Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Banach Lattices and Free Objects
Retículos de Banach y Objetos Libres

Memoria para optar al grado de doctor presentada por

Enrique García Sánchez

Director:

Pedro Tradacete Pérez

Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas

Programa de Doctorado en Investigación Matemática

Instituto de Ciencias Matemáticas (ICMAT)



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UNIVERSIDAD COMPLUTENSE MADRID

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Resumen

El retículo de Banach libre generado por un espacio de Banach es un objeto libre que asigna a cada espacio de Banach E un retículo de Banach, denotado FBL[E], y un embebimiento lineal de E en FBL[E] con la propiedad universal de extender de forma única cualquier operador de E en un retículo de Banach a un homomorfismo de retículos definido en FBL[E]. En esta tesis, estudiamos la conexión entre este objeto libre y varias nociones de retículos de Banach y construcciones universales. Concretamente, proponemos una solución al problema de encontrar una representación explícita del retículo de Banach con p-estimaciones superiores libre, un análogo del retículo de Banach libre en la categoría de retículos de Banach con p-estimaciones superiores, y construimos *push-outs* en las categorías de retículos de Banah p-convexos y con p-estimaciones superiores, explorando por el camino algunas diferencias fundamentales entre estas dos clases de retículos de Banach tan relevantes. También investigamos cómo los biduales, que se comportan como objetos libres entre las categorías de espacios de Banach y espacios de Banach duales, interaccionan con los retículos de Banach libres, lo que nos lleva a introducir el retículo de Banach dual libre generado por un espacio de Banach. A continuación, empleamos técnicas de interpolación y extrapolación con el fin de comprender mejor los espacios de funciones positivamente homogéneas en los que se construyen los retículos de Banach libres. Finalmente, nos centramos en la cuestión más aplicada de la recuperación de fase estable, un problema proveniente de la cristalografía de rayos X que consiste en recuperar de forma estable una señal a partir de la medida de su módulo. Recientemente se han propuesto los retículos de Banach como el marco abstracto adecuado para modelizar este problema. Siguiendo esta aproximación, proporcionamos una caracterización de los subespacios de un retículo de Banach complejo que hacen recuperación de fase estable y estudiamos la existencia de subespacios infinito-dimensionales haciendo recuperación de fase estable en espacios de funciones continuas.

Abstract

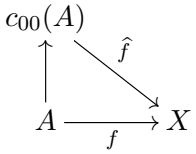
The free Banach lattice generated by a Banach space is a free object that assigns to every Banach space E a Banach lattice, denoted FBL$[E]$, and a linear embedding from E into FBL$[E]$ with the universal property that every operator from E to a Banach lattice extends uniquely to a lattice homomorphism on FBL$[E]$. In this dissertation, we study the connections of this free object with several Banach lattice notions and universal constructions. Namely, we provide a solution to the problem of finding an explicit representation for the free Banach lattice with upper p-estimates, which is an analogue of the free Banach lattice in the category of Banach lattices with upper p-estimates, and we construct push-outs in the categories of p-convex Banach lattices and Banach lattices with upper p-estimates, exploring some fundamental differences between these two relevant classes of Banach lattices along the way. We also investigate how biduals, that behave as free objects between the categories of Banach spaces and dual Banach spaces, interact with free Banach lattices, which leads us to introduce the free dual Banach lattice generated by a Banach space. Next, we apply interpolation and extrapolation techniques in order to better understand the spaces of positively homogeneous functions in which free Banach lattices are constructed. Finally, we focus on the more applied topic of stable phase retrieval, a problem from X-ray crystallography consisting in recovering stably a signal from the measurement of its modulus. Banach lattices have been recently suggested as the suitable framework to model this problem. Following this approach, we provide a characterization of the subspaces of complex Banach lattices doing stable phase retrieval and study the existence of infinite-dimensional subspaces doing stable phase retrieval in spaces of continuous functions.

Notation

- E, F, G will generally denote Banach spaces, whereas X, Y, Z will be used for Banach lattices.
- B_E and S_E denote the unit ball and the unit sphere of a Banach space E, respectively.
- X_+ denotes the positive cone of a Banach lattice X, that is, the set of positive elements.
- $\operatorname{span}(A)$ and $\operatorname{lat}(A)$ denote the subspace and the sublattice generated by the set A, respectively.
- An operator is a linear and bounded map between Banach spaces.
- An embedding is an invertible operator with continuous inverse, that is, an isomorphism onto its image.
- $id_E : E \to E$ denotes the identity operator on a Banach space E.
- E^* denotes the dual of a Banach space E, and E^{**} the bidual. $J_E : E \to E^{**}$ is the canonical isometric embedding from E into its bidual.
- $C(K)$ denotes the space of continuous functions over a compact Hausdorff space K, and $\mathbb{1}$ or $\mathbb{1}_K$ will denote the constant 1 function on K.
- If $1 \leqslant p \leqslant \infty$, p^* denotes its conjugate exponent, defined by $\frac{1}{p} + \frac{1}{p^*} = 1$.
- $H[E]$ denotes the space of positively homogeneous functions from E^* to $\mathbb{R}$. For $1 \leqslant p \leqslant \infty$, $H_p[E]$ is the ideal of $H[E]$ of weakly p-summing functions.

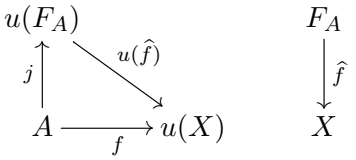
Introducción

Los objetos libres son una construcción común en matemáticas, de la que podemos encontrar ejemplos en álgebra, topología o análisis. Estos objetos categóricos permiten extender aplicaciones entre objetos en una categoría grande (típicamente la de conjuntos, pero también espacios topológicos o espacios de Banach) a morfismos en una categoría más pequeña con alguna estructura adicional, como linealidad, topología, orden, etc. Consideremos por ejemplo un espacio vectorial generado por una base de Hamel $(e_a)_{a\in A}$ indexada por un conjunto A. Este espacio es esencialmente único, salvo permutaciones de los elementos de la base, por lo que podemos fijar como representación el espacio $c_{00}(A)$ de sucesiones de soporte finito indexadas en A. Además, este espacio vectorial satisface la siguiente propiedad universal: dado un espacio vectorial X y una aplicación $f : A \to X$, existe una aplicación lineal $\widehat{f} : c_{00}(A) \to X$ tal que $\widehat{f}(e_a) = f(a)$ para todo $a \in A$, que viene dada por la expresión $\widehat{f}(\sum_{a\in A} x_a e_a) = \sum_{a\in A} x_a f(a)$.



Nótese que una vez que las imágenes de los elementos de la base e_a han sido fijadas por la aplicación f, la extensión $\widehat{f}$ queda unívocamente determinada por la propia linealidad de la aplicación. Esta propiedad universal es la que caracteriza los objetos libres, en este caso el *espacio vectorial libre sobre el conjunto A*.

En el lenguaje de la teoría de categorías, un objeto libre se define de la siguiente manera. Sean $\mathcal{C}$ y $\mathcal{D}$ dos categorías y $u : \mathcal{C} \to \mathcal{D}$ un funtor fiel (*faithful*), esto es, un funtor que es inyectivo en los conjuntos de homomorfismos entre objetos. Considérese por ejemplo la situación típica en la que la categoría $\mathcal{D}$ contiene a $\mathcal{C}$ y el funtor u representa un funtor olvido que "olvida" la estructura adicional que tenía $\mathcal{C}$ respecto de $\mathcal{D}$ (en el ejemplo anterior, $\mathcal{D}$ sería la categoría de conjuntos y aplicaciones, $\mathcal{C}$ la de espacios vectoriales y aplicaciones lineales, y $u : \mathcal{C} \to \mathcal{D}$ el funtor olvido que representa los espacios vectoriales como conjuntos sin estructura adicional y las aplicaciones lineales como simples aplicaciones entre conjuntos). Entonces, dado un objeto A en $\mathcal{D}$, un *objeto libre en $\mathcal{C}$ respecto de A* es un objeto F_A de $\mathcal{C}$, junto con un morfismo $j : A \to u(F_A)$ de $\mathcal{D}$, que satisface la siguiente propiedad universal: para todo objeto X en $\mathcal{C}$ y todo morfismo $f : A \to u(X)$ de $\mathcal{D}$, existe un único morfismo $\widehat{f} : F_A \to X$ de $\mathcal{C}$ tal que $u(\widehat{f}) \circ j = f$.



Cabe destacar que la propiedad universal implica que si un objeto libre existe, es único (salvo isomorfismo de la categoría).

La principal fuente de ejemplos de objetos libres es el álgebra. En efecto, un teorema clásico del álgebra universal (véase [71, Theorem 2.9]) garantiza la existencia de objetos libres generados por conjuntos en una amplia variedad de categorías, que incluyen espacios vectoriales, grupos, anillos, retículos vectoriales... Por otra parte, obtener una descripción concreta de un objeto libre como la que nos proporciona $c_{00}(A)$ para el espacio vectorial libre no es en general tan sencillo, y puede requerir un esfuerzo adicional. Sin embargo, en aquellos contextos en los que se dispone de una representación explícita del objeto libre, esta suele ser una construcción bastante natural. Así sucede por ejemplo con el *grupo libre sobre un conjunto*, que asocia a cada conjunto A un grupo G_A formado por todas las palabras finitas cuyos caracteres son elementos de A y sus inversos formales (módulo cancelación), el *espacio vectorial generado por una base* presentado anteriormente, que asocia a cada conjunto A un espacio vectorial generado por una base de Hamel de cardinalidad $|A|$, representado por $c_{00}(A)$, o el *anillo de polinomios sobre un conjunto de variables*, que asigna a cada conjunto A un anillo de polinomios con coeficientes en $\mathbb{Z}$ y variables indexadas por los elementos de A.

También se pueden encontrar objetos libres en topología, generalmente bajo la forma de compleciones o compactificaciones. Por ejemplo, la *compactificación de Stone–Čech de un espacio topológico* X (ver [59]) es un espacio compacto y Hausdorff βX junto con una aplicación continua $j : X \to \beta X$ que satisfacen que para cualquier aplicación continua $f : X \to K$, con K compacto y Hausdorff, existe una única función continua $\widehat{f} : \beta X \to K$ tal que $\widehat{f}j = f$. En este ejemplo, los espacios compactos y Hausdorff (con funciones continuas) ocupan el papel de la categoría pequeña $\mathcal{C}$, mientras que la categoría grande $\mathcal{D}$ se corresponde con los espacios topológicos arbitrarios (también con funciones continuas). Un procedimiento similar se puede usar para construir compleciones y clausuras en diferentes contextos, como la compleción de un espacio métrico (añadiendo los límites de todas las sucesiones de Cauchy) o la compleción en orden de un retículo o álgebra booleana (añadiendo los supremos e ínfimos de todo conjunto acotado en orden).

Pasando al análisis, nos damos cuenta de que los objetos libres no han sido tan estudiados como en álgebra. Pese a que el resultado de álgebra universal mencionado anteriormente se puede adaptar para probar la existencia de objetos libres en ciertos contextos de interés para los analistas (véase por ejemplo [77]), obtener una representación explícita se vuelve más delicado, en parte porque las categorías consideradas suelen combinar una estructura algebraica y topológica. Por otra parte, hay construcciones clásicas en análisis que satisfacen la propiedad universal de un objeto libre y sin embargo rara vez se las presenta como tales. Un ejemplo bastante naíf de lo anterior es $\ell_1(A)$, el espacio de Banach de las sucesiones absolutamente sumables indexadas por el conjunto A, que se comporta como el *espacio de Banach libre generado por el conjunto* A. En efecto, este espacio tiene la propiedad de que para todo espacio de Banach E y toda aplicación *acotada* $T : A \to E$, existe una única extensión $\widehat{T} : \ell_1(A) \to E$ como operador (lineal y acotado) prescrita por la relación $\widehat{T}e_a = Ta$ para cada $a \in A$, que además satisface que $\|\widehat{T}\| = \sup_{a\in A} \|Ta\|$. Aunque la restricción adicional de tener que considerar únicamente aplicaciones acotadas de A en E, necesaria para que el objeto libre exista, se puede reformular empleando en lugar de la categoría de espacios de Banach la de las bolas unidad de espacios de Banach con restricciones de operadores contractivos entre los correspondientes espacios como morfismos, este simple ejemplo ilustra cómo trabajar en categorías "analíticas" no es tan automático como en el marco puramente algebraico.

Otro ejemplo ubicuo de objeto libre en la teoría de espacios de Banach que raramente se considera desde esta perspectiva se puede encontrar en la categoría $\mathcal{B}an^*$ de espacios de Banach duales con operadores débil*-débil* continuos (equivalentemente, operadores adjuntos). En este contexto, es fácil comprobar que dado un espacio de Banach E, su segundo dual o bidual E^{**} junto con el embebimiento canónico $J_E : E \to E^{**}$ se comporta como el *espacio de Banach dual libre generado por* E, dado que para todo operador sobre un espacio de Banach dual $T : E \to F^*$ existe una única extensión débil*-débil* continua a E^{**} dada por $(T^*J_F)^* = J_F^*T^{**}$:

$$\begin{array}{ccc} E^{**} & \xrightarrow{T^{**}} & F^{***} \\ {\scriptstyle J_E}\uparrow & & \downarrow{\scriptstyle J_F^*} \\ E & \xrightarrow{T} & F^* \end{array}$$

A pesar de lo anterior, existen algunos objetos libres que han llamado la atención de la comunidad de espacios de Banach en las últimas décadas (véase la revisión bibliográfica [53]). Uno de ellos es el *espacio de Banach Lipschitz libre sobre un espacio métrico* (también llamado *espacio de Arens–Eells* o *espacio de costes de transporte*). Este espacio ha atraído el interés de investigadores de la teoría no lineal de espacios de Banach y la geometría métrica, especialmente desde su aparición en [61] y [122], dado que establecen un puente entre la categoría de espacios métricos con punto base (esto es, espacios métricos M con un punto distinguido $0 \in M$) y aplicaciones Lipschitz que preservan el 0, y la de espacios de Banach con operadores. Más concretamente, $\mathcal{F}(M)$ es un espacio de Banach junto con una isometría $\delta_M : M \to \mathcal{F}(M)$ tal que toda aplicación Lipschitz f de M en un espacio de Banach E que mande $0 \in M$ a $0 \in E$ se puede extender de forma única a un operador $\widehat{f} : \mathcal{F}(M) \to E$, de tal forma que $\widehat{f}\delta_M = f$ y $\|\widehat{f}\| = \|f\|_{\mathrm{Lip}}$, la mejor constante Lipschitz de f. Dado un espacio métrico M, $\mathcal{F}(M)$ admite una representación explícita como el subespacio del dual de $\mathrm{Lip}_0(M)$ (el espacio de Banach de las funciones Lipschitz de M a $\mathbb{R}$ que preservan el 0, equipado con la norma $\|\cdot\|_{\mathrm{Lip}}$) generado por los funcionales evaluación δ_p con $p \in M$ dados por $\delta_p(f) = f(p)$ para cada $f \in \mathrm{Lip}_0(M)$. Se tiene que $\mathcal{F}(M)$ es de hecho un predual de $\mathrm{Lip}_0(M)$, aunque todavía no se sabe si $\mathcal{F}(M)$ es el único predual. Remitimos al lector al monográfico [60] y las referencias que contiene para más información sobre el tema, y a [105] para algunas notas históricas sobre el origen de estos espacios. Mencionamos también que algunas construcciones similares han sido consideradas en el marco de los espacios cuasi métricos y cuasi normados [4], y los espacios asimétricos (aquellos en los que la distancia no es simétrica) [41].

Otro ejemplo de objeto libre, proveniente, esta vez, del análisis complejo, es el *espacio de Banach holomorfo libre*. Esta construcción, que fue introducida por Mujica en [99, Theorem 2.1], inspirado por el teorema de Dixmier–Ng (véase también [44, p. 417]), asigna a cada abierto U de un espacio de Banach complejo E un espacio de Banach complejo $\mathcal{G}^\infty(U)$ y una función holomorfa y acotada $\delta_U : U \to \mathcal{G}^\infty(U)$ que satisfacen la siguiente propiedad universal: para todo espacio de Banach complejo F y toda función holomorfa acotada $f : U \to F$ existe un único operador $\mathbb{C}$-lineal $\widehat{f} : \mathcal{G}^\infty(U) \to F$ tal que $\widehat{f}\delta_U = f$ y $\|\widehat{f}\| = \|f\|_\infty$. Este espacio se puede identificar de forma explícita como el subespacio de $\mathcal{H}^\infty(U)^*$, el dual del espacio de las funciones holomorfas acotadas de U en $\mathbb{C}$, que son continuas al restringirlas a la bola unidad de $\mathcal{H}^\infty(U)$ equipada con la topología compacto-abierta. Al igual que en el caso Lipschitz, $\mathcal{G}^\infty(U)$ es un predual de $\mathcal{H}^\infty(U)$, pero sigue siendo un problema abierto determinar si este predual es único.

La última construcción a considerar es el *retículo de Banach libre generado por un espacio de Banach*, sobre el cual versa esta tesis. Dado un espacio de Banach E, el retículo de Banach libre sobre E es un retículo de Banach $\mathrm{FBL}[E]$ junto con una isometría lineal $\phi_E : E \to \mathrm{FBL}[E]$ que satisface que para todo retículo de Banach X y todo operador $T : E \to X$ existe un único homomorfismo de retículos $\widehat{T} : \mathrm{FBL}[E] \to X$ tal que $\widehat{T}\phi_E = T$ y $\|\widehat{T}\| = \|T\|$. Este objeto fue introducido por Avilés, Rodríguez y Tradacete [11], generalizando los retículos de Banach libres generados por conjuntos estudiados por de Patger y Wickstead [106]. Tal y como veremos más adelante, los autores de [11] también dieron una construcción explícita de $\mathrm{FBL}[E]$ partiendo de las representaciones de los retículos vectoriales libres como espacios de funciones [16, 28]. Desde la aparición de [11] ha habido un número creciente de publicaciones que extienden el conocimiento existente sobre este objeto y lo generalizan a otros contextos, como los retículos de Banach libres con condiciones de convexidad [75], los retículos de Banach libres complejos [68] o los retículos de Banach libres generados por retículos [12].

Los ejemplos de objetos libres presentados anteriormente han atraído el interés de la comunidad investigadora porque proporcionan un puente entre áreas relevantes del análisis. Estas nuevas

conexiones nos permiten desarrollar diccionarios entre propiedades de diferentes categorías que se encuentran en correspondencia, estudiar nuevas familias de espacios con propiedades determinadas, encontrar contraejemplos a conjeturas existentes, o transferir técnicas de un área a otra. En esta tesis doctoral, nos centraremos en el estudio del retículo de Banach libre generado por un espacio de Banach y algunas de las construcciones relacionadas con el mismo. La estructura del trabajo es la siguiente:

El Capítulo 1 contiene nociones preliminares. Comenzaremos recordando los aspectos básicos de la teoría de retículos de Banach necesarios para el resto de la tesis. A continuación, introduciremos las nociones de p-convexidad y p-estimaciones superiores, que jugarán un papel central en los siguientes capítulos, y resumiremos las propiedades principales de estas dos clases de retículos de Banach. En analogía con el caso p-convexo, donde los espacios L_p juegan un papel central, señalaremos la importancia de $L_{p,\infty}$ entre los retículos de Banach con p-estimaciones superiores y enumeraremos algunas de sus propiedades. Finalmente, repasaremos la construcción del retículo de Banach libre generado por un espacio de Banach, así como otros desarrollos recientes del área, tales como los retículos de Banach p-convexos libres.

En el Capítulo 2 abordaremos varias cuestiones relacionadas con las p-estimaciones superiores. Históricamente, la p-convexidad se ha estudiado de forma más detallada que las p-estimaciones superiores, en parte por su conexión con los espacios L_p y la mayor variedad de herramientas que estos espacios traen consigo. Un ejemplo de esta asimetría puede encontrarse en [75] donde se prueba la existencia tanto del retículo de Banach p-convexo libre como del retículo de Banach con p-estimaciones superiores libre a través de un argumento abstracto, pero sin embargo solo se proporciona una construcción explícita para el caso p-convexo. Quedó formulado como un problema abierto en [75, Remark 6.2], que fue posteriormente reiterado en [103, p. 91], determinar si el retículo de Banach con p-estimaciones superiores libre, o en general cualquier otro retículo de Banach libre, se podía representar como un retículo de funciones. El problema es el siguiente. Dado un espacio de Banach E, el retículo vectorial libre generado por E, denotado $\mathrm{FVL}[E]$, es denso en $\mathrm{FBL}^{p,\infty}[E]$, el retículo de Banach con p-estimaciones superiores libre, o equivalentemente, $\mathrm{FBL}^{p,\infty}[E]$ es la compleción de $\mathrm{FVL}[E]$ bajo cierta norma. Por otra parte, $\mathrm{FVL}[E]$ se puede representar fácilmente como subretículo de $C_{ph}(B_{E^*})$, el espacio de las funciones positivamente homogéneas y débil* continuas en B_{E^*}, y la inclusión es continua cuando se equipa a $\mathrm{FVL}[E]$ y $C_{ph}(B_{E^*})$ con la norma de $\mathrm{FBL}^{p,\infty}[E]$ y la norma del supremo, respectivamente. Sin embargo, no es cierto en general que un homomorfismo de retículos inyectivo definido sobre un retículo vectorial normado se extienda de forma inyectiva a su compleción. Esta propiedad queda caracterizada en el Teorema 2.1.6, y no se sabe si $\mathrm{FVL}[E]$ satisface dicha propiedad con respecto a $\mathrm{FBL}^{p,\infty}[E]$, ni siquiera cuando consideramos exclusivamente la inclusión a $C_{ph}(B_{E^*})$.

Motivados por este problema, y con el objetivo de entender mejor las p-estimaciones superiores, introducimos en la Sección 2.1.1 el retículo de Banach libre generado por un espacio de Banach E asociado a una clase $\mathcal{C}$ de retículos de Banach, denotado $\mathrm{FBL}^{\mathcal{C}}[E]$, y deducimos algunas de sus propiedades. Partiendo de las clases $\mathcal{C}_p$ y $\mathcal{C}_{p,\infty}$ de retículos de Banach p-convexos y con p-estimaciones superiores, respectivamente, nuestra aproximación al problema en la Sección 2.2 consiste en buscar clases más simples de retículos de Banach que induzcan normas equivalentes en los correspondientes retículos de Banach libres hasta alcanzar clases que admitan una fórmula explícita para la norma (típicamente, estas clases estarán formadas por retículos de Banach finito-dimensionales). Usando los teoremas de factorización clásicos de Maurey (Teorema 1.2.3) y Pisier (Teorema 1.2.4), demostramos un teorema de representación de retículos p-convexos y con p-estimaciones superiores, respectivamente, (Teorema 2.2.1) que nos permite reducir las clases $\mathcal{C}_p$ y $\mathcal{C}_{p,\infty}$ a las clases de ℓ_∞-sumas de espacios L_p y $L_{p,\infty}$, respectivamente. Estas clases se pueden reducir de nuevo hasta llegar a las clases de un solo elemento $\{L_p[0,1]\}$ y $\{L_{p,\infty}[0,1]\}$ usando propiedades generales de la Sección 2.1.1, y un argumento técnico de la Sección 2.2.3 permite simplificar las clases una vez más, obteniendo como resultado final las clases $\{\ell_p^n : n \in \mathbb{N}\}$ y $\{\ell_{p,\infty}^n : n \in \mathbb{N}\}$. Las

conclusiones de este argumento se resumen en los Teoremas 2.2.13 y 2.2.14, donde recuperamos la norma de $\mathrm{FBL}^p[E]$ hallada en [75, Theorem 6.1] y proporcionamos una norma explícita equivalente para $\mathrm{FBL}^{p,\infty}[E]$. También caracterizamos las clases $\mathcal{D}$ cuyo retículo de Banach libre asociado es isométrico o isomorfo a $\mathrm{FBL}^p[E]$ o $\mathrm{FBL}^{p,\infty}[E]$, respectivamente. Nótese que la norma explícita obtenida en el Teorema 2.2.14 no puede ser isométrica a la norma original de $\mathrm{FBL}^{p,\infty}[E]$, tal y como probamos en 2.2.15. Más concretamente, proporcionamos un ejemplo de un retículo de Banach que satisface una p-estimación superior con constante 1 pero no se embebe isométricamente en ninguna ℓ_∞-suma de espacios $L_{p,\infty}$ (véase el Ejemplo 2.2.9).

A continuación, dedicamos la Sección 2.3 a estudiar *push-outs* y otras nociones categóricas en subcategorías de retículos de Banach como $\mathcal{C}_p$ o $\mathcal{C}_{p,\infty}$. Reproduciendo la construcción de [14], demostramos la existencia de *push-outs* isométricos en cualquier categoría $\mathcal{C}$ que sea cerrada bajo cocientes y ℓ_∞-sumas (Teorema 2.3.1). Más aún, demostramos en el Teorema 2.3.2 que los *push-outs* en la categoría de retículos de Banach p-convexos preservan isometrías. Sin embargo, no se puede aplicar el mismo argumento en el caso de las p-estimaciones superiores, dado que emplea técnicas de p-convexificación y p-concavificación, que no tienen un análogo para p-estimaciones superiores. Pese a esto, demostramos que los *push-outs* con p-estimaciones superiores transforman isometrías en isomorfismos con distorsión controlada por una constante universal, proporcionando dos demostraciones diferentes de este resultado (Teorema 2.3.3). En la Sección 2.3.2 usamos los *push-outs* construidos en la sección anterior para dar una solución al problema del embebimiento para retículos de Banach libres con p-estimaciones superiores (ver Teorema 2.3.6). Dado un embebimiento isométrico entre espacios de Banach $\iota : F \to E$, este problema consiste en caracterizar cuando el homomorfismo de retículos $\bar{\iota} = \widehat{\phi_E \iota} : \mathrm{FBL}^{p,\infty}[F] \to \mathrm{FBL}^{p,\infty}[E]$ es también un embebimiento. El mismo argumento usado para los retículos de Banach con p-estimaciones superiores, junto con una propiedad de ℓ_p estrechamente ligada a la inyectividad en la categoría de retículos de Banach p-convexos con operadores positivos (Teorema 2.3.7), nos permite recuperar la solución original de [103, Theorem 3.7] al problema del embebimiento para el caso p-convexo. Esto señala otra diferencia entre la p-convexidad y las p-estimaciones superiores, dado que el Teorema 2.3.10 demuestra que $\ell_{p,\infty}$ no satisface la correspondiente propiedad de inyectividad para los retículos de Banach con p-estimaciones superiores. Concluimos el capítulo con la Sección 2.4, dedicada a extender resultados de Reisner [111] sobre factorizaciones de operadores por operadores p-convexos y q-cóncavos al contexto de operadores que factorizan por operadores (p, p_2)-convexos y (q, q_2)-cóncavos, para $p_2 \in \{p, \infty\}$ y $q_2 \in \{1, q\}$.

En el Capítulo 3 estudiamos nociones de dualidad en retículos de Banach libres. El punto de partida es la concepción del bidual de un espacio de Banach como un objeto libre en la categoría de espacios de Banach duales con operadores débil*-débil* continuos (equivalentemente, adjuntos). Esto motiva investigar la relación entre duales libres y retículos de Banach libres en dos direcciones diferentes. La primera es demostrar que el retículo de Banach libre sobre un bidual se embebe reticular e isométricamente en el bidual del retículo de Banach libre generado por el espacio de Banach original (Teorema 3.2.2). Para ello, desarrollamos en la Sección 3.1 una representación simplificada del retículo de Banach libre generado por un espacio de Banach dual E^* usando el Principio de Reflexividad Local. Más específicamente, probamos que $\mathrm{FBL}[E^*]$ se puede representar como un espacio de funciones sobre E en lugar de sobre E^{**}. Esto nos permite establecer el Teorema 3.2.2, obteniendo como consecuencia resultados de representabilidad finita para el retículo de Banach libre sobre un bidual con respecto al retículo de Banach libre sobre el espacio de Banach original. También observamos que un argumento similar se puede emplear para obtener resultados análogos en otros contextos, como espacios $C(K)$ libres (Teorema 3.2.9) y ultraproductos de retículos de Banach libres (Teorema 3.2.10). La otra línea de investigación seguida en este capítulo consiste en la construcción de un objeto libre en la categoría de duales de retículos de Banach con homomorfismos de retículos débil*-débil* continuos. Probamos que dicho objeto existe en el caso p-convexo con $1 < p \leqslant \infty$, y coincide con el bidual del retículo de Banach p-convexo libre. En el caso general ($p = 1$), demostramos que este objeto libre coincide con el bidual del retículo de Banach libre si y solo si el

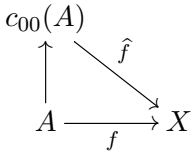
espacio de Banach generador no contiene una copia complementada de ℓ_1. Pese a algunos resultados parciales, queda como problema abierto determinar la existencia de este objeto libre cuando el espacio de Banach contiene una copia complementada de ℓ_1. A continuación, continuamos en la Sección 3.5 con el estudio iniciado en [103, Section 3] de la correspondencia entre las propiedades de un operador entre espacios de Banach y las del homomorfismo de retículos entre retículos de Banach libres que este induce, caracterizando la propiedad de preservar copias complementadas de ℓ_1. Finalmente, en la Sección 3.6 proporcionamos una aplicación de los retículos de Banach libres al estudio de los homomorfismos de retículos que alcanzan la norma, demostrando que todo homomorfismo de retículos en un AM-espacio alcanza la norma. Este resultado se puede comparar con el hecho trivial de que todo homomorfismo de retículos en un espacio $C(K)$ alcanza su norma en la función constantemente 1, siendo la principal diferencia que en los AM-espacios no hay, en principio, unidad fuerte.

El Capítulo 4 está dedicado al estudio de los espacios de funciones positivamente homogéneas $H_p[E]$, $1 \leqslant p \leqslant \infty$, en los que se construyen los retículos de Banach p-convexos libres, y sus generalizaciones, los espacios $H_{p;q}[E]$, $1 \leqslant q \leqslant p \leqslant \infty$. Tras recordar algunas propiedades de estos espacios, aplicaremos técnicas de interpolación y extrapolación para comprender mejor cómo se ordenan estos espacios con respecto a la inclusion, y cuándo estas inclusiones son estrictas. Concretamente, aplicaremos el método real de interpolación (Proposición 4.2.2) y el método de Calderón–Lozanovskii (Teorema 4.2.7) a ciertos pares de retículos de Banach compatibles dentro de esta familia, obteniendo estimaciones para las normas de estos nuevos espacios y proporcionando ejemplos en los que las inclusiones inducidas por estas desigualdades son estrictas. También demostramos en el Teorema 4.3.1 que $H_p[E]$ se puede extrapolar a partir de la familia $(H_q[E])_{q>p}$ usando el Δ método, y damos más ejemplos de inclusiones estrictas entre espacios $H_{p;q}$. Concluimos el capítulo estudiando en detalle las inclusiones entre espacios $H_{p;q}[E]$ en el caso $E = \ell_1$, obteniendo que todas las inclusiones entre espacios $H_{p;q}[\ell_1]$ son estrictas (Proposición 4.3.5).

Finalmente, el Capítulo 5 cambia el tema de los retículos de Banach libres por otro más aplicado dentro de la teoría de retículos de Banach: la recuperación de fase estable (SPR por sus siglas en inglés, *stable phase retrieval*). Este problema, que tiene aplicaciones en áreas tales como la cristalografía de rayos X, la astronomía, la física cuántica o el reconocimiento de voz, consiste en estudiar cuándo una señal se puede recuperar a partir de su módulo de forma estable bajo perturbaciones. Comenzamos el capítulo analizando el problema y su conexión con el marco abstracto de los retículos de Banach y, a continuación, proporcionamos una caracterización de los subespacios de un retículo de Banach complejo que hacen SPR (Teorema 5.2.7), en línea con el correspondiente resultado para el caso real [48, Theorem 6.1]. Tras esto, extendemos algunos resultados de [48] sobre la existencia de subespacios infinito-dimensionales de espacios $C(K)$ reales haciendo SPR al marco complejo. En particular, caracterizamos los espacios $C(K)$ que contienen un subespacio infinito-dimensional haciendo SPR (Teorema 5.3.7). En la Sección 5.4 estudiamos la Pregunta 6.4 de [48], donde se conjetura una versión cuantitativa de [48, Theorem 6.1] (o el Teorema 5.3.7, respectivamente), proporcionando una respuesta afirmativa tanto en el caso real como en el complejo. Finalmente, en la Sección 5.5 iniciamos un estudio preliminar de cómo la propiedad de contener un subespacio haciendo SPR se preserva bajo ciertas operaciones con retículos de Banach, como sumas o cocientes.

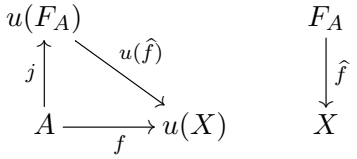
Introduction

Free objects are a common construction in mathematics. We can find examples of them in algebra, topology or analysis. These categorical objects allow us to extend maps from objects in a large category (typically sets, but also topological spaces or Banach spaces) to morphisms in a smaller category with some additional structure, such as linearity, topology, order, etc. Consider for instance a vector space generated by a Hamel basis $(e_a)_{a\in A}$ indexed by a set A. This space will be essentially unique, up to permutations of the elements of the basis, so we can fix as a representation the space $c_{00}(A)$ of finitely supported sequences indexed by A. Moreover, this vector space satisfies the following universal property: given any vector space X and any map $f : A \to X$, there exists a linear map $\widehat{f} : c_{00}(A) \to X$ such that $\widehat{f}(e_a) = f(a)$ for every $a \in A$, given by the expression $\widehat{f}(\sum_{a\in A} x_a e_a) = \sum_{a\in A} x_a f(a)$.



Note that once the images of the basis elements e_a are prescribed by the map f, the extension map $\widehat{f}$ is uniquely determined by linearity. This universal property is what characterizes a free object, in this case the free vector space over a set A.

In the language of category theory, a free object can be defined as follows. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $u : \mathcal{C} \to \mathcal{D}$ a faithful functor, i.e., a functor that is injective on the sets of morphisms between objects. Consider for instance the typical situation where the category $\mathcal{D}$ is larger than $\mathcal{C}$ and the functor u represents a forgetful functor that "forgets" the additional structure of $\mathcal{C}$ with respect to $\mathcal{D}$ (in the example above, $\mathcal{D}$ would be the category of sets and maps, $\mathcal{C}$ corresponds to the category of vector spaces and linear maps, and $u : \mathcal{C} \to \mathcal{D}$ is the forgetful functor that represents vector spaces as plain sets with no additional structure and linear maps as maps between sets). Then, given an object A of $\mathcal{D}$, a *free object in $\mathcal{C}$ with respect to A* is an object F_A of $\mathcal{C}$, together with a morphism $j : A \to u(F_A)$ of $\mathcal{D}$, satisfying the following universal property: for every object X of $\mathcal{C}$ and every morphism $f : A \to u(X)$ of $\mathcal{D}$, there exists a unique morphism $\widehat{f} : F_A \to X$ of $\mathcal{C}$ such that $u(\widehat{f}) \circ j = f$.



It should be noted that this universal property implies that free objects are unique (up to isomorphism of the category) whenever they exist.

The main source of examples of free objects is algebra. Indeed, a well-known theorem from universal algebra (cf. [71, Theorem 2.9]) guarantees the existence of free objects generated by sets

in a wide range of categories, which include vector spaces, groups, rings, vector lattices... On the other hand, obtaining a concrete representation for the free object as the one provided by $c_{00}(A)$ for the free vector space is not as straightforward in general, and might require an additional effort. However, in those contexts where an explicit description of the free object is available, they tend to be very natural constructions. This is the case of the *free group over a set*, which associates to every set A a group G_A that consists of all the finite words whose characters are elements of A and their formal inverses (modulo cancellation), the *vector space generated by a basis* discussed above, which associates to every set A the vector space generated by a Hamel basis of cardinality $|A|$, represented by $c_{00}(A)$, or the *ring of polynomials over a set of variables*, which assigns to any set A the ring of polynomials with coefficients on $\mathbb{Z}$ and variables indexed by the elements of A.

Free objects can also be found in topology, usually under the form of completions or compactifications. For instance, the *Stone–Čech compactification of a topological space* X (cf. [59]) is a compact Hausdorff space βX together with a continuous map $j : X \to \beta X$ satisfying that for every continuous function $f : X \to K$ with K a compact Hausdorff space there exists a unique continuous function $\widehat{f} : \beta X \to K$ such that $\widehat{f}j = f$. In this example, compact Hausdorff spaces (with continuous functions) play the role of the category $\mathcal{C}$, whereas the larger category $\mathcal{D}$ corresponds to arbitrary topological spaces (with continuous functions). A similar scheme can be used to construct different instances of completions or closures, such as the completion of a metric space (adding limits of Cauchy sequences) or the order completion of lattices or boolean algebras (adding infima and suprema of order bounded sets).

Moving on to analysis, we notice that free objects are not as well studied as in algebra. Even though the above-mentioned result from universal algebra can be adapted to show the existence of free objects in some contexts of interest for analysts (see for instance [77]), obtaining an explicit representation becomes more intricate, partly because the categories considered are a combination of algebraic and topological structures. On the other hand, there are classical constructions of analysis that satisfy the universal property of a free object between certain categories, even though they are not usually seen as such. A very naive example of this is $\ell_1(A)$, the Banach space of all the absolutely summable sequences indexed by a set A, which behaves as the *free Banach space generated by the set* A. Indeed, this space has the property that for every Banach space E and every *bounded* map $T : A \to E$, there exists a unique extension $\widehat{T} : \ell_1(A) \to E$ as a (linear and bounded) operator prescribed by the relation $\widehat{T}e_a = Ta$ for every $a \in A$, which moreover satisfies $\|\widehat{T}\| = \sup_{a\in A} \|Ta\|$. Though the additional restriction of considering only bounded maps from A to E, necessary for the free object to exist, can be restated by replacing the category of Banach spaces with the category of unit balls of Banach spaces with restrictions of contractive operators between the corresponding spaces as morphisms, this simple example illustrates how working with "analytical" categories is not as straightforward as the purely algebraic setting.

Another ubiquitous example of a free object in Banach space theory that is rarely considered from this perspective can be found in the category $\mathcal{B}an^*$ of dual Banach spaces with weak*-to-weak* continuous operators (equivalently, adjoint operators). In this setting, it is easy to check that for a Banach space E, the second dual or bidual E^{**} together with the canonical embedding $J_E : E \to E^{**}$ behaves as the *free dual Banach space generated by* E, as for every operator into a dual Banach space $T : E \to F^*$ there is a unique weak*-to-weak* continuous extension to E^{**} given by $(T^*J_F)^* = J_F^*T^{**}$:

$$\begin{array}{ccc} E^{**} & \xrightarrow{T^{**}} & F^{***} \\ {\scriptstyle J_E}\uparrow & & \downarrow{\scriptstyle J_F^*} \\ E & \xrightarrow{T} & F^* \end{array}$$

Notwithstanding, there exist some free objects that have raised the attention of the Banach space theory community in the recent decades (see the survey [53]). One of them is the *Lipschitz free Banach space over a metric space* (also known as *Arens–Eells space* or *transportation cost space*). These spaces have attracted the interest of researchers in the non-linear theory of Banach spaces and metric geometry, specially since their appearance in [61] and [122], as they establish a bridge between the categories of pointed metric spaces (i.e., metric spaces M with a distinguished point $0 \in M$) with Lipschitz maps that preserve 0, and Banach spaces with (linear and bounded) operators. More specifically, $\mathcal{F}(M)$ is a Banach space together with an isometry $\delta_M : M \to \mathcal{F}(M)$ such that every Lipschitz map f from M to a Banach space E sending $0 \in M$ to $0 \in E$ can be extended uniquely to an operator $\widehat{f} : \mathcal{F}(M) \to E$, in such a way that $\widehat{f}\delta_M = f$ and $\|\widehat{f}\| = \|f\|_{\mathrm{Lip}}$, the best Lipschitz constant of f. Given a metric space M, $\mathcal{F}(M)$ admits an explicit construction as the subspace of the dual of $\mathrm{Lip}_0(M)$ (the Banach space of Lipschitz functions from M to $\mathbb{R}$ that preserve the 0 endowed with the norm $\|\cdot\|_{\mathrm{Lip}}$) generated by the evaluation functionals δ_p with $p \in M$ given by $\delta_p(f) = f(p)$ for every $f \in \mathrm{Lip}_0(M)$. It turns out that $\mathcal{F}(M)$ is actually a predual of $\mathrm{Lip}_0(M)$, even though it is still unknown whether $\mathcal{F}(M)$ is a unique predual. We refer the reader to the survey [60] and the references therein for further information on the topic, and [105] for historical notes on the origin of these spaces. We also mention that similar constructions have been considered in the setting of quasi-metric and quasi-normed spaces [4], and asymmetric spaces (those where the distance is not symmetric) [41].

Another example of free object coming, this time, from complex analysis is the *holomorphic free Banach space*. This construction, that was introduced by Mujica in [99, Theorem 2.1], inspired by the Dixmier–Ng theorem (see also [44, p. 417]), assigns to every open subset U of a complex Banach space E a complex Banach space $\mathcal{G}^\infty(U)$ and a bounded holomorphic function $\delta_U : U \to \mathcal{G}^\infty(U)$ with the following universal property: for every complex Banach space F and every bounded holomorphic function $f : U \to F$ there exists a unique $\mathbb{C}$-linear operator $\widehat{f} : \mathcal{G}^\infty(U) \to F$ such that $\widehat{f}\delta_U = f$ and $\|\widehat{f}\| = \|f\|_\infty$. This space can be explicitly identified as the subspace of functionals of $\mathcal{H}^\infty(U)^*$, the dual of the space of bounded holomorphic functions from U to $\mathbb{C}$, that are continuous when they are restricted to the unit ball of $\mathcal{H}^\infty(U)$ endowed with the compact-open topology. As in the Lipschitz case, $\mathcal{G}^\infty(U)$ is a predual of $\mathcal{H}^\infty(U)$, but it is still an open problem whether this predual is unique.

The last construction that will be discussed is the *free Banach lattice generated by a Banach space*, to which this dissertation is devoted. Given a Banach space E, the free Banach lattice over E is a Banach lattice $\mathrm{FBL}[E]$ together with a linear isometry $\phi_E : E \to \mathrm{FBL}[E]$ satisfying that for every Banach lattice X and operator $T : E \to X$ there exists a unique lattice homomorphism $\widehat{T} : \mathrm{FBL}[E] \to X$ such that $\widehat{T}\phi_E = T$ and $\|\widehat{T}\| = \|T\|$. This object was introduced by Avilés, Rodríguez and Tradacete [11], generalizing the free Banach lattice generated by a set studied by de Patger and Wickstead [106]. As we will see, the authors of [11] also provided an explicit construction of $\mathrm{FBL}[E]$ based on the representations of free vector lattices as lattices of functions [16, 28]. Since the appearence of [11] there has been an increasing number of publications extending the available knowledge about this object and generalizing it to other settings, such as free Banach lattices with convexity conditions [75], complex free Banach lattices [68] or free Banach lattices generated by lattices [12].

The examples of free objects presented above have attracted the interest of researchers because they provide a bridge between relevant areas of analysis. These new connections allow us to develop dictionaries of properties within different categories that are in correspondence, study new families of spaces with prescribed properties, find counterexamples to existing conjectures, or transfer techniques from one area into the other. In this dissertation, we will focus on the study of the free Banach lattice generated by a Banach space and some of its related constructions. The structure is the following:

Chapter 1 will be devoted to preliminary notions. We will start recalling the basic theory of Banach lattices that will be required for the rest of the dissertation. Next, we will introduce the notions of p-convexity and upper p-estimates, which will play a central role in subsequent chapters, and discuss the main properties of these classes of Banach lattices. Analogously to the p-convex setting, where L_p-spaces play a prominent role, we will point out the importance of $L_{p,\infty}$ among Banach lattices with upper p-estimates and describe some of its features. Finally, we will review the construction of the free Banach lattice generated by a Banach space, as well as some of the recent developments in the topic, such as free p-convex Banach lattices.

In Chapter 2 we will study several notions related with upper p-estimates. Traditionally, p-convexity has been much better understood than upper p-estimates, partly because of the connection with L_p-spaces and the broader toolbox that comes with them. A particular example of this asymmetry can be found in [75], where the existence of both the free p-convex Banach lattice and the free Banach lattice with upper p-estimates was established via an abstract argument, but the authors were able to provide an explicit construction only in the p-convex setting. It was stated as an open question in [75, Remark 6.2] and later reiterated in [103, p. 91] whether the free Banach lattice with upper p-estimates, or in general any other free Banach lattice, could be represented as a lattice of functions. The issue is the following: If E is a Banach space, the free vector lattice generated by E, $\mathrm{FVL}[E]$, is dense in $\mathrm{FBL}^{p,\infty}[E]$, the free Banach lattice with upper p-estimates, or equivalently, $\mathrm{FBL}^{p,\infty}[E]$ is the completion of $\mathrm{FVL}[E]$ under a certain norm. Moreover, $\mathrm{FVL}[E]$ can be easily represented as a sublattice of $C_{ph}(B_{E^*})$, the space of positively homogeneous and weak* continuous functions on B_{E^*}, and the inclusion is continuous when $\mathrm{FVL}[E]$ and $C_{ph}(B_{E^*})$ are endowed with the norm of $\mathrm{FBL}^{p,\infty}[E]$ and the supremum norm, respectively. However, it is not always true that an injective lattice homomorphism on a normed vector lattice extends injectively to its completion. This property is characterized in Theorem 2.1.6, and it was not know whether $\mathrm{FVL}[E]$ satisfies such property with respect to $\mathrm{FBL}^{p,\infty}[E]$, not even when we only consider the inclusion into $C_{ph}(B_{E^*})$.

Motivated by this problem and the goal of better understanding upper p-estimates, we introduce in Section 2.1.1 the free Banach lattice generated by a Banach space E associated to an arbitrary class $\mathcal{C}$ of Banach lattices, denoted $\mathrm{FBL}^{\mathcal{C}}[E]$, and deduce some of their properties. Considering the classes $\mathcal{C}_p$ of p-convex Banach lattices and $\mathcal{C}_{p,\infty}$ of Banach lattices with upper p-estimates as our starting point, our approach in Section 2.2 is to find simpler classes of Banach lattices which induce equivalent norms on the corresponding free Banach lattices until we obtain a class that admits an explicit formula for the norm (typically, this class will be composed of finite-dimensional Banach lattices). Using the classical factorization theorems of Maurey (Theorem 1.2.3) and Pisier (Theorem 1.2.4), we show a representation theorem for p-convex Banach lattices and Banach lattices with upper p-estimates (Theorem 2.2.1) that allows us to reduce the classes $\mathcal{C}_p$ and $\mathcal{C}_{p,\infty}$ to the classes of ℓ_∞-sums of L_p and $L_{p,\infty}$-spaces, respectively. These classes can be further reduced to the singleton classes $\{L_p[0,1]\}$ and $\{L_{p,\infty}[0,1]\}$ using general properties from Section 2.1.1, and a technical argument from Section 2.2.3 simplifies the classes one more time, obtaining as a final result the classes $\{\ell_p^n : n \in \mathbb{N}\}$ and $\{\ell_{p,\infty}^n : n \in \mathbb{N}\}$. The conclusions of the previous argument are summarized in Theorems 2.2.13 and 2.2.14, where we recover the norm of $\mathrm{FBL}^p[E]$ established in [75, Theorem 6.1] and provide an equivalent explicit renorming for $\mathrm{FBL}^{p,\infty}[E]$. We also characterize all the classes $\mathcal{D}$ whose associated free Banach lattice is isometric or isomorphic to $\mathrm{FBL}^p[E]$ or $\mathrm{FBL}^{p,\infty}[E]$, respectively. It should be noted that the explicit norm obtained in Theorem 2.2.14 cannot be isometric to the original norm of $\mathrm{FBL}^{p,\infty}[E]$, as we show in Proposition 2.2.15. This is done by constructing an example of a Banach lattice satisfying upper p-estimates with constant 1 that does not isometrically embed into an ℓ_∞-sum of $L_{p,\infty}$-spaces (see Example 2.2.9).

Next, we devote Section 2.3 to study push-outs and other categorical notions in restricted categories of Banach lattices such as $\mathcal{C}_p$ or $\mathcal{C}_{p,\infty}$. Replicating the construction of [14], we show the existence of isometric push-outs in a category $\mathcal{C}$ whenever it is closed under quotients and ℓ_∞-sums

(Theorem 2.3.1). Moreover, we show in Theorem 2.3.2 that push-outs in the category of p-convex Banach lattices preserve isometries. However, the same argument cannot be adapted to the setting of upper p-estimates, as it uses techniques of p-convexification and p-concavification, which do not have an upper p-estimate analogue. In spite of this, we show that upper p-estimates push-outs transform isometries into isomorphisms with distortion bounded by a universal constant, and we provide two different proofs of this fact (see Theorem 2.3.3). In Section 2.3.2 we use the push-outs that we have constructed in order to provide a solution to the embedding problem for free Banach lattices with upper p-estimates (Theorem 2.3.6). Provided $\iota : F \to E$ an isometric embedding between Banach spaces, this problem consists in characterizing when the lattice homomorphism $\bar{\iota} = \widehat{\phi_E \iota} : \mathrm{FBL}^{p,\infty}[F] \to \mathrm{FBL}^{p,\infty}[E]$ is an embedding as well. The same argument used for Banach lattices with upper p-estimates, together with a property of ℓ_p closely related to injectivity in the category of p-convex Banach lattices with positive operators (Theorem 2.3.7), allows us to recover the original solution from [103, Theorem 3.7] to the embedding problem for the p-convex case. This points out another difference between p-convexity and upper p-estimates, since Theorem 2.3.10 shows that $\ell_{p,\infty}$ does not satisfy the corresponding injectivity property for Banach lattices with upper p-estimates. We finish the chapter with Section 2.4, which is dedicated to extend some results from Reisner [111] on factorization of operators through p-convex and q-concave operators to the setting of operators which factor through (p, p_2)-convex and (q, q_2)-concave operators, for $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$.

We devote Chapter 3 to investigate duality notions in free Banach lattices. The starting point is the understanding of the bidual of a Banach space as a free object in the category of dual Banach spaces with weak*-to-weak* continuous (i.e. adjoints) operators. This motivates us to investigate the relation between free duals and free Banach lattices in two different directions. The first one is to show that the free Banach lattice of a bidual space embeds lattice isometrically into the bidual of the free Banach lattice of the original Banach space (Theorem 3.2.2). To do so, we develop in Section 3.1 a simpler representation of the free Banach lattice over a dual Banach space E^* using the Principle of Local Reflexivity. More specifically, we prove that $\mathrm{FBL}[E^*]$ can be realized as a space of functions over E instead of E^{**}. This allows us to establish Theorem 3.2.2, obtaining as a consequence some finite representability results for the free Banach lattice of a bidual space with respect to the free Banach lattice of the original Banach space. We also observe that a similar argument can be used to obtain analogous results in other contexts, such as free $C(K)$-spaces (Theorem 3.2.9) and ultraproducts of free Banach lattices (Theorem 3.2.10). The other line of research we pursue in this chapter is the construction of a free object in the category of duals of Banach lattices with weak*-to-weak* continuous lattice homomorphisms. We show that such object exists in the p-convex setting for $1 < p \leqslant \infty$, and coincides with the bidual of the free p-convex Banach lattice. In the general case ($p = 1$), we show that the free object coincides with the bidual of the free Banach lattice if and only if the generating space does not contain a complemented copy of ℓ_1. In spite of some partial results, it is left as an open question whether it exists when the Banach space contains ℓ_1 complemented. Next, in Section 3.5 we continue the study initiated in [103, Section 3] of the correspondence between the properties of an operator between Banach spaces and the lattice homomorphism between free Banach lattices that it induces, characterizing the property of preserving complemented copies of ℓ_1. Finally, in Section 3.6 we provide an application of free Banach lattices to the study of norm attaining lattice homomorphisms, showing that every lattice homomorphism on an AM-space attains its norm. This should be compared with the trivial fact that every lattice homomorphism on a $C(K)$-space attains its norm on the constant 1 function, the main difference being that in general there is no strong unit in an AM-space.

Chapter 4 is dedicated to the study of the spaces of positively homogeneous functions $H_p[E]$, $1 \leqslant p \leqslant \infty$, in which free p-convex Banach lattices are constructed, and the spaces $H_{p;q}[E]$, $1 \leqslant q \leqslant p \leqslant \infty$, that generalize them. After reviewing some known facts concerning these spaces, we will apply interpolation and extrapolation techniques to better understand how these spaces are ordered with respect to inclusion, and when these inclusions are strict. More specifically, we

apply the real method (Proposition 4.2.2) and the Calderón–Lozanovskii method (Theorem 4.2.7) to certain compatible pairs of Banach lattices within this family, obtaining bounds for the norms of these new spaces and providing examples where the inclusions induced by these inequalities are strict. We also show in Theorem 4.3.1 that $H_p[E]$ can be extrapolated from the family $(H_q[E])_{q>p}$ using the Δ method, and give more examples of strict inclusions between $H_{p;q}$-spaces. We conclude the chapter studying in detail the inclusions between $H_{p;q}[E]$ spaces for the case $E = \ell_1$, obtaining that all the inclusions between $H_{p;q}[\ell_1]$ spaces are strict (Proposition 4.3.5).

Finally, Chapter 5 changes the topic of free objects for a more applied subject within Banach lattice theory: stable phase retrieval (SPR). This problem, that has applications in many areas such as X-ray crystallography, astronomy, quantum physics or speech recognition, consist in studying when a signal can be recovered from its modulus in a stable way. We begin the chapter reviewing the problem and its connections with the abstract framework of Banach lattices, and next, we provide a characterization of the subspaces of complex Banach lattices that do SPR (Theorem 5.2.7), in line with the corresponding result for real scalars [48, Theorem 6.1]. After this, we extend some results from [48] concerning the existence of infinite-dimensional SPR subspaces in real $C(K)$-spaces to the complex setting. In particular, we characterize the $C(K)$-spaces containing an infinite-dimensional SPR subspace (Theorem 5.3.7). In Section 5.4 we study Question 6.4 from [48], where a quantitative version of [48, Theorem 6.1] (respectively, Theorem 5.3.7) is conjectured, providing an affirmative answer for both the real and the complex case. Finally, in Section 5.5 we initiate a preliminary study of the stability of the property of containing SPR subspaces under certain operations with Banach lattices, such as direct sums or quotients.

Chapter 1

Preliminary notions on Banach lattices and free Banach lattices

1.1 Banach lattices

A *vector lattice* is a vector space X over $\mathbb{R}$ endowed with a partial order $\leqslant$ that is compatible with the linear structure and admits lattice operations, i.e.,

- if $x, y \in X$ are such that $x \leqslant y$, then $x + z \leqslant y + z$ for every $z \in X$ and $\lambda x \leqslant \lambda y$ for every scalar $\lambda > 0$;
- for every $x, y \in X$ there exist the *least upper bound* and the *greatest lower bound* of the set $\{x, y\}$, called the *supremum* and *infimum*, respectively, and denoted $x \vee y$ and $x \wedge y$.

A vector lattice X with a norm $\|\cdot\|$ compatible with the order, that is, $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$, where $|x| = x \vee (-x)$ is the *absolute value*, is called a *normed vector lattice*. If the norm is complete, X is said to be a *Banach lattice*. Many classical Banach spaces are also examples of Banach lattices. This is the case, among many others, for the sequence spaces ℓ_p, the spaces of p-integrable functions $L_p(\mu)$, the Lorentz spaces $L_{p,q}(\mu)$, and the spaces $C(K)$ of continuous functions over a compact Hasdorff space K, when they are endowed with the coordinatewise, almost everywhere and pointwise order and lattice operations, respectively. Banach lattices will be the main object of study of this dissertation. Throughout the text, we will follow the standard notation used in classical textbooks such as [88] or [96].

Vector lattices have a much richer subspace structure than vector spaces. Let X be a vector lattice and Y subspace of X. If Y is closed under the lattice operations (supremum, infimum, absolute value...), it is called a *sublattice*. If Y is *solid*, that is, whenever $|x| \leqslant |y|$ and $y \in Y$ we have $x \in Y$, then Y is said to be an *ideal*. An ideal Y satisfying that whenever $x = \sup A$ with $A \subseteq Y$, then $x \in Y$ is called a *band*. Moreover, if there is a linear projection $P : X \to X$ such that $P(X) = Y$ and $0 \leqslant Px \leqslant x$ for every $x \in X_+$, where $X_+ := \{x \in X : x \geqslant 0\}$ denotes the *positive cone*, then Y is a *projection band*. Given a subset A of X, $\operatorname{lat}(A)$ and $I(A)$ will denote the *sublattice* and the *ideal generated by* A, that is, the smallest sublattice or ideal that contains the set A, respectively. If V is a vector subspace, then we can represent the sublattice generated by V (cf. [5, Exercise 8, p. 204]) as

$$\operatorname{lat}(V) = \left\{ \bigvee_{i=1}^{m} x_i - \bigvee_{j=1}^{n} y_j : (x_i)_{i=1}^m, (y_j)_{j=1}^n \subseteq V \right\}. \tag{1.1}$$

We will constantly use this fact to represent elements of the free vector lattice generated by a Banach space (see Section 1.3).

The lattice order makes possible a wider diversity of operators as well. Let X and Y be vector lattices and $T : X \to Y$ a linear map. T is called *positive* if $x \leqslant y$ implies that $Tx \leqslant Ty$, or equivalently, if $T(X_+) \subseteq Y_+$, and it is said to be *regular* if it is the difference of positive operators.

Every positive operator from a Banach lattice to a normed vector lattice is bounded. If T preserves the lattice operations (it suffices to check that it preserves just one of the operations, for instance, the absolute value: $T|x| = |Tx|$ for every $x \in X$) it is called a *lattice homomorphism*. Observe that every lattice homomorphism is positive. T is said to be *interval preserving* if it is positive and $T[0,x] = [0,Tx]$ for every $x \in X_+$, where $[0,x] = \{y \in X : 0 \leqslant y \leqslant x\}$ is the *order interval* generated by x. There is a duality relation between lattice homomorphisms and interval preserving maps that will be exploited in Chapter 3, see [96, Theorem 1.4.19]. Finally, observe that the range of a lattice homomorphism is a sublattice and its kernel is an ideal, whereas the image of an interval preserving operator is an ideal.

If X is a Banach lattice, its dual X^* can also be endowed with a lattice order compatible with the original dual norm. Given $x^*, y^* \in X^*$, we say that $x^* \leqslant y^*$ if $y^* - x^*$ is a positive operator from X to $\mathbb{R}$. The lattice operations with respect to this order are described by the Riesz–Kantorovich formulas: if $x^*, y^* \in X^*$ and $x \in X_+$, then

$$(x^* \vee y^*)(x) = \sup\{x^*(x-y) + y^*(y) : 0 \leqslant y \leqslant x\},$$

and

$$(x^* \wedge y^*)(x) = \inf\{x^*(x-y) + y^*(y) : 0 \leqslant y \leqslant x\}.$$

Actually, these formulas can be used to endow of a vector lattice structure the space of regular operators between two (Archimedean) vector lattices X and Y whenever the target space Y is *Dedekind complete*, that is, there exists $\sup(A)$ for every subset $A \subset Y$ satisfying that there is $x \in Y$ with $y \leqslant x$ for every $y \in A$ (cf. [96, Theorem 1.3.2]). In particular, the dual of every Banach lattice is a Dedekind complete Banach lattice.

Another related property of Banach lattices is the order continuity of the norm. A Banach lattice X is *order continuous* if every decreasing net in X_+ whose infimum is 0 satisfies that the infimum of the norms is also 0. There are many equivalent characterizations of order continuity (see [96, Section 2.4]), such as X sitting as an ideal in its bidual or every order interval of X being weakly compact.

As it can be deduced from the examples, Banach lattices are an abstraction of classical spaces of functions. It turns out that, in a certain way, the spaces of functions are the only type of Banach lattices that exists. For instance, if X is a Banach lattice, it was established in [89, Lemma 3.4] that there exist a family of probability measures Γ and a lattice isometric embedding (i.e., a lattice homomorphism that is also an isometric embedding into the image) of X into the space

$$\ell_\infty(L_1(\mu))_{\mu\in\Gamma} = \left\{\overline{f} = (f_\mu)_{\mu\in\Gamma} : f_\mu \in L_1(\mu), \|\overline{f}\| = \sup_{\mu\in\Gamma} \|f_\mu\|_{L_1(\mu)} < \infty\right\}.$$

Moreover, every Banach lattice resembles locally a $C(K)$-space: if X is a Banach lattice and e is a non-zero positive element, then $I(\{e\}) = \{x \in X : |x| \leqslant \lambda e \text{ for some } \lambda > 0\}$, the *principal ideal generated by* e, can be identified with a $C(K)$-space by means of Kakutani's representation theorem for AM-spaces (cf. [96, Theorem 2.1.3]). Recall that a Banach lattice X is said an *AM-space* if $\|x \vee y\| = \|x\| \vee \|y\|$ for every $x, y \in X_+$.

This similarity with function spaces endows Banach lattices with an interesting feature, usually referred to as *Krivine's functional calculus*. Given a Banach lattice X and some vectors $x_1, \ldots, x_n \in X$, this functional calculus enables to define uniquely in X expressions of the form $h(x_1, \ldots, x_n)$ for any h continuous and positively homogeneous function over $\mathbb{R}^n$, that is, $h(\lambda t_1, \ldots, \lambda t_n) = \lambda h(t_1, \ldots, t_n)$ for every $\lambda \geqslant 0$ and $(t_1, \ldots, t_n) \in \mathbb{R}^n$. Examples of these expressions are $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ for $1 < p < \infty$, $|x_1|^{1-\theta}|x_2|^\theta$ for $0 < \theta < 1$, or in general functions that cannot be obtained by just using a finite number of linear and lattice operations over the vectors $x_1, \ldots, x_n$. This correspondence allows us to immediately translate well-known equalities and inequalities of real numbers to Banach lattices. Moreover, the uniqueness of functional calculus implies that it is

preserved under lattice homomorphisms. Note that if the Banach lattice X happens to be a space of functions with the usual pointwise or almost everywhere order, the corresponding functional calculus for vectors $f_1, \dots, f_n \in X$ and a positively homogeneous expression $h : \mathbb{R}^n \to \mathbb{R}$ consists of the function given by $h(f_1, \dots, f_n)(t) = h(f_1(t), \dots, f_n(t))$ for every t in the domain. The details of the construction can be found in [43, Chapter 16], [88, Section 1.d] or [96, Theorem 2.1.20].

The local similarity with the spaces of functions is also used in the definition of complex Banach lattices. Let X be a real Banach lattice. The complexification of X, denoted by $X_{\mathbb{C}}$, is the complex vector space $X \times X \equiv X \oplus iX$ endowed with the usual coordinatewise addition and the multiplication by complex scalars given by $(a+ib)(x_1+ix_2) = ax_1 - bx_2 + i(bx_1 + ax_2)$ for $x_1, x_2 \in X$, $a, b \in \mathbb{R}$. It is well known that the modulus of a complex-valued function $f = f_1 + if_2$ is given by the expression

$$|f| = \sqrt{|f_1|^2 + |f_2|^2} = \sup\big\{f_1 \cos\theta + f_2 \sin\theta \ : \ \theta \in [0, 2\pi]\big\}.$$

Analogously, we can define the modulus of an element $x = x_1 + ix_2 \in X_{\mathbb{C}}$ as

$$|x| = \sqrt{|x_1|^2 + |x_2|^2} = \sup\big\{x_1 \cos\theta + x_2 \sin\theta \ : \ \theta \in [0, 2\pi]\big\},$$

using a $C(K)$-representation to show that this supremum exists. Observe that $|x|$ is an element of X, so we can define a norm in the whole complexification $X_{\mathbb{C}}$ by $\|x\|_{X_{\mathbb{C}}} = \||x|\|$, in such a way that $(X_{\mathbb{C}}, \|\cdot\|_{X_{\mathbb{C}}})$ is a complex Banach space. We call *complex Banach lattice* to the complexification $X_{\mathbb{C}}$ of a real Banach lattice X endowed with the modulus and the norm defined in this way. Note that Banach spaces of complex-valued functions such as $C(K;\mathbb{C})$ or $L_p(\mu;\mathbb{C})$ are particular cases of complex Banach lattices, as they are the complexification of their real-valued version $C(K;\mathbb{R})$ or $L_p(\mu;\mathbb{R})$. We will only work with complex Banach lattices in Chapter 5. Throughout the rest of the chapters, all the Banach lattices and Banach spaces will be assumed to be real.

1.2 Convexity in Banach lattices

A representation theorem due to Kakutani (see for instance [88, Theorem 1.b.2] or [96, Theorem 2.7.1]) establishes that if a Banach lattice X has a p-additive norm for some $1 \leqslant p < \infty$, that is, if $\|x+y\|^p = \|x\|^p + \|y\|^p$ for every pair of disjoint elements $x, y \in X$ ($x, y \in X$ are called *disjoint* when $|x| \wedge |y| = 0$), then X is lattice isometric to $L_p(\mu)$ for some measure μ. One can wonder what can be said about X if the equality above is replaced by an inequality. This leads to the definition of p-convexity, p-concavity, upper and lower p-estimates, and other related notions (see [43, Chapter 16] and [88, Sections 1.d and 1.f]). Recall that in a Banach lattice we can define expressions such as $\big(\sum_{i=1}^n |x_i|^p\big)^{\frac{1}{p}}$ via functional calculus.

Definition 1.2.1. *Let X be a Banach lattice, E a Banach space and $1 \leqslant p, q \leqslant \infty$.*

(1) An operator $T : E \to X$ is (p,q)-convex *if there exists a constant $K > 0$ such that*

$$\left\| \left(\sum_{i=1}^n |Tx_i|^q \right)^{\frac{1}{q}} \right\| \leqslant K \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \tag{1.2}$$

for every $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$. The best constant K satisfying (1.2) is called the (p,q)-convexity constant of T *and is denoted $K^{(p,q)}(T)$.*

(2) An operator $T : X \to E$ is (p,q)-concave *if there exists a constant $K > 0$ such that*

$$\left(\sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leqslant K \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\| \tag{1.3}$$

for every $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. The best constant K satisfying (1.3) is called the (p,q)-concavity constant of T *and is denoted* $K_{(p,q)}(T)$.

(3) *A Banach lattice X is* (p,q)-convex *(respectively,* (p,q)-concave*) if the identity operator $i_X : X \to X$ is (p,q)-convex (respectively, (p,q)-concave), and $K^{(p,q)}(X) = K^{(p,q)}(i_X)$ is called the* (p,q)-convexity constant of X *(respectively, $K_{(p,q)}(X) = K_{(p,q)}(i_X)$ is called the* (p,q)-concavity constant of X*).*

(4) *If $p = q$, we will simply say that T or X is p-convex (respectively, p-concave) and write $K^{(p)}(T)$ or $K^{(p)}(X)$ (respectively, $K_{(p)}(T)$ or $K_{(p)}(X)$) instead.*

A similar property can be introduced for Banach lattices when we require the inequalities above to be witnessed only on disjoint vectors.

Definition 1.2.2. *Let X be a Banach lattice and $1 \leqslant p \leqslant \infty$.*

(1) *X is said to satisfy an* upper p-estimate *if there exists a constant $K \geqslant 1$ such that*

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqslant K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \tag{1.4}$$

for every (positive) pairwise disjoint $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. The best constant K satisfying (1.4) is called the upper p-estimates constant of X *and is denoted $K^{(\uparrow p)}(X)$.*

(2) *X is said to satisfy a* lower p-estimate *if there exists a constant $K \geqslant 1$ such that*

$$\left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \leqslant K \left\| \sum_{i=1}^{n} x_i \right\| \tag{1.5}$$

for every (positive) pairwise disjoint $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. The best constant K satisfying (1.5) is called the lower p-estimates constant of X *and is denoted $K_{(\downarrow p)}(X)$.*

Note that every (p,q)-convex or (p,q)-concave operator must be bounded, and the norm is bounded by the corresponding (p,q)-convexity or (p,q)-concavity constant. Clearly, every bounded operator $T : E \to X$ is 1-convex with $K^{(1)}(T) = \|T\|$, as every Banach lattice satisfies the triangular inequality, and every operator $T : X \to E$ is ∞-concave with $K_{(\infty)}(T) = \|T\|$, since the norm in X is monotone with respect to the order. The same argument yields that every Banach lattice satisfies an upper 1-estimate and a lower ∞-estimate. Moreover, it can be seen that upper ∞-estimates and ∞-convexity are both equivalent to being lattice isomorphic to an AM-space (i.e., being ∞-convex with constant 1). Similarly, lower 1-estimates and 1-concavity are equivalent to being lattice isomorphic to an AL-space (i.e., being 1-concave with constant 1). Therefore, Definition 1.2.2 is only relevant for $1 < p < \infty$. Finally, observe that if $q < p$ and $T : E \to X$ is (p,q)-convex, then necessarily $T = 0$. Indeed, if $x \in E$, it follows that $n^{\frac{1}{q}} \|Tx\| \leqslant n^{\frac{1}{p}} K^{(p,q)}(T) \|x\|$ for every $n \in \mathbb{N}$, so $Tx = 0$. Similarly, if $p < q$ and $T : X \to E$ is (p,q)-concave, then $T = 0$. Actually, the only relevant cases for (p,q)-convexity are $q = p$ and $q = \infty$; and $q = p$ and $q = 1$ for (p,q)-concavity (cf. [43, Corollary 16.6]).

As one could expect, the notions of convexity and concavity are related by duality: an operator T is (p,q)-convex (respectively, (p,q)-concave), if and only if T^* is (p^*,q^*)-concave (respectively, (p^*,q^*)-convex), with the same constant [43, Theorem 16.21] (hereby, $p^* = \frac{p}{p-1}$ denotes the conjugate exponent of p). A similar statement holds for upper and lower p-estimates [88, Proposition 1.f.5]. Observe that (p,q)-convexity, (p,q)-concavity, upper p-estimates and lower p-estimates are lattice isomorphic properties, and they are inherited by closed sublattices and lattice quotients. Moreover, an arbitrary ℓ_∞ sum of p-convex Banach lattices (respectively, Banach lattices with upper p-estimates) with uniformly bounded constants is again p-convex (satisfies an upper p-estimate).

Similarly, an arbitrary ℓ_1 sum of p-concave Banach lattices (respectively, Banach lattices with lower p-estimates) with uniformly bounded constants is again p-concave (satisfies a lower p-estimate).

It is clear that every p-convex (p-concave) Banach lattice satisfies an upper (lower) p-estimate with the same constant. It is also true (see [88, Proposition 1.f.6]) that for a Banach lattices the notions of upper (lower) p-estimates and (p,∞)-convexity ($(p,1)$-concavity) coincide. It can be shown that every p-convex (p-concave) operator is q-convex (q-concave) for every $q < p$ ($q > p$), and the q-convexity constant is non-decreasing (the q-concavity is non-increasing) [88, Proposition 1.d.5]. Moreover, every (p,∞)-convex ($(p,1)$-concave) operator is q-convex (q-concave) for every $q < p$ ($q > p$), with constant $K^{(q)}(T) \leqslant C_{p,q}K^{(p,\infty)}(T)$ ($K_{(q)}(T) \leqslant C_{p^*,q^*}K_{(p,1)}(T)$), where $C_{p,q}$ is a constant that explodes as $(p-q)^{-\frac{1}{p}}$ when $q \to p^-$. This a consequence of [108, Theorem 2.4], [120, Theorem 21.3] and [43, Theorem 16.5], which establish a connection between (p,q)-concave operators and (p,q)-summing operators on $C(K)$-spaces, which have been much more studied (see also [88, Theorem 1.f.7] for a proof using probabilistic arguments). In Corollary 2.2.2 we will provide an alternative proof of this fact via a representation theorem for Banach lattices with upper p-estimates. We will also show that the converse is not true (see Example 2.2.3).

Following with some examples, the prototypical Banach lattice that is p-convex and p-concave is $L_p(\mu)$. By the results mentioned above, it follows that $L_p(\mu)$ is also r-convex for every $1 \leqslant r < p$ and s-concave for every $p < s \leqslant \infty$. Actually, it can be checked that if $L_p(\mu)$ is infinite-dimensional, then it is r-convex if and only if $1 \leqslant r \leqslant p$, and it is s-concave if and only if $p \leqslant s \leqslant \infty$. This follows from the straightforward fact that if an infinite-dimensional Banach lattice satisfies an upper r-estimate and a lower s-estimate, then $r \leqslant s$. Another renowned family of examples are Lorentz spaces. In [39] it is established that for $1 \leqslant q < p$, $L_{p,q}(\mu)$ is q-convex and satisfies a lower p-estimate, but is not p-concave, whereas for $p < q \leqslant \infty$ we have that $L_{p,q}(\mu)$ is q-concave and satisfies an upper p-estimate, but is not p-convex.

Several results on factorization of p-convex/concave operators [94] and representation of p-convex/concave Banach lattices [54, 78] point towards L_p as the model space for p-convexity and p-concavity. On the other hand, for upper and lower p-estimates, the canonical spaces seem to be $L_{p,\infty}$, sometimes called weak-L_p, and $L_{p,1}$, respectively [54, 108]. Let us recall some facts about these spaces. Let (Ω,Σ,μ) be a measure space and f a measurable function in Ω, we define its decreasing rearrangement $f^* : [0,\infty) \to [0,\infty)$ by

$$f^*(t) := \inf\{\lambda > 0 : \mu\{|f| > \lambda\} \leqslant t\}.$$

Given $1 < p < \infty$, the space $L_{p,1}(\mu)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that the norm

$$|||f|||_{L_{p,1}} := \int_0^\infty t^{\frac{1}{p}} f^*(t)\frac{dt}{t}$$

is finite. Similarly, the space $L_{p,\infty}(\mu)$ is the set of all measurable functions f such that the quasinorm

$$|||f|||_{L_{p,\infty}} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda>0} \lambda\mu(\{|f| > \lambda\})^{\frac{1}{p}}$$

is finite. Note, in particular, that every $f \in L_{p,\infty}(\mu)$ has σ-finite support. It is easy to check that the quasinorm of $L_{p,\infty}(\mu)$ is equivalent to the dual norm of $L_{p^*,1}(\mu)$ with equivalence constant p^*, in other words, $L_{p,\infty}(\mu)$ is the dual of $L_{p^*,1}(\mu)$ (up to a lattice isomorphic renorming). Actually, $L_{p,\infty}(\mu)$ admits many different lattice renormings: for each $1 \leqslant r < p$, let

$$\|f\|_{L_{p,\infty}^{[r]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-\frac{1}{r}} \left(\int_A |f|^r \, d\mu\right)^{\frac{1}{r}}.$$

These norms are all equivalent to the quasinorm [63, Exercise 1.1.12]:

$$|||f|||_{L_{p,\infty}} \leqslant \|f\|_{L^{[r]}_{p,\infty}} \leqslant \left(\frac{p}{p-r}\right)^{\frac{1}{r}} |||f|||_{L_{p,\infty}} \quad \text{for every} \quad f \in L_{p,\infty}(\mu). \tag{1.6}$$

Moreover, given $f \in L_{p,\infty}(\mu)$, the norms $\|f\|_{L^{[r]}_{p,\infty}}$ are increasing in $r \in [1,p)$. Observe that for each $1 \leqslant r < p$, $L^{[r]}_{p,\infty}(\mu) = (L_{p,\infty}(\mu), \|\cdot\|_{L^{[r]}_{p,\infty}})$ is a Banach lattice that satisfies an upper p-estimate with constant 1. Hence, $L_{p^*,1}(\mu)$ satisfies a lower p^*-estimate. Unless stated otherwise, throughout the dissertation we will assume that $L_{p,\infty}(\mu)$ is endowed with the norm $\|\cdot\|_{L^{[1]}_{p,\infty}}$, and we will simply denote this norm by $\|\cdot\|_{L_{p,\infty}}$ or $\|\cdot\|_{p,\infty}$. As usual, when the measure space is the natural numbers with the counting measure we will write $\ell_{p,\infty}$. Similarly, when referring to the predual space $L_{p^*,1}(\mu)$, we will write $L^{[r]}_{p^*,1}(\mu)$ when it is endowed with the dual norm induced by $\|\cdot\|_{L^{[r]}_{p,\infty}}$, and, we will simply write $L_{p^*,1}(\mu)$ in the default case $r = 1$.

As we mentioned earlier, the spaces L_p, $L_{p,\infty}$ and $L_{p,1}$ play a special role in the theory of concavity and convexity, as the following results due to Maurey [94] and Pisier [108] illustrate:

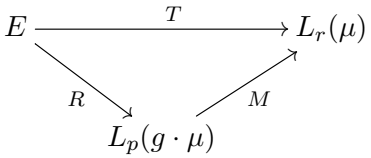
Theorem 1.2.3 (Maurey's Factorization Theorem)**.** *Let $1 \leqslant r < p < \infty$, E a Banach space, $T : E \to L_r(\mu)$ a linear operator and $C > 0$. The following are equivalent:*

(1) T is p-convex with $K^{(p)}(T) \leqslant C$.

(2) There exists a normalized $g \in L_1(\mu)_+$ such that for every $x \in E$,

$$\{g = 0\} \subseteq \{Tx = 0\} \quad \text{and} \quad \|g^{-\frac{1}{r}} Tx\|_{L_p(g\cdot\mu)} \leqslant C\|x\|.$$

(3) There exists a normalized $g \in L_1(\mu)_+$ such that $\{g = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$, and $T = MR$, where $R : E \to L_p(g \cdot \mu)$ given by $Rx = g^{-\frac{1}{r}} Tx$ is bounded with $\|R\| \leqslant C$ and $M : L_p(g \cdot \mu) \to L_r(\mu)$ is the multiplication operator by $g^{\frac{1}{r}}$.



Theorem 1.2.4 (Pisier's Factorization Theorem)**.** *Let $1 \leqslant r < p < \infty$, E a Banach space and $T : E \to L_r(\mu)$ a linear operator. The following are equivalent:*

(1) There exists $C > 0$ such that T is (p,∞)-convex with $K^{(p,\infty)}(T) \leqslant C$.

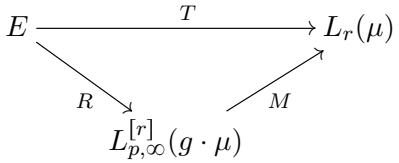
(2) There exists $C' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $x \in E$ and measurable subset $B \subseteq \Omega$

$$\|Tx\chi_B\|_{L_r(\mu)} \leqslant C'\|x\| \left(\int_B g\, d\mu\right)^{\frac{1}{r}-\frac{1}{p}}.$$

(3) There exists $C'' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $x \in E$,

$$\{g = 0\} \subseteq \{Tx = 0\} \quad \text{and} \quad \|g^{-\frac{1}{r}} Tx\|_{L^{[r]}_{p,\infty}(g\cdot\mu)} \leqslant C''\|x\|.$$

(4) There exists $C'' > 0$ and a normalized $g \in L_1(\mu)_+$ such that $\{g = 0\} \subseteq \{Tx = 0\}$ for every $x \in E$, and $T = MR$, where $R : E \to L^{[r]}_{p,\infty}(g \cdot \mu)$ given by $Rx = g^{-\frac{1}{r}} Tx$ is bounded with $\|R\| \leqslant C''$ and $M : L^{[r]}_{p,\infty}(g \cdot \mu) \to L_r(\mu)$ is the multiplication operator by $g^{\frac{1}{r}}$.



Moreover, $C \leqslant C'' \leqslant C' \leqslant (1-\frac{r}{p})^{\frac{1}{p}-\frac{1}{r}}C$.

Both results have their corresponding dual statements for concave operators:

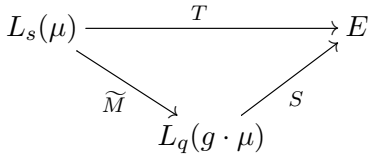
Theorem 1.2.5. *Let* $1 < q < s < \infty$, *E a Banach space,* $T : L_s(\mu) \to E$ *a linear operator and* $C > 0$. *The following are equivalent:*

(1) T *is* q*-concave with* $K_{(q)}(T) \leqslant C$.

(2) There exists a normalized $g \in L_1(\mu)_+$ *such that for every* $f \in L_s(\mu)$,

$$\|Tf\| \leqslant C\|fg^{-\frac{1}{s}}\|_{L_q(g\cdot\mu)}.$$

(3) There exists a normalized $g \in L_1(\mu)_+$ *such that* $T = S\widetilde{M}$, *where* $\widetilde{M} : L_s(\mu) \to L_q(g\cdot\mu)$ *is the multiplication operator by* $g^{-\frac{1}{s}}$ *and* $S : L_q(g\cdot\mu) \to E$ *is bounded with* $\|S\| \leqslant C$.



Theorem 1.2.6. *Let* $1 < q < s < \infty$, *E a Banach space and* $T : L_s(\mu) \to E$ *a linear operator. The following are equivalent:*

(1) There exists $C > 0$ *such that* T *is* $(q,1)$*-concave with* $K_{(q,1)}(T) \leqslant C$.

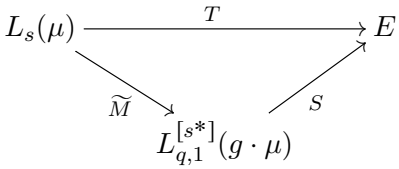
(2) There exists $C' > 0$ *and a normalized* $g \in L_1(\mu)_+$ *such that for every* $f \in L_s(\mu)$ *and measurable subset* $B \subseteq \Omega$

$$\|T(f\chi_B)\| \leqslant C'\|f\|_{L_s(\mu)}\left(\int_B g\,d\mu\right)^{\frac{1}{q}-\frac{1}{s}}.$$

(3) There exists $C'' > 0$ *and a normalized* $g \in L_1(\mu)_+$ *such that for every* $f \in L_s(\mu)$,

$$\|Tf\| \leqslant C''\|fg^{-\frac{1}{s}}\|_{L^{[s^*]}_{q,1}(g\cdot\mu)}.$$

(4) There exists $C'' > 0$ *and a normalized* $g \in L_1(\mu)_+$ *such that* $T = S\widetilde{M}$, *where* $\widetilde{M} : L_s(\mu) \to L^{[s^*]}_{q,1}(g\cdot\mu)$ *is the multiplication operator by* $g^{-\frac{1}{s}}$ *and* $S : L^{[s^*]}_{q,1}(g\cdot\mu) \to E$ *is bounded with* $\|S\| \leqslant C''$.



Moreover, $C \leqslant C'' \leqslant C' \leqslant (1-\frac{s^*}{q^*})^{\frac{1}{q^*}-\frac{1}{s^*}}C$.

For concave operators defined on L_∞, or more generally, on $C(K)$-spaces, the analogues of the two last results are Pietsch's Factorization Theorem (see for instance [120, Theorem 9.2]) and Pisier's generalization for $(p,1)$-concave operators [108, Theorem 2.4]. This is a consequence of the simple fact that an operator defined on a $C(K)$-space is (p,q)-concave if and only if it is (p,q)-summing. The previous factorization theorems will be used intensively in Chapter 2, to establish, for instance, a representation result for arbitrary Banach lattices with upper p-estimates in the spirit of [89, Lemma 3.4], or an explicit renorming for the free Banach lattice with upper p-estimates.

Focusing now on p-convexity, there exists an abstract procedure to generate a p-convex Banach lattice from an arbitrary Banach lattice, called *p-convexification*. The idea behind this technique is to emulate the fact that we can generate $L_p(\mu)$ for any $1 < p < \infty$ from L_1 by sending every function $f \in L_1$ to $f^{\frac{1}{p}} := \operatorname{sgn}(f)|f|^{\frac{1}{p}}$. This map is clearly non linear, but the image of L_1 is L_p. When instead of L_1 we have an arbitrary Banach lattice X, we cannot raise every element of X to the power $\frac{1}{p}$, as this is not a positively homogeneous map. However, there is a way of circumventing this issue by redefining the linear operations of sum and multiplication by scalars in X using functional calculus, so that the resulting space is a p-convex Banach lattice. This process can be reversed, in the same way that we can recover L_1 from L_p by raising every element to the power p (preserving the sign). It should be noted that this procedure, called *p-concavification*, will provide a Banach lattice only if the input space is p-convex. In general, it will only generate a quasi-Banach lattice. This is the case, for instance, of L_q for $q < p$: the p-concavification of L_q is $L_{\frac{q}{p}}$, which cannot be endowed with a Banach lattice norm. The following proposition summarizes both constructions. The details can be found in [88, Section 1.d]. Recall that in this context, for every $0 < \sigma < \infty$ we write $t^\sigma := \operatorname{sgn}(t)|t|^\sigma$.

Proposition 1.2.7. *Let X be a Banach lattice. For every $0 < \sigma < \infty$, define for every $x, y \in X$ and $\alpha \in \mathbb{R}$ the operations*

$$x \oplus_\sigma y := (x^\sigma + y^\sigma)^{\frac{1}{\sigma}} \quad \text{and} \quad \alpha \odot_\sigma x := \alpha^{\frac{1}{\sigma}} \cdot x.$$

Let $1 < p < \infty$.

(1) The set X endowed with the linear operations $\oplus_{\frac{1}{p}}$ and $\odot_{\frac{1}{p}}$, the same order of X and the norm $|||\cdot||| := \|\cdot\|^{\frac{1}{p}}$ is a p-convex Banach lattice with p-convexity constant 1, called the p-convexification *of X and denoted $X^{(p)}$.*

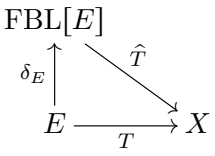
(2) If X is p-convex, the set X endowed with the linear operations $\oplus_p$ and $\odot_p$ and the same order of X can be provided with a norm $|||\cdot|||$ that is $K^{(p)}(X)^p$-equivalent to $\|\cdot\|^p$, in such a way that the resulting space is a Banach lattice, called the p-concavification *of X and denoted $X_{(p)}$.*

It should be noted that if a Banach lattice X has already some additional concavity or convexity properties, or satisfies an upper or lower estimate, then $X^{(p)}$ and $X_{(p)}$ have the same properties modified by a factor of p and $\frac{1}{p}$, respectively, with some natural adaptations on the constants. These techniques of p-convexification and p-concavification can be used to obtain renorming results that allow us to assume that the constants of convexity, concavity and upper and lower estimates of our Banach lattice are 1 (see [88, Proposition 1.d.8 and Lemma 1.f.11]). For instance, if we start with a p-convex Banach lattice X, then $(X_{(p)})^{(p)}$ provides a $K^{(p)}(X)$-equivalent renorming of X with p-convexity constant 1. In particular, if $K^{(p)}(X) = 1$, both norms coincide. Finally, observe that since lattice homomorphisms preserve functional calculus, a lattice homomorphism between Banach lattices is also a lattice homomorphism between the corresponding p-convexifications or p-concavifications. Another useful property is the following: if $T : X \to Y$ is a lattice homomorphism between (p-convex) Banach lattices and we p-convexify (p-concavify) the spaces, the map T is again a lattice homomorphism with respect to the new Banach lattice structure. This is a consequence of the fact that lattice homorphisms preserve functional calculus.

1.3 Free Banach lattices

Free Banach lattices were first introduced by de Patger and Wickstead in 2015. In [106], they show the existence of the *free Banach lattice generated by a set* A, which is a Banach lattice $\mathrm{FBL}(A)$ together with a bounded map $j : A \to \mathrm{FBL}(A)$ satisfying that for every Banach lattice X and bounded map $T : A \to X$ there exists a unique lattice homomorphism extension $\widehat{T} : \mathrm{FBL}(A) \to A$ satisfying that $\widehat{T}j = T$ and $\|\widehat{T}\| = \sup_{a\in A} \|Ta\|$. The construction is based on the *free vector lattice over a set* A, a well known construction from the 60's [16, 28] that admits an explicit representation as the sublattice of $\mathbb{R}^{\mathbb{R}^A}$ generated by the evaluation functions $\delta_a(x^*) = x^*(a)$, $x^* \in \mathbb{R}^A$, with $a \in A$. The authors defined a norm on $\mathrm{FVL}(A)$ such that the completion satisfies the desired universal property. In this seminal paper, they also study some general properties of $\mathrm{FBL}(A)$, characterize the number of generators of A in terms of Banach lattice properties of $\mathrm{FBL}(A)$, and use the new tool to study projectivity in the category of Banach lattices with lattice homomorphisms (see [106, Definition 10.1]).

Motivated by this work, Avilés, Rodríguez and Tradacete introduced in [11] the *free Banach lattice generated by a Banach space* E. This object consist of a Banach lattice $\mathrm{FBL}[E]$ and a linear isometric embedding $\phi_E : E \to \mathrm{FBL}[E]$ satisfying that for every Banach lattice X and every operator $T : E \to X$ there exists a unique extension $\widehat{T} : \mathrm{FBL}[E] \to X$ as a lattice homomorphism such that $\widehat{T}\phi_E = T$ and moreover $\|\widehat{T}\| = \|T\|$.



This object generalizes the free Banach lattice generated by a set introduced by de Patger and Wickstead: if A is a set, then $\mathrm{FBL}(A)$ is lattice isometric to $\mathrm{FBL}[\ell_1(A)]$. In other words, the free Banach lattice generated by a set is the composition of two free objects: the free Banach space generated by the set and the free Banach lattice generated by a Banach space.

Another important contribution of [11] is that an explicit expression for the norm of $\mathrm{FBL}[E]$ is provided. This allows them to represent $\mathrm{FBL}[E]$ as a Banach lattice of functions. The construction goes as follows: Let

$$H[E] := \{f : E^* \to \mathbb{R} \,:\, f \text{ positively homogeneous}\}$$

(recall the definition of a positively homogeneous function from Section 1.1), which becomes a vector lattice when it is endowed with the pointwise order and lattice operations (i.e., $f \in H[E]_+$ if and only if $f(x^*) \geqslant 0$ for every $x^* \in E^*$). Observe that the functions of $H[E]$ are uniquely determined by their values on S_{E^*}, so $H[E]$ can be identified with $\mathbb{R}^{S_{E^*}}$. We define for every $f \in H[E]$ the expression

$$\|f\|_{H_1[E]} := \sup\left\{\sum_{i=1}^m |f(x_i^*)| : m \in \mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E} \sum_{i=1}^m |x_i^*(x)| \leqslant 1\right\}. \tag{1.7}$$

and consider the set

$$H_1[E] := \{f \in H[E] : \|f\|_{H_1[E]} < \infty\}.$$

It is standard to check that $(H_1[E], \|\cdot\|_{H_1[E]})$ is a Banach lattice. Moreover, the map

$$\begin{aligned} \phi_E : E &\longrightarrow H_1[E] \\ x &\longmapsto \delta_x(x^*) = x^*(x), x^* \in E^* \end{aligned}$$

defines a linear isometric embedding of E into $H_1[E]$. It turns out that the sublattice generated by the image of ϕ_E satisfies the universal property of the *free vector lattice generated by a vector space*: every linear map T from E to an (Archimedean) vector lattice X extends uniquely to a lattice homomorphism from $\widehat{T} : \mathrm{lat}(\phi_E(E)) \to X$ in such a way that $\widehat{T}\phi_E = T$. Hence, we will denote the sublattice by $\mathrm{FVL}[E] := \mathrm{lat}(\phi_E(E))$. The extension mechanism is already prescribed by the structure of $\mathrm{FVL}[E]$: if $f = F(\delta_{z_1}, \ldots, \delta_{z_k})$ is described by a finite lattice-linear expression F evaluated on a finite number of elements $\delta_{z_1}, \ldots, \delta_{z_k} \in \phi_E(E)$, then its image by the lattice homomorphism $\widehat{T}$ must be $\widehat{T}f = F(Tz_1, \ldots, Tz_k)$. However, the representation of f is not necessarily unique, but it can be showed that the map $\widehat{T}$ is well defined. An additional argument shows that if X is a Banach lattice and $T : E \to X$ is bounded, then $\widehat{T}$ is bounded from $(\mathrm{FVL}[E], \|\cdot\|_{H_1[E]})$ to X. Hence, it can be extended to the closure of $\mathrm{FVL}[E]$ in $H_1[E]$. Therefore, the Banach lattice $\overline{\mathrm{FVL}[E]}^{H_1[E]}$ satisfies the universal property of the free Banach lattice generated by E [11, Theorem 2.5], so we denote it by $\mathrm{FBL}[E]$.

The next step in the history of free Banach lattices comes with the introduction of the free Banach lattice generated by a lattice [12]. Recall that a *lattice* is a partially ordered set with an order that admits two operations of supremum ($\vee$) and infimum ($\wedge$), and a *lattice homomorphism* in this setting is a map between lattices that preserve the lattice operations. In contrast with the situation of the free Banach lattice generated by a Banach space, where the forgetful functor deletes the lattice structure of a Banach lattice to present it as a Banach space, for the free Banach lattice generated by a lattice the "forgotten" information corresponds to the linear and normed structure. More specifically, the *free Banach lattice generated by a lattice* $\mathbb{L}$ is a Banach lattice $\mathrm{FBL}\langle\mathbb{L}\rangle$ and a bounded lattice homomorphism $\phi : \mathbb{L} \to \mathrm{FBL}\langle\mathbb{L}\rangle$ such that every bounded lattice homomorphism $T : \mathbb{L} \to X$, with X a Banach lattice, extends uniquely to a linear lattice homomorphism $\widehat{T} : \mathrm{FBL}\langle\mathbb{L}\rangle \to X$ such that $\widehat{T}\phi = T$ and $\|\widehat{T}\| = \sup_{x\in\mathbb{L}} \|Tx\|$. The authors of [12] provide an explicit description of $\mathrm{FBL}\langle\mathbb{L}\rangle$, which is actually isomorphic to an AM-space (see [8, Theorem 3.9] or [27, Proposition 7.4] for a direct proof). Recent works on free Banach lattices over lattices have focused on studying projectivity [7, 13], the embedding problem (see Section 2.3.2 for a statement in the setting of free Banach lattices over Banach spaces) [8] and norm-attaining lattice homomorphisms [27].

Moving back to free Banach lattices over Banach spaces, there has been some modifications of the original definition that have yielded free Banach lattices in other relevant classes of Banach lattices. Namely, in [75] the authors showed the existence of *free Banach lattices under convexity conditions generated by a Banach space* E, which included as particular cases the *free p-convex Banach lattice over* E, denoted $\mathrm{FBL}^p[E]$, and the *free Banach lattice with upper p-estimates over* E, denoted $\mathrm{FBL}^{p,\infty}[E]$. As one could expect, given $1 \leqslant p \leqslant \infty$, $\mathrm{FBL}^p[E]$ is a p-convex Banach lattice with constant 1, with a linear isometric embedding $\phi_E : E \to \mathrm{FBL}^p[E]$, satisfying that for every operator $T : E \to X$, where X is a p-convex Banach lattice with constant 1, there exists a unique extension as a lattice homomorphism $\widehat{T} : \mathrm{FBL}^p[E] \to X$ with $\|\widehat{T}\| = \|T\|$. The same definition, replacing p-convexity by upper p-estimates, applies for $\mathrm{FBL}^{p,\infty}[E]$. Note that the case $p = 1$ reduces to the original $\mathrm{FBL}[E]$, since every Banach lattice is 1-convex with constant 1. As it is customary, the p-convex setting turned out to be more accessible than the upper p-estimates setting, since the authors of [75] were able to provide an explicit construction for $\mathrm{FBL}^p[E]$, but not for $\mathrm{FBL}^{p,\infty}[E]$. The description they found was similar to [11]: for every $f \in H[E]$, they defined the norm

$$\|f\|_{H_p[E]} := \sup\left\{ \left(\sum_{i=1}^{m} |f(x_i^*)|^p \right)^{\frac{1}{p}} : m \in \mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E} \left(\sum_{i=1}^{m} |x_i^*(x)|^p \right)^{\frac{1}{p}} \leqslant 1 \right\} \tag{1.8}$$

for $1 \leqslant p < \infty$ and

$$\|f\|_{H_\infty[E]} := \sup_{x^*\in B_{E^*}} |f(x^*)| \tag{1.9}$$

for $p = \infty$, and considered the Banach lattice

$$H_p[E] := \{f \in H[E] : \|f\|_{H_p[E]} < \infty\},$$

which is p-convex with constant 1. This Banach lattice can be understood as a space of non-linear positively homogeneous p-summing functions from E^* to $\mathbb{R}$. The operator ϕ_E previously defined is again an isometric embedding, and the closure of $\mathrm{FVL}[E]$ under the norm $\|\cdot\|_{H_p[E]}$ behaves as the free p-convex Banach lattice over E. In particular, this construction allows them to show that $\mathrm{FBL}^\infty[E]$ coincides with $C_{ph}(B_{E^*})$, the space of weak* continuous and positively homogeneous functions on B_{E^*} endowed with the supremum norm, and contains $\mathrm{FBL}^p[E]$ for every $1 \leqslant p < \infty$. On the other hand, the proof of the existence of $\mathrm{FBL}^{p,\infty}[E]$, which is valid in general for any of such convexity conditions, in particular p-convexity, follows an abstract approach, more similar to the procedure described in [121] or the one that will be used in Section 2.1.1 to construct free Banach lattices associated to arbitrary classes of Banach lattice. The idea of the proof is to define a lattice norm on $\mathrm{FVL}[E]$ which is maximal among all the seminorms satisfying certain convexity condition (in this case, the upper p-estimates inequality) and complete it to obtain the free Banach lattice under that particular convexity condition. As it was explained in the introduction, it cannot be guaranteed that the inclusion map of $\mathrm{FVL}[E]$ into $C_{ph}(B_{E^*})$ will still be injective after taking the completion. This problem of finding an explicit representation for $\mathrm{FBL}^{p,\infty}[E]$ was later reiterated in [103], an extensive memoir on free Banach lattices where the spaces $\mathrm{FBL}^p[E]$ are studied in full detail. Chapter 2 in this dissertation will be partly devoted to providing a solution to this problem. Another variation of the free Banach lattice over a Banach space that has appeared in the last years is the *free complex Banach lattice*, denoted $\mathrm{FBL}_{\mathbb{C}}[E]$, which adapts the construction carried out in the reals to the setting of complex scalars [68].

Throughout this dissertation, we will focus on free Banach lattices generated by Banach spaces, leaving aside their analogues for lattices and sets. This is the object that has attracted most of the attention of the Banach lattice community in recent years. For instance, it is shown in [10] that $\mathrm{FBL}[E]$ has the *countable chain condition (ccc)*, that is, every disjoint family in $\mathrm{FBL}[E]$ must be countable. Another example is [9], where free Banach lattices generated by Banach spaces that are also Banach lattices are studied. If X is a Banach lattice, the universal property of $\mathrm{FBL}[X]$ implies the existence of an onto lattice homomorphism $\widehat{id_X} : \mathrm{FBL}[X] \to X$ such that $\widehat{id_X}\phi_X = id_X$. Motivated by the work of Godefroy and Kalton on Lipschitz free spaces [61], the authors of [9] study the *lattice lifting property*, that is, the Banach lattices X that admit a lattice embedding $\alpha : X \to \mathrm{FBL}[X]$ such that $\widehat{id_X}\alpha = id_X$. They show that projective Banach lattices, in particular certain $C(K)$-spaces, Banach spaces with a 1-unconditional basis that induces the order and free Banach lattices satisfy this lattice lifting property.

Another interesting feature of free objects in general, and in particular free Banach lattices, is that they establish a bridge that allows us to characterize certain properties of one category in terms of properties of the other category. The memoir [103] contains some equivalences between Banach space properties of E and Banach lattice properties of $\mathrm{FBL}[E]$ that can be added to this "dictionary":

Theorem 1.3.1. *Let E be a Banach space. Then:*

(1) E has finite dimension if and only if $\mathrm{FBL}[E]$ has a strong unit.

(2) E is separable if and only if $\mathrm{FBL}[E]$ has a quasi-interior point.

(3) E contains a complemented subspace isomorphic to ℓ_1 if and only if $\mathrm{FBL}[E]$ contains a sublattice isomorphic to ℓ_1.

(4) The identity operator $id_{E^*} : E^* \to E^*$ is $(p^*, 1)$-summing if and only if $\mathrm{FBL}[E]$ satisfies an upper p-estimate.

This program can also be extended to the study of properties of the corresponding morphisms. Given Banach spaces E and F and $T : E \to F$ a linear and bounded operator, we can consider the lattice homomorphism $\overline{T} = \widehat{\phi_F T} : \mathrm{FBL}[E] \to \mathrm{FBL}[F]$, which makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{FBL}[E] & \xrightarrow{\overline{T}} & \mathrm{FBL}[F] \\ {\scriptstyle \phi_E}\uparrow & & \uparrow{\scriptstyle \phi_F} \\ E & \xrightarrow[T]{} & F \end{array}$$

Then, T and $\overline{T}$ are related in the following way [103, Section 3.1]:

Theorem 1.3.2. *Let E and F be Banach spaces and $T : E \to F$ and $\overline{T} : \mathrm{FBL}[E] \to \mathrm{FBL}[F]$ as above. Then:*

(1) $\overline{T}$ is injective if and only if T is injective.

(2) $\overline{T}$ has dense range if and only if T has dense range.

(3) $\overline{T}$ is onto if and only if T is onto.

(4) $\overline{T}$ is an embedding if and only if T is an embedding and there exists some $C > 0$ such that for every $n \in \mathbb{N}$, $\varepsilon > 0$ and $S : E \to \ell_1^n$ there exists an extension $\widetilde{S} : E \to \ell_1^n$ with $\|\widetilde{S}\| \leqslant C(1+\varepsilon)\|S\|$.

In Corollary 3.5.3 we will establish a similar characterization for the operators that do not preserve a copy of ℓ_1 complementably. Note that the previous result implies that if E and F are isomorphic, then $\mathrm{FBL}[E]$ and $\mathrm{FBL}[F]$ are lattice isomorphic. Nevertheless, the reverse question is still open: namely, suppose that $\mathrm{FBL}[E]$ and $\mathrm{FBL}[F]$ are lattice isomorphic, are E and F necessarily linearly isomorphic? Note that in the complex case, this question has a negative answer (see [68]).

We conclude this introduction to free Banach lattices reviewing some of their applications to Banach lattice theory. More detailed explanations can be found in the survey paper [53]. As mentioned earlier, free Banach lattices over sets were used in [106, Theorem 10.3] to characterize projective Banach lattices. Moreover, $\mathrm{FBL}[\ell_2(\Gamma)]$ with Γ uncountable was the first example of a Banach lattice that is weakly compactly generated as a Banach lattice (LWCG), but not as a Banach space (WCG) [11, Section 5]. This answered negatively a question asked by Diestel during a conference in 2011. Another counterexample, this time to a question raised in [119, Remark 4.4] concerning the existence of a subspace of a Banach lattice with no bibasic sequence, was provided in [103, Theorem 7.5]. Certain free Banach lattices have also been the first examples of Banach lattices with lattice homomorphisms to $\mathbb{R}$ that do not attain their norm [42]. Finally, free Banach lattices have been used to construct other categorical objects in the category of Banach lattices, such as isometric push-outs [14]. This topic will be further developed in Section 2.3.

Chapter 2

Free Banach lattices and upper p-estimates

In general, much more is known about p-convexity and p-concavity than upper and lower p-estimates. This is partly due to the fact that L_p-spaces, which model p-convexity and p-concavity, are better understood and easier to work with than $L_{p,\infty}$ of $L_{p,1}$, the spaces that play an analogous role for upper and lower p-estimates, respectively. Additionally, some techniques that are very useful in the former setting, such as p-convexifications and p-concavifications, do not have an analogue in the latter. The aim of this chapter is to clarify some questions whose answers are well-known in the p-convex setting, but remain unknown in the framework of upper p-estimates. It is based on:

[54] E. García-Sánchez, D. H. Leung, M.A. Taylor, P. Tradacete, *Banach lattices with upper p-estimates: free and injective objects*. Math. Ann. **391** (2025), 3363–3398.

[55] E. García-Sánchez, D. H. Leung, M.A. Taylor, P. Tradacete, *Banach lattices with upper p-estimates: factorizations*. In preparation.

Our main motivation stems from the absence of an explicit construction for the free Banach lattice with upper p-estimates, whose existence was shown in [75] by abstract methods. To tackle this question, we define in Section 2.1 free Banach lattices over a Banach space E associated to arbitrary classes of Banach lattices and study their properties. These Banach lattices are constructed by completing certain lattice norms defined on $\mathrm{FVL}[E]$, the free vector lattice over E, following a similar approach to [75]. Therefore, we face the same problem of determining whether the injective map that represents $\mathrm{FVL}[E]$ as a sublattice of $C_{ph}(B_{E^*})$, the space of continuous and positively homogenous functions on B_{E^*} endowed with the weak* topology, admits an injective extension to such completion (see [75, Remark 6.2]). This question can be understood as a weaker instance of the general problem of characterizing those normed vector lattices such that continuous and injective lattice homomorphisms defined on them extend injectively to the completion. We will study this problem in Section 2.1.2.

In Section 2.2 we use free Banach lattices associated to certain classes of Banach lattices to provide an explicit representation of, $\mathrm{FBL}^{p,\infty}[E]$, the free Banach lattice with upper p-estimates. This is done by observing that the class of Banach lattices with upper p-estimates coincides with the class of sublattices of ℓ_∞-sums of $L_{p,\infty}$-spaces (see Theorem 2.2.1). However, this identification is only isomorphic, not isometric, as we show in Section 2.2.2. This fact, combined with an argument developed in Section 2.2.3 that allows us to represent explicitly the free Banach lattice associated to the singleton class $\{L_{p,\infty}(\mu)\}$, yields the main result of the section, Theorem 2.2.14, which provides an explicit equivalent renorming for the free Banach lattice with upper p-estimates. The same analysis is carried out for the class of p-convex Banach lattices, providing an alternative proof of [75, Theorem 6.1]. Additionally, in Section 2.2.5 we characterize all classes of Banach lattices whose associated free Banach lattice is $\mathrm{FBL}^{p,\infty}[E]$ or $\mathrm{FBL}^{p}[E]$, respectively. Finally, we devote Section 2.2.6 to determine whether the multiple renormings that can be equipped on $L_{p,\infty}$ are responsible for the universal isomorphism constant that appears in Theorem 2.2.1 and Theorem 2.2.14.

Next, we devote Section 2.3 to solve the embedding problem for free Banach lattices with upper p-estimates. If $\iota : E \to F$ is an isometric embedding between Banach spaces, this problem aims to characterize when the canonical extension $\bar{\iota} : \mathrm{FBL}^{p,\infty}[E] \to \mathrm{FBL}^{p,\infty}[F]$ is an embedding too. Our approach requires the construction of push-outs in the class of Banach lattices with upper p-estimates that preserve embeddings, which is carried out in Section 2.3.1. We then use these push-outs to show Theorem 2.3.6. The last part of the section is devoted to analyzing the differences between the solution obtained in [103, Theorem 3.7] for p-convex Banach lattices and our solution, which stems from fundamental differences in the injectivity properties of ℓ_p and $\ell_{p,\infty}$ (see Theorems 2.3.7 and 2.3.10).

Finally, in Section 2.4 we extend some results from Reisner [111] on factorization through p-convex and q-concave operators to the setting of operators which factor through (p, p_2)-convex and (q, q_2)-concave operators, for $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$.

2.1 An abstract construction of free Banach lattices and the problem of the explicit representation

2.1.1 The free Banach lattice associated to a class of Banach lattices

Let $\mathcal{C}$ be a (non-empty and non-zero) class of Banach lattices. For instance, $\mathcal{C}$ could be the class of $L_{p,\infty}$ spaces or the class of Banach lattices satisfying an upper p-estimate with constant 1. Given a Banach space E, define a norm $\rho_{\mathcal{C}}$ on $\mathrm{FVL}[E]$ by

$$\rho_{\mathcal{C}}(f) = \sup\big\{\|\widehat{T}f\|_X : X \in \mathcal{C},\ T : E \to X \text{ is a linear contraction}\big\}.$$

Note that $\rho_{\mathcal{C}}(f)$ is finite for every $f \in \mathrm{FVL}[E]$. Indeed, by (1.1), every $f \in \mathrm{FVL}[E]$ can be represented as

$$f = \bigvee_{i=1}^{m} \delta_{x_i} - \bigvee_{j=1}^{n} \delta_{y_j}$$

for some $(x_i)_{i=1}^m, (y_j)_{j=1}^n \subseteq E$. Therefore,

$$|\widehat{T}f| = \left|\bigvee_{i=1}^{m} Tx_i - \bigvee_{j=1}^{n} Ty_j\right| \leqslant \sum_{i=1}^{m} |Tx_i| + \sum_{j=1}^{n} |Ty_j|$$

and hence

$$\|\widehat{T}f\|_X \leqslant \sum_{i=1}^{m} \|x_i\|_X + \sum_{j=1}^{n} \|y_j\|_X < \infty,$$

which does not depend on the map T. It is standard to check that $\rho_{\mathcal{C}}$ is a lattice norm on $\mathrm{FVL}[E]$. The *free Banach lattice generated by E associated to $\mathcal{C}$* is the completion of $(\mathrm{FVL}[E], \rho_{\mathcal{C}})$ and is denoted by $\mathrm{FBL}^{\mathcal{C}}[E]$. It is easy to see that $\phi_E : E \to \mathrm{FBL}^{\mathcal{C}}[E]$ is a linear isometric embedding. Define an enlarged class of Banach lattices $\overline{\mathcal{C}}$ so that $Y \in \overline{\mathcal{C}}$ if and only if Y is lattice isometric to a closed sublattice of a Banach lattice of the form

$$\ell_\infty(X_\gamma)_{\gamma\in\Gamma} = \{(x_\gamma)_{\gamma\in\Gamma} : x_\gamma \in X_\gamma, \|(x_\gamma)_{\gamma\in\Gamma}\|_\infty := \sup_{\gamma\in\Gamma} \|x_\gamma\| < \infty\},$$

where $X_\gamma \in \mathcal{C}$ for all $\gamma \in \Gamma$. Note that we do not require the X_γ's to be distinct.

Remark 2.1.1. *Let $\mathcal{C}_1$ be the class of all Banach lattices. Then, $\mathrm{FBL}^{\mathcal{C}_1}[E]$ coincides with $\mathrm{FBL}[E]$, the free Banach lattice generated by E introduced in [11].*

Remark 2.1.2. *Observe that for every $f \in \mathrm{FVL}[E] \subseteq H[E]$, $\rho_{\mathcal{C}}(f)$ dominates $\|f\|_{H_\infty[E]}$ defined in (1.9), so there exists a continuous and injective lattice homomorphism $j : (\mathrm{FVL}[E], \rho_{\mathcal{C}}) \to C_{ph}(B_{E^*})$ given by the restriction to B_{E^*}.*

Proposition 2.1.3. *Suppose that $Y \in \overline{\mathcal{C}}$ and $T : E \to Y$ is a bounded linear operator. Then $\widehat{T} : (\mathrm{FVL}[E], \rho_{\mathcal{C}}) \to Y$ is a continuous vector lattice homomorphism such that $\|\widehat{T}\| = \|T\|$.*

Proof. By the universal property of the free vector lattice, there exists a lattice homomorphism $\widehat{T} : \mathrm{FVL}[E] \to Y$ that extends T. Note that $\phi_E : E \to (\mathrm{FVL}[E], \rho_{\mathcal{C}})$ is an isometric embedding and $\widehat{T}\phi_E = T$. Hence, $\|T\| \leqslant \|\widehat{T}\|$. On the other hand, suppose that Y is a closed sublattice of X, where $X = \ell_\infty(X_\gamma)_{\gamma\in\Gamma}$, with $X_\gamma \in \mathcal{C}$ for all γ. Let π_γ be the projection from X onto X_γ. Then $\pi_\gamma T : E \to X_\gamma$ is a bounded linear operator with $\|\pi_\gamma T\| \leqslant \|T\|$. By definition of $\rho_{\mathcal{C}}$, $\|\widehat{\pi_\gamma T} f\| \leqslant \|T\|\, \rho_{\mathcal{C}}(f)$ for any $f \in \mathrm{FVL}[E]$. As π_γ is a lattice homomorphism, by the uniqueness of $\widehat{\pi_\gamma T}$, we have $\widehat{\pi_\gamma T} = \pi_\gamma \widehat{T}$. Thus, $\|\widehat{T} f\| = \sup_\gamma \|\pi_\gamma \widehat{T} f\| \leqslant \|T\|\, \rho_{\mathcal{C}}(f)$ for any $f \in \mathrm{FVL}[E]$. This shows that $\|\widehat{T}\| \leqslant \|T\|$. □

Obviously, the operator $\widehat{T}$ in Proposition 2.1.3 extends to a lattice homomorphism from $\mathrm{FBL}^{\mathcal{C}}[E]$ to Y, still denoted by $\widehat{T}$.

Corollary 2.1.4. *Let E be a Banach space. For any non-empty class $\mathcal{C}$ of Banach lattices, the norms $\rho_{\mathcal{C}}$ and $\rho_{\overline{\mathcal{C}}}$ agree on $\mathrm{FVL}[E]$. In particular, $\mathrm{FBL}^{\mathcal{C}}[E] = \mathrm{FBL}^{\overline{\mathcal{C}}}[E]$ as Banach lattices.*

Proof. Since $\mathcal{C} \subseteq \overline{\mathcal{C}}$ it is clear that $\rho_{\mathcal{C}} \leqslant \rho_{\overline{\mathcal{C}}}$. On the other hand, suppose that $f \in \mathrm{FVL}[E]$ and $\varepsilon > 0$ are given. There exists $Y \in \overline{\mathcal{C}}$ and a linear contraction $T : E \to Y$ such that $\rho_{\overline{\mathcal{C}}}(f) \leqslant (1+\varepsilon)\|\widehat{T} f\|$. By Proposition 2.1.3, $\|\widehat{T} f\| \leqslant \|T\|\, \rho_{\mathcal{C}}(f) = \rho_{\mathcal{C}}(f)$. It follows that $\rho_{\overline{\mathcal{C}}} \leqslant \rho_{\mathcal{C}}$. □

The class $\overline{\mathcal{C}}$ can be understood as a "completion" of the class $\mathcal{C}$ so that it remains closed under free objects:

Proposition 2.1.5. *The Banach lattice $\mathrm{FBL}^{\mathcal{C}}[E]$ belongs to the class $\overline{\mathcal{C}}$.*

Proof. For any $f \in \mathrm{FVL}[E]$ and $n \in \mathbb{N}$, there exists $X_{f,n} \in \mathcal{C}$ and a linear contraction $T_{f,n} : E \to X_{f,n}$ so that $\rho_{\mathcal{C}}(f) \leqslant (1 + \frac{1}{n})\|\widehat{T_{f,n}} f\|$. Let $X = \ell_\infty(X_{f,n})_{f\in\mathrm{FVL}[E],n\in\mathbb{N}}$. Clearly, $X \in \overline{\mathcal{C}}$. We define $j : \mathrm{FBL}^{\mathcal{C}}[E] \to X$ by $jg = (\widehat{T_{f,n}} g)_{f,n}$. Obviously, j is a lattice homomorphism. Let $g \in \mathrm{FVL}[E]$. By definition, $\rho_{\mathcal{C}}(g) \geqslant \|\widehat{T_{f,n}} g\|$ for all (f, n). Hence, j is a linear contraction. On the other hand, $\|jg\| \geqslant \|\widehat{T_{g,n}} g\| \geqslant \frac{n}{n+1}\, \rho_{\mathcal{C}}(g)$ for all n, implying that $\|jg\| \geqslant \rho_{\mathcal{C}}(g)$. This shows that $j : \mathrm{FVL}[E] \to X$ is a lattice isometry. Consequently, $j : \mathrm{FBL}^{\mathcal{C}}[E] \to X$ is as well. □

The above discussion motivates the question of finding necessary and sufficient conditions on classes of Banach lattices $\mathcal{C}$ and $\mathcal{D}$ so that $\rho_{\mathcal{C}}$ and $\rho_{\mathcal{D}}$ coincide, or at least, are equivalent. In Section 2.2.5, after some tools have been developed, we will be able to provide a complete solution to this question when $\mathcal{C}$ is either the class of p-convex Banach lattices with constant 1, or the class of Banach lattices satisfying an upper p-estimate with constant 1.

2.1.2 Extension of injective homomorphisms to the norm completion

It is not obvious in general that a representation of a normed free object can be extended to its completion. In particular, in Section 2.1.1 we defined $\mathrm{FBL}^{\mathcal{C}}[E]$ as the completion of $(\mathrm{FVL}[E], \rho_{\mathcal{C}})$,

for which there is a continuous lattice homomorphic injection $(\mathrm{FVL}[E], \rho_{\mathcal{C}}) \hookrightarrow C(B_{E^*})$ (see Remark 2.1.2). However, it is not clear in general whether the continuous extension to $\mathrm{FBL}^{\mathcal{C}}[E]$ of this canonical inclusion will still be an injection. This is known to be true when $\mathrm{FBL}^{\mathcal{C}}[E] = \mathrm{FBL}^{p}[E]$ with $1 \leqslant p \leqslant \infty$, that is, when $\mathcal{C}$ is the class of all Banach lattices [11] or when it is the class of p-convex Banach lattices with p-convexity constant 1 [75] (see Theorem 2.2.17 for a complete characterization of the classes such that $\mathrm{FBL}^{\mathcal{C}}[E] = \mathrm{FBL}^{p}[E]$). Theorem 2.2.12 below also provides new cases where $\mathrm{FBL}^{\mathcal{C}}[E]$ is fully contained in $C(B_{E^*})$, but in all of these situations the argument heavily relies on an explicit representation of the norm $\rho_{\mathcal{C}}$. This is the reason why the question of finding a function lattice representation for $\mathrm{FBL}^{p,\infty}[E]$ asked in [75, Remark 6.2] has remained unanswered until the present. In [37] it is shown that under certain order continuity assumptions (which are not fulfilled in our situation) continuous lattice homomorphic injections retain their injectivity after being extended to the norm completion. Moreover, it is wrongly claimed in [116, Lemma 14] that *any* continuous lattice homomorphic injection from a normed vector lattice to a Banach lattice lifts injectively to the completion. Note that if this were the case, it would provide a straightforward solution to the question above. In this subsection, we give a complete characterization of when continuous injective lattice homomorphisms extend injectively to the completion, and, in particular, show that this need not always be the case. Afterwards, in Section 2.2.4, we will provide an explicit representation for the norm of $\mathrm{FBL}^{p,\infty}[E]$, solving affirmatively the question raised in [75, Remark 6.2].

Throughout this subsection X will be a vector lattice equipped with a lattice norm and Y will be a Banach lattice. We denote the norm completion of X by $\widehat{X}$ and consider polar sets with respect to the duality $\langle \widehat{X}, X^* \rangle$: for $A \subseteq \widehat{X}$, set $A^\circ = \{x^* \in X^* : x^*(a) \leqslant 1,\ \forall a \in A\}$.

Theorem 2.1.6. *Let X be a normed vector lattice equipped with a lattice norm and let $\widehat{X}$ be its norm completion. The following are equivalent.*

(1) For any Banach lattice Y, any bounded injective lattice homomorphism $T : X \to Y$ extends to an injective lattice homomorphism $\widehat{T} : \widehat{X} \to Y$.

(2) For any $0 < x \in \widehat{X}$, $I_x \cap X \neq \{0\}$, where I_x is the closed ideal in $\widehat{X}$ generated by x.

Proof. (1)⇒(2) Suppose that there exists $0 < x \in \widehat{X}$ so that $I_x \cap X = \{0\}$. By [114, Proposition II.4.7], $(I_x)^\circ$ is an ideal in X^*. Define $\rho : X \to \mathbb{R}$ by

$$\rho(u) = \sup_{x^* \in (I_x)^\circ \cap B_{X^*}} |x^*(u)|.$$

Since $(I_x)^\circ \cap B_{X^*}$ is a solid subset of B_{X^*}, ρ is a lattice semi-norm on X such that $\rho(\cdot) \leqslant \|\cdot\|$. If $0 < u \in X$, then $u \notin I_x$. Since I_x is a closed subspace of $\widehat{X}$, there exists $x^* \in (I_x)^\circ \cap B_{X^*}$ such that $x^*(u) \neq 0$. Hence, $\rho(u) > 0$. This shows that ρ is a norm on X.

Denote by X_ρ the vector lattice X normed by ρ and let $i : X \to X_\rho \subseteq \widehat{X_\rho}$ be the formal identity, where $\widehat{X_\rho}$ is the norm completion of X_ρ. By construction, i is a bounded linear injective lattice homomorphism. By (1), i extends to an injective bounded linear operator $\widehat{i} : \widehat{X} \to \widehat{X_\rho}$. Denote by $\widehat{\rho}$ the norm on $\widehat{X_\rho}$. By injectivity of $\widehat{i}$, $\widehat{i}x \neq 0$. Let $c = \widehat{\rho}(\widehat{i}x) > 0$. There exists (x_n) in X that converges to x in $\widehat{X}$. By continuity, (ix_n) converges in $\widehat{X_\rho}$ to $\widehat{i}x$. Choose N large enough so that $\widehat{\rho}(ix_N - \widehat{i}x) < \frac{c}{2}$. Then $\rho(ix_N) > \frac{c}{2}$. By definition of ρ, there exists $x^* \in (I_x)^\circ \cap B_{X^*}$ so that $|x^*(x_N)| > \frac{c}{2}$. By definition of ρ again, x^* defines a bounded linear functional on X_ρ with norm $\rho^*(x^*) \leqslant 1$, that extends to a continuous functional z^* defined on $\widehat{X_\rho}$ with $\rho^*(z^*) \leqslant 1$. Moreover,

$$z^*(\widehat{i}x) = \lim_n x^*(ix_n) = \lim_n x^*(x_n) = x^*(x).$$

Hence,

$$|x^*(x)| = |z^*(\widehat{i}x)| \geqslant |x^*(ix_N)| - \widehat{\rho}(ix_N - \widehat{i}x) \cdot \rho^*(z^*) > \frac{c}{2} - \frac{c}{2} = 0.$$

This is impossible since $x \in I_x$ and $x^* \in (I_x)^\circ$.

(2)⇒(1) Let $T : X \to Y$ be as in (1), and assume that its extension $\widehat{T}$ is not injective. There exists some $x \in \widehat{X}$, $x \neq 0$, such that $\widehat{T}x = 0$. Since $\widehat{T}$ is a lattice homomorphism, we can assume that $x > 0$. Let I_x be the closed ideal in $\widehat{X}$ generated by x. By (2), there is a non-zero element $z \in I_x \cap X$. Since I_x is the closure of the ideal generated by x, there exists a sequence $(z_n)_n \subseteq \widehat{X}$ converging to z in $\widehat{X}$ and some positive scalars $(\lambda_n)_n$ such that $0 \leqslant |z_n| \leqslant \lambda_n x$. Therefore,

$$0 \leqslant |\widehat{T}z_n| \leqslant \lambda_n \widehat{T}x = 0$$

and

$$Tz = \widehat{T}z = \lim_n \widehat{T}z_n = 0,$$

so by the injectivity of T we conclude that z must be zero. This is a contradiction. □

The following example taken from [25] illustrates the above situation.

Example 2.1.7. *Let $X \subseteq c_0$ consist of all $(a_n) \in c_0$ such that there exists $m \in \mathbb{N}$ so that $a_n = \frac{a_1}{n}$ for all $n \geqslant m$. Clearly, X is a vector sublattice of c_0 with $e_n \in X$ if $n \geqslant 2$. Moreover, for any $n \geqslant 2$, $x_n := e_1 + \sum_{k=n}^{\infty} \frac{e_k}{k} \in X$ and (x_n) converges to e_1 in c_0. Thus, $e_1 \in \overline{X}$. It follows that $\overline{X} = c_0$, that is, the norm completion of X is c_0. Let $I \subseteq c_0$ be the span generated by e_1, which coincides with the ideal generated by e_1. Clearly $I \cap X = \{0\}$. On the other hand, if P_1 is the band projection onto I and T is the restriction of $id_{c_0} - P_1$ to the sublattice X, then $T : X \to c_0$ is an injective lattice homomorphism, but its extension $\widehat{T} = id_{c_0} - P_1 : \widehat{X} = c_0 \to c_0$ vanishes on e_1, so it is not injective.*

2.2 An explicit representation of the free Banach lattice with upper p-estimates

2.2.1 Representations of p-convex Banach lattices and Banach lattices with upper p-estimates

Taking as a starting point the classical result from [89] that allows us to represent any Banach lattice as a sublattice of an ℓ_∞-sum of L_1-spaces, we will use Maurey's and Pisier's factorization theorems, stated in Section 1.2, to improve it in order to represent p-convex Banach lattices and Banach lattices with upper p-estimates inside of ℓ_∞-sums of L_p-spaces and $L_{p,\infty}$-spaces, respectively.

Theorem 2.2.1. *Let X be a Banach lattice and $1 < p < \infty$.*

(1) If X is p-convex, then there are a family Γ of probability measures and a lattice isomorphism

$$J : X \to \ell_\infty(L_p(\mu))_{\mu\in\Gamma}$$

such that $\|x\| \leqslant \|Jx\| \leqslant K^{(p)}(X)\|x\|$ for all $x \in X$.

(2) If X satisfies an upper p-estimate, then there are a family Γ of probability measures and a lattice isomorphism

$$J : X \to \ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$$

such that $\|x\| \leqslant \|Jx\| \leqslant \gamma_p K^{(\uparrow p)}(X)\|x\|$ for all $x \in X$, where $\gamma_p := (1 - \frac{1}{p})^{\frac{1}{p}-1}$.

Proof. Let $x \in S = \{z \in X_+ : \|z\| = 1\}$. Choose $x^* \in X_+^*$ such that $\|x^*\| = 1$ and $x^*(x) = 1$. Define $\rho_x : X \to \mathbb{R}$ by $\rho_x(z) = x^*(|z|)$. Then ρ_x induces an AL-norm $\widehat{\rho}_x$ on $X/\ker\rho_x$. Let $q_x : X \to X/\ker\rho_x$ be the quotient map. By Kakutani's Representation Theorem for AL-spaces [88, Theorem 1.b.2], we know that the completion of $(X/\ker\rho_x, \widehat{\rho}_x)$ is lattice isometric to $L_1(\mu_x)$ for some measure μ_x, so let us denote by $i_x : X/\ker\rho_x \to L_1(\mu_x)$ the formal inclusion, that satisfies $\|i_x q_x x\|_{L_1(\mu_x)} = \widehat{\rho}_x(q_x x) = \rho_x(x) = 1$. Since X is p-convex or satisfies an upper p-estimate (i.e., it is (p,∞)-convex), depending on the case, and $i_x q_x$ is a contractive lattice homomorphism, it follows that $i_x q_x$ is p-convex or (p,∞)-convex, respectively, with the same constant as X. Therefore, Theorems 1.2.3 and 1.2.4 yield that there exists $g_x \in L_1(\mu_x)_+$ with $\|g_x\|_{L_1(\mu_x)} = 1$, such that $i_x q_x$ factors as $M_x R_x$, with $R_x z = g_x^{-1} i_x q_x z$ for any $z \in X$, M_x is the operator of multiplication by g_x, and

$$\|R_x z\|_{L_p(g_x\cdot\mu_x)} \leqslant K^{(p)}(X)\|x\|, \quad \text{respectively,} \quad \|R_x z\|_{L_{p,\infty}(g_x\cdot\mu_x)} \leqslant \gamma_p K^{(\uparrow p)}(X)\|x\|.$$

Define

$$J : X \to \ell_\infty(L_p(g_x\cdot\mu_x))_{x\in S}, \quad \text{respectively} \quad J : X \to \ell_\infty(L_{p,\infty}(g_x\cdot\mu_x))_{x\in S}$$

by $Jz = (R_x z)_{x\in S}$. It is clear that R_x, and thus, J, are lattice homomorphisms. Moreover, $\|J\| = \sup_{x\in S}\|R_x\|$ is bounded by $K^{(p)}(X)$ or $\gamma_p K^{(\uparrow p)}(X)$, respectively. On the other hand, since M_x is a contraction in both cases, for every $x \in S$ we have that, in the p-convex setting,

$$\|Jx\| \geqslant \|R_x x\|_{L_p(g_x\cdot\mu_x)} \geqslant \|i_x q_x x\|_{L_1(\mu_x)} = 1,$$

and in the (p,∞)-convex setting

$$\|Jx\| \geqslant \|R_x x\|_{L_{p,\infty}(g_x\cdot\mu_x)} \geqslant \|i_x q_x x\|_{L_1(\mu_x)} = 1.$$

It follows that $\|Jx\| \geqslant \|x\|$ for all $x \in X$ in both cases. □

As a corollary, we obtain an alternative proof of the fact that every (p,∞)-convex Banach lattice (and, by [96, Theorem 2.8.7] every (p,∞)-convex operator) is q-convex for every $q < p$, with a constant that explodes as $(p-q)^{-\frac{1}{p}}$ when $q \to p^-$. To our knowledge, there existed two proofs of this fact: The proof presented in [43, Corollary 16.7] exploits a connection between (p,q)-concave operators and (p,q)-summing operators on $C(K)$-spaces (see [43, Theorem 16.5]) and a result from Pisier [108, Theorem 2.4] that generalizes Pietsch's factorization theorem for (p,q)-summing operators (see also [120, Theorem 21.3]). On the other hand, the proof presented in [88, Theorem 1.f.7] uses a probabilistic argument. The argument presented here also relies on results from Pisier (namely, [108, Theorem 1.1] instead of [108, Theorem 2.4]), but has some advantages with respect to the other proofs. First of all, it is more direct than the one presented in [43], as it does not require to go through (p,q)-summing operators. On the other hand, the proof is quite simple and does not require techniques from other areas.

Corollary 2.2.2. *Let $1 < p < \infty$ and assume that X is a Banach lattice satisfying an upper p-estimate. Then it is q-convex for every $1 \leqslant q < p$ with constant $K^{(q)}(X) \leqslant \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \gamma_p K^{(\uparrow p)}(X)$.*

Proof. First of all, by Theorem 2.2.1 we know that X lattice embeds with distortion $\gamma_p K^{(\uparrow p)}(X)$ into a Banach lattice of the form $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$, with μ a probability measure for every $\mu \in \Gamma$. Now, let us fix $1 \leqslant q < p$ and observe that by (1.6) each of the factors $L_{p,\infty}(\mu)$ is lattice isomorphic to $L_{p,\infty}^{[q]}(\mu)$, with distortion $\left(\frac{p}{p-q}\right)^{\frac{1}{q}}$. Now, it suffices to show that $L_{p,\infty}^{[q]}(\mu)$ is q-convex with constant 1: given $f_1,\dots,f_n \in L_{p,\infty}^{[q]}(\mu)$ and a measurable set A with $\mu(A) > 0$, it follows that

$$\mu(A)^{\frac{1}{p}-\frac{1}{q}}\left(\int_A \sum_{i=1}^n |f_i|^q\,d\mu\right)^{\frac{1}{q}} = \left(\sum_{i=1}^n \left(\mu(A)^{\frac{1}{p}-\frac{1}{q}}\left(\int_A |f_i|^q\,d\mu\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}} \leqslant \left(\sum_{i=1}^n \|f\|^q_{L_{p,\infty}^{[q]}(\mu)}\right)^{\frac{1}{q}}.$$

Taking the supremum over all the sets A we obtain that

$$\left\| \left(\sum_{i=1}^{n} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L_{p,\infty}^{[q]}(\mu)} \leqslant \left(\sum_{i=1}^{n} \|f\|^q_{L_{p,\infty}^{[q]}(\mu)} \right)^{\frac{1}{q}}.$$

Since q-convexity is preserved under taking ℓ_∞-sums, sublattices and isomorphisms, we conclude that X is q-convex with constant $K^{(q)}(X) \leqslant \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \gamma_p K^{(\uparrow p)}(X)$. □

The following example illustrates that the reverse implication is not true:

Example 2.2.3. *Let $1 < p < \infty$. The space $X = \ell_{p,\infty}(\ell_p)$, formed by the sequences of vectors $\overline{x} = (x_k)_{k\in\mathbb{N}}$ such that $x_k \in \ell_p$ for every $k \in \mathbb{N}$ and $\|\overline{x}\|_X := \|(\|x_k\|_{\ell_p})_k\|_{\ell_{p,\infty}}$ is finite, does not satisfy an upper p-estimate, but is q-convex with constant $\left(\frac{p}{p-q}\right)^{\frac{1}{q}}$ for every $1 \leqslant q < p$.*

Proof. We start by showing that X is q-convex for any $1 \leqslant q < p$. Recall that both ℓ_p and $\ell_{p,\infty}$ are q-convex with constant 1 and $\left(\frac{p}{p-q}\right)^{\frac{1}{q}}$, respectively. Given $\overline{x}^{(i)} = (x_k^{(i)})_{k\in\mathbb{N},} \in X$, $i = 1, \dots, n$, we have

$$\begin{aligned}
\left\| \left(\sum_{i=1}^{n} |\overline{x}^{(i)}|^q \right)^{\frac{1}{q}} \right\|_X &= \left\| \left(\left(\sum_{i=1}^{n} |x_k^{(i)}|^q \right)^{\frac{1}{q}} \right)_k \right\|_X = \left\| \left(\left\| \left(\sum_{i=1}^{n} |x_k^{(i)}|^q \right)^{\frac{1}{q}} \right\|_{\ell_p} \right)_k \right\|_{\ell_{p,\infty}} \\
&\leqslant \left\| \left(\left(\sum_{i=1}^{n} \|x_k^{(i)}\|^q_{\ell_p} \right)^{\frac{1}{q}} \right)_k \right\|_{\ell_{p,\infty}} = \left\| \left(\sum_{i=1}^{n} \left| \left(\|x_k^{(i)}\|_{\ell_p} \right)_k \right|^q \right)^{\frac{1}{q}} \right\|_{\ell_{p,\infty}} \\
&\leqslant \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} \left\| \left(\|x_k^{(i)}\|_{\ell_p} \right)_k \right\|^q_{\ell_{p,\infty}} \right)^{\frac{1}{q}} = \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} \|\overline{x}^{(i)}\|^q_X \right)^{\frac{1}{q}},
\end{aligned}$$

where the first and the second inequalities follow from the q-convexity of ℓ_p and $\ell_{p,\infty}$, respectively.

To show that X does not satisfy an upper p-estimate (i.e., is not (p,∞)-convex), let us assume that the coordinates in ℓ_p and $\ell_{p,\infty}$ are indexed starting on 0. Let us fix $n \geqslant 1$ and denote by $(e_j)_{j=0}^{\infty}$ the canonical basis of ℓ_p. For $k = 0, \dots, n-1$, we write $a^{(k)} = \sum_{j=0}^{n-1} \alpha_{(k+j)_n} e_j$ for the cyclic permutations of the coefficients $\alpha_k = (k+1)^{\frac{1}{p^*}} - k^{\frac{1}{p^*}}$ over the set $\{0, \dots, n-1\}$. Here, by i_n we mean $i \mod n$. We consider for $i = 0, \dots, n-1$ the (finite) sequences $\overline{x}^{(i)} = (x_k^{(i)})_{k=0}^{\infty} \in X$ given by $x_k^{(i)} = \alpha_{(k+i)_n} e_i$ for $k = 0, \dots, n-1$ and $x_k^{(i)} = 0$ for $k \geqslant n$. It is clear that for every i,

$$\|\overline{x}^{(i)}\|_X = \|(\alpha_{(k+i)_n})_{k=0}^{n-1}\|_{\ell_{p,\infty}} = \|(\alpha_k)_{k=0}^{n-1}\|_{\ell_{p,\infty}} = 1, \quad \text{so } \left(\sum_{i=0}^{n-1} \|\overline{x}^{(i)}\|_X^p \right)^{\frac{1}{p}} = n^{\frac{1}{p}}.$$

On the other hand, for every $k = 0, \dots, n-1$,

$$\bigvee_{i=0}^{n-1} |x_k^{(i)}| = \sum_{i=0}^{n-1} \alpha_{(k+i)_n} e_i = a^{(k)},$$

so

$$\left\| \bigvee_{i=0}^{n-1} |\overline{x}^{(i)}| \right\|_X = \left\| \left(\left\| \bigvee_{i=0}^{n-1} |x_k^{(i)}| \right\|_{\ell_p} \right)_{k=0}^{n-1} \right\|_{\ell_{p,\infty}} = \left\| \left(\|a^{(k)}\|_{\ell_p} \right)_{k=0}^{n-1} \right\|_{\ell_{p,\infty}} = A_n n^{\frac{1}{p}},$$

where $A_n = \left(\sum_{j=0}^{n-1} \alpha_j^p\right)^{\frac{1}{p}}$ is the norm of $a^{(k)}$ in ℓ_p for every k, and $n^{\frac{1}{p}}$ is the norm in $\ell_{p,\infty}$ of the constant 1 vector on the set $\{0,\dots,n-1\}$. It is not difficult to check that A_n diverges as $n \to \infty$. More specifically, $\alpha_k \approx (k+1)^{-\frac{1}{p}}$ uniformly on k, so $A_n \approx \left(\sum_{k=1}^{n} k^{-1}\right)^{\frac{1}{p}}$, which diverges. To conclude the argument, observe that the quotients $\left\|\bigvee_{i=0}^{n-1} |\overline{x}^{(i)}|\right\|_X / \left(\sum_{i=0}^{n-1} \|\overline{x}^{(i)}\|_X^p\right)^{\frac{1}{p}} = A_n$ are not uniformly bounded, so X cannot be (p,∞)-convex. □

We can formulate a dual version of Theorem 2.2.1 for q-concave Banach lattices and Banach lattices with lower q-estimates:

Theorem 2.2.4. *Let X be a Banach lattice and $1 < q < \infty$.*

(1) If X is q-concave, then there are a family Γ of probability measures and an onto interval preserving positive operator

$$Q : \ell_1(L_q(\mu))_{\mu\in\Gamma} \to X$$

such that $B_X \subseteq Q(B_Y) \subseteq K_{(q)}(X)B_X$, where $Y = \ell_1(L_q(\mu))_{\mu\in\Gamma}$.

(2) If X satisfies an lower q-estimate, then there are a family Γ of probability measures and an onto interval preserving positive operator

$$Q : \ell_1(L_{q,1}(\mu))_{\mu\in\Gamma} \to X$$

such that $B_X \subseteq Q(B_Y) \subseteq \gamma_{q^*} K_{(\downarrow q)}(X)B_X$, where $Y = \ell_1(L_{q,1}(\mu))_{\mu\in\Gamma}$ (recall that here, $L_{q,1}$ is endowed with the dual norm induced by $\|\cdot\|_{L^{[1]}_{q^*,\infty}}$).

For the proof, we will need the following general facts:

Lemma 2.2.5.

(1) Let $T_\gamma : Y_\gamma \to X$ be almost interval preserving operators for every $\gamma \in \Gamma$, such that $\sup_{\gamma\in\Gamma} \|T_\gamma\| < \infty$, and let $Y = \ell_1(Y_\gamma)_{\gamma\in\Gamma}$. Then,

$$\begin{aligned} T : \quad & Y \longrightarrow X \\ & \overline{y} = (y_\gamma)_{\gamma\in\Gamma} \longmapsto T\overline{y} = \textstyle\sum_{\gamma\in\Gamma} T_\gamma y_\gamma \end{aligned}$$

is an almost interval preserving operator with $\|T\| \leqslant \sup_{\gamma\in\Gamma} \|T_\gamma\|$.

(2) Let Y be order continuous, X a KB-space and $T : Y \to X$ an almost interval preserving operator. Then T is interval preserving.

Proof. (1) It is standard to check (cf. [88, p. 47]) that $Y^* = \ell_\infty(Y_\gamma^*)_{\gamma\in\Gamma}$ and $T^*x^* = (T_\gamma^* x^*)_{\gamma\in\Gamma}$, which is clearly a lattice homomorphism. Therefore, T must be almost interval preserving by [96, Theorem 1.4.19].

(2) Y is order continuous, so $J_Y : Y \to Y^{**}$ is interval preserving [96, Theorem 2.4.2]. Moreover, X is a KB-space, so $J_X(X)$ is a projection band in X^{**} [96, Theorem 2.4.12 and Theorem 1.2.9], so there exists an interval preserving operator $P : X^{**} \to X$ such that PJ_X is the identity on X. Since T is almost interval preserving, $T^{**} : Y^{**} \to X^{**}$ is interval preserving. But observe that

$$T = PJ_XT = PT^{**}J_Y,$$

so T must be interval preserving, as it is the composition of interval preserving operators. □

Proof of Theorem 2.2.4. We start as in the proof of Theorem 2.2.1 by fixing $x \in S = \{z \in X_+ : \|z\| = 1\}$ and considering the quotient $q_x : X^* \to X^*/\ker \rho_x$, where $\rho_x(x^*) = |x^*|(x)$, and $i_x : X^*/\ker \rho_x \to L_1(\mu_x)$ is the inclusion into the completion of $(X^*/\ker \rho_x, \rho_x)$, which is an AL-space denoted $L_1(\mu_x)$. It is clear that $i_x q_x$ is a q^*-convex operator with constant $K_{(q)}(X)$ (respectively, (q^*,∞)-convex operator with constant $K_{(\downarrow q)}(X)$). By Theorem 1.2.3 (respectively, Theorem 1.2.4), there exists a normalized $g_x \in L_1(g_x)_+$ such that $i_x = M_x R_x$, where $R_x x^* = \frac{1}{g_x} \cdot (i_x q_x x^*)$ is defined from X^* into $L_{q^*}(g_x \cdot \mu_x)$ with norm $K_{(q)}(X)$ (respectively, $L_{q^*,\infty}(g_x \cdot \mu_x)$ with norm $\gamma_{q^*} K_{(\downarrow q)}(X)$) and M_x is the operator of multiplication by g_x.

Observe that in both cases X is a KB-space. This is due to the fact that if X satisfies a lower q-estimate, it cannot contain a sublattice isomorphic to c_0, which is equivalent to X being a KB-space by [96, Theorem 2.4.12]. By [96, Theorem 2.4.12 and Theorem 1.2.9], there exists an interval preserving operator $P : X^{**} \to X$ such that PJ_X is the identity on X. Let us define $Y = \ell_1(L_q(g_x \cdot \mu_x))_{x\in S}$ and $Q_x = PR_x^* : L_q(g_x \cdot \mu_x) \to X$ when X is q-concave, and $Y = \ell_1(L_{q,1}(g_x \cdot \mu_x))_{x\in S}$ and $Q_x = PR_x^* J_{L_{q,1}} : L_{q,1}(g_x \cdot \mu_x) \to X$ when X satisfies a lower q-estimate, and denote by j_x and P_x, $x \in S$, the embeddings and projections between Y and each of its factors, which are clearly interval preserving. In both cases, the operators Q_x are interval preserving (recall that $L_{q,1}(g_x \cdot \mu_x)$ is a KB-space, so in particular it is order continuous), so the compositions $Q_x P_x$ are interval preserving as well, and their norms are uniformly bounded. Applying Lemma 2.2.5(1) we obtain an almost interval preserving operator $Q : Y \to X$ given by $Q\overline{f} = \sum_{x\in S} Q_x f_x$ for every $\overline{f} = (f_x)_{x\in S} \in Y$, with norm bounded by either $K_{(q)}(X)$ or $\gamma_{q^*} K_{(\downarrow q)}(X)$. Since Y is q-concave or satisfies an upper q-estimate, it is order continuous, and applying Lemma 2.2.5(2) we get that Q is actually interval preserving.

To check the surjectivity of Q, observe that $Q(j_x \mathbb{1}_x) = x$ for every $x \in S$, where $\mathbb{1}_x$ denotes the constant one function in the corresponding $L_q(g_x \cdot \mu_x)$ or $L_{q,1}(g_x \cdot \mu_x)$ (we only check it in the lower q-estimate setting, as the q-concave setting is slightly easier): for every $x^* \in X_+^*$,

$$\begin{aligned}(R_x^* J_{L_{q,1}} \mathbb{1}_x)(x^*) = (J_{L_{q,1}} \mathbb{1}_x)(R_x x^*) = (R_x x^*)(\mathbb{1}_x) &= \int \frac{i_x q_x x^*}{g_x} \mathbb{1}_x g_x \, d\mu_x \\ &= \|i_x q_x x^*\|_{L_1(\mu_x)} = \rho_x(x^*) = x^*(x) = J_X x(x^*),\end{aligned}$$

so $R_x^* J_{L_{q,1}} \mathbb{1}_x = J_X x$ and hence $Q(j_x \mathbb{1}_x) = Q_x \mathbb{1}_x = x$. This fact alone already implies that Q is onto, but actually we can show that $B_X \subseteq Q(B_Y)$. Indeed, let $0 \neq z \in B_X$. If $z \geqslant 0$, then $z = Q(\|z\| j_x \mathbb{1}_x) \in Q(B_Y)$ for $x = z/\|z\| \in S$. Otherwise, let $x = |z|/\|z\| \in S$ and use the fact that Q_x is interval preserving to find $f_\pm \in [0, \mathbb{1}_x]$ such that $Q_x f_\pm = z_\pm \in [0, x]$. Clearly, $f = f_+ - f_- \in [-\mathbb{1}_x, \mathbb{1}_x]$, so $z = Q(j_x f) \in Q(B_Y)$, and the proof is concluded. □

2.2.2 Banach lattices with upper p-estimates vs. sublattices of $\ell_\infty(L_{p,\infty})$

Theorem 2.2.1 points out a fundamental difference between p-convexity and upper p-estimates: while the class of p-convex Banach lattices (with constant 1) and the class of closed sublatices of ℓ_∞-sums of L_p-spaces can be identified isometrically, the class of Banach lattices with upper p-estimates (with constant 1) and the class of closed sublattices of ℓ_∞-sums of $L_{p,\infty}$-spaces in principle are not isometrically the same classes, but only isomorphically. Indeed, in this section we provide an example of a Banach lattice that satisfies an upper p-estimate with constant 1 but does not embed isometrically into any ℓ_∞-sums of $L_{p,\infty}$-spaces. This is equivalent to determining whether the constant γ_p that arises in Theorem 1.2.4 and Theorem 2.2.1 can be improved to 1.

Let X, Y be Banach lattices and let $1 \leqslant C < \infty$. We say that X *C-lattice embeds into* Y if there exists a lattice isomorphic injection $T : X \to Y$ such that $C^{-1}\|x\| \leqslant \|Tx\| \leqslant \|x\|$ for all $x \in X$. In this case, we call T a C-lattice embedding. If $C = 1$, then T is a lattice isometric embedding.

A Banach space with a normalized 1-unconditional basis is regarded as a Banach lattice in the coordinatewise order.

Theorem 2.2.6. *Let X be a Banach lattice with a normalized 1-unconditional basis $(e_i)_i$. Denote the biorthogonal functionals by $(e_i^*)_i$. For any $C \geqslant 1$, the following statements are equivalent.*

(1) For each normalized finitely supported $a = \sum_{i=1}^n a_i e_i \in X_+$ and every $\varepsilon > 0$, there exist $b = \sum_{i=1}^n b_i e_i^* \in X_+^*$ and $d = (d_i)_{i=1}^n \in \mathbb{R}_+^n$ so that $\sum_{i=1}^n a_i b_i > 1-\varepsilon$, $\sum_{i=1}^n d_i = 1$ and $\|\sum_{i\in I} b_i e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i\in I} d_i$ for any $I \subseteq \{1,\dots,n\}$.

(2) There is a set Γ of (probability) measures such that X C-lattice embeds into $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$.

Proof. (1)⇒(2) Let $a \in X_+$ be normalized and finitely supported, and let $\varepsilon > 0$ be given. Obtain b and d from (1). Let $(U_i)_{i=1}^n$ be a measurable partition of $(0,1)$ such that $\lambda(U_i) = d_i$, $1 \leqslant i \leqslant n$, where λ denotes the Lebesgue measure. Define

$$S_{a,\varepsilon} : X \to L_{p,\infty}(0,1) \text{ by } S_{a,\varepsilon}\left(\sum_{i=1}^\infty c_i e_i\right) = \sum_{i=1}^n \frac{c_i b_i \chi_{U_i}}{d_i}.$$

We have

$$\|S_{a,\varepsilon} a\| \geqslant \lambda(0,1)^{-\frac{1}{p^*}} \int_0^1 |S_{a,\varepsilon} a|\, d\lambda = \sum_{i=1}^n a_i b_i > 1-\varepsilon.$$

On the other hand, if $c = \sum_{i=1}^\infty c_i e_i \in B_X$ then there exists $I \subseteq \{1,\dots,n\}$ so that

$$\begin{aligned}\|S_{a,\varepsilon} c\| &= \lambda\left(\bigcup_{i\in I} U_i\right)^{-\frac{1}{p^*}} \int_{\bigcup_{i\in I} U_i} |S_{a,\varepsilon} c|\, d\lambda = \left(\sum_{i\in I} d_i\right)^{-\frac{1}{p^*}} \sum_{i\in I} |c_i| b_i \\ &= \left(\sum_{i\in I} d_i\right)^{-\frac{1}{p^*}} \left\langle \sum_{i=1}^\infty |c_i| e_i, \sum_{i\in I} b_i e_i^* \right\rangle \leqslant \left(\sum_{i\in I} d_i\right)^{-\frac{1}{p^*}} \left\|\sum_{i\in I} b_i e_i^*\right\| \leqslant C.\end{aligned}$$

Let Γ be the Cartesian product of the set of normalized finitely supported elements in X_+ and the interval $(0, \frac{1}{2})$. It is now clear that the map $T : X \to \ell_\infty(L_{p,\infty}(0,1))_{(a,\varepsilon)\in\Gamma}$ given by

$$Tc = (C^{-1} S_{a,\varepsilon} c)_{(a,\varepsilon)\in\Gamma}$$

is a C-lattice embedding.

(2)⇒(1) Let $T : X \to \ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$ be a C-lattice embedding. Suppose that $a = \sum_{i=1}^n a_i e_i$ is normalized and $\varepsilon > 0$. For each $\mu_0 \in \Gamma$, let P_{μ_0} be the projection of $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$ onto the μ_0-th component. There exists μ_0 such that $C\|P_{\mu_0} Ta\| > 1-\varepsilon$. Let $f_i = P_{\mu_0} Te_i$. Choose a μ_0-measurable set $U \subseteq \operatorname{supp} Ta$ such that $0 < \mu_0(U) < \infty$ and

$$C \int_U P_{\mu_0} Ta\, d\mu_0 > (1-\varepsilon)\mu_0(U)^{\frac{1}{p^*}}.$$

Set $U_i = U \cap \operatorname{supp} f_i$ and $m = (\sum_{i=1}^n \mu_0(U_i))^{\frac{1}{p^*}}$. Define

$$b_i = \frac{C}{m} \int_{U_i} f_i\, d\mu_0, \quad d_i = \frac{\mu_0(U_i)}{m^{p^*}}, \quad 1 \leqslant i \leqslant n.$$

We have $d_i \geqslant 0$, $\sum_{i=1}^n d_i = 1$ and

$$\sum_{i=1}^n a_i b_i = \frac{C}{m} \int_{\bigcup_{i=1}^n U_i} \sum_{i=1}^n a_i f_i\, d\mu_0 = \frac{C}{m} \int_{\bigcup_{i=1}^n U_i} P_{\mu_0} Ta > 1-\varepsilon.$$

On the other hand, suppose that $I \subseteq \{1,\dots,n\}$. If $\|\sum_{i=1}^\infty c_i e_i\| \leqslant 1$, then

$$\left\langle \sum_{i=1}^\infty c_i e_i, \sum_{i\in I} b_i e_i^* \right\rangle = \sum_{i\in I} c_i b_i = \frac{C}{m} \int_{\bigcup_{i\in I} U_i} \sum_{i\in I} c_i f_i \, d\mu_0$$
$$\leqslant \frac{C}{m} \left\| P_{\mu_0} T \left(\sum_{i=1}^\infty c_i e_i \right) \right\| \mu_0 \left(\bigcup_{i\in I} U_i \right)^{\frac{1}{p^*}} \leqslant C \left(\sum_{i\in I} d_i \right)^{\frac{1}{p^*}}.$$

This proves that $\|\sum_{i\in I} b_i e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i\in I} d_i$. □

Remark 2.2.7. *If X is a two dimensional Banach lattice that satisfies an upper p-estimate with constant 1, then $C = 1$ works in Theorem 2.2.6. Indeed, let (e_1, e_2) be a normalized 1-unconditional basis for X. Suppose that $a = a_1e_1 + a_2e_2$ is normalized and non-negative. Choose $b = b_1e_1^* + b_2e_2^* \in X_+^*$ such that $\langle a, b\rangle = 1 = \|b\|$. Define $d_i = b_i^{p^*}/(b_1^{p^*} + b_2^{p^*})$, $i = 1, 2$. Note that X^* satisfies a lower p^*-estimate with constant 1 and e_1^* and e_2^* are disjoint, so*

$$b_1^{p^*} + b_2^{p^*} = \|b_1e_1^*\|^{p^*} + \|b_2e_2^*\|^{p^*} \leqslant \|b\|^{p^*} = 1.$$

It is easy to check that $\|\sum_{i\in I} b_i e_i^*\|^{p^*} \leqslant \sum_{i\in I} d_i$ for any $I \subseteq \{1, 2\}$ and $d_1 + d_2 = 1$.

Corollary 2.2.8. *Suppose that X is a Banach lattice with a normalized 1-unconditional basis $(e_i)_i$ that C-lattice embeds into $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$. For any normalized element $b = \sum_{i=1}^n b_i e_i^* \in X_+^*$ and any $I_j \subseteq \{1,\dots,n\}$, $1 \leqslant j \leqslant m$, such that $\sum_{j=1}^m \chi_{I_j} = l\chi_{\{1,\dots,n\}}$ for some $l \in \mathbb{N}$, we have*

$$\sum_{j=1}^m \left\| \sum_{i\in I_j} b_i e_i^* \right\|^{p^*} \leqslant C^{p^*} l.$$

Proof. Choose a normalized $a = \sum_{i=1}^n a_i e_i \in X_+$ so that $\sum_{i=1}^n a_i b_i = 1$. First assume that $[(e_i)_{i=1}^n]$ is uniformly convex. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ so that if $b' = \sum_{i=1}^n b_i' e_i^* \in X_+^*$, $\|b'\| = 1$ and $\langle a, b'\rangle > 1 - \delta$, then $\|b - b'\| < \varepsilon$. By Theorem 2.2.6, there exists $b' \in X_+^*$ and $d = (d_i)_{i=1}^n \in \mathbb{R}_+^n$ so that $\langle a, b'\rangle > 1 - \delta$ and $\|\sum_{i\in I} b_i' e_i^*\|^{p^*} \leqslant C^{p^*} \sum_{i\in I} d_i$ for any $I \subseteq \{1,\dots,n\}$. It follows that for any $I \subseteq \{1,\dots,n\}$,

$$\left\| \sum_{i\in I} b_i e_i^* \right\| \leqslant \left\| \sum_{i\in I} |b_i - b_i'| e_i^* \right\| + \left\| \sum_{i\in I} b_i' e_i^* \right\| \leqslant \varepsilon |I| + C \left(\sum_{i\in I} d_i \right)^{\frac{1}{p^*}}.$$

Now assume that $I_j \subseteq \{1,\dots,n\}$, $1 \leqslant j \leqslant m$, and that $\sum_{j=1}^m \chi_{I_j} = l\chi_{\{1,\dots,n\}}$ for some $l \in \mathbb{N}$. Set $A_j = \varepsilon|I_j|$ and $B_j = C(\sum_{i\in I_j} d_i)^{\frac{1}{p^*}}$. Then

$$\left(\sum_{j=1}^m \left\| \sum_{i\in I_j} b_i e_i^* \right\|^{p^*} \right)^{\frac{1}{p^*}} \leqslant \left(\sum_{j=1}^m (A_j + B_j)^{p^*} \right)^{\frac{1}{p^*}} \leqslant \left(\sum_{j=1}^m A_j^{p^*} \right)^{\frac{1}{p^*}} + \left(\sum_{j=1}^m B_j^{p^*} \right)^{\frac{1}{p^*}}$$
$$\leqslant \sum_{j=1}^m A_j + \left(\sum_{j=1}^m B_j^{p^*} \right)^{\frac{1}{p^*}} = \varepsilon \sum_{j=1}^m |I_j| + C \left(\sum_{j=1}^m \sum_{i\in I_j} d_i \right)^{\frac{1}{p^*}} \leqslant \varepsilon n l + C l^{\frac{1}{p^*}}.$$

Taking $\varepsilon \downarrow 0$ gives the desired result.

We now return to a general X with a normalized 1-unconditional basis (e_i). Given a normalized element $b = \sum_{i=1}^n b_i e_i^* \in X_+^*$ and $\varepsilon > 0$, there is a uniformly convex lattice norm $|||\cdot|||$ on $[(e_i)_{i=1}^n]$ such that $\|x\| \leqslant |||x||| \leqslant (1+\varepsilon)\|x\|$ for any $x \in [(e_i)_{i=1}^n]$ (take for instance $|||\cdot||| = \|\cdot\| + \delta\|\cdot\|_{\ell_2^n}$ for $\delta > 0$ small enough, which is strictly convex and hence uniformly convex, since $[(e_i)_{i=1}^n]$ is finite-dimensional).

Set $u_i = e_i/|||e_i|||$, $u_i^* = e_i^*/|||e_i^*|||$ and $b' = \sum_{i=1}^n b_i' u_i^* = b/|||b|||$. Note that $(u_i)_{i=1}^n$ is a 1-unconditional basis for $[(e_i)_{i=1}^n] = [(u_i)_{i=1}^n]$ and b' is $|||\cdot|||$-normalized. Moreover, if $T : [(e_i)_{i=1}^n] \to \ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$ is a C-lattice embedding, then

$$\widetilde{T}x = Tx : [(u_i)_{i=1}^n] \to \ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$$

is a $(1+\varepsilon)C$-lattice embedding. Assume that $I_j \subseteq \{1,\dots,n\}$, $1 \leqslant j \leqslant m$, such that $\sum_{j=1}^m \chi_{I_j} = l\chi_{\{1,\dots,n\}}$ for some $l \in \mathbb{N}$. By the first part,

$$\sum_{j=1}^m \left|\!\left|\!\left| \sum_{i\in I_j} b_i' u_i^* \right|\!\right|\!\right|^{p^*} \leqslant (1+\varepsilon)^{p^*} C^{p^*} l.$$

Since $|||b||| \leqslant 1+\varepsilon$, for any $I \subseteq \{1,\dots,n\}$ we have

$$\left|\!\left|\!\left| \sum_{i\in I} b_i' u_i^* \right|\!\right|\!\right| = \frac{1}{|||b|||} \left|\!\left|\!\left| \sum_{i\in I} b_i e_i^* \right|\!\right|\!\right| \geqslant \frac{1}{1+\varepsilon} \left\| \sum_{i\in I} b_i e_i^* \right\|.$$

Thus,

$$\sum_{j=1}^m \left\| \sum_{i\in I} b_i e_i^* \right\|^{p^*} \leqslant (1+\varepsilon)^{2p^*} C^{p^*} l.$$

Taking $\varepsilon \downarrow 0$ gives the desired result. □

We conclude this subsection with an example that shows that γ_p cannot be made to be 1 in Theorems 2.2.1 and 1.2.4.

Example 2.2.9. *Suppose that $1 < p < \infty$. Denote by C_p the infimum of the set of constants C so that every Banach lattice satisfying an upper p-estimate with constant 1 C-lattice embeds into $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$ for some set of measures Γ (this set is non-empty due to Theorem 2.2.1). Then $C_p^{p^*} \geqslant \frac{3\cdot 2^{p^*}}{2(1+2^{p^*})} > 1$.*

Proof. Let $X = \mathbb{R}^3$ so that the norm on X^* is

$$\|(b_1,b_2,b_3)\|_{X^*} = \max\left\{ \left(|b_i|^{p^*} + (|b_j| + |b_k|)^{p^*}\right)^{\frac{1}{p^*}} : \{i,j,k\} = \{1,2,3\} \right\}.$$

It is clear that the coordinate unit vectors form a normalized 1-unconditional basis for X. Suppose that $b = (b_1,b_2,b_3) \in X^*$ and I, J are disjoint non-empty subsets of $\{1,2,3\}$. Without loss of generality, we may assume that $I = \{i\}$ and $J = \{j,k\}$, where i,j,k are distinct. Then

$$\|b\chi_I\|_{X^*}^{p^*} + \|b\chi_J\|_{X^*}^{p^*} = |b_i|^{p^*} + (|b_j| + |b_k|)^{p^*} \leqslant \|b\|_{X^*}^{p^*}.$$

Thus, X^* satisfies a lower p^*-estimate with constant 1. Hence, X satisfies an upper p-estimate with constant 1. Suppose that X C-lattice embeds into some $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$. The vector $(1+2^{p^*})^{-\frac{1}{p^*}}(1,1,1)$ is normalized in X_+^*. By Corollary 2.2.8,

$$\frac{1}{1+2^{p^*}} \left(\|(1,1,0)\|^{p^*} + \|(1,0,1)\|^{p^*} + \|(0,1,1)\|^{p^*}\right) \leqslant 2C^{p^*}.$$

Thus,

$$\frac{3\cdot 2^{p^*}}{2(1+2^{p^*})} \leqslant C^{p^*}.$$

Taking infimum over the eligible C's gives the desired result. Direct verification shows that $\frac{3\cdot 2^{p^*}}{2(1+2^{p^*})} > 1$ if $1 < p < \infty$. □

2.2.3 The norm of the free Banach lattice associated to a rearrangement invariant space

In this subsection, let (Ω, Σ, μ) be a non-atomic σ-finite measure space and let X be a rearrangement invariant (r.i.) Banach function space on Ω in the sense of [20, Definition II.4.1]. To be specific, X is a vector lattice ideal of $L_0(\Omega, \Sigma, \mu)$, equipped with a complete lattice norm $\|\cdot\|$, so that $f \in X$, $g \in L_0(\Omega, \Sigma, \mu)$ and $\mu\{|f| > t\} = \mu\{|g| > t\}$ for all $t \geqslant 0$ imply that $g \in X$ and $\|f\| = \|g\|$. We further assume that X has the *Fatou property*: Whenever $f_n \in X$, $0 \leqslant f_n \uparrow f$ a.e. and $\sup \|f_n\| < \infty$ then $f \in X$ and $\|f\| = \sup_n \|f_n\|$. Note that we are using the definition of the Fatou property from [88, p. 30].

Let $\mathcal{C}$ be the class consisting of X only. Given a Banach space E, the aim of this subsection will be to produce a formula for the norm $\rho_{\mathcal{C}}$ on $\mathrm{FVL}[E]$ and show that $\mathrm{FBL}^{\mathcal{C}}[E] \subseteq C(B_{E^*})$.

Given a sequence $\mathcal{U} = (U_n)_n$ of disjoint measurable subsets of Ω so that $0 < \mu(U_n) < \infty$ for all n, we define for each $f \in L_0(\Omega, \Sigma, \mu)$ the conditional expectation over $\mathcal{U}$ by

$$P_{\mathcal{U}} f = \sum_n \frac{\int_{U_n} f \, d\mu}{\mu(U_n)} \chi_{U_n}.$$

Note that the above sum has at most one non-zero term at any given point of Ω, so is well-defined as long as $\int_{U_n} f \, d\mu$ is finite for every $n \in \mathbb{N}$. This will be the case when $f \in X$.

Proposition 2.2.10. *Let $\mathcal{U} = (U_n)_n$ be a sequence of disjoint measurable subsets of Ω so that $0 < \mu(U_n) < \infty$ for all n. Then $P_{\mathcal{U}} f \in X$ if $f \in X$, and $P_{\mathcal{U}}$ is a positive contraction on X.*

Proof. First note that, since X is a Banach function space, $\int_{U_n} f \, d\mu$ is finite for every $n \in \mathbb{N}$ (see property 5 in [20, Definition I.1.1]), so $P_{\mathcal{U}} f$ is well-defined. It is straightforward to check that $P_{\mathcal{U}}$ is a positive contraction from $L_1(\Omega, \Sigma, \mu)$ to $L_1(\Omega, \Sigma, \mu)$ and from $L_\infty(\Omega, \Sigma, \mu)$ to $L_\infty(\Omega, \Sigma, \mu)$. Thus, $P_{\mathcal{U}}$ is an admissible operator for the compatible pair $(L_1(\Omega, \Sigma, \mu), L_\infty(\Omega, \Sigma, \mu))$. Moreover, since X is a r.i. space over a non-atomic σ-finite (in particular, resonant) measure space, it follows from [20, Theorem II.2.2] that X is an exact interpolation space, so $P_{\mathcal{U}} : X \to X$ is a positive contraction. □

Lemma 2.2.11. *Let $h_1, \ldots, h_n \in X$ and set $h = \sum_{k=1}^n |h_k|$. Given $\varepsilon > 0$, there is a sequence $\mathcal{U} = (U_n)_n$ of disjoint measurable subsets of Ω so that $0 < \mu(U_n) < \infty$ for all n and $|h_k - \mathcal{P}_{\mathcal{U}} h_k| \leqslant \varepsilon h$, $1 \leqslant k \leqslant n$.*

Proof. We may assume that $\varepsilon < 1$. Let $V_i = \{\varepsilon^i \leqslant h < \varepsilon^{i-1}\}$. Then $(V_i)_{i \in \mathbb{Z}}$ is a sequence of disjoint measurable subsets of Ω so that $h = h_k = 0$ on $(\bigcup_i V_i)^c$ for any k. Since the measure space is σ-finite, for each i there is an at most countable measurable partition $(U_{ij})_{j \in J_i}$ of V_i with $0 < \mu(U_{ij}) < \infty$ and $b_{kij} \in \mathbb{R}$ so that

$$\left| h_k \chi_{V_i} - \sum_{j \in J_i} b_{kij} \chi_{U_{ij}} \right| \leqslant \frac{\varepsilon^{i+1}}{2}, \ i \in \mathbb{Z}, \ 1 \leqslant k \leqslant n.$$

Thus, for each $1 \leqslant k \leqslant n$ we have

$$\left| h_k - \sum_{i,j} b_{kij} \chi_{U_{ij}} \right| \leqslant \sum_i \frac{\varepsilon^{i+1}}{2} \chi_{V_i} \leqslant \frac{\varepsilon h}{2} \tag{2.1}$$

and

$$\left| \frac{1}{\mu(U_{ij})} \int_{U_{ij}} h_k \, d\mu - b_{kij} \right| = \left| \frac{1}{\mu(U_{ij})} \int_{U_{ij}} h_k - b_{kij} \chi_{U_{ij}} \, d\mu \right| \leqslant \frac{\varepsilon^{i+1}}{2}.$$

Take $\mathcal{U} = (U_{ij})_{i\in\mathbb{Z},j\in J_i}$. Note that $\mathcal{P}_{\mathcal{U}}h_k = 0$ on the set $(\bigcup_{ij} U_{ij})^c$. For $1 \leqslant k \leqslant n$,

$$\left|\mathcal{P}_{\mathcal{U}}h_k - \sum_{i,j} b_{kij}\chi_{U_{ij}}\right| \leqslant \sum_{i,j} \frac{\varepsilon^{i+1}}{2}\chi_{U_{ij}} = \sum_i \frac{\varepsilon^{i+1}}{2}\chi_{V_i} \leqslant \frac{\varepsilon h}{2}. \tag{2.2}$$

Summing (2.1) and (2.2) gives the desired result. □

Let $I = \{(m,r) \in \mathbb{N}^2 : \frac{m}{r} \leqslant \mu(\Omega)\}$. If $(m,r) \in I$, take any sequence $(V_i)_{i=1}^m$ of disjoint measurable subsets of Ω such that $\mu(V_i) = r^{-1}$, $1 \leqslant i \leqslant m$. Define a norm ρ_{mr} on $\mathbb{R}^m$ by

$$\rho_{mr}(a_1,\dots,a_m) = \left\|\sum_{i=1}^m a_i\chi_{V_i}\right\|_X.$$

Let E be a Banach space and define $\rho : C(B_{E^*}) \to [0,\infty]$ by

$$\rho(f) = \sup\left\{\rho_{mr}(f(x_1^*),\dots,f(x_m^*)) : (m,r) \in I, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E} \rho_{mr}(x_1^*(x),\dots,x_m^*(x)) \leqslant 1\right\}.$$

By definition, $\rho(\delta_x) = \|x\|$ for any $x \in E$. Moreover, it is easy to check that ρ is a complete lattice norm on the vector lattice $\{f \in C(B_{E^*}) : \rho(f) < \infty\}$, which contains $\mathrm{FVL}[E]$. For our next result, recall that we are considering the case $\mathcal{C} = \{X\}$.

Theorem 2.2.12. *The norms ρ and $\rho_{\mathcal{C}}$ are equal on $\mathrm{FVL}[E]$. In particular, $\mathrm{FBL}^{\mathcal{C}}[E] \subseteq \{f \in C(B_{E^*}) : \rho(f) < \infty\}$.*

Proof. Suppose that $f \in \mathrm{FVL}[E]$. There are $x_1,\dots,x_n \in E$ and a lattice-linear function G so that $f = G(\delta_{x_1},\dots,\delta_{x_n})$. Let $\varepsilon > 0$ be given. There are $(m,r) \in I$ and $x_1^*,\dots,x_m^* \in E^*$ so that $\sup_{x\in B_E} \rho_{mr}(x_1^*(x),\dots,x_m^*(x)) \leqslant 1$ and

$$\rho(f) \leqslant \rho_{mr}(f(x_1^*),\dots,f(x_m^*)) + \varepsilon = \left\|\sum_{i=1}^m f(x_i^*)\chi_{V_i}\right\| + \varepsilon,$$

where $(V_i)_{i=1}^m$ is the sequence of disjoint measurable subsets of Ω chosen above, so that $\mu(V_i) = r^{-1}$, $1 \leqslant i \leqslant m$. Define a linear operator $S : \mathrm{FVL}[E] \to X$ by $Sg = \sum_{i=1}^m g(x_i^*)\chi_{V_i}$. It is clear that S is a lattice homomorphism. By definition of ρ, if $\rho(g) \leqslant 1$, then $\rho_{mr}(g(x_1^*),\dots,g(x_m^*)) \leqslant 1$ and thus $\|Sg\| \leqslant 1$. Let $T = S\phi_E : E \to X$. Then $\|T\| \leqslant 1$ and $\widehat{T} = S$ is the unique extension of T to $\mathrm{FVL}[E]$ as a vector lattice. Hence

$$\rho_{\mathcal{C}}(f) \geqslant \|\widehat{T}f\| = \|Sf\|,$$

where the last step holds since S is a lattice homomorphism. Thus, $\rho_{\mathcal{C}}(f) \geqslant \rho(f) - \varepsilon$ for any $\varepsilon > 0$. Therefore, $\rho_{\mathcal{C}}(f) \geqslant \rho(f)$.

Conversely, given $\varepsilon > 0$, there exists a linear contraction $T : E \to X$ so that $\rho_{\mathcal{C}}(f) \leqslant \|\widehat{T}f\| + \varepsilon$. Let $h_k = Tx_k$, $1 \leqslant k \leqslant n$, and $h = \sum_{k=1}^n |h_k|$. By Lemma 2.2.11, there is a sequence $\mathcal{U} = (U_i)_i$ of disjoint measurable subsets of Ω so that $0 < \mu(U_i) < \infty$ for all i and $|h_k - \mathcal{P}_{\mathcal{U}}h_k| \leqslant \varepsilon h$, $1 \leqslant k \leqslant n$. Since G is a finite lattice-linear function, it is Lipschitz, so there is a constant $C_G < \infty$, depending only on G, so that

$$|\widehat{T}f - G(\mathcal{P}_{\mathcal{U}}h_1,\dots,\mathcal{P}_{\mathcal{U}}h_n)| = |G(h_1,\dots,h_n) - G(\mathcal{P}_{\mathcal{U}}h_1,\dots,\mathcal{P}_{\mathcal{U}}h_n)| \leqslant C_G\varepsilon h.$$

Express $\mathcal{P}_{\mathcal{U}}h_k$ as $\sum_i a_{ki}\chi_{U_i}$, where $T^*\chi_{U_i} = y_i^*$ and

$$a_{ki} = \frac{1}{\mu(U_i)}\int_{U_i} h_k\,d\mu = \frac{y_i^*(x_k)}{\mu(U_i)}.$$

Then

$$G(\mathcal{P}_{\mathcal{U}}h_1,\dots,\mathcal{P}_{\mathcal{U}}h_n) = G\left(\sum_i a_{1i}\chi_{U_i},\dots,\sum_i a_{ni}\chi_{U_i}\right) = \sum_i G(a_{1i},\dots,a_{ni})\chi_{U_i}$$
$$= \sum_i G(\delta_{x_1},\dots,\delta_{x_n})(y_i^*)\,\frac{\chi_{U_i}}{\mu(U_i)} = \sum_i \frac{f(y_i^*)}{\mu(U_i)}\,\chi_{U_i}.$$

Since X has the Fatou property, there exists $l\in\mathbb{N}$ so that

$$\left\|\sum_{i=1}^{l}\frac{f(y_i^*)}{\mu(U_i)}\,\chi_{U_i}\right\| > \|G(\mathcal{P}_{\mathcal{U}}h_1,\dots,\mathcal{P}_{\mathcal{U}}h_n)\| - \varepsilon \geqslant \|\widehat{T}f\| - (1+C_G\|h\|)\varepsilon.$$

Using the Fatou property once more, choose disjoint Lebesgue measurable sets $V_1,\dots,V_l$ so that $V_i\subseteq U_i$, $\mu(V_i)\in\mathbb{Q}$ and

$$\|\widehat{T}f\| - (1+C_G\|h\|)\varepsilon < \left\|\sum_{j=1}^{l}\frac{f(y_i^*)}{\mu(U_i)}\,\chi_{V_i}\right\|.$$

Write $\mu(V_i)=\frac{j_i}{r}$, where $j_i,r\in\mathbb{N}$. Decompose each V_i as a disjoint union $V_i=\bigcup_{s=1}^{j_i}V_{is}$, with $\mu(V_{is})=r^{-1}$. Take $j_0=\sum_{i=1}^{l}j_i$. It follows that $(j_0,r)\in I$. Consider the sequence where each $\frac{y_i^*}{\mu(U_i)}$ is repeated j_i times. For any $x\in B_E$, let b be the element of $\mathbb{R}^{j_0}$ so that each $\frac{y_i^*(x)}{\mu(U_i)}$ occurs j_i times. Then

$$\rho_{j_0 r}(b) = \left\|\sum_{i=1}^{l}\frac{y_i^*(x)}{\mu(U_i)}\,\chi_{V_i}\right\| \leqslant \|\mathcal{P}_{\mathcal{U}}(Tx)\| \leqslant \|T\| = 1.$$

Therefore, the sequence of functionals of E^* of length j_0 where each $\frac{y_i^*}{\mu(U_i)}$ occurs j_i times satisfies the normalization condition required in the definition of ρ. It follows, taking a to be the element of $\mathbb{R}^{j_0}$ where each $\frac{f(y_i^*)}{\mu(U_i)}$ occurs j_i times, that

$$\rho(f) \geqslant \rho_{j_0 r}(a) = \left\|\sum_{i=1}^{l}\sum_{s=1}^{j_i}\frac{f(y_i^*)}{\mu(U_i)}\,\chi_{V_{is}}\right\| = \left\|\sum_{i=1}^{l}\frac{f(y_i^*)}{\mu(U_i)}\,\chi_{V_i}\right\|$$
$$> \|\widehat{T}f\| - (1+C_G\|h\|)\varepsilon \geqslant \rho_{\mathcal{C}}(f) - (2+C_G\|h\|)\varepsilon.$$

Thus, $\rho(f)\geqslant\rho_{\mathcal{C}}(f)$. □

2.2.4 An explicit expression for the norm of $\mathbf{FBL}^{p,\infty}$

Theorem 2.2.1(1) has the following interpretation: Assume that $1<p<\infty$ and let $\mathcal{C}_p$ be the class of all p-convex Banach lattices with constant 1 and $\mathcal{X}_p$ the class of all $L_p(\mu)$ spaces with μ a probability measure. Then, by Theorem 2.2.1(1), $\mathcal{C}_p\subseteq\overline{\mathcal{X}}_p$ and therefore $\mathcal{C}_p=\overline{\mathcal{X}}_p$. This fact, together with the abstract framework developed in the previous sections provides an alternative proof of [75, Theorem 6.1], where an explicit representation of $\mathrm{FBL}^{\mathcal{C}_p}[E]\equiv\mathrm{FBL}^p[E]$ is given:

Theorem 2.2.13. *Let E be a Banach space and let $1<p<\infty$. The norm $\rho_{\mathcal{C}_p}$ on $\mathrm{FVL}[E]$ is given by*

$$\rho_{\mathcal{C}_p}(f) = \sup\left\{\|(f(x_k^*))_{k=1}^n\|_{\ell_p^n} : n\in\mathbb{N},\, x_1^*,\dots,x_n^*\in E^*,\, \sup_{x\in B_E}\|(x_k^*(x))_{k=1}^n\|_{\ell_p^n}\leqslant 1\right\}$$

for any $f\in\mathrm{FVL}[E]$.

Proof. For short, let us denote the expression on the right-hand side of the statement by ρ_p. As observed above, by Theorem 2.2.1(1), $\mathcal{C}_p = \overline{\mathcal{X}}_p$, so by Corollary 2.1.4 we obtain that for any Banach space E, $\rho_{\mathcal{C}_p} = \rho_{\mathcal{X}_p}$ on $\mathrm{FVL}[E]$. Now, if μ is a probability measure, there is always a non-atomic probability measure ν such that $L_p(\mu)$ embeds lattice isometrically into $L_p(\nu)$. Thus, $\rho_{\{L_p(\mu)\}} \leqslant \rho_{\{L_p(\nu)\}}$, where both norms are associated to the singleton classes $\{L_p(\mu)\}$ and $\{L_p(\nu)\}$, respectively. Moreover, by Theorem 2.2.12, the norm $\rho_{\{L_p(\nu)\}}$ coincides with ρ_p. In particular, $\rho_p \leqslant \rho_{\mathcal{C}_p}$, since $L_p(\nu)$ is in the class $\mathcal{C}_p$. Putting everything together we get

$$\rho_{\mathcal{C}_p} = \rho_{\mathcal{X}_p} = \sup_{\mu \text{ finite}} \rho_{\{L_p(\mu)\}} \leqslant \rho_p \leqslant \rho_{\mathcal{C}_p}.$$

Hence, $\rho_{\mathcal{C}_p} = \rho_p$, as claimed. □

The situation presented in Theorem 2.2.1(2) is slightly different due to the isomorphism constant γ_p. Again, fix $1 < p < \infty$ and let $\mathcal{C}_{p,\infty}$ be the class of all (p,∞)-convex Banach lattices with constant 1 and $\mathcal{X}_{p,\infty}$ the class of all $L_{p,\infty}(\mu)$ spaces with μ a σ-finite measure. Clearly, $\overline{\mathcal{X}}_{p,\infty} \subseteq \mathcal{C}_{p,\infty}$. On the other hand, statement (2) in Theorem 2.2.1 establishes that any Banach lattice $X \in \mathcal{C}_{p,\infty}$ is γ_p-lattice isomorphic to an element of $\overline{\mathcal{X}}_{p,\infty}$. This correspondence allows us to find an explicit representation of an equivalent norm in $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E] \equiv \mathrm{FBL}^{p,\infty}[E]$ for any Banach space E, solving the question raised in [75, Remark 6.2] in the upper p-estimates setting:

Theorem 2.2.14. *Let E be a Banach space and let $1 < p < \infty$. The norm $\rho_{\mathcal{X}_{p,\infty}}$ on $\mathrm{FVL}[E]$ is given by*

$$\rho_{\mathcal{X}_{p,\infty}}(f) = \sup\left\{ \|(f(x_k^*))_{k=1}^n\|_{\ell_{p,\infty}^n} : n \in \mathbb{N},\, x_1^*, \dots, x_n^* \in E^*,\, \sup_{x \in B_E} \|(x_k^*(x))_{k=1}^n\|_{\ell_{p,\infty}^n} \leqslant 1 \right\}$$

for any $f \in \mathrm{FVL}[E]$. Moreover, $\rho_{\mathcal{X}_{p,\infty}} \leqslant \rho_{\mathcal{C}_{p,\infty}} \leqslant \gamma_p\, \rho_{\mathcal{X}_{p,\infty}}$, where $\gamma_p = (1 - \frac{1}{p})^{\frac{1}{p}-1}$.

Proof. The proof of the first part of the statement is similar to that of Theorem 2.2.13. Now, since $\mathcal{X}_{p,\infty} \subseteq \mathcal{C}_{p,\infty}$, $\rho_{\mathcal{X}_{p,\infty}} \leqslant \rho_{\mathcal{C}_{p,\infty}}$. On the other hand, let $X \in \mathcal{C}_{p,\infty}$. By Theorem 2.2.1, there exists $Y \in \overline{\mathcal{X}}_{p,\infty}$ and a lattice isomorphism $J : X \to Y$ so that $\|x\| \leqslant \|Jx\| \leqslant \gamma_p \|x\|$ for all $x \in X$. Let $T : E \to X$ be a linear contraction. Then $\gamma_p^{-1} JT : E \to Y$ is a linear contraction. Since J is a lattice homomorphism, $\widehat{\gamma_p^{-1} JT} = \gamma_p^{-1} J\widehat{T}$. For any $f \in \mathrm{FVL}[E]$,

$$\|\widehat{T} f\| \leqslant \gamma_p \|\gamma_p^{-1} J\widehat{T} f\| = \gamma_p \|\widehat{\gamma_p^{-1} JT} f\| \leqslant \gamma_p \rho_{\mathcal{X}_{p,\infty}}(f).$$

Taking supremum over all linear contractions $T : E \to X$ for any $X \in \mathcal{C}_{p,\infty}$ shows that $\rho_{\mathcal{C}_{p,\infty}} \leqslant \gamma_p \rho_{\mathcal{X}_{p,\infty}}$. □

Note that $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E]$ endowed with the equivalent norm $\rho_{\mathcal{X}_{p,\infty}}$ satisfies the following universal property: given any Banach lattice X satisfying an upper p-estimate with constant 1 and a linear operator $T : E \to X$, there exists a unique lattice homomorphic extension $\widehat{T} : \mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E] \to X$ with $\|\widehat{T}\| \leqslant \gamma_p \|T\|$. Moreover, if $X \in \overline{\mathcal{X}}_{p,\infty}$ then actually $\|\widehat{T}\| = \|T\|$.

Finally, observe that Theorems 2.2.13 and 2.2.14 state that $\rho_{\mathcal{C}_p} = \rho_{\{\ell_p\}}$ and that $\rho_{\{\ell_{p,\infty}\}} \leqslant \rho_{\mathcal{C}_{p,\infty}} \leqslant \gamma_p\, \rho_{\{\ell_{p,\infty}\}}$. As a result, $\mathrm{FBL}^{\mathcal{C}_p}[E] = \mathrm{FBL}^{\{\ell_p\}}[E]$ and $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E] = \mathrm{FBL}^{\{\ell_{p,\infty}\}}[E]$ and it is clear that both of these spaces are vector sublattices of $C(B_{E^*})$. The proof of Theorem 2.2.13 works in exactly the same way if $p = 1$, where we obtain that the free Banach lattice generated by a Banach space, which by Remark 2.1.1 is equal to the free Banach lattice associated to the class $\mathcal{C}_1$ of all (1-convex) Banach lattices, can be represented using the norm obtained in [11, Theorem 2.5].

2.2.5 Classes of Banach lattices generating $\mathbf{FBL}^p[E]$ and $\mathbf{FBL}^{p,\infty}[E]$.

Recall from the previous subsection that $\mathcal{C}_p$ and $\mathcal{C}_{p,\infty}$ denote the classes of p-convex Banach lattices with constant 1 and Banach lattices satisfying an upper p-estimate with constant 1, respectively. We will characterize the classes of Banach lattices $\mathcal{D}$ so that $\mathrm{FBL}^p[E] \equiv (\mathrm{FBL}^{\mathcal{C}_p}[E], \rho_{\mathcal{C}_p})$ and $(\mathrm{FBL}^{\mathcal{D}}[E], \rho_{\mathcal{D}})$ agree for any Banach space E, and the classes $\mathcal{D}'$ so that $\mathrm{FBL}^{p,\infty}[E] \equiv (\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E], \rho_{\mathcal{C}_{p,\infty}})$ and $(\mathrm{FBL}^{\mathcal{D}'}[E], \rho_{\mathcal{D}'})$ are lattice isomorphic. But first, let us show that the constant γ_p in Theorem 2.2.14 cannot be made 1, i.e., that the free Banach lattices $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E]$ and $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[E]$ are not isometric for some Banach spaces E.

For the proofs of the claims above, we need an extension of the isometric lattice-lifting property which may be of independent interest. Recall from Section 1.3 that a Banach lattice X is said to have the *isometric lattice-lifting property* if there exists a lattice isometric embedding $\alpha : X \to \mathrm{FBL}[X]$ such that $\widehat{id_X}\alpha = id_X$. This property was introduced in the article [9], inspired by previous works of Godefroy and Kalton [61] on Lipschitz-free spaces. Notably, Banach lattices ordered by a 1-unconditional basis have this property (see [9, Theorem 4.1]). In [103, Theorem 8.3] an alternative proof of this fact was given, which also worked when $\mathrm{FBL}[E]$ was replaced by $\mathrm{FBL}^p[E]$. We now show how to generalize these results to any class $\mathcal{C}$.

Proposition 2.2.15. *Let $\mathcal{C}$ be a class of Banach lattices, X a Banach space with a normalized 1-unconditional basis $(e_i)_i$, viewed as a Banach lattice with the pointwise order induced by the basis, and $C \geqslant 1$. The following are equivalent.*

(1) X C-lattice embeds into some $\ell_\infty(X_\gamma)_{\gamma\in\Gamma}$ with $X_\gamma \in \mathcal{C}$ for every $\gamma \in \Gamma$.

(2) For every operator $T : X \to X$, there exists a unique extension $\widehat{T} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ satisfying $\|\widehat{T}\| \leqslant C\|T\|$.

(3) There exists a unique extension $\widehat{id_X} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ of the identity $id_X : X \to X$ satisfying $\|\widehat{id_X}\| \leqslant C$.

(4) X C-lattice embeds into $\mathrm{FBL}^{\mathcal{C}}[X]$.

Moreover, the C-lattice embedding $\alpha : X \to \mathrm{FBL}^{\mathcal{C}}[X]$ in (4) can be chosen so that $\widehat{id_X}\alpha = id_X$.

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) Recall that by either [9, Theorem 4.1] or [103, Theorem 8.3] with $p = 1$ there is a lattice isometric embedding $\alpha_0 : X \to \mathrm{FBL}[X]$, the free Banach lattice associated to $\mathcal{C}_1$, the class of all Banach lattices (recall Remark 2.1.1), such that $\widehat{id_X}\alpha_0 = id_X$ (here, $\widehat{id_X} : \mathrm{FBL}[X] \to X$). Moreover, since $\mathcal{C} \subseteq \mathcal{C}_1$, the formal identity $j : \mathrm{FBL}^{\mathcal{C}_1}[X] \to \mathrm{FBL}^{\mathcal{C}}[X]$ is a continuous norm one injection such that $\widehat{id_X}j\alpha_0 = id_X$ (now, we are abusing the notation and considering the extension $\widehat{id_X} : \mathrm{FBL}^{\mathcal{C}}[X] \to X$ given in (3)). We claim that $\alpha := j\alpha_0 : X \to \mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]$ defines a C-lattice embedding. To prove the claim, simply note that for every $x \in X$

$$\frac{1}{C}\|x\| \leqslant \frac{1}{\|\widehat{id_X}\|}\|\widehat{id_X}\alpha x\| \leqslant \rho_{\mathcal{C}}(\alpha x) \leqslant \rho_{\mathcal{C}_1}(\alpha_0 x) \leqslant \|x\|.$$

This also proves the additional claim.

(4) $\Rightarrow$ (1) It follows from Proposition 2.1.5, as $\mathrm{FBL}^{\mathcal{C}}[X]$ belongs to $\overline{\mathcal{C}}$. □

Taking $C = 1$ in the previous proposition we obtain:

Corollary 2.2.16. *Let $\mathcal{C}$ be a class of Banach lattices and $X \in \overline{\mathcal{C}}$ a Banach space with a normalized 1-unconditional basis $(e_i)_i$, viewed as a Banach lattice with the pointwise order induced by the basis. Then $\mathrm{FBL}^{\mathcal{C}}[X]$ contains a lattice isometric copy of X. Moreover, there exists a contractive lattice homomorphic projection onto this sublattice.*

We can apply Proposition 2.2.15 to the Banach lattice X constructed in Example 2.2.9 to conclude that, since X does not 1-lattice embed into $\ell_\infty(L_{p,\infty}(\mu))_{\mu\in\Gamma}$ for any choice of measures Γ, the extension $\widehat{id_X}$ of the identity on X to $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]$ cannot be contractive. However, if we consider instead $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[X]$ (which by Theorem 2.2.14 coincides as a set with $\mathrm{FBL}^{\mathcal{X}_{p,\infty}}[X]$, with an equivalent renorming), we observe that in this setting the norm of the extension $\widehat{id_X}$ becomes $\|id_X\| = 1$, since X belongs to the class $\mathcal{C}_{p,\infty}$. Therefore, the norms $\rho_{\mathcal{X}_{p,\infty}}$ and $\rho_{\mathcal{C}_{p,\infty}}$ cannot agree on $\mathrm{FVL}[X]$.

Now that this is clarified, we characterize the classes of Banach lattices that generate (isometrically) the free Banach lattice $\mathrm{FBL}^{\mathcal{C}_p}[E]$, and those that generate (isomorphically) the free Banach lattice $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E]$.

Theorem 2.2.17. *The following are equivalent for a class of Banach lattices $\mathcal{D}$:*

(1) For any Banach space E, $\mathrm{FBL}^{\mathcal{C}_p}[E] = \mathrm{FBL}^{\mathcal{D}}[E]$ as sets and the norms $\rho_{\mathcal{C}_p}$ and $\rho_{\mathcal{D}}$ agree there.

(2) $\mathcal{D} \subseteq \mathcal{C}_p$ and $\ell_p \in \overline{\mathcal{D}}$.

Proof. (1)⇒(2) By Corollary 2.2.16, $\mathrm{FBL}^{\mathcal{D}}[\ell_p] = \mathrm{FBL}^{\mathcal{C}_p}[\ell_p]$ contains a lattice isometric copy of ℓ_p. By Proposition 2.1.5, $\mathrm{FBL}^{\mathcal{D}}[\ell_p] \in \overline{\mathcal{D}}$. Hence, $\ell_p \in \overline{\mathcal{D}}$.

Suppose that $X \in \mathcal{D}$. Let $\phi_X : X \to \mathrm{FBL}^{\mathcal{D}}[X]$ be the canonical embedding. The identity $id_X : X \to X$ induces a lattice homomorphic contraction $\widehat{id_X} : \mathrm{FBL}^{\mathcal{D}}[X] \to X$ so that $\widehat{id_X}\phi_X = id_X$. Since $\mathrm{FBL}^{\mathcal{D}}[X] = \mathrm{FBL}^{\mathcal{C}_p}[X]$ lattice isometrically, this space belongs to $\overline{\mathcal{C}}_p = \mathcal{C}_p$ by Proposition 2.1.5, i.e., it is p-convex with constant 1. Hence, so does $\mathrm{FBL}^{\mathcal{D}}[X]/\ker\widehat{id_X}$. Define $j : \mathrm{FBL}^{\mathcal{D}}[X]/\ker\widehat{id_X} \to X$ by $j[f] = \widehat{id_X} f$, where $[f]$ is the equivalence class of f. The map j is a lattice isomorphism so that $\|j\| \leqslant 1$ and $x = j[\delta_x]$ for any $x \in X$. Hence, $B_X = j(B_{\mathrm{FBL}^{\mathcal{D}}[X]/\ker\widehat{id_X}})$. This shows that j is an onto lattice isometry. Therefore, $X \in \mathcal{C}_p$.

(2)⇒(1) Since $\mathcal{D} \subseteq \mathcal{C}_p$, for any Banach space E, $\rho_{\mathcal{D}} \leqslant \rho_{\mathcal{C}_p}$ on $\mathrm{FVL}[E]$. Similarly, since $\ell_p \in \overline{\mathcal{D}}$, $\rho_{\{\ell_p\}} \leqslant \rho_{\overline{\mathcal{D}}}$. However, $\rho_{\mathcal{C}_p} = \rho_{\{\ell_p\}}$ by Theorem 2.2.13 and $\rho_{\overline{\mathcal{D}}} = \rho_{\mathcal{D}}$ by Corollary 2.1.4. Therefore, we have the reverse inequality $\rho_{\mathcal{C}_p} \leqslant \rho_{\mathcal{D}}$. □

For the case of $\mathcal{C}_{p,\infty}$, the following result can be proven in essentially the same way as Theorem 2.2.17. The only difference is that, since $\ell_{p,\infty}$ does not have a basis, Corollary 2.2.16 cannot be applied directly. However, we can still use Corollary 2.2.16 to embed $\ell_{p,\infty}$ lattice isometrically into $\ell_\infty(\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[\ell_{p,\infty}^m])_{m\in\mathbb{N}}$ and continue with the rest of the proof with slight adaptations.

Theorem 2.2.18. *The following are equivalent for a class of Banach lattices $\mathcal{D}$:*

(1) There is a constant $M_1 < \infty$ so that for any Banach space E, $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E] = \mathrm{FBL}^{\mathcal{D}}[E]$ as sets and $\frac{1}{M_1}\rho_{\mathcal{D}} \leqslant \rho_{\mathcal{C}_{p,\infty}} \leqslant M_1\rho_{\mathcal{D}}$ on that set.

(2) There is a constant $M_2 < \infty$ so that any $X \in \mathcal{D}$ satisfies an upper p-estimate with constant M_2, and $\ell_{p,\infty}$ is M_2-lattice isomorphic to a space $Y \in \overline{\mathcal{D}}$.

2.2.6 The renormings of weak-L_p

Recall from Section 1.2 that we chose to equip $L_{p,\infty}(\mu)$ with the norm

$$\|f\|_{L_{p,\infty}^{[1]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-1} \int_A |f|\, d\mu.$$

However, for each $1 \leqslant r < p$, we could have also chosen to equip $L_{p,\infty}(\mu)$ with the norm

$$\|f\|_{L_{p,\infty}^{[r]}} := \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-\frac{1}{r}} \left(\int_A |f|^r \, d\mu \right)^{\frac{1}{r}},$$

as it also defines a lattice norm on $L_{p,\infty}(\mu)$ with upper p-estimate constant 1. For this reason, it may be natural to ask whether the ambiguity in choosing a norm on $L_{p,\infty}(\mu)$ is the reason why γ_p cannot be chosen to be 1 in Theorems 2.2.1 and 2.2.14.

The aim of this subsection is to show that $(L_{p,\infty}(\mu), \|\cdot\|_{L_{p,\infty}^{[r]}})$ lattice isometrically embeds into a Banach lattice of the form $\ell_\infty(L_{p,\infty}^{[1]}(\nu))_{\nu\in\Gamma}$. As a consequence, we deduce that the above renormings cannot eliminate the constant γ_p in the aforementioned results.

Before starting, it is convenient to better understand the differences between these renormings. As we mentioned in the previous chapter, it is well known that every Banach lattice with upper p-estimates is r-convex for every $1 \leqslant r < p$, with a constant that might explode as $(p-r)^{-\frac{1}{p}}$ when r approaches p. Moreover, every r-convex Banach lattice can be renormed so that the r-convexity constant becomes 1. As we saw in the proof of Corollary 2.2.2, it turns out that in the case of $L_{p,\infty}$, the renorming $L_{p,\infty}^{[r]}$ satisfies that both the upper p-estimates constant and the r-convexity constant are simultaneously 1. Now that we have clarified the role of the renormings, we can proceed with the main goal of the section. We begin with some technical lemmata.

Lemma 2.2.19. *Let $e = (1,\dots,1) \in \mathbb{R}^n$ and let $(e_i)_{i=1}^n$ be the coordinate unit vector basis for $\mathbb{R}^n$. Let $\beta, b \in \mathbb{R}_+^n$, $\|\beta\|_1, \|b\|_1 \leqslant 1$, and let $0 < s < 1$. Define $\alpha : \mathbb{R}_+^n \to \mathbb{R}$ by $\alpha(x) = (\beta\cdot x)^{1-s}\,(b\cdot x)^s$, where $\cdot$ is the dot product on $\mathbb{R}^n$. Then α is positively homogeneous, $\alpha(e) = \|\beta\|_1^{1-s}\cdot\|b\|_1^s \leqslant 1$ and $\sum_{j=1}^m \alpha(x_j) \leqslant \alpha(\sum_{j=1}^m x_j)$ for any sequence $(x_j)_{j=1}^m \subseteq \mathbb{R}_+^n$.*

Proof. Let $(x_j)_{j=1}^m \subseteq \mathbb{R}_+^n$. Then

$$\begin{aligned}\sum_{j=1}^m \alpha(x_j) &\leqslant \|((\beta\cdot x_j)^{1-s})_j\|_{\ell^m_{\frac{1}{1-s}}} \, \|((b\cdot x_j)^s)_j\|_{\ell^m_{\frac{1}{s}}} \\ &= \left(\beta\cdot\sum_{j=1}^m x_j\right)^{1-s}\left(b\cdot\sum_{j=1}^m x_j\right)^s = \alpha\left(\sum_{j=1}^m x_j\right).\end{aligned}$$

The remaining assertions are clear. □

Let $x \in \mathbb{R}^n$ be expressed as $(x(j))_{j=1}^n$. In the next result, for fixed $i \in \mathbb{N}$, we write $x < i$ if $\max(\operatorname{supp} x) < i$.

Proposition 2.2.20. *There is a sequence $(d_i)_{i=1}^n$ of non-negative real numbers such that for each i,*

$$\sum_{j=1}^{i-1} x(j)d_j + \alpha(z) - m - \sum_{j=1}^{i-1} y(j)d_j \leqslant d_i \leqslant m' + \sum_{j=1}^{i-1} y'(j)d_j - \sum_{j=1}^{i-1} x'(j)d_j - \alpha(z')$$

provided that $m, m' \in \mathbb{R}_+$, $x, y, z, x', y', z' \in \mathbb{R}_+^n$,

$$x, y, x', y' < i, \quad x + z = e_i + me + y \quad \text{and} \quad e_i + x' + z' = m'e + y'. \tag{2.3}$$

Proof. The proof is by induction on i. For $i = 1$, $x = y = x' = y' = 0$. Thus it suffices to show that

$$\alpha(z) - m \leqslant m' - \alpha(z') \quad \text{if} \quad z = e_1 + me \ \text{ and } \ e_1 + z' = m'e.$$

To see this, note that

$$m + m' \geqslant \alpha((m + m')e) = \alpha(z + z') \geqslant \alpha(z) + \alpha(z'),$$

from which the desired conclusion follows.

Assume now that d_j, $1 \leqslant j < i$, have been chosen to satisfy the proposition. To be able to choose d_i satisfying the proposition, we need to show that

$$\sum_{j=1}^{i-1} x(j)d_j + \alpha(z) - m - \sum_{j=1}^{i-1} y(j)d_j \leqslant m' + \sum_{j=1}^{i-1} y'(j)d_j - \sum_{j=1}^{i-1} x'(j)d_j - \alpha(z')$$

if $m, m', x, y, z, x', y', z'$ satisfy (2.3). Equivalently, we must show that

$$\sum_{j=1}^{i-1} (x(j) + x'(j))d_j + \alpha(z) + \alpha(z') \leqslant m + m' + \sum_{j=1}^{i-1} (y(j) + y'(j))d_j. \tag{2.4}$$

If $x + x' = y + y'$, then (2.3) gives $z + z' = (m + m')e$. Thus, $m + m' \geqslant \alpha(z) + \alpha(z')$, yielding (2.4). Otherwise, let $u = (x + x') \wedge (y + y') \in \mathbb{R}^n_+$. Then $u < i$ and $v_1 := x + x' - u$, $v_2 := y + y' - u$ are disjoint in $\mathbb{R}^n_+$, not both 0. Assume that $j_0 = \max \operatorname{supp} v_1 > \max \operatorname{supp} v_2$, as the other case is similar. Set $w = v_1 - v_1(j_0)e_{j_0}$. Then $w, v_2 < j_0$, $w(j) = v_1(j)$ if $j < j_0$ and

$$v_1(j_0)e_{j_0} + w + (z + z') = (m + m')e + v_2.$$

Using the right inequality in the inductive hypothesis for the elements w, v_2 and $z + z'$ and the positive homogeneity of α, we see that

$$v_1(j_0)d_{j_0} \leqslant (m + m') + \sum_{j=1}^{j_0-1} v_2(j)d_j - \sum_{j=1}^{j_0-1} v_1(j)d_j - \alpha(z + z').$$

Adding $\sum_{j=1}^{i-1} u(j)d_j$ to both sides and rearranging gives

$$\sum_{j=1}^{i-1} (x(j) + x'(j))d_j + \alpha(z + z') \leqslant m + m' + \sum_{j=1}^{i-1} (y(j) + y'(j))d_j.$$

The desired inequality (2.4) then follows since $\alpha(z) + \alpha(z') \leqslant \alpha(z + z')$. □

Corollary 2.2.21. *In the above notation, we have* $d = (d_i)_{i=1}^n \in \mathbb{R}^n_+$, $\alpha(e) \leqslant \|d\|_1 \leqslant 1$ *and* $\alpha(x) \leqslant x \cdot d$ *for any* $x \in \mathbb{R}^n_+$.

Proof. Let

$$x = 0,\ y = e - e_n,\ z = e,\ m = 0;\ x' = e - e_n,\ y' = 0,\ z' = 0,\ m' = 1.$$

Note that $x, y, x', y' < n$, $x + z = e_n + me + y$ and $e_n + x' + z' = m'e + y'$. Thus, taking $i = n$ in Proposition 2.2.20 we have

$$\alpha(e) - \sum_{j=1}^{n-1} d_j \leqslant d_n \leqslant 1 - \sum_{j=1}^{n-1} d_j.$$

Hence, $\alpha(e) \leqslant \|d\|_1 \leqslant 1$. Finally, if $u \in \mathbb{R}^n_+$, let $x = 0$, $z = u$, $m = 0$ and $y = u - u(n)e_n$. Then $x + z = u(n)e_n + me + y$. By Proposition 2.2.20 and the positive homogeneity of α,

$$\alpha(u) - \sum_{j=1}^{n-1} u(j)d_j \leqslant u(n)d_n, \quad \text{so} \quad \alpha(u) \leqslant u \cdot d.$$

□

Lemma 2.2.22. *Let (Ω, Σ, μ) be a measure space and $1 < r < p$. Suppose that $(a_i)_{i=1}^n \in \mathbb{R}^n_+$ and $(U_i)_{i=1}^n$ is a disjoint sequence in Σ with $0 < \mu(U_i) < \infty$ so that $C := (\sum_{i=1}^n \mu(U_i))^{-\frac{1}{r}+\frac{1}{p}} \|\sum_{i=1}^n a_i\chi_{U_i}\|_{L_r(\mu)} \leqslant 1$. There is a measure ν on Σ and a lattice homomorphism $S : L^{[r]}_{p,\infty}(\mu) \to L^{[1]}_{p,\infty}(\nu)$ so that $\|S\| \leqslant 1$ and $\|S(\sum_{i=1}^n a_i\chi_{U_i})\| \geqslant C^r$.*

Proof. Define $M = \sum_{j=1}^n \mu(U_j)$, $b_i = \frac{\mu(U_i)}{M}$, $b = (b_i)_{i=1}^n$ and $\beta_i = M^{\frac{r}{p}-1}\mu(U_i)a_i^r$, $\beta = (\beta_i)_{i=1}^n$. Clearly, $\beta, b \in \mathbb{R}^n_+$ and $\|b\|_1 = 1$. Moreover,

$$\|\beta\|_1 = M^{\frac{r}{p}-1}\left\|\sum_{i=1}^n a_i\chi_{U_i}\right\|^r_{L_r(\mu)} = C^r \leqslant 1.$$

Take $s = p^*(\frac{1}{r} - \frac{1}{p})$ and define α as in Lemma 2.2.19. Note that $\alpha(e) = \|\beta\|_1^{1-s}\|b\|_1^s = C^{r(1-s)} \leqslant 1$. Obtain d using Proposition 2.2.20 and define a measure ν on Σ by $\nu(A) = \sum_{i=1}^n \frac{d_i\,\mu(A\cap U_i)}{\mu(U_i)}$ and a linear operator

$$S : L^{[r]}_{p,\infty}(\mu) \to L^{[1]}_{p,\infty}(\nu) \quad \text{by} \quad Sf = M^{\frac{r}{p}}\sum_{i=1}^n \frac{a_i^{r-1}b_i}{d_i} f\chi_{U_i}.$$

Clearly, S is a lattice homomorphism. Note that $\nu(\bigcup_{i=1}^n U_i) = \sum_{i=1}^n d_i \leqslant 1$. Hence,

$$\begin{aligned}\left\|S\left(\sum_{i=1}^n a_i\chi_{U_i}\right)\right\| &\geqslant M^{\frac{r}{p}}\left\|\sum_{i=1}^n \frac{a_i^r b_i}{d_i}\chi_{U_i}\right\|_{L_1(\nu)} = M^{\frac{r}{p}}\sum_{i=1}^n a_i^r b_i \\ &= M^{\frac{r}{p}-1}\left\|\sum_{i=1}^n a_i\chi_{U_i}\right\|^r_{L_r(\mu)} = C^r.\end{aligned}$$

On the other hand, let $f \in L^{[r]}_{p,\infty}(\mu)$ with $\|f\| \leqslant 1$. If $A \in \Sigma$ with $0 < \nu(A) < \infty$ then

$$\begin{aligned}\int_A |Sf|\,d\nu &= M^{\frac{r}{p}}\sum_{i=1}^n \frac{a_i^{r-1}b_i}{d_i}\int_{A\cap U_i}|f|\,d\nu = M^{\frac{r}{p}-1}\sum_{i=1}^n a_i^{r-1}\int_{A\cap U_i}|f|\,d\mu \\ &\leqslant M^{\frac{r}{p}-1}\left\|\sum_{i=1}^n a_i^{r-1}\chi_{A\cap U_i}\right\|_{L_{r^*}(\mu)}\left\|f\chi_{\bigcup_{i=1}^n (A\cap U_i)}\right\|_{L_r(\mu)} \\ &\leqslant \left(\sum_{i=1}^n \beta_i\frac{\mu(A\cap U_i)}{\mu(U_i)}\right)^{\frac{1}{r^*}}\left(\sum_{i=1}^n b_i\frac{\mu(A\cap U_i)}{\mu(U_i)}\right)^{\frac{1}{r}-\frac{1}{p}}.\end{aligned}$$

Let $x = \left(\frac{\mu(A\cap U_i)}{\mu(U_i)}\right)_{i=1}^n \in \mathbb{R}^n_+$. By direct calculation, $1 - s = \frac{p^*}{r^*}$. Thus,

$$\int_A |Sf|\,d\nu \leqslant (\beta\cdot x)^{\frac{1-s}{p^*}}(b\cdot x)^{\frac{s}{p^*}} = \alpha(x)^{\frac{1}{p^*}} \leqslant (x\cdot d)^{\frac{1}{p^*}} = \nu(A)^{\frac{1}{p^*}}.$$

Hence, $\|Sf\| \leqslant 1$. □

We are now in disposition of proving the claim that we made at the beginning of the subsection.

Theorem 2.2.23. *Suppose that (Ω, Σ, μ) is a measure space and $1 < r < p$. There is a set Γ of measures on Σ so that $L_{p,\infty}^{[r]}(\mu)$ lattice isometrically embeds into $\ell_\infty(L_{p,\infty}^{[1]}(\nu))_{\nu\in\Gamma}$.*

Proof. Let Γ be the set of simple functions $f = \sum_{i=1}^n a_i\chi_{U_i}$, where $n \in \mathbb{N}$, $a_i > 0$, $0 < \mu(U_i) < \infty$ and $(U_i)_{i=1}^n$ is a disjoint sequence in Σ satisfying

$$C_f = \left(\sum_{i=1}^n \mu(U_i)\right)^{-\frac{1}{r}+\frac{1}{p}} \left\|\sum_{i=1}^n a_i\chi_{U_i}\right\|_{L_r(\mu)} \leqslant 1.$$

For each $f \in \Gamma$, use the above results to choose a lattice homomorphism $S_f : L_{p,\infty}^{[r]}(\mu) \to L_{p,\infty}^{[1]}(\nu_f)$ so that $\|S_f\| \leqslant 1$ and $\|S_f f\| \geqslant C_f^r$. Define

$$\begin{aligned} T : L_{p,\infty}^{[r]}(\mu) &\longrightarrow \ell_\infty(L_{p,\infty}^{[1]}(\nu_f))_{f\in\Gamma} \\ g &\longmapsto Tg = (S_f g)_{f\in\Gamma} \end{aligned}.$$

Clearly, T is a lattice homomorphism and $\|T\| \leqslant 1$. Suppose that $g \in L_{p,\infty}^{[r]}(\mu)_+$ with $\|g\|_{L_{p,\infty}^{[r]}} = 1$. For any $\varepsilon > 0$, there exists a non-negative simple function h such that $g \geqslant h$ and $\|h\|_{L_{p,\infty}^{[r]}} \geqslant 1 - \varepsilon$. Write $h = \sum_{i=1}^n a_i\chi_{U_i}$. There exists $I \subseteq \{1, \ldots, n\}$ so that

$$\left(\sum_{i\in I} \mu(U_i)\right)^{-\frac{1}{r}+\frac{1}{p}} \left\|\sum_{i\in I} a_i\chi_{U_i}\right\|_r = \|h\|_{L_{p,\infty}^{[r]}} \geqslant 1 - \varepsilon.$$

Then $f := \sum_{i\in I} a_i\chi_{U_i} \in \Gamma$ with $C_f \geqslant 1 - \varepsilon$. Thus,

$$\|Tg\| \geqslant \|S_f g\| \geqslant \|S_f f\| \geqslant C_f^r \geqslant (1 - \varepsilon)^r,$$

so that $\|Tg\| \geqslant 1$. This shows that T is an isometric embedding. □

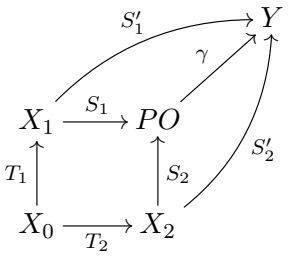
2.3 A solution to the embedding problem via push-outs of Banach lattices

2.3.1 Push-outs in a class $\mathcal{C}$ of Banach lattices

Recall that given a category and objects X_0, X_1, X_2 and morphisms $T_i : X_0 \to X_i$, $i = 1, 2$, a *push-out diagram* is an object $PO = PO(T_1, T_2)$ together with morphisms $S_i : X_i \to PO$, $i = 1, 2$, making the following diagram commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{S_1} & PO \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle S_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

and with the universal property that if $S_i' : X_i \to Y$ are such that $S_1'T_1 = S_2'T_2$, then there is a unique $\gamma : PO \to Y$ such that $\gamma S_i = S_i'$ for $i = 1, 2$, as follows:



In the category $\mathcal{BL}$ of Banach lattices and lattice homomorphisms, *isometric push-outs* were shown to exist in [14]. These are push-outs satisfying the extra condition that $\max\{\|S_1\|, \|S_2\|\} \leqslant 1$ and which guarantee the inequality $\|\gamma\| \leqslant \max\{\|S_1'\|, \|S_2'\|\}$ in the universal property. The construction of such push-outs can be adapted to Banach lattices within certain classes $\overline{\mathcal{C}}$, as follows.

Theorem 2.3.1. *Let $\mathcal{C}$ be a class of Banach lattices such that $\overline{\mathcal{C}}$ is closed under lattice quotients. Given Banach lattices X_0, X_1, X_2 in $\overline{\mathcal{C}}$ and lattice homomorphisms $T_i : X_0 \to X_i$ for $i = 1, 2$, there is a Banach lattice $PO^{\overline{\mathcal{C}}}$ in $\overline{\mathcal{C}}$ and lattice homomorphisms S_1, S_2 so that the following is an isometric push-out diagram in $\overline{\mathcal{C}}$:*

$$\begin{array}{ccc} X_1 & \xrightarrow{S_1} & PO^{\overline{\mathcal{C}}} \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle S_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

Proof. Let $X_1 \oplus_1 X_2$ denote the direct sum equipped with the norm $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$ and let $j_i : X_i \to X_1 \oplus_1 X_2$ denote the canonical embedding for $i = 1, 2$. Let $\phi : X_1 \oplus_1 X_2 \to \mathrm{FBL}^{\mathcal{C}}[X_1 \oplus_1 X_2]$ be the canonical embedding and Z be the (closed) ideal in $\mathrm{FBL}^{\mathcal{C}}[X_1 \oplus_1 X_2]$ generated by the families $(\phi j_1|x| - |\phi j_1 x|)_{x \in X_1}$, $(\phi j_2|y| - |\phi j_2 y|)_{y \in X_2}$ and $(\phi j_1 T_1 z - \phi j_2 T_2 z)_{z \in X_0}$. Let

$$PO^{\overline{\mathcal{C}}} = \mathrm{FBL}^{\mathcal{C}}[X_1 \oplus_1 X_2]/Z,$$

and let $S_i = q\phi j_i : X_i \to PO^{\overline{\mathcal{C}}}$, for $i = 1, 2$, where $q : \mathrm{FBL}^{\mathcal{C}}[X_1 \oplus_1 X_2] \to PO^{\overline{\mathcal{C}}}$ denotes the canonical quotient map.

Since $\overline{\mathcal{C}}$ is closed under lattice quotients and $\mathrm{FBL}^{\mathcal{C}}[X_1 \oplus_1 X_2]$ is in $\overline{\mathcal{C}}$, so is $PO^{\overline{\mathcal{C}}}$. The rest of the proof follows exactly as in [14, Theorem 4.3]. □

Theorem 2.3.1 applies, in particular, when $\mathcal{C}$ is the class $\mathcal{C}_p$ of p-convex Banach lattices or the class $\mathcal{C}_{p,\infty}$ of Banach lattices with upper p-estimates (with constant 1). For these classes, we are also interested in extending [14, Theorem 4.4], where the authors show that the isometric push-out in the category $\mathcal{BL}$ that they have constructed preserves isometries. Namely, if in the definition of push-out diagram above $T_1 : X_0 \to X_1$ is contractive and $T_2 : X_0 \to X_2$ is an isometric embedding, they show that $S_1 : X_1 \to PO$ is an isometric embedding. Extending this result to the class $\mathcal{C}_p$ is straightforward using p-convexification/p-concavification techniques:

Theorem 2.3.2. *Let $1 < p < \infty$ and $\mathcal{C}_p$ be the class of p-convex Banach lattices (with constant 1). Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{S_1} & PO^{\mathcal{C}_p} \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle S_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

be an isometric push-out diagram in $\mathcal{C}_p$. If T_1 is contractive and T_2 is an isometric embedding, then S_1 is an isometric embedding.

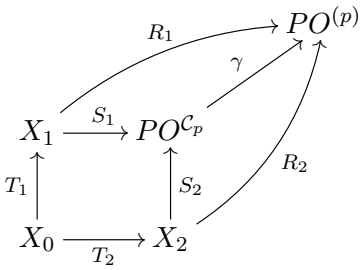
Proof. Since X_0, X_1 and X_2 are p-convex Banach lattices, we can p-concavify the spaces. The maps $T_i : X_{0(p)} \to X_{i(p)}$, $i = 1, 2$, will still be contractive lattice homomorphisms, and moreover T_2 will still be an isometry:

$$\|T_2 z\|_{X_{2(p)}} = \|T_2 z\|_{X_2}^p = \|z\|_{X_0}^p = \|z\|_{X_{0(p)}}.$$

Let PO be a push-out in the category of all Banach lattices for this new diagram:

$$\begin{array}{ccc} X_{1(p)} & \xrightarrow{R_1} & PO \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle R_2} \\ X_{0(p)} & \xrightarrow[T_2]{} & X_{2(p)} \end{array}$$

By [14, Theorem 4.4] we know that $R_1 : X_{1(p)} \to PO$ is an isometric embedding. If we p-convexify this diagram, we obtain that for every $i = 0, 1, 2$, $(X_{i(p)})^{(p)} = X_i$, as X_i has p-convexity constant 1. Arguing as before, $R_i : X_i \to PO^{(p)}$, $i = 1, 2$, are contractive, and R_1 is an isometry. We now apply the minimality property of $PO^{\mathcal{C}_p}$ to find a lattice homomorphism $\gamma : PO^{\mathcal{C}_p} \to PO^{(p)}$ such that $R_i = \gamma S_i$, $i = 1, 2$, and $\|\gamma\| \leqslant \max\{\|R_1\|, \|R_2\|\} \leqslant 1$:



It follows that

$$\|y\|_{X_1} = \|R_1 y\|_{PO^{(p)}} = \|\gamma S_1 y\|_{PO^{(p)}} \leqslant \|S_1 y\|_{PO^{\mathcal{C}_p}} \leqslant \|y\|_{X_1},$$

so S_1 is an isometry. □

Unfortunately, this technique cannot be used in the upper p-estimates setting, so alternative approaches using factorization techniques are needed. However, in this result the isometry becomes an isomorphism with a universal constant K_p:

Theorem 2.3.3. *Let $1 < p < \infty$ and $\mathcal{C}_{p,\infty}$ be the class of Banach lattices with upper p-estimates (with constant 1). Let*

$$\begin{array}{ccc} X_1 & \xrightarrow{S_1} & PO^{\mathcal{C}_{p,\infty}} \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle S_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

be an isometric push-out diagram in $\mathcal{C}_{p,\infty}$. There exists a universal constant $K_p \geqslant 1$ depending only on p such that if T_1 is contractive and T_2 is an isometric embedding, then S_1 is a K_p-embedding.

At the present time, we have come up with two different proofs of this result. The first one, presented in [54], consist of a factorization through $L_{p,\infty}$ argument. It uses the original statement of Pisier [108, Theorem 1.1], which is slightly more general than Theorem 1.2.4:

Theorem 2.3.4. *Let $1 \leqslant r < p < \infty$, $A \subseteq L_r(\mu)$. The following are equivalent:*

(1) There exists $C > 0$ such that for every $f_1, \ldots, f_n \in A$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$,

$$\left\| \bigvee_{i=1}^n |\alpha_i f_i| \right\|_{L_r(\mu)} \leqslant C \left(\sum_{i=1}^n |\alpha_i|^p \right)^{\frac{1}{p}}.$$

(2) There exists $C' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $f \in A$ and measurable subset $B \subseteq \Omega$

$$\|f \chi_B\|_{L_r(\mu)} \leqslant C' \left(\int_B g \, d\mu \right)^{\frac{1}{r} - \frac{1}{p}}.$$

(3) There exists $C'' > 0$ and a normalized $g \in L_1(\mu)_+$ such that for every $f \in A$,

$$\{g = 0\} \subseteq \{f = 0\} \quad \text{and} \quad \|g^{-\frac{1}{r}} f\|_{L^{[r]}_{p,\infty}(g\cdot\mu)} \leqslant C''.$$

Moreover, $C \leqslant C'' \leqslant C' \leqslant (1 - \frac{r}{p})^{\frac{1}{p}-\frac{1}{r}} C$.

The constant obtained in this case is $K_p = 2^{\frac{1}{p^*}} \gamma_p$.

First proof of Theorem 2.3.3. Fix $y \in X_1$. We aim to show that $\|y\| \leqslant K_p \|S_1 y\|$. The reverse inequality is trivially true since $\|S_1\| \leqslant 1$. Since we are dealing with lattice norms and lattice homomorphisms, we can suppose that y is positive. By the push-out universal property, as given in Theorem 2.3.1, it is enough to find a commutative diagram in $\mathcal{C}_{p,\infty}$

$$\begin{array}{ccc} X_1 & \xrightarrow{\widehat{T}_1} & Z \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle \widehat{T}_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

such that $\|y\| \leqslant \|\widehat{T}_1 y\|$ and $\max\{\|\widehat{T}_1\|, \|\widehat{T}_2\|\} \leqslant K_p$.

To this end, by [14, Theorem 4.4] we have a Banach lattice isometric push-out diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{R_1} & PO \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle R_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

such that R_1 is an isometric embedding. Hence, by [14, Lemma 3.3], we can pick a positive σ-finite element $z^* \in PO^*$ of norm one such that $z^*(R_1 y) = \|y\|$.

The functional z^* induces a lattice semi-norm on PO given by $\|z\|_{z^*} = z^*(|z|)$. After making a quotient by the elements of norm 0, this becomes a norm that satisfies $\||z_1|+|z_2|\|_{z^*} = \|z_1\|_{z^*} + \|z_2\|_{z^*}$ for all $z_1, z_2 \in PO$. This identity extends to the completion of this normed lattice, which, by Kakutani's representation theorem [88, Theorem 1.b.2], is lattice isometric to a Banach lattice of the form $L_1(\nu_{z^*})$. The quotient induces a homomorphism $\psi_{z^*} : PO \to L_1(\nu_{z^*})$ of norm one.

For $i = 1, 2$, let $A_i = \psi_{z^*} R_i(B_{X_i})$ and set $A = A_1 \cup A_2 \subseteq L_1(\nu_{z^*})$. Since the X_i satisfy an upper p-estimate and the $\Psi_{z^*} R_i$ are contractive lattice homomorphisms we have that for all finitely supported sequences $(\alpha_i)_{i\in I}$ of real numbers and $(f_i)_{i\in I} \subseteq A$,

$$\begin{aligned} \left\| \bigvee_{i\in I} |\alpha_i f_i| \right\|_{L_1(\nu_{z^*})} &\leqslant \left\| \bigvee_{f_i\in A_1} |\alpha_i f_i| \right\|_{L_1(\nu_{z^*})} + \left\| \bigvee_{f_i\in A_2} |\alpha_i f_i| \right\|_{L_1(\nu_{z^*})} \\ &\leqslant \left(\sum_{f_i\in A_1} |\alpha_i|^p \right)^{\frac{1}{p}} + \left(\sum_{f_i\in A_2} |\alpha_i|^p \right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p^*}} \left(\sum_{i\in I} |\alpha_i|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hence, by Theorem 2.3.4, there is $g \in L_1(\nu_{z^*})_+$ with $\|g\|_{L_1(\nu_{z^*})} \leqslant 1$ and lattice homomorphisms $\widehat{R}_i : X_i \to L_{p,\infty}(g\nu_{z^*})$ that make the following diagram commutative

$$\begin{array}{ccccc} X_i & \xrightarrow{R_i} & PO & \xrightarrow{\psi_{z^*}} & L_1(\nu_{z^*}) \\ & {\scriptstyle \widehat{R}_i}\searrow & & \nearrow{\scriptstyle M_g} & \\ & & L_{p,\infty}(g\nu_{z^*}) & & \end{array}$$

with $\|\widehat{R}_i\| \leqslant 2^{\frac{1}{p^*}}\gamma_p =: K_p$ and M_g the operator of multiplication by g, which is contractive. It follows that

$$\begin{array}{ccc} X_1 & \xrightarrow{\widehat{R}_1} & L_{p,\infty}(g\nu_{z^*}) \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle \widehat{R}_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

is a commutative diagram in $\mathcal{C}_{p,\infty}$ and

$$\|y\| = z^*(|R_1 y|) = \|\psi_{z^*} R_1 y\|_{L_1(\nu_{z^*})} = \|M_g \widehat{R}_1 y\|_{L_1(\nu_{z^*})} \leqslant \|\widehat{R}_1 y\|_{L_{p,\infty}(g\nu_{z^*})}.$$

□

The second proof was obtained during the preparation of [55]. In this paper, we extend the results from [110] (see also [111]) on minimal and maximal factorizations of p-convex and p-concave operators, respectively, to the setting of (p,∞)-convex and $(p,1)$-concave operators. For this, we analyze the Banach lattices generated by certain convex sets which are induced by these (p,∞)-convex and $(p,1)$-concave operators. These arguments can be adapted to factor simultaneously two operators as we did in the previous proof. This has the advantage of not relying on Theorem 2.3.4, which makes the constant γ_p disappear, but the best constant we can obtain is $K_p = 2^{\frac{1}{p^*}}$.

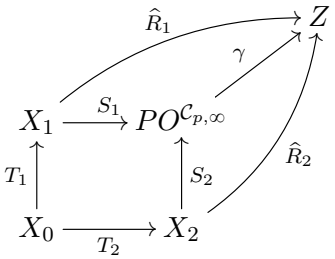
Second proof of Theorem 2.3.3. We begin the proof by considering the isometric push-out diagram in the category $\mathfrak{BL}$ from [14, Theorem 4.4], so that R_1 is an isometric embedding:

$$\begin{array}{ccc} X_1 & \xrightarrow{R_1} & PO \\ {\scriptstyle T_1}\uparrow & & \uparrow{\scriptstyle R_2} \\ X_0 & \xrightarrow[T_2]{} & X_2 \end{array}$$

Our aim is to construct a (p,∞)-convex Banach lattice Z with constant 1 and a lattice homomorphism $\psi : Z \to PO$ with $\|\psi\| \leqslant 2^{\frac{1}{p^*}} =: K_p$ such that for $i = 1, 2$, R_i factors through Z as $\psi\widehat{R}_i$, with $\widehat{R}_i : X_i \to Z$ a contractive lattice homomorphism. Once this is established, it follows that $\widehat{R}_1$ must be an isomorphism, since

$$\|y\|_{X_1} = \|R_1 y\|_{PO} \leqslant \|\psi\|\|\widehat{R}_1 y\|_Z \leqslant K_p \|\widehat{R}_1 y\|_Z$$

for every $y \in X_1$. Moreover, by the universal property of the push-out in $\mathcal{C}_{p,\infty}$ established in Theorem 2.3.1, there exists a contractive lattice homomorphism $\gamma : PO^{\mathcal{C}_{p,\infty}} \to Z$ that makes the following diagram commute:



It follows that

$$\|y\|_{X_1} \leqslant K_p \|\widehat{R}_1 y\|_Z = K_p \|\gamma S_1 y\|_Z \leqslant K_p \|S_1 y\|_{PO^{\mathcal{C}_{p,\infty}}},$$

as we wanted to show.

Let us construct such Banach lattice Z. Our candidate is $Z = \operatorname{span} B$, for $B \subseteq PO$ a certain closed, convex and solid set (it is standard to show that such Z endowed with the Minkowski functional is a Banach lattice) such that Z is (p,∞)-convex with constant 1 and $R_i(B_{X_i}) \subseteq B \subseteq K_p B_{PO}$. The latter will imply that R_i can be factored through Z as $R_i = \psi \widehat{R}_i$, with $\widehat{R}_i : X_i \to Z$ given by $\widehat{R}_i x = R_i x$ and $\psi : Z \to PO$ the formal identity, so that $\widehat{R}_i$ is contractive and $\|\psi\| \leqslant K_p$. Inspired by the techniques of [55], let us choose $B = C^{\circ\circ}$, where

$$C = \left\{ u \in PO \,: \exists (v_j)_{j=1}^m \subseteq X_1, (w_j)_{j=m+1}^n \subseteq X_2 \text{ such that } |u| \leqslant \left(\bigvee_{j=1}^m |R_1 v_j|\right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j|\right) \right.$$
$$\left. \text{and } \left(\sum_{j=1}^m \|v_j\|^p + \sum_{j=m+1}^n \|w_j\|^p\right)^{\frac{1}{p}} \leqslant 1 \right\}.$$

and the polars are taken with respect to $\langle PO, PO^* \rangle$. B is the closed convex hull of the solid set C, so it is closed, convex and solid. Clearly $R_i(B_{X_i}) \subseteq C \subseteq B$. Moreover, $C \subseteq K_p B_{PO}$. Indeed if $u \in C$, there exist $(v_j)_{j=1}^m \subseteq X_1$ and $(w_j)_{j=m+1}^n \subseteq X_2$ such that

$$|u| \leqslant \left(\bigvee_{j=1}^m |R_1 v_j|\right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j|\right) \text{ and } \left(\sum_{j=1}^m \|v_j\|^p + \sum_{j=m+1}^n \|w_j\|^p\right)^{\frac{1}{p}} \leqslant 1.$$

Therefore, using that X_i satisfy upper p-estimates with constant 1, and hence R_i are (p,∞)-convex with constant 1, we obtain that

$$\|u\| \leqslant \left\| \left(\bigvee_{j=1}^m |R_1 v_j|\right) \vee \left(\bigvee_{j=m+1}^n |R_2 w_j|\right) \right\| \leqslant \left\| \bigvee_{j=1}^m |R_1 v_j| \right\| + \left\| \bigvee_{j=m+1}^n |R_2 w_j| \right\|$$
$$\leqslant \left(\sum_{j=1}^m \|v_j\|^p\right)^{\frac{1}{p}} + \left(\sum_{j=m+1}^n \|w_j\|^p\right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p^*}}.$$

Since B_{PO} is already closed and convex, it follows that $B \subseteq K_p B_{PO}$. It remains to show that Z satisfies an upper p-estimate (equivalently, is (p,∞)-convex) with constant 1. This can be done by reproducing the arguments from [55], where the techniques and results from [110] and [96, Section 2.8] are explored from a general point of view. More specifically, we have to replicate the results of duality of convex sets and factorization of convex and concave operator for the (p,∞)-convex case, that is, choosing σ to be the c_0-norm and τ the ℓ_p-norm, replacing in all the proofs the roles of the sets $C_T^{\tau,\sigma}$ and $D_{\tau^*,\sigma^*}^{T^*}$ by C and

$$D = \left\{ u^* \in PO^* \,: |u^*| \geqslant \left(\sum_{j=1}^n |u_j^*|^{p_2^*}\right)^{\frac{1}{p_2}} \text{ implies that } \right.$$
$$\left. \forall J \subseteq \{1,\dots,n\} \left(\sum_{j\in J} \|R_1^* u_j^*\|^{p^*} + \sum_{j\notin J} \|R_2^* u_j^*\|^{p^*}\right)^{\frac{1}{p^*}} \leqslant 1 \right\},$$

respectively. Since we have already provided a proof of Theorem 2.3.3 with a different value of the constant K_p and it is not relevant to our purposes whether K_p takes the value $2^{\frac{1}{p^*}}$ or $2^{\frac{1}{p^*}}\gamma_p$, as both values are strictly greater than 1, we will omit the details of this last step in order not to make the proof disproportionately long. □

The contrast between Theorems 2.3.2 and 2.3.3 makes us wonder whether the constant K_p in the latter result is just an artifact of the proof or, on the contrary, it is a fundamental difference between p-convexity and upper p-estimates. It should be noted that if K_p could be taken to be

1 in Theorem 2.3.3, this would allow us to adapt other categorical constructions from [14], such as Banach lattices of universal disposition or separably injective Banach lattices, to the category of Banach lattices with upper p-estimates. Another related question that we have investigated is whether there exists a separable Banach lattice with upper p-estimates that is universal for the class of separable Banach lattices with upper p-estimates (with constant 1). That is, a separable Banach lattice X in $\mathcal{C}_{p,\infty}$ such that any other separable Y in $\mathcal{C}_{p,\infty}$ lattice isometrically (or isomorphically) embeds into X. The existence of separable universal spaces is known for several classes of Banach lattices [47, 87]. For instance, in [87] the authors show that $C(\Delta, L_1[0,1])$ is universal for the class of all separable Banach lattices, where Δ denotes the Cantor set. Moreover, Theorem 1.2.4 can be improved so that the operator T factors through $L^\circ_{p,\infty}$, the order continuous part of $L_{p,\infty}$, when E is separable. This points towards $C(\Delta, L^\circ_{p,\infty}[0,\infty))$ as a natural candidate.

Question 2.3.5. Is there a separable Banach lattice with upper p-estimates (with constant 1) that is universal for this class?

2.3.2 The embedding problem

As a consequence of the above, we get a solution to the embedding problem for free Banach lattices with upper p-estimates. Recall that if $\mathcal{C}$ is a class of Banach lattices and $\iota : F \to E$ is an isometric embedding between Banach spaces, we can consider the unique extension $\bar{\iota} = \widehat{\phi_E \iota} : \mathrm{FBL}^{\mathcal{C}}[F] \to \mathrm{FBL}^{\mathcal{C}}[E]$.

$$\begin{array}{ccc} \mathrm{FBL}^{\mathcal{C}}[F] & \xrightarrow{\bar{\iota}} & \mathrm{FBL}^{\mathcal{C}}[E] \\ {\scriptstyle \phi_F}\uparrow & & \uparrow{\scriptstyle \phi_E} \\ F & \xrightarrow[\iota]{} & E \end{array}$$

The *embedding problem* consist in characterizing the pairs of Banach spaces $F \subseteq E$ such that $\bar{\iota}$ is an embedding. This problem was introduced in [103, Section 3], where it was studied for the classes $\mathcal{C}_p$. Theorems 2.3.2 and 2.3.3 allow us to provide an alternative proof of this fact in the p-convex setting and solve it for the class $\mathcal{C}_{p,\infty}$.

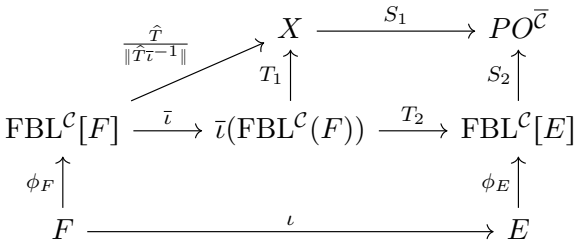
Theorem 2.3.6. *Suppose that $\iota : F \to E$ is an isometric embedding and let $\mathcal{C}$ denote either the class $\mathcal{C}_p$ of p-convex Banach lattices or the class $\mathcal{C}_{p,\infty}$ of Banach lattices with upper p-estimates. Let $K_{\mathcal{C}} = 1$ if $\mathcal{C} = \mathcal{C}_p$ and $K_{\mathcal{C}} = K_p$ if $\mathcal{C} = \mathcal{C}_{p,\infty}$. The following are equivalent:*

(1) There exists $c_1 \geqslant 1$ such that $\bar{\iota} : \mathrm{FBL}^{\mathcal{C}}[F] \to \mathrm{FBL}^{\mathcal{C}}[E]$ is a c_1-lattice embedding.

(2) There exists $c_2 > 0$ such that for every operator $T : F \to X$ with X in $\mathcal{C}$, there is Y in $\mathcal{C}$, a (norm one) $K_{\mathcal{C}}$-lattice embedding $j : X \to Y$ and $S : E \to Y$ so that $jT = S\iota$ and $\|S\| \leqslant c_2\|T\|$.

Moreover, the constants c_1 and c_2 can be taken so that $c_2 \leqslant c_1 \leqslant K_{\mathcal{C}} c_2$.

Proof. (1)⇒(2) Let $X \in \mathcal{C}$, $T : F \to X$ an operator and $\widehat{T} : \mathrm{FBL}^{\mathcal{C}}[F] \to X$ the lattice homomorphism extending T. By Theorem 2.3.1, let us consider the isometric push-out diagram in $\overline{\mathcal{C}}$ for $X_0 = \bar{\iota}(\mathrm{FBL}^{\mathcal{C}}(F))$, $X_1 = X$, $X_2 = \mathrm{FBL}^{\mathcal{C}}(E)$, $T_1 = \frac{\widehat{T}\bar{\iota}^{-1}}{\|\widehat{T}\bar{\iota}^{-1}\|} : X_0 \to X_1$ and T_2 the formal identity from X_0 to X_2.



Let us write $Y = PO^{\overline{\mathcal{C}}}$ and $j = S_1$. Since T_2 is an isometric embedding, we can apply Theorem 2.3.2 or Theorem 2.3.3 to obtain that j is a norm one $K_{\mathcal{C}}$-embedding. On the other hand, let $S = \|\widehat{T}\bar{\iota}^{-1}\|S_2\phi_E : E \to Y$. It follows that

$$jT = j\widehat{T}\phi_F = \|\widehat{T}\bar{\iota}^{-1}\|jT_1\bar{\iota}\phi_F = \|\widehat{T}\bar{\iota}^{-1}\|S_2T_2\bar{\iota}\phi_F = \|\widehat{T}\bar{\iota}^{-1}\|S_2\phi_E\iota = S\iota,$$

and $\|S\| \leqslant \|\widehat{T}\|\|\bar{\iota}^{-1}\| \leqslant c_1\|T\|$, so we can take $c_2 \leqslant c_1$.

(2)⇒(1) First, note that $\|\bar{\iota}\| = 1$. Now, take $f \in \mathrm{FVL}[F]$ and a contraction $T : F \to X$ with X in $\mathcal{C}$. By the hypothesis, there exists Y in $\mathcal{C}$, a norm one $K_{\mathcal{C}}$-lattice embedding $j : X \to Y$ and $S : E \to Y$ so that $jT = S\iota$ and $\|S\| \leqslant c_2$. It follows that $j\widehat{T} = \widehat{S}\bar{\iota}$, so

$$\|\widehat{T}f\|_X = \|j^{-1}\widehat{S}\bar{\iota}f\|_X \leqslant \|j^{-1}\|\|S\|\rho_{\mathcal{C}}(\bar{\iota}f) \leqslant K_{\mathcal{C}}c_2\rho_{\mathcal{C}}(\bar{\iota}f).$$

Taking the supremum over all possible contractions T we conclude that $\bar{\iota}$ is a $(K_{\mathcal{C}}c_2)$-embedding. □

Recall that a Banach lattice X is *injective* if for every Banach lattice Z, every closed linear sublattice Y of Z and every positive linear operator $T : Y \to X$ there is a positive linear extension $\widetilde{T} : Z \to X$ with $\|\widetilde{T}\| = \|T\|$. Equivalently [89, Proposition 3.5], X is injective if whenever X lattice isometrically embeds into a Banach lattice Y there is a positive contractive projection from Y onto X. Note that this notion of injectivity is closer to the category of Banach lattices with positive linear maps, which is considerably different from the category $\mathcal{BL}$ of Banach lattices with lattice homomorphisms (where in fact there are no injective objects [14]). The study of injective Banach lattices is classical [33, 65, 82, 89, 93]. However, it seems that not much is known on injectivity (or projectivity) in subcategories of Banach lattices.

Next, we discuss the notion of injectivity in the class of p-convex Banach lattices with positive operators, with an aim towards providing an alternative argument to recover the POE-p (Property of operator extension into L_p, see [103, Definition 3.8]). The original proof of [103, Theorem 3.7] requires a result due to Pisier [109, Theorem 4] which implicitly implies the injectivity of L_p spaces in the class of p-convex Banach lattices. This fact additionally allows one to establish an analogue of [89, Proposition 3.5] in this restricted context (see Proposition 2.3.9 below). However, it turns out that Theorem 2.3.6 together with Theorem 2.3.7 recover [103, Theorem 3.7] without making use of Pisier's arguments. The first step in this proof is to show that ℓ_p satisfies an *a priori* weaker property than injectivity in the class of p-convex Banach lattices.

Theorem 2.3.7. *Let $1 \leqslant p < \infty$ and X be a p-convex Banach lattice, and suppose that (x_n) is a disjoint positive sequence in X equivalent to the ℓ_p-basis. Then $[(x_n)]$ is the image of a positive projection on X.*

Proof. We may assume that X is a closed sublattice of $\ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ by Theorem 2.2.1. It suffices to show that $[(x_n)]$ is the range of a positive projection on $\ell_\infty(L_p(\mu))_{\mu\in\Gamma}$. Assume that $c\|(a_n)\|_p < \|\sum a_nx_n\| \leqslant \|(a_n)\|_p$ for any non-zero $(a_n) \in \ell_p$. Let $m \in \mathbb{N}$. There exists a finite set Γ_m, $k \in \mathbb{N}$, and a positive contraction $T : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to \ell_\infty(\ell_p^k)_{\Gamma_m}$ (by $\ell_\infty(\ell_p^k)_{\Gamma_m}$ we mean the ℓ_∞-sum of $|\Gamma_m|$ copies of ℓ_p^k) so that $(Tx_n)_{n=1}^m$ is a positive disjoint sequence satisfying $c\|(a_n)_{n=1}^m\|_p < \|\sum_{n=1}^m a_nTx_n\| \leqslant$

$\|(a_n)_{n=1}^m\|_p$ for any non-zero $(a_n)_{n=1}^m \in \ell_p^m$. Indeed, let us fix $\delta > 0$ to be chosen later and observe that for every $\mu \in \Gamma$ we can find simple functions $(s_{\mu,n})_{n=1}^m \subseteq L_p(\mu)$ such that $0 \leqslant s_{\mu,n} \leqslant P_\mu x_n$ and $\|P_\mu x_n - s_{\mu,n}\|_{L_p(\mu)} \leqslant \delta \|P_\mu x_n\|_{L_p(\mu)}$, where $P_\mu : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to L_p(\mu)$ denotes the projection onto the μ-th factor. The functions $(s_{\mu,n})_{n=1}^m$ lie in the span of a finite number of disjoint characteristic functions $(\chi_{B_l})_{l=1}^L \subseteq L_p(\mu)$. Let $Q_\mu : L_p(\mu) \to \operatorname{span}(\chi_{B_l})_{l=1}^L$ be the conditional expectation operator associated to the collection $(B_l)_{l=1}^L$, which is positive, contractive, and satisfies that

$$\begin{aligned}\left\|Q_\mu\left(\sum_{n=1}^m a_n P_\mu x_n\right)\right\|_{L_p(\mu)} &\geqslant \left\|Q_\mu\left(\sum_{n=1}^m a_n s_{\mu,n}\right)\right\|_{L_p(\mu)} \\ &= \left\|\sum_{n=1}^m a_n s_{\mu,n}\right\|_{L_p(\mu)} \geqslant (1-\delta)\left\|\sum_{n=1}^m a_n P_\mu x_n\right\|_{L_p(\mu)}.\end{aligned}$$

It is clear that $\operatorname{span}(\chi_{B_l})_{l=1}^L$ is lattice isometric to the weighted space $\ell_p^L(\omega_l)$, where $\omega_l = \mu(B_l)^{\frac{1}{p}}$. Choosing natural numbers k_μ and $(s_l)_{l=1}^L$ such that $\omega_l \geqslant \frac{s_l}{k_\mu}\sum_{j=1}^L \omega_j \geqslant (1-\delta)\omega_l$, we can construct a contractive lattice embedding $J_\mu : \operatorname{span}(\chi_{B_l})_{l=1}^L \to \ell_p^{k_\mu}$ with distortion $(1-\delta)^{-1}$. Therefore, the operator $T_\mu = J_\mu Q_\mu P_\mu : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to \ell_p^{k_\mu}$ is contractive, positive, sends $(x_n)_{n=1}^m$ to a disjoint sequence, and satisfies that

$$\left\|T_\mu\left(\sum_{n=1}^m a_n x_n\right)\right\|_{\ell_p^{k_\mu}} \geqslant (1-\delta)^2\left\|\sum_{n=1}^m a_n P_\mu x_n\right\|_{L_p(\mu)}.$$

Next, select a δ-net $(y_j)_{j=1}^J$ in the unit sphere of $[(x_n)_{n=1}^m]$ and for each $j = 1, \ldots, J$ fix a measure $\mu_j \in \Gamma$ such that $1 = \|y_j\| \leqslant (1+\delta)\|P_{\mu_j} y_j\|_{L_p(\mu_j)}$. Taking $\Gamma_m = (\mu_j)_{j=1}^J$ and $k = \sup_j k_{\mu_j}$, and defining

$$\begin{aligned} T : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} &\longrightarrow \ell_\infty(\ell_p^k)_{\Gamma_m} \\ x &\longmapsto (T_{\mu_j} x)_{j=1}^J\end{aligned}$$

(the inclusions from $\ell_p^{k_{\mu_j}}$ into ℓ_p^k are implicit), it is easy to check that

$$\left\|\sum_{n=1}^m a_n x_n\right\| \geqslant \left\|\sum_{n=1}^m a_n T x_n\right\| \geqslant (1-\delta)^2\left(\frac{1}{1+\delta} - \delta\right)\left\|\sum_{n=1}^m a_n x_n\right\|.$$

Recall that $c\|(a_n)\|_p < \|\sum a_n x_n\|$ for any non-zero $(a_n) \in \ell_p$, so in particular

$$c < \inf_{(a_n)\in\ell_p^m} \frac{\left\|\sum_{n=1}^m a_n x_n\right\|}{\|(a_n)_{n=1}^m\|_p} = \min_{(a_n)\in S_{\ell_p^m}} \left\|\sum_{n=1}^m a_n x_n\right\| = c_m.$$

Therefore, choosing δ so that $(1-\delta)^2\left(\frac{1}{1+\delta} - \delta\right) > \frac{c}{c_m}$, we get that $c\|(a_n)_{n=1}^m\|_p < \|\sum_{n=1}^m a_n T x_n\|$, as we wanted to show.

Now, write $Tx_n = (y_n(\gamma))_{\gamma\in\Gamma_m}$ with $y_n(\gamma) \in \ell_p^k$. Define $S : \ell_1^m \to \ell_\infty(\Gamma_m)$ by

$$S(b_1, \ldots, b_m) = \left(\sum_{n=1}^m b_n \|y_n(\gamma)\|_p^p\right)_{\gamma\in\Gamma_m}.$$

Since $(y_n(\gamma))_{n=1}^m$ is a disjoint sequence for each γ, if $b = (b_1, \ldots, b_m) \geqslant 0$, then

$$\|Sb\| = \max_{\gamma\in\Gamma_m}\left\|\sum_{n=1}^m b_n^{\frac{1}{p}} y_n(\gamma)\right\|_p^p = \left\|\sum_{n=1}^m b_n^{\frac{1}{p}} T x_n\right\|^p \geqslant c^p \|(b_n^{\frac{1}{p}})_{n=1}^m\|_p^p = c^p\|b\|_1.$$

We claim that $B^+_{\ell_\infty^m} \subseteq c^{-p}\operatorname{co}\operatorname{so}\{S^*e^*_\gamma : \gamma \in \Gamma_m\}$. Here, for a subset A of a Banach lattice X, $\operatorname{so} A = \bigcup_{a\in A}[-|a|,|a|]$ denotes the solid hull of A and, as usual, $\operatorname{co} A$ denotes the convex hull. To prove the claim, note that if $0 \leqslant y^* \notin c^{-p}\operatorname{co}\operatorname{so}\{S^*e^*_\gamma : \gamma \in \Gamma_m\}$ then by [104, Proposition 3.1] there exists $y \in (\ell_1^m)_+$ so that

$$\|y^*\|_\infty \|y\|_1 \geqslant y^*(y) > c^{-p}\max_{\gamma\in\Gamma_m}(S^*e^*_\gamma)(y) = c^{-p}\|Sy\| \geqslant \|y\|_1.$$

Hence, $\|y^*\|_\infty > 1$. This completes the proof of the claim.

By the claim we can find a convex combination $\sum_{\gamma\in\Gamma_m}\alpha_\gamma S^*e^*_\gamma \geqslant c^p\overbrace{(1,\dots,1)}^{m}$. Choose $y^*_n(\gamma) \in (\ell^k_{p^*})_+$ so that $\operatorname{supp} y^*_n(\gamma) = \operatorname{supp} y_n(\gamma) =: I_n$, $\|y^*_n(\gamma)\|_{p^*} = 1$ and $\langle y_n(\gamma), y^*_n(\gamma)\rangle = \|y_n(\gamma)\|_p$. If $x \in \ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ and $Tx = (y(\gamma))_{\gamma\in\Gamma_m}$, let

$$z^*_n(x) = \sum_{\gamma\in\Gamma_m}\alpha_\gamma\|y_n(\gamma)\|_p^{p-1}\langle y(\gamma), y^*_n(\gamma)\rangle.$$

It is easy to see that z^*_n is a positive linear functional and for any $x \in \ell_\infty(L_p(\mu))_{\mu\in\Gamma}$,

$$\begin{aligned}
\|(z^*_n(x))_{n=1}^m\|_p &\leqslant \sum_{\gamma\in\Gamma_m}\alpha_\gamma\big\|(\|y_n(\gamma)\|_p^{p-1}\|y(\gamma)\chi_{I_n}\|_p)_{n=1}^m\big\|_p\\
&\leqslant \sum_{\gamma\in\Gamma_m}\alpha_\gamma\sup_n\|y_n(\gamma)\|_p^{p-1}\Big\|\sum_{n=1}^m y(\gamma)\chi_{I_n}\Big\|_p\\
&\leqslant \sup_{\gamma\in\Gamma_m}\sup_n\|y_n(\gamma)\|_p^{p-1}\|y(\gamma)\|_p\\
&\leqslant \sup_n\|Tx_n\|^{p-1}\cdot\|Tx\| \leqslant \|Tx\| \leqslant \|x\|.
\end{aligned}$$

On the other hand, $z^*_l(x_n) = 0$ if $l \neq n$ and

$$z^*_n(x_n) = \sum_{\gamma\in\Gamma_m}\alpha_\gamma\|y_n(\gamma)\|_p^{p-1}\langle y_n(\gamma), y^*_n(\gamma)\rangle = \sum_{\gamma\in\Gamma_m}\alpha_\gamma\|y_n(\gamma)\|_p^p \geqslant c^p,$$

since $\sum_{\gamma\in\Gamma_m}\alpha_\gamma\|y_n(\gamma)\|_p^p$ is the n-th coordinate of the sum $\sum_{\gamma\in\Gamma_m}\alpha_\gamma S^*e^*_\gamma$. Therefore, the map P_m on $\ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ given by

$$P_m x = \sum_{n=1}^m \frac{z^*_n(x)}{z^*_n(x_n)}x_n$$

is a positive projection from $\ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ onto $[(x_n)_{n=1}^m]$ such that $\|P_m\| \leqslant c^{-p}$. Finally, let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If we denote by $j : [x_n] \to \ell_p$ the lattice isomorphism that sends (x_n) to the canonical basis of ℓ_p, it follows that the operator $R : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to \ell_p$ given by $Rx = w^*\text{-}\lim_{m\to\mathcal{U}} jP_mx$ is a positive operator such that $R|_{[x_n]} = j$. Therefore, the map $P = j^{-1}R : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to [(x_n)]$ is a positive projection onto $[(x_n)]$. □

As a corollary of Theorems 2.3.6 and 2.3.7, we recover the solution to the embedding problem for the free p-convex Banach lattice given in [103, Theorem 3.7].

Corollary 2.3.8. *Let $\iota : F \hookrightarrow E$ be an isometric embedding, $1 \leqslant p < \infty$ and $c \geqslant 1$. The following are equivalent.*

(1) $\bar{\iota} : \mathrm{FBL}^p[F] \to \mathrm{FBL}^p[E]$ is a c-lattice embedding.

(2) For every operator $T : F \to \ell_p$ there is $R : E \to \ell_p$ so that $T = R\iota$ and $\|R\| \leqslant c\|T\|$.

Proof. We use the identification $\mathrm{FBL}^p = \mathrm{FBL}^{\mathcal{C}_p}$ where $\mathcal{C}_p$ denotes the class of p-convex Banach lattices and invoke Theorem 2.3.6. The implication (2) $\Rightarrow$ (1) is clear. For the converse, we get that there exist a p-convex Banach lattice Y, a lattice isometric embedding $j : \ell_p \to Y$ and $S : E \to Y$ such that $jT = S\iota$ with $\|S\| \leqslant c\|T\|$. By Theorem 2.3.7, there is a (positive) projection $P : Y \to j(\ell_p) \subseteq Y$ with $\|P\| \leqslant 1$ (as j is an isometric embedding). Thus, we can take $R = j^{-1}PS$. □

Let us conclude this section with a note on how, in general, the property stated in Theorem 2.3.7 is actually equivalent to injectivity in the class of p-convex Banach lattices, provided we make use of Pisier's result [109, Theorem 4].

Proposition 2.3.9. *Let X be a p-convex Banach lattice with constant 1. The following are equivalent:*

(1) X is injective in the class of p-convex Banach lattices, i.e., given Z a p-convex Banach lattice, Y a closed sublattice of Z and $T : Y \to X$ a positive operator, there exists a positive extension $\widetilde{T} : Z \to X$.

(2) If X lattice isomorphically embeds into a p-convex Banach lattice Z, there exists a positive projection from Z onto X.

Proof. (1) $\Rightarrow$ (2) Let $J : X \to Z$ be a lattice isomorphic embedding. It suffices to apply the injectivity property of X to the sublattice $Y = J(X) \subseteq Z$ and the operator $J^{-1} : Y \to X$.

(2) $\Rightarrow$ (1) Let Z be a p-convex Banach lattice, Y a closed sublattice of Z and $T : Y \to X$ a positive operator. By Theorem 2.2.1, there is a lattice isometric embedding $J : X \to \ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ for a certain family of measures Γ, and by the hypothesis, X is complemented inside this space by means of a positive projection $P : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to X$. For every $\mu \in \Gamma$ let $P_\mu : \ell_\infty(L_p(\mu))_{\mu\in\Gamma} \to L_p(\mu)$ be the (positive) projection onto the factor indexed by μ. Using [109, Theorem 4] we can extend each operator $P_\mu JT$ to a positive operator $S_\mu : Z \to L_p(\mu)$. Gluing them together we obtain $S : Z \to \ell_\infty(L_p(\mu))_{\mu\in\Gamma}$ given by $Sz = (S_\mu z)_{\mu\in\Gamma}$. The operator $\widetilde{T} = PS : Z \to X$ is the desired extension. □

2.3.3 $\ell_{p,\infty}$ is not injective for Banach lattices with upper p-estimates

We now show that for any $1 < p < \infty$, $\ell_{p,\infty}$ is not injective in the class of Banach lattices with upper p-estimates. As in the previous section, we say that a Banach lattice X satisifying an upper p-estimate with constant 1 is *injective in the class of Banach lattices with upper p-estimates* if given any Banach lattice with upper p-estimates Z, a closed sublattice Y of Z and $T : Y \to X$ a positive operator, there exists a positive extension $\widetilde{T} : Z \to X$.

Theorem 2.3.10. *For any $1 < p < \infty$ there is a Banach lattice X with an upper p-estimate and an uncomplemented closed sublattice Y of X that is lattice isomorphic to $\ell_{p,\infty}$. In particular, $\ell_{p,\infty}$ is not injective in the class of Banach lattices with upper p-estimates.*

Proof. We begin with some notation. Let X be the Banach lattice $\ell_\infty(\ell_{p,\infty})_{\mathbb{N}}$. Clearly, X satisfies an upper p-estimate with constant 1. We may express any $x \in X$ as $x = (x(j))_{j=1}^\infty = (x(i,j))_{i,j=1}^\infty$, where $(x(j))_{j=1}^\infty$ is a bounded sequence in $\ell_{p,\infty}$ and $x(j) = (x(i,j))_{i=1}^\infty \in \ell_{p,\infty}$ for each j. Given $m \in \mathbb{N}$, define P_m on X by

$$(P_m x)(i,j) = \begin{cases} x(m,j) & \text{if } i = m, \\ 0 & \text{if } i \neq m. \end{cases}$$

Clearly, P_m is a band projection on X and the projection band $P_m(X)$ is lattice isometric to ℓ_∞ via the map

$$L_m : \ell_\infty \to P_m(X),\ (L_m y)(i,j) = \begin{cases} y(j) & \text{if } i = m, \text{ where } y = (y(j))_{j=1}^\infty, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}$ be the Schreier family, i.e., all subsets I of $\mathbb{N}$ so that $|I| \leqslant \min I$. Since $\mathcal{S}$ is countable, we can list its elements in a sequence $(S_j)_{j=1}^\infty$. Given $a = (a_i)_i \in \ell_{p,\infty}$, let

$$x_a(i,j) = \begin{cases} a_i & \text{if } i \in S_j, \\ 0 & \text{otherwise.} \end{cases}$$

Denote the sequence of coordinate unit vectors in $\ell_{p,\infty}$ by $(e_i)_i$. For any j,

$$\|(x_a(i,j))_{i=1}^\infty\|_{p,\infty} = \Big\|\sum_{i\in S_j} a_i e_i\Big\|_{p,\infty} \leqslant \|(a_i)\|_{p,\infty}.$$

Hence, $T : \ell_{p,\infty} \to X$, $a \mapsto x_a$ is a bounded linear map which is clearly a lattice homomorphism. On the other hand, if $\|a\|_{p,\infty} > 1$ then by (1.6) there exists a non-empty finite set L in $\mathbb{N}$ so that $|a_i| \geqslant (p^*|L|^{\frac{1}{p}})^{-1}$ for all $i \in L$. Let $S_j \in \mathcal{S}$ be such that $S_j \subseteq L$ and $|S_j| \geqslant |L|/2$. Then,

$$\|x_a\| \geqslant \Big\|\sum_{i\in S_j} a_i e_i\Big\|_{p,\infty} \geqslant (p^*|L|^{\frac{1}{p}})^{-1}\Big\|\sum_{i\in S_j} e_i\Big\|_{p,\infty} = \frac{1}{p^*}\left(\frac{|S_j|}{|L|}\right)^{\frac{1}{p}} \geqslant \frac{1}{p^* 2^{\frac{1}{p}}}.$$

This shows that T is a lattice isomorphism from $\ell_{p,\infty}$ into X.

Let $Y = T(\ell_{p,\infty})$. We will show that there is no bounded projection from X onto Y. Assume on the contrary that there is a projection P from X onto Y. Then there is a sequence (x_i^*) in X^* so that $(x_i^*(x))_i \in \ell_{p,\infty}$ for any $x \in X$ and $x_i^*(Ta) = a_i$ for any i and $a = (a_k)_k \in \ell_{p,\infty}$. It follows readily that $(x_i^*)_i \subseteq X^*$ is equivalent to the unit vector basis of $\ell_{p^*,1}$. In particular, it is an unconditional basic sequence in a Banach lattice that is q-concave for some $q < \infty$. Since (P_i^*) is a disjoint sequence of band projections on X^*, the sequence $(P_i^* x_i^*)$ is disjoint and hence also unconditional in X^*. By [88, Theorem 1.d.6(i)], for any finitely nonzero real sequence (b_i),

$$\|(b_i)\|_{p^*,1} \sim \Big\|\sum b_i x_i^*\Big\| \sim \Big\|\sqrt{\sum |b_i x_i^*|^2}\Big\| \geqslant \Big\|\sqrt{\sum |b_i P_i^* x_i^*|^2}\Big\| \sim \Big\|\sum b_i P_i^* x_i^*\Big\|.$$

Thus, the linear map $Q : X \to \ell_{p,\infty}$ defined by $Qx = (P_i^* x_i^*(x))_i$ is bounded. On the other hand, observe that $Te_k \in P_k(X)$. Therefore,

$$\langle Te_k, P_i^* x_i^*\rangle = \begin{cases} \langle Te_i, x_i^*\rangle = 1 & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Hence, $QTe_k = e_k$ for any $k \in \mathbb{N}$.

Regard L_i as a lattice isometric embedding from ℓ_∞ into X. Define $y_i^* = L_i^* P_i^* x_i^*$ in ℓ_∞^*. Let I be a finite subset of $\mathbb{N}$. Suppose that $(u_i)_{i\in I}$ is a finite sequence in ℓ_∞ so that $\|(\sum_{i\in I} |u_i|^p)^{\frac{1}{p}}\|_\infty \leqslant 1$. Set $u = \sum_{i\in I} L_i u_i \in X$. Then $\|u\| \leqslant \|(\sum_{i\in I} |u_i|^p)^{\frac{1}{p}}\|_\infty \leqslant 1$. Thus,

$$\begin{aligned} \sum_{i\in I} y_i^*(u_i) &= \sum_{i\in I} L_i^* P_i^* x_i^*(u_i) = \sum_{i\in I} (P_i^* x_i^*)(u) \\ &\leqslant \|(P_i^* x_i^*(u))\|_{p,\infty} \cdot |I|^{\frac{1}{p^*}} = \|Qu\| \cdot |I|^{\frac{1}{p^*}} \lesssim |I|^{\frac{1}{p^*}}. \end{aligned}$$

Making use of the duality described on [88, p. 47], we conclude that there is an absolute constant $C < \infty$ so that

$$\left\| \left(\sum_{i \in I} |y_i^*|^{p^*} \right)^{\frac{1}{p^*}} \right\| \leqslant C |I|^{\frac{1}{p^*}} \tag{2.6}$$

for any finite subset I of $\mathbb{N}$. Since ℓ_∞^* is an AL-space, the bounded sequence $(y_i^*)_i$ has a subsequence $(y_{i_k}^*)_k$ that has a splitting

$$y_{i_k}^* = u_k^* + v_k^*, \ |u_k^*| \wedge |v_k^*| = 0, \ (u_k^*) \text{ is almost order bounded, } (v_k^*) \text{ is disjoint.}$$

See, e.g., [125]. Since $|v_k^*| \leqslant |y_{i_k}^*|$ for all k, it follows from (2.6) that $\|v_k^*\| \to 0$. Thus, the whole sequence $(y_{i_k}^*)_k$ is almost order bounded in ℓ_∞^* and hence it is relatively weakly compact.

Since $x_i := Te_i \in P_i(X)$ and $L_i : \ell_\infty \to P_i(X)$ is a surjective lattice isometry, we may find a sequence $(u_k)_k$ in ℓ_∞ so that $L_{i_k} u_k = x_{i_k}$ for each k. If we write $u_k = (u_k(j))_{j=1}^\infty$ then $u_k(j) = 1$ if $k \in S_j$ and 0 otherwise. Hence, for any finitely supported real sequence $(a_k)_k$,

$$\left\| \sum a_k u_k \right\|_\infty = \sup_j \left| \sum_{k \in S_j} a_k \right|.$$

The expression on the right is the norm of the Schreier space. The preceding equation says that $(u_k)_k$ is equivalent to the unit vector basis of the Schreier space. Thus, $(u_k)_k$ is a weakly null sequence (in ℓ_∞). Since ℓ_∞ has the Dunford-Pettis property, the weakly null sequence $(u_k)_k$ converges uniformly to 0 on the relatively weakly compact set $(y_{i_k}^*)_k$. In particular,

$$0 = \lim y_{i_k}^*(u_k) = \lim L_{i_k}^* P_{i_k}^* x_{i_k}^*(u_k) = \lim P_{i_k}^* x_{i_k}^*(x_{i_k}).$$

However,

$$e_{i_k} = QTe_{i_k} = Qx_{i_k} = (P_j^* x_j^*(x_{i_k}))_j.$$

Taking the i_k-th component yields that $P_{i_k}^* x_{i_k}^*(x_{i_k}) = 1$ for any k, which is the desired contradiction. □

2.4 Operators factoring through a (p, p_2)-convex and a (q, q_2)-concave operator

In [111], Reisner studies the ideal of operators that factor through a p-convex and a q-concave operator, that he denotes $M_{p,q}$. He shows that this ideal is perfect, that is, it coincides with its double adjoint, and identifies it with the ideal $I_{p,q}$ of operators factoring through a diagonal operator from $L_p(\mu)$ to $L_q(\mu)$ defined in [62]. In this section we replicate these results for the ideal of operators factoring through a (p, p_2)-convex and a (q, q_2)-concave operator, where $1 < p, q < \infty$, $p \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_2 \leqslant q$. Note that in light of [43, Corollary 16.7] we know that the most relevant cases will be $p_2 \in \{p, \infty\}$ and $q_2 \in \{1, q\}$.

Throughout the section, we will use the definitions and notation from tensor products and operator ideals given in [62]. For instance, recall that an *operator ideal* is a class A of operators between Banach spaces such that for each pair of Banach spaces E and F the set $A(E, F) = A \cap \mathcal{L}(E, F)$ satisfies that:

(1) If $x^* \in E^*$ and $y \in F$, then $x^* \otimes y \in A(E, F)$, where $x^* \otimes y(x) = x^*(x)y$ is a rank one operator.

(2) $A(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$.

(3) If X and Y are Banach spaces, $U \in \mathcal{L}(X, E)$, $T \in A(E, F)$ and $V \in \mathcal{L}(F, Y)$, then $VTU \in A(X, Y)$.

Moreover, a function $\alpha : A \to \mathbb{R}_+$ is an *ideal norm* if one has:

(4) $\alpha(x^* \otimes y) = \|x^*\|\|y\|$ for every $x^* \in E^*$ and $y \in F$.

(4) $\alpha(S + T) \leqslant \alpha(S) + \alpha(T)$ for every $S, T \in A(E, F)$.

(4) If X and Y are Banach spaces, $U \in \mathcal{L}(X, E)$, $T \in A(E, F)$ and $V \in \mathcal{L}(F, Y)$, then $\alpha(VTU) \leqslant \|V\|\alpha(T)\|U\|$.

A *normed (operator) ideal* $[A, \alpha]$ is an ideal A with an ideal norm α, and we say that it is a *Banach ideal* if each $A(E, F)$ is a Banach space when endowed with α.

Given a normed ideal $[A, \alpha]$, the *adjoint ideal* $[A, \alpha]^* = [A^*, \alpha^*]$ is defined as follows (observe the different use of $*$ with respect to the notation that we have been using throughout the dissertation, where it denotes the dual of a Banach space): for each pair of Banach spaces E and F, $A^*(E, F)$ is the set of all $T \in \mathcal{L}(E, F)$ such that there exists $\rho > 0$ satisfying that for all finite-dimensional Banach spaces X, Y and all $U \in A(Y, X)$, $V \in \mathcal{L}(X, E)$ and $W \in \mathcal{L}(F, Y)$,

$$|\operatorname{tr} WTVU| \leqslant \rho\|W\|\|V\|\alpha(U),$$

where tr denotes the trace of an operator defined from a finite-dimensional Banach space to itself. The norm $\alpha^*(T)$ is given by the infimum of all ρ satisfying the above. α^* is always a complete ideal norm. An operator ideal $[A, \alpha]$ is *perfect* if $[A, \alpha] = [A, \alpha]^{**}$. This notion is equivalent to the ideal being *finitely generated*, *maximal* or being the adjoint of some other ideal (cf. [112, Theorem 8.11]).

Regarding tensor products, recall that if E and F are Banach spaces, a norm η on $E \otimes F$ is a *reasonable crossnorm* if it satisfies the following properties:

(1) $\eta(x \otimes y) \leqslant \|x\|\|y\|$ for every $x \in E$ and $y \in F$.

(2) For every $x^* \in E^*$ and $y^* \in F^*$, the linear functional $x^* \otimes y^*$ on $E \otimes F$ that takes values $x^* \otimes y^*(x \otimes y) = x^*(x)y^*(y)$ on elementary products $x \otimes y \in E \otimes F$ is bounded, and $\|x^* \otimes y^*\| \leqslant \|x^*\|\|y^*\|$.

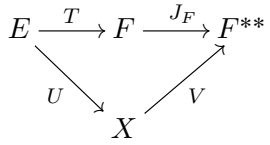
A *uniform crossnorm* is an assignment to each pair of Banach spaces of a reasonable crossnorm on the tensor product, in such a way that whenever E, F, G and H are Banach spaces and $S : E \to G$ and $T : F \to H$ are operators, then the operator $S \otimes T : E \otimes F \to G \otimes H$ is bounded and $\|S \otimes T\| \leqslant \|S\|\|T\|$. It is easy to check that every uniform crossnorm induces an ideal norm over the operator ideal $\mathcal{F}$ of all finite rank operators via the identification $\mathcal{F}(E, F) = E^* \otimes F$ described in property (1) in the definition of operator ideal. We will denote this normed ideal by $[\mathcal{F}, \eta]$. A uniform crossnorm η is *finitely generated* (see [112, p. 129], this is what [111] calls a $\otimes$-norm), if for all $u \in E \otimes F$,

$$\eta(u) = \inf \eta(u; M, N),$$

where the infimum is taken over all finite-dimensional subspaces $M \subseteq E$ and $N \subseteq F$ such that $u \in M \otimes N$, and $\eta(u; M, N)$ denotes the η-norm of u as an element of $M \otimes N$.

Let us fix $1 < p, q < \infty$, $p \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_2 \leqslant q$. We introduce the following definition, that generalizes the ideal $[M_{p,q}, \mu_{p,q}]$ of operators that factor though a p-convex and a q-concave operator introduced in [111, Section 2]:

Definition 2.4.1. *Let E and F be Banach spaces. An operator $T : E \to F$ belongs to the class $M_{(p,p_2;q,q_2)}(E, F)$ (i.e. factors $(p, p_2; q, q_2)$ through a Banach lattice) if there exists a Banach lattice X, a (p, p_2)-convex operator $U : E \to X$ and a (q, q_2)-concave operator $V : X \to F^{**}$ such that $J_F T = VU$, where $J_F : F \to F^{**}$ denotes the canonical embedding:*



We define $\mu_{(p,p_2;q,q_2)}(T) = \inf\{K^{(p,p_2)}(U)K_{(q,q_2)}(V)\}$*, where the infimum is taken over all possible factorizations above.*

As is customary, when $p_2 = p$ we will simply write p instead of (p, p_2), and similarly when $q_2 = q$. Our first goal is to show that $M_{(p,p_2;q,q_2)}$ is a perfect ideal. This will be done by showing that it is the adjoint of $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^*$, where $\eta_{(p,p_2;q,q_2)}$ is a uniform crossnorm defined as follows:

Definition 2.4.2. *Given* $u \in E \otimes F$*, we define*

$$\eta_{(p,p_2;q,q_2)}(u) = \inf\left\{\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n) : (x_i)_{i=1}^n \subseteq E, (y_i)_{i=1}^n \subseteq F, u = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where

$$\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n) = \sup\left\{\sum_{i=1}^n \left(\sum_{k=1}^\infty |x_k^*(x_i)|^{p_2}\right)^{\frac{1}{p_2}} \left(\sum_{j=1}^\infty |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} : \right.$$
$$\left. \|(x_k^*)_{k=1}^\infty\|_{\ell_p(E^*)} \leqslant 1, \|(y_j^*)_{j=1}^\infty\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}$$

and q^* *and* q_2^* *denote the conjugate exponents of* q *and* q_2*, respectively (when* $p_2 = \infty$ *or* $q_2 = 1$*, the* p_2 *and* q_2^**-sum must be understood as a supremum).*

Following the steps of [111], the first thing to show is that:

Proposition 2.4.3. $\eta_{(p,p_2;q,q_2)}$ *is finitely generated.*

Proof. It suffices to check that whenever we fix a representation $u = \sum_{i=1}^n x_i \otimes y_i$ with $(x_i)_{i=1}^n \subseteq M$ and $(y_i)_{i=1}^n \subseteq N$ for some finite-dimensional subspaces $M \subseteq E$ and $N \subseteq F$, then

$$\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) = \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N).$$

Clearly, $\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) \leqslant \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N)$, since every pair of sequences $(x_k^*)_{k=1}^\infty \in B_{\ell_p(E^*)}$ and $(y_j^*)_{j=1}^\infty \in B_{\ell_{q^*}(F^*)}$ can be restricted to M and N, respectively, to induce sequences in $B_{\ell_p(M^*)}$ and $B_{\ell_{q^*}(N^*)}$, and the values of the sums considered in the computation of $\theta_{(p,p_2;q,q_2)}$ remain unchanged.

On the other hand, $\theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; E, F) \geqslant \theta_{(p,p_2;q,q_2)}(\{x_i \otimes y_i\}_{i=1}^n; M, N)$ is a consequence of Hahn–Banach Theorem: every pair of sequences $(v_k^*)_{k=1}^\infty \in B_{\ell_p(M^*)}$ and $(w_j^*)_{j=1}^\infty \in B_{\ell_{q^*}(N^*)}$ can be extended to sequences in $B_{\ell_p(E^*)}$ and $B_{\ell_{q^*}(F^*)}$ without altering their values on $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$. □

The next proposition shows the connection between the $\eta_{(p,p_2;q,q_2)}$-norm and (p, p_2)-convex and (q, q_2)-concave operators.

Proposition 2.4.4. *For every* $u \in E^* \otimes F = \mathcal{F}(E, F) \subseteq L(E, F)$*,*

$$\eta_{(p,p_2;q,q_2)}(u) = \inf\{K^{(p,p_2)}(R)K_{(q,q_2)}(S)\},$$

where the infimum is taken over all possible factorizations $u = SR$ *with* Z *a finite-dimensional Banach space with a 1-unconditional basis (endowed with the lattice order induced by the basis),* $R : E \to Z$ *and* $S : Z \to F$*.*

Proof. The proof is just an adaptation of [111, Proposition 1.3] with the corresponding changes in the exponents. We include it for the sake of completeness.

Let $u \in E^* \otimes F$ and assume that it factors as $u = RS$ with Z a finite-dimensional Banach space with a 1-unconditional basis, $R : E \to Z$ and $S : Z \to F$. Let $(e_i, e_i^*)_{i=1}^n$ be the corresponding basis of Z and its biorthogonal functionals. We define $x_i^* = R^* e_i^*$ and $y_i = Se_i$, so that $u = SR = \sum_{i=1}^n x_i^* \otimes y_i$. By definition,

$$K^{(p,p_2)}(R) = \sup\left\{\left\|\sum_{i=1}^n \left(\sum_k |x_i^*(x_k)|^{p_2}\right)^{\frac{1}{p_2}} e_i\right\| : \|(x_k)_{k=1}^\infty\|_{\ell_p(E)} \leqslant 1\right\},$$

$$K_{(q,q_2)}(S) = K^{(q^*,q_2^*)}(S^*) = \sup\left\{\left\|\sum_{i=1}^n \left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} e_i^*\right\| : \|(y_j^*)_{k=1}^\infty\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}.$$

Therefore, given $\varepsilon > 0$, we can choose $(x_k)_{k=1}^\infty \in \ell_p(E)$ and $(y_j^*)_{k=1}^\infty \in \ell_{q^*}(F^*)$ that ε-almost attain the supremum in the definition of $\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n)$, so that

$$\begin{aligned}
\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n) &\leqslant \sum_{i=1}^n \left(\sum_{k=1}^\infty |x_k^*(x_i)|^{p_2}\right)^{\frac{1}{p_2}} \left(\sum_{j=1}^\infty |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} + \varepsilon \\
&\leqslant \left\langle \sum_{i=1}^n \left(\sum_k |x_i^*(x_k)|^{p_2}\right)^{\frac{1}{p_2}} e_i, \sum_{i=1}^n \left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} e_i^* \right\rangle + \varepsilon \\
&\leqslant K^{(p,p_2)}(R) K_{(q,q_2)}(S) + \varepsilon.
\end{aligned}$$

Hence, $\eta_{(p,p_2;q,q_2)}(u) \leqslant K^{(p,p_2)}(R) K_{(q,q_2)}(S)$.

For the reverse inequality, let $\varepsilon > 0$ and suppose that $u = \sum_{i=1}^n x_i^* \otimes y_i$ is a representation such that

$$\theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n) \leqslant \eta_{(p,p_2;q,q_2)}(u) + \varepsilon.$$

Let Z be $\mathbb{R}^n$ endowed with the norm

$$|||z||| = \sup\left\{\sum_{i=1}^n \left| z_i \left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} \right| : \|(y_j^*)_{k=1}^\infty\|_{\ell_{q^*}(F^*)} \leqslant 1\right\}$$

for $z = (z_i)_{i=1}^n \in \mathbb{R}^n$. Then, the canonical basis $(e_i)_{i=1}^n$ of $\mathbb{R}^n$ is 1-unconditional for this norm, as $|||z|||$ is only determined by $(|z_i|)$. We define $R : E \to Z$ by $R = \sum_{i=1}^n x_i^* \otimes e_i$ and $S : Z \to F$ by $Sz = \sum_{i=1}^n z_i y_i$. It follows that $u = SR$,

$$\begin{aligned}
K^{(p,p_2)}(R) &= \sup \\
&= \theta_{(p,p_2;q,q_2)}(\{x_i^* \otimes y_i\}_{i=1}^n) \leqslant \eta_{(p,p_2;q,q_2)}(u) + \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
K_{(q,q_2)}(S) = K^{(q^*,q_2^*)}(S^*) &= \sup \\
&= \sup\left\{\sum_{i=1}^n \left| z_i \left(\sum_k |y_j^*(y_i)|^{q_2^*}\right)^{\frac{1}{q_2^*}} \right| : |||z||| \leqslant 1, \|(y_j^*)_{k=1}^\infty\|_{\ell_{q^*}(F^*)} \leqslant 1\right\} \\
&= \sup\{|||z||| : |||z||| \leqslant 1\} = 1.
\end{aligned}$$

Therefore, $K^{(p,p_2)}(R) K_{(q,q_2)}(S) \leqslant \eta_{(p,p_2;q,q_2)}(u)$. □

Next, we show that the ideal $M_{(p,p_2;q,q_2)}$ defined in Definition 2.4.1 is perfect.

Proposition 2.4.5.

(1) $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}]$ *is a perfect normed ideal of operators.*

(2) $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] = [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$.

Proving Proposition 2.4.5 requires some previous lemmata. Namely, we need [111, Lemma 2], which is also valid for the case $p = \infty$ even though Reisner originally stated it for $1 \leqslant p < \infty$, and an analogue of [111, Lemma 3]:

Lemma 2.4.6. *Let G be a finite-dimensional subspace of an order complete Banach lattice X. Given $\delta > 0$ there are pairwise disjoint $(x_i)_{i=1}^n \subseteq X_+$ and an operator $W : G \to \operatorname{span}(x_i)_{i=1}^n$ such that for all $(y_j)_{j=1}^m \subseteq G$ and $1 \leqslant p \leqslant \infty$,*

$$\left\| \left(\sum_{j=1}^m |y_j|^p \right)^{\frac{1}{p}} - \left(\sum_{j=1}^m |Wy_j|^p \right)^{\frac{1}{p}} \right\| \leqslant \delta \left(\sum_{j=1}^m \|y_j\|^p \right)^{\frac{1}{p}}.$$

Lemma 2.4.7. *Assume E or F are finite-dimensional and $T \in \mathcal{L}(E,F)$ is such that $J_F T = VU$ with X a Banach lattice, $U : E \to X$ a (p,p_2)-convex operator and $V : X \to F^{**}$ a (q,q_2)-concave operator. Then, for every $\varepsilon > 0$ there exist a finite-dimensional Z with a 1-unconditional basis and a factorization*

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ & {\scriptstyle R} \searrow \quad \nearrow {\scriptstyle S} & \\ & Z & \end{array}$$

with $K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1+\varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V)$.

Now, we are ready to prove Proposition 2.4.5:

Proof of Proposition 2.4.5. The first statement follows from the second by [112, Theorem 8.11].

Let us first show that $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \subseteq [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$. Let E and F be Banach spaces and $T \in M_{(p,p_2;q,q_2)}(E,F)$. Assume $J_F T$ factors as in Definition 2.4.1. By Lemma 2.4.7, for every finite-dimensional $G \subseteq E$ and $H \subseteq F^*$ (denote by $i : G \to E$ and $j : H \to F^*$ the embeddings) the operator $j^* J_F T i$ admits a factorization

$$\begin{array}{ccc} G & \xrightarrow{\quad j^* J_F T i \quad} & H^* \\ & {\scriptstyle R} \searrow \quad \nearrow {\scriptstyle S} & \\ & Z & \end{array}$$

with Z finite-dimensional with a 1-unconditional basis, such that $K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1+\varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V)$. Since G and H^* are finite-dimensional, using [107, Proposition 9.2.2] and Proposition 2.4.4 we get

$$\eta^{**}_{(p,p_2;q,q_2)}(j^* J_F T i) = \eta_{(p,p_2;q,q_2)}(j^* J_F T i) \leqslant K^{(p,p_2)}(R)K_{(q,q_2)}(S) \leqslant (1+\varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V).$$

Observe that $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ is already a perfect ideal, as it is the adjoint of $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^*$. Therefore, by [112, Theorem 8.11] it is a finitely generated ideal, so

$$\begin{aligned} \eta^{**}_{(p,p_2;q,q_2)}(T) &= \sup\{\eta^{**}_{(p,p_2;q,q_2)}(j^* J_F T i) : G \subseteq E, H \subseteq F^* \text{ finite-dimensional}\} \\ &\leqslant (1+\varepsilon)K^{(p,p_2)}(U)K_{(q,q_2)}(V). \end{aligned}$$

It now follows that $\eta^{**}_{(p,p_2;q,q_2)}(T) \leqslant \mu_{(p,p_2;q,q_2)}(T)$.

In order to show that $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \supseteq [\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ we will make use of a standard ultraproduct argument, so we will omit some of the details. We refer to [66] for basic terminology and notation on ultraproducts (see also Section 3.2.1, where ultraproducts will be considered in more detail). Again, let us fix Banach spaces E and F and an operator in $[\mathcal{F}(E;F), \eta_{(p,p_2;q,q_2)}]^{**}$. First of all, we make use of the fact that every Banach space embeds isometrically into the ultraproduct of a collection of its finite-dimensional subspaces (see [66, Proposition 6.2]). In particular, there exist index sets K and L, ultrafilters $\mathfrak{U}_E$ and $\mathfrak{U}_{F^*}$ over K and L (respectively), collections $\{G_k\}_{k\in K} \subseteq \{G : G \subseteq E \text{ finite-dimensional}\}$ and $\{H_l\}_{l\in L} \subseteq \{H : H \subseteq F^* \text{ finite-dimensional}\}$ and isometric embeddings $i_E : E \to (G_k)_{\mathfrak{U}_E}$ and $j_{F^*} : F^* \to (H_l)_{\mathfrak{U}_{F^*}}$. It follows that

$$\mathfrak{U}_\times = \{C \subseteq K \times L : \exists A \in \mathfrak{U}_E, B \in \mathfrak{U}_{F^*} \text{ such that } A \times B \subseteq C\}$$

defines a filter on $K \times L$, so by [66, Proposition 1.1] there exists an ultrafilter $\mathfrak{U}$ that dominates $\mathfrak{U}_\times$. If we put $G_{k,l} = G_k$ for every $l \in L$ and $H_{k,l} = H_l$ for every $k \in K$, there exist canonical isometric embeddings $\widehat{i} : (G_k)_{\mathfrak{U}_E} \to (G_{k,l})_{\mathfrak{U}}$ and $\widehat{j} : (H_l)_{\mathfrak{U}_{F^*}} \to (H_{k,l})_{\mathfrak{U}}$. Finally, note that by [66, Section 7] there exists a canonical isometric embedding $\varphi : (H^*_{k,l})_{\mathfrak{U}} \to (H_{k,l})^*_{\mathfrak{U}}$.

Now, let us fix $\varepsilon > 0$ and consider for each pair $(k,l) \in K \times L$ the finite-dimensional subspaces $G_{k,l} \subseteq E$ and $H_{k,l} \subseteq F^*$ (this time we denote by $i_{k,l} : G_{k,l} \to E$ and $j_{k,l} : H_{k,l} \to F^*$ the embeddings). By [107, Proposition 9.2.2], $T_{k,l} = j^*_{k,l} J_F T i_{k,l} \in [\mathcal{F}(G_{k,l}, H^*_{k,l}), \eta_{(p,p_2;q,q_2)}]^{**} = [\mathcal{F}(G_{k,l}, H^*_{k,l}), \eta_{(p,p_2;q,q_2)}]$, so we can use Proposition 2.4.4 to find a finite-dimensional $Z_{k,l}$ with a 1-unconditional basis and operators $R_{k,l} : G_{k,l} \to Z_{k,l}$ and $S_{k,l} : Z_{k,l} \to H_{k,l}$ such that $T_{k,l}$ factors as $S_{k,l}R_{k,l}$ and

$$K^{(p,p_2)}(R_{k,l}) K_{(q,q_2)}(S_{k,l}) \leqslant (1+\varepsilon)\eta_{(p,p_2;q,q_2)}(j^*_{k,l} J_F T i_{k,l}) \leqslant (1+\varepsilon)\eta^{**}_{(p,p_2;q,q_2)}(T)$$

(the last inequality holds because $[\mathcal{F}, \eta_{(p,p_2;q,q_2)}]^{**}$ is a finitely generated ideal, see [112, Theorem 8.11]). By rescaling the norm of $Z_{k,l}$ we can assume without loss of generality that $K_{(q,q_2)}(S_{k,l}) = 1$. It follows from [66, Proposition 3.2] that $X = (X_{k,l})_{\mathfrak{U}}$ is a Banach lattice, and it is easy to check that $R = (R_{k,l})_{\mathfrak{U}} : (G_{k,l})_{\mathfrak{U}} \to X$ is (p,p_2)-convex with $K^{(p,p_2)}(R) \leqslant \sup_{k,l} K^{(p,p_2)}(R_{k,l}) \leqslant (1+\varepsilon)\eta^{**}_{(p,p_2;q,q_2)}(T)$ and $S = (S_{k,l})_{\mathfrak{U}} : X \to (H^*_{k,l})_{\mathfrak{U}}$ is (q,q_2)-concave with $K_{(q,q_2)}(S) \leqslant \sup_{k,l} K_{(q,q_2)}(S_{k,l}) = 1$. Therefore, we have obtained the following commutative diagram:

$$\begin{array}{ccccccccccc} E & \xrightarrow{i_E} & (G_k)_{\mathfrak{U}_E} & \xrightarrow{\widehat{i}} & (G_{k,l})_{\mathfrak{U}} & \xrightarrow{(T_{k,l})_{\mathfrak{U}}} & (H^*_{k,l})_{\mathfrak{U}} & \xrightarrow{\varphi} & (H_{k,l})^*_{\mathfrak{U}} & \xrightarrow{\widehat{j}^*} & (H_l)^*_{\mathfrak{U}_{F^*}} \xrightarrow{j^*_{F^*}} F^{**} \\ & & & & & \searrow^{R} \quad \nearrow_{S} & & & & & \\ & & & & & X & & & & & \end{array}$$

A careful computation shows that $j^*_{F^*}\widehat{j}^*\varphi(T_{k,l})_{\mathfrak{U}}\widehat{i}i_E = J_F T$, so for every $\varepsilon > 0$ we have found a factorization of $J_F T$ through a Banach lattice X with operators $U = R\widehat{i}i_E$ and $V = j^*_{F^*}\widehat{j}^*\varphi S$ such that

$$\mu_{(p,p_2;q,q_2)}(T) \leqslant K^{(p,p_2)}(R) K_{(q,q_2)}(S) \leqslant (1+\varepsilon)\eta^{**}_{(p,p_2;q,q_2)}(T).$$

□

We devote the last part of the section to identify the ideal $M_{(p,p_2;q,q_2)}$. From now on, we consider only $p_2 \in \{p,\infty\}$ and $q_2 \in \{1,q\}$, as they are the most relevant cases, even though the next proposition can be proven for any $p_2 \geqslant p$ and $q_2 \leqslant q$ using the results from [55]:

Proposition 2.4.8. *Let $T : E \to F$. Then, $T \in M_{(p,p_2;q,q_2)}(E,F)$ if and only if $J_F T$ factors*

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \xrightarrow{J_F} & F^{**} \\ {\scriptstyle U}\downarrow & & & & \uparrow{\scriptstyle V} \\ X & & \xrightarrow[Q]{} & & Y \end{array}$$

where X is a (p,p_2)-convex Banach lattice, Y is a (q,q_2)-concave Banach lattice, both with constant 1, U and V are contractive operators and Q is an (almost interval preserving) lattice homomorphism. Moreover, $\mu_{(p,p_2;q,q_2)}(T) = \inf \|Q\|$, where the infimum is computed over all possible factorizations.

Proof. This is just a consequence of the fact that every (p,p_2)-convex (respectively, (q,q_2)-concave) operator factors through a (p,p_2)-convex (respectively, (q,q_2)-concave) Banach lattice (see [111] or [110] and [96, Theorem 2.8.7]), and the renorming results from [88, Sections 1.d and 1.f]. □

The following definition is an analogue in our setting of the ideal $I_{p,q}$ introduced in [62].

Definition 2.4.9. *An operator $T : E \to F$ belongs to $I_{(p,p_2;q,q_2)}(E,F)$ (i.e. factors through a multiplication operator from $L_{p,p_2}(g_1 \cdot \mu)$ to $L_{q,q_2}(g_2 \cdot \mu)$) if there exists a positive measure μ, functions $g_1, g_2 \in L_1(\mu)$ and operators $A : E \to L_{p,p_2}(g_1 \cdot \mu)$, $D : L_{p,p_2}(g_1 \cdot \mu) \to L_{q,q_2}(g_2 \cdot \mu)$ and $B : L_{q,q_2}(g_2 \cdot \mu) \to F^{**}$ such that $J_F T = BDA$ and D is a multiplication operator, that is, $Df = gf$ for some measurable function g:*

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \xrightarrow{J_F} & F^{**} \\ {\scriptstyle A}\downarrow & & & & \uparrow{\scriptstyle B} \\ L_{p,p_2}(g_1 \cdot \mu) & & \xrightarrow[D]{} & & L_{q,q_2}(g_2 \cdot \mu) \end{array}$$

We define $i_{(p,p_2;q,q_2)}(T) = \inf \{\|A\|\|D\|\|B\|\}$, where the infimum is taken over all possible factorizations above.

Theorem 2.4.10. *Assume $p > q$. Then $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}]$ and $[I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$ are isomorphic ideals.*

Proof. $[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \supseteq [I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$: Assume that $J_F T$ factors as in Definition 2.4.9. Since $L_{q,q_2}(g_2 \cdot \mu)$ is (q,q_2)-concave with constant 1, it follows that B is (q,q_2)-concave with constant $\|B\|$. On the other hand, observe that $L_{q,q_2}(g_2 \cdot \mu)$ is Dedekind complete, so the operator D has a modulus $|D|$, which is given by $|D|f = |g|f$ for every $f \in L_{p,p_2}(g_1 \cdot \mu)$. Moreover, $\||D|\| = \|D\|$ and $|D|$ is a lattice homomorphism. Therefore,

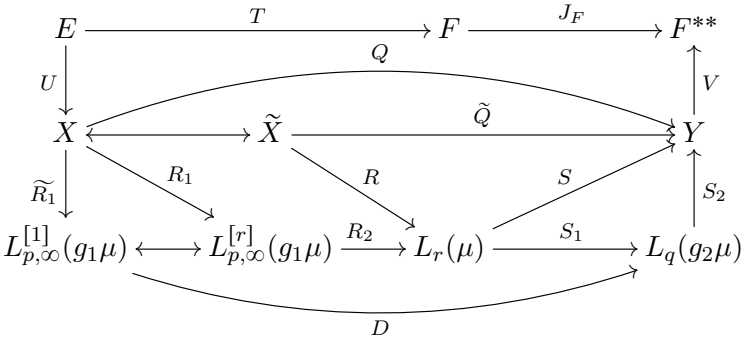
$$\begin{aligned} \left\| \left(\sum_{i=1}^n |Df_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{q,q_2} &= \left\| \left(\sum_{i=1}^n ||D|f_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{q,q_2} \\ &\leqslant \||D|\| \left\| \left(\sum_{i=1}^n |f_i|^{p_2} \right)^{\frac{1}{p_2}} \right\|_{p,p_2} \leqslant \|D\| \left(\sum_{i=1}^n \|f_i\|_{p,p_2}^{p_2} \right)^{\frac{1}{p_2}} \end{aligned}$$

and D is (p,p_2)-convex with constant $\|D\|$. It follows that DA is (p,p_2)-convex with constant $\|A\|\|D\|$, so $T \in M_{(p,p_2;q,q_2)}(E,F)$ and $\mu_{(p,p_2;q,q_2)}(T) \leqslant \|A\|\|D\|\|B\|$.

$[M_{(p,p_2;q,q_2)}, \mu_{(p,p_2;q,q_2)}] \subseteq [I_{(p,p_2;q,q_2)}, i_{(p,p_2;q,q_2)}]$: We start by considering a factorization $J_F T = VQU$ as in Proposition 2.4.8 and fixing $q < r < p$. The case $p_2 = p$, $q_2 = q$ was established in [111]. Here, we show the other three cases, which are summarized in the commutative diagrams below:

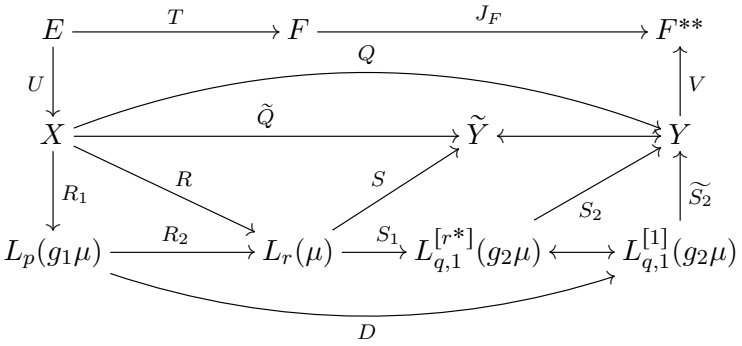
(1) **Case 1:** $p_2 = \infty$, $q_2 = q$. X is (p, ∞)-convex, so it is r-convex with constant $C_{p,r}$ [88, Theorem 1.f.7]. Let $\widetilde{X}$ be a latice renorming of X so that the r-convexity constant is 1 [88, Proposition 1.d.8]. By rescaling, we can assume that $\widetilde{Q} : \widetilde{X} \to Y$, the composition of Q with the renorming isomorphism, is a contractive lattice homomorphism. Moreover, Y is already r-concave with constant 1, so by Krivine's Factorization Theorem (cf. [80, Théorème 2] or [110, Lemma 17]), $\widetilde{Q}$ can be factored through some $L_r(\mu)$ with contractive lattice homomorphisms $R : \widetilde{X} \to L_r(\mu)$ and $S : L_r(\mu) \to Y$. Note that S is q-concave, so, by Theorem 1.2.5, it factors through $L_q(g_2 \cdot \mu)$ for some change of density $0 \leqslant g_2 \in S_{L_1(\mu)}$, with contractive factors $S_1 : L_r(\mu) \to L_q(g_2 \cdot \mu)$ and $S_2 : L_q(g_2 \cdot \mu) \to Y$, where S_1 is the operator of multiplication by $g_2^{-\frac{1}{r}}$. Similarly, by Theorem 1.2.4, there exists a change of density $0 \leqslant g_1 \in S_{L_1(\mu)}$ such that the composition of the renorming of X with R (which is (p, ∞)-convex) factors through $L^{[r]}_{p,\infty}(g_1 \cdot \mu)$ with $R_1 : X \to L^{[r]}_{p,\infty}(g_1 \cdot \mu)$ of norm bounded by $(1 - \frac{r}{p})^{\frac{1}{p} - \frac{1}{r}} C_{p,r} \|Q\|$ and $R_2 : L^{[r]}_{p,\infty}(g_1 \cdot \mu) \to L_r(\mu)$ the contractive operator of multiplication by $g_1^{\frac{1}{r}}$. Composing the renorming isomorphism from $L^{[1]}_{p,\infty}(g_1 \cdot \mu)$ into $L^{[r]}_{p,\infty}(g_1 \cdot \mu)$ given by (1.6) with the multiplication operators R_2 and S_1, we obtain the operator $D : L^{[1]}_{p,\infty}(g_1 \cdot \mu) \to L_q(g_2 \cdot \mu)$ of multiplication by $\left(\frac{g_1}{g_2}\right)^{\frac{1}{r}} \in L_r(g_2 \cdot \mu)_+$, which has norm bounded by $\left(\frac{p}{p-r}\right)^{\frac{1}{r}}$. Putting everything together, we conclude that $T \in I_{(p,\infty;q)}(E, F)$ and, since the factorization through Q and the exponent $r \in (q, p)$ were arbitrary, we can take the infimum over all possibilities to get that

$$i_{(p,\infty;q)}(T) \leqslant \inf_{q<r<p} \left\{ \left(\frac{p}{p-r} \right)^{\frac{2}{r}-1} C_{p,r} \right\} \mu_{(p,\infty;q)}(T).$$



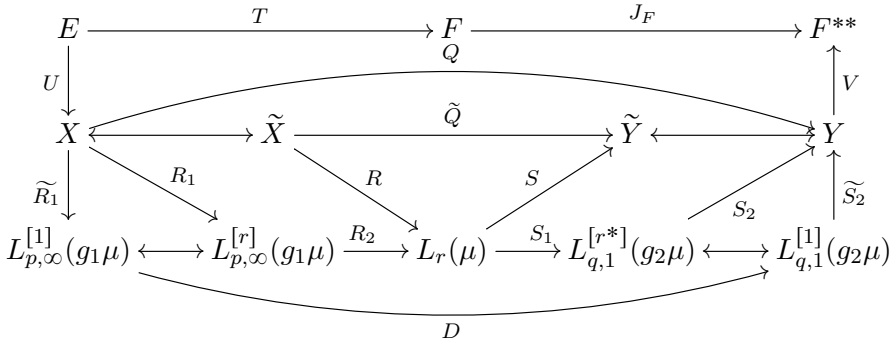
(2) **Case 2:** $p_2 = p$, $q_2 = 1$. The procedure is similar, but this time, we need to renorm Y with distortion C_{q^*,r^*} so that it becomes r-concave with constant 1, and after factoring through $L_r(\mu)$ with Krivine's Theorem, we need to apply Theorem 1.2.3 and Theorem 1.2.6, and renorm $L^{[r^*]}_{q,1}(g_2 \cdot \mu)$ to recover the standard norm of $L^{[1]}_{q,1}(g_2 \cdot \mu)$. Finally, we get that

$$i_{(p;q,1)}(T) \leqslant \inf_{q<r<p} \left\{ \left(\frac{q^*}{q^* - r^*} \right)^{\frac{2}{r^*}-1} C_{q^*,r^*} \right\} \mu_{(p;q,1)}(T).$$



(3) **Case 3:** $p_2 = \infty$, $q_2 = 1$. This time, both X and Y need to be renormed, introducing distortions $C_{p,r}$ and $C_{p*,r*}$, respectively, and, after applying Theorem 1.2.4 and Theorem 1.2.6, we need to renorm the corresponding $L^{[r]}_{p,\infty}(g_1 \cdot \mu)$ and $L^{[r^*]}_{q,1}(g_2 \cdot \mu)$ to obtain a multiplication operator D from $L^{[1]}_{p,\infty}(g_1 \cdot \mu)$ into $L^{[1]}_{q,1}(g_2 \cdot \mu)$. This time, the relation between the ideal norms is given by

$$i_{(p,\infty;q,1)}(T) \leqslant \inf_{q<r<p} \left\{ \left(\frac{p}{p-r} \right)^{\frac{2}{r}-1} \left(\frac{q^*}{q^*-r^*} \right)^{\frac{2}{r^*}-1} C_{p,r} C_{q^*,r^*} \right\} \mu_{(p,\infty;q,1)}(T).$$



□

Chapter 3
Duality on free Banach lattices

As it was mentioned in the introduction, the bidual of a Banach space satisfies the universal property of a free object in the category of dual Banach spaces with adjoint operators with respect to the category of Banach spaces with operators. However, despite the fact that duals of Banach lattices also have a canonical Banach lattice structure, the bidual of a Banach lattice does not in general satisfy the analogous universal property within the category of Banach lattices. Therefore, the aim of this chapter is to clarify the relation between free Banach lattices, free dual spaces and free dual Banach lattices generated by a Banach space. It is based on:

[56] E. García-Sánchez, P. Tradacete, *Free dual spaces and free Banach lattices*. J. Math. Anal. Appl. **532** (2024), 127931.

Moreover, some parts of the chapter rely on new results that have not been published yet, and Section 3.6 is based on a theorem from

[27] E. Bilokopytov, E. García-Sánchez, D. de Hevia, G. Martínez-Cervantes, P. Tradacete, *Norm-attaining lattice homomorphisms and renormings of Banach lattices*. J. Funct. Anal. (accepted).

Recall that the bidual or second dual E^{**} of a Banach space E (together with the canonical embedding $J_E : E \to E^{**}$) behaves as the *free dual Banach space* generated by E, that is, a free object in the category $\mathcal{B}an^*$ of dual Banach spaces with weak*-to-weak* continuous (equivalently, adjoint) operators, with respect to the category $\mathcal{B}an$ of Banach spaces with (bounded and linear) operators, as it satisfies the following universal property: for every operator into a dual Banach space $T : E \to F^*$ there is a unique weak*-to-weak* continuous extension to E^{**} given by $J_F^* T^{**}$:

$$\begin{array}{ccc} E^{**} & \xrightarrow{T^{**}} & F^{***} \\ {\scriptstyle J_E}\uparrow & & \downarrow{\scriptstyle J_F^*} \\ E & \xrightarrow{T} & F^* \end{array}$$

If we try to replicate this construction in the subcategory of Banach lattices, the situation is somewhat more involved. Indeed, some problem appear already in the very definition of the category to consider. For instance, it is an open problem whether every Banach lattice which is a dual Banach space is actually the dual of a Banach lattice. For separable dual spaces this was solved in the affirmative by Talagrand in [117], but the non-separable case remains open. Moreover, it should be noted that the preduals might not necessarily be isomorphic as Banach spaces, so the weak* topologies they induce might be different. Therefore, the first aspect we should clarify is whether we are considering the class of *dual Banach lattices*, that is, Banach lattices which are also dual Banach spaces, or the class of *duals of Banach lattices*. Throughout this chapter we will rely on certain Banach lattice properties of the predual space and the preadjoint operator, so we will

always work with the latter. Passing now to the morphisms, it should be noted that, whereas an operator between Banach lattices is positive if and only if its adjoint is, the adjoint of a lattice homomorphism is an interval preserving operator and, conversely, the adjoint of an (almost) interval preserving operator is a lattice homomorphism (see [96, Theorem 1.4.19]).

If we consider positive operators, the same construction above can be carried out to yield that the bidual X^{**} of a Banach lattice X is a free object in the category of duals of Banach lattices with weak*-to-weak* continuous positive operators, with respect to the category of Banach lattices with positive operators, to which X belongs. Indeed, if Y is a Banach lattice and $T : X \to Y^*$ is a positive map, then both $T^{**} : X^{**} \to Y^{***}$ and $J_Y^* : Y^{***} \to Y^*$ are positive, as J_Y is a lattice homomorphism and hence, positive. A more intricate situation appears if we consider lattice homomorphisms instead (see Remark 3.3.6). Henceforth, we will consider the category $\mathcal{BL}^*$, whose objects are duals of Banach lattices and the morphisms are weak*-to-weak* continuous lattice homomorphisms, that is, adjoints of (almost) interval preserving operators between the corresponding preduals. It should be noted that whenever we choose an element in this category, we are fixing in particular its weak* topology and its Banach lattice structure, so we are actually identifying its predual X as well. More specifically, if X^* and Y^* are isomorphic (isometric) in this category, this means that there exists an isomorphic (isometric) weak*-to-weak* continuous lattice homomorphism $T : X^* \to Y^*$. Since onto lattice homomorphism are also interval preserving (cf. [114, Proposition II.2.5]), the preadjoint operator is also an isomorphic (isometric) interval preserving lattice homomorphism between the predual spaces X and Y.

This chapter is mainly devoted to the study of the interplay between duality and the categories of Banach spaces and Banach lattices, from the point of view of the universal properties of certain constructions. First of all, in Section 3.2 we will compare the free dual space of the free Banach lattice generated by a Banach space with the free Banach lattice generated by the corresponding free dual space. Namely, we will show that for every Banach space E, there is a lattice isometric embedding of $\mathrm{FBL}^p[E^{**}]$ into $\mathrm{FBL}^p[E]^{**}$. The construction of such embedding is canonical. Indeed, consider the embedding $\phi_E : E \to \mathrm{FBL}^p[E]$, take its second adjoint $\phi_E^{**} : E^{**} \to \mathrm{FBL}^p[E]^{**}$, and finally, extend it to the unique lattice homomorphism $\widehat{\phi_E^{**}} : \mathrm{FBL}^p[E^{**}] \to \mathrm{FBL}^p[E]^{**}$. By means of a careful application of the Principle of Local Reflexivity carried out in Section 3.1, we will prove that $\widehat{\phi_E^{**}}$ is a lattice isometric embedding. Moreover, it turns out that a similar argument holds if we replace free Banach lattices with free $C(K)$-spaces, or if instead of taking biduals we take ultraproducts (see Section 3.2.1).

Next, we will explore the existence of free objects in $\mathcal{BL}^*$ generated by a Banach space. The main result, Theorem 3.3.7, establishes that $\mathrm{FBL}[E]^{**}$ can be identified with the free dual Banach lattice generated by the Banach space E precisely when E does not contain any complemented subspace isomorphic to ℓ_1. This should be compared with the p-convex setting for $p > 1$, where $\mathrm{FBL}^p[E]^{**}$ always behaves as the free dual p-convex Banach lattice over E (see Proposition 3.3.2), or the category $\mathcal{AL}^*$ of duals of AL-spaces (Section 3.4). Section 3.3.1 will be dedicated to initiate a preliminary study of the connections between the existence of the free dual Banach lattice generated by ℓ_1 and the understanding of w^*-closures of sublattices of duals of Banach lattices.

The last two sections are not necessarily related with the main topic of the chapter, but will explore some other duality notions on (free) Banach lattices. Namely, Section 3.5 contains a result in the spirit of [103, Proposition 3.2] for operators T between Banach spaces that do not preserve a copy of ℓ_1 complementably, or equivalently, such that their adjoint T^* is order weakly compact. Finally, in Section 3.6 we will present the (chronologically) first proof of one of the main result of [27], that asserts that every lattice homomorphism on an AM-space attains its norm. Even though a more elementary proof was obtained later, the argument that will be presented has interest on its own as an example of how free Banach lattices can be applied to the study of problems in the theory of general Banach lattices.

3.1 An alternative representation of FBL$^p[E^*]$ via local reflexivity

For convenience and generality, we will state the results of the following sections in the setting of free p-convex Banach lattices. Let E, F be Banach spaces and $1 \leqslant p \leqslant \infty$. For a function $f : E \to F$ consider the following expression for $1 \leqslant p < \infty$

$$\|f\|_{\mathcal{H}_p(E,F)} := \sup\left\{ \left(\sum_{i=1}^m |y^*(f(x_i))|^p \right)^{\frac{1}{p}} : \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x^*(x_i)|^p \right)^{\frac{1}{p}} \leqslant 1 \right\}, \tag{3.1}$$

where $y^* \in B_{F^*}$, $(x_i)_{i=1}^m \subseteq E$ and $m \in \mathbb{N}$ is arbitrary, and

$$\|f\|_{\mathcal{H}_\infty(E,F)} := \sup_{x \in B_E} \|f(x)\|_F \tag{3.2}$$

when $p = \infty$. Let us denote

$$\mathcal{H}_p(E,F) = \{f : E \to F : f \text{ is positively homogeneous and } \|f\|_{\mathcal{H}_p(E,F)} < \infty\}.$$

These spaces of non-linear positively homogeneous p-summing maps, which are Banach spaces with the norm given by (3.1) (respectively (3.2)), are relevant in the study of lattice homomorphisms between FBL^p spaces (see [84, 103]). When $F = \mathbb{R}$, we can equip $\mathcal{H}_p(E, \mathbb{R})$ with the pointwise order and lattice operations, and it becomes a Banach lattice with the above norm. In particular, for every Banach space E we have that $H_p[E] = \mathcal{H}_p(E^*, \mathbb{R})$.

Given a Banach space E, its dual E^* embeds into $\mathcal{H}_p(E^{**}, \mathbb{R})$ through the isometric embedding ϕ_{E^*} defined by $(\phi_{E^*}x^*)(x^{**}) = x^{**}(x^*)$ for $x^* \in E^*$ and $x^{**} \in E^{**}$. In the same way, E^* can be embedded into $\mathcal{H}_p(E, \mathbb{R})$ by means of the identity map

$$\begin{aligned} \psi_{E^*} : E^* &\longrightarrow \mathcal{H}_p(E, \mathbb{R}) \\ x^* &\longmapsto \psi_{E^*}x^* = x^*, \end{aligned}$$

which is again a linear isometry. Let us consider the composition operator

$$\begin{aligned} C_{J_E} : \mathcal{H}_p(E^{**}, \mathbb{R}) &\longrightarrow \mathcal{H}_p(E, \mathbb{R}) \\ h &\longmapsto C_{J_E}h = h \circ J_E, \end{aligned}$$

where $J_E : E \to E^{**}$ is the canonical embedding. This operator is well defined and has norm one, since

$$\sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x^*(x_i)|^p \right)^{\frac{1}{p}} = \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |J_E x_i(x^*)|^p \right)^{\frac{1}{p}} \tag{3.3}$$

for every $(x_i)_{i=1}^m \subseteq E$. It is also a lattice homomorphism, because $\mathcal{H}_p(E, \mathbb{R})$ and $\mathcal{H}_p(E^{**}, \mathbb{R})$ are equipped with the pointwise lattice operations, and it satisfies that $C_{J_E}\phi_{E^*} = \psi_{E^*}$.

We can also define the "extension by zero" operator $\kappa_E : \mathcal{H}_p(E, \mathbb{R}) \to \mathcal{H}_p(E^{**}, \mathbb{R})$ given by

$$\kappa_E g(x^{**}) = \begin{cases} g(J_E^{-1}x^{**}) & \text{if } x^{**} \in J_E(E), \\ 0 & \text{if } x^{**} \notin J_E(E), \end{cases} \qquad \text{for } g \in \mathcal{H}_p(E, \mathbb{R}).$$

Lemma 3.1.1. *For every Banach space E, $\kappa_E : \mathcal{H}_p(E, \mathbb{R}) \to \mathcal{H}_p(E^{**}, \mathbb{R})$ is a lattice isometric embedding such that $C_{J_E}\kappa_E = id_{\mathcal{H}_p(E,\mathbb{R})}$. In particular, $\mathcal{H}_p(E^{**}, \mathbb{R})$ contains a lattice isometric copy of $\mathcal{H}_p(E, \mathbb{R})$ complemented by a lattice projection.*

Proof. Let $1 \leqslant p < \infty$ (the case $p = \infty$ follows with the obvious modifications). Firstly, let us check that κ_E is well defined. Given $g \in \mathcal{H}_p(E, \mathbb{R})$, $\kappa_E g$ is always positively homogeneous, since $J_E(E)$

is a linear subspace of E^{**} and g is already positively homogeneous on E. In order to estimate $\|\kappa_E g\|_{\mathcal{H}_p(E^{**},\mathbb{R})}$, we choose an arbitrary sequence $(x_i^{**})_{i=1}^m \subseteq E^{**}$, $m \in \mathbb{N}$, that we can assume without loss of generality to be of the form $\{J_E x_1, \dots, J_E x_n, x_{n+1}^{**}, \dots, x_m^{**}\}$ for some $0 \leqslant n \leqslant m$. It follows by (3.3) that

$$\begin{aligned}\left(\sum_{i=1}^m |\kappa_E g(x_i^{**})|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n |\kappa_E g(J_E x_i)|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^n |g(x_i)|^p\right)^{\frac{1}{p}} \\ &\leqslant \|g\|_{\mathcal{H}_p(E,\mathbb{R})} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{\frac{1}{p}} \\ &= \|g\|_{\mathcal{H}_p(E,\mathbb{R})} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |J_E x_i(x^*)|^p\right)^{\frac{1}{p}} \\ &\leqslant \|g\|_{\mathcal{H}_p(E,\mathbb{R})} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x_i^{**}(x^*)|^p\right)^{\frac{1}{p}},\end{aligned}$$

so $\|\kappa_E g\|_{\mathcal{H}_p(E^{**},\mathbb{R})} \leqslant \|g\|_{\mathcal{H}_p(E,\mathbb{R})}$ and κ_E is well defined and bounded. On the other hand, given a sequence $(x_i)_{i=1}^m \subseteq E$ we observe making use of (3.3) again that

$$\begin{aligned}\left(\sum_{i=1}^m |g(x_i)|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^m |\kappa_E g(J_E x_i)|^p\right)^{\frac{1}{p}} \\ &\leqslant \|\kappa_E g\|_{\mathcal{H}_p(E^{**},\mathbb{R})} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |J_E x_i(x^*)|^p\right)^{\frac{1}{p}} \\ &= \|\kappa_E g\|_{\mathcal{H}_p(E^{**},\mathbb{R})} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x^*(x_i)|^p\right)^{\frac{1}{p}},\end{aligned}$$

so we can conclude that κ_E is an isometry. It can be easily checked that κ_E is a lattice homomorphism and that $C_{J_E}\kappa_E = id_{\mathcal{H}_p(E,\mathbb{R})}$. □

Remark 3.1.2. *Note that C_{J_E} is injective if and only if E is reflexive. Indeed, one implication is clear. To check the converse, assume that E is not reflexive. By the Hahn–Banach Theorem there exists a non-zero functional $x^{***} \in E^{***}$ such that $x^{***} = 0$ on $J_E(E)$. Since E^{***} embeds canonically into $\mathcal{H}_p(E^{**},\mathbb{R})$, we can see that $C_{J_E}x^{***} = x^{***} \circ J_E = 0$.*

Nevertheless, if we consider instead $\mathrm{FBL}^p[E^*]$ as a sublattice of $\mathcal{H}_p(E^{**},\mathbb{R})$, we will see in Proposition 3.1.5 that C_{J_E} restricted to $\mathrm{FBL}^p[E^*]$ is an isometric embedding. For the proof of this fact, we need first to recall the Principle of Local Reflexivity (cf. [46, Theorem 6.3]) and make a careful analysis of the approximation parameters involved in the version given next.

Theorem 3.1.3. *Let E be a Banach space. For any finite-dimensional subspaces $U \subseteq E^{**}$ and $V \subseteq E^*$ and $\epsilon > 0$, there exists a linear isomorphism S of U onto $S(U) \subseteq E$ such that $\|S\|\,\|S^{-1}\| \leqslant 1+\epsilon$, $x^*(Sx^{**}) = x^{**}(x^*)$ for every $x^* \in V$ and $x^{**} \in U$, and S is the identity on $U \cap J_E(E)$.*

In the proof given in [46, Theorem 6.3], in order to construct the operator S, a δ-net $\{b_1^{**}, \dots, b_n^{**}\}$ of the unit sphere S_U satisfying some additional properties is needed, with δ chosen in such a way that

$$\theta(\delta) := \frac{1+\delta}{1-\delta}\left(\frac{1}{1+\delta} - \delta\frac{1+\delta}{1-\delta}\right)^{-1} < 1+\epsilon.$$

The operator S can then be defined, so that

$$(1+\delta)^{-1} \leqslant \|Sb_i^{**}\| \leqslant 1+\delta$$

for each $i = 1, \dots, n$. Under these conditions, it is straightforward to check (see [46, Ex. 1.76]) that

$$\|S\| \leqslant \frac{1+\delta}{1-\delta} \text{ and } \|S^{-1}\| \leqslant \left(\frac{1}{1+\delta} - \delta\frac{1+\delta}{1-\delta}\right)^{-1}. \tag{3.4}$$

In particular, S is invertible and $\|S\|\,\|S^{-1}\| \leqslant 1+\epsilon$. The above estimates on the norm of S and its inverse lead us to study the behaviour of $\delta \equiv \delta(\epsilon)$ when ϵ is small. The following technical lemma shows more precisely how this dependence on ϵ can be described.

Lemma 3.1.4. *There exists some $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there is a $\delta \equiv \delta(\epsilon) \in \left(0, \frac{\epsilon}{1+\epsilon}\right)$ such that $\theta(\delta) = 1+\epsilon$.*

Proof. Putting $z = 1+\delta$ and $\lambda \equiv \lambda(\epsilon) = \frac{\epsilon}{1+\epsilon}$ we can transform the equation $\theta(\delta) = 1+\epsilon$ into $g(z) = 0$, where $g(z) = z^3 - \lambda z^2 + z - 2$. Our goal is to show that, for every ϵ small enough, g has a real root $z_0 \equiv z_0(\epsilon)$ such that $1 < z_0 < 1+\lambda$. To do so, we will make use of Rouché's Theorem (cf. [50, Section VIII.2]) to compare the polynomial $g(z)$ with $f(z) = z^3 + z - 2 = (z-1)(z^2+z+2)$, which has a simple root when $z = 1$ and two complex-conjugate roots in the half-plane $\mathfrak{Re}\,(z) < 0$.

Let us consider the auxiliary function

$$\phi_\lambda(r) = -r^3 - (3+\lambda)r^2 + (4-2\lambda)r - \lambda.$$

Given $r \in (0,1)$, if $\phi_\lambda(r) > 0$, then $|g(z) - f(z)| < |f(z)|$ for all $z \in \partial D(1,r)$, where $D(1,r)$ is the open disc of radius r centered at 1, and $\partial D(1,r)$ is its boundary. Indeed, $\phi_\lambda(r) > 0$ if and only if

$$\lambda(1+r)^2 < r(4-3r-r^2).$$

In this situation,

$$\begin{aligned} |g(z) - f(z)| = \lambda|z|^2 \leqslant \lambda(1+r)^2 < r(4-3r-r^2) \\ \leqslant |z-1|\,|z^2+z+2| = |f(z)| \end{aligned}$$

for all $z \in \partial D(1,r)$, where the last inequality is obtained as follows: since $\mathfrak{Re}\,(z) \geqslant 1-r > 0$ for every $z \in \partial D(1,r)$, we have that

$$\begin{aligned} |z^2+z+2| = |(z-1)^2 + 3z + 1| \geqslant |3z+1| - |z-1|^2 \\ \geqslant |\,\mathfrak{Re}\,(3z+1)| - r^2 \geqslant 4 - 3r - r^2. \end{aligned}$$

Now, evaluating at $r = \lambda$ we get that $\phi_\lambda(\lambda) = 3\lambda - 5\lambda^2 - 2\lambda^3$, which is positive whenever λ is positive and small enough. Let $\epsilon_0 \in (0,1)$ be such that $\phi_\lambda(\lambda) > 0$ for every $\lambda \in (0, \epsilon_0)$. Then, for any $\epsilon \in (0, \epsilon_0)$ the radius $r = \lambda(\epsilon) \in (0, \epsilon_0)$ satisfies $\phi_\lambda(r) > 0$, so $|g(z) - f(z)| < |f(z)|$ for every $z \in \partial D(1,r)$. Since f has only a simple root in $D(1,r)$, Rouché's Theorem allows us to conclude that g has a single root in $D(1,r)$ as well, denoted z_0.

Such z_0 must be real, because if $z_0 \in \mathbb{C}\backslash\mathbb{R}$, then its conjugate $\bar{z}_0$ would be a different root of g in $D(1,r)$, which leads to a contradiction. Thus, $z_0 \in (1-r, 1+r)$. In addition, $z_0 > 1$: if $z_0 \leqslant 1$, then we would have $0 = g(z_0) = z_0^3 - \lambda z_0^2 + z_0 - 2 \leqslant -\lambda z_0^2 < 0$, which is also a contradiction. Taking $\delta(\epsilon) = z_0 - 1 > 0$ the proof is concluded. □

In particular, from Lemma 3.1.4 it follows that in the proof of [46, Theorem 6.3] δ can always be chosen so that $\delta \to 0$ whenever $\epsilon \to 0$. This will be necessary for the next result, that states that for a dual Banach space E^*, the norm of a function $h \in \mathrm{FBL}^p[E^*]$ can be computed in a simpler way by evaluating h only on elements of E, in contrast with the general formula (3.1), that requires evaluations in the whole E^{**}.

Proposition 3.1.5. *Let $1 \leqslant p \leqslant \infty$. The restriction of the operator C_{J_E} to $\mathrm{FBL}^p[E^*]$ is a lattice isometric embedding, that is, the norm of any element $h \in \mathrm{FBL}^p[E^*]$ can be computed as*

$$\|h\|_{\mathrm{FBL}^p[E^*]} = \sup\left\{\left(\sum_{i=1}^m |h(J_E x_i)|^p\right)^{\frac{1}{p}} : (x_i)_{i=1}^m \subseteq E, \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x^*(x_i)|^p\right)^{\frac{1}{p}} \leqslant 1\right\}.$$

Proof. Let $1 \leqslant p < \infty$. First, we will show that $\|h\|_{\mathrm{FBL}^p[E^*]} = \|C_{J_E} h\|_{\mathcal{H}_p(E,\mathbb{R})}$ for every $h \in \mathrm{FVL}[E^*]$. The inequality $\|h\|_{\mathrm{FBL}^p[E^*]} \geqslant \|C_{J_E} h\|_{\mathcal{H}_p(E,\mathbb{R})}$ follows from $\|C_{J_E}\| \leqslant 1$. In order to obtain the converse inequality, we apply Theorem 3.1.3 to the space E. On the one hand, given $(x_i^{**})_{i=1}^m \subseteq E^{**}$, let us consider the finite-dimensional subspace $U = \operatorname{span}\{x_1^{**}, \ldots, x_m^{**}\} \subseteq E^{**}$. On the other hand, we can express a fixed $h \in \mathrm{FVL}[E^*]$ as

$$h = \bigvee_{j=1}^r \phi_{E^*} x_j^* - \bigvee_{k=1}^r \phi_{E^*} y_k^*$$

for some $(x_j^*)_{j=1}^r, (y_k^*)_{k=1}^r \subseteq E^*$. Let $V = \operatorname{span}\{x_1^*, y_1^*, \ldots, x_r^*, y_r^*\} \subseteq E^*$. Under these circumstances, Theorem 3.1.3 yields that for every $\epsilon > 0$ there exists an isomorphism $S : U \to S(U) \subseteq E$ such that $\|S\|\,\|S^{-1}\| \leqslant 1 + \epsilon$ and $x^*(Sx^{**}) = x^{**}(x^*)$ for every $x^* \in V$ and $x^{**} \in U$. From this last property it follows that

$$\begin{aligned} C_{J_E} h(Sx_i^{**}) &= \bigvee_{j=1}^r x_j^*(Sx_i^{**}) - \bigvee_{k=1}^r y_k^*(Sx_i^{**}) \\ &= \bigvee_{j=1}^r x_i^{**}(x_j^*) - \bigvee_{k=1}^r x_i^{**}(y_k^*) = h(x_i^{**}). \end{aligned}$$

Moreover,

$$\begin{aligned} \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x^*(Sx_i^{**})|^p\right)^{\frac{1}{p}} &= \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |S^* x^*(x_i^{**})|^p\right)^{\frac{1}{p}} \\ &\leqslant \|S^*\| \sup_{x^{***} \in B_{E^{***}}} \left(\sum_{i=1}^m |x^{***}(x_i^{**})|^p\right)^{\frac{1}{p}} \\ &= \|S\| \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x_i^{**}(x^*)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Thus, if we choose $(x_i^{**})_{i=1}^m \subseteq E^{**}$ satisfying $\sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^m |x_i^{**}(x^*)|^p\right)^{\frac{1}{p}} \leqslant 1$, it follows that

$$\left(\sum_{i=1}^m |h(x_i^{**})|^p\right)^{\frac{1}{p}} = \|S\| \left(\sum_{i=1}^m \left|C_{J_E} h\left(\frac{Sx_i^{**}}{\|S\|}\right)\right|^p\right)^{\frac{1}{p}} \leqslant \frac{1+\delta}{1-\delta} \|C_{J_E} h\|_{\mathcal{H}_p(E,\mathbb{R})}$$

for every $\epsilon > 0$ small enough, where δ has been chosen according to Lemma 3.1.4. Hence, first letting $\epsilon \to 0$, and then taking the supremum over all the possible choices of functionals $(x_i^{**})_{i=1}^m \subseteq E^{**}$, $m \in \mathbb{N}$, satisfying the p-summability condition above, we conclude that $\|h\|_{\mathrm{FBL}^p[E^*]} = \|C_{J_E} h\|_{\mathcal{H}_p(E,\mathbb{R})}$ for every $h \in \mathrm{FVL}[E^*]$. The result now follows from the fact that $\mathrm{FVL}[E^*]$ is dense in $\mathrm{FBL}^p[E^*]$.

The case $p = \infty$ is straightforward, since every function $h \in \mathrm{FBL}^\infty[E^*]$ is weak* continuous in $B_{E^{**}}$ [75, Theorem 5.3] and, by Goldstine's Theorem (cf. [46, Theorem 3.96]), $J_E(B_E)$ is weak* dense in $B_{E^{**}}$. □

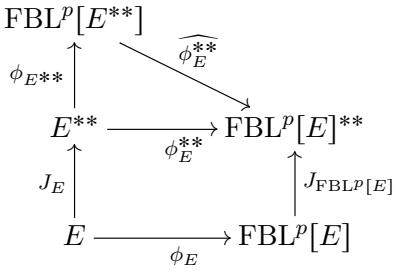
Remark 3.1.6. *Note that* $R = \kappa_E \circ C_{J_E}|_{\mathrm{FBL}^p[E^*]}$ *is a lattice isometric embedding of* $\mathrm{FBL}^p[E^*]$ *into* $H_p[E^*] = \mathcal{H}_p(E^{**}, \mathbb{R})$. *When* E *is not reflexive, the image of this embedding is different from the canonical copy of* $\mathrm{FBL}^p[E^*]$. *Indeed, by construction every element of* $R(\mathrm{FBL}^p[E^*])$ *vanishes outside of* $J_E(E)$. *Nevertheless, given* $x^{**} \in E^{**}\backslash J_E(E)$, *there is always a functional* $x^* \in E^*$ *such that* $\phi_{E^*}x^*(x^{**}) = x^{**}(x^*) \neq 0$, *so* $\phi_{E^*}x^* \in \mathrm{FBL}^p[E^*]$ *but* $\phi_{E^*}x^* \notin R(\mathrm{FBL}^p[E^*])$ *and hence*

$$\mathrm{FBL}^p[E^*] \neq R(\mathrm{FBL}^p[E^*]).$$

3.2 FBL$^p[E^{**}]$ embeds into FBL$^p[E]^{**}$

Using Proposition 3.1.5 we will show that $\mathrm{FBL}^p[E^{**}]$ embeds canonically into $\mathrm{FBL}^p[E]^{**}$ for any $1 \leqslant p \leqslant \infty$. Before the proof of this assertion, we need some observations. First, note that [11, Corollary 2.7] also holds for $\mathrm{FBL}^p[E]$, $1 \leqslant p \leqslant \infty$, i.e., the lattice homomorphisms from $\mathrm{FBL}^p[E]$ to $\mathbb{R}$ (or, equivalently, the atoms of $\mathrm{FBL}^p[E]^*$) are precisely the extensions $\widehat{x^*}$ of $x^* \in E^*$.

Now, consider the embedding $\phi_E : E \to \mathrm{FBL}^p[E]$ and its second adjoint $\phi_E^{**} : E^{**} \to \mathrm{FBL}^p[E]^{**}$. Since $\mathrm{FBL}^p[E]^{**}$ is a p-convex Banach lattice with p-convexity constant 1 [88, Proposition 1.d.4], it follows that $\widehat{\phi_E^{**}} : \mathrm{FBL}^p[E^{**}] \to \mathrm{FBL}^p[E]^{**}$ is a lattice homorphism with $\|\widehat{\phi_E^{**}}\| = 1$.



This lattice homomorphism satisfies the following property.

Lemma 3.2.1. *Let* $1 \leqslant p \leqslant \infty$. *For every* $h \in \mathrm{FBL}^p[E^{**}]$ *and* $x^* \in E^*$

$$\widehat{\phi_E^{**}}h(\widehat{x^*}) = C_{J_{E^*}}h(x^*).$$

Proof. Given $x^* \in E^*$, the functional $\psi_{x^*} = J_{\mathrm{FBL}^p[E]^*}\widehat{x^*} \in \mathrm{FBL}^p[E]^{***}$ is a lattice homomorphism. Hence, $\psi_{x^*}\widehat{\phi_E^{**}}$ is a lattice homomorphism between $\mathrm{FBL}^p[E^{**}]$ and $\mathbb{R}$ such that for every $x^{**} \in E^{**}$ we have

$$\begin{aligned}\psi_{x^*}\widehat{\phi_E^{**}}\phi_{E^{**}}(x^{**}) &= \psi_{x^*}\phi_E^{**}(x^{**}) = \phi_E^{**}x^{**}(\widehat{x^*}) \\ &= x^{**}(\phi_E^*\widehat{x^*}) = x^{**}(x^*) = J_{E^*}x^*(x^{**}).\end{aligned}$$

Hence, $\psi_{x^*}\widehat{\phi_E^{**}}\phi_{E^{**}} = J_{E^*}x^*$. The uniqueness of extension allows us to conclude that $\psi_{x^*}\widehat{\phi_E^{**}} = \widehat{J_{E^*}x^*}$. Thus, for every $h \in \mathrm{FBL}^p[E^{**}]$ we have

$$\widehat{\phi_E^{**}}h(\widehat{x^*}) = \psi_{x^*}\widehat{\phi_E^{**}}(h) = \widehat{J_{E^*}x^*}(h) = h(J_{E^*}x^*) = C_{J_{E^*}}h(x^*),$$

as claimed. □

This property, together with Proposition 3.1.5, allows us to show the next result.

Theorem 3.2.2. *Let* E *be a Banach space and* $1 \leqslant p \leqslant \infty$. *The canonical extension* $\widehat{\phi_E^{**}} : \mathrm{FBL}^p[E^{**}] \to \mathrm{FBL}^p[E]^{**}$ *is a lattice isometric embedding.*

Proof. We begin by proving the case $p < \infty$. First, recall that $\widehat{\phi_E^{**}}$ has norm one, so for every $h \in \mathrm{FBL}^p[E^{**}]$ we have

$$\|\widehat{\phi_E^{**}} h\| \leqslant \|h\|_{\mathrm{FBL}^p[E^{**}]}.$$

To obtain the reverse inequality, let us fix a function $h \in \mathrm{FBL}^p[E^{**}]$ and $(x_i^*)_{i=1}^m \subseteq E^*$ such that

$$\sup_{x \in B_E} \left(\sum_{i=1}^m |x_i^*(x)|^p \right)^{\frac{1}{p}} \leqslant 1.$$

We claim that

$$\left(\sum_{i=1}^m |C_{J_{E^*}} h(x_i^*)|^p \right)^{\frac{1}{p}} \leqslant \|\widehat{\phi_E^{**}} h\|. \tag{3.5}$$

Indeed, by Lemma 3.2.1 we have that

$$\begin{aligned} \left(\sum_{i=1}^m |C_{J_{E^*}} h(x_i^*)|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^p \right)^{\frac{1}{p}} = \sum_{i=1}^m \theta_i \widehat{\phi_E^{**}} h(\widehat{x_i^*}) \\ &= \widehat{\phi_E^{**}} h \left(\sum_{i=1}^m \theta_i \widehat{x_i^*} \right) \leqslant \|\widehat{\phi_E^{**}} h\| \left\| \sum_{i=1}^m \theta_i \widehat{x_i^*} \right\|, \end{aligned}$$

where the coefficients θ_i are defined as

$$\theta_i = \operatorname{sgn}(\widehat{\phi_E^{**}} h(\widehat{x_i^*}))\, |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^{p-1} \left(\sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^p \right)^{\frac{1}{p}-1}.$$

In order to estimate the norm of $\sum_{i=1}^m \theta_i \widehat{x_i^*} \in \mathrm{FBL}^p[E]^*$, let us evaluate it on $f \in \mathrm{FBL}^p[E]$ and then apply Hölder's inequality with exponent p as follows:

$$\begin{aligned} \left| \sum_{i=1}^m \theta_i \widehat{x_i^*}(f) \right| &\leqslant \sum_{i=1}^m |\theta_i f(x_i^*)| \\ &= \left(\sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^p \right)^{\frac{1}{p}-1} \sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^{p-1} |f(x_i^*)| \\ &\leqslant \left(\sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^p \right)^{\frac{1}{p}-1} \left(\sum_{i=1}^m |\widehat{\phi_E^{**}} h(\widehat{x_i^*})|^{p^*(p-1)} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^m |f(x_i^*)|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^m |f(x_i^*)|^p \right)^{\frac{1}{p}} \leqslant \|f\|_{\mathrm{FBL}^p[E]}. \end{aligned}$$

Therefore, we have that $\|\sum_{i=1}^m \theta_i \widehat{x_i^*}\| \leqslant 1$, and inequality (3.5) follows. By Proposition 3.1.5, we conclude that

$$\|h\|_{\mathrm{FBL}^p[E^{**}]} = \|C_{J_{E^*}} h\|_{H_p[E]} \leqslant \|\widehat{\phi_E^{**}} h\|$$

for every $h \in \mathrm{FBL}^p[E^{**}]$, so $\widehat{\phi_E^{**}}$ is a lattice isometric embedding.

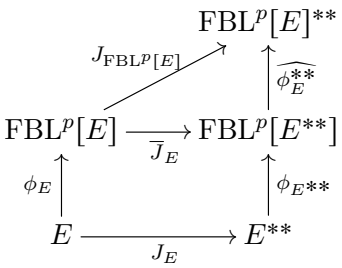
In order to prove the case $p = \infty$, just note that by Proposition 3.1.5 and Lemma 3.2.1 we have

$$\begin{aligned} \|\widehat{\phi_E^{**}} h\| \leqslant \|h\|_{\mathrm{FBL}^\infty[E^{**}]} &= \|C_{J_{E^*}} h\|_{H_\infty[E]} = \sup_{x^* \in B_{E^*}} |C_{J_{E^*}} h(x^*)| \\ &= \sup_{x^* \in B_{E^*}} |\widehat{\phi_E^{**}} h(\widehat{x^*})| \leqslant \|\widehat{\phi_E^{**}} h\|. \end{aligned}$$

□

Remark 3.2.3. *Note that the map $\widehat{\phi_E^{**}}$ considered in Theorem 3.2.2 is not surjective as long as $\dim(E) > 1$. Indeed, this can be seen as a consequence of the fact that $\mathrm{FBL}^p[E]$ is not Dedekind complete [103, Proposition 2.11].*

Remark 3.2.4. *It can be checked that the canonical embedding $J_{\mathrm{FBL}^p[E]} : \mathrm{FBL}^p[E] \to \mathrm{FBL}^p[E]^{**}$ can be factored as $J_{\mathrm{FBL}^p[E]} = \widehat{\phi_E^{**}}\overline{J}_E$:*



Indeed, for every $x \in E$ it follows that

$$\widehat{\phi_E^{**}}\overline{J}_E\phi_E(x) = \widehat{\phi_E^{**}}\phi_{E^{**}}J_E(x) = \phi_E^{**}J_E(x) = J_{\mathrm{FBL}^p[E]}\phi_E(x).$$

We have that $J_{\mathrm{FBL}^p[E]}$ and $\widehat{\phi_E^{**}}\overline{J}_E$ are lattice homomorphisms which coincide on the image of ϕ_E, and thus they coincide on $\mathrm{FBL}^p[E]$. This immediately yields that $\widehat{\phi_E^{**}}$ is an isometry on the image of $\overline{J}_E$. Hence, Theorem 3.2.2 can be considered as a strengthening of this simple fact.

The following corollary is just a direct consequence of Theorem 3.2.2 and the Principle of Local Reflexivity (Theorem 3.1.3).

Corollary 3.2.5. *Let E be a Banach space and $1 \leqslant p \leqslant \infty$. Then $\mathrm{FBL}^p[E^{**}]$ is finitely representable in $\mathrm{FBL}^p[E]$, i.e., for every finite-dimensional subspace $F \subseteq \mathrm{FBL}^p[E^{**}]$ and every $\epsilon > 0$ there is a linear isomorphism T of F onto $T(F) \subseteq \mathrm{FBL}^p[E]$ such that $\|T\|\,\|T^{-1}\| \leqslant 1+\epsilon$. Moreover, $T = \overline{J}_E^{-1}$ on $\overline{J}_E(\mathrm{FBL}^p[E]) \cap F$.*

Using [22, Theorem 2] instead we can obtain a lattice version of Corollary 3.2.5. This answers a natural question raised by an anonymous referee of the paper [56].

Corollary 3.2.6. *Let E be a Banach space and $1 \leqslant p \leqslant \infty$. Then $\mathrm{FBL}^p[E^{**}]$ is lattice finitely representable in $\mathrm{FBL}^p[E]$, i.e., for every finite-dimensional sublattice $X \subseteq \mathrm{FBL}^p[E^{**}]$ and every $\epsilon > 0$ there is a lattice isomorphism T of X onto $T(X) \subseteq \mathrm{FBL}^p[E]$ such that $\|T\|\,\|T^{-1}\| \leqslant 1+\epsilon$.*

Note that, in contrast with the Banach space case described in Corollary 3.2.5, the Principle of Local Reflexivity for Banach lattices stated in [22] cannot be used to conclude that $T = \overline{J}_E^{-1}$ on $\overline{J}_E(\mathrm{FBL}^p[E]) \cap X$. However, Proposition 3.1.5 together with an argument from [102] for computing the norm locally allow us to prove an improved version of Corollary 3.2.6.

Theorem 3.2.7. *Let E be a Banach space and $1 \leqslant p \leqslant \infty$. Then $\mathrm{FBL}^p[E^{**}]$ is lattice finitely representable in $\mathrm{FBL}^p[E]$, i.e., for every finite-dimensional sublattice $X \subseteq \mathrm{FBL}^p[E^{**}]$ and every $\epsilon > 0$ there is a lattice isomorphism T of X onto $T(X) \subseteq \mathrm{FBL}^p[E]$ such that $\|T\|\,\|T^{-1}\| \leqslant 1+\epsilon$. Moreover, if $X \subseteq \mathrm{FVL}[E^{**}]$ then $T = \overline{J}_E^{-1}$ on $\overline{J}_E(\mathrm{FBL}^p[E]) \cap X$.*

Proof. Let us assume first that the statement is true for every finite-dimensional sublattice $Y \subseteq \mathrm{FVL}[E^{**}]$. Given a finite-dimensional sublattice $X \subseteq \mathrm{FBL}^p[E^{**}]$, we consider $f_1, \ldots, f_N \in X$ positive, normalized and pairwise disjoint elements that span X.

Claim: Given a Banach space F and $f_1, \ldots, f_N \in \mathrm{FBL}^p[F]$ positive and pairwise disjoint vectors, for every $\delta > 0$ there exist $g_1, \ldots, g_N \in \mathrm{FVL}[F]$ positive and pairwise disjoint such that $\|f_n - g_n\|_{\mathrm{FBL}^p[F]} < \delta$ for every $n = 1, \ldots, N$.

We will prove this claim by induction on N. The case $N = 1$ is trivial, since $\mathrm{FVL}[F]_+$ is dense in $\mathrm{FBL}^p[F]_+$. Now, assume the claim holds for $N \geqslant 1$ vectors, and let us show that it is also true for $f_1, \dots, f_{N+1} \in \mathrm{FBL}^p[F]$ positive and pairwise disjoint vectors. Apply the induction hypothesis to find $h_1, \dots, h_N, h_{N+1} \in \mathrm{FVL}[F]_+$ such that $h_1, \dots, h_N$ are pairwise disjoint and $\|f_n - h_n\|_{\mathrm{FBL}^p[F]} < \frac{\delta}{2N+1}$ for every $n = 1, \dots, N+1$. Define $g_n = h_n - h_n \wedge h_{N+1}$ for $n = 1, \dots, N$ and $g_{N+1} = h_{N+1} - h \wedge h_{N+1}$, where $h = \sum_{n=1}^N h_n$. It is easy to check that the vectors $g_1, \dots, g_{N+1}$ are positive and pairwise disjoint. Moreover, we have the following inequalities:

$$\begin{aligned} |h_n \wedge h_{N+1}| &\leqslant |f_n - h_n| \wedge h_{N+1} + f_n \wedge |f_{N+1} - h_{N+1}| + f_n \wedge f_{N+1} \\ &\leqslant |f_n - h_n| + |f_{N+1} - h_{N+1}|, n = 1, \dots, N, \\ |h \wedge h_{N+1}| &\leqslant \sum_{n=1}^N |h_n \wedge h_{N+1}| \leqslant \sum_{n=1}^N |f_n - h_n| + N|f_{N+1} - h_{N+1}|. \end{aligned}$$

Therefore,

$$\begin{aligned} |f_n - g_n| &\leqslant |f_n - h_n| + |h_n \wedge h_{N+1}| \\ &\leqslant 2|f_n - h_n| + |f_{N+1} - h_{N+1}|, n = 1, \dots, N, \\ |f_{N+1} - g_{N+1}| &\leqslant |f_{N+1} - h_{N+1}| + |h \wedge h_{N+1}| \\ &\leqslant \sum_{n=1}^N |f_n - h_n| + (N+1)|f_{N+1} - h_{N+1}|. \end{aligned}$$

Taking norms we obtain the claim.

Now, let us fix $\epsilon > 0$ and consider $\delta = \delta(\epsilon) > 0$ small. We can apply the claim to obtain $g_1, \dots, g_N \in \mathrm{FVL}[E^{**}]$ positive and pairwise disjoint such that $\|f_n - g_n\|_{\mathrm{FBL}^p[E^{**}]} < \delta$ for every $n = 1, \dots, N$. Denote by $Y \subseteq \mathrm{FVL}[E^{**}]$ the subspace generated by these elements, which is already a sublattice. We can define a surjective lattice isomorphism $R : X \to Y$ given by

$$R\left(\sum_{n=1}^N \lambda_n f_n\right) = \sum_{n=1}^N \lambda_n g_n.$$

Moreover, using our initial assumption we can find a lattice embedding $Q : Y \to \mathrm{FBL}^p[E]$ with $\|Q\| \, \|Q^{-1}\| \leqslant 1 + \delta$. Choosing δ sufficiently small we can obtain $T = QR$ from X onto $T(X) \subseteq \mathrm{FBL}^p[E]$ with distortion below $1 + \epsilon$.

Thus, it suffices to show that for every finite-dimensional sublattice $Y \subseteq \mathrm{FVL}[E^{**}]$ and every $\epsilon > 0$ there is a lattice embedding T of Y into $\mathrm{FBL}^p[E]$ such that $\|T\| \, \|T^{-1}\| \leqslant 1 + \epsilon$.

Let Y and ϵ be as above and consider $\delta > 0$ to be chosen later. Since $Y \subseteq \mathrm{FVL}[E^{**}]$, we can find a finite-dimensional subspace $F \subseteq E^{**}$ such that $Y \subseteq \mathrm{lat}(\phi_{E^{**}}(F))$. Analogously, if we denote $Y_0 = \overline{J}_E(\mathrm{FVL}[E]) \cap Y$, there exists a finite-dimensional subspace $F_0 \subseteq E$ such that $Y_0 \subseteq \mathrm{lat}(\phi_{E^{**}} J_E(F_0))$. We can assume without loss of generality that $J_E(F_0) \subseteq F$. Now, using [102, Proposition 4.4 and Remark 4.7] we can find a finite-dimensional subspace $G \subseteq E^{**}$ containing F such that

$$\|h\|_{\mathrm{FBL}^p[E^{**}]} \leqslant \|h\|_{\mathrm{FBL}^p[G]} \leqslant (1 + \delta)\|h\|_{\mathrm{FBL}^p[E^{**}]}$$

for every $h \in Y$. Now, consider a δ-net $(h_j)_{j=1}^J$ in the compact set $S_Y = Y \cap S_{\mathrm{FBL}^p[E^{**}]}$. We can use Proposition 3.1.5 to choose functionals $x^*_{j,1}, \dots, x^*_{j,K} \in E^*$ such that

$$\sup_{x \in B_E} \left(\sum_{k=1}^K |x^*_{j,k}(x)|^p\right)^{\frac{1}{p}} \leqslant 1,$$

and

$$1 = \|h_j\|_{\mathrm{FBL}^p[E^{**}]} \leqslant \left(\sum_{k=1}^{K} |h_j \circ J_{E^*}(x^*_{j,k})|^p\right)^{\frac{1}{p}} + \delta$$

for every $j = 1, \dots, J$. Let V be the finite-dimensional subspace of E^* generated by these $(x^*_{j,k})$ and apply Theorem 3.1.3 with G and V to obtain a linear isomorphism $S : G \to E$ such that $\|S\| \, \|S^{-1}\| \leqslant 1 + \delta$, $SJ_E = id_E$ on $J_E^{-1}(G)$ (in particular, on F_0), and $x^*(Sx^{**}) = x^{**}(x^*)$ for every $x^* \in V$ and $x^{**} \in G$. Moreover, according to Lemma 3.1.4 we can assume that

$$\|S\| \leqslant \frac{1+\delta}{1-\delta}.$$

Let $\overline{S} : \mathrm{FBL}^p[G] \to \mathrm{FBL}^p[E]$ be the lattice homomorphism extension of $\widehat{\phi_E S}$. We claim that $T = \overline{S}\big|_Y : (Y, \|\cdot\|_{\mathrm{FBL}^p[E^{**}]}) \to \mathrm{FBL}^p[E]$ is a lattice embedding with distortion close to 1 for δ small enough. To check this, first note that $\|\overline{S}\| = \|S\|$ and

$$\|\overline{S}h\|_{\mathrm{FBL}^p[E]} \leqslant \|\overline{S}\| \|h\|_{\mathrm{FBL}^p[G]} \leqslant (1+\delta)\|S\| \|h\|_{\mathrm{FBL}^p[E^{**}]}$$

for every $h \in Y$, so T is bounded. On the other hand, given $h \in Y$ and $x^* \in V$ it is straightforward to check that $\overline{S}h(x^*) = h \circ J_{E^*}(x^*)$. In particular, for every $j = 1, \dots, J$ we have

$$\|\overline{S}h_j\|_{\mathrm{FBL}^p[E]} \geqslant \left(\sum_{k=1}^{K} |\overline{S}h_j(x^*_{j,k})|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{K} |h_j \circ J_{E^*}(x^*_{j,k})|^p\right)^{\frac{1}{p}} \geqslant 1 - \delta.$$

Since (h_j) is a δ-net on S_Y, for every $h \in S_Y$ there is a j such that $\|h - h_j\|_{\mathrm{FBL}^p[E^{**}]} \leqslant \delta$, and thus

$$\begin{aligned} \|\overline{S}h\|_{\mathrm{FBL}^p[E]} &\geqslant \|\overline{S}h_j\|_{\mathrm{FBL}^p[E]} - \|\overline{S}(h - h_j)\|_{\mathrm{FBL}^p[E]} \\ &\geqslant 1 - \delta - \delta(1+\delta)\|S\| \geqslant 1 - 3\delta \end{aligned}$$

for δ small enough. We conclude that

$$\|T\| \leqslant \frac{(1+\delta)^2}{1-\delta}, \ \|T^{-1}\| \leqslant \frac{1}{1-3\delta} \text{ and } \|T\| \, \|T^{-1}\| \leqslant \frac{(1+\delta)^2}{(1-\delta)(1-3\delta)},$$

so choosing δ sufficiently small we can obtain T from Y onto $T(Y) \subseteq \mathrm{FBL}^p[E]$ with distortion below $1 + \epsilon$.

Finally, in order to obtain the last part of the statement, consider $h \in Y_0$, that can be expressed as

$$h = \overline{J}_E \left(\bigvee_{i=1}^{m} \phi_E x_i - \bigvee_{i=1}^{m} \phi_E y_i\right)$$

with $(x_i)_{i=1}^m, (y_i)_{i=1}^m \subseteq F_0$. Using that $\overline{S}\,\overline{J}_E \phi_E = SJ_E = id_E$ on F_0 we conclude that $Th = \overline{J}_E^{-1} h$ for every $h \in Y_0$. Noting that Y_0 is dense in $\overline{J}_E(\mathrm{FBL}^p[E]) \cap Y$ the argument is concluded. □

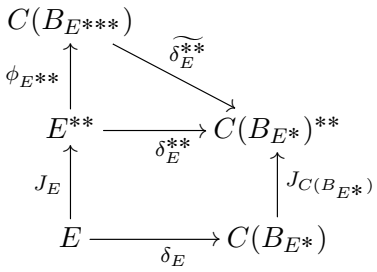
3.2.1 Related constructions

Recall that in the setting of ∞-convex Banach lattices, $\mathrm{FBL}^\infty[E]$ coincides with the lattice of positively homogeneous weak* continuous functions over B_{E^*}, denoted by $C_{ph}(B_{E^*})$ [103, Proposition 2.2]. Therefore, Theorem 3.2.2 asserts that $C_{ph}(B_{E^{***}})$ embeds isometrically into $C_{ph}(B_{E^*})^{**}$. A

similar statement holds if we replace the spaces of positively homogeneous functions with the whole space of weak* continuous functions over the corresponding unit balls. Indeed, for any Banach space E the Banach lattice $C(B_{E^*})$ behaves as the free object over E in the subcategory of $C(K)$-spaces (i.e., AM-spaces with an order unit that induces the norm) in the following sense (cf. [75, Theorem 5.4]):

Theorem 3.2.8. *Let E be a Banach space. For every compact Hausdorff space K and every operator $T : E \to C(K)$, there exists a unique lattice homomorphism $\widetilde{T} : C(B_{E^*}) \to C(K)$ such that $\widetilde{T}\delta_E = T$ and $\widetilde{T}\mathbb{1}_{E^*} = \|T\|\,\mathbb{1}_K$, where $\delta_E : E \to C(B_{E^*})$ denotes the isometric embedding that sends each $x \in E$ to the evaluation function $\delta_E x(x^*) = x^*(x)$ for $x^* \in B_{E^*}$, and $\mathbb{1}_{E^*}$ and $\mathbb{1}_K$ denote the constant functions over B_{E^*} and K, respectively. Moreover, $\|\widetilde{T}\| = \|T\|$.*

In particular, given a Banach space E and its canonical embedding δ_E, we can consider the operator $\delta_E^{**} : E^{**} \to C(B_{E^*})^{**}$, which is an isometric embedding too. Note that $C(B_{E^*})^{**}$ is an AM-space with order unit $J_{C(B_{E^*})}\mathbb{1}_{E^*}$. Moreover, the norm in $C(B_{E^*})^{**}$ coincides with the order unit norm induced by $J_{C(B_{E^*})}\mathbb{1}_{E^*}$, so by Kakutani's Theorem (cf. [96, Theorem 2.1.3]) $C(B_{E^*})^{**}$ is lattice isometric to some $C(K)$-space. Consequently, Theorem 3.2.8 yields that there exists a unique lattice homomorphism $\widetilde{\delta_E^{**}} : C(B_{E^{***}}) \to C(B_{E^*})^{**}$ such that $\widetilde{\delta_E^{**}}\delta_{E^{**}} = \delta_E^{**}$ and $\widetilde{\delta_E^{**}}\mathbb{1}_{E^{***}} = J_{C(B_{E^*})}\mathbb{1}_{E^*}$. Moreover, $\widetilde{\delta_E^{**}}$ is contractive.



As it was mentioned above, there is an analogue of Theorem 3.2.2 for this operator.

Theorem 3.2.9. *Let E be a Banach space. Then the operator $\widetilde{\delta_E^{**}} : C(B_{E^{***}}) \to C(B_{E^*})^{**}$ is a lattice isometric embedding.*

Proof. On the one hand, we observe that the composition operator

$$\begin{aligned} C_{J_{E^*}} : C(B_{E^{***}}) &\longrightarrow \ell_\infty(B_{E^*}) \\ h &\longmapsto C_{J_{E^*}} h = h \circ J_{E^*} \end{aligned}$$

is a lattice isometric embedding as a consequence of Goldstine's Theorem (cf. [46, Theorem 3.96]). On the other hand, we observe that if we denote by δ_{x^*} the evaluation functional on $x^* \in B_{E^*}$ defined as

$$\begin{aligned} \delta_{x^*} : C(B_{E^*}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \delta_{x^*}(f) = f(x^*), \end{aligned}$$

then for every $h \in L := \operatorname{lat}(\delta_{E^{**}}(E^{**}) \cup \{\mathbb{1}_{E^{***}}\}) \subseteq C(B_{E^{***}})$ and $x^* \in B_{E^*}$ the following identity holds:

$$\widetilde{\delta_E^{**}} h(\delta_{x^*}) = C_{J_{E^*}} h(x^*).$$

Indeed, the equation is trivially true if $h = \mathbb{1}_{E^{***}}$, and it follows easily when $h = \delta_{E^{**}} x^{**}$ for any $x^{**} \in E^{**}$ since $\delta_E^* \delta_{x^*} = x^*$. Moreover, δ_{x^*} is a lattice homomorphism from $C(B_{E^*})$ to $\mathbb{R}$ and thus, an atom in $C(B_{E^*})^*$, so $J_{C(B_{E^*})^*}\delta_{x^*}$ is also a lattice homomorphism. Since any $h \in L$ can be represented as a finite lattice-linear expression depending only on some $(\delta_{E^{**}} x_i^{**})_{i=1}^m$ and $\mathbb{1}_{E^{***}}$, and $C_{J_{E^*}}$ is a lattice homomorphism, we obtain the identity.

Now, for every $h \in L$ it follows that

$$\|h\|_{C(B_{E^{***}})} = \|C_{J_{E^*}} h\|_{\ell_\infty(B_{E^*})} = \sup_{x^* \in B_{E^*}} |C_{J_{E^*}} h(x^*)| = \sup_{x^* \in B_{E^*}} |\widetilde{\delta_E^{**}} h(\delta_{x^*})|$$
$$\leqslant \|\widetilde{\delta_E^{**}} h\|_{C(B_{E^*})^{**}} \leqslant \|\widetilde{\delta_E^{**}}\| \|h\|_{C(B_{E^{***}})} = \|h\|_{C(B_{E^{***}})}.$$

Since the sublattice L is dense in $C(B_{E^{***}})$ by Stone–Weierstraß's Theorem (cf. [96, Theorem 2.1]), we conclude that

$$\|h\|_{C(B_{E^{***}})} = \|\widetilde{\delta_E^{**}} h\|_{C(B_{E^*})^{**}}$$

for every $h \in C(B_{E^{***}})$. □

Going back to free Banach lattices, we can show an analogue of Theorem 3.2.2 when the biduals are replaced by ultraproducts. Throughout this subsection, we will be using the notation of [66]. Recall that given an index set I, a family of Banach spaces $(E_i)_{i\in I}$ and an ultrafilter $\mathfrak{U}$ over the set I, the *ultraproduct* $(E_i)_\mathfrak{U}$ *of the family of Banach spaces* $(E_i)_{i\in I}$ *with respect to the ultrafilter* $\mathfrak{U}$ is defined as the quotient

$$\ell_\infty(E_i)_{i\in I}/N_\mathfrak{U},$$

where $\ell_\infty(E_i)_{i\in I}$ is the space of norm-bounded sequences $(x_i)_{i\in I}$ with $x_i \in E_i$ and $N_\mathfrak{U}$ denotes the subspace of sequences such that $\lim_\mathfrak{U} \|x_i\| = 0$. The equivalence class in $(E_i)_\mathfrak{U}$ of a sequence $(x_i)_{i\in I}$ will be denoted by $(x_i)_\mathfrak{U}$, and its quotient norm can be computed easily as

$$\|(x_i)_\mathfrak{U}\| = \lim_\mathfrak{U} \|x_i\|.$$

If $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ are families of Banach spaces and $(T_i)_{i\in I}$ is a family of operators $T_i : E_i \to F_i$ such that $\sup_{i\in I} \|T_i\| < \infty$, then the *ultraproduct of the family of operators* $(T_i)_{i\in I}$ is defined as the operator

$$\begin{aligned}(T_i)_\mathfrak{U} : (E_i)_\mathfrak{U} &\longrightarrow (F_i)_\mathfrak{U} \\ (x_i)_\mathfrak{U} &\longmapsto (T_i x_i)_\mathfrak{U}.\end{aligned}$$

It can be checked that

$$\|(T_i)_\mathfrak{U}\| = \lim_\mathfrak{U} \|T_i\|.$$

We refer the reader to [66] and the bibliography therein for a complete introduction to ultraproducts and a selection of the main results in the topic. For instance, we will be constantly using the fact that the class of Banach lattices is stable under ultraproducts (see [66, Proposition 3.2]).

Now, let $(E_i)_{i\in I}$ be a family of Banach spaces index by I, $\mathfrak{U}$ an ultrafilter over I and $1 \leqslant p \leqslant \infty$ a fixed exponent. Consider the canonical embeddings $\phi_i \equiv \phi_{E_i} : E_i \to \mathrm{FBL}^p[E_i]$. Since $\sup_{i\in I} \|\phi_i\| = 1$, we can define

$$\begin{aligned}\Phi \equiv (\phi_i)_\mathfrak{U} : \quad (E_i)_\mathfrak{U} &\longrightarrow (\mathrm{FBL}^p[E_i])_\mathfrak{U} \\ x = (x_i)_\mathfrak{U} &\longmapsto \Phi x = (\phi_i x_i)_\mathfrak{U},\end{aligned}$$

which is an isometric embedding:

$$\|\Phi x\| = \lim_\mathfrak{U} \|\phi_i x_i\| = \lim_\mathfrak{U} \|x_i\| = \|x\|.$$

Note that $(\mathrm{FBL}^p[E_i])_\mathfrak{U}$ is a p-convex Banach lattice with p-convexity constant 1 (the functional calculus in the ultraproduct of Banach lattices can be defined as the limit along $\mathfrak{U}$ of the corresponding functional calculus in each factor, and the p-sum and the limit along $\mathfrak{U}$ commute, as the p-sum is a continuous function of a finite number of variables), so there exists a unique lattice homomorphism $\widehat{\Phi} : \mathrm{FBL}^p[(E_i)_\mathfrak{U}] \to (\mathrm{FBL}^p[E_i])_\mathfrak{U}$ such that $\widehat{\Phi}\phi_{(E_i)_\mathfrak{U}} = \Phi$ and $\|\widehat{\Phi}\| = \|\Phi\| = 1$. We can compute explicitly $\widehat{\Phi} f$ for any $f \in \mathrm{FVL}[(E_i)_\mathfrak{U}]$. Indeed, if we put

$$f = \bigvee_{j=1}^n \phi_{(E_i)_\mathfrak{U}} x_j - \bigvee_{j=1}^n \phi_{(E_i)_\mathfrak{U}} y_j$$

for some $x_j = (x_{j,i})_{\mathfrak{U}}, y_j = (y_{j,i})_{\mathfrak{U}} \in (E_i)_{\mathfrak{U}}$, $j = 1, \dots, n$, it follows that

$$\begin{aligned}\widehat{\Phi} f &= \bigvee_{j=1}^{n} \widehat{\Phi}\phi_{(E_i)_{\mathfrak{U}}} x_j - \bigvee_{j=1}^{n} \widehat{\Phi}\phi_{(E_i)_{\mathfrak{U}}} y_j = \bigvee_{j=1}^{n} \Phi x_j - \bigvee_{j=1}^{n} \Phi y_j \\ &= \bigvee_{j=1}^{n} (\phi_i x_{j,i})_{\mathfrak{U}} - \bigvee_{j=1}^{n} (\phi_i y_{j,i})_{\mathfrak{U}} = \left(\bigvee_{j=1}^{n} \phi_i x_{j,i} - \bigvee_{j=1}^{n} \phi_i y_{j,i} \right)_{\mathfrak{U}} = (f_i)_{\mathfrak{U}},\end{aligned}$$

where

$$f_i = \bigvee_{j=1}^{n} \phi_i x_{j,i} - \bigvee_{j=1}^{n} \phi_i y_{j,i} \in \mathrm{FVL}[E_i]$$

is the same expression used to represent f, this time evaluated in the i-th component of each x_j, y_j.

Theorem 3.2.10. *The map $\widehat{\Phi} : \mathrm{FBL}^p[(E_i)_{\mathfrak{U}}] \to (\mathrm{FBL}^p[E_i])_{\mathfrak{U}}$ is a lattice isometric embedding.*

Proof. First of all, recall (see [66, Section 7]) that there is a canonical isometric embedding of $(E_i^*)_{\mathfrak{U}}$, the ultraproduct of the duals, into $(E_i)^*_{\mathfrak{U}}$, the dual of the ultraproduct, given by

$$\begin{aligned} J : \quad (E_i^*)_{\mathfrak{U}} &\longrightarrow (E_i)^*_{\mathfrak{U}} \\ \overline{x^*} = (x_i^*)_{\mathfrak{U}} &\longmapsto J\overline{x^*}(x_i)_{\mathfrak{U}} = \lim_{\mathfrak{U}} x_i^*(x_i) \quad \forall (x_i)_{\mathfrak{U}} \in (E_i)_{\mathfrak{U}}. \end{aligned}$$

Abusing the notation, we will simply assume that $(E_i^*)_{\mathfrak{U}}$ is contained in $(E_i)^*_{\mathfrak{U}}$. Now, let us fix

$$f = \bigvee_{j=1}^{n} \phi_{(E_i)_{\mathfrak{U}}} x_j - \bigvee_{j=1}^{n} \phi_{(E_i)_{\mathfrak{U}}} y_j \in \mathrm{FVL}[(E_i)_{\mathfrak{U}}]$$

and $(f_i)_{i \in I}$ as above. We can compute f evaluated in a functional $\overline{x^*} = (\overline{x_i^*})_{\mathfrak{U}} \in (E_i^*)_{\mathfrak{U}}$ as

$$\begin{aligned} f(\overline{x^*}) &= \bigvee_{j=1}^{n} \overline{x^*}(x_j) - \bigvee_{j=1}^{n} \overline{x^*}(y_j) = \bigvee_{j=1}^{n} \lim_{\mathfrak{U}} \overline{x_i^*}(x_{j,i}) - \bigvee_{j=1}^{n} \lim_{\mathfrak{U}} \overline{x_i^*}(y_{j,i}) \\ &= \lim_{\mathfrak{U}} \left(\bigvee_{j=1}^{n} \overline{x_i^*}(x_{j,i}) - \bigvee_{j=1}^{n} \overline{x_i^*}(y_{j,i}) \right) = \lim_{U} f_i(\overline{x_i^*}). \end{aligned} \tag{3.6}$$

Moreover, since continuous functions (in particular, p-sums) commute with $\lim_{\mathfrak{U}}$, we get that

$$\left(\sum_{k=1}^{m} |f(\overline{x_k^*})|^p \right)^{\frac{1}{p}} = \lim_{\mathfrak{U}} \left(\sum_{k=1}^{m} |f_i(\overline{x_{k,i}^*})|^p \right)^{\frac{1}{p}}$$

for every $\overline{x_k^*} = (\overline{x_{k,i}^*})_{\mathfrak{U}} \in (E_i^*)_{\mathfrak{U}}$, $k = 1, \dots, m$. Any such choice of functionals in $(E_i^*)_{\mathfrak{U}}$ will also satisfy that

$$\lim_{\mathfrak{U}} \left(\sup_{x_i \in B_{E_i}} \left(\sum_{k=1}^{m} |\overline{x_{k,i}^*}(x_i)|^p \right)^{\frac{1}{p}} \right) \leqslant \sup_{x \in B_{(E_i)_{\mathfrak{U}}}} \left(\sum_{k=1}^{m} |\overline{x_k^*}(x)|^p \right)^{\frac{1}{p}}. \tag{3.7}$$

Indeed, for every $\delta > 0$ and $i \in I$ we can pick $z_i \in B_{E_i}$ such that

$$\sup_{x_i \in B_{E_i}} \left(\sum_{k=1}^{m} |\overline{x_{k,i}^*}(x_i)|^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{k=1}^{m} |\overline{x_{k,i}^*}(z_i)|^p \right)^{\frac{1}{p}} + \delta,$$

so

$$\lim_{\mathfrak{U}}\left(\sup_{x_i\in B_{E_i}}\left(\sum_{k=1}^m|\overline{x_{k,i}^*}(x_i)|^p\right)^{\frac{1}{p}}\right)\leqslant\lim_{\mathfrak{U}}\left(\left(\sum_{k=1}^m|\overline{x_{k,i}^*}(z_i)|^p\right)^{\frac{1}{p}}\right)+\delta$$
$$\leqslant\left(\sum_{k=1}^m|\overline{x_k^*}(z)|^p\right)^{\frac{1}{p}}+\delta\leqslant\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|\overline{x_k^*}(x)|^p\right)^{\frac{1}{p}}+\delta,$$

where $z=(z_i)_{\mathfrak{U}}\in B_{(E_i)_{\mathfrak{U}}}$. Taking $\delta\to 0$ we recover (3.7).

Next, let us fix $(x_k^*)_{k=1}^m\subseteq(E_i)_{\mathfrak{U}}^*$ and $\varepsilon>0$, and apply the Theorem of local duality of ultraproducts (cf. [66, Theorem 7.3]) to find a $(1+\varepsilon)$-isomorphism $T:M=\operatorname{span}\{x_k^*\}_{k=1}^m\to T(M)\subseteq(E_i^*)_{\mathfrak{U}}$ satisfying that $T|_{M\cap(E_i^*)_{\mathfrak{U}}}=id_{M\cap(E_i^*)_{\mathfrak{U}}}$ and $Tx^*(x)=x^*(x)$ for every $x^*\in M$ and $x\in\{x_1,y_1,\ldots,x_n,y_n\}$. Moreover, according to the proof provided in [66], T can be chosen contractive. It follows that

$$\begin{aligned}\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|Tx_k^*(x)|^p\right)^{\frac{1}{p}}&=\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|T^*J_{(E_i)_{\mathfrak{U}}}x(x_k^*)|^p\right)^{\frac{1}{p}}\\&\leqslant\sup_{y^{**}\in B_{(E_i)_{\mathfrak{U}}^{**}}}\left(\sum_{k=1}^m|y^{**}(x_k^*)|^p\right)^{\frac{1}{p}}\\&=\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|x_k^*(x)|^p\right)^{\frac{1}{p}},\end{aligned}\tag{3.8}$$

where in the inequality we have used that $T^*J_{(E_i)_{\mathfrak{U}}}x\in\|T^*\|B_{M^*}=B_{M^*}$ extends to some $y^{**}\in B_{(E_i)_{\mathfrak{U}}^{**}}$. Moreover, for every $x^*\in M$, we have

$$\begin{aligned}f(x^*)&=\bigvee_{j=1}^n\phi_{(E_i)_{\mathfrak{U}}}x_j(x^*)-\bigvee_{j=1}^n\phi_{(E_i)_{\mathfrak{U}}}y_j(x^*)=\bigvee_{j=1}^nx^*(x_j)-\bigvee_{j=1}^nx^*(y_j)\\&=\bigvee_{j=1}^nTx^*(x_j)-\bigvee_{j=1}^nTx^*(y_j)=\bigvee_{j=1}^n\phi_{(E_i)_{\mathfrak{U}}}x_j(Tx^*)-\bigvee_{j=1}^n\phi_{(E_i)_{\mathfrak{U}}}y_j(Tx^*)=f(Tx^*).\end{aligned}\tag{3.9}$$

Therefore, denoting by $Tx_k^*=\overline{x_k^*}=(\overline{x_{k,i}^*})_{\mathfrak{U}}\in(E_i^*)_{\mathfrak{U}}$ and combining (3.9), (3.6), (3.7) and (3.8), together with the fact that $(\mathbb{R})_{\mathfrak{U}}$ is a Banach algebra (cf. [66, Proposition 3.1]), we obtain that

$$\begin{aligned}\left(\sum_{k=1}^m|f(x_k^*)|^p\right)^{\frac{1}{p}}&=\left(\sum_{k=1}^m|f(Tx_k^*)|^p\right)^{\frac{1}{p}}=\lim_{\mathfrak{U}}\left(\sum_{k=1}^m|f_i(\overline{x_{k,i}^*})|^p\right)^{\frac{1}{p}}\\&\leqslant\lim_{\mathfrak{U}}\|f_i\|\cdot\lim_{\mathfrak{U}}\left(\sup_{x_i\in B_{E_i}}\left(\sum_{k=1}^m|\overline{x_{k,i}^*}(x_i)|^p\right)^{\frac{1}{p}}\right)\\&\leqslant\|\widehat{\Phi}f\|\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|\overline{x_k^*}(x)|^p\right)^{\frac{1}{p}}=\|\widehat{\Phi}f\|\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|Tx_k^*(x)|^p\right)^{\frac{1}{p}}\\&\leqslant\|\widehat{\Phi}f\|\sup_{x\in B_{(E_i)_{\mathfrak{U}}}}\left(\sum_{k=1}^m|x_k^*(x)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

It follows that $\|f\|\leqslant\|\widehat{\Phi}f\|$ for every $f\in\mathrm{FVL}[(E_i)_{\mathfrak{U}}]$. Since $\|\widehat{\Phi}\|=1$, we conclude that $\widehat{\Phi}$ is an isometry in $\mathrm{FVL}[(E_i)_{\mathfrak{U}}]$, and hence in $\mathrm{FBL}^p[(E_i)_{\mathfrak{U}}]$. □

3.3 Free p-convex dual Banach lattices

The present section will be devoted to studying the existence of free objects in the category of duals of Banach lattices (and the subcategories of p-convex Banach lattices). Given $1 \leqslant p \leqslant \infty$, we denote by $\mathcal{BL}^{*(p)}$ the category whose objects are p-convex Banach lattices which are duals of Banach lattices, and whose morphisms are weak*-to-weak* continuous (equivalently, adjoint) lattice homomorphisms.

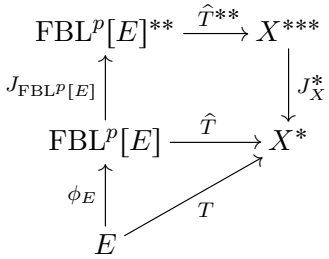
Remark 3.3.1. *Observe that when $p > 1$ every object X^* of the category $\mathcal{BL}^{*(p)}$ is canonically lattice complemented in its bidual X^{***}. Consider the contractive adjoint operator $J_X^* : X^{***} \to X^*$. Indeed, since X^* is p-convex, X must be p^*-concave, where $p^* = \frac{p}{p-1} < \infty$ (cf. [88, Proposition 1.d.4]). The p^*-concavity of X for finite p^* implies that X is order continuous, and thus it is an ideal in X^{**} (cf. [5, Theorems 4.9 and 4.14]). In other words, the canonical embedding J_X is an interval preserving isometric embedding, so J_X^* is a contractive weak*-to-weak* continuous onto lattice homomorphism (cf. [96, Theorem 1.4.19]). Finally, the composition with the canonical embedding J_{X^*} yields a lattice projection.*

The remark above will be key to show that $\mathrm{FBL}^p[E]^{**}$, the bidual of $\mathrm{FBL}^p[E]$, satisfies the universal property of the free object in the category $\mathcal{BL}^{*(p)}$ with respect to the Banach space E when $p > 1$:

Proposition 3.3.2. *Let E be a Banach space and $1 < p \leqslant \infty$. Then, for every p-convex Banach lattice X^* which is the dual of some Banach lattice and every operator $T : E \to X^*$ there exists a unique weak*-to-weak* continuous lattice homomorphism $\check{T} : \mathrm{FBL}^p[E]^{**} \to X^*$, such that $\check{T} J_{\mathrm{FBL}^p[E]} \phi_E = T$. Moreover, $\|\check{T}\| \leqslant M^{(p)}(X^*)\|T\|$.*

We can then say that $\mathrm{FBL}^p[E]^{**}$ is the *free p-convex dual Banach lattice generated by E*.

Proof. Since X^* is p-convex, we can consider the following commutative diagram, where $\widehat{T}$ is the unique extension of T to $\mathrm{FBL}^p[E]$ as a lattice homomorphism:



Since J_X^* is a norm one, weak*-to-weak* continuous lattice homomorphism satisfying $J_X^* J_{X^*} = id_{X^*}$, it follows that the operator $\check{T} := J_X^* \widehat{T}^{**}$ is also a weak*-to-weak* continuous lattice homomorphism such that $\check{T} J_{\mathrm{FBL}^p[E]} \phi_E = T$ and $\|\check{T}\| \leqslant M^{(p)}(X)\|T\|$. Now, suppose there exists another weak*-to-weak* continuous lattice homomorphism $\check{S} : \mathrm{FBL}^p[E]^{**} \to X^*$ satisfying $\check{S} J_{\mathrm{FBL}^p[E]} \phi_E = T$. The uniqueness of extension of T to $\mathrm{FBL}^p[E]$ implies that $\check{S} J_{\mathrm{FBL}^p[E]} = \widehat{T}$. Since $J_{\mathrm{FBL}^p[E]}$ has weak* dense range and both $\check{T}$ and $\check{S}$ are weak*-to-weak* continuous, they are uniquely determined by their values on $J_{\mathrm{FBL}^p[E]}(\mathrm{FBL}^p[E])$. Hence, $\check{T} = \check{S}$. □

Remark 3.3.3. *The previous argument cannot be applied to the case $p = 1$ because there exists objects of $\mathcal{BL}^*$ that do not have an order continuous predual (see [51] for a characterization of this property). Consider for instance $\mathcal{M}[0,1]$, the space of Borel measures over the interval $[0,1]$, which is the dual of the Banach lattice $C[0,1]$ but does not have any order continuous predual. Indeed, $\mathcal{M}[0,1]$ is an AL-space, so by [96, Proposition 1.4.7], all its predual Banach lattices must be AM-spaces. Since the only order continuous AM-spaces are of the form $c_0(\Gamma)$ for some set Γ*

[96, Lemma 2.7.12] and the dual of $c_0(\Gamma)$ is $\ell_1(\Gamma)$, which is never isomorphic to $\mathcal{M}[0,1]$ (note that $L_1[0,1]$ lattice embeds into $\mathcal{M}[0,1]$ but not into $\ell_1(\Gamma)$), we conclude that $\mathcal{M}[0,1]$ does not have an order continuous predual.

However, we can still define a free object in the category $\mathcal{BL}^*$ of duals of Banach lattices in the following sense.

Definition 3.3.4. *Let E be a Banach space. The* free dual Banach lattice over E *is a Banach lattice* $\mathrm{FBL}^*[E]$ *which is the dual of some other Banach lattice Z_E, together with a linear isometric embedding $\iota_E : E \to \mathrm{FBL}^*[E]$, such that for every operator T from E to X^*, the dual of a Banach lattice X, there exists a unique weak*-to-weak* continuous lattice homomorphism $\check{T} : \mathrm{FBL}^*[E] \to X^*$, such that $\check{T}\iota_E = T$, and moreover, $\|\check{T}\| = \|T\|$.*

By standard arguments it can be proved that whenever this object exists, it is unique in the category $\mathcal{BL}^*$. However, we do not know whether such a free object exists for every Banach space E. The natural candidate to satisfy this definition is $\mathrm{FBL}[E]^{**}$. Nevertheless, we cannot hope for Proposition 3.3.2 to be true in general when $p = 1$, as the following example shows.

Proposition 3.3.5. *Let X^* be the dual of a Banach lattice such that X^* does not admit an order continuous predual (take for instance $X^* = \mathcal{M}(0,1)$ as mentioned in Remark 3.3.3). The pair $(\mathrm{FBL}[X^*]^{**}, K_{X^*})$ does not have the universal property of Definition 3.3.4, where $K_{X^*} = J_{\mathrm{FBL}[X^*]}\phi_{X^*}$. More specifically, the identity operator id_{X^*} on X^* cannot be extended to a weak*-to-weak* continuous lattice homomorphism $\check{id}_{X^*} : \mathrm{FBL}[X^*]^{**} \to X^*$ such that $\check{id}_{X^*}K_{X^*} = id_{X^*}$.*

Proof. Suppose that there exists a weak*-to-weak* continuous lattice homomorphism $\check{id}_{X^*} : \mathrm{FBL}[X^*]^{**} \to X^*$ such that $\check{id}_{X^*}K_{X^*} = id_{X^*}$. Let $S : X \to \mathrm{FBL}[X^*]^*$ be such that $S^* = \check{id}_{X^*}$. Note that $\check{id}_{X^*}$ is an onto lattice homomorphism, so S must be an interval preserving embedding as a consequence of the Open Mapping Principle (cf. [46, Exercise 2.49]) and [96, Theorem 1.4.19].

On the other hand, since $\check{id}_{X^*}K_{X^*} = id_{X^*}$ and $\check{id}_{X^*}J_{\mathrm{FBL}[X^*]}$ is a lattice homomorphism, the uniqueness of extension of id_{X^*} to $\mathrm{FBL}[X^*]$ implies that $\check{id}_{X^*}J_{\mathrm{FBL}[X^*]} = \hat{id}_{X^*}$. Considering the adjoint of this operator and composing it by the right with J_X, the canonical embedding of X into its bidual, we get that

$$\begin{aligned}\hat{id}^*_{X^*}J_X &= J^*_{\mathrm{FBL}[X^*]}\check{id}^*_{X^*}J_X = J^*_{\mathrm{FBL}[X^*]}S^{**}J_X \\ &= J^*_{\mathrm{FBL}[X^*]}J_{\mathrm{FBL}[X^*]^*}S = S,\end{aligned}$$

where in the last equality we have used that $J^*_{\mathrm{FBL}[X^*]}J_{\mathrm{FBL}[X^*]^*} = id_{\mathrm{FBL}[X^*]^*}$. Using again the Open Mapping Principle (cf. [46, Exercise 2.49]) and [96, Theorem 1.4.19] it is easy to check that $\hat{id}^*_{X^*}$ is an interval preserving embedding (in particular, it is injective). Thus, for every $x \in X_+$ it follows that

$$\hat{id}^*_{X^*}(J_X[0,x]) = S[0,x] = [0,Sx] = [0,\hat{id}^*_{X^*}(J_Xx)] = \hat{id}^*_{X^*}[0,J_Xx],$$

so $J_X[0,x] = [0,J_Xx]$ and J_X is interval preserving. This is equivalent to $J_X(X)$ being an ideal of X^{**}, so by [96, Theorem 2.4.2] X must be order continuous, which is a contradiction. □

Remark 3.3.6. *A similar argument can be used to show that the bidual X^{***} of a Banach lattice X^* that does not admit an order continuous predual, together with the embedding $J_{X^*} : X^* \to X^{***}$, cannot be a free object in the category $\mathcal{BL}^*$ with respect to the category $\mathcal{BL}$: the existence of a weak*-to-weak* continuous lattice homomorphism $\check{id}_{X^*} : X^{***} \to X^*$ such that $\check{id}_{X^*}J_{X^*} = id_{X^*}$ would imply that the canonical embedding of the predual, $J_X : X \to X^{**}$, is the preadjoint of $\check{id}_{X^*}$, and hence, an interval preserving embedding, which is not possible.*

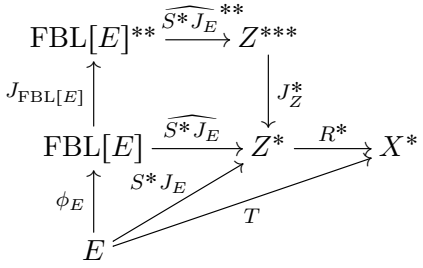
This shows that, in general, $\mathrm{FBL}[E]^{**}$ is not the free object over the Banach space E in the sense of Definition 3.3.4. However, we can provide a characterization of when $\mathrm{FBL}[E]^{**} = \mathrm{FBL}^*[E]$.

Theorem 3.3.7. *Let E be a Banach space. The following are equivalent:*

(1) E does not contain a complemented subspace isomorphic to ℓ_1.

(2) The pair $(\mathrm{FBL}[E]^{**}, K_E)$, with $K_E = J_{\mathrm{FBL}[E]}\phi_E$, coincides in the category $\mathcal{BL}^*$ with the free dual Banach lattice over E, $(\mathrm{FBL}^*[E], \iota_E)$.

Proof. (1) $\Rightarrow$ (2) Let X be a Banach lattice and $T : E \to X^*$ an operator. By [23, Theorem 4], E does not contain a complemented copy of ℓ_1 if and only if E^* does not contain a copy of c_0. Thus, applying [58, Theorem I.2] we obtain that the operator $T^*J_X : X \to E^*$ can be factored as $T^*J_X = SR$, where $R : X \to Z$, $S : Z \to E^*$ and Z is an order continuous Banach lattice. Moreover, Z is the norm completion of a quotient of X by an ideal, and R is the canonical quotient, so it is a lattice homomorphism, and, by [114, Proposition II.2.5], it is almost interval preserving. Note that, in particular, $T = R^*S^*J_E$, and we can consider the following commutative diagram:

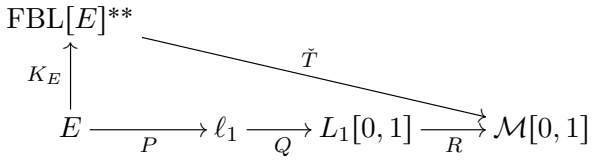


Since Z is order continuous, by [96, Theorem 2.4.2] J_Z must be interval preserving, so using [96, Theorem 1.4.19] we conclude that the operator $\check{T} = R^*J_Z^*(\widehat{S^*J_E})^{**}$ is a weak*-to-weak* continuous lattice homomorphism such that $\check{T}J_{\mathrm{FBL}[E]}\phi_E = T$. That this extension is unique follows from the universal property of $\mathrm{FBL}[E]$ and the weak* density of $\mathrm{FBL}[E]$ inside its bidual. Since $\|R\| \leqslant \|T\|$ and $\|S\| \leqslant 1$ by construction (see the proof in [58, Theorem I.2] for the details), it is straightforward to check that $\|\check{T}\| = \|T\|$. Thus, we have proved that the pair $(\mathrm{FBL}[E]^{**}, J_{\mathrm{FBL}[E]}\phi_E)$ satisfies the conditions of Definition 3.3.4.

(2) $\Rightarrow$ (1) Let us suppose that the pair $(\mathrm{FBL}[E]^{**}, K_E)$ has the universal property of Definition 3.3.4 but E contains a complemented copy of ℓ_1. In particular, there exists a quotient $P : E \to \ell_1$. Let us consider the separable space $L_1[0,1]$, which is a quotient of ℓ_1 through a certain $Q : \ell_1 \to L_1[0,1]$, and the operator $R : L_1[0,1] \to \mathcal{M}[0,1]$ given by

$$Rg(A) = \int_A g d\lambda,$$

for each Borel set $A \subseteq [0,1]$ and $g \in L_1[0,1]$, where λ denotes the Lebesgue measure over $[0,1]$. Note that $R(B_{L_1[0,1]})$ is weak* dense in $B_{\mathcal{M}[0,1]}$. Indeed, given $t \in [0,1]$, let $(I_{t,n})_n$ be a sequence of intervals of $[0,1]$ of length $\frac{1}{n}$ decreasing to $\{t\}$. Then, it is straightforward to check that the measures $\mu_{t,n} = \lambda(I_{t,n})^{-1}R\chi_{I_{t,n}} \in R(B_{L_1[0,1]})$ converge weak* to δ_t, the pointwise evaluation on t. Thus, $\{\pm\delta_t : t \in [0,1]\}$, which is the set of extreme points of $B_{\mathcal{M}[0,1]}$ [96, Proposition 2.1.2], is contained in the w^*-closure of $R(B_{L_1[0,1]})$. By Krein-Milman Theorem, we conclude that the whole $B_{\mathcal{M}[0,1]}$ lies in the w^*-closure of $R(B_{L_1[0,1]})$.



Now, consider the operator $T = RQP : E \to \mathcal{M}[0,1]$. By our assumption, there exists a weak*-to-weak* continuous lattice homomorphism $\check{T} : \mathrm{FBL}[E]^{**} \to \mathcal{M}[0,1]$ such that $\check{T}K_E = T$. We claim that $\check{T}$ is onto. Indeed, fix some $\mu \in B_{\mathcal{M}[0,1]}$. We have seen above that $R(B_{L_1[0,1]})$ is weak* dense in $B_{\mathcal{M}[0,1]}$, so there exists some net $(g_\alpha)_\alpha \subseteq B_{L_1[0,1]}$ such that $(Rg_\alpha)_\alpha$ converges weak* to μ. Since QP is onto, by the Open Mapping Theorem [46, Corollary 2.26] there exists some $r > 0$ such that $B_{L_1[0,1]} \subseteq QP(rB_E)$. Hence, we can find a bounded net $(x_\alpha)_\alpha \subseteq E$ such that $QPx_\alpha = g_\alpha$. Now, using the weak* compactness of $B_{\mathrm{FBL}[E]^{**}}$, we can extract a subnet $(x_\beta)_\beta \subseteq (x_\alpha)_\alpha$ such that $(K_E x_\beta)_\beta$ is weak* convergent to some $f^{**} \in \mathrm{FBL}[E]^{**}$. Evaluating on $\check{T}$ we obtain that $\check{T}K_E x_\beta = Rg_\beta$ converges weak* to both μ and $\check{T}f^{**}$, so $\mu = \check{T}f^{**}$ and $\check{T}$ is an onto lattice homomorphism. By [114, Proposition II.2.5], this implies that $\check{T}$ is also interval preserving. Let $S : C[0,1] \to \mathrm{FBL}[E]^*$ be an operator such that $\check{T} = S^*$. By the Open Mapping Principle (cf. [46, Exercise 2.49]) and [96, Theorem 1.4.19] it follows that S must be an interval preserving lattice embedding of $C[0,1]$ into $\mathrm{FBL}[E]^*$.

Note that $\mathrm{FBL}[E]^*$ is a dual of a Banach lattice, so in particular it is (σ-)Dedekind complete. Clearly, $C[0,1]$ is not σ-Dedekind complete. Since any ideal of a σ-Dedekind complete Banach lattice is again σ-Dedekind complete, it follows that $C[0,1]$ cannot be embedded as an ideal in $\mathrm{FBL}[E]^*$. Thus, E cannot contain a complemented copy of ℓ_1. □

It remains an open question whether $\mathrm{FBL}^*[E]$ exists when E contains a complemented subspace isomorphic to ℓ_1. The following result provides some properties that $\mathrm{FBL}^*[E]$ must satisfy, in case it exists. Note that all of them become trivial when $\mathrm{FBL}^*[E]$ happens to coincide with $\mathrm{FBL}[E]^{**}$, that is, when E does not contain a complemented copy of ℓ_1.

Lemma 3.3.8. *Let E be a Banach space, and assume that $\mathrm{FBL}^*[E]$ exists. Then:*

(1) The unique extension $\widehat{\iota}_E : \mathrm{FBL}[E] \to \mathrm{FBL}^*[E]$ of ι_E as a lattice homomorphism is an isometric embedding.

(2) $\check{K}_E : \mathrm{FBL}^*[E] \to \mathrm{FBL}[E]^{**}$ is an onto interval preserving lattice homomorphism, where $K_E = J_{\mathrm{FBL}[E]}\phi_E$.

(3) $\overline{\widehat{\iota}_E(\mathrm{FBL}[E])}^{w^*}$ is a subspace of $\mathrm{FBL}^*[E]$ linearly isometric to $\mathrm{FBL}[E]^{**}$, and it is complemented in $\mathrm{FBL}^*[E]$ by means of a positive contractive weak-to-weak* continuous projection given by $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E$.*

(4) $\mathrm{FBL}[E]^*$ embeds isometrically in Z_E as an ideal, and it is complemented in Z_E by means of a positive contractive projection.

Proof. (1) Note that

$$\check{K}_E\widehat{\iota}_E\phi_E = \check{K}_E\iota_E = K_E = J_{\mathrm{FBL}[E]}\phi_E.$$

Since both $\check{K}_E\widehat{\iota}_E$ and $J_{\mathrm{FBL}[E]}$ are lattice homomorphisms, the uniqueness of extension of operators to $\mathrm{FBL}[E]$ implies that $\check{K}_E\widehat{\iota}_E = J_{\mathrm{FBL}[E]}$. Now, using that $J_{\mathrm{FBL}[E]}$ is an isometric embedding and both $\widehat{\iota}_E$ and $\check{K}_E$ have norm one, we obtain that

$$\|f\|_{\mathrm{FBL}[E]} = \|J_{\mathrm{FBL}[E]}f\|_{\mathrm{FBL}[E]^{**}} \leqslant \|\widehat{\iota}_E f\|_{\mathrm{FBL}^*[E]} \leqslant \|f\|_{\mathrm{FBL}[E]}$$

for every $f \in \mathrm{FBL}[E]$.

(2) First, let us show that $\check{K}_E$ is onto. Given some $f^{**} \in B_{\mathrm{FBL}[E]^{**}}$, by Goldstine's Theorem we know that there exists a net $(f_\alpha)_\alpha \subseteq B_{\mathrm{FBL}[E]}$ such that $(J_{\mathrm{FBL}[E]}f_\alpha)_\alpha$ is weak* convergent to f^{**} in $\mathrm{FBL}[E]^{**}$. Now, observe that $(\widehat{\iota}_E f_\alpha)_\alpha \subseteq B_{\mathrm{FBL}^*[E]}$, which is a weak* compact set, so there exists a subnet $(\widehat{\iota}_E f_\beta)_\beta$ weak* convergent to some $z^* \in B_{\mathrm{FBL}^*[E]}$. Using that $\check{K}_E$ is weak*-to-weak* continuous and satisfies $\check{K}_E\widehat{\iota}_E = J_{\mathrm{FBL}[E]}$, we conclude that $f^{**} = \check{K}_E z^*$. Thus, $\check{K}_E$ is an

onto lattice homomorphism, and by [114, Proposition II.2.5], it follows that it is interval preserving.

(3) Consider the operator $J^*_{Z_E}\widehat{\iota}^{**}_E : \mathrm{FBL}[E]^{**} \to \mathrm{FBL}^*[E]$. Using that $J^*_{Z_E}J_{Z^*_E}$ is the identity on $Z^*_E = \mathrm{FBL}^*[E]$, it follows that

$$\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E J_{\mathrm{FBL}[E]} = \check{K}_E J^*_{Z_E} J_{Z^*_E}\widehat{\iota}_E = \check{K}_E\widehat{\iota}_E = J_{\mathrm{FBL}[E]}.$$

Since $\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E$ is weak*-to-weak* continuous and $J_{\mathrm{FBL}[E]}$ has weak* dense range, we conclude that $\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E = id_{\mathrm{FBL}[E]^{**}}$. This implies that $J^*_{Z_E}\widehat{\iota}^{**}_E$ is an isometric embedding. Indeed, given any $f^{**} \in \mathrm{FBL}[E]^{**}$, we have

$$\begin{aligned}\left\|f^{**}\right\|_{\mathrm{FBL}[E]^{**}} &= \|\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E f^{**}\|_{\mathrm{FBL}[E]^{**}} \\ &\leqslant \|J^*_{Z_E}\widehat{\iota}^{**}_E f^{**}\|_{\mathrm{FBL}^*[E]} \leqslant \left\|f^{**}\right\|_{\mathrm{FBL}[E]^{**}}.\end{aligned}$$

The next step is to show that the range of $J^*_{Z_E}\widehat{\iota}^{**}_E$ is the subspace $U_E = \overline{\widehat{\iota}_E(\mathrm{FBL}[E])}^{w^*}$. The inclusion $J^*_{Z_E}\widehat{\iota}^{**}_E(\mathrm{FBL}[E]^{**}) \subseteq U_E$ is straightforward, since $J^*_{Z_E}\widehat{\iota}^{**}_E$ is weak*-to-weak* continuous and $J_{\mathrm{FBL}[E]}$ has weak* dense range. To check the reverse inclusion, let $z^* \in U_E$ and $(f_\alpha)_\alpha \subseteq \mathrm{FBL}[E]$ such that $(\widehat{\iota}_E f_\alpha)_\alpha$ is weak* convergent to z^*. Using the weak* continuity of $\check{K}_E$ and the identity $\check{K}_E\widehat{\iota}_E = J_{\mathrm{FBL}[E]}$ it follows that $(J_{\mathrm{FBL}[E]}f_\alpha)_\alpha$ is weak* convergent to $\check{K}_E z^*$. Now, recall that $J^*_{Z_E}\widehat{\iota}^{**}_E J_{\mathrm{FBL}[E]} = \widehat{\iota}_E$, so $(\widehat{\iota}_E f_\alpha)_\alpha$ weak* converges to $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E z^*$. The uniqueness of the weak* limit implies that $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E z^* = z^*$, so $U_E \subseteq J^*_{Z_E}\widehat{\iota}^{**}_E(\mathrm{FBL}[E]^{**})$. We conclude that $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E$ is a positive contractive weak*-to-weak* continuous projection over U_E.

(4) By definition $\check{K}_E$ is weak*-to-weak* continuous, so it is the adjoint operator of some $M_E : \mathrm{FBL}[E]^* \to Z_E$. Since $\check{K}_E$ is an onto interval preserving lattice homomorphism, by the Open Mapping Principle (cf. [46, Exercise 2.49]) and [96, Theorem 1.4.19] it follows that M_E is an interval preserving lattice embedding. From the proof of (3) we know that

$$(\widehat{\iota}^*_E J_{Z_E} M_E)^* = \check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E = id_{\mathrm{FBL}[E]^{**}},$$

so the operator $\widehat{\iota}^*_E J_{Z_E} M_E$ must be the identity on $\mathrm{FBL}[E]^*$. Proceeding as before, we can see that M_E is an isometry, and that $M_E\widehat{\iota}^*_E J_{Z_E}$ is a positive contractive projection. □

In addition to the previous result, we present a compilation of statements equivalent to $\mathrm{FBL}[E]^{**}$ being isometric to $\mathrm{FBL}^*[E]$ in the category $\mathcal{BL}^*$.

Proposition 3.3.9. *Let E be a Banach space, and assume that $\mathrm{FBL}^*[E]$ exists. The following are equivalent:*

(1) The pair $(\mathrm{FBL}[E]^{**}, K_E)$ coincides in the category $\mathcal{BL}^*$ with the free dual Banach lattice over E, $(\mathrm{FBL}^*[E], \iota_E)$.

(2) $\widehat{\iota}_E(B_{\mathrm{FBL}[E]})$ is w^*-dense in $B_{\mathrm{FBL}^*[E]}$.

(3) $\widehat{\iota}_E : \mathrm{FBL}[E] \to \mathrm{FBL}^*[E]$ has weak dense range.*

(4) $\overline{\widehat{\iota}_E(\mathrm{FBL}[E])}^{w^*}$ is a sublattice of $\mathrm{FBL}^*[E]$.

(5) $J^*_{Z_E}\widehat{\iota}^{**}_E : \mathrm{FBL}[E]^{**} \to \mathrm{FBL}^*[E]$ is a lattice homomorphism.

(6) $\check{K}_E : \mathrm{FBL}^*[E] \to \mathrm{FBL}[E]^{**}$ is an onto weak-to-weak* continuous lattice isometry.*

(7) There exists a Banach lattice Y such that $\mathrm{FBL}^*[E] = Y^{**}$ and $\iota_E(E) \subseteq J_Y(Y)$.

(8) The unique extension $\check{j}_E : \mathrm{FBL}^*[E] \to \mathrm{FBL}^*[E]^{**}$ of $j_E = J_{\mathrm{FBL}^*[E]}\iota_E$ as a weak-to-weak* continuous lattice homomorphism is an isometric embedding.*

Proof. (1) $\Rightarrow$ (2) Let $\widetilde{\iota}_E : \mathrm{FBL}[E]^{**} \to \mathrm{FBL}^*[E]$ be the unique extension of ι_E to $\mathrm{FBL}[E]^{**}$ as a contractive weak*-to-weak* continuous lattice homomorphism. By standard arguments we can show that $\check{K}_E$ and $\widetilde{\iota}_E$ are mutually inverse. In particular, $\widetilde{\iota}_E(B_{\mathrm{FBL}[E]^{**}}) = B_{\mathrm{FBL}^*[E]}$. Moreover, $\widehat{\iota}_E = \widetilde{\iota}_E J_{\mathrm{FBL}[E]}$, $J_{\mathrm{FBL}[E]}(B_{\mathrm{FBL}[E]})$ is w^*-dense in $B_{\mathrm{FBL}[E]^{**}}$ and $\widetilde{\iota}_E$ is weak*-to-weak* continuous, so $\widehat{\iota}_E(B_{\mathrm{FBL}[E]})$ is w^*-dense in $B_{\mathrm{FBL}^*[E]}$.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (4) Trivial, since $\overline{\widehat{\iota}_E(\mathrm{FBL}[E])}^{w^*} = \mathrm{FBL}^*[E]$.

(4) $\Rightarrow$ (5) Recall from Lemma 3.3.8 that $J^*_{Z_E}\widehat{\iota}^{**}_E(\mathrm{FBL}[E]^{**}) = \overline{\widehat{\iota}_E(\mathrm{FBL}[E])}^{w^*}$, and denote this subspace by U_E. Since it is a sublattice, for any $f_1^{**} \in \mathrm{FBL}[E]^{**}$ there must exist some $f_0^{**} \in \mathrm{FBL}[E]^{**}$ such that

$$|J^*_{Z_E}\widehat{\iota}^{**}_E f_1^{**}| = J^*_{Z_E}\widehat{\iota}^{**}_E f_0^{**}.$$

Applying the lattice homomorphism $\check{K}_E$ and using that $\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E = id_{\mathrm{FBL}[E]^{**}}$ we obtain that $f_0^{**} = |f_1^{**}|$. Thus,

$$|J^*_{Z_E}\widehat{\iota}^{**}_E f_1^{**}| = J^*_{Z_E}\widehat{\iota}^{**}_E\,|f_1^{**}|\,,$$

so $J^*_{Z_E}\widehat{\iota}^{**}_E$ must be a lattice homomorphism.

(5) $\Rightarrow$ (6) On the one hand we know from Lemma 3.3.8 that $\check{K}_E J^*_{Z_E}\widehat{\iota}^{**}_E = id_{\mathrm{FBL}[E]^{**}}$. On the other hand, it follows from the hypothesis that $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E$ is a weak*-to-weak* continuous lattice homomorphism such that $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E\iota_E = \iota_E$. The uniqueness of extension to $\mathrm{FBL}^*[E]$ yields that $J^*_{Z_E}\widehat{\iota}^{**}_E\check{K}_E = id_{\mathrm{FBL}^*[E]}$, so $\check{K}_E$ is a lattice isomorphism. Since both $\check{K}_E$ and $J^*_{Z_E}\widehat{\iota}^{**}_E$ have norm one, it is actually an isometry.

(6) $\Rightarrow$ (7) Just take $Y = \widehat{\iota}_E(\mathrm{FBL}[E])$.

(7) $\Rightarrow$ (8) Let us show that $\check{\jmath}_E = J^{**}_Y$, which is an isometric embedding. Indeed, the condition $\iota_E(E) \subseteq J_Y(Y)$ implies that the operator $J_Y^{-1}\iota_E : E \to Y$ is well-defined. Thus,

$$J^{**}_Y\iota_E = J^{**}_Y J_Y J_Y^{-1}\iota_E = J_{Y^{**}} J_Y J_Y^{-1}\iota_E = J_{\mathrm{FBL}^*[E]}\iota_E = j_E.$$

The uniqueness of extension of j_E to $\mathrm{FBL}^*[E]$ implies that $\check{\jmath}_E = J^{**}_Y$, as we wanted.

(8) $\Rightarrow$ (6) We know from Lemma 3.3.8 that $\check{K}_E$ is onto. Let us see that it is also an isometric embedding, and hence an isomorphism. To do so, we will check that $\widehat{\iota}^{**}_E\check{K}_E = \check{\jmath}_E$. Indeed, $\widehat{\iota}^{**}_E\check{K}_E$ is a weak*-to-weak* continuous lattice homomorphism satisfying

$$\widehat{\iota}^{**}_E\check{K}_E\iota_E = \widehat{\iota}^{**}_E J_{\mathrm{FBL}[E]}\phi_E = J_{\mathrm{FBL}^*[E]}\widehat{\iota}_E\phi_E = J_{\mathrm{FBL}^*[E]}\iota_E = j_E,$$

so the uniqueness of extension of j_E implies that $\widehat{\iota}^{**}_E\check{K}_E = \check{\jmath}_E$. Since both $\widehat{\iota}^{**}_E$ and $\check{\jmath}_E$ are isometric embeddings, $\check{K}_E$ is too.

(6) $\Rightarrow$ (1) Trivial, since $\check{K}_E$ is an onto isometry in the category $\mathcal{BL}^*$. □

Proposition 3.3.9 provides some useful information in the search for an answer to the question of the existence of $\mathrm{FBL}^*[E]$ when E contains a complemented copy of ℓ_1. For instance, we can

affirm that if $\mathrm{FBL}^*[E]$ exists, then it cannot be the bidual of a Banach lattice and cannot contain $\mathrm{FBL}[E]^{**}$ as a sublattice (at least in the canonical way described in Lemma 3.3.8).

Next, let us further simplify the problem. We claim that in order to show the existence of $\mathrm{FBL}^*[E]$ for every E that contains an (isomorphic) complemented copy of ℓ_1, it suffices to show the existence of $\mathrm{FBL}^*[F]$ for every F that contains an isometric 1-complemented copy of ℓ_1. First, we show the following:

Lemma 3.3.10. *Suppose that E and F are isomorphic Banach spaces. Then $\mathrm{FBL}^*[E]$ exists if and only if $\mathrm{FBL}^*[F]$ exists.*

Proof. The roles of E and F are interchangeable, so it suffices to show that if $\mathrm{FBL}^*[E]$ exists, then $\mathrm{FBL}^*[F]$ exists as well. Let $R : E \to F$ be an onto isomorphism, and assume without loss of generality that $\|x\| \leqslant \|Rx\| \leqslant \|R\|\|x\|$, i.e., $\|R^{-1}\| \leqslant 1$. We define for every $z^* \in \mathrm{FBL}^*[E]$ the following lattice norm:

$$|||z^*||| := \sup\left\{\|(\check{TR})z^*\| : X \text{ Banach lattice, } T : F \to X^* \text{ contractive}\right\}.$$

Clearly, $|||z^*||| \leqslant \|R\|\|z^*\|$, and choosing the contraction $\iota_E R^{-1} : F \to \mathrm{FBL}^*[E]$ we get that $(\iota_E \check{R^{-1}} R) = \check{\iota}_E = id_{\mathrm{FBL}^*[E]}$, so $|||z^*||| \geqslant \|z^*\|$. We claim that the Banach lattice $Y = (\mathrm{FBL}^*[E], |||\cdot|||)$ together with the map $\iota_E R^{-1} : F \to Y$ satisfies Definition 3.3.4 for the Banach space F. Indeed, $|||\cdot|||$ induces an equivalent renorming over Z_E, the predual of $\mathrm{FBL}^*[E]$, given by

$$|||z||| := \sup\{|z^*(z)| : |||z^*||| \leqslant 1\}$$

for every $z \in Z_E$. Clearly, $(Z_E, |||\cdot|||)^* = Y$, so Y is an object of $\mathcal{BL}^*$. The operator $\iota_E R^{-1} : F \to Y$ is a linear isometry. Indeed,

$$\begin{aligned} |||\iota_E R^{-1} x||| &= \sup\left\{\|(\check{TR})\iota_E R^{-1} x\| : X \text{ Banach lattice, } T : F \to X^* \text{ contractive}\right\} \\ &= \sup\left\{\|TRR^{-1}x\| : X \text{ Banach lattice, } T : F \to X^* \text{ contractive}\right\}, \end{aligned}$$

so

$$\|x\| = \|J_{\mathrm{FBL}[F]}\phi_F x\| \leqslant |||\iota_E R^{-1} x||| \leqslant \|x\|.$$

Finally, let X^* be the dual of a Banach lattice X and $T : F \to X^*$ a contraction. Then, the lattice homomorphism $(\check{TR}) : Y \to X^*$ is weak*-to-weak* continuous (the isomorphism between Y and $\mathrm{FBL}^*[E]$ is actually the adjoint of the isomorphism between Z_E and $(Z_E, |||\cdot|||)$, so it is weak*-to-weak* continuous), $(\check{TR})\iota_E R^{-1} = T$ and $\|(\check{TR})z^*\| \leqslant |||z^*|||$ for every $z^* \in Y$. We conclude that $\mathrm{FBL}^*[F]$ exists and is given by the pair $(Y, \iota_E R^{-1})$. □

The previous lemma allows us to establish the following:

Proposition 3.3.11. *$\mathrm{FBL}^*[E]$ exists for every Banach space E containing an (isomorphic) complemented copy of ℓ_1 if and only if $\mathrm{FBL}^*[F]$ exists for every Banach space F containing an isometric 1-complemented copy of ℓ_1.*

Proof. The necessary implication is trivial. In order to check the sufficiency, let E be a Banach space that contains a complemented isomorphic copy of ℓ_1, and consider an embedding $j : \ell_1 \to E$ (that we can assume to be contractive) and an onto operator $Q : E \to \ell_1$ such that jQ is a projection onto $j(\ell_1)$ and Qj is the identity on ℓ_1. We define for every $x \in E$ the equivalent renorming

$$|||x||| = \|x\| \vee \|Qx\|,$$

so that the Banach space $F = (E, |||\cdot|||)$ is isomorphic to E. It follows that $j : \ell_1 \to F$ is an isometric embedding, as for every $a \in \ell_1$ we have

$$\|a\| = \|Qja\| \leqslant |||ja||| = \|ja\| \vee \|a\| \leqslant (\|j\| \vee 1)\|a\| = \|a\|.$$

Moreover, $jQ : F \to F$ defines a contractive projection onto $j(\ell_1)$: for every $x \in F$,

$$|||jQx||| = \|jQx\| \vee \|QjQx\| \leqslant (\|j\| \vee 1)\|Qx\| \leqslant |||x|||.$$

Therefore, F contains an isometric 1-complemented copy of ℓ_1, so the hypothesis yields that $\mathrm{FBL}^*[F]$ exists. Since E and F are isomorphic, by Lemma 3.3.10 it follows that $\mathrm{FBL}^*[E]$ exists as well. □

Therefore, we have reduced the problem to the following question:

Question 3.3.12. Does $\mathrm{FBL}^*[E]$ exists for every Banach space E containing an isometric 1-complemented copy of ℓ_1?

3.3.1 w^*-closures of sublattices

A possible approach to solve Question 3.3.12 is to better understand the process of generation of $\mathrm{FBL}^*[E]$ from the Banach space E. This mechanism is clear for other related free objects, such as the bidual (by Goldstine Theorem, the unit ball of E^{**} can be seen as a suitable completion and compactification of B_E endowed with the weak topology induced by E), the free Banach lattice (the process of constructing $\mathrm{FVL}[E]$ and completing it under the appropriate norm is well understood, see for instance [11, 75, 121] or Sections 1.3 and 2.1) or even the free p-convex dual Banach lattice or the free dual Banach lattice generated by a Banach space E that does not contain a complemented copy of ℓ_1 studied above, as they can be represented as the bidual of the corresponding free (p-convex) Banach lattice.

From these examples we deduce that two main processes operate in the generation of $\mathrm{FBL}^*[E]$ from E: lattice-linear expressions and w^*-limits. It is well known that in general lattice operations on the dual of a Banach lattice are not weak* continuous, but there are some Banach lattices, such as ℓ_∞, where the w^*-closure of any sublattice is again a sublattice (see the discussions [123, 124] on https://mathoverflow.net/). Condition (4) in Proposition 3.3.9 shows that in the case that concerns us the w^*-closure of the canonical copy of $\mathrm{FBL}[E]$ inside $\mathrm{FBL}^*[E]$ will not be a sublattice. This motivates the following questions, which have interest on their own: given a subset of the dual X^* of a Banach lattice X, how many times can we iterate the operation of taking the sublattice generated by the set and then its w^*-closure before it stabilizes on a w^*-closed sublattice of X^*? How large can this sublattice be in terms of the original size (cardinality, character of density...) of the generating set?

In this subsection we initiate a preliminary study of these problems and establish some partial results that might help solve Question 3.3.12. The first thing that should be noted is that free Banach lattices (or rather their duals) serve as a canonical model to tackle this problem. This does not come as a surprise, as free objects usually can be used as model spaces for certain problems. Take as an example the Complemented Subspace Problem for Banach lattices [69, Proposition 3.1] or $C(K)$-spaces [19, Lemma, p. 247]: a Banach space E is isomorphic to a complemented subspace of a Banach lattice (respectively, $C(K)$-space) if and only if it is complemented in $\mathrm{FBL}[E]$ (respectively, $C(B_{E^*})$).

Proposition 3.3.13. *Let X be a Banach lattice. Then, the operator $\widehat{id}_X^* : X^* \to \mathrm{FBL}[X]^*$ is a weak*-to-weak* continuous interval preserving lattice isometric embedding and a homeomorphism onto its image for the weak* topologies. In particular, for every $A \subseteq X^*$,*

$$\mathrm{lat}\left(\widehat{id}_X^*(A)\right) = \widehat{id}_X^*(\mathrm{lat}(A)) \quad \text{and} \quad \overline{\widehat{id}_X^*(A)}^{w^*} = \widehat{id}_X^*\left(\overline{A}^{w^*}\right).$$

Proof. Given a Banach lattice X, we can consider the onto lattice homomorphism $\widehat{id}_X : \mathrm{FBL}[X] \to X$. By [114, Proposition II.2.5], $\widehat{id}_X$ is also interval preserving, so by the Open Mapping Principle (cf. [46, Exercise 2.49]) and [96, Theorem 1.4.19] $\widehat{id}_X^* : X^* \to \mathrm{FBL}[X]^*$ is a weak*-to-weak* continuous interval preserving lattice isometric embedding. Moreover, $\widehat{id}_X^*(X^*)$ is w^*-closed in $\mathrm{FBL}[X]^*$. Indeed, since $\widehat{id}_X \phi_X = id_X$, given a net $(x_\alpha^*)_\alpha \subset X^*$ such that $\widehat{id}_X^* x_\alpha^* \xrightarrow{w^*} f^*$ we have that

$$\widehat{id}_X^* x_\alpha^* = \widehat{id}_X^* \phi_X^* (\widehat{id}_X^* x_\alpha^*) \xrightarrow{w^*} \widehat{id}_X^* (\phi_X^* f^*),$$

so the uniqueness of the limit implies that $f^* = \widehat{id}_X^* (\phi_X^* f^*) \in \widehat{id}_X^*(X^*)$. Finally, if we denote by $w^*|_{\widehat{id}_X^*(X^*)} = w(\mathrm{FBL}[X]^*, \mathrm{FBL}[X])|_{\widehat{id}_X^*(X^*)}$ we have that $\widehat{id}_X^* : (X^*, w^*) \to (\widehat{id}_X^*(X^*), w^*|_{\widehat{id}_X^*(X^*)})$ is an homeomorphism. It is clearly continuous and invertible, so it suffices to check that it is open. To do so, consider any w^*-open neighbourhood $W = \{x^* \in X^* : |x^*(x_i)| < \varepsilon, i = 1, \dots, n\}$ with $\varepsilon > 0$ and $(x_i)_{i=1}^n \subset X$, and observe that $\widehat{id}_X \phi_X x_i = x_i$. Then

$$\widehat{id}_X^*(W) = \{f^* \in \mathrm{FBL}[X]^* : |f^*(\phi_X x_i)| < \varepsilon, i = 1, \dots, n\} \cap \widehat{id}_X^*(X^*),$$

so the image by $\widehat{id}_X^*$ of any w^*-open subset of X^* is a $w^*|_{\widehat{id}_X^*(X^*)}$-open subset of $\widehat{id}_X^*(X^*)$. Therefore, given any $A \subseteq X^*$, we get that $\mathrm{lat}\left(\widehat{id}_X^*(A)\right) = \widehat{id}_X^*(\mathrm{lat}(A))$, since $\widehat{id}_X^*$ is a lattice homomorphism, and $\overline{\widehat{id}_X^*(A)}^{w^*} = \widehat{id}_X^*\left(\overline{A}^{w^*}\right)$. □

Next, we provide an example of a separable sublattice Z_0 of an AL-space such that its w^*-closure Z_1 is not a sublattice, but the w^*-closure of the sublattice generated by Z_1 is a w^*-closed sublattice that is not w^*-separable.

Theorem 3.3.14. *There exist a three level scattered compact Hausdorff space K and a sublattice $Z_0 \subseteq C(K)^*$ such that:*

- *Z_0 is generated by a countable set of disjoint elements;*
- *$Z_1 := \overline{Z_0}^{w^*}$ is not a sublattice;*
- *$Z_2 := \overline{\mathrm{lat}(Z_1)}^{w^*}$ is a w^*-closed sublattice;*
- *Z_2 is not w^*-separable.*

Proof. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be an uncountable almost disjoint family of subsets of $\mathbb{N}$, that is, a family such that $|A_\alpha \cap A_{\alpha'}| < \infty$ for every $\alpha \neq \alpha'$, and for every $\alpha \in \Lambda$, let $B_{\alpha,0}$ and $B_{\alpha,1}$ be disjoint subsets of A_α such that $A_\alpha = B_{\alpha,0} \cup B_{\alpha,1}$. Let K_1 be the Mrowka–Isbell compact space induced by the famillly $\{A_\alpha\}_{\alpha \in \Lambda}$ as follows: in the ground (0) level we have a (discrete) copy of $\mathbb{N}$, denoted by $\{a_i\}_{i \in \mathbb{N}}$; in the first level we have limit points $\{a_\alpha\}_{\alpha \in \Lambda}$ equipped with a base of open neighbourhoods given by the sets $\{a_i : i \in A\} \cup \{a_\alpha\}$ such that $A \subseteq \mathbb{N}$ differs from A_α in a finite quantity of elements; finally, in the last level, we have the compactification by one point, say a_∞, of the above topological space. Let K_2 be the analogue construction, this time with the almost disjoint family $\{B_{\alpha,\gamma}\}_{\alpha \in \Lambda, \gamma \in \{0,1\}}$, and denote by $\{b_i\}_{i \in \mathbb{N}}$, $\{b_{\alpha,\gamma}\}_{\alpha \in \Lambda, \gamma \in \{0,1\}}$ and b_∞ the points in the level 0, 1 and 2, respectively. Define K as the disjoint union of K_1 and K_2, and note that K is a scattered compact Hausdorff space, so by [83, Section 2.8, Theorems 9 and 10] $C(K)^* = \ell_1(K) = \overline{\mathrm{span}}\{\delta_t : t \in K\}$, where δ_t denote the evaluation functional in the point $t \in K$.

Define $Z_0 := \overline{\mathrm{span}}\{\delta_{a_i} + \delta_{b_i} : i \in \mathbb{N}\} \subseteq C(K)^*$, which is a (norm) closed sublattice, as the generating elements are pairwise disjoint (actually, Z_0 is an isometric copy of ℓ_1). Let $Z_1 := \overline{Z_0}^{w^*}$. We claim that Z_1 is not a sublattice of $C(K)^*$. Indeed, note that for any $\alpha \in \Lambda$ and $\gamma \in \{0,1\}$, $\delta_{a_\alpha} + \delta_{b_{\alpha,\gamma}} = w^* - \lim_{i \in B_{\alpha,\gamma}} (\delta_{a_i} + \delta_{b_i}) \in Z_1$, since $\lim_{i \in B_{\alpha,\gamma}} a_i = a_\alpha$ and $\lim_{i \in B_{\alpha,\gamma}} b_i = b_{\alpha,\gamma}$ in K.

Therefore, $\delta_{a_\alpha} = (\delta_{a_\alpha} + \delta_{b_{\alpha,0}}) \wedge (\delta_{a_\alpha} + \delta_{b_{\alpha,1}}) \in \mathrm{lat}(Z_1)$. However, $\delta_{a_\alpha} \notin Z_1$. To check this, note that the function $f = \chi_{K_1} - \chi_{K_2} \in C(K)$ defines a w^*-closed subspace $[f]^\circ := \{\mu \in C(K)^* : \mu(f) = 0\}$ that contains Z_0, and thus Z_1, but $\delta_{a_\alpha} \notin [f]^\circ$ as $f(a_\alpha) = 1$.

Actually, we can show that

$$Z_1 = \overline{\mathrm{span}}\{\delta_{a_i} + \delta_{b_i}, \delta_{a_\alpha} + \delta_{b_{\alpha,\gamma}}, \delta_{a_\infty} + \delta_{b_\infty} : i \in \mathbb{N}, \alpha \in \Lambda, \gamma \in \{0,1\}\}. \tag{3.10}$$

The inclusion ($\supseteq$) has already been proven above. In order to prove the reverse inclusion, denote by Y the subspace in the right side of (3.10). It is clear that $Z_0 \subseteq Y$. If we prove that Y is w^*-closed, it would follow that $Z_1 \subseteq Y$. Denote by $f_i = \chi_{\{a_i\}} - \chi_{\{b_i\}}$, $i \in \mathbb{N}$, $f_\alpha = \chi_{U_\alpha} - \chi_{V_\alpha}$, $\alpha \in \Lambda$, and $f_\infty = \chi_{K_1} - \chi_{K_2}$, where $U_\alpha = \{a_i : i \in A_\alpha\} \cup \{a_\alpha\}$ and $V_\alpha = \{b_i : i \in A_\alpha\} \cup \{b_{\alpha,0}, b_{\alpha,1}\}$. Note that these functions are continuous on K, as they are combinations of characteristic functions over clopens of K. We have

$$Y = \left(\bigcap_{i\in\mathbb{N}} [f_i]^\circ\right) \cap \left(\bigcap_{\alpha\in\Lambda} [f_\alpha]^\circ\right) \cap [f_\infty]^\circ.$$

The inclusion ($\subseteq$) is trivial. For the reverse inclusion, express an element $\mu \in C(K)^* = \ell_1(K)$ as a linear combination of the evaluation functionals

$$\mu = \sum_{i\in\mathbb{N}} (r_i \delta_{a_i} + s_i \delta_{b_i}) + \sum_{\alpha\in\Lambda} r_\alpha \delta_{a_\alpha} + \sum_{\alpha\in\Lambda,\gamma\in\{0,1\}} s_{\alpha,\gamma}\delta_{b_{\alpha,\gamma}} + r_\infty \delta_{a_\infty} + s_\infty \delta_{b_\infty}$$

and note that if μ belongs to the set in the right side, then it follows that $r_i = s_i$ for every $i \in \mathbb{N}$, $r_\alpha = s_{\alpha,0} + s_{\alpha,1}$ for every $\alpha \in \Lambda$ and $r_\infty = s_\infty$, so $\mu \in Y$. Therefore, Y is the intersection of w^*-closed subspaces, so it is w^*-closed, and thus $Z_1 \subseteq Y$.

If we iterate a second time this process of taking the w^*-closure of the sublattice generated by Z_1 we obtain $Z_2 := \overline{\mathrm{lat}(Z_1)}^{w^*}$, which is already a sublattice. Actually, we claim that

$$Z_2 = \overline{\mathrm{span}}\{\delta_{a_i} + \delta_{b_i}, \delta_{a_\alpha}, \delta_{b_{\alpha,\gamma}}, \delta_{a_\infty}, \delta_{b_\infty} : i \in \mathbb{N}, \alpha \in \Lambda, \gamma \in \{0,1\}\}, \tag{3.11}$$

so in particular it is a sublattice, as it is generated by disjoint elements. The inclusion ($\supseteq$) is clear, since $\delta_{a_\alpha}, \delta_{b_{\alpha,\gamma}} \in \mathrm{lat}(Z_1)$ and δ_{a_∞} and δ_{b_∞} can be obtained through appropriate w^*-limits of these elements. For the reverse inclusion, we can proceed as above to show that the space in the right side of the identity coincides with the w^*-closed space $\bigcap_{i\in\mathbb{N}} [f_i]^\circ$.

Finally, we show that Z_2 is not w^*-separable. Assume there exists a countable w^*-dense subset $\{\mu_j\}_{j=1}^\infty \subseteq Z_2$. Each $\mu \in Z_2$ can be expressed as a linear combination

$$\mu = \sum_{i\in\mathbb{N}} r_i(\delta_{a_i} + \delta_{b_i}) + \sum_{\alpha\in\Lambda} r_\alpha \delta_{a_\alpha} + \sum_{\alpha\in\Lambda,\gamma\in\{0,1\}} s_{\alpha,\gamma}\delta_{b_{\alpha,\gamma}} + r_\infty \delta_{a_\infty} + s_\infty \delta_{b_\infty}$$

with at most a countable quantity of non zero-coefficients. Therefore, the union of the supports of the μ_j is countable, so there exists some $\alpha_0 \in \Lambda$ such that $a_{\alpha_0}, b_{\alpha_0,0}, b_{\alpha_0,1}$ do not belong to this set. It follows that the function $f_{\alpha_0} = \chi_{U_{\alpha_0}} - \chi_{V_{\alpha_0}}$, with $U_{\alpha_0} = \{a_i : i \in A_{\alpha_0}\} \cup \{a_{\alpha_0}\}$ and $V_{\alpha_0} = \{b_i : i \in A_{\alpha_0}\} \cup \{b_{\alpha_0,0}, b_{\alpha_0,1}\}$, satisfies that $\mu_j(f_{\alpha_0}) = 0$ for every j but $\delta_{a_{\alpha_0}}(f_{\alpha_0}) = 1$, so $\delta_{a_{\alpha_0}} \in Z_2 \backslash \overline{\{\mu_j\}_j}^{w^*}$. □

We can use this example to show the following:

Corollary 3.3.15. *Assume* FBL$^*[\ell_1]$ *exists. Then, it is not w^*-separable.*

Proof. Let us consider the map $T : \ell_1 \to Z_0 \subseteq C(K)^*$ given by $Tx = \frac{1}{2}\sum_{i=1}^\infty x_i(\delta_{a_i} + \delta_{b_i})$ for every $x = (x_i)_{i=1}^\infty \in \ell_1$, which is actually a lattice isometric embedding. In particular, $B_{Z_0} = T(B_{\ell_1})$. We

will show that $Z_2 \subseteq \check{T}(\mathrm{FBL}^*[\ell_1])$. Using (3.10) it is easy to check that $B_{Z_1} = \overline{B_{Z_0}}^{w^*}$. Therefore, for every $\mu \in B_{Z_1}$ there exists a net $(x_\sigma)_\sigma \subseteq B_{\ell_1}$ such that $Tx_\sigma \xrightarrow{w^*} \mu$. We can substract a net $(x_\tau)_\tau \subseteq (x_\sigma)_\sigma$ such that $\iota_{\ell_1} x_\tau$ w^*-converges to some $z^* \in B_{\mathrm{FBL}^*[\ell_1]}$. It follows that $\check{T}z^* = \mu$, so $B_{Z_1} \subseteq \check{T}(B_{\mathrm{FBL}^*[\ell_1]})$. For each $\alpha \in \Lambda$ and $\gamma \in \{0,1\}$, let $z^*_{\alpha,\gamma} \in 2B_{\mathrm{FBL}^*[\ell_1]}$ such that $\check{T}z^*_{\alpha,\gamma} = \delta_{a_\alpha} + \delta_{b_{\alpha,\gamma}}$, and consider $y^*_\alpha = z^*_{\alpha,0} \wedge z^*_{\alpha,1} \in 2B_{\mathrm{FBL}^*[\ell_1]}$ and $y^*_{\alpha,\gamma} = z^*_{\alpha,\gamma} - y^*_\alpha \in 2B_{\mathrm{FBL}^*[\ell_1]}$. It follows that $\delta_{a_\alpha} = \check{T}y^*_\alpha$ and $\delta_{b_{\alpha,\gamma}} = \check{T}y^*_{\alpha,\gamma}$. Extracting w^*-convergent subnets of $(y^*_\alpha)_{\alpha\in\Lambda}$ and $(y^*_{\alpha,\gamma})_{\alpha\in\Lambda,\gamma\in\{0,1\}}$ we get that $\delta_{a_\infty}, \delta_{a_\infty} \in 2\check{T}(B_{\mathrm{FBL}^*[\ell_1]})$. Using (3.11) we conclude that $B_{Z_2} \subseteq 2\check{T}(B_{\mathrm{FBL}^*[\ell_1]})$, so $Z_2 \subseteq \check{T}(\mathrm{FBL}^*[\ell_1])$. Since Z_2 is not w^*-separable and $\check{T}$ is weak*-to-weak* continuous, it follows that $\mathrm{FBL}^*[\ell_1]$ is not w^*-separable either. □

The previous result opens some possible paths towards proving that $\mathrm{FBL}^*[\ell_1]$ does not exist. One would be showing by other methods that $\mathrm{FBL}^*[\ell_1]$ has to be w^*-separable, yielding a contradiction. This is a reasonable claim, as the weak* topology of $\mathrm{FBL}^*[\ell_1]$ has to be generated somehow by ℓ_1, which is separable. Moreover, the assertion is true in similar contexts: if E is separable, $\mathrm{FBL}^p[E]$ is separable by [103, Proposition 9.4], so $\mathrm{FBL}^p[E]^{**}$ is w^*-separable. Therefore, the free p-convex dual Banach lattice over a separable Banach space ($p > 1$) and the free dual Banach lattice over a separable Banach space that does not contain a complemented copy of ℓ_1 are w^*-separable. Another approach would be to keep investigating w^*-closures of sublattices, seeking for countable sets that could generate arbitrarily big w^*-closed sublattices. If this could be done in such a way that the extension argument above could be reproduced to show that the range of certain extension operators on $\mathrm{FBL}^*[\ell_1]$ contain these large sublattices, we would get a contradiction, as the cardinality and the character of w^*-density of $\mathrm{FBL}^*\ell_1$ must be bounded. Further research need to be done in both directions.

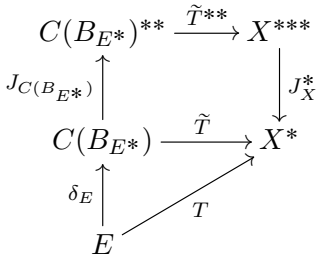
3.4 Free duals of AL-spaces

The results from the previous section can be adapted to the context of *AM-spaces with order unit*, that is, AM-spaces satisfying that there exists an element $e \in X_+$ such that $[-e, e] = B_X$. This element is unique whenever it exists, so we will refer to it as *the order unit of the AM-space* X throughout this section. Recall that by Kakutani's Representation Theorem (cf. [96, Theorem 2.1.3]), AM-spaces with order unit are in correspondence with $C(K)$-spaces. In this setting, we can define the following category, denoted $\mathcal{AL}^*$. On the one hand, the objects of $\mathcal{AL}^*$ are AM-spaces which are duals of Banach lattices. This *a priori* general assumption is actually quite informative. Indeed, by [96, Proposition 1.4.7], X^* is an AM-space and the dual of a Banach lattice X if and only if X is an AL-space, and this in turn implies that X^* must have an order unit $e \in X^*_+$ such that the norm in X^* coincides with the order unit norm $\|\cdot\|_e$ (equivalently, $[-e, e] = B_{X^*}$). This is why we have chosen to denote the category by $\mathcal{AL}^*$, as the objects are duals of AL-spaces. Even more, making use of Kakutani's and Dixmier's Theorems (cf. [96, Theorems 2.1.3 and 2.1.7]), an object is in $\mathcal{AL}^*$ if and only if it can be identified with a dual $C(K)$-space, and these are precisely the $C(K)$-spaces with K hyper-stonean (see [40, Theorem 6.4.1] for a complete characterization of this property). On the other hand, the morphisms of the category $\mathcal{AL}^*$ are weak*-to-weak* continuous lattice homomorphisms that preserve the order unit, meaning that normalized morphisms send the order unit of the domain to the order unit of the image.

As AM-spaces are in particular ∞-convex, an analogue of Remark 3.3.1 holds for the category $\mathcal{AL}^*$. It should also be checked that J^*_X preserves the order unit, but this is a consequence of the equality $J^*_X J_{X^*} = id_{X^*}$, together with the simple fact that if e is the order unit of the AM-space X^*, then $J_{X^*}e$ is the order unit of X^{***}. Thus, we can state a version of Proposition 3.3.2 in this new setting. The proof is essentially the same, so we omit the details.

Proposition 3.4.1. *Let E be a Banach space. Then, for every object X^* in $\mathcal{AL}^*$ and every operator $T : E \to X^*$ there exists a unique weak*-to-weak* continuous lattice homomorphism $\check{T} : C(B_{E^*})^{**} \to X^*$, such that $\check{T} J_{C(B_{E^*})} \delta_E = T$ and $\check{T} J_{C(B_{E^*})} \mathbb{1}_{E^*} = \|T\| e_{X^*}$, where $\mathbb{1}_{E^*}$ is the constant function on B_{E^*} and e_{X^*} is the order unit of X^*. Moreover, $\|\check{T}\| = \|T\|$.*

Proof. Since clearly $C(B_{E^*})^{**}$ is an object of $\mathcal{AL}^*$, it suffices to consider the following commutative diagram, where $\widetilde{T}$ is the extension of T to $C(B_{E^*})$ given by Theorem 3.2.8:



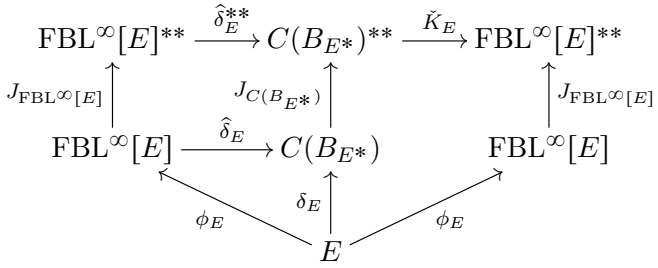
The rest of the proof is similar to the one in Proposition 3.3.2. □

To finish this section, let us focus on the relation between $C(B_{E^*})$ and $\mathrm{FBL}^\infty[E]$, which can be represented as $C_{ph}(B_{E^*})$ in a canonical way [103, Proposition 2.2]. These Banach lattices are never lattice isomorphic. Indeed, $C(B_{E^*})$ always has an order unit, while in $\mathrm{FBL}^\infty[E]$ this only happens when E has finite dimension [103, Proposition 9.1]. In that case, $\mathrm{FBL}^\infty[E]$ is lattice isometric to $C(S_{E^*})$, which by Banach–Stone's Theorem can never be lattice isomorphic to $C(B_{E^*})$ (cf. [96, Corollary 3.2.11]). Neither can $\mathrm{FBL}^\infty[E]$ be complemented in $C(B_{E^*})$ by means of a positive projection when E has infinite dimension: if there was such a positive projection $P : C(B_{E^*}) \to C(B_{E^*})$ over $\mathrm{FBL}^\infty[E] \equiv C_{ph}(B_{E^*})$, it would follow that $P\mathbb{1}_{E^*}$ should be an order unit of $\mathrm{FBL}^\infty[E]$, which by [103, Proposition 9.1] is not possible.

The latter situation no longer holds when we compare the bidual spaces instead: $\mathrm{FBL}^\infty[E]^{**}$ is complemented in $C(B_{E^*})^{**}$ with a positive and contractive projection. This can be seen as a consequence of the general fact that every Dedekind complete AM-space is injective (cf. [96, Theorem 3.2.4]), since every dual of a Banach lattice is Dedekind complete. Nevertheless, in this particular situation we have a canonical construction of the projection, and moreover, it is a lattice homomorphism.

Proposition 3.4.2. *Let $K_E = J_{\mathrm{FBL}^\infty[E]} \phi_E$ denote the canonical embedding of E into $\mathrm{FBL}^\infty[E]^{**}$ and $\check{K}_E : C(B_{E^*})^{**} \to \mathrm{FBL}^\infty[E]^{**}$ the extension of K_E given by Proposition 3.4.1. Then the operator $\widehat{\delta}_E^{**} \check{K}_E : C(B_{E^*})^{**} \to C(B_{E^*})^{**}$ is a contractive lattice projection over $\mathrm{FBL}^\infty[E]^{**}$, where $\widehat{\delta}_E$ is the canonical embedding of $\mathrm{FBL}^\infty[E]$ into $C(B_{E^*})$ given by the restriction to B_{E^*}.*

Proof. It is clear that $\widehat{\delta}_E^{**} : \mathrm{FBL}^\infty[E]^{**} \to C(B_{E^*})^{**}$ is a lattice isometric embedding. From the following commutative diagram,



it is clear that

$$\check{K}_E \widehat{\delta}_E^{**} K_E = \check{K}_E J_{C(B_{E^*})} \widehat{\delta}_E \phi_E = \check{K}_E J_{C(B_{E^*})} \delta_E = K_E.$$

By the uniqueness of extension of K_E to $\mathrm{FBL}^\infty[E]^{**}$ established in Proposition 3.3.2, we conclude that $\check{K}_E\widehat{\delta}_E^{**}$ must be the identity on $\mathrm{FBL}^\infty[E]^{**}$, so the result follows. □

Remark 3.4.3. *Note that $\widehat{\delta}_E^{**}$ is an embedding, but it is never surjective. Indeed, if $\widehat{\delta}_E^{**}$ were a surjective isomorphism, since it is also a lattice homomorphism, it would preserve the order unit. Let us denote by e the order unit of $\mathrm{FBL}^\infty[E]^{**}$. Then, $\widehat{\delta}_E^{**}e$ is also an order unit in $C(B_{E^*})^{**}$, so there exists some $\lambda \geqslant 0$ such that $J_{C(B_{E^*})}\mathbb{1}_{E^*} \leqslant \lambda\widehat{\delta}_E^{**}e$. In particular, evaluating both elements in the evaluation functional $\delta_0 \in C(B_{E^*})^*$ we obtain that*

$$\begin{aligned} 1 = \mathbb{1}_{E^*}(0) = \delta_0(\mathbb{1}_{E^*}) = J_{C(B_{E^*})}\mathbb{1}_{E^*}(\delta_0) \\ \leqslant (\lambda\widehat{\delta}_E^{**}e)(\delta_0) = \lambda e(\widehat{\delta}_E^*\delta_0) = \lambda e(\widehat{0}) = \lambda e(0) = 0, \end{aligned}$$

where we have used that $\widehat{\delta}_E^*\delta_{x^*} = \widehat{x^*}$ for every $x^* \in B_{E^*}$ and that $\widehat{0}$, the extension of the null functional $0 \in E^*$ to $\mathrm{FBL}^\infty[E]$ as a lattice homomorphism, is the null functional $0 \in \mathrm{FBL}^\infty[E]^*$. This leads us to a contradiction, so $\widehat{\delta}_E^{**}$ cannot be an onto isomorphism.

3.5 Order weakly compact operators

Recall that an operator T from a Banach lattice X to a Banach space E is *order weakly compact* if $T[0,x]$ is relatively weakly compact for every $x \in X_+$. In particular, every operator defined on an order continuous Banach lattice is order weakly compact, as every order interval is weakly compact (see [96, Theorem 2.4.2]). An operator $T : E \to F$ is said to *preserve a copy of G_0* if there is a subspace $G \subseteq E$ isomorphic to G_0 such that $T|_G$ is an isomorphism. It is said to *preserve a copy of G_0 complementably* if, moreover, $T(G)$ is complemented in F (in particular, G must be complemented in E).

The following is a well-known fact:

Lemma 3.5.1. *Let X be a Banach lattice. The following are equivalent:*

(1) X contains ℓ_1 as a complemented subspace.

(2) X contains ℓ_1 as a sublattice.

(3) X^* is not order continuous.

Proof. Note that by [23, Theorem 4] X contains ℓ_1 as a complemented subspace if and only if X^* contains c_0 as a subspace. By [95, Corollary I.8], this is equivalent to X^* containing c_0 as a sublattice, which at the same time is equivalent to X containing ℓ_1 as a sublattice or X^* not being order continuous [96, Theorem 2.4.14]. □

The equivalence of the first two conditions is also folklore, but we include a proof for the sake of completeness.

Proposition 3.5.2. *Let E be a Banach space, X a Banach lattice, and $T : E \to X$ an operator. The following are equivalent:*

(1) T does not preserve a copy of ℓ_1 complementably,

(2) T^* is order weakly compact,

(3) $\hat{T}$ does not preserve a copy of ℓ_1 complementably,

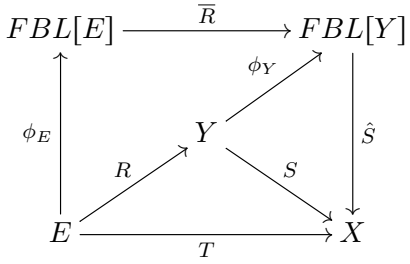
(4) $\hat{T}^*$ is order weakly compact.

Proof. (1) ⇒ (2) By [58, Theorem I.6], there exist a Banach lattice Y not containing a complemented copy of ℓ_1 and operators $R : E \to Y$ and $S : Y \to X$ such that $T = SR$ and S is an interval preserving lattice homomorphism. By Lemma 3.5.1 it follows that Y^* has order continuous norm. Since $T^* = R^*S^*$ factors through Y^* and S^* is interval preserving, T^* is order weakly compact.

(2) ⇒ (1) Suppose that T^* is order weakly compact but T preserves a copy of ℓ_1 complementably. By [5, Theorem 5.58], T^* factors through an order continuous Banach lattice Y. On the other hand, T^* preserves a copy of ℓ_∞ [96, Theorem 5.1.6], so in particular Y must contain a copy of ℓ_∞. The results contained in [96, Section 5.1] yield a contradiction: ℓ_∞ cannot have property (u), but on the other hand every Banach lattice with order continuous norm has property (u), and so it does every closed subspace of a Banach space with property (u).

(3) ⇔ (4) Apply the previous implications to $\hat{T}$ instead of T.

(1) ⇒ (4) By [58, Theorem I.6] T factors through a Banach lattice Y not containing a complemented copy of ℓ_1, so that we can consider the following commutative diagram.



Note that since $\overline{R}$ and $\hat{S}$ are lattice homomorphisms, the uniqueness of extension of T to $FBL[E]$ implies that $\hat{T} = \hat{S}\overline{R}$. On the other hand, from [103, Theorem 9.18] it follows that $FBL[Y]$ does not contain ℓ_1 as a sublattice, and thus $FBL[Y]^*$ has order continuous norm by Lemma 3.5.1. In particular, $\overline{R}^* : FBL[Y]^* \to FBL[E]^*$ is order weakly compact [96, Theorem 2.4.2]. Since $\hat{S}$ is a lattice homomorphism, [96, Theorem 1.4.19] implies that $\hat{S}^*$ is interval preserving, so we can conclude that $\hat{T}^* = \overline{R}^*\hat{S}^*$ is order weakly compact.

(4) ⇒ (2) It is straightforward, since $T^* = \phi_E^*\hat{T}^*$ and ϕ_E^* is weak-to-weak continuous. □

As a consequence, we can add another property to the dictionary of equivalences between properties of operator T between Banach spaces and properties of the corresponding extensions $\overline{T}$ established in [103, Proposition 3.2].

Corollary 3.5.3. *Let E and F be Banach spaces. Then, an operator $T : E \to F$ does not preserve a copy of ℓ_1 complementably if and only if $\overline{T} : FBL[E] \to FBL[F]$ does not preserve a copy of ℓ_1 complementably.*

Proof. The necessary implication follows from Proposition 3.5.2 and the fact that $\phi_F T$ does not preserve a copy of ℓ_1 complementably either.

To show the reverse implication, assume that T preserves a copy of ℓ_1 complementably. Using the properties of the extension $\overline{T}$ included in [103, Section 3] it can be easily seen that $\overline{T}$ preserves a lattice copy of $FBL[\ell_1]$ lattice complementably. Since $FBL[\ell_1]$ contains ℓ_1 as a complemented sublattice [103, Theorem 9.20], it follows that $\overline{T}$ preserves a copy of ℓ_1 complementably. □

3.6 An application of free Banach lattices to norm attaining lattice homomorphisms

We end this chapter with an application of free (∞-convex) Banach lattices to the general theory of Banach lattices, more specifically, to the problem of determining when a lattice homomorphism attains its norm. We say that a linear functional $x^* : E \to \mathbb{R}$ is *norm-attaining* if there exists an element $x \in B_E$ such that $x^*(x) = \|x^*\|$. The study of norm-attaining functionals in Banach spaces has provided profound results, such as James' Theorem [73], which characterizes reflexive Banach spaces as those Banach spaces such that all their functionals attain their norm.

Banach lattices have a richer structure that allows us to study the norm attainment of positive functionals and lattice homomorphisms as well. For instance, recently there has been an increasing interest in the study of positive functionals, see for example [1], [92] or [104], and in the setting of lattice homomorphisms, with contributions such as [42] and [27]. It is worth mentioning that free Banach lattices have a prominent role in the lattice homomorphism version of the problem. For instance, in [42] the authors use free Banach lattices to provide the first example of a lattice homomorphism that does not attain its norm.

In this section, we will focus on lattice homomorphisms on AM-spaces. It is clear that every lattice homomorphism on a $C(K)$-space attains its norm at the constant 1 function $\mathbb{1}$. However, if we enlarge the class and consider AM-spaces instead, the situation is not as straightforward, as we can no longer assume the existence of an order unit. Theorem 3.6.2 establishes that, despite this, every lattice homomorphism on an AM-space also attains its norm. In [27] two very different proofs of this fact are presented. In this section, we present what was (chronologically) the first one. Even though it uses some advanced tools from the theory of free Banach lattices, in contrast with the proof presented in [27, Theorem 4.3], which is much more elementary, it has interest on its own, as it illustrates how free Banach lattices can sometimes enlighten the way for solving general Banach lattice questions. We need first the following auxiliary lemma.

Lemma 3.6.1. *Let $1 \leqslant p \leqslant \infty$ and let X be a p-convex Banach lattice with p-convexity constant 1. If every lattice homomorphism of $\mathrm{FBL}^p[X]$ is norm-attaining, then so is every lattice homormorphism of X.*

Proof. Let $x^* \colon X \to \mathbb{R}$ be a lattice homomorphism such that $\|x^*\| = 1$. If ϕ_X denotes the canonical isometric embedding of X into $\mathrm{FBL}^p[X]$, the universal property of $\mathrm{FBL}^p[X]$ ensures the existence of a unique lattice homomorphism $\widehat{id}_X : \mathrm{FBL}^p[X] \to X$ satisfying $\widehat{id}_X \phi_X = id_X$ and such that $\|\widehat{id}_X\| = 1$.

If every lattice homomorphism of $\mathrm{FBL}^p[X]$ is norm-attaining, then, in particular, there exists $f \in B_{\mathrm{FBL}^p[X]}$ such that $\widehat{x^*}(f) = 1$. Now, consider $x^* \circ \widehat{id}_X$, which is a lattice homomorphism of $\mathrm{FBL}^p[X]$. Since $\widehat{x^*}$ and $x^* \circ \widehat{id}_X$ are both lattice homomorphisms of $\mathrm{FBL}^p[X]$ which agree on $\phi_X(X)$, then, in fact, $\widehat{x^*} = x^* \circ \widehat{id}_X$. In particular,

$$1 = \widehat{x^*}(f) = (x^* \circ \widehat{id}_X)(f) = x^*(\widehat{id}_X f),$$

and, as $\widehat{id}_X f \in B_X$, this shows that x^* is norm-attaining. □

Theorem 3.6.2. *Every lattice homomorphism on an AM-space attains its norm.*

Proof. To show that every lattice homomorphism on an AM-space attains its norm, we will check the following statements:

(1) Every lattice homomorphism of $\mathrm{FBL}^\infty[\ell_2]$ attains its norm.

(2) Given any separable Banach space E, every lattice homomorphism of $\mathrm{FBL}^\infty[\mathrm{FBL}^\infty[E]]$ attains its norm.

(3) Given any separable Banach space E, every lattice homomorphism of $\mathrm{FBL}^\infty[E]$ attains its norm.

(4) Given any Banach space E, every lattice homomorphism of $\mathrm{FBL}^\infty[E]$ attains its norm.

(5) Every lattice homomorphism of an AM-space attains its norm.

(1) Observe that since ℓ_2 is reflexive then, by James' Theorem, all its functionals are norm-attaining. As the lattice homomorphisms of $\mathrm{FBL}^\infty[\ell_2]$ are of the form $\widehat{x^*}$, for some $x^* \in \ell_2$ (and are defined by $\widehat{x^*}(f) = f(x^*)$), this shows that every lattice homomorphism of $\mathrm{FBL}^\infty[\ell_2]$ attains its norm.

(2) If E is a separable Banach space, then $\mathrm{FBL}^\infty[E]$ is a separable AM-space. By [86, Theorem 2], $\mathrm{FBL}^\infty[E]$ has a monotone Schauder basis, so by [103, Theorem 10.24] $\mathrm{FBL}^\infty[\mathrm{FBL}^\infty[E]]$ is lattice isometric to $\mathrm{FBL}^\infty[\ell_2]$ and the conclusion follows from (1).

(3) Since $\mathrm{FBL}^\infty[E]$ is an AM-space (∞-convex with constant 1), this implication is an immediate consequence of Lemma 3.6.1 and (2).

(4) Let E be an arbitrary Banach space. Let $\widehat{x^*}$ be a lattice homomorphism of $\mathrm{FBL}^\infty[E]$ such that $\|\widehat{x^*}\| = \|x^*\| = 1$. There is a sequence $(x_n)_n \subseteq B_E$ such that $x^*(x_n) \to 1$ as $n \to \infty$. We define

$$F := \overline{\mathrm{span}}\{x_n \,:\, n \in \mathbb{N}\},$$

which is a separable subspace of E. By [103, Proposition 3.10], the mapping $\bar{\imath} : \mathrm{FBL}^\infty[F] \to \mathrm{FBL}^\infty[E]$ (where $\imath : F \hookrightarrow E$ is the canonical inclusion of F into E as a subspace) is a lattice isometric embedding.

Now observe that $y^* = \widehat{x^*}|_{\bar{\imath}(\mathrm{FBL}^\infty[F])}$ is a norm-one lattice homomorphism of $\bar{\imath}(\mathrm{FBL}^\infty[F])$. Indeed, for every $n \in \mathbb{N}$ we have

$$y^*(\bar{\imath}\delta_{x_n}) = \widehat{x^*}(\bar{\imath}\delta_{x_n}) = \widehat{x^*}(\delta_{x_n} \circ \imath^*) = (\delta_{x_n} \circ \imath^*)(x^*) = \delta_{x_n}(\imath^* x^*) = \imath^* x^*(x_n) = x^*(\imath x_n) = x^*(x_n).$$

Since F is separable, we know by (3) that every lattice homomorphism of $\mathrm{FBL}^\infty[F]$ attains its norm, and as $\bar{\imath}$ is a lattice isometry, y^* is norm-attaining. That is, there is $f \in \bar{\imath}(\mathrm{FBL}^\infty[F]) \subseteq \mathrm{FBL}^\infty[E]$, $\|f\| = 1$ such that $y^*(f) = 1$. Thus, $\widehat{x^*}$ also attains its norm at f.

(5) It follows directly from the above and the preceding lemma. □

Chapter 4

Interpolation and extrapolation of spaces of positively homogeneous functions

The families of free p-convex Banach lattices over a Banach space E and the corresponding spaces $H_p[E]$ that contain them, with p ranging from 1 to ∞, are ordered by inclusion. This nice property motivates the study of the interpolation and extrapolation spaces that these families generate, to which this chapter is dedicated. It is based on:

[55] E. García-Sánchez, P. Tradacete, *Interpolation and extrapolation of spaces of positively homogeneous functions*. In preparation.

Recall from Section 1.3 that given a Banach space E we denote by $H[E]$ the space of positively homogeneous functions from E^* to $\mathbb{R}$, and for $f \in H[E]$ we define the norms

$$\begin{aligned}\|f\|_{H_p[E]} &:= \sup\left\{\left(\sum_{i=1}^m |f(x_i^*)|^p\right)^{\frac{1}{p}} : m\in\mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E}\left(\sum_{i=1}^m |x_i^*(x)|^p\right)^{\frac{1}{p}} \leqslant 1\right\}\\ &= \sup\left\{\|(f(x_i^*))_{i=1}^m\|_{\ell_p^m} : m\in\mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E}\|(x_i^*(x))_{i=1}^m\|_{\ell_p^m} \leqslant 1\right\}\end{aligned}$$

for $1 \leqslant p < \infty$, and

$$\|f\|_{H_\infty[E]} := \sup_{x^*\in B_{E^*}} |f(x^*)|$$

for $p = \infty$ (we will simply put $\|f\|_{H_p}$ when there is no risk of confusion) and the Banach lattices

$$H_p[E] := \{f \in H[E] : \|f\|_{H_p} < \infty\},$$

which are p-convex with constant 1 and contain an isometric representation of $\mathrm{FBL}^p[E]$. If we consider instead the expression

$$\|f\|_{H_{p,\infty}[E]} := \sup\left\{\|(f(x_i^*))_{i=1}^m\|_{\ell_{p,\infty}^m} : m\in\mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E}\|(x_i^*(x))_{i=1}^m\|_{\ell_{p,\infty}^m} \leqslant 1\right\},$$

where

$$\|(a_i)_{i=1}^m\|_{\ell_{p,\infty}^m} := \sup_{A\neq\varnothing} |A|^{\frac{1}{p}-1}\sum_{i\in A}|a_i|$$

is the norm of the m-dimensional weak-L_p space $\ell_{p,\infty}^m$, we obtain the Banach lattice $H_{p,\infty}[E]$, which satisfies an upper p-estimate with constant 1 and contains an isomorphic copy of the free Banach lattice with upper p-estimates over E, denoted $\mathrm{FBL}^{p,\infty}[E]$ (see Chapter 2).

Motivated by their connection to free Banach lattices, the spaces $H_p[E]$, which can also be understood as spaces of non-linear p-summing maps from E^* into $\mathbb{R}$, were studied recently in [84]. In this work, the authors study the lattice homomorphisms between $H_p[E]$ spaces and characterize

some properties of the generating Banach space E such as finite dimension or reflexivity in terms of the inclusions between some distinguished sublattices of $H_p[E]$.

Another direction of study consist in relating concavity and convexity properties of the Banach space E, such as type and cotype, with inclusions of certain sublattices of $H[E]$ [75, 103]. This was initiated in [75, Section 7], where the authors introduced for every $1 \leqslant p, q \leqslant \infty$ and $f \in H[E]$ the norm

$$\begin{aligned}\|f\|_{H_{p;q}[E]} &:= \sup\left\{\left(\sum_{i=1}^{m}|f(x_i^*)|^p\right)^{\frac{1}{p}} : m \in \mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E}\left(\sum_{i=1}^{m}|x_i^*(x)|^q\right)^{\frac{1}{q}} \leqslant 1\right\}\\ &= \sup\left\{\|(f(x_i^*))_{i=1}^m\|_{\ell_p^m} : m \in \mathbb{N}, (x_i^*)_{i=1}^m \subseteq E^*, \sup_{x\in B_E}\|(x_i^*(x))_{i=1}^m\|_{\ell_q^m} \leqslant 1\right\},\end{aligned}$$

(when p or q are infinite, replace the respective sum by a supremum) and the Banach lattice $H_{p;q}[E] = \left\{f \in H[E] : \|f\|_{H_{p;q}} < \infty\right\}$ (if $q = \infty$, do not mistake $H_{p;\infty}[E]$ with $H_{p,\infty}[E]$ introduced above). When $p = q$, we recover $\|\cdot\|_{H_p[E]}$ and $H_p[E]$. These norms generalize the (p, q)-summing norms to the context of non-linear positively homogeneous functions. In [75] the authors study the hierarchy of inclusions of these spaces and show that under certain convexity hypothesis on E (E^* having finite cotype) some of these norms are isomorphically equivalent.

In this chapter, we study the family of spaces of positively homogeneous functions from the point of view of interpolation and extrapolation theory. We will start reviewing some basic interpolation and extrapolation methods, as well as some known facts concerning the spaces $H_{p;q}[E]$.

Next, Section 4.2 will be devoted to the study of the interpolation spaces that lie between $H_1[E]$ and $H_\infty[E]$. After generalizing a result from Miyazaki [98] about the real interpolation of linear (p, q)-summing operators to the setting of non-linear positively homogeneous functions, we focus on the Calderón–Lozanovskii method, which is exclusive for compatible pairs of Banach lattices. We show that the Calderón–Lozanovskii space $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$, which can be seen as the $\frac{1}{1-\theta}$-convexification of $H_{p_0}[E]$, lies between $H_{p_\theta}[E]$ and $H_{p_\theta;p_0}[E]$, with $p_\theta = \frac{p_0}{1-\theta}$ (Theorem 4.2.7), and show that these inclusions are strict in general and $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ is not in general an $H_{p;q}$-space.

Finally, in Section 4.3 we show that $H_{p;q}[E]$ can be recovered from a suitable family of $H_{s;t}[E]$ spaces using the Δ-method of extrapolation (see Theorem 4.3.1). In particular, $H_p[E]$ can be computed applying the Δ method to the family $(H_s[E])_{s>p}$. We also provide examples where some inclusions between $H_{p;q}$-spaces are strict. We conclude the chapter applying the results from Sections 4.2 and 4.3 to show that all the inclusions between spaces $H_{p;q}[\ell_1]$ are strict (see Proposition 4.3.5).

4.1 Preliminaries

We start this section recalling some basic notions from interpolation theory, adapted to the setting of Banach lattices. We will be using the notation and terminology from [21]. A pair of Banach spaces (X_0, X_1) is called a *compatible couple of Banach spaces* if there exists an ambient topological vector space U and linear continuous injections $i_j : X_j \to U$, $j = 0, 1$. A pair of Banach lattices (X_0, X_1) is called a *compatible couple of Banach lattices* if the space U can be chosen to be a uniformly complete vector lattice and the maps i_j are interval preserving lattice homomorphisms. Given a compatible pair of Banach spaces (respectively, lattices) (X_0, X_1), we can define the sum

$$X_0 + X_1 := \{x_0 + x_1 \in U : x_j \in X_j, j = 0, 1\}$$

endowed with the norm

$$\|x\|_{X_0+X_1} := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_j \in X_j, j = 0, 1\},$$

and the intersection

$$X_0 \cap X_1 := \{x \in X_0 + X_1 : x \in X_j, j = 0, 1\}$$

endowed with the norm

$$\|x\|_{X_0\cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\}.$$

Both are Banach spaces (respectively, lattices) such that $X_0 \cap X_1 \subseteq X_j \subseteq X_0 + X_1$, $j = 0, 1$ with contractive (respectively, contractive interval preserving lattice) inclusions.

The *real method of interpolation* is constructed as follows. Let (X_0, X_1) be a compatible pair of Banach spaces (respectively, lattices). Given $t > 0$ and $x \in X_0 + X_1$, we define the *K-functional*

$$K(t, x) \equiv K(t, x; X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_j \in X_j, j = 0, 1\},$$

and consider for every $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$ the expression

$$\|x\|_{\theta,q} := \left(\int_0^\infty (t^{-\theta}K(t, x))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and the Banach space (respectively, lattice)

$$(X_0, X_1)_{\theta,q} := \{x \in X_0 + X_1 : \|x\|_{\theta,q} < \infty\}.$$

Then, $(\cdot, \cdot)_{\theta,q}$ is an exact interpolation functor of exponent θ, that is:

$$X_0 \cap X_1 \subseteq (X_0, X_1)_{\theta,q} \subseteq X_0 + X_1$$

with continuous (respectively, contractive interval preserving lattice) inclusions, and for every compatible pair of Banach spaces (Y_0, Y_1) and operator $T : X_0 + X_1 \to Y_0 + Y_1$ such that $T|_{X_j} : X_j \to Y_j$ is bounded, $j = 0, 1$, it follows that $T|_{(X_0,X_1)_{\theta,q}} : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ is bounded with norm

$$\|T|_{(X_0,X_1)_{\theta,q}}\| \leqslant \|T|_{X_0}\|^{1-\theta}\|T|_{X_1}\|^{\theta}.$$

The *Calderón–Lozanovskii construction* was introduced by Calderón [31] for Banach lattices of functions, and later generalized to arbitrary Banach lattices by Lozanovskii (see [90, 91] for details). Given a compatible pair of Banach lattices (X_0, X_1) and $0 < \theta < 1$, the space $X_0^{1-\theta}X_1^{\theta}$ is defined via Krivine's functional calculus as

$$X_0^{1-\theta}X_1^{\theta} := \{x \in X_0 + X_1 : |x| \leqslant \lambda|x_0|^{1-\theta}|x_1|^{\theta}, \lambda \geqslant 0, x_j \in B_{X_j}, j = 0, 1\},$$

with the norm

$$\|x\|_{X_0^{1-\theta}X_1^{\theta}} := \inf\{\lambda : |x| \leqslant \lambda|x_0|^{1-\theta}|x_1|^{\theta}, x_j \in B_{X_j}, j = 0, 1\}.$$

In general, the process described above does not provide an interpolation space, but $X_0^{1-\theta}X_1^{\theta}$ is always an *intermediate space* between X_0 and X_1, that is, a Banach lattice such that

$$X_0 \cap X_1 \subseteq X_0^{1-\theta}X_1^{\theta} \subseteq X_0 + X_1$$

with continuous interval preserving lattice inclusions.

The idea behind extrapolation theory is the opposite of that of interpolation (see for instance [97]). Our starting point is a family of Banach spaces (respectively, lattices) $(X_\theta)_{\theta\in\Theta}$ that are *strongly compatible*, i.e., there exist Banach spaces (respectively, lattices) Δ_X and Σ_X such that $\Delta_X \subseteq X_\theta \subseteq \Sigma_X$ for every $\theta \in \Theta$ with continuous (respectively, contractive interval preserving lattice) inclusions.

If $(X_\theta)_{\theta\in\Theta}$ and $(Y_\theta)_{\theta\in\Theta}$ are two extrapolation families, an operator $T : (X_\theta)_{\theta\in\Theta} \to (Y_\theta)_{\theta\in\Theta}$ will refer to a map $T : \Sigma_X \to \Sigma_Y$ such that $T|_{X_\theta} : X_\theta \to Y_\theta$ is uniformly bounded for every $\theta \in \Theta$. In this setting, X and Y are called *extrapolation spaces* (with respect to the families $(X_\theta)_{\theta\in\Theta}$ and $(Y_\theta)_{\theta\in\Theta}$) if $\Delta_X \subseteq X \subseteq \Sigma_X$, $\Delta_Y \subseteq Y \subseteq \Sigma_Y$, and $T|_X : X \to Y$ is bounded for every $T : (X_\theta)_{\theta\in\Theta} \to (Y_\theta)_{\theta\in\Theta}$.

In this chapter, we will only study one of the simplest extrapolation methods, the Δ method. This method can be applied to families $(X_\theta)_{\theta\in\Theta}$ such that the norms of the inclusions $\Delta_X \to X_\theta$ are uniformly bounded in $\theta \in \Theta$. If this is the case, we define

$$\underset{\theta\in\Theta}{\Delta}(X_\theta) := \left\{ x \in \bigcap_{\theta\in\Theta} X_\theta : \|x\|_{\Delta(X_\theta)} := \sup_{\theta\in\Theta} \|x\|_{X_\theta} < \infty \right\}.$$

We next review how the spaces $H_{p;q}[E]$ are ordered with respect to inclusion. The first thing that should be noted is that if $p < q$, then $H_{p;q}[E] = \{0\}$ (see [75, Lemma 7.2]), so we will always assume that $(p, q) \in \mathcal{T} := \{(p, q) : 1 \leqslant q \leqslant p \leqslant \infty\}$. Another trivial observation is that

$$H_{p;q_1}[E] \subseteq H_{p;q_0}[E]$$

with norm 1 inclusion (equivalently, $\|f\|_{H_{p;q_1}} \geqslant \|f\|_{H_{p;q_0}}$ for every $f \in H[E]$) whenever $1 \leqslant q_0 \leqslant q_1 \leqslant p \leqslant \infty$. This is just a consequence of the fact that ℓ_{q_0} is contained contractively in ℓ_{q_1}. In particular, it follows that

$$H_{\infty;q}[E] = H_\infty[E]$$

isometrically for every $1 \leqslant q < \infty$. Indeed, one inclusion was already established, and it is clear that $\|f\|_{H_\infty} \leqslant \|f\|_{H_{p;q}}$ for every $1 \leqslant p, q \leqslant \infty$. Regarding the space $H_\infty[E]$, it should be noted that it can be identified isometrically with $\ell_\infty(S_{E^*})$ via the operator of restriction to S_{E^*}, so that the norm on E^* corresponds to the order unit (see [84, Section 3]). It is also true [75, Proposition 7.3] that if $(p_j, q_j) \in \mathcal{T}$, $j = 0, 1$, satisfy $p_1 \leqslant p_0$, $q_1 \leqslant q_0$ and $\frac{1}{q_1} - \frac{1}{p_1} \leqslant \frac{1}{q_0} - \frac{1}{p_0}$, then $\|f\|_{H_{p_0;q_0}} \leqslant \|f\|_{H_{p_1;q_1}}$. We can summarize all as follows:

Proposition 4.1.1. *Let $(p_j, q_j) \in \mathcal{T}$, $j = 0, 1$, be such that $p_1 \leqslant p_0$ and $\frac{1}{q_1} - \frac{1}{p_1} \leqslant \frac{1}{q_0} - \frac{1}{p_0}$. Then $\|f\|_{H_{p_0;q_0}} \leqslant \|f\|_{H_{p_1;q_1}}$. In particular,*

$$H_{p_1;q_1}[E] \subseteq H_{p_0;q_0}[E].$$

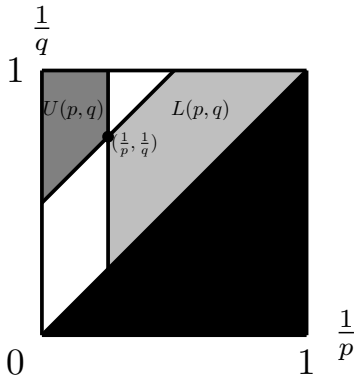


Figure 4.1: Representation of the sets $U(p, q)$ (dark gray) and $L(p, q)$ (light gray) in the square $[0, 1]^2$ (via the transformation $(p, q) \to (\frac{1}{p}, \frac{1}{q})$).

Given $(p,q) \in \mathcal{T}$, let us denote by $U(p,q) := \{(p',q') \in \mathcal{T} : p' \geqslant p, \frac{1}{q'} - \frac{1}{p'} \geqslant \frac{1}{q} - \frac{1}{p}\}$ and $L(p,q) := \{(p',q') \in \mathcal{T} : p' \leqslant p, \frac{1}{q'} - \frac{1}{p'} \leqslant \frac{1}{q} - \frac{1}{p}\}$ (see Figure 4.1). Then the previous result can be understood as follows: if $(p',q') \in U(p,q)$, then $H_{p;q}[E] \subseteq H_{p';q'}[E]$ with norm 1 inclusion for every Banach space E; conversely, if $(p',q') \in L(p,q)$, then $H_{p;q}[E] \supseteq H_{p';q'}[E]$ with norm 1 inclusion for every Banach space E. Moreover, putting $p = q$ we obtain that the spaces $H_p[E]$ are ordered by inclusion, being $H_1[E]$ the smallest and $H_\infty[E]$ the largest. Finally, it was established in [75, Proposition 7.4] that if E^* has finite cotype $r \geqslant 2$, then

$$H_{p;q}[E] = H_\infty[E]$$

with equivalence of norms for every $(p,q) \in \mathcal{T}$ with $\frac{1}{q} - \frac{1}{p} \geqslant 1 - \frac{1}{r}$.

4.2 Interpolation of $H_{p;q}$-spaces

The aim of this section is to study the spaces of positively homogeneous functions over the dual of a Banach space E using interpolation tools. As we mentioned earlier, the families $(\mathrm{FBL}^p[E])_{1\leqslant p\leqslant\infty}$ of free Banach lattices and $(H_p[E])_{1\leqslant p\leqslant\infty}$ of Banach lattices of non-linear p-summing functions are ordered by inclusion. Therefore, any pair of spaces of these families would immediately satisfy the definition of a compatible pair of Banach spaces, taking $\mathrm{FBL}^\infty[E]$ or $H_\infty[E]$ as an ambient space. However, we would like to preserve the Banach lattice structure of these spaces while interpolating, so we also need the inclusions to be interval preserving lattice homomorphisms. In general, it is not true that the inclusion from $\mathrm{FBL}^p[E]$ into $\mathrm{FBL}^q[E]$ with $p < q$ is interval preserving (see for instance Theorem 4.3.2, where we show that for certain Banach spaces E, there exist elements $x \in E$ such that the order interval generated by $|\delta_x|$ in $\mathrm{FBL}^p[E]$ and $\mathrm{FBL}^q[E]$ for $p < q$ do not agree), so the family $(\mathrm{FBL}^p[E])_{1\leqslant p\leqslant\infty}$ is not suitable for Banach lattice interpolation. On the other hand, the spaces $H_p[E]$ are ideals in $H[E]$, so the corresponding inclusions are interval preserving lattice homomorphisms. This makes the family $(H_p[E])_{1\leqslant p\leqslant\infty}$ more adequate for our purposes. Moreover, since $\mathrm{FVL}[E]$ lies in $H_p[E]$ for every p, it will also be contained in any intermediate space that we generate. Therefore, the closure of $\mathrm{FVL}[E]$ in each intermediate space will also provide an interesting space to study.

Once this has been clarified, it should be noted that when E is finite-dimensional, the task of distinguishing the intermediate spaces between $H_1[E]$ and $H_\infty[E]$ (or any other pair $(H_{p_0}[E], H_{p_1}[E])$ with $1 \leqslant p_0 < p_1 \leqslant \infty$) becomes trivial, as they are all isomorphic.

Proposition 4.2.1. *Let E be a finite-dimensional Banach space. Then $H_1[E] = H_\infty[E]$ lattice isomorphically. In particular, any intermediate Banach lattice between $H_1[E]$ and $H_\infty[E]$ is lattice isomorphic to $H_\infty[E]$.*

Proof. By [84, Theorem 3.5], if E is finite-dimensional we have that for every $1 \leqslant p < \infty$ the space $H_p[E]$ coincides as a set with $H_\infty[E]$. Actually, the authors show that $H_p[E] = I[E]$, where $I[E]$ denotes the ideal in $H[E]$ generated by $\phi_E(E)$, but it is straightforward to see that $\|\cdot\|$, the norm on E^*, belongs to $I[E]$ (every norm on a finite-dimensional Banach space is equivalent to the ℓ_∞ norm of the corresponding dimension, which is a finite supremum of evaluation functions). Since $H_\infty[E]$ is the ideal generated by $\|\cdot\|$, we conclude that $H_\infty[E] \subseteq I[E]$, and the reverse inclusion is clear. Hence, the identity map from $H_p[E]$ into $H_\infty[E]$ is a contractive bijective lattice homomorphism, so it must be bounded below, i.e., all the H_p-norms are equivalent.

Next, observe that $H_1[E] \subseteq H_\infty[E]$ with norm 1 inclusion, so $H_1[E] \cap H_\infty[E] = H_1[E]$ and $H_1[E] + H_\infty[E] = H_\infty[E]$ with equality of norms. Therefore, if X is an intermediate Banach lattice between $H_1[E]$ and $H_\infty[E]$, it follows that

$$\frac{1}{c_1}\|\cdot\|_{H_\infty} = \frac{1}{c_1}\|\cdot\|_{H_1+H_\infty} \leqslant \|\cdot\|_X \leqslant c_0\|\cdot\|_{H_1\cap H_\infty} = c_0\|\cdot\|_{H_1}.$$

Since $\|\cdot\|_{H_1}$ is equivalent to $\|\cdot\|_{H_\infty}$, the statement holds. □

We continue with a result on real interpolation of $H_{p;q}[E]$ spaces that reproduces an argument from [98] for (p,q)-summing operators.

Proposition 4.2.2. *Let $1 \leqslant q \leqslant p_0 < p_1 \leqslant \infty$ and $0 < \theta < 1$, and define $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$(H_{p_0;q}[E], H_{p_1;q}[E])_{\theta,p_\theta} \subseteq H_{p_\theta;q}[E]$$

with continuous inclusion.

Proof. First, recall from Proposition 4.1.1 that $\|\cdot\|_{H_{p_1;q}[E]} \leqslant \|\cdot\|_{H_{p_0;q}[E]}$, so $H_{p_0;q}[E] + H_{p_1;q}[E] = H_{p_1;q}[E]$ isometrically. Now, fix $f \in H_{p_1;q}[E]$ and $t > 0$ and let us obtain a lower bound for $K(t, f; H_{p_0;q}[E], H_{p_1;q}[E])$. To do so, consider an arbitrary decomposition $f = f_0 + f_1$, with $f_j \in H_{p_j,q}[E]$, $j = 0, 1$. Then, given a finite sequence $(x_i^*)_{i=1}^m \subseteq E^*$ such that

$$\sup_{x\in B_E}\left(\sum_{i=1}^m |x_i^*(x)|^q\right)^{\frac{1}{q}} \leqslant 1$$

we have

$$\begin{aligned}\|f_0\|_{H_{p_0;q}[E]} + t\|f_1\|_{H_{p_1;q}[E]} &\geqslant \|(f_0(x_i^*))_{i=1}^m\|_{\ell_{p_0}^m} + t\|(f_1(x_i^*))_{i=1}^m\|_{\ell_{p_1}^m}\\ &\geqslant K(t, (f(x_i^*))_{i=1}^m; \ell_{p_0}^m, \ell_{p_1}^m).\end{aligned}$$

Taking the infimum over all decompositions (f_0, f_1) we get that

$$K(t, f; H_{p_0;q}[E], H_{p_1;q}[E]) \geqslant K(t, (f(x_i^*))_{i=1}^m; \ell_{p_0}^m, \ell_{p_1}^m),$$

so in particular, applying the reiteration theorem (cf. [21, Theorem 3.5.3]) for the pair $(\ell_{p_0}^m, \ell_{p_1}^m)$ we obtain

$$\|f\|_{(H_{p_0;q},H_{p_1;q})_{\theta,p_\theta}} \geqslant \|(f(x_i^*))_{i=1}^m\|_{(\ell_{p_0}^m,\ell_{p_1}^m)_{\theta,p_\theta}} \approx \|(f(x_i^*))_{i=1}^m\|_{\ell_{p_\theta}^m}.$$

Finally, taking the supremum over all the sequences $(x_i^*)_{i=1}^m$ we conclude that

$$\|f\|_{(H_{p_0;q},H_{p_1;q})_{\theta,p_\theta}} \gtrsim \|f\|_{H_{p_\theta;q}[E]}.$$

□

As it was observed in Section 4.1, $H_{\infty;q}[E] = H_\infty[E]$ isometrically for every $1 \leqslant q \leqslant \infty$, so the following corollary can be deduced:

Corollary 4.2.3. *Let $1 \leqslant p_0 < \infty$ and $0 < \theta < 1$, and define $p_\theta = \frac{p_0}{1-\theta}$. Then*

$$(H_{p_0}[E], H_\infty[E])_{\theta,p_\theta} \subseteq H_{p_\theta;p_0}[E].$$

We do not know whether the reverse inclusion in Proposition 4.2.2 holds, or if there exists some $q \leqslant r \leqslant p_\theta$ such that $H_{p_\theta;r}[E] \subseteq (H_{p_0;q}[E], H_{p_1;q}[E])_{\theta,p_\theta}$.

Next, we will consider the Calderón–Lozanovskii method applied to the Banach lattices $H_{p_0}[E]$, $1 \leqslant p_0 < \infty$ and $H_\infty[E]$. This method is strongly related to the p-convexification process described in Section 1.2, as we will explain below. Recall from Section 4.1 that we can identify isometrically $H_\infty[E]$ with $\ell_\infty(S_{E^*})$ by restricting the functions to the unit sphere, so that $H_{p_0}[E] \subseteq H_\infty[E]$ can

also be seen as an ideal of functions over S_{E^*} with the counting measure. In this setting, for every $1 < p < \infty$ the p-convexification $H_{p_0}[E]^{(p)}$ of $H_{p_0}[E]$ can be identified with the space

$$H_{p_0}[E]^{(p)} = \left\{ \left(f|_{S_{E^*}} \right)^{\frac{1}{p}} : f \in H_{p_0}[E] \right\} \subseteq \ell_\infty(S_{E^*}),$$

endowed with the norm $\|g\|_{H_{p_0}[E]^{(p)}} = \|g^p\|^{\frac{1}{p}}_{H_{p_0}[E]}$ for $g \in H_{p_0}[E]^{(p)}$. Here, by $f^{\frac{1}{p}}$ we understand $\operatorname{sgn}(f)|f|^{\frac{1}{p}}$. By extending these functions from S_{E^*} to B_{E^*} in a positively homogeneous way, we can represent $H_{p_0}[E]^{(p)}$ as a subset of $H[E]$:

$$H_{p_0}[E]^{(p)} = \left\{ f^{\frac{1}{p}} \| \cdot \|^{\frac{1}{p^*}} : f \in H_{p_0}[E] \right\}, \tag{4.1}$$

where $\| \cdot \| \in H_\infty[E]$ denotes the norm on E^*, and now the norm on $H_{p_0}[E]^{(p)}$ is given by the relation

$$\|g\|_{H_{p_0}[E]^{(p)}} = \|f\|^{\frac{1}{p}}_{H_{p_0}} = \left\| \frac{g^p}{\| \cdot \|^{p-1}} \right\|^{\frac{1}{p}}_{H_{p_0}},$$

provided $g = f^{\frac{1}{p}} \| \cdot \|^{\frac{1}{p^*}} \in H_{p_0}[E]^{(p)}$ for a certain $f \in H_{p_0}[E]$.

Actually, it is well known that the p-convexification $X^{(p)}$ of a Banach lattice X of measurable functions over a measure space (Ω, Σ, μ) coincides isometrically with the Calderón–Lozanovskii space $X^{1-\theta} L_\infty(\mu)^\theta$ for $\theta = 1 - \frac{1}{p}$. We reproduce the details for the particular case of $H_{p_0}[E]$. We want to show that the identification

$$H_{p_0}[E]^{(p)} = H_{p_0}[E]^{1-\theta} H_\infty[E]^\theta$$

holds isometrically. The inclusion from left to right is clear from the representation of $H_{p_0}[E]^{(p)}$ given above, as $\| \cdot \|$ is the order unit of $H_\infty[E]$ and has norm 1. On the other hand, given $g \in H_{p_0}[E]^{1-\theta} H_\infty[E]^\theta$, we can express it as

$$g = \left(\frac{g^p}{\| \cdot \|^{p-1}} \right)^{\frac{1}{p}} \| \cdot \|^{\frac{1}{p^*}}.$$

We claim that $\frac{g^p}{\| \cdot \|^{p-1}} \in H_{p_0}[E]$. Indeed, given arbitrary $\lambda > 0$ and positive $f_0 \in B_{H_{p_0}[E]}$ and $f_1 \in B_{H_\infty[E]}$ such that $|g| \leqslant \lambda f_0^{\frac{1}{p}} f_1^{\frac{1}{p^*}}$, it follows that

$$\frac{|g|^p}{\| \cdot \|^{p-1}} \leqslant \lambda^p f_0 \left(\frac{f_1}{\| \cdot \|} \right)^{\frac{p}{p^*}} \leqslant \lambda^p f_0,$$

since f_1 has norm 1 and $\| \cdot \|$ is the order unit of $H_\infty[E]$. Therefore,

$$\left\| \frac{g^p}{\| \cdot \|^{p-1}} \right\|_{H_{p_0}} \leqslant \lambda^p.$$

Taking the infimum over all possible λ we obtain that

$$\|g\|_{H_{p_0}[E]^{(p)}} = \left\| \frac{g^p}{\| \cdot \|^{p-1}} \right\|^{\frac{1}{p}}_{H_{p_0}} \leqslant \|g\|_{H_{p_0}^{1-\theta} H_\infty^\theta},$$

so the remaining inclusion is established. From this identification we obtain an explicit representation for the norm of the Calderón–Lozanovskii space $H_{p_0}[E]^{1-\theta} H_\infty[E]^\theta$:

$$\begin{aligned}\|g\|_{H_{p_0}^{1-\theta}H_\infty^\theta} &= \left\| \frac{g^p}{\|\cdot\|^{p-1}} \right\|_{H_{p_0}}^{\frac{1}{p}} \\ &= \sup\left\{ \left(\sum_{i=1}^m \left(\frac{|g(x_i^*)|^p}{\|x_i^*\|^{p-1}} \right)^{p_0} \right)^{\frac{1}{pp_0}} : \sup_{x\in B_E} \left(\sum_{i=1}^m |x_i^*(x)|^{p_0} \right)^{\frac{1}{p_0}} \leqslant 1 \right\} \\ &= \sup\left\{ \left(\sum_{i=1}^m \left| \frac{g(x_i^*)}{\|x_i^*\|^\theta} \right|^{p_\theta} \right)^{\frac{1}{p_\theta}} : \sup_{x\in B_E} \left(\sum_{i=1}^m |x_i^*(x)|^{p_0} \right)^{\frac{1}{p_0}} \leqslant 1 \right\},\end{aligned}$$

where $p_\theta = pp_0 = \frac{p_0}{1-\theta}$. It could be said that $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ behaves as a weighted $H_{p_\theta;p_0}$-space, or more specifically, a space of non-linear maps that send weakly p_0-summing sequences of E^* to sequences in $\ell_{p_\theta}(\|\cdot\|^{-\theta})$.

Proceeding in a similar way we can conclude that the p-convexification of $\mathrm{FBL}^{p_0}[E]$ can be represented as

$$\mathrm{FBL}^{p_0}[E]^{(p)} = \left\{ f^{\frac{1}{p}} \|\cdot\|^{\frac{1}{p^*}} : f \in \mathrm{FBL}^{p_0}[E] \right\},$$

and the norm is given by

$$\|g\|_{\mathrm{FBL}^{p_0}[E]^{(p)}} = \|g\|_{H_{p_0}^{1-\theta}H_\infty^\theta} = \|f\|_{H_{p_0}}^{\frac{1}{p}}$$

for $g = f^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} \in \mathrm{FBL}^{p_0}[E]^{(p)}$ with $f \in \mathrm{FBL}^{p_0}[E]$. Recall that

$$\begin{aligned}\phi_E : E &\longrightarrow \mathrm{FBL}^{p_0}[E] \\ x &\longmapsto \phi_E x = \delta_x\end{aligned}$$

is an isometric embedding, where $\delta_x(x^*) = x^*(x)$, and $\mathrm{FBL}^{p_0}[E] = \overline{\mathrm{lat}}\{\delta_x : x \in E\}$. Even though the correspondence

$$\begin{aligned}\phi_E^{(p)} : E &\longrightarrow \mathrm{FBL}^{p_0}[E]^{(p)} \\ x &\longmapsto \delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}\end{aligned}$$

described above is no longer linear, these functions generate $\mathrm{FBL}^{p_0}[E]^{(p)}$, as will be shown below.

Proposition 4.2.4. *Consider* $\mathrm{FBL}^{p_0}[E]^{(p)}$ *as a sublattice of* $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$, *with* $\theta = 1 - \frac{1}{p}$. *Then*

$$\mathrm{FBL}^{p_0}[E]^{(p)} = \overline{\mathrm{lat}}\left\{ \delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} : x \in E \right\}.$$

Proof. One inclusion is trivial, since $\mathrm{FBL}^{p_0}[E]^{(p)}$ is a closed sublattice that contains $\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}$ for every $x \in E$. For the reverse inclusion, let $f \in \mathrm{FVL}[E] = \mathrm{lat}\left\{\delta_x : x \in E\right\}$, that can be represented as

$$f = \bigvee_{i=1}^m \delta_{x_i} - \bigvee_{j=1}^n \delta_{y_j}$$

for some $(x_i)_i, (y_j)_j \subseteq E$. We know that $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ is endowed with the pointwise order and lattice operations, so it is clear that

$$\left(\bigvee_{i=1}^m \delta_{x_i} \right)^{\frac{1}{p}} \|\cdot\|^{\frac{1}{p^*}} = \bigvee_{i=1}^m \left(\delta_{x_i}^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} \right) \in \mathrm{lat}\left\{ \delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} : x \in E \right\}.$$

Since $\overline{\mathrm{lat}}\left\{ \delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} : x \in E \right\}$ is a Banach lattice, the continuous positively homogeneous expression $F(g,h) = (g^p - h^p)^{\frac{1}{p}}$ is well defined for every g, h in this lattice via functional calculus, so in particular

$$f^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} = F\left(\left(\bigvee_{i=1}^{m}\delta_{x_i}\right)^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}, \left(\bigvee_{j=1}^{n}\delta_{y_j}\right)^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}\right)$$

belongs to $\overline{\text{lat}}\left\{\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} : x\in E\right\}$.

Now, let $f\in \text{FBL}^{p_0}[E]$, and take a sequence $(f_k)_k\subseteq \text{FVL}[E]$ such that $\|f-f_k\|_{H_{p_0}}\to 0$. We claim that $g_k = f_k^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}$ converges to $g=f^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}$ in $\text{FBL}^{p_0}[E]^{(p)}$. Indeed, it can be checked using the Mean Value Theorem that the inequality

$$|t^{\frac{1}{p}}-s^{\frac{1}{p}}|^p\leqslant C|t-s|$$

holds in $\mathbb{R}^2$, so it is also true in the Banach lattice $\text{FBL}^{p_0}[E]$. In particular,

$$|f^{\frac{1}{p}}-f_k^{\frac{1}{p}}|^p\leqslant C|f-f_k|,$$

so

$$\|g-g_k\|^p_{H_{p_0}^{1-\theta}H_\infty^\theta} = \|(f^{\frac{1}{p}}-f_k^{\frac{1}{p}})^p\|_{H_{p_0}}\leqslant C\|f-f_k\|_{H_{p_0}}\to 0$$

and therefore $f^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}\in\overline{\text{lat}}\left\{\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} : x\in E\right\}$ for every $f\in\text{FBL}^{p_0}[E]$. □

Arguing as in the last part of the proof we also get:

Corollary 4.2.5. *The inclusion $\phi_E^{(p)}: E\to \text{FBL}^{p_0}[E]^{(p)}$ is Hölder continuous with exponent $\frac{1}{p}$, that is*

$$\|\phi_E^{(p)}x-\phi_E^{(p)}y\|_{H_{p_0}^{1-\theta}H_\infty^\theta}\leqslant C\|x-y\|^{\frac{1}{p}}.$$

However, there is a linear way of embedding E into $H_{p_0}[E]^{(p)} = H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$. It suffices to consider the canonical operator $\phi_E: E\to H_{p_0}[E]\subseteq H_\infty[E]$, which is an isometric embedding into both $H_{p_0}[E]$ and $H_\infty[E]$. Given $x\in B_E$, we can write $|\delta_x| = |\delta_x|^{1-\theta}|\delta_x|^\theta$, so $\|\delta_x\|_{H_{p_0}^{1-\theta}H_\infty^\theta}\leqslant 1$ and $\phi_E: E\to H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ is bounded. Furthermore, given $x\in E$, let $x^*\in S_{E^*}$ such that $x^*(x)=\|x\|$. Then,

$$\|\delta_x\|_{H_{p_0}^{1-\theta}H_\infty^\theta} = \left\|\frac{\delta_x^p}{\|\cdot\|^{p-1}}\right\|^{\frac{1}{p}}_{H_{p_0}}\geqslant\left(\frac{|x^*(x)|^p}{\|x^*\|^{p-1}}\right)^{\frac{1}{p}} = \|x\|,$$

so it is an isometric embedding into $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$. Let us denote by

$$Z_p = \overline{\text{lat}}\left\{\delta_x : x\in E\right\} = \overline{\text{FVL}[E]}\subseteq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta.$$

We have the following dichotomy.

Lemma 4.2.6. *Let E be a Banach space, and consider both $\text{FBL}^{p_0}[E]^{(p)}$ and Z_p as closed sublattices of $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$, where $\theta = 1-\frac{1}{p}$.*

(1) If E has finite dimension, then $\text{FBL}^{p_0}[E]^{(p)} = Z_p$.

(2) If E has infinite dimension, then $\text{FBL}^{p_0}[E]^{(p)}\not\subseteq Z_p$ and $\text{FBL}^{p_0}[E]^{(p)}\not\supseteq Z_p$.

Proof. (1) Observe that if E has finite dimension, then $\text{FBL}^{p_0}[E]$ can be identified with $C(S_{E^*})$ through the restriction operator, and $\|\cdot\|\in\text{FBL}^{p_0}[E]$ (see for instance [103, Section 9.1]). It follows from functional calculus that the functions $\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}$ are contained in $\text{FBL}^{p_0}[E]$ for every $x\in E$. Next, note that

$$H_{p_0}[E] = H_{p_0}[E]\cap H_\infty[E]\subseteq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$$

with a norm 1 inclusion. In particular, $\mathrm{FBL}^{p_0}[E]$, the closure of $\mathrm{FVL}[E]$ under the norm of $H_{p_0}[E]$, will be contained in Z_p, the closure of $\mathrm{FVL}[E]$ in $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$. Hence, $\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}} \in Z_p$ for every $x \in E$, and by Proposition 4.2.4, $\mathrm{FBL}^{p_0}[E]^{(p)} \subseteq Z_p$.

On the other hand, the functions $\frac{\delta_x^p}{\|\cdot\|^{p-1}}$ are continuous on the unit sphere, so they belong to $\mathrm{FBL}^{p_0}[E]$. It follows that $\delta_x \in \mathrm{FBL}^{p_0}[E]^{(p)}$ for every $x \in E$, so $Z_p \subseteq \mathrm{FBL}^{p_0}[E]^{(p)}$.

(2) If E is infinite-dimensional, consider any $x \in E\backslash\{0\}$ and take some $z^* \in E^*$ such that $\|z^*\| < 1$ and $z^*(x) \neq 0$. By [46, Exercise 3.39], there is a sequence $(z_n^*)_n \subseteq S_{E^*}$ such that $z_n^* \xrightarrow{w^*} z^*$. In particular,

$$(\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}})(z_n^*) = (z_n^*(x))^{\frac{1}{p}} \to z^*(x)^{\frac{1}{p}} > z^*(x)^{\frac{1}{p}}\|z^*\|^{\frac{1}{p^*}} = (\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}})(z^*),$$

so $\delta_x^{\frac{1}{p}}\|\cdot\|^{\frac{1}{p^*}}$ is not w^*-continuous on B_{E^*}. In particular, it does not belong to $Z_p \subseteq \mathrm{FBL}^\infty[E] \subseteq C(B_{E^*}, w^*)$, so $Z_p \not\ni \mathrm{FBL}^{p_0}[E]^{(p)}$. Similarly, $\frac{\delta_x^p}{\|\cdot\|^{p-1}}$ is not w^*-continuous on B_{E^*}, so it does not belong to $\mathrm{FBL}^{p_0}[E]$ and consequently $\delta_x \notin \mathrm{FBL}^{p_0}[E]^{(p)}$ and $Z_p \not\subseteq \mathrm{FBL}^{p_0}[E]^{(p)}$. □

Next, we study where the space $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ is located within the ideals of $H[E]$:

Theorem 4.2.7. *Let E be a Banach space, $1 \leqslant p_0 < \infty$, $0 < \theta < 1$ and $p_\theta = \frac{p_0}{1-\theta}$. Then*

$$H_{p_\theta}[E] \subseteq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta \subseteq H_{p_\theta;p_0}[E],$$

with norm 1 inclusions.

Proof. Let $f \in H_{p_\theta}[E]$. We can write f as

$$|f| = g^{1-\theta}\|\cdot\|^\theta, \quad \text{with} \quad g = \frac{|f|^{\frac{p_\theta}{p_0}}}{\|\cdot\|^{\frac{p_\theta}{p_0}-1}}.$$

It is clear that $\|\cdot\| \in H_\infty[E]$ and has norm 1. Let us see that $g \in H_{p_0}[E]$. Indeed, g is positively homogeneous, and given $(x_i^*)_{i=1}^m \subseteq E^*$ such that

$$\sup_{x\in B_E}\left(\sum_{i=1}^m |x_i^*(x)|^{p_0}\right)^{\frac{1}{p_0}} \leqslant 1,$$

we have that

$$\left(\sum_{i=1}^m |g(x_i^*)|^{p_0}\right)^{\frac{1}{p_0}} \leqslant \|f\|_{H_{p_\theta}[E]}^{\frac{p_\theta}{p_0}} \sup_{x\in B_E}\left(\sum_{i=1}^m \frac{|x_i^*(x)|^{p_\theta-p_0+p_0}}{\|x_i^*\|^{p_\theta-p_0}}\right)^{\frac{1}{p_0}} \leqslant \|f\|_{H_{p_\theta}[E]}^{\frac{p_\theta}{p_0}}.$$

Therefore,

$$\|f\|_{H_{p_0}^{1-\theta}H_\infty^\theta} = \|g\|_{H_{p_0}}^{1-\theta} \leqslant \|f\|_{H_{p_\theta}}$$

and the first inclusion holds.

Now, given $f \in H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$, for every $\lambda > \|f\|_{H_{p_0}^{1-\theta}H_\infty^\theta}$ there are positive $f_0 \in B_{H_{p_0}[E]}$, $f_1 \in B_{H_\infty[E]}$ such that $|f| \leqslant \lambda f_0^{1-\theta}f_1^\theta$. If we fix $(x_i^*)_{i=1}^m \subseteq E^*$ such that

$$\sup_{x\in B_E}\left(\sum_{i=1}^m |x_i^*(x)|^{p_0}\right)^{\frac{1}{p_0}} \leqslant 1,$$

we get that

$$\left(\sum_{i=1}^{m}|f(x_i^*)|^{p_\theta}\right)^{\frac{1}{p_\theta}} \leqslant \lambda\left(\sum_{i=1}^{m} f_0(x_i^*)^{(1-\theta)p_\theta} f_1(x_i^*)^{\theta p_\theta}\right)^{\frac{1}{p_\theta}}$$
$$\leqslant \lambda\|f_1\|_{H_\infty[E]}^{\theta}\left(\sum_{i=1}^{m} f_0(x_i^*)^{p_0}\right)^{\frac{1-\theta}{p_0}} \leqslant \lambda.$$

Taking the infimum over all possible λ and the supremum over the sequences $(x_i^*)_{i=1}^m$ we conclude that

$$\|f\|_{H_{p_\theta;p_0}} \leqslant \|f\|_{H_{p_0}^{1-\theta}H_\infty^\theta}.$$

□

However, the inclusions above are strict in general, as the following results show.

Proposition 4.2.8. *Let $1 < q < \infty$ and E be an infinite-dimensional Banach space such that E^* contains an isomorphic copy of ℓ_{q^*}. Then, if $1 \leqslant p_0 < q \leqslant p_\theta$ and $\theta = 1 - \frac{p_0}{p_\theta}$, there exists $f \in H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ such that $f \notin H_{p_\theta;q}[E]$. In particular,*

$$H_{p_\theta}[E] \subsetneq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta \quad \text{and} \quad H_{p_\theta;q}[E] \subsetneq H_{p_\theta;p_0}[E].$$

Proof. Let $T : \ell_{q^*} \to E^*$ be an isomorphic embedding such that $C^{-1}\|a\|_{\ell_{q^*}} \leqslant \|Ta\| \leqslant \|a\|_{\ell_{q^*}}$ for every $a \in \ell_{q^*}$, and denote by $x_k^* = Te_k^*$, where $(e_k^*)_{k=1}^\infty$ denotes the canonical basis of ℓ_{q^*}. Note that

$$\sup_{x\in B_E}\left(\sum_{k=1}^{\infty}|x_k^*(x)|^q\right)^{\frac{1}{q}} = \|T^*J_E\| = \|T\| \leqslant 1.$$

Since T is an embedding, it follows from [46, Exercise 2.49] that $T^* : E^{**} \to \ell_q$ is an onto map. Even more, there exists $x^{**} \in E^{**}$ such that $T^*x^{**} = b := (k^{-\frac{1}{p_0}})_{k=1}^\infty \in \ell_q \backslash \ell_{p_0}$. We consider $f = |x^{**}|^{1-\theta}\|\cdot\|^\theta$, which clearly belongs to $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ (recall that E^{**} embeds into $H_{p_0}[E]$ by [84, Lemma 5.1]). It follows that for every $n \in \mathbb{N}$,

$$\|f\|_{H_{p_\theta;q}[E]} \geqslant \left(\sum_{k=1}^{n}|f(x_k^*)|^{p_\theta}\right)^{\frac{1}{p_\theta}} = \left(\sum_{k=1}^{n}|x^{**}(x_k^*)|^{p_0}\|x_k^*\|^{\theta p_\theta}\right)^{\frac{1}{p_\theta}}$$
$$\geqslant C^{-\theta}\left(\sum_{k=1}^{n}|e_k^*(b)|^{p_0}\right)^{\frac{1}{p_\theta}} = C^{-\theta}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{\frac{1}{p_\theta}},$$

so $f \notin H_{p_\theta;q}[E]$. Since $H_{p_\theta}[E] \subseteq H_{p_\theta;q}[E]$ and $H_{p_\theta}[E] \subseteq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta \subseteq H_{p_\theta;p_0}[E]$, it follows that $H_{p_\theta}[E] \subsetneq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ and $H_{p_\theta;q}[E] \subsetneq H_{p_\theta;p_0}[E]$. □

Remark 4.2.9. *A weaker version of the argument above can be established with a more general assumption: if E^* contains uniform copies of $\ell_{q^*}^n$, then $H_{p_\theta}[E] \subsetneq H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ whenever $1 \leqslant p_0 < q \leqslant p_\theta$ and $\theta = 1 - \frac{p_0}{p_\theta}$.*

Otherwise, the formal identity from $H_{p_\theta}[E]$ into $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ would be onto, and hence, an isomorphism, so the norms on both spaces would be equivalent. However, if $T_n : \ell_{q^*}^n \to E^*$ denote the uniform embeddings for $n \in \mathbb{N}$, there exists a bounded sequence $(x_n^{**})_{n=1}^\infty \subseteq E^{**} \subseteq H_{p_0}[E]$ such that $T_n^*x_n^{**} = b_n := (k^{-\frac{1}{p_0}})_{k=1}^n \in \ell_q^n$. Therefore, the functions $f_n = |x_n^{**}|^{1-\theta}\|\cdot\|^\theta$ are uniformly bounded in $H_{p_0}[E]^{1-\theta}H_\infty[E]^\theta$ but

$$\|f\|_{H_{p_\theta}[E]}^{p_\theta} \gtrsim \sum_{k=1}^{n}\frac{1}{k},$$

which is a contradiction with the fact that both norms are equivalent.

Example 4.2.10. *Next we provide some examples of spaces where Proposition 4.2.8 and Remark 4.2.9 can be applied:*

(1) If E is an infinite-dimensional Banach space, then by Dvoretzky's Theorem (see for instance [46, Theorem 6.15]) we can find for every $\varepsilon > 0$ and $n \in \mathbb{N}$ a subspace $F \subseteq E^*$ of dimension n which is $(1+\varepsilon)$-isomorphic to ℓ_2^n. In particular, E^* contains uniform copies of ℓ_2^n, so for $p_0 < 2 \leqslant p_\theta$ we have a strict inclusion $H_{p_\theta}[E] \subsetneq H_{p_0}[E]^{1-\theta} H_\infty[E]^\theta$.

(2) If X is an infinite-dimensional Banach lattice, the indices of X *are defined as*

$$\sigma(X) := \inf\{r : X \text{ satisfies a lower } r\text{-estimate}\}$$

and

$$s(X) := \sup\{r : X \text{ satisfies an upper } r\text{-estimate}\}.$$

Observe that $1 \leqslant s(X) \leqslant \sigma(X) \leqslant \infty$, $\sigma(X^*) = s(X)^*$ and $s(X^*) = \sigma(X)^*$. It was shown by Krivine [81] (see also [115]) that X contains a lattice $(1+\varepsilon)$-isomorphic copy of ℓ_r^n for every $n \in \mathbb{N}$ and $\varepsilon > 0$ when $r = \sigma(X)$ and $r = s(X)$. Applying this result to X^*, it follows that $H_{p_\theta}[X] \subsetneq H_{p_0}[X]^{1-\theta} H_\infty[X]^\theta$ whenever $p_0 < \sigma(X) \leqslant p_\theta$ or $p_0 < s(X) \leqslant p_\theta$. Note that taking for instance X as ℓ_q or $L_q[0,1]$, that satisfy $\sigma(X) = s(X) = q$, we can cover the cases $p_0 < q \leqslant p_\theta < 2$ and $2 \leqslant p_0 < q \leqslant p_\theta$, which were not covered by the first example.

(3) The space ℓ_∞ contains an isometric copy of every separable Banach space. In particular, it contains ℓ_{q^*} for every $1 < q < \infty$, so $H_{p_\theta}[\ell_1] \subsetneq H_{p_0}[\ell_1]^{1-\theta} H_\infty[\ell_1]^\theta$ for any choice of $1 \leqslant p_0 < \infty$ and $0 < \theta < 1$. The same can be said about $L_1(\mu)$.

The second inclusion of Theorem 4.2.7 can be strict as well. We provide an example in ℓ_1, for which the following well-known property will be useful.

Lemma 4.2.11. *Let $(e_n)_{n=1}^\infty$ denote the canonical basis of ℓ_1, $1 \leqslant p < \infty$ and $x_1^*, \dots, x_m^* \in \ell_\infty$. Then*

$$\sup_{x \in B_{\ell_1}} \left(\sum_{i=1}^m |x_i^*(x)|^p \right)^{\frac{1}{p}} = \sup_{n \in \mathbb{N}} \left(\sum_{i=1}^m |x_i^*(e_n)|^p \right)^{\frac{1}{p}}.$$

Proof. Since the vectors e_n are normalized, the right term is always lower or equal than the left term. To show the reverse inequality, observe that the function $\phi(t) = |t|^p$ is convex and continuous, so it satisfies

$$\left| \sum_{k=1}^\infty \lambda_k t_k \right|^p \leqslant \sum_{k=1}^\infty \lambda_k |t_k|^p$$

for any $(t_k)_{k=1}^\infty \subseteq \mathbb{R}$ and $(\lambda_k)_{k=1}^\infty \subseteq [0,1]$ such that $\sum_{k=1}^\infty \lambda_k = 1$. Therefore, given $x = \sum_{n=1}^\infty x_n e_n \in B_{\ell_1}$, we have

$$\begin{aligned} \left(\sum_{i=1}^m |x_i^*(x)|^p \right)^{\frac{1}{p}} &= \|x\|_{\ell_1} \left(\sum_{i=1}^m \left| \frac{x_i^*(x)}{\|x\|_{\ell_1}} \right|^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^m \left| \sum_{n=1}^\infty \frac{x_n x_i^*(e_n)}{\|x\|_{\ell_1}} \right|^p \right)^{\frac{1}{p}} \\ &\leqslant \left(\sum_{n=1}^\infty \frac{|x_n|}{\|x\|_{\ell_1}} \sum_{i=1}^m |x_i^*(e_n)|^p \right)^{\frac{1}{p}} \leqslant \sup_{n \in \mathbb{N}} \left(\sum_{i=1}^m |x_i^*(e_n)|^p \right)^{\frac{1}{p}}, \end{aligned}$$

as we wanted to show. □

Proposition 4.2.12. *Let $1 \leqslant p_0 < \infty$, $0 < \theta < 1$ and $p_\theta = \frac{p_0}{1-\theta}$. Then, $H_{p_0}[\ell_1]^{1-\theta} H_\infty[\ell_1]^\theta \subsetneq H_{p_\theta;p_0}[\ell_1]$.*

Proof. Let us denote by $(e_j)_{j=1}^{\infty}$ the canonical basis of ℓ_1, and choose a countable set $(a_j = (a_{j,k})_{k=1}^{\infty})_{j=1}^{\infty}$ that is dense in $(S_{\ell_{\frac{1}{\theta}}})_+$ and contains the canonical basis of $\ell_{\frac{1}{\theta}}$ as a subsequence $(a_{j_i})_{i=1}^{\infty}$. For each $k \in \mathbb{N}$, we define the functionals $x_k^* \in B_{\ell_\infty}$ by $x_k^*(e_j) = a_{j,k}^{\frac{1}{p_0}}$ for every $j \in \mathbb{N}$. Note that $x_k^*(e_{j_k}) = a_{j_k,k}^{\frac{1}{p_0}} = 1$ for every $k \in \mathbb{N}$, so actually $x_k^* \in S_{\ell_\infty}$. We claim that there is no pair of functionals in the sequence $(x_k^*)_{k=1}^{\infty}$ such that one is a non-negative multiple of the other. Assume there are different $k_1, k_2 \in \mathbb{N}$ and $\mu \geqslant 0$ (without loss of generality, $\mu \leqslant 1$) such that $x_{k_1}^* = \mu x_{k_2}^*$. This implies that $a_{j,k_1} = \mu^{p_0} a_{j,k_2}$ for every $j \in \mathbb{N}$. In particular, if d_{k_1} denotes the k_1-th vector of the canonical basis of $\ell_{\frac{1}{\theta}}$, it follows that

$$\begin{aligned}\|a_j - d_{k_1}\|_{\ell_{\frac{1}{\theta}}} &\geqslant (|1 - a_{j,k_1}|^{\frac{1}{\theta}} + |a_{j,k_2}|^{\frac{1}{\theta}})^{\theta} = (|1 - \mu^{p_0} a_{j,k_2}|^{\frac{1}{\theta}} + |a_{j,k_2}|^{\frac{1}{\theta}})^{\theta} \\ &\geqslant \min_{t \in [0,1]} ((1 - \mu^{p_0} t)^{\frac{1}{\theta}} + t^{\frac{1}{\theta}})^{\theta} > 0\end{aligned}$$

for every $j \in \mathbb{N}$, which is a contradiction with the fact that $(a_j)_{j=1}^{\infty}$ is dense in $(S_{\ell_{\frac{1}{\theta}}})_+$. Now, we define a function in $H[\ell_1]$ by

$$f(x^*) = \begin{cases} \lambda & \text{if } x^* = \lambda x_k^* \text{ for some } \lambda \geqslant 0 \text{ and } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked that $\|f\|_{H_{p_\theta;p_0}}$ can be computed by evaluating the corresponding expression only at weakly p_0-summing sequences which are of the form $(\lambda_k x_k^*)_{k=1}^{\infty}$ with $\lambda_k \geqslant 0$ for every $k \in \mathbb{N}$, i.e.,

$$\|f\|_{H_{p_\theta;p_0}} = \sup\left\{ \left(\sum_{k=1}^{\infty} \lambda_k^{p_\theta} \right)^{\frac{1}{p_\theta}} : \lambda_k \geqslant 0,\ \sup_{x \in B_{\ell_1}} \left(\sum_{k=1}^{\infty} |\lambda_k x_k^*(x)|^{p_0} \right)^{\frac{1}{p_0}} \leqslant 1 \right\}.$$

Given any of such sequences, it follows using that $(a_j)_{j=1}^{\infty} \subseteq S_{\ell_{\frac{1}{\theta}}}$ is a norming set for the positive cone of $\ell_{\frac{1}{1-\theta}} = \ell_{\frac{p_\theta}{p_0}}$ and Lemma 4.2.11 that

$$\begin{aligned}\left(\sum_{k=1}^{\infty} \lambda_k^{p_\theta} \right)^{\frac{1}{p_\theta}} &= \|(\lambda_k)_k\|_{\ell_{p_\theta}} = \|(\lambda_k^{p_0})_k\|_{\ell_{\frac{p_\theta}{p_0}}}^{\frac{1}{p_0}} = \sup_{j \in \mathbb{N}} \left(\sum_{k=1}^{\infty} a_{j,k} \lambda_k^{p_0} \right)^{\frac{1}{p_0}} \\ &= \sup_{j \in \mathbb{N}} \left(\sum_{k=1}^{\infty} (\lambda_k x_k^*(e_j))^{p_0} \right)^{\frac{1}{p_0}} = \sup_{x \in B_{\ell_1}} \left(\sum_{k=1}^{\infty} (\lambda_k x_k^*(x))^{p_0} \right)^{\frac{1}{p_0}},\end{aligned}$$

so $\|f\|_{H_{p_\theta;p_0}} \leqslant 1$. Next, recall that $f \in H_{p_0}[\ell_1]^{1-\theta} H_\infty[\ell_1]^{\theta}$ if and only if $f^{\frac{p_\theta}{p_0}} \| \cdot \|^{1-\frac{p_\theta}{p_0}} \in H_{p_0}[\ell_1]$ by (4.1). Using the relation above we get that

$$\begin{aligned}\|f\|_{H_{p_0}^{1-\theta} H_\infty^{\theta}}^{\frac{p_\theta}{p_0}} &= \|f^{\frac{p_\theta}{p_0}} \| \cdot \|^{1-\frac{p_\theta}{p_0}}\|_{H_{p_0}} \\ &= \sup\left\{ \left(\sum_{k=1}^{\infty} f(y_k^*)^{p_\theta} \|y_k^*\|^{p_0 - p_\theta} \right)^{\frac{1}{p_0}} : \sup_{x \in B_{\ell_1}} \left(\sum_{k=1}^{\infty} |y_k^*(x)|^{p_0} \right)^{\frac{1}{p_0}} \leqslant 1 \right\} \\ &= \sup\left\{ \left(\sum_{k=1}^{\infty} \lambda_k^{p_0} \right)^{\frac{1}{p_0}} : \lambda_k \geqslant 0,\ \sup_{x \in B_{\ell_1}} \left(\sum_{k=1}^{\infty} |\lambda_k x_k^*(x)|^{p_0} \right)^{\frac{1}{p_0}} \leqslant 1 \right\} \\ &= \sup\left\{ \left(\sum_{k=1}^{\infty} \lambda_k^{p_0} \right)^{\frac{1}{p_0}} : \lambda_k \geqslant 0,\ \left(\sum_{k=1}^{\infty} \lambda_k^{p_\theta} \right)^{\frac{1}{p_\theta}} \leqslant 1 \right\} = \infty,\end{aligned}$$

so $f \notin H_{p_0}[\ell_1]^{1-\theta} H_\infty[\ell_1]^{\theta}$. □

As a consequence of the above result and Example 4.2.10 (3) we obtain that $H_{p_0}[\ell_1]^{1-\theta}H_\infty[\ell_1]^\theta$ does not coincide with any of the $H_{p;q}[\ell_1]$ spaces that lie between $H_{p_\theta}[\ell_1]$ and $H_{p_\theta;p_0}[\ell_1]$.

Corollary 4.2.13. *Let $1 \leqslant p_0 < \infty$, $0 < \theta < 1$ and $p_\theta = \frac{p_0}{1-\theta}$. Then, $H_{p_0}[\ell_1]^{1-\theta}H_\infty[\ell_1]^\theta \neq H_{p_\theta;q}[\ell_1]$ for every $p_0 \leqslant q \leqslant p_\theta$.*

4.3 Extrapolation of $H_{p;q}$-spaces

The first thing that we will show in this section is that $H_{p;q}[E]$ coincides isometrically with the extrapolation space obtained by applying the Δ method to the family $(H_{s;t(s)}[E])_{s>p}$, with $\frac{1}{t(s)} := \frac{1}{q} - \frac{1}{p} + \frac{1}{s}$.

Theorem 4.3.1. *Let $1 \leqslant q \leqslant p < \infty$. Then*

$$\|f\|_{H_{p;q}} = \sup_{s>p} \|f\|_{H_{s;t(s)}}$$

for every $f \in H[E]$ and $H_{p;q}[E] = \Delta_{s>p}(H_{s;t(s)}[E])$ isometrically. In particular, $H_p[E] = \Delta_{s>p}(H_s[E])$.

Proof. Let us write

$$|||f|||_{p,q} := \sup_{s>p} \|f\|_{H_{s;t(s)}}$$

and observe that

$$\Delta_{s>p}(H_{s;t(s)}[E]) = \{f \in H[E] : |||f|||_{p,q} < \infty\}.$$

Clearly $\|f\|_{H_{s;t(s)}} \leqslant |||f|||_{p,q} \leqslant \|f\|_{H_{p;q}}$ for every $s > p$ by Proposition 4.1.1, so $H_{p;q}[E] \subseteq \Delta_{s>p}(H_{s;t(s)}[E]) \subseteq H_{s;t(s)}[E]$.

To show that $|||f|||_{p,q} \geqslant \|f\|_{H_{p;q}}$, let us fix $f \in \Delta_{s>p}(H_{s;t(s)}[E])$ and $n \in \mathbb{N}$, and choose functionals $x_1^*, \ldots, x_n^* \in E^*$ such that $\sup_{x\in B_E} \left(\sum_{i=1}^n |x_i^*(x)|^q\right)^{\frac{1}{q}} \leqslant 1$. Then, by Hölder's inequality we have that for every $s > p$ (and hence $t(s) > q$)

$$\left(\sum_{i=1}^n |f(x_i^*)|^p\right)^{\frac{1}{p}} \leqslant n^{\frac{1}{p}-\frac{1}{s}} \left(\sum_{i=1}^n |f(x_i^*)|^s\right)^{\frac{1}{s}} \leqslant n^{\frac{1}{p}-\frac{1}{s}} \|f\|_{H_{s;t(s)}} \sup_{x\in B_E} \left(\sum_{i=1}^n |x_i^*(x)|^{t(s)}\right)^{\frac{1}{t(s)}} \leqslant n^{\frac{1}{p}-\frac{1}{s}} |||f|||_{p,q},$$

so taking $s \to p^+$ we get that

$$\left(\sum_{i=1}^n |f(x_i^*)|^p\right)^{\frac{1}{p}} \leqslant |||f|||_{p,q}.$$

It now follows that $\|f\|_{H_{p;q}} \leqslant |||f|||_{p,q}$, as we wanted to show. □

Note that in general, $H_{p;q}[E] = \Delta_{s>p}(H_{s;t(s)}[E])$ does not coincide with the intersection of all $H_{s;t(s)}[E]$ with $s > p$, as the following example, that generalizes a remark from Timur Oikhberg [101] for $E = \ell_1$, shows.

Theorem 4.3.2. *Let $1 \leqslant q \leqslant p < \infty$ and E be a Banach space with a normalized basis $(e_k)_{k=1}^\infty \subseteq E$. Suppose that the sequence of biorthogonal functionals $(e_k^*)_{k=1}^\infty \subseteq E^*$ is weakly q-summing. Then*

$$H_{p;q}[E] \subsetneq \bigcap_{s>p} H_{s;t(s)}[E].$$

Moreover, when $p = q$,

$$\mathrm{FBL}^p[E] \subsetneq \bigcap_{s>p} \mathrm{FBL}^s[E].$$

Proof. We first show that $H_{p;q}[E] \subsetneq \bigcap_{s>p} H_{s;t(s)}[E]$. Take any $0 \leqslant \omega = (\omega_k)_{k=1}^{\infty} \in (\bigcap_{s>p} \ell_s) \backslash \ell_p$ (for instance, $\omega = (k^{-\frac{1}{p}})_{k=1}^{\infty}$) and let

$$f = \bigvee_{k=1}^{\infty} \omega_k |\delta_{e_k}|.$$

Clearly, f is finite on every $x^* \in E^*$ and positively homogeneous. Let $C = \sup_{x \in B_E} (\sum_{k=1}^{\infty} |e_k^*(x)|^q)^{\frac{1}{q}} \geqslant 1$. It follows that

$$C\|f\|_{H_{p;q}} \geqslant \left(\sum_{k=1}^{n} |f(e_k^*)|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n} |\omega_k e_k^*(e_k)|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n} \omega_k^p \right)^{\frac{1}{p}} \longrightarrow \infty$$

and $f \notin H_{p;q}[E]$.

On the other hand, fix $s > p$ and observe that $f(x^*) = \sup_n f_n(x^*)$ for every $x^* \in E^*$, where $f_n = \bigvee_{k=1}^{n} \omega_k |\delta_{e_k}|$ is an increasing sequence in $\mathrm{FVL}[E]$, and hence, in $H_{s;t(s)}[E]$. It is easy to check that $H_{s;t(s)}[E]$ is s-convex with constant 1, so in particular it satisfies an upper s-estimate, and thus

$$\|f_n\|_{H_{s;t(s)}} \leqslant \left(\sum_{k=1}^{n} \|\omega_k \delta_{e_k}\|_{H_{s;t(s)}}^s \right)^{\frac{1}{s}} = \left(\sum_{k=1}^{n} \omega_k^s \right)^{\frac{1}{s}} \leqslant \|\omega\|_{\ell_s}.$$

Using [15, Lemma 4.1], which can be adapted to hold also in $H_{s;t(s)}[E]$, we conclude that $\|f\|_{H_{s;t(s)}} = \sup_n \|f_n\|_{H_{s;t(s)}} < \infty$, so $f \in H_{s;t(s)}[E]$ for every $s > p$.

Next, assume that $p = q$ and let us show that $\mathrm{FBL}^p[E] \subsetneq \bigcap_{s>p} \mathrm{FBL}^s[E]$. Consider

$$f = |\delta_{e_1}| \wedge \left(\bigvee_{k=2}^{\infty} k^{-\frac{1}{p}} |\delta_{e_k}| \right).$$

Clearly, $f \in H_p[E]$, since $0 \leqslant f \leqslant |\delta_{e_1}| \in H_p[E]$. We claim that $\|f - h\|_{H_p} \geqslant (8C)^{-1}$ for every $h \in \mathrm{FVL}[E]$. Indeed, fix $h \in \mathrm{FVL}[E]$, that can be expressed as

$$h = \bigvee_{i=1}^{m} \delta_{x_i} - \bigvee_{j=1}^{m} \delta_{y_j}$$

for some $(x_i)_i, (y_j)_j \subseteq E$ by (1.1), and observe that, since $(e_k)_{k=1}^{\infty}$ is a base of E and the lattice operations are continuous, there exists $n \in \mathbb{N}$ such that $\|h - g\|_{H_p} < \frac{1}{8C}$, where

$$g = \bigvee_{i=1}^{m} \delta_{x_i^n} - \bigvee_{j=1}^{m} \delta_{y_j^n} \in \mathrm{lat}(\{\delta_{e_1}, \ldots, \delta_{e_n}\})$$

and $(x_i^n)_i, (y_j^n)_j$ are the projections onto $\mathrm{span}\{e_1, \ldots, e_n\}$ of the vectors $(x_i)_i, (y_j)_j$. Now, for every $k = 1, \ldots, n$ we define the functionals

$$x_k^* = \frac{1}{n^{\frac{1}{p}}} e_1^* + e_{n+k}^* \text{ and } y_k^* = \frac{1}{n^{\frac{1}{p}}} e_1^*.$$

It follows by the hypothesis that

$$\sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p^*}} \sup_{x \in B_E} \left(|e_1^*(x)|^p + \sum_{k=1}^{n} |e_{n+k}^*(x)|^p \right)^{\frac{1}{p}} \leqslant 2^{\frac{1}{p^*}} C,$$

and

$$\sup_{x\in B_E}\left(\sum_{k=1}^{n}|y_k^*(x)|^p\right)^{\frac{1}{p}} = \sup_{x\in B_E}|e_1^*(x)| = \|e_1^*\| = 1 \leqslant 2^{\frac{1}{p^*}}C.$$

Substituting we obtain that $f(x_k^*) = (n+k)^{-\frac{1}{p}}$, while $f(y_k^*) = 0$. On the other hand, $g(x_k^*) = g(y_k^*)$, as g only depends on the first n coordinates and x_k^* and y_k^* coincide when restricted to $\operatorname{span}\{e_1,\dots,e_n\}$. Applying these properties we obtain that

$$\begin{aligned}2^{\frac{1}{p^*}}C\|f-g\|_{H_p} &\geqslant \max\left\{\left(\sum_{k=1}^{n}|f(x_k^*)-g(x_k^*)|^p\right)^{\frac{1}{p}},\left(\sum_{k=1}^{n}|f(y_k^*)-g(y_k^*)|^p\right)^{\frac{1}{p}}\right\}\\ &\geqslant \frac{1}{2}\left(\sum_{k=1}^{n}|f(x_k^*)-f(y_k^*)|^p\right)^{\frac{1}{p}} = \frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{n+k}\right)^{\frac{1}{p}} \geqslant \frac{1}{2^{1+\frac{1}{p}}}.\end{aligned}$$

Hence, $\|f-g\|_{H_p} \geqslant \frac{1}{4C}$ and $\|f-h\|_{H_p} \geqslant \frac{1}{8C}$, as we wanted to show.

To check that $f \in \mathrm{FBL}^s[E]$ for every $s > p$, define

$$f_n = |\delta_{e_1}| \wedge \left(\bigvee_{k=2}^{n} k^{-\frac{1}{p}}|\delta_{e_k}|\right) \in \mathrm{FVL}[E]$$

for every $n \geqslant 2$ and note that

$$0 \leqslant f - f_n \leqslant h_n = \bigvee_{k=n+1}^{\infty} k^{-\frac{1}{p}}|\delta_{e_k}|.$$

Arguing as before, we know by [15, Lemma 4.1] that

$$\begin{aligned}\|f-f_n\|_{H_s} \leqslant \|h_n\|_{H_s} &= \sup_{m>n}\left\|\bigvee_{k=n+1}^{m} k^{-\frac{1}{p}}|\delta_{e_k}|\right\|_{H_s}\\ &\leqslant \sup_{m>n}\left(\sum_{k=n+1}^{m}k^{-\frac{s}{p}}\right)^{\frac{1}{s}} = \left(\sum_{k=n+1}^{\infty}k^{-\frac{s}{p}}\right)^{\frac{1}{s}} \longrightarrow 0,\end{aligned}$$

so $f \in \overline{\mathrm{FVL}[E]}^{H_s} = \mathrm{FBL}^s[E]$ for every $s > p$. □

Corollary 4.3.3. *Let $1 \leqslant q \leqslant p < \infty$, F a Banach space, and E a Banach space with a normalized basis $(e_k)_{k=1}^{\infty} \subseteq E$. Suppose that the sequence of biorthogonal functionals $(e_k^*)_{k=1}^{\infty} \subseteq E^*$ is weakly q-summing and there exists a surjective operator from F onto E. Then*

$$H_{p;q}[F] \subsetneq \bigcap_{s>p} H_{s;t(s)}[F].$$

Moreover, when $p = q$,

$$\mathrm{FBL}^p[F] \subsetneq \bigcap_{s>p}\mathrm{FBL}^s[F].$$

Proof. Let us denote by $T : F \to E$ the surjective map onto E and $C = \sup_{x\in B_E}(\sum_{k=1}^{\infty}|e_k^*(x)|^q)^{\frac{1}{q}}$. By the Open Mapping Theorem (cf. [46, Corollary 2.26]), there is some $\delta > 0$ such that $\delta B_E \subseteq T(B_F)$, so let $(z_k)_{k=1}^{\infty} \subseteq \delta^{-1}B_F$ such that $Tz_k = e_k$. Moreover, the sequence $(T^*e_k^*)_{k=1}^{n} \subseteq F^*$ is weakly q-summing for every $n \in \mathbb{N}$:

$$\sup_{y\in B_F}\left(\sum_{k=1}^{\infty}|T^*e_k^*(y)|^q\right)^{\frac{1}{q}} \leqslant \|T\|\sup_{x\in B_E}\left(\sum_{k=1}^{\infty}|e_k^*(x)|^q\right)^{\frac{1}{q}} = \|T\|C.$$

Emulating the argument from the proof of Theorem 4.3.2, we can conclude that

$$f = \bigvee_{k=1}^{\infty} \omega_k |\delta_{z_k}| \in \left(\bigcap_{s>p} H_{s;t(s)}[F] \right) \backslash H_{p;q}[F]$$

for any $0 \leqslant \omega = (\omega_k)_{k=1}^{\infty} \in (\bigcap_{s>p} \ell_s) \backslash \ell_p$.

On the other hand, when $p = q$, we can consider the function

$$f = |\delta_{z_1}| \wedge \left(\bigvee_{k=2}^{\infty} k^{-\frac{1}{p}} |\delta_{z_k}| \right) \in H_p[F]$$

and the lattice homomorphism $\overline{T} = \widehat{\phi_E T} : \mathrm{FBL}^p[F] \to \mathrm{FBL}^p[E]$ that sends every $g \in \mathrm{FBL}^p[F]$ to $\overline{T}g = g \circ T^*$. It is clear that $\overline{T}(\mathrm{FVL}[F]) \subseteq \mathrm{FVL}[E]$. Moreover, in [84, Remark 6.6] the authors explain that $\overline{T}$ can be extended to a lattice homomorphism

$$\begin{aligned} C_{T^*} : H_p[F] &\longrightarrow H_p[E] \\ g &\longmapsto g \circ T^* \end{aligned}$$

with norm $\|C_{T^*}\| = \|T\|$. In particular,

$$C_{T^*} f = |\delta_{e_1}| \wedge \left(\bigvee_{k=2}^{\infty} k^{-\frac{1}{p}} |\delta_{e_k}| \right)$$

is the function that we constructed in the second part of the proof of Theorem 4.3.2 to show that $\mathrm{FBL}^p[E] \subsetneq \bigcap_{s>p} \mathrm{FBL}^s[E]$. It follows that

$$\|T\| \|f - h\|_{H_p[F]} \geqslant \|C_{T^*} f - \overline{T} h\|_{H_p[E]} \geqslant \frac{1}{8C}$$

for every $h \in \mathrm{FVL}[F]$, so $f \notin \mathrm{FBL}^p[F]$. Checking that $f \in \mathrm{FBL}^s[F]$ for every $s > p$ can be done repeating the argument from Theorem 4.3.2. □

Remark 4.3.4. *Note that for $H_{p;q}[F] \subsetneq \bigcap_{s>p} H_{s;t(s)}[F]$ to hold it suffices to assume that there exists an isomorphic embedding of E^* into F^* (it does not need to be an adjoint operator). Indeed, the adjoint of this embedding will induce an onto map from F^{**} onto E^{**}, so we can choose uniformly bounded $(z_k^{**})_{k=1}^{\infty} \subseteq F^{**}$ whose images are the elements $(e_k)_{k=1}^{\infty} \subseteq E$. Since F^{**} always embeds isometrically into $H_{p;q}[F]$ for every $1 \leqslant q \leqslant p \leqslant \infty$ (see [84, Lemma 5.1] for a proof in the case $p = q$), we can construct f as above so that it belongs to every $H_{s;t(s)}[F]$ for $s > p$, but not to $H_{p;q}[F]$.*

The simplest example of a Banach space E satisfying the hypothesis of Theorem 4.3.2 is ℓ_r for $1 \leqslant r \leqslant q$. Therefore, we can apply Corollary 4.3.3 to any Banach space F that has a map onto ℓ_r.

We conclude the chapter applying the previous results to show that all the inclusions between $H_{p;q}$-spaces described in Proposition 4.1.1 are strict when $E = \ell_1$. This contrast with [75, Proposition 7.4], where some $H_{p;q}$-spaces coincide whenever E^* has finite cotype.

Proposition 4.3.5. *Let $(p, q) \in \mathcal{T}$. Then, $H_{p;q}[\ell_1] \subsetneq H_{p';q'}[\ell_1]$ for every $(p', q') \in U(p, q) \backslash \{(p, q)\}$ and $H_{p;q}[\ell_1] \supsetneq H_{p';q'}[\ell_1]$ for every $(p', q') \in L(p, q) \backslash \{(p, q)\}$.*

Proof. Let $(p', q') \in U(p, q) \backslash \{(p, q)\}$. If $p' = \infty$, then $H_{p';q'}[\ell_1] = H_\infty[\ell_1]$. Observe that the norm on E^*, denoted $h = \|\cdot\|$, belongs to $H_\infty[\ell_1]$, but its distance to $H_{p;q}[\ell_1]$ in $H_\infty[\ell_1]$ is bounded below by 1. Indeed, the biorthogonal functionals $(e_k^*)_k$ of the canonical basis of ℓ_1 are weakly q-summing,

so for any $f \in H_{p;q}[\ell_1]$ we have that $\sum_k |f(e_k^*)|^p \leqslant \|f\|^p_{H_{p;q}}$ and hence $f(e_k^*)$ tends to 0. It follows that $\|h - f\|_{H_\infty} \geqslant |h(e_k^*) - f(e_k^*)| \to 1$.

If $p' < \infty$, then either $q' < q$ or $q' \geqslant q$ and hence $p' > p$. In the first case, since ℓ_∞ contains a copy of ℓ_{q*}, we can use Example 4.2.10 (3) to obtain that $H_{p;q}[\ell_1] \subsetneq H_{p;q'}[\ell_1] \subseteq H_{p';q'}[\ell_1]$. In the second case, we use Theorem 4.3.2 to conclude that $H_{p;q}[\ell_1] \subsetneq H_{p';q_0}[\ell_1] \subseteq H_{p';q'}[\ell_1]$, where $\frac{1}{q_0} = \frac{1}{q} - \frac{1}{p} + \frac{1}{p'}$.

The second part of the statement follows interchanging the roles of (p, q) and (p', q'). □

Chapter 5

Stable phase retrieval in Banach lattices

Phase retrieval problems are a family of inverse problems that aim to provide mathematical methods of recovering a real or complex signal f from the measurement of its modulus $|f|$. This last chapter of the dissertation is devoted to the study of these problems in the abstract framework of Banach lattices. It is based on:

[32] M. Camúñez, E. García-Sánchez, D. de Hevia, *A characterization of complex stable phase retrieval in Banach lattices*. Proc. Amer. Math. Soc. (accepted).

[52] E. García-Sánchez, D. de Hevia, M.A. Taylor, *Subspaces of $C(K)$-spaces doing stable phase retrieval*. In preparation.

These works were originated during the supervised research project "Open problems in stable phase retrieval" of the ICMAT-IMAG Doc-course in functional analysis that took place during the month of June of 2023 at ICMAT (Madrid) and IMAG (Granada). During that month, the instructor of the course, Mitchell Taylor, presented some open problems in stable phase retrieval (SPR), such as the characterization of stable phase retrieval in complex Banach lattices, or the extension of some results on the existence of SPR subspaces from the real to the complex setting. Here, we present the solutions to some of these problems.

After introducing the phase retrieval (PR) and stable phase retrieval (SPR) problems and motivating why the abstract framework of Banach lattices is suitable for their study in Section 5.1, we review the existing characterizations of PR and SPR for real Banach lattices and establish the corresponding results for the complex setting (Proposition 5.2.2 and Theorem 5.2.7). It turns out that what causes the failure of complex SPR is the existence of *almost perpendicular pairs* (see Definition 5.2.6), in contrast with the real case, where this role is played by what the authors of [48] call *almost disjoint pairs*.

As the paper [48] illustrates, the theory of Banach lattices has provided very useful tools to tackle SPR-related problems in the setting of functions spaces. For instance, this abstract approach has been successfully applied in L_p-spaces to construct infinite-dimensional subspaces doing stable phase retrieval [30, 36, 48]. In contrast, $C(K)$-spaces, another classical family of function spaces that include in particular L_∞-spaces, have not received attention apart from [48]. Thus, the aim of Sections 5.3 and 5.4 is to continue the study of stable phase retrieval in $C(K)$-spaces initiated in [48]. In Section 5.3, we use a separation lemma (Lemma 5.3.1) to extend results from [48] on the existence of SPR subspaces of ℓ_∞ and $C[0,1]$ to the complex case. Moreover, we apply Theorem 5.2.7 in order to find a sufficient condition for building SPR subspaces of complex $C(K)$-spaces (Lemma 5.3.5), which in turn allows us to generalize to the complex setting the characterization of the compact spaces K such that $C(K)$ contains an infinite-dimensional SPR subspace established in [48, Theorem 6.1]. This equivalence can also be viewed as part of a classical program of research in $C(K)$-spaces initiated in [45] that consists on characterizing Banach space/lattice/al-

gebra properties of $C(K)$ in terms of topological properties of K (see [113, p. 132] for a summary of properties).

Section 5.4 is devoted to [48, Question 6.4], which asks for a quantitative version of the aforementioned characterization. That is, if a compact space K is "large" in the sense of having α-many non-empty Cantor–Bendixson derivatives, where α is an ordinal number, can we embed bigger SPR subspaces inside $C(K)$, namely $C[1, \omega^\alpha]$? We show that this problem reduces to the study of SPR embeddings between spaces of the form $C[1, \omega^\alpha]$ (see Lemma 5.4.2 and Proposition 5.4.4). This simplification allows us to provide a positive answer to [48, Question 6.4] for separable ordinals in both the real and the complex framework, see Corollary 5.4.9 and Corollary 5.4.11 respectively.

Finally, the last section is dedicated to understand whether the existence of SPR subspaces of Banach lattices is preserved under certain standard operations between Banach lattices, such as sums, tensor products or quotients. This comes motivated by the study of SPR subspaces of Banach lattices of the form $C(K, X)$, where K is a compact Hausdorff space and X is a Banach lattice, which have interest in the study of SPR for solutions to certain PDEs, such as the Schrödinger equation. The results contained in this section are preliminary, but they bring up interesting problems that will require further research.

5.1 Motivation

Phase retrieval problems are of particular interest in applied harmonic analysis due to applications in multiple branches of physics and engineering, such as optics, astronomy, quantum mechanics, speech recognition, and, especially, diffraction imaging (see the surveys [64, 72] and the references therein). In all these fields, the measuring process of a certain physical signal entails a certain loss of information that needs to be reconstructed. More specifically, we want to recover an unknown function from the measurements of a certain positive magnitude, such as its modulus or the modulus of its Fourier transform. Note that there will always be a trivial ambiguity when retrieving the phase (or sign) of a function from its modulus, since a function f and λf have the same modulus whenever λ is a scalar of modulus one. Hence, the problem is to uniquely determine (up to this trivial ambiguity) the phase of such target function from the information given by these positive measurements and any other additional restriction that our target is known to satisfy *a priori*. Moreover, the measurements can be influenced by the experimental procedure, that can introduce small errors. Therefore, it becomes necessary to study the stable version of the problem, that is, quantifying the robustness under small perturbations of the phase retrieval procedure.

Since the above problems involve linear, order theoretic and metric notions, the authors of [48] (see also [36, 118]) suggested the framework of Banach lattices, a natural generalization of function spaces, as the most suitable abstract setting for many phase retrieval problems, such as Fourier phase retrieval, Gabor phase retrieval or wavelet phase retrieval. In these examples, the problem consists in uniquely recovering an element f of a Banach space F from $|Tf|$, which is the modulus of the image of f by an invertible operator T from F into a suitable target Banach lattice X. In the case of Fourier phase retrieval, $F = X = L_2$ and T is the Fourier transform, whereas for the Gabor and wavelet phase retrieval, T corresponds to the analysis operator of a certain frame on the Banach space F. These problems can be split into two parts: the first one is to retrieve Tf (up to a global phase) from $|Tf|$ within the subspace $E = T(F) \subseteq X$, and the second one is to invert T, a problem which is much better understood. This motivates the following definition:

Definition 5.1.1 ([48, Definition 3.1])**.** *A subspace E of a (real or complex) Banach lattice X does* phase retrieval (PR) *if whenever $f, g \in E$ satisfy $|f| = |g|$, it follows that $f = \lambda g$ for some unimodular scalar λ.*

Consider the equivalence relation $\sim$ given by $f \sim g$ if and only if $f = \lambda g$ for some unimodular scalar λ. The fact that a subspace $E \subseteq X$ does PR is equivalent to the injectivity, or equivalently, the invertibility, of the map that sends any equivalence class of $E/\sim$ to the absolute value in X of a representative of the class. However, in applications it is also necessary to understand how stable is the inverse map under small perturbations, such as experimental errors. That is, studying if the inverse map has some kind of continuity, and more specifically, Lipschitz continuity:

Definition 5.1.2 ([48, Definition 3.2]). *A subspace E of a (real or complex) Banach lattice X does* C-stable phase retrieval *(C-SPR) if*

$$\min_{|\lambda|=1} \|f - \lambda g\| \leqslant C \, \big\| |f| - |g| \big\| \tag{5.1}$$

for every $f, g \in E$. We will say that E does stable phase retrieval (SPR) *if it does C-SPR for some constant $C \geqslant 1$.*

Note that if a subspace does SPR, its closure does SPR as well. It is also clear that SPR implies PR. By an *isometric* (resp., *isomorphic*) *SPR-embedding* of a Banach space E into a Banach lattice X we will mean a linearly isometric (resp., isomorphic) embedding of E into X such that the image of E does SPR in X. One focus of recent research is to characterize PR and SPR in terms of simpler properties of the subspace or the operator that is being studied. For instance, in the setting of frames, it has been established [2, 17, 18, 29] that real PR is equivalent to the *complement property*, whereas, in the complex setting, the complement property is a necessary condition for PR, but a characterization is not yet available. Analogously, the *strong complement property* is equivalent to SPR in the real case, but it is only a necessary condition in the complex case. In the next section, we will review the problem of characterizing phase retrieval and stable phase retrieval in both real and complex Banach lattices, and prove one of the main results of the chapter, Theorem 5.2.7, that provides a characterization of complex stable phase retrieval.

Before passing to this subject, let us briefly recall some facts about *ordinal numbers* that we will extensively use from Section 5.3 onwards (we refer the reader to standard textbooks such as [76] for a detailed discussion of the topic). Loosely speaking, ordinal numbers are a class of sets that can be linearly ordered by inclusion in such a way that every set of the class is the collection of all the sets that precede it; this family has a *smallest* element: the empty set, which will be denoted by 0. More specifically, ordinal numbers satisfy that $\alpha < \beta$ if and only if α is an element of β, and every ordinal can be expressed as $\alpha = \{\beta : \beta < \alpha\}$.

Given an ordinal α, we can define its *successor ordinal* as $\alpha + 1 := \alpha \cup \{\alpha\}$. If $\alpha \neq 0$ is *not* a successor ordinal, then it is called a *limit ordinal*, and it can be written as $\alpha = \bigcup_{\beta<\alpha} \beta$. Moreover, ordinal numbers admit some arithmetical operations, such as addition, multiplication and exponentiation. Starting at 0, which is the empty set, we can first generate every natural number using the successor operation. Once they have been created, it is time to introduce the first infinite ordinal, $\omega = \mathbb{N}$, which is a limit ordinal. After ω comes $\omega + 1, \omega + 2, \ldots, \omega + n, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega \cdot n, \ldots, \omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega, \ldots, \omega^2 \cdot 2, \ldots, \omega^n, \ldots, \omega^\omega$, and the list keeps going (see [38, Chapter 10] for a light introduction to the topic). The first uncountable ordinal is denoted ω_1. In this chapter we will only work with countable ordinals, so we will not be using ordinals beyond ω_1. Every ordinal $\alpha > 0$ can be represented uniquely as

$$\alpha = w^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n, \tag{5.2}$$

where $n \geqslant 1$, $\alpha \geqslant \beta_1 > \ldots > \beta_n \geqslant 0$ and $k_1, \ldots, k_n$ are non-zero natural numbers. This representation is known as *Cantor's normal form*. We can associate to every ordinal α a totally disconnected compact Hausdorff topological space called the *ordinal interval* $[0, \alpha]$, which can be described as the set $[0, \alpha] = \{\beta : 0 \leqslant \beta \leqslant \alpha\}$ endowed with the topology generated by the base of open sets $\{(\beta_1, \beta_2] : 0 \leqslant \beta_1 < \beta_2 \leqslant \alpha\}$, where $(\beta_1, \beta_2] = \{\beta \in [0, \alpha] : \beta_1 < \beta \leqslant \beta_2\}$. Note that the interval $(\beta_1, \beta_2]$ is a clopen set in this topology for every $\beta_1 < \beta_2$.

5.2 Characterizations of stable phase retrieval in Banach lattices

One of the main goals of the current research on (stable) phase retrieval is to find characterizations of the sets and subspaces doing phase retrieval or stable phase retrieval that may be used to prove further results and develop algorithms for experimental purposes. In many of the situations, the existence of disjointly supported functions (or similar conditions) seems to be the main obstruction for PR (see [34, 35] for some recent examples). Propositions 5.2.1 and 5.2.2 synthesize these ideas into characterizations of PR for general Banach lattices. Before proceeding with the proofs, let us state some facts regarding the functional calculus that will be needed.

Let X be a real Banach lattice. Recall that, given $f, g \in X$, Krivine's functional calculus allows us to define the expression $|fg|^{\frac{1}{2}} = |f|^{\frac{1}{2}}|g|^{\frac{1}{2}}$. When X is a Banach lattice of functions with the usual pointwise or almost everywhere order, then $(|fg|^{\frac{1}{2}})(t) = |f(t)g(t)|^{\frac{1}{2}}$ for every t in the domain. Moreover, the equality

$$|fg|^{\frac{1}{2}} = (|f| \vee |g|)^{\frac{1}{2}}(|f| \wedge |g|)^{\frac{1}{2}} \tag{5.3}$$

holds in $\mathbb{R}$, so it can be transferred to X.

Similarly, if X is a complex Banach lattice, let $\hat{X}$ be the real Banach lattice such that $X = \hat{X} + i\hat{X}$ and let $f = f_1 + if_2, g = g_1 + ig_2 \in X$ with $f_j, g_j \in \hat{X}$, $j = 1, 2$. We can define the conjugate of g as $\overline{g} = g_1 - ig_2$. If X were a Banach lattice of functions, the meaning of the expression $|\,\mathfrak{Re}\,(f\overline{g})\,|^{\frac{1}{2}} \in \hat{X}$ would be clear. For a general complex Banach lattice, we need to define this expression in $\hat{X}$ by means of functional calculus as

$$|\,\mathfrak{Re}\,(f\overline{g})\,|^{\frac{1}{2}} = |f_1g_1 + f_2g_2|^{\frac{1}{2}} = |\,\mathfrak{Re}\,(f)\,\mathfrak{Re}\,(g) + \mathfrak{Im}\,(f)\,\mathfrak{Im}\,(g)\,|^{\frac{1}{2}}.$$

Note that the equalities

$$\left|\,\mathfrak{Re}\left((\lambda f)\overline{(\mu g)}\right)\right|^{\frac{1}{2}} = |\lambda|^{\frac{1}{2}}|\mu|^{\frac{1}{2}}|\,\mathfrak{Re}\,(f\overline{g})\,|^{\frac{1}{2}} \tag{5.4}$$

and

$$2|\,\mathfrak{Re}\,(f\overline{g})\,|^{\frac{1}{2}} = \left||f+g|^2 - |f-g|^2\right|^{\frac{1}{2}} = \big||f+g| + |f-g|\big|^{\frac{1}{2}}\,\big||f+g| - |f-g|\big|^{\frac{1}{2}} \tag{5.5}$$

are true in $\mathbb{R}$, provided $\lambda, \mu \in \mathbb{R}$. Therefore, they also hold in $\hat{X}$.

Now that we have extended the expressions above to the setting of Banach lattices, let us go back to the original question of characterizing when a subspace of a Banach lattice does phase retrieval. In the real setting, the main and only obstruction to phase retrieval turns out to be the existence of disjoint pairs of elements in our subspace. This fact appears implicitly in [48], but here we provide a complete proof.

Proposition 5.2.1 (Real phase retrieval)**.** *Let E be a subspace of a real Banach lattice X. The following are equivalent:*

(1) E fails PR.

(2) There exist non-zero $f, g \in E$ such that $|f| \wedge |g| = 0$.

(3) There exist non-zero $f, g \in E$ such that $|fg|^{\frac{1}{2}} = 0$.

Proof. (1) $\Rightarrow$ (2) Suppose that E fails PR. Then, there exist $f, g \in E$ with $|f| = |g|$ and $f \neq \pm g$. In particular, $f - g$ and $f + g$ are non-zero elements of E, and using some of the identities stated in [96, Theorem 1.1.1] we deduce that

$$\begin{aligned}|f+g|\wedge|f-g| =&\big((f+g)\vee(-f-g)\big)\wedge\big((f-g)\vee(-f+g)\big)\\ =&\big((f+g)\wedge(f-g)\big)\vee\big((f+g)\wedge(-f+g)\big)\\ &\vee\big((-f-g)\wedge(f-g)\big)\vee\big((-f-g)\wedge(-f+g)\big)\\ =&(f-|g|)\vee(-|f|+g)\vee(-|f|-g)\vee(-f-|g|)\\ =&\big((f-|g|)\vee(-f-|g|)\big)\vee\big((-|f|+g)\vee(-|f|-g)\big)\\ =&(|f|-|g|)\vee(-|f|+|g|)=\big||f|-|g|\big|=0.\end{aligned}$$

(2) $\Rightarrow$ (1) Let $f,g\in E$ be two disjoint vectors. Then, in particular, they are linearly independent and so are $f+g$ and $f-g$. Now taking into account that f and g are disjoint we get that $|f+g|=|f|+|g|=|f-g|$, so E fails PR.

(2) $\Leftrightarrow$ (3) Let $f,g\in E$, and consider the principal ideal generated by $h=|f|+|g|$ in X, X_h, which can be identified with some $C(K)$ by means of a bijective lattice homomorphism $J:X_h\to C(K)$ [96, Theorem 2.1.3]. On one hand, if f and g are disjoint in X, so are Jf and Jg in $C(K)$, where being disjoint means having disjoint supports. Hence, $Jf\cdot Jg=0$ and, in particular, $|Jf\cdot Jg|^{\frac{1}{2}}=0$. Since J is bijective and the functional calculus is preserved by lattice homomorphisms, we recover that $|fg|^{\frac{1}{2}}=0$. Similarly, if $|fg|^{\frac{1}{2}}=0$, then $|Jf\cdot Jg|^{\frac{1}{2}}=0$, so Jf and Jg must have disjoint supports and thus f and g are disjoint in X. □

The complex case is slightly more involved, as disjointness is not the only obstruction for phase retrieval. Indeed, if a subspace E of a complex Banach lattice $X=\hat{X}+i\hat{X}$ contains two real and linearly independent elements $f,g\in E\cap\hat{X}$, then E automatically fails phase retrieval, as it contains the elements $f+ig$ and $f-ig$, which are linearly independent but have the same modulus. However, E might not contain disjoint pairs (take for instance $E=\operatorname{span}\{f,g\}$ with f and g non-disjoint satisfying the previous conditions). In view of this, it seems that it is not condition (2) from Proposition 5.2.1 but rather (3) that should be used in order to get a generalization to the complex setting.

Proposition 5.2.2 (Complex phase retrieval)**.** *Let E be a subspace of a complex Banach lattice X. The following are equivalent:*

(1) E fails PR.

(2) There exist linearly independent $f,g\in E$ such that $|f-g|=|f+g|$.

(3) There exist linearly independent $f,g\in E$ such that $|\mathfrak{Re}\,(f\overline{g})\,|^{\frac{1}{2}}=0$.

(4) There exist linearly independent $f,g\in E$ such that $|f+g|=(|f|^2+|g|^2)^{\frac{1}{2}}$.

Proof. (1) $\Leftrightarrow$ (2) E fails PR if and only if there exist $u,v\in E$ linearly independent such that $|u|=|v|$. Considering the linearly independent combinations $f=\frac{u+v}{2}$ and $g=\frac{u-v}{2}$ we can write $u=f+g$ and $v=f-g$, so conditions (1) and (2) are equivalent.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Assume that $X=\hat{X}+i\hat{X}$ and let $f,g\in E$, $h=|f|+|g|\in\hat{X}$ and $\hat{X}_h$ be the principal ideal generated by h in $\hat{X}$. Then, by Kakutani's representation Theorem [96, Theorem 2.1.3] we can identify $\hat{X}_h$ with some real $C(K;\mathbb{R})$, as we did in Proposition 5.2.1, and $X_h=\hat{X}_h+i\hat{X}_h$ with $C(K;\mathbb{C})$. Therefore, we can assume without loss of generality that $f,g\in C(K;\mathbb{C})$. It is easy to check that

$$|f+g|^2=(f+g)(\bar{f}+\bar{g})=|f|^2+|g|^2+2\,\mathfrak{Re}\,(f\overline{g})$$

and

$$|f-g|^2=(f-g)(\bar{f}-\bar{g})=|f|^2+|g|^2-2\,\mathfrak{Re}\,(f\overline{g})\,,$$

so (2), (3) and (4) are clearly equivalent on $C(K;\mathbb{C})$. The uniqueness of functional calculus allows us to bring this equivalence back to the complex Banach lattice X. □

To illustrate the intuition behind condition (3), let a and b be complex numbers such that $\mathfrak{Re}\left(a\bar{b}\right) = 0$. This implies that $\arg(a) = \arg(b) \pm \frac{\pi}{2}$, that is, they are perpendicular when viewed as vectors in $\mathbb{C} = \mathbb{R}^2$. Similarly, if f and g are functions on a domain Ω with complex values, the condition $\mathfrak{Re}\,(f\bar{g}) = 0$ implies that at each point $\omega \in \Omega$, the values $f(\omega)$ and $g(\omega)$ are perpendicular in $\mathbb{C} = \mathbb{R}^2$. In analogy with this, the pairs f, g in a Banach lattice X satisfying $|\,\mathfrak{Re}\,(f\bar{g})\,|^{\frac{1}{2}} = 0$ will be called *perpendicular pairs*.

Note that phase retrieval is a purely algebraic property, so Propositions 5.2.1 and 5.2.2 could be stated for vector lattices, as the norm does not play any role (the only requirement is that they admit a functional calculus, see [85] for a characterization of this property). In contrast, for stable phase retrieval the situation is not as straightforward. The authors of [48] provided a full characterization of real stable phase retrieval by changing in Proposition 5.2.1 the notion of disjointness by that of almost disjointness. Given $\varepsilon > 0$, we say that $f, g \in S_E$ is a (normalized) *ε-almost disjoint pair* if $\||f| \wedge |g|\| \leqslant \varepsilon$. Almost disjoint pairs were used to stablish a quantitative characterization of real SPR in [48, Theorem 3.4 and Remark 3.5] (see the equivalence (1) ⇔ (2) and the additional statement below), which was later refined in [26, Proposition 3.4]. It turns out that an equivalent condition can be obtained looking at products (in the functional calculus sense) instead of infima.

Theorem 5.2.3 (Real stable phase retrieval)**.** *Let E be a subspace of a real Banach lattice X. The following are equivalent:*

(1) E fails SPR.

(2) For every $\varepsilon > 0$, E contains an ε-almost disjoint pair.

(3) For every $\varepsilon > 0$, there exist $f, g \in S_E$ such that $\||fg|^{\frac{1}{2}}\| < \varepsilon$.

Moreover, E does C-SPR if and only if E does not contain any $\frac{1}{C}$-almost disjoint pair.

Proof. (1) ⇔ (2) As we already mentioned, the last part of the statement was established in [48, Theorem 3.4 and Remark 3.5], and later improved in [26, Proposition 3.4]. In particular, this yields the equivalence between (1) and (2).

(2) ⇒ (3) For every $\varepsilon > 0$, take $f, g \in S_E$ such that $\||f| \wedge |g|\| < \varepsilon$. By equation (5.3) and [88, Proposition 1.d.2(i)] we get that

$$\||fg|^{\frac{1}{2}}\| \leqslant \||f| \vee |g|\|^{\frac{1}{2}} \||f| \wedge |g|\|^{\frac{1}{2}} \leqslant {}^{1}\sqrt{2\varepsilon}.$$

(3) ⇒ (2) It is clear from equation (5.3) that $|f| \wedge |g| \leqslant |fg|^{\frac{1}{2}}$, so

$$\||f| \wedge |g|\| \leqslant \||fg|^{\frac{1}{2}}\| \leqslant \varepsilon.$$ □

Note that (2) ⇒ (1) holds in the complex setting as well. More specifically, if a subspace E of a complex Banach lattice contains a normalized ε-almost disjoint pair, then it fails SPR with constant $\frac{1}{2\sqrt{2\varepsilon}}$. This is probably a known fact in the community, but the proof is not as straightforward as in the real case. For the sake of completeness, we reproduce in Proposition 5.2.5 an argument provided by Jesús Illescas-Fiorito in his Master's thesis [70, Remark II.3.3]. The proof is based on [48, Theorem 3.10], whose statement we recall:

[1] Here we are using that $\||f| \vee |g|\| \leqslant \|f\| + \|g\| \leqslant 2$. Note that if X is an AM-space, then the factor $\sqrt{2}$ can be removed in this inequality.

Theorem 5.2.4. *Let $(X, \|\cdot\|)$ be a (real or complex) Banach lattice. Fix linearly independent $f, g \in X$, and suppose that $Y = \operatorname{span}\{f, g\}$ is equipped with a Hilbert space norm $\|\cdot\|_H$ which is K-equivalent to $\|\cdot\|$, with $1 \leqslant K \leqslant \sqrt{2}$. Then there exist $f', g' \in Y$ so that*

$$\min_{|\lambda|=1} \|f - \lambda g\| \leqslant K \min_{|\lambda|=1} \|f' - \lambda g'\|$$

, and

$$(\|f'\|^2 + \|g'\|^2)^{\frac{1}{2}} \leqslant K \min_{|\lambda|=1} \|f' - \lambda g'\|,$$

and

$$\big\| |f'| - |g'| \big\| \leqslant \big\| |f| - |g| \big\|.$$

Theorem 5.2.4 establishes that SPR is witnessed on "almost orthogonal" vectors, meaning that if f and g satisfy the SPR inequality (5.1), then we can find f' and g' in $\operatorname{span}\{f, g\}$ that tighten the inequality (up to a constant $K \leqslant \sqrt{2}$) and moreover $\min_{|\lambda|=1} \|f' - \lambda g'\|$ is bounded below by $K^{-1}(\|f'\|^2 + \|g'\|^2)^{\frac{1}{2}}$. In particular, this allows us to assume that both terms in the SPR inequality are large. Moreover, in the proof of [48, Theorem 3.10] we can find a method for obtaining such f' and g' from f and g, that we will use to show:

Proposition 5.2.5. *Let E be a subspace of a complex Banach lattice X and $\varepsilon > 0$. If there exist $u, v \in S_E$ such that $\||u| \wedge |v|\| < \varepsilon$, then E fails $\frac{1}{\sqrt{2}\varepsilon}$-SPR.*

Proof. Given $u, v \in S_E$ such that $\||u| \wedge |v|\| < \varepsilon$, fix a Hilbert norm $\|\cdot\|_H$ on $\operatorname{span}\{u, v\}$ induced by a scalar product $\langle \cdot, \cdot \rangle$ such that $\|\cdot\| \leqslant \|\cdot\|_H \leqslant \sqrt{2}\|\cdot\|$, and assume without loss of generality that $\|u\|_H \geqslant \|v\|_H$. Let $\mu = \frac{\langle u,v \rangle}{|\langle u,v \rangle|}$ (if $\langle u, v \rangle = 0$ take $\mu = 1$ instead) and denote $f = u + \mu v$ and $g = u - \mu v$. Observe that $\langle f, g \rangle = \|u\|_H^2 - \|v\|_H^2 \geqslant 0$, which implies that

$$\min_{|\lambda|=1} \|f - \lambda g\|_H = \|f - g\|_H.$$

By the proof of Theorem 5.2.4, there exists some $R \in [0, \frac{1}{2}]$ such that $f' = f - R(f + g)$ and $g' = g - R(f + g)$ satisfy that

$$\big\| |f'| - |g'| \big\| \leqslant \big\| |f| - |g| \big\|$$

and $\langle f', g' \rangle = 0$, so

$$\min_{|\lambda|=1} \|f' - \lambda g'\|_H = \|f' - g'\|_H.$$

In particular, the fact that $f' - g' = f - g = 2\mu v$ implies that

$$\min_{|\lambda|=1} \|f' - \lambda g'\| \geqslant \frac{1}{\sqrt{2}} \min_{|\lambda|=1} \|f' - \lambda g'\|_H = \frac{1}{\sqrt{2}} \|f' - g'\|_H = \frac{2}{\sqrt{2}} \|v\|_H \geqslant \frac{2}{\sqrt{2}} \|v\| = \frac{2}{\sqrt{2}}.$$

On the other hand, observe that $|f + g| = 2\,|u|$ and $|f - g| = 2\,|v|$, so

$$\big\| |f| - |g| \big\| \leqslant |f + g| \wedge |f - g| \leqslant 2\,|u| \wedge |v|.$$

Putting everything together, we conclude that

$$\| |f'| - |g'| \| \leqslant \| |f| - |g| \| < 2\varepsilon \leqslant \sqrt{2}\varepsilon \min_{|\lambda|=1} \|f' - \lambda g'\|,$$

so E fails $\frac{1}{\sqrt{2}\varepsilon}$-SPR. □

Using again the orthogonal reduction argument provided in Theorem 5.2.4, we can obtain the main result of the paper: a characterization of stable phase retrieval in the complex setting. In a similar way to what happens in the real case, where the restriction of not containing disjoint pairs of vectors in Proposition 5.2.1 became the property of not containing almost disjoint pairs, in the complex, case condition (3) in Proposition 5.2.2 will also have a *stable analogue*:

Definition 5.2.6. *Let X be a complex Banach lattice and $\varepsilon > 0$. We say that two elements $u, v \in X$ form a* (normalized) ε-almost perpendicular pair *if $\|u\| = \|v\| = 1$ and $\| |\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}} \| < \varepsilon$.*

Theorem 5.2.7 states that a subspace of a complex Banach lattice fails SPR if and only if it contains uniformly separated ε-almost perpendicular pairs for every $\varepsilon > 0$.

Theorem 5.2.7 (Complex stable phase retrieval)**.** *Let E be a subspace of a complex Banach lattice X. The following are equivalent:*

(1) E fails SPR.

(2) For every $0 < m < \frac{1}{\sqrt{2}} - \frac{1}{2}$ and $\varepsilon > 0$, there exist $u, v \in S_E$ such that $\min_{|\lambda|=1} \|u - \lambda v\| \geqslant m$ and $\| |\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}} \| < \varepsilon$.

(3) There exists $m > 0$ satisfying that for every $\varepsilon > 0$, there exist $u, v \in S_E$ such that $\min_{|\lambda|=1} \|u - \lambda v\| \geqslant m$ and $\| |\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}} \| < \varepsilon$.

Proof. (1) $\Rightarrow$ (2) Let us fix $0 < m < \frac{1}{\sqrt{2}} - \frac{1}{2}$ and $\varepsilon > 0$. E fails SPR, so given any $C \geqslant 1$ there exist linearly independent $f, g \in E$ such that

$$\min_{|\lambda|=1} \|f - \lambda g\| > C \| |f| - |g| \|.$$

Following the proof of Theorem 5.2.4 we can find a pair of linearly independent vectors $f', g' \in Y = \operatorname{span}\{f, g\}$ such that $\|g'\| \leqslant \|f'\| = 1$,

$$\min_{|\lambda|=1} \|f' - \lambda g'\| > \frac{C}{\sqrt{2}} \| |f'| - |g'| \|$$

and

$$\min_{|\lambda|=1} \|f' - \lambda g'\| \geqslant \frac{1}{\sqrt{2}} \left(\|f'\|^2 + \|g'\|^2 \right)^{\frac{1}{2}}.$$

Moreover, there exists an Euclidean norm $\|\cdot\|_H$ on Y such that $\|h\| \leqslant \|h\|_H \leqslant \sqrt{2}\|h\|$ for every $h \in Y$ and $\langle f', g' \rangle = 0$. Note that, in particular,

$$0 \leqslant \|f'\| - \|g'\| \leqslant \| |f'| - |g'| \| \leqslant \frac{\sqrt{2}}{C} \min_{|\lambda|=1} \|f' - \lambda g'\| \leqslant \frac{2\sqrt{2}}{C}.$$

Let $0 < \theta < 1$ be a parameter depending only m whose value will be established later in the proof, and assume that $C > \frac{2\sqrt{2}}{\theta}$. Then,

$$1 = \|f'\| \geqslant \|g'\| \geqslant \|f'\| - \frac{2\sqrt{2}}{C} \geqslant 1 - \theta$$

and

$$\min_{|\lambda|=1} \|f' - \lambda g'\| \geqslant \frac{1}{\sqrt{2}} \left(1 + (1-\theta)^2 \right)^{\frac{1}{2}}.$$

Now, let us define $u' = \frac{f'+g'}{2}$ and $v' = \frac{f'-g'}{2}$. Clearly, they are linearly independent, since they generate $f' = u' + v'$ and $g' = u' - v'$. Moreover, by equation (5.5) and [88, Proposition 1.d.2(i)] it follows that

$$\left\| |\operatorname{\mathfrak{Re}}(u'\overline{v'})|^{\frac{1}{2}} \right\| = \frac{1}{2} \left\| \left| |f'| + |g'| \right|^{\frac{1}{2}} \left| |f'| - |g'| \right|^{\frac{1}{2}} \right\| \leqslant \frac{1}{2} \| |f'| + |g'| \|^{\frac{1}{2}} \| |f'| - |g'| \|^{\frac{1}{2}} \leqslant \frac{2^{\frac{1}{4}}}{\sqrt{C}}. \tag{5.6}$$

On the other hand, for every unimodular $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
2\|u'-\lambda v'\|^2 &\geqslant \|u'-\lambda v'\|_H^2 \\
&= \frac{1}{4}\|(1-\lambda)f'+(1+\lambda)g'\|_H^2 \\
&= \frac{1}{4}\left(|1-\lambda|^2\,\|f'\|_H^2+|1+\lambda|^2\,\|g'\|_H^2\right) \\
&= \frac{1}{2}\left((1-\mathfrak{Re}\lambda)\|f'\|_H^2+(1+\mathfrak{Re}\lambda)\|g'\|_H^2\right) \\
&\geqslant \min\{\|f'\|_H^2,\|g'\|_H^2\} \geqslant \min\{\|f'\|^2,\|g'\|^2\} = \|g'\|^2 \geqslant (1-\theta)^2,
\end{aligned}$$

and thus

$$\min_{|\lambda|=1}\|u'-\lambda v'\| \geqslant \frac{1-\theta}{\sqrt{2}}.$$

Finally, note that

$$\frac{1}{2\sqrt{2}}\left(1+(1-\theta)^2\right)^{\frac{1}{2}} \leqslant \frac{1}{2}\min_{|\lambda|=1}\|f'-\lambda g'\| \leqslant \|u'\| \leqslant \frac{1}{2}(\|f'\|+\|g'\|) \leqslant 1,$$

and the same holds for $\|v'\|$. Define $u=\frac{u'}{\|u'\|}$ and $v=\frac{v'}{\|v'\|}$. Then, u and v are normalized vectors, and they satisfy that

$$\begin{aligned}
\|u-\lambda v\| &\geqslant \frac{1}{\|u'\|}\left(\|u'-\lambda v'\|-\left\|\lambda\left(1-\frac{\|u'\|}{\|v'\|}\right)v'\right\|\right) \\
&\geqslant \min_{|\lambda|=1}\|u'-\lambda v'\|-\big|\|u'\|-\|v'\|\big| \\
&\geqslant \frac{1-\theta}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\left(1+(1-\theta)^2\right)^{\frac{1}{2}}-1
\end{aligned}$$

for every unimodular scalar λ. Choosing $\theta\in(0,1)$ so that

$$m=\frac{1-\theta}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\left(1+(1-\theta)^2\right)^{\frac{1}{2}}-1$$

(this is always possible, since $0<m<\frac{1}{\sqrt{2}}-\frac{1}{2}$), and making

$$C>\max\left\{\frac{8\sqrt{2}}{(1+(1-\theta)^2)\varepsilon^2},\frac{2\sqrt{2}}{\theta}\right\},$$

we can use (5.4) and (5.6) to conclude that

$$\|\,|\,\mathfrak{Re}\,(u\overline{v})\,|^{\frac{1}{2}}\| = \frac{1}{\|u'\|^{\frac{1}{2}}\|v'\|^{\frac{1}{2}}}\|\,|\,\mathfrak{Re}\left(u'\overline{v'}\right)|^{\frac{1}{2}}\| \leqslant \left(\frac{8\sqrt{2}}{(1+(1-\theta)^2)C}\right)^{\frac{1}{2}} < \varepsilon.$$

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) We want to show that the SPR condition fails in E for any constant $C>0$, that is, for every C there is a pair of linearly independent elements $f,g\in E$ such that

$$\min_{|\lambda|=1}\|f-\lambda g\| > C\||f|-|g|\,\|.$$

Let us fix a constant $C>0$ and find $m>0$ satisfying the hypothesis. Then, we take $0<\varepsilon<\frac{\delta m}{2C}$, where $\delta>0$ is a parameter depending only on m that will be fixed later, and $u,v\in E$ as in the

statement. Let $f = u + v$ and $g = u - v$, which are clearly linearly independent. It follows using functional calculus and (5.5) that

$$\big||f| - |g|\big| \leqslant \big||f| - |g|\big|^{\frac{1}{2}} \big||f| + |g|\big|^{\frac{1}{2}} = 2|\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}}, \tag{5.7}$$

so $\||f| - |g|\| \leqslant 2\|\,|\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}}\| \leqslant 2\varepsilon$.

Now, let us provide a lower bound for the expressions $\|f - \lambda g\|$, where λ is a unimodular scalar. To do so, note that $|1-\lambda|^2 + |1+\lambda|^2 = 4$, so

$$\|f - \lambda g\| = \|(1-\lambda)u + (1+\lambda)v\| \geqslant 2M,$$

where

$$M = \inf\Big\{\|\alpha u + \beta v\| : |\alpha|^2 + |\beta|^2 = 1\Big\}.$$

In order to bound M from below, we select $\delta = (1 + (1 + \frac{m}{2})^2)^{-\frac{1}{2}} > 0$, so that $\frac{\sqrt{1-\delta^2}}{\delta} = 1 + \frac{m}{2}$, and we distinguish three cases:

- If $|\alpha| \geqslant \sqrt{1-\delta^2}$ (equivalently, $|\beta| \leqslant \delta$), then

$$\|\alpha u + \beta v\| \geqslant \|\alpha u\| - \|\beta v\| = |\alpha| - |\beta| \geqslant \sqrt{1-\delta^2} - \delta = \frac{\delta m}{2}.$$

- If $|\alpha| \leqslant \delta$ (equivalently, $|\beta| \geqslant \sqrt{1-\delta^2}$), then

$$\|\alpha u + \beta v\| \geqslant \|\beta v\| - \|\alpha u\| = |\beta| - |\alpha| \geqslant \sqrt{1-\delta^2} - \delta = \frac{\delta m}{2}.$$

- If $\delta \leqslant |\alpha| \leqslant \sqrt{1-\delta^2}$ (equivalently, $\delta \leqslant |\beta| \leqslant \sqrt{1-\delta^2}$), then $\frac{\delta}{\sqrt{1-\delta^2}} \leqslant r = \frac{|\beta|}{|\alpha|} \leqslant \frac{\sqrt{1-\delta^2}}{\delta}$ and $|r-1| \leqslant \frac{m}{2}$. Writing $\frac{\beta}{\alpha} = re^{i\varphi}$ we get

$$\|\alpha u + \beta v\| = |\alpha|\,\|u + e^{i\varphi}v + (r-1)e^{i\varphi}v\| \geqslant |\alpha|\Big(\|u + e^{i\varphi}v\| - |r-1|\Big) \geqslant \frac{\delta m}{2}.$$

We conclude that $M \geqslant \frac{\delta m}{2}$, so we have found a pair of linearly independent vectors f and g in E such that

$$C\||f| - |g|\| \leqslant 2\varepsilon C < \delta m \leqslant 2M \leqslant \min_{|\lambda|=1} \|f - \lambda g\|$$

and the proof is concluded. □

Remark 5.2.8. *Note that the restriction* $\min_{|\lambda|=1}\|u-\lambda v\| \geqslant m > 0$ *cannot be dropped in conditions* (2) *and* (3) *of Theorem 5.2.7. Indeed, there exists a two dimensional subspace of* $\mathbb{C}^4$ *that does stable phase retrieval, but for every* $\varepsilon > 0$ *it contains a pair of linearly independent vectors* $u, v \in S_E$ *such that* $\|\,|\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}}\| < \varepsilon$.

Proof. Let $u = (1,1,1,0), w = (1,i,0,1) \in \ell_\infty^4$ (over $\mathbb{C}$) and $E = \operatorname{span}\{u, w\}$. Given $0 < \delta < 1$, let $v = A(iu + \delta w) = A(i+\delta, (1+\delta)i, i, \delta)$, where $A = (1+\delta)^{-1}$. It is clear that, for every δ, u and v are normalized and linearly independent. Moreover, $|\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}} = (\sqrt{A\delta}, 0, 0, 0)$, so for every $\varepsilon > 0$ we can choose δ such that $1 - A = A\delta = \frac{\delta}{1+\delta} < \varepsilon^2$, so that $\|\,|\operatorname{\mathfrak{Re}}(u\overline{v})|^{\frac{1}{2}}\|_\infty < \varepsilon$. However, we can observe that

$$\min_{|\lambda|=1}\|u - \lambda v\|_\infty \leqslant \left\|u - \frac{i+1}{i-1}v\right\|_\infty = \frac{1}{\sqrt{2}}\|(i-1)(1-A)u - (i+1)A\delta w\|_\infty \leqslant 2A\delta < 2\varepsilon^2,$$

so when ε is small the pair u, v does not satisfy the separation restriction of Theorem 5.2.7.

Next, let us show that E does SPR. To do so, it suffices to prove that E does PR, since by [48, Corollary 3.14] PR and SPR are equivalent for finite-dimensional subspaces. By Proposition 5.2.2, we need to check that for every pair of linearly independent vectors $f, g \in E$, the expression $|\operatorname{\mathfrak{Re}}(f\overline{g})|^{\frac{1}{2}}$ is non-zero. Assume the contrary: there exist two linearly independent vectors $f = a_1 u + b_1 w$, $g = a_2 u + b_2 w \in E$ such that $|\operatorname{\mathfrak{Re}}(f\overline{g})|^{\frac{1}{2}} = 0$. Since ℓ_∞^4 is a space of sequences, the functional calculus coincides with the usual functional calculus in $\mathbb{R}$ coordinatewise, so, in particular, we can write $\operatorname{\mathfrak{Re}}(f\overline{g}) = 0$. A simple computation shows that

$$\begin{cases} 0 = (\operatorname{\mathfrak{Re}}(a_1) + \operatorname{\mathfrak{Re}}(b_1))(\operatorname{\mathfrak{Re}}(a_2) + \operatorname{\mathfrak{Re}}(b_2)) + (\operatorname{\mathfrak{Im}}(a_1) + \operatorname{\mathfrak{Im}}(b_1))(\operatorname{\mathfrak{Im}}(a_2) + \operatorname{\mathfrak{Im}}(b_2)), \\ 0 = (\operatorname{\mathfrak{Re}}(a_1) - \operatorname{\mathfrak{Im}}(b_1))(\operatorname{\mathfrak{Re}}(a_2) - \operatorname{\mathfrak{Im}}(b_2)) + (\operatorname{\mathfrak{Im}}(a_1) + \operatorname{\mathfrak{Re}}(b_1))(\operatorname{\mathfrak{Im}}(a_2) + \operatorname{\mathfrak{Re}}(b_2)), \\ 0 = \operatorname{\mathfrak{Re}}(a_1)\operatorname{\mathfrak{Re}}(a_2) + \operatorname{\mathfrak{Im}}(a_1)\operatorname{\mathfrak{Im}}(a_2), \\ 0 = \operatorname{\mathfrak{Re}}(b_1)\operatorname{\mathfrak{Re}}(b_2) + \operatorname{\mathfrak{Im}}(b_1)\operatorname{\mathfrak{Im}}(b_2). \end{cases}$$

Without loss of generality, we can assume that $a_1 = 1$, so that the third equation implies that $\operatorname{\mathfrak{Re}}(a_2) = 0$, i.e., $a_2 = \alpha i$ for some $\alpha \in \mathbb{R}$. Substituting the information from the third and fourth equations in the other two equations we obtain

$$\begin{cases} 0 = \alpha \operatorname{\mathfrak{Im}}(b_1) + \operatorname{\mathfrak{Re}}(b_2), \\ 0 = \alpha \operatorname{\mathfrak{Re}}(b_1) - \operatorname{\mathfrak{Im}}(b_2), \end{cases}$$

so, in particular,

$$a_1 b_2 - a_2 b_1 = \alpha \operatorname{\mathfrak{Im}}(b_1) + \operatorname{\mathfrak{Re}}(b_2) - i(\alpha \operatorname{\mathfrak{Re}}(b_1) - \operatorname{\mathfrak{Im}}(b_2)) = 0.$$

However, by our assumption f and g are linearly independent, so $a_1 b_2 - a_2 b_1 \neq 0$, which leads to a contradiction. Therefore, $\operatorname{\mathfrak{Re}}(f\overline{g}) \neq 0$, so E does PR, and hence SPR. □

5.3 Existence of SPR subspaces in complex $C(K)$-spaces

In this section we extend to the complex setting some results from [48] concerning the SPR subspaces of $C(K)$-spaces. Unless stated otherwise, throughout this section all the spaces considered will be complex Banach spaces or lattices. We begin by proving a separation lemma.

Lemma 5.3.1. *Let E be a (real or complex) Banach space and $0 < m < 2$. Then for every $x \in S_E$ and $y \in B_E$ satisfying $\min_{|\lambda|=1} \|x - \lambda y\| \geqslant m$ there exists $x^* \in S_{E^*}$ such that $\big||x^*(x)| - |x^*(y)|\big| \geqslant \frac{m}{3}$.*

Proof. We distinguish two cases. If $\|y\| \leqslant 1 - \frac{m}{3}$, then we choose $x^* \in S_{E^*}$ to be a norming functional for x, so that $x^*(x) = 1$ and $|x^*(y)| \leqslant \|x^*\|\|y\| \leqslant 1 - \frac{m}{3}$. Otherwise, if $\|y\| > 1 - \frac{m}{3}$, then the distance of y to the span of x, denoted $\mathbb{F}x$ (here, $\mathbb{F}$ denotes the scalar field), must be bounded below. Indeed, given $\mu \in \mathbb{F}$, we can have one of the following three situations:

- If $|\mu| > 1 + \frac{m}{3}$,
$$\|y - \mu x\| \geqslant \|\mu x\| - \|y\| \geqslant \frac{m}{3}.$$
- If $|\mu| < 1 - \frac{2m}{3}$,
$$\|y - \mu x\| \geqslant \|y\| - \|\mu x\| \geqslant \frac{m}{3}.$$

◦ If $1-\frac{2m}{3} \leqslant |\mu| \leqslant 1+\frac{m}{3}$, then $\big|1-|\mu|\big| \leqslant \frac{2m}{3}$ and

$$\|y-\mu x\| \geqslant \left\|y-\frac{\mu}{|\mu|}x\right\| - \left\|\left(\frac{\mu}{|\mu|}-\mu\right)x\right\| \geqslant m-\frac{2m}{3} = \frac{m}{3}.$$

Thus, $d(y,\mathbb{F}x) \geqslant \frac{m}{3}$. Using Hahn–Banach Theorem, we can find a functional $x^* \in S_{E^*}$ such that $x^*(x)=0$ and $x^*(y)=d(y,\mathbb{F}x)$, so the proof is concluded. □

The next result generalizes [48, Lemma 4.4] to the complex setting following a similar argument.

Lemma 5.3.2. *For any cardinal α there is an isometric 12-SPR embedding of $\ell_\infty(\alpha)$ into itself.*

Proof. Let us denote by $E=\ell_\infty(\alpha)$ and $E_*=\ell_1(\alpha)$. Let $(z_i)_{i\in I}$ be a dense set in S_{E_*} with $|I|=\alpha$, and define the operator $J: E \to \ell_\infty(I)$ by $Jx=(x(z_i))_{i\in I}$. Using Goldstine's Theorem and the density of $(z_i)_{i\in I}$ it can be checked that J is an isometric embedding. Let us show that $J(E)$ does SPR in $\ell_\infty(I)$, i.e., that there is some $C>0$ such that for every $x,y\in E$,

$$\min_{|\lambda|=1}\|Jx-\lambda Jy\| \leqslant C\||Jx|-|Jy|\|.$$

Without loss of generality we can assume that $x\in S_E$, $y\in B_E$ and $\min_{|\lambda|=1}\|x-\lambda y\| \geqslant \frac{1}{\sqrt{2}}$. Indeed, suppose the inequality above is stablished (with some constant C') for x,y satisfying these conditions. Then, given any $f,g\in E$ we can apply Theorem 5.2.4 to obtain f',g' satisfying the properties listed in the statement. In particular, denoting $x=\frac{f'}{\|f'\|}$ and $y=\frac{g'}{\|f'\|}$, we get that

$$\begin{aligned}\min_{|\lambda|=1}\|Jf-\lambda Jg\| &\leqslant \sqrt{2}\min_{|\lambda|=1}\|Jf'-\lambda Jg'\| = \sqrt{2}\|f'\|\min_{|\lambda|=1}\|Jx-\lambda Jy\| \\ &\leqslant \sqrt{2}\|f'\|C'\||Jx|-|Jy|\| = \sqrt{2}C'\big\||Jf'|-|Jg'|\big\| \leqslant \sqrt{2}C'\||Jf|-|Jg|\|,\end{aligned}$$

so $J(E)$ does SPR with constant $C=\sqrt{2}C'$.

Now, if $x\in S_E$, $y\in B_E$ and $\min_{|\lambda|=1}\|x-\lambda y\| \geqslant \frac{1}{\sqrt{2}}$, we can apply Lemma 5.3.1 to obtain $x^*\in E^*$ such that $\big||x^*(x)|-|x^*(y)|\big| \geqslant \frac{1}{3\sqrt{2}}$. Using again Goldstine's Theorem and the density of $(z_i)_{i\in I}$ we can find for every $\varepsilon>0$ some z_i such that $|x^*(x)-x(z_i)|<\varepsilon$ and $|x^*(y)-y(z_i)|<\varepsilon$, and thus $\big||x(z_i)|-|y(z_i)|\big| \geqslant \frac{1}{3\sqrt{2}}-2\varepsilon$. It follows that

$$\||Jx|-|Jy|\| \geqslant \big||x(z_i)|-|y(z_i)|\big| \geqslant \frac{1}{3\sqrt{2}}-2\varepsilon$$

for every $\varepsilon>0$, so

$$6\sqrt{2}\||Jx|-|Jy|\| \geqslant 2 \geqslant \min_{|\lambda|=1}\|Jx-\lambda Jy\|$$

and we can take $C'=6\sqrt{2}$ (and $C=12$). □

Remark 5.3.3. *The same argument can be used to show that any E embeds isometrically into $C(B_{E^*})$ as a 12-SPR subspace by means of the cannonical embedding that sends every $x\in E$ to the evaluation function $x^*\mapsto x^*(x)$.*

As one could expect, [48, Proposition 6.2] can also be generalized to the complex setting using Lemma 5.3.1. We include the proof for the sake of completeness.

Proposition 5.3.4. *Every separable Banach space embeds isometrically into $C(\Delta)$, and into $C[0,1]$, as a 12-SPR subspace (here Δ is the Cantor set).*

Proof. Given a separable Banach space E, we denote $K = (B_{E^*}, w^*)$ and $j : E \to C(K)$ the isometric embedding introduced in Remark 5.3.3. Recall that $\||jx| - |jy|\| \geqslant \frac{1}{3\sqrt{2}}$ for every $x \in S_E$, $y \in B_E$ such that $\min_{|\lambda|=1} \|x - \lambda y\| \geqslant \frac{1}{\sqrt{2}}$. Since E is separable, K is a metrizable compact, so [79, Theorem 4.18] yields a continuous surjection $h : \Delta \to K$ that defines a composition operator $H : C(K) \to C(\Delta)$, which is in particular a lattice isometric embedding. Now, since $\Delta \subseteq [0,1]$, [3, Theorem 4.4.4] provides an extension operator $T : C(\Delta) \to C[0,1]$ such that $Tf|_\Delta = f$, $T\mathbb{1} = \mathbb{1}$, $\|T\| = 1$ and $Tf \geqslant 0$ for every $f \geqslant 0$ (it is straightforward to check that this result is also valid in the complex setting). In particular, T is an isometric embedding. Moreover, given $x \in S_E$, $y \in B_E$ such that $\min_{|\lambda|=1} \|x - \lambda y\| \geqslant \frac{1}{\sqrt{2}}$, Lemma 5.3.1 and the surjectivity of h implies that there is some $t \in \Delta$ such that

$$\begin{aligned}\frac{1}{3\sqrt{2}} &\leqslant \||jx| - |jy|\| = \big||jx(h(t))| - |jy(h(t))|\big| = \big||(h(t))(x)| - |(h(t))(y)|\big| \\ &= \big||Hjx(t)| - |Hjy(t)|\big| = \big||THjx(t)| - |THjy(t)|\big| \leqslant \||THjx| - |THjy|\|.\end{aligned}$$

Repeating the argument used in Lemma 5.3.2 we conclude that $THj(E)$ does 12-SPR in $C[0,1]$ (and in $C(\Delta)$). □

Next, we show a technical lemma that will be the base of the main results of the paper. Inspired by the idea from [48, Theorem 6.1] of embedding c_0 into a $C(K)$-space by creating overlapping points for any pair of natural numbers, we use the characterization of complex SPR from Theorem 5.2.7 to establish that if an embedding between complex $C(K)$-spaces satisfies a certain overlapping property, then it will do complex SPR.

Lemma 5.3.5. *Let K, L be compact Hausdorff spaces, and assume that the field of scalars is $\mathbb{C}$. Suppose that $T : C(L) \to C(K)$ is a contraction that satisfies that for every pair $(s,t) \in L \times L$ there are $r_{s,t}, \widetilde{r}_{s,t} \in K$ such that $Tf(r_{s,t}) = \frac{1}{2}(f(s) + f(t))$ and $Tf(\widetilde{r}_{s,t}) = \frac{1}{2}(f(s) + if(t))$. Then, T is an isometric SPR embedding.*

Remark 5.3.6. *It should be noted that the proof works with a weaker assumption, namely, that for every point $s \in L$ there exists $r_s \in K$ such that $Tf(r_s) = f(s)$ and for every pair of distinct points $s, t \in L$ there are $r_{s,t}, \widetilde{r}_{s,t} \in K$ such that $Tf(r_{s,t}) = \frac{1}{2}(f(s) + f(t))$ and $Tf(\widetilde{r}_{s,t})$ is either $\frac{1}{2}(f(s) + if(t))$ or $\frac{1}{2}(f(t) + if(s))$.*

Proof of Lemma 5.3.5. The fact that T is an isometry follows from the fact that T is contractive and $Tf(r_{s,s}) = f(s)$ for every $s \in L$. We claim that $T(C(L))$ does complex SPR. By Theorem 5.2.7 (choosing $m = \frac{1}{5}$), this is equivalent to showing that there exists some $\delta > 0$ such that if $f, g \in S_{C(L)}$ satisfy $\min_{|\lambda|=1} \|f - \lambda g\| \geqslant \frac{1}{5}$, then $\|\,|\mathfrak{Re}\left(Tf\overline{Tg}\right)|^{\frac{1}{2}}\| \geqslant \delta$, or equivalently, $\|\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant \delta^2$. Let us fix f and g as above, and put $f(s) = A_s e^{i\theta_s}$ and $g(s) = B_s e^{i\sigma_s}$ for every $s \in L$. Lastly, let $c > 0$ be a small parameter whose value will be established later. It will be useful for the rest of the proof to write now the explicit values of $\mathfrak{Re}\left(Tf\overline{Tg}\right)$ at the points $r_{s,t}$ and $\widetilde{r}_{s,t}$ for $s, t \in L$:

$$\begin{aligned}\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,t}) = \frac{1}{4}[&A_sB_s\cos(\theta_s - \sigma_s) + A_tB_t\cos(\theta_t - \sigma_t) \\ &+A_tB_s\cos(\theta_t - \sigma_s) + A_sB_t\cos(\theta_s - \sigma_t)],\end{aligned}$$

$$\begin{aligned}\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,t}) = \frac{1}{4}[&A_sB_s\cos(\theta_s - \sigma_s) + A_tB_t\cos(\theta_t - \sigma_t) \\ &-A_tB_s\sin(\theta_t - \sigma_s) + A_sB_t\sin(\theta_s - \sigma_t)].\end{aligned}$$

Since both f and g have norm one, there exist some $s, t \in L$ such that $A_s = 1 = B_t$. We distinguish two main cases:

Case A: $A_t > \frac{9}{10}$ or $B_s > \frac{9}{10}$ (without loss of generality, assume the latter).

- If $|\cos(\theta_s - \sigma_s)| \geqslant c$, then

$$\|\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)\| \geqslant |\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)(r_{s,s})| = A_s B_s |\cos(\theta_s - \sigma_s)| \geqslant \frac{9}{10}c.$$

- If $|\cos(\theta_s - \sigma_s)| < c$, then we use the fact that $\min_{|\lambda|=1}\|f - \lambda g\| \geqslant \frac{1}{5}$ to find some $u \in L$ such that

$$\frac{1}{5} \leqslant |f(u) - e^{i(\theta_s - \sigma_s)}g(u)| \leqslant A_u + B_u. \tag{5.8}$$

  Note that this u must be different from s, since $|f(s) - e^{i(\theta_s-\sigma_s)}g(s)| = |A_s - B_s| = |1 - B_s| < \frac{1}{10}$.

  - If $B_u < c$, then necessarily $A_u \geqslant \frac{1}{5} - c$. Since either $|\cos(\theta_u - \sigma_s)|$ or $|\sin(\theta_u - \sigma_s)|$ is greater or equal than $\frac{1}{\sqrt{2}}$, evaluating at $r_{s,u}$ or $\widetilde{r}_{s,u}$, respectively, and applying the triangular inequality, we get that

$$4\|\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)\| \geqslant \frac{9}{10\sqrt{2}}A_u - 3c \geqslant \frac{9}{50\sqrt{2}} - \left(3 + \frac{9}{10\sqrt{2}}\right)c > 0.$$

  - If $A_u < c$, we proceed analogously.
  - If $A_u, B_u \geqslant c$, we distinguish two cases again:
    - If $|\cos(\theta_u - \sigma_u)| \geqslant c$, then

$$\|\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)\| \geqslant |\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)(r_{u,u})| = A_u B_u |\cos(\theta_u - \sigma_u)| \geqslant c^3.$$

    - Otherwise, we are in a situation where $A_u, B_u \geqslant c$ and $|\cos(\theta_s - \sigma_s)|, |\cos(\theta_u - \sigma_u)| < c$. This case requires more analysis than the previous ones, so we treat it separately:

Let $\Delta = (\theta_s - \sigma_s) + (\theta_u - \sigma_u) = (\theta_s - \sigma_u) + (\theta_u - \sigma_s)$. Using trigonometric relations, it is clear that $|\cos\Delta| \geqslant 1 - 2c^2$ and $|\sin\Delta| \leqslant 2c$. Moreover, we can write

$$\cos(\theta_s - \sigma_u) = \cos\Delta\cos(\theta_u - \sigma_s) + \sin\Delta\sin(\theta_u - \sigma_s)$$

and

$$\sin(\theta_s - \sigma_u) = \sin\Delta\cos(\theta_u - \sigma_s) - \cos\Delta\sin(\theta_u - \sigma_s).$$

When $\cos\Delta > 0$, we have that $0 \leqslant 1 - \cos\Delta \leqslant 2c^2$, so

$$\begin{aligned} |\cos(\theta_s - \sigma_u) - \cos(\theta_u - \sigma_s)| &\leqslant 4c, \\ |\sin(\theta_s - \sigma_u) + \sin(\theta_u - \sigma_s)| &\leqslant 4c; \end{aligned} \tag{5.9}$$

whereas when $\cos\Delta < 0$, it follows that $0 \leqslant 1 + \cos\Delta \leqslant 2c^2$, so we get

$$\begin{aligned} |\cos(\theta_s - \sigma_u) + \cos(\theta_u - \sigma_s)| &\leqslant 4c, \\ |\sin(\theta_s - \sigma_u) - \sin(\theta_u - \sigma_s)| &\leqslant 4c. \end{aligned} \tag{5.10}$$

According to this dichotomy, we have that:

- If $\cos\Delta > 0$, then evaluating at $r_{s,u}$ and using (5.8), (5.9) and $B_s > \frac{9}{10}$ we obtain that

$$\begin{aligned} 4|\operatorname{\mathfrak{Re}}\left(Tf\overline{Tg}\right)(r_{s,u})| &\geqslant |B_u\cos(\theta_s - \sigma_u) + A_u B_s\cos(\theta_u - \sigma_s)| - 2c \\ &\geqslant (A_u + B_u)|\cos(\theta_s - \sigma_u)| - A_u(1 - B_s)|\cos(\theta_u - \sigma_s)| \\ &\quad - A_u|\cos(\theta_u - \sigma_s) - \cos(\theta_s - \sigma_u)| - 2c \\ &\geqslant \frac{1}{5}|\cos(\theta_s - \sigma_u)| - \frac{1}{10} - 6c, \end{aligned}$$

while at $\widetilde{r}_{s,u}$ we get

$$\begin{aligned}4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,u})| &\geqslant |B_u\sin(\theta_s-\sigma_u)-A_uB_s\sin(\theta_u-\sigma_s)|-2c\\ &\geqslant (A_u+B_u)|\sin(\theta_s-\sigma_u)|-A_u(1-B_s)|\sin(\theta_u-\sigma_s)|\\ &\quad -A_u|\sin(\theta_u-\sigma_s)+\sin(\theta_s-\sigma_u)|-2c\\ &\geqslant \frac{1}{5}|\sin(\theta_s-\sigma_u)|-\frac{1}{10}-6c.\end{aligned}$$

Therefore,

$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\|\geqslant\frac{1}{5}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)-6c>0.$$

- If $\cos\Delta<0$, then again by the trigonometric relations it follows that

$$\cos((\theta_s-\sigma_s)-(\theta_u-\sigma_u))=\cos(\Delta-2(\theta_u-\sigma_u))\geqslant 1-6c^2. \tag{5.11}$$

Then, we can split one last time in two cases:

  - If $|A_u-B_u|\geqslant\frac{\sqrt{24}}{25}$, we can evaluate at $r_{s,u}$ and $\widetilde{r}_{s,u}$ and apply (5.10) to obtain

$$\begin{aligned}4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,u})| &\geqslant |B_u\cos(\theta_s-\sigma_u)+A_uB_s\cos(\theta_u-\sigma_s)|-2c\\ &\geqslant |A_u-B_u||\cos(\theta_s-\sigma_u)|-A_u(1-B_s)|\cos(\theta_u-\sigma_s)|\\ &\quad -A_u|\cos(\theta_u-\sigma_s)+\cos(\theta_s-\sigma_u)|-2c\\ &\geqslant \frac{\sqrt{24}}{25}|\cos(\theta_s-\sigma_u)|-\frac{1}{10}-6c,\end{aligned}$$

and

$$\begin{aligned}4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,u})| &\geqslant |B_u\sin(\theta_s-\sigma_u)-A_uB_s\sin(\theta_u-\sigma_s)|-2c\\ &\geqslant |A_u-B_u||\sin(\theta_s-\sigma_u)|-A_u(1-B_s)|\sin(\theta_u-\sigma_s)|\\ &\quad -A_u|\sin(\theta_u-\sigma_s)-\sin(\theta_s-\sigma_u)|-2c\\ &\geqslant \frac{\sqrt{24}}{25}|\sin(\theta_s-\sigma_u)|-\frac{1}{10}-6c,\end{aligned}$$

respectively. Again, this implies that

$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\|\geqslant\frac{1}{5}\left(\frac{\sqrt{12}}{5}-\frac{1}{2}\right)-6c>0.$$

  - On the contrary, if $|A_u-B_u|<\frac{\sqrt{24}}{25}$, by (5.8) and (5.11) we have

$$\begin{aligned}\frac{1}{25}&\leqslant |A_u-B_ue^{i((\theta_s-\sigma_s)-(\theta_u-\sigma_u))}|^2\\ &=A_u^2+B_u^2-2A_uB_u\cos((\theta_s-\sigma_s)-(\theta_u-\sigma_u))\\ &=|A_u-B_u|^2+2A_uB_u[1-\cos((\theta_s-\sigma_s)-(\theta_u-\sigma_u))]\\ &<\frac{24}{25^2}+12c^2<\frac{1}{25},\end{aligned}$$

which becomes a contradiction as long as we choose $c<\frac{1}{25\sqrt{12}}$. Therefore, this situation is not possible, and we have exhausted all the possibilities within Case A.

Note that choosing $0 < c < \frac{1}{30}(\frac{\sqrt{12}}{5} - \frac{1}{2})$ and $0 < \delta^2 \leqslant \min\{c^3, \frac{1}{4}(\frac{1}{5}(\frac{\sqrt{12}}{5} - \frac{1}{2}) - 6c)\}$, all the conditions that have appeared in Case A are satisfied.

Case B: $A_t < \frac{9}{10}$ and $B_s < \frac{9}{10}$ (in particular, $s \neq t$).

- If $B_s < c$, we consider A_t:
  - If $A_t < c$, then
$$4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,t})| \geqslant |\cos(\theta_s - \sigma_t)| - 3c$$
and
$$4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,t})| \geqslant |\sin(\theta_s - \sigma_t)| - 3c,$$
so
$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant \frac{1}{\sqrt{2}} - 3c > 0.$$
  - If $A_t \geqslant c$, we split again:
    - If $|\cos(\theta_t - \sigma_t)| \geqslant c$, then
$$\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant |\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{t,t})| \geqslant c^2.$$
    - If $|\cos(\theta_t - \sigma_t)| < c$, then evaluating again at either $r_{s,t}$ or $\widetilde{r}_{s,t}$ we obtain that
$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant \frac{1}{\sqrt{2}} - 3c > 0.$$
- If $B_s \geqslant c$, we look at $|\cos(\theta_s - \sigma_s)|$:
  - If $|\cos(\theta_s - \sigma_s)| \geqslant c$, it follows that
$$\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant |\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,s})| \geqslant c^2.$$
  - If $|\cos(\theta_s - \sigma_s)| < c$, we split again:
    - If $A_t < c$, evaluating again at the points $r_{s,t}$ and $\widetilde{r}_{s,t}$ yields
$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant \frac{1}{\sqrt{2}} - 3c > 0.$$
    - If $A_t \geqslant c$, then we examine $|\cos(\theta_t - \sigma_t)|$:
      - If $|\cos(\theta_t - \sigma_t)| \geqslant c$, then
$$\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\| \geqslant |\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{t,t})| \geqslant c^2.$$
      - If $|\cos(\theta_t - \sigma_t)| < c$, we arrive to a situation similar to Case A, but not exactly the same, since we might not have an analogue of the condition (5.8). Therefore, to deal with the case $A_t, B_s \geqslant c$ and $|\cos(\theta_s - \sigma_s)|, |\cos(\theta_t - \sigma_t)| < c$, we proceed as follows:

Let $\Gamma = (\theta_s - \sigma_s) + (\theta_t - \sigma_t)$, and observe that $|\cos\Gamma| \geqslant 1 - 2c^2$ and we can obtain again the estimates (5.9) and (5.10) (with the obvious modifications). We now split as in Case A:

- If $\cos\Gamma > 0$, then if we evaluate at $r_{s,t}$ we obtain that

$$\begin{aligned} 4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,t})| &\geqslant |\cos(\theta_s-\sigma_t)+A_tB_s\cos(\theta_t-\sigma_s)|-2c \\ &\geqslant (1+A_tB_s)|\cos(\theta_s-\sigma_t)|-A_tB_s|\cos(\theta_t-\sigma_s)-\cos(\theta_s-\sigma_t)|-2c \\ &\geqslant |\cos(\theta_s-\sigma_t)|-6c, \end{aligned}$$

  while at $\widetilde{r}_{s,t}$ we get

$$\begin{aligned} 4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,t})| &\geqslant |\sin(\theta_s-\sigma_t)-A_tB_s\sin(\theta_t-\sigma_s)|-2c \\ &\geqslant (1+A_tB_s)|\sin(\theta_s-\sigma_t)|-A_tB_s|\sin(\theta_t-\sigma_s)+\sin(\theta_s-\sigma_t)|-2c \\ &\geqslant |\sin(\theta_s-\sigma_t)|-6c, \end{aligned}$$

  so

$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\|\geqslant\frac{1}{\sqrt{2}}-6c>0.$$

- If $\cos\Gamma < 0$, then we use that $A_t, B_s < \frac{9}{10}$, so $1-A_tB_s > \frac{1}{10}$, and we apply the analogue of (5.10) to obtain that

$$\begin{aligned} 4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(r_{s,t})| &\geqslant |\cos(\theta_s-\sigma_t)+A_tB_s\cos(\theta_t-\sigma_s)|-2c \\ &\geqslant (1-A_tB_s)|\cos(\theta_s-\sigma_t)|-A_tB_s|\cos(\theta_t-\sigma_s)+\cos(\theta_s-\sigma_t)|-2c \\ &\geqslant \frac{1}{10}|\cos(\theta_s-\sigma_t)|-6c, \end{aligned}$$

  and

$$\begin{aligned} 4|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)(\widetilde{r}_{s,t})| &\geqslant |\sin(\theta_s-\sigma_t)-A_tB_s\sin(\theta_t-\sigma_s)|-2c \\ &\geqslant (1-A_tB_s)|\sin(\theta_s-\sigma_t)|-A_tB_s|\sin(\theta_t-\sigma_s)-\sin(\theta_s-\sigma_t)|-2c \\ &\geqslant \frac{1}{10}|\sin(\theta_s-\sigma_t)|-6c, \end{aligned}$$

  and therefore

$$4\|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)\|\geqslant\frac{1}{10\sqrt{2}}-6c>0.$$

Observe that the Case B has provided new restrictions for c and δ, but it can be checked that the choice of parameters made in Case A satisfies this new conditions. Hence, we have shown that there is a $\delta > 0$ such that $\|\,|\,\mathfrak{Re}\left(Tf\overline{Tg}\right)|^{\frac{1}{2}}\|\geqslant\delta$ whenever $f,g\in S_{C(L)}$ satisfy $\min_{|\lambda|=1}\|f-\lambda g\|\geqslant\frac{1}{5}$, so the proof is concluded. □

The final part of the section is devoted to characterizing by means of Lemma 5.3.5 when a complex $C(K)$-space contains an infinite-dimensional subspace doing SPR, generalizing [48, Theorem 6.1], where the real case was considered. From now on, by *SPR subspace* we mean an infinite-dimensional SPR subspace, unless stated otherwise. Our objective is to show the following characterization:

Theorem 5.3.7. *Suppose K is a compact Hausdorff space. Then $C(K;\mathbb{C})$ has an SPR subspace if and only if K' is infinite.*

Recall that K' is the set of accumulation points of K, also known as the *derived set* of K. This operation over the set K can be iterated α times for any ordinal α, yielding the *α-th Cantor–Bendixon derivative of K*, denoted $K^{(\alpha)}$. The real case was already proved in [48, Theorem 6.1], and later expanded in [26, Theorem 5.3]. Here, we dissect the proof from [48], which contains some claims that have interest on their own, and adapt it to the complex setting. For instance, the necessary condition consists of a standard gliding hump argument that holds in a general context:

Proposition 5.3.8. *Let X be a (real or complex) Banach lattice that contains an infinite-dimensional SPR subspace. Then, the closed span of the atoms of X has infinite codimension in X.*

Recall that an *atom* of a (real) Banach lattice X is a non-zero positive element $x_0 \in X$ such that whenever $|x| \leqslant x_0$, it follows that $x = \lambda x_0$ for some scalar λ. The span of an atom is always a projection band [83, p. 8]. Its corresponding band projection $P_{x_0} : X \to X$ is given by $P_{x_0}x = \lambda_{x_0}(x)x_0$, where λ_{x_0} is a lattice homomorphism from X to the scalar field. This lattice homomorphism is called the *coordinate functional* associated to x_0.

Proof of Proposition 5.3.8. Let $E \subseteq X$ be an infinite-dimensional SPR subspace, and assume that X_0, the closed span of the atoms of X, has finite codimension. It follows that $E_0 = E \cap X_0$ has infinite dimension. Let $x \in S_{E_0}$ and $\varepsilon > 0$. There exists a linear combination of atoms $y = \sum_{i=1}^{n} \lambda_{x_i}(y)x_i$, where $(x_i)_{i=1}^n$ are atoms and $(\lambda_{x_i})_{i=1}^n$ their corresponding coordinate functionals, such that $\|x - y\| < \varepsilon$. Again, $E_0 \cap \left(\bigcap_{i=1}^n \ker \lambda_{x_i}\right)$ must be infinite-dimensional, so in particular we can find a norm-one vector z in this subspace. Note that z is disjoint from y (i.e., supported on different atoms). Hence,

$$|x| \wedge |z| \leqslant \big(|x - y| + |y|\big) \wedge |z| \leqslant |x - y| \wedge |z| + |y| \wedge |z| = |x - y| \wedge |z| \leqslant |x - y|,$$

and x and z form a normalized ε-almost disjoint pair in E. Since ε was arbitrary, Proposition 5.2.5 yields a contradiction. □

Next, we show that $C_0[1, \omega^2)$, that can be identified with the closed sublattice of $C[1, \omega^2]$ of functions that vanish at ω^2, contains an isometric copy of c_0 doing SPR. The construction for the complex case is just an adaptation of the real one, extracted from [48], so that it satisfies the overlapping condition of Lemma 5.3.5. Throughout the proof, $\mathbb{1}_{m_1}$ will represent the characteristic function of the clopen set $(\omega \cdot (m_1 - 1), \omega \cdot m_1]$ for any natural number $m_1 \geqslant 1$. Analogously, given natural numbers $m_1, m_2 \geqslant 1$, $\mathbb{1}_{m_1,m_2}$ will stand for the characteristic of the singleton $\{\omega \cdot (m_1 - 1) + m_2\}$.

Lemma 5.3.9. *There exists an isometric SPR embedding of c_0 into $C_0[1, \omega^2)$.*

Proof. This time, for each $n \in \mathbb{N}$, we define the expression

$$x^{(n)} = \mathbb{1}_{1,n} + \frac{1}{2}\mathbb{1}_{n+1} + \frac{1}{2}\sum_{m=2}^{n} (\mathbb{1}_{m,2n-1} + i\mathbb{1}_{m,2n})$$

(see Figure 5.1), which is continuous on $[1, \omega^2]$ and vanishes at ω^2.

The sequence $(x^{(n)})_n$ spans an isometric copy of c_0. Indeed, given $(\alpha_n)_n \in c_{00}$ with $\bigvee_n |\alpha_n| = 1$, it follows that $x = \sum_n \alpha_n x^{(n)}$ takes the value α_n at the point n for every $n \in \mathbb{N}$, so $\|x\| \geqslant 1$. On the other hand, observe that x vanishes outside of $[1, \omega \cdot (k + 1)]$ for k big enough. Moreover, the function x takes the values α_n at the point n, 0 at ω, $\frac{1}{2}\alpha_n$ at $[\omega \cdot n + 1, \omega \cdot n + 2n] \cup \{\omega \cdot (n + 1)\}$, and $\frac{1}{2}(\alpha_n + \alpha_m)$ at $\omega \cdot n + 2m - 1$ and $\frac{1}{2}(\alpha_n + i\alpha_m)$ at $\omega \cdot n + 2m$ for every $m > n$ and $n \in \mathbb{N}$. In all the cases above, $|x| \leqslant 1$, so we can conclude that $\|x\| = 1$.

Moreover, it is clear that if we take $r_n = n$ for every natural number $n \in \mathbb{N}$, and $r_{m,n} = \omega \cdot m + 2n - 1$ and $\widetilde{r}_{m,n} = \omega \cdot m + 2n$ for every pair $m < n$, then the relaxed condition from Remark 5.3.6 holds, so Lemma 5.3.5 guarantees that $E = \overline{\text{span}}\{x^{(n)} : n \in \mathbb{N}\}$ does complex SPR (the fact that c_0 is not a $C(K)$-space, but only an AM-space, is not relevant in the proof). □

Finally, we recall the following topological fact (see [100, Theorem 29.2]). We will use this lemma to build a lattice copy of $C_0[1, \omega^2)$ inside any $C(K)$ with $K'' \neq \varnothing$.

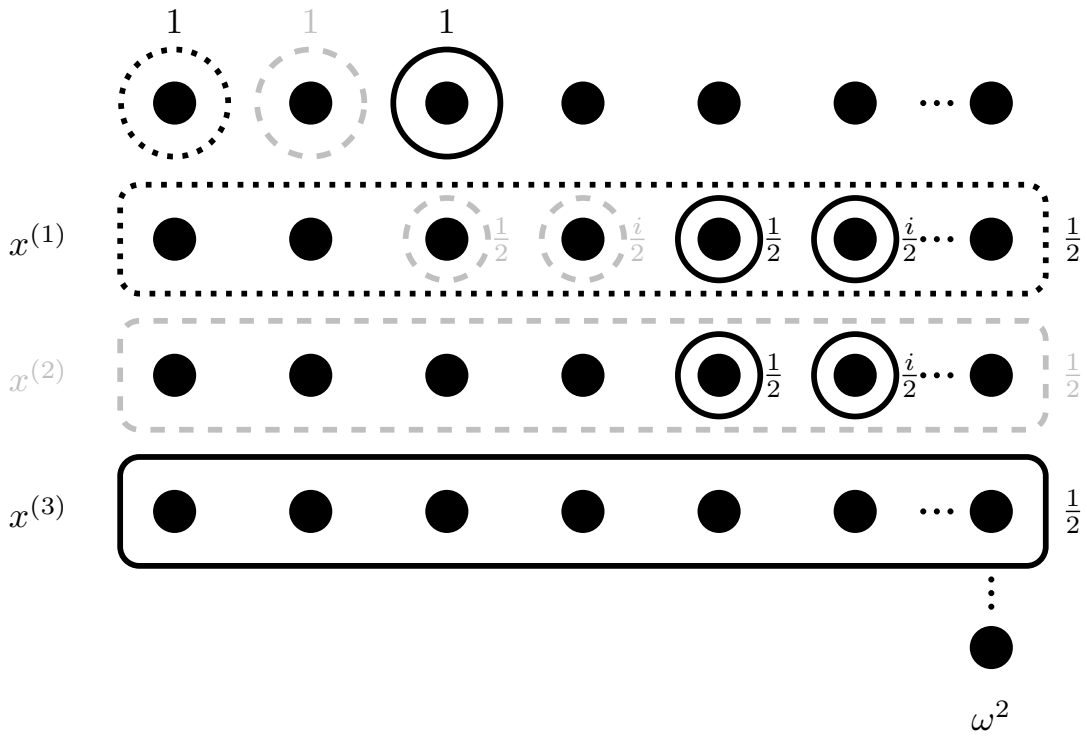


Figure 5.1: Representation of $x^{(n)}$ in the complex setting. The points in $[1, \omega^2]$ are ordered from the left to the right and from the top to the bottom. Each type of line (dotted and black, dashed and gray, solid and black) defines a function $x^{(1)}, x^{(2)}$ and $x^{(3)}$, respectively.

Lemma 5.3.10. *Suppose K is a compact Hausdorff space, and $t \in U \subseteq K$, where U is an open set. Then there exists an open set V so that $t \in V \subseteq \overline{V} \subseteq U$.*

We also recall that a *perfect set* P of a topological space T is a non-empty closed set which has no isolated points, and a topological space T is said to be *scattered* if it does not contain any perfect sets.

Proof of Theorem 5.3.7. The necessary condition follows from Proposition 5.3.8, equivalence $(i) \Leftrightarrow (ii)$ in [26, Theorem 5.3], and the fact that in a $C(K)$-space (or more generally, in an AM-space), the span of the atoms and the order continuous part coincide (see for instance [27, Proposition 5.11]).

To show the sufficient condition, we distinguish two cases. If K contains a perfect set, then by [83, Theorem 2, Section 4, p.29], $C[0,1]$ embeds lattice isometrically into $C(K)$ by means of a composition operator associated to a continuous surjection $\phi : K \to [0,1]$, and $C[0,1]$ contains SPR subspaces by Proposition 5.3.4. On the other hand, if K is scattered but K' is infinite, then we can show the following claim:

Claim: $C_0[1, \omega^2)$ embeds lattice isometrically into $C(K)$.

Indeed, reproducing the argument from [48, Theorem 6.1], it is clear that $K' \backslash K''$ is infinite and every point in $K' \backslash K''$ is an accumulation point of $K \backslash K'$. Therefore, applying Lemma 5.3.10 we can find a sequence of distinct points $(t_k)_{k=1}^{\infty} \subseteq K' \backslash K''$, pairwise disjoint open sets $(U_k)_{k=1}^{\infty}$ of K, and open sets $(V_k)_{k=1}^{\infty}$, such that $t_k \in V_k \subseteq \overline{V_k} \subseteq U_k$ for every $k \in \mathbb{N}$. Since every t_k is an accumulation point of $K \backslash K'$, there exists a sequence of distinct points $(s_{j,k})_{j=1}^{\infty} \subseteq (K \backslash K') \cap V_k$ for every k.

Let $g_k \in C(K)$ be such that $0 \leqslant g_k \leqslant \mathbb{1}$, $g_k|_{\overline{V_k}} = 1$ and $g_k|_{K\setminus U_k} = 0$, and observe that $\chi_{\{s_{j,k}\}}$ is continuous for every j and k, as $s_{j,k}$ are isolated points. Given $f \in C_0[1,\omega^2)$, we define

$$Tf := \sum_{k=1}^{\infty} \left(f(\omega\cdot k)g_k + \sum_{j=1}^{\infty} \big(f(\omega\cdot(k-1)+j) - f(\omega\cdot k)\big)\chi_{\{s_{j,k}\}} \right).$$

We claim that Tf is a continuous function on K with $\|Tf\| = \|f\|$. Indeed, the sequence of partial sums $\sum_{j=1}^{J} \big(f(\omega\cdot(k-1)+j) - f(\omega\cdot k)\big)\chi_{\{s_{j,k}\}}$ is Cauchy, so it converges. Moreover, each of the summands

$$F_k = f(\omega\cdot k)g_k + \sum_{j=1}^{\infty} \big(f(\omega\cdot(k-1)+j) - f(\omega\cdot k)\big)\chi_{\{s_{j,k}\}}$$

is supported on U_k and has norm $\|f|_{(\omega(k-1),\omega k]}\|$. Similarly, $(\sum_{k=1}^{K} F_k)_{K=1}^{\infty}$ is a Cauchy sequence, so it converges in $C(K)$ to what we have called Tf. Clearly, the operator $T : C_0[1,\omega^2) \to C(K)$ that we have defined is a linear isometry. It is also a lattice homomorphism, since for every $f \in C_0[1,\omega^2)$ we have that

$$(T|f|)(s_{j,k}) = |f(\omega\cdot k)| + |f(\omega\cdot(k-1)+j)| - |f(\omega\cdot k)| = |f(\omega\cdot(k-1)+j)| = |Tf|(s_{j,k})$$

for every $j,k \in \mathbb{N}$, $(T|f|)(t) = |f(\omega\cdot k)|g_k(t) = |Tf|(t)$ for every $t \in U_k\setminus\{s_{j,k} : j \in \mathbb{N}\}$ and $k \in \mathbb{N}$, and $T|f| = 0 = |Tf|$ on $K\setminus\bigcup_{k=1}^{\infty} U_k$. Hence, $T|f| = |Tf|$, and T is a lattice isometric embedding, as we wanted to show. Now that the claim is established, Lemma 5.3.9 yields that $C(K)$ contains an SPR copy of c_0. □

5.4 Quantitative SPR embeddings of $C(K)$-spaces

After Theorem 5.3.7 was established in [48] (for real scalars), showing that if K' is infinite then $C(K)$ contains an SPR copy of c_0, the authors asked whether a *quantitative version* of the result was possible:

Question 5.4.1. [48, Question 6.4] If K is "large" enough (in terms of the smallest ordinal α for which $K^{(\alpha)}$ is finite), what SPR subspaces (other than c_0) does $C(K)$ have? Note that c_0 is isomorphic to $c = C[0,\omega]$ (ω is the first infinite ordinal). If $K^{(\alpha)}$ is infinite, does $C(K)$ contain an SPR copy of $C[0,\omega^\alpha]$?

Our aim in this section is to investigate this problem for infinite countable ordinals. It will be convenient for us to work with $C[1,\omega^\alpha]$ instead of $C[0,\omega^\alpha]$ throughout this section in order to obtain a slightly simpler notation when using Cantor's normal forms; note that $[1,\omega^\alpha]$ is homeomorphic to $[0,\omega^\alpha]$ for all $\alpha \geqslant 1$, and so $C[1,\omega^\alpha]$ and $C[0,\omega^\alpha]$ are lattice isometric in this case. We will show the following (see Corollaries 5.4.9 and 5.4.11):

- When $0 < \alpha < \omega_1$ and the field of scalars is $\mathbb{R}$, we have that if $K^{(\alpha)} \neq \varnothing$ and $\beta \in \omega_1$ is such that $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$, then $C(K)$ contains an isometric SPR copy of $C[1,\omega^\gamma]$ for every $\gamma < \omega^\beta$. Moreover, if $\omega^\beta\cdot 2 \leqslant \alpha$, then $C(K)$ contains an isomorphic SPR copy of $C[1,\omega^\alpha]$.
- When $0 < \alpha < \omega_1$ and the field of scalars is $\mathbb{C}$, we have that if $K^{(\alpha)} \neq \varnothing$ and $\beta \in \omega_1$ is such that $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$, then $C(K)$ contains an isometric SPR copy of $C[1,\omega^\gamma]$ for every $\gamma < \omega^\beta$. Moreover, if $|K^{(\omega^\beta\cdot 2)}| \geqslant 2$ (that is, if $\alpha > \omega^\beta\cdot 2$ or $\alpha = \omega^\beta\cdot 2$ and $|K^{(\alpha)}| \geqslant 2$), then $C(K)$ contains an isomorphic SPR copy of $C[1,\omega^\alpha]$.

The first relevant result of the section is Proposition 5.4.4, which generalizes the ideas from the claim in the proof of Theorem 5.3.7 in order to show that if $K^{(\alpha)} \neq \varnothing$, then we can find a *nice*

isometric embedding of $C[1,\omega^\alpha]$ into $C(K)$. By *nice* we mean that, even though the embedding may not respect the lattice operations, it does preserve SPR subspaces (see Lemma 5.4.2 below). This fact allows us to reduce Question 5.4.1 to the study of SPR embeddings between $C[1,\omega^\alpha]$ spaces, which are easier to manipulate. Unless we state the opposite, in every result of this section the field of scalars will be assumed to be $\mathbb{R}$ or $\mathbb{C}$ indistinctly.

Lemma 5.4.2. *Let K, L be compact Hausdorff spaces and $T : C(L) \to C(K)$ an operator with the following property:*

(*) for every $t \in L$ there is $s_t \in K$ such that $Tf(s_t) = f(t)$ for every $f \in C(L)$.

Then, if E is an SPR subspace of $C(L)$, $T(E)$ is an SPR subspace of $C(K)$.

Remark 5.4.3. *Note that if an operator $T : C(L) \to C(K)$ has the property (*) defined above, then $\|Tf\| \geqslant \|f\|$ for every $f \in C(L)$. Therefore, if E is a subspace of $C(L)$, then it is $\|T\|$-isomorphic to $T(E)$; in particular, if $\|T\| = 1$, T is a linear isometry.*

Proof of Lemma 5.4.2. Let us suppose that E is a C-SPR subspace of $C(L)$ for some constant $C \geqslant 1$, that is, for any $f, g \in E$ we have

$$\min_{|\lambda|=1} \|f - \lambda g\| \leqslant C\||f| - |g|\|.$$

Given any $f, g \in C(L)$ and any $t \in L$,

$$\big(|Tf| - |Tg|\big)(s_t) = |Tf(s_t)| - |Tg(s_t)| = |f(t)| - |g(t)| = \big(|f| - |g|\big)(t),$$

and hence $\||f| - |g|\| \leqslant \||Tf| - |Tg|\|$. With this in mind, we obtain for any $f, g \in E$ the following:

$$\min_{|\lambda|=1} \|Tf - \lambda Tg\| \leqslant \|T\| \min_{|\lambda|=1} \|f - \lambda g\| \leqslant C\|T\|\||f| - |g|\| \leqslant C\|T\|\||Tf| - |Tg|\|.$$

This shows that $T(E) \subseteq C(K)$ does $C\|T\|$-SPR. □

It turns out that every compact Hausdorff space with non-empty $K^{(\alpha)} \neq \varnothing$ has to contain an isometric copy of $C[1,\omega^\alpha]$ satisfying property $(*)$. This fact will allow us to simplify Question 5.4.1 substantially.

Proposition 5.4.4. *Let $\alpha < \omega_1$ be a countable ordinal and K a compact Hausdorff space. Then, if $K^{(\alpha)} \neq \varnothing$, there exists a positive linear isometry $T : C[1,\omega^\alpha] \to C(K)$ that preserves the unit, that is, $T\mathbb{1} = \mathbb{1}$, and has the property (*), i.e., for every $t \in [1,\omega^\alpha]$ there is $s_t \in K$ such that $Tf(s_t) = f(t)$ for every $f \in C[1,\omega^\alpha]$.*

Proof. First, we will show by induction on $\alpha < \omega_1$ that if $K^{(\alpha)} \neq \varnothing$ and V is an open subset of K such that $V \cap K^{(\alpha)} \neq \varnothing$, then there exists a positive linear isometry $S : C_0[1,\omega^\alpha) \to C_0(V)$ with property $(*)$, i.e., for every $t \in [1,\omega^\alpha]$ there exists $s_t \in V$ such that $Sf(s_t) = f(t)$ for every $f \in C_0[1,\omega^\alpha)$ (when $t = \omega^\alpha$, this should be read as $Sf(s_t) = 0$). As usual, $C_0(V)$ stands for

$$\big\{f \in C(V) \,:\, \forall \varepsilon > 0\ \exists\, L \subseteq V \text{ compact s.t. } |f(s)| < \varepsilon\ \forall s \notin L\big\},$$

so, in particular, $C_0[1,\omega^\alpha)$ can be identified with the space of continuous functions on $[1,\omega^\alpha]$ vanishing at ω^α. Note that if $f \in C_0(V)$, then it can be extended to K by simply defining $\widetilde{f}(t) = f(t)$ for $t \in V$ and $\widetilde{f}(t) = 0$ for $t \in K\backslash V$. Indeed, it is clear that $\widetilde{f}$ is continuous on the open subsets V and $K\backslash\overline{V}$. Now, fix any $s_0 \in \overline{V}\backslash V$ and $\varepsilon > 0$. By definition, there is a compact subset $L \subseteq V$ such that $|f(t)| < \varepsilon$ for all $s \in V\backslash L$. Thus, $K\backslash L$ is an open neighborhood of s_0 such that $|\widetilde{f}(t)| < \varepsilon$ whenever $t \in K\backslash L$.

If $\alpha = 0$, our assertion is trivial. Let us assume that it is true for $\alpha < \omega_1$, and suppose that $K^{(\alpha+1)} \cap V \neq \varnothing$, for some open subset $V \subseteq K$. Fix any $s_0 \in K^{(\alpha+1)} \cap V$. Given that $K^{(\alpha+1)} = (K^{(\alpha)})'$ by definition, and using the normality of K, it is easy to construct a sequence of pairwise disjoint open subsets $V_n \subseteq V$ such that for every $n \geqslant 1$ there exists $s_n \in V_n \cap K^{(\alpha)}$ and $s_0 \notin V_n$. By Lemma 5.3.10, we can further assume that $\overline{V_n} \subseteq V$. For every natural number n, we can use Lemma 5.3.10 again to find an open set U_n such that $s_n \in U_n \subseteq \overline{U_n} \subseteq V_n$.

Let us consider $C_0(U_n)$. Since $[1, \omega^\alpha]$ may be identified with $(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$, by the induction hypothesis, given any natural number n there exists a positive linear isometry $S_n : C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n) \to C_0(U_n)$ with property $(*)$. As we have mentioned before, any function in $C_0(U_n)$ may be extended by zero to $C_0(V)$. We will write $\widetilde{S_n} f := \widetilde{S_n(f)}$, so that $\widetilde{S_n}$ is a linear isometry from $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n)$ into $C_0(V)$. Note that for any $f \in C[1, \omega^\alpha]$, the function $f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)}$ belongs to $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n)$.

By Urysohn's lemma, for every $n \geqslant 1$ there is a continuous function $g_n : K \to [0,1]$ such that $g_n|_{\overline{U_n}} = 1$ and $g_n|_{K \setminus V_n} = 0$. Note that $g_n|_V \in C_0(V)$, since $\overline{V_n} \subseteq V$. For simplicity, it will be convenient for us to assume that the domain of each g_n is V.

Finally, given $f \in C_0[1, \omega^{\alpha+1})$, we will define Sf as

$$Sf := \sum_{n=1}^{\infty} \Big(f(\omega^\alpha \cdot n) g_n + \widetilde{S_n}\big(f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)} \big) \Big).$$

For S to be well defined, we need to show that the sequence of partial sums converges. Once this is established, it will be clear that S is a linear operator. Let $f \in C_0[1, \omega^{\alpha+1})$, and write

$$h_n = f(\omega^\alpha \cdot n) g_n + \widetilde{S_n}\big(f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)} \big). \tag{5.12}$$

We have that $\|h_n\|_\infty \leqslant 3 \sup_{t \in (\omega^\alpha(n-1), \omega^\alpha n]} |f(t)|$, and the summands $(h_n)_n$ are pairwise disjoint. Therefore, if $n < m$,

$$\Big\| \sum_{k=n}^{m} h_k \Big\|_\infty = \Big\| \bigvee_{k=n}^{m} h_k \Big\|_\infty = \bigvee_{k=n}^{m} \|h_k\|_\infty \leqslant 3 \sup_{t \in (\omega^\alpha(n-1), \omega^\alpha m]} |f(t)| \xrightarrow{n,m \to \infty} 0,$$

so the sequence of partial sums is Cauchy and converges in $C_0(V)$, and Sf is well defined for every $f \in C_0[1, \omega^{\alpha+1})$. Next, let us see that S is positive. Let f be a positive function in $C_0[1, \omega^{\alpha+1})$. It is enough to show that for any natural number n the summand h_n defined in (5.12) is positive. If $f(\omega^\alpha \cdot n) = 0$, the term h_n is positive, as $\widetilde{S_n}$ is positive on $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n)$. Suppose now that $f(\omega^\alpha \cdot n) > 0$ and write

$$g = f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)} \in C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n).$$

We decompose $g = g^+ - g^-$, and keeping in mind that f is positive, we deduce that $\|g^-\|_\infty \leqslant f(\omega^\alpha \cdot n)$. Since $\widetilde{S_n}$ is a norm-preserving operator which sends g^- to a function whose support is contained in U_n, then $f(\omega^\alpha \cdot n) g_n \geqslant \widetilde{S_n}(g^-)$. Now, observe that $h_n = \big(f(\omega^\alpha \cdot n) g_n - \widetilde{S_n}(g^-) \big) + \widetilde{S_n}(g^+) \geqslant 0$.

It remains to show that S is norm-preserving and has property $(*)$. We will start by checking that $\|S\| \leqslant 1$. Let $f \in C_0[1, \omega^{\alpha+1})$ with $\|f\|_\infty \leqslant 1$. We want to see that $\|Sf\|_\infty \leqslant 1$. It should be observed that since S is positive we may assume that $f \geqslant 0$ (by replacing f by its modulus $|f|$). As the addends (5.12) used to define S are pairwise disjoint, it will be enough to verify that each of them is bounded by 1. If $f(\omega^\alpha \cdot n) = 0$, then the corresponding summand is bounded by 1 because $\widetilde{S_n}$ is norm-preserving, so suppose that $f(\omega^\alpha \cdot n) > 0$. In this case, we have $\widetilde{S_n}(g^-) \geqslant 0$ because $\widetilde{S_n}$ is positive, and $g^+ \leqslant 1 - f(\omega^\alpha \cdot n)$, so $\|\widetilde{S_n}(g^+)\|_\infty \leqslant 1 - f(\omega^\alpha \cdot n)$, where

$$g = f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)}.$$

Thus,

$$0 \leqslant \big(f(\omega^\alpha \cdot n)g_n - \widetilde{S_n}(g^-)\big)(s) + \widetilde{S_n}(g^+)(s) \leqslant f(\omega^\alpha \cdot n) + 1 - f(\omega^\alpha \cdot n) = 1$$

for every $s \in U_n$, and

$$0 \leqslant f(\omega^\alpha \cdot n)g_n(s) + \widetilde{S_n}(g)(s) \leqslant f(\omega^\alpha \cdot n)g_n(s) \leqslant 1$$

for every $s \in V \backslash U_n$, since $\widetilde{S_n}(g)$ is supported on U_n. Therefore, any term of S is bounded by 1, and $\|S\| \leqslant 1$. On the other hand, fix any $t \in [1, \omega^{\alpha+1})$, so that $t \in (\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n]$ for some $n \geqslant 1$. By the induction hypothesis, there exists $s_t \in U_n$ such that $\widetilde{S_n} f(s_t) = S_n f(s_t) = f(t)$ for every $f \in C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n)$. Now take any $f \in C_0[1, \omega^{\alpha+1})$ and consider the function

$$f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)}$$

which belongs to $C_0(\omega^\alpha \cdot (n-1), \omega^\alpha \cdot n)$, so it must satisfy

$$\widetilde{S_n}\left(f|_{(\omega^\alpha(n-1), \omega^\alpha n)} - f(\omega^\alpha \cdot n)\chi_{(\omega^\alpha(n-1), \omega^\alpha n)}\right)(s_t) = f(t) - f(\omega^\alpha \cdot n).$$

Since $s_t \in U_n$, $g_n(s_t) = 1$, and $Sf(s_t) = f(t)$ for every $f \in C_0[1, \omega^{\alpha+1})$. Moreover, taking into account that we have constructed the sequence $(V_n)_{n=1}^\infty$ in such a way that $s_0 \notin V_n$ for any $n \geqslant 1$, it follows that $Sf(s_0) = 0$ for every $f \in C_0[1, \omega^{\alpha+1})$. Finally, recall that property $(*)$ together with the fact that $\|S\| \leqslant 1$ implies that S is norm-preserving.

Next, we need to prove the transfinite step. Let $\alpha < \omega_1$ be a limit ordinal and assume that the induction hypothesis holds for every $\beta < \alpha$. Since α is a countable limit ordinal, it has cofinality ω, that is, there exists a sequence of strictly increasing ordinals $(\alpha_n)_{n\in\omega}$ such that $\alpha = \lim_n \alpha_n = \sup_n \alpha_n$. In particular, writing $\omega^{\alpha_0} = 0$ we can decompose

$$[1, \omega^\alpha) = \bigcup_{n=1}^\infty (\omega^{\alpha_{n-1}}, \omega^{\alpha_n}],$$

where each $(\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$ is a clopen set homeomorphic to $[1, \omega^{\alpha_n}]$. Similarly, we have $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)} = \bigcap_{n\in\omega} K^{(\alpha_n)}$. Now, let V be an open set of K that intersects $K^{(\alpha)}$, and fix $s_0 \in V \cap K^{(\alpha)}$. We can choose for every n a point $s_n \in V \cap K^{(\alpha_n)}$ different from $s_0, s_1, \ldots, s_{n-1}$. Using the normality of K, we can find pairwise disjoint open sets V_n such that $s_n \in V_n \subseteq V$ and $s_0 \notin V_n$. Applying Lemma 5.3.10 twice, we can further assume that $\overline{V_n} \subseteq V$ and select open neighborhoods U_n such that $s_n \in U_n \subseteq \overline{U_n} \subseteq V_n$. By the induction hypothesis, there exists a positive isometry $S_n : C_0(\omega^{\alpha_{n-1}}, \omega^{\alpha_n}) \to C_0(U_n)$ with property $(*)$, let $\widetilde{S_n}$ be its extension by zero to $C_0(V)$. By Urysohn lemma, we can find $g_n \in C(K)$ such that $0 \leqslant g_n \leqslant 1$, $g_n|_{\overline{U_n}} = 1$ and $g_n|_{K\backslash V_n} = 0$. As in the previous step, we can assume that $g_n \in C_0(V)$. For every $f \in C_0[1, \omega^\alpha)$ we can define

$$Sf := \sum_{n=1}^\infty \left(f(\omega^{\alpha_n})g_n + \widetilde{S_n}\big(f|_{(\omega^{\alpha_{n-1}}, \omega^{\alpha_n})} - f(\omega^{\alpha_n})\chi_{(\omega^{\alpha_{n-1}}, \omega^{\alpha_n})}\big)\right).$$

Proceeding as in the previous inductive step, it can be checked that S is a well-defined positive isometry with property $(*)$. Hence, the induction hypothesis holds for every countable ordinal α.

To conclude the proof, suppose that K is a compact Hausdorff space such that $K^{(\alpha)} \neq \varnothing$. We have proved that given $s_0 \in K^{(\alpha)}$ there is a positive linear isometry $S : C_0[1, \omega^\alpha) \to C(K)$ with the property that for every $t \in [1, \omega^\alpha]$ there exists $s_t \in V$ such that $Sf(s_t) = f(t)$ and $Sf(s_0) = 0$ for every $f \in C_0[1, \omega^\alpha)$. If we define

$$Tf := f(\omega^\alpha)\mathbb{1}_K + S\big(f|_{[1,\omega^\alpha)} - f(\omega^\alpha)\chi_{[1,\omega^\alpha)}\big), \qquad f \in C[1, \omega^\alpha],$$

it is not difficult to verify that T defines a positive linear isometric embedding of $C[1, \omega^\alpha]$ into $C(K)$ with the desired properties. □

Remark 5.4.5. *The proof of Proposition 5.4.4 can be adapted to show that, if $|K^{(\alpha)}| \geqslant 2$, there exists a positive linear isometric embedding of $C[1,\omega^\alpha\cdot 2]$ into $C(K)$ with property $(*)$. Indeed, it suffices to consider distinct points $s_1, s_2 \in K^{(\alpha)}$ and open neighborhoods $V_i, W_i \subseteq K$, $i=1,2$, such that W_1 and W_2 are disjoint and $s_i \in V_i \subseteq \overline{V_i} \subseteq W_i$, $i=1,2$. As $V_i \cap K^{(\alpha)} \neq \varnothing$, by the inductive step there exist positive linear isometries $S_i : C_0[1,\omega^\alpha) \to C_0(V_i)$ satisfying property $(*)$, $i=1,2$ (denote by $\widetilde{S}_i$ the extensions by 0 to $C(K)$), and we can find $h_i \in C(K)$ such that $0 \leqslant h_i \leqslant 1$, $h_i|_{\overline{V_i}} = 1$ and $h_i|_{K\setminus W_i}$, $i=1,2$. Therefore, the operator $T : C[1,\omega^\alpha\cdot 2] \to C(K)$ defined as*

$$\begin{aligned} Tf = {} & f(\omega^\alpha)h_1 + \widetilde{S_1}(f|_{[1,\omega^\alpha)} - f(\omega^\alpha)\chi_{[1,\omega^\alpha)}) \\ & + f(\omega^\alpha\cdot 2)h_2 + \widetilde{S_2}(f|_{(\omega^\alpha,\omega^\alpha\cdot 2)} - f(\omega^\alpha)\chi_{(\omega^\alpha,\omega^\alpha\cdot 2)}) \end{aligned}$$

for every $f \in C[1,\omega^\alpha\cdot 2]$ will satisfy the properties mentioned above. In particular, T will preserve the SPR subspaces of $C[1,\omega^\alpha\cdot 2]$.

Remark 5.4.6. *It should be noticed that the assumption $K^{(\alpha)} \neq \varnothing$ does not guarantee the existence of a lattice isometric embedding $T : C[1,\omega^\alpha] \to C(K)$. Indeed, suppose that there exists a lattice isometry $T : c \to C[0,1]$. Let us denote $f_n = Te_n$ for every natural number n. Note that the functions f_n are pairwise disjoint norm-one elements of $C[0,1]$. Hence, for every n there is $t_n \in [0,1]$ such that $f_n(t_n) = 1$. Passing to a subsequence, we may assume that $t_n \to t_0$ in $[0,1]$. Note that $f_n(t_0) = 0$ for every n. Moreover, as $T\mathbb{1}(t_n) = 1$, it follows by the continuity of this function that $T\mathbb{1}(t_0) = 1$. Now, let I be an open interval of $[0,1]$ such that $t_0 \in I$ and $T\mathbb{1}(t) > \frac{1}{2}$ for every $t \in I$, and let N be such that $t_N \in I$. Now, observe that $\mathbb{1} - e_N$ and e_N are disjoint in c, so $T(\mathbb{1}-e_N)$ and Te_N are disjoint in $C[0,1]$. Thus, $T(\mathbb{1}-e_N)$ is zero on $\operatorname{supp}(Te_N)$ and is equal to $T\mathbb{1}$ outside that set. Then, I can be represented as the union of the following two disjoint non-empty open sets*

$$A_1 := \{Te_N > 0\} \cap I \ni t_N \qquad \text{and} \qquad A_2 := \big\{T(\mathbb{1}-e_N) > 0\big\} \cap I \ni t_0,$$

which is a contradiction with the fact that I is connected.

Next, we will construct SPR embeddings of $C[1,\omega^\alpha]$ inside $C[1,\omega^\beta]$ for $\beta > \alpha$ big enough. Recall that Cantor's normal form allows us to express uniquely every ordinal $0 < \alpha < \omega_1$ as a finite combination of ordinals of the form ω^β with non-zero natural coefficients. Since $\omega^\beta \cdot 0 = 0$ for every β, the representation (5.2) can be understood as a bijection from ω_1 into $c_{00}(\omega_1,\mathbb{Z}_+) := \big\{x : \omega_1 \to \mathbb{Z}_+ \,:\, |\{\gamma \,:\, x(\gamma)\neq 0\}| < \aleph_0\big\}$, that is, the set of sequences of non-negative integers indexed in ω_1 with finite support. Observe that, in general, the coordinate-wise sum in the set $c_{00}(\omega_1,\mathbb{Z}_+)$ does not coincide with the usual sum of ordinal numbers. Given $\alpha,\beta \in \omega_1$, we denote by $\alpha\oplus\beta$ the ordinal obtained by adding the normal forms of α and β in $c_{00}(\omega_1,\mathbb{Z}_+)$. This operation $\oplus : \omega_1\times\omega_1 \to \omega_1$ is associative, symmetric (in contrast with the usual sum of ordinals) and compatible with the order (if $\alpha \leqslant \beta$, then $\alpha\oplus\gamma \leqslant \beta\oplus\gamma$ for every $\gamma \in \omega_1$). Moreover, it satisfies that $\alpha\oplus 1 = \alpha + 1$.

We will use this operation to show that for every countable ordinal α, $C[1,\omega^\alpha]$ embeds isometrically into $C[1,\omega^{\alpha\odot 2}]$ doing SPR in the real setting, and $C[1,\omega^\alpha]$ embeds isometrically into $C[1,\omega^{\alpha\odot 2}\cdot 2]$ doing SPR in the complex setting, where $\alpha\odot 2 = \alpha\oplus\alpha$, with an SPR constant independent of α. Our first step is the following construction:

Lemma 5.4.7. *For every pair of ordinals $\alpha,\beta \in \omega_1$ there is a map $U_{\alpha,\beta} : C[1,\omega^\alpha]\times C[1,\omega^\beta] \to C[1,\omega^{\alpha\oplus\beta}]$ satisfying the following properties:*

(i) for every $f_1, f_2 \in C[1,\omega^\alpha]$ and $g_1, g_2 \in C[1,\omega^\beta]$,

$$U_{\alpha,\beta}(f_1+f_2, g_1+g_2) = U_{\alpha,\beta}(f_1,g_1) + U_{\alpha,\beta}(f_2,g_2);$$

(ii) for every $f \in C[1,\omega^\alpha]$, $g \in C[1,\omega^\beta]$ and $\lambda \in \mathbb{R}$,

$$U_{\alpha,\beta}(\lambda f, \lambda g) = \lambda U_{\alpha,\beta}(f,g);$$

(iii) for every $f \in C[1,\omega^\alpha]$ and $g \in C[1,\omega^\beta]$,

$$\|U_{\alpha,\beta}(f,g)\|_\infty \leqslant \frac{1}{2}\big(\|f\|_\infty + \|g\|_\infty\big);$$

(iv) for every $s \in [1,\omega^\alpha]$ and $t \in [1,\omega^\beta]$ there exists $r_{s,t} \in [1,\omega^{\alpha\oplus\beta}]$ such that

$$[U_{\alpha,\beta}(f,g)](r_{s,t}) = \frac{1}{2}\big(f(s)+g(t)\big).$$

(v) for every $r \in [1,\omega^{\alpha\oplus\beta}]$ there exists $s_r \in [1,\omega^\alpha]$ and $t_r \in [1,\omega^\beta]$ such that

$$[U_{\alpha,\beta}(f,g)](r) = \frac{1}{2}\big(f(s_r)+g(t_r)\big).$$

Proof. First of all, we observe that for any pair $0 \leqslant \alpha < \beta < \omega_1$, the construction of the map $U_{\beta,\alpha}$ is automatic once the existence of the map $U_{\alpha,\beta}$ is established. It suffices to define

$$U_{\beta,\alpha}(f,g) := U_{\alpha,\beta}(g,f)$$

for $f \in C[1,\omega^\beta]$ and $g \in C[1,\omega^\alpha]$. The existence of the map $U_{\alpha,\beta}$ for every $\alpha \leqslant \beta$ is proved by means of two nested transfinite induction arguments, one for $\alpha \in \omega_1$ and another one for $\beta \geqslant \alpha$:

Induction Hypothesis (IH): Given $\alpha \in \omega_1$, if $U_{\gamma,\delta}$ is defined for every $\gamma < \alpha$ and $\delta \geqslant \gamma$, then $U_{\alpha,\beta}$ exists for every $\beta \geqslant \alpha$.

1. **First step ($\alpha = 0$):** Given $\beta \geqslant 0$, it is straightforward to check that

$$\begin{aligned} U_{0,\beta} : C(\{1\}) \times C[1,\omega^\beta] &\longrightarrow C[1,\omega^\beta] \\ (f,g) &\longmapsto \tfrac{1}{2}(f(1)+g) \end{aligned}$$

   is well defined and satisfies the properties of the statement.

2. **Inductive step ($\alpha \to \alpha+1$):** Assume that (IH) has been established for α. Let us show that it also holds for $\alpha+1$ by a transfinite induction argument on $\beta \geqslant \alpha+1$:

   Induction Hypothesis (IH-I): Fix $\alpha \in \omega_1$, and assume that $U_{\gamma,\delta}$ has been constructed for every $\gamma \leqslant \alpha$ and $\delta \geqslant \gamma$. Let $\beta \geqslant \alpha+1$. If $U_{\alpha+1,\gamma}$ is defined for every $\alpha+1 \leqslant \gamma < \beta$, then $U_{\alpha+1,\beta}$ exists.

   2.1) **First step ($\beta = \alpha+1$):** We construct $U_{\alpha+1,\alpha+1} : C[1,\omega^{\alpha+1}] \times C[1,\omega^{\alpha+1}] \to C[1,\omega^{(\alpha+1)\odot 2}]$ in the following way (the order-preserving homeomorphisms that transform the clopen subsets of $[1,\omega^\gamma]$ into intervals of the form $[1,\omega^\delta]$ for a suitable δ by means of a translation are not written explicitly):

$$\begin{aligned} &U_{\alpha+1,\alpha+1}(f,g)|_{(\omega^{\alpha\odot 2+1}(2n-2),\omega^{\alpha\odot 2+1}(2n-1)]} \\ &\qquad = U_{\alpha+1,\alpha}\big(f|_{(\omega^\alpha(n-1),\omega^{\alpha+1}]}, g|_{(\omega^\alpha(n-1),\omega^\alpha n]}\big), n \in \mathbb{N}; \\ &U_{\alpha+1,\alpha+1}(f,g)|_{(\omega^{\alpha\odot 2+1}(2n-1),\omega^{\alpha\odot 2+1}2n]} \\ &\qquad = U_{\alpha,\alpha+1}\big(f|_{(\omega^\alpha(n-1),\omega^\alpha n]}, g|_{(\omega^\alpha n,\omega^{\alpha+1}]}\big), n \in \mathbb{N}; \\ &U_{\alpha+1,\alpha+1}(f,g)|_{\{\omega^{(\alpha+1)\odot 2}\}} \\ &\qquad = \frac{1}{2}\big(f(\omega^{\alpha+1}) + g(\omega^{\alpha+1})\big). \end{aligned}$$

Clearly, the function defined above satisfies conditions (i), (ii), (iii) and (v). Let us check that it is continuous on $[1, \omega^{(\alpha+1)\odot 2}]$. By the assumption, it is continuous on each clopen set $(\omega^{\alpha\odot 2+1}\cdot(m-1), \omega^{\alpha\odot 2+1}\cdot m]$ with $m\in\mathbb{N}$. Moreover, it is continuous on $\omega^{(\alpha+1)\odot 2}$: for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $|f(\omega^{\alpha+1})-f(s)|<\varepsilon$ and $|g(\omega^{\alpha+1})-g(t)|<\varepsilon$ for every $s,t\in(\omega^\alpha\cdot n_0,\omega^{\alpha+1}]$, so using (v) we know that for every $r\in(\omega^{\alpha\odot 2+1}\cdot 2n_0, \omega^{(\alpha+1)\odot 2}]$ there are $s_r,t_r\in(\omega^\alpha\cdot n_0,\omega^{\alpha+1}]$ such that $[U_{\alpha,\beta}(f,g)](r)=\frac{1}{2}\big(f(s_r)+g(t_r)\big)$, so we conclude that

$$\Big|[U_{\alpha+1,\alpha+1}(f,g)](\omega^{(\alpha+1)\odot 2})-[U_{\alpha+1,\alpha+1}(f,g)](r)\Big|<\varepsilon.$$

In other words, $U_{\alpha+1,\alpha+1}(f,g)$ is continuous on $\omega^{(\alpha+1)\odot 2}$ because of property (v) and the fact that for every $n\in\mathbb{N}$, $U_{\alpha+1,\alpha+1}(f,g)$ restricted to $(\omega^{\alpha\odot 2+1}\cdot 2n, \omega^{(\alpha+1)\odot 2}]$ depends only on $f|_{(\omega^\alpha n,\omega^{\alpha+1}]}$ and $g|_{(\omega^\alpha n,\omega^{\alpha+1}]}$. Finally, property (iv) follows from the fact that $U_{\alpha+1,\alpha+1}$ and $U_{\alpha,\alpha+1}$ also satisfy (iv), and $[1,\omega^{(\alpha+1)\odot 2}]^2$ can be decomposed as

$$\begin{aligned}[1,\omega^{(\alpha+1)\odot 2}]^2 &= \bigg(\bigcup_{n=1}^{\infty}(\omega^\alpha\cdot(n-1),\omega^{\alpha+1}]\times(\omega^\alpha\cdot(n-1),\omega^\alpha\cdot n]\bigg)\\ &\quad\cup\bigg(\bigcup_{n=1}^{\infty}(\omega^\alpha\cdot(n-1),\omega^\alpha\cdot n]\times(\omega^\alpha\cdot n,\omega^{\alpha+1}]\bigg)\cup\{(\omega^{(\alpha+1)\odot 2},\omega^{(\alpha+1)\odot 2})\}.\end{aligned}$$

The continuity of the function $U_{\alpha,\beta}(f,g)$ and properties (i) to (v) are checked in a similar way in the rest of the steps, so we will omit the details.

2.2) **Inductive step ($\beta\to\beta+1$):** Assuming that $U_{\alpha+1,\beta}$ has been established, let us construct $U_{\alpha+1,\beta+1}$ as follows:

$$\begin{aligned}U_{\alpha+1,\beta+1}(f,g)|_{(\omega^{\alpha\oplus\beta+1}(2n-2),\omega^{\alpha\oplus\beta+1}(2n-1)]}&\\ = U_{\alpha+1,\beta}\big(f|_{(\omega^\alpha(n-1),\omega^{\alpha+1}]},g|_{(\omega^\beta(n-1),\omega^\beta n]}\big),\ n\in\mathbb{N};&\\ U_{\alpha+1,\beta+1}(f,g)|_{(\omega^{\alpha\oplus\beta+1}(2n-1),\omega^{\alpha\oplus\beta+1}2n]}&\\ = U_{\alpha,\beta+1}\big(f|_{(\omega^\alpha(n-1),\omega^\alpha n]},g|_{(\omega^\beta n,\omega^{\beta+1}]}\big),\ n\in\mathbb{N};&\\ U_{\alpha+1,\beta+1}(f,g)|_{\{\omega^{(\alpha+1)\oplus(\beta+1)}\}}&\\ = \frac{1}{2}\big(f(\omega^{\alpha+1})+g(\omega^{\beta+1})\big).&\end{aligned}$$

2.3) **Limit step (β limit ordinal):** If $\beta>\alpha+1$ is a limit ordinal and $U_{\alpha+1,\gamma}$ exists for every $\alpha+1\leqslant\gamma<\beta$, then we can define $U_{\alpha+1,\beta}$ in the following way. First, observe that β is countable, so it has cofinality ω and there is an strictly increasing sequence of ordinals $(\beta_n)_n$ indexed in $n\in\omega$ such that $\beta=\sup_n\beta_n$. Next, note that $\gamma=\sup_n(\alpha+1)\oplus\beta_n$ exists and $\gamma\leqslant\alpha\oplus\beta$. Indeed, the ordinals are a well-ordered set, so the subset of upper bounds of $\{(\alpha+1)\oplus\beta_n : n\in\omega\}$ must have a minimum, i.e., a least upper bound for the sequence, denoted γ. Moreover, $\beta_n<\beta_{n+1}$ implies that $(\alpha+1)\oplus\beta_n=\alpha\oplus(\beta_n+1)\leqslant\alpha\oplus\beta_{n+1}\leqslant\alpha\oplus\beta$ for every $n\in\omega$, so $\gamma\leqslant\alpha\oplus\beta$. Using this sequence we can make the following decomposition (hereby, ω^{β_0} and $\omega^{(\alpha+1)\oplus\beta_0}$ must be understood as the ordinal 0):

$$\begin{aligned}[1,\omega^{(\alpha+1)\oplus\beta}] &= \bigg(\bigcup_{n=1}^{\infty}(\omega^{(\alpha+1)\oplus\beta_{n-1}},\omega^{(\alpha+1)\oplus\beta_n}]\bigg)\cup[\omega^\gamma,\omega^{\alpha\oplus\beta}]\\ &\quad\cup\bigg(\bigcup_{n=1}^{\infty}(\omega^{\alpha\oplus\beta}\cdot n,\omega^{\alpha\oplus\beta}\cdot(n+1)]\bigg)\cup\{\omega^{(\alpha+1)\oplus\beta}\}.\end{aligned}$$

Since $((\alpha+1)\oplus\beta_n)_n$ is strictly increasing, every clopen set $(\omega^{(\alpha+1)\oplus\beta_{n-1}},\omega^{(\alpha+1)\oplus\beta_n}]$ is homeomorphic to $[1,\omega^{(\alpha+1)\oplus\beta_n}]$. We define

$$\begin{aligned}
&U_{\alpha+1,\beta}(f,g)|_{(\omega^{(\alpha+1)\oplus\beta_{n-1}},\omega^{(\alpha+1)\oplus\beta_n}]}\\
&\qquad = U_{\alpha+1,\beta_n}\big(f|_{(\omega^\alpha(n-1),\omega^{\alpha+1}]}, g|_{(\omega^{\beta_{n-1}},\omega^{\beta_n}]}\big), n\in\mathbb{N};\\
&U_{\alpha+1,\beta}(f,g)|_{[\omega^\gamma,\omega^{\alpha\oplus\beta}]}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha+1})+g(\omega^\beta)\big);\\
&U_{\alpha+1,\beta}(f,g)|_{(\omega^{\alpha\oplus\beta}n,\omega^{\alpha\oplus\beta}(n+1)]}\\
&\qquad = U_{\alpha,\beta}\big(f|_{(\omega^\alpha(n-1),\omega^\alpha n]}, g|_{(\omega^{\beta_n},\omega^\beta]}\big), n\in\mathbb{N};\\
&U_{\alpha+1,\beta}(f,g)|_{\{\omega^{(\alpha+1)\oplus\beta}\}}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha+1})+g(\omega^\beta)\big).
\end{aligned}$$

We have defined $U_{\alpha+1,\beta}$ for every $\beta\geqslant\alpha+1$, so the proof of the inductive step (IH-I) is concluded.

3. **Limit step (α limit ordinal):** Assume that α is a limit ordinal, and (IH) has been established for every $\gamma<\alpha$. Since α has cofinality ω, there is an strictly increasing sequence of ordinals $(\alpha_n)_{n\in\omega}$ such that $\alpha=\sup_n\alpha_n$. We construct $U_{\alpha,\beta}$ by transfinite induction on $\beta\geqslant\alpha$:

   Induction Hypothesis (IH-L): Fix $\alpha\in\omega_1$ a limit ordinal, and assume that $U_{\gamma,\delta}$ has been constructed for every $\gamma<\alpha$ and $\delta\geqslant\gamma$. Let $\beta\geqslant\alpha$. If $U_{\alpha,\gamma}$ is defined for every $\alpha\leqslant\gamma<\beta$, then $U_{\alpha,\beta}$ exists.

   3.1) **First step ($\beta=\alpha$):** We can construct $U_{\alpha,\alpha}$ in the following way (as before, ω^{α_0} and $\omega^{\alpha_0\oplus\alpha}$ denote the 0 ordinal):

$$\begin{aligned}
&U_{\alpha,\alpha}(f,g)|_{(\omega^{\alpha_{n-1}\oplus\alpha}\cdot 2,\omega^{\alpha_n\oplus\alpha}]}\\
&\qquad = U_{\alpha,\alpha_n}\big(f|_{(\omega^{\alpha_{n-1}},\omega^\alpha]}, g|_{(\omega^{\alpha_{n-1}},\omega^{\alpha_n}]}\big), n\in\mathbb{N};\\
&U_{\alpha,\alpha}(f,g)|_{(\omega^{\alpha_n\oplus\alpha},\omega^{\alpha_n\oplus\alpha}\cdot 2]}\\
&\qquad = U_{\alpha_n,\alpha}\big(f|_{(\omega^{\alpha_{n-1}},\omega^{\alpha_n}]}, g|_{(\omega^{\alpha_n},\omega^\alpha]}\big), n\in\mathbb{N};\\
&U_{\alpha,\alpha}(f,g)|_{\{\omega^{\alpha\odot 2}\}}\\
&\qquad = \frac{1}{2}\big(f(\omega^\alpha)+g(\omega^\alpha)\big).
\end{aligned}$$

   3.2) **Inductive step ($\beta\to\beta+1$):** Assume that $U_{\alpha,\beta}$ is known. We need to construct $U_{\alpha,\beta+1}$. First observe that the sequence $\alpha_n\oplus(\beta+1)$ is bounded below by $\alpha\oplus\beta$. Let $\gamma=\sup_n\alpha_n\oplus(\beta+1)\leqslant\alpha\oplus\beta$. Then, we can make the following decomposition (we are using the convention $\omega^{\alpha_0}=\omega^{\alpha_0\oplus(\beta+1)}=0$):

$$\begin{aligned}
[1,\omega^{\alpha\oplus(\beta+1)}] &= \left(\bigcup_{n=1}^{\infty}(\omega^{\alpha_{n-1}\oplus(\beta+1)},\omega^{\alpha_n\oplus(\beta+1)}]\right)\cup[\omega^\gamma,\omega^{\alpha\oplus\beta}]\\
&\cup\left(\bigcup_{n=1}^{\infty}(\omega^{\alpha\oplus\beta}\cdot n,\omega^{\alpha\oplus\beta}\cdot(n+1)]\right)\cup\{\omega^{(\alpha+1)\oplus\beta}\}.
\end{aligned}$$

   Now, we define $U_{\alpha,\beta+1}$ piecewise as:

$$
\begin{aligned}
&U_{\alpha,\beta+1}(f,g)|_{(\omega^{\alpha_{n-1}\oplus(\beta+1)},\omega^{\alpha_n\oplus(\beta+1)}]}\\
&\qquad = U_{\alpha_n,\beta+1}\big(f|_{(\omega^{\alpha_{n-1}},\omega^{\alpha_n}]}, g|_{(\omega^{\beta}n,\omega^{\beta+1}]}\big), n\in\mathbb{N};\\
&U_{\alpha,\beta+1}(f,g)|_{[\omega^{\gamma},\omega^{\alpha\oplus\beta}]}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha})+g(\omega^{\beta+1})\big);\\
&U_{\alpha,\beta+1}(f,g)|_{(\omega^{\alpha\oplus\beta}n,\omega^{\alpha\oplus\beta}(n+1)]}\\
&\qquad = U_{\alpha,\beta}\big(f|_{(\omega^{\alpha_{n-1}},\omega^{\alpha}]}, g|_{(\omega^{\beta}(n-1),\omega^{\beta}n]}\big), n\in\mathbb{N};\\
&U_{\alpha,\beta+1}(f,g)|_{\{\omega^{\alpha\oplus(\beta+1)}\}}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha})+g(\omega^{\beta+1})\big).
\end{aligned}
$$

3.3) **Limit step (β limit ordinal):** Assume that $\beta>\alpha$ is a limit ordinal and let $(\beta_n)_n$ be an strictly increasing sequence indexed in ω whose supremum is β. Provided $U_{\alpha,\gamma}$ exists for every $\alpha\leqslant\gamma<\beta$, we want to define $U_{\alpha,\beta}$. Let $\gamma_n=\max\{\alpha_n\oplus\beta,\alpha\oplus\beta_n\}$, which defines a strictly increasing sequence whose supremum, denoted γ, is bounded by $\alpha\oplus\beta$. We make the following decomposition of $[1,\omega^{\alpha\oplus\beta}]$ (here, $\omega^{\gamma_0}=0$):

$$
[1,\omega^{\alpha\oplus\beta}] = \left(\bigcup_{n=1}^{\infty}(\omega^{\gamma_{n-1}}\cdot 2,\omega^{\gamma_n}]\cup(\omega^{\gamma_n},\omega^{\gamma_n}\cdot 2]\right)\cup[\omega^{\gamma},\omega^{\alpha\oplus\beta}].
$$

Additionally, we can break each of the clopen sets above as follows:

$$
\begin{aligned}
(\omega^{\gamma_{n-1}}\cdot 2,\omega^{\gamma_n}] &= (\omega^{\gamma_{n-1}}\cdot 2,\omega^{\gamma_{n-1}}\cdot 2+\omega^{\alpha\oplus\beta_n}]\cup(\omega^{\gamma_{n-1}}\cdot 2+\omega^{\alpha\oplus\beta_n},\omega^{\gamma_n}], n\in\mathbb{N};\\
(\omega^{\gamma_n},\omega^{\gamma_n}\cdot 2] &= (\omega^{\gamma_n},\omega^{\gamma_n}+\omega^{\alpha_n\oplus\beta}]\cup(\omega^{\gamma_n}+\omega^{\alpha_n\oplus\beta},\omega^{\gamma_n}\cdot 2], n\in\mathbb{N};
\end{aligned}
$$

where the last set in each line is either empty or a clopen set. Now, we define $U_{\alpha,\beta}$ piecewise as follows:

$$
\begin{aligned}
&U_{\alpha,\beta}(f,g)|_{(\omega^{\gamma_{n-1}}\cdot 2,\omega^{\gamma_{n-1}}\cdot 2+\omega^{\alpha\oplus\beta_n}]}\\
&\qquad = U_{\alpha,\beta_n}\big(f|_{(\omega^{\alpha_{n-1}},\omega^{\alpha}]}, g|_{(\omega^{\beta_{n-1}},\omega^{\beta_n}]}\big), n\in\mathbb{N};\\
&U_{\alpha,\beta}(f,g)|_{(\omega^{\gamma_{n-1}}\cdot 2+\omega^{\alpha\oplus\beta_n},\omega^{\gamma_n}]}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha})+g(\omega^{\beta_n})\big), n\in\mathbb{N};\\
&U_{\alpha,\beta}(f,g)|_{(\omega^{\gamma_n},\omega^{\gamma_n}+\omega^{\alpha_n\oplus\beta}]}\\
&\qquad = U_{\alpha_n,\beta}\big(f|_{(\omega^{\alpha_{n-1}},\omega^{\alpha_n}]}, g|_{(\omega^{\beta_n},\omega^{\beta}]}\big), n\in\mathbb{N};\\
&U_{\alpha,\beta}(f,g)|_{(\omega^{\gamma_n}+\omega^{\alpha_n\oplus\beta},\omega^{\gamma_n}\cdot 2]}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha_n})+g(\omega^{\beta})\big), n\in\mathbb{N};\\
&U_{\alpha,\beta}(f,g)|_{[\omega^{\gamma},\omega^{\alpha\oplus\beta}]}\\
&\qquad = \frac{1}{2}\big(f(\omega^{\alpha})+g(\omega^{\beta})\big).
\end{aligned}
$$

With this last construction we have defined $U_{\alpha,\beta}$ for every $\beta\geqslant\alpha$, so (IH-L) holds, and hence so does (IH) for every countable ordinal α. □

Now that we have built an overlapping map for every ordinal α, we can use them to obtain the following isometric and isomorphic results.

Theorem 5.4.8. *Let K be a compact Hausdorff space, $\alpha\in\omega_1$ a countable ordinal and assume the field of scalars is $\mathbb{R}$. If $K^{(\alpha\odot 2)}$ is non-empty, then there exists an isometric 3-SPR embedding of $C[1,\omega^{\alpha}]$ into $C(K)$.*

Proof. By Proposition 5.4.4, $C(K)$ contains an isometric copy of $C[1,\omega^{\alpha\odot 2}]$ that preserve SPR subspaces, so without loss of generality we can assume that $K=[1,\omega^{\alpha\odot 2}]$. By Lemma 5.4.7, there is a map $U_{\alpha,\alpha}: C[1,\omega^\alpha]\times C[1,\omega^\alpha]\to C[1,\omega^{\alpha\odot 2}]$ satisfying properties (i) to (v). Let

$$\begin{aligned} T: C[1,\omega^\alpha] &\longrightarrow C[1,\omega^{\alpha\odot 2}] \\ f &\longmapsto Tf = U_{\alpha,\alpha}(f,f). \end{aligned}$$

Properties (i) and (ii) imply that T is a linear map. Moreover, (iii) implies that T is contractive, and (iv) yields that for every $s\in[1,\omega^\alpha]$ there exists $r_{s,s}\in[1,\omega^{\alpha\odot 2}]$ such that $Tf(r_{s,s})=f(s)$, so T is an isometric embedding. It remains to show that $T(C[1,\omega^\alpha])$ does 3-SPR in $C[1,\omega^{\alpha\odot 2}]$. Let f and g be normalized functions in $C[1,\omega^\alpha]$, and find $s,t\in[1,\omega^\alpha]$ such that $|f(s)|=|g(t)|=1$.

- If $s=t$, then $|Tf(r_{s,s})|\wedge|Tg(r_{s,s})|=|f(s)|\wedge|g(s)|=1$.
- If $s\neq t$, we distinguish three cases:
  - If $|Tf(r_{s,t})|<\frac{1}{3}$, then

$$|f(t)|\geqslant|f(s)|-|f(s)+f(t)|=1-2|Tf(r_{s,t})|>\frac{1}{3},$$

    so $|Tf(r_{t,t})|\wedge|Tg(r_{t,t})|=|f(t)|\wedge|g(t)|\geqslant\frac{1}{3}$.
  - If $|Tg(r_{s,t})|<\frac{1}{3}$, then

$$|g(s)|\geqslant|g(t)|-|g(s)+g(t)|=1-2|Tg(r_{s,t})|>\frac{1}{3},$$

    so $|Tf(r_{s,s})|\wedge|Tg(r_{s,s})|=|f(s)|\wedge|g(s)|\geqslant\frac{1}{3}$.
  - Otherwise, $|Tf(r_{s,t})|\wedge|Tg(r_{s,t})|\geqslant\frac{1}{3}$.

Hence, there is always a point $r\in[1,\omega^{\alpha\odot 2}]$ such that $\||Tf|\wedge|Tg|\|_\infty\geqslant|Tf(r)|\wedge|Tg(r)|\geqslant\frac{1}{3}$, and we conclude that $T(C[1,\omega^\alpha])$ does not contain normalized $\frac{1}{3}$-almost disjoint pairs, so, by [26, Proposition 3.4], it does 3-SPR. □

Corollary 5.4.9. *Let $0<\alpha<\omega_1$ be a countable ordinal, K a compact Hausdorff space with $K^{(\alpha)}\neq\varnothing$, and assume the field of scalars is $\mathbb{R}$. Let $\beta\in\omega_1$ such that $\omega^\beta\leqslant\alpha<\omega^{\beta+1}$. Then, $C(K)$ contains an isometric SPR copy of $C[1,\omega^\gamma]$ for every $\gamma<\omega^\beta$. Moreover, if $\omega^\beta\cdot 2\leqslant\alpha$, then $C(K)$ contains an isomorphic SPR copy of $C[1,\omega^\alpha]$.*

Proof. Using Cantor's normal form (5.2) it can be easily checked that for every $\gamma<\omega^\beta$, we have $\gamma\odot 2<\omega^\beta\leqslant\alpha$, so using Theorem 5.4.8 we obtain an isometric 3-SPR embedding of $C[1,\omega^\gamma]$ into $C[1,\omega^{\gamma\odot 2}]$, which in turn is a sublattice of $C[1,\omega^\alpha]$. By Proposition 5.4.4, $C(K)$ contains an isometric 3-SPR copy of $C[1,\omega^\gamma]$ as well.

Next, assume that $\omega^\beta\cdot 2\leqslant\alpha$. It was established in [24, Theorem 1] that for ordinals $\omega\leqslant\lambda\leqslant\mu<\omega_1$, $C[1,\lambda]$ is isomorphic (as a Banach space) to $C[1,\mu]$ if and only if $\mu<\lambda^\omega$. In particular, it can be checked that $\omega^\alpha<(\omega^{\omega^\beta})^\omega$, so $C[1,\omega^\alpha]$ and $C[1,\omega^{\omega^\beta}]$ are linearly isomorphic. By Theorem 5.4.8, we know that $C[1,\omega^{\omega^\beta\cdot 2}]$, which is a sublattice of $C[1,\omega^\alpha]$, contains an SPR copy of $C[1,\omega^{\omega^\beta}]$, and hence, of $C[1,\omega^\alpha]$. Proposition 5.4.4 yields the result. □

In the complex case, the situation is not as straightforward, but it still allows us for a quantitative answer to Question 5.4.1.

Theorem 5.4.10. *Let K be a compact Hausdorff space, $\alpha \in \omega_1$ a countable ordinal and assume the field of scalars is $\mathbb{C}$. If $|K^{(\alpha\odot 2)}| \geqslant 2$, then there exists an isometric SPR embedding of $C[1,\omega^\alpha]$ into $C(K)$, and the SPR constant is uniform on α.*

Proof. First, we observe that arguing as in Remark 5.4.5 we can assume without loss of generality that $K = [1, \omega^{\alpha\odot 2} \cdot 2]$. Next, we use Lemma 5.4.7 to build the operator

$$\begin{aligned} T : C[1,\omega^\alpha] &\longrightarrow C[1,\omega^{\alpha\odot 2}\cdot 2] = C[1,\omega^{\alpha\odot 2}] \oplus_\infty C[1,\omega^{\alpha\odot 2}] \\ f &\longmapsto Tf = (U_{\alpha,\alpha}(f,f), U_{\alpha,\alpha}(f,if)). \end{aligned}$$

It is clear that T is a contractive operator that satisfies the condition of Lemma 5.3.5, so it is an isometric SPR embedding. Moreover, the SPR constant is given by the proofs of Lemma 5.3.5 and Theorem 5.2.7, so it does not depend on α. □

Repeating the argument of Corollary 5.4.9 we obtain the following:

Corollary 5.4.11. *Let $0 < \alpha < \omega_1$ be a countable ordinal, K a compact Hausdorff space with $K^{(\alpha)} \neq \varnothing$, and assume the field of scalars is $\mathbb{C}$. Let $\beta \in \omega_1$ such that $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$. Then, $C(K)$ contains an isometric SPR copy of $C[1,\omega^\gamma]$ for every $\gamma < \omega^\beta$. Moreover, if $|K^{(\omega^\beta\cdot 2)}| \geqslant 2$, that is, if $\alpha > \omega^\beta \cdot 2$ or $\alpha = \omega^\beta \cdot 2$ and $|K^{(\alpha)}| \geqslant 2$, then $C(K)$ contains an isomorphic SPR copy of $C[1,\omega^\alpha]$.*

5.5 Stability of SPR under operations with Banach lattices

For some applications, it is interesting to study the (stable) phase retrieval behavior of solutions to certain PDEs, such as the Schrödinger equation [74]. In this situation, the space where the solutions lie is usually some $C(K,X)$ space, that is, a space of continuous functions $f : K \to X$, where K is a compact Hausdorff space (typically an interval of the real line) and X is a real or complex Banach lattice. The norm in these spaces is given by

$$\|f\| = \sup_{t\in K} \|f(t)\|_X.$$

One could wonder if the existence of (infinite-dimensional) SPR subspaces in $C(K,X)$ could be characterized in a similar way to the $C(K)$ setting. Clearly, if K' is infinite (and $X \neq \{0\}$), by Theorem 5.3.7 or its real version [48, Theorem 6.1], there exists an SPR subspace E of $C(K)$, so choosing some $x \in S_X$, we can build the subspace $\widetilde{E} = \{f \otimes x : f \in E\}$, where $(f\otimes x)(k) = f(k)x$ for every $k \in K$, so that $\widetilde{E}$ is an SPR subspace of $C(K,X)$. Additionally, if X already contains an SPR subspace $E \subseteq X$, then the subspace $\widetilde{E} = \{\mathbb{1}\otimes x : x \in E\}$ is an isometric copy of X that does SPR in $C(K,X)$. However, is it necessary that any of these two conditions holds for $C(K,X)$ to have an SPR subspace? The answer is negative in general. Indeed, take $K = [0,\omega]$, that is, the one point compactification of the natural numbers, and $X = C[0,\omega]$. It is easy to see that if K and L are compact spaces, then $C(K,C(L))$ is lattice isometric to $C(K\times L)$. In particular, $C([0,\omega],C[0,\omega])$ can be identified with $C([0,\omega]^2)$. Clearly, $[0,\omega]$ has only one accumulation point (ω itself), so $C[0,\omega]$ does not have SPR subspaces either. However, $[0,\omega]^2$ has an infinite number of accumulation points, so $C([0,\omega]^2) = C([0,\omega],C[0,\omega])$ contains SPR subspaces.

This example invites to explore the existence of SPR subspaces in Banach lattices that are generated from other Banach lattices by means of standard operations, such as sums, tensor products, quotients, ultraproducts... For instance, a similar argument yields that $X_1 \otimes_{|\pi|} X_2$, where $\otimes_{|\pi|}$ denotes the Fremlin projective tensor product [49], can have an SPR subspace, even though X_1 and X_2 might not contain SPR subspaces. Indeed, we know by [49] that $C(K)\otimes_{|\pi|} C(L)$ is lattice isometric to $C(K\times L)$, so considering $X_1 = X_2 = C[0,\omega]$ we get that $X_1 \otimes_{|\pi|} X_2 = C([0,\omega]^2)$

contains SPR subspaces, while $C[0,\omega]$ does not.

The situation with sums of Banach lattices is more involved. It is false in general that the sum of an infinite family of Banach lattices contains an SPR subspace if and only if one of the factors contains an SPR subspace. As a counterexample, consider ℓ_∞ as the ℓ_∞-sum of a countable number of copies of $\mathbb{R}$ (see Lemma 5.3.2), or the c_0-sum of a countable number of copies of c. Indeed, $c_0(c)$ can be identified with $C_0[0,\omega^2)$, and it is shown in the claim of Lemma 5.3.9 that $C_0[0,\omega^2)$ contains an SPR copy of c_0. However, when we restrict the problem to finite sums, $C(K)$-spaces do not provide a source of counterexamples, but rather the contrary: if K and L are Hausdorff compact spaces, then $C(K)\oplus_\infty C(L)$ contains an SPR subspace if and only if $C(K)$ or $C(L)$ do. Just observe that $C(K)\oplus_\infty C(L)$ is isometric to $C(K\sqcup L)$, where $K\sqcup L$ is the disjoint union of K and L. Clearly, $(K\sqcup L)' = K'\sqcup L'$, so Theorem 5.3.7 and [48, Theorem 6.1] prove the claim. For arbitrary Banach lattices the situation is unclear, but still some information can be provided.

For instance, note that if one of the Banach lattices, X_1 for instance, contains an SPR-subspace, denoted $E\subseteq X_1$, then for any bounded operator $T: E\to X_2$ (in particular, the null operator) we can build the graph of T, $\widetilde{E} = \{(x_1, Tx_1) : x_1\in E\}\subseteq X_1\oplus X_2$, which does SPR too. To check this, assume without loss of generality that T is contractive (this can be done by rescaling the norm in X_2) and $X_1\oplus X_2$ is endowed with the ∞-sum norm. Then, $\widetilde{E}$ is isometric to E, and for every $x=(x_1,Tx_1), y=(y_1,Ty_1)\in\widetilde{E}$ we have that

$$\min_{|\lambda|=1}\|x-\lambda y\| = \min_{|\lambda|=1}\|x_1-\lambda y_1\| \leqslant C\||x_1|-|y_1|\| \leqslant \||x|-|y|\|,$$

where C denotes the SPR constant of E. It turns out that every SPR susbpace of $X_1\oplus X_2$ must be the graph of an operator between subspaces of the factors.

Proposition 5.5.1. *Let X_1 and X_2 be Banach lattices and $P_i: X_1\oplus X_2\to X_i$ denote the canonical projection, $i=1,2$. If E is an SPR subspace of $X_1\oplus X_2$, then one of the following holds:*

(1) E is isomorphic to $P_1(E)$ and there exists a bounded operator $T: P_1(E)\to P_2(E)$ such that $E=\{(x_1,Tx_1) : x_1\in P_1(E)\}$.

(2) E is isomorphic to $P_2(E)$ and there exists a bounded operator $T: P_2(E)\to P_1(E)$ such that $E=\{(Tx_2,x_2) : x_2\in P_2(E)\}$.

Proof. Since SPR is an isomorphic property, without loss of generality assume that $X_1\oplus X_2$ is endowed with the ∞-sum norm. We claim that there is a constant $C\geqslant 1$ such that one of the following holds:

(1) for every $x=(x_1,x_2)\in E$, $\|x_2\|\leqslant C\|x_1\|$, or

(2) for every $x=(x_1,x_2)\in E$, $\|x_1\|\leqslant C\|x_2\|$.

Assume the contrary. Then, for every $C\geqslant 1$ there exist $x,y\in E$ such that $\|x_2\|>C\|x_1\|$ and $\|y_1\|>C\|y_2\|$. Without loss of generality, we can assume that $x,y\in S_E$, so that

$$\||x_1|\wedge|y_1|\| \leqslant \|x_1\|\wedge\|y_1\| \leqslant \frac{\|x_2\|}{C}\wedge 1 \leqslant \frac{1}{C}$$

and

$$\||x_2|\wedge|y_2|\| \leqslant \|x_2\|\wedge\|y_2\| \leqslant 1\wedge\frac{\|y_1\|}{C} \leqslant \frac{1}{C}.$$

It follows that

$$\||x|\wedge|y|\| = \max\{\||x_1|\wedge|y_1|\|, \||x_2|\wedge|y_2|\|\} \leqslant \frac{1}{C},$$

so Proposition 5.2.5 (respectively [48, Theorem 3.4(i)]) yields that E fails $\frac{C}{\sqrt{2}}$-SPR (respectively, C-SPR). Since C was arbitrary, we get a contradiction.

Next, assume we are in situation (1) above (the other case is analogue). This implies that for every $x_1 \in P_1(E)$, there exists a unique $x_2 \in P_2(E)$ such that $(x_1, x_2) \in E$. Indeed, if for a certain $x_1 \in P_1(E)$ there are $x_2, x_2' \in P_2(E)$ such that (x_1, x_2) and (x_1, x_2') belong to E, in particular $(0, x_2 - x_2') \in E$, so $\|x_2 - x_2'\| \leqslant C\|0\| = 0$ and we conclude that $x_2 = x_2'$. Let $T : P_1(E) \to P_2(E)$ be the map that assigns to every $x_1 \in P_1(E)$ the corresponding $x_2 \in P_2(E)$ such that $(x_1, x_2) \in E$. Clearly, T is lineal, $\|T\| \leqslant C$, and $E = \{(x_1, Tx_1) : x_1 \in P_1(E)\}$. In particular,

$$\|x_1\| \leqslant \|(x_1, Tx_1)\| \leqslant C\|x_1\|,$$

so the operator $I \oplus T : P_1(E) \to E$ is an onto isomorphism and $P_1(E)$ is closed. □

One could conjecture that the subspace $P_1(E)$ (respectively, $P_2(E)$) in the situation above should do SPR, so that a subspace of $X_1 \oplus X_2$ does SPR if and only if it is the graph of an SPR subspace of one of the factors. However, this is not true in general, as the following example shows.

Example 5.5.2. *Let $2 \leqslant p < \infty$. In [48, Theorem 5.3] the authors build an SPR subspace of the (real) $L_p[0,2] = L_p[0,1] \oplus_p L_p[1,2]$ given by the closed span of the functions*

$$g_j = r_j + 2^{\frac{j}{p}} \chi_{[1+2^{-j}, 1+2^{1-j})}, \; j \in \mathbb{N},$$

where $(r_j)_{j=1}^{\infty}$ denotes the Rademacher sequence of independent, mean-zero, ± 1 random variables on $[0,1]$. Observe that, even though E does SPR in $L_p[0,2]$, both projection images $P_1(E)$ and $P_2(E)$ fail SPR (even more, they fail PR): On the one hand, $P_1(E)$ contains the Rademacher functions, all of them having the same modulus and being linearly independent. On the other hand, $P_2(E)$ contains a disjoint sequence of characteristic functions, so by Proposition 5.2.1 it fails PR.

Note that the example above is not a counterexample to the original conjecture that a sum of Banach lattices contains an SPR subspace if and only if one of the factors does, since $L_p[0,1]$ contains SPR subspaces [48, Section 4]. Nor does it provide a negative answer to the more specific question of whether the domain of the operator T (in this case, $P_1(E) = \overline{\text{span}}\{r_j\} \subseteq L_p[0,1]$) should contain a smaller subspace doing SPR, since $P_1(E)$ is dispersed (this is a consequence of Khintchine's inequality and Kadec-Pełczynski dichotomy, see [48, Theorem 2.1] for a summary of the literature on this topic), and in an order continuous Banach lattice such as $L_p[0,1]$ every dispersed subspace contains an SPR subspace [48, Theorem 5.1].

Question 5.1. If $X_1 \oplus X_2$ contains an infinite-dimensional SPR subspace, does it imply that X_1 or X_2 contains an infinite-dimensional SPR subspace? Moreover, in the setting of Proposition 5.5.1, can we ensure that the domain of T contains an infinite-dimensional SPR subspace?

Regarding Banach lattice quotients, we can observe that if X is a Banach lattice and I is an ideal in X, the fact that X contains an SPR subspace does not imply that X/I contains an SPR subspace, as the following examples show: $\ell_2 \oplus L_2[0,1]$ contains SPR subspaces (see for instance [36, Examples 2 and 3]), but the quotient through $L_2[0,1]$ is atomic, so it cannot contain SPR subspaces by Proposition 5.3.8. The same happens in $C(K)$-spaces: $C[0, \omega^2]$ contains an SPR subspace, but $C[0, \omega]$, which can be seen for instance as the quotient of $C[0, \omega^2]$ by its order continuous part (the closed span of the characteristic functions of isolated points), does not contain an infinite-dimensional SPR subspace. However, the converse seems to be true in this setting:

Proposition 5.5.3. *Let K be a compact Hausdorff space and I an ideal in $C(K)$. If $C(K)/I$ contains an SPR subspace, then so does $C(K)$.*

Proof. By [96, Proposition 2.1.9], there exists some closed subset $L \subseteq K$ such that $I = \{f \in C(K) : f|_L = 0\}$. It can be checked that $C(K)/I$ is lattice isometric to $C(L)$. Therefore, Theorem 5.3.7 and [48, Theorem 6.1] imply that L' is infinite. Hence, K' must be infinite too, and $C(K)$ must contain an SPR subspace. □

Question 5.2. Let X be a Banach lattice and I an ideal in X. Suppose that X/I contains an infinite-dimensional SPR subspace. Does X contain an infinite-dimensional SPR subspace?

Finally, we conclude this section with a quick remark on ultraproducts, and more specifically, ultrapowers (see [66] for a survey on the topic). It was established in [48, Proposition 4.11] that the ultraproduct of a collection of SPR subspaces $(E_i)_{i\in I}$ of Banach lattices $(X_i)_{i\in I}$ is again an SPR subspace of the ultraproduct of the ambient Banach lattices. In the particular case of ultrapowers, this fact becomes straightforward, as any Banach lattice X embeds lattice isometrically into the ultrapower $X^{\mathcal{U}}$ with respect to the ultrafilter $\mathcal{U}$. Hence, if X contains an SPR subspace, so does $X^{\mathcal{U}}$. However, the converse is not true, as one might already expect: it is shown in [67, Theorem 2.4] that $\ell_2^{\mathcal{U}}$, which can be identified with $L_2(\mu)$ for some measure μ, has a non-trivial purely non-atomic part, and therefore it contains an infinite-dimensional SPR subspace (the non-atomic part contains a lattice copy of L_2 over an interval, which contains SPR subspaces by [36, Examples 2 and 3]), even though ℓ_2 does not (Proposition 5.3.8).

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