and $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$ such that

$$u_n \to u_\infty \text{ strongly in } W^{1,p}(\Omega) \text{ and } v_n \to v_\infty \text{ strongly in } W^{1,q}(\Omega),$$

and, in addition, the limit pair (u_∞, v_∞) satisfies

$$u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f) \text{ and } v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g).$$

Proof. Notice that from the definition of the sequences as solutions to the obstacle problem we have

$$v_n \leq u_n$$

a.e. Ω for each $n \in \mathbb{N}$. Since we assumed that the boundary data are ordered, $f \geq g$, and functions in $W^{1,p}(\Omega)$ and in $W^{1,q}(\Omega)$ have a trace on the boundary, we can also say that $u_n \geq v_n$ a.e. on the boundary (with respect to the $(N-1)$-dimensional measure). From the construction of the sequences we obtain that the sets $\underline{\Lambda}^p_{f,v_n}$ and $\overline{\Lambda}^q_{g,u_n}$ are not empty, in fact we have that $u_{n-1} \in \underline{\Lambda}^p_{f,v_n}$ (since v_n solves an obstacle problem from above with obstacle u_{n-1}, we have $v_n \leq u_{n-1}$) and $v_n \in \overline{\Lambda}^q_{g,u_n}$ (since u_n solves an obstacle problem from below with obstacle v_n, we have $v_n \leq u_n$). Hence the sequences $\{u_n\}$ and $\{v_n\}$ are well defined.

We will divide our arguments in several steps.

Step 1: First, we prove a monotonicity result. The two sequences are non-decreasing.

Let us see that $v_0 \leq v_1$ a.e. Ω. This is due to the fact that v_0 is a weak subsolution of $\mathcal{L}_q v = 0$ and $v_0 \in \overline{\Lambda}^q_{g,u_0}$ and by (5.2.4) we have

$$v_1 = \sup\left\{ w \in \overline{\Lambda}^q_{g,u_0} \ : \ \mathcal{L}_q w \leq 0 \text{ weakly in } \Omega \right\}.$$

Then, we conclude that $v_0 \leq v_1$.

Also we can see that $u_0 \leq u_1$ a.e. Ω. In fact, from (5.2.3), we have

$$u_0 = \inf\left\{ w \in \underline{\Lambda}^p_{f,v_0} \ : \ \mathcal{L}_p w \geq 0 \text{ weakly in } \Omega \right\}$$

and then, using that $v_1 \geq v_0$ a.e. Ω, we obtain that $u_1 \in \underline{\Lambda}^q_{f,v_0}$ and u_1 is a weak supersolution for $\mathcal{L}_p u = 0$. Then, $u_0 \leq u_1$ a.e. in Ω.

Now, to obtain the general case, we just use an inductive argument. Suppose $v_{n-1} \leq v_n$ and $u_{n-1} \leq u_n$ a.e. Ω. Since $v_n \leq u_n$, we have $v_n \in \overline{\Lambda}^q_{g,u_n}$. From (5.2.4) we have

$$v_{n+1} = \sup\left\{w \in \overline{\Lambda}^q_{g,u_n} \ : \ \mathcal{L}_q w \leq 0 \ \text{ weakly in } \ \Omega\right\} \ \text{ and } \ \mathcal{L}_q v_n \leq 0 \ \text{ weakly in } \ \Omega$$

which implies $v_n \leq v_{n+1}$ a.e. Ω. On the other hand, the last inequality and the fact that $v_{n+1} \leq u_{n+1}$ give us $u_{n+1} \in \underline{\Lambda}^p_{f,v_n}$. Hence, (5.2.3) implies that

$$u_n = \inf\left\{w \in \underline{\Lambda}^p_{f,v_n} \ : \ \mathcal{L}_p w \geq 0 \ \text{ weakly in } \ \Omega\right\} \ \text{ and } \ \mathcal{L}_p u_{n+1} \geq 0 \ \text{ weakly in } \ \Omega,$$

and we conclude that $u_n \leq u_{n+1}$ a.e. Ω.

Step 2: Our next step is to show that the sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are bounded in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. To prove this fact let us take $w \in W^{1,q}(\Omega)$ the solution to $\mathcal{L}_q w = 0$ with boundary datum g. That is, w satisfies

$$\mathcal{L}_q w = 0 \quad \text{weakly in } \ \Omega \ \text{ and } \ w = g \ \text{ in } \ \partial\Omega \ \text{ in the sense of traces.}$$

By (5.2.4), we have that $\mathcal{L}_q v_n \leq 0$ weakly in Ω, and thanks to $v_n - w \in W^{1,q}_0(\Omega)$ and the comparison principle for $\mathcal{L}_q$, we obtain that

$$v_n \leq w$$

a.e. Ω. Let us consider $z = \underline{O}(\mathcal{L}_p, w, f)$. Since $v_n \leq w$ we have $z \in \underline{\Lambda}^p_{f,v_n}$ for all $n \in \mathbb{N}$. Now, since u_n is the solution to the obstacle problem, u_n minimizes the energy among functions in the set $\underline{\Lambda}^p_{f,v_n}$. Then, as z is in the former set, we get

$$E_p(u_n) \leq E_p(z)$$

for all $n \in \mathbb{N}$, that is, we have

$$\fint\!\!\!\!\!\!\int_\Omega \frac{|\nabla u_n|^p}{p} + \int_\Omega h_p u_n \leq \fint\!\!\!\!\!\!\int_\Omega \frac{|\nabla z|^p}{p} + \int_\Omega h_p z.$$

We can rewrite this inequality to obtain

$$\fint\!\!\!\!\!\!\int_\Omega |\nabla u_n|^p \leq C(z, p, h_p) + C(p, h_p)\|u_n\|_{L^p(\Omega)}. \tag{5.2.6}$$

Using that $u_n - f \in W^{1,p}_0(\Omega)$, from Poincare's inequality, we obtain

$$\|u_n - f\|_{L^p(\Omega)} \leq C\|\nabla u_n - \nabla f\|_{L^p(\Omega)} \ \text{ for all } \ n \in \mathbb{N}.$$

Then, by the triangle inequality we get

$$\|u_n\|_{L^p(\Omega)} \leq C(f) + C\|\nabla u_n\|_{L^p(\Omega)} \ \text{ for all } \ n \in \mathbb{N}.$$

If we come back to (5.2.6), we obtain

$$\fint\!\!\!\!\!\!\int_{\Omega}|\nabla u_n|^p \leq C(z,p,h_p,f) + C(p,h_p,f)\|\nabla u_n\|_{L^p(\Omega)}.$$

Now, if we use Young's inequality $ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}$ with $\varepsilon = \frac{1}{2}$ in the last term, we conclude

$$\fint\!\!\!\!\!\!\int_{\Omega}|\nabla u_n|^p \leq C(z,p,h_p,f) \ \text{ for all } \ n \in \mathbb{N}.$$

Thus, we get the desired bound for u_n, there exists a constant C independent of n such that

$$\|u_n\|_{W^{1,p}(\Omega)} \leq C \ \text{ for all } \ n \in \mathbb{N}.$$

Now, we prove that $\{v_n\}$ is bounded in $W^{1,q}(\Omega)$. Since $v_0 \leq v_n \leq u_{n-1}$ we have that $v_0 \in \underline{\Lambda}^q_{g,u_{n-1}}$ for all $n \in \mathbb{N}$. This implies

$$E_q(v_n) \leq E_q(v_0).$$

That is,

$$\fint\!\!\!\!\!\!\int_{\Omega}\frac{|\nabla v_n|^q}{q} + \int_{\Omega} h_q v_n \leq \fint\!\!\!\!\!\!\int_{\Omega}\frac{|\nabla v_0|^q}{q} + \int_{\Omega} h_q v_0.$$

Now, we just proceed as before to obtain

$$\|v_n\|_{W^{1,q}(\Omega)} \leq C \ \text{ for all } \ n \in \mathbb{N}.$$

Step 3: We show strong convergence to a solution of the two membranes problem.

Since $\{u_n\}$ and $\{v_n\}$ are bounded in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ respectively, there exist subsequences $\{u_{n_j}\} \subset \{u_n\}$ and $\{v_{n_j}\} \subset \{v_n\}$ such that

$$u_{n_j} \rightharpoonup u_\infty \ \text{ weakly in } \ W^{1,p}(\Omega) \quad \text{and} \quad v_{n_j} \rightharpoonup v_\infty \ \text{ weakly in } \ W^{1,q}(\Omega),$$

as $n_j \to \infty$.

By the Rellich-Kondrachov theorem, there exists a subsequence $\{u_{n_j}\}$ that converges strongly in $L^p(\Omega)$ and a subsequence $\{v_{n_j}\}$ that converges strongly in $L^q(\Omega)$. Extracting again a subsequence if needed, we get

$$u_{n_j} \longrightarrow u_\infty \quad \text{and} \quad v_{n_j} \longrightarrow v_\infty \ \text{ a.e. } \Omega,$$

as $n_j \to \infty$. Using that $\{u_n\}$ and $\{v_n\}$ are increasing we obtain that the entire sequences converge pointwise (and weakly) to the unique limits (that are given by the supremums of the sequences). This implies that $u_\infty \geq v_\infty$ a.e. Ω, $u_\infty = f$ and $v_\infty = g$ a.e. $\partial\Omega$ in the sense of traces. This gives us $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ and $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$.

Let us define $u = \underline{\mathrm{O}}(\mathcal{L}_p, v_\infty, f)$ and $v = \overline{\mathrm{O}}(\mathcal{L}_q, u_\infty, g)$. Our next step is to prove that $u = u_\infty$ and $v = v_\infty$ a.e. Ω and to get that the convergence is strong in the corresponding Sobolev spaces.

First, we begin with the case of the upper membranes u_n. Since $u = \underline{O}(\mathcal{L}_p, v_\infty, f)$ and $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ we have that $E_p(u) \leq E_p(u_\infty)$. On the other hand, since $\{v_n\}$ is an increasing sequence we have $u_\infty \geq v_\infty \geq v_n$ for each $n \in \mathbb{N}$ wich implies $u \in \underline{\Lambda}^p_{f,v_n}$. Then, we have that $E_p(u_n) \leq E_p(u)$.

On the other hand, by the semicontinuity of the norm, $u_n = u_\infty$ in $\partial\Omega$ and by weak convergence, we obtain

$$\|\nabla u_\infty\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^p(\Omega)}, \quad \int_\Omega h_p u_n \longrightarrow \int_\Omega h_p u_\infty \ \text{ as } \ n \to \infty.$$

Then, we have

$$E_p(u_\infty) \leq \liminf_{n\to\infty} E_p(u_n) \leq \limsup_{n\to\infty} E_p(u_n) \leq E_p(u) \leq E_p(u_\infty). \tag{5.2.7}$$

Thus, $E_p(u_\infty) = E_p(u)$. And since the energy minimizer is unique, $u_\infty = u$ a.e. Ω. Moreover, by (5.2.7) and the weak convergence, we have obtained that $\{u_n\}$ is a minimizing sequence of the energy and the gradient of u_n converges strongly to the gradient of u_∞ as n goes to infinity. Then, we conclude that

$$u_n \longrightarrow u_\infty \ \text{ strongly in } \ W^{1,p}(\Omega) \ \text{ as } \ n \to \infty.$$

Once we have seen that $u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f)$, let us prove that $v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g)$. Again, since $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$, we have that $E_p(v) \leq E_p(v_\infty)$.

Note that we cannot repeat the previous argument step by sept because we do not know if v is in $\overline{\Lambda}^q_{g,u_{n-1}}$. Though, it is enough to change a little our strategy (we use at this point that $p \geq q$). We construct a subsequence $\{\tilde{v}_n\}$ such that $\tilde{v}_n \in \overline{\Lambda}^q_{g,u_{n-1}}$ and $E_p(\tilde{v}_n)$ converges to $E_p(v)$ as n goes to infinity. We define $\{\tilde{v}_n\}$ as

$$\tilde{v}_n = v - u_\infty + u_n.$$

Since $p \geq q$ we have that $\tilde{v}_n \in W^{1,q}(\Omega)$, $\tilde{v}_n - g \in W^{1,q}_0(\Omega)$ and $E_q(\tilde{v}_n)$ goes to $E_q(v)$ as $n \to \infty$ due to the Rellick-Kondrachov Compactness Theorem and the fact that u_n converges to u_∞ in $W^{1,p}(\Omega)$. Also we have the inequality $\tilde{v}_n \leq u_n$ because $v \leq u_\infty$. Then, $\tilde{v}_n \in \overline{\Lambda}^q_{g,u_n}$ which implies $E_q(v_{n+1}) \leq E_q(\tilde{v}_n)$. Besides, by the semicontinuity of the norm, the fact that $v_n = v_\infty$ on $\partial\Omega$ and the weak convergence, we obtain

$$\|\nabla v_\infty\|_{L^q(\Omega)} \leq \liminf_{n\to\infty} \|\nabla v_{n+1}\|_{L^q(\Omega)}, \quad \int_\Omega h_q v_{n+1} \longrightarrow \int_\Omega h_q v_\infty \ \text{ as } \ n \to \infty.$$

As a consequence,

$$\begin{aligned} E_q(v_\infty) &\leq \liminf_{n\to\infty} E_q(v_{n+1}) \leq \limsup_{n\to\infty} E_q(v_{n+1}) \\ &\leq \limsup_{n\to\infty} E_q(\tilde{v}_n) = E_q(v) \leq E_p(v_\infty). \end{aligned}$$

Then, $E_q(v_\infty) = E_q(v)$. By the uniqueness of the minimizer, $v_\infty = v$ a.e. Ω. Furthermore, from the same reasons as in the case of the upper membranes, we have that the entire sequence converges strongly,

$$v_n \longrightarrow v_\infty \text{ strongly in } W^{1,q}(\Omega) \text{ as } n \to \infty.$$

This ends the proof. □

If $p < q$ we can also construct a pair of sequences that converges in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ to a solution of the two membranes problem. In this case we just have to start the iterations of the obstacle problems with u_0 a weak supersolution to $\mathcal{L}_p u = 0$.

Theorem 5.2.10. *For $p < q$ let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces and take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. Take u_0 a weak supersolution to $\mathcal{L}_p u = 0$ in Ω such that $u_0 - f \in W_0^{1,p}(\Omega)$ and then let*

$$u_n = \underline{O}(\mathcal{L}_p, v_{n-1}, f), \qquad v_n = \overline{O}(\mathcal{L}_q, u_n, g).$$

Both sequences $\{u_n\}_{n=0}^\infty \subset W^{1,p}(\Omega)$ and $\{v_n\}_{n=0}^\infty \subset W^{1,q}(\Omega)$ converge strongly in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Moreover, the limits of the sequences are a solution of the two membranes problem. That is, there exists a pair of functions $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ and $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$ such that

$$u_n \to u_\infty \text{ strongly in } W^{1,p}(\Omega) \text{ and } v_n \to v_\infty \text{ strongly in } W^{1,q}(\Omega),$$

and, in addition, the limit pair (u_∞, v_∞) satisfies

$$u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f) \text{ and } v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g).$$

Proof. The difference between these sequences and the sequences defined in proposition (5.2.9) is that, here, $\{u_n\}$ and $\{v_n\}$ are decreasing sequences.

Due to the monotonicity of the sequences, we can prove that u_n converges weakly to some u_∞ in $W^{1,p}(\Omega)$ and v_n converges to some v_∞ weakly in $W^{1,q}(\Omega)$ as n goes to infinity.

The strong convergence of v_n to v_∞ is given in the same way that we got strong convergence for u_n to u_∞ in proposition (5.2.9) thanks to $\{v_n\}$ is a decreasing sequence. On the other hand, for $\{u_n\}$ we can reproduce the proof for $\{v_n\}$ in proposition (5.2.9) because $W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ continuously. □

Remark 5.2.11. An alternative idea to deal with the case $p < q$ runs as follows: It can be easily proved that

$$u = \underline{\mathrm{O}}(-\Delta_p + h_p, \varphi, f) \text{ if and only if } u = \overline{\mathrm{O}}(-\Delta_p - h_p, -\varphi, -f).$$

Then, if $p < q$ we consider

$$\begin{aligned} p' &= q, \quad h_{p'} = -h_q, \quad f' = -g, \\ q' &= p, \quad h_{q'} = -h_p, \quad g' = -f. \end{aligned}$$

For the problem with the new set of parameters, p', q', $h_{p'}$, $h_{q'}$, f' and g', we can apply the iterative method described in Theorem (5.2.9) and get a pair (u'_∞, v'_∞) such that $u_\infty \in \underline{\Lambda}^{p'}_{f',v'_\infty}$ and $v'_\infty \in \overline{\Lambda}^{q'}_{g',u'_\infty}$ such that $u'_\infty = \underline{O}(-\Delta_{p'} + h_{p'}, v'_\infty, f')$ and $v'_\infty = \overline{O}(-\Delta_{q'} + h_{q'}, u'_\infty, g')$. Thus, $u_\infty = -v'_\infty$ and $v_\infty = -u'_\infty$ are in $\underline{\Lambda}^{p}_{f,v_\infty}$ and $\overline{\Lambda}^{q}_{g,u_\infty}$ respectively and

$$u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f), \quad \text{and} \quad v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g),$$

i.e, (u_∞, v_∞) is a solution for the original two membranes problem.

Remark 5.2.12. We remark that the limit depends strongly on the initial function from where we start the iterations. We would like to highlight that $\{u_n\}$ and $\{v_n\}$ converge to u_∞ and v_∞ strongly in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ respectively due to the fact that both sequences are monotone. The monotonicity, in turn, depends on the initial function for the iteration. Specifically, when $p \geq q$, the monotonicity arises because the initial datum v_0 is a weak subsolution of $\mathcal{L}_q v = 0$. On the other hand, when $p \leq q$ it stems from the fact that u_0 is a weak supersolution of $\mathcal{L}_p u = 0$.

If one considers the pair (u, v) obtained as follows: let u be the solution to $\mathcal{L}_p u = 0$ in Ω with $u = f$ on $\partial\Omega$ and v the solution to the corresponding obstacle problem $v = \overline{O}(\mathcal{L}_q, u, g)$, then, as we have mentioned in the introduction this pair (u, v) is a solution to the two membranes problem. Now, if one starts the iteration procedure with $u_0 = u$ we obtain that the sequences converge after only one step. In fact we get $v_1 = \overline{O}(\mathcal{L}_q, u, g)$ and next the iteration gives again, that is, we have $u = \underline{O}(\mathcal{L}_p, v_1, f)$. This is due to the fact that u is a solution to $\mathcal{L}_p u = 0$ in the whole Ω with $u \geq v_1$ and therefore it is the solution to the obstacle problem (with v_1 as obstacle from below). Analogously, if one starts with v the solution to $\mathcal{L}_q v = 0$ in Ω with $v = g$ on $\partial\Omega$ we obtain u as the solution to the corresponding obstacle problem $u = \underline{O}(\mathcal{L}_p, v, f)$. This pair (u, v) is also a solution to the two membranes problem, but, in general the two pairs are different.

Extension to the two membranes problem for nonlocal operators If we consider

$$\mathcal{L}_p := (-\Delta_p)^s + h_p \quad \text{and} \quad \mathcal{L}_q := (-\Delta_q)^t + h_q$$

with

$$(-\Delta_p)^s(u)(x) = \nint_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{N+sp}} dy,$$

the fractional $p-$Laplacian. Notice that $\mathcal{L}_p$ and $\mathcal{L}_q$ are associated with the energies

$$E^s_p(w) = \frac{1}{p} \int_{\mathbb{R}^N} \nint_{\mathbb{R}^N} \frac{|u(y) - u(x)|^p}{|x - y|^{N+sp}} dy\, dx + \int_\Omega h_p w$$

and

$$E^t_q(w) = \frac{1}{q} \int_{\mathbb{R}^N} \nint_{\mathbb{R}^N} \frac{|u(y) - u(x)|^q}{|x - y|^{N+tq}} dy\, dx + \int_\Omega h_q w.$$

Our iterative method of the Theorem (5.2.9) gives us a pair (u_∞, v_∞) that is solution of the "new" two membranes problem whenever

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega) \text{ continuosly.}$$

To be able to define a variational solution of the obstacle problem for these new $\mathcal{L}_p$ and $\mathcal{L}_q$ is nedeed that $\underline{\Lambda}^p_{f,\varphi}$ and $\overline{\Lambda}^q_{g,\varphi}$ are closed sets under the weak topology. This is given by Mazur's Theorem if $W^{s,p}(\Omega)$ and $W^{t,q}(\Omega)$ are reflexive Banach space and $\underline{\Lambda}^p_{f,\varphi}$ and $\overline{\Lambda}^q_{g,\varphi}$ are closed convex sets (with the norm) in $W^{s,p}(\Omega)$ and $W^{t,q}(\Omega)$ respectively.

5.2.2 Viscosity solutions

In this section we aim to prove similar convergence results for iterations of the obstacle problem to a pair that solves the two membranes problems when the involved operators are not variational. Here we will consider two nonlinear elliptic operators and we understand solutions in the viscosity sense. To define the sequences as solutions to the corresponding obstacle problems we need to assume that the involved operators verify the following set of conditions:

HYPOTHESIS: The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ verify:

- $\mathcal{L}w = F(D^2w, \nabla w) - h(x)$ with F continuous in both coordinates, such that the comparison principle holds.
- If f is continuous, there exists a unique $w \in C(\overline{\Omega})$ solution to $\mathcal{L}w = 0$ in Ω, $w = f$ in $\partial\Omega$.
- If φ, ψ and f, g are continuous, then the corresponding solutions to the obstacle problems, $u = \underline{\mathcal{O}}(\mathcal{L}_1, \varphi, f)$ and $v = \overline{\mathcal{O}}(\mathcal{L}_2, \psi, g)$ are continuous. See Definition 5.2.18 below.

To be precise, working in the viscosity sense, we need to introduce the definition of semicontinuous functions.

Definition 5.2.13. $f : \Omega \longrightarrow \mathbb{R}$ is a lower semicontinuous function, l.s.c, at $x \in \Omega$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) \leq f(y) + \varepsilon \text{ for all } y \in B_\delta(x).$$

If a function f is l.s.c. at every $x \in \Omega$, we say that f is l.s.c. in Ω. The lower semicontinuous envelope of f is

$$f_* = \sup\{h : \Omega \longrightarrow \mathbb{R} \text{ l.s.c. } : \ h \leq f.\}$$

On the other hand, $g : \Omega \longrightarrow \mathbb{R}$ is a upper semicontinuous function, u.s.c, at $x \in \Omega$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$g(y) \leq g(x) + \varepsilon \text{ for all } y \in B_\delta(x).$$

If a functions g is u.s.c. at every $x \in \Omega$, we say that it is u.s.c. in Ω. The upper semicontinuous envelope of g is

$$g^* = \inf\{h : \Omega \longrightarrow \mathbb{R} \text{ u.s.c. } : \ h \geq f.\}$$

Now, we give the precise definition of sub and supersolutions in the viscosity sense.

Definition 5.2.14. The function $u : \Omega \longrightarrow \mathbb{R}$ is called a $\mathcal{L}$-viscosity supersolution if its lower semicontinuous envelope function, u_*, satisfies the following: for every $\phi \in C^2(\overline{\Omega})$ such that ϕ touches u_* at $x \in \Omega$ strictly from below, that is, $u_* - \phi$ has a strict minimum at x with $u_*(x) = \phi(x)$, we have

$$\mathcal{L}\phi(x) \geq 0.$$

In this case, we write $\mathcal{L}u \geq 0$ in the viscosity sense.

Conversely, u is called a $\mathcal{L}$-viscosity subsolution if its upper semicontinuous envelope function, u^*, satisfies that for every $\phi \in C^2(\overline{\Omega})$ such that ϕ touches u^* at $x \in \Omega$ strictly from above, that is, $\phi - u^*$ has a strict minimum at x with $u^*(x) = \phi(x)$, we have

$$\mathcal{L}\phi(x) \leq 0$$

and we write $\mathcal{L}u \leq 0$ in the viscosity sense.

Finally, u is a $\mathcal{L}$-viscosity solution if it is both a $\mathcal{L}$-viscosity supersolution and a $\mathcal{L}$-viscosity subsolution and we denote $\mathcal{L}u = 0$ in the viscosity sense.

Now, let us introduce three results concerning the limit of non-decreasing sequences of viscosity sub or supersolutions. Although these results are not difficult to prove, they will be crucial for the proof of our main Theorem (5.2.24) in this section.

Proposition 5.2.15. *Let be $\{v_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$ a non-decreasing sequence of functions, continuous up to the boundary such that v_n is a $\mathcal{L}$-viscosity subsolution for each $n \in \mathbb{N}$ and $v_n \longrightarrow v_\infty$ pointwise. Then, v_∞ is a $\mathcal{L}$-viscosity subsolution.*

Proof. Let be $\phi \in C^2(\overline{\Omega})$ such that $\phi - v_\infty^*$ has a strict minimum at $x_0 \in \Omega$ and $\phi(x_0) = v_\infty^*(x_0)$.

Since v_∞^* is upper semicontinuous, $\phi - v_\infty^*$ is lower semicontinuous. Then, fixed $r > 0$ small enough, $\phi - v_\infty^*$ reaches a minimun in $D_r := \overline{\Omega} \setminus B_r(x_0)$. Say in $z_\infty \in D_r$. Thanks to $-v_\infty^* = (-v_\infty)_*$ and that the sequence $\{v_n\}$ is non-decreasing, we have

$$0 < (\phi - v_\infty^*)(z_\infty) \leq (\phi - v_\infty^*)(z) \leq (\phi - v_\infty)(z) \leq (\phi - v_n)(z) \tag{5.2.8}$$

for each $z \in D_r$ and for each $n \in \mathbb{N}$.

On the other hand, by the definition of pointwise value of the lower semicontinuous envelope,

$$\begin{aligned} 0 = (\phi - v_\infty^*)(x_0) &= (\phi + (-v_\infty)_*)(x_0) \\ &= \inf_{\left\{\{x_k\}_{k=0}^{\infty} \subset \Omega:\ x_k \to_{k\to\infty} x_0\right\}} \lim_{k\to\infty} (\phi - v_\infty)(x_k). \end{aligned}$$

Then, given $\varepsilon > 0$ there exists a sequence $\{y_k\}$ within $B_{r/2}(x_0)$ such that

$$\lim_{k\to\infty} |(\phi - v_\infty)(y_k)| \leq \varepsilon/3.$$

Thus, there is $k_0 \in \mathbb{N}$ such that $|(\phi - v_\infty)(y_{k_0})| \leq 2\varepsilon/3$. Since v_n converges to v_∞ pointwise, there exists $n_0 \in \mathbb{N}$ such that $|(\phi - v_\infty)(y_{k_0}) - (\phi - v_n)(y_{k_0})| \leq \varepsilon/3$ for all $n \geq n_0$.

In short, for each $\varepsilon > 0$ there exists a $y \in B_{r/2}(x_0)$ and $n_0 \in \mathbb{N}$ such that

$$(\phi - v_n)(y) \leq \varepsilon \ \text{ for all } \ n \geq n_0.$$

Since, $\phi - v_n$ is a continuous function, given $\varepsilon < (\phi - v_\infty^*)(z_\infty)$ one can see that the minimum of $\phi - v_n$ in $\overline{\Omega}$ is reached at some $x_n^r \in B_{r/2}(x_0)$ for all $n \geq n_0$ thanks to the above inequality and (5.2.8). Note that n_0 depends on r, so we write $n_0 = n_0(r)$. We have $n_0(r) \to \infty$ and $x_n^r \to x_0$ as $r \to 0^+$. Due to the fact that v_n is a $\mathcal{L}$- viscosity subsolution, we have

$$\mathcal{L}\phi(x_n^r) \leq 0.$$

Since $\mathcal{L}$ is a continuous operator we have

$$0 \geq \lim_{r\to\infty} (\mathcal{L}\phi)(x_n^r) = (\mathcal{L}\phi)\big(\lim_{r\to\infty} x_n^r\big) = (\mathcal{L}\phi)(x_0).$$

Therefore v_∞ is a $\mathcal{L}$-viscosity subsolution. □

Lemma 5.2.16. *Let $\{u_n\} \subset C(\overline{\Omega})$ a non-decreasing sequence of continuous up to the boundary functions. If u_n converge pointwise to some function u_∞, then u_∞ is a lower semicontinuous function in $\overline{\Omega}$.*

Proof. Let be $x_0 \in \Omega$. Since $u_n \longrightarrow u_\infty$ pointwise, we have that for each $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that $u_\infty(x_0) \leq u_{n_0}(x_0) + \varepsilon$. On the other hand, due to the fact that u_{n_0} is continuous there exist a δ such that $u_{n_0}(x_0) \leq u_{n_0}(y) + \varepsilon$ for all $y \in B_\delta(x_0)$. Moreover, thanks to the fact that the sequence $\{u_n\}$ is non-decreasing we get that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$u_\infty(x_0) \leq u_\infty(y) + \varepsilon \ \text{ for all } \ y \in B_\delta(x_0).$$

The proof is finished. □

Proposition 5.2.17. *Let $\{u_n\}_{n=0}^\infty \subset C(\overline{\Omega})$ be a non-decreasing sequence of continuous up to the boundary functions such that u_n is a $\mathcal{L}$-viscosity supersolution for each $n \in \mathbb{N}$ and $u_n \longrightarrow u_\infty$ pointwise. Then, u_∞ is a $\mathcal{L}$-viscosity supersolution.*

Proof. By Lemma 5.2.16 we know that u_∞ is a lower semicontinuous function. To prove that u_∞ is a $\mathcal{L}$-viscosity supersolution let us consider $\phi \in C^2(\overline{\Omega})$ such that $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$ such that $(u_\infty - \phi)(x_0) = 0$. We want to obtain that $(\mathcal{L}\phi)(x_0) \geq 0$.

Let $r > 0$ be a fixed small radius. Since $u_\infty - \phi$ is lower semicontinuous, these functions reach a positive minimum in $\overline{\Omega} \setminus B_r(x_0)$, say in $z_\infty \in \overline{\Omega} \setminus B_r(x_0)$, then, we have

$$0 < (u_\infty - \phi)(z_\infty) \leq (u_\infty - \phi)(x) \ \text{ for all } \ x \in \overline{\Omega} \setminus B_r(x_0).$$

Also, since $u_n - \phi$ is continuous, there exists $z_n \in \overline{\Omega} \setminus B_r(x_0)$ where $u_n - \phi$ reaches a minimum in $\overline{\Omega} \setminus B_r(x_0)$. We claim that $(u_n - \phi)(z_n)$ converges to $(u_\infty - \phi)(z_\infty)$ as n

to infinity. Let us continue the proof assuming this fact and prove it after finishing our argument.

The claim implies that there exists $n_0 = n_0(r)$ such that

$$0 < \frac{1}{2}(u_\infty - \phi)(z_\infty) \leq (u_n - \phi)(z_n) \leq (u_n - \phi)(x) \tag{5.2.9}$$

for all $x \in \overline{\Omega} \setminus B_r(x_0)$ and $n \geq n_0$.

On the other hand, since u_n converges to u_∞ pointwise and $(u_\infty - \phi)(x_0) = 0$, there exists $n_1 = n_1(r)$ such that

$$(u_n - \phi)(x_0) \leq \frac{1}{2}(u_\infty - \phi)(z_\infty) \text{ for all } n \geq n_1.$$

Thus, the above and (5.2.9) imply $u_n - \phi$ reaches its minimum in $\overline{\Omega}$ in the interior of the ball $B_r(x_0)$ for all $n \geq n_3 = \max\{n_0, n_1\}$. Call this point x_n^r. Then, since u_n is a $\mathcal{L}$-viscosity supersolution, we get

$$(\mathcal{L}\phi)(x_n^r) \geq 0 \text{ for all } n \geq n_3(r) \text{ with } x_n^r \in B_r(x_0).$$

Due to the fact that $x_n^r \to x_0$ as $r \to 0^+$, we have

$$(\mathcal{L}\phi)(x_0) = \lim_{r \to 0^+} (\mathcal{L}\phi)(x_n^r) \geq 0.$$

Therefore, u_∞ is a $\mathcal{L}$-viscosity supersolution in Ω.

Now, we will prove the claim. Let us define $w_\infty = u_\infty - \phi$, $w_n = u_n - \phi$ and $a_n = w_n(z_n)$. Thanks to $\{u_n\}$ is a pointwise increasing sequence, $\{a_n\}$ is also an increasing sequence,

$$a_n = w_n(z_n) \leq w_n(z_{n+1}) \leq w_{n+1}(z_{n+1}) = a_{n+1}.$$

Moreover, the sequence is bounded above by $w_\infty(z_\infty)$ due to the fact that u_n converges pointwise to u_∞. Hence,

$$a_n \leq w_n(z_n) \leq w_n(z_\infty) \leq w_\infty(z_\infty).$$

Then, there exists a number $a_\infty \leq w_\infty(z_\infty)$ such that a_n converges to a_∞ as n tends to infinity.

Suppose that $a_\infty < w_\infty(z_\infty)$. Then, because $\{z_n\}$ is a colection of points in a compact set, there exists a subsequence $\{z_{n_j}\} \subset \{z_n\}$ and a point $\widetilde{z} \in \overline{\Omega} \setminus B_r(x_0)$ sucht that

$$z_{n_j} \longrightarrow \widetilde{z} \text{ as } n_j \to \infty.$$

Since w_n is a lower semicontinuous function, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, \widetilde{z})$ such that

$$w_\infty(\widetilde{z}) \leq w_\infty(y) + \varepsilon \text{ for all } y \in B_\delta(\widetilde{z}).$$

Due to the fact that z_{n_j} converges to $\widetilde{z}$, we can suppose $z_{n_j} \in B_\delta(\widetilde{z})$ for all n_j. Then

$$w_\infty(\widetilde{z}) \leq w_\infty(z_{n_j}) + \varepsilon \text{ for all } n_j.$$

For each n_j there exists a $m_j \in \mathbb{N}$ such that

$$w_\infty(z_{n_j}) \leq w_k(z_{n_j}) + \varepsilon \text{ for all } k \geq m_j.$$

Observe that we can construct a sequence $\{(n_j, k_j)\}$ where $k_j \geq m_j$ for all $j \in \mathbb{N}$ and $(n_j, k_j) \longrightarrow (\infty, \infty)$ as $j \to \infty$ such that

$$w_\infty(z_{n_j}) \leq w_{k_j}(z_{n_j}) + \varepsilon \text{ for all } j \in \mathbb{N}.$$

Since $a_\infty < w_\infty(z_\infty)$ there exists $\varepsilon_0 > 0$ such that

$$a_\infty + 2\varepsilon_0 < w_\infty(z_\infty).$$

For that ε_0 the above argument implies that there exists a sequence $\{(n_j, k_j)\}$ that goes to (∞, ∞) as $j \to \infty$ such that

$$a_\infty + 2\varepsilon_0 < w_\infty(z_\infty) \leq w_{k_j}(z_{n_j}) + 2\varepsilon_0 \text{ for all } j \in \mathbb{N}$$

and therefore,

$$a_\infty < w_{k_j}(z_{n_j}) \text{ for all } j \in \mathbb{N}.$$

On the other hand, since $\{a_n\}$ is a increasing sequence and $a_{k_j} = w_{k_j}(z_{k_j})$ we have that

$$w_{k_j}(z_{k_j}) \leq a_\infty < w_{k_j}(z_{n_j}) \text{ for all } j \in \mathbb{N}.$$

Thanks to the fact that for each $j \in \mathbb{N}$ the functions w_{k_j} are continuous, $\overline{\Omega} \setminus B_r(x_0)$ is connected and z_{n_j} and z_{k_j} are in $\overline{\Omega} \setminus B_r(x_0)$, there exists $y_j \in \overline{\Omega} \setminus B_r(x_0)$ such that

$$w_{k_j}(y_j) = a_\infty.$$

Thus, there is a subsequence $\{y_{j_i}\} \subset \{y_j\}$ and a point $y \in \overline{\Omega} \setminus B_r(x_0)$ such that $y_{j_i} \to y$ as $i \to \infty$. Due to the fact that w_∞ is a lower semicontinuous function we have

$$w_\infty(y) \leq \lim_{y_{j_i} \to y} w_{k_{j_i}}(y_{j_i}) = a_\infty.$$

However, this is a contradiction because $y \in \overline{\Omega} \setminus B_r(x_0)$ and that implies

$$a_\infty < w_\infty(z_\infty) \leq w_\infty(y).$$

Therefore, we have the desired equality $a_\infty = w_\infty(z_\infty)$. □

Now, we are ready to introduce the definition of a solution to the obstacle problem in the viscosity sense.

Definition 5.2.18. Given $\varphi : \overline{\Omega} \to \mathbb{R}$ and $f \in C(\partial\Omega)$ such that $f \geq \varphi^*$ in $\partial\Omega$. We say that $u : \Omega \longrightarrow \mathbb{R}$ is a viscosity solution of the $\mathcal{L}$-lower obstacle problem with obstacle φ and boundary datum f and we denote $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ if u satisfies:

$$\begin{cases} u_* \geq \varphi^* & \text{in } \Omega, \\ u_* = f & \text{in } \partial\Omega, \\ \mathcal{L}u \geq 0 & \text{in } \Omega \text{ in the viscosity sense}, \\ \mathcal{L}u = 0 & \text{in } \{u_* > \varphi^*\} \text{ in the viscosity sense}. \end{cases}$$

Whereas, given ψ and g such that $f \leq \psi_*$ on $\partial\Omega$, we say that v is a viscosity solution of the $\mathcal{L}$-upper obstacle problem with obstacle ψ and boundary datum g, and we denote $v = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$, if v satisfies:

$$\begin{cases} v^* \leq \psi_* & \text{in } \Omega, \\ v^* = g & \text{in } \partial\Omega, \\ \mathcal{L}v \leq 0 & \text{in } \Omega \text{ in the viscosity sense}, \\ \mathcal{L}v = 0 & \text{in } \{v^* < \psi_*\} \text{ in the viscosity sense}. \end{cases}$$

Remark 5.2.19. Note that, when u is a viscosity solution of the $\mathcal{L}$-lower obstacle problem with obstacle φ and boundary datum f, then, its lower and upper semicontinuous envelopes, u_* and u^*, are also viscosity solutions for the same problem.

The same applies to the upper obstacle, if $v = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$, then, $v_* = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$ and $v^* = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$.

However, we have uniqueness of solutions to the obstacle problem up to semicontinuous envelopes.

Lemma 5.2.20. *Given φ and f defined as before*

(a) if $f \geq \varphi^*$ in $\partial\Omega$, there exists at most one lower semicontinuous function u such that $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$.

(b) if $f \leq \varphi^*$ in $\partial\Omega$, there exists at most one upper semicontinuous function u such that $u = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$.

Proof. We will only prove item (a), the other item is analogous.

(a) Suppose that there exists two lower semicontinuous functions solutions u_1 and u_2, let us prove that $u_2 \geq u_1$.

In the set $\{u_1 = \varphi^*\}$ we have $u_2 \geq \varphi^* = u_1$. In the open set $\{u_1 > \varphi^*\}$ we have that $\mathcal{L}u_2 \geq 0$ and $\mathcal{L}u_1 = 0$ and $u_2 \geq u_1$ in $\partial\{u_1 > \varphi^*\}$. Then, by the Comparison Principle we obtain that $u_2 \geq u_1$ in $\{u_1 > \varphi^*\}$. Thus, we conclude that

$$u_2 \geq u_1 \text{ in } \Omega.$$

The reverse inequality follows interchanging the roles of u_1 and u_2. □

In view of the previous lemma, from now we will suppose that $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ is lower semicontinuous and $v = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ is upper semicontinuous.

Next, we show that when the obstacles and the boundary data are ordered then the solutions to the obstacle problems are also ordered.

Lemma 5.2.21.

(a) Given φ_1 and φ_2 functions such that $\varphi_1 \leq \varphi_2$ (a.e.) and $f_1, f_2 \in C(\partial\Omega)$ such that $f_1 \leq f_2$, if $f_1 \geq \varphi_1^*$ and $f_2 \geq \varphi_2^*$ on $\partial\Omega$, let us consider $u_1 = \underline{\mathcal{O}}(\mathcal{L}, \varphi_1, f)_1$ and $u_2 = \underline{\mathcal{O}}(\mathcal{L}, \varphi_2, f_2)$, then

$$u_1 \leq u_2 \ \text{ in } \ \Omega.$$

(b) Given ψ_1 and ψ_2 functions such that $\psi_1 \leq \psi_2$ (a.e.) and $g_1, g_2 \in C(\partial\Omega)$ such that $g_1 \leq g_2$, if $g_1 \leq \psi_{1*}$ and $g_2 \leq \psi_{2*}$ on $\partial\Omega$, let us consider $v_1 = \overline{\mathcal{O}}(\mathcal{L}, \psi_1, g_1)$ and $v_2 = \overline{\mathcal{O}}(\mathcal{L}, \psi_2, g_2)$, then

$$v_1 \leq v_2 \ \text{ in } \ \Omega.$$

Proof. We will prove item (a), the other case is analogous.

(a) We will consider two cases:

In the set $\{u_1 = \varphi_1^*\}$ we have $u_2 \geq \varphi_2^* \geq \varphi_1^* = u_1$.

In the open set $\{u_1 > \varphi_1^*\}$ we have that $\mathcal{L}u_2 \geq 0$ and $\mathcal{L}u_1 = 0$ and $u_2 \geq u_1$ in $\partial\{u_1 > \varphi_1^*\}$. Then, by the Comparison Principle we obtain that $u_2 \geq u_1$ in $\{u_1 > \varphi^*\}$.

Thus $u_2 \geq u_1$ in Ω. □

Lemma 5.2.22. *Given φ and f as before, let us consider $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$, then*

$$u = \min\Big\{w : \Omega \longrightarrow \mathbb{R} \ \textit{lower semicontinuous}: \\ \begin{array}{l} w \geq \varphi^* \ \textit{in} \ \Omega, \ \ w \geq f \ \textit{on} \ \partial\Omega, \\ \mathcal{L}w \geq 0 \ \textit{in} \ \Omega \ \ \textit{in the viscosity sense} \end{array}\Big\}.$$

In the same way, if we take $v = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$, then

$$v = \max\Big\{w : \Omega \longrightarrow \mathbb{R} \ \textit{upper semicontinuous}: \\ \begin{array}{l} w \leq \varphi_* \ \textit{in} \ \Omega, \ \ w \leq f \ \textit{on} \ \partial\Omega, \\ \mathcal{L}w \leq 0 \ \textit{in} \ \Omega \ \ \textit{in the viscosity sense} \end{array}\Big\}.$$

Proof. Let us prove the first claim. Consider

$$\overline{u} = \min\Big\{w : \Omega \longrightarrow \mathbb{R} \ \text{lower semicontinuous}: \\ \begin{array}{l} w \geq \varphi^* \ \text{in} \ \Omega, \ \ w \geq f \ \text{on} \ \partial\Omega, \\ \mathcal{L}w \geq 0 \ \text{in} \ \Omega \ \ \text{in the viscosity sense} \end{array}\Big\}.$$

Since $u \geq \varphi^*$ in Ω, $u = f$ on $\partial\Omega$ and $\mathcal{L}u \geq 0$ in Ω in viscosity sense, we get

$$\overline{u} \leq u.$$

Now, let us consider

$$w \in \Big\{ w : \Omega \longrightarrow \mathbb{R} \text{ lower semicontinuous : } \begin{array}{l} w \geq \varphi^* \text{ in } \Omega, \quad w \geq f \text{ on } \partial\Omega, \\ \mathcal{L}w \geq 0 \text{ in } \Omega \quad \text{in the viscosity sense} \end{array} \Big\}.$$

In the set $\{u = \varphi^*\}$ we have $w \geq \varphi^* = u$.

In the open set $\{u > \varphi^*\}$ we have that $\mathcal{L}w \geq 0$ and $\mathcal{L}u = 0$ in the viscosity sense and $w \geq u$ in $\partial\{u > \varphi^*\}$. Then, by the Comparison Principle we obtain that $w \geq u$ in $\{u_1 > \varphi^*\}$.

Then, $u \leq w$ in Ω. Taking minimum we get

$$u \leq \overline{u}.$$

The other case is analogous. □

Now, we introduce the definition of a solution to the two membranes problem in the viscosity sense (this is just Definition 5.2.1 with solutions understood in the viscosity sense).

Definition 5.2.23. Given $f, g \in C(\partial\Omega)$ with $f \geq g$, let $\mathcal{L}_1$ and $\mathcal{L}_2$ by two operators that satisfy the hypothesis at the begining of the section. We say a pair of functions (u, v) is a solution of the two membranes problem with boundary data (f, g) if

$$u = \underline{\mathcal{O}}(\mathcal{L}_1, v, f) \text{ and } v = \overline{\mathcal{O}}(\mathcal{L}_2, u, g).$$

Our main result is the following.

Theorem 5.2.24. *Given f, g and $\mathcal{L}_1, \mathcal{L}_2$, let us consider $v_0 \in C(\overline{\Omega})$ a $\mathcal{L}_2$-viscosity subsolution such that $v_0 \leq g$ in $\partial\Omega$ and then define inductively the sequences*

$$u_n = \underline{\mathcal{O}}(\mathcal{L}_1, v_n, f), \quad \text{and} \quad v_n = \overline{\mathcal{O}}(\mathcal{L}_2, u_{n-1}, g).$$

Both sequences of functions $\{u_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$, $\{v_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$ converge to some limit functions u_∞ and v_∞, respectively. Moreover, theses limits are a solution of the two membranes problem with boundary data (f, g), that is,

$$u_\infty = \underline{\mathcal{O}}(\mathcal{L}_1, v_\infty, f) \quad \text{and} \quad v_\infty = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g).$$

In addition, the functions u_∞ and v_∞ are lower semicontinuous in $\bar{\Omega}$ and continuous in the interior of the set where u_∞ touches v_∞, i.e, in the interior of $\{u_\infty = v_\infty\}$.

Proof. As before, we divide the proof in several steps.

STEP 1: Let us start proving that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are non-decreasing.

Let us see that $v_0 \leq v_1$. Recall that $v_1 = \overline{\mathcal{O}}(\mathcal{L}_2, u_0, g)$ and $u_0 = \underline{\mathcal{O}}(\mathcal{L}_1, v_0, f)$, then $v_1 \leq u_0$ and $v_0 \leq u_0$.

In the set $\{v_1 = u_0\}$ we have $v_0 \leq u_0 = v_1$.

In the open set $\{v_1 < u_0\}$ we have that $\mathcal{L}_2 v_1 = 0$ and $\mathcal{L}_2 v_0 \leq 0$ and $v_1 \geq v_0$ in $\partial\{v_1 > u_0\}$. Then, by the Comparison Principle we obtain that $v_1 \geq v_0$ in $\{v_1 > u_0\}$.

Thus, $v_0 \leq v_1$ in Ω.

Now, using Lemma 5.2.21 we have $u_0 \leq u_1$. By induction we can continue and obtain $u_n \leq u_{n+1}$ and $v_n \leq v_{n+1}$ for all $n \geq 1$.

STEP 2: Let us prove that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded.

Let $w \in C(\overline{\Omega})$ be a $\mathcal{L}_2$-solution with $w = g$ in $\partial\Omega$. Then, since $\mathcal{L}_2 w = 0$, we get $\mathcal{L}_2 v_n \leq 0$. Moreover, we have $w \geq v_n$ in $\partial\Omega$. Hence, applying the Comparison Principle, we get $w \geq v_n$ in $\overline{\Omega}$ for all $n \geq 0$. This implies, together with the fact that $v_n \geq v_0$, that $\{v_n\}$ is bounded.

Now, if we consider $z = \underline{\mathcal{O}}(\mathcal{L}_1, w, f)$, using that $v_n \leq w$ by Lemma 5.2.21 $u_n \leq z$ and $z \in C(\overline{\Omega})$. This implies that $\{u_n\}$ is bounded.

STEP 3: Let us define

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) \qquad \text{and} \qquad \lim_{n\to\infty} v_n(x) = v_\infty(x)$$

for all $x \in \overline{\Omega}$. Using that $\{u_n\}$ are continuous and the sequence is increasing we obtain that u_∞ is lower semincontinuous. We also have $u_\infty = f$ and $v_\infty = g$ in $\partial\Omega$. Let us consider

$$v = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g).$$

This function v is well defined and upper semicontinuous. Our goal is to prove that $v = v_\infty$.

Using that $u_n \leq u_\infty$, we get $v_n \leq v$ for all $n \geq 0$ by Lemma 5.2.21. Then, we have that $v_\infty \leq v$ and $v_\infty^* \leq v^* = v$. This implies that $v_\infty^* \leq u_\infty$. On the other hand, by Proposition 5.2.15, we have that $\mathcal{L}_2 v_\infty^* \leq 0$ in Ω. We also have $v_\infty^* \geq g$ in $\partial\Omega$. Now, we consider $w \in C(\Omega)$ a solution to

$$\begin{cases} \mathcal{L}_2 w = 0 & \text{in } \Omega \text{ in the viscosity sense,} \\ w = g & \text{in } \partial\Omega. \end{cases}$$

By the Comparison Principle we have that $v_n \leq w$ in $\overline{\Omega}$. Then $v_\infty \leq w$ in $\overline{\Omega}$ and $v_\infty^* \leq w^* = w$ in $\overline{\Omega}$. This implies $v_\infty^* \leq g$ on $\partial\Omega$ and, therefore, $v_\infty^* = g$ on $\partial\Omega$. Finally, we will prove that $\mathcal{L}_2 v_\infty^* = 0$ in the (open) set $\{v_\infty^* < u_\infty\}$. Let us consider $x_0 \in \{v_\infty^* < u_\infty\}$, and call $\delta = u_\infty(x_0) - v_\infty^*(x_0)$. Let us consider $n_0 \geq 0$ such that $u_\infty(x_0) - u_{n_0}(x_0) < \frac{\delta}{4}$. Using that u_{n_0} is continuous, there exists $\eta_1 = \eta_1(n_0) > 0$ such that $|u_{n_0}(x_0) - u_{n_0}(y)| < \frac{\delta}{4}$ for all $y \in B_{\eta_1}(x_0)$. On the other hand, there exists $\eta_2 > 0$ such that $v_\infty^*(y) < v_\infty^*(x_0) + \frac{\delta}{4}$ for all

$y \in B_{\eta_2}(x_0)$. Gathering the previous estimates we obtain

$$u_{n_0}(y) - v_\infty^*(y) = \underbrace{u_{n_0}(y) - u_{n_0}(x_0)}_{>-\frac{\delta}{4}} + \underbrace{u_{n_0}(x_0) - u_\infty(x_0)}_{>-\frac{\delta}{4}} + \underbrace{u_\infty(x_0) - v_\infty^*(x_0)}_{=\delta} + \underbrace{v_\infty^*(x_0) - v_\infty^*(y)}_{>-\frac{\delta}{4}} > \frac{\delta}{4}$$

for all $y \in B_\eta(x_0)$ with $\eta = \min\{\eta_1, \eta_2\}$. Now, we have that $u_n(y) \geq u_{n_0}(y)$ and $v_n(y) \leq v_\infty^*(y)$ for all $n \geq n_0$. Thus

$$u_n(y) - v_n(y) \geq u_{n_0}(y) - v_\infty^*(y) > \frac{\delta}{4} > 0$$

for all $n \geq n_0$ and for all $y \in B_\eta(x_0)$. Then $B_\eta(x_0) \subseteq \{u_n > v_n\}$ for all $n \geq n_0$. Using that $\mathcal{L}_2 v_n = 0$ in $B_\eta(x_0)$ and taking the limit we obtain $\mathcal{L}_2 v_\infty = 0$ in $B_\eta(x_0)$ which is the same that $\mathcal{L}_2 v_\infty^* = 0$ in $B_\eta(x_0)$ because v_∞ is lower semicontinuous. As a consequence, $\mathcal{L}_2 v_n \geq 0$ in $B_\eta(x_0)$. In particular, $\mathcal{L}_2 v_n \geq 0$ in $B_\eta(x_0)$ for all $n \geq n_0$. Then, by Proposition 5.2.17, $\mathcal{L}_2 v_\infty \geq 0$ in $B_\eta(x_0)$ in the viscosity sense. Since previously we just proved that v_∞ is a $\mathcal{L}_2$-viscosity subsolution in Ω, we obtain that v_∞ is a $\mathcal{L}_2$-viscosity solution in $B_\eta(x_0)$. Therefore, v_∞ is a $\mathcal{L}_2$-viscosity solution in the set $\{v_\infty^* < u_\infty\}$.

Putting all together, since u_∞ is lower semicontinuous, we obtain

$$\begin{cases} v_\infty^* \leq u_\infty & \text{in } \Omega \\ v_\infty^* = g & \text{in } \partial\Omega, \\ \mathcal{L}_2 v_\infty \leq 0 & \text{in } \Omega \text{ in the viscosity sense,} \\ \mathcal{L}_2 v_\infty = 0 & \text{in } \{v_\infty^* < u_\infty\} \text{ in the viscosity sense.} \end{cases}$$

By uniqueness of the obstacle problem we get $v_\infty = v = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g)$.

Now, let us define $u = \underline{\mathcal{O}}(\mathcal{L}_1, v_\infty, f)$. Using $v_n \leq v_\infty^*$, we have $u_{n-1} \leq u$. Then, taking the limit, $u_\infty \leq u$. On the other hand we have that $\mathcal{L}_1 u_\infty \geq 0$ in Ω in viscosity sense due to Proposition 5.2.17 and $u_\infty = f$ in $\partial\Omega$. Moreover, since u_∞ is lower semicontinuous, taking lim sup in the inequality $u_n \geq v_n$, we obtain $u_\infty \geq v_\infty^*$ in $\overline{\Omega}$. Then, using the Lemma 5.2.22, we get $u \leq u_\infty$. Thus, we conclude that $u_\infty = u$.

Finally, we have that u_∞ and v_∞ are lower semicontinuous functions in $\bar{\Omega}$ by Lemma 5.2.16. Moreover, u_∞ is continuous in the interior of $\{u_\infty = v_\infty^*\}$ because u_∞ is a lower semicontinuous and, by definition, v_∞^* is an upper semicontinous function. This also implies that v_∞^* is a continuous function in the interior of $\{u_\infty = v_\infty^*\}$. Then, $v_\infty = v_\infty^*$ there and therefore v_∞ is a continuous function in the interior of the set $\{u_\infty = v_\infty\}$. □

Remark 5.2.25. If u_∞ or v_∞ are continuous at $\partial\{u_\infty = v_\infty^*\}$, then, both functions u_∞ and v_∞ are continuous in $\bar{\Omega}$ and therefore the sequences defined in the previous theorem converge uniformly in the whole $\overline{\Omega}$.

In fact, without loss of generality, suppose that u_∞ is continuous in $\partial\{u_\infty = v_\infty^*\}$. We will prove the above using the Comparison Principle for $\mathcal{L}_1$. Since $\mathcal{L}_1 u_\infty = 0$ in $\{u_\infty > v_\infty^*\}$ in the viscosity sense, we have in particular that u_∞^* is a $\mathcal{L}_1$-subsolution and $u_{\infty *}$ a $\mathcal{L}_1$-supersolution in $\{u_\infty > v_\infty^*\}$ in the viscosity sense. Moreover, since $f > g$ in $\partial\Omega$, the boundary $\partial\{u_\infty > v_\infty^*\}$ is the disjoint union of $\partial\Omega$ and $\partial\{u_\infty = v_\infty^*\}$. In $\partial\Omega$, we have $u_\infty^* = u_{\infty *}$ by construction of u_n. And in $\partial\{u_\infty = v_\infty^*\}$ we have also that $u_\infty^* = u_{\infty *}$ because we have supposed u_∞ continuous across that bounday. Then, $u_\infty^* = u_{\infty *}$ in $\partial\{u_\infty > v_\infty^*\}$. Therefore, by the Comparison Principle of $\mathcal{L}_1$, $u_\infty^* \leq u_{\infty *}$ in $\overline{\{u_\infty > v_\infty^*\}}$ and therefore u_∞ is continuous in $\overline{\{u_\infty > v_\infty^*\}}$. Moreover, as we have seen in the proof of the above theorem, u_∞ is also continuous in $\{u_\infty > v_\infty^*\}$. Then u_∞ is continuous in $\overline{\Omega}$. As a consequence, using the latest hypothesis concerning the operator $\mathcal{L}_2$, v_∞ is continuous in the whole $\overline{\Omega}$, because v_∞ is the solution of the upper obstacle problem with continuous boundary datum and continuous obstacle.

Finally, since the continuous functions u_∞ and v_∞ are the limit of the sequences of continuous functions $\{u_n\}$ and $\{v_n\}$ in a compact set $\bar{\Omega}$, the convergence is uniform.

Remark 5.2.26. As happens in the variational setting, we also have here that the limit depends strongly on the initial function from where we start the iterations. Moreover, the convergence of (u_n, v_n) to a solution of the two membranes problem relies on the monotonicity of the sequences $\{u_n\}$ and $\{v_n\}$. This property comes from the fact that the initial function v_0 is a $\mathcal{L}_2$-viscosity subsolution. For this discussion in the variational context, we refer to the arguments given in Remark 5.2.12.

5.3 The two membranes problem on trees

Our main goal of this section is to introduce and analyze the obstacle problem and the two membranes problem when the ambient space is an infinite graph with a regular structure (a regular tree) and the involved operators are given by mean value formulas.

A regular tree. Let us first describe the ambient space. Regular trees can be viewed as discrete models of the unit ball of $\mathbb{R}^N$. A tree is, informally, an infinite graph (that we will denote by $\mathbb{T}$) in which each node but one (the root of the tree denoted by $\emptyset$) has exactly $m+1$ connected nodes, m successors and one predecessor (the root has only m successors). An element x in $\mathbb{T}$ is called a vertex and it is represented as a k-tuple for some $k \in \mathbb{N}$ of natural numbers between 0 and m, that is, $x = (0, a_1, ..., a_k)$ where $a_i \in \{0, ..., m-1\}$ for all $1 \leq j \leq k$. Here k is the level of x, that we denote by $|x|$, according to the distance (in nodes) to the root. The root has zero level, its successors have level one, etc. Each vertex x has m successors, that we will denote by $S^1(x)$, and are described by

$$S^1(x) = \Big\{(0, a_1, ..., a_k, i) \colon i \in \{0, 1, \dots, m-1\}\Big\}$$

where $x = (0, a_1, ..., a_k)$. Notice that we used a digit $i \in \{0, 1, \dots, m-1\}$ to enumerate the successors of x. In general, we will denote

$$S^l(x) = \Big\{(0, a_1, ..., a_k, i_1, \dots, i_l) \colon i_j \in \{0, 1, \dots, m-1\} \quad \text{for} \quad 1 \leq j \leq l\Big\}$$

the set of successors of $x = (0, a_1, ..., a_k)$ with $|x| + l$ level. By convention, $S^0(x) = \{x\}$. If x is not the root then x has a unique immediate predecessor, which we will denote $\hat{x}$. We use the notation x^j to denote the predecessor of x with $|x^j| = j$, that is, for a node x with $|x| = k$, we have $x = (0, a_1, \dots, a_j, \dots, a_k)$, then $x^j = (0, a_1, \dots, a_j)$. Let $\mathbb{T}^x \subset \mathbb{T}$ be the subtree with root x (the subset of the tree composed of all the consecutive successors of x). A branch of $\mathbb{T}$ (that we denote by z) is an infinite sequence of vertices starting at the root, where each of the vertices in the sequence is followed by one of its immediate successors. The collection of all branches forms the boundary of $\mathbb{T}$, denoted by $\partial\mathbb{T}$. Observe that the mapping $\psi : \partial\mathbb{T} \to [0, 1]$ defined as

$$\psi(z) = \sum_{k=0}^{+\infty} \frac{a_k}{m^k} \tag{5.3.1}$$

is surjective, where $z = (0, a_1, \dots, a_k, \dots) \in \partial\mathbb{T}$ and $a_k \in \{0, 1, \dots, m-1\}$ for all $k \in \mathbb{N}$. When $x = (0, a_1, \dots, a_k)$ is a vertex, we set $\psi(x) = \psi((0, a_1, \dots, a_k, 0, \dots))$.

Mean value formulas and operators. Given a parameter $0 \le \beta \le 1$ we define the averaging operator L acting on functions $u : \mathbb{T} \mapsto \mathbb{R}$ as follows:

$$L(u)(x) = u(x) - \beta u(\hat{x}) - (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big), \qquad x \neq \emptyset \tag{5.3.2}$$

and

$$L(u)(\emptyset) = u(\emptyset) - \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big). \tag{5.3.3}$$

The operator L acting on u at a vertex $x \in \mathbb{T}$ is given by the difference between the value of u at that node and the mean value of u at the vertices that are connected with x (with a weight given by β for the predecessor and $(1-\beta)/m$ for the successors). Note that here the distinction in the definition of Lu based on whether $x = \emptyset$ or not relies on the fact that the root has no predecessor.

Now, given a function $h : \mathbb{T} \mapsto \mathbb{R}$, a solution to

$$L(u)(x) = h(x)$$

is a function $u : \mathbb{T} \mapsto \mathbb{R}$ that verifies

$$u(x) = \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x), \qquad x \neq \emptyset,$$

and the equation at the root of the tree that only involves the successors and is given by

$$u(\emptyset) = \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big) + h(\emptyset).$$

Notice that the equation $L(u)(x) = h(x)$ is an analogous in the tree of the partial differential equation $-\Delta u(x) = h(x)$ in the unit ball of $\mathbb{R}^N$ (recall the mean value formula for harmonic functions).

Observe that, in contrast with the introduction of this chapter, we prefer not to merge the average operator and the source terms under the notation L. This choice is made to emphasize and analyze the role of the source term h in the existence of solutions.

Next, in order to impose boundary conditions, we want to make precise what we understand for $u|_{\partial\mathbb{T}} = f$.

Definition 5.3.1. Given $f : [0,1] \to \mathbb{R}$ we say that u verifies the boundary condition $u|_{\partial\mathbb{T}} = f$ if

$$\lim_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)),$$

that is, if $z = (0, a_1, \dots, a_k, \dots) \in \partial\mathbb{T}$, then

$$\lim_{n\to\infty} u((0, a_1, ..., a_n)) = f\Big(\sum_{k=1}^{+\infty} \frac{a_k}{m^k}\Big).$$

Here and in what follows we will take this limit uniformly, that is, given $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $|u(x) - f(\psi(z))| < \varepsilon$ for every $|x| \geq K$ and every $z \in \partial\mathbb{T}^x$.

Note that, if $u|_{\partial\mathbb{T}} = f$, due to the fact that the limit is uniform, we have that

$$\lim_{N\to\infty} \frac{1}{m^N} \sum_{y \in S^N(x)} u(y) = \int_{\partial\pi_m^x} f(s)\mathrm{d}s \tag{5.3.4}$$

Now, let us state the precise definition of being a subsolution/supersolution to $Lu = h$ in $\mathbb{T}$, with a boundary condition $u|_{\partial\mathbb{T}} = f$.

Definition 5.3.2. Given an operator L defined as before, a function $h : \mathbb{T} \to \mathbb{R}$ and a continuous function $f : [0,1] \to \mathbb{R}$, we say that $u : \mathbb{T} \to \mathbb{R}$ is a subsolution (resp. supersolution) to $Lu = h$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ if it verifies

$$\begin{cases} L(u)(x) \leq h(x) \quad (resp \geq), & x \in \mathbb{T}, \\ \limsup_{x \to z \in \partial\mathbb{T}} u(x) \leq f(\psi(z)) \quad (resp \liminf \geq). \end{cases}$$

We say that u is a solution if it is both a subsolution and a supersolution.

With this definition at hand we can introduce the obstacle problem (see also Section 5.3.1).

Definition 5.3.3. Given an operator L, a function $h : \mathbb{T} \to \mathbb{R}$, a continuous boundary datum $f : [0,1] \to \mathbb{R}$ and a function $\varphi : \mathbb{T} \to \mathbb{R}$ (the obstacle), we say that $u : \mathbb{T} \to \mathbb{R}$

is the solution to the obstacle problem for $L-h$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} \geq f$ and obstacle φ from below (φ is assumed to satisfy $\limsup_{x\to z\in\partial\mathbb{T}} \varphi(x) < f(\psi(z))$) if it verifies

$$u(x) = \inf \left\{ w(x) : \begin{array}{l} L(w)(x) \geq h(x), \quad x \in \mathbb{T}, \\ w(x) \geq \varphi(x), \quad x \in \mathbb{T}, \\ \text{and } \liminf\limits_{x\to z\in\partial\mathbb{T}} w(x) \geq f(\psi(z)) \end{array} \right\}.$$

Under adequate conditions on h (see below), this is function $u(x)$ is the unique function that satisfies

$$\begin{cases} 0 = \max\Big\{ -L(u)(x) + h(x), \varphi(x) - u(x)\Big\}, \quad x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

Analogously, given an operator L, a function $h : \mathbb{T} \to \mathbb{R}$, a continuous boundary datum $g : [0,1] \to \mathbb{R}$ and a function $\varphi : \mathbb{T} \to \mathbb{R}$ (the obstacle), we say that $v : \mathbb{T} \to \mathbb{R}$ is the solution to the obstacle problem for $L-h$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle φ from above (here u is assumed to satisfy $\liminf_{x\to z\in\partial\mathbb{T}} \varphi(x) > g(\psi(z))$) if it verifies

$$v(x) = \sup \left\{ w(x) : \begin{array}{l} L(w)(x) \leq h(x), \quad x \in \mathbb{T}, \\ w(x) \leq \varphi(x), \quad x \in \mathbb{T}, \\ \text{and } \limsup\limits_{x\to z\in\partial\mathbb{T}} w(x) \leq g(\psi(z)) \end{array} \right\}.$$

Under adequate conditions on h, this can be written as

$$\begin{cases} 0 = \min\Big\{ -L(v)(x) + h(x), \varphi(x) - v(x)\Big\}, \quad x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases}$$

With the definition of the obstacle problem we can describe the two membranes problem in $\mathbb{T}$.

Definition 5.3.4. Let L_1 and L_2 be two averaging operators defined as in (5.3.2) and (5.3.3) (with different β_1, β_2), $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ two functions and $f, g : [0,1] \to \mathbb{R}$ two continuous boundary conditions. A pair $(u,v) : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ is a solution to the two membranes problem if it solves the system

$$\begin{cases} 0 = \max\Big\{ -L_1(u)(x) + h_1(x), v(x) - u(x)\Big\}, \ x \in \mathbb{T}, \\ 0 = \min\Big\{ -L_2(v)(x) + h_2(x), u(x) - v(x)\Big\}, \ x \in \mathbb{T}, \end{cases} \tag{5.3.5}$$

with

$$\begin{cases} \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)), \\ \lim\limits_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases} \tag{5.3.6}$$

Based on the previous Definition 5.3.3, a pair (u, v) is a solution to the two membranes problem (5.3.5) with initial datum (5.3.6) if and only if u is the solution to the obstacle problem for $L_1 - h_1$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ and obstacle v from below and v is the solution to the obstacle problem for $L_2 - h_2$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle u from above.

Recall that u is the solution to the obstacle problem for $L_1 - h_1$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ and obstacle v from below if it is the infimum of supersolutions for $L_1 - h_1$ with boundary datum f that are above the obstacle v, and analogously, v is the solution to the obstacle problem for $L_2 - h_2$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle v from above if it is the supremum of subsolutions for $L_2 - h_2$ with boundary datum g that are above the obstacle u.

Let us point out that (5.3.5) has a probabilistic interpretation that we will describe in Section 5.3.4.

Now, for short, we introduce a notation, for a node x, let

$$\mathcal{S}_h(x) := \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\left(\frac{\beta}{1-\beta}\right)^{k-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j}.$$

With this notation at hand, we are ready to state our main result for the two membranes problem.

Theorem 5.3.5. *Given two averaging operators, L_1 and L_2 defined as in (5.3.2) and (5.3.3) (that involve two different parameters β_1, β_2), two different functions $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ and two continuous boundary conditions $f, g : [0,1] \to \mathbb{R}$ with $f > g$ in $[0,1]$, such that*

$$0 \le \beta_1 < \frac{1}{2}, \qquad 0 \le \beta_2 < \frac{1}{2},$$

$$\lim_{x\in\mathbb{T},|x|=k\to\infty}\sum_{j=1}^{k}\left(\frac{\beta}{1-\beta}\right)^{k-j}S_{h_1}(x^j) = 0,$$

$$\lim_{x\in\mathbb{T},|x|=k\to\infty}\sum_{j=1}^{k}\left(\frac{\beta}{1-\beta}\right)^{k-j}S_{h_2}(x^j) = 0,$$

then, there exists a pair $(u, v) : \mathbb{T} \mapsto \mathbb{R}$ that is a solution to the two membranes problem. That is, (u, v) solves the system (5.3.5).

Moreover, under these conditions the coincidence set $\{x \in \mathbb{T} : u(x) = v(x)\}$ is finite.

To prove this result we first deal with a single equation and find necessary and sufficient conditions for the existence of a solution. We have the following result that we believe that has its own interest.

Theorem 5.3.6. *Given an averaging operator, L, a function $h : \mathbb{T} \to \mathbb{R}$ and a continuous boundary condition $f : [0,1] \to \mathbb{R}$, there exists a unique bounded solution to*

$$\begin{cases} L(u)(x) = h(x), & x \in \mathbb{T}, \\ \lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

if and only if

$$\begin{aligned} &0 \le \beta < \frac{1}{2}, \\ &\lim_{x\in\mathbb{T},|x|=k\to\infty} \sum_{j=1}^{k} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) = 0. \end{aligned} \tag{5.3.7}$$

Under the conditions in the statement of the previous theorem we can also construct sub and supersolutions that are the key to obtain solvability of the obstacle problem (from above or below).

Notice that when $\beta < \frac{1}{2}$ if h is such that

$$\lim_{x\in\mathbb{T},|x|=k\to\infty} S_h(x) = 0$$

then the second condition in (5.3.7) holds. In fact, we have

$$\begin{aligned} \sum_{j=1}^{k-1} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) &= \sum_{j=1}^{k_0} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) + \sum_{j=k_0}^{k-1} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) \\ &\le C\Big(\frac{\beta}{1-\beta}\Big)^{k} \sum_{j=1}^{k_0} \Big(\frac{\beta}{1-\beta}\Big)^{-j} + \max_{j\ge k_0} |S_h(x^j)| \sum_{j=k_0}^{k-1} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} \end{aligned}$$

that is small if we first choose k_0 large (in order to make $\max_{j\ge k_0} |S_h(x^j)|$ small) and then send k to infinity.

To look for examples of functions h that satisfy our condition we take a function that depends only on the level k, that is, $h(y) = \widehat{H}(k)$. Then, for $0 < \beta < \frac{1}{2}$ the condition $S_h(x) \to 0$ reads as

$$\lim_{x\in\mathbb{T},|x|\to\infty} \frac{1}{\beta} \sum_{k=1}^{\infty} H(k) \sum_{j=0}^{k-1} \Big(\frac{\beta m}{1-\beta}\Big)^{k-j} = 0.$$

Therefore, for H constant, $H = cte$, this condition is not verified. When $\frac{\beta m}{1-\beta} < 1$ any $H(k)$ with $\lim_{k\to\infty} H(k) = 0$ satisfies the condition, but any $H(k) > 0$ with $\liminf_{k\to\infty} H(k) > 0$ does not. On the other hand, when $\frac{\beta m}{1-\beta} \ge 1$ we need that $H(k)$ goes to zero very fast in order to verify the condition.

With the result for a single equation at hand, to prove our main result concerning the two membranes problem, we use the strategy of iterate the obstacle problem from above or below that we describe in the introduction of this chapter.

5.3.1 Existence of bounded solutions to a single equation

Our aim is to find conditions on β and the function $h : \mathbb{T} \to \mathbb{R}$ such that for every continuous function $f : [0,1] \to \mathbb{R}$ there exists a bounded solution to the Dirichlet problem

$$\begin{cases} L(u)(x) = h(x), & x \in \mathbb{T}, \\ \lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases} \tag{5.3.8}$$

Proof of Theorem 5.3.6. First, suppose that such a bounded solution exists, then we have a function $u : \mathbb{T} \mapsto \mathbb{R}$ such that

$$u(x) = \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u(y)\Big) + h(x),$$

for $x \in \mathbb{T} \setminus \{\emptyset\}$ and $|x| = k$, if we write $u(x) = \beta u(x) + (1-\beta)u(x)$ we get

$$\beta\Big(u(x) - u(\hat{x})\Big) = (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} (u(y) - u(x))\Big) + h(x).$$

We define $w(x)$ for $x \in \mathbb{T} \setminus \{\emptyset\}$ as the increment of u between x and its predecessor, i.e,

$$w(x) := u(x) - u(\hat{x}).$$

Then, the previous equation written in terms of w reads as

$$\begin{aligned}
\beta w(x) &= (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} w(y)\Big) + h(x)\\
w(x) &= \Big(\frac{1-\beta}{\beta}\Big)^i \frac{1}{m^i}\sum_{y\in S^i(x)} w(y) + \frac{1}{\beta}\sum_{j=0}^{i-1}\Big(\frac{1-\beta}{\beta}\Big)^j \sum_{y\in S^j(x)} \frac{h(y)}{m^j}\\
w(x)\Big(\frac{\beta}{1-\beta}\Big)^i &= \frac{1}{m^i}\sum_{y\in S^i(x)} w(y) + \frac{1}{\beta}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{i-j} \sum_{y\in S^j(x)} \frac{h(y)}{m^j}.
\end{aligned}$$

Adding from $i = 1$ to infinity, we get

$$w(x)\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^i = \sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i} + \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{i-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j}. \tag{5.3.9}$$

If $h \equiv 0$ we have existence of a solution such that

$$w(x)\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^i = \sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i}$$

and since w is bounded (this follows from the fact that u is assumed to be bounded) we have that the right hand side is finite. Hence, we obtain the convergence of the series

$$\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^i$$

that is equivalent to the condition

$$\frac{\beta}{1-\beta} < 1 \iff 0 < \beta < \frac{1}{2}.$$

Now, we use the notations

$$\mathcal{S}_h(x) := \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{i-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j},$$
$$c_\beta := \sum_{i=0}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{i} = \frac{1-\beta}{1-2\beta}, \quad \text{for } \beta\in\Big(0,\frac{1}{2}\Big).$$

With this notation it is straightforward to check that,

$$c_\beta - 1 = \sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{i} > 0.$$

On the other hand, we claim that

$$\sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i} = \fint_{\partial\pi_m^x} f(s)\mathrm{d}s - u(x). \tag{5.3.10}$$

Assuming this claim, (5.3.9) can be rewritten as

$$w(x)\Big(c_\beta - 1\Big) = \fint_{\partial\pi_m^x} f(s)\mathrm{d}s - u(x) + \mathcal{S}_h(x).$$

From this, we obtain that the solution $u:\mathbb{T}\mapsto\mathbb{R}$ verifies

$$u(x) = \frac{c_\beta-1}{c_\beta}u(\hat{x}) + \frac{1}{c_\beta}\fint_{\partial\pi_m^x} f(s)\mathrm{d}s + \frac{1}{c_\beta}\mathcal{S}_h(x), \quad x\neq\emptyset. \tag{5.3.11}$$

Now, we have

$$u(\emptyset) = \frac{1}{m}\sum_{y\in S^1(\emptyset)} u(y) + h(\emptyset),$$

and, using (5.3.11) for $x\in S^1(\emptyset)$ we get

$$u(\emptyset) = \frac{1}{m}\sum_{y\in S^1(\emptyset)}\Big(\frac{c_\beta-1}{c_\beta}u(\emptyset) + \frac{1}{c_\beta}\fint_{\partial\pi_m^y} f(s)\mathrm{d}s + \frac{1}{c_\beta}\mathcal{S}_h(y)\Big) + h(\emptyset),$$

that is

$$u(\emptyset) = \frac{1}{m}\sum_{y\in S^1(\emptyset)}\Big(\fint_{\partial\pi_m^y} f(s)\mathrm{d}s + \mathcal{S}_h(y)\Big) + c_\beta h(\emptyset).$$

Note that

$$\fint_0^1 f(s)ds = \frac{1}{m}\sum_{y\in S^1(\emptyset)}\fint_{\partial\pi_m^y} f(s)\mathrm{d}s$$

and then we obtain

$$u(\emptyset) = \fint_0^1 f(s)\mathrm{d}s + \frac{1}{m}\sum_{y\in S^1(\emptyset)} \mathcal{S}_h(y) + c_\beta h(\emptyset).$$

Therefore, we have that u satisfies the recurrence

$$\begin{cases} u(x) = \dfrac{c_\beta - 1}{c_\beta} u(\hat{x}) + \dfrac{1}{c_\beta} \displaystyle\fint_{\partial\pi_m^x} f(s)\mathrm{d}s + \frac{1}{c_\beta}\mathcal{S}_h(x), \quad x \neq \emptyset, \\ u(\emptyset) = \displaystyle\fint_0^1 f(s)\mathrm{d}s + \frac{1}{m}\sum_{y\in S^1(\emptyset)} \mathcal{S}_h(y) + c_\beta h(\emptyset). \end{cases} \tag{5.3.12}$$

If we iterate this recurrence up to the root we get the following formula for u; given $x \in \mathbb{T}$ with $|x| = k$ for $k \geq 1$,

$$\begin{aligned} u(x) = &\Big(\frac{\beta}{1-\beta}\Big)^k \fint_0^1 f(s)ds + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \fint_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s \\ &+\Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) \end{aligned} \tag{5.3.13}$$

where x^j denotes the predecessor of x with $|x^j| = j$, that is, in terms of the notation given in the introduction, for a node x with $|x| = k$, we have $x = (a_1, \ldots, a_j, \ldots, a_k)$ with $0 \leq a_j \leq m-1$, then $x^j = (a_1, \ldots, a_j)$.

This function u given by (5.3.13) is well-defined, finite, and solves the equation $L(u)(x) = h(x)$ for $x \in \mathbb{T}$.

Now we just take $f \equiv 0$ and we obtain that u, given by

$$u(x) = \Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j)$$

must satisfy

$$\lim_{x\in\mathbb{T},|x|\to\infty} u(x) = 0$$

uniformly. Since $\beta < \frac{1}{2}$, it is clear that

$$\lim_{k\to\infty}\Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] = 0.$$

Hence, we must have

$$\lim_{k\to\infty}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) = 0$$

as we wanted to show.

Now we proceed to prove the claim (5.3.10). We will prove by induction in N that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} = \int_{\partial\pi_m^x} f(s)\mathrm{d}s - u(x).$$

For $N=2$ we have that

$$\begin{aligned}
\sum_{k=1}^{2}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} &= \sum_{y\in S^1(x)}\frac{w(y)}{m^1} + \sum_{y\in S^2(x)}\frac{w(y)}{m^2}\\
&= \frac{1}{m}\Big(\sum_{y\in S^1(x)}\Big(u(y)-u(x)\Big)\Big) + \frac{1}{m^2}\Big(\sum_{y\in S^1(x)}\sum_{z\in S^1(y)}\Big(u(z)-u(y)\Big)\Big)\\
&= \frac{1}{m}\Big(\sum_{y\in S^1(x)}u(y)\Big) - u(x) + \frac{1}{m^2}\Big(\sum_{y\in S^2(x)}u(y)\Big) - \frac{1}{m}\Big(\sum_{y\in S^1(x)}u(y)\Big)\\
&= \frac{1}{m^2}\Big(\sum_{y\in S^2(x)}u(y)\Big) - u(x).
\end{aligned}$$

Now, our inductive hypothesis is

$$\sum_{k=1}^{N}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} = \frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big) - u(x),$$

and we have to prove that for $N+1$,

$$\begin{aligned}
\sum_{k=1}^{N+1}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} &= \sum_{y\in S^{N+1}(x)}\frac{w(y)}{m^{N+1}} + \frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big) - u(x)\\
&= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^N(x)}\sum_{z\in S^1(y)}\Big(u(z)-u(y)\Big)\Big) + \frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big) - u(x)\\
&= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^{N+1}(x)}u(y)\Big) - \frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big) + \frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big) - u(x)\\
&= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^{N+1}(x)}u(y)\Big) - u(x).
\end{aligned}$$

Since u have supposed that $u|_{\partial\pi} = f$, we have that

$$\lim_{N\to\infty}\frac{1}{m^N}\sum_{y\in S^N(x)}u(y) = \int_{\partial\pi_m^x} f(s)\mathrm{d}s$$

due to the fact in the definition of $u|_{\partial\pi} = f$ the convergence is uniform. This ends the proof of the claim.

Conversely, assuming that the conditions on β and h, (5.3.7) hold, from our previous computations, (5.3.13) gives us a way to construct solutions to our problem. In fact, we can define the function $u : \mathbb{T} \mapsto \mathbb{R}$ given by

$$\begin{aligned} u(x) = &\Big(\frac{\beta}{1-\beta}\Big)^k \fint_{\mathbb{T}} f(s)ds + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \fint_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s \\ &+\Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) \end{aligned} \tag{5.3.14}$$

where, as before, x^j denotes the predecessor of x with $|x^j| = j$.

One can check that u, given by the explicit expression (5.3.14), satisfies

$$Lu(x) = h(x), \qquad x \in \mathbb{T}.$$

Now, let us check that, when we have the solvability conditions

$$0 \le \beta < \frac{1}{2},$$

$$\lim_{x\in\mathbb{T},|x|=k\to\infty} \sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) = 0,$$

it holds that

$$\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z))$$

uniformly.

From (5.3.14) we get

$$\begin{aligned} &u(x) - f(\psi(z)) \\ &= \Big(\frac{\beta}{1-\beta}\Big)^k \fint_{\mathbb{T}} f(s)ds + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \fint_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s - f(\psi(z)) \\ &\quad +\Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j). \end{aligned}$$

Since $\beta < \frac{1}{2}$, it holds that

$$\lim_{k\to\infty}\Big(\frac{\beta}{1-\beta}\Big)^k \fint_{\mathbb{T}} f(s)ds = 0$$

and also

$$\lim_{k\to\infty}\Big(\frac{\beta}{1-\beta}\Big)^k\Big[\frac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] = 0.$$

Now, given $\gamma > 0$, since f is continuous, we have that there exists k_0 such that

$$\int_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s - f(\psi(z)) \ < \gamma$$

for every $j > k_0$. Now, we use that

$$c_\beta = \sum_{l=0}^{\infty} \Big(\frac{\beta}{1-\beta}\Big)^l$$

and that f is bounded to obtain

$$\begin{aligned}
&\frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \int_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s - f(\psi(z)) \\
&\quad = \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \int_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s - \frac{1}{c_\beta}\sum_{l=0}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{l} f(\psi(z)) \\
&\quad \leq C\sum_{j=1}^{k_0-1}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \\
&\qquad + \frac{1}{c_\beta}\sum_{l=0}^{k-k_0}\Big(\frac{\beta}{1-\beta}\Big)^{l} \int_{\partial\pi_m^{x^{k-l}}} f(s)\mathrm{d}s - \frac{1}{c_\beta}\sum_{l=0}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{l} f(\psi(z)) \\
&\quad \leq C\sum_{j=1}^{k_0-1}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} + \gamma\frac{1}{c_\beta}\sum_{l=0}^{k-k_0}\Big(\frac{\beta}{1-\beta}\Big)^{l} + C\frac{1}{c_\beta}\sum_{l=k-k_0+1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{l} \leq \frac{\varepsilon}{2},
\end{aligned}$$

for γ small enough and k large enough. Hence, we have that

$$\lim_{k\to\infty} \frac{1}{c_\beta}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} \int_{\partial\pi_m^{x^j}} f(s)\mathrm{d}s - f(\psi(z)) \ = 0.$$

Finally, the condition

$$\lim_{x\in\mathbb{T},|x|=k\to\infty} \sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) = 0,$$

is just what we need to ensure that

$$\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)).$$

This shows that there exists a solution.

Uniqueness follows for our previous arguments. In fact, if u is a solution, the increments $w(x) = u(x) - u(\hat{x})$ solve (5.3.9) from where we obtain (by our previous computations) that u verifies (5.3.13). Hence, any solution to our problem is given by (5.3.13) and therefore we conclude uniqueness of solutions. In fact, (5.3.13) is a representation formula for the solution to (5.3.8) in terms of the functions h and f. Uniqueness also follows from the comparison principle (see Lemma 5.3.9 below). □

Remark 5.3.7. Under our hypothesis on β and h, for $f : [0,1] \to \mathbb{R}$ continuous the function $w_1 : \mathbb{T} \to \mathbb{R}$ defined as

$$\begin{aligned} w_1(x) &= \Big(\frac{\beta}{1-\beta}\Big)^k \Big[\frac{1}{m} \sum_{y \in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] \\ &\quad + \frac{1}{c_\beta} \sum_{j=1}^{k} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j), \quad x \neq \emptyset, |x| = k, \\ w_1(\emptyset) &= \frac{1}{m} \sum_{y \in S^1(\emptyset)} \mathcal{S}_h(y) + c_\beta h(\emptyset), \end{aligned}$$

is the solution to

$$\begin{cases} L(w_1)(x) = h(x), & x \in \mathbb{T}, \\ \lim\limits_{x \to z \in \partial \mathbb{T}} w_1(x) = 0. \end{cases}$$

Moreover, the function $w_2 : \mathbb{T} \to \mathbb{R}$ defined as

$$\begin{aligned} w_2(x) &= \Big(\frac{\beta}{1-\beta}\Big)^k \fint_0^1 f(s)ds + \frac{1}{c_\beta} \sum_{j=1}^{k} \Big(\frac{\beta}{1-\beta}\Big)^{k-j} \fint_{\partial \pi_m^{x^j}} f(s)\mathrm{d}s, \quad x \neq \emptyset, |x| = k, \\ w_2(\emptyset) &= \fint_0^1 f(s)\mathrm{d}s, \end{aligned}$$

is the solution to

$$\begin{cases} L(w_2)(x) = 0, & x \in \mathbb{T}, \\ \lim\limits_{x \to z \in \partial \mathbb{T}} w_2(x) = f(\psi(z)). \end{cases}$$

Therefore, we can write

$$u(x) = w_1(x) + w_2(x)$$

being w_1 the solution to (5.3.8) for a given h with $f = 0$ and w_2 the solution for a given f with $h = 0$.

Remark 5.3.8. Notice that analogous arguments allow us to show that, under the same conditions, we can build supersolutions as large as we want enlarging h (and also large subsolutions) to the equation that satisfy the boundary condition

$$\lim_{x \to z \in \partial \mathbb{T}} u(x) = f(\psi(z)).$$

To this end, consider L defined as in (5.3.2) and (5.3.3) and $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ two functions in the tree such that $h_1 \leq h_2$ and f a boundary continuous function. Moreover, suppose that β, h_1 and h_2 satisfy the solvability condition. Let u_i be the unique solution for the Dirichlet problem of the single equation associated to the operator $L - h_i$ and boundary function f, that is, u_i is given by (5.3.12) for $i = 1, 2$. Then, u_1 a supersolution the Dirichlet problem with $L - h_2$ and u_2 is a subsolution for the Dirichlet problem with $L - h_1$ and f. Now, notice that we can make u_1 as large as we want in a finite number of nodes just taking $h_1(\emptyset)$ large enough.

Let us prove the uniqueness for solutions to (5.3.8). To this end we prove that the operator L verifies the comparison principle.

Lemma 5.3.9. *(Comparison Principle) Given $u, v : \mathbb{T} \to \mathbb{R}$ such that*

$$\begin{cases} L(u)(x) \geq h(x), & x \in \mathbb{T}, \\ \liminf_{x \to z \in \partial\mathbb{T}} u(x) \geq f(\psi(z)). \end{cases}$$

and

$$\begin{cases} L(v)(x) \leq h(x), & x \in \mathbb{T}, \\ \limsup_{x \to z \in \partial\mathbb{T}} v(x) \leq g(\psi(z)). \end{cases}$$

Then, if $f \geq g$ in $[0,1]$, we get

$$u(x) \geq v(x), \qquad x \in \mathbb{T}.$$

Proof. Let us argue by contradiction, i.e., assume that

$$\sup_{x \in \mathbb{T}} (v-u)(x) = \theta > 0.$$

Let us suppose that there exists $k \geq 1$, and $|x_0| = k$ such that $(v-u)(x_0) \geq \frac{\theta}{2}$ and $(v-u)(y) < \frac{\theta}{2}$ for all $|y| < k$. At that point we have

$$L(v)(x_0) - L(u)(x_0) \leq 0.$$

Solving this inequality we arrive to

$$\begin{aligned} 0 < \frac{\theta}{2} \leq (v-u)(x_0) &\leq \beta(v-u)(\hat{x_0}) + \frac{1-\beta}{m} \sum_{y \in S(x_0)} (v-u)(y) \\ &\leq \beta(v-u)(x_0) + \frac{1-\beta}{m} \sum_{y \in S(x_0)} (v-u)(y), \end{aligned}$$

then

$$(v-u)(x_0) \leq \frac{1}{m} \sum_{y \in S(x_0)} (v-u)(y).$$

This implies that there exists $x_1 \in S(x_0)$ such that $(v-u)(x_1) \geq (v-u)(x_0)$. If we repeat this argument we will obtain a sequence $(x_n)_{n \geq 0}$ such that $x_{n+1} \in S(x_n)$ and $(v-u)(x_{n+1}) \geq (v-u)(x_n)$. Thus, we get

$$\begin{aligned} 0 < \frac{\theta}{2} &\leq \liminf_{x_k \to z \in \partial\mathbb{T}} (v-u)(x_k) \\ &\leq \limsup_{x_k \to z \in \partial\mathbb{T}} v(x_k) - \liminf_{x_k \to z \in \partial\mathbb{T}} u(x_k) \leq g(\psi(z)) - f(\psi(z)) \leq 0, \end{aligned}$$

which is a contradiction. Now, let us consider the case

$$(v-u)(\emptyset) \geq \frac{\theta}{2}.$$

In this case we have

$$0 < \frac{\theta}{2} \leq (v-u)(\emptyset) \leq \frac{1}{m} \sum_{y \in S(\emptyset)} (v-u)(y),$$

and this implies that there exists $x_1 \in S(\emptyset)$ such that $(v-u)(x_1) \geq (v-u)(\emptyset)$. If from this point we argue as before we obtain a contradiction. This ends the proof. □

5.3.2 The obstacle problem on trees.

We defined a solution for the obstacle problem from below and from above in the introduction, see Definition 5.3.3. Now, we claim that the problem can be regarded from two perspectives.

Definition 5.3.10. Given an operator L defined as before and a function $h : \mathbb{T} \to \mathbb{R}$, such that the conditions for solvability, (5.3.7), hold, $\varphi : \mathbb{T} \to \mathbb{R}$ a bounded function, and $f : [0,1] \to \mathbb{R}$ a continuous function such that

$$\limsup_{x \to z \in \partial\mathbb{T}} \varphi(x) < f(\psi(z)),$$

we say that u is a solution to the obstacle problem from below, and we note $u = \underline{O}(L, h, \varphi, f)$ if u verifies

$$u(x) = \inf \left\{ w(x) : \begin{array}{c} L(w)(x) \geq h(x), \quad x \in \mathbb{T}, \\ w(x) \geq \varphi(x), \quad x \in \mathbb{T}, \\ \text{and } \liminf_{x \to z \in \partial\mathbb{T}} w(x) \geq f(\psi(z)) \end{array} \right\}, \tag{5.3.15}$$

that is equivalent to

$$\begin{cases} u(x) \geq \varphi(x), & x \in \mathbb{T}, \\ L(u)(x) \geq h(x), & x \in \mathbb{T}, \\ L(u)(x) = h(x), & x \in \{u > \varphi\}, \\ \lim_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases} \tag{5.3.16}$$

Notice that (5.3.16) can be written as

$$\begin{cases} 0 = \max\Big\{ -L(u)(x) + h(x), \varphi(x) - u(x) \Big\}, & x \in \mathbb{T}, \\ \lim_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

Analogously, for $g : [0,1] \to \mathbb{R}$ and $\varphi : \mathbb{T} \to \mathbb{R}$ such that

$$\liminf_{x \to z \in \partial\mathbb{T}} \varphi(x) > g(\psi(z))$$

we define v a solution from above and denote $v = \overline{\mathrm{O}}(L, h, \varphi, g)$ if

$$v(x) = \sup \left\{ w(x) : \begin{array}{c} L(w)(x) \leq h(x), \quad x \in \mathbb{T}, \\ w(x) \leq \varphi(x), \quad x \in \mathbb{T}, \\ \text{and } \limsup_{x \to z \in \partial\mathbb{T}} w(x) \leq g(\psi(z)) \end{array} \right\}, \tag{5.3.17}$$

that is equivalent to

$$\begin{cases} v(x) \leq \varphi(x), & x \in \mathbb{T}, \\ L(v)(x) \leq h(x), & x \in \mathbb{T}, \\ L(v)(x) = h(x), & x \in \{v < \varphi\}, \\ \lim_{x \to z \in \partial\mathbb{T}} v(x) = g(\psi(z)), \end{cases} \tag{5.3.18}$$

that can be written as

$$\begin{cases} 0 = \min\Big\{ - L(v)(x) + h(x), \varphi(x) - v(x) \Big\}, & x \in \mathbb{T}, \\ \lim_{x \to z \in \partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases}$$

Let us prove that both definitions (the one as inf/sup of super/subsolutions and the one solving inequalities) of being a solution to the obstacle problem are equivalent.

Proposition 5.3.11. *Consider an operator L defined as in (5.3.2) and (5.3.3), and a function $h : \mathbb{T} \to \mathbb{R}$ such that the conditions for solvability, (5.3.7), hold, $\varphi : \mathbb{T} \to \mathbb{R}$ a bounded function, and $f : [0,1] \to \mathbb{R}$ a continuous function such that*

$$\limsup_{x \to z \in \partial\mathbb{T}} \varphi(x) < f(\psi(z)).$$

Then, the function u defined by (5.3.15) is well defined and is solution to (5.3.16). Conversely, a solution u to (5.3.16) is also the minimizer in (5.3.15).

Analogously, for a given $g : [0,1] \to \mathbb{R}$ a continuous function such that

$$\liminf_{x \to z \in \partial\mathbb{T}} \varphi(x) > g(\psi(z)),$$

we have that the function v defined by (5.3.17) is well defined and is solution to (5.3.18). Conversely, a solution v to (5.3.18) is the maximizer in (5.3.17).

Proof. Let us consider the set

$$\underline{\Lambda}_{f,h,\varphi} = \Big\{ w : L(w) \geq h \ , \ w \geq \varphi \ , \ \liminf_{x \to z \in \partial \mathbb{T}} w(x) \geq f(\psi(z)) \Big\}. \tag{5.3.19}$$

This set is non empty and is bounded from below. In fact, if we consider $M = \max\{\|f\|_\infty, \|\varphi\|_\infty\}$, and w_0 the unique bounded solution to

$$\begin{cases} L(w_0)(x) = h(x), & x \in \mathbb{T}, \\ \lim\limits_{x \to z \in \partial \mathbb{T}} w_0 = f(\psi(z)). \end{cases}$$

Note that $\underline{\Lambda}_{f,h,\varphi}$ is not empty due to the fact that the function $w_0 + M \in \underline{\Lambda}_{f,h,\varphi}$. Moreover, if $w \in \underline{\Lambda}_{f,h,\varphi}$, $w(x) \geq \varphi(x) \geq -M$, then, is bounded from below. Let us define

$$u(x) = \inf_{w \in \underline{\Lambda}_{f,h,\varphi}} w(x). \tag{5.3.20}$$

Let us show that this function u verifies (5.3.16). In fact, using that $w \geq \varphi$ for all $w \in \underline{\Lambda}_{f,h,\varphi}$, taking infimum we get $u \geq \varphi$. In addition, given $w \in \underline{\Lambda}_{f,h,\varphi}$ we have

$$L(w)(x) \geq h(x) \quad \Rightarrow \quad w(x) \geq \beta w(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} w(y)\Big) + h(x), \quad x \neq \emptyset.$$

As a consequence, if we take infimum in the right hand of the above inequalty we obtain

$$w(x) \geq \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x), \quad x \neq \emptyset.$$

Furthermore, taking infimum in the left hand

$$u(x) \geq \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x) \quad \Rightarrow \quad L(u)(x) \geq h(x), \quad x \neq \emptyset.$$

Analogously, we can do the same computation on the root $\emptyset$ and obtain

$$u(\emptyset) \geq \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big) + h(\emptyset) \quad \Rightarrow \quad L(u)(\emptyset) \geq h(x).$$

Finally, we have

$$\liminf_{x \to z \in \partial \mathbb{T}} w(x) \geq f(\psi(z))$$

for all $w \in \underline{\Lambda}_{f,h,\varphi}$, taking infimum we get

$$\liminf_{x \to z \in \partial \mathbb{T}} u(x) \geq f(\psi(z)).$$

We just proved that $u \in \underline{\Lambda}_{f,h,\varphi}$. Let us prove now that

$$L(u)(x) = 0$$

if $x \in \{u > \varphi\}$. Suppose that this is not true, given $x_0 \in \{u > \varphi\}$ such that $L(u)(x_0) > h(x_0)$, i.e.

$$u(x_0) > \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y \in S^1(x)} u(y)\Big) + h(x).$$

Let us define

$$\delta_1 = u(x_0) - \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y \in S^k(x)} u(y)\Big) + h(x),$$

$$\delta_2 = u(x_0) - \varphi(x_0)$$

and

$$\delta_0 = \min\{\delta_1, \delta_2\}.$$

If we consider

$$u_0(x) = \begin{cases} u(x), & x \neq x_0, \\ u(x) - \dfrac{\delta_0}{2}, & x = x_0. \end{cases}$$

This function verifies $u_0 \geq \varphi$,

$$\liminf_{x \to z \in \partial\mathbb{T}} u_0(x) \geq f(\psi(z))$$

and $L(u_0)(x) \geq h(x)$. In fact, $L(u_0)(x_0) \geq h(x)$ and then $L(u_0)(\hat{x_0}) \geq h(\hat{x_0})$, and for $y_0 \in S(x_0)$, $L(u_0)(y_0) \geq h(y_0)$. Thus, $u_0 \in \underline{\Lambda}_{f,h,\varphi}$ and $u_0 < u$ which is a contradiction. We have proved that

$$L(u)(x) = h(x)$$

in the set $\{u > \varphi\}$.

Let us verify that

$$\limsup_{x \to z \in \partial\mathbb{T}} u(x) \leq f(\psi(z)).$$

Suppose that this is not true, i.e.

$$\limsup_{x \to z \in \partial\mathbb{T}} u(x) > f(\psi(z)).$$

Using that

$$\limsup_{x \to z \in \partial\mathbb{T}} \varphi(x) < f(\psi(z)),$$

there exists $k \in \mathbb{N}$ such that $|x| \geq k$ and $\varphi(x) < f(\psi(z))$. Then, there exists x_0 such that $|x_0| \geq k$ and $u(x_0) > f(\psi(z))$. We can argue as before to make a function $u_0 \in \underline{\Lambda}_{f,h,\varphi}$ and $u_0 < u$ which is a contradiction.

Conversely, assume that u solves (5.3.16) and let us prove that u is the minimizer in (5.3.15). Since u solves (5.3.16) we have that it satisfies $L(u)(x) \geq h(x)$ for $x \in \mathbb{T}$, $\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z))$ and $w(x) \geq \varphi(x)$ and then u is a competitor in the minimization problem (5.3.15). Therefore,

$$u(x) \geq \inf_{w\in\underline{\Lambda}_{f,h,\varphi}} w(x).$$

To prove the reverse inequality, call z the minimizer, that is,

$$z(x) = \inf_{w\in\underline{\Lambda}_{f,h,\varphi}} w(x).$$

Since we have that $u \geq z$ we get an inclusion for the sets where these functions touch the obstacle,

$$\{x : u(x) = \varphi(x)\} \subset \{x : z(x) = \varphi(x)\}.$$

Outside the set $\{x : u(x) = \varphi(x)\}$ we have that u solves $L(u)(x) = h(x)$ and z is a supersolution to this equation, while the boundary conditions give

$$\liminf_{x\to z\in\partial\mathbb{T}} z(x) \geq f(\psi(z)) = \lim_{x\to z\in\partial\mathbb{T}} u(x).$$

From the proof of the comparison principle, we get that

$$u(x) \leq z(x),$$

and we conclude that $u \equiv z$, the minimizer of (5.3.15).

Analogously, we consider

$$\overline{\Lambda}_{g,h,\varphi} = \Big\{w : L(w) \leq h \ , \ w \leq \varphi \ , \ \limsup_{x\to z\in\partial\mathbb{T}} w(x) \leq g(\psi(z))\Big\}. \tag{5.3.21}$$

This set is non empty and bounded from above. Then, we define

$$v(x) = \sup_{w\in\overline{\Lambda}_{g,h,\varphi}} w(x), \tag{5.3.22}$$

This function v verifies (5.3.18), i.e $v = \overline{O}(L, h, \varphi, g)$, and conversely, a function that verifies (5.3.18) is given by (5.3.22). This ends the proof. □

Corollary 5.3.12. *Given an operator L defined as in (5.3.2) and (5.3.3), and a function $h : \mathbb{T} \to \mathbb{R}$ such that the conditions for solvability, (5.3.7), hold, $\varphi : \mathbb{T} \to \mathbb{R}$ a bounded function, and $f : [0, 1] \to \mathbb{R}$ a continuous function. If*

$$\limsup_{x\to z\in\partial\mathbb{T}} \varphi(x) < f(\psi(z)),$$

there exists a unique solution for the obstacle problem from below,

$$\underline{O}(L, h, \varphi, f) = \inf_{w\in\underline{\Lambda}_{f,h,\varphi}} w.$$

where $\underline{\Lambda}_{f,h,\varphi}$ *is defined in* (5.3.19).

Analogously, if

$$\liminf_{x\to z\in\partial\mathbb{T}} \varphi(x) > f(\psi(z)),$$

there exists a unique solution for the obstacle problem from above,

$$\overline{O}(L,h,\varphi,g) = \sup_{w\in\overline{\Lambda}_{f,h,\varphi}} w.$$

where $\overline{\Lambda}_{g,h,\varphi}$ *is defined in* (5.3.21).

Proof. The existence of $\underline{O}(L,h,\varphi,f)$ and of $\overline{O}(L,h,\varphi,g)$ relies on the fact that the sets $\underline{\Lambda}_{f,h,\varphi}$ and $\overline{\Lambda}_{g,h,\varphi}$ are not empty, and there exists a $M>0$ sucht that $w\geq -M$ for all $w\in\underline{\Lambda}_{f,h,\varphi}$ and $v\leq M$ for all $v\in\overline{\Lambda}_{f,h,\varphi}$, as we showed in the proof of the equivalence of the definitions of being a solution to the obstacle problem. The uniqueness is a direct consequence of taking the infimum or the supremum on these sets. □

5.3.3 Existence in the two membranes problem via iterations of the obstacle problem.

Let us show that the two membranes problem in the tree has a solution.

Proof of Theorem 5.3.5. Let us consider L_1 and L_2 two operators given by the mean value formula and $h_1,h_2:\mathbb{T}\to\mathbb{R}$ that verify the solvability condition with β_1, h_1 and β_2, h_2 respectively, and $f,g:[0,1]\to\mathbb{R}$ two continuous functions such that $f>g$. Let us start the method with v_0 a bounded subsolution of the operator with boundary condition g, that is,

$$L_2(v_0)(x)\leq h_2(x), \qquad x\in\mathbb{T},$$

$$\limsup_{x\to z\in\partial\mathbb{T}} v_0(x)\leq g(\psi(z)).$$

With this function v_0 we let u_1 be the solution to the obstacle problem from below for the operator L_1, right hand side h_1, boundary datum f and obstacle v_0, that is,

$$u_1 = \underline{O}(L_1,h_1,v_0,f).$$

Notice that the set

$$\underline{\Lambda}_{f,h_1v_0} = \left\{w : L_1(w)\geq h_1\ ,\ w\geq v_0\ ,\ \liminf_{x\to z\in\partial\mathbb{T}} w(x)\geq f(\psi(z))\right\}$$

is not empty, since by the results in Section 5.3.1 we can construct a supersolution to our problem with L_1-h_1 and boundary condition f as large as we want (see Remark 5.3.8). Therefore, the function u_1 can be obtained solving the minimization problem $\inf\{w(x) : w\in\underline{\Lambda}_{f,h_1v_0}\}$.

With this function u_1, we take

$$v_1 = \overline{\mathrm{O}}(L_2, h_2, u_1, g).$$

Here, notice that the corresponding set

$$\overline{\Lambda}_{g,h_2,u_1} = \left\{ w : L_2(w) \leq h_2 \ , \ w \leq u_1 \ , \ \limsup_{x \to z \in \partial\mathbb{T}} w(x) \leq g(\psi(z)) \right\}$$

is not empty (here we recall that we can construct large subsolutions to $L_2 - h_2$) and hence v_1 can be obtained from the maximization problem $\sup\{w(x) : \ w \in \overline{\Lambda}_{g,h_2,u_1}\}$

Now, we iterate this procedure and define

$$u_n = \underline{\mathrm{O}}(L_1, h_1, v_{n-1}, f) \quad \text{and} \quad v_n = \overline{\mathrm{O}}(L_2, h_2, u_n, g).$$

In this way we obtain two sequences $\{u_n\}_{n\geq 1}, \{v_n\}_{n\geq 1}$. Our goal is to show that these sequences are monotone and that they converge to a pair of functions (u, v) that is a solution to the two membranes problem.

Step 1: Let us start by proving that the sequences are increasing, i.e. $u_n \geq u_{n-1}, v_n \geq v_{n-1}$:

We begin with v_{n-1}. By definition of being a solution to the obstacle problem, v_{n-1} satisfies $u_{n-1} \geq v_{n-1}$, $L_2(v_{n-1}) \leq h_2$ and $\lim_{x\to z\in\partial\mathbb{T}} v_{n-1}(x) = g(\psi(z))$. Hence, $v_{n-1} \in \overline{\Lambda}_{g,h_2,u_{n-1}}$. Then, using again the definition of being a solution to the obstacle problem we get

$$v_n = \sup_{w \in \overline{\Lambda}_{g,h_2,u_{n-1}}} w \geq v_{n-1}.$$

On the other hand, $u_n \geq v_n \geq v_{n-1}$, $L_1(u_n) \geq h_1$ and $\lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z))$, then, using one more time the definition of being a solution to the obstacle problem, we get

$$u_{n-1} = \inf_{w \in \underline{\Lambda}_{f,h_1,v_{n-1}}} w \leq u_n.$$

This ends the proof of the claim.

Step 2: Now we will prove that the sequences are bounded.

Let us start with $\{u_n\}_{n\geq 1}$. Let w be the solution to

$$\begin{cases} L_2(w)(x) = h_2(x), & x \in \mathbb{T}, \\ \lim_{x\to z\in\partial\mathbb{T}} w(x) = g(\psi(z)). \end{cases}$$

By the comparison principle $v_n \leq w$. Let us consider

$$\rho = \underline{\mathrm{O}}(L_1, h_1, w, f).$$

Note that ρ is bounded since w is bounded and the operators satisfy the solvability conditions. Since $v_n \leq w$ we have that $\rho \in \underline{\Lambda}_{g,h_2,v_n}$ for all $n \in \mathbb{N}$. Using (5.3.20) we get

$$u_n \leq \rho$$

for all $n \in \mathbb{N}$.

Now we observe that $\{v_n\}_{n\geq 1}$ is bounded, since it holds that

$$v_n \leq u_{n-1} \leq \rho$$

for all $n \in \mathbb{N}$. This ends the proof the second claim.

Thanks to Step 1 and Step 2 we can take limits as $n \to \infty$ to obtain

$$u_n(x) \to u_\infty(x) \quad \text{and} \quad v_n(x) \to v_\infty(x),$$

for all $x \in \mathbb{T}$.

Step 3 Now, we will chech that the limit pair (u_∞, v_∞) solves the two membranes problem:

Let us prove that

$$u_\infty = \underline{\mathrm{O}}(L_1, h_1, v_\infty, f) \quad \text{and} \quad v_\infty = \overline{\mathrm{O}}(L_2, h_2, u_\infty, g).$$

First, we observe that we have

$$\begin{cases} u_n(x) = \max\Big\{\beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(x)} u_n(y)\Big) + h_1(x), v_{n-1}(x)\Big\}, \ x \neq \emptyset, \\ u_n(\emptyset) = \max\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(\emptyset)} u_n(y)\Big) + h_1(\emptyset), v_{n-1}(x)\Big\}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)). \end{cases}$$

Therefore, from the monotonicity of the sequence u_n we get

$$u_\infty(x) \geq u_n(x) \geq \beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u_n(y)\Big) + h_1(x),$$

$$u_\infty(\emptyset) \geq u_n(\emptyset) \geq \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} u_n(y)\Big) + h_1(\emptyset),$$

and

$$u_\infty(x) \geq u_n(x) \geq v_{n-1}(x).$$

Letting $n \to \infty$ in the right hand sides we get

$$\begin{aligned} u_\infty(x) &\geq \beta_1 u_\infty(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u_\infty(y)\Big) + h_1(x), \\ u_\infty(\emptyset) &\geq \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} u_\infty(y)\Big) + h_1(\emptyset), \end{aligned} \tag{5.3.23}$$

and

$$u_\infty(x) \geq v_\infty(x). \tag{5.3.24}$$

Now, for v_n we have

$$v_n(x) \leq \beta_2 v_n(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum_{y\in S^1(x)} v_n(y)\Big) + h_2(x),$$

$$v_n(\emptyset) \leq \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} v_n(y)\Big) + h_2(\emptyset),$$

and using again the monotonicity of the sequence we get (after passing to the limit as $n \to \infty$),

$$\begin{aligned} &v_\infty(x) \leq \beta_2 v_\infty(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum_{y\in S^1(x)} v_\infty(y)\Big) + h_2(x), \quad x \neq \emptyset, \\ &v_\infty(\emptyset) \leq \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} v_\infty(y)\Big) + h_2(\emptyset). \end{aligned} \tag{5.3.25}$$

Now, for a point x in the set $\{x : u_\infty(x) > v_\infty(x)\}$ with $x \neq \emptyset$ we have that $u_n(x) > v_n(x)$ for n large and hence we obtain, for $x \in \{x : u_\infty(x) > v_\infty(x)\}$,

$$u_n(x) = \beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum_{y\in S^k(x)} u_n(y)\Big) + h_1(x),$$

and

$$v_n(x) = \beta_2 v_n(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum_{y\in S^k(x)} v_n(y)\Big) + h_2(x).$$

Passing to the limit as $n \to \infty$ we conclude that

$$u_\infty(x) = \beta_1 u_\infty(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u_\infty(y)\Big) + h_1(x), \tag{5.3.26}$$

and

$$v_\infty(x) = \beta_2 v_\infty(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum_{y\in S^1(x)} v_\infty(y)\Big) + h_2(x). \tag{5.3.27}$$

An analogous computation can be done when $u_\infty(\emptyset) > v_\infty(\emptyset)$ to obtain

$$u_\infty(\emptyset) = \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} u_\infty(y)\Big) + h_1(\emptyset), \tag{5.3.28}$$

and

$$v_\infty(\emptyset) = \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} v_\infty(y)\Big) + h_2(\emptyset). \tag{5.3.29}$$

Now, concerning the boundary condition of u_∞, we have

$$\lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)).$$

Hence, from the monotonicity of the sequence, we obtain

$$\liminf_{x\to z\in\partial\mathbb{T}} u_\infty(x) \geq \lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)).$$

Now, since from Section 5.3.1 we know that we can construct a large supersolution for L_1 with boundary datum f that we call $\overline{u}$ and we have a comparison principle, we obtain

$$u_n(x) \leq \overline{u}(x)$$

for every $x \in \mathbb{T}$ and every n and hence we get

$$u_\infty(x) \leq \overline{u}(x).$$

Thanks to this inequality we obtain

$$\limsup_{x\to z\in\partial\mathbb{T}} u_\infty(x) \leq \lim_{x\to z\in\partial\mathbb{T}} \overline{u}(x) = f(\psi(z))$$

and hence we conclude that

$$\lim_{x\to z\in\partial\mathbb{T}} u_\infty(x) = f(\psi(z)). \tag{5.3.30}$$

A similar argument shows that

$$\lim_{x\to z\in\partial\mathbb{T}} v_\infty(x) = g(\psi(z)). \tag{5.3.31}$$

To conclude we just observe that (5.3.23), (5.3.24), (5.3.26), (5.3.28) and (5.3.30) show that

$$u_\infty = \underline{\mathrm{O}}(L_1, h_1, v_\infty, f).$$

In the same way (5.3.25), (5.3.24), (5.3.27), (5.3.29) and (5.3.31) imply that

$$v_\infty = \overline{\mathrm{O}}(L_2, h_2, u_\infty, g).$$

Finally, we observe that, since we have a subsolution (called $\underline{u}$) for L_1 with datum f and a supersolution (called $\overline{v}$) for L_2 with datum g and we assumed that $f > g$, we have

$$\lim_{x\to z\in\partial\mathbb{T}} \overline{v}(x) = g(\psi(z)) < f(\psi(z)) = \lim_{x\to z\in\partial\mathbb{T}} \underline{u}(x).$$

Therefore we have that

$$v_\infty(x) \leq \overline{v}(x) < \underline{u}(x) \leq u_\infty(x)$$

for every x with $|x|$ large enough, say $|x| > C$. This proves that the contact set $\{x : u_\infty(x) = v_\infty(x)\}$ does not contain nodes in $\{x : |x| > C\}$ and therefore the contact set is finite. □

5.3.4 A probabilistic interpretation

We will provide a probabilist interpretation in terms of a theoretical game of the system

$$\begin{cases} u(x) = \max\Big\{\beta_1 u(\hat{x}) + (1-\beta_1)\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(x)} u(y)\Big) + h_1(x), v(x)\Big\}, \ x \neq \emptyset, \\ u(\emptyset) = \max\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(\emptyset)} u(y)\Big) + h_1(\emptyset), v(\emptyset)\Big\}, \\ v(x) = \min\Big\{\beta_2 v(\hat{x}) + (1-\beta_2)\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(x)} v(y)\Big) + h_2(x), u(x)\Big\}, \ x \neq \emptyset, \\ v(\emptyset) = \min\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(\emptyset)} v(y)\Big) + h_2(\emptyset), u(\emptyset)\Big\}, \end{cases} \tag{5.3.32}$$

that is, for the solutions of (5.3.5).

The related game is a two player, zero game player in two borardes. Its rules are the following:

- The boards: each board is a m-regular $\mathbb{T}$ tree. The fist board is labeled with $i = 1$ and the second with $j = 2$. In the first board there is a running payoff $h_1 : \mathbb{T} \to \mathbb{R}$ and the final payoff $f : [0,1] \to \mathbb{R}$ while in the second, the running payoff $h_2 : \mathbb{T} \to \mathbb{R}$ and the final payoff $g : [0,1] \to \mathbb{R}$.

- The players: there are two players: Paul and Carol. The aim of Paul is to maxime the who aims to maximize the *total payoff* and the goal's Carol is to minimizes it. To see the definition of total payoff see the bulled below.

- The start: the game starts with a token of some node in one of the two trees, (x_0, j) with $x_0 \in \mathbb{T}$ and $j = 1, 2$ (we add an index to denote at which board is the position of the game)

  If $j = 1$ is in the first board then Paul decides to remain in the same board and to play one round of the game or to jump to the other board, $j = 2$. On the other hand, $j = 2$, Carol decides to remain in the same board and play one round of the game or to jump to the first board $j = 1$.

- Allowed movements: In each turn the players have a choice to play at the same board or to change boards.

  - Movements in the j- board: given a token (x, j), the player moves to the predecessor of x with probability β_j or to one of the m successors with probability $(1-\beta_j)/m$. She or he collects a running payoff $h_i(x)$. At the root of the tree the token moves to one of the successors with probability $1/m$.
  - How to change the board: in each turn, if the player consider that is more adventagentaus for his/her purpose can choice not to play in the board where the token is and moce to the other.

- THE END: let $M \in \mathbb{N}$ a fixed large number. When the the position arrives a node of level M called x_{τ_M} after a priori unknown τ_M turns, the game ends.

  In the last round, apart of the accumulated running payoff, if x_τ is in the first board, Paul has to pay to Carol the amound encoded by $f(\psi(x_\tau))$ while if x_{τ_M} in the second board the final payoff is given by $g(\psi(x_{\tau_M}))$.

  Recall that, as we have explained at the beginning of the subsection, since $x_{\tau_M} = (0, a_1, a_2, ..., a_{\tau_M})$ is a vertex, we set $\psi(x_{\tau_M}) = \psi((0, a_1, a_2, ..., a_{\tau_M}, 0, ...))$

- THE TOTAL PAYOFF: It is given by

$$\begin{aligned}\text{total payoff}_M\,(x_0, j) := f(\psi(x_\tau))\chi_{\{j=1\}}(j_{\tau_M}) + g(\psi(x_{\tau_M}))\chi_{\{j=2\}}(j_\tau)\\ -\sum_{k=0}^{\tau-1}\Big(-h_1(x_k)\chi_{\{j=1\}}(j_{k+1}) - h_2(x_k)\chi_{\{j=2\}}(j_{k+1})\Big).\end{aligned}$$

  Notice that the total payoff a priori depends of the initial token (x_0, j). It is the sum of the final payoff (given by $f(x_\tau)$ or by $g(x_\tau)$ according to the board at which the position leaves the domain) and the running payoff that is given by $h_1(x_k)$ and $h_2(x_k)$ corresponding to the board in which they play at step $k+1$.

Value funtion and the two membranes problem.

On the one hand, Let S_P a strategy for Paul and S_C a strategy for Carol. The *value function* $w_M : \mathbb{T} \times \{1, 2\} \to [0, 1]$ for the game explaned before is given by

$$w_M(x, i) = \inf_{S_P}\sup_{S_C}\mathbb{E}(\text{total payoff}_M\,(x, j)) = \sup_{S_C}\inf_{S_P}\mathbb{E}(\text{total payoff}_M\,(x, j)).$$

Here the inf and sup are taken among all possible strategies of the players. In this definition of the value of the game we penalize games that never end (both players may choose to change boards at the same node for ever producing a game that never ends). The value of the game $w_M(x, i)$ encodes the amount that the players expect to get/pay playing their best with final payoffs f and g at level M. We refer to [16] for more details and precise definitions of strategies and to the books [16] and [66] for more information concerning games and mean value properties.

On the other hand, let us defined the pair

$$(u_M(x), v_M(x)) = \begin{cases}(w_M(x, 1), w_M(x, 2)) & \text{if } |x| < M,\\ (f(\psi(x)), g(\psi(x))) & \text{if } |x| = M.\end{cases} \tag{5.3.33}$$

That is, the above pair is a solution to the system (5.3.32) in the finite subgraph of the tree composed by nodes of level less than M and with boundary datum $(f(\psi), g(\psi))$ at the level M.

Observe that that the first and the second equation in (5.3.32) encodes all the possibilities for the next position of the game in the first board . Moreover, that the maximum that

appears in both equations capture the Paul's choice (who want to maximaze) to play or change to the other board. Similarly, the two last equationsin (5.3.32) takes into account all the possibilities for the game in the second board. They include a minimum since, in this case, Carol decides to play or to change boards.

Now our goal is to stady the convergence of the pair (u_M, v_M) given by(5.3.33) tas $M \to \infty$ iWe will obtain that the limit is a solution to the two membranes problem in a tree (5.3.5) under the boundary conditions

$$\begin{cases} \lim\limits_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)), \\ \lim\limits_{x \to z \in \partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases} \tag{5.3.34}$$

Theorem 5.3.13. *Consider two continuous functions $f, g : [0,1] \to \mathbb{R}$ and the function $\psi : \partial\mathbb{T} \to [0,1]$ given by (5.3.1). Let (u_M, v_M) is a solution to the system (5.3.32) in the finite subgraph of the tree composed by nodes of level less than M and with boundary datum $(f(\psi), g(\psi))$ at the level M given by (5.3.33).*

Then (u_M, v_M) converge in subsequences to a solution of the two membranes problem in the whole tree (5.3.32) with boundary data (5.3.34) as M goes to infinity.

Proof. From the estimates that we have proved in the previous sections for a solution (u_M, v_M) we know that these functions are uniformly bounded in M. Therefore, we can extract a subsequence $M_j \to \infty$ such that

$$u_{M_j}(x) \to u(x) \qquad \text{and} \qquad v_{M_j}(x) \to v(x),$$

for every $x \in \mathbb{T}$. Passing to the limit in the equations

$$\begin{cases} u_M(x) = \max\Big\{\beta_1 u_M(\hat{x}) + (1-\beta_1)\Big(\dfrac{1}{m}\displaystyle\sum_{y \in S^1(x)} u_M(y)\Big) + h_1(x), v_M(x)\Big\}, \\ u_M(\emptyset) = \max\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y \in S^1(\emptyset)} u_M(y)\Big) + h_1(\emptyset), v_M(\emptyset)\Big\}, \\ v_M(x) = \min\Big\{\beta_2 v_M(\hat{x}) + (1-\beta_2)\Big(\dfrac{1}{m}\displaystyle\sum_{y \in S^1(x)} v_M(y)\Big) + h_2(x), u_M(x)\Big\}, \\ v_M(\emptyset) = \min\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y \in S^1(\emptyset)} v_M(y)\Big) + h_2(\emptyset), u_M(\emptyset)\Big\}, \end{cases}$$

we get that the limit (u, v) solves

$$\begin{cases} u(x) = \max\Big\{\beta_1 u(\hat{x}) + (1-\beta_1)\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(x)} u(y)\Big) + h_1(x), v(x)\Big\}, \\ u(\emptyset) = \max\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(\emptyset)} u(y)\Big) + h_1(\emptyset), v(\emptyset)\Big\}, \\ v(x) = \min\Big\{\beta_2 v(\hat{x}) + (1-\beta_2)\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(x)} v(y)\Big) + h_2(x), u(x)\Big\}, \\ v(\emptyset) = \min\Big\{\Big(\dfrac{1}{m}\displaystyle\sum_{y\in S^1(\emptyset)} v(y)\Big) + h_2(\emptyset), u(\emptyset)\Big\}, \end{cases}$$

in the whole tree.

On the other hand, since f and g are continuous, given $\eta > 0$ there exists M large enough such that

$$|u_M(x) - \max_{I_x} f| = |f(\psi(x)) - \max_{I_x} f| < \eta,$$

and

$$|v_M(x) - \max_{I_x} g| = |g(\psi(x)) - \max_{I_x} g| < \eta$$

for every x at level M with M large enough. Therefore, we get

$$u_M(x) \leq \max_{I_x} f + \eta, \qquad \text{and} \qquad v_M(x) \leq \max_{I_x} g + \eta$$

for every x at level M large. Using that (u_M, v_M) converge as $M \to \infty$ to (u, v) we conclude that

$$\begin{cases} \limsup_{x\to z\in\partial\mathbb{T}} u(x) \leq f(\psi(z)) + \eta, \\ \limsup_{x\to z\in\partial\mathbb{T}} v(x) \leq g(\psi(z)) + \eta. \end{cases}$$

A similar argument using that

$$u_M(x) \geq \min_{I_x} f - \eta, \qquad \text{and} \qquad v_M(x) \geq \min_{I_x} g - \eta$$

for M large gives that

$$\begin{cases} \liminf_{x\to z\in\partial\mathbb{T}} u(x) \geq f(\psi(z)) - \eta, \\ \liminf_{x\to z\in\partial\mathbb{T}} v(x) \geq g(\psi(z)) - \eta. \end{cases}$$

Therefore, since η is arbitrary, we conclude that (u, v) satisfies

$$\begin{cases} \lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)), \\ \lim_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)), \end{cases}$$

and the proof is completed. □

Conclusiones

La presente tesis doctoral consta de una introducción seguida de cuatro capítulos principales, en los que se estudian las conexiones entre las Ecuaciones en Derivadas Parciales con la Probabilidad, la Geometría y la Teoría de Juegos dando lugar a contribuciones significativas en estas áreas, las cuales se recopilan en cinco artículos de investigación [48, 49, 50, 51, 52].

En primer lugar, en el Capítulo 2, desarrollamos una teoría de tipo Widder para ecuaciones del calor no locales asociadas a ciertos operadores de Lévy. Demostramos la unicidad de soluciones clásicas y muy débiles no negativas con una traza inicial dada, la existencia de una traza inicial perteneciente a una clase admisible y la existencia de una solución, expresada mediante una fórmula de representación, para cualquier traza inicial admisible. Estos resultados se establecen inicialmente para operadores de Lévy puramente no locales definidos a través de núcleos de Lévy positivos y simétricos, comparables con funciones radiales de crecimiento polinómico mixto, y posteriormente se extienden a operadores más generales, incluyendo operadores anisotrópicos y operadores que combinan una parte local con una no local.

En segundo lugar, en el Capítulo 3, estudiamos el problema de Cauchy en el espacio hiperbólico para la ecuación del calor con un término de reacción tipo Fisher-KPP. Analizamos la dicotomía entre propagación y extinción, demostrando nuevos resultados que incluyen el caso crítico en el que, a priori, no domina ni el término de reacción ni el ínfimo del espectro del operador de Laplace-Beltrami (considerado aquí como un operador positivo). En el caso de propagación, mostramos que si el dato inicial posee un cierto tipo de invariancia, la solución converge asintóticamente a una onda viajera de velocidad mínima en un sistema de referencia móvil. La elección de este sistema de referencia depende de la invariancia del dato inicial, la cual, a su vez, está estrechamente relacionada con los tres tipos de isometrías en el espacio hiperbólico: elípticas, hiperbólicas y parabólicas. Además, demostramos que los términos dependientes del tiempo en el sistema de referencia en movimiento están influenciados por el desplazamiento desde el infinito del espacio hiperbólico.

Después, en el Capítulo 4, buscamos la envolvente convexa de un conjunto mediante la evolución por curvatura mínima de una hipersuperficie que la contiene. Para determinar la envolvente convexa, analizamos el comportamiento en tiempos grandes de las soluciones a un problema de obstáculo en la formulación de conjunto de nivel del flujo geométrico dictado por el mínimo de las curvaturas principales. Demostramos que el superconjunto

de nivel donde la solución de este problema de obstáculo es positiva converge, cuando el tiempo tiende a infinito, hacia la envolvente convexa del obstáculo. Nuestro enfoque se basa en una aproximación de tipo teórico de juegos para este flujo geométrico, inspirada en resultados previos para el flujo por curvatura media.

Finalmente, en el Capítulo 5, estudiamos un problema elíptico. Analizamos iteraciones del problema del obstáculo para operadores diferenciales en un dominio de Lipschitz y para operadores definidos mediante fórmulas de valor medio en un árbol regular. Resolvemos de manera iterativa el problema del obstáculo desde arriba o desde abajo y demostramos que el par de funciones que se obtiene en el límite es una solución del problema de las dos membranas para los dos operadores considerados.

Cuando el espacio ambiente es un dominio de Lipschitz, realizamos nuestro análisis tanto en el marco variacional como en el de soluciones viscosas. En cambio, cuando el espacio ambiente es un árbol regular, demostramos la existencia de soluciones bajo condiciones adecuadas sobre los datos de frontera y los términos fuente involucrados. También probamos que, si los datos de frontera están estrictamente separados, el conjunto de coincidencia se mantiene alejado de la frontera y, en consecuencia, contiene únicamente un número finito de nodos.

En conclusión, esta tesis contribuye a una comprensión más amplia de las Ecuaciones en Derivadas Parciales, aportando resultados matemáticos profundos sobre la ecuación del calor para operadores de Lévy generales, ecuaciones de reacción-difusión en el espacio hiperbólico, juegos teóricos para estudiar el flujo por curvatura mínima y métodos iterativos para el problema de las dos membranas.

Conclusions

This PhD dissertation has presented an introduction followed by four main chapters, exploring the connections between Partial Differential Equations and Probability, Geometry and Game Theory having led to significant contributions in these areas, which are summarized in five research papers [48, 49, 50, 51, 52].

First, in Chapter 2 we develop a Widder-type theory for nonlocal heat equations for certain Lévy operators. We prove uniqueness of nonnegative classical and very weak solutions with a given initial trace, the existence of an initial trace, belonging to certain admissibility class and the existence of a solution, given by a representation formula, for any admissible initial trace. Such results are obtained first for purely nonlocal Lévy operators defined through positive symmetric Lévy kernels comparable to radial functions with mixed polynomial growth, and then extended to more general operators, including anisotropic ones and operators that have both a local and a nonlocal part.

Second, in Chapter 3 we study the Cauchy problem on the hyperbolic space for the heat equation with Fisher-KPP type forcing term. We address the question of propagation versus extinction, proving new results that include the critical case where either the forcing term or the infimum of the spectrum of the Laplace-Beltrami operator (spectral gap) (considered here as a positive operator) a-priori dominates. In the case of propagation, we prove that if the initial datum possesses a certain type of invariance, the solution converges asymptotically to a traveling wave of minimal speed in a moving frame. The choice of this moving frame depends on the invariance of the initial datum, which, in turn, is deeply related to the three types of isometries in the hyperbolic space: elliptic, hyperbolic, and parabolic. Moreover, we show that the time-dependent terms of the moving frame are affected by the drift from infinity of the hyperbolic space.

Latter, in Chapter 4 we look for the convex hull of a set using the geometric evolution by minimal curvature of a hypersurface that surrounds the set. To find the convex hull, we study the large time behavior of solutions to an obstacle problem for the level set formulation of the geometric flow driven by the minimum of the principal curvatures. We prove that the superlevel set where the solution to this obstacle problem is positive converges as time goes to infinity to the convex hull of the obstacle. Our approach is based on a game-theoretic approximation for this geometric flow that is inspired by previous results for the mean curvature flow.

Finally, in Chapter 5 we study an elliptic problem. Here we analyze iterations of the obstacle problem for differential operators in a Lipschitz domain and for an operators given

by mean value formulas setting in a regular tree. We solve iteratively the obstacle problem from above or below and we prove that the pair of functions that we obtain at the limit is a solution to the two membranes problem for the two different operators. When the ambient space is a Lipschitz domain we perform our analysis in both the variational and the viscosity settings. While the ambient space is a regular tree we show existence of solutions under adequate conditions on the boundary data and the involved source terms. We also show that, when the boundary data are strictly separated, the coincidence set is separated from the boundary and thus it contains only a finite number of nodes.

In conclusion, this PhD dissertation contributes to a broader understanding of Partial Differential Equations, providing deep mathematical results on the heat equation for general Lévy operators, reaction-diffusion equations on hyperbolic space, game-theoretic approaches to model minimal curvature flow, and iterative methods for the two membranes problem.

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