UNIVERSIDAD COMPLUTENSE DE MADRID
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TESIS DOCTORAL

Haantjes Geometry and Integrable Systems:
A Tensorial Approach to Geometric Mechanics

Geometría de Haantjes y Sistemas Integrables:
Un Enfoque Tensorial para la Mecánica Geométrica

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Universidad Complutense de Madrid

Facultad de Ciencias Físicas



UNIVERSIDAD
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Memoria para optar al grado de Doctor

Programa de Doctorado en Física

Doctorando:
Daniel Reyes Nozaleda

Director:
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A María y Antonio, mis abuelos.
Os quiero porque os quiero.

Que la vida iba en serio
uno lo empieza a comprender más tarde.

Jaime Gil de Biedma, *No volveré a ser joven*

El sueño va sobre el tiempo
flotando como un velero.
Nadie puede abrir semillas
en el corazón del sueño.

Federico García Lorca, *Así que pasen cinco años*

Me dijeron:
—O te subes al carro
o tendrás que empujarlo.

Ni me subí ni lo empujé.
Me senté en la cuneta
y alrededor de mí,
a su debido tiempo,
brotaron las amapolas.

Gloria Fuertes, *Autobio*

Agradecimientos

Il y a des êtres qui justifient le monde, qui aident
à vivre par leur seule présence.

Albert Camus, *Le Premier Homme*

Gracias. Si en algo he tenido suerte en esta vida es por las personas con las que he tenido la oportunidad de coincidir, un brillante y acogedor remanso en este mar de fueguitos que es el mundo. Sois tantos y tendría tantas cosas que decir, que simplemente pido disculpas por todos los que sois pero no estáis.

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También pasan muchas cosas en los laboratorios. Agradezco a Paco V. por meterle tanta caña a este fermión desde que pisó por primera vez la facultad siendo apenas mayor de edad. A Manuel, quien me introdujo en LaTeX e incluso hace apenas unas semanas me preguntó si estaba escribiendo la tesis así. A mi apañero Juan, con quien aprendí a pensar con la cabeza para salvar la vida de tantos fusibles inocentes. A *elektrolatino42*, David y Marco, el equipo de duendes nocturnos que redactan prácticas mientras uno duerme. A Álvaro, quien además de responder a todas nuestras dudas enrevesadas nos dejaba tiempo de más para que tomáramos medidas tan exhaustivas que casi alcanzáramos el continuo. A Juan, quien me contagió el gusto por la termodinámica aunque nunca me acercara a ningún resultado de los tabulados: seguiré insistiendo en que la culpa es de los instrumentos de medida.

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During my PhD, I also had the opportunity to spend three months in Prague, which was a fantastic experience. Beyond the city's beauty, I thoroughly enjoyed my time there. I'd like to thank Libor and Antonella for their academic hospitality and for introducing me to Czech culture. To Fazlul, practically my office-mate, for all his kindness. I also want to thank Štěpánka for hosting me during that period and for being so attentive. There, I had an incredible buddy, Kristýna, who taught me why we love *svíčková* far more than *trdelníks*–and she even came to Madrid for *chocolate con churros*. I'm grateful to Marco and Iris for those unforgettable evenings around the city. Now I know Český Krumlov is a place you must linger in. Mockrát Vám za všechno děkuji!

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No le encuentro sentido a limitarme a las personas que me han acompañado activamente durante estos años de doctorado, pues el camino ha sido algo más largo y eso también se refleja en esta tesis. En el fondo, considero que nuestro contacto con los demás, incluso cuando el encuentro es sucinto y parece de lo más intrascendente, de alguna manera nos moldea y nos marca. Por ello, también quería aprovechar para agradecer a las dos Teresas, que me enseñaron mucho más que a contar y a leer, y aún habiendo pasado un cerro de años siguen alegrándose de mis éxitos más in-

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Abbreviations

AA (variables)	Action-angle (variables)
AKN	Arnold, Kozlov and Neishtadt
DH (coordinates)	Darboux-Haantjes (coordinates)
DN (coordinates)	Darboux-Nijenhuis (coordinates)
FN (bracket)	Frölicher-Nijenhuis (bracket)
GSE	Generalized separation equations
HJ (equations)	Hamilton-Jacobi (equations)
ISM	Inverse Scattering Method
KAM (theorem)	Kolmogorov-Arnold-Moser (theorem)
KdV (equation)	Korteweg-de Vries (equation)
LH (theorem)	Liouville-Haantjes (theorem)
ODE(s)	Ordinary differential equation(s)
PDE(s)	Partial differential equation(s)
PW (system)	Post-Winternitz (system)
SE	Separation equations
SN (bracket)	Schouten-Nijenhuis (bracket)
SoV	Separation of variables
SW (systems)	Smorodinsky-Winternitz (systems)
TTW (systems)	Tremblay-Turbiner-Winternitz (systems)
$\omega\mathscr{H}$	Symplectic-Haantjes
ωN	Symplectic-Nijenhuis

Publications

In this thesis, I discuss the main results from the following articles:

1. D. Reyes Nozaleda, P. Tempesta, and G. Tondo. "Classical multiseparable Hamiltonian systems, superintegrability and Haantjes geometry." In: *Communications in Nonlinear Science and Numerical Simulation* 104 (2022). ISSN: 10075704. DOI: 10.1016/j.cnsns.2021.106021 (Chapters 2, 3 and 4)

2. D. Reyes, P. Tempesta, and G. Tondo. "Generalized Nijenhuis Torsions and Block-Diagonalization of Operator Fields." In: *Journal of Nonlinear Science* 33 (2 2023). ISSN: 14321467. DOI: 10.1007/s00332-022-09877-8 (Chapter 7)

3. D. Reyes, P. Tempesta, and G. Tondo. "Partial separability and symplectic-Haantjes manifolds." In: *Annali di Matematica Pura ed Applicata (1923 -)* 203.6 (Dec. 2024), pp. 2677–2710. ISSN: 0373-3114, 1618-1891. DOI: 10.1007/s10231-024-01462-y (Chapters 3 and 5)

4. O. Kubů et al. "Hamiltonian integrable systems in a magnetic field and symplectic-Haantjes geometry." In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 480.2301 (Nov. 2024), p. 20240076. ISSN: 1364-5021, 1471-2946. DOI: 10.1098/rspa.2024.0076 (Chapters 5 and 6)

5. A. Marchesiello, D. Reyes, and L. Šnobl. "Superintegrable families of magnetic monopoles with non-radial potential in curved background." In: *Journal of Geometry and Physics* 203 (Sept. 2024), p. 105261. ISSN: 03930440. DOI: 10.1016/j.geomphys.2024.105261 (Chapter 6)

6. D. Reyes, M. A. Rodríguez, and P. Tempesta. "A Frobenius-Type Theory for Discrete Systems." In: *Studies in Applied Mathematics* 154.3 (Mar. 2025), e70037. ISSN: 0022-2526, 1467-9590. DOI: 10.1111/sapm.70037 (Chapter 8)

Resumen

Esta tesis propone un marco teórico unificado para estudiar la integrabilidad de sistemas hamiltonianos de dimensión finita mediante la geometría de Haantjes. Combinando los fundamentos de la mecánica geométrica con la teoría clásica de integrabilidad de Arnold-Liouville, demostramos el papel crucial de los operadores de Nijenhuis y Haantjes en la estructura de los sistemas dinámicos. La clave reside en que la anulación de la torsión de estos operadores asegura la integrabilidad de las distribuciones de sus autovectores y permite construir cartas locales donde estos operadores adoptan formas diagonales o por bloques. Estos resultados ofrecen un criterio elegante y potente para la separación de variables en la ecuación de Hamilton-Jacobi.

En este contexto, realizamos un estudio exhaustivo de las álgebras de Haantjes y analizamos sus propiedades algebraicas y geométricas. Nuestros resultados muestran que estas estructuras permiten definir de forma natural a las coordenadas de Darboux-Haantjes, que cumplen una doble función: simplifican la estructura simpléctica a su forma canónica y descomponen la ecuación de Hamilton-Jacobi en subsistemas más abordables, simplificando así el problema de integración. Esta metodología evita los complejos análisis espectrales tradicionales y los sustituye por técnicas tensoriales y algebraicas precisas, proporcionando herramientas eficaces para estudiar sistemas integrables.

Uno de los aportes centrales de este trabajo es el tratamiento de la superintegrabilidad. Demostramos que los sistemas hamiltonianos con más integrales de movimiento que grados de libertad poseen múltiples estructuras de Haantjes, donde cada una de ellas está asociada a distintas coordenadas de separación. Esta multiplicidad no solo revela simetrías ocultas del sistema, sino que amplía la teoría clásica de separación de variables, permitiendo integrar explícitamente modelos complejos que antes resultaban inaccesibles. La claridad y la generalidad de estos resultados resalta la utilidad práctica del enfoque de Haantjes.

La tesis discute además el problema de la separabilidad parcial. Cuando no es posible separar todas las variables en la ecuación de Hamilton-Jacobi, demostramos que incluso en variedades simplécticas de Haantjes no semisimples o no maximales puede lograrse desacoplar un subconjunto de variables. Las coordenadas de Darboux-Haantjes obtenidas, aunque no permiten una separación completa, aportan información crucial sobre la dinámica de sistemas complejos, ampliando así el abanico de técnicas disponibles para estudiar su integrabilidad.

Estudiamos también la aplicación del formalismo de Haantjes en sistemas hamiltonianos en presencia de campos magnéticos. Al incorporar geométricamente los efectos de los potenciales vectoriales, desarrollamos un método que permite analizar modelos integrables con interacciones magnéticas sin necesidad de fijar un gauge específico. Este enfoque, relacionado con la geometría de

Stäckel, reafirma la versatilidad del enfoque de Haantjes y ofrece nuevas perspectivas sobre las simetrías de sistemas en presencia de campos magnéticos.

Ampliando la teoría clásica, generalizamos las torsiones de Nijenhuis y Haantjes a niveles superiores, lo que permite definir las álgebras de Haantjes generalizadas. En estas álgebras, la anulación de las torsiones de nivel superior garantiza la diagonalización por bloques simultánea de familias de operadores que conmutan. Este avance representa un salto cualitativo en el estudio las formas normales de campos de operadores, con aplicaciones tanto teóricas como computacionales.

Como contribución complementaria, en el capítulo final desarrollamos un método de discretización al estilo Frobenius para ecuaciones diferenciales de segundo orden con coeficientes analíticos, basado en el cálculo de operadores finitos y en las álgebras de Rota. Este proceso de discretización preserva propiedades clave de los modelos continuos como son la regla de Leibniz y la relación con las soluciones analíticas exactas, creando un puente entre sistemas integrables continuos y discretos. En particular, obtuvimos discretizaciones exactas para ecuaciones clásicas (Airy, Hermite, Bessel), abriendo nuevos caminos para modelizar sistemas con propiedades inherentemente discretas.

En definitiva, esta tesis establece un enfoque unificado para estudiar la integrabilidad y separación de variables en sistemas hamiltonianos. Al combinar geometría diferencial avanzada con técnicas algebraicas modernas, nuestros resultados no solo consolidan principios fundamentales de la mecánica geométrica, sino que abren nuevas líneas de investigación en áreas como son los operadores con espectro complejo, la adaptación del formalismo de Haantjes a variedades de contacto o cosimplécticas para sistemas disipativos o dependientes del tiempo, y la exploración de álgebras de Haantjes generalizadas en el contexto de los modelos integrables. Este trabajo sienta bases sólidas para futuras investigaciones teóricas y para aplicaciones prácticas, ampliando significativamente las herramientas disponibles para analizar y discretizar sistemas dinámicos complejos.

Abstract

This thesis is framed within the quest for a unified theoretical approach to address the integrability of finite-dimensional Hamiltonian systems through an in-depth study of Haantjes geometry. Building upon the fundamentals of geometric mechanics and the classical theory of integrability in the Arnold-Liouville sense, our work explores the decisive role played by Nijenhuis and Haantjes operators in structuring dynamical systems. The key of this approach lies in the vanishing of the torsion of these operators, which guarantees the integrability of the eigendistributions generated by their eigenvectors and enables the construction of local charts in which these operators assume diagonal or block-diagonal forms. Such tensorial properties provide a powerful and elegant criterion for achieving the separation of variables in the Hamilton-Jacobi equation.

Within this framework, we rigorously investigate the algebraic and geometric properties of Haantjes algebras. Our analysis shows that these algebras naturally lead to the definition of Darboux-Haantjes coordinates, which both bring the symplectic structure into its canonical form and decouple the Hamilton-Jacobi equation into subsystems that depend on fewer variables, thereby significantly simplifying the integration process. Our methodology, based on refined tensorial and algebraic techniques, circumvents the need for exhaustive spectral analysis while providing clear and effective tools for the study of integrable systems.

A fundamental aspect of our study is the treatment of superintegrability. We prove that Hamiltonian systems possessing more integrals of motion than degrees of freedom are endowed with multiple Haantjes structures, each associated with its own set of separable coordinates. This multiplicity not only reveals hidden symmetries inherent to the system but also extends the traditional separation theory, offering novel methods for the explicit integration of complex models. The clarity and generality of these results underscore the practical utility of the Haantjes framework in addressing problems that were previously intractable.

In addition, the thesis addresses the topic of partial separability. When total separation of variables in the Hamilton-Jacobi equation is unattainable, our results demonstrate that even non-semisimple or non-maximal symplectic-Haantjes manifolds allow for the effective decoupling of a subset of variables. In these cases, the resulting Darboux-Haantjes coordinates, though not yielding complete separation, still offer valuable insights into the dynamics of complex systems, thereby broadening the scope of separability techniques available for Hamiltonian systems.

Another central contribution of this work is the application of the Haantjes formalism to Hamiltonian systems immersed in magnetic fields. By naturally incorporating the effects of vector potentials into the geometric structure, we develop a gauge-independent method for analyzing integrable

models in the presence of magnetic interactions. This treatment, which is closely related to Stäckel geometry, not only reinforces the versatility of the Haantjes approach but also provides new perspectives on the integrability and symmetry properties of magnetic systems.

Furthermore, we extend the classical theory by generalizing Nijenhuis and Haantjes torsions to higher levels. This generalization leads to the definition of generalized Haantjes algebras, wherein the vanishing of higher-level torsions sets tensorial conditions that ensure the simultaneous block-diagonalization of families of commuting operators. These results represent a significant advance in our ability to obtain normal forms for operator fields on differentiable manifolds, offering both theoretical insight and practical computational advantages.

Finally, an auxiliary but noteworthy contribution of this thesis is its extension into the discrete domain. In the final chapter, we present a Frobenius-type theory for the discretization of second-order differential equations with analytic coefficients, based on finite operator calculus and Rota algebras. This discretization scheme preserves key features of the continuous models–such as the Leibniz rule and the inheritance of exact analytic solutions–thus establishing a clear bridge between continuous and discrete integrable structures. In particular, we obtain exact discretizations for classical equations including those of Airy, Hermite, and Bessel, thereby opening new possibilities for modeling and analysis in contexts where data are naturally discrete.

In summary, this thesis provides a unified and technically robust framework for the study of integrability and variable separation in Hamiltonian systems. By synthesizing advanced concepts from differential geometry with modern algebraic techniques, our results not only consolidate fundamental principles in geometric mechanics but also set the stage for further research in areas such as operators with complex eigenvalues, the adaptation of the symplectic-Haantjes formalism to contact and cosymplectic manifolds for dissipative or time-dependent systems, and the deep exploration of generalized Haantjes algebras within integrable models. This work lays a solid foundation for future theoretical and applied investigations, broadening the horizon of methods available for the integration and discretization of complex dynamical systems.

Las cosas podían haber sucedido de cualquier otra manera y, sin embargo, sucedieron así.

Miguel Delibes, *El camino*

Prelude

Everything is in motion, even that which appears still.

In the documentary "El sol del membrillo" (*Dream of Light*), by Víctor Erice, the painter Antonio López observes the quince tree in his garden with an almost mystical patience, as though he wished to unravel the secrets of the universe woven into the branches and fruits. Armed with his brush, he embarks on the quest to capture an instant where time seems to stand still. Yet, he soon becomes aware of the futility of this endeavor: movement reveals itself even in the details we often overlook. The sunlight that caresses the tree slips by—imperceptible yet relentless—transforming shadows, softening hues, redrawing contours; the breeze whispers through the branches, setting the leaves in a subtle sway; the quinces, ripe with the promise of descent, hint at their inevitable fall. In spite of the painter's ardent and ritualistic effort, he appears not so much to capture the scene as to engage in a silent dialogue with it—a dance in which what he seeks to fix has already changed. Day after day, his brush lags behind a nature that never ceases, a world in perpetual metamorphosis.

This struggle against the inevitable echoes words spoken centuries ago by Heraclitus: "Everything flows" (*Πάντα ῥεῖ*). Nothing remains; all that exists is immersed in a river of ceaseless transformations. The eternal, unyielding flow spares no moment: what is today will be no more tomorrow. All is in becoming, even that which our senses deem immobile. Even in the serene guise of the quince tree, even in the stillness captured by Erice's camera, this truth emerges. The tree grows, ages, lives, and breathes, albeit in a cadence too subtle for the naked eye. There is no repose in the universe; a secret choreography transforms every instant into the next, every form into its becoming.

With the creation of this thesis, I invite you to contemplate and heighten your awareness of movement in all its manifestations—unraveling the laws that govern everything from the oscillating rhythm of a pendulum to the orbits of distant stars, from the delicate flutter of a leaf to the grand cycles of the seasons—finding in the simple and everyday a gateway to the profound.

In what concerns me, with the fervor of Antonio López's brush and in pursuit of my own dialogue with change, I am writing these pages with the aim of devising methods to solve the equations of motion that describe a myriad of physical systems. More specifically, my aim is advancing the theory of separability of the Hamilton-Jacobi equation, where the tools I develop hold promising applications in other branches of mathematics.

With this objective in mind, this thesis is organized into four parts. In the first part we introduce the core concepts of geometric mechanics. Chapter 1 presents the Hamilton-Jacobi framework and discusses the differential geometric tools–such as the Schouten-Nijenhuis bracket and the structures of symplectic, Poisson, and Jacobi manifolds–that allow us to approach the problem of motion, as well as the formulation of Lax systems and generalized Stäckel systems. Chapter 2 then lays the algebraic foundations by reviewing the major contributions of the Dutch school in differential geometry, with a focus on the Frölicher-Nijenhuis bracket, Nijenhuis and Haantjes tensors, and the construction of Haantjes algebras, emphasizing their diagonalization properties.

The second part centers on symplectic Haantjes manifolds as the natural setting for integrable dynamics and the separation of variables. In this section, four chapters develop the subject further. Chapter 3 revisits different geometric approaches to the separability (bi-Hamiltonian manifolds, Nijenhuis symplectic manifolds, and Haantjes manifolds), but it is focussed in symplectic Haantjes manifolds and their main properties in relation to integrable systems. Chapter 4 examines systems displaying superintegrability, where multiple Haantjes algebras can be associated with a single system, leading to multiseparability. In Chapter 5, we explore cases in which full separability is not possible but a partial decoupling of variables can still be achieved, owing to the presence of non-semisimple or non-maximal Haantjes algebras. Chapter 6 extends the framework to encompass Hamiltonian systems influenced by magnetic fields, demonstrating that the symplectic Haantjes setting naturally accommodates such interactions without the need to fix a gauge.

The third part is devoted to generalized aspects of the theory. In Chapter 7 we study generalized Nijenhuis torsions and the corresponding generalized Haantjes algebras, examining their block-diagonalization properties and the role of torsion polarizations. This chapter not only broadens the classical approach but also provides new algebraic structures that are instrumental in understanding the integrability of more complex systems.

The fourth and final part addresses topics related to the main line of inquiry, yet distinct in their own right. Chapter 8 explores a Frobenius-type theory for discrete systems, presenting a rigorous discretization scheme for second-order differential equations with analytic coefficients. This approach preserves key features of the continuous models and establishes a clear connection between continuous and discrete integrable structures.

The thesis concludes with general remarks on the topics discussed and potential directions for future research, along with three appendices containing calculations whose detailed presentation would have interrupted the flow of the discussion.

Before delving into the main content of the thesis, it is important to note that all parts except Part I are almost *entirely original*. Specifically, Chapters 4 through 8 are completely original, and Chapter 3–where I introduce various geometric frameworks for studying the separation of variables in

the Hamilton-Jacobi equation–includes several significant original contributions. This progression reflects a deepening understanding of symplectic-Haantjes manifolds and their application to new scenarios. Notable original contributions in Chapter 3 include the analysis of commutativity in $\omega\mathscr{H}$ algebras (Proposition 3.3.1), the discussion on the existence and construction of Darboux–Haantjes coordinates (Sections 3.3.3 and 3.3.6), the formulation of the Jacobi-Haantjes theorem (Theorem 3.4.2), the treatment of separation of variables in the Drach-Holt type system (Section 3.5), the introduction of a new geometric lift for constructing Haantjes operators in the cotangent space (Section 3.6), and the derivation of necessary conditions for a function to generate a Haantjes chain (Theorem 3.3.3 and Corollary 3.3.1).

As Eugene Wigner observed, the way in which mathematics so aptly describes the laws of physics is a remarkable gift — one we neither fully understand nor deserve. The following chapters explore how this connection unfolds in the study of motion.

Part I

INTRODUCTION

El mundo hay que fabricárselo uno mismo, hay que crear peldaños que te suban, que te saquen del pozo. Hay que inventar la vida porque acaba siendo verdad.

Ana María Matute, *El libro de Ana María Matute: antología de literatura y vida*

1

Integrability and Hamilton-Jacobi theory

To describe the world in its first principles, you must unsee it. Let the crimson of autumn fade to shadow; let the roar of rivers dissolve into silence. Strip the bark from the tree, the flesh from the bone, the pigment from the eye. What remains is not emptiness, but architecture: the curve of a planet's arc through void, the torsion of a pendulum's swing, the geodesic etched by light across spacetime. Geometry does not ignore life–it unknots life's tangles. Galileo traced parabolas in falling stones; Euler wrote the moon's dance in differentials. Here, in this rarefied space, the universe is not a cacophony of things but a silent lexicon of lines, symmetries, and connections. To wield these tools is to converse with the unseen: the scaffolding beneath the skin of reality.

This chapter explores the geometric foundations of classical mechanics, where smooth manifolds and algebraic structures encode the principles of motion and integrability. At its core lies symplectic geometry, a framework that equips phase spaces with a canonical structure to describe Hamiltonian dynamics, linking symmetries to conserved quantities through the geometry of Poisson brackets. Moving beyond the rigidity of symplectic manifolds, we generalize to Poisson manifolds, where degenerate brackets accommodate constrained systems, and further to Jacobi manifolds, which relax classical axioms to unify contact geometry and non-conservative dynamics.

The pursuit of integrability threads through these structures. Liouville's theorem asserts that systems with sufficient conserved quantities unravel into solvable motion, while Arnold's refinement frames such systems as linear flows on invariant tori, parameterized by action-angle coordinates. Together, they transmute dynamics into orderly evolution, charting trajectories as predictably as celestial mechanics.

Central to this discourse is the Hamilton-Jacobi (HJ) formalism, a method that recasts dynamical trajectories through generating functions. The principal advantage of this approach arises when complex systems admit separation into independent coordinates. Owing to the topic's significance, separability criteria have been rigorously developed, including the Levi-Civita test, Benenti's theorem, and the Jacobi-Sklyanin separation equations.

The chapter culminates with the study of two families of completely integrable systems, where theory converges with application. Generalized Stäckel systems, constructed from separable potentials and diagonal metrics, exemplify classical separation techniques, while Lax pairs –matrix equations preserving spectral invariants– reveal integrability in systems ranging from particle collisions to nonlinear waves. These frameworks are not mere pedagogical examples but foundational constructs, anticipating their recurrence in quantum mechanics and field theory.

This exposition is structured as follows. Section 1.1 introduces symplectic manifolds as the foundational framework for Hamiltonian mechanics, revisits the Schouten-Nijenhuis bracket as a cornerstone of Poisson geometry, and extends the formalism to Poisson and Jacobi manifolds, establishing a unified geometric language for mechanical systems. Section 1.2 details the Liouville-Arnold theorem and its implications for classical integrability, emphasizing action-angle variables. Section 1.3 derives the Hamilton-Jacobi equation through canonical and geometric perspectives, including separability criteria. Section 1.4 analyzes integrable systems, including Stäckel models and Lax pairs, with the Jacobi-Calogero model as a paradigmatic case. The discussion synthesizes insights from [1, 5, 53, 86, 101, 111, 130].

1.1 PRELIMINARIES ON GEOMETRIC MECHANICS

Geometric mechanics provides a unifying framework for classical mechanics by describing physical systems in terms of geometric structures on smooth manifolds. In this setting, the phase space of a mechanical system is modeled by differentiable manifolds endowed with either a Poisson structure or, in the nondegenerate case, a symplectic structure. This approach not only encapsulates the algebraic properties of observables via the Poisson bracket but also clarifies the role of Hamiltonian vector fields in generating the system's time evolution.

In what follows, let M be a smooth real manifold. We denote by TM the tangent bundle of M and by T^*M its cotangent bundle. The sections of TM form the Lie algebra of smooth vector fields, denoted by $\mathfrak{X}(M)$, while the sections of T^*M constitute the space of smooth 1-forms, denoted by $\Omega^1(M)$. For each nonnegative integer p, the module $\Omega^p(M) := \Gamma\left(\bigwedge^p(T^*M)\right)$ consists of differential p-forms on M, and $\mathfrak{X}^p(M) := \Gamma\left(\bigwedge^p(TM)\right)$ denotes the space of p-vector fields (or multivector fields). We will assume that all functions and tensor fields are smooth, though this standard regularity condition can often be weakened in many results.

1.1.1 *Symplectic Manifolds*

A symplectic structure endows a manifold with rich geometric features that provide a natural framework for formulating the equations of motion and interpreting the evolution of a system. In particular, it establishes a one-to-one correspondence between smooth functions and vector fields via the symplectic form. This correspondence is essential for linking the energy function (or Hamiltonian) of a system with the flow describing its time evolution.

Definition 1.1.1. *A* symplectic structure *on a differentiable manifold M is a closed and nondegenerate 2-form $\omega \in \Omega^2(M)$.*

Since ω is nondegenerate, such a structure can exist only on even-dimensional manifolds. Thus, we define:

Definition 1.1.2. *A* symplectic manifold *is a pair (M, ω), where M is an even-dimensional differentiable manifold endowed with a symplectic structure ω.*

Henceforth, we assume that any symplectic manifold has dimension $2n$. Moreover, these manifolds are orientable since ω defines a volume form known as the *symplectic* or *Liouville volume*:

$$\Omega_\omega := \frac{(-1)^{n(n-1)/2}}{n!}\,\omega^n = \frac{(-1)^{n(n-1)/2}}{n!}\,\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}\,. \tag{1.1}$$

A fundamental result in symplectic geometry is the Darboux theorem.

Theorem 1.1.1 (Darboux)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every point $x \in M$ there exists an open neighborhood $U \subset M$ and a local coordinate chart $(\boldsymbol{q}, \boldsymbol{p}) := (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$ on U such that the symplectic form ω takes the standard form*

$$\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i\,. \tag{1.2}$$

The coordinates (q^i, p_i) are called Darboux *or* symplectic coordinates*.*

Darboux's theorem implies that locally all symplectic manifolds are isomorphic to the standard symplectic vector space; hence, the only local invariant (up to symplectomorphism) is the dimension. In Darboux coordinates, the Liouville volume is expressed as

$$\Omega_\omega = \mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}p_1 \wedge \mathrm{d}p_2 \wedge \cdots \wedge \mathrm{d}p_n\,. \tag{1.3}$$

The nondegeneracy of ω induces a canonical bundle isomorphism $\omega^\flat \colon \mathfrak{X}(M) \to \Omega^1(M)$, defined by

$$\omega^\flat(X) := \omega(X, \cdot\,), \qquad X \in \mathfrak{X}(M)\,. \tag{1.4}$$

Its inverse, denoted by $\omega^\sharp := (\omega^\flat)^{-1}$, allows us to introduce the following notion.

Definition 1.1.3. *The* symplectic gradient *of a function $H \in C^\infty(M)$ is defined as the vector field*

$$X_H := \omega^\sharp(\mathrm{d}H)\,, \tag{1.5}$$

or, equivalently, as the unique vector field satisfying that the interior product of ω with X_H yields

$$\iota_{X_H}\omega = \mathrm{d}H\,. \tag{1.6}$$

The triple (M, ω, H) is then called a Hamiltonian system *with Hamiltonian function H.*

Note that if two functions differ by a locally constant term, they yield the same symplectic gradient. Moreover, by the antisymmetry of ω, one immediately sees that $X_H(H) = 0$; that is, the flow of X_H preserves the level sets of H. In this thesis we consider only *autonomous* Hamiltonian systems, i.e., those in which H is time-independent, corresponding physically to energy-conserving systems.

We also introduce the related notion of a Hamiltonian vector field.

Definition 1.1.4. *A vector field $X \in \mathfrak{X}(M)$ is called a* Hamiltonian vector field *if there exists a function $H \in C^\infty(M)$ such that $X = X_H$.*

Proposition 1.1.1. *For a vector field $X \in \mathfrak{X}(M)$, the following statements are equivalent:*

(i) X is a Hamiltonian vector field.

(ii) $\iota_X\omega$ is exact.

(iii) the Lie derivative of ω with respect to X vanishes, i.e., $\mathcal{L}_X\omega = 0$.

A remarkable result concerning Hamiltonian vector fields is that they preserve the powers of the symplectic form. In particular, we have the following theorem.

Theorem 1.1.2 (Liouville)**.** *The Liouville volume is invariant under the flow of any Hamiltonian vector field, i.e., $\mathcal{L}_{X_H}\Omega_\omega = 0$.*

In Darboux coordinates, the symplectic gradient of a Hamiltonian function H is given by

$$X_H = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\right). \tag{1.7}$$

Thus, an integral curve $\gamma(t) = (\boldsymbol{q}(t), \boldsymbol{p}(t))$ of X_H satisfies

$$\dot\gamma := \frac{\mathrm{d}\gamma}{\mathrm{d}t} = X_H \circ \gamma\,, \tag{1.8}$$

if and only if it fulfills Hamilton's equations:

$$\dot q^i(t) = \frac{\partial H}{\partial p_i}\,, \quad \dot p_i(t) = -\frac{\partial H}{\partial q^i}\,, \qquad i = 1, 2, \ldots, n\,. \tag{1.9}$$

Therefore, in the geometric formulation of mechanics, the evolution of a physical system is governed by a Hamiltonian vector field that both generates the system's flow and encodes its dynamics.

We now recall a fundamental algebraic concept.

Definition 1.1.5. *[130] A* Poisson bracket *is a bilinear, antisymmetric biderivation* $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$*, satisfying the Jacobi identity*

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0\,, \tag{1.10}$$

for all $f,g,h\in C^\infty(M)$.

The symplectic structure on M naturally induces a Poisson bracket. For any $f,g\in C^\infty(M)$, one defines

$$\{f,g\}:=-\iota_{X_f}\iota_{X_g}\omega=\omega(X_f,X_g)\,. \tag{1.11}$$

This can also be expressed as

$$\{f,g\}=\mathcal{L}_{X_g}f\,, \tag{1.12}$$

which represents the variation of f along the flow generated by X_g.

Thus, the Poisson bracket endows $C^\infty(M)$ with an $\mathbb{R}$-algebra structure. Moreover, since ω is closed, the set of Hamiltonian vector fields is closed under the Lie bracket:

$$[X_f,X_g]+X_{\{f,g\}}=0\,. \tag{1.13}$$

In Darboux coordinates, the Poisson bracket takes the form

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)\,, \tag{1.14}$$

so that the canonical coordinates satisfy

$$\{q^i,q^j\}=0\,,\quad \{p_i,p_j\}=0\,,\quad \{q^i,p_j\}=\delta^i_j\,, \tag{1.15}$$

where δ^i_j is the Kronecker delta.

To further clarify the geometry of submanifolds in a symplectic manifold, we introduce the notion of the *symplectic orthogonal complement*. Let $N\subset M$ be a submanifold of the symplectic manifold (M,ω). For each point $x\in N$, define the symplectic orthogonal complement of the tangent space T_xN as

$$(T_xN)^\omega:=\{X\in T_xM\colon \omega(X,Y)=0\quad \forall\, Y\in T_xN\}\,.$$

In many contexts, one refers to the collection

$$N^\omega:=\bigcup_{x\in N}(T_xN)^\omega$$

as the symplectic orthogonal complement of N.

It is a standard fact that for every $x\in N$ we have

$$\dim T_xN+\dim(T_xN)^\omega=\dim T_xM\quad\text{and}\quad\left((T_xN)^\omega\right)^\omega=T_xN\,.$$

These identities naturally raise the question of whether one always has the direct-sum decomposition $T_xN\oplus(T_xN)^\omega=T_xM$. In general, this decomposition need not hold globally, as T_xN and $(T_xN)^\omega$ may have a nontrivial intersection.

Based on these properties, one classifies submanifolds as follows:

Definition 1.1.6. *Let (M, ω) be a symplectic manifold and $N \subset M$ a submanifold. Then:*

- *N is* isotropic *if $T_xN \subset (T_xN)^\omega$ for every $x \in N$.*
- *N is* coisotropic *if $(T_xN)^\omega \subset T_xN$ for every $x \in N$.*
- *N is* Lagrangian *if $T_xN = (T_xN)^\omega$ for every $x \in N$.*
- *N is a* symplectic submanifold *if $T_xN \cap (T_xN)^\omega = \{0\}$ for every $x \in N$.*

The existence of Darboux coordinates naturally gives rise to a natural Lagrangian foliation. In these coordinates, the projection onto the q-variables defines a foliation whose leaves, determined by constant values of the momenta $(p_1, \dots, p_n)$, are Lagrangian submanifolds. In other words, locally the phase space decomposes into invariant Lagrangian leaves, which play a fundamental role in the integrability and quantization of the system. Conversely, a Lagrangian foliation guarantees the existence of local Darboux coordinates adapted to the foliation, with the leaves identified as level sets of either the p_i or the q^i, depending on the chosen orientation.

1.1.2 *The Schouten-Nijenhuis bracket*

The *Schouten-Nijenhuis (SN) bracket*, also known as the Schouten bracket, was originally introduced in 1940 by Jan A. Schouten [171, 172] as an extension of the Lie bracket to tensor fields on a differentiable manifold. His motivation was related to the theory of invariants, so he searched for differential brackets of tensorial quantities and succeeded in defining a bracket for two arbitrary contravariant tensors, by imposing it to be a derivative in each of its arguments. In this way he obtained a bracket for contravariant tensors which are completely symmetric or completely antisymmetric (also known as multivectors). Although the work in both cases is similar in spirit, the latter case has become more popular because of its direct applications in differential geometry and mathematical physics.

The SN bracket for multivectors was later studied in more detail in 1955 by Albert Nijenhuis [143], who determined its main properties. In the 1970s, the mathematician André Lichnerowicz linked this bracket to classical mechanics. Specifically, he defined the Poisson and Jacobi structures [112, 113] and observed that one could deduce intrinsic conditions under which a field of bivectors defines a Poisson bracket, using the SN bracket for this purpose (see Proposition 1.1.2).

In this section we will only introduce the SN bracket for completely antisymmetric contravariant tensors, whose properties and applications can be found, for instance, in [193].

Definition 1.1.7. *The SN bracket for multivectors is uniquely defined as the bilinear operation $[\cdot,\cdot]_{\mathrm{SN}}$: $\mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$ which satisfies the following properties for all $P \in \mathfrak{X}^p(M)$, $Q \in \mathfrak{X}^q(M)$, $R \in \mathfrak{X}^r(M)$:*

- *Graded antisymmetry:*

$$[P, Q]_{\mathrm{SN}} = (-1)^{pq}[Q, P]_{\mathrm{SN}}\,; \tag{1.16}$$

- *Graded Leibniz rule:*

$$[P, Q \wedge R]_{\mathrm{SN}} = [P, Q]_{\mathrm{SN}} \wedge R + (-1)^{(p+1)q} Q \wedge [P, R]_{\mathrm{SN}}\,; \tag{1.17}$$

- *Graded Jacobi identity:*

$$(-1)^{p(r-1)}[P,[Q,R]_{\mathrm{SN}}]_{\mathrm{SN}} + (-1)^{q(p-1)}[Q,[R,P]_{\mathrm{SN}}]_{\mathrm{SN}} + (-1)^{r(q-1)}[R,[P,Q]_{\mathrm{SN}}]_{\mathrm{SN}} = 0\,. \tag{1.18}$$

Consider two multivectors $P \in \mathfrak{X}^p(M)$ and $Q \in \mathfrak{X}^q(M)$. To obtain the local expression in coordinates of the SN bracket $[P, Q]_{\mathrm{SN}}$, we can first assume that they are decomposable, i.e. $P = X_1 \wedge X_2 \wedge \ldots \wedge X_p$ and $Q = Y_1 \wedge Y_2 \wedge \ldots \wedge Y_q$, where $X_i, Y_j \in \mathfrak{X}(M)$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$. In this case we get

$$[P, Q]_{\mathrm{SN}} = (-1)^{p+1} \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots \wedge Y_q\,, \tag{1.19}$$

where $[X_i, Y_j]$ is the Lie bracket of the vector fields X_i and Y_j, and $\widehat{X}_i$ (resp. $\widehat{Y}_j$) denotes the omission of X_i (resp. Y_j) in the product.

Since in general P and Q can be written in a local chart $(x^1, x^2, \ldots, x^n)$ of M as

$$\begin{aligned} P &= \frac{1}{p!} \sum_{i_1, i_2, \ldots, i_p = 1}^{n} P^{i_1 i_2 \ldots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_2}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}}\,, \\ Q &= \frac{1}{q!} \sum_{j_1, j_2, \ldots, j_q = 1}^{n} Q^{j_1 j_2 \ldots j_q} \frac{\partial}{\partial x^{j_1}} \wedge \frac{\partial}{\partial x^{j_2}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_q}}\,; \end{aligned} \tag{1.20}$$

we conclude from the expression (1.19) that

$$\begin{aligned} [P, Q]_{\mathrm{SN}} = \frac{(-1)^p}{p!\, q!} \sum_{k=1}^{n} \Bigg(& \sum_{i_1, \ldots, i_p = 1}^{n} \sum_{j_2, \ldots, j_q = 1}^{n} q\, Q^{k j_2 \ldots j_q} \frac{\partial P^{i_1 \ldots i_p}}{\partial x^k} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}} \\ & \wedge \frac{\partial}{\partial x^{j_2}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_q}} \\ & + \sum_{i_2, \ldots, i_p = 1}^{n} \sum_{j_1, \ldots, j_q = 1}^{n} (-1)^p\, p\, P^{k i_2 \ldots i_p} \frac{\partial Q^{j_1 \ldots j_q}}{\partial x^k} \frac{\partial}{\partial x^{i_2}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}} \\ & \wedge \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_q}} \Bigg)\,. \end{aligned} \tag{1.21}$$

Remark 1.1.1. *When $X \in \mathfrak{X}(M)$ (and $Q \in \mathfrak{X}^q(M)$), the SN bracket reduces to the Lie derivative of Q with respect to X, i.e. $[X, Q]_{\mathrm{SN}} \equiv \mathcal{L}_X Q$. Likewise, when $X, Y \in \mathfrak{X}(M)$, this operation reduces to the Lie bracket, $[X, Y]_{\mathrm{SN}} \equiv [X, Y]$.*

1.1.3 *Poisson Manifolds*

Poisson brackets, when regarded as abstract algebraic structures, admit a rich geometric interpretation that provides deeper insight into dynamical systems. In Section 1.1.1 we saw that a Poisson bracket can be naturally defined on a symplectic manifold; however, one may consider more general differentiable manifolds endowed with arbitrary Poisson brackets. This framework will be useful in Section 3.1 when introducing bi-Hamiltonian systems.

Definition 1.1.8. *A* Poisson manifold *is a differentiable manifold M equipped with a Poisson bracket* $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$.

Since the Poisson bracket is a biderivation, its value at any point $x \in M$ depends only on the differentials $\mathrm{d}f(x)$ and $\mathrm{d}g(x)$. As stated in [2, Theorem 4.2.16], this observation motivates the definition of an antisymmetric bivector field $\boldsymbol{P}\colon \Omega^1(M)\times\Omega^1(M)\to C^\infty(M)$ such that

$$\{f,g\} = \boldsymbol{P}(\mathrm{d}f,\mathrm{d}g)\,. \tag{1.22}$$

The bivector $\boldsymbol{P}$ is commonly called the *Poisson bivector*. In local coordinates $(x^1,x^2,\dots,x^n)$, its components are given by $P^{ij}=\{x^i,x^j\}$, so that the Poisson bracket can be written as

$$\{f,g\} = \sum_{i,j=1}^{n} P^{ij}\,\frac{\partial f}{\partial x^i}\,\frac{\partial g}{\partial x^j}\,. \tag{1.23}$$

An important question is under what conditions an arbitrary bivector field defines a Poisson bracket. This is answered by the following proposition.

Proposition 1.1.2. *A bivector field $\boldsymbol{P}\in\mathfrak{X}^2(M)$ is a Poisson bivector if and only if its SN bracket with itself vanishes, i.e.,* $[\boldsymbol{P},\boldsymbol{P}]_{\mathrm{SN}}=\boldsymbol{0}$.

On a symplectic manifold (M,ω), the inverse of the isomorphism in Eq. (1.4) yields the Poisson bivector induced by ω, namely, $\boldsymbol{P}:=\omega^\sharp\colon\Omega^1(M)\to\mathfrak{X}(M)$. However, not every Poisson manifold admits a symplectic structure; the two notions coincide if and only if the Poisson bivector is nondegenerate.

Using the Poisson bivector $\boldsymbol{P}$, one may express the symplectic gradient of a function $H\in C^\infty(M)$ as

$$X_H = \boldsymbol{P}\,\mathrm{d}H\,. \tag{1.24}$$

This formulation underlies the equivalence between the symplectic and Poisson descriptions of Hamiltonian dynamics in the nondegenerate case, while the Poisson framework is more general and extends the notion of Hamiltonian vector fields.

1.1.4 *Jacobi Manifolds*

Although not the primary focus of this thesis, another generalization that extends the concepts of symplectic and Poisson manifolds is provided by *Jacobi manifolds*. As explained in [129], both Kirillov [91] and Lichnerowicz [112] independently arrived at equivalent definitions of this structure.

In Lichnerowicz's formulation, a differentiable manifold M is equipped with a bivector field Λ and a vector field E. A *Jacobi structure* on M is given by a bilinear map (the *Jacobi bracket*) $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, defined for $f,g\in C^\infty(M)$ by

$$\{f,g\} := \Lambda(\mathrm{d}f,\mathrm{d}g) + \langle f\,\mathrm{d}g - g\,\mathrm{d}f, E\rangle\,, \tag{1.25}$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between a 1-form and a vector field, and subject to the compatibility conditions

$$[E,\Lambda]_{SN} = \mathcal{L}_E\Lambda = 0 \quad \text{and} \quad [\Lambda,\Lambda]_{SN} = 2E\wedge\Lambda\,. \tag{1.26}$$

A manifold endowed with the triple (M,Λ,E) is called a *Jacobi manifold*.

Alternatively, Kirillov introduced an equivalent definition via the notion of "local Lie algebras," requiring that the Jacobi bracket be antisymmetric, satisfy the Jacobi identity, and be local in the sense that the support of $\{f,g\}$ is contained in the intersection of the supports of f and g.

The compatibility conditions in Eq. (1.26) ensure that the Jacobi bracket is a Lie bracket. However, it does not satisfy the standard Leibniz rule but rather a *weak Leibniz rule*:

$$\{f,gh\} = \{f,g\}h + \{f,h\}g + gh\,E(f)\,, \tag{1.27}$$

for all $f,g,h\in C^\infty(M)$. In particular, a Jacobi manifold reduces to a Poisson manifold if and only if $E=0$.

In this context, the Hamiltonian vector field associated with a function $H\in C^\infty(M)$ is defined by

$$X_H = \Lambda^\sharp\circ\mathrm{d}H + H\,E\,, \tag{1.28}$$

where the bundle map $\Lambda^\sharp\colon\Omega^1(M)\to\mathfrak{X}(M)$ is defined by

$$\langle\beta,\Lambda^\sharp(\alpha)\rangle := \Lambda(\alpha,\beta) \qquad \forall\alpha,\beta\in\Omega^1(M)\,.$$

Jacobi manifolds are relevant in classical mechanics since, in the case where Λ is nondegenerate and E is a Reeb vector field, the structure defines a *contact manifold*.

1.2 INTEGRABILITY IN CLASSICAL HAMILTONIAN MECHANICS

We can describe the evolution of a function defined on a symplectic manifold in terms of the Poisson bracket naturally induced on it. Indeed, let f be a smooth function on M and let $\gamma(t)$ be the trajectory of the Hamiltonian system (M,ω,H). Then, we have

$$\dot f(t) = \frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t} = \mathcal{L}_{\dot\gamma}f = \mathcal{L}_{X_H\circ\gamma}f = \{f,H\}\circ\gamma\,. \tag{1.29}$$

Therefore, if a function f commutes with the Hamiltonian function H, $\{f, H\} = 0$, it will be constant and we will say that it is a *(first) integral of motion*.

In general we will say that two functions $f, g \in C^\infty(M)$ are *in involution* if they Poisson-commute, $\{f, g\} = 0$. This concept will be central to the discussion of the integrability of Hamilton's equations, i.e. the solvability of the dynamics of a system.

One of the most important results on this topic dates from the mid-nineteenth century, and it was proved by Edmond Bour [29] for autonomous systems and generalized by Joseph Liouville [114] for the non-autonomous case.

Theorem 1.2.1 (Liouville)**.** *If a Hamiltonian system with n degrees of freedom has n integrals of the motion $(H_1 = H, H_2, \dots, H_n)$ in involution and functionally independent on the intersection of the level sets of the n functions $H_i = c_i$, where c_i are n real constants, then the solution of the associated Hamilton's equations can be found by quadratures.*

By *integrability by quadratures*, we mean that we can determine the solutions of the system by a finite number of steps involving algebraic operations, including the inversion of functions; and the integration of known functions, called quadratures. This systems are said to be *integrable in the Liouville sense*.

Liouville's theorem was generalized by Kozlov in 1983, following the ideas of Mishchenko and Fomenko on the "noncommutative Liouville theorem" [134]. Instead of requiring that the first integrals be in involution, Kozlov imposed that they form a solvable algebra. This result also holds for non-autonomous systems.

Theorem 1.2.2 (Theorem 1, [95])**.** *Let M be a symplectic manifold. Suppose that a Hamiltonian system has $n = \frac{1}{2} \dim M$ first integrals $(H_1 = H, H_2, \dots, H_n) \colon M \times \mathbb{R} \to \mathbb{R}$, satisfying*

$$\frac{\partial H_i}{\partial t} + \{H_i, H\} = 0, \quad i = 1, \dots, n,$$

and such that their Poisson brackets are given by

$$\{H_i, H_j\} = \sum_{k=1}^{n} c_{ij}^k H_k, \quad c_{ij}^k \in \mathbb{R}, \quad i, j = 1, \dots, n.$$

If

(i) on the level set

$$M_{\boldsymbol{h}} := \{(x, t) \in M \times \mathbb{R} \mid H_i(x, t) = h_i,\ 1 \le i \le n\},$$

the functions $H_1, H_2, \dots, H_n$ are functionally independent;

(ii) the structure constants satisfy

$$\sum_{k=1}^{n} c_{ij}^k H_k = 0 \quad \textit{for all } i, j = 1, \dots, n;$$

(iii) the Lie algebra generated by the linear combinations $\sum_{i=1}^{n} \lambda_i H_i$*, with* $\lambda_i \in \mathbb{R}$*, is solvable;*

then the solutions of the Hamiltonian system that lie on $M_{\boldsymbol{h}}$ *can be obtained by quadratures.*

In 1978, Vladimir I. Arnold also extended and refined Liouville's theorem:

Theorem 1.2.3 (Arnold-Liouville, [5])**.** *Let* (M, ω) *be a symplectic manifold of dimension* $2n$*. Suppose there exist* n *smooth functions* $(H_1, \dots, H_n) \in C^\infty(M)$ *that are in involution. Consider the joint level set of these functions,*

$$M_{\boldsymbol{h}} := \{x \in M \mid H_i(x) = h_i,\ 1 \leq i \leq n\}\,, \tag{1.30}$$

and assume that the n *functions* H_i *are functionally independent on* $M_{\boldsymbol{h}}$*. Then:*

(i) $M_{\boldsymbol{h}}$ *is a smooth manifold, invariant under the phase flow of the Hamiltonian* $H = H_1$*.*

(ii) If $M_{\boldsymbol{h}}$ *is compact and connected, it is diffeomorphic to the* n*-dimensional torus*

$$T^n = \{(\phi^1, \dots, \phi^n) \bmod 2\pi\}\,. \tag{1.31}$$

(iii) The phase flow of H *induces conditionally periodic motion on* $M_{\boldsymbol{h}}$*; in angular coordinates* $\boldsymbol{\phi} = (\phi^1, \dots, \phi^n)$*, this motion satisfies*

$$\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{\nu}(\boldsymbol{h})\,. \tag{1.32}$$

(iv) The canonical equations with Hamiltonian H *are integrable by quadratures.*

Remark 1.2.1. *If* $M_{\boldsymbol{h}}$ *is compact and connected, there exists a canonical transformation* $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{\phi}, \boldsymbol{J})$*, where* $(\boldsymbol{\phi}, \boldsymbol{J}) \in T^n \times V$ *(with* V *an open subset of* $\mathbb{R}^n$*) is a set of Darboux coordinates called* action-angle variables*. In this chart, the angles* $\boldsymbol{\phi} = (\phi^1, \phi^2, \dots, \phi^n)$ *parametrize the invariant torus* T^n*, while the actions* $\boldsymbol{J} = (J_1, J_2, \dots, J_n)$ *are constants of the motion. Moreover, the Hamiltonian* H *depends solely on the actions* $\boldsymbol{J}$*, making the dynamics separable. Hamilton's equations reduce to*

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{\partial H}{\partial \phi^i} = 0, \quad \frac{\mathrm{d}\phi^i}{\mathrm{d}t} = \frac{\partial H}{\partial J_i} = \textit{constant}, \qquad i = 1, 2, \dots, n, \tag{1.33}$$

and can be trivially integrated as

$$\boldsymbol{J}(t) = \boldsymbol{J}(0), \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}(0) + \boldsymbol{\nu}(\boldsymbol{J})\, t \quad (\textit{mod}\ 2\pi), \tag{1.34}$$

where $\nu_k(\boldsymbol{J}) := \frac{\partial H}{\partial J_k}$ *are the frequencies of the system.*

Remark 1.2.2. *This theorem establishes that the phase space of an autonomous Hamiltonian system with as many integrals of motion as degrees of freedom is* foliated into invariant tori*, provided the trajectories are bounded. The motion on each torus is conditionally periodic, with trajectories that are either periodic or dense depending on the ratios of the system's frequencies.*

Remark 1.2.3. *In completely integrable systems, the Liouville-Arnold theorem provides action-angle coordinates (a type of Darboux coordinates) where Lagrangian tori (leaves of the foliation) are defined by constant action variables. This exemplifies how Darboux coordinates encode foliations into symplectically orthogonal, non-intersecting leaves.*

The geometric structure of invariant tori is remarkably robust under small perturbations, as described by the Kolmogorov-Arnold-Moser (KAM) theorem. Specifically, the KAM theorem ensures that, for sufficiently smooth and non-degenerate Hamiltonians, many of these invariant tori persist under small perturbations, albeit they may become slightly deformed. This result, introduced in [92] (1954) and rigorously proved and extended later in [4] (1963) and [137] (1962), underscores the stability of integrable systems under realistic physical conditions.

Definition 1.2.1. *A completely integrable Hamiltonian system is* non-degenerate *if*

$$\det\left[\frac{\partial \nu_k}{\partial J_i}\right] \equiv \det\left[\frac{\partial^2 H}{\partial J_i\, \partial J_k}\right] \neq 0\,, \qquad i,k = 1,2,\ldots,n\,. \tag{1.35}$$

Proposition 1.2.1. *If a system is non-degenerate, the invariant tori are uniquely determined in the phase space as geometric structures, independent of the specific choice of action-angle coordinates* $(\boldsymbol{\phi}, \boldsymbol{J})$*. However, the parametrization of these coordinates remains non-unique, reflecting the flexibility in describing the dynamics within each torus.*

Remark 1.2.4. *Superintegrable systems, which will be discussed in Chapter 4, violate the non-degeneracy condition due to their additional integrals of motion. These extra constraints, however, often lead to regular dynamics such as closed periodic trajectories or other invariant structures. Although this differs from the assumptions of the KAM theorem, the stability of superintegrable systems arises from their over-constrained nature.*

1.3 HAMILTON-JACOBI THEORY

Hamilton-Jacobi theory is a fundamental milestone in theoretical physics. It was first developed by Sir William Rowan Hamilton (1805–1865), who sought to unify the concepts of geometrical optics and mechanics through an elegant mathematical formulation. Inspired by the trajectories of light rays, Hamilton extended these principles to the motion of mechanical systems, reducing the study of complex dynamics to a partial differential equation (PDE) known as the Hamilton-Jacobi (HJ) equation [68, 69].

Shortly afterwards, Carl Gustav Jacob Jacobi (1804–1851) further developed this theory by resolving certain mathematical issues and applying it to new problems [84]. This formulation not only revolutionized the study of classical mechanics but also influenced the development of quantum mechanics, serving as an important precursor to de Broglie's and Schrödinger's wave theories.

1.3.1 *Hamilton-Jacobi Equation*

We begin by considering a non-autonomous system with Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p}, t)$. One way to address the problem of integrating the corresponding Hamilton's equations (1.9) is to find a canonical transformation from the variables $(\boldsymbol{q}, \boldsymbol{p})$ to new variables $(\boldsymbol{Q}, \boldsymbol{P})$, with generating function $S(\boldsymbol{q}, \boldsymbol{P}, t)$, such that the Hamiltonian in the new coordinates is identically zero, i.e., $K(\boldsymbol{Q}, \boldsymbol{P}, t) \equiv 0$. In this case, the canonical equations become trivial and both the positions and momenta remain constant in the new coordinates.

Since the variables and the Hamiltonians before and after the canonical transformation are related by the identities

$$p_i = \frac{\partial S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial q^i}, \quad Q^i = \frac{\partial S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial P_i}, \qquad i = 1, 2, \ldots, n, \tag{1.36}$$

$$K(\boldsymbol{Q}, \boldsymbol{P}, t) = H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial S(\boldsymbol{q}, \boldsymbol{P}, t)}{\partial t}, \tag{1.37}$$

we deduce that the generating function $S(\boldsymbol{q}, \boldsymbol{P}, t)$ must satisfy the following nonlinear first-order PDE, known as the *HJ equation*:

$$H\left(\boldsymbol{q}, \frac{\partial S}{\partial \boldsymbol{q}}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{1.38}$$

The function S is known as *Hamilton's principal function* and it satisfies the invertibility condition

$$\det\left[\frac{\partial^2 S}{\partial q^i \, \partial h_j}\right] \neq 0, \tag{1.39}$$

where we set $h_i := P_i$ for $i = 1, 2, \ldots, n$; these represent the constant momenta in the new coordinates.

Remark 1.3.1. *To solve the system's dynamics, determining the* general integral *of the HJ equation–a solution involving arbitrary functions of the independent variables–is unnecessary. Instead, a* complete integral *suffices. This is defined as a solution containing n independent constants (where n is the number of degrees of freedom) and an arbitrary additive constant (irrelevant to the dynamics, as S appears only through its derivatives).*

Since, in this work, we deal only with autonomous systems, we can seek a solution S of the HJ equation (1.38) of the form

$$S(\boldsymbol{q}, \boldsymbol{h}, t) = W(\boldsymbol{q}, \boldsymbol{h}) - E(\boldsymbol{h})\, t, \tag{1.40}$$

where E is a smooth function of the n constants $\boldsymbol{h} = (h_1, h_2, \ldots, h_n)$ satisfying $\partial E / \partial \boldsymbol{h} \neq \boldsymbol{0}$. With the *ansatz* (1.40), the HJ equation reduces to its stationary version

$$H\left(\boldsymbol{q}, \frac{\partial W}{\partial \boldsymbol{q}}\right) = E(\boldsymbol{h}), \tag{1.41}$$

with E identified as the total energy. The function W, known as *Hamilton's characteristic function*, satisfies the nondegeneracy condition

$$\det\left[\frac{\partial^2 W}{\partial q^i\,\partial h_j}\right] = \det\left[\frac{\partial^2 S}{\partial q^i\,\partial h_j}\right] \neq 0. \tag{1.42}$$

Let Q be the configuration manifold associated with the Hamiltonian system (T^*Q, ω, H). As a consequence of the following theorem, W can be understood geometrically.

Proposition 1.3.1 (Theorem 4.1, [16]). *The image of a 1-form $\sigma\colon Q \to T^*Q$, viewed as a section of T^*Q, is a Lagrangian submanifold of T^*Q if and only if σ is closed.*

In particular, when a 1-form σ is exact, every differentiable function $G\colon Q \to \mathbb{R}$ generates a Lagrangian submanifold $L \subset T^*Q$ as the image of its differential $\mathrm{d}G \equiv \sigma$, considered as a section of T^*Q. In this case, we say that G is the *generating function of* L. Consequently, if $(\boldsymbol{q}, \boldsymbol{p})$ are canonical coordinates, the submanifold L is defined by the equations

$$p_i = \frac{\partial G}{\partial q^i}(\boldsymbol{q}), \qquad i = 1, \ldots, n. \tag{1.43}$$

Therefore, geometrically, W can be interpreted as the generating function of a Lagrangian foliation of T^*Q; this foliation is both transverse to the fibers and compatible with H. It is explicitly defined by the n parametric equations for the momenta:

$$p_i = \frac{\partial W}{\partial q^i}(\boldsymbol{q}; h_1, h_2, \ldots, h_n), \qquad i = 1, \ldots, n. \tag{1.44}$$

Owing to condition (1.42), these equations can be solved for the constants $(h_1, \ldots, h_n)$, yielding

$$h_i = H_i(\boldsymbol{q}, \boldsymbol{p}), \tag{1.45}$$

where $\boldsymbol{p} = (p_1, \ldots, p_n)$. These relations provide n independent integrals of motion that are in involution with each other and with H. In particular, these integrals are *vertically independent*, meaning that

$$\det\left[\frac{\partial H_i}{\partial p_k}\right] \neq 0. \tag{1.46}$$

Thus, a complete integral W for H determines a completely integrable Hamiltonian system $(H_1, H_2, \ldots, H_n)$ in the sense of Liouville and Arnold, and it provides a common solution of the stationary HJ equations associated with the Hamiltonians of the system. Conversely, a family of independent functions $(H_1, \ldots, H_n)$ that are in involution and vertically independent determines a complete integral of the HJ equations by solving Eqs. (1.45) with respect to the momenta $(p_1, \ldots, p_n)$.

Although this is the classical method to derive the HJ equation, it can also be interpreted in geometrical terms. In the upcoming section, we discuss autonomous systems using a geometric approach, following [167].

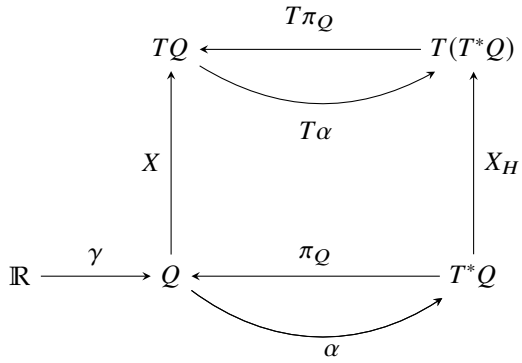
1.3.2 *Geometric approach to the Hamilton-Jacobi problem*

The geometric Hamilton-Jacobi theory provides a powerful framework for understanding Hamiltonian systems through symplectic geometry. Consider a Hamiltonian system (T^*Q, ω, H), where T^*Q is the cotangent bundle of the configuration manifold Q, ω is the canonical symplectic form, and H is the Hamiltonian function. The dynamics are governed by the Hamiltonian vector field X_H satisfying $\iota_{X_H}\omega = \mathrm{d}H$.

At its core, the HJ problem seeks to simplify these dynamics through geometric constraints. We begin with the generalized formulation:

Definition 1.3.1. *A pair (X, α), consisting of a vector field $X \in \mathfrak{X}(Q)$ and a 1-form $\alpha \in \Omega^1(Q)$, is said to solve the* generalized Hamiltonian Hamilton-Jacobi problem *if every integral curve $\gamma : \mathbb{R} \to Q$ of X lifts via α to an integral curve of X_H. In other words, if $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = X \circ \gamma$, then $\frac{\mathrm{d}}{\mathrm{d}t}(\alpha \circ \gamma) = X_H(\alpha \circ \gamma)$.*

This relationship is captured diagrammatically by the commutation of the associated maps:



Theorem 1.3.1 (Theorem 1, [167])**.** *The following characterizations are equivalent:*

(i) (X, α) *is a solution of the generalized HJ problem.*

(ii) The vector field satisfies $X_H \circ \alpha = T\alpha \circ X$, so that $X = T\pi_Q \circ X_H \circ \alpha$, in which case X is called the vector field associated with α.

(iii) The submanifold $\mathrm{Im}(\alpha) \subset T^*Q$ is invariant under X_H, meaning that X_H is tangent to $\mathrm{Im}(\alpha)$.

(iv) The integral curves of X_H with initial conditions in $\mathrm{Im}(\alpha)$ project onto the integral curves of X.

(v) The differential condition $\iota_X \mathrm{d}\alpha = -\mathrm{d}(\alpha^* H)$ holds, where α^* denotes the pullback by α.

The standard HJ problem further requires that α be closed, ensuring the local existence of generating functions and establishing a connection with Lagrangian submanifolds:

Definition 1.3.2. *A 1-form α is said to solve the* Hamiltonian HJ problem *if it satisfies the generalized problem and is closed. Locally, one may write $\alpha = \mathrm{d}W$ for some function $W : Q \to \mathbb{R}$, called the* generating function.

Theorem 1.3.2. *For $\alpha = \mathrm{d}W$, the following conditions are equivalent:*

(i) *α is a solution of the standard Hamilton-Jacobi problem.*

(ii) *$\mathrm{Im}(\alpha)$ is a Lagrangian submanifold of T^*Q invariant under X_H, and W serves as a local generating function for this Lagrangian submanifold.*

(iii) *The equation $\mathrm{d}(\alpha^* H) = 0$ holds, implying that the function $H \circ \mathrm{d}W : Q \to \mathbb{R}$ is locally constant.*

This local constancy appears in coordinates as the classical HJ equation:

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E \equiv \text{constant},$$

which corresponds to Eq. (1.41).

Complete solutions extend this framework by parameterizing families of solutions. A complete solution $\{\alpha_{\boldsymbol{h}}\}$ induces a foliation of the phase space where each leaf corresponds to conserved quantities. More precisely, if we define the map

$$\Phi : Q \times \mathbb{R}^n \to T^*Q, \quad \Phi(q, \boldsymbol{h}) = \alpha_{\boldsymbol{h}}(q),$$

then Φ is a local diffeomorphism.

This geometric perspective unifies several fundamental concepts: the symplectic structure encodes dynamics via X_H, closed 1-forms generate invariant Lagrangian submanifolds, and complete solutions correspond to the action-angle variables of integrable systems. In this way, the Hamilton-Jacobi equation emerges not merely as a computational tool but as a structural feature of the symplectic landscape, linking local generating functions to global foliations of the phase space.

1.3.3 *Separation of variables for the Hamilton-Jacobi equation*

Solving the HJ equation is generally challenging because no universal method exists for its solution. However, the technique of *separation of variables (SoV)* has proven to be highly effective [86, 101].

To illustrate this method, consider the case where the stationary HJ equation involves a specific variable q^1 and its associated derivative $\frac{\partial W}{\partial q^1}$ in a combination that is independent of the remaining variables, derivatives, and time. This combination is represented by a function $\varphi\left(q^1, \frac{\partial W}{\partial q^1}\right)$. In this scenario, the variable q^1 is said to be *separable*, and the HJ equation takes the form

$$H\left(\varphi\left(q^1, \frac{\partial W}{\partial q^1}\right), q^2, \ldots, q^n, \frac{\partial W}{\partial q^2}, \ldots, \frac{\partial W}{\partial q^n}\right) = E. \tag{1.47}$$

In this case, a solution may be sought in the form

$$W = W_1(q^1; \boldsymbol{h}) + \tilde{W}(q^2, \dots, q^n; \boldsymbol{h}), \tag{1.48}$$

where, because q^1 appears only through the function φ, one sets

$$\varphi\left(q^1, \frac{\partial W_1}{\partial q^1}\right) = h_1, \tag{1.49}$$

with h_1 constant. Meanwhile, the function $\tilde{W}$ satisfies the reduced equation

$$H\left(h_1, q^2, \dots, q^n, \frac{\partial \tilde{W}}{\partial q^2}, \dots, \frac{\partial \tilde{W}}{\partial q^n}\right) = E. \tag{1.50}$$

The equation for W_1 is an ordinary differential equation (ODE) that can be solved by quadratures:

$$W_1(q^1; h_1, \dots, h_n) = \int^{q^1} \frac{\partial W_1}{\partial \xi}(\xi; h_1, \dots, h_n)\, \mathrm{d}\xi, \tag{1.51}$$

while the reduced equation for $\tilde{W}$ is a PDE in fewer variables than the original HJ equation.

In favorable cases, this procedure can be applied iteratively to fully solve the problem by successive quadratures. A system is said to be *(totally) separable* if there exists a coordinate system in which Hamilton's characteristic function can be expressed as

$$W = \sum_{k=1}^{n} W_k(q^k; h_1, \dots, h_n) = \sum_{k=1}^{n} \int^{q^k} \frac{\partial W_k}{\partial \xi^k}(\xi^k; h_1, \dots, h_n)\, \mathrm{d}\xi^k. \tag{1.52}$$

When only a subset of the coordinates can be separated, the system is referred to as *partially separable* [44, 150, 160, 177]. This intermediate case will be explored in detail in Chapter 5.

Given the fundamental importance of separability in the context of the HJ equation, it has been a central topic of study in classical mechanics. On one hand, criteria have been developed to determine whether a Hamiltonian system is separable in a particular coordinate system, as will be discussed in Section 1.3.4. On the other hand, novel geometric and algebraic techniques have been proposed to identify the coordinate systems in which the HJ equation of an integrable Hamiltonian system becomes separable. These approaches will be examined in Chapter 3 and throughout this thesis.

1.3.4 *Separability criteria*

The problem of finding separating variables for integrable Hamiltonian systems has been extensively investigated. In 1904, Levi-Civita proposed a test which permits to establish whether a given Hamiltonian separates in an assigned coordinate system.

Theorem 1.3.3 (Levi-Civita, [105]). *The HJ equation* (1.38) *related to a Hamiltonian function H is separable in a specific chart $(\boldsymbol{q}, \boldsymbol{p})$ if and only if the following set of $\frac{1}{2}n(n-1)$ equations is identically satisfied:*

$$\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial q^i \partial q^j} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial q^i \partial p_j} - \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial q^j} + \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial q^j}\frac{\partial^2 H}{\partial p_i \partial p_j} = 0\,, \qquad 1 \le i < j \le n, \quad (1.53)$$

where there is no summation over the indices.

In the same spirit, but using more modern geometric techniques, we have the subsequent results. First, Benenti's analysis led to the following theorem, which will be of crucial importance throughout the thesis.

Theorem 1.3.4 (Benenti, [10, 13]). *An integrable family of Hamiltonian functions $(H_1, H_2, \dots, H_n)$ is totally separable in a set of canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ if and only if they are in* separable involution, *i.e., if they satisfy the relations*

$$\{H_i, H_j\}|_k = \frac{\partial H_i}{\partial q^k}\frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q^k} = 0\,, \qquad k = 1, 2, \dots, n\,, \quad (1.54)$$

where no summation over k is understood. In this case, we shall say that the Hamiltonians $(H_1, \dots, H_n)$ are in total separable involution *or $\mathcal{T}$-involution (this notion was originally named "separable involution" or $\mathcal{S}$-involution in [10]).*

However, such theorem as well as the Levi-Civita test are not constructive, and do not allow us to determine a complete integral of the HJ equation. A constructive approach for the determination of separating variables was given by Sklyanin [174] within the framework of Lax systems.

Theorem 1.3.5 (Jacobi-Sklyanin). *The integrable family of Hamiltonian functions $(H_1, H_2, \dots, H_n)$ is separable in a set of canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ if there exist n suitable equations, called the Jacobi-Sklyanin separation equations (SE) for $\{H_i\}_{1 \le i \le n}$, having the form*

$$\Phi_i(q^i, p_i; H_1, H_2, \dots, H_n) = 0\,, \qquad \det\left[\frac{\partial \Phi_i}{\partial H_j}\right] \neq 0\,, \quad i = 1, 2, \dots, n\,. \quad (1.55)$$

When $H_1 = h_1, \dots, H_n = h_n$, the SE represent implicitly the Lagrangian transversal foliation (1.44). Moreover, the SE allow us to construct a separated solution to the HJ equations. In fact, by solving Eqs. (1.55) with respect to the momenta p_k, one gets the Lagrangian foliation in the form of a set of separated first-order ODEs:

$$p_k = \frac{\partial W_k}{\partial q^k}(q^k; h_1, \dots, h_n)\,, \qquad k = 1, \dots, n\,, \quad (1.56)$$

which implies that W acquires the additively separated expression (1.52).

Nevertheless, the three above-mentioned criteria of separability are not intrinsic: in order to be applied, they require the explicit knowledge of the local chart $(\boldsymbol{q}, \boldsymbol{p})$. To overcome such a drawback, in the last decades the modern theory of SoV has been conceived in the context of symplectic and Poisson geometry; in particular, the bi-Hamiltonian theory has offered a fundamental geometric insight into the theory of integrable systems [52, 125]. These approaches, together with more modern ones, will be analyzed in the following chapter.

1.4 REMARKABLE FAMILIES OF INTEGRABLE SYSTEMS

1.4.1 *Generalized and classical Stäckel systems*

Among all the possible SE (1.55), we can consider, as an important example, the case where they are *affine* in the functions $(H_1, \dots, H_n)$. Within this scenario lies the large class of *generalized Stäckel systems* introduced by Arnold, Kozlov, and Neishtadt (AKN) in [6], which provides us with a novel insight in the theory of completely separable systems. More precisely, they considered the SE

$$\sum_{j=1}^{n} S_{ij}(q^i)\, H_j - f_i(q^i, p_i) = 0\,, \qquad i = 1, \dots, n\,, \tag{1.57}$$

leading to the AKN Hamiltonian functions

$$H_j = \sum_{k=1}^{n} \frac{\tilde{S}_{jk}}{\det \boldsymbol{S}}\, f_k(q^k, p_k)\,, \qquad j = 1, \dots, n\,. \tag{1.58}$$

In the previous expressions, S_{ij} are the elements of a *Stäckel matrix* $\boldsymbol{S}(\boldsymbol{q})$ (i.e., an invertible matrix whose ith row depends only on the coordinate q^i), $\tilde{S}_{jk}$ denotes the cofactor of the element S_{kj}, and $f_k(q^k, p_k)$ are the so-called *Stäckel functions*. Equivalently, Eqs. (1.58) can be written in matrix form as

$$\begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix} = \boldsymbol{S}^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ \vdots \\ f_n(q^n, p_n) \end{bmatrix}, \tag{1.59}$$

where

$$|\boldsymbol{F}\rangle := \begin{bmatrix} f_1(q^1, p_1) \\ \vdots \\ f_n(q^n, p_n) \end{bmatrix} \tag{1.60}$$

is said to be a *Stäckel vector*.

It can be shown that the functions H_j are in involution among each other; consequently, they represent a completely integrable Hamiltonian system. Recently, a new approach to totally separable systems within the framework of Stäckel geometry has been developed in [124].

A specific yet very important case of this family of systems is that of the *classical separable Stäckel systems* [148, 177]. These are recovered from Eqs. (1.58) when the Stäckel functions take the form

$$f_k(q^k, p_k) = \frac{1}{2}\, p_k^2 + W_k(q^k)\,, \qquad k = 1, \dots, n\,. \tag{1.61}$$

Thus, we obtain the Hamiltonian function

$$H_1 = \frac{1}{2} \sum_{j=1}^{n} g^{jj}(\boldsymbol{q})\, p_j^2 + V(\boldsymbol{q})\,, \tag{1.62}$$

which is quadratic and orthogonal in the momenta. Here, the functions $g^{jj}(\boldsymbol{q}) = \frac{\tilde{S}_{1j}}{\det S}$ can be regarded as the diagonal components of the inverse of a metric tensor over the configuration space Q:

$$\boldsymbol{G} := \sum_{j=1}^{n} g^{jj}(\boldsymbol{q})\, \frac{\partial}{\partial q^j} \otimes \frac{\partial}{\partial q^j}\,. \tag{1.63}$$

Moreover, $V(\boldsymbol{q}) = \sum_{j=1}^{n} g^{jj}(\boldsymbol{q})\, W_j(q^j)$ is the corresponding potential energy.

1.4.2 *Lax pairs*

The Lax pair method was developed by the mathematician Peter Lax in 1968 [104], and it allows us to obtain invariants of differential equations of the form $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = f(\boldsymbol{x})$, with $\boldsymbol{x} = (x^1, x^2, \dots, x^n)$, when we are able to express them in the form of the Lax equation, which is defined as follows:

Definition 1.4.1. *A* Lax pair *is a pair of linear operators $\boldsymbol{L}$, $\boldsymbol{M}$ that explicitly depend on an evolution parameter t (and also possibly on the $\boldsymbol{x}$-coordinates), act on a Hilbert space and satisfy the* Lax equation*:*

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = [\boldsymbol{M}, \boldsymbol{L}]\,, \tag{1.64}$$

where $[\boldsymbol{M}, \boldsymbol{L}] = \boldsymbol{M}\boldsymbol{L} - \boldsymbol{L}\boldsymbol{M}$ is the usual commutator of two operators.

Remark 1.4.1. *The Lax equation* (1.64) *is invariant under the transformation*

$$\boldsymbol{L} \mapsto \boldsymbol{L}' = \boldsymbol{N}\,\boldsymbol{L}\,\boldsymbol{N}^{-1}\,, \qquad \boldsymbol{M} \mapsto \boldsymbol{M}' = \boldsymbol{N}\,\boldsymbol{M}\,\boldsymbol{N}^{-1} + \frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}t}\,\boldsymbol{N}^{-1}\,, \tag{1.65}$$

where $\boldsymbol{N}(\boldsymbol{x}, t)$ is an invertible linear operator. Therefore, the Lax pair associated with a problem is not unique.

Remark 1.4.2. *We can interpret the Lax equation as the compatibility condition of the system*

$$\begin{cases} \boldsymbol{L}\,X = \lambda\, X\,, \\ \dfrac{\mathrm{d}X}{\mathrm{d}t} = \boldsymbol{M}\,X; \end{cases} \tag{1.66}$$

where the parameter λ is time-independent. This spectral problem was of great help in the development of the Inverse Scattering Method (ISM), originally introduced in 1967 by Gardner, Greene, Kruskal and Miura to solve the Korteweg-de Vries (KdV) equation [63].

Theorem 1.4.1. *The operator $\boldsymbol{L}$ is isospectral, i.e. its spectrum is independent of t.*

Proof. To prove this result, we first deduce that $\boldsymbol{L}(t)$ is similar to $\boldsymbol{L}(0)$. We do this by defining the invertible matrix $\boldsymbol{U}(t)$ that satisfies $\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \boldsymbol{M}\,\boldsymbol{U}$ with initial condition $\boldsymbol{U}(0) = \boldsymbol{I}$. Thus, by direct computation, we can prove that the operator $\boldsymbol{L}(t) = \boldsymbol{U}(t)\,\boldsymbol{L}(0)\,\boldsymbol{U}^{-1}(t)$ is the solution of the Lax equation with initial condition $\boldsymbol{L}(0)$, so that $\boldsymbol{L}(t)$ is indeed similar to $\boldsymbol{L}(0)$.

Then we observe that two matrices related by a similarity transformation have the same eigenvalues. Thus, if $\boldsymbol{L}(0)\,X = \lambda\,X$, then

$$\boldsymbol{L}(t)\,(\boldsymbol{U}\,X) = \boldsymbol{U}\,\boldsymbol{L}(0)\,\boldsymbol{U}^{-1}\,(\boldsymbol{U}\,X) = \lambda\,(\boldsymbol{U}\,X)\,. \tag{1.67}$$

It is therefore concluded that the eigenvalues of $\boldsymbol{L}$ are conserved along the evolution. □

However, there is another way in which we can use the operator $\boldsymbol{L}$ to calculate the constants of motion:

Theorem 1.4.2. *The traces of the powers of the operator $\boldsymbol{L}$ provide n conserved quantities, which are not necessarily independent:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(tr\,\boldsymbol{L}^k\right) = 0\,, \qquad k = 1,\,2,\,\ldots\,. \tag{1.68}$$

Proof. This result can be derived by direct calculation using the Lax equation (1.64) and the cyclic property of the trace. □

In practice, this approach is more useful for obtaining conserved quantities associated with a system. Nevertheless, to be integrable in the Liouville sense, there would have to be enough of them and they would have to be in involution, which is not always true but is easy to check.

Example 1.4.1 (Calogero model)**.** *To exemplify the use of Lax pairs, we will introduce here the Calogero model, which will also serve to illustrate the concepts presented in Chapters 4 and 5 of this thesis. This system was introduced by Francesco Calogero in 1969 [33] and further studied in [35]; and it describes the behaviour of n particles in a line interacting with an inverse square potential. Thus, the Hamiltonian function of the model is*

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i,\,j=1;\,i<j}^{n} \frac{1}{(q^i - q^j)^2} \\ &\equiv \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_{Cal}\,, \end{aligned} \tag{1.69}$$

where $V_{Cal} := \sum_{i,j=1;i<j}^{n} \frac{1}{(q^i-q^j)^2}$ *is the potential energy of the system. Hamilton's canonical equations* (1.9) *associated with this system can be expressed in terms of the Lax equation* (1.64) *by introducing the operators* $\boldsymbol{L}$ *and* $\boldsymbol{M}$ *[138, 146]:*

$$\boldsymbol{L}_{jk} = \delta_{jk}\, p_j + (1-\delta_{jk})\,\frac{i}{q^j-q^k}, \qquad j,k = 1,2,\ldots,n; \tag{1.70}$$

$$\boldsymbol{M}_{jk} = -\delta_{jk}\sum_{l=1,\,l\neq j}^{n}\frac{i}{(q^j-q^l)^2} + (1-\delta_{jk})\frac{i}{(q^j-q^k)^2}, \qquad j,k = 1,2,\ldots,n, \quad j\neq k. \tag{1.71}$$

Our subsequent discussions will be restricted to the three-particle case ($n = 3$)*, which we shall refer to as the* Jacobi-Calogero model*. This case had already been studied by Jacobi more than a century earlier [83], but it fell into oblivion until its rediscovery in [148]. We can derive constants of motion for the Jacobi-Calogero model from the operator* $\boldsymbol{L}$ *as*

$$H_k = \frac{1}{k}\, tr\, \boldsymbol{L}^k, \qquad k = 1,2,3, \tag{1.72}$$

which explicitly take the expressions

$$H_1 = p_1 + p_2 + p_3\,, \tag{1.73}$$

$$H_2 = \frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right) + \frac{1}{(q^1-q^2)^2} + \frac{1}{(q^1-q^3)^2} + \frac{1}{(q^2-q^3)^2} \equiv H\,, \tag{1.74}$$

$$H_3 = \frac{1}{3}\left(p_1^3+p_2^3+p_3^3\right) + \frac{p_1+p_2}{(q^1-q^2)^2} + \frac{p_1+p_3}{(q^1-q^3)^2} + \frac{p_2+p_3}{(q^2-q^3)^2}\,. \tag{1.75}$$

These quantities are sufficient to ensure integrability in the Liouville sense, since they are functionally independent and in involution. Additionally, the function H_2 *coincides with the Hamiltonian of the system.*

¿Quién puede entender los mil hilos que unen las almas de los hombres
y el alcance de sus palabras?

Carmen Laforet, *Nada*

2

Nijenhuis and Haantjes operators

Time has a way of burying things–not out of neglect, but like the earth holds a seed, waiting for the right season to stir. Some ideas, like stories half-told, slip into silence, only to resurface decades later in the hands of strangers. You might call it chance. Or perhaps it's the quiet persistence of curiosity: a thread dropped by one thinker, frayed by years, yet still strong enough for another to grasp and follow. There's a kinship in this–how a life's work, shelved and dusted by history, can suddenly feel alive again, as if it had been waiting all along for someone to notice, to lean in and ask, *What if?* It's not resurrection. It's conversation. A dialogue across eras, where the past doesn't speak to us, but *through us*, in the language of questions we've only just learned to hear. Such is the story of Haantjes' discoveries.

In the early 20th century, a distinct geometric school emerged in the Netherlands, anchored by Jan Arnoldus Schouten's (1883–1971) pioneering work in tensor calculus. A polymath whose career bridged engineering and pure mathematics, Schouten redefined the language of differential geometry through his systematic study of "direct quantities"–what we now call tensors. His *Ricci Calculus* (1954) became a cornerstone, synthesizing decades of insights into a cohesive framework. Yet Schouten's legacy extended beyond notation: his 1940 discovery of a differential concomitant for pairs of contravariant tensors–later named the Schouten-Nijenhuis bracket–laid the groundwork for modern Poisson geometry. This bracket, a graded generalization of the Lie derivative, would quietly permeate mathematics, from deformation theory to symplectic geometry, though its full significance took decades to unfold.

Schouten's intellectual heir, Albert Nijenhuis (1926–2015), refined and expanded this vision. In his 1951 thesis, Nijenhuis introduced a tensor measuring the obstruction to integrability of geometric structures–aptly termed *torsion* for its conceptual kinship with the torsion of complex manifolds. Just as torsion in complex geometry quantifies the failure of an almost complex structure to be holomorphic, the *Nijenhuis torsion* encodes the deviation of a general operator from integrability. This choice of nomenclature bridged disciplines, embedding his work into both differential and complex geometry. Nijenhuis' collaboration with Alfred Frölicher further yielded the *Frölicher-Nijenhuis bracket*, a graded Lie algebra structure on vector-valued forms that became crucial in deformation theory. Unlike Schouten's bracket, which persisted as a quiet undercurrent, Nijenhuis' torsion tensor swiftly gained prominence, underpinning foundational results such as the Newlander-Nirenberg theorem for complex structures.

Yet not all threads of the Netherlands school flourished equally. In 1955, Johannes Haantjes (1916–1956)–a former assistant to Schouten–introduced operators of profound subtlety in an article that languished in obscurity, cited only twice before 1996. The *Haantjes torsion*, a higher-order obstruction to integrability, demanded intricate tensor calculations impractical in the pre-computational era. Its neglect mirrored the fate of Haantjes himself, who died young, leaving his work dormant until Evgeny Ferapontov's late 20th-century revival. The concept reappeared in algebraic contexts through Bogoyavlenskij's investigations [19, 22], while its resurgence emerged mainly in the framework of infinite-dimensional hydrodynamic chains at the hands of Ferapontov. Today, Haantjes' operators resurface as keystones in classical Hamiltonian systems, their algebraic elegance offering new pathways to integrability. This resurgence underscores a recurring truth: ideas, like seeds, often require the right season to germinate.

The interest in the geometry of Nijenhuis and Haantjes tensors recently has considerably increased. Apart from their relevance in classical mechanics, which will be discussed in this thesis, new applications have been found in many different contexts: for instance, the characterization of integrable chains of partial differential equations of hydrodynamic type (see e.g. [55, 56]) and their integrable reductions [57], as well as the study of topological field theories [119]. Also, Nijenhuis geometry, which plays a crucial role in the theory of almost complex structures [141], has been extended to contexts as the study of analytic matrix functions, linearization theory, operator algebras [26], and cofactor and bi-cofactor structures [15].

Situated within this lineage of inquiry, this chapter revisits key contributions of the Netherlands school and subsequent developments. While Section 1.1.2 from the previous chapter addressed the SN bracket, Section 2.1 introduces the FN bracket. Section 2.2 examines Nijenhuis operators and their principal algebraic properties, followed by Section 2.3, which reintroduces Haantjes operators–from obscurity to their resurgence in relation to the integrability of their eigendistributions. Section 2.4 constructs Haantjes algebras that unify multiple operators.

Recalling the notation from the previous chapter, let M be a smooth real manifold. We denote its tangent bundle by TM and its cotangent bundle by T^*M. The smooth sections of TM form the Lie algebra of vector fields, written as $\mathfrak{X}(M)$, while the smooth sections of T^*M form the space of 1-forms, denoted $\Omega^1(M)$. Moreover, for each nonnegative integer p, the module $\Omega^p(M)$ comprises

the differential p-forms on M, and $\mathfrak{X}^p(M)$ represents the space of p-vector fields (or multivector fields). Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a smooth $(1,1)$-tensor field, i.e., a linear operator on vector fields. The *transpose* of $\boldsymbol{K}$, denoted $\boldsymbol{K}^\top\colon \Omega^1(M) \to \Omega^1(M)$, is defined by the duality pairing:

$$\langle \boldsymbol{K}^\top \alpha, X\rangle = \langle \alpha, \boldsymbol{K}X\rangle \quad \forall \alpha \in \Omega^1(M),\ X \in \mathfrak{X}(M).$$

Throughout this work, all tensor and vector fields are assumed to be smooth. The identity operator on $\mathfrak{X}(M)$ is denoted by $\boldsymbol{I}$.

2.1 THE FRÖLICHER-NIJENHUIS BRACKET

In the mid-twentieth century, Alfred Frölicher and Albert Nijenhuis introduced a bracket operation on vector-valued differential forms-now known as the *Frölicher–Nijenhuis (FN) bracket*-that generalizes the classical Lie bracket of vector fields.

Definition 2.1.1. *Let R be a vector-valued r-form and S a vector-valued s-form on a smooth manifold M. The* FN bracket $[R,S]_{\mathrm{FN}}$ *is defined as the unique vector-valued $(r+s)$-form satisfying*

$$\mathcal{L}_{[R,S]_{\mathrm{FN}}} := [\mathcal{L}_R, \mathcal{L}_S], \tag{2.1}$$

where $\mathcal{L}_R$ and $\mathcal{L}_S$ denote the Lie derivatives along R and S, respectively.

This bracket endows the space of vector-valued differential forms on M with the structure of a graded Lie algebra, as it satisfies the following properties:

- Graded antisymmetry:
$$[R,S]_{\mathrm{FN}} = -(-1)^{rs}[S,R]_{\mathrm{FN}}, \tag{2.2}$$
- Graded Jacobi identity:
$$[R,[S,T]_{\mathrm{FN}}]_{\mathrm{FN}} = [[R,S]_{\mathrm{FN}},T]_{\mathrm{FN}} + (-1)^{rs}[S,[R,T]_{\mathrm{FN}}]_{\mathrm{FN}}, \tag{2.3}$$

for any vector-valued r-form R, s-form S, and t-form T.

In particular, when R and S are ordinary vector fields (i.e., elements of $\mathfrak{X}(M)$), the FN bracket coincides with the classical Lie bracket:

$$[X,Y]_{\mathrm{FN}} \equiv [X,Y] \quad \text{for } X,Y \in \mathfrak{X}(M).$$

Of particular interest is the case where both arguments of the bracket are operator fields, which will be the primary focus of our subsequent analysis.

Theorem 2.1.1 ([60])**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $(1,1)$-tensor fields on M. Their FN bracket is the vector-valued 2-form defined by*

$$\begin{aligned}[\boldsymbol{A},\boldsymbol{B}]_{\mathrm{FN}}(X,Y) = {}& \big(\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}\boldsymbol{A}\big)[X,Y] + [\boldsymbol{A}X,\boldsymbol{B}Y] + [\boldsymbol{B}X,\boldsymbol{A}Y] \\ & - \boldsymbol{A}\big([X,\boldsymbol{B}Y]+[\boldsymbol{B}X,Y]\big) - \boldsymbol{B}\big([X,\boldsymbol{A}Y]+[\boldsymbol{A}X,Y]\big), \quad X,Y \in \mathfrak{X}(M)\,. \end{aligned} \tag{2.4}$$

2.2 NIJENHUIS OPERATORS

This section reviews fundamental properties and results related to the Nijenhuis torsion and Nijenhuis operators [142, 143].

2.2.1 *The Nijenhuis torsion*

Let M be a real differentiable manifold of dimension n, $\mathfrak{X}(M)$ the Lie algebra of vector fields on M and $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ a smooth $(1,1)$-tensor field, i.e. a linear operator field. Henceforth, all tensor and vector fields will be considered to be smooth.

Definition 2.2.1. *The* Nijenhuis torsion *of $\boldsymbol{K}$ is the vector-valued 2-form defined by*

$$\tau_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2\,[X,Y] + [\boldsymbol{K}X,\boldsymbol{K}Y] - \boldsymbol{K}\left([X,\boldsymbol{K}Y] + [\boldsymbol{K}X,Y]\right), \tag{2.5}$$

where $X, Y \in \mathfrak{X}(M)$ and $[\,\cdot\,,\cdot\,]$ denotes the commutator of two vector fields.

In local coordinates $\boldsymbol{x} = (x^1,\dots,x^n)$, it can be represented as the skew-symmetric $(1,2)$-tensor field

$$(\tau_{\boldsymbol{K}})^i_{j\,k} = \sum_{\alpha=1}^{n}\left(\frac{\partial \boldsymbol{K}^i_k}{\partial x^\alpha}\,\boldsymbol{K}^\alpha_j - \frac{\partial \boldsymbol{K}^i_j}{\partial x^\alpha}\,\boldsymbol{K}^\alpha_k + \left(\frac{\partial \boldsymbol{K}^\alpha_j}{\partial x^k} - \frac{\partial \boldsymbol{K}^\alpha_k}{\partial x^j}\right)\boldsymbol{K}^i_\alpha\right), \tag{2.6}$$

containing $\frac{1}{2}\,n^2\,(n-1)$ independent components.

The Nijenhuis torsion is also directly related to the FN bracket. Indeed, choosing both entries equal to $\boldsymbol{K}$ in Eq. 2.4 it follows that

$$\tau_{\boldsymbol{K}}(X,Y) = \frac{1}{2}\,[\boldsymbol{K},\boldsymbol{K}]_{\mathrm{FN}}(X,Y)\,. \tag{2.7}$$

2.2.2 *Definition and examples*

The vanishing of the Nijenhuis torsion is a sufficient condition for its eigendistributions to be integrable. Although this fact is of central importance for the theory, we will explain in detail what it means and what its consequences are when we deal with the Haantjes operators (see Section 2.3.2), since they are the main topic of this thesis. This result motivates the definition of the Nijenhuis operators:

Definition 2.2.2. *A* Nijenhuis operator *is a $(1,1)$-tensor field whose Nijenhuis torsion identically vanishes.*

We can look at a number of simple cases to illustrate this concept.

Example 2.2.1. *Since all the summands of Eq. (2.6) involve derivatives of the components of the tensor $\boldsymbol{K}$, we deduce that the Nijenhuis torsion of any constant operator field vanishes.*

Example 2.2.2. *[27] Consider a two-dimensional manifold M, on which we define an operator $\boldsymbol{K}$: $\mathfrak{X}(M) \to \mathfrak{X}(M)$. In such a case*

- *if its trace is constant, then its determinant is also constant and $\boldsymbol{K}$ is a Nijenhuis operator;*
- *if $\boldsymbol{K}$ is a nilpotent operator, then it is a Nijenhuis operator.*

Example 2.2.3. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a smooth tensor field that in some local coordinate chart $\boldsymbol{x} = (x^1, x^2, \dots, x^n)$ is represented in the diagonal form*

$$\boldsymbol{K}(\boldsymbol{x}) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) \, \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i \,, \tag{2.8}$$

where $\lambda_i(\boldsymbol{x}) := \lambda^i_i(\boldsymbol{x})$ are the eigenvalue fields of $\boldsymbol{K}$. The components of its Nijenhuis torsion are

$$(\tau_{\boldsymbol{K}})^i_{j\,k} = (\lambda_j - \lambda_k) \left(\frac{\partial \lambda_j}{\partial x^k} \, \delta^i_j + \frac{\partial \lambda_k}{\partial x^j} \, \delta^i_k \right). \tag{2.9}$$

It is clear that $(\tau_{\boldsymbol{K}})^i_{j\,k} = 0$ when i, j and k are all distinct or when $j = k$. Therefore, we can focus on examining the $n(n-1)$ components

$$(\tau_{\boldsymbol{K}})^i_{j\,k} = (\lambda_j - \lambda_k) \, \frac{\partial \lambda_j}{\partial x^k} \,, \qquad j \neq k \,. \tag{2.10}$$

If $\frac{\partial \lambda_j}{\partial x^k} \neq 0$, each component is zero if and only if $\lambda_i(\boldsymbol{x}) \equiv \lambda_j(\boldsymbol{x})$. Consequently, we can state the following:

Lemma 2.2.1. *Let $\boldsymbol{K}$ be the diagonal operator field (2.8), and assume that its Nijenhuis torsion vanishes. Besides the trivial case where each eigenvalue is constant, there are several further cases. For example,*

(i) Each eigenvalue λ_j depends only on the variable x^j:

$$\lambda_j(\boldsymbol{x}) = \lambda_j(x^j) \,, \qquad j = 1, 2, \dots n \,; \tag{2.11}$$

(ii) $\boldsymbol{K}$ has a single eigenvalue of multiplicity n, i.e. it is a multiple of the identity operator:

$$\lambda_j(\boldsymbol{x}) = \lambda(\boldsymbol{x}) \,, \qquad j = 1, 2, \dots n \,. \tag{2.12}$$

2.2.3 *General properties*

Proposition 2.2.1. *The condition for an operator field $\boldsymbol{K}$ to have its Nijenhuis torsion vanishing can be expressed equivalently in terms of its Lie derivative along the flow of any vector field X as follows:*

$$\mathcal{L}_{\boldsymbol{K}X} \boldsymbol{K} - \boldsymbol{K} \mathcal{L}_X \boldsymbol{K} = \boldsymbol{0} \,, \qquad \forall X \in \mathfrak{X}(M) \,. \tag{2.13}$$

Proof. See, for instance, Ref. [64]. □

Proposition 2.2.2. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a Nijenhuis operator. If its inverse $\boldsymbol{K}^{-1}$ exists, it is also a Nijenhuis operator.*

Proof. For instance, refer to Proposition 13.26, p. 469 of [64]. Analogously to the proof of Proposition 14 from [184] (see Prop. 2.3.5), this can also be deduced from the identity

$$\tau_{\boldsymbol{K}^{-1}}(X,Y) = \boldsymbol{K}^{-2}\,\tau_{\boldsymbol{K}}(\boldsymbol{K}^{-1}X, \boldsymbol{K}^{-1}Y)\,, \tag{2.14}$$

which follows directly from Eq. (2.5). □

Proposition 2.2.3. *Let $\boldsymbol{K}$ be a Nijenhuis operator. For any polynomial in $\boldsymbol{K}$ with constant coefficients a_j, that is $p(\boldsymbol{x},\boldsymbol{K}) = \sum_j a_j\,\boldsymbol{K}^j$, the associated Nijenhuis tensor vanishes:*

$$\tau_{\boldsymbol{K}}(X,Y) = \boldsymbol{0} \quad\Rightarrow\quad \tau_{p(\boldsymbol{K})}(X,Y) = \boldsymbol{0}\,. \tag{2.15}$$

Proof. For a demonstration, see e.g. Proposition 2.1, pp. 10-11 of [26]. □

2.2.4 *Splitting theorem*

One of the most remarkable properties of the Nijenhuis operators is the *splitting theorem*. This result implies that, under certain conditions, these operators provide a local decomposition into invariant subvarieties of a manifold. This greatly simplifies the analysis of their local properties, which can be reduced to the study of operators with a single eigenvalue at each point.

Theorem 2.2.1 (Splitting theorem)**.** *Consider a Nijenhuis tensor field $\boldsymbol{K}$ whose characteristic polynomial allows an admissible factorisation in a neighbourhood of a point $p \in M$, i.e. in this neighbourhood its characteristic polynomial $\chi_{\boldsymbol{K}}(\boldsymbol{x},\lambda)$ can be decomposed as a product of two factors without common roots:*

$$\chi_{\boldsymbol{K}}(\boldsymbol{x},\lambda) = \chi_1(\boldsymbol{x},\lambda)\,\chi_2(\boldsymbol{x},\lambda)\,. \tag{2.16}$$

In that case, the distributions $\mathcal{D}_i(\boldsymbol{x}) = \ker \chi_i(\boldsymbol{x},\boldsymbol{K})$, $i = 1,2$, are integrable. Note that they provide a natural decomposition of the tangent space $T_pM = \mathcal{D}_1(p) \oplus \mathcal{D}_2(p)$.

Furthermore, in an adapted coordinate system $\boldsymbol{x} = (x^{1,1}, x^{1,2}, \ldots, x^{1,k}, x^{2,1}, x^{2,2}, \ldots, x^{2,n-k})$ such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^{1,1}}, \frac{\partial}{\partial x^{1,2}}, \ldots, \frac{\partial}{\partial x^{1,k}} \right\rangle,$$
$$\mathcal{D}_2 = \left\langle \frac{\partial}{\partial x^{2,1}}, \frac{\partial}{\partial x^{2,2}}, \ldots, \frac{\partial}{\partial x^{2,n-k}} \right\rangle;$$

the operator $\boldsymbol{K}$ takes the block diagonal form

$$\boldsymbol{K}(\boldsymbol{x}) = \left[\begin{array}{c|c} \boldsymbol{K}_1(x^{1,1}, x^{1,2}, \ldots, x^{1,k}) & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{K}_2(x^{2,1}, x^{2,2}, \ldots, x^{2,n-k}) \end{array}\right], \tag{2.17}$$

where $\boldsymbol{K}_1$ is a block of dimension $k \times k$ and $\boldsymbol{K}_2$ of dimension $(n-k) \times (n-k)$. Thus, $\boldsymbol{K}$ splits into a direct sum of two Nijenhuis operators $\boldsymbol{K} = \boldsymbol{K}_1 \oplus \boldsymbol{K}_2$.

Proof. Refer for instance Theorem 3.1, pp.19-20 of [26]. □

The theorem holds regardless of whether the eigenvalues of K are real or complex. Therefore, by induction we can deduce that

Corollary 2.2.1. *Every Nijenhuis operator can be decomposed locally in a neighbourhood of any point $p \in M$ into the direct sum of Nijenhuis operators $\boldsymbol{K} = \boldsymbol{K}_1 \oplus \boldsymbol{K}_2 \oplus \ldots$, each of which has either a* single real eigenvalue *or a* single complex conjugate pair of eigenvalues *at the point p.*

2.2.5 *Normal forms for Nijenhuis operators*

The characterization of normal forms of operator fields consists in finding standard representations of these tensors via coordinate transformations. These representations preserve the fundamental properties of the operators while facilitating their systematic study.

One of the most well-known normal forms for Nijenhuis operators is the block-diagonal form in Jordan blocks, or even diagonal when possible, as discussed in the previous section and further detailed in Section 2.3.3 (which is devoted to Haantjes operators but is relevant here, as Nijenhuis operators form a subclass of them). However, beyond the Jordan form, other normal forms of Nijenhuis operators have been extensively studied. We recall the normal forms for cyclic operators or in neighborhoods of differentially non-degenerate points.

Among the various equivalent definitions, we adopt a practical one for cyclic operators.

Definition 2.2.3. *A linear operator $\boldsymbol{K}$ on an n-dimensional differentiable manifold is said to be* cyclic *if the set $\{\boldsymbol{I}, \boldsymbol{K}, \ldots, \boldsymbol{K}^{n-1}\}$ is linearly independent.*

Equivalently, the minimal polynomial of $\boldsymbol{K}$ coincides with its characteristic polynomial, ensuring that each eigenvalue has exactly one Jordan block. For this type of operator, two specific normal forms are known, called companion forms. For brevity, we present only one of them.

Theorem 2.2.2. *Let $\boldsymbol{K}$ be a cyclic Nijenhuis tensor field with characteristic polynomial*

$$p_{\boldsymbol{K}}(\boldsymbol{x}, \lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{K}) = \lambda^n + \sigma_1(\boldsymbol{x})\, \lambda^{n-1} + \cdots + \sigma_{n-1}(\boldsymbol{x})\, \lambda + \sigma_n(\boldsymbol{x}). \tag{2.18}$$

Then, in a suitable local coordinate system $\boldsymbol{y} = (y^1, y^2, \ldots, y^n)$, $\boldsymbol{K}$ takes the so-called companion form

$$\boldsymbol{K} = J^{-1}\, \boldsymbol{K}_{comp}\, J\,, \qquad \boldsymbol{K}_{comp} = \begin{bmatrix} -\sigma_1(\boldsymbol{y}) & 1 & 0 & \cdots & 0 \\ -\sigma_2(\boldsymbol{y}) & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\sigma_{n-1}(\boldsymbol{y}) & 0 & \cdots & 0 & 1 \\ -\sigma_n(\boldsymbol{y}) & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{2.19}$$

where $J = \frac{\partial(\sigma_1,\sigma_2,\ldots,\sigma_n)}{\partial(y^1,y^2,\ldots,y^n)}$ *is the corresponding Jacobian matrix.*

Corollary 2.2.2. *If the coefficients* $\sigma_1, \ldots, \sigma_n$ *of the characteristic polynomial of* $\boldsymbol{K}$ *are functionally independent at a point* $p \in M$ *(where* p *is called a* differentially non-degenerate *point), then* $\boldsymbol{K}$ *can be uniquely reconstructed from these coefficients in a neighborhood of* p*. Furthermore, there exists a coordinate system* $\tilde{\boldsymbol{y}} = (\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^n)$ *in which it takes the canonical form:*

$$\tilde{\boldsymbol{K}}_{comp} = \begin{bmatrix} \tilde{y}^1 & 1 & 0 & \cdots & 0 \\ \tilde{y}^2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \tilde{y}^{n-1} & 0 & \cdots & 0 & 1 \\ \tilde{y}^n & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.20}$$

Moreover, if two Nijenhuis operators $\boldsymbol{K}_1$ *and* $\boldsymbol{K}_2$ *share the same characteristic polynomial on* M*, and its coefficients are functionally independent almost everywhere, then* $\boldsymbol{K}_1 = \boldsymbol{K}_2$*.*

2.2.6 *The Newlander-Nirenberg theorem*

The *Newlander-Nirenberg theorem*, a cornerstone of complex geometry, establishes a fundamental criterion for the integrability of almost complex structures. It states that an almost complex manifold $(M, \boldsymbol{J})$, equipped with a fiber-preserving endomorphism $\boldsymbol{J}: TM \to TM$ satisfying $\boldsymbol{J}^2 = -\boldsymbol{I}$, admits a complex structure if and only if the Nijenhuis tensor associated with $\boldsymbol{J}$ vanishes identically. Proven by Albert Newlander and Louis Nirenberg in 1957, this deep result generalizes the classical Frobenius theorem to the complex setting, ensuring that the vanishing of the Nijenhuis tensor guarantees the existence of local holomorphic coordinates, thereby integrating $\boldsymbol{J}$ into a genuine complex manifold structure. This theorem not only bridges almost complex and complex geometries but also underpins modern studies of complex manifolds, deformation theory, and geometric analysis, highlighting the profound interplay between algebraic conditions and analytic structures. Such a result has been proved in [141], but in this discussion we are going to use a more modern point of view [103, 197]. To do this, we must first define the concepts of complex and almost complex manifolds.

Definition 2.2.4. *A* complex manifold *M of complex dimension n is a topological space M along with a complex atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ that satisfies the following properties:*

(i) Each U_α is an open subset of M and each ϕ_α is a homeomorphism from U_α onto an open subset $V_\alpha \subset \mathbb{C}^n$. That is, the chart (U_α, ϕ_α) assigns the complex coordinates $\phi_\alpha(p) = (z^1, z^2, \ldots, z^n)$ in $\mathbb{C}^n$ to each point $p \in U_\alpha$.

(ii) For any two charts (U_α, ϕ_α) and (U_β, ϕ_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the transition functions

$$\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta) \tag{2.21}$$

are holomorphic. This means that each component of $\phi_{\alpha\beta}$ is a holomorphic function of the complex variables $(z^1, z^2, \ldots, z^n)$.

Definition 2.2.5. *An* almost complex manifold *is a differentiable manifold M of real dimension $2n$ equipped with an additional geometric structure called an almost complex structure.*

An almost complex structure *on M is a tensor field $\boldsymbol{J}$ of type $(1,1)$ such that, at each point $p \in M$, the operator $\boldsymbol{J}_p : T_pM \to T_pM$ satisfies the following condition:*

$$\boldsymbol{J}_p^2 = -\boldsymbol{I}_p\,, \tag{2.22}$$

where $T_p(M)$ is the tangent space of M at the point p, and $\boldsymbol{I}_p$ is the identity operator on T_pM.

We can show that every complex manifold M of dimension n has a natural almost complex structure, so it is additionally an almost complex manifold. In the local coordinates $(z^1, z^2, \ldots, z^n)$ in $\mathbb{C}^n$, each z^j can be written as $z^j = x^j + iy^j$, where x^j and y^j are real coordinates. This establishes an isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, given by:

$$(z^1, z^2, \ldots, z^n) \mapsto (x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n). \tag{2.23}$$

Similarly, the complex tangent space $T_p^{\mathbb{C}}M$ can be regarded as a real vector space $T_p^{\mathbb{R}}M$ of dimension $2n$, so we consider the basis vectors corresponding to the coordinates x^j and y^j:

$$\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}\right\}. \tag{2.24}$$

Thus, we can define the endomorphism $\boldsymbol{J}_p$ acting on the real coordinates as follows:

$$\boldsymbol{J}_p\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}\,, \qquad \boldsymbol{J}_p\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}\,. \tag{2.25}$$

The almost complex structure $\boldsymbol{J}$ satisfies $\boldsymbol{J}_p^2 = -\boldsymbol{I}_p$ at every point p. Crucially, this structure is chart-independent–a property ensured by the holomorphicity of the transition functions, which guarantees compatibility between overlapping coordinate systems.

We can also extend the definition of $\boldsymbol{J}_p$ to the complex tangent space $T_p^{\mathbb{C}}M$. Given that

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right); \tag{2.26}$$

from (2.25) we deduce

$$\boldsymbol{J}_p\left(\frac{\partial}{\partial z^j}\right) = i\,\frac{\partial}{\partial z^j}\,, \qquad \boldsymbol{J}_p\left(\frac{\partial}{\partial \bar{z}^j}\right) = -i\,\frac{\partial}{\partial \bar{z}^j}\,. \tag{2.27}$$

It is noteworthy that this result shows that the complex tangent space is decomposed into two disjoint subspaces:

$$T_p^{\mathbb{C}}M = T_pM^+ \oplus T_pM^-, \tag{2.28}$$

where

$$T_pM^{\pm} = \{Z \in T_pM \mid \boldsymbol{J}_p Z = \pm i\, Z\}. \tag{2.29}$$

In general, any almost complex structure $\boldsymbol{J}$ provides a decomposition of the complexified tangent bundle $T_p^{\mathbb{C}}M$ into a direct sum of two subbundles consisting of eigenvectors of the operator $\boldsymbol{J}$ (linearly extended to $T_p^{\mathbb{C}}M$) with eigenvalues i and $-i$.

The components of $\boldsymbol{J}_p$ we have defined are constant at each point p, which allows us to construct a globally smooth tensor field $\boldsymbol{J}$ on the manifold using the local expression in (2.25). However, this global definition is not guaranteed for arbitrary even-dimensional manifolds. While every such manifold admits *local* almost complex structures, these local structures may fail to patch together consistently across overlapping charts. For $\boldsymbol{J}$ to be *globally* well defined, the manifold must be equipped with a complex structure–a system of holomorphic transition functions that ensure compatibility between the local definitions of $\boldsymbol{J}$. This compatibility is precisely what characterizes a complex manifold.

In short, we have discussed that a complex manifold is always itself almost complex. However, it is not trivial for an even-dimensional differential manifold to admit an almost complex structure, but certain restrictions on its topology must be satisfied. For instance, S^4 is an even-dimensional manifold, but it is not an almost complex manifold [175]. Similarly, not every manifold admitting an almost complex structure is a complex manifold; for example, S^6 presents an open problem in this regard [3]. The conditions for this are given in the Newlander-Nirenberg theorem.

Theorem 2.2.3 (Newlander-Nirenberg, [141])**.** *Let M be a differentiable manifold of real dimension $2n$. An almost complex structure $\boldsymbol{J}$ on M is integrable (i.e., it comes from a complex structure) if and only if the Nijenhuis tensor $\tau_{\boldsymbol{J}}(X,Y) = \mathbf{0}$ for any $X, Y \in \mathfrak{X}(M)$.*

This result holds trivially for the complex structure $\boldsymbol{J}$ defined from (2.25) that we have discussed, since it is a constant operator. However, the theorem provides a useful tool for arbitrary almost complex structures to be complex.

2.3 HAANTJES OPERATORS

In this section, we shall review some basic notions concerning the geometry of Haantjes torsions, following the original papers [67, 142] and the related ones [60, 61, 143, 144]. Here we shall focus only on the aspects of the theory which are relevant for the subsequent discussion.

2.3.1 *The Haantjes torsion*

Definition 2.3.1. *The* Haantjes torsion *of $\boldsymbol{K}$ is the vector-valued 2-form defined by*

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2\, \tau_{\boldsymbol{K}}(X,Y) + \tau_{\boldsymbol{K}}(\boldsymbol{K}X, \boldsymbol{K}Y) - \boldsymbol{K}\left(\tau_{\boldsymbol{K}}(X, \boldsymbol{K}Y) + \tau_{\boldsymbol{K}}(\boldsymbol{K}X, Y)\right). \tag{2.30}$$

This expression can also be written explicitly as [93]

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) = \boldsymbol{K}^4\,[X,Y] - 2\,\boldsymbol{K}^3\big(\,[X,\boldsymbol{K}Y] + [\boldsymbol{K}X,Y]\,\big) + \boldsymbol{K}^2\big(\,[X,\boldsymbol{K}^2Y] + 4\,[\boldsymbol{K}X,\boldsymbol{K}Y] + [\boldsymbol{K}^2X,Y]\,\big) - 2\,\boldsymbol{K}\,\big(\,[\boldsymbol{K}X,\boldsymbol{K}^2Y] + [\boldsymbol{K}^2X,\boldsymbol{K}Y]\,\big) + [\boldsymbol{K}^2X,\boldsymbol{K}^2Y]\,. \quad (2.31)$$

The Nijenhuis torsion being skew-symmetric implies that the Haantjes torsion must also be skew-symmetric. In local coordinates, it can be expressed as

$$(\mathcal{H}_{\boldsymbol{K}})^i_{j\,k} = \sum_{\alpha,\beta=1}^{n}\Big(\boldsymbol{K}^i{}_\alpha\,\boldsymbol{K}^\alpha{}_\beta\,(\tau_{\boldsymbol{K}})^\beta{}_{jk} + (\tau_{\boldsymbol{K}})^i{}_{\alpha\beta}\,\boldsymbol{K}^\alpha{}_j\,\boldsymbol{K}^\beta{}_k - \boldsymbol{K}^i{}_\alpha\Big((\tau_{\boldsymbol{K}})^\alpha{}_{\beta k}\,\boldsymbol{K}^\beta{}_j + (\tau_{\boldsymbol{K}})^\alpha{}_{j\beta}\,\boldsymbol{K}^\beta{}_k\Big)\Big)\,; \quad (2.32)$$

or, more explicitly, as

$$(\mathcal{H}_{\boldsymbol{K}})^i_{j\,k} = \sum_{\alpha=1}^{n}\Big(-2(\boldsymbol{K}^3)^i{}_\alpha\,\partial_{[j}\boldsymbol{K}^\alpha{}_{k]} + (\boldsymbol{K}^2)^\alpha{}_{[j}\,\partial_{|\alpha|}(\boldsymbol{K}^2)^i{}_{k]} + (\boldsymbol{K}^2)^i{}_\alpha\Big(\partial_{[j}(\boldsymbol{K}^2)^\alpha{}_{k]} + 4\sum_{\beta=1}^{n}\boldsymbol{K}^\beta{}_{[j}\,\partial_{|\beta|}\boldsymbol{K}^\alpha{}_{k]}\Big) - 2\,\boldsymbol{K}^i{}_\alpha\sum_{\beta=1}^{n}\Big(\boldsymbol{K}^\beta{}_{[j}\,\partial_{|\beta|}(\boldsymbol{K}^2)^\alpha{}_{k]} + (\boldsymbol{K}^2)^\beta{}_{[j}\,\partial_{|\beta|}(\boldsymbol{K})^\alpha{}_{k]}\Big)\Big)\,. \quad (2.33)$$

For conciseness, we denote $\partial_j := \frac{\partial}{\partial x^j}$; and it is implied that indices within square brackets should be skew-symmetrized, excluding those within $|\cdot|$.

2.3.2 *Integrable frames, Haantjes theorem and Haantjes operators*

In order to formulate our approach to separability, we shall review the relationship between Haantjes geometry and integrability. A more detailed treatment as well as the proofs of the statements reviewed here are available in Refs. [184, 185].

The significance of the Haantjes tensor lies mainly in the Haantjes theorem, which essentially implies that the vanishing of the torsion $\mathcal{H}_{\boldsymbol{K}}$ leads to the integrability of the eigendistributions of the operator $\boldsymbol{K}$. Since this statement is fundamental to the theory, it will be explained in detail.

A *reference frame* is a collection of n vector fields $\{X_1, X_2, \ldots, X_n\}$ such that, at each point $\boldsymbol{x}$ in an open set $U \subseteq M$, they form a basis for the tangent space of U at that point, $T_{\boldsymbol{x}}U$. For instance, in a local chart $\{U, (x^1, x^2, \ldots, x^n)\}$, we can choose the reference frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\}$, which we refer to as the *natural frame*. However, this choice is just one of many possible options, so we could consider two different frames, $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$, and we say that they are *equivalent* if there exist n nowhere-vanishing smooth functions f_i such that

$$X_i = f_i(\boldsymbol{x})\,Y_i\,, \qquad i = 1, 2, \ldots, n\,. \quad (2.34)$$

Definition 2.3.2. *A reference frame equivalent to a natural frame is called* integrable.

The following result, obtained by applying the classical Fitting lemma [85] to the module of operator fields in $\mathfrak{X}(M)$ over the ring $C^\infty(M)$, is crucial for the spectral study of Haantjes operators.

Lemma 2.3.1 (Lemma 4, [184])**.** *Let $\boldsymbol{K}$ be a $(1,1)$-tensor field. For any $\boldsymbol{x} \in M$ there exists a minimum positive integer $\rho(\boldsymbol{x})$, called the* Riesz index *of $\boldsymbol{K}$ at $\boldsymbol{x}$, such that*

$$\ker \boldsymbol{K}^{\rho(\boldsymbol{x})}(\boldsymbol{x}) = \ker \boldsymbol{K}^{\rho(\boldsymbol{x})+1}(\boldsymbol{x}), \qquad \operatorname{Im} \boldsymbol{K}^{\rho(\boldsymbol{x})}(\boldsymbol{x}) = \operatorname{Im} \boldsymbol{K}^{\rho(\boldsymbol{x})+1}(\boldsymbol{x}) \tag{2.35}$$

and

$$T_{\boldsymbol{x}}M = \ker \boldsymbol{K}^{\rho(\boldsymbol{x})}(\boldsymbol{x}) \oplus \operatorname{Im} \boldsymbol{K}^{\rho(\boldsymbol{x})}(\boldsymbol{x}). \tag{2.36}$$

In the forthcoming considerations, given an operator $\boldsymbol{K}$, we denote its *pointwise spectrum* by

$$\operatorname{spec}(\boldsymbol{K}) := \big\{\lambda_1(\boldsymbol{x}), \dots, \lambda_s(\boldsymbol{x})\big\}, \quad s \in \mathbb{N} \setminus \{0\},$$

where the eigenvalues $\lambda_i(\boldsymbol{x})$ are distinct at every $x \in M$, smooth, and real-valued by assumption. For each eigenvalue $\lambda_i(\boldsymbol{x})$, the corresponding *generalized eigendistribution* is defined as

$$\mathcal{D}_i := \ker\big(\boldsymbol{K} - \lambda_i \boldsymbol{I}\big)^{\rho_i}, \quad i = 1, \dots, s, \tag{2.37}$$

where $\rho_i \in \mathbb{N} \setminus \{0\}$ is the Riesz index associated with λ_i. Critically, by hypothesis, ρ_i is *constant* over M (i.e., independent of $\boldsymbol{x}$). When $\rho_i = 1$, the distribution $\mathcal{D}_i$ reduces to the *proper eigendistribution* spanned by eigenvector fields of $\boldsymbol{K}$ corresponding to λ_i.

Definition 2.3.3. *An operator field $\boldsymbol{K}$ is* semisimple *if in each open neighborhood $U \subseteq M$ there exists a (possibly non-integrable) reference frame formed by proper eigenvector fields of $\boldsymbol{K}$. Moreover, $\boldsymbol{K}$ is* simple *if all of its eigenvalues are pointwise distinct, namely $\lambda_i(\boldsymbol{x}) \neq \lambda_j(\boldsymbol{x})$, $i, j = 1, \dots, n$, $\forall \boldsymbol{x} \in M$.*

A frame of proper eigenvectors is said to be a proper eigenframe of $\boldsymbol{K}$. If the frame contains generalized eigenvectors, it will be said to be a generalized eigenframe.

Definition 2.3.4. *We say that a set of distributions $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}$ is* pairwise integrable *if every pair $\mathcal{D}_i + \mathcal{D}_j$, $i \neq j$, is integrable.*

Definition 2.3.5. *Consider a set of distributions $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}$. We say that such distributions are* mutually integrable *if*

i. each of them is integrable,

ii. they are pairwise integrable.

In 1955, J. Haantjes proved a result of crucial significance, which serves as the cornerstone of the theory developed in this thesis:

Theorem 2.3.1 (Haantjes, [67]). *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator field and assume that the rank of each of its generalized eigendistributions $\mathcal{D}_i$, $i = 1, \dots, s$, is independent of $\boldsymbol{x} \in M$. The vanishing of the Haantjes torsion*

$$\mathcal{H}_{\boldsymbol{K}}(X, Y) = \mathbf{0}\,, \qquad \forall X, Y \in \mathfrak{X}(M) \tag{2.38}$$

is a sufficient condition to ensure the mutual integrability of the generalized eigendistributions $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_s\}$. Moreover, if $\boldsymbol{K}$ is semisimple, the condition (2.38) is also necessary.

Therefore, by analogy with to the definition of the Nijenhuis operators in Def. 2.2.2, the Haantjes torsion leads to the introduction of the Haantjes operators, which is motivated by the Haantjes theorem. This notion was introduced in [184], and the entire dissertation will be based on these operators.

Definition 2.3.6. *A* Haantjes operator *is a $(1,1)$-tensor field whose Haantjes torsion identically vanishes.*

2.3.3 *Diagonalization of Haantjes operators*

An interesting consequence of Haantjes theorem (see Theorem 2.3.1) is that the vanishing of the Haantjes torsion of an operator $\boldsymbol{K}$ not only characterizes Haantjes operators but also guarantees the existence of a local coordinate chart in which $\boldsymbol{K}$ assumes a *block-diagonal* form.

Let $\boldsymbol{K}$ be a Haantjes operator. We denote by r_i the rank of the distribution $\mathcal{D}_i$ associated with $\boldsymbol{K}$. We also introduce the distribution (of corank r_i)

$$\mathcal{E}_i := \operatorname{Im}\left(\boldsymbol{K} - \lambda_i \boldsymbol{I}\right)^{\rho_i} = \bigoplus_{j=1,\, j\neq i}^{s} \mathcal{D}_j, \qquad i = 1, 2, \dots, s, \tag{2.39}$$

which is spanned by all the generalized eigenvectors of $\boldsymbol{K}$, except those associated with the eigenvalue λ_i (recall that, by hypothesis, $\boldsymbol{K}$ has real eigenvalues). We shall refer to $\mathcal{E}_i$ as a *characteristic distribution* of $\boldsymbol{K}$. Let $\mathcal{E}_i^\circ$ denote the annihilator of $\mathcal{E}_i$. The cotangent spaces of M decompose as

$$T^*_{\boldsymbol{x}} M = \bigoplus_{i=1}^{s} \mathcal{E}_i^\circ(\boldsymbol{x})\,. \tag{2.40}$$

As a consequence of Haantjes theorem, each characteristic distribution $\mathcal{E}_i$ is integrable. We denote by E_i the foliation associated with $\mathcal{E}_i$ and by $E_i(\boldsymbol{x})$ the connected leaf through $\boldsymbol{x}$ belonging to E_i. Given the set of distributions $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_s\}$, we obtain an equal number of foliations $\{\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_s\}$. This set of foliations is referred to as the *characteristic web* of $\boldsymbol{K}$. The leaves $E_i(\boldsymbol{x})$ of each foliation E_i are called the *characteristic fibers* of the web.

Definition 2.3.7. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a Haantjes operator. A collection of r_i smooth functions is said to be adapted to the foliation E_i of the characteristic web of $\boldsymbol{K}$ if their level sets coincide with the characteristic fibers of E_i.*

Definition 2.3.8. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a Haantjes operator. A parametrization of the characteristic web of $\boldsymbol{K}$ is an ordered set of n independent smooth functions, denoted by $(\boldsymbol{f}^1, \ldots, \boldsymbol{f}^i, \ldots, \boldsymbol{f}^s)$, such that for any $i = 1, \ldots, s$, the ordered subset $\boldsymbol{f}^i = \left(f^{i,1}, \ldots, f^{i,r_i}\right)$ is adapted to the ith characteristic foliation of the web:*

$$f^{i,k}\big|_{E_i(\boldsymbol{x})} = c^{i,k} \qquad \forall E_i(\boldsymbol{x}) \in \mathrm{E}_i, \quad k = 1, \ldots, r_i. \tag{2.41}$$

Here, $c^{i,k}$ are real constants depending only on the indices i and k. In this case, we say that the collection of these functions is adapted to the web and that each of them is a characteristic function.

Thus, an equivalent formulation of Haantjes theorem can be stated in terms of the existence of integrable generalized eigen-frames of $\boldsymbol{K}$.

Proposition 2.3.1. *The vanishing of the Haantjes torsion of an operator field $\boldsymbol{K}$ is a sufficient condition to guarantee that $\boldsymbol{K}$ admits an equivalence class of integrable generalized eigenframes, in which $\boldsymbol{K}$ assumes a block-diagonal form. Furthermore, if $\boldsymbol{K}$ is semisimple, the condition is also necessary, and $\boldsymbol{K}$ assumes a diagonal form; if $\boldsymbol{K}$ is* simple*, then each of its proper eigen-frames is integrable.*

This result motivates the following definition [184].

Definition 2.3.9. *Let $\boldsymbol{K}$ be a Haantjes operator. A local chart whose natural frame is a generalized eigen-frame is called a* Haantjes chart *for $\boldsymbol{K}$.*

2.3.4 *Basic examples of Haantjes operators*

A number of concrete cases that fall into the definition of Haantjes operators can be found in a simple way, as can be seen from the examples that follow.

Example 2.3.1. *When* $\dim M = 2$*, the Haantjes torsion of any smooth* $(1,1)$*-tensor field vanishes. This can be proved by a straightforward calculation.*

Example 2.3.2. *Every Nijenhuis operator is a Haantjes operator. This follows naturally from Eq.* (2.30)*, where each of the summands cancels out. Consequently, all the examples discussed in Section 2.2.2 are themselves Haantjes operators and the properties discussed in this section also hold.*

Example 2.3.3. *The Haantjes torsion of any diagonal operator field of the form* (2.8) *vanishes. In fact, its components are*

$$(\mathcal{H}_{\boldsymbol{K}})^i_{j\,k} = (\lambda_i - \lambda_j)\,(\lambda_i - \lambda_k)\,(\tau_{\boldsymbol{K}})^i_{j\,k} = 0\,, \tag{2.42}$$

where $(\tau_{\boldsymbol{K}})^i_{j\,k}$ are given by Eq. (2.9)*.*

This last example is the motivation for the following:

Proposition 2.3.2. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a smooth tensor field. If there exists a local coordinate chart $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$ in which $\boldsymbol{K}$ assumes the diagonal form* (2.8)*, then its Haantjes torsion identically vanishes.*

Nijenhuis and Haantjes operators find natural applications in classical mechanics, as we will illustrate with a particularly ubiquitous instance in the dynamics of rigid bodies. Consider a finite system of point masses in the n-dimensional affine Euclidean space E_n, represented as $M = \{(P_\gamma, m_\gamma)\}$ where $P_\gamma \in E_n$ and $m_\gamma \in \mathbb{R}$ (with masses m_γ possibly negative). Within this framework, two key tensor fields emerge: the *planar inertia tensor* (or Euler tensor in continuum mechanics) and the full *inertia tensor*. The planar inertia tensor, defined at a point P as the $(1,1)$-tensor field

$$\boldsymbol{E}_P(\boldsymbol{v}) := \sum_\gamma m_\gamma \left((P_\gamma - P)\cdot \boldsymbol{v}\right)(P_\gamma - P), \quad \boldsymbol{v} \in T_P E_n, \tag{2.43}$$

governs rotational properties within subspaces and constitutes a *Nijenhuis operator*. In contrast, the full inertia tensor, given by

$$\mathbb{I}_P(\boldsymbol{v}) := \sum_\gamma m_\gamma \Big(|P_\gamma - P|^2 \boldsymbol{v} - \left((P_\gamma - P)\cdot \boldsymbol{v}\right)(P_\gamma - P)\Big), \tag{2.44}$$

encodes the system's full inertial behavior and is a *Haantjes operator* [184]. These operators exemplify the synergy between geometric structures and mechanical systems. Further examples from classical mechanics and Riemannian geometry are detailed in [157, 159, 184–187, 192]. Concrete instances of these ideas will be explored in subsequent chapters, illustrating their broader relevance.

2.3.5 *Main properties*

Some of the most important properties of these operators will be presented below. They will also be used later to introduce interesting algebraic structures.

Proposition 2.3.3 (Proposition 1, [20]). *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator field. The following identity holds*

$$\mathcal{H}_{f\boldsymbol{I}+g\boldsymbol{K}}(X,Y) = g^4\,\mathcal{H}_{\boldsymbol{K}}(X,Y)\,, \tag{2.45}$$

where $f, g \in C^\infty(M)$ are real functions and $\boldsymbol{I}$ denotes the identity operator on $\mathfrak{X}(M)$.

It follows from this statement that if $\boldsymbol{K}$ is a Haantjes operator, then so is $f\boldsymbol{I} + g\boldsymbol{K}$. This property makes an important difference with respect to the Nijenhuis torsion, where the product of any function by a Nijenhuis operator generally no longer remains a Nijenhuis operator. This can be seen from the following equality:

$$\begin{aligned}\tau_{f\boldsymbol{K}}(X,Y) = {} & f^2\,\tau_{\boldsymbol{K}}(X,Y) \\ & + f\left((\boldsymbol{K}X)(f)\,\boldsymbol{K}Y - (\boldsymbol{K}Y)(f)\,\boldsymbol{K}X + Y(f)\,\boldsymbol{K}^2X - X(f)\,\boldsymbol{K}^2Y\right),\end{aligned} \tag{2.46}$$

where $X(f)$ denotes the Lie derivative of the arbitrary function $f \in C^\infty(M)$ with respect to the vector field X. Overall, the differential and algebraic properties of Haantjes operators are much richer, as evidenced by the following properties.

Proposition 2.3.4 (Corollary 3.3, [19]). *Let $\boldsymbol{K}$ be a Haantjes operator. For any polynomial in $\boldsymbol{K}$ with coefficients $a_j \in C^\infty(M)$, that is $p(\boldsymbol{x},\boldsymbol{K}) = \sum_j a_j(\boldsymbol{x})\,\boldsymbol{K}^j$, the associated Haantjes tensor also vanishes:*

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) = \boldsymbol{0} \quad \Rightarrow \quad \mathcal{H}_{p(\boldsymbol{x},\boldsymbol{K})}(X,Y) = \boldsymbol{0}\,. \tag{2.47}$$

Proposition 2.3.5. *Let $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a Haantjes operator. If its inverse $\boldsymbol{K}^{-1}$ exists, it is also a Haantjes operator.*

Proof. As discussed in Proposition 14 of [184], this follows from the identity

$$\mathcal{H}_{\boldsymbol{K}^{-1}}(X,Y) = \boldsymbol{K}^{-4}\,\mathcal{H}_{\boldsymbol{K}}(\boldsymbol{K}^{-2}X, \boldsymbol{K}^{-2}Y)\,, \tag{2.48}$$

which can be easily derived from Eq. (2.31). For an alternative proof, refer to Proposition 2 on page 257 of [20]. □

2.4 HAANTJES ALGEBRAS

2.4.1 *Definition*

Another crucial notion is that of Haantjes algebras, which has been introduced in [184] and further studied and generalized in [159].

Definition 2.4.1. *A* Haantjes algebra *of rank r is a pair $(M,\mathscr{H})$ satisfying the following properties:*

(i) M is a differentiable manifold of dimension n;

(ii) $\mathscr{H}$ is a set of Haantjes operators $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ that generate:

- *a free module of rank r over the ring of smooth functions on M:*

$$\begin{aligned} \mathcal{H}_{(f\boldsymbol{K}_1 + g\boldsymbol{K}_2)}(X,Y) = \boldsymbol{0}\,, \qquad & \forall\, X,Y \in \mathfrak{X}(M)\,, \\ & \forall\, f,\, g \in C^\infty(M)\,, \quad \forall\, \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}\,; \end{aligned} \tag{2.49}$$

- *a ring w.r.t. the composition operation:*

$$\mathcal{H}_{(\boldsymbol{K}_1\boldsymbol{K}_2)}(X,Y) = \boldsymbol{0}\,, \qquad \forall\, \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}\,, \quad \forall\, X,Y \in \mathfrak{X}(M)\,. \tag{2.50}$$

If

$$\boldsymbol{K}_1\boldsymbol{K}_2 = \boldsymbol{K}_2\boldsymbol{K}_1\,, \qquad \forall\, \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}\,, \tag{2.51}$$

the algebra $(M,\mathscr{H})$ is said to be an Abelian Haantjes algebra. Moreover, if the identity operator $\boldsymbol{I} \in \mathscr{H}$, then $(M,\mathscr{H})$ will be said to be a Haantjes algebra with identity.

In other words, the requirements (2.49) and (2.50) show that $\mathscr{H}$ is an *associative algebra of Haantjes operators*.

We find a natural and particularly important realization of these algebras in the sets of operators that take a diagonal form at a given coordinate system. Consider a local chart $\{U, \boldsymbol{x} = (x^1, x^2, \dots, x^n)\}$ with a set of operators of the form

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \lambda_i^{(\alpha)}(\boldsymbol{x}) \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i\,, \qquad \alpha = 1, \dots, r\,, \tag{2.52}$$

where the arbitrary smooth functions $\lambda_i^{(\alpha)}(\boldsymbol{x}) := \lambda_i^{i\,(\alpha)}(\boldsymbol{x})$ act as their eigenvalues. Indeed, on the one hand, the Haantjes torsion of these operators vanishes identically and the assumption (2.49) is also satisfied under Proposition 2.3.2. On the other hand, they constitute a commutative ring which fulfils Eq. (2.50). This type of structure will be of great relevance throughout the thesis.

Definition 2.4.2. *The algebra generated by operators of the form* (2.52) *is said to be a* diagonal *Haantjes algebra.*

2.4.2 *Haantjes coordinates*

As we shall see, a crucial result for the theory of separability of Hamiltonian systems is the theorem, proved in [184], ensuring that an algebra of Haantjes operators can be represented in suitable coordinates in a block-diagonal form.

Proposition 2.4.1 (Proposition 33, [184])**.** *Let $\mathcal{K} = \{\boldsymbol{K}_1, \dots, \boldsymbol{K}_r\}$, $\boldsymbol{K}_\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\alpha = 1, \dots, r$ be a family of commuting operator fields; we assume that one of them, say $\boldsymbol{K}_1$, is a Haantjes operator. Then, there exist local charts adapted to the spectral decomposition of $\boldsymbol{K}_1$ in which all of the operators $\boldsymbol{K}_\alpha$ can be written simultaneously in a block-diagonal form. In addition, if we assume that*

(i) all the operators of the family are Haantjes operators

(ii) all possible nontrivial intersections of their generalized eigendistributions

$$\mathcal{V}_a(\boldsymbol{x}) := \bigoplus_{i_1,\dots,i_r}^{s_1,\dots,s_r} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \bigcap \dots \bigcap \mathcal{D}_{i_r}^{(r)}(\boldsymbol{x})\,, \qquad a = 1, \dots, v \leq n\,, \tag{2.53}$$

are pairwise integrable (i.e. any direct sum of $\mathcal{V}_a$ is integrable), then there exist sets of local coordinates, adapted to the decomposition

$$T_{\boldsymbol{x}}M = \bigoplus_{a=1}^{v} \mathcal{V}_a(\boldsymbol{x})\,, \qquad \boldsymbol{x} \in M\,, \tag{2.54}$$

in which all operators $\boldsymbol{K}_\alpha$ admit simultaneously a block-diagonal form (with possibly finer blocks).

A simple but relevant consequence of the latter Proposition is the following result.

Corollary 2.4.1. *Under the hypotheses of Proposition 2.4.1, the simultaneous block-diagonalization property extends to the entire $C^\infty(M)$-algebra generated by the family $\mathcal{K}$. Specifically, if $\mathcal{K}$ consists of operators that share a common block-diagonal structure in a local coordinate system, then all $C^\infty(M)$-linear combinations and products of elements of $\mathcal{K}$ inherit this block decomposition.*

Remark 2.4.1. *Each set of coordinates allowing us to write down the family $\mathcal{K}$ in a block-diagonal form will be said to be a set of* Haantjes coordinates. *In particular, if we deal with a semisimple Haantjes algebra, then each set of Haantjes coordinates allows us to write simultaneously all the operators of the algebra in a diagonal form.*

As will be discussed in Chapter 7, this theorem has been extended in [159] to the case of *generalized Haantjes algebras of level m*, which are algebras whose elements are represented by generalized Nijenhuis operators of level m.

2.4.3 *Unigenerated Haantjes algebras*

As discussed in [184], a simple but particularly relevant class of Abelian Haantjes algebras is that generated from a *single* operator $\boldsymbol{L}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$. This is a consequence of Proposition 2.3.4, which ensures that an algebra whose basis consists of powers of the same Haantjes operator fulfils all the conditions of Definition 2.4.1.

Let l be the degree of the minimal polynomial of $\boldsymbol{L}$. Then, we can construct a Haantjes algebra $\mathscr{L}$ of rank $r \leq l$ from the first $l-1$ powers of the tensor field $\boldsymbol{L}$ together with $\boldsymbol{L}^0 := \boldsymbol{I}$,

$$\mathscr{L}(\boldsymbol{L}) := \operatorname{span}\{\boldsymbol{I}, \boldsymbol{L}, \ldots, \boldsymbol{L}^{l-1}, \ldots\} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{L}, \ldots, \boldsymbol{L}^{l-1}\}\,, \tag{2.55}$$

which will be Abelian by construction. Thus we can introduce the notion of unigenerated[1] Haantjes algebras.

Definition 2.4.3. *Let $(M, \mathscr{H})$ be an Abelian Haantjes algebra of rank r. A $(1,1)$-tensor field $\boldsymbol{L}$, with minimal polynomial of degree $l \geq r$, is a generator of $\mathscr{H}$ if*

$$\mathscr{H} \subseteq \mathscr{L}(\boldsymbol{L})\,. \tag{2.56}$$

If this is the case, we say that the algebra $\mathscr{H}$ is a unigenerated *algebra of Haantjes operators over the ring of smooth functions on M.*

Remark 2.4.2. *In virtue of the Hamilton-Cayley theorem, the degree of the minimal polynomial of $\boldsymbol{L}$ –and thus the rank of the corresponding unigenerated algebra– cannot exceed the dimension of the manifold, i.e., $r \leq l \leq n = \dim(M)$.*

1 We borrow the terminology of unigenerated polynomial algebras from [96], although in our context it is defined for the more general case of polynomial algebras with variable coefficients. In previous works [157, 184, 185] the same notion was equivalently denominated "cyclic algebras".

Thus, all the operators of the algebra are of the form $\boldsymbol{K}_\alpha = p_\alpha(\boldsymbol{L})$, where $p_\alpha(\boldsymbol{x}, \lambda)$ is a suitable polynomial in λ of degree not greater than $l - 1$ with coefficients in $C^\infty(M)$.

Unigenerated algebras play an essential role in this theory, and the main reason for this lies in the following result.

Proposition 2.4.2 (Proposition 47, [184]). *Every semisimple Abelian Haantjes algebra is unigenerated and admits a Nijenhuis generator.*

Thus, the rank of an Abelian Haantjes algebra (or any unigenerated Haantjes algebra) is determined by the minimal polynomial degree of its generator, contingent on the identity operator's presence.

Lemma 2.4.1 (Lemma 2, [157]). *Let $(M, \mathscr{H})$ be an Abelian Haantjes algebra of rank r. An operator $\boldsymbol{L} \in \mathscr{H}$ is a generator of $\mathscr{H}$ if and only if its minimal polynomial has degree $l = r$ when $\boldsymbol{I} \in \mathscr{H}$, and $l = r + 1$ otherwise.*

Proof. First, consider the case $\boldsymbol{I} \in \mathscr{H}$. Since $\boldsymbol{L} \in \mathscr{H}$, the unigenerated algebra $\mathscr{L}(\boldsymbol{L})$ satisfies $\mathscr{L}(\boldsymbol{L}) \subseteq \mathscr{H}$. Assume $\boldsymbol{L}$ is a generator of $\mathscr{H}$. By definition, $\mathscr{H} \subseteq \mathscr{L}(\boldsymbol{L})$, so the degree of the minimal polynomial of $\boldsymbol{L}$ is at least r. Therefore, $\mathscr{L}(\boldsymbol{L}) = \mathscr{H}$ and the minimal polynomial has degree exactly r. Conversely, suppose that an operator $\boldsymbol{L} \in \mathscr{H}$ has a minimal polynomial of degree r. Then $\mathscr{L}(\boldsymbol{L})$ has rank r, so $\mathscr{L}(\boldsymbol{L}) = \mathscr{H}$, proving $\boldsymbol{L}$ generates $\mathscr{H}$.

If $\boldsymbol{I} \notin \mathscr{H}$, construct the extended Haantjes algebra by adding the identity operator to $\mathscr{H}$, which has rank $r + 1$. By the previous case, $\boldsymbol{L} \in \mathscr{H}$ generates the extended algebra (and therefore $\mathscr{H}$) if and only if its minimal polynomial has degree $r + 1$. □

Part II

SYMPLECTIC-HAANTJES GEOMETRY AND SEPARABILITY

El todo puede ser mayor, menor o igual que la suma de las partes, todo depende de la interacción que se establezca entre estas últimas. Pensad bien en lo que acabo de decir porque ésta es una frase muy importante, y lo es en sí misma y porque desemboca en esta otra, sólo podemos afirmar con certeza que el todo es igual a la suma de las partes cuando las partes se ignoran entre sí.

Almudena Grandes, *El corazón helado*

3

The Haantjes approach to separation of variables

There is a train station in every life–a moment where the air thickens with the weight of parting, where the same scene fractures into a mosaic of truths. Two faces at a platform: one feels the shudder of the engine in their ribs, counts seconds by the slackening grip of a hand, tastes the metallic ache of farewell sharp as the whistle's cry. The other watches the clock's indifferent turn, hears only the rhythm of footsteps dissolving into schedules, feels the bag on their shoulder as a fact, not an anchor. The train departs. A single event, split by the prism of presence. Physics, too, lives here–in the quiet rebellion of reality, how it bends its light to the eye of the beholder yet remains, stubbornly, itself. The universe does not care if you measure time by heartbeats or atomic clocks, if you chart motion in tears or parsecs. It only asks that we find a language fluent enough to hold all these versions at once, to trace the invisible threads that stitch one observer's sunrise to another's twilight. Differential geometry is this language: a mathematics of perspective, where laws are not written in stone but in the curvature between *here* and *there*, a calculus of how truths transform–and yet, beneath every coordinate system, every shifting frame, there pulses an unchanging core, a geometry of whispers that outlasts the storm of subjectivity.

The construction of separation variables for the HJ equation remains a cornerstone challenge in classical mechanics, tracing its lineage to foundational contributions by Liouville, Jacobi, Stäckel, Levi-Civita, and Arnold. These works crystallized the geometric principles of integrability and separability, framing them through symplectic structures and canonical transformations. In modern

terms, this problem has matured into a unifying framework intertwining differential geometry, algebraic topology, and spectral theory, offering fresh insights into the intrinsic geometry of integrable dynamics.

Central to modern developments is the theory of *bi-Hamiltonian systems*, which posits that a manifold endowed with two compatible Poisson structures can generate hierarchies of integrals of motion through the Magri-Lenard scheme. This framework, pioneered by Magri, reveals how integrability emerges from the compatibility of geometric structures. Closely related is the concept of *symplectic-Nijenhuis (ωN) manifolds*, where a symplectic form and a Nijenhuis tensor coexist under specific compatibility conditions. These manifolds provide a natural setting for Liouville-Arnold integrability, as the Nijenhuis tensor encodes both the integrals of motion and their recursive relations.

Building on these ideas, Magri introduced *Haantjes manifolds*, obtained by replacing the Nijenhuis torsion with the weaker Haantjes torsion condition and by requiring the presence of an additional geometric structure expressed in terms of a family of suitable differential forms. Haantjes tensors, while less restrictive, retain the ability to diagonalize in suitable coordinates, making them potent tools for studying integrability. In parallel, in [185] the notion of *symplectic-Haantjes ($\omega\mathscr{H}$) manifolds* emerged, unifying symplectic geometry with Haantjes algebras. These manifolds, equipped with compatible $(1,1)$-tensor fields, generalize the ωN framework by allowing an algebra of Haantjes generators, thereby accommodating both semisimple and non-semisimple integrable systems. The $\omega\mathscr{H}$ theory, as shown in recent works, establishes that a Hamiltonian system is completely integrable if and only if it admits an $\omega\mathscr{H}$ structure, with Haantjes chains replacing the traditional Lenard-Magri chains.

A key advancement lies in the connection between $\omega\mathscr{H}$ manifolds and SoV. For semisimple systems, the existence of a Haantjes structure guarantees the presence of *Darboux-Haantjes coordinates*, which simultaneously diagonalize the Haantjes operators and simplify the HJ equation into separable form. Remarkably, these coordinates coincide with Darboux-Nijenhuis coordinates in the ωN framework, bridging the two theories. However, in non-semisimple cases—exemplified by systems like the Post-Winternitz model—the $\omega\mathscr{H}$ formalism proves indispensable, as multiple Haantjes generators are required, a feature absent in the ωN approach.

This chapter explores these geometric frameworks, emphasizing their role in integrability and SoV. After reviewing bi-Hamiltonian and ωN structures, we delve into Haantjes manifolds and their symplectic counterparts, demonstrating their utility through concrete examples. The theory is then applied to solve SoV for Drach-Holt systems and extended via a novel operator lift from configuration space to phase space, generalizing the classical Yano lift.

This chapter systematically explores the interplay between differential geometry and the separability of Hamilton-Jacobi equations. In Section 3.1, beginning with foundational concepts such as bi-Hamiltonian systems and symplectic-Nijenhuis manifolds, we establish the geometric prerequisites for integrability. In Sections 3.2 and 3.3, we introduce Haantjes and symplectic-Haantjes manifolds, highlighting their role in generalizing Nijenhuis geometry. Section 3.4 unifies these ideas through the Liouville-Haantjes theorem, proving that integrability is equivalent to the existence of an $\omega\mathscr{H}$ structure. Section 3.5 demonstrates the practical efficacy of this framework by solving

the separation problem for Drach-Holt systems, a class previously deemed non-separable, through the construction of Darboux-Haantjes coordinates in phase space. Finally, Section 3.6 introduces a novel geometric lift of operators from configuration space to phase space, extending the Yano lift to preserve Haantjes structures.

3.1 BI-HAMILTONIAN AND SYMPLECTIC-NIJENHUIS MANIFOLDS

In many respects, the *bi-Hamiltonian* approach represents one of the most relevant geometric frameworks for the study of Hamiltonian systems. A bi-Hamiltonian manifold is a differentiable manifold endowed with a pencil of Poisson structures [121]. In these settings, the theory of ωN *manifolds*, introduced in [52, 125], is especially relevant in the construction of separation variables for the HJ equations associated with a Hamiltonian integrable system. Such manifolds carry on a symplectic structure ω (assigned as the inverse of a non-degenerate Poisson bivector) and a compatible $(1,1)$-tensor field $\boldsymbol{N}$, also called the recursion or hereditary operator, whose Nijenhuis torsion [142] vanishes. This approach has been applied in [183] to the study of multiseparable systems. We mention that this theory has also been generalized to the case of Poisson-Nijenhuis structures [94] and their Lie algebroids [38, 65].

3.1.1 *Bi-Hamiltonian manifolds*

Bi-Hamiltonian manifolds naturally arise in the study of integrable systems and Poisson geometry, providing a geometric framework for understanding the hierarchical structure of conserved quantities. A manifold is said to be bi-Hamiltonian if it is equipped with two compatible Poisson structures, meaning that any linear combination of them is again a Poisson structure. This compatibility condition plays a fundamental role in the theory of integrability, as it allows the construction of a hierarchy of commuting Hamiltonian flows via the so-called Magri-Lenard scheme. The existence of two Poisson structures enables the recursion relations that generate conserved integrals, often leading to the complete integrability of the system. Bi-Hamiltonian structures appear prominently in the theory of soliton equations, such as the Korteweg-de Vries (KdV) and Gel'fand–Dikii hierarchies.

Definition 3.1.1. *Two Poisson brackets* $\{\cdot,\cdot\}_0$ *and* $\{\cdot,\cdot\}_1$ *are* compatible *if any linear combination*

$$\{\cdot,\cdot\}_{\lambda,\mu} = \lambda\{\cdot,\cdot\}_0 + \mu\{\cdot,\cdot\}_1\,, \qquad \lambda,\mu\in\mathbb{R}\,, \tag{3.1}$$

is itself a Poisson bracket.

Equivalently, we can write this compatibility condition by means of the Poisson bivectors $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ associated with the brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$, respectively, as

$$[\boldsymbol{P}_0,\boldsymbol{P}_0]_{\mathrm{SN}} = [\boldsymbol{P}_1,\boldsymbol{P}_1]_{\mathrm{SN}} = 0\,, \tag{3.2}$$

$$[\boldsymbol{P}_0,\boldsymbol{P}_1]_{\mathrm{SN}} = 0\,. \tag{3.3}$$

Definition 3.1.2. *A* bi-Hamiltonian manifold $(M, \boldsymbol{P}_0, \boldsymbol{P}_1)$ *is a manifold equipped with two compatible Poisson brackets given by* $\boldsymbol{P}_0$ *and* $\boldsymbol{P}_1$.

As a natural generalization to the concept of Hamiltonian vector field (see Eq. (1.24)), we define a *bi-Hamiltonian system* as one that admits two compatible Hamiltonian decompositions:

$$X = \boldsymbol{P}_0 \mathrm{d}H_0 = \boldsymbol{P}_1 \mathrm{d}H_1 \,. \tag{3.4}$$

The vector field X is said to be a bi-Hamiltonian vector field.

3.1.2 *Symplectic-Nijenhuis manifolds*

Symplectic-Nijenhuis (or ωN) manifolds provide a powerful geometric framework for studying integrable systems, particularly the SoV in the HJ equation. They arise naturally from bi-Hamiltonian manifolds where one of the Poisson brackets is non-degenerate, inducing a symplectic structure. The compatibility between this symplectic form and a second Poisson structure is encoded in a Nijenhuis tensor, which acts as a recursion operator to generate conserved quantities and eigenfunctions essential for variable separation.

Definition 3.1.3. *Consider a bi-Hamiltonian system satisfying Eq.* (3.4)*, and suppose that the bivector* $\boldsymbol{P}_1$ *is non-degenerate. The* recursion operator *associated with this system is the tensor field* $\boldsymbol{N}$: $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ *defined as*

$$\boldsymbol{N} := \boldsymbol{P}_0 \boldsymbol{P}_1^{-1} \,. \tag{3.5}$$

A bi-Hamiltonian system $(M, \boldsymbol{P}_0, \boldsymbol{P}_1)$ *with a non-degenerate Poisson bivector is termed a* ωN manifold.

The compatibility condition between the Poisson bivectors (Eq. (3.3)) leads to the following fundamental property of this operator.

Proposition 3.1.1. *The Nijenhuis torsion of the recursion operator associated to a bi-Hamiltonian system identically vanishes.*

One of the main properties of the recursion operator is that it maps symmetries of the operator into other symmetries. Suppose that X is a symmetry of $\boldsymbol{N}$, so that $\mathcal{L}_X \boldsymbol{N} = \boldsymbol{0}$. According to the Proposition 2.2.1, the vector field $\boldsymbol{N}X$ is also a symmetry. By applying this argument repeatedly, we can conclude that if X is a symmetry of $\boldsymbol{N}$, then the vector fields $\boldsymbol{N}^k X$, $\forall k \in \mathbb{N}$, are also symmetries of $\boldsymbol{N}$. This property justifies the use of the name of recursion operator for $\boldsymbol{N}$, as successive elements inherit the properties of the first.

Using this result, we can analyze the symmetries of bi-Hamiltonian systems that fulfil (3.4) when one of the bivectors, say $\boldsymbol{P}_1$, is invertible. To do this, we consider the sequence of vector fields

$$X_k = \boldsymbol{P}_0 \mathrm{d}H_k = \boldsymbol{P}_1 \mathrm{d}H_{k+1} \,. \tag{3.6}$$

Then, through the recursion operator $\boldsymbol{N}$, we obtain the relation $X_{k+1} = \boldsymbol{N}X_k$. Thus, if the vector field X_0 is a symmetry of $\boldsymbol{N}$, it follows that all X_k, $\forall k \in \mathbb{N}$, will also be symmetries.

Therefore, the recursion operator generates two infinite chains of vector fields and 1-forms, namely

$$\begin{aligned} \mathrm{d}H_{k+1} &= \boldsymbol{N}^T \mathrm{d}H_k\,, & \mathrm{d}H_0 &\in \Omega^1(M)\,, \\ X_{k+1} &= \boldsymbol{N}X_k\,, & X_0 &\equiv X \in \mathfrak{X}(M)\,; \end{aligned} \tag{3.7}$$

where $\boldsymbol{N}^T : \Omega^1(M) \to \Omega^1(M)$ is the transpose operator of $\boldsymbol{N}$. These are known as the *Lenard chains* associated with a bi-Hamiltonian system, and can be represented schematically as follows:

$$\mathrm{d}H_0 \xrightarrow{\boldsymbol{N}^T} \mathrm{d}H_1 \xrightarrow{\boldsymbol{N}^T} \mathrm{d}H_2 \xrightarrow{\boldsymbol{N}^T} \mathrm{d}H_3 \xrightarrow{\boldsymbol{N}^T} \cdots$$

$$X_0 \xrightarrow{\boldsymbol{N}} X_1 \xrightarrow{\boldsymbol{N}} X_2 \xrightarrow{\boldsymbol{N}} X_3 \xrightarrow{\boldsymbol{N}} \cdots$$

We can summarize the inheritance of properties among the elements of the chain in the following way:

Proposition 3.1.2. *[121] Let $(M, \boldsymbol{P}_0, \boldsymbol{P}_1)$ be a bi-Hamiltonian manifold, where $\boldsymbol{P}_1$ is non-degenerate. Starting from the exact 1-forms $\mathrm{d}H_0$ and $\mathrm{d}H_1$, all the forms $\mathrm{d}H_k$ are also exact. Additionally, the vector fields X_k are bi-Hamiltonian and commute in pairs.*

This result is closely related to the integrability of the system. The functions H_k, derived from the exact 1-forms $\mathrm{d}H_k$, are in involution with respect to both Poisson brackets associated with the bivectors $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$,

$$\{H_i, H_j\}_0 = \{H_i, H_j\}_1 = 0\,, \qquad i, j = 1, 2, \ldots,$$

meaning they are integrals of motion of the system. However, for total integrability, a sufficient number of functionally independent integrals is required, which cannot be guaranteed *a priori*.

3.1.3 *Darboux-Nijenhuis coordinates and separability on ωN manifolds*

In ωN manifolds, there exists a class of canonical coordinates of particular importance known as *Darboux-Nijenhuis coordinates*, which play a central role in the SoV for the HJ equation.

Definition 3.1.4. *Given an ωN manifold, a set of local coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$ is called a set of* Darboux-Nijenhuis (DN) coordinates *if the symplectic form assumes the Darboux form and the recursion operator $\boldsymbol{N}$ takes the diagonal form*

$$\boldsymbol{N} = \sum_{i=1}^{n} l_i(\boldsymbol{q}, \boldsymbol{p}) \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right), \tag{3.8}$$

where $l_i \equiv \lambda_i = \lambda_{n+i}$ are the eigenvalue fields of $\boldsymbol{N}$. Consequently, the only non-vanishing Poisson brackets are

$$\{q^i, p_j\}_1 = \delta^i_j, \quad \{q^i, p_j\}_0 = \lambda_i \delta^i_j, \qquad i, j = 1, \ldots, n. \tag{3.9}$$

The eigendistributions associated with each l_i are even-dimensional, as the kernel of $\boldsymbol{P}_0 - l_i\boldsymbol{P}_1$ is also even-dimensional. Moreover, if the eigenvalues λ_i are functionally independent, they and their conjugate momenta form a set of DN coordinates.

An ωN manifold M is *regular* if the eigenvalues of $\boldsymbol{N}$ are functionally independent on M. For such manifolds, the following holds:

Proposition 3.1.3. *In a neighborhood of a point in a regular ωN manifold where the eigenvalues of $\boldsymbol{N}$ are distinct, there exist n functions $(\mu_1, \dots, \mu_n)$ obtainable by quadratures, which together with $(\lambda_1, \dots, \lambda_n)$ constitute DN coordinates. These coordinates are termed* special Darboux-Nijenhuis coordinates.

A defining property of DN coordinates $(\boldsymbol{q}, \boldsymbol{p})$ is that their differentials span an eigenspace of $\boldsymbol{N}^T$:

$$\boldsymbol{N}^T \mathrm{d}q^i = \lambda_i \mathrm{d}q^i, \quad \boldsymbol{N}^T \mathrm{d}p_i = \lambda_i \mathrm{d}p_i, \qquad i = 1, \dots, n. \tag{3.10}$$

These properties make it possible to characterize separability for integrable Hamiltonian systems within this framework:

Theorem 3.1.1 (Theorem 3.1, [52])**.** *Let M be an ωN manifold with a simple recursion operator $\boldsymbol{N}$ (ensuring local existence of DN coordinates), and let $(H_1, H_2, \dots, H_n)$ be a set of functionally independent Hamiltonians on M. The following are equivalent:*

(i) The foliation defined by $(H_1, H_2, \dots, H_n)$ is separable in DN coordinates (and thus Lagrangian with respect to ω);

(ii) The distribution tangent to the foliation defined by $(H_1, H_2, \dots, H_n)$ is Lagrangian with respect to ω and invariant under $\boldsymbol{N}$;

(iii) The Hamiltonians $(H_1, H_2, \dots, H_n)$ are in bi-involution, i.e., $\{H_i, H_j\}_0 = 0$ and $\{H_i, H_j\}_1 = 0$ for all $i, j = 1, 2, \dots, n$.

To introduce Haantjes manifolds more intuitively in the next section, it is helpful to summarize some properties of ωN manifolds, as explained in [117]. In a bi-Hamiltonian manifold $(M, \boldsymbol{P}_0, \boldsymbol{P}_1)$ where $\boldsymbol{P}_1$ is invertible, we have three distinguished geometric objects on the manifold M: the vector field X, the 1-form now denoted as $\theta \equiv \mathrm{d}H_0$, and the recursion operator $\boldsymbol{N} = \boldsymbol{P}_0\boldsymbol{P}_1^{-1}$. These objects satisfy the following relations:

(i) $\boldsymbol{N}$ is a Nijenhuis operator:

$$\tau_{\boldsymbol{N}}(X, Y) = \boldsymbol{0}, \qquad \forall\, X, Y \in \mathfrak{X}(M); \tag{3.11}$$

(ii) The 1-form θ is closed:

$$\mathrm{d}\theta = 0; \tag{3.12}$$

(iii) The first iterated 1-form $\boldsymbol{N}^T\theta$ is closed:

$$\mathrm{d}(\boldsymbol{N}^T\theta) = 0\,; \tag{3.13}$$

(iv) X is a symmetry of the recursion operator $\boldsymbol{N}$:

$$\mathcal{L}_X \boldsymbol{N} = \boldsymbol{0}\,, \tag{3.14}$$

where $\mathcal{L}_X \boldsymbol{N}$ denotes the Lie derivative of $\boldsymbol{N}$ with respect to X;

(v) θ remains constant throughout the flow determined by the vector field X:

$$\mathcal{L}_X \theta = \boldsymbol{0}\,. \tag{3.15}$$

3.2 HAANTJES MANIFOLDS

Haantjes manifolds are a concept introduced by Franco Magri in a series of papers written since 2012 [117–119, 123], where the geometry of Haantjes operators has been related with integrable systems. The main idea is to extend the recursion operators defined for ωN manifolds so that they can have non-vanishing Nijenhuis torsion, but in a controlled way. For this purpose, some conditions are introduced, which lead to the following definition.

Definition 3.2.1. *A manifold M is a* Haantjes manifold *if it possesses a distinguished 1-form θ and a distinguished vector-valued 1-form $\boldsymbol{K}$ that fulfil the conditions*

(i) $\boldsymbol{K}$ is a Haantjes operator:

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) = \boldsymbol{0}\,, \qquad \forall\, X,Y \in \mathfrak{X}(M)\,; \tag{3.16}$$

(ii) The 1-form θ is closed:

$$\mathrm{d}\theta = 0\,; \tag{3.17}$$

(iii) The first iterated 1-form $\boldsymbol{K}^T\theta$ is closed:

$$\mathrm{d}(\boldsymbol{K}^T\theta) = 0\,; \tag{3.18}$$

(iv) The 1-form θ annihilates the Nijenhuis torsion of $\boldsymbol{K}$ viewed as a vector-valued 2-form on M:

$$\theta\big(\tau_{\boldsymbol{K}}(X,Y)\big) = 0 \qquad \forall\, X,Y \in \mathfrak{X}(M)\,. \tag{3.19}$$

Furthermore, if there exists a vector field X that satisfies the conditions

(v) $\boldsymbol{K}$ is invariant under the flow generated by the vector field X:

$$\mathcal{L}_X \boldsymbol{K} = \boldsymbol{0}\,; \tag{3.20}$$

(vi) θ remains unchanged under the flow generated by the vector field X:

$$\mathcal{L}_X \theta = 0\,; \tag{3.21}$$

with $\mathcal{L}_X$ representing the Lie derivative with respect to X, then the manifold is said to be a Haantjes manifold with symmetry.

To define his Haantjes manifolds, F. Magri starts by relaxing the requirement that the recursion operator $\boldsymbol{K}$ is a Nijenhuis tensor field. Instead, we simply require that it is a Haantjes tensor, which significantly broadens its applicability. However, this modification comes with a trade-off: we can no longer construct Lenard chains using powers of $\boldsymbol{K}$. To address this limitation, we impose condition (iv), which is crucial for the following result.

Lemma 3.2.1. *Let M be a manifold, θ a 1-form, and $\boldsymbol{K}$ a $(1,1)$-tensor field. If θ annihilates the Nijenhuis torsion of $\boldsymbol{K}$ and $\boldsymbol{K}^T$ maps θ to a closed 1-form, then the 1-form $(\boldsymbol{K}^T)^2\theta$ is closed.*

Condition (iv) has the drawback of needing an explicit study for each specific case, yet it remains interesting in its applicability.

It is noteworthy that on Haantjes manifolds, there exist two distinguished tensor fields of type $(1,1)$: the identity operator $\boldsymbol{I}$ and the operator $\boldsymbol{K}$. In light of this fact and Lemma 3.2.1, the Lenard-Magri chain in this context is defined as a family of tensor fields sharing the properties of these distinguished operators, which constitute the core of the chain. The theory of these chains is a fundamental piece of the geometric approach to soliton hierarchies. They have been introduced to construct integrals of motion in involution for infinite-dimensional Hamiltonian systems.

Definition 3.2.2. *A* Lenard-Magri chain *of length p on a Haantjes manifold of dimension n is a family of p operator fields $\boldsymbol{K}_i$ that includes $\boldsymbol{K}_0 = \boldsymbol{I}$ and $\boldsymbol{K}_1 = \boldsymbol{K}$, and meets the conditions*

(i) It consists of a set of Haantjes operators:

$$\mathcal{H}_{\boldsymbol{K}_i}(X,Y) = \boldsymbol{0}\,, \qquad \forall\, X, Y \in \mathfrak{X}(M)\,; \tag{3.22}$$

(ii) The elements of the chain pairwise commute:

$$\left[\boldsymbol{K}_i, \boldsymbol{K}_j\right] = \boldsymbol{0}\,, \qquad i, j = 0, \ldots, p-1\,; \tag{3.23}$$

(iii) All of the first iterated 1-forms $\boldsymbol{K}_i^T\theta$ are closed:

$$\mathrm{d}(\boldsymbol{K}_i^T\theta) = 0\,; \tag{3.24}$$

(iv) The 1-form θ annihilates the Nijenhuis torsion of every $\boldsymbol{K}_i$:

$$\theta\big(\tau_{\boldsymbol{K}_i}(X,Y)\big) = 0\,, \qquad \forall\, X, Y \in \mathfrak{X}(M)\,. \tag{3.25}$$

The main difference regarding Lenard chains in bi-Hamiltonian systems is that, while the 1-form $(\boldsymbol{K}^T)^2\theta$ is closed, it is not possible to connect an additional element to the chain because the 1-form $(\boldsymbol{K}^T)^3\theta$ is not necessarily closed. As a result, in Haantjes manifolds, Lenard-Magri chains break at the third iteration. This set of three 1-forms $(\theta, \boldsymbol{K}^T\theta, (\boldsymbol{K}^T)^2\theta)$ is referred to as the short Lenard chain generated by θ and $\boldsymbol{K}$. This can be represented schematically as follows:

$$\theta \xrightarrow{\boldsymbol{K}^T} \boldsymbol{K}^T\theta \xrightarrow{\boldsymbol{K}^T} (\boldsymbol{K}^T)^2\theta$$

A significant advantage of the Lenard-Magri chains, as defined, is that a chain of length p involves p operators $\boldsymbol{K}_i$. Thus, one can additionally consider the 1-forms obtained by applying two of these operators, resulting in expressions of the form

$$\theta_{ij} = \boldsymbol{K}_i^T \boldsymbol{K}_j^T \theta\,. \tag{3.26}$$

This family of forms is referred to as the *square* associated with the Lenard-Magri chain.

Proposition 3.2.1. *[117] All the forms within this square are closed. Therefore, a Lenard-Magri chain of length p in a Haantjes manifold of dimension n generates a square of $\frac{1}{2}p(p+1)$ locally exact 1-forms.*

The most interesting situation arises when the chain length is equal to the manifold dimension M. This scenario is connected to the *WDVV* equations [123] and the concept of the (dispersionless) Hirota *tau*-function.

3.3 SYMPLECTIC-HAANTJES MANIFOLDS

Inspired by Magri's construction, in [185] the family of *symplectic-Haantjes* (or $\omega\mathscr{H}$) manifolds was introduced. They are symplectic manifolds endowed with a compatible Haantjes algebra [184], namely a set of Haantjes $(1,1)$-tensor fields closed under linear $C^\infty(M)$-combinations and products. In [93], a comparison between Magri's Haantjes manifolds and $\omega\mathscr{H}$ manifolds has been made by Kosmann-Schwarzbach.

We recall here the main properties of $\omega\mathscr{H}$ manifolds, which offer a natural theoretical framework for the formulation of the theory of Hamiltonian integrable systems. Under suitable hypotheses, given a Hamiltonian integrable system there exists an $\omega\mathscr{H}$ manifold associated (as stated in the Liouville-Haantjes (LH) theorem [185]). Many examples of these structures have been exhibited in [192], [185], and [157]. The classical theory of SoV can also be formulated in the context of Haantjes geometry. The Jacobi-Haantjes theorem, stated in [157], establishes under mild hypotheses the equivalence between the notions of total separability of a Hamiltonian system and the existence of a semisimple, Abelian $\omega\mathscr{H}$ manifold of *maximal rank*. Indeed, in the semisimple case, there exists a special class of canonical coordinates, the *Darboux-Haantjes* coordinates, which represent separation variables for the HJ equations and diagonalize simultaneously all the operators of the Haantjes algebra associated with a totally separable model [157].

We mention that a generalization of the notion of $\omega\mathscr{H}$ manifolds is that of $P\mathscr{H}$ manifolds, introduced in [189], where P is a Poisson bivector compatible with the Haantjes algebra $\mathscr{H}$. When P is invertible, a $P\mathscr{H}$ structure reduces to a $\omega\mathscr{H}$ one.

3.3.1 *Definition*

Symplectic-Haantjes (or $\omega\mathscr{H}$) manifolds are symplectic manifolds endowed with an algebra of Haantjes operators compatible with the symplectic form. Apart from their mathematical properties, these geometric structures are relevant because they provide a simple but sufficiently general setting in which the theory of Hamiltonian integrable systems can be naturally formulated.

Definition 3.3.1. *A* symplectic-Haantjes (or $\omega\mathscr{H}$) manifold *of class r is a triple $(M,\omega,\mathscr{H})$ which satisfies the following properties:*

(i) (M,ω) *is a symplectic manifold of dimension $2n$;*

(ii) $\mathscr{H}$ *is an Abelian Haantjes algebra of rank r;*

(iii) $(\omega,\mathscr{H})$ *are algebraically compatible, that is*

$$\omega(X,\boldsymbol{K}Y)=\omega(\boldsymbol{K}X,Y)\,,\qquad \forall\,\boldsymbol{K}\in\mathscr{H}\,, \tag{3.27}$$

or equivalently

$$\boldsymbol{\Omega}\,\boldsymbol{K}=\boldsymbol{K}^{T}\boldsymbol{\Omega},\qquad \forall\,\boldsymbol{K}\in\mathscr{H}\,, \tag{3.28}$$

where $\boldsymbol{\Omega}:=\omega^{\flat}$ is the bundle isomorphism defined by Eq. (1.4).

If the identity operator $\boldsymbol{I}$ belongs to $\mathscr{H}$, then we shall say that $(M,\omega,\mathscr{H})$ is an $\omega\mathscr{H}$ manifold with identity.

The commutativity condition in (ii) of the latter Definition is motivated by the following result.

Proposition 3.3.1 (Proposition 2, [160])**.** *The set of (non-necessarily Haantjes) operators compatible with the symplectic form ω is a $C^{\infty}(M)$ module. Moreover, this module is an algebra if and only if it is Abelian with respect to the product of operators.*

Proof. The first statement follows immediately from the linearity of the compatibility condition (3.28). Concerning the second one, we note that if $\boldsymbol{K}_1,\boldsymbol{K}_2$ are operators such that

$$\boldsymbol{\Omega}\boldsymbol{K}_1=\boldsymbol{K}_1^{T}\boldsymbol{\Omega}\,,\qquad \boldsymbol{\Omega}\boldsymbol{K}_2=\boldsymbol{K}_2^{T}\boldsymbol{\Omega}\,, \tag{3.29}$$

then

$$\boldsymbol{\Omega}\boldsymbol{K}_1\boldsymbol{K}_2=\boldsymbol{K}_1^{T}\boldsymbol{\Omega}\boldsymbol{K}_2=\boldsymbol{K}_1^{T}\boldsymbol{K}_2^{T}\boldsymbol{\Omega}=(\boldsymbol{K}_2\boldsymbol{K}_1)^{T}\boldsymbol{\Omega}\,. \tag{3.30}$$

Thus, $\boldsymbol{K}_1\boldsymbol{K}_2$ is compatible with the symplectic form ω if and only if $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ commute. □

As in the case of Haantjes algebras (see Sect. 2.4.1), we can also identify a natural and straightforward realization of Haantjes manifolds. Specifically, consider a local coordinate system $\boldsymbol{x} = (x^1, x^2, \dots, x^{2n})$, where the operators

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{2n} \lambda_i^{(\alpha)}(\boldsymbol{x}) \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i\,, \qquad \alpha = 1, \dots, r\,,$$

generate a diagonal Haantjes algebra for any functions $\lambda_i^{(\alpha)}(\boldsymbol{x}) \in C^\infty(M)$. By imposing the algebraic compatibility condition 3.28 on a Darboux chart $\boldsymbol{x} = (q^1, \dots q^n, p_1, \dots, p_n)$, we find that the desired realization is provided by the operators

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} l_i^{(\alpha)}(\boldsymbol{x}) \Big(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \Big), \qquad \alpha = 1, \dots, r\,, \tag{3.31}$$

where $l_i^{(\alpha)} = \lambda_i^{(\alpha)}(\boldsymbol{x}) = \lambda_{n+i}^{(\alpha)}(\boldsymbol{x})$, $i = 1, \dots, n$.

We present the following definition, which is especially relevant in light of Proposition 2.4.2.

Definition 3.3.2. *An $\omega\mathscr{H}$ manifold $(M, \omega, \mathscr{H})$ is* semisimple *if $\mathscr{H}$ is a semisimple Haantjes algebra.*

The compatibility of the Haantjes operators with the symplectic form within this framework has significant consequences for their eigendistributions and spectral properties, as we will discuss below.

Proposition 3.3.2 (Corollary 20, [185])**.** *Given a $2n$-dimensional $\omega\mathscr{H}$ manifold M, every generalized eigendistribution* $\ker(\boldsymbol{K} - \lambda_i \boldsymbol{I})^{\rho_i}$, $\rho_i \in \mathbb{N}$, *is of* even rank. *Therefore, the geometric and algebraic multiplicities of each eigenvalue $\lambda_i(\boldsymbol{x})$ are even.*

Let us consider the spectral decomposition of the tangent spaces of M given by

$$T_{\boldsymbol{x}} M = \bigoplus_{i=1}^{s} \mathcal{D}_i(\boldsymbol{x}), \qquad \mathcal{D}_i(\boldsymbol{x}) := \ker \left(\boldsymbol{K} - \lambda_i \boldsymbol{I}\right)^{\rho_i}(\boldsymbol{x}), \tag{3.32}$$

where ρ_i is the Riesz index of the eigenvalue λ_i. Due to the latter Proposition, as rank $\mathcal{D}_i \geq 2$, $i = 1, \dots s$ we conclude that the number of distinct eigenvalues of any Haantjes operator $\boldsymbol{K}$ of an $\omega\mathscr{H}$ manifold is not greater than n. In particular, if the number of distinct eigenvalues of an operator $\boldsymbol{K} \in \mathscr{H}$ is exactly n, the operator will be said to be *maximal*. This is equivalent to require that the minimal polynomial of $\boldsymbol{K}$, $m_{\boldsymbol{K}}(\boldsymbol{x}, \lambda) = \prod_{i=1}^{n} \left(\lambda - \lambda_i(\boldsymbol{x})\right)^{\rho_i}$, has degree $l = n$.

Several other interesting results can be stated in the $\omega\mathscr{H}$ geometry. We shall assume that the distributions $\mathcal{D}_j$, $\mathcal{E}_i$, $\mathcal{E}_i^\circ$ have constant rank. Thus, they are integrable by virtue of the Haantjes theorem and fulfill the properties summed up in the following

Proposition 3.3.3. *Given a semisimple $\omega\mathscr{H}$ manifold of class n, the eigen-distributions $\mathcal{D}_j$ of each generator $\boldsymbol{L}$ are integrable and of rank 2. Their integral leaves are two-dimensional symplectic submanifolds of M and are symplectically orthogonal to each other:*

$$\boldsymbol{\Omega}(\mathcal{D}_j) = \mathcal{E}_j^\circ \Leftrightarrow \mathcal{D}_j = \boldsymbol{P}(\mathcal{E}_j^\circ) = \mathcal{E}_j^\perp , \tag{3.33}$$

$$\boldsymbol{\Omega}(\mathcal{E}_j) = \mathcal{D}_j^\circ \Leftrightarrow \mathcal{E}_j = \boldsymbol{P}(\mathcal{D}_j^\circ) = \mathcal{D}_j^\perp , \tag{3.34}$$

$$\omega(\mathcal{D}_j, \mathcal{D}_j) = \textit{symplectic}, \tag{3.35}$$

$$\omega(\mathcal{D}_j, \mathcal{D}_k) = \mathbf{0} \qquad j \neq k . \tag{3.36}$$

Here $\mathcal{E}_j^\perp$ and $\mathcal{D}_j^\perp$ are the symplectic orthogonal distributions of $\mathcal{E}_j$ and $\mathcal{D}_j$, respectively.

Proof. This result is a consequence of Propositions 23 and 24 of [185]. □

In general, given a (not necessarily semisimple) $\omega\mathscr{H}$ manifold, the distributions $\mathcal{D}_j$, $j = 1, \dots, s$, associated with each $\boldsymbol{K} \in \mathscr{H}$ are integrable and have even rank. Furthermore, the integral leaves of these distributions are symplectic submanifolds of M and are symplectically orthogonal to each other, namely $\omega(\mathcal{D}_j, \mathcal{D}_k) = 0$ for $j \neq k$.

3.3.2 *Haantjes chains*

The notion of Haantjes chains, which generalizes that of Lenard-Magri chains [116, 121] has been proposed in [185, 192]. In the forthcoming analysis, Haantjes chains will enable us to build a bridge between the Haantjes geometry and the theory of integrable systems. Other generalizations have also been proposed in the literature during the last decades [51, 52, 135, 136, 190].

Definition 3.3.3. *Let $(M, \mathscr{H})$ be a Haantjes algebra of rank r. A smooth function H is said to generate a* Haantjes chain *of closed 1-forms of length r if there exists a distinguished basis $\{\tilde{\boldsymbol{K}}_1, \tilde{\boldsymbol{K}}_2, \dots, \tilde{\boldsymbol{K}}_r\}$ of $\mathscr{H}$ such that*

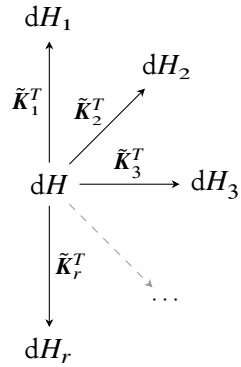
$$\mathrm{d}(\tilde{\boldsymbol{K}}_\alpha^T \, \mathrm{d}H) = \mathbf{0} , \qquad \alpha = 1, \dots, r , \tag{3.37}$$

where $\tilde{\boldsymbol{K}}_\alpha^T \colon \Omega^1(M) \to \Omega^1(M)$ is the transposed operator of $\tilde{\boldsymbol{K}}_\alpha$. The (locally) exact 1-forms $\mathrm{d}H_\alpha$ such that

$$\mathrm{d}H_\alpha = \tilde{\boldsymbol{K}}_\alpha^T \, \mathrm{d}H , \tag{3.38}$$

which are supposed to be linearly independent, are called the elements of the Haantjes chain of length r generated by H. The functions $H_\alpha \in C^\infty(M)$ are said to be the potential functions of the chain.

Unlike Lenard and Lenard-Magri chains, which rely on iterating powers of a single operator, Haantjes chains in symplectic-Haantjes manifolds are generated "nonlinearly": a generator 1-form $\mathrm{d}H$ produces all elements of the chain via a *family* of distinct operators $\{\tilde{\boldsymbol{K}}_1, \tilde{\boldsymbol{K}}_2, \dots, \tilde{\boldsymbol{K}}_r\}$. A schematic diagram illustrating these chains is shown below.



The importance of strings in $\omega\mathscr{H}$ manifolds is due to the fact that they play the role of Poisson brackets in this framework, as the following result illustrates.

Lemma 3.3.1 (Lemma 32, [185]). *Let $(M, \omega, \mathscr{H})$ be an Abelian $\omega\mathscr{H}$ manifold. Then, the potential functions H_α, whose differentials belong to the Haantjes chain generated by a single function H, are in involution among each other and with H, w.r.t. the Poisson bracket defined by the Poisson operator $\boldsymbol{P} = \boldsymbol{\Omega}^{-1}$.*

In order to enquire about the existence of Haantjes chains for an assigned Haantjes algebra, we have to consider the codistribution, of rank $r \leq k$, generated by a given function H via an arbitrary basis $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \ldots, \boldsymbol{K}_k\}$ of $\mathscr{H}$, i.e.

$$\mathcal{D}_H^\circ := \operatorname{span}\{\boldsymbol{K}_1^T \mathrm{d}H, \boldsymbol{K}_2^T \mathrm{d}H, \ldots, \boldsymbol{K}_k^T \mathrm{d}H\}, \tag{3.39}$$

and the distribution $\mathcal{D}_H$ of the vector fields annihilated by them, which has rank $n - r$. Note that such distributions do not depend on the particular choice of the basis of $\mathscr{H}$. The following theorems offer a geometric characterization of the existence of a Haantjes chain generated by a smooth function H in terms of the Frobenius integrability of its associated codistribution.

Theorem 3.3.1 (Theorem 16, [185]). *Let $(M, \mathscr{H})$ be a Haantjes algebra of rank r, and H a smooth function on M. Let $\mathcal{D}_H^\circ$ be the codistribution (3.39), assumed of rank r (independent on $\boldsymbol{x}$), and let $\mathcal{D}_H$ be the distribution of the vector fields annihilated by the 1-forms of $\mathcal{D}_H^\circ$. Then, the function H generates a Haantjes chain (3.37) if and only if $\mathcal{D}_H^\circ$ (or equivalently $\mathcal{D}_H$) is Frobenius-integrable.*

The following result clarifies the compatibility between a given $\omega\mathscr{H}$ manifold and a set of functions in involution.

Theorem 3.3.2 (Theorem 35, [185]). *Let $(M, \omega, \mathscr{H})$ be a $2n$-dimensional $\omega\mathscr{H}$ Abelian manifold of class r. Let $(H_1, H_2, \ldots, H_r)$ be r independent functions in involution and $\mathcal{D}^\circ = \operatorname{span}\{\mathrm{d}H_1, \mathrm{d}H_2, \ldots, \mathrm{d}H_r\}$. The differentials $(\mathrm{d}H_1, \mathrm{d}H_2, \ldots, \mathrm{d}H_r)$ form a Haantjes chain generated by a smooth function H in involution with them if and only if H satisfies the condition*

$$\mathcal{D}_H^\circ = \mathcal{D}^\circ. \tag{3.40}$$

In addition, the subsequent theorem characterises the Hamiltonian functions that generate Haantjes chains.

Theorem 3.3.3 (Theorem 3, [160]). *Let $(H_1, H_2, \dots, H_r)$ be r independent functions in involution and $\mathcal{D}^\circ = \mathrm{span}\{\mathrm{d}H_1, \mathrm{d}H_2, \dots, \mathrm{d}H_r\}$. The set $(H_1, H_2, \dots, H_r)$ are the potential functions of a Haantjes chain formed by $(\mathrm{d}H_1, \mathrm{d}H_2, \dots, \mathrm{d}H_r)$ if and only if, for all $\boldsymbol{K} \in \mathscr{H}$, the following invariance condition holds*

$$\boldsymbol{K}^T(\mathcal{D}^\circ) \subseteq \mathcal{D}^\circ. \tag{3.41}$$

Proof. Let H be functionally dependent on $(H_1, \dots, H_r)$. Then $\mathrm{d}H \in \mathcal{D}^\circ$, and H is in involution with $(H_1, \dots, H_r)$. Moreover, for each $\boldsymbol{K} \in \mathscr{H}$ we have

$$\boldsymbol{K}^T \mathrm{d}H = \boldsymbol{K}^T\left(\sum_{i=1}^{r} f_i \,\mathrm{d}H_i\right) = \sum_{i=1}^{r} f_i \,\boldsymbol{K}^T \mathrm{d}H_i. \tag{3.42}$$

If the invariance condition (3.41) is fulfilled, then $\mathcal{D}_H^\circ \subseteq \mathcal{D}^\circ$; in particular, having the same rank by assumption, the two distributions coincide. Thus, we deduce that condition (3.40) is satisfied. Conversely, if condition (3.40) is satisfied, then $\boldsymbol{K}^T \mathrm{d}H$ is a linear combination of $(\mathrm{d}H_1, \dots, \mathrm{d}H_r)$. Thus, since $\mathrm{d}H \in \mathcal{D}^\circ$, we conclude that the invariance condition (3.41) is fulfilled. □

From Theorems 3.3.2 and 3.3.3, we deduce the following

Corollary 3.3.1 (Corollary 1, [160]). *A function H generates a Haantjes chain only if it satisfies the invariance condition*

$$\boldsymbol{K}^T(\mathcal{D}_H^\circ) \subseteq \mathcal{D}_H^\circ, \qquad \forall \boldsymbol{K} \in \mathscr{H}. \tag{3.43}$$

3.3.3 *Darboux-Haantjes coordinates*

In analogy to the Darboux-Nijenhuis coordinates–which simultaneously trivialize a symplectic structure into canonical form and diagonalize (or block-diagonalize) a Nijenhuis operator–one may define *Darboux-Haantjes coordinates* in symplectic-Haantjes manifolds.

Definition 3.3.4. *Given an $\omega\mathscr{H}$ manifold, the local coordinates where all Haantjes operators take simultaneously a block-diagonal form and the symplectic form takes the Darboux form are called* Darboux-Haantjes (DH) coordinates.

The existence of DH coordinates for Haantjes manifolds–a central element of this work–directly follows from the theoretical framework developed in [160]. In fact, Proposition 2.4.1 guarantees their existence, as demonstrated by the reasoning presented in that article. Precisely, the fact that all the operators considered are Haantjes operators ensures that there exist coordinates allowing for a simultaneous block-diagonalization. Notice that all eigendistributions $\mathcal{V}_a(\boldsymbol{x})$ are integrable, as they are intersections of integrable distributions (see (2.53)). Moreover, they are of even rank, as stated in 3.3.2. If they are also pairwise integrable, then the block-diagonalization can be realized

with finer blocks. By definition, the operators are also algebraically compatible with the symplectic form. As each of these blocks is invariant, we can restrict the symplectic form on them. Then, over the leaves of each distribution $\mathcal{V}_a$, one can find Darboux coordinates for the corresponding restriction of the symplectic form, which is still symplectic. This is due to the fact that the distributions are integrable and of even rank, so their integral leaves are symplectic submanifolds and are symplectically orthogonal to each other (Proposition 3.3.3). Therefore, one can collect such coordinates to obtain a local chart in M of the form

$$(\boldsymbol{q}^1, \boldsymbol{p}_1, \dots, \boldsymbol{q}^v, \boldsymbol{p}_v)\,, \tag{3.44}$$

where the subsets

$$\begin{aligned}(\boldsymbol{q}^1, \boldsymbol{p}_1) &= (q^{1,1}, \dots, q^{1,\sigma_1}, p_{1,1}, \dots, p_{1,\sigma_1})\,,\\ &\dots\,,\\ (\boldsymbol{q}^v, \boldsymbol{p}_v) &= (q^{v,1}, \dots, q^{v,\sigma_v}, p_{v,1}, \dots, p_{v,\sigma_v})\,,\end{aligned}$$

are adapted to the decomposition (2.54). In this chart, the symplectic form ω takes the Darboux form

$$\omega = \sum_{a=1}^{v} \sum_{i_a=1}^{\sigma_a} \mathrm{d}q^{a,i_a} \wedge \mathrm{d}p_{a,i_a}\,, \tag{3.45}$$

being

$$\sigma_a := \frac{1}{2}\,\mathrm{rank}\,(\mathcal{V}_a)\,.$$

Also, as a consequence of Proposition 2.4.1, each Haantjes operator $\boldsymbol{K} \in \mathscr{H}$ possesses a block-diagonal form. Then, in the chart (3.44) the operators $\boldsymbol{K}_\alpha \in \mathscr{H}$ can be written as

$$\begin{aligned}\boldsymbol{K}_\alpha = \sum_{a=1}^{v} \sum_{j_a,k_a=1}^{\sigma_a} \Bigg(&\Big(\boldsymbol{A}_a^{(\alpha)}(\boldsymbol{q},\boldsymbol{p})\Big)_{a,k_a}^{a,j_a} \left(\frac{\partial}{\partial q^{a,j_a}} \otimes \mathrm{d}q^{a,k_a}\right)\\ &+ \Big(\boldsymbol{D}_a^{(\alpha)}(\boldsymbol{q},\boldsymbol{p})\Big)_{a,k_a}^{a,j_a} \left(\frac{\partial}{\partial p_{a,j_a}} \otimes \mathrm{d}p_{a,k_a}\right) + \Big(\boldsymbol{B}_a^{(\alpha)}(\boldsymbol{q},\boldsymbol{p})\Big)_{a,k_a}^{a,j_a} \left(\frac{\partial}{\partial q^{a,j_a}} \otimes \mathrm{d}p_{a,k_a}\right)\\ &+ \Big(\boldsymbol{C}_a^{(\alpha)}(\boldsymbol{q},\boldsymbol{p})\Big)_{a,k_a}^{a,j_a} \left(\frac{\partial}{\partial p_{a,j_a}} \otimes \mathrm{d}q^{a,k_a}\right)\Bigg)\,,\end{aligned} \tag{3.46}$$

where the blocks $\boldsymbol{A}_a^{(\alpha)}, \boldsymbol{B}_a^{(\alpha)}, \boldsymbol{C}_a^{(\alpha)}, \boldsymbol{D}_a^{(\alpha)}$ are $\sigma_a \times \sigma_a$ matrix-valued functions suitably chosen in order for $\boldsymbol{K}_\alpha$ to have a vanishing Haantjes torsion. Due to the compatibility condition (3.28), the relations

$$\begin{aligned}(\boldsymbol{D}_a^{(\alpha)})^T = \boldsymbol{A}_a^{(\alpha)}\,, \quad \boldsymbol{B}_a^{(\alpha)} + (\boldsymbol{B}_a^{(\alpha)})^T = 0\,,\\ \boldsymbol{C}_a^{(\alpha)} + (\boldsymbol{C}_a^{(\alpha)})^T = 0\,, \qquad \alpha = 1, \dots, r\,,\end{aligned} \tag{3.47}$$

must also be satisfied. Moreover, as the Haantjes algebra $\mathscr{H}$ is Abelian, the abovementioned blocks have to satisfy the further conditions

$$[\boldsymbol{A}_a^{(\alpha)}, \boldsymbol{A}_a^{(\beta)}] + \boldsymbol{B}_a^{(\alpha)}\boldsymbol{C}_a^{(\beta)} - \boldsymbol{B}_a^{(\beta)}\boldsymbol{C}_a^{(\alpha)} = 0\,, \tag{3.48}$$

$$\boldsymbol{A}_a^{(\alpha)}\boldsymbol{B}_a^{(\beta)} - \boldsymbol{A}_a^{(\beta)}\boldsymbol{B}_a^{(\alpha)} + \boldsymbol{B}_a^{(\alpha)}(\boldsymbol{A}_a^{(\beta)})^T - \boldsymbol{B}_a^{(\beta)}(\boldsymbol{A}_a^{(\alpha)})^T = 0\,, \tag{3.49}$$

$$\boldsymbol{C}_a^{(\alpha)}\boldsymbol{A}_a^{(\beta)} - \boldsymbol{C}_a^{(\beta)}\boldsymbol{A}_a^{(\alpha)} + (\boldsymbol{A}_a^{(\alpha)})^T\boldsymbol{C}_a^{(\beta)} - (\boldsymbol{A}_a^{(\beta)})^T\boldsymbol{C}_a^{(\alpha)} = 0\,. \tag{3.50}$$

Combining the conditions (3.47) with Eqs. (3.49) and (3.50), we deduce a useful result.

Lemma 3.3.2. *The operators*

$$\boldsymbol{A}_a^{(\alpha)}\boldsymbol{B}_a^{(\beta)} - \boldsymbol{A}_a^{(\beta)}\boldsymbol{B}_a^{(\alpha)}\,, \qquad \boldsymbol{C}_a^{(\alpha)}\boldsymbol{A}_a^{(\beta)} - \boldsymbol{C}_a^{(\beta)}\boldsymbol{A}_a^{(\alpha)} \tag{3.51}$$

are skew-symmetric.

In particular, if the Haantjes algebra $\mathscr{H}$ is *semisimple*, the blocks $\boldsymbol{B}_a^{(\alpha)}$ and $\boldsymbol{C}_a^{(\alpha)}$ vanish, whereas the blocks $\boldsymbol{A}_a^{(\alpha)}$ take the form $\boldsymbol{A}_a^{(\alpha)} = \lambda_a^{(\alpha)}\boldsymbol{I}_{\sigma_a}$, where $\lambda_a^{(\alpha)}$ are the eigenvalues of $\boldsymbol{K}_\alpha$.

Corollary 3.3.2. *[157] Given a semisimple Abelian $\omega\mathscr{H}$ manifold $(M, \omega, \mathscr{H})$, on a set of Darboux–Haantjes coordinates each $\boldsymbol{K} \in \mathscr{H}$ takes the diagonal form* (3.31).

3.3.4 *ωN and $\omega\mathscr{H}$ manifolds*

An interesting family of $\omega\mathscr{H}$ manifolds endowed with a unigenerated Haantjes algebra is represented by ωN manifolds [122, 125]. In that context, unigenerated Haantjes chains convert into Nijenhuis chains, as in [51], or generalized Lenard chains as in [183, 191]. In the following, we shall discuss the relation between these two geometric structures. Precisely, we shall prove that in the Abelian, semisimple case, there is a one-to-one correspondence between $\omega\mathscr{H}$ manifolds and equivalence classes of ωN manifolds. This correspondence does not hold in the more general, non-semisimple case.

Let us recall that an ωN manifold $(M, \omega, \boldsymbol{N})$ is a symplectic manifold endowed with a Nijenhuis operator $\boldsymbol{N}$ compatible with ω, that is

$$\boldsymbol{\Omega}\boldsymbol{N} = \boldsymbol{N}^T\boldsymbol{\Omega}\,, \tag{3.52}$$

$$\mathrm{d}(\boldsymbol{\Omega}\boldsymbol{N}) = \boldsymbol{0}\,, \tag{3.53}$$

where $\boldsymbol{\Omega}\colon \mathfrak{X}(M) \to \Omega^1(M)$ is the linear map defined by $\boldsymbol{\Omega} := \omega^\flat$. Condition (3.52) is equivalent to the fact that the composed linear map $\boldsymbol{\Omega}\boldsymbol{N}$ is again skew-symmetric, therefore the tensor field ω_1 induced by the linear map $\omega_1^\flat := \boldsymbol{\Omega}\boldsymbol{N}$, is a 2-form. Thus, we can compute its exterior derivative, which is imposed to vanish by condition (3.53).

First, let us consider an ωN manifold, and assume that its Nijenhuis operator $\boldsymbol{N}$ admits a minimal polynomial of degree l. Then, the manifold M carries an associated unigenerated Haantjes algebra, and the resulting $\omega\mathscr{H}$ structure is given by

$$(M, \omega, \boldsymbol{K}_1 = \boldsymbol{I}, \boldsymbol{K}_2 = \boldsymbol{N}, \ldots, \boldsymbol{K}_l = \boldsymbol{N}^{l-1})\,, \tag{3.54}$$

with a Haantjes algebra of rank $l \leq \dim(M)$. In fact, each Nijenhuis operator $\boldsymbol{N}$ is also a Haantjes operator; therefore, it generates the unigenerated Haantjes algebra $\mathscr{L}(\boldsymbol{N})$. Furthermore, the algebraic compatibility condition (3.52) assures that for all Haantjes operators

$$\boldsymbol{K} = p_{\boldsymbol{K}}(\boldsymbol{x}, \boldsymbol{N}) = \sum_{i=0}^{l-1} a_i(\boldsymbol{x})\, \boldsymbol{N}^i, \tag{3.55}$$

condition *(iii)* of Definition 3.3.1 is fulfilled.

Conversely, let us construct an ωN structure starting from an Abelian, semisimple $\omega\mathscr{H}$ manifold. As a consequence of Propositions 37 and 38 in [185], Abelian semisimple Haantjes algebras are always unigenerated ones. Besides, their generator can be chosen to be a Nijenhuis operator. In this case, DH coordinates coincide with DN coordinates and one can take, as usual in the ωN theory, the eigenvalue fields $(l_1, \ldots l_n)$ of the Nijenhuis generator of the Haantjes algebra, assumed to be functionally independent, as half of the set of DH coordinates. Regarding the other half, we propose in Sec. 3.3.6 a simple method for computing them, based on the analysis developed in [185] about the characteristic web of a $\omega\mathscr{H}$ manifold.

Let us recall that the spectral decomposition of the tangent spaces associated to a Haantjes generator $\boldsymbol{L}$ of a semisimple Abelian Haantjes algebra is given by

$$T_{\boldsymbol{x}}M = \bigoplus_{i=1}^{n} \mathcal{D}_i(\boldsymbol{x}) = \mathcal{D}_i(\boldsymbol{x}) \oplus \mathcal{E}_i(\boldsymbol{x}) \tag{3.56}$$

where

$$\mathcal{D}_i = \ker\left(\boldsymbol{L} - \lambda_i \boldsymbol{I}\right), \qquad \mathcal{E}_i = \operatorname{Im}\left(\boldsymbol{L} - \lambda_i \boldsymbol{I}\right) = \bigoplus_{j=1, j\neq i}^{n} \mathcal{D}_j, \qquad i = 1, \ldots, n. \tag{3.57}$$

Correspondingly,

$$\mathcal{E}_i^{\circ} = \ker\left(\boldsymbol{L}^T - \lambda_i \boldsymbol{I}\right) \tag{3.58}$$

and the cotangent spaces decompose as

$$T_{\boldsymbol{x}}^* M = \bigoplus_{i=1}^{n} \mathcal{E}_i^{\circ}(\boldsymbol{x}). \tag{3.59}$$

The ranks of the eigendistributions $\mathcal{D}_i$, of the characteristic distributions $\mathcal{E}_i$ and of the annhilators $\mathcal{E}_i^{\circ}, i = 1, \ldots, n$ are $2, 2n-2, 2$, respectively .

As a further step, we can prove that given an Abelian semisimple $\omega\mathscr{H}$ manifold of class n, there exists an ωN manifold associated. To this aim, for the sake of clarity, we specialize Proposition 38, proved in [185], to the case $l = n$.

Proposition 3.3.4. *Let $(M, \omega, \mathscr{H})$ be an Abelian $2n$-dimensional semisimple $\omega\mathscr{H}$ manifold of class n. Let us consider the spectral decomposition (3.56) and a DH chart $\{U, (q^i, p_i)\}$, $i = 1, \dots, n$, adapted to the decomposition (3.56), namely*

$$\mathcal{D}_i = \left\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i} \right\rangle . \tag{3.60}$$

Then, each operator defined by

$$\boldsymbol{L} = \sum_{i=1}^{n} l_i(\boldsymbol{q}, \boldsymbol{p}) \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \tag{3.61}$$

is a generator of $\mathscr{H}$, provided that $\{l_1(\boldsymbol{q}, \boldsymbol{p}), \dots, l_n(\boldsymbol{q}, \boldsymbol{p})\}$ are arbitrary, pointwise distinct smooth functions. Therefore, every operator $\boldsymbol{K} \in \mathscr{H}$ can be written in the form

$$\boldsymbol{K} = p_{\boldsymbol{K}}(\boldsymbol{x}, \boldsymbol{L}) = \sum_{i=1}^{n} \lambda_i \frac{\Pi_{j \neq i}(\boldsymbol{L} - l_j \boldsymbol{I})}{\Pi_{j \neq i}(l_i - l_j)} , \tag{3.62}$$

where $\lambda_i = \lambda_i(\boldsymbol{q}, \boldsymbol{p})$ are the eigenvalue fields of $\boldsymbol{K}$. In particular, if

$$l_i(\boldsymbol{q}, \boldsymbol{p}) = l_i(q^i, p_i) \qquad i = 1, \dots, n , \tag{3.63}$$

each generator $\boldsymbol{L}$ is a Nijenhuis operator. Each of these operators endows the manifold M with an ωN structure, for which the DH chart $\{U, (q^i, p_i)\}$, $i = 1, \dots, n$ is a DN chart.

Proof. The first part of the Theorem has been proved in [185].

Let us prove now that both conditions (3.52) and (3.53) are fulfilled by a Nijenhuis generator, that is a generator (3.61) of the algebra $\mathscr{H}$ which satisfies condition (3.63). We denote such Nijenhuis generator by $\boldsymbol{N}$. Condition (3.52) is satisfied as $\boldsymbol{N}$ takes a diagonal form in the DH coordinates $(\boldsymbol{q}, \boldsymbol{p})$. Concerning condition (3.53), it suffices to observe that in any set of DH coordinates, the 2-form $\boldsymbol{\Omega N}$ takes the local expression

$$\boldsymbol{\Omega N} = \sum_{i=1}^{n} l_i(q^i, p_i)\, \mathrm{d}p_i \wedge \mathrm{d}q^i . \tag{3.64}$$

Therefore, its exterior derivative vanishes, as can be computed by means of a direct calculation. □

We can now make precise the meaning of the correspondence between $\omega\mathscr{H}$ manifolds and equivalence classes of ωN manifolds.

Given a semisimple $\omega\mathscr{H}$ manifold, it admits infinitely many Nijenhuis generators. However, they are all related. Indeed, the corresponding ωN structures, although *a priori* different, possess common DN coordinates which in turn are also DH coordinates for the algebra $\omega\mathscr{H}$. We shall say that two semisimple ωN structures are equivalent if they possess common DN coordinates. It is easy to show that this property defines an equivalence relation of ωN manifolds. Thus, given

an $\omega\mathscr{H}$ manifold, there exists associated an equivalence class of ωN manifolds. The vice versa is obvious, since all Nijenhuis operators of the class by construction generate the same $\omega\mathscr{H}$ structure, which is endowed with a unigenerated Haantjes algebra.

More generally, given an equivalence class of Abelian, semisimple ωN manifolds, one can associate a unique $\omega\mathscr{H}$ manifold. Indeed, the powers of the Nijenhuis operators of the class all diagonalize in the same DN chart. Therefore, from the $C^\infty(M)$ modules generated by them one can define an algebra of diagonal Haantjes operators compatible with ω, i.e., an $\omega\mathscr{H}$ manifold. Obviously, the DN chart is a DH chart for the $\omega\mathscr{H}$ manifold.

From an applicative point of view, the $\omega\mathscr{H}$ structures represent a very flexible tool, which can be used either to construct in a consistent way separating coordinates, or to enhance the applicability of Nijenhuis geometry. In this regard, we have clarified the relationship between $\omega\mathscr{H}$ and ωN structures. The approach *à la Nijenhuis* represents an important theoretical framework, allowing for the construction of separating variables via the eigenvalues of a suitable semisimple Nijenhuis operator associated with a given integrable system [52]. The Haantjes geometry can complement and integrate, for practical purposes, the Nijenhuis approach. Indeed, a semisimple Haantjes algebra always admits a Haantjes generator; in addition, such a generator can be chosen to be a Nijenhuis one. Therefore, a possible strategy for finding separating variables is the following: given a Hamiltonian system, first we can construct in a natural way a $\omega\mathscr{H}$ manifold by solving the equations of its Haantjes chains (without the need for generalized ones, as is often necessary in Nijenhuis geometry [183]). Then, assuming that the algebra $\mathscr{H}$ is semisimple and Abelian, we can always select in $\mathscr{H}$ a Nijenhuis generator, providing us with separating variables (the so called Darboux-Nijenhuis (DN) coordinates). In the multiseparable case, this procedure can be repeated for all the structures allowed by the considered system.

We mention that finding a Nijenhuis operator without the help of the $\omega\mathscr{H}$ structure can be computationally cumbersome. Usually, one tries to construct Nijenhuis operators as reductions of bi-Hamiltonian structures associated with systems defined in a higher-dimensional space. Another possibility is to generate ωN structures as Yano liftings of Nijenhuis operators defined in the configuration space. However, these procedures are not generally applicable: only in some cases one can determine a system (easily tractable within the Nijenhuis approach), that by reduction gives the original system under study (and consequently, its ωN structure). Also, there are systems, as the Drach-Holt and the Post-Winternitz ones, whose Nijenhuis structure is not obtainable as a Yano complete lift.

Nevertheless, as shown in this thesis (and in many examples of [185, 192]), for determining an $\omega\mathscr{H}$ structure a direct, computationally affordable procedure is available, since the differential equations for the Haantjes chains are in general quite manageable.

In short, given an integrable Hamiltonian system, separating coordinates can be found in two different ways: as DH coordinates associated with its $\omega\mathscr{H}$ semisimple structures or, equivalently, via the eigenvalues of the Nijenhuis operator generating the same $\omega\mathscr{H}$ structures (DN coordinates).

Interestingly enough, $\omega\mathscr{H}$ manifolds appear to be a quite ubiquitous geometric structure in the class of superintegrable systems: indeed, they exist also in cases where orthogonal separating

variables are not allowed in T^*Q and are realized by *non-semisimple* Haantjes operators. This is the case for the anisotropic oscillator described above, and for the Post-Winternitz system [152], which admits two Abelian algebras of non-semisimple Haantjes operators.

3.3.5 *Haantjes and $\omega\mathscr{H}$ manifolds*

The symplectic-Haantjes and Magri's Haantjes formulations provide two complementary geometric approaches to integrability that both employ the Haantjes condition, yet they do so through fundamentally different methodologies. In the $\omega\mathscr{H}$ framework, the Haantjes algebra becomes intrinsically linked to the symplectic structure by means of the compatibility condition (3.28), which ensures that the operators respect the Hamiltonian phase space geometry. In this context, the spectral data of the Haantjes operators play a direct role in unraveling the system's integrability, with eigenvalues corresponding to action variables and the associated eigendistributions generating Lagrangian foliations. This construction is particularly effective for Hamiltonian systems, as the symplectic structure enables the systematic construction of DH coordinates and establishes direct connections to the Liouville-Arnold theorem.

Magri's formulation, on the other hand, positions the Haantjes formalism as an algebraic generalization of bi-Hamiltonian theory. Its key innovation lies in relaxing the Nijenhuis condition while preserving the torsionless Haantjes property. In this approach, the interaction between a 1-form θ and an operator $\boldsymbol{K}$ is governed by the conditions $\theta(\tau_{\boldsymbol{K}}) = \mathbf{0}$ and $d(\boldsymbol{K}^T\theta) = \mathbf{0}$. These constraints establish a self-consistent algebraic framework that enables the generation of integrals of motion through truncated Lenard-Magri chains.

The geometric commitments of these approaches differ significantly. While the $\omega\mathscr{H}$ theory embeds integrability rigorously within Hamiltonian mechanics by explicitly encoding phase space geometry, Magri's framework develops abstract algebraic machinery that applies to broader classes of recursive systems. The symplectic approach offers direct computational advantages for classical integrable systems through its explicit spectral correspondence, whereas the algebraic formulation provides deeper insights into structural singularities and recursive properties independent of specific geometric constraints. For a more detailed discussion, refer [93].

3.3.6 *A procedure to construct DH coordinates*

We can diagonalize algebras of semisimple Haantjes operators by constructing suitable coordinate charts. To this aim, we propose the following procedure [99, 159, 184, 185].

1) Given an algebra $\mathscr{H}$, determine the non-trivial intersections $\mathcal{V}_a$ of the eigendistributions of its operators (joint eigendistributions).

2) For each of the annihilators $(\mathcal{V}_a)^\circ$, construct a basis of closed 1-forms.

3) Find the characteristic coordinates by integrating them and imposing the canonical commutation relations.

The coordinate chart so determined represents a set of DH coordinates for the Haantjes operators of $\mathscr{H}$, namely, they simultaneously diagonalize all the operators of $\mathscr{H}$ and are Darboux coordinates for the (compatible) symplectic form.

The step 3) is not always easy to accomplish, so we wish to recall the strategy for the explicit construction of families of DH coordinates introduced in [157]. Under the same assumptions of Proposition 3.3.3, DH coordinates can be determined as pairs of characteristic functions of the Haantjes web of the generator $\boldsymbol{L}$, namely functions (x^i, y_i) which are constant on the characteristic eigendistributions $\mathcal{E}_i$. These functions are the same as the potential functions of the exact eigen-forms of $\boldsymbol{L}^T$ (due to Eq. (3.58)). Also, they fulfill the involution relations

$$\{x^i, x^j\} = 0\,, \quad \{y_i, y_j\} = 0\,, \qquad i = 1, \dots, n\,, \tag{3.65}$$

$$\{x^i, y_j\} = 0\,, \qquad i \neq j = 1, \dots, n \tag{3.66}$$

as a consequence of properties (3.33) and (3.36). Therefore, our problem is to find pairs of characteristic functions (x^i, y_i) that, in addition, are canonically conjugated:

$$\{x^i, y_i\} = 1\,, \qquad i = 1, \dots, n\,.$$

They represent Darboux coordinates for each of bidimensional symplectic distributions $\mathcal{E}_i^\circ$. We present now an effective procedure for the determination of these coordinates, which can be summed up in the following steps.

1) We determine a basis of 1-forms $\{\gamma_i, \delta_i\}$ for the characteristic co-distributions $\mathcal{E}_i^\circ$ of the Haantjes generator $\boldsymbol{L}$.

2) For each $i = 1, \dots, n$, we search for an exact 1-form $\alpha_i = \mathrm{d}x^i \in \mathcal{E}_i^\circ$, by representing it as

$$\alpha_i = f_i\, \gamma_i + g_i\, \delta_i\,, \tag{3.67}$$

and requiring that $\mathrm{d}\alpha_i = 0$. This step can be managed with a suitable ansatz about the form of the two integrating factors f_i and g_i.

3) We find the potential function x^i of the exact 1-form α_i. The potential functions so determined are automatically in involution w.r.t. the Poisson bracket induced by ω.

4) We search for another exact 1-form $\beta_i = \mathrm{d}y_i \in \mathcal{E}_i^\circ$, linearly independent of $\mathrm{d}x^i$. To this aim, we can represent β_i as

$$\beta_i = h_i\, \mathrm{d}x^i + r_i\, \gamma_i\,, \tag{3.68}$$

where we have assumed, without loss of generality, that $\gamma_i \in \mathcal{E}_i^\circ$ is linearly independent of $\mathrm{d}x^i$.

5) By requiring that the potential function y_i is canonically conjugated with x_i, that is

$$1 = \{x^i, y_i\} = \langle \mathrm{d}x^i, \boldsymbol{P}\, \mathrm{d}y_i \rangle = h_i \cancel{\langle \mathrm{d}x^i, \boldsymbol{P}\, \mathrm{d}x^i \rangle} + r_i \langle \mathrm{d}x^i, \boldsymbol{P}\, \gamma_i \rangle ,$$

we get the normalizing factor

$$r_i = \frac{1}{\langle \mathrm{d}x^i, \boldsymbol{P}\gamma_i \rangle} .$$

6) Substituting r_i into Eq. (3.68) and imposing that the 1-form β_i is a closed 1-form, we find the last normalizing factor h_i.

7) Finally, we find the potential function y_i of the exact 1-form β_i.

The previous task is much easier if one already knows half of the DH coordinates, say $\{x^i\}_{1\leq i\leq n}$. In this case, one skips the second and third step of the previous procedure. For instance, this occurs if one has a Haantjes generator $\boldsymbol{L}$ which is also Nijenhuis operator. In fact, in this case, the eigenvalues fields of $\boldsymbol{L}$ are just characteristic functions of the Haantjes web. Then, once the eigenvalues of $\boldsymbol{L}$ have been computed by algebraic methods, the conjugated momenta $\{y_i\}_{1\leq i\leq n}$ can be determined by means of the steps 1), 4), 5), 6), 7). We shall apply this shorter procedure in Section 3.5.

3.4 COMPLETE INTEGRABILITY AND SEPARATION OF VARIABLES IN $\omega\mathscr{H}$ MANIFOLDS

3.4.1 *Haantjes theorem for integrable systems*

A central feature of the theory of $\omega\mathscr{H}$ manifolds is its ability to reinterpret Liouville-Arnold integrability within this framework.

Theorem 3.4.1 (Liouville-Haantjes, [185]). *Let M be a $2n$-dimensional Abelian $\omega\mathscr{H}$ manifold of class n and $(H_1, H_2, \ldots, H_n)$ be smooth potential functions whose differentials belong to the Haantjes chain generated by a single function H. Then, the foliation generated by these functions is Lagrangian. Consequently, each Hamiltonian function H and H_α, $1 \leq \alpha \leq n$, is integrable by quadratures.*

Conversely, consider a completely integrable system with n degrees of freedom, defined by a Hamiltonian H and a set of n integrals of motion $(H_1, H_2, \ldots, H_n)$, in involution and independent among each other. Let $\{\phi^k, J_k\}$, $k = 1, 2, \ldots, n$, denote a set of action-angle variables, with associated frequencies $\nu_k(\boldsymbol{J}) = \frac{\partial H}{\partial J_k}$. If H is non-degenerate (see Definition 1.2.1), then M admits, in a tubular neighbourhood of an Arnold torus, a semisimple $\omega\mathscr{H}$ structure whose Haantjes algebra is generated by the operators

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\nu_i^{(\alpha)}(\boldsymbol{J})}{\nu_i(\boldsymbol{J})} \left(\frac{\partial}{\partial \phi^i} \otimes \mathrm{d}\phi^i + \frac{\partial}{\partial J_i} \otimes \mathrm{d}J_i \right) , \qquad \alpha = 1, 2, \ldots, n , \tag{3.69}$$

where $\nu_i^{(\alpha)}(\boldsymbol{J}) := \frac{\partial H_\alpha}{\partial J_i}$ are the frequencies of the αth linear flow.

From the preceding result, a natural relationship emerges between action-angle variables and DH coordinates within Haantjes geometry.

Proposition 3.4.1 (Proposition 40, [185]). *Every set of action-angle variables for a completely integrable system is a set of DH coordinates on the $\omega\mathscr{H}$ manifold defined by the symplectic form ω and the Haantjes algebra generated by the operators* (3.69).

3.4.2 *Separation of variables in $\omega\mathscr{H}$ manifolds*

The next theorem represents our main result concerning the existence of separation variables in the theory of $\omega\mathscr{H}$ manifolds.

Theorem 3.4.2 (Jacobi-Haantjes, [157]). *Let M be an Abelian semisimple $\omega\mathscr{H}$ manifold of class n and $(H_1, H_2, \dots, H_n)$ be a set of smooth potential functions belonging to a Haantjes chain generated by a function $H \in C^\infty(M)$ via the basis of operators $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_n\} \in \mathscr{H}$. Then, each set $(\boldsymbol{q}, \boldsymbol{p})$ of* DH *coordinates provides us with separation variables for the HJ equation associated with each function H_j.*

Conversely, if M is a symplectic manifold and $(H_1, H_2, \dots, H_n)$ are n independent, smooth functions separable in a set of Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$, then they belong to the Haantjes chain generated by the operators

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right), \qquad \alpha = 1, \dots, n, \tag{3.70}$$

where H is any of the functions $(H_1, \dots, H_n)$, with $\frac{\partial H}{\partial p_i} \neq 0$, $i = 1, \dots, n$. These operators generate a semisimple $\omega\mathscr{H}$ structure on M.

Proof. Theorem 28 of [185] guarantees the existence of sets of DH coordinates for a semisimple $\omega\mathscr{H}$ manifold. Therefore, it suffices to show that the functions H_j in such coordinates are in separable involution, according to Eq. (1.54). To this aim, let us note that, due to the diagonal form of $\boldsymbol{K}_\alpha^T$ in a DH local chart, the relations

$$\frac{\partial H_j}{\partial q^k} = \lambda_k^{(j)} \frac{\partial H}{\partial q^k}, \qquad j, k = 1, \dots, n, \tag{3.71}$$

$$\frac{\partial H_j}{\partial p_k} = \lambda_k^{(j)} \frac{\partial H}{\partial p_k}, \qquad j, k = 1, \dots, n, \tag{3.72}$$

hold. Here $\lambda_k^{(j)}$ denotes the eigenvalues of the Haantjes operator $\boldsymbol{K}_j^T$. Therefore,

$$\{H_i, H_j\}_{|k} = \lambda_k^{(i)} \frac{\partial H}{\partial q^k} \lambda_k^{(j)} \frac{\partial H}{\partial p_k} - \lambda_k^{(j)} \frac{\partial H}{\partial q^k} \lambda_k^{(i)} \frac{\partial H}{\partial p_k} = 0. \tag{3.73}$$

In order to prove the converse statement, without loss of generality we can assume that $\frac{\partial H}{\partial p_i} \neq 0, i = 1, \dots, n$. The operators (3.70), being diagonal in the separated coordinates, are Haantjes operators. Also, they commute with each other and generate an Abelian, semisimple Haantjes algebra

$\mathscr{H}$. The algebraic compatibility conditions (3.28) of the operators (3.70) with the symplectic form are equivalent to the further conditions

$$\lambda_{n+i}^{(\alpha)} = \lambda_i^{(\alpha)}, \qquad i = 1, \dots, n. \tag{3.74}$$

Thus, the Haantjes operators (3.70) must possess at least double eigenvalues.

Finally, we impose that the integrals of motion $(H_1, H_2, \dots, H_n)$ form a Haantjes chain, which will be generated by any of these functions, denoted by H. Since every $\boldsymbol{K}_\alpha, \alpha = 1, \dots, n$, is diagonal in the $(\boldsymbol{q}, \boldsymbol{p})$ variables, such conditions are equivalent, for each α, to the overdetermined system of $2n$ algebraic equations in the n functions $\lambda_i^{(\alpha)}$

$$\lambda_i^{(\alpha)} \frac{\partial H}{\partial q^i} = \frac{\partial H_\alpha}{\partial q^i}, \tag{3.75}$$

$$\lambda_i^{(\alpha)} \frac{\partial H}{\partial p_i} = \frac{\partial H_\alpha}{\partial p_i}, \tag{3.76}$$

$i = 1, \dots, n$. In fact, the above equations are compatible, because the Benenti conditions (1.54) of separate involution ensure that

$$\frac{\partial H}{\partial q^i}\frac{\partial H_\alpha}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{\partial H_\alpha}{\partial q^i}, \quad 1 \le i \le n. \tag{3.77}$$

Consequently, the equations (3.76) provide us with the unique solution (3.70). □

Remark 3.4.1. *Note that in the first implication of the previous theorem, the set of functions* $(H_1, H_2, \dots, H_n)$ *is in involution by virtue of Lemma 3.3.1.*

Corollary 3.4.1. *If* M *is a symplectic manifold and* $(H_1, H_2, \dots, H_n)$ *are* n *independent,* $C^\infty(M)$ *functions separable in a set of Darboux coordinates* $(\boldsymbol{q}, \boldsymbol{p})$*, the Haantjes operators* (3.70) *can be written as*

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \frac{\Pi_{j\neq i}(\boldsymbol{N} - \lambda_j \boldsymbol{I})}{\Pi_{j\neq i}(\lambda_i - \lambda_j)}, \tag{3.78}$$

where $\boldsymbol{N}$ *is the operator defined by*

$$\boldsymbol{N} = \sum_{i=1}^{n} l_i(q^i, p_i) \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \tag{3.79}$$

and $\{l_i(q^i, p_i)\}_{1\le i\le n}$ *are arbitrary, pointwise distinct smooth functions on* M*. The operator* $\boldsymbol{N}$ *is a Nijenhuis operator compatible with* ω *and, therefore, a generator of a* $\omega\mathscr{H}$ *structure on* M *endowed with a unigenerated Haantjes algebra, as defined by* (3.54)*.*

Remark 3.4.2. *Given* n *arbitrary smooth functions* $(H_1, \dots, H_n)$ *on a* $2n$*-dimensional manifold* M*, it is always possible to determine* n *diagonal Haantjes operators* $\boldsymbol{K}_i$ *which satisfy the chain equations* $\boldsymbol{K}_i^T \, dH = dH_i$*, where* H *is any of the previous functions. However, if* M *is a symplectic manifold,*

the compatibility condition (3.28) *of the Haantjes operators* $\boldsymbol{K}_i$ *with the symplectic structure defined on* M *imposes* n *additional constraints on the eigenvalues of these operators. Thus, the systems of* $2n$ *algebraic equations* (3.75) *and* (3.76) *can be solved if and only if* n *of these equations are automatically satisfied. This requirement is equivalent to the Benenti conditions* (1.54)*. Consequently, the existence for an integrable system of a Haantjes chain with semisimple operators (and consequently of a* $\omega\mathscr{H}$ *structure), is not at all a trivial property. The case of the superintegrable Post-Winternitz system is illustrative of this aspect: no separation variables are known for this system, and the unique known* $\omega\mathscr{H}$ *structures are* non-semisimple ones*.*

3.4.3 *Haantjes structures for generalized Stäckel systems*

As demonstrated in [192], generalized Stäckel systems (see Section 1.4.1) can be effectively described within the framework of symplectic-Haantjes geometry. In particular, it is shown that the Haantjes chain

$$\boldsymbol{K}_j^T \mathrm{d}H_1 = \mathrm{d}H_j\,, \qquad j = 1, \dots, n \tag{3.80}$$

associated with the AKN Hamiltonian functions

$$H_j = \sum_{k=1}^{n} \frac{\tilde{S}_{jk}}{\det \boldsymbol{S}}\, f_k(q^k, p_k)\,, \qquad j = 1, \dots, n$$

admits a unique semisimple solution, given by the operators

$$\boldsymbol{K}_j := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left(\frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r \right), \qquad j = 1, \dots, n\,. \tag{3.81}$$

Remark 3.4.3. *It is straightforward to verify that multiplying the* i*th row of the Stäckel matrix by any function* $F_i(q^i)$ *produces a different Stäckel matrix associated with the same coordinate web. Notably, while this transformation modifies the Hamiltonian functions, the corresponding Haantjes operators remain unchanged. This invariance highlights that the Haantjes operators serve as the tensorial representation of a given Stäckel web in* T^*Q*.*

Remark 3.4.4. *The Haantjes operators* (3.81) *are independent of the specific choice of Stäckel functions* $f_k(q^k, p_k)$*.*

A noteworthy property of the Haantjes operators associated with an AKN system is that they can be mapped onto Q via the canonical projection $\pi\colon T^*Q \to Q$, $(\boldsymbol{q}, \boldsymbol{p}) \mapsto \boldsymbol{q}$. Specifically, we have the following:

Proposition 3.4.2. *The family of Haantjes operators* (3.81) *can be projected along the fibers of* T^*Q *into the following operators:*

$$\tilde{\boldsymbol{K}}_j := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r\,, \qquad j = 1, \dots, n-1. \tag{3.82}$$

Remark 3.4.5. *In classical Stäckel systems, the contravariant forms of the operators* (3.82) *are diagonal Killing tensors associated with the metric* (1.63)*. This metric provides the structure needed to define these contravariant forms, which are in involution with respect to the SN bracket of symmetric contravariant tensors.*

3.4.4 *A general procedure to construct $\omega\mathscr{H}$ manifolds*

In order to determine the $\omega\mathscr{H}$ structures admitted by a separable system, we need to construct the Haantjes chains associated with it. Precisely, we wish to solve the chain equations

$$\mathrm{d}(\boldsymbol{K}_\alpha^T\,\mathrm{d}H) = \mathbf{0}\,, \qquad \alpha = 1,\dots,m \tag{3.83}$$

for suitable Haantjes operators $\boldsymbol{K}_\alpha^T$. Generally speaking, these equations do not admit a unique solution. Notice that the operators $\boldsymbol{K}_\alpha^T$ we are interested in must be compatible with the symplectic form ω (see Eq. (3.28)). The most general operator $\boldsymbol{M}$ compatible with the symplectic form, in Darboux coordinates $(\boldsymbol{q},\boldsymbol{p})$ reads

$$\boldsymbol{M} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{B}(\boldsymbol{q},\boldsymbol{p}) \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{A}^T(\boldsymbol{q},\boldsymbol{p}) \end{array}\right], \quad \boldsymbol{B}+\boldsymbol{B}^T = \mathbf{0}\,, \quad \boldsymbol{C}+\boldsymbol{C}^T = \mathbf{0}\,, \tag{3.84}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $n\times n$ matrices with coefficients smoothly depending on the Darboux coordinates. However, the operator (3.84) in general is not a Haantjes operator, unless some specific choices for its arbitrary functions are made. Therefore, our task is to determine solutions of Eqs. (3.83) in the class of Haantjes operators of the form (3.84). For $n = 2$, the operators of the family (3.84) can depend on up to 6 arbitrary functions; for $n = 3$, up to 15 arbitrary functions are admissible (see Appendix A).

The procedure for the determination of the $\omega\mathscr{H}$ structure associated with an n-dimensional Hamiltonian system can be summarized in the following steps. Assume that we are given a set of functions $(H_1,\dots,H_m)$, which are functionally independent and in involution with the potential function H of the generator of the chain. Note that m is not necessarily equal to n: it can be larger than n (superintegrable systems), equal (integrable systems) or smaller (almost integrable models).

1) Within the class of commuting operators algebraically compatible with the symplectic structure, namely of the family (3.84), determine the operators $\boldsymbol{K}_\alpha$ that solve the chain equations (3.83).

2) Among the solutions found, choose the operators with vanishing Haantjes torsion: $\mathcal{H}_{\boldsymbol{K}}(X,Y) = \mathbf{0}, \forall\, X,Y\in\mathfrak{X}(M)$.

3) From these Haantjes operators, construct semisimple and/or non-semisimple Abelian $\omega\mathscr{H}$ manifolds.

4) Find the DH coordinates associated with each $\omega\mathscr{H}$ manifold. They will represent either partially or totally separated coordinates for the Hamiltonian system under scrutiny.

This procedure is completely general. The case of superintegrable systems will be considered in Chapter 4, and partial separability will be discussed in Chapter 5.

An alternative strategy is to use a suitable ansatz for the form of the operators $\boldsymbol{K}$ that are supposed to solve Eqs. (3.83). Precisely, one can start from a sub-family of operators depending on arbitrary functions, which are both compatible with the symplectic form and have vanishing Haantjes torsion. In this way, the requirement of step 2) is already fulfilled. Then, one can try to fix the arbitrary functions available by solving Eqs. (3.83) (step 1). This approach is less general, since it presupposes the determination *a priori* of special subclasses of operators of the form (3.84) which are Haantjes' as well. This task, which in general is computationally nontrivial, is affordable in the case $n = 2$; consequently, this procedure has been adopted, for instance, in Section 4.3.

3.5 SEPARATION OF VARIABLES FOR A DRACH-HOLT TYPE SYSTEM

The approach proposed in this paper offers an effective procedure to construct algorithmically separating variables admitted by Hamiltonian integrable systems. As a paradigmatic example, following [157], we shall study the case of a system showing an irrational dependence on its coordinates. Precisely, we shall consider a three-parametric deformation of the Holt potential, that has been introduced in [36]:

$$H_1 = \frac{1}{2}(p_x^2 + p_y^2) + k_1 \frac{4x^2 + 3y^2}{y^{2/3}} + k_2 \frac{x}{y^{2/3}} + \frac{k_3}{y^{2/3}}. \tag{3.85}$$

It is integrable in the manifold $M = T^*(E^2 \setminus \{y = 0\})$, with a third-order integral

$$H_2 = 2p_x^3 + 3p_x p_y^2 + 12k_1\left(\frac{2x^2 - 3y^2}{y^{2/3}} p_x + 6xy^{1/3} p_y\right) + k_2\left(\frac{6x}{y^{2/3}} p_x + 9y^{1/3} p_y\right) + 6\frac{k_3}{y^{2/3}} p_x\,. \tag{3.86}$$

When $k_1 \to 0$, the Hamiltonian H_1 converts into the (generalized) Post-Winternitz (gPW) superintegrable potential [152]. A crucial aspect is that H_1 in the literature has been regarded as an example of *nonseparable system*. Indeed, it is not of Stäckel type; thus is not separable by means of an extended–point transformation. A natural question is whether there is a full canonical transformation (in general difficult to find) redeeming its separability. Our theory of integrability *à la Haantjes* enables us to solve this problem, since it provides us with a set of DH separation coordinates, according to Theorem 3.4.2.

Remark 3.5.1. *The set of conditions we obtain provides us with an* overdetermined *system of equations. In general, solving an overdetermined system is computationally affordable; however, an interesting open question, theoretically relevant, is to find a minimal set of* independent *conditions ensuring separability.*

3.5.1 *The $\omega\mathscr{H}$ structure*

Following the procedure of Section 3.4.4 we get a Haantjes operator linear in the momenta. In the coordinates (x, y, p_x, p_y), it reads

$$\boldsymbol{K}^{(\mathrm{DH})} = 3\left[\begin{array}{cc|cc} 2p_x & p_y & 0 & 3y \\ 0 & 2p_x & -3y & 0 \\ \hline 0 & -24k_1 y^{1/3} & 2p_x & 0 \\ 24k_1 y^{1/3} & 0 & p_y & 2p_x \end{array}\right]. \tag{3.87}$$

Due to Lemma 2.4.1, the Haantjes operator (3.87) is a generator of the algebra $\langle \boldsymbol{I}, \boldsymbol{K}^{(\mathrm{DH})}\rangle$ since it is semisimple and its minimal polynomial $m_{\boldsymbol{K}}(z) = z^2 - b_1(\boldsymbol{x})z - b_2(\boldsymbol{x})$ is of degree two. Moreover, its eigenvalues $l_1(\boldsymbol{x})$ and $l_2(\boldsymbol{x})$ fulfil the following recursion relations

$$(\boldsymbol{K}^{(\mathrm{DH})})^T \mathrm{d}l_1 = l_2\,\mathrm{d}l_1\,, \qquad (\boldsymbol{K}^{(\mathrm{DH})})^T\,\mathrm{d}l_2 = l_1 \mathrm{d}l_2\,, \tag{3.88}$$

that is, they are potential functions of exact eigen 1-forms of $(\boldsymbol{K}^{(DH)})^T$. Furthermore, the recursion relations (3.88) imply the remarkable relation

$$(\boldsymbol{K}^{(\mathrm{DH})})^T \mathrm{d}\left(\frac{1}{2}\,\mathrm{tr}\,(\boldsymbol{K}^{(\mathrm{DH})})\right) = \mathrm{d}\sqrt{\det(\boldsymbol{K}^{(\mathrm{DH})})}\,, \tag{3.89}$$

which reminds us the main property of the Kowaleskaya operator appearing in the recent new theory of SoV due to F. Magri [120]. Equation (3.89) implies that the two functions $\frac{1}{2}\,\mathrm{tr}\,(\boldsymbol{K}^{(\mathrm{DH})}) = b_1(\boldsymbol{x})$ and $\sqrt{\det(\boldsymbol{K}^{(\mathrm{DH})})} = -b_2(\boldsymbol{x})$ are potential functions of exact 1-forms belonging to a Haantjes chain. Therefore, they are in involution with respect to the Poisson bracket induced by the symplectic form ω, thanks to Lemma 3.3.1. Consequently, by rewriting Eq. (3.89) in terms of the eigenvalue fields of $\boldsymbol{K}^{(\mathrm{DH})}$ it is easy to prove that

$$\{l_1(\boldsymbol{x}), l_2(\boldsymbol{x})\} = 0\,. \tag{3.90}$$

Remark 3.5.2. *Let $\boldsymbol{L}$ be an operator expressed as an affine function of a Haantjes operator $\boldsymbol{K}$:*

$$\boldsymbol{L} = f\boldsymbol{I} + g\boldsymbol{K}\,, \tag{3.91}$$

where f and g are arbitrary smooth functions, with g nowhere vanishing. Thus, the operator $\boldsymbol{L}$ is a Haantjes operator as a consequence of identity (2.45). From Proposition 1.2.4 of [96] it follows that the minimal polynomials of the two operators are related by

$$m_{\boldsymbol{L}}(z) = g^{\deg(m_K(z))} m_{\boldsymbol{K}}\left(\frac{1}{g}(z-f)\right),$$

so that they have the same number of irreducible factors with the same multiplicity. Therefore, the eigenvalues λ_i of $\boldsymbol{L}$, the eigenvalues l_i of $\boldsymbol{K}$ and their eigendistributions are related by the equations

$$\lambda_i = f + g\,l_i \tag{3.92}$$

$$\ker\left(\boldsymbol{L}-\lambda_i\boldsymbol{I}\right)^{\rho_i}=\ker\left(\boldsymbol{K}-l_i\boldsymbol{I}\right)^{\rho_i} \tag{3.93}$$

for $i=1,\dots,s$.

Remark 3.5.3. *For the large class represented by the quasi-bi-Hamiltonian systems [31, 135, 136, 190] with two degrees of freedom, we can establish a simple connection with the present theory. In fact, due to the results of [192], we can prove that if the Haantjes operator* $\boldsymbol{L}$ *in Eq.* (3.91) *is also a Nijenhuis operator with*

$$f=-\frac{1}{2}\operatorname{tr}(\boldsymbol{K})\,,\qquad\qquad g=1\,, \tag{3.94}$$

then the Hamiltonian system under scrutiny admits a quasi-bi-Hamiltonian formulation. Therefore, the eigenvalues of the Haantjes generator $\boldsymbol{K}$ *are themselves characteristic functions of the Haantjes web since, by plugging the solution* (3.94) *into Eq.* (3.92)*, we obtain*

$$\lambda_1=-l_2\,,\qquad\qquad \lambda_2=-l_1\,. \tag{3.95}$$

In other words, the spectrum of the Haantjes generator $\boldsymbol{K}$ *coincides with the spectrum of the Nijenhuis generator* $\boldsymbol{L}$*. Therefore, the eigenvalues of the Haantjes operator* (3.87) *are characteristic functions of the characteristic web associated with* $\boldsymbol{K}^{(\mathrm{DH})}$*. This explains, in the case of the Drach-Holt system, the occurrence of the recurrence relations* (3.88) *and the involution properties* (3.90)*.*

3.5.2 *Separation Coordinates*

These functions read

$$\begin{aligned}\lambda_1&=-6\left(p_x+3\sqrt{2k_1}y^{2/3}\right),\\ \lambda_2&=-6\left(p_x-3\sqrt{2k_1}y^{2/3}\right).\end{aligned} \tag{3.96}$$

They are in involution with respect to the Poisson bracket induced by ω, due to Eq. (3.90). In order to get a system of DH coordinates, we apply the procedure of Section 3.3.6.

1) We have that $T^*_{\boldsymbol{x}}M=\mathcal{E}^\circ_1(\boldsymbol{x})\oplus\mathcal{E}^\circ_2(\boldsymbol{x})$, where

$$\begin{aligned}\mathcal{E}^\circ_1&=\ker\left((\boldsymbol{K}^{(\mathrm{DH})})^T-l_1\boldsymbol{I}\right)=\langle \mathrm{d}\lambda_1,\gamma_1\rangle\,,\\ \mathcal{E}^\circ_2&=\ker\left((\boldsymbol{K}^{(\mathrm{DH})})^T-l_2\boldsymbol{I}\right)=\langle \mathrm{d}\lambda_2,\gamma_2\rangle\,,\end{aligned} \tag{3.97}$$

with

$$\begin{aligned}\gamma_1&=-6\sqrt{2k_1}\,y^{1/3}\mathrm{d}x+\frac{p_y\,y^{1/3}+4\sqrt{2k_1}x}{y^{2/3}}\mathrm{d}y+2\frac{x}{y^{1/3}}\,\mathrm{d}p_x+3y^{2/3}\mathrm{d}p_y\,,\\ \gamma_2&=6\sqrt{2k_1}\,y^{1/3}\mathrm{d}x+\frac{p_y\,y^{1/3}-4\sqrt{2k_1}x}{y^{2/3}}\mathrm{d}y+2\frac{x}{y^{1/3}}\,\mathrm{d}p_x+3y^{2/3}\mathrm{d}p_y\,.\end{aligned}$$

2) We find two exact 1-forms $\alpha_1=\mathrm{d}\mu_1=f_1\mathrm{d}\lambda_1+g_1\gamma_1$, $\alpha_2=\mathrm{d}\mu_2=f_2\mathrm{d}\lambda_2+g_2\gamma_2$ such that their potential functions (μ_1,μ_2) fulfill the canonical relations

$$\{\lambda_1,\mu_1\}=\langle \mathrm{d}\lambda_1,\boldsymbol{P}_0\mathrm{d}\mu_1\rangle=1\,,\qquad \{\lambda_2,\mu_2\}=\langle \mathrm{d}\lambda_2,\boldsymbol{P}_0\mathrm{d}\mu_2\rangle=1\,. \tag{3.98}$$

3) Such algebraic equations provide us with the solutions

$$g_1 = -\frac{1}{72\sqrt{2k_1}y^{1/3}}\,, \qquad g_2 = \frac{1}{72\sqrt{2k_1}y^{1/3}}\,.$$

4) Now, requiring that the 1-forms α_1 and α_2 are closed, we find a family of solutions for (f_1, f_2). The simplest choices are

$$f_1 = -\frac{x}{216\sqrt{2k_1}y^{2/3}}\,, \qquad f_2 = \frac{x}{216\sqrt{2k_1}y^{2/3}}\,. \tag{3.99}$$

5) Thus, we get the following momenta

$$\mu_1 = \frac{1}{24\sqrt{2k_1}}\left(2\sqrt{2k_1}x - p_y y^{1/3}\right), \qquad \mu_2 = \frac{1}{24\sqrt{2k_1}}\left(2\sqrt{2k_1}x + p_y y^{1/3}\right). \tag{3.100}$$

They are, by construction, canonically conjugated to the coordinates (λ_1, λ_2).

3.5.3 *Separation Equations of Jacobi–Sklyanin*

The Jacobi-Sklyanin approach represents a fundamental piece in the theory of separable systems. Here we will establish a connection between the Haantjes geometry and the Jacobi-Sklyanin SE for the case of the Drach-Holt system.

These equations allow one to construct a solution (W, E) of the HJ equation, according to Eq. (1.52). The set of coordinates (3.96), (3.100), being DN coordinates, are separation variables for both the Hamiltonian functions H_1 and H_2. Besides, they fulfill the Jacobi-Sklyanin SE

$$\begin{aligned} &b_1\mu_1^2 + b_2\mu_1 + b_3\lambda_1^3 + \lambda_1 H_1 + H_2 + b_4 = 0\,,\\ &b_1\mu_2^2 + b_2\mu_2 - b_3\lambda_2^3 - \lambda_2 H_1 - H_2 + b_4 = 0\,, \end{aligned}$$

where $b_i, i = 1, \ldots, 4$ are the constants given by

$$b_1 = 10368\sqrt{2k_1^3}\,, \quad b_2 = 216\sqrt{2k_1}\,k_2\,, \quad b_3 = -\frac{1}{216}\,, \quad b_4 = 18\sqrt{2k_1}k_3\,.$$

We arrive therefore at the separated solutions of the HJ equation

$$\begin{aligned} W_1(\lambda_1; h_1, h_2) &= \frac{1}{2b_1}\int^{\lambda_1}\left(-b_2 \pm \sqrt{b_2^2 - 4b_1 P_3(\lambda)}\right)\mathrm{d}\lambda\,,\\ W_2(\lambda_2; h_1, h_2) &= \frac{1}{2b_1}\int^{\lambda_2}\left(-b_2 \pm \sqrt{b_2^2 - 4b_1 Q_3(\lambda)}\right)\mathrm{d}\lambda\,, \end{aligned}$$

where h_1, h_2 are the values of H_1, H_2 on the Lagrangian tori, and

$$\begin{aligned} P_3(\lambda_1; h_1, h_2) &= b_3\lambda_1^3 + \lambda_1 h_1 + h_2 + b_4\,,\\ Q_3(\lambda_2; h_1, h_2) &= -b_3\lambda_2^3 - \lambda_2 h_1 - h_2 + b_4\,. \end{aligned}$$

3.6 CONSTRUCTION OF HAANTJES OPERATORS IN T^*Q: A NOVEL GEOMETRIC LIFT

The problem of constructing Haantjes operators on an n-dimensional manifold M is, in general, a hard one, as it entails solving a system of $\frac{1}{2}n^2(n-1)$ nonlinear PDE of first order in the n^2 unknown components of the operators we wish to determine.

However, when $M \equiv T^*Q$, being Q the configuration space of a given mechanical system, one can plan to simplify the problem of constructing a $(T^*Q, \omega, \mathscr{H})$ manifold by means of a geometric procedure which allows us to lift a Haantjes operator $\boldsymbol{A}: TQ \to TQ$ to a suitable Haantjes operator $\widehat{\boldsymbol{A}}: T(T^*Q) \to T(T^*Q)$. The approach envisaged, in order to be effective, should preserve the vanishing condition of Haantjes tensors. The procedure we propose to this aim is inspired by the one introduced in [76] for the construction of a suitable Nijenhuis operator for the Benenti systems; later it has been successfully formulated and applied in [52] in the context of the ωN geometry. In those works, from a technical point of view the *complete lift* from a manifold to its cotangent bundle, introduced by Yano in [198], has been adopted. Although the Yano lift preserves the vanishing of Nijenhuis tensors, unfortunately this property does not hold true for the Haantjes case: the Yano lift of a Haantjes operator need not be another Haantjes operator.

In order to overcome this drawback, in [157] a novel geometric lifting procedure is proposed, which generalizes Yano's one. To this aim, let us consider an operator $\boldsymbol{A}: TQ \to TQ$ and the canonical projection map $\pi: T^*Q \to Q$. Let us denote by $\widehat{\boldsymbol{A}}$ a lift of $\boldsymbol{A}$ to the cotangent bundle T^*Q which is required to be *projectable* onto $\boldsymbol{A}$, that is to say, $\boldsymbol{A}\,\pi_* = \pi_*\,\widehat{\boldsymbol{A}}$. Such a condition is fulfilled if and only if $\widehat{\boldsymbol{A}}$ takes the following block-matrix form in a Darboux chart $(\boldsymbol{q}, \boldsymbol{p})$:

$$\widehat{\boldsymbol{A}} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q}) & \boldsymbol{0} \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{D}(\boldsymbol{q},\boldsymbol{p}) \end{array}\right], \tag{3.101}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are $n \times n$ matrices depending possibly on all coordinates. In addition, we must impose the compatibility condition with the symplectic form (3.84), which reduces the form of $\widehat{\boldsymbol{A}}$ to

$$\widehat{\boldsymbol{A}} = \left[\begin{array}{c|c} \boldsymbol{A}(\boldsymbol{q}) & \boldsymbol{0} \\ \hline \boldsymbol{C}(\boldsymbol{q},\boldsymbol{p}) & \boldsymbol{A}^T(\boldsymbol{q}) \end{array}\right] \qquad \boldsymbol{C} + \boldsymbol{C}^T = \boldsymbol{0}\,. \tag{3.102}$$

Assuming that $\boldsymbol{A}$ is a Haantjes operator, we wish to determine the form of the matrix $\boldsymbol{C}$ in such a way that $\widehat{\boldsymbol{A}}$ is also a Haantjes operator. In the subsequent discussion, we shall prove our result for configuration spaces of dimension $n = 2$ only.

As clarified in Example 2.3.1, when $\dim Q = 2$, any operator $\boldsymbol{A}(\boldsymbol{q})$ is a Haantjes operator. Then, its lifted operator takes the form

$$\widehat{\boldsymbol{A}} = \left[\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & r & a & c \\ -r & 0 & b & d \end{array}\right], \tag{3.103}$$

where all the entries are functions of (q^1, q^2) only, except for the smooth function $r = r(q^1, q^2, p_1, p_2)$. Let us assume that the eigenvalues of $\boldsymbol{A}(\boldsymbol{q})$ are pointwise distinct; by requiring that $\widehat{\boldsymbol{A}}$ is a Haantjes operator we find that the most general solution for r is an affine function of (p_1, p_2). Precisely,

$$r = f(q^1, q^2)p_1 + g(q^1, q^2)p_2 + h(q^1, q^2),$$

where

$$f = \frac{\partial a}{\partial q^2} - \frac{\partial b}{\partial q^1} - \frac{a-d}{\Delta}\tau^1_{12} - \frac{2b}{\Delta}\tau^2_{12} \tag{3.104}$$

$$g = \frac{\partial c}{\partial q^2} - \frac{\partial d}{\partial q^1} + \frac{a-d}{\Delta}\tau^2_{12} - \frac{2c}{\Delta}\tau^1_{12}\,, \tag{3.105}$$

$\Delta := (a-d)^2 + 4bc$ is the discriminant of the minimal polynomial of $\boldsymbol{A}(\boldsymbol{q})$, τ^1_{12} and τ^2_{12} are the two independent components of the Nijenhuis torsion of $\boldsymbol{A}(\boldsymbol{q})$ and $h(q^1, q^2)$ is an arbitrary smooth function. In particular, when $\boldsymbol{A}(\boldsymbol{q})$ is a Nijenhuis operator and $h(q^1, q^2) = 0$, the operator $\widehat{\boldsymbol{A}}$ coincides with the Yano complete lift of $\boldsymbol{A}(\boldsymbol{q})$, therefore it is still a Nijenhuis operator.

The lifting procedure presented here can be extended to the n-dimensional case, $n \geq 3$, but the details concerning it will be published elsewhere.

Certain things they should stay the way they are. You ought to be able to stick them in one of those big glass cases and just leave them alone. I know that's impossible, but it's too bad anyway.

Jerome David Salinger, *The Catcher in the Rye*

4

Multiseparable Systems and Superintegrability

A *tanguillo* from Cádiz is a joyful interplay of balance and rhythm–a convergence of polyrhythmic pulses that, though seemingly pulling in different directions, merge with inevitable precision. A light melody drifts over a dynamic foundation where syncopated accents fall just ahead of or behind the beat, creating a constant, unresolved tension that never disrupts the flow. Each voice, distinct in its contrasting rhythms yet harmoniously interwoven, claims its place in a structure that is both liberating and governed by an unseen order. It is a dance in which every step is as spontaneous as it is guided, revealing complexity and clarity as two harmonious facets of the same movement.

Much like this intricate dance, superintegrable classical Hamiltonian systems reveal a realm of exceptional stability and elegant complexity. These systems, the focus of this chapter, constitute a special class of integrable systems distinguished by their remarkably rich algebraic and geometric properties [88, 133, 134, 139, 152, 180, 188]. Fundamentally, they possess more independent integrals of motion than degrees of freedom. A system with n degrees of freedom is termed *maximally superintegrable* if it admits $2n - 1$ functionally independent integrals of motion–the maximum possible–whereas it is called *minimally superintegrable* if it has $n + 1$ integrals.

Among the most famous examples of superintegrable models we find the classical harmonic oscillator, the Kepler potential, the Calogero-Moser potential, the Smorodinsky-Winternitz systems and the Euler top. The presence of "hidden" symmetries, expressed by integrals which are second (or higher) degree polynomials in the momenta, usually allows us to determine the dynamical be-

haviour of the superintegrable models. In the maximal case, the bounded orbits are closed and periodic according to the Nekhoroshev's theorem [139]. As is well known, the phase space topology is also very rich: it can be described in terms of a symplectic bifoliation, determined by the standard Liouville-Arnold invariant fibration [6] of Lagrangian tori, and complemented by a co-isotropic polar foliation [54, 139].

Although this work focuses on classical systems, it is worth noting that quantum superintegrable systems also exhibit many relevant properties. The exact solvability of a quantum Hamiltonian systems, associated with suitable Lie algebras of raising and lowering operators, can be regarded as the quantum analog of the classical notion of maximal superintegrability [188].

The primary aim of this chapter is to reveal a novel connection between the theory of separable Hamiltonian systems and Haantjes geometry. In contrast to standard integrable systems, superintegrable systems possess more integrals of motion than degrees of freedom. While these integrals are not all in mutual involution–resulting in non-vanishing Poisson brackets–the excess of integrals allows for the selection of distinct subsets that *do* commute. Within the framework of symplectic Haantjes structures, these commuting subsets correspond to independent operator subalgebras. Crucially, each subalgebra defines a distinct set of separation coordinates, thereby enabling multiple integrable formulations for the same physical system.

This chapter is structured as follows. In Section 4.1, Jacobi-Haantjes theorem is extended to the case of multiseparable (and superintegrable) models. Section 4.2 is devoted to the study of the Haantjes structures for the Smorodinsky-Winternitz systems in the plane, whereas Section 4.3 deals with the study of the anisotropic oscillator. In Section 4.4, the $\omega\mathscr{H}$ manifolds associated with certain important multiseparable systems in three dimensions are determined.

4.1 MULTISEPARABLE SYSTEMS AND HAANTJES GEOMETRY

A particularly interesting instance of the previous theory is represented by the case of multiseparable systems. They are Hamiltonian systems that can be separated in more than one coordinate system in their phase space. Fundamental physical examples of multiseparable systems are the n-dimensional harmonic oscillator and the Kepler system. Another important class is represented by the four Smorodinsky-Winternitz systems, which are the only maximally superintegrable systems in the Euclidean plane admitting orthogonal separation variables.

Multiseparable systems are superintegrable ones, provided that the sets of separation functions in the Hamilton-Jacobi equation related to different separation coordinates are functionally independent. However, to our knowledge, there is no general theoretical result establishing a relation between the two notions of superintegrability and separability. For instance, for the classical Post-Winternitz (PW) system [152], which is maximally superintegrable, no separation coordinates are known in phase space. Besides, due to the presence of integrals of motion of degree higher than two as polynomials in the momenta, the PW system does not admit orthogonal separation coordinates in its configuration space. A superintegrable system can also be simply separable, without being a multiseparable one. This is the case for the anisotropic oscillator discussed in Section 4.3. In this ar-

ticle, we shall focus mainly on the case of superintegrable multiseparable systems, which is perhaps the most interesting one from a geometric and physical point of view.

In order to extend the Haantjes geometry to the case of multiseparable systems, an important, preliminary aspect should be pointed out.

Let us consider an integrable Hamiltonian system with Hamiltonian function H; we denote by $\{(J_k, \phi_k)\}$, $k = 1, \ldots, n$, a set of action-angle variables for the system, with associated frequencies $\nu_k(\boldsymbol{J}) := \frac{\partial H}{\partial J_k}$. As discussed in 3.4.1, the LH theorem was proved under the hypothesis of *nondegeneracy* for H, that is

$$\det\left[\frac{\partial \nu_k}{\partial J_i}\right] = \det\left[\frac{\partial^2 H}{\partial J_i\, \partial J_k}\right] \neq 0\,, \tag{4.1}$$

and it states that there exists a semisimple $\omega\mathscr{H}$ manifold in any tubular neighbourhood of an Arnold torus. However, superintegrable systems just violate the condition (4.1); therefore, the LH theorem cannot be applied to them. Nevertheless, this does not imply that $\omega\mathscr{H}$ structures cannot exist for superintegrable systems. Indeed, we can construct such structures by means of a different approach, based on a simple consequence of Theorem 3.4.2.

Proposition 4.1.1. *[157] An integrable Hamiltonian system possesses as many inequivalent separation coordinate systems as the number of the independent semisimple Abelian $\omega\mathscr{H}$ manifolds of class n that it admits.*

Proof. It suffices to observe that, according to Theorem 3.4.2, for each semisimple $\omega\mathscr{H}$ structure of class n admitted by the Hamiltonian system there exists a set of DH coordinates, which play the role of separating coordinates for the corresponding HJ equation. □

In the following sections, we shall exhibit explicitly the Haantjes structures associated with celebrated examples of both maximally superintegrable and multiseparable Hamiltonian systems.

4.2 HAANTJES STRUCTURES FOR MULTISEPARABLE SYSTEMS IN E_2

The Smorodinsky-Winternitz (SW) systems are a family of superintegrable systems defined in the Euclidean plane E_2, which were introduced first as quantum-mechanical systems in [59, 126, 194] and later studied from a group theoretical point of view in [173, 188]. They are all multiseparable in E_2 and admit three independent integrals of motion, expressed in terms of second-degree polynomials in the momenta. Also, they are separable in at least two different orthogonal coordinate systems in their configuration space.

In [183], the SW systems were analyzed in the context of Nijenhuis geometry. Precisely, it was shown that a ωN structure can be associated with each of them; this can be achieved by renouncing the standard notion of Lenard chain, and using a generalized version of it. In this section, we will show that a natural and more general framework for studying the geometry of SW systems is offered by the Haantjes geometry. As we will show, one can introduce Haantjes chains and construct two different $\omega\mathscr{H}$ structures for each of the SW systems. In turn, according to Theorem

3.4.2 and Proposition 4.1.1, these structures guarantee the existence of separating variables for the SW systems.

4.2.1 *Smorodinsky-Winternitz system SWI*

The Hamiltonian function is

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}a(x^2 + y^2) + \frac{c_1}{x^2} + \frac{c_2}{y^2}. \tag{4.2}$$

This system is separable in *cartesian*, *polar* and *elliptic* coordinates, and admits the integrals

$$H_2 = \frac{p_y^2}{2} + \frac{a}{2}y^2 + \frac{c_2}{y^2}, \tag{4.3}$$

$$H_3 = 2\left(c^2 p_x^2 + (xp_y - yp_x)^2 + ac^2x^2 + 2c_1\frac{y^2 + c^2}{x^2} + 2c_2\left(\frac{x}{y}\right)^2\right), \tag{4.4}$$

with a, c, c_1 and $c_2 \in \mathbb{R}$. A first Haantjes structure is $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWI})})$, where $\boldsymbol{I}$ is the identity operator and

$$\boldsymbol{K}_2^{(\mathrm{SWI})} = \mathrm{diag}\,[0, 1, 0, 1]\,. \tag{4.5}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWI})} \rangle$ admits the Cartesian coordinates as DH coordinates: therefore they are obviously separation coordinates for the SWI system.

We can also obtain a second Haantjes structure $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWI})})$ with

$$\boldsymbol{K}_3^{(\mathrm{SWI})} = 4\left[\begin{array}{cc|cc} y^2 + c^2 & -xy & 0 & 0 \\ -xy & x^2 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & y^2 + c^2 & -xy \\ xp_y - yp_x & 0 & -xy & x^2 \end{array}\right]. \tag{4.6}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWI})} \rangle$ diagonalizes in elliptic coordinates (separation coordinates). If $c = 0$, this algebra diagonalizes in polar coordinates.

4.2.2 *The Smorodinsky-Winternitz system SWII*

The Hamiltonian function reads:

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2) + a(4x^2 + y^2) + c_1 x + \frac{c_2}{y^2}\,, \tag{4.7}$$

with $a, c_1, c_2 \in \mathbb{R}$. The associated integrals are

$$H_2 = \frac{p_y^2}{2} + ay^2 + \frac{c_2}{y^2}\,, \tag{4.8}$$

$$H_3 = p_y(yp_x - xp_y) + 2axy^2 + \frac{c_1}{2}y^2 - 2c_2\frac{x}{y^2}\,. \tag{4.9}$$

We obtain the manifolds $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWII})})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWII})})$, where

$$\boldsymbol{K}_2^{(\mathrm{SWII})} = \mathrm{diag}\,[0, 1, 0, 1]\,, \tag{4.10}$$

and

$$\boldsymbol{K}_3^{(\mathrm{SWII})} = \left[\begin{array}{cc|cc} 0 & y & 0 & 0 \\ y & -2x & 0 & 0 \\ \hline 0 & p_y & 0 & y \\ -p_y & 0 & y & -2x \end{array}\right]. \tag{4.11}$$

The first structure admits the Cartesian coordinates as DH coordinates, whereas the second one diagonalizes in parabolic coordinates.

4.2.3 *The Smorodinsky-Winternitz system SWIII*

The system is defined in polar coordinates $(r, \theta, p_r, p_\theta)$ by the Hamiltonian

$$H = H_1 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{\alpha}{r} + \frac{1}{r^2}\frac{\beta + \gamma\cos\theta}{\sin^2\theta}, \tag{4.12}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. The integrals of motion are

$$H_2 = \frac{p_\theta^2}{2} + \frac{\beta + \gamma\cos\theta}{\sin^2\theta}\,, \tag{4.13}$$

$$H_3 = -p_\theta\left(\frac{p_\theta\cos\theta}{r} + p_r\sin\theta\right) - \alpha\cos\theta - \frac{\gamma + 2\beta\cos\theta + \gamma\cos^2\theta}{r\sin^2\theta}\,. \tag{4.14}$$

We can construct the Haantjes manifolds $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWIII})})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWIII})})$, with

$$\boldsymbol{K}_2^{(\mathrm{SWIII})} = \mathrm{diag}\,[0, r^2, 0, r^2]\,, \tag{4.15}$$

$$\boldsymbol{K}_3^{(\mathrm{SWIII})} = -\left[\begin{array}{cc|cc} 0 & r^2\sin\theta & 0 & 0 \\ \sin\theta & 2r\cos\theta & 0 & 0 \\ \hline 0 & p_\theta\cos\theta & 0 & \sin\theta \\ -p_\theta\cos\theta & 0 & r^2\sin\theta & 2r\cos\theta \end{array}\right]. \tag{4.16}$$

The algebra $\langle \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWIII})}\rangle$ ensures the separability of the system in polar coordinates, which are indeed DH coordinates. Notice that if we re-write the operator $\boldsymbol{K}_2^{(\mathrm{SWIII})}$ in Cartesian coordinates, it coincides (up to an irrelevant multiplicative factor) with the operator $\boldsymbol{K}_3^{(\mathrm{SWI})}$, for $c = 0$. Besides, the algebra $\langle \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWIII})}\rangle$ diagonalizes in parabolic coordinates. If we write the expression of $\boldsymbol{K}_3^{(\mathrm{SWIII})}$ in cartesian coordinates, it converts into the operator $\boldsymbol{K}_3^{(\mathrm{SWII})}$.

4.2.4 *The Smorodinsky-Winternitz system SWIV*

The Hamiltonian function of this system is

$$H = H_1 = \frac{1}{2}\frac{p_\xi^2 + p_\eta^2}{\xi^2+\eta^2} + \frac{2\alpha + \beta\xi + \gamma\eta}{\xi^2+\eta^2}, \tag{4.17}$$

where we have used the parabolic coordinates

$$x = \frac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta. \tag{4.18}$$

The corresponding integrals read

$$H_2 = \frac{\gamma\xi^3 + \xi^2(p_\xi p_\eta - \beta\eta) - \xi\eta(p_\xi^2 + p_\eta^2 + 4\alpha + \gamma\eta) + \eta^2(p_\xi p_\eta + \beta\eta)}{\xi^2+\eta^2}, \tag{4.19}$$

$$H_3 = \frac{\xi^2\left(p_\eta^2 + 2(\alpha+\gamma\eta) - 2\beta\xi\eta^2 - \eta^2(p_\xi^2 + 2\alpha)\right)}{\xi^2+\eta^2}. \tag{4.20}$$

In the coordinates $(\xi, \eta, p_\xi, p_\eta)$ we get the Haantjes structures $(\omega, \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{SWIV})})$ and $(\omega, \boldsymbol{I}, \boldsymbol{K}_3^{(\mathrm{SWIV})})$, with

$$\boldsymbol{K}_2^{(\mathrm{SWIV})} = \left[\begin{array}{cc|cc} -2\xi\eta & \xi^2+\eta^2 & 0 & 0 \\ \xi^2+\eta^2 & -2\xi\eta & 0 & 0 \\ \hline 0 & 0 & -2\xi\eta & \xi^2+\eta^2 \\ 0 & 0 & \xi^2+\eta^2 & -2\xi\eta \end{array}\right], \tag{4.21}$$

$$\boldsymbol{K}_3^{(\mathrm{SWIV})} = \mathrm{diag}\left[-2\eta^2, 2\xi^2, -2\eta^2, 2\xi^2\right]. \tag{4.22}$$

These two structures are related with two different parabolic coordinate systems, with distinct axes, which are separating coordinates for the systems SWIV. Notice that the operator $\boldsymbol{K}_3^{(\mathrm{SWIV})}$ in Cartesian coordinates converts into the operator $-2\boldsymbol{K}_3^{(\mathrm{SWII})}$.

Remark 4.2.1. *The form of the Haantjes operators presented above can be geometrically interpreted, in all cases, as the application of the generalized lifting procedure described in Section 3.6 to a suitable Haantjes operator on the configuration space E_2, related to a specific coordinate system in which it diagonalizes. Its lifted version to the 4-dimensional phase space is again a Haantjes operator, at most linearly depending on the momenta.*

4.3 ANISOTROPIC OSCILLATOR WITH ROSOCHATIUS TERMS

We shall determine now the $\omega\mathscr{H}$ structures of an important physical model: the two-dimensional anisotropic oscillator with Rosochatius terms [168]. This system in the general case with n degrees of freedom has been studied in [161, 164, 196], where its maximal superintegrability was established.

In particular, in [164] the higher-order (missing) integral was determined by means of a geometric approach based on the Marsden-Weinstein reduction procedure. The anisotropic oscillator on curved spaces has been introduced and analyzed in [7].

The Hamiltonian function of the system reads

$$H_{\mathrm{AO}} = \frac{1}{2}\left(p_x^2 + p_y^2 + \nu^2(n_1x^2 + n_2y^2) + \frac{c_1}{x^2} + \frac{c_2}{y^2}\right), \tag{4.23}$$

where $n_1, n_2 \in \mathbb{N}\backslash\{0\}$, $c_1, c_2, \nu \in \mathbb{R}$. The integrals of motion corresponding to the one-dimensional energies are

$$E_1 = \frac{1}{2}\left(p_x^2 + \nu^2 n_1 x^2 + \frac{c_1}{x^2}\right), \tag{4.24}$$

$$E_2 = \frac{1}{2}\left(p_y^2 + \nu^2 n_2 y^2 + \frac{c_2}{y^2}\right). \tag{4.25}$$

The system (4.23) is separable in Cartesian coordinates, and admits a Haantjes algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_1^{(\mathrm{AO})}\rangle$, with

$$\boldsymbol{K}_1^{(\mathrm{AO})} = \mathrm{diag}\,[1, 0, 1, 0]\,. \tag{4.26}$$

However, the system admits a further integral, not related with separating coordinates. Let us introduce the complex quantities

$$\Delta_j = p_j^2 + \frac{c_j}{x_j^2} - \nu^2 n_j^2 x_j^2 - 2i\nu n_j p_j x_j, \qquad \Psi = \Delta_1^{n_2}\overline{\Delta}_2^{n_1}.$$

The additional, real integral is given by

$$\mathfrak{R}(\Psi) = \mathfrak{R}(\Delta_1^{n_2}\overline{\Delta}_2^{n_1}), \tag{4.27}$$

where $\mathfrak{R}(z)$ denotes the real part of $z \in \mathbb{C}$.

We can obtain a second Haantjes algebra in the following way. First, we shall consider the family of operators of the form

$$\boldsymbol{K} = \left[\begin{array}{cc|cc} m_d & 0 & 0 & 0 \\ m_{21} & m_d & 0 & 0 \\ \hline 0 & m_{32} & m_d & m_{21} \\ -m_{32} & 0 & 0 & m_d \end{array}\right], \tag{4.28}$$

which are Haantjes operators for any choice of the three arbitrary functions m_d, m_{21}, m_{32} of (x, y, p_x, p_y). This family generalizes (up to a transposition of coordinates with momenta) the form of the two Haantjes operators admitted by the PW system

$$H^{(\mathrm{PW})} = \frac{1}{2}(p_x^2 + p_y^2) + a\frac{x}{y^{2/3}}, \qquad a \in \mathbb{R}. \tag{4.29}$$

Notice that the operators of the form (4.28) generically are non-semisimple ones. We shall show that the class (4.28) also contains the explicit form of the specific Haantjes operator related to the integral of motion (4.27).

Now, let us impose the chain equation $(\boldsymbol{K}_2^{(\mathrm{AO})})^T \, \mathrm{d}H_{\mathrm{AO}} = \mathrm{d}\mathfrak{R}(\Psi)$, where $\boldsymbol{K}_2^{(\mathrm{AO})}$ has the form (4.28). We find the unique solution

$$(\boldsymbol{K}_2^{(\mathrm{AO})})^T \, \mathrm{d}H = \left(\nu^2(m_d n_1^2 x + m_{21} n_2^2 y) - m_{32} p_y - \left(\frac{m_d c_1}{x^3} + \frac{m_{21} c_2}{y^3}\right)\right) \mathrm{d}x + \left(\nu^2 m_d n_2^2 y + m_{32} p_x - \frac{m_d c_2}{y^3}\right) \mathrm{d}y + (m_d p_x)\mathrm{d}p_x + (m_{21} p_x + m_d p_y)\mathrm{d}p_y \,,$$

where

$$m_d = \frac{1}{p_x} \frac{\partial \mathfrak{R}(\Psi)}{\partial p_x} \,; \tag{4.30}$$

$$m_{21} = \frac{1}{p_x}\left(-\frac{p_y}{p_x}\frac{\partial \mathfrak{R}(\Psi)}{\partial p_x} + \frac{\partial \mathfrak{R}(\Psi)}{\partial p_y}\right); \tag{4.31}$$

$$m_{32} = \frac{1}{p_x}\left(-\nu^2\left(n_2^2 y + \frac{c_2}{y^3}\right)\frac{1}{p_x}\frac{\partial \mathfrak{R}(\Psi)}{\partial p_x} + \frac{\partial \mathfrak{R}(\Psi)}{\partial y}\right). \tag{4.32}$$

The explicit form of the coefficients m_d, m_{21}, m_{32} is reported in Appendix B. Thus, we have proved that the system (4.23) admits a second, non-semisimple $\omega\mathscr{H}$ structure, with $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2^{(\mathrm{AO})} \rangle$. It is an open problem to ascertain whether the system (4.23) admits other separating coordinates in phase space, apart from the Cartesian ones.

4.4 $\omega\mathscr{H}$ STRUCTURES FOR MULTISEPARABLE SYSTEMS IN E_3

We shall analyze in detail three relevant examples of multiseparable systems in the Euclidean space E_3, in the context of Haantjes geometry. One of them is maximally superintegrable, the other two are minimally superintegrable ones [50]. For a different treatment, in the framework of the Killing-Stäckel theory, see [14, 41]. In the following analysis, $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_z$ denote the components of the angular momentum in the Cartesian frame.

4.4.1 *The Kepler system with a Rosochatius-type term*

We shall consider the Hamiltonian function

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} \,, \tag{4.33}$$

whose integrals of motion are

$$H_2 = \frac{1}{2}|\boldsymbol{\mathcal{L}}|^2 + (x^2 + y^2 + z^2)\left(\frac{k_1}{x^2} + \frac{k_2}{y^2}\right), \tag{4.34}$$

$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + (x^2+y^2)\left(\frac{k_1}{x^2}+\frac{k_2}{y^2}\right), \tag{4.35}$$

$$H_4 = \frac{1}{2}\mathcal{L}_y^2 + \frac{k_1 z^2}{x^2}, \tag{4.36}$$

$$H_5 = \mathcal{L}_x p_y - p_x \mathcal{L}_y - 2z\left(-\frac{k}{2\sqrt{x^2+y^2+z^2}} + \frac{k_1}{x^2}+\frac{k_2}{y^2}\right). \tag{4.37}$$

These integrals form three families of functions in involution: $\{H_1, H_2, H_3\}, \{H_1, H_2, H_4\}, \{H_1, H_3, H_5\}$. The equations of the Haantjes chains associated are: $\boldsymbol{K}_i^T \mathrm{d}H = \mathrm{d}H_i$ $(i = 1, \ldots, 5)$, with the Haantjes operators

$$\boldsymbol{K}_2 = \left[\begin{array}{ccc|ccc} y^2+z^2 & -xy & -xz & 0 & 0 & 0 \\ -xy & x^2+z^2 & -yz & 0 & 0 & 0 \\ -xz & -yz & x^2+y^2 & 0 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & zp_x - xp_z & y^2+z^2 & -xy & -xz \\ xp_y - yp_x & 0 & -(yp_z - zp_y) & -xy & x^2+z^2 & -yz \\ -(zp_x - xp_z) & yp_z - zp_y & 0 & -xz & -yz & x^2+y^2 \end{array}\right], \tag{4.38}$$

$$\boldsymbol{K}_3 = \left[\begin{array}{ccc|ccc} y^2 & -xy & 0 & 0 & 0 & 0 \\ -xy & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -(xp_y - yp_x) & 0 & y^2 & -xy & 0 \\ xp_y - yp_x & 0 & 0 & -xy & x^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{4.39}$$

$$\boldsymbol{K}_4 = \left[\begin{array}{ccc|ccc} z^2 & 0 & -xz & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -xz & 0 & x^2 & 0 & 0 & 0 \\ \hline 0 & 0 & zp_x - xp_z & z^2 & 0 & -xz \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -(zp_x - xp_z) & 0 & 0 & -xz & 0 & x^2 \end{array}\right], \tag{4.40}$$

and

$$\boldsymbol{K}_5 = \left[\begin{array}{ccc|ccc} -2z & 0 & x & 0 & 0 & 0 \\ 0 & -2z & y & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -p_x & -2z & 0 & x \\ 0 & 0 & -p_y & 0 & -2z & y \\ p_x & p_y & 0 & x & y & 0 \end{array}\right]. \tag{4.41}$$

We shall also take into account the integral

$$H_6 := \frac{1}{2}\mathcal{L}_x^2 + \frac{k_2 z^2}{y^2},$$

which is not functionally independent with respect to the other ones; however, it will play a relevant role in the construction of separating coordinates. Imposing the chain equation $\boldsymbol{K}_6^T \mathrm{d}H = \mathrm{d}H_6$, we get the further Haantjes operator

$$\boldsymbol{K}_6 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^2 & -yz & 0 & 0 & 0 \\ 0 & -yz & y^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & zp_y - yp_z & 0 & z^2 & -yz \\ 0 & yp_z - zp_y & 0 & 0 & -yz & y^2 \end{array}\right]. \tag{4.42}$$

Let us discuss now the separating coordinates admitted by the system (4.33). First, we observe that there exist three Abelian semisimple Haantjes algebras related with the operators (4.38)-(4.41): $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$, $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_4 \rangle$ and $\mathscr{H}_3 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_6 \rangle$. The algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$, as was discussed in Ref. [184], diagonalizes in the *spherical polar* coordinates with the z-axis as the polar axis; therefore, according to the previous discussion, they are DH coordinates and separating coordinates for the Kepler system (4.33). The algebra $\mathscr{H}_2$ diagonalizes in spherical polar coordinates with the y-axis as the polar axis; besides, the algebra $\mathscr{H}_3$ ensures the separability of the system in spherical polar coordinates with the x-axis as the polar axis.

We also have the algebra $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$, which diagonalizes in the *rotational parabolic* coordinates.

Finally, the system (4.33) separates in *spherical conical* coordinates. Precisely, the algebra $\mathscr{H}_5 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_7 = a\boldsymbol{K}_6 + b\boldsymbol{K}_4 + c\boldsymbol{K}_3 \rangle$ diagonalizes in the spherical conical coordinates $u_1 \in \mathbb{R}$, $u_2 \in \mathbb{R}$, $u_3 > 0$, where $a \leq u_1 \leq b \leq u_2 \leq c$ and $a, b, c \in \mathbb{R}$.

4.4.1.1 *Generators of the Haantjes algebras*

In order to find a generator of a unigenerated semisimple Abelian Haantjes algebra of rank n, it is sufficient to choose inside the algebra a Haantjes operator whose minimal polynomial is of degree n, as explained in Section 2.4.3. As a consequence of this observation, we obtain that a simple (not unique) choice for the generator for the algebra $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$ reads

$$\boldsymbol{L}_1 = \boldsymbol{K}_2 + \boldsymbol{K}_3 \,. \tag{4.43}$$

We have

$$\boldsymbol{K}_2 = \alpha_1^{(1)} \boldsymbol{L}_1 + \alpha_2^{(1)} \boldsymbol{L}_1^2 \,, \qquad \alpha_1^{(1)} = \frac{3x^2 + 3y^2 + 2z^2}{2x^2 + 2y^2 + z^2} \,, \qquad \alpha_2^{(1)} = -\frac{1}{2x^2 + 2y^2 + z^2} \,, \tag{4.44}$$

and

$$\boldsymbol{K}_3 = \beta_1^{(1)} \boldsymbol{L}_1 + \beta_2^{(1)} \boldsymbol{L}_1^2 \,, \qquad \beta_1^{(1)} = -\frac{x^2 + y^2 + z^2}{2x^2 + 2y^2 + z^2} \,, \qquad \beta_2^{(1)} = \frac{1}{2x^2 + 2y^2 + z^2} \,. \tag{4.45}$$

A generator of the algebra $\mathscr{H}_2 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_4 \rangle$ is given by

$$\boldsymbol{L}_2 = \boldsymbol{K}_2 + \boldsymbol{K}_4 \,. \tag{4.46}$$

We have

$$\boldsymbol{K}_2 = \alpha_1^{(2)} \boldsymbol{L}_2 + \alpha_2^{(2)} \boldsymbol{L}_2^2 \,, \qquad \alpha_1^{(2)} = \frac{3x^2 + 2y^2 + 3z^2}{2x^2 + y^2 + 2z^2} \,, \qquad \alpha_2^{(2)} = -\frac{1}{2x^2 + y^2 + 2z^2} \,, \tag{4.47}$$

and

$$\boldsymbol{K}_4 = \beta_1^{(2)} \boldsymbol{L}_2 + \beta_2^{(2)} \boldsymbol{L}_2^2 \,, \qquad \beta_1^{(2)} = -\frac{x^2 + y^2 + z^2}{2x^2 + y^2 + 2z^2} \,, \qquad \beta_2^{(2)} = \frac{1}{2x^2 + y^2 + 2z^2} \,. \tag{4.48}$$

A generator of the algebra $\mathscr{H}_3 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_6 \rangle$ is $\boldsymbol{L}_3 = \boldsymbol{K}_2 + \boldsymbol{K}_6$, with

$$\boldsymbol{K}_2 = \alpha_1^{(3)} \boldsymbol{L}_3 + \alpha_2^{(3)} \boldsymbol{L}_3^2 \,, \qquad \alpha_1^{(3)} = \frac{2x^2 + 3y^2 + 3z^2}{x^2 + 2y^2 + 2z^2} \,, \qquad \alpha_2^{(3)} = -\frac{1}{x^2 + 2y^2 + 2z^2} \,, \tag{4.49}$$

and

$$\boldsymbol{K}_6 = \beta_1^{(3)} \boldsymbol{L}_3 + \beta_2^{(3)} \boldsymbol{L}_3^2 \,, \qquad \beta_1^{(3)} = -\frac{x^2 + y^2 + z^2}{x^2 + 2y^2 + 2z^2} \,, \qquad \beta_2^{(3)} = \frac{1}{x^2 + 2y^2 + 2z^2} \,. \tag{4.50}$$

A generator of the algebra $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$ is just $\boldsymbol{L}_4 = \boldsymbol{K}_5$, since the minimal polynomial of $\boldsymbol{K}_5$ is of degree 3. We have

$$\boldsymbol{K}_3 = \alpha_0^{(4)} \boldsymbol{I} + \alpha_1^{(4)} \boldsymbol{L}_4 + \alpha_2^{(4)} \boldsymbol{L}_4^2 \,,$$

with

$$\alpha_0^{(4)} = x^2 + y^2, \qquad \alpha_1^{(4)} = -2z, \qquad \alpha_3^{(4)} = -1\,.$$

A generator of the algebra $\mathscr{H}_5 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_7 \rangle$ is simply $\boldsymbol{L}_5 = \boldsymbol{K}_7$, with

$$\boldsymbol{K}_2 = \alpha_1^{(5)} \boldsymbol{L}_5 + \alpha_2^{(5)} \boldsymbol{L}_5^2\,,$$

where

$$\alpha_1^{(5)} = \frac{(b+c)x^2 + (a+c)y^2 + (a+b)z^2}{bcx^2 + acy^2 + abz^2} \qquad \alpha_2^{(5)} = -\frac{1}{c(bx^2+ay^2)+abz^2}\,.$$

4.4.2 *A class of generalized Kepler systems*

We consider a family of deformations of the Kepler system which depends on an arbitrary function $F\left(\frac{y}{x}\right)$. The Hamiltonian function of this family has the form

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2+y^2+z^2}} + \frac{k_1 z}{\sqrt{x^2+y^2+z^2}(x^2+y^2)} + \frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{4.51}$$

The associated integrals of motion are

$$H_2 = \frac{1}{2}|\boldsymbol{\mathcal{L}}|^2 + \frac{k_1 z\sqrt{x^2+y^2+z^2} + (x^2+y^2+z^2)\,F\left(\frac{y}{x}\right)}{x^2+y^2}, \tag{4.52}$$

$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + F\left(\frac{y}{x}\right), \tag{4.53}$$

$$H_4 = \mathcal{L}_x p_y - p_x \mathcal{L}_y + \frac{kz}{\sqrt{x^2+y^2+z^2}} - \frac{k_1(x^2+y^2+2z^2)}{\sqrt{x^2+y^2+z^2}(x^2+y^2)} - 2z\frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{4.54}$$

These integrals form two families of functions in involution: $\{H_1, H_2, H_3\}$ and $\{H_1, H_3, H_4\}$. The equations of the Haantjes chains are: $\boldsymbol{K}_2^T \mathrm{d}H = \mathrm{d}H_2$, $\boldsymbol{K}_3^T \mathrm{d}H = \mathrm{d}H_3$, $\boldsymbol{K}_5^T \mathrm{d}H = \mathrm{d}H_4$. The Haantjes operators associated are $\boldsymbol{K}_2$, Eq. (4.38), $\boldsymbol{K}_3$, Eq. (4.39) and $\boldsymbol{K}_5$, Eq. (4.41). We obtain again the two Abelian, unigenerated algebras $\mathscr{H}_1 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3 \rangle$ and $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5 \rangle$, whose generators have been determined above. They provide us with the two coordinate systems admitted by the class of Hamiltonian systems (4.51), namely the spherical polar and the rotational parabolic ones.

4.4.3 *A class of anisotropic oscillators*

An interesting family of deformations of the anisotropic oscillator is given by the Hamiltonian function

$$H = H_1 = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + k(x^2+y^2) + 4kz^2 + \frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{4.55}$$

where again F is an arbitrary function of its argument. It admits the following integrals of motion:

$$H_2 = \frac{1}{2}p_z^2 + 4kz^2, \tag{4.56}$$

$$H_3 = \frac{1}{2}\mathcal{L}_z^2 + F\left(\frac{y}{x}\right), \tag{4.57}$$

$$H_4 = \mathcal{L}_x p_y - p_x \mathcal{L}_y + 2kz(x^2+y^2) - 2z\frac{F\left(\frac{y}{x}\right)}{x^2+y^2}. \tag{4.58}$$

Therefore, this class of systems is minimally superintegrable. These integrals form two families of functions in involution: $\{H_1, H_2, H_3\}$ and $\{H_1, H_3, H_4\}$. The equations for the Haantjes chains are: $\boldsymbol{K}_1^T \mathrm{d}H = \mathrm{d}H_2$, $\boldsymbol{K}_3^T \mathrm{d}H = \mathrm{d}H_3$, $\boldsymbol{K}_5^T \mathrm{d}H = \mathrm{d}H_4$. The corresponding Haantjes operators are:

$$\boldsymbol{K}_1 = \mathrm{diag}\,[0,0,1,0,0,1]\,, \tag{4.59}$$

$\boldsymbol{K}_3$, Eq. (4.39) and $\boldsymbol{K}_5$, Eq. (4.41). The family of systems (4.55) admits the two Abelian algebras $\mathscr{H}_6 = \langle \boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_3\rangle$ and $\mathscr{H}_4 = \langle \boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_5\rangle$. It separates in polar cylindrical and rotational parabolic coordinates.

The algebra $\mathscr{H}_6 = \langle \boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_3\rangle$ admits the generator

$$\boldsymbol{L}_6 = 2\boldsymbol{K}_1 + \frac{1}{x^2+y^2}\boldsymbol{K}_3\,. \tag{4.60}$$

We have

$$\boldsymbol{K}_1 = \alpha_1^{(6)}\boldsymbol{L}_6 + \alpha_2^{(6)}\boldsymbol{L}_6^2, \qquad \alpha_1^{(6)} = -\frac{1}{2}, \qquad \alpha_2^{(6)} = \frac{1}{2}\,, \tag{4.61}$$

and

$$\boldsymbol{K}_3 = \beta_1^{(6)}\boldsymbol{L}_6 + \beta_2^{(6)}\boldsymbol{L}_6^2, \qquad \beta_1^{(6)} = 2(x^2+y^2), \qquad \beta_2^{(6)} = -(x^2+y^2)\,. \tag{4.62}$$

4.4.4 *The Jacobi-Calogero model*

To conclude the chapter, we briefly discuss the Jacobi-Calogero model, introduced in Section 1.4.2 to illustrate the application of Lax pairs in obtaining integrals of motion. In addition to being integrable in the Liouville sense, it is a maximally superintegrable system that admits five integrals of motion [133, 180]. Moreover, it is multiseparable; in [11] it was shown that, in addition to circular cylindrical coordinates [34], four other coordinate systems allow the separability of the HJ equation–namely, spherical, parabolic, oblate spheroidal, and prolate spheroidal coordinates.

In [192] the multiseparability of the system in circular cylindrical, spherical, and parabolic coordinates is studied from the perspective of $\omega\mathscr{H}$ manifolds. To this end, besides the integrals (1.73)–(1.75), three additional integrals are considered: the integral $H_{\mathcal{L}}$ associated with the axial angular momentum, the integral H_{sph} corresponding to the square of the angular momentum modulus, and the integral H_{par} related to the product of the rotational symmetry axis with the radial linear momentum.

Not all of these integrals are functionally independent; however, they are chosen to generate alternative Haantjes structures that yield different separation coordinates, as demonstrated in Section 4.4.1.

We recall that the Hamiltonian of the system is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + V_{\text{Cal}}, \tag{4.63}$$

where

$$V_{\text{Cal}} = \frac{1}{(x-y)^2} + \frac{1}{(x-z)^2} + \frac{1}{(y-z)^2}. \tag{4.64}$$

The integral $H_{\mathcal{L}}$ is given by

$$H_{\mathcal{L}} = \frac{1}{6}\left[(y\,p_z - z\,p_y) + (z\,p_x - x\,p_z) + (x\,p_y - y\,p_x)\right]^2 + \frac{1}{3}\left[(x-y)^2 + (x-z)^2 + (y-z)^2\right] V_{\text{Cal}}, \tag{4.65}$$

and it plays a privileged role by being separable in all three coordinate systems under consideration. The following operator solves the corresponding chain equation,

$$\boldsymbol{K}_7^T \mathrm{d}H = \mathrm{d}H_{\mathcal{L}},$$

namely,

$$\boldsymbol{K}_7 = \frac{1}{3}\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{0}_3 \\ \hline \boldsymbol{B} & \boldsymbol{A} \end{array}\right], \tag{4.66}$$

where

$$\boldsymbol{A} = \begin{bmatrix} (y+z)^2 & (y-z)(z-x) & (y-z)(x-y) \\ (y-z)(z-x) & (x+z)^2 & (z-x)(x-y) \\ (y-z)(x-y) & (z-x)(x-y) & (x+y)^2 \end{bmatrix},$$

and

$$\boldsymbol{B} = \left[(y-z)\,p_x + (z-x)\,p_y + (x-y)\,p_z\right] \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

The conservation of linear momentum, $H_1 = p_x + p_y + p_z$ (see Eq. (1.73)), is associated with separability in circular cylindrical coordinates. The Haantjes chain $\boldsymbol{K}_8^T \mathrm{d}H = \mathrm{d}H_1$ is solved by choosing the operator

$$\boldsymbol{K}_8 = \frac{1}{p_x + p_y + p_z} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]. \tag{4.67}$$

The integral H_{sph} is given by

$$H_{\text{sph}} = \frac{1}{2}\left[(y\,p_z - z\,p_y)^2 + (z\,p_x - x\,p_z)^2 + (x\,p_y - y\,p_x)^2\right] + (x^2 + y^2 + z^2)\,V_{\text{Cal}}, \tag{4.68}$$

which separates in spherical coordinates; the operator $\boldsymbol{K}_2$ introduced in Eq. (4.38) solves the chain equation $\boldsymbol{K}_2^T \mathrm{d}H = \mathrm{d}H_{\text{sph}}$.

Finally, the integral

$$H_{\text{par}} = \frac{1}{2}\left[(p_x + p_y + p_z)(x\,p_x + y\,p_y + z\,p_z) - (x + y + z)(p_x^2 + p_y^2 + p_z^2)\right] + (x + y + z)\,V_{\text{Cal}} \tag{4.69}$$

is separable in parabolic coordinates. Its corresponding chain equation, $\boldsymbol{K}_9^T \mathrm{d}H = \mathrm{d}H_{\text{par}}$, is solved by means of the Haantjes operator

$$\boldsymbol{K}_9 = \frac{1}{2}\left[\begin{array}{ccc|ccc} -2(y+z) & x+y & x+z & 0 & 0 & 0 \\ x+y & -2(x+z) & y+z & 0 & 0 & 0 \\ x+z & y+z & -2(x+y) & 0 & 0 & 0 \\ \hline 0 & p_y - p_x & p_z - p_x & -2(y+z) & x+y & x+z \\ -(p_y - p_x) & 0 & p_z - p_y & x+y & -2(x+z) & y+z \\ -(p_z - p_x) & -(p_z - p_y) & 0 & x+z & y+z & -2(x+y) \end{array}\right]. \tag{4.70}$$

The Haantjes operators introduced above can be grouped into three Abelian algebras:

$$\mathscr{H}_7 = \langle \boldsymbol{I}, \boldsymbol{K}_7, \boldsymbol{K}_8 \rangle, \quad \mathscr{H}_8 = \langle \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_7 \rangle, \quad \mathscr{H}_9 = \langle \boldsymbol{I}, \boldsymbol{K}_7, \boldsymbol{K}_9 \rangle.$$

These algebras are associated with the separability of the system in circular cylindrical, spherical, and parabolic coordinates, respectively.

For the integral H_3 from Eq. (1.75), no semisimple Haantjes operator was found as a solution of the associated chain equation. Nevertheless, a non-semisimple operator was computed in [160], leading to a coordinate chart in which the system partially separates, as will be discussed in Section 5.4.1.

The art of losing isn't hard to master;
so many things seem filled with the intent
to be lost that their loss is no disaster.

Elizabeth Bishop, *One Art*

5

Partial separability

Loss is inherent in living. One day we misplace our keys; memories inevitably fade, and bit by bit, we relinquish what once seemed indispensable. Sometimes we are conscious of the absence, like watching a bus disappear down the road; other times, life accumulates quiet forfeitures–great and small–without our awareness until much later, if ever. In this chapter, I seek to lose with intention: to recognize absence not as failure, but as an essential stitch in the fabric of existence.

Conceptually, we present a twofold generalization of the Jacobi-Haantjes theorem (Section 3.4.2). For Hamiltonian systems, the existence of semisimple, maximal-rank (Abelian) Haantjes operator algebras is equivalent to the total separability of the associated HJ equations. Thus, we relax either the semisimplicity hypothesis or the maximal rank condition. Each scenario leads to partial separability of the system under study.

The notion of *partial separability* refers to the existence of a set of coordinates allowing one to separate the HJ equations into a system of both ODEs and PDEs, the latter ones involving just a subset of independent variables. This relevant subject, in contrast to the theory of total separability, has received so far much less attention. The first study on partial separability dates back to 1896 and is due to G. Di Pirro [150], who considered a generalization of the notion of Stäckel matrix. In his construction, it was assumed that a single row of the $(n-r)\times(n-r)$ generalized Stäckel matrix, say the first one, depends on a *block of coordinates* $(q^1,\dots,q^r)$, whereas the remaining rows depend each on one coordinate only, picked up from the set $(q^{r+1},\dots,q^n)$. In this way, Di Pirro defined new Hamiltonian systems admitting $n-r$ first integrals, which are homogeneous, quadratic in the momenta, and orthogonally separated.

Taking into account these results, in 1897, Stäckel generalized his original construction [177] by proposing a block-Stäckel invertible matrix, in which more than one row may depend on a subset of coordinates (which are different for each row). The number of quadratic integrals so obtained coincides with the number of subsets of coordinates. Further results have been contributed by Havas [71], who considered for the first time partial separation of the HJ equation in non-orthogonal coordinate systems. Subsequently, Boyer et al. obtained in [30] a complete list of all partial separable coordinate systems for the Helmholtz and the HJ equations in non-orthogonal coordinates for the case of four-dimensional Riemannian spaces. Recently, in [39] the notion of partial separability has been discussed in the context of Riemannian geometry; in [44], partially separated Painlevé metrics and their conformal deformations have been related to the classical inverse Calderón problem.

When considering non-semisimple algebras of operators, we deal with a complete family of Hamiltonian functions in involution $(H_1, \ldots, H_n)$ for which, in a given set of coordinates, a *partial involutivity condition* ($\mathcal{P}$-involution) holds. This means that in these coordinates, the Poisson bracket between two integrals splits into several equations, which generalize Benenti's conditions (1.54) (see Definition 5.1.2 below). As we shall prove in Theorem 5.2.2, this, in turn, implies that the Hamiltonian family is partially separable.

Our main result is Theorem 5.2.1. Given the family $(H_1, \ldots, H_n)$, we shall prove that if one can associate it with an Abelian algebra of *non-semisimple Haantjes operators of maximal rank*, then the DH coordinates (allowing us to write all the operators simultaneously in a block-diagonal form) also provide us with a set of partially separable coordinates for the associated HJ equations. This result paves the way for the systematic application of non-semisimple Haantjes algebras in classical mechanics.

In the other scenario, we shall deal with *non-maximal-rank*, but semisimple Haantjes algebras. We propose two results. First, we construct a novel class of partially separated systems, which we obtain by further generalizing the AKN construction to the case of block-Stäckel matrices. We recover as particular instances of our construction all the partially separable systems that have been proposed in the literature so far. Second, we interpret our novel class of Hamiltonian systems in the framework of Haantjes geometry. Precisely, in Theorem 5.3.2, we prove the existence of non-maximal-rank, semisimple $\omega\mathscr{H}$ manifolds for these systems, and we provide the explicit expression of their Haantjes algebras.

As a general comment, we wish to clarify that partial and total separability are not exclusive notions. As a matter of fact, a given integrable Hamiltonian system may admit more than one $\omega\mathscr{H}$ structure, which leads either to total or to partial separability depending on whether the structure is semisimple or non-semisimple. From a computational point of view, partial separability can also be regarded as a useful intermediate step toward the construction of total-separation variables.

Another relevant point is that the new systems coming from Stäckel-type matrices represent an (a priori) *incomplete* family of commuting Hamiltonians: we have as many Hamiltonians as the number of sets of variables involved in the partial separation process. We mention that the original constructions of partially separable systems by Di Pirro [150], Stäckel [178] and Painlevé [44] lead as well to incomplete sets of Hamiltonian systems.

However, we prove in Theorem 5.3.1 that for our non-maximal rank Stäckel-type Hamiltonian systems, still the Benenti conditions of $\mathcal{T}$-involution (1.54) hold in the set of coordinates in which the family is defined.

To summarize, Haantjes geometry provides us with two general classes of partially separated Hamiltonian systems. The first class possesses non-semisimple, maximal rank Haantjes algebras associated. These Hamiltonian systems are completely integrable and are in $\mathcal{P}$-involution. The associated HJ equations are partially separable in DH coordinates.

The second class involves "incomplete" families of Hamiltonian systems: the number of integrals explicitly known is less than n, half of the dimension of the phase space. These systems possess semisimple, non-maximal-rank Haantjes algebras, are in standard $\mathcal{T}$-involution, and, once again, the associated HJ equations are partially separable in DH coordinates. Our main conclusion is that symplectic-Haantjes manifolds represent a natural geometric setting for studying partially separated Hamiltonian systems.

This chapter is based in [160], and its structure is the following. In Section 5.1, we define the notion of partial separability and a generalization of Benenti conditions, namely the concept of $\mathcal{P}$-involution. In Section 5.2, we address the problem of partial separability for completely integrable models; we shall show that the DH coordinates which block-diagonalize the non-semisimple Haantjes algebra associated with a given integrable model also guarantee partial separability of the associated HJ equations. In Section 5.3, we define a new class of generalized Stäckel systems and show that they admit semisimple, non-maximal-rank symplectic-Haantjes manifolds. In Section 5.4, two examples illustrating the application of the theory proposed are presented. Another example can be found in Appendix D.

5.1 PARTIAL SEPARABILITY

5.1.1 *Main definition*

We shall propose a definition of partial separability inspired by the original theory of Jacobi. In essence, we require that, given a Hamiltonian system, the corresponding Hamilton-Jacobi equations decouple into a system of PDEs, each of them depending on a different subset of independent variables. More precisely, we have the following

Definition 5.1.1 (Generalized Separation Equations)**.** *We shall say that a $2n$-dimensional Hamiltonian system with Hamiltonian function H is partially separable if there exist:*

(1) $m \leq n$ independent smooth functions $(H_1, \ldots, H_m)$ in involution, on which H is functionally dependent;

(2) $1 < \nu \leq n$ subsets of Darboux coordinates $(\boldsymbol{q}^1, \boldsymbol{p}_1, \ldots, \boldsymbol{q}^\nu, \boldsymbol{p}_\nu)$ defined in the phase space;

(3) m separation equations of the form

$$\phi_{a,j_a}(\boldsymbol{q}^a, \boldsymbol{p}_a; H_1, \dots, H_m) = 0\,, \quad a = 1, \dots, v, \quad j_a = 1, \dots r_a, \quad \det\left[\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{H}}\right] \neq 0 \tag{5.1}$$

where $r_a \in \mathbb{N}\backslash\{0\}$ *represents the number of equations associated with*

$$(\boldsymbol{q}^a, \boldsymbol{p}_a) = (q^{a,1}, \dots, q^{a,\sigma_a}, p_{a,1}, \dots, p_{a,\sigma_a}),$$

$\boldsymbol{\phi} = (\phi_{1,1} \dots, \phi_{1,r_1}, \dots, \phi_{v,1}, \dots, \phi_{v,r_v})^T$, $\boldsymbol{H} = (H_1, \dots, H_m)$ *and* σ_a *represents the number of coordinates (or momenta) involved in the pair* $(\boldsymbol{q}^a, \boldsymbol{p}_a)$.

These equations are a generalization of the Jacobi-Sklyanin SE (1.55) and will be referred to as generalized separation equations (GSE). In fact, if for each $a \in \{1, \dots, v\}$, $\sigma_a = 1$ (which implies $v = n$) and $r_a = 1$ (then $v = m = n$), one recovers the classical Jacobi's theory of separability. A Hamiltonian system such that $\sigma_a = r_a = 1$ will be said to be *totally separable*.

In this chapter, we shall study two cases in which $\sigma_a > 1$ for some $a \in \{1, \dots, v\}$ (thus, $v < n$):

– Case **I**: $r_a = \sigma_a$, implying $m = n$;

– Case **II**: $r_a = 1\ \forall a \in \{1, \dots, v\}$, implying $m = v$.

The Case **I** corresponds to the maximum possible number of (independent) separation equations for each subset of Darboux coordinates, whereas the Case **II** corresponds to the minimum one. Other possible cases, corresponding to different values of the indices (σ_a, r_a) are left for future work.

Remark 5.1.1. *Definition 5.1.1 can be regarded as a generalization of the approach to partial separability developed in [150], [178], [44], [71], [39], etc. By solving, when possible, the PDEs (5.1), one obtains a complete integral shared by the HJ equations of* $(H_1, \dots, H_n)$ *which is partially additively separated, i.e., of the form*

$$W = \sum_{a=1}^{v} W_a(\boldsymbol{q}^a; h_1, \dots, h_n), \qquad \det\left(\frac{\partial^2 W}{\partial q^i \partial h_j}\right) \neq 0. \tag{5.2}$$

For instance, this is guaranteed in Case ***I****, under the hypotheses of the following result.*

Theorem 5.1.1. *Let* $(H_1, \dots, H_n)$ *be a family of n independent functions satisfying the GSE (5.1) with* $r_a = \sigma_a$, *and* $\mathcal{L}_{\boldsymbol{h}}$ *be the foliation defined by the level sets* $\mathcal{L}_{\boldsymbol{h}} := \{H_1 = h_1, \dots, H_n = h_n\}$. *Assuming that*

$$\det\left[\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}}\right] \neq 0\,, \tag{5.3}$$

the Hamiltonian system $(H_1, \dots, H_n)$ *admits a complete integral of the form (5.2) if and only if the* $n \times n$ *matrix*

$$\left[\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}}\right]^{-1}\Bigg|_{\mathcal{L}_{\boldsymbol{h}}} \left[\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}}\right]\Bigg|_{\mathcal{L}_{\boldsymbol{h}}} = \left[\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}\right]\Bigg|_{\mathcal{L}_{\boldsymbol{h}}} \tag{5.4}$$

is a symmetric matrix. Under this assumption, the functions $(H_1, \dots, H_n)$ *are in involution with each other. Vice versa, if there exists a complete integral of the HJ equations of the form* (5.2)*, there exist* m *separation equations of the form* (5.1)*.*

Proof. Condition (5.3) allows us to solve the GSE (5.1) with respect to $\boldsymbol{p}$ and to get a system of n partially separated first-order PDEs for the characteristic Hamiltonian function $W(\boldsymbol{q}; h_1, \dots, h_n)$, which read

$$\frac{\partial W}{\partial q^{a,j_a}} = p_{a,j_a}(q^{a,1}, \dots, q^{a,\sigma_a}; h_1, \dots, h_n) \qquad a = 1, \dots, v, \quad j_a = 1, \dots, \sigma_a \,. \tag{5.5}$$

The integrability of the latter system is equivalent to the compatibility conditions

$$\frac{\partial p_{b,j_b}}{\partial q^{a,j_a}} = \frac{\partial p_{a,j_a}}{\partial q^{b,j_b}} \,. \tag{5.6}$$

In turn, such conditions are equivalent to the symmetry of the matrix product in the l.h.s. of Eq. (5.4). In fact, from the GSE (5.1) it follows that for each $k = 1, \dots, n$,

$$0 = \mathrm{d}\phi_{k|_{\mathcal{L}_{\boldsymbol{h}}}} = \sum_{i=1}^{n} \left(\frac{\partial \phi_k}{\partial q^i} \mathrm{d}q^i + \frac{\partial \phi_k}{\partial p_i} \mathrm{d}p_i \right)_{|_{\mathcal{L}_{\boldsymbol{h}}}} = \sum_{i=1}^{n} \left(\frac{\partial \phi_k}{\partial q^i} + \sum_{j=1}^{n} \frac{\partial \phi_k}{\partial p_j} \frac{\partial p_j}{\partial q^i} \right) \mathrm{d}q^i_{|_{\mathcal{L}_{\boldsymbol{h}}}} \,. \tag{5.7}$$

Thanks to condition (5.3), Eqs. (5.7) can be solved for the matrix $\left[\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} \right]_{|_{\mathcal{L}_{\boldsymbol{h}}}}$, which gives us Eq. (5.4). Thus, the functions $(H_1, \dots, H_n)$ are in involution with each other. Consequently, $\mathcal{L}_{\boldsymbol{h}}$ is a Lagrangian foliation.

Conversely, if there exists a complete integral of the form (5.2), then the set of equations

$$p_{a,j_a} = \frac{\partial W_a}{\partial q^{a,j_a}}(q^{a,1}, \dots, q^{a,\sigma_a}; h_1, \dots, h_n) \qquad a = 1, \dots, v, \quad j_a = 1, \dots, \sigma_a \tag{5.8}$$

are generalized separation equations of the form (5.1), and satisfy the condition $\det \left[\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{H}} \right] \neq 0$ as a consequence of relation (1.42).

□

Remark 5.1.2. *In the case of totally separable systems, the condition of symmetry of matrix* (5.4) *is obviously fulfilled as both the Jacobian matrices of* $\boldsymbol{\phi}$ *with respect to* $\boldsymbol{q}$ *and* $\boldsymbol{p}$ *are diagonal.*

5.1.2 *Partial separability and Benenti-type conditions*

We wish to show that the notion of partial separability for completely integrable Hamiltonian systems can be characterized by means of a generalization of Benenti's relations of $\mathcal{T}$-involution (1.54).

Definition 5.1.2 ($\mathcal{P}$-involution). *A pair of functions (H_α, H_β) in involution will be said to be in partially separated involution (or $\mathcal{P}$-involution) with respect to the set of coordinates $(\boldsymbol{q}^1, \boldsymbol{p}_1, \dots, \boldsymbol{q}^v, \boldsymbol{p}_v)$, if the reduction of the Poisson brackets to each subset of coordinates $(\boldsymbol{q}^a, \boldsymbol{p}_a)$ vanishes:*

$$\{H_\alpha, H_\beta\}|_a = \sum_{j_a=1}^{\sigma_a} \left(\frac{\partial H_\alpha}{\partial q^{a,j_a}} \frac{\partial H_\beta}{\partial p_{a,ja}} - \frac{\partial H_\alpha}{\partial p_{a,ja}} \frac{\partial H_\beta}{\partial q^{a,ja}} \right) = 0\,, \qquad a = 1, \dots, v\,. \tag{5.9}$$

In particular, if each addend in the above sums vanishes, we recover Benenti's case of $\mathcal{T}$-involution.

The previous definition is motivated by the following

Theorem 5.1.2 (Theorem 5, [160]). *Given a Hamiltonian function H, a system of Darboux coordinates $(\boldsymbol{q}^1, \boldsymbol{p}_1, \dots, \boldsymbol{q}^v, \boldsymbol{p}_v)$ partially separates the complete integral W (5.2) if and only if there exists a family of independent functions $(H_1, \dots, H_n)$ which are vertically independent (i.e., they fulfill Eq. (1.46)) and in $\mathcal{P}$-involution with each other and with H.*

Proof. As is well known, the existence of a complete integral W of the HJ equation of H is equivalent to the existence of a family of n independent functions $(H_1, \dots, H_n)$, vertically independent and in involution with each other and with H. We will show that W is additively separated if and only if the functions $(H_1, \dots, H_n)$ are in $\mathcal{P}$-involution. By differentiating Eqs. (1.45) with respect to the coordinates $(q^1, \dots, q^n)$ and taking into account Eqs. (1.44), one gets

$$\frac{\partial H_\alpha}{\partial q_j} + \sum_{k=1}^{n} \frac{\partial H_\alpha}{\partial p_k} \frac{\partial^2 W}{\partial q^j \partial q^k} = 0, \qquad \alpha, j = 1, \dots, n\,. \tag{5.10}$$

We shall denote by $\Gamma_{jk} = \frac{\partial^2 W}{\partial q^j \partial q^k}$ the elements of the Hessian matrix of W. Let us write down the *partial* Poisson brackets of two functions (H_α, H_β) in matrix form:

$$\begin{aligned} \{H_\alpha, H_\beta\}|_a &= \left\langle \frac{\partial H_\alpha}{\partial \boldsymbol{q}^a}, \frac{\partial H_\beta}{\partial \boldsymbol{p}_a} \right\rangle - \left\langle \frac{\partial H_\beta}{\partial \boldsymbol{q}^a}, \frac{\partial H_\alpha}{\partial \boldsymbol{p}_a} \right\rangle \\ &= \left\langle \frac{\partial H_\alpha}{\partial \boldsymbol{q}} \middle| \boldsymbol{\Pi}_a \middle| \frac{\partial H_\beta}{\partial \boldsymbol{p}} \right\rangle - \left\langle \frac{\partial H_\beta}{\partial \boldsymbol{q}} \middle| \boldsymbol{\Pi}_a \middle| \frac{\partial H_\alpha}{\partial \boldsymbol{p}} \right\rangle \end{aligned} \tag{5.11}$$

where $\boldsymbol{\Pi}_a$ are the $n \times n$ block-diagonal matrices given by

$$\boldsymbol{\Pi}_a = \operatorname{diag}\left[\mathbf{0}, \dots, \boldsymbol{I}_{\sigma_a}, \dots, \mathbf{0}\right] \tag{5.12}$$

with v blocks $\sigma_a \times \sigma_a$. By solving Eq. (5.10) for $\frac{\partial H_\alpha}{\partial \boldsymbol{q}^a}$ and substituting it into Eq. (8.14), thanks to the symmetry of $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}_a$, we get

$$\begin{aligned} \{H_\alpha, H_\beta\}|_a &= \left\langle -\frac{\partial H_\alpha}{\partial \boldsymbol{p}} \middle| \boldsymbol{\Gamma}\, \boldsymbol{\Pi}_a \middle| \frac{\partial H_\beta}{\partial \boldsymbol{p}} \right\rangle + \left\langle \frac{\partial H_\beta}{\partial \boldsymbol{p}} \middle| \boldsymbol{\Gamma}\, \boldsymbol{\Pi}_a \middle| \frac{\partial H_\alpha}{\partial \boldsymbol{p}} \right\rangle \\ &= -\left\langle \frac{\partial H_\alpha}{\partial \boldsymbol{p}} \middle| \left[\boldsymbol{\Gamma}, \boldsymbol{\Pi}_a\right] \middle| \frac{\partial H_\beta}{\partial \boldsymbol{p}} \right\rangle \end{aligned} \tag{5.13}$$

for $a = 1, \dots, v$. For each $\alpha, \beta = 1, \dots, n$, we can collect together the partial Poisson brackets (5.13) to form an $n \times n$ matrix $\boldsymbol{P}_a$. By introducing the Jacobian matrix $\boldsymbol{\mathcal{J}}$ of the functions $(H_1, \dots, H_n)$ with respect to the momenta $(p_1, \dots, p_n)$, namely, $[\boldsymbol{\mathcal{J}}]_{\alpha k} := \frac{\partial H_\alpha}{\partial p_k}$, we can rewrite Eqs. (5.13) in the following matrix form

$$\boldsymbol{P}_a = -\boldsymbol{\mathcal{J}} \left[\boldsymbol{\Gamma}, \boldsymbol{\Pi}_a \right] \boldsymbol{\mathcal{J}}^T . \tag{5.14}$$

Thus, the condition of $\mathcal{P}$-involution with respect to the set of coordinates $\boldsymbol{q}^a$ is equivalent to the vanishing of the matrix function $\boldsymbol{P}_a$. In turn, due to the vertical independence (1.46) of the integrals of motion $(H_1, \dots, H_n)$, such a matrix vanishes if and only if $\boldsymbol{\Gamma}$ commutes with $\boldsymbol{\Pi}_a$, that is

$$\left[\boldsymbol{\Gamma}, \boldsymbol{\Pi}_a \right] = \mathbf{0}_n, \qquad a = 1, \dots, v . \tag{5.15}$$

Taking into account that: (i) the commutator in the left-hand side of Eq. (5.15) has possibly non-vanishing elements only in the rows $a, 1; \dots; a, \sigma_a$ and columns $a, 1; \dots; a, \sigma_a$ (each of them containing n elements), with a null $\sigma_a \times \sigma_a$ square block defined by the intersection of these rows and columns, and (ii) Eq. (5.15) must hold for $a = 1, \dots, v$, we can conclude that the condition of $\mathcal{P}$-involution is equivalent to the fact that the Hessian matrix $\boldsymbol{\Gamma}$ has the block-diagonal structure

$$\boldsymbol{\Gamma} = \operatorname{diag} \left[\boldsymbol{\Gamma}_1, \dots, \boldsymbol{\Gamma}_v \right] , \tag{5.16}$$

where $\boldsymbol{\Gamma}_a$ are symmetric blocks $\sigma_a \times \sigma_a$, $a = 1, \dots, v$. In turn, this is equivalent to the fact that W is partially separated in v addends, as in Eq. (5.2).

Therefore, by substituting W into the HJ equations associated with H, we obtain the partial separation of this equation. □

In [99] we proposed the following result, which completes and generalizes the statement of Proposition 4.1.1 on *multiseparable Hamiltonian systems*.

Proposition 5.1.1. *An integrable Hamiltonian system possesses as many inequivalent separation coordinate systems as the number of independent semisimple $\omega\mathcal{H}$ manifolds of class n that it admits. Also, for each semisimple $\omega\mathcal{H}$ manifold of class $r < n$, there exists an associated set of coordinates assuring partial separation of the HJ equations admitted by the Hamiltonian system.*

Proof. This result is a direct consequence of Corollary 3.3.2 and of Theorem 5.1.2. □

5.2 NON-SEMISIMPLE HAANTJES ALGEBRAS AND PARTIAL SEPARABILITY

5.2.1 *Darboux-Haantjes coordinates and partial separability*

A crucial step for the application of our geometric theory of partial separability is the construction of suitable sets of Darboux-Haantjes coordinates. In [185], we proved that, given an integrable Hamiltonian system, if we associate it with an Abelian, *semisimple* Haantjes algebra of maximal

rank, then we can totally separate the HJ equations in the Darboux-Haantjes coordinates diagonalizing the algebra. The results of this section offer a new, geometric interpretation of the *non-semisimple* scenario (Case **I** of Definition 5.1.1). Precisely, if an integrable system admits an Abelian, non-semisimple Haantjes algebra, then the DH variables ensuring block-diagonalization of the operators of the algebra also allow us to partially separate the related HJ equations.

Theorem 5.2.1 ($\mathcal{P}$-involution)**.** *Let M be an Abelian $2n$-dimensional (possibly non-semisimple) $\omega\mathscr{H}$ manifold of class $m \leq n$ and $(H_1, ..., H_m)$ be a family of $C^\infty(M)$ functions belonging to a Haantjes chain generated by a function H, via a basis of Haantjes operators $\{\boldsymbol{K}_1, ..., \boldsymbol{K}_m\} \in \mathscr{H}$. Then, the m functions $(H_1, ..., H_m)$ are in $\mathcal{P}$-involution with respect to any set $(\boldsymbol{q}, \boldsymbol{p})$ of DH coordinates.*

Proof. Let $\{U, (q^{a,j_a}, p_{a,j_a})\}$, $a = 1, ..., \nu$, $j_a = 1, ..., \sigma_a = \frac{1}{2}\operatorname{rank}(\mathcal{V}_a)$ be a Darboux-Haantjes chart adapted to the spectral decomposition (2.54), where

$$\mathcal{V}_a(\boldsymbol{x}) := \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \cap ... \cap \mathcal{D}^{(m)}_{i_m}(\boldsymbol{x}) \equiv \operatorname{span}\left\{\frac{\partial}{\partial q^{a,j_a}}, \frac{\partial}{\partial p_{a,j_a}}\right\}. \tag{5.17}$$

We recall that

$$\mathcal{D}^{(\alpha)}_{i_\alpha}(\boldsymbol{x}) := \ker(\boldsymbol{K}_\alpha - \lambda^{(\alpha)}_{i_\alpha}\boldsymbol{I})^{\rho^{(\alpha)}_{i_\alpha}}(\boldsymbol{x}), \qquad \alpha = 1, ..., m, \quad i_\alpha = 1, ..., s_\alpha, \tag{5.18}$$

where $\rho^{(\alpha)}_{i_\alpha}$ is the Riesz index of the eigenvalue $\lambda^{(\alpha)}_{i_\alpha}$ and s_α is the number of distinct eigenvalues of $\boldsymbol{K}_\alpha$. Due to Eq. (3.46), the corresponding chain equations $\mathrm{d}H_\alpha = \boldsymbol{K}^T_\alpha \mathrm{d}H$ are

$$\frac{\partial H_\alpha}{\partial q^{a,j_a}} = \sum_{k_a=1}^{\sigma_a}\left(\frac{\partial H}{\partial q^{a,k_a}}\left(\boldsymbol{A}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,k_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,k_a}}\left(\boldsymbol{C}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,k_a}_{a,j_a}\right), \tag{5.19}$$

$$\frac{\partial H_\alpha}{\partial p_{a,j_a}} = \sum_{k_a=1}^{\sigma_a}\left(\frac{\partial H}{\partial q^{a,k_a}}\left(\boldsymbol{B}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,k_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,k_a}}\left((\boldsymbol{A}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p}))^T\right)^{a,k_a}_{a,j_a}\right). \tag{5.20}$$

Therefore, the partial brackets involved in Benenti's theorem read

$$\begin{aligned}\{H_\alpha, H_\beta\}\big|_{a,j_a} = &\sum_{i_a,k_a=1}^{\sigma_a}\left(\frac{\partial H}{\partial q^{a,i_a}}\left(\boldsymbol{A}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,i_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,i_a}}\left(\boldsymbol{C}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,i_a}_{a,j_a}\right)\\ &\left(\frac{\partial H}{\partial q^{a,k_a}}\left(\boldsymbol{B}^{(\beta)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,k_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,k_a}}\left((\boldsymbol{A}^{(\beta)}_a(\boldsymbol{q},\boldsymbol{p}))^T\right)^{a,k_a}_{a,j_a}\right)\\ &- \sum_{i_a,k_a=1}^{\sigma_a}\left(\frac{\partial H}{\partial q^{a,k_a}}\left(\boldsymbol{B}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,k_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,k_a}}\left((\boldsymbol{A}^{(\alpha)}_a(\boldsymbol{q},\boldsymbol{p}))^T\right)^{a,k_a}_{a,j_a}\right)\\ &\left(\frac{\partial H}{\partial q^{a,i_a}}\left(\boldsymbol{A}^{(\beta)}_a(\boldsymbol{q},\boldsymbol{p})\right)^{a,i_a}_{a,j_a} + \frac{\partial H}{\partial p_{a,i_a}}\left((\boldsymbol{C}^{(\beta)}_a(\boldsymbol{q},\boldsymbol{p}))^T\right)^{a,i_a}_{a,j_a}\right).\end{aligned} \tag{5.21}$$

After a long but straightforward calculation (reported in Appendix B), we get

$$
\begin{aligned}
\{H_\alpha, H_\beta\}\big|_a = \\
\sum_{i_a,k_a=1}^{\sigma_a} \Bigg(& \frac{\partial H}{\partial q^{a,i_a}} \frac{\partial H}{\partial p_{a,k_a}} \left(\left[\boldsymbol{A}_a^{(\alpha)}, \boldsymbol{A}_a^{(\beta)} \right]_{k_a}^{i_a} - \left(\boldsymbol{B}_a^{(\alpha)} (\boldsymbol{C}_a^{(\beta)})^T - \boldsymbol{B}_a^{(\beta)} (\boldsymbol{C}_a^{(\alpha)})^T \right)_{k_a}^{i_a} \right) \\
& + \frac{\partial H}{\partial q^{a,i_a}} \frac{\partial H}{\partial q^{a,k_a}} \left(\boldsymbol{A}_a^{(\alpha)} (\boldsymbol{B}_a^{(\beta)})^T - \boldsymbol{A}_a^{(\beta)} (\boldsymbol{B}_a^{(\alpha)})^T \right)_{k_a}^{i_a} \\
& + \frac{\partial H}{\partial p_{a,i_a}} \frac{\partial H}{\partial p_{a,k_a}} \left(\boldsymbol{C}_a^{(\alpha)} \boldsymbol{A}_a^{(\beta)} - \boldsymbol{C}_a^{(\beta)} \boldsymbol{A}_a^{(\alpha)} \right)_{k_a}^{i_a} \Bigg) . \qquad (5.22)
\end{aligned}
$$

From the compatibility conditions (3.47) with $\boldsymbol{\Omega}$, and Eqs. (3.48), it follows that

$$
\frac{\partial H}{\partial q^{a,i_a}} \frac{\partial H}{\partial p_{a,k_a}} \left(\left[\boldsymbol{A}_a^{(\alpha)}, \boldsymbol{A}_a^{(\beta)} \right]_{k_a}^{i_a} - \left(\boldsymbol{B}_a^{(\alpha)} (\boldsymbol{C}_a^{(\beta)})^T - \boldsymbol{B}_a^{(\beta)} (\boldsymbol{C}_a^{(\alpha)})^T \right)_{k_a}^{i_a} \right) = 0 .
$$

Moreover, from Lemma 3.3.2 we deduce that each of the two remaining addends also vanish. This completes the proof. □

In the case of a semisimple $\omega\mathscr{H}$ manifold, we can deduce a direct consequence of the previous result.

Corollary 5.2.1. *If the Haantjes algebra $\mathscr{H}$ is semisimple, then the m functions $(H_1, \dots, H_m)$ are in $\mathcal{T}$-involution with respect to any set $(\boldsymbol{q}, \boldsymbol{p})$ of DH coordinates, that is*

$$
\{H_\alpha, H_\beta\}\big|_{a,j_a} = 0 , \qquad a = 1, \dots, \nu , \qquad j_a = 1, \dots, \sigma_a . \qquad (5.23)
$$

Proof. As the algebra $\mathscr{H}$ is semisimple, the blocks $\boldsymbol{B}_a^{(\alpha)}$ and $\boldsymbol{C}_a^{(\alpha)}$ vanish whereas the blocks $\boldsymbol{A}_a^{(\alpha)}$ are all diagonal and have the eigenvalues $\lambda_a^{(\alpha)}$ of $\boldsymbol{K}_\alpha$ as diagonal elements, as can be ascertained from Eq. (5.21). □

The converse of this corollary holds as well, as we prove in the following

Theorem 5.2.2 (Jacobi-Haantjes, [160])**.** *Let (M, ω) be a symplectic manifold and $(H_1, \dots, H_m)$ be $m \leq n$ vertically independent $C^\infty(M)$ functions which are in $\mathcal{T}$-involution w.r.t. a set of Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$. Then, these functions belong to the (short) Haantjes chains generated by the Haantjes operators*

$$
\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right), \qquad \alpha = 1, \dots, m , \qquad (5.24)
$$

where H is any function depending on $(H_1, \dots, H_m)$ and fulfilling $\frac{\partial H}{\partial p_i} \neq 0$, $i = 1, \dots, n$. These operators generate a semisimple $\omega\mathscr{H}$ structure on M of rank m.

Proof. The proof proceeds, exactly, as the proof of Theorem 3.4.2 by simply interchanging m with n. □

5.3 PARTIAL SEPARABILITY AND GENERALIZED STÄCKEL SYSTEMS

The theory of partial separability can also be formulated in the case of Hamiltonian systems admitting a priori less than n integrals of motion [39, 44, 71, 147, 150, 178] (Case **II** of Definition 5.1.1). We shall present here a novel formulation of this theory in the context of Haantjes geometry, which widely extends the approach we presented in [192]. More precisely, in that article, we have determined the semisimple symplectic-Haantjes manifolds associated with the AKN systems (1.58). In essence, they are families of n Hamiltonians in involution, which are generated from a standard Stäckel matrix $\boldsymbol{S}$, but keeping arbitrary the dependence of the functions f_k on the momentum p_k. By specializing the Stäckel matrix and vector from Section 1.4.1 for the AKN systems conveniently, we have also derived directly both families of quasi-bi-Hamiltonian and classical Stäckel systems. In other words, Haantjes geometry allows us to *totally* separate the AKN systems using the construction of related DH coordinates.

We shall now extend these results to a much more general family of Hamiltonian models, which are *partially separable*.

5.3.1 *A new family of generalized Stäckel systems*

In order to introduce the class of systems we are interested, we shall first review the notion of generalized Stäckel matrix, defined in 1897 by Stäckel in [178] and by Painlevé in [147].

Definition 5.3.1. *Let $(q^1, \dots, q^n)$ be a coordinate chart, and $\sigma_1, \dots, \sigma_m \in \mathbb{N}\backslash\{0\}$ be a set of integers such that $\sigma_1 + \dots + \sigma_m = n$. An invertible $m \times m$ matrix of the form*

$$\boldsymbol{S} = \begin{bmatrix} S_{11}(\boldsymbol{q}^1) & \cdots & S_{1m}(\boldsymbol{q}^1) \\ S_{21}(\boldsymbol{q}^2) & \cdots & S_{2m}(\boldsymbol{q}^2) \\ \vdots & \ddots & \vdots \\ S_{m1}(\boldsymbol{q}^m) & \cdots & S_{mm}(\boldsymbol{q}^m) \end{bmatrix} \tag{5.25}$$

where $\boldsymbol{q}^1 = (q^{1,1}, \dots, q^{1,\sigma_1}), \dots, \boldsymbol{q}^m = (q^{m,1}, \dots, q^{m,\sigma_m})$, is said to be a generalized Stäckel matrix.

Inspired by the previous definition, we propose a novel family of Hamiltonian models.

Definition 5.3.2. *Let (M, ω) be a symplectic manifold and (q^i, p_i) a set of Darboux coordinates. We introduce the class of generalized Stäckel Hamiltonians defined by*

$$\begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix} = \boldsymbol{S}^{-1} \begin{bmatrix} f_1(\boldsymbol{q}^1, \boldsymbol{p}_1) \\ \vdots \\ f_m(\boldsymbol{q}^m, \boldsymbol{p}_m) \end{bmatrix}, \tag{5.26}$$

where $(H_1, \dots, H_m)$ *are* m *functions on* M, $\boldsymbol{q}^1 = (q^{1,1}, \dots, q^{1,\sigma_1})$, $\boldsymbol{p}_1 = (p_{1,1}, \dots, p_{1,\sigma_1}), \dots,$ $\boldsymbol{q}^m = (q^{m,1}, \dots, q^{m,\sigma_m})$, $\boldsymbol{p}_m = (p_{m,1}, \dots, p_{m,\sigma_m})$ *and* $\boldsymbol{S}$ *is a generalized* $m \times m$ *Stäckel matrix, whose* a*th row depends on the variables* $(\boldsymbol{q}^a)$ *only. Also,*

$$|\boldsymbol{F}\rangle := \begin{bmatrix} f_1(\boldsymbol{q}^1, \boldsymbol{p}_1) \\ \vdots \\ f_m(\boldsymbol{q}^m, \boldsymbol{p}_m) \end{bmatrix} \tag{5.27}$$

will be said to be a generalized Stäckel vector.

Remark 5.3.1. *The family of Hamiltonian systems given by generalized Stäckel Hamiltonians* (5.26) *extend significantly the family of known systems obtainable by the Stäckel geometry. The AKN systems are realized when* $m = n$ *(i.e.* $\sigma_1 = \dots = \sigma_n = 1$*), namely when* $\boldsymbol{S}$ *is a standard Stäckel matrix (see Section 1.4.1). In addition, if we impose that the functions* f_k *depend just on a pair* (q^k, p_k) *of canonical coordinates and that their dependence on* p_k *is* quadratic*, then we get back the original Stäckel systems [148, 177]. The models introduced by Di Pirro [150] belong to the class* (5.26) *if* $m = n - 1$ *and* f_k *are quadratic in the momentum* p_k*. The more general systems considered first in [178] and then in [71], [39] and [44], related to the Painlevé metrics [147], are obtained from the class* (5.26) *when* $\boldsymbol{S}$ *is of the form* (5.25) *with* $m \leq n - 1$*, and the Hamiltonians* $(H_1, \dots, H_m)$ *are still quadratic in the momenta.*

5.3.2 *Partial separability and semisimple Haantjes manifolds*

In this section, we prove the existence of $\omega\mathscr{H}$ manifolds associated with partially separable Hamiltonian systems. We start with the following

Theorem 5.3.1 ($\mathcal{T}$-Involution)**.** *Let* $(H_1, \dots, H_m)$ *be* m *generalized Stäckel Hamiltonians. Then, any pair of functions* (H_a, H_b) *are in* $\mathcal{T}$*-involution, that is*

$$\{H_a, H_b\}\big|_{c,j_c} = 0, \qquad c = 1, \dots, m, \quad j_c = 1, \dots, \sigma_c\,. \tag{5.28}$$

Proof. Observe that $\frac{\partial f_c}{\partial q^{d,j_d}} = 0$ if $c \neq d$. Let us denote by S^{-1}_{ij} the element in the i-th row and the j-th column of $\boldsymbol{S}^{-1}$. We obtain

$$\{H_a, H_b\}\big|_{c,j_c} = \sum_{d=1}^{m} \left(\frac{\partial S^{-1}_{ad}}{\partial q^{c,j_c}} S^{-1}_{bc} - \frac{\partial S^{-1}_{bd}}{\partial q^{c,jc}} S^{-1}_{ac} \right) f_d \frac{\partial f_c}{\partial p_{c,j_c}}. \tag{5.29}$$

Taking into account the identity

$$\partial_x(\boldsymbol{S}^{-1}) = -\boldsymbol{S}^{-1} \partial_x(\boldsymbol{S}) \boldsymbol{S}^{-1}\,, \tag{5.30}$$

where x is any variable on which $\boldsymbol{S}$ depends, we deduce

$$\frac{\partial S^{-1}_{ad}}{\partial q^{c,j_c}} = -S^{-1}_{ac} \left(\sum_{k=1}^{m} \frac{\partial S_{ck}}{\partial q^{c,j_c}} S^{-1}_{kd} \right). \tag{5.31}$$

This relation implies

$$\frac{\partial S^{-1}_{ad}}{\partial q^{c,j_c}} S^{-1}_{bc} = -S^{-1}_{ac}\left(\sum_{k=1}^{m} \frac{\partial S_{ck}}{\partial q^{c,j_c}} S^{-1}_{kd}\right) S^{-1}_{bc} \,. \tag{5.32}$$

It is immediate to observe that the latter identity is symmetric with respect to the interchange of a with b. Therefore, each of the terms in the sum of Eq. (5.29) vanishes. □

Remark 5.3.2. *Interestingly enough, Theorem 5.3.1 holds even in the case when each a-th row of the Stäckel matrix $\boldsymbol{S}$ may depend also on the variables $(\boldsymbol{q}^a, \boldsymbol{p}_a)$, as well as the a-th row of the generalized Stäckel vector $|\boldsymbol{F}\rangle$ (we omit the proof of this observation for simplicity). In this more general scenario, both the elements $S_{ij}(\boldsymbol{q}^a, \boldsymbol{p}_a)$ and the functions $f_a(\boldsymbol{q}^a, \boldsymbol{p}_a)$ will be called* generalized Stäckel functions *as they are in $\mathcal{P}$-involution and are characteristic functions of the Haantjes web, by analogy with the standard case already discussed in [185, 192].*

We can now prove one of our main results.

Theorem 5.3.2. *Let (M, ω) be a symplectic manifold and $(H_1, \ldots, H_m)$ be m independent, $C^\infty(M)$ functions of the form (5.26). Then, they belong to the Lenard–Haantjes chain*

$$\boldsymbol{K}_\alpha^T \mathrm{d}H_1 = \mathrm{d}H_\alpha, \qquad \alpha = 1, \ldots, m \,, \tag{5.33}$$

where $\boldsymbol{K}_\alpha$ are the Haantjes operators defined by

$$\boldsymbol{K}_\alpha := \sum_{i=1}^{m} \frac{\tilde{S}_{\alpha i}}{\tilde{S}_{1i}} \sum_{k=1}^{\sigma_i} \left(\frac{\partial}{\partial q^{i,k}} \otimes \mathrm{d}q^{i,k} + \frac{\partial}{\partial p_{i,k}} \otimes \mathrm{d}p_{i,k}\right), \qquad \alpha = 1, \ldots, m \,. \tag{5.34}$$

They generate a semisimple $\omega\mathscr{H}$ manifold of class m.

Proof. The statement is a consequence of Theorem 5.2.2. In particular, the Haantjes operators (5.34) are a realization of the operators (5.24) for the case of the generalized Stäckel systems defined by the Hamiltonians (5.26). In fact, it is sufficient to observe that from Eq. (5.26) we have

$$\frac{\frac{\partial H_\alpha}{\partial p_{a,k_a}}}{\frac{\partial H_1}{\partial p_{a,k_a}}} = \frac{\tilde{S}_{\alpha a}}{\tilde{S}_{1a}} \qquad k_a = 1, \ldots, \sigma_a \,. \tag{5.35}$$

□

5.3.3 *Generalized Stäckel systems and Hamilton-Jacobi equations*

We now study the problem of partial separability of the HJ equations for the generalized Stäckel systems (5.26). The general form of the time-independent HJ equation is

$$H\left(\boldsymbol{q}^1, \ldots, \boldsymbol{q}^m, \boldsymbol{p}_1 = \frac{\partial W}{\partial \boldsymbol{q}^1}, \ldots, \boldsymbol{p}_m = \frac{\partial W}{\partial \boldsymbol{q}^m}\right) = h, \tag{5.36}$$

where h is an arbitrary constant. Moreover, the m GSE (1.55) become

$$\begin{bmatrix} f_1(\boldsymbol{q}^1, \boldsymbol{p}_1) \\ \vdots \\ f_m(\boldsymbol{q}^m, \boldsymbol{p}_m) \end{bmatrix} - \boldsymbol{S} \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{5.37}$$

They allow to split the HJ equations into the system

$$f_a \left(\boldsymbol{q}^a, \frac{\partial W}{\partial \boldsymbol{q}^a} \right) - \langle \boldsymbol{S}_a \mid \boldsymbol{h} \rangle = 0 \qquad\qquad a = 1, \dots, m \,, \tag{5.38}$$

where $\boldsymbol{S}_a$ is the a-th row of the Stäckel matrix $\boldsymbol{S}$, and $\boldsymbol{h} = [h_1, \dots, h_m]^T$. Notice that in this context the number of subsets of Darboux coordinates, previously denoted by v, coincides with the number m of Hamiltonian functions of the system.

At this stage, one is left with m independent PDEs to solve. This task may be a nontrivial one. However, one can study these PDEs using several techniques, including numerical methods or Lie symmetry analysis, to solve or further reduce them.

5.3.3.1 *Systems partially separable in (2,1,1) coordinates*

For the sake of concreteness, we illustrate a realization of the class (5.26). Let us consider the system

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \boldsymbol{S}^{-1} \begin{bmatrix} f_1(q^1, p_1, q^2, p_2) \\ f_2(q^3, p_3) \\ f_3(q^4, p_4) \end{bmatrix} , \tag{5.39}$$

where f_1, f_2 and f_3 are arbitrary functions and the generalized Stäckel matrix takes the form

$$\boldsymbol{S} = \begin{bmatrix} S_{11}(q^1, q^2) & S_{12}(q^1, q^2) & S_{13}(q^1, q^2) \\ S_{21}(q^3) & S_{22}(q^3) & S_{23}(q^3) \\ S_{31}(q^4) & S_{32}(q^4) & S_{33}(q^4) \end{bmatrix} . \tag{5.40}$$

The functions S_{ij} are arbitrary, except for the condition that $\boldsymbol{S}$ must be invertible. Thus, from Eq. (5.39) we obtain a huge family of Hamiltonian systems (H_1, H_2, H_3) in involution. It explicitly reads

$$H_1 = \frac{1}{\det \boldsymbol{S}} \left[(S_{22}S_{33} - S_{23}S_{32}) f_1 + (S_{13}S_{32} - S_{12}S_{33}) f_2 + (S_{12}S_{23} - S_{13}S_{22}) f_3 \right] , \tag{5.41}$$

$$H_2 = \frac{1}{\det \boldsymbol{S}} \left[(S_{23}S_{31} - S_{21}S_{33}) f_1 + (S_{11}S_{33} - S_{13}S_{31}) f_2 + (S_{13}S_{21} - S_{11}S_{23}) f_3 \right] , \tag{5.42}$$

$$H_3 = \frac{1}{\det \boldsymbol{S}} \left[(S_{21}S_{32} - S_{22}S_{31}) f_1 + (S_{12}S_{31} - S_{11}S_{32}) f_2 + (S_{11}S_{22} - S_{12}S_{21}) f_3 \right] .$$

(5.43)

Each of these Hamiltonian systems is partially separable in the original coordinates and satisfies Benenti's $\mathcal{T}$-separability conditions. From relations (5.40) we obtain the set of three associated HJ equations:

$$\begin{cases} f_1\left(q^1, \frac{\partial W}{\partial q^1}, q^2, \frac{\partial W}{\partial q^2}\right) - S_{11}h_1 - S_{12}h_2 - S_{13}h_3 = 0 \\ f_2\left(q^3, \frac{\partial W}{\partial q^3}\right) - S_{21}h_1 - S_{22}h_2 - S_{23}h_3 = 0 \\ f_3\left(q^4, \frac{\partial W}{\partial q^4}\right) - S_{31}h_1 - S_{32}h_2 - S_{33}h_3 = 0. \end{cases} \tag{5.44}$$

A Hamiltonian system representing a realization of formula (5.39) is given in Section 5.4.2.

5.4 EXAMPLES OF PARTIALLY SEPARABLE HAMILTONIAN MODELS

As proved in the previous sections, the construction of sets of partially or totally separated variables for a given Hamiltonian model is directly related to the determination of the $\omega\mathscr{H}$ manifolds associated with the model. To construct these manifolds, one can adopt the procedure detailed in Section 3.4.4. In the following, we will discuss some examples of its application, leading to the partial separability of the systems studied.

5.4.1 *The Jacobi-Calogero three-particle system (Case **I**)*

As was pointed out in [157, 185], a Hamiltonian system can support multiple distinct $\omega\mathscr{H}$ structures. A key feature lies in the coexistence of semisimple and non-semisimple Haantjes operators: while semisimple structures enable full separability in certain charts, non-semisimple ones provide a framework for partial separability in others. This interplay demonstrates that non-semisimple operators are not secondary tools but intrinsic geometric objects, essential in cases where semisimple algebras alone cannot account for a system's separable behavior. The theory's richness stems from this duality, where both operator classes–each with distinct spectral properties–contribute uniquely to the global geometry of integrable systems.

To illustrate this point and to consider systems falling into Case **I** of Definition 5.1.1, we shall consider the three-particle Jacobi-Calogero model, with the Hamiltonian function

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + V_{\text{Cal}}, \tag{5.45}$$

where

$$V_{\text{Cal}} = \frac{1}{(q^1 - q^2)^2} + \frac{1}{(q^1 - q^3)^2} + \frac{1}{(q^2 - q^3)^2}. \tag{5.46}$$

This system is both superintegrable and multiseparable, and it was previously discussed in [192] from the perspective of the symplectic-Haantjes theory. In particular, Haantjes algebras associated

with the cylindrical, spherical, and parabolic separation coordinates were obtained. Nevertheless, the existence of an $\omega\mathscr{H}$ manifold associated with the algebra of integrals $\{H, I_1, I_3\}$, where

$$\begin{aligned} I_1 &= p_1 + p_2 + p_3, \\ I_3 &= \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + \frac{p_1 + p_2}{(q^1 - q^2)^2} + \frac{p_1 + p_3}{(q^1 - q^3)^2} + \frac{p_2 + p_3}{(q^2 - q^3)^2}, \end{aligned} \tag{5.47}$$

remained as an open problem. We address here and solve this problem by showing that there exists an Abelian, non-semisimple Haantjes algebra associated with the family $\{H, I_1, I_3\}$.

In order to solve the chain equations $\boldsymbol{L}_i^T \mathrm{d}H = \mathrm{d}I_i$, $i = 1, 3$, we start from the most general operator compatible with the symplectic form (see Eq. (A.3)). Then, we need to fix the arbitrary functions to satisfy the chain equations and ensure a resulting tensor with vanishing Haantjes torsion.

The algebraic chain equation $\boldsymbol{L}_1^T \mathrm{d}H = \mathrm{d}I_1$ can be solved by choosing for instance the operator

$$\boldsymbol{L}_1 = \frac{1}{p_1 + p_2 + p_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right], \tag{5.48}$$

whose Nijenhuis (and consequently, Haantjes) torsion trivially vanishes. It is semisimple, its minimal polynomial is of second degree,

$$m_{\boldsymbol{L}_1}(\{\boldsymbol{q}, \boldsymbol{p}\}, \lambda) = \lambda \left(\lambda - \frac{3}{p_1 + p_2 + p_3}\right), \tag{5.49}$$

and its generalized eigenvalues and eigendistributions are

$$\begin{aligned} &\lambda_1 = \frac{3}{p_1 + p_2 + p_3},\ \rho_1 = 1, \qquad \mathcal{D}_1 = \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right\rangle, \\ &\lambda_2 = 0,\ \rho_2 = 1, \qquad \mathcal{D}_2 = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3} \right\rangle. \end{aligned} \tag{5.50}$$

Similarly, the algebraic chain equation $\boldsymbol{L}_3^T \mathrm{d}H = \mathrm{d}I_3$ can be solved by choosing the Haantjes operator

$$\boldsymbol{L}_3 = \frac{1}{2p_1 - p_2 - p_3} \left[\begin{array}{ccc|ccc} f_1 & f_2 & f_2 & 0 & 0 & 0 \\ f_3 & f_4 & f_5 & 0 & 0 & 0 \\ f_6 & f_7 & f_8 & 0 & 0 & 0 \\ \hline 0 & f_9 & -f_9 & f_1 & f_3 & f_6 \\ -f_9 & 0 & f_9 & f_2 & f_4 & f_7 \\ f_9 & -f_9 & 0 & f_2 & f_5 & f_8 \end{array}\right], \tag{5.51}$$

where

$$
\begin{aligned}
f_1 &= \frac{1}{p_1+p_2+p_3}\left[2p_1\left(p_1^2+\frac{1}{(q^1-q^2)^2}+\frac{1}{(q^1-q^3)^2}\right)\right.\\
&\qquad\left.+(p_2+p_3)\left(p_1^2-p_2^2-p_3^2-\frac{2}{(q^2-q^3)^2}\right)\right],\\
f_2 &= \frac{1}{p_2+p_3}\left[\left(p_1^2+\frac{1}{(q^1-q^2)^2}+\frac{1}{(q^1-q^3)^2}\right)(2p_1-p_2-p_3)-p_1 f_1\right],\\
f_3 &= \left(\frac{3(p_1-p_2)}{2p_1-p_2-p_3}-\frac{p_2}{p_2+p_3}\right)f_1+2\left(p_2^2+\frac{1}{(q^1-q^2)^2}+\frac{1}{(q^2-q^3)^2}\right)\\
&\qquad-\frac{p_2+3p_3}{p_2+p_3}\left(p_1^2+\frac{1}{(q^1-q^2)^2}+\frac{1}{(q^1-q^3)^2}\right),\\
f_9 &= \frac{2}{p_2-p_3}\left[(2p_1-p_2-p_3)\left(\frac{p_1+p_2}{(q^1-q^2)^3}+\frac{p_1+p_3}{(q^1-q^3)^3}\right)\right.\\
&\qquad+2\left(\frac{1}{(q^1-q^2)^3}-\frac{1}{(q^1-q^3)^3}-\frac{2}{(q^2-q^3)^3}\right)\\
&\qquad\qquad\cdot\left(p_2^2+\frac{1}{(q^1-q^2)^2}+\frac{1}{(q^2-q^3)^2}\right)\\
&\qquad-\left(\frac{2p_2}{2p_1-p_2-p_3}\left(\frac{1}{(q^1-q^2)^3}-\frac{1}{(q^1-q^3)^3}-\frac{2}{(q^2-q^3)^3}\right)\right.\\
&\qquad\qquad\left.+\frac{1}{(q^1-q^2)^3}+\frac{1}{(q^1-q^3)^3}\right)f_1\\
&\qquad+\left(\frac{p_2+3p_3}{2p_1-p_2-p_3}\left(-\frac{1}{(q^1-q^2)^3}+\frac{1}{(q^1-q^3)^3}+\frac{2}{(q^2-q^3)^3}\right)\right.\\
&\qquad\qquad\left.\left.+\frac{2}{(q^1-q^3)^3}+\frac{2}{(q^2-q^3)^3}\right)f_2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
f_4 &= f_1+\frac{1}{2}(f_2-f_3),\\
f_5 &= \frac{1}{2}(3f_2-f_3),\\
f_6 &= 2f_2-f_3,\\
f_7 &= \frac{1}{2}(f_2+f_3),\\
f_8 &= f_1-\frac{1}{2}(f_2-f_3).
\end{aligned}
\tag{5.52}
$$

The operator (5.51) is non-semisimple. Its minimal polynomial is of third degree and reads

$$
m_{\boldsymbol{L}_3}(\{\boldsymbol{q},\boldsymbol{p}\},\lambda)=\left(\lambda-\frac{f_1+2f_2}{2p_1-p_2-p_3}\right)\left(\lambda-\frac{f_1-f_2}{2p_1-p_2-p_3}\right)^2. \tag{5.53}
$$

Its generalized eigenvalues and eigendistributions are

$$\begin{aligned}
\lambda_3 &= \frac{f_1 + 2f_2}{2p_1 - p_2 - p_3},\ \rho_3 = 1,\\
&\qquad \mathcal{D}_3 = \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right\rangle \equiv \mathcal{D}_1,\\
\lambda_4 &= \frac{f_1 - f_2}{2p_1 - p_2 - p_3},\ \rho_4 = 2,\\
&\qquad \mathcal{D}_4 = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3} \right\rangle \equiv \mathcal{D}_2.
\end{aligned} \tag{5.54}$$

The algebra $\mathscr{H}_1 = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{L}_1, \boldsymbol{L}_3\}$ is a non-semisimple Abelian algebra of Haantjes operators of rank 3; thus, they can be simultaneously block-diagonalized in a DH chart.

As the generalized eigen-distributions coincide, their intersections are trivial and do not provide a finer block-diagonalization for the operators of this algebra. The corresponding annihilators are:

$$\begin{aligned}
(\mathcal{D}_1)^\circ &= \langle \mathrm{d}q^1 - \mathrm{d}q^2, \mathrm{d}q^1 - \mathrm{d}q^3, \mathrm{d}p_1 - \mathrm{d}p_2, \mathrm{d}p_1 - \mathrm{d}p_3 \rangle,\\
(\mathcal{D}_2)^\circ &= \langle \mathrm{d}q^1 + \mathrm{d}q^2 + \mathrm{d}q^3, \mathrm{d}p_1 + \mathrm{d}p_2 + \mathrm{d}p_3 \rangle.
\end{aligned} \tag{5.55}$$

By integrating them, we can construct a set of Haantjes coordinates. Among them, we can select those satisfying the canonical commutation relations, obtaining the following DH coordinates:

$$\begin{aligned}
&Q^1 = \frac{1}{6}\left(2\,q^1 - q^2 - q^3\right), \qquad Q^2 = \frac{1}{2}\left(q^2 - q^3\right), \qquad Q^3 = \frac{1}{3}\left(q^1 + q^2 + q^3\right),\\
&P_1 = 2\,p_1 - p_2 - p_3, \qquad P_2 = p_2 - p_3, \qquad P_3 = p_1 + p_2 + p_3.
\end{aligned} \tag{5.56}$$

Notice that these simple relations become non-trivial if we express the relation between partial and total separation variables. More precisely, the transformation relating partial and total separation coordinates $(\mathfrak{q}^1, \mathfrak{q}^2, \mathfrak{q}^3)$ is not of the form $Q^1 = F_1(\mathfrak{q}^1, \mathfrak{q}^2)$, $Q^2 = F_2(\mathfrak{q}^1, \mathfrak{q}^2)$, $Q^3 = F_3(\mathfrak{q}^3)$, which a priori guarantees partial separability (with F_1, F_2, F_3 arbitrary, invertible functions). Indeed, once written, for instance, in terms of spherical coordinates, the partial separation ones read

$$\begin{aligned}
Q^1 &= -\frac{1}{6}\, r\,(\sin\theta\,(\sin\varphi - 2\cos\varphi) + \cos\theta),\\
Q^2 &= \frac{1}{2}\, r\,(\sin\theta\sin\varphi - \cos\theta),\\
Q^3 &= \frac{1}{3}\, r\,(\sin\theta\,(\sin\varphi + \cos\varphi) + \cos\theta).
\end{aligned} \tag{5.57}$$

Similar expressions hold for the cases of cylindrical and parabolic coordinates.

Remark 5.4.1. *There is much freedom in the representation of the foliation obtained by integrating the distribution* $(\mathcal{D}_1)^\circ$. *Indeed, more generally, any choice of a transformation of coordinates of the form*

$$Q^1 = \mathfrak{f}_1(q^1 - q^2, q^1 - q^3), \quad Q^2 = \mathfrak{f}_2(q^1 - q^2, q^1 - q^3), \quad Q^3 = \mathfrak{f}_3(q^1 + q^2 + q^3), \tag{5.58}$$

would provide Haantjes coordinates for the algebra $\mathscr{H}_1$, where $\mathfrak{f}_1$, $\mathfrak{f}_2$ and $\mathfrak{f}_3$ are arbitrary functions that make the transformation invertible.

In the chart (5.56), the Hamiltonian function and the integrals of the motion read

$$
\begin{aligned}
H &= \frac{1}{12}\left(P_1^2+3P_2^2+2P_3^2\right)+\frac{1}{4(Q^2)^2}+\frac{1}{(3Q^1+Q^2)^2}+\frac{1}{(3Q^1-Q^2)^2},\\
I_1 &= P_3,\\
I_3 &= \frac{1}{108}\left[P_1^3+4P_3^3+6P_1^2P_3-9P_1P_2^2+18P_2^2P_3-\frac{9(P_1-2P_3)}{(Q^2)^2}\right.\\
&\quad\left.+18(P_1+4P_3)\left(\frac{1}{(3Q^1+Q^2)^2}+\frac{1}{(3Q^1-Q^2)^2}\right)-54P_2\left(\frac{1}{(3Q^1+Q^2)^2}-\frac{1}{(3Q^1-Q^2)^2}\right)\right];
\end{aligned}
\tag{5.59}
$$

and the operators $\boldsymbol{L}_1$ and $\boldsymbol{L}_3$ in the coordinates $(Q^1,Q^2,P_1,P_2,Q^3,P_3)$ take the *block-diagonal form*

$$
\boldsymbol{L}_1=\frac{3}{P_3}\left[\begin{array}{cccc|cc}
0&0&0&0&0&0\\
0&0&0&0&0&0\\
0&0&0&0&0&0\\
0&0&0&0&0&0\\
\hline
0&0&0&0&1&0\\
0&0&0&0&0&1
\end{array}\right]
\tag{5.60}
$$

and

$$
\boldsymbol{L}_3=\frac{1}{2(Q^2)^2\left(9(Q^1)^2-(Q^2)^2\right)^2}\left[\begin{array}{cccc|cc}
g_1&0&0&0&0&0\\
g_2&g_1&0&0&0&0\\
0&g_3&g_1&g_2&0&0\\
-g_3&0&0&g_1&0&0\\
\hline
0&0&0&0&g_4&0\\
0&0&0&0&0&g_4
\end{array}\right],
\tag{5.61}
$$

where

$$
\begin{aligned}
g_1 &= \frac{1}{3P_1}\left[(Q^2)^2(9(Q^1)^2-(Q^2)^2)^2(P_1^2-3P_2^2+4P_1P_3)\right.\\
&\qquad\left.+9(-27(Q^1)^4+18(Q^1)^2(Q^2)^2+(Q^2)^4)\right],\\
g_2 &= \frac{3}{P_1^2}\left[(Q^2)^2(9(Q^1)^2-(Q^2)^2)^2(-P_1^2+P_2^2)P_2+24Q^1(Q^2)^3P_1\right.\\
&\qquad\left.+3(27(Q^1)^4-18(Q^1)^2(Q^2)^2-(Q^2)^4)P_2\right],
\end{aligned}
$$

$$
\begin{aligned}
g_3 = &-\frac{3}{(Q^2)^3\,(9(Q^1)^2-(Q^2)^2)^3\,P_1^2}\Big[72Q^1(Q^2)^5(9(Q^1)^2-(Q^2)^2)^2(3(Q^1)^2+(Q^2)^2)P_1P_2\\
&-(Q^2)^{10}(27+(Q^2)^2(P_1^2-9P_2^2))-162(Q^1)^4(Q^2)^6(63+5(Q^2)^2(P_1^2+3P_2^2))\\
&+9(Q^1)^2(Q^2)^8(-117+(Q^2)^2(5P_1^2+3P_2^2))+1458(Q^1)^6(Q^2)^4(3+(Q^2)^2(5P_1^2+7P_2^2))\\
&+6561(Q^1)^8(9(Q^2)^2+5(Q^2)^4(-P_1^2+P_2^2)+9(Q^1)^2(-1+(Q^2)^2(P_1^2-P_2^2)))\Big],\\
g_4 = &\frac{1}{3P_3}\left[27(3(Q^1)^2+(Q^2)^2)^2+(Q^2)^2(9(Q^1)^2-(Q^2)^2)^2(P_1^2+3P_2^2+2P_3^2)\right].
\end{aligned}
$$

The HJ equations associated with this system read

$$
\begin{aligned}
&\left(\frac{\partial W_1}{\partial Q^1}\right)^2+3\left(\frac{\partial W_1}{\partial Q^2}\right)^2+\frac{3}{(Q^2)^2}+\frac{12}{(3Q^1+Q^2)^2}+\frac{12}{(3Q^1-Q^2)^2}=h,\\
&\frac{\mathrm{d}W_2}{\mathrm{d}Q^3}=h_1,\\
&\frac{\partial W_1}{\partial Q^1}\left[\left(\frac{\partial W_1}{\partial Q^2}\right)^2+\frac{1}{(Q^2)^2}-\frac{1}{2}\,\frac{1}{(3Q^1+Q^2)^2}-\frac{1}{2}\,\frac{1}{(3Q^1-Q^2)^2}-h+\frac{1}{6}\,h_1^2\right]\\
&\qquad\qquad-\frac{54\,Q^1Q^2}{(9(Q^1)^2-(Q^2)^2)^2}\frac{\partial W_1}{\partial Q^2}=h_2,
\end{aligned}
\tag{5.62}
$$

where h, h_1, h_2 are real constants. This reformulation simplifies the original system, reducing it to a coupled pair of PDEs in two variables and an auxiliary ODE. The streamlined structure now lends itself to systematic analysis–either through symmetry-based methods or computational tools–exploiting its reduced complexity to advance toward solutions.

5.4.2 *Construction of a new generalized Stäckel system (Case **II**)*

We can construct directly new families of partially separable systems by specializing the arbitrary functions involved in definitions (5.25) and (5.26). For instance, working in an eight-dimensional phase space, we can specialize directly Eqs. (5.39) and (5.40) by making the choice

$$
\boldsymbol{S}=\begin{bmatrix} q^1q^2 & 0 & 0\\ 0 & 1 & q^3\\ q^4 & 0 & 1\end{bmatrix}
\tag{5.63}
$$

for the generalized Stäckel matrix and

$$
\begin{aligned}
f_1 &= q^1q^2(p_1^2+p_2^2+q^1q^2),\\
f_2 &= p_3^2+q^3,\\
f_3 &= p_4^2+q^4,
\end{aligned}
\tag{5.64}
$$

for the elements of the Stäckel vector. In this way, we obtain the Hamiltonian system

$$\begin{aligned} H_1 &= p_1^2 + p_2^2 + q^1 q^2, \\ H_2 &= q^3 q^4 (p_1^2 + p_2^2) + p_3^2 - q^3 p_4^2 + q^3 q^4 (q^1 q^2 - 1) + q^3, \\ H_3 &= -q^4 (p_1^2 + p_2^2) + p_4^2 + q^4 (1 - q^1 q^2). \end{aligned} \tag{5.65}$$

The three GSE (5.37) explicitly read

$$\begin{cases} q^1 q^2 H_1 - q^1 q^2 (p_1^2 + p_2^2 + q^1 q^2) = 0, \\ H_2 + q^3 H_3 - p_3^2 - q^3 = 0, \\ q^4 H_1 + H_3 - p_4^2 - q^4 = 0. \end{cases} \tag{5.66}$$

Therefore, assuming that

$$\begin{aligned} W(q^1, q^2, q^3, q^4; h_1, h_2, h_3) = W_1(q^1, q^2; h_1, h_2, h_3) \\ + W_2(q^3; h_1, h_2, h_3) + W^3(q^4; h_1, h_2, h_3), \end{aligned} \tag{5.67}$$

we obtain the partially separated HJ equations

$$\begin{cases} \left(\dfrac{\partial W_1}{\partial q^1}\right)^2 + \left(\dfrac{\partial W_1}{\partial q^2}\right)^2 + q^1 q^2 = h_1, \\ \left(\dfrac{\partial W_2}{\partial q^3}\right)^2 + q^3 - h_3 q^3 = h_2, \\ \left(\dfrac{\partial W_3}{\partial q^4}\right)^2 + q^4 - h_1 q^4 = h_3. \end{cases} \tag{5.68}$$

In the case under scrutiny, we can determine an Abelian and semisimple Haantjes algebra $\mathscr{H}_2 = \text{span}\{\boldsymbol{K}_1, \boldsymbol{K}_2 = \boldsymbol{I}, \boldsymbol{K}_3\}$, allowing for partial separation. Applying Theorem 5.3.2 by choosing $H = H_2$ to be the generator function of the Haantjes chains, we obtain in the chart $(q^1, q^2, q^3, q^4, p_1, p_2, p_3, p_4)$ that the Haantjes operators

$$\boldsymbol{K}_1 = \frac{1}{q^3 q^4} \operatorname{diag}[1, 1, 0, 0, 1, 1, 0, 0], \quad \boldsymbol{K}_3 = -\frac{1}{q^3} \operatorname{diag}[1, 1, 0, 1, 1, 1, 0, 1] \tag{5.69}$$

satisfy the chain equations $\mathrm{d}H_1 = \boldsymbol{K}_1^T \mathrm{d}H$ and $\mathrm{d}H_3 = \boldsymbol{K}_3^T \mathrm{d}H$, respectively.

Remark 5.4.2. *In [192], it has been shown that Killing tensors for the AKN systems [6] can be naturally constructed within the theory of $\omega\mathscr{H}$ manifolds by projecting the Haantjes operators associated with the AKN systems from phase space into the configuration space. This, in turn, provides a direct connection between the Haantjes geometry and Eisenhart's theory of Killing tensors for Stäckel systems [49].*

Para ver una cosa hay que comprenderla. El sillón presupone el cuerpo humano, sus articulaciones y partes; las tijeras, el acto de cortar. ¿Qué decir de una lámpara o de un vehículo? El salvaje no puede percibir la biblia del misionero; el pasajero no ve el mismo cordaje que los hombres de a bordo. Si viéramos realmente el universo, tal vez lo entenderíamos.

Jorge Luis Borges, *There Are More Things*

6

Hamiltonian integrable systems in a magnetic field

On a quiet afternoon, the breeze threads through the leaves, coaxing them into a dance only it can hear. You watch their arcs and tremors–chaotic, yet somehow deliberate–as if each twist were pulled by currents unseen. There's a quiet thrill in realizing this: beneath the flutter of randomness lies a rhythm written in the language of air and inertia, a choreography so precise it turns whispers into revelation. To study this is to decipher the wind's whispered stanzas, to trace the same hidden order that guides a leaf's shiver or a particle's path through the silent pull of a magnetic hymn–proof that even the imperceptible hums with laws waiting to be known.

The purpose of this chapter is to analyze classical integrable and superintegrable Hamiltonian systems in magnetic fields through a novel geometric framework rooted in Haantjes geometry. Our approach achieves two goals: first, it provides new geometric insights into known physical systems; second, it enables the systematic construction of new magnetic models from geometric principles.

We consider Hamiltonian systems defined on a Riemannian manifold (Q, g), where g is the metric tensor on an n-dimensional differentiable manifold Q (we shall focus in $n = 3$). In canonical coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*Q, the Hamiltonian function takes the form (with units $e = -1, m = 1$):

$$H = \frac{1}{2} g^{ij}(p_i + A_i)(p_j + A_j) + V(\boldsymbol{q}) = \frac{1}{2} g^{ij} p_i p_j + A^i p_i + U(\boldsymbol{q}), \tag{6.1}$$

where A is a smooth vector field (the vector potential), $B = \nabla \times A$ the magnetic field associated with the model, $V(\boldsymbol{q})$ the scalar potential, and

$$U := \frac{1}{2} A^i A_i + V. \tag{6.2}$$

The study of integrable Hamiltonian models immersed in magnetic fields, or with velocity dependent potentials, originated in the 1960s with monopole-related models [81, 100, 131, 133, 199]. Systematic exploration began in the 1980s, focusing on two-dimensional Euclidean systems with first-degree integrals in the momenta [47]. Subsequent work extended this to second-degree integrals in the 2000s, leading to a partial classification [32, 42, 132, 154, 155]; while the analysis of three-dimensional systems with the work [127], but it is still in progress (see e.g. [73, 97] and references therein). Separability on curved Riemannian spaces has also been considered (refer for instance [12]). Recently, in [17] a novel approach for the study of the orbits of superintegrable systems in a magnetic field, based on Lie group theory, has been proposed.

The main idea of this chapter is to study the interplay between magnetic systems and their underlying geometry. In the first part, we shall consider families of integrable Hamiltonian systems immersed in a magnetic field, which are of physical interest, and we unravel their underlying geometric structure. More precisely, we shall construct the symplectic-Haantjes manifolds associated with the magnetic systems under investigation. Thus, we prove that the DH coordinates for these geometric structures provide coordinate charts where the HJ equations associated with our systems separate into systems of ODEs (*total separability*), or ODEs and PDEs (*partial separability*). Also, we shall prove that there is a direct relation between the Haantjes geometry and the Stäckel theory of separability. We stress that, in contrast to the usual methods based on the configuration space separation, the full generality of the Haantjes geometry, directly defined on the phase space, bypasses the necessity of fixing a specific gauge from the very beginning for the associated vector potentials, usually done in an *ad hoc* manner. Indeed, it is the geometry of the full phase space that naturally determines the gauge needed for separation.

In the second part, we go back from geometry to magnetic systems. We will suitably generalize the Haantjes and Stäckel geometries determined in the first part in order to construct new families of integrable Hamiltonian systems, defined on Riemannian manifolds, immersed in suitable magnetic fields.

A novel observation is that we can regard our new magnetic models as generalized Stäckel systems in the sense of AKN (see Section 1.4.1). This important class of Hamiltonian integrable models possesses an interesting geometry (see Section 3.4.3), which will be fundamental in the construction of the new models.

This chapter presents the results of [99], and it is structured as follows. Section 6.1 establishes preliminary results on Haantjes operators for cyclic coordinates. Section 6.2 reviews magnetic fields in Riemannian geometry. Section 6.3 derives the most general 3D systems of type (6.1) separable in cylindrical coordinates, alongside their Haantjes algebras. Section 6.4 analyzes a constant magnetic field system, revealing three distinct symplectic-Haantjes manifolds linked to separation coordinates, and examines the helical undulator. Section 6.5 constructs Stäckel geometries for the previous

superintegrable systems via their Jacobi-Sklyanin equations; and Section 6.6 solves an inverse problem to generalize these geometries, and defines new magnetic systems. Finally, Section 6.7 explores superintegrability in a broad class of systems immersed in a magnetic monopole field.

6.1 HAANTJES OPERATORS ASSOCIATED TO CYCLIC COORDINATES

The theory of $\omega\mathscr{H}$ manifolds provides a natural framework for analyzing Hamiltonian systems in the presence of magnetic fields, as we demonstrate in subsequent sections. A first example of this framework is presented below, where we derive Haantjes operators associated with *cyclic* coordinates

Let us consider the Hamiltonian function (6.1) and assume it is independent of the coordinate q^a (i.e., q^a is *cyclic or ignorable*). Then, as is well-known, the conjugate momenta p_a is an integral of motion, as it is in involution with H. More precisely, it is in total separable involution with H. In such a situation, we can write down a Haantjes operator $\boldsymbol{K}_a$ such that $\boldsymbol{K}_a^T \mathrm{d}H = \mathrm{d}p_a$. We obtain

$$\boldsymbol{K}_a = \frac{1}{g^{ai}(p_i + A_i(\boldsymbol{q}))} \left(\frac{\partial}{\partial q^a} \otimes \mathrm{d}q^a + \frac{\partial}{\partial p_a} \otimes \mathrm{d}p_a \right). \tag{6.3}$$

In fact, from Eq. (3.70) it follows immediately

$$\lambda_b^{(a)} = \frac{\frac{\partial H_a}{\partial p_b}}{\frac{\partial H}{\partial p_b}} = \frac{\delta_{ab}}{g^{bj}(p_j + A_j)},$$

where $H_a := p_a$.

6.2 MAGNETIC FIELDS AND RIEMANNIAN GEOMETRY

Let us recall some notions that we will use in what follows.

Definition 6.2.1. *Let Q be a Riemannian manifold and $\boldsymbol{G}$ be the inverse of its metric tensor. It defines the $C^\infty(Q)$-linear map (denoted, with a slight abuse of notation, by $\boldsymbol{G}$) and defined by $\boldsymbol{G}$: $\Omega^1(Q) \to \mathfrak{X}(Q), \alpha \mapsto X = \boldsymbol{G}\,\alpha$ where X is the unique vector field such that*

$$g(X, Y) = \langle \alpha, Y \rangle, \qquad \forall Y \in \mathfrak{X}(Q).$$

Such isomorphism that maps 1-forms onto vector fields is also denoted as $X = (\alpha)^\sharp$. The inverse of such isomorphism $\boldsymbol{G}^{-1}: \mathfrak{X}(Q) \to \Omega^1(Q), X \mapsto \alpha = \boldsymbol{G}^{-1} X$ is also denoted as $\alpha = X^\flat$.

Let us recall that, in a 3D Riemaniann manifold, the Hodge star operator $*: \Omega^k(Q) \to \Omega^{3-k}(Q)$ evaluated on the basis 2-forms gives us

$$*(\mathrm{d}q^i \wedge \mathrm{d}q^j) = \iota_{\boldsymbol{G}\mathrm{d}q^j} \iota_{\boldsymbol{G}\mathrm{d}q^i} \sqrt{\det(g)}\, \mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3, \tag{6.4}$$

where ι_X denotes the interior product of a k-form with a vector field X.

If A denotes the vector field of the magnetic potential, the magnetic vector field can be defined as $B := \nabla \times A = (*\,\mathrm{d}(A)^\flat)^\sharp = \boldsymbol{G} * \mathrm{d}(\boldsymbol{G}^{-1}A)$. In general curvilinear coordinates (q^1, q^2, q^3), we can compute B through the formula

$$\nabla \times A = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathsf{e}_1 & h_2 \mathsf{e}_2 & h_3 \mathsf{e}_3 \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & \frac{\partial}{\partial q^3} \\ A_1 & A_2 & A_3 \end{vmatrix}, \tag{6.5}$$

where $(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3})$ are the vector fields of the natural reference frame, $h_i = \sqrt{g_{ii}}$ their modules and $(\mathsf{e}_1 = \frac{1}{h_1}\frac{\partial}{\partial q^1}, \mathsf{e}_2 = \frac{1}{h_2}\frac{\partial}{\partial q^2}, \mathsf{e}_3 = \frac{1}{h_3}\frac{\partial}{\partial q^3})$ the unit vector fields in the same directions. In the following, we shall denote by (A_1, A_2, A_3) the covariant components of a vector field $A(q^1, q^2, q^3)$ along the basis of the gradients $(\nabla q^1 = \boldsymbol{G}\,\mathrm{d}q^1, \nabla q^2 = \boldsymbol{G}\,\mathrm{d}q^2, \nabla q^3 = \boldsymbol{G}\,\mathrm{d}q^3)$.

6.3 SEPARABLE SYSTEMS WITH MAGNETIC FIELD IN CYLINDRICAL COORDINATES

In this section, we shall determine the symplectic-Haantjes manifolds associated with a large class of magnetic systems in $\mathbb{E}^3$: those separating in cylindrical coordinates. Our strategy will be based on some natural ansätze together with the Jacobi-Haantjes Theorem 3.4.2.

In cylindrical coordinates (r, φ, z), the Euclidean metric reads $g = \mathrm{diag}\,[1, r^2, 1]$ and the Hamiltonian (6.1) takes the form

$$\begin{aligned} H &= \frac{1}{2}\left[(p_r + A_r(r,\varphi,z))^2 + \frac{(p_\varphi + A_\varphi(r,\varphi,z))^2}{r^2} + (p_z + A_z(r,\varphi,z))^2\right] + V(r,\varphi,z) \\ &= \frac{1}{2}\left(\Pi_r^2 + \frac{\Pi_\varphi^2}{r^2} + \Pi_z^2\right) + V(r,\varphi,z)\,, \end{aligned} \tag{6.6}$$

where $\Pi_r := p_r + A_r(r,\varphi,z), \Pi_\varphi := p_\varphi + A_\varphi(r,\varphi,z)$ and $\Pi_z := p_z + A_z(r,\varphi,z)$ and (A_r, A_φ, A_z) are, respectively, the covariant components of the linear momentum $\Pi = v$ and the vector magnetic potential A, along the reference frame $(\nabla r, \nabla\varphi, \nabla z)$. From Eq. (6.5) we obtain the expression of the magnetic field:

$$B(r,\varphi,z) = \frac{1}{r}\left(\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\mathsf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathsf{e}_\varphi + \frac{1}{r}\left(\frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\mathsf{e}_z, \tag{6.7}$$

where $(\mathsf{e}_r, \mathsf{e}_\varphi, \mathsf{e}_z)$ are the unit vectors tangent to the coordinate lines. We introduce the Haantjes algebra $\mathscr{H}_{cyl}$ associated with the cylindrical web, which is generated by the set of Haantjes (Nijenhuis) operators

$$\boldsymbol{L}_1 := \frac{\partial}{\partial r} \otimes \mathrm{d}r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r\,, \tag{6.8}$$

$$\boldsymbol{L}_2 := \frac{\partial}{\partial \varphi} \otimes \mathrm{d}\varphi + \frac{\partial}{\partial p_\varphi} \otimes \mathrm{d}p_\varphi\,, \tag{6.9}$$

$$\boldsymbol{L}_3 := \frac{\partial}{\partial z} \otimes \mathrm{d}z + \frac{\partial}{\partial p_z} \otimes \mathrm{d}p_z \,. \tag{6.10}$$

In the Cartesian frame (x, y, z, p_x, p_y, p_z), they take the matrix form

$$\boldsymbol{L}_1 = \frac{1}{x^2+y^2} \left[\begin{array}{ccc|ccc} x^2 & xy & 0 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x\,p_y - y\,p_x & 0 & x^2 & xy & 0 \\ -(x\,p_y - y\,p_x) & 0 & 0 & xy & y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{6.11}$$

$$\boldsymbol{L}_2 = \left[\begin{array}{ccc|ccc} y^2 & -xy & 0 & 0 & 0 & 0 \\ -xy & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -(x\,p_y - y\,p_x) & 0 & y^2 & -xy & 0 \\ x\,p_y - y\,p_x & 0 & 0 & -xy & x^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{6.12}$$

$$\boldsymbol{L}_3 = \mathrm{diag}\,[0, 0, 1, 0, 0, 1] \,. \tag{6.13}$$

Note that they can also be obtained using the complete Yano lift [76, 198] of, respectively, the evaluation of the Euler tensor at the equatorial plane $z = 0$ (up to the factor $(x^2 + y^2)^{-1}$) and the inertia tensor (in $\mathbb{E}^2$) of a single particle [184].

We obtained three cases of general systems in a magnetic field separable in cylindrical coordinates. In the following discussion $H_1 = H$.

Case 1. Let us look for an integrable system that admits as integrals of motion the momenta conjugated to the coordinates (φ, z):

$$H_2 := p_\varphi \,, \qquad H_3 := p_z \,. \tag{6.14}$$

With this ansatz, the coordinates (φ, z) are ignorable in the Hamiltonian function (6.6). Therefore,

$$A(r) = A_r(r)\,\nabla r + A_\varphi(r)\,\nabla\varphi + A_z(r)\,\nabla z \,, \qquad V = V(r). \tag{6.15}$$

From Eq. (8.8), we obtain the magnetic field

$$B = -A_z'(r)\,\mathsf{e}_\varphi + \frac{A_\varphi'(r)}{r}\,\mathsf{e}_z \,, \tag{6.16}$$

which is tangent to the cylindrical surfaces of the cylindrical web. As it is independent of A_r, we can set $A_r = 0$ without loss of generality. In view of Eq. (6.6), the Hamiltonian function of this system turns out to be

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + \frac{A_\varphi(r)}{r^2}\,p_\varphi + A_z(r)\,p_z + V(r) + \frac{1}{2}\left(\frac{A_\varphi^2(r)}{r^2} + A_z^2(r)\right). \quad (6.17)$$

Thus, the Hamiltonian functions (6.14) and (6.17) are not only in involution but even in total separable involution. Then, Eq. (6.3) implies that the corresponding semisimple Haantjes operators that solve the chain equations $\boldsymbol{K}_2^T \mathrm{d}H = \mathrm{d}H_2$ and $\boldsymbol{K}_3^T \mathrm{d}H = \mathrm{d}H_3$ are

$$\boldsymbol{K}_2 = \frac{r^2}{p_\varphi + A_\varphi(r)}\,\boldsymbol{L}_2, \qquad \boldsymbol{K}_3 = \frac{1}{p_z + A_z(r)}\,\boldsymbol{L}_3. \quad (6.18)$$

Therefore, the HJ equations associated with the Hamiltonian system are

$$\begin{cases} \frac{1}{2}\left(\left(\frac{\partial W}{\partial r}\right)^2 + \frac{c_2^2}{r^2} + c_3^2\right) + \frac{A_\varphi(r)}{r^2}\,c_2 + A_z(r)\,c_3 + V(r) + \frac{1}{2}\left(\frac{A_\varphi^2(r)}{r^2} + A_z^2(r)\right) = c_1\,, \\ \frac{\partial W}{\partial \varphi} = c_2\,, \\ \frac{\partial W}{\partial z} = c_3\,. \end{cases}$$

By putting $p_r = \frac{\partial W}{\partial r}$, $p_z = \frac{\partial W}{\partial z}$, $p_\varphi = \frac{\partial W}{\partial \varphi}$ in the previous formulas, one obtains the Jacobi-Skyanin separation relations.

Since it is separated, we can split W as $W(r,\varphi,z) = W_1(r) + W_2(\varphi) + W_3(z)$. Thus, the resulting system consists of three ODEs, and solving them yields:

$$W(r,\varphi,z;c_1,c_2,c_3) = c_2\,\varphi + c_3\,z \pm \int \sqrt{-2\left(\frac{c_2^2}{2r^2} + \frac{c_3^2}{2} - c_1 + \frac{A_\varphi(r)}{r^2}c_2 + A_z(r)\,c_3 + U(r)\right)}\,\mathrm{d}r\,, \quad (6.19)$$

where $U(r) = V(r) + \frac{1}{2}\left(\frac{A_\varphi^2(r)}{r^2} + A_z^2(r)\right)$, c_1, c_2, c_3 are the values taken by the integrals of motion (H, H_2, H_3) on the leaves of the Lagrangian foliation.

Case 2. Let us seek an integrable system possessing the momentum conjugated to the coordinate φ as an integral of motion:

$$H_2 := p_\varphi. \quad (6.20)$$

With this ansatz, the coordinate φ must be ignorable in the Hamiltonian (6.6). This implies that the vector and the scalar potentials are, respectively, of the form

$$A(r,z) = A_r(r,z)\,\nabla r + A_\varphi(r,z)\,\nabla\varphi + A_z(r,z)\,\nabla z\,, \qquad V = V(r,z)\,, \quad (6.21)$$

where the functions A_r, A_φ, A_z, V are smooth arbitrary functions of their arguments. We deduce for the magnetic field the expression

$$B = -\frac{1}{r}\frac{\partial A_\varphi}{\partial z}\,\mathsf{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathsf{e}_\varphi + \frac{1}{r}\frac{\partial A_\varphi}{\partial r}\,\mathsf{e}_z\,, \tag{6.22}$$

so that the Hamiltonian function turns out to be

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + A_r(r,z)\,p_r + \frac{A_\varphi(r,z)}{r^2}\,p_\varphi + A_z(r,z)\,p_z + V(r,z) + \frac{1}{2}\left(A_r^2(r,z) + \frac{A_\varphi^2(r,z)}{r^2} + A_z^2(r,z)\right), \tag{6.23}$$

which is in separable involution with the integral (6.20). From Eq. (6.3) we get the semisimple Haantjes operator for the chain $\boldsymbol{K}_2^T \mathrm{d}H = \mathrm{d}H_2$:

$$\boldsymbol{K}_2 = \frac{r^2}{p_\varphi + A_\varphi(r,z)}\,\boldsymbol{L}_2\,. \tag{6.24}$$

In order to determine another integral of motion in separable involution with (6.23) and (6.20) and a corresponding Haantjes operator $\boldsymbol{K}_3$, we can assume that φ is ignorable in $\boldsymbol{K}_3$ and that, for simplicity, it depends on p_φ, p_z but not on p_r:

$$\boldsymbol{K}_3 = \mathrm{diag}\,[a_1, a_2, a_3, a_1, a_2, a_3]\,,$$

where $a_i = a_i(r, z, p_\varphi, p_z), i = 1, 2, 3$. By requiring the local exactness condition $\mathrm{d}(\boldsymbol{K}_3^T \mathrm{d}H) = \mathbf{0}$, we get an overdetermined system of six PDEs in three unknowns. We obtain the particular solution

$$A(r,z) = \big(f_2(r) + r^2\,f_3(z)\big)\,\nabla\varphi, \quad B = -r f_3'(z)\mathsf{e}_r + \left(\frac{f_2'(r)}{r} + 2f_3(z)\right)\mathsf{e}_z\,, \tag{6.25}$$

$$V = f_5(r) + f_4(z) - \frac{r^2 f_3^2(z)}{2} - f_2(r) f_3(z), \tag{6.26}$$

$$\boldsymbol{K}_3 = \frac{r^2\,f_3(z)}{p_\varphi + A_\varphi}\,\boldsymbol{L}_2 + \boldsymbol{L}_3\,, \tag{6.27}$$

$$H_3 = \frac{1}{2}\,p_z^2 + f_3(z)\,p_\varphi + f_4(z), \tag{6.28}$$

where f_2, f_3, f_4, f_5 are arbitrary functions.

The complete integral of the set of HJ equations is

$$W(r,\varphi,z;c_1,c_2,c_3) = c_2\varphi \pm \int \sqrt{-2\left(\frac{c_2^2}{2r^2} - c_1 + \frac{f_2(r)}{r^2}c_2 + c_3 + f_6(r)\right)}\,\mathrm{d}r$$

$$\pm \int \sqrt{-2\left(f_3(z)c_2 - c_3 + f_4(z)\right)}\,\mathrm{d}z\,, \quad (6.29)$$

where $f_6(r) = f_5(r) + f_2(r)^2/2r^2$, c_1, c_2, c_3 are the values taken by the integrals of motion (H, H_2, H_3) on the leaves of the Lagrangian foliation.

Case 3. Let us proceed as in Case 2, by interchanging φ with z. To this aim, let us look for a Hamiltonian system admitting the integral

$$H_3 := p_z\,. \quad (6.30)$$

With this ansatz, the coordinate z is ignorable in the Hamiltonian function (6.6). This implies that the vector and the scalar potential are of the form

$$A(r,\varphi) = A_r(r,\varphi)\,\nabla r + A_\varphi(r,\varphi)\,\nabla\varphi + A_z(r,\varphi)\,\nabla z\,, \qquad V = V(r,\varphi), \quad (6.31)$$

where the functions A_r, A_φ, A_z, V are arbitrary (smooth) functions of r and periodic functions of φ. The associated magnetic field is

$$B = \frac{1}{r}\frac{\partial A_z}{\partial\varphi}\,\mathsf{e}_r - \frac{\partial A_z}{\partial r}\,\mathsf{e}_\varphi + \frac{1}{r}\left(\frac{\partial A_\varphi}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right)\mathsf{e}_z\,. \quad (6.32)$$

The Hamiltonian function of this system reads

$$\begin{aligned} H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + A_r(r,\varphi)\,p_r + \frac{A_\varphi(r,\varphi)}{r^2}\,p_\varphi + A_z(r,\varphi)\,p_z \\ + V(r,\varphi) + \frac{1}{2}\left(A_r^2(r,\varphi) + \frac{A_\varphi^2(r,\varphi)}{r^2} + A_z^2(r,\varphi)\right), \end{aligned} \quad (6.33)$$

that is in separable involution with the integral (6.30). From Eq. (6.3), we get that the following semisimple Haantjes operator solves the chain equation $\boldsymbol{K}_3^T\mathrm{d}H = \mathrm{d}H_3$:

$$\boldsymbol{K}_3 = \frac{1}{p_z + A_z(r,\varphi)}\,\boldsymbol{L}_3\,. \quad (6.34)$$

By analogy with the previous case, we can determine another integral of motion in separable involution with the integrals (6.33) and (6.30), and the associated Haantjes operator $\boldsymbol{K}_2$ by assuming that z is ignorable in $\boldsymbol{K}_2$ and that it depends on p_φ, p_z but not on p_r:

$$\boldsymbol{K}_2 = \mathrm{diag}\,[b_1, b_2, b_3, b_1, b_2, b_3],$$

where $b_i = b_i(r, \varphi, p_\varphi, p_z)$, $i = 1, 2, 3$. The local exactness condition $\mathrm{d}(\boldsymbol{K}_2^T\mathrm{d}H) = \boldsymbol{0}$ provides us with an overdetermined system of PDEs, from which we obtain

$$A = \left(f_1(r) + \frac{f_2(\varphi)}{r^2}\right)\nabla z\,, \quad B = \frac{f_2'(\varphi)}{r^3}\mathsf{e}_r + \left(-f_1'(r) + 2\frac{f_2(\varphi)}{r^3}\right)\mathsf{e}_\varphi\,, \quad (6.35)$$

$$V = f_3(r) - \frac{f_1(r) f_2(\varphi)}{r^2} - \frac{f_2^2(\varphi)}{2r^4} + \frac{f_4(\varphi)}{r^2}\,, \tag{6.36}$$

$$\boldsymbol{K}_2 = r^2\, \boldsymbol{L}_2 + \frac{f_2(\varphi)}{p_z + A_z} \boldsymbol{L}_3\,, \tag{6.37}$$

$$H_2 \;=\; \frac{p_\varphi^2}{2} + f_2(\varphi)\, p_z + f_4(\varphi)\,. \tag{6.38}$$

The complete integral of the set of HJ equations is

$$\begin{aligned} W(r, \varphi, z; c_1, c_2, c_3) = c_3\, z &\pm \int \sqrt{-2\left(\frac{c_3^2}{2} - c_1 + \frac{c_2}{r^2} + f_1(r) c_3 + \frac{1}{2} f_1^2(r) + f_3(r)\right)}\, \mathrm{d}r \\ &\pm \int \sqrt{-2\left(-c_2 + f_2(\varphi) c_3 + f_4(\varphi)\right)}\, \mathrm{d}\varphi\,. \end{aligned} \tag{6.39}$$

Remark 6.3.1. *Let us note that, if we trade the triple of the Hamiltonian functions* (H_1, H_2, H_3) *with the triple* $(\tilde{H}_1 := H_1 - \frac{1}{2} H_3^2, \tilde{H}_2 := H_2, \tilde{H}_3 := H_3)$*, we get an AKN system with the following Stäckel matrix and vector*

$$\boldsymbol{S}(\boldsymbol{q}) = \begin{bmatrix} 1 & -\frac{1}{r^2} & -f_1(r) \\ 0 & 1 & -f_2(\varphi) \\ 0 & 0 & 1 \end{bmatrix}, \quad |\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{p})\rangle = \begin{bmatrix} \frac{p_r^2}{2} + \frac{f_1^2(r)}{2} + f_3(r) \\ \frac{p_\varphi^2}{2} + f_4(\varphi) \\ p_z \end{bmatrix}.$$

The modified Hamiltonian functions $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$ *form a Haantjes chain by means of the modified Haantjes operators, constructed through Eq.* (5.34)*:*

$$\tilde{\boldsymbol{K}}_2 = r^2 \boldsymbol{L}_2 + \frac{f_2(\varphi)}{A_z} \boldsymbol{L}_3\,, \qquad \tilde{\boldsymbol{K}}_3 = \frac{1}{A_z} \boldsymbol{L}_3;$$

however, they belong to the same Haantjes algebra $\mathscr{H}_{cyl}$*. Thus, we can say that the Lagrangian foliation generated by* (H_1, H_2, H_3) *admits the Stäckel basis* $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$*. We have not been able to find the Stäckel basis for Cases 1 and 2. The general problem of constructing a Stäckel basis for a given Lagrangian foliation will be addressed elsewhere.*

In [12], it has been proved that the three cases discussed above give the most general systems in a magnetic field separable in cylindrical coordinates. Thus, our analysis in the geometric setting of Haantjes algebras is coherent with the results obtained in [12] with a different approach.

6.4 SUPERINTEGRABILITY AND MULTISEPARABILITY

The purpose of this section is to construct the symplectic-Haantjes manifolds associated with several cases of physically relevant superintegrable magnetic systems. According to Proposition 5.1.1,

we shall determine as many Haantjes structures as the number of separation variables admitted by the system under analysis.

We shall first consider the case of the constant magnetic field with vanishing scalar potential. On the one hand, this case can be regarded as a specialization of the general discussion concerning separability in cylindrical coordinates and Haantjes geometry developed in Section 6.3. On the other hand, the presence of a constant magnetic field allows the system to possess two additional integrals of motion. This, in turn, implies that there exist additional symplectic-Haantjes structures related to the extra integrals, providing total or partial separation coordinates [160].

Remark 6.4.1. *In Cartesian coordinates, the physical Hamiltonian that describes a system defined in a Euclidean 3D space with a magnetic field and vanishing potential is*

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(p_i + A_i(\boldsymbol{x})\right)^2, \tag{6.40}$$

with $\boldsymbol{x} = (x, y, z)$. *The components of the kinetic momentum of the system, which coincides with the velocity vector in our units, are given by [82]*

$$\Pi_i(\boldsymbol{x}) := p_i + A_i(\boldsymbol{x})\,. \tag{6.41}$$

6.4.1 *A superintegrable model in a constant magnetic field*

The simplest magnetic system we can study in three dimensions is the one in a constant magnetic field, that without loss of generality we can define as

$$B(x, y, z) = b\,\mathrm{e}_z\,, \tag{6.42}$$

where $b \in \mathbb{R}$ is an arbitrary constant. The Hamiltonian model (6.40) is maximally superintegrable [127]. The integrals of the motion of first degree in the momenta are

$$\begin{aligned} H_1 &= \Pi_x + b\,y\,,\\ H_2 &= \Pi_y - b\,x\,,\\ H_3 &= \Pi_z\,,\\ H_4 &= \Lambda_z - \frac{b}{2}(x^2 + y^2)\,. \end{aligned} \tag{6.43}$$

Here, $\Lambda_z = x\,\Pi_y - y\,\Pi_x$ is the z-component of the total angular momentum. The set of these integrals jointly with the Hamiltonian (6.40) is not functionally independent; however, another independent, non-polynomial integral is given by

$$H_5 = -\Pi_x \cos\left(\frac{b\,z}{\Pi_z}\right) - \Pi_y \sin\left(\frac{b\,z}{\Pi_z}\right). \tag{6.44}$$

It commutes with H_1 and H_2. In the subsequent discussion, we shall construct Haantjes manifolds associated with commutative Poisson subalgebras generated by the previous integrals.

6.4.1.1 *Semisimple Haantjes operators and spectral analysis*

In the following, we shall provide a set of *semisimple* Haantjes operators that solve the chain equation $\boldsymbol{K}_i^T \mathrm{d}H = \mathrm{d}H_i$ jointly with their spectral properties. We obtain for the first chain the following operator and the corresponding eigen-distributions:

$$\boldsymbol{K}_1 = \frac{1}{\Pi_x}\boldsymbol{R}_1, \tag{6.45}$$

with

$$\boldsymbol{R}_1 := \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \partial_x A_y & \partial_x A_z & 1 & 0 & 0 \\ -\partial_x A_y & 0 & 0 & 0 & 0 & 0 \\ -\partial_x A_z & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{6.46}$$

and

$$\lambda_1^{(1)} = 0\,, \quad \rho_1^{(1)} = 1\,, \; \mathcal{D}_1 = \left\langle \partial_x A_z \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle,$$
$$\lambda_2^{(1)} = \frac{1}{\Pi_x}\,, \quad \rho_2^{(1)} = 1\,, \; \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x} - \partial_x A_y \frac{\partial}{\partial p_y} - \partial_x A_z \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x} \right\rangle.$$

For the second chain, we have

$$\boldsymbol{K}_2 = \frac{1}{\Pi_y}\boldsymbol{R}_2, \tag{6.47}$$

with

$$\boldsymbol{R}_2 := \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -\partial_y A_x & 0 & 0 & 0 & 0 \\ \partial_y A_x & 0 & \partial_y A_z & 0 & 1 & 0 \\ 0 & -\partial_y A_z & 0 & 0 & 0 & 0 \end{array}\right], \tag{6.48}$$

and

$$\lambda_1^{(2)} = 0\,, \quad \rho_1^{(2)} = 1\,, \; \mathcal{D}_3 = \left\langle \partial_y A_z \frac{\partial}{\partial x} - \partial_y A_x \frac{\partial}{\partial z}, \frac{\partial}{\partial x} - \partial_y A_x \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_z} \right\rangle,$$
$$\lambda_2^{(2)} = \frac{1}{\Pi_y}\,, \quad \rho_2^{(2)} = 1\,, \; \mathcal{D}_4 = \left\langle \frac{\partial}{\partial y} - \partial_y A_x \frac{\partial}{\partial p_x} - \partial_y A_z \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_y} \right\rangle.$$

Solving the third chain, we obtain

$$\boldsymbol{K}_3 = \frac{1}{\Pi_z}\boldsymbol{R}_3, \tag{6.49}$$

with

$$\boldsymbol{R}_3 := \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -\partial_z A_x & 0 & 0 & 0 \\ 0 & 0 & -\partial_z A_y & 0 & 0 & 0 \\ \partial_z A_x & \partial_z A_y & 0 & 0 & 0 & 1 \end{array}\right], \tag{6.50}$$

$$\lambda_1^{(3)} = 0\,, \quad \rho_1^{(3)} = 1\,, \; \mathcal{D}_5 = \left\langle \partial_z A_y \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y} \right\rangle,$$
$$\lambda_2^{(3)} = \frac{1}{\Pi_z}\,, \quad \rho_2^{(3)} = 1\,, \; \mathcal{D}_6 = \left\langle \frac{\partial}{\partial z} - \partial_z A_x \frac{\partial}{\partial p_x} - \partial_z A_y \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle.$$

The operators $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$ introduced above will play a significant role in the discussion of Section 6.6. Finally,

$$\boldsymbol{K}_4 = \frac{1}{\Lambda_z} \left[\begin{array}{ccc|ccc} y^2 & -xy & 0 & 0 & 0 & 0 \\ -xy & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & f_4 & g_4 & y^2 & -xy & 0 \\ -f_4 & 0 & h_4 & -xy & x^2 & 0 \\ -g_4 & -h_4 & 0 & 0 & 0 & 0 \end{array}\right], \tag{6.51}$$

where

$$\begin{aligned} f_4 &= -\Lambda_z + xy\,(\partial_x A_x - \partial_y A_y) - x^2\,\partial_y A_x + y^2\,\partial_x A_y\,, \\ g_4 &= y\,\left(-x\,\partial_y A_z + y\,\partial_x A_z\right)\,, \\ h_4 &= x\,\left(x\,\partial_y A_z - y\,\partial_x A_z\right)\,. \end{aligned} \tag{6.52}$$

The spectral analysis for $\boldsymbol{K}_4$ can be performed in full analogy with the previous cases:

$$\begin{aligned} &\lambda_1^{(4)} = 0\,, \quad \rho_1^{(4)} = 1\,, \\ &\mathcal{D}_7 = \left\langle x\left(x\partial_y A_z - y\partial_x A_z\right)\frac{\partial}{\partial x} + y\left(x\partial_y A_z - y\partial_x A_z\right)\frac{\partial}{\partial y} + f_4\frac{\partial}{\partial z}, \right. \\ &\qquad \left. xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - f_4\frac{\partial}{\partial p_x}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + f_4\frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle, \end{aligned} \tag{6.53}$$

$$\begin{aligned} &\lambda_2^{(4)} = \frac{x^2+y^2}{\Lambda_z}\,, \quad \rho_2^{(4)} = 1\,, \\ &\mathcal{D}_8 = \left\langle xy\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y} - f_4\frac{\partial}{\partial p_x} + x\left(x\partial_y A_z - y\partial_x A_z\right)\frac{\partial}{\partial p_z}, y\frac{\partial}{\partial p_x} - x\frac{\partial}{\partial p_y} \right\rangle. \end{aligned} \tag{6.54}$$

6.4.1.2 *Construction of the DH coordinates*

The operators $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_4\}$ generate several commutative Haantjes algebras $\mathscr{H}_i$. We shall determine the DH coordinates for these algebras, which both diagonalize all semisimple Haantjes operators of each algebra and leave the symplectic form invariant. For the sake of clarity, we discuss in full detail the case of the algebra $\mathscr{H}_1 = \{\boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_3\}$. Defining the distributions

$$\mathcal{V}_1 = \mathcal{D}_1 \cap \mathcal{D}_5, \quad \mathcal{V}_2 = \mathcal{D}_1 \cap \mathcal{D}_6, \quad \mathcal{V}_3 = \mathcal{D}_2 \cap \mathcal{D}_5, \tag{6.55}$$

we can work out the annihilators

$$\begin{aligned} (\mathcal{V}_1 \oplus \mathcal{V}_2)^\circ &= \langle \mathrm{d}x\,, \partial_x A_y\, \mathrm{d}y + \partial_z A_x\, \mathrm{d}z + \mathrm{d}p_x \rangle, \\ (\mathcal{V}_1 \oplus \mathcal{V}_3)^\circ &= \langle \mathrm{d}z\,, \partial_x A_z\, \mathrm{d}x + \partial_y A_z\, \mathrm{d}y + \mathrm{d}p_z \rangle, \\ (\mathcal{V}_2 \oplus \mathcal{V}_3)^\circ &= \langle \mathrm{d}y\,, \partial_x A_y\, \mathrm{d}x + \partial_z A_y\, \mathrm{d}z + \mathrm{d}p_y \rangle\,. \end{aligned} \tag{6.56}$$

Integrating them, we find the set of DH coordinates

$$\begin{aligned} q^1 &= x\,, & p_1 &= \Pi_x + b\, y\,, \\ q^2 &= y\,, & p_2 &= \Pi_y\,, \\ q^3 &= z\,, & p_3 &= \Pi_z\,. \end{aligned} \tag{6.57}$$

In this chart, the operators of the algebra $\mathscr{H}_1$ read

$$\boldsymbol{K}_1 = \frac{1}{p_1 - b\, q^2}\, \mathrm{diag}\,[1,0,0,1,0,0], \qquad \boldsymbol{K}_3 = \frac{1}{p_3}\, \mathrm{diag}\,[0,0,1,0,0,1]\,; \tag{6.58}$$

and the Hamiltonian function and the integrals of the motion read

$$\begin{aligned} H &= \frac{1}{2}\left[(p_1 - b\, q^2)^2 + p_2^2 + p_3^2\right], \\ H_1 &= p_1\,, \\ H_3 &= p_3\,. \end{aligned} \tag{6.59}$$

We shall similarly analyse the remaining algebras, skipping most technical details. For the algebra $\mathscr{H}_2 = \{\boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3\}$, we define the distributions

$$\mathcal{V}_4 = \mathcal{D}_3 \cap \mathcal{D}_5\,, \quad \mathcal{V}_5 = \mathcal{D}_3 \cap \mathcal{D}_6\,, \quad \mathcal{V}_6 = \mathcal{D}_4 \cap \mathcal{D}_5\,. \tag{6.60}$$

Integrating the corresponding annihilators, we find the set of DH coordinates

$$\begin{aligned} q^1 &= x\,, & p_1 &= \Pi_x\,, \\ q^2 &= y\,, & p_2 &= \Pi_y - b\, x\,, \\ q^3 &= z\,, & p_3 &= \Pi_z\,. \end{aligned} \tag{6.61}$$

In this chart, the operators of the algebra $\mathscr{H}_2$ read

$$\boldsymbol{K}_2 = \frac{1}{p_2 + b\,q^1}\,\mathrm{diag}\,[0,1,0,0,1,0]\,, \qquad \boldsymbol{K}_3 = \frac{1}{p_3}\,\mathrm{diag}\,[0,0,1,0,0,1]\,; \tag{6.62}$$

and the Hamiltonian function and the integrals of the motion read

$$\begin{aligned} H &= \frac{1}{2}\,\left[p_1^2 + (p_2 + b\,q^1)^2 + p_3^2\right]\,, \\ H_2 &= p_2\,, \\ H_3 &= p_3\,. \end{aligned} \tag{6.63}$$

Finally, for the algebra $\mathscr{H}_3 = \{\boldsymbol{I}, \boldsymbol{K}_3, \boldsymbol{K}_4\}$, the corresponding annihilators can be easily integrated in several ways, providing us with different choices for the DH coordinates. The most natural choice is that of cylindrical coordinates:

$$\begin{aligned} q^1 &= \sqrt{x^2+y^2}\,, & p_1 &= \frac{x\,\Pi_x + y\,\Pi_y}{\sqrt{x^2+y^2}}\,, \\ q^2 &= \arctan\left(\frac{y}{x}\right)\,, & p_2 &= x\left(\Pi_y - \frac{b}{2}x\right) - y\left(\Pi_x + \frac{b}{2}y\right)\,, \\ q^3 &= z\,, & p_3 &= \Pi_z\,; \end{aligned} \tag{6.64}$$

where the operators $\boldsymbol{K}_3$ and $\boldsymbol{K}_4$ take the form

$$\boldsymbol{K}_3 = \frac{1}{p_3}\boldsymbol{L}_3\,, \qquad \boldsymbol{K}_4 = \frac{2(q^1)^2}{2\,p_2 + b\,(q^1)^2}\boldsymbol{L}_2\,. \tag{6.65}$$

In this chart, the Hamiltonian function and the integrals of the motion read

$$\begin{aligned} H &= \frac{1}{2}\left[p_1^2 + \frac{1}{(q^1)^2}\left(p_2 + \frac{b}{2}\,(q^1)^2\right)^2 + p_3^2\right]\,, \\ H_3 &= p_3\,, \\ H_4 &= p_2\,. \end{aligned} \tag{6.66}$$

This case can also be obtained by specializing Case 1 of Section 6.3 by choosing $B = b\,\mathrm{e}_z$ in Eq. (8.14).

6.4.2 *The helical undulator*

This interesting magnetic model [127] is described by the Hamiltonian (6.40) with a magnetic field of the form

$$B(x,y,z) = -b_3\cos\left(\frac{2z}{a}\right)\mathrm{e}_x + b_3\sin\left(\frac{2z}{a}\right)\mathrm{e}_y\,, \tag{6.67}$$

where b_3 and a are arbitrary constants.[1] The model is minimally superintegrable and the associated integrals of the motion are

$$\begin{aligned} H_1 &= \Pi_x + \frac{b_3\, a}{2}\, \cos\left(\frac{2z}{a}\right), \\ H_2 &= \Pi_y - \frac{b_3\, a}{2}\, \sin\left(\frac{2z}{a}\right), \\ H_3 &= \Lambda_z - \frac{a}{2}\,\Pi_z - \frac{1}{2}\left[b_3\, a\, x\, \sin\left(\frac{2z}{a}\right) + b_3\, a\, y\, \cos\left(\frac{2z}{a}\right)\right]. \end{aligned} \tag{6.68}$$

Their commutation relations read

$$\{H_1, H_2\} = 0\,, \qquad \{H_1, H_3\} = -H_2\,, \qquad \{H_2, H_3\} = H_1\,. \tag{6.69}$$

The Haantjes operators associated with this system are $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_5\}$, where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are given by Eqs. (6.45) and (6.47) respectively, and

$$\boldsymbol{K}_5 = \frac{1}{a\Lambda_z - 2\,(x^2+y^2)\,\Pi_z} \left[\begin{array}{ccc|ccc} ay^2 & -axy & 2y\,(x^2+y^2) & 0 & 0 & 0 \\ -axy & ax^2 & -2x\,(x^2+y^2) & 0 & 0 & 0 \\ \frac{a^2}{2}y & -\frac{a^2}{2}x & a\,(x^2+y^2) & 0 & 0 & 0 \\ \hline 0 & f_5 & g_5 & ay^2 & -axy & \frac{a^2}{2}y \\ -f_5 & 0 & h_5 & -axy & ax^2 & -\frac{a^2}{2}x \\ -g_5 & -h_5 & 0 & 2y\,(x^2+y^2) & -2x\,(x^2+y^2) & a\,(x^2+y^2) \end{array}\right], \tag{6.70}$$

with

$$\begin{aligned} f_5 &= \frac{a}{2}\,\left[a\,x\,\partial_z A_x + a\,y\,\partial_z A_y + 2\left(x\,y\,\left(\partial_x A_x - \partial_y A_y\right) + \left(y^2 - x^2\right)\,\partial_x A_y - \Lambda_z\right)\right], \\ g_5 &= \frac{1}{2}\,a^2\,y\,\partial_z A_z - a\left(x^2+y^2\right)\,\partial_z A_x + a\,y^2\,\partial_x A_z - a\,x\,y\,\partial_y A_z - 2\,y\,(y^2+x^2)\,\partial_x A_x \\ &\quad + 2\,x\,\left(x^2+y^2\right)\partial_x A_y + 2\,\left(x^2+y^2\right)\,\Pi_y\,, \\ h_5 &= -\frac{1}{2}\,a^2\,x\,\partial_z A_z - a\left(x^2+y^2\right)\partial_z A_y + a\,x^2\,\partial_y A_z - a\,x\,y\,\partial_x A_z + 2\,x\,(x^2+y^2)\partial_y A_y \\ &\quad - 2\,y\,\left(x^2+y^2\right)\partial_x A_y - 2\,\left(x^2+y^2\right)\,\Pi_x\,. \end{aligned}$$

1 A slightly more general system can be obtained by adding a constant contribution along the z-axis to the magnetic field (6.67). The system so obtained is still superintegrable, but not separable in any orthogonal coordinate system on Q (see [98]).

Integrating the annihilators associated with the algebra $\mathscr{H}_1 = \{\boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_2\}$, we find the set of DH coordinates

$$\begin{aligned} q^1 &= x\,, \qquad p_1 = \Pi_x + \frac{1}{2} a\, b_3\, \cos\left(\frac{2z}{a}\right), \\ q^2 &= y\,, \qquad p_2 = \Pi_y - \frac{1}{2} a\, b_3\, \sin\left(\frac{2z}{a}\right), \\ q^3 &= z\,, \qquad p_3 = \Pi_z\,. \end{aligned} \tag{6.71}$$

In this chart, the operators of the algebra $\mathscr{H}_1$ read

$$\begin{aligned} \boldsymbol{K}_1 &= \frac{1}{p_1 - \frac{1}{2} a\, b_3\, \cos\left(\frac{2q^3}{a}\right)} \operatorname{diag}\left[1,0,0,1,0,0\right], \\ \boldsymbol{K}_2 &= \frac{1}{p_2 + \frac{1}{2} a\, b_3\, \sin\left(\frac{2q^3}{a}\right)} \operatorname{diag}\left[0,1,0,0,1,0\right]; \end{aligned} \tag{6.72}$$

and the Hamiltonian function and the integrals of the motion take the form

$$\begin{aligned} H &= \frac{1}{2}\left[\left(p_1 - \frac{1}{2} a\, b_3\, \cos\left(\frac{2q^3}{a}\right)\right)^2 + \left(p_2 + \frac{1}{2} a\, b_3\, \sin\left(\frac{2q^3}{a}\right)\right)^2 + p_3^2\right], \\ H_1 &= p_1\,, \\ H_2 &= p_2\,. \end{aligned} \tag{6.73}$$

6.4.2.1 *Partial separability of the helical undulator*

We also mention that it is possible to construct a set of partial separation coordinates [39, 44, 160] for the helical undulator. This set is related to an incomplete system of commuting Hamiltonians, i.e. a system possessing fewer integrals than the dimension of the configuration space. In our case, the incomplete Poisson algebra is (H, H_3), to which corresponds the Haantjes algebra $\{\boldsymbol{I}, \boldsymbol{K}_5\}$. The operator $\boldsymbol{K}_5$ is semisimple and diagonalizes in the chart

$$\begin{aligned} q^1 &= r\,, & p_1 &= \Pi_r\,, \\ q^2 &= \frac{a}{2}\,\varphi + z\,, & p_2 &= \frac{2}{a}\Pi_\varphi + \Pi_z + b_3 r \sin\left(\varphi + \frac{2z}{a}\right), \\ q^3 &= \frac{a}{2}\,\varphi - z\,, & p_3 &= \frac{2}{a}\Pi_\varphi - \Pi_z - b_3 r \sin\left(\varphi + \frac{2z}{a}\right). \end{aligned} \tag{6.74}$$

These coordinates represent a set of partial separation variables. The integrals of the motion take the form

$$\begin{aligned} H = \frac{1}{8(q^1)^2}\Bigg[& a^2(p_2+p_3)^2 + 4(q^1)^2\left(p_1^2 + (p_2-p_3)^2\right) \\ & + 4b_3\,(q^1)^3 \sin\left(\frac{2q^2}{a}\right)\left(b_3\, q^1 \sin\left(\frac{2q^2}{a}\right) - 2p_2 + 2p_3\right)\Bigg], \\ H_3 = a\, p_3\,. & \end{aligned} \tag{6.75}$$

The corresponding system of HJ equations splits into a system of a PDE and an ODE, which *a priori* provide an integral W for the undulator:

$$\begin{cases} a^2\left(\dfrac{\partial W}{\partial q^2}+c_3\right)^2+4(q^1)^2\left(\left(\dfrac{\partial W}{\partial q^1}\right)^2+\left(\dfrac{\partial W}{\partial q^2}-c_3\right)^2\right) \\ \qquad\qquad +4b_3(q^1)^3\sin\left(\dfrac{2q^2}{a}\right)\left(b_3q^1\sin\left(\dfrac{2q^2}{a}\right)-2\dfrac{\partial W}{\partial q^2}+2c_3\right)=8(q^1)^2c, \\ \dfrac{\partial W}{\partial q^3}=c_3, \end{cases}$$

where $W=W_{12}(q^1,q^2)+W_3(q^3)$ is the Hamilton's characteristic function and c, c_3 are arbitrary constants.

6.5 RELATION WITH THE STÄCKEL GEOMETRY: THE AKN SYSTEMS

In this section, we shall establish a new connection between the Haantjes geometry for Hamiltonian systems in a magnetic field and the Stäckel geometry. In particular, we shall prove that the magnetic models discussed above belong to the generalized class of Stäckel systems introduced in [6], namely, they are AKN systems.

To prove this result, we shall determine the *Jacobi-Sklyanin SE* [174] and the associated Stäckel matrices for two specific systems, namely, the Hamiltonian superintegrable model for a constant magnetic field and the helical undulator.

For each set of separation coordinates, there exists a corresponding set of SE and an associated Stäckel matrix, as we shall see.

6.5.1 *Stäckel analysis for the system in a constant magnetic field*

We shall distinguish three cases.

(I) We shall first work in the coordinates (6.57) that diagonalize the Haantjes algebra $\mathscr{H}_1$. In order to obtain the corresponding Stäckel matrix, we need to rewrite the system (6.59) with a suitable functional combination of the Hamiltonians:

$$\begin{aligned} \tilde{H}_1 &= 2H-H_1^2=-2b\,q^2\,p_1+b^2\,(q^2)^2+p_2^2+p_3^2\,, \\ \tilde{H}_2 &= H_1=p_1\,, \\ \tilde{H}_3 &= H_3^2=p_3^2\,. \end{aligned} \tag{6.76}$$

Now, it is easy to see that the equations

$$\boldsymbol{S}\begin{bmatrix}\tilde{H}_1\\ \tilde{H}_2\\ \tilde{H}_3\end{bmatrix}=\begin{bmatrix}f_1(q^1,p_1)\\ f_2(q^2,p_2)\\ f_3(q^3,p_3)\end{bmatrix}, \qquad \boldsymbol{S}=\begin{bmatrix}S_{11}(q^1) & S_{12}(q^1) & S_{13}(q^1)\\ S_{21}(q^2) & S_{22}(q^2) & S_{23}(q^2)\\ S_{31}(q^3) & S_{32}(q^3) & S_{33}(q^3)\end{bmatrix} \tag{6.77}$$

can be solved by choosing

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2\,b\,q^2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.78}$$

and therefore $f_1(q^1, p_1) = p_1$, $f_2(q^2, p_2) = p_2^2 + b^2\,(q^2)^2$ and $f_3(q^3, p_3) = p_3^2$. Note that Eqs. (6.77) represent the SE for our Hamiltonian model (6.76).

(II) Let us consider the DH coordinates associated with the algebra $\mathscr{H}_2$. As before, we suitably combine the integrals of the motion to get the Stäckel basis

$$\begin{aligned} \tilde{H}_1 &= 2\,H - H_2^2 - H_3^2 + H_3 = p_1^2 + 2\,b\,q^1\,p_2 + p_3 + b^2\,(q^1)^2\,, \\ \tilde{H}_2 &= H_2 = p_2\,, \\ \tilde{H}_3 &= H_3 = p_3\,. \end{aligned} \tag{6.79}$$

From Eqs. (6.77), we deduce the Stäckel matrix

$$\boldsymbol{S} = \begin{bmatrix} 1 & -2\,b\,q^1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.80}$$

and the Stäckel functions $f_1(q^1, p_1) = p_1^2 + b^2\,(q^1)^2$, $f_2(q^2, p_2) = p_2$ and $f_3(q^3, p_3) = p_3$.

(III) The cylindrical coordinates (6.64) diagonalize the algebra $\mathscr{H}_3$. Adopting the same procedure as before, we obtain the Stäckel basis

$$\begin{aligned} \tilde{H}_1 &= 2\,H - b\,H_4 = p_1^2 + \frac{p_2^2}{(q^1)^2} + p_3^2 + \frac{1}{4}\,b^2\,(q^1)^2\,, \\ \tilde{H}_2 &= H_4^2 = p_2^2\,, \\ \tilde{H}_3 &= H_3^2 = p_3^2\,. \end{aligned} \tag{6.81}$$

The Stäckel matrix and functions are, respectively

$$\boldsymbol{S} = \begin{bmatrix} 1 & -\frac{1}{(q^1)^2} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.82}$$

and $f_1(q^1, p_1) = p_1^2 + \frac{1}{4}\,b^2\,(q^1)^2$, $f_2(q^2, p_2) = p_2^2$, $f_3(q^3, p_3) = p_3^2$.

6.5.2 *Stäckel analysis for the helical undulator*

By analogy with the previous discussion, we will work in the chart that diagonalizes the algebra $\mathscr{H}_1$ of the helical undulator. From the equations

$$\begin{aligned}
\tilde{H}_1 &= 2\,H - H_1^2 - H_2^2 - \frac{1}{4}\,a^2\,b_3^2 = -a\,b_3\,p_1\cos\left(\frac{2\,q^3}{a}\right) + a\,b_3\,p_2\sin\left(\frac{2\,q^3}{a}\right) + p_3^2\,,\\
\tilde{H}_2 &= H_1 = p_1\,,\\
\tilde{H}_3 &= H_2 = p_2\,,
\end{aligned} \tag{6.83}$$

using (6.77) we deduce the Stäckel matrix

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a\,b_3\,\cos\left(\frac{2q^3}{a}\right) & -a\,b_3\,\sin\left(\frac{2q^3}{a}\right) \end{bmatrix}, \tag{6.84}$$

and the Stäckel functions $f_1(q^1, p_1) = p_1$, $f_2(q^2, p_2) = p_2$ and $f_3(q^3, p_3) = p_3^2$.

6.6 NEW FAMILIES OF MAGNETIC SYSTEMS ON A RIEMANNIAN SPACE

In this section, we show that the previous geometric construction leads to new integrable models. Precisely, we wish to solve an inverse problem: given a Stäckel matrix S, find the most general Hamiltonian family of systems admitting a given Haantjes web. This means that the family shares S as the associated Stäckel matrix and the same $\omega\mathscr{H}$ manifold. Systems belonging to the same Haantjes web will automatically be separable in the same coordinate system.

The model with a constant magnetic field, being multiseparable, provides us with three different Stäckel matrices. We will generalize them in order to construct families of separable models.

(A) Let us consider the Stäckel matrix

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & b\,\xi_1(q^2) & \xi_2(q^2) \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.85}$$

where ξ_1 and ξ_2 are arbitrary functions. It clearly generalizes the Stäckel matrix (6.78). We define the auxiliary integrable model

$$\begin{aligned}
\hat{H}_1 &= -b\,\xi_1(q^2)\,\mu_1(q^1)\,p_1 + \mu_2(q^2)\,p_2^2 + \xi_2(q^2)\mu_3(q^3)\,p_3^2 + b^2\,\mu_4(q^2)\,,\\
\hat{H}_2 &= \mu_1(q^1)\,p_1\,,\\
\hat{H}_3 &= \mu_3(q^3)\,p_3^2\,,
\end{aligned} \tag{6.86}$$

where μ_1, μ_2, μ_3, μ_4 are additional arbitrary functions. We can consider this model as a generalization of the system (6.76). The Stäckel functions for the system (D.2) are $f_1 = \mu_1(q^1)\,p_1$,

$f_2 = \mu_2(q^2)\, p_2^2 + b^2\, \mu_4(q^2)$, $f_3 = \mu_3(q^3)\, p_3^2$. In order to construct the "magnetic" version of the latter model, we introduce the associated Hamiltonians

$$(\hat{H}_1 + \hat{H}_2^2, \hat{H}_2, \hat{H}_3)\,. \tag{6.87}$$

Then, using the transformation

$$\begin{aligned} q^1 &= x\,, \qquad p_1 = \Pi_x + \frac{b\,\xi_1(y)}{2\,\mu_1(x)}\,, \quad \mu_1(x) \neq 0 \\ q^2 &= y\,, \qquad p_2 = \Pi_y\,, \\ q^3 &= z\,, \qquad p_3 = \Pi_z\,, \end{aligned} \tag{6.88}$$

system (6.87) provides us with the family of integrable magnetic models which are the goal of our construction:

$$\begin{aligned} H &= \mu_1^2(x)\,\Pi_x^2 + \mu_2(y)\,\Pi_y^2 + \xi_2(y)\mu_3(z)\,\Pi_z^2 + b^2\left(\mu_4(y) - \frac{\xi_1^2(y)}{4}\right), \\ H_1 &= \mu_1(x)\left(\Pi_x + \frac{b}{2}\frac{\xi_1(y)}{\mu_1(x)}\right), \qquad H_2 = \mu_3(z)\,\Pi_z^2\,. \end{aligned} \tag{6.89}$$

Note that this system admits a Riemannian metric $g^{ii} = \mathrm{diag}\,[\mu_1^2(x), \mu_2(y), \xi_2(y)\mu_3(z)]$, where we assume $\mu_2(y) > 0$ and the non-vanishing functions $\xi_2(y)$, $\mu_3(z)$ have the same sign. The associated magnetic field is $B = \frac{b}{2}\xi_1'(y)\sqrt{\mu_2(y)}\,\mathrm{e}_z$; this condition also implies that the transformation (7.45) is canonical.

This system admits a semisimple Haantjes algebra of operators:

$$\boldsymbol{K}_1 = \frac{1}{2\,\mu_1(x)\,\Pi_x}\,\boldsymbol{R}_1\,, \qquad \boldsymbol{K}_2 = \frac{1}{\xi_2(y)}\,\boldsymbol{R}_3, \qquad \xi_2(y) \neq 0\,, \tag{6.90}$$

with $\boldsymbol{R}_1$ and $\boldsymbol{R}_3$ defined as in Eqs. (6.46) and (6.50).

(B) As in the previous case, the form of both Stäckel matrices (7.26) and (6.82) suggests the following generalization:

$$\boldsymbol{S} = \begin{bmatrix} 1 & -\psi_1(q^1) & \psi_2(q^1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.91}$$

where $\psi_1(q^1)$ and $\psi_2(q^1)$ are arbitrary functions. Following the same strategy (which entails some algebraic combinations of the involved Hamiltonians), we are led to the auxiliary integrable model

$$\begin{aligned} \hat{H}_1 &= \nu_1(q^1)\, p_1^2 + \psi_1(q^1)\, \nu_2^2(q^2)\, p_2^2 + \psi_2(q^1)\, \nu_3(q^3)\, p_3^2 + \nu_4(q^1)\,, \\ \hat{H}_2 &= \nu_2^2(q^2)\, p_2^2\,, \qquad \hat{H}_3 = \nu_3(q^3)\, p_3^2\,, \end{aligned} \tag{6.92}$$

with Stäckel functions $f_1 = \nu_1(q^1)\, p_1^2 + \nu_4(q^1)$, $f_2 = \nu_2^2(q^2)\, p_2^2$, $f_3 = \nu_3(q^3)\, p_3^2$. It widely generalizes the Stäckel basis (6.81), since ν_1, ν_2, ν_3, ν_4 are also arbitrary functions. To construct, as before, the "magnetic" version of system (6.92), we first observe that $\hat{H}_4 = \nu_2(q^2)\, p_2$ is a first integral. Then, we consider the associated model

$$(\hat{H}_1 + \hat{H}_4, \hat{H}_2, \hat{H}_3)\,. \tag{6.93}$$

By means of the transformation

$$\begin{aligned} q^1 &= x\,, \qquad & p_1 &= \Pi_x\,, \\ q^2 &= y\,, \qquad & p_2 &= \Pi_y - \frac{1}{2\,\psi_1(x)\,\nu_2(y)}\,, \qquad & \psi_1(x), \nu_2(y) &\neq 0 \\ q^3 &= z\,, \qquad & p_3 &= \Pi_z\,, \end{aligned} \tag{6.94}$$

we get the new integrable Hamiltonian family of magnetic systems

$$\begin{aligned} H &= \nu_1(x)\,\Pi_x^2 + \psi_1(x)\,\nu_2^2(y)\,\Pi_y^2 + \psi_2(x)\nu_3(z)\,\Pi_z^2 + \nu_4(x) - \frac{1}{4\,\psi_1(x)}\,, \\ H_1 &= \nu_2^2(y)\,\left(\Pi_y - \frac{1}{2\,\psi_1(x)\,\nu_2(y)}\right)^2\,, \\ H_2 &= \nu_3(z)\,\Pi_z^2\,. \end{aligned} \tag{6.95}$$

This system admits as well a Riemannian metric $g^{ii} = \mathrm{diag}\,[\nu_1(x), \psi_1(x)\,\nu_2^2(y), \psi_2(x)\nu_3(z)]$, when $\psi_1(x), \nu_1(x) > 0$ and the non-vanishing functions $\nu_3(z)$, $\psi_2(x)$ have the same sign. The associated magnetic field is $B = -\frac{\psi_1'(x)\sqrt{\nu_1(x)}}{2\,\psi_1^{3/2}(x)}\,\mathrm{e}_z$; this ensures that the transformation (D.10) is canonical.

Once again, we derive the associated semisimple algebra of Haantjes operators:

$$\boldsymbol{K}_2 = \frac{2\,\psi_1(x)\,\nu_2(y)\,\Pi_y - 1}{2\,\psi_1^2(x)\,\nu_2(y)\,\Pi_y}\,\boldsymbol{R}_2\,, \qquad \boldsymbol{K}_3 = \frac{1}{\psi_2(x)}\,\boldsymbol{R}_3\,, \tag{6.96}$$

with $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$ defined as in Eqs. (6.48) and (6.50).

Remark 6.6.1. *Some of the restrictions on the arbitrary functions appearing in the models previously defined can be relaxed if one wishes to define these systems on a pseudo-Riemannian geometry.*

6.7 SUPERINTEGRABLE FAMILIES OF MAGNETIC MONOPOLES WITH NON-RADIAL POTENTIAL IN CURVED BACKGROUND

Superintegrable systems with magnetic monopole interaction have received much attention in mathematical physics due to their rich symmetry structures and connections to fundamental problems in classical and quantum mechanics. To conclude this chapter, I will discuss the superintegrability of a general family of systems immersed in a magnetic monopole field, following [128]. This work extends known results on monopole dynamics in flat and curved spaces, demonstrating that minimal superintegrability persists even when radial symmetry is broken.

6.7.1 *Superintegrable systems and magnetic monopoles: an overview*

In the three-dimensional Euclidean space, any rotationally invariant electrostatic potential in the presence of a monopole is at least minimally superintegrable [127, 149]. Known maximally superintegrable exceptions are both the MIC-oscillator [131] and the hydrogen atom (MIC-Kepler monopole [80, 131, 199]). Their superintegrability has been investigated and confirmed on various curved spaces, including e.g. 3D Taub-NUT spaces of Kepler-type and oscillator-type [75, 78] and the 3D sphere [24].

Generalizations of superintegrable systems from flat to curved spaces have been extensively studied [8, 140, 162], resulting in the discovery of several families of superintegrable systems, often depending on a rational parameter related to the metric on the configuration space [37, 40, 87, 115, 156]. In the following, in analogy to the 3D generalization of the Tremblay-Turbiner-Winternitz (TTW) and PW systems obtained in [87], we present a family of superintegrable monopole systems in 3D curved background that include multi-fold Kepler systems [8, 79] as special cases.

Let us consider the 4D generalized Taub-NUT metric [78]

$$\mathrm{d}s^2 = f(R)(\mathrm{d}R^2 + \mathrm{d}\Theta^2 + R^2\sin^2(\Theta)\mathrm{d}\Phi^2) + g(R)(\mathrm{d}\Psi + \cos(\Theta)\mathrm{d}\Phi)^2, \tag{6.97}$$

for $R > 0, \Theta \in (0,\pi), \Phi \in [0,2\pi], \Psi \in [0,4\pi]$ and

$$f(R) = R^{\frac{1}{\nu}-2}(\alpha_1 + \beta_1 R^{\frac{1}{\nu}}), \qquad g(R) = \frac{(\alpha_1 + \beta_1 R^{\frac{1}{\nu}})R^{\frac{1}{\nu}}}{\alpha_2 R^{\frac{2}{\nu}} + \beta_2 R^{\frac{1}{\nu}} + 1}, \tag{6.98}$$

where ν, α_i, β_i are real parameters. By the conservation of the quantity $k = g(R)(\dot{\Psi} + \cos(\Theta)\dot{\Phi})$, the corresponding free dynamics reduces to monopole systems in 3D Iwai-Katayama spaces, the so-called multi-fold Kepler systems introduced in [79]. In particular, we have monopoles in oscillator-type and Kepler-type Taub-NUT spaces for $\nu = \frac{1}{2}$ and $\nu = 1$ [78], respectively. For $\nu = \frac{1}{2}, \beta_1 = 0$, $\alpha_1 = 1$, we have the MIC-oscillator monopole and for $\nu = 1, \alpha_1 = 0, \beta_1 = 1$, the MIC-Kepler monopole system in Euclidean space.

Let us now consider the 3D space with the metric

$$\begin{aligned} \mathrm{d}s^2 = r^{-1}(\alpha_1 + \beta_1 r)(\mathrm{d}r^2 + r^2 m^2 \mathrm{d}\theta^2 + r^2\sin^2(\theta)\mathrm{d}\varphi^2), \\ \theta \in (0,\pi),\ r > 0,\ \varphi \in \left[0, \frac{2\pi}{\nu}\right], \end{aligned} \tag{6.99}$$

where m is a nonvanishing constant. For $\nu = m^2 = 1$ we have just the Iwai-Katayama space for $\nu = 1$. For $m^2 \neq 1$ we have a curved 3D space with non-constant scalar curvature given by $\frac{2(1-m^2)}{m^2(\alpha_1+\beta_1 r)r} + \frac{3\alpha_1^2}{2(\alpha_1+\beta_1 r)^3 r}$. For $\alpha_1 = 0$ and $\beta_1 = \nu = 1$ this space is the one introduced in [87]. However, the most natural choice $\nu = 1$ for the domain of the angular variable φ does not exhaust all the systems of physical interest.

The Cotton-York tensor of the metric (6.99) vanishes identically for all m, therefore the underlying space is conformally flat. The canonical coordinate transformations

$$\begin{aligned} r &= R^{m\delta}, & \theta &= \Theta, & \varphi &= m\delta\Phi, \\ p_r &= \frac{R^{1-m\delta}P_R}{m\delta}, & p_\theta &= P_\Theta, & p_\varphi &= \frac{P_\Phi}{m\delta}, \end{aligned} \qquad \delta \in \{-1, 1\} \tag{6.100}$$

brings the metric (6.99) into the conformally flat form

$$\mathrm{d}s^2 = m^2 R^{m\delta-2}(\alpha_1 + \beta_1 R^{m\delta})(\mathrm{d}R^2 + R^2\mathrm{d}\Theta^2 + R^2\sin^2(\Theta)\mathrm{d}\Phi^2), \qquad R > 0,\ \Phi \in \left[0, \frac{2\pi}{m\nu}\right], \tag{6.101}$$

that corresponds to Iwai-Katayama spaces for $\nu = (m\delta)^{-1}$ (after proper renaming of α_1, β_1).

The corresponding multi-fold Kepler systems with magnetic monopole are known to be maximally superintegrable [8, 79]. In the following Section 6.7.2, we introduce a Hamiltonian system with non-rotationally invariant potential in the metric space (6.99) that, in the special case when the non-radial part of the potential vanishes, it reduces to the family of multi-fold Kepler systems for $\nu = (m\delta)^{-1}$. We will show in Section 6.7.3 that for $m \in \mathbb{Q}$, this new family of systems has (at least) four independent integrals of motion and it is therefore minimally superintegrable. Thus, after breaking the radial symmetry of the multi-fold Kepler system, minimal superintegrability is preserved. The system exhibits three integrals of order at most quadratic in the momenta and the fourth integral is generally of higher order, with its order depending on m. We conclude this discussion with some final remarks and conclusions in Section 6.7.4.

6.7.2 *New family of systems*

We consider a Hamiltonian system in the 3D space with the metric (6.99) in the field of a magnetic monopole of strength k

$$B(r, \theta, \varphi) = \frac{k}{r^2}\,\mathrm{e}_r\,, \qquad k \neq 0 \tag{6.102}$$

and an electrostatic potential of the form

$$W(r, \theta, \varphi) = W_1(r) + \frac{W_2(\theta)}{r(\alpha_1 + \beta_1 r)}. \tag{6.103}$$

Thus, its Hamiltonian function reads

$$H = \frac{r}{2(\alpha_1 + \beta_1 r)}\left(\Pi_r^2 + \frac{\Pi_\theta^2}{m^2 r^2} + \frac{\Pi_\varphi^2}{r^2\sin^2(\theta)}\right) + W_1(r) + \frac{W_2(\theta)}{r(\alpha_1 + \beta_1 r)}, \quad m \in \mathbb{R},\ m \neq 0, \tag{6.104}$$

where $\Pi_j = p_j + A_j$, $j = r, \theta, \varphi$. The spatial coordinates take values $r > 0$, $\theta \in (0, \pi)$, $\varphi \in \left[0, \frac{2\pi}{\nu}\right]$, where $\nu > 0$. The gauge for the magnetic field (6.102) can be chosen e.g. so that

$$A_r = A_\theta = 0\,, \quad A_\varphi = \ell - k\cos(\theta)\,, \tag{6.105}$$

where we prefer to leave the constant $\ell \in \mathbb{R}$ arbitrary for the moment.

For $m^2 = \nu = 1$ and $\alpha_1 = 0$ we have a charged particle in the three-dimensional Euclidean space moving under the influence of the magnetic field (6.102) and potential (6.103).

The system is rotationally invariant w.r.t to the angular variable φ and indeed in proper gauge choice, e.g. (6.105), the coordinate φ is cyclic and p_φ is a constant of motion. If we set p_φ equal to a constant, the corresponding reduced system is separable in the coordinates (r, θ). This reflects itself in the existence of the integrals

$$X_1 = \Pi_\varphi + k\cos(\theta) \quad \text{and} \quad X_2 = \Pi_\theta^2 + m^2\left(\frac{\Pi_\varphi^2}{\sin^2(\theta)} + 2W_2(\theta)\right) \tag{6.106}$$

that correspond to rotational invariance w.r.t to the angular variable φ and conservation of a generalization of the total angular momentum which incorporates terms due to the presence of the magnetic field.

Let us restrict our potential to the structure

$$W_1(r) = \frac{\alpha_2 r^2 + \beta_2 r + k^2}{2r(\alpha_1 + \beta_1 r)} \quad \text{and} \quad W_2(\theta) = \frac{4\left(a\cos^2\left(\frac{\theta}{2}\right) + b\sin^2\left(\frac{\theta}{2}\right)\right) + c}{\sin^2(\theta)}, \tag{6.107}$$

where $a, b, c, \alpha_i, \beta_i$ are constants. The motivation for this assumption comes from the fact that for $\nu = m^2 = 1$ we have a monopole system in Kepler-type Taub-NUT space and modified Hartmann potential [70].

Let us choose the gauge (6.105) with $\ell = k$ and use the fact that in this gauge choice p_φ is a constant. Thus, we set $p_\varphi = p_0$, p_0 constant and, after the canonical scaling

$$\theta \to \frac{\theta}{m}, \qquad p_\theta \to m p_\theta, \tag{6.108}$$

we see that the system (6.104) reduces to the $2D$ system

$$\tilde{H}(p_r, p_\theta, r, \theta) = \frac{r}{2(\alpha_1 + \beta_1 r)}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{1}{r(\alpha_1 + \beta_1 r)}\left(\frac{\alpha}{\sin^2(\mu\theta)} + \frac{\beta}{\cos^2(\mu\theta)}\right) + W_0(r), \tag{6.109}$$

where $m = \frac{1}{2\mu}$, $8\alpha = 8a + 2c + p_0^2$, $8\beta = 8b + 2c + (p_0 + 2k)^2$, $W_0(r) = W_1(r) - \frac{k^2}{2r(\alpha_1 + \beta_1 r)}$. Thus, for $\alpha_1 = 0, \beta_1 = 1$, the system becomes the PW system [151], known to be superintegrable for any $m \in \mathbb{Q}$. Therefore, besides p_φ, X_2 as in (6.106) and the Hamiltonian, we have another integral coming from the 2D reduced system that renders the original system minimally superintegrable.

The order of this integral depends on m. To our knowledge, this relationship between the MIC-Kepler monopole and the PW system has not been noticed before, though the 3D system (6.104)-(6.107) is known to be superintegrable for $c = 0, m^2 = \nu = 1$ [75]. In the following, we will prove its minimal superintegrability for any value of the constants $a, b, c, \alpha_i, \beta_i$ and any $m \in \mathbb{Q}$.

By assuming $\nu = (m\delta)^{-1}, \delta \in \{-1, 1\}$, the coordinate transformation (6.100) brings the system (6.104) with potential (6.107) to a generalized Taub-NUT space with new Hamiltonian given by

$$\mathcal{H} = \frac{R^{2-m\delta}}{2m^2(\alpha_1 + \beta_1 R^{m\delta})}\left(\Pi_R^2 + \frac{\Pi_\Theta^2}{R^2} + \frac{\Pi_\Phi^2}{R^2\sin^2(\Theta)} + \frac{\alpha_2 R^{2m\delta} + \beta_2 R^{m\delta} + k^2}{R^2} + \frac{4\left(a\cos^2\left(\frac{\Theta}{2}\right) + b\sin^2\left(\frac{\Theta}{2}\right)\right) + c}{R^2\sin^2(\Theta)}\right) \tag{6.110}$$

for $\theta \in (0, \pi)$, $\Phi \in [0, 2\pi]$. We will see in the following section 6.7.3 that system (6.104)-(6.107) is minimally superintegrable for every rational value of m and ν. As a consequence, we derive also the superintegrability of system (6.110) and of (6.109). This extends the known results on the superintegrability of the PW system and of the MIC-Kepler monopole system with modified Hartmann potential (6.107) to the more general Taub-NUT metric space corresponding to (6.97)-(6.98).

6.7.3 *Analysis of minimal superintegrability*

We will work from now on with the Hamiltonian (6.104) in the coordinates (r, θ, φ), with potential specified by (6.107). As we saw in the previous section, the system (6.104) has integrals X_1, X_2 as in (6.106). By following [87], we look for an additional integral that renders the system minimally superintegrable in the form

$$X = M(r, H, X_2) - N(\theta, X_2, p_\varphi) \equiv M(r) - N(\theta) \tag{6.111}$$

where, in order to achieve $\{X, H\} = 0$, N and M have to satisfy

$$\frac{p_\theta N'(\theta) - m^2 p_r r^2 M'(r)}{m^2 r(\alpha_1 + \beta_1 r)} = 0. \tag{6.112}$$

By using the conservation of $p_\varphi = p_0$ and $H = E_0$, $X_2 = E_1$, cf. (6.106), we can express p_r and p_θ as $p_r(r, E_0, E_1)$ and $p_\theta(\theta, E_1, p_0)$, respectively. This allows us to solve equation (6.112) by separation of variables, looking for N and M such that

$$m^2 p_r(r, E_0, E_1) r^2 M'(r) = 1 \quad \text{and} \quad p_\theta(\theta, E_1, p_0) N'(\theta) = 1. \tag{6.113}$$

Let us choose the gauge as in (6.105). It is more convenient in the following computation to fix the constants so that $\ell = 0$. This implies

$$M(r) = \pm \int \frac{1}{mr\sqrt{-\left(m^2\left(r(\beta_2 - 2E_0(\alpha_1 + \beta_1 r) + \alpha_2 r) + k^2\right)\right) - E_1}}\, dr$$

and

$$N(\theta) = \pm \int \frac{\sin\theta}{\sqrt{E_1 - m^2\left(4a+4b+2c+p_0^2\right) - 2m^2\cos(\theta)(2a-2b-kp_0) - \cos^2(\theta)\left(E_1+k^2m^2\right)}}\, d\theta\,.$$

By choosing the solution with a minus sign and integrating, we find

$$M(r) = -\frac{\arccos\left(\mathcal{T}_1\right)}{m\sqrt{E_1+k^2m^2}}, \qquad N(\theta) = \frac{\arcsin\left(\mathcal{T}_2\right)}{\sqrt{E_1+k^2m^2}}, \tag{6.114}$$

where

$$\mathcal{T}_1 = \frac{m^2\left(r(\beta_2 - 2\alpha_1 E_0) + 2k^2\right) + 2E_1}{mr\sqrt{4\alpha_1^2E_0^2m^2 + 8\beta_1E_0E_1 + 4E_0m^2\left(2\beta_1k^2 - \alpha_1\beta_2\right) - 4\alpha_2E_1 + m^2\left(\beta_2^2 - 4\alpha_2k^2\right)}}, \tag{6.115}$$

$$\mathcal{T}_2 = \frac{\cos(\theta)\left(E_1+k^2m^2\right) - m^2(-2a+2b+kp_0)}{\sqrt{2m^4\left(2a^2 - 2a(2b+k(k+p_0)) + 2b^2 - 2bk(k-p_0) - ck^2\right) - E_1m^2\left(4a+4b+2c-k^2+p_0^2\right) + E_1^2}}. \tag{6.116}$$

In order to obtain polynomial and globally defined integrals, we consider for $m = \frac{m_1}{m_2} \in \mathbb{Q}$, $g.c.d.(m_1, m_2) = 1$,

$$\mathcal{I} = 2ie^{\frac{1}{2}i\pi m_2}\sin\left(m_1\sqrt{E_1+k^2m^2}\left(N(\theta) - M(r)\right)\right),$$

that formally reads

$$\mathcal{I} = e^{i\pi m_2}\left(\sqrt{1-\mathcal{T}_2^2} + i\,\mathcal{T}_2\right)^{m_1}\left(\sqrt{1-\mathcal{T}_1^2} - i\,\mathcal{T}_1\right)^{m_2} - \left(\sqrt{1-\mathcal{T}_2^2} - i\,\mathcal{T}_2\right)^{m_1}\left(\sqrt{1-\mathcal{T}_1^2} + i\,\mathcal{T}_1\right)^{m_2}. \tag{6.117}$$

At this point, by using the fact that both square roots in the denominators of $\mathcal{T}_1$ and $\mathcal{T}_2$ are constants of motion, by multiplying $\mathcal{I}$ by suitable powers of those constants, we can obtain the simpler integral

$$\begin{aligned}\mathcal{X} = &\left(-2\sqrt{E_1+k^2m^2}\,p_r + \frac{2E_1 + m^2\left(r(\beta_2 - 2\alpha_1E_0) + 2k^2\right)}{mr}i\right)^{m_2} \\ &\cdot\left(\sin(\theta)\sqrt{E_1+k^2m^2}\,p_\theta + \left(m^2(2a-2b-kp_0) + \cos(\theta)\left(E_1+k^2m^2\right)\right)i\right)^{m_1} \\ &- \left(2\sqrt{E_1+k^2m^2}\,p_r + \frac{\left(m^2\left(-2\alpha_1E_0r + 2k^2 + \beta_2r\right) + 2E_1\right)}{mr}i\right)^{m_2} \\ &\cdot\left(\sin(\theta)\sqrt{E_1+k^2m^2}\,p_\theta - \left(m^2(2a-2b-kp_0) + \cos(\theta)\left(E_1+k^2m^2\right)\right)i\right)^{m_1},\end{aligned} \tag{6.118}$$

that except for the term $\sqrt{E_1 + k^2 m^2}$ would be polynomial in the momenta, once substituting the explicit polynomial expressions of $E_0 = H$ and $E_1 = X_2$ in terms of the momenta. However, let us notice that given any complex numbers $z_1 = a + bi$ and $z_2 = c + di$,

$$(-\bar{z}_1)^{m_2}(\bar{z}_2)^{m_1} - z_1^{m_2} z_2^{m_1} = (-\bar{z}_1)^{m_2}(\bar{z}_2)^{m_1} - (-1)^{m_2}(-z_1)^{m_2} z_2^{m_1}.$$

This expression equals to $-2i\,\mathrm{Im}((-z_1)^{m_2} z_2^{m_1})$ for m_2 even and to $2\mathrm{Re}((-z_1)^{m_2} z_2^{m_1})$ for m_2 odd. By expanding the product $(-z_1)^{m_2} z_2^{m_1}$ we obtain

$$(-z_1)^{m_2} z_2^{m_1} = (-1)^{m_2} \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} i^{j+k} \begin{pmatrix} m_1 \\ j \end{pmatrix} \begin{pmatrix} m_2 \\ k \end{pmatrix} a^{m_2-k} b^k c^{m_1-j} d^j.$$

Consequently, upon expanding the products, the integral $\mathcal{X}$ will include contributions of the form $(E_1 + k^2 m^2)^{\frac{m_1+m_2-(k+j)}{2}}$. If m_2 is even, only exponents with $j + k$ odd would be present (i.e. we take only the imaginary part of the expansion as above). Since g.c.d.$(m_1, m_2) = 1$ by assumption, for m_2 even m_1 must be odd. Therefore $m_1 + m_2 - (k + j)$ is even and only integer powers of $(E_1 + k^2 m^2)$ appear in $\mathcal{X}$. Otherwise, for m_2 odd, only terms for $j + k$ even would be in the integral (i.e. only the real part). Therefore, for m_1 odd, we have again that $m_1 + m_2 - (k + j)$ is an even number and only integer powers of $(E_1 + k^2 m^2)$ are present in the integral. For m_1 even, $m_1 + m_2 - (k + j)$ is odd. However, we can divide $\mathcal{X}$ by $\sqrt{E_1 + k^2 m^2}$ (which is a constant of motion) and obtain an integral with only even powers of $(E_1 + k^2 m^2)$. Therefore, we can always achieve a polynomial integral.

6.7.4 *Final remarks*

In this section, we proved the minimal superintegrability of the monopole system (6.104) with potential specified by (6.107). For particular values of its parameters, this system is related via conformal transformation to the MIC-Kepler, the MIC-oscillator and the more general multi-fold Kepler system (6.110) modified by the addition of a non-radial term in the potential.

In recent years, there has been active research on (super)integrable systems in curved backgrounds or, alternatively, with position-dependent mass in Euclidean space [74, 89, 145], often resulting in the proof of superintegrability of Kepler-related systems and oscillator-related systems in curved spaces [9, 37, 75, 87]. Here we saw that this is also true in presence of a magnetic monopole in the metric space corresponding to (6.99). This extends known results for Kepler-type and monopole-type systems in generalized Taub-NUT spaces and shows that minimal superintegrability is preserved even if the non-radial term specified by (6.107) is added to the potential.

We attempted to search for a fifth integral by the method used in the previous section, however with the negative result that such an integral would be polynomial only for $a = b = c = 0$, i.e. for vanishing angular term in the potential, and globally defined only for $\nu \in \mathbb{Q}$. Therefore we just obtain the known result on the maximal superintegrability of the multi-fold Kepler system. Numerical investigation seems to confirm that for generic values of the parameters a, b, c there

can exist bounded orbits which do not close, therefore we expect the system to be in general only minimally superintegrable.

Let us notice that given the Hamiltonian in the form

$$H = \frac{1}{2} f(r) \left(p_r^2 + \frac{L}{r^2} \right), \tag{6.119}$$

where L is a constant of motion, by scaling $L \to \frac{L}{m^2}$ the new Hamiltonian

$$\tilde{H} = \frac{1}{2} f(r) \left(p_r^2 + \frac{L}{m^2 r^2} \right) \tag{6.120}$$

is still integrable [115]. However, it corresponds to a system in general with different metric and there is no guarantee that superintegrability is preserved. Therefore, the integrability of the system (6.104) with potential (6.107) for any nonvanishing m comes from the integrability of the corresponding Kepler-type monopole system for $m = 1$. However, the superintegrability of the system for general value of m can not be deduced as a consequence of the superintegrability of the original system, as to our knowledge there is no proof that all the integrals survive the scaling, though this is often the case if m is rational for many examples known [115]. It might be worth further investigation in this direction.

We conclude by noting that the system (6.110) belongs to the Rodríguez-Tempesta family of systems, which has been conjectured to be superintegrable [162]. In the limit for vanishing strength of the monopole and for $\alpha_1 = c = 0$, the Hamiltonian (6.110) is included in it for special choices of the parameters therein. In this perspective, it might come as a surprise that only the PW system appeared here, as a 2D reduction of the system (6.104), though also the TTW system is included in the family of systems presented in [162]. The TTW system can indeed be obtained from Hamiltonian (6.110) (canonically conjugated to (6.104)) after setting $m = \frac{2}{\delta}$, p_φ to constant and then performing the scaling (6.108). This seems to suggest that also in presence of a magnetic field, a similar family of superintegrable systems as the one introduced in [162] that includes the monopole might exist.

Part III

GENERALIZED NIJENHUIS GEOMETRY

Un cronopio pequeñito buscaba la llave de la puerta de calle en la mesa de luz, la mesa de luz en el dormitorio, el dormitorio en la casa, la casa en la calle. Aquí se detenía el cronopio, pues para salir a la calle precisaba la llave de la puerta.

Julio Cortázar, *Historias de cronopios y de famas*

7

Generalized Haantjes algebras and block-diagonalization

A definition without purpose is like a path that goes nowhere–it leaves us trudging in endless circles, chasing after woozles and wizzles that vanish the moment we think we've grasped them. But when a concept is crafted to reveal its essence, it becomes a compass: something that doesn't just guide our thinking, but roots us in what truly matters. This act of purposeful defining helps us sidestep those dizzying loops of confusion, replacing them with clarity—and in that clarity, we find beauty. Because beauty isn't a luxury; it's what we need to endure life's weight. That's how definitions stop being bland labels and start acting like lighthouses, cutting through the fog to show us shores of understanding we might've missed.

The notion of *generalized Nijenhuis torsions*, introduced in [186], establishes a graded hierarchy of tensorial conditions for $(1,1)$-tensor fields, indexed by non-negative integers. Beginning with the Lie bracket $[X,Y]$ as the *torsion of level o*, the sequence advances to the classical *Nijenhuis torsion* (level 1) and *Haantjes torsion* (level 2). For each $m \geq 1$, the torsion of level m is defined recursively through algebraic operations involving the tensor field and the torsion of level $m-1$. A $(1,1)$-tensor field is termed a *generalized Nijenhuis operator of level m* if its torsion of level m vanishes, thereby unifying classical integrability criteria–such as Nijenhuis ($m=1$) and Haantjes ($m=2$) operators—into a coherent infinite hierarchy.

These generalized Nijenhuis operators enable the study of the integrability of operator fields beyond the framework of Haantjes geometry. As an illustrative example [186, Example 44], consider

the following operator defined in the chart (x^1, x^2, x^3), which is *not* a Haantjes operator but possesses integrable distributions, as one can verify directly. In fact, its torsion of level three $\tau_{\boldsymbol{A}}^3(X, Y)$ vanishes:

$$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & f & 0 \\ 0 & \lambda_1 & g \\ 0 & 0 & \lambda_2 \end{bmatrix},$$

where $\lambda_1, \lambda_2, f, g \in C^\infty(M)$ and may depend on any combination of the variables (x^1, x^2, x^3). To analyze the integrability of its eigendistributions, we must rely on the Theorem 7.1.1 since the classical Haantjes theorem is not applicable. Furthermore, existing results provide no insight into the simultaneous diagonalization of operator sets that include even one such non-Haantjes operator. To bridge these gaps, this chapter introduces *generalized Haantjes algebras* and systematically investigates their structural and integrability properties. The purpose of this chapter is to establish a new geometric setting for the block-diagonalization of operator fields on a differentiable manifold.

The problem of finding normal forms of $(1, 1)$-tensor fields on a differentiable manifold M is relevant in many theoretical and applicative contexts. Indeed, it is useful in the determination of a reduced form for a system of partial differential equations [20, 21, 23] and in the theory of hydrodynamic-type systems [55, 56]. Besides, it finds applications in classical mechanics, since it may allow to construct separation variables for completely integrable Hamiltonian systems [39]. In this chapter we address this problem with a tensorial approach, based on the notion of Nijenhuis and Haantjes torsions and their generalizations, recently introduced in [186].

From a theoretical point of view, a first result is Theorem 7.3.1. It states that for a set of commuting operators on a differentiable manifold, if at least one of them is a generalized Nijenhuis operator, then there exists a local coordinate chart where they take simultaneously a block-diagonal form.

This statement is further refined by requiring that all of the commuting operators are generalized Nijenhuis operators of a (common) level m and their joint eigendistributions are *pairwise integrable* (see Definition 2.3.4). Thus, our main result is Theorem 7.3.2 ensuring that, under these hypotheses, on a local coordinate chart all of them take a block-diagonal form with finer blocks.

These results naturally suggest the notion of *generalized Haantjes algebra* of level m. It consists of a set of generalized Nijenhuis operators of the same level m, forming a $C^\infty(M)$-module; besides, this set is closed under the product of operators. We will discuss in detail the properties of the class of unigenerated Haantjes algebras of level m, which are generated by the independent powers of a single generalized Nijenhuis operator.

Another crucial result is that given an operator $\boldsymbol{A}$ whose generalized Nijenhuis torsion of level m vanishes, any polynomial $P(\boldsymbol{A})$ whose coefficients are functions in $C^\infty(M)$ also has the same level-m torsion vanishing. This property allows us to construct generalized Haantjes algebras in a very natural and direct way. We shall present some nontrivial examples of unigenerated generalized Haantjes algebras of level three and four. In particular, we will show how to construct local coordinate charts allowing the simultaneous block-diagonalization of all the operators forming these algebras.

The approach proposed in this work offers a new perspective on the classical problem of the normal form of operator fields. We remind that a relevant contribution to this problem, in the spirit of tensor analysis, was given by Bogoyavlenskij in [21, 23], where necessary and sufficient conditions have been proposed for the existence of local charts were a given operator acquires a block-diagonal form. The main difference with respect to those results is that we solve the problem for a *complete family* of commuting operators. At the same time, no knowledge *a priori* of the eigendistributions of the given operators is required. This aspect is crucial, since the spectral analysis of operator fields computationally becomes rapidly intractable by increasing the dimension of the underlying differentiable manifold. Thus, we offer sufficient tensorial conditions for the simultaneous block-diagonalization of a family of operators which can be easily checked with the aid of computer algebra, without entering the study of their eigendistributions. Of course, for the explicit construction of local charts where the simultaneous block-diagonalization of the family of operators given takes place, it is necessary to determine their spectral properties. However, our result can be interpreted as a tensorial test, ensuring *a priori* the block-diagonalizability of the whole family. Once this property is ascertained for a concrete family, then one can proceed to the construction of the required local charts.

A thorough study of the rich algebraic properties of the generalized Haantjes algebras represents an interesting research perspective, that will be developed in future works.

This chapter includes the results of [159] and is organized as follows. First, Section 7.1 discusses an infinite family of higher-order torsions of level m, generalizing both the Nijenhuis and the Haantjes ones. In Section 7.2, the new notion of generalized Haantjes algebra is presented. In Section 7.3, the main theorems of the work, concerning simultaneous block-diagonalization of non-semisimple operator fields, are proved. In Sections 7.4 and 7.5, examples of generalized Haantjes algebras are discussed and local charts ensuring simultaneous block-diagonalization are explicitly determined. Finally, Section 7.6 investigates polarizations of these torsions, linking them to Haantjes modules.

7.1 GENERALIZED NIJENHUIS OPERATORS

A novel class of generalized Nijenhuis operators of level m has been introduced via a recursive procedure in [184] and further discussed in [186, 187]. When $m = 1$, they coincide with the standard Nijenhuis operators; for $m = 2$, they are Haantjes operators. For $m > 2$, we have genuinely new operators. In this chapter, we will generalize the notion of Haantjes algebras to consider situations in which they may contain generalized Nijenhuis operators. We will also discuss the block-diagonalization of these generalized algebras.

Definition 7.1.1 (Definition 7, [184]). *Let* $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ *be a* $(1,1)$*-tensor field. The* generalized Nijenhuis torsion of $\boldsymbol{A}$ of level m*, for each integer* $m \geq 1$*, is the skew-symmetric* $(1,2)$*-tensor field defined by*

$$\tau_{\boldsymbol{A}}^{(m)}(X,Y) = \boldsymbol{A}^2\,\tau_{\boldsymbol{A}}^{(m-1)}(X,Y) + \tau_{\boldsymbol{A}}^{(m-1)}(\boldsymbol{A}X,\boldsymbol{A}Y) - \boldsymbol{A}\Big(\tau_{\boldsymbol{A}}^{(m-1)}(X,\boldsymbol{A}Y) + \tau_{\boldsymbol{A}}^{(m-1)}(\boldsymbol{A}X,Y)\Big), \quad X,Y \in \mathfrak{X}(M). \tag{7.1}$$

Here, we adopt the notation $\tau_{\boldsymbol{A}}^{(0)}(X,Y) := [X,Y]$, $\tau_{\boldsymbol{A}}^{(1)}(X,Y) := \tau_{\boldsymbol{A}}(X,Y)$, *and* $\tau_{\boldsymbol{A}}^{(2)}(X,Y) := \mathcal{H}_{\boldsymbol{A}}(X,Y)$.

Definition 7.1.2. *Let* $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ *be a* $(1,1)$*-tensor field. If* $\tau_{\boldsymbol{A}}^{(m)}(X,Y) = \mathbf{0}$ *for some* $m \in \mathbb{N}\setminus\{0\}$, *we say that* $\boldsymbol{A}$ *is a* generalized Nijenhuis operator of level m.

The significance of these definitions lies in the following key result, which comprises and generalizes Haantjes' theorem. It will play a central role in the subsequent analysis.

Theorem 7.1.1 (Generalized Haantjes theorem, [186])**.** *Let* $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ *be an operator field. Assume that*

$$\tau_{\boldsymbol{A}}^{(m)}(X,Y) = \mathbf{0}\,, \qquad \forall\, X,Y \in \mathfrak{X}(M)\,, \tag{7.2}$$

for some $m \geq 1$. *Then each eigen-distribution of* $\boldsymbol{A}$, *as well as every direct sum of its eigen-distributions, is integrable.*

In [93], an alternative notion of generalized Nijenhuis torsion was independently proposed. Instead of employing a geometric and recursive approach, an algebraic construction is introduced that considers generalized torsions of order m associated with an arbitrary vector-valued skew-symmetric bilinear map on a real vector space. However, this other definition will not be used in this work.

7.1.1 *Main properties*

One of the most significant results concerning generalized Nijenhuis tensors is the following extension of Proposition 2.3.3, which will be crucial for defining generalized Haantjes algebras in Chapter 7.

Proposition 7.1.1 (Corollary 14, [186])**.** *Let* $\boldsymbol{A}$ *be an operator field. Then, for all* $f, g \in C^\infty(M)$, *it holds that*

$$\tau_{f\boldsymbol{I}+g\boldsymbol{A}}^{(m)}(X,Y) = g^{2m}\,\tau_{\boldsymbol{A}}^{(m)}(X,Y), \quad \forall\, m \in \mathbb{N}\setminus\{0,1\}. \tag{7.3}$$

It is also useful, from an applicative point of view, to consider the explicit action of a generalized Nijenhuis torsion on a pair of generalized eigenvectors of $\boldsymbol{A}$. In [186], the following formula is proved by induction for integers $m \geq 2$:

Proposition 7.1.2. *Let* $\boldsymbol{A}$ *be a* $(1,1)$*-tensor field and let* $X_\alpha \in \mathcal{D}_\mu$ *and* $Y_\beta \in \mathcal{D}_\nu$ *be generalized eigenvectors of* $\boldsymbol{A}$ *corresponding to the eigenvalues* μ *and* ν, *respectively. Then, for any integer* $m \geq 2$, *the following formula holds:*

$$\tau_{\boldsymbol{A}}^{(m)}(X_\alpha, Y_\beta) = \sum_{i,j=0}^{m} (-1)^{i+j} \binom{m}{i}\binom{m}{j} (\boldsymbol{A}-\mu\boldsymbol{I})^{m-i} (\boldsymbol{A}-\nu\boldsymbol{I})^{m-j} [X_{\alpha-i}, Y_{\beta-j}], \tag{7.4}$$

where

$$\boldsymbol{A}X_\alpha = \mu\, X_\alpha + X_{\alpha-1}, \qquad \boldsymbol{A}Y_\beta = \nu\, Y_\beta + Y_{\beta-1}, \qquad 1 \leq \alpha \leq \rho_\mu, \quad 1 \leq \beta \leq \rho_\nu, \tag{7.5}$$

and by definition, X_0 *and* Y_0 *are null vector fields.*

7.1.2 *Block-diagonalization*

As a nontrivial application of Theorem 7.1.1, one can show that for a given operator $\boldsymbol{A}$, condition (7.2) is sufficient to guarantee the existence of a local chart in which $\boldsymbol{A}$ can be *block-diagonalized*. We foresee relevant applications in, for example, the theory of hydrodynamic-type systems [21], the study of partial separability of Hamiltonian systems [39], and more generally in the context of Courant's problems for first-order hyperbolic systems of partial differential equations [43].

In the case of a single operator with vanishing higher-order torsion, an analysis analogous to that in Section 2.3.3 leads to the following result, which provides a simple and general tensorial criterion for the existence of local coordinates ensuring block-diagonalizability.

Theorem 7.1.2 (Proposition 43, [186]). *Let $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. If*

$$\tau^{(m)}_{\boldsymbol{A}}(X,Y) = \mathbf{0} \qquad \forall\, X, Y \in \mathfrak{X}(M), \tag{7.6}$$

for some $m \geq 1$, then $\boldsymbol{A}$ admits local charts adapted to the spectral decomposition of the tangent spaces into generalized eigenspaces, in which it assumes a block-diagonal form.

7.2 GENERALIZED HAANTJES ALGEBRAS

One of the main achievements in this context is to generalize this result to the case of families of commuting operators. To this aim, we shall introduce a new class of operator algebras.

7.2.1 *Definitions*

The notion of Haantjes algebra has been introduced and discussed in [184]. In this section, we define a class of new, generalized Haantjes algebras.

Definition 7.2.1. *A generalized Haantjes algebra of level m and rank r is a pair $(M, \mathscr{H}^{(m)})$ with the following properties:*

(i) M is a differentiable manifold of dimension n;

(ii) $\mathscr{H}^{(m)}$ is a set of operators $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ whose Nijenhuis torsion of level m vanishes: $\tau^{(m)}_{\boldsymbol{K}} = \mathbf{0}$. Also, they generate:

- *a free module of rank r over the ring of smooth functions on M:*

$$\begin{aligned} \tau^{(m)}_{(f\boldsymbol{K}_1 + g\boldsymbol{K}_2)}(X,Y) = \mathbf{0}\,, \qquad & \forall\, X, Y \in \mathfrak{X}(M)\,, \\ & \forall\, f, g \in C^\infty(M)\,, \quad \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}^{(m)}\,; \end{aligned} \tag{7.7}$$

- *a ring w.r.t. the composition operation:*

$$\tau^{(m)}_{(\boldsymbol{K}_1 \boldsymbol{K}_2)}(X,Y) = \mathbf{0}\,, \qquad \forall\, \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}^{(m)}\,, \quad \forall\, X, Y \in \mathfrak{X}(M)\,. \tag{7.8}$$

If

$$\boldsymbol{K}_1 \boldsymbol{K}_2 = \boldsymbol{K}_2 \boldsymbol{K}_1 , \qquad \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}^{(m)} , \tag{7.9}$$

the algebra $(M, \mathscr{H}^{(m)})$ *will be said to be an Abelian generalized Haantjes algebra. Moreover, if the identity operator* $\boldsymbol{I} \in \mathscr{H}^{(m)}$*, then* $(M, \mathscr{H}^{(m)})$ *will be said to be a generalized Haantjes algebra with identity.*

In other words, the set $\mathscr{H}^{(m)}$ can be regarded as an associative algebra of generalized Haantjes operators.

The case $m = 2$, namely that of standard Haantjes algebras [184], possesses several important properties. Among them, we recall that for a given Abelian Haantjes algebra $\mathscr{H}^{(2)} \equiv \mathscr{H}$ there exists an associated set of coordinates, called *Haantjes coordinates*, by means of which all $\boldsymbol{K} \in \mathscr{H}$ can be written simultaneously in a block-diagonal form. In particular, if $\mathscr{H}$ is also semisimple, on each set of Haantjes coordinates all $\boldsymbol{K} \in \mathscr{H}$ can be written simultaneously in a purely diagonal form [184].

The following result simplifies the study of the integrability of the eigendistributions of a family of operators forming an Abelian, generalized Haantjes algebra.

Lemma 7.2.1. *Let* $(M, \mathscr{H}^{(m)})$ *be an Abelian generalized Haantjes algebra of level* m*. We shall assume that the rank of the eigendistributions of the operators belonging to* $\mathscr{H}^{(m)}$ *is independent of* $\boldsymbol{x} \in M$*. Then, each nontrivial intersection of these eigendistributions is integrable.*

Proof. Let $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_w\}$ be a basis of $(M, \mathscr{H}^{(m)})$, and $(\mathcal{D}^{(1)}_{i_1}, \dots, \mathcal{D}^{(w)}_{i_w})$, $i_1 = 1, \dots, s_1, \dots$, $i_w = 1, \dots, s_w$, be the set of their eigendistributions. Let

$$\mathcal{V}_a = \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \bigcap \dots \bigcap \mathcal{D}^{(w)}_{i_w}(\boldsymbol{x}) \qquad a = 1, \dots, v, \qquad v \leq n \tag{7.10}$$

denote a nontrivial intersection of eigendistributions of the operators $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_w\}$, which we call a *joint eigendistribution*. Consequently, the distribution $\mathcal{V}_a$, being the intersection of distributions which are involutive due to Theorem 7.1.1 [186], is also involutive.

□

7.2.2 *Generalized and unigenerated Haantjes algebras*

A special class of commutative generalized Haantjes algebras are those generated by a single operator $\boldsymbol{L}$ and its independent powers. We denote by h the degree of the minimal polynomial of $\boldsymbol{L}$.

Definition 7.2.2. *A generalized Haantjes algebra* $(M, \mathscr{H}^{(m)})$ *will be said to be unigenerated if there exists an operator* $\boldsymbol{L}: \mathfrak{X}(M) \to \mathfrak{X}(M)$*, with* $\tau^{(m)}_{\boldsymbol{L}}(X, Y) = 0$*, such that* $\mathscr{H}^{(m)} \subseteq \mathscr{L}(\boldsymbol{L})$*, where* $\mathscr{L}(\boldsymbol{L})$ *is the set of all polynomials* $\sum_{k=0}^{h} c_k(\boldsymbol{x}) \boldsymbol{L}^k$ *in the powers of* $\boldsymbol{L}$ *with coefficients* $c_k(\boldsymbol{x}) \in C^\infty(M)$*.*

Given a unigenerated algebra $(M, \mathscr{H}^{(m)})$ generated by an operator $\boldsymbol{L}$, its rank is less than or equal to h. If $h = n - 1$, $\boldsymbol{L}$ is a *cyclic operator* and the algebra will be said to be *cyclic*.

Remark 7.2.1. *Unigenerated Haantjes algebras of level m as algebraic structures are not rare. Indeed, as we will prove in Theorem 7.2.1, given an operator $\boldsymbol{L}$ with $\tau_{\boldsymbol{L}}^{(m)}(X,Y) = 0$, any polynomial in $\boldsymbol{L}$ having coefficients in $C^\infty(M)$ is still an operator having a vanishing torsion of the same level. Thus, the independent powers of any operator with a given vanishing generalized torsion provide naturally a basis of a unigenerated generalized algebra.*

In the following, we shall prove this fundamental fact. To this aim, we shall use and adapt to our general case several algebraic techniques and results introduced and proved by Bogoyavlenskij in [19] for the case of Nijenhuis and Haantjes operators. In particular, we shall use a representation of (1,2)-tensors in the ring S_3 of polynomials of three independent variables z, λ, μ with coefficients depending on $\boldsymbol{x} \in M$. A generic polynomial of this ring has the form

$$S(z,\lambda,\mu) := \sum_{i,j,k}^{N} s_{ijk}(\boldsymbol{x}) z^i \lambda^j \mu^k, \qquad \boldsymbol{x} \in M$$

with $N \in \mathbb{N}\backslash\{0\}$, $s_{ijk}(\boldsymbol{x}) \in C^\infty(M)$. Given an operator $\boldsymbol{A}$ and a (1,2)-tensor $T(X,Y)$, we introduce the representation defined by [19]

$$R_S(T)(X,Y) = \sum_{i,j,k}^{N} s_{ijk}(\boldsymbol{x}) \boldsymbol{A}^i T(\boldsymbol{A}^j X, \boldsymbol{A}^k Y)\,. \tag{7.11}$$

Thus, the action of λ and μ is associated with the first and second arguments of $T(X,Y)$, whereas z is associated with the powers of $\boldsymbol{A}$ acting on the values of $T(X,Y)$. Representation (7.11) satisfies the basic properties

$$R_{S_1+S_2} = R_{S_1} + R_{S_2}, \tag{7.12}$$

$$R_{S_1 \cdot S_2} = R_{S_2 \cdot S_1} = R_{S_1} \cdot R_{S_2}\,. \tag{7.13}$$

This framework allows us to represent the "tower" of generalized Nijenhuis torsions in a direct way.

Lemma 7.2.2. *Let $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. We have:*

$$\tau_{\boldsymbol{A}}^{(m+1)}(X,Y) = R_\sigma \tau_{\boldsymbol{A}}^{(m)}(X,Y) \tag{7.14}$$

where σ is the polynomial $\sigma(z,\lambda,\mu) = (z-\lambda)(z-\mu)$.

Proof. It is an immediate consequence of the formula (7.1), which indeed can be written as

$$\begin{aligned} R_{(z^2 - z\lambda - z\mu + \lambda\mu)} \tau_{\boldsymbol{A}}^{(m)}(X,Y) =& \boldsymbol{A}^2 \tau_{\boldsymbol{A}}^{(m)}(X,Y) + \tau_{\boldsymbol{A}}^{(m)}(\boldsymbol{A}X, \boldsymbol{A}Y) \\ & - \boldsymbol{A}\left(\tau_{\boldsymbol{A}}^{(m)}(X, \boldsymbol{A}Y) + \tau_{\boldsymbol{A}}^{(m)}(\boldsymbol{A}X, Y)\right) \\ =& \tau_{\boldsymbol{A}}^{(m+1)}(X,Y)\,. \end{aligned} \tag{7.15}$$

□

When $m = 2$, we recover the formula

$$\mathcal{H}_{\boldsymbol{A}}(X,Y) = R_\sigma \tau_{\boldsymbol{A}}(X,Y)$$

first stated in [19]. We also remind that, given a polynomial $P(z) = \sum_{k=0}^{N} c_k(\boldsymbol{x}) z^k$, the *Bézout identity* holds:

$$P(z) - P(\lambda) = (z-\lambda) Q_P(z,\lambda) \tag{7.16}$$

where

$$Q_P(z,\lambda) = \sum_{k=1}^{N} c_k(\boldsymbol{x}) \sum_{p+q=k-1} z^p \lambda^q \,.$$

We prove now a useful technical result.

Lemma 7.2.3. *Let $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator. Let $\boldsymbol{P} := P(\boldsymbol{A}) = \sum_{k=0}^{N} c_k(\boldsymbol{x}) \boldsymbol{A}^k$ be a polynomial in $\boldsymbol{A}$ with variable coefficients. We have*

$$\tau^{(m)}_{\boldsymbol{P}}(X,Y) = R_{(Q_P(z,\lambda))^m (Q_P(z,\mu))^m} \tau^{(m)}_{\boldsymbol{A}}(X,Y), \qquad m \geq 2 \,. \tag{7.17}$$

Proof. We shall proceed by induction over $m \geq 2$. The case $m = 2$, corresponding to the Haantjes torsion, has been proved in [19]. Thus, we assume that the property is true for the case of a torsion of level $m-1$ and we prove that it holds true for the case of a torsion of level m. According to Lemma 7.2.2, we have

$$\tau^{(m)}_{\boldsymbol{A}}(X,Y) = R_\sigma \tau^{(m-1)}_{\boldsymbol{A}}(X,Y) \tag{7.18}$$

where $\sigma = (z-\lambda)(z-\mu)$. Also, in terms of the operator $\boldsymbol{P}$ expression (7.18) can be written as

$$\tau^{(m)}_{\boldsymbol{P}}(X,Y) = R_{(P(z)-P(\lambda))(P(z)-P(\mu))} \tau^{(m-1)}_{\boldsymbol{P}}(X,Y) \,. \tag{7.19}$$

Thus, applying twice the Bezout identity (7.16), we get

$$(P(z)-P(\lambda))(P(z)-P(\mu)) = (z-\lambda)(z-\mu) Q_P(z,\lambda) Q_P(z,\mu) \,.$$

Exploiting property (7.13), we obtain

$$\tau^{(m)}_{\boldsymbol{P}}(X,Y) = R_{Q_P(z,\lambda) Q_P(z,\mu)} R_\sigma \tau^{(m-1)}_{\boldsymbol{P}}(X,Y) \,. \tag{7.20}$$

By the induction hypothesis, we have

$$\tau^{(m-1)}_{\boldsymbol{P}}(X,Y) = R_{(Q_P(z,\lambda))^{m-1} (Q_P(z,\mu))^{m-1}} \tau^{(m-1)}_{\boldsymbol{A}}(X,Y) \,.$$

Consequently, due to Eq. (7.20) and Lemma 7.2.2, we deduce that

$$\begin{aligned} \tau^{(m)}_{\boldsymbol{P}}(X,Y) &= R_{(Q_P(z,\lambda))^m (Q_P(z,\mu))^m} R_\sigma \tau^{(m-1)}_{\boldsymbol{A}}(X,Y) \\ &= R_{(Q_P(z,\lambda))^m (Q_P(z,\mu))^m} \tau^{(m)}_{\boldsymbol{A}}(X,Y) \,, \end{aligned} \tag{7.21}$$

which completes the proof. □

We can now state the main result of this section.

Theorem 7.2.1. *Let $\boldsymbol{A}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an operator with $\tau^{(m)}_{\boldsymbol{A}}(X,Y) = \mathbf{0}$, $m \geq 2$. Then*

$$\tau^{(m)}_{\left(\sum_{k=0}^{N} c_k(\boldsymbol{x})\boldsymbol{A}^k\right)}(X,Y) = \mathbf{0}, \qquad X,Y \in \mathfrak{X}(M)\,, \tag{7.22}$$

where $c_k(\boldsymbol{x}) \in C^\infty(M)$.

Proof. Let $\boldsymbol{P} := P(\boldsymbol{A}) = \sum_{k=0}^{N} c_k(\boldsymbol{x})\boldsymbol{A}^k$. It is sufficient to apply Lemma 7.2.3, taking into account the assumption $\tau^{(m)}_{\boldsymbol{A}}(X,Y) = \mathbf{0}$. □

Interestingly enough, Theorem 7.2.1 does not hold in general for the case of the Nijenhuis torsion (i.e., $m = 1$). However, as is well-known, relation (7.22) is valid for the Nijenhuis torsion if the coefficients $c_k(\boldsymbol{x})$ are all constant.

7.3 SIMULTANEOUS BLOCK-DIAGONALIZATION

In this section, we state our main theoretical results. We consider first the case of a single generalized Nijenhuis operator belonging to an arbitrary family of commuting operators.

Theorem 7.3.1. *Let $\mathcal{S} = \{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_w\}$, $\boldsymbol{K}_\alpha\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\alpha = 1, \ldots, w$ be a set of commuting operator fields; we assume that one of them, say $\boldsymbol{K}_1$, satisfies the condition*

$$\tau^{(m)}_{\boldsymbol{K}_1}(X,Y) = \mathbf{0}, \qquad X,Y \in \mathfrak{X}(M)\,, \tag{7.23}$$

for some $m \geq 1$. Then, there exist local charts in which all of the operators $\boldsymbol{K}_\alpha$ can be written simultaneously in a block-diagonal form.

Proof. Assuming that condition (7.23) is satisfied, Proposition 7.1.2 ensures the existence of an equivalence class of integrable frames and local charts where the operator $\boldsymbol{K}_1$ takes a block-diagonal form. Such coordinates are adapted to the characteristic web associated with the spectral decomposition of $\boldsymbol{K}_1$:

$$T_{\boldsymbol{x}}M = \mathcal{D}_i(\boldsymbol{x}) \oplus \mathcal{E}_i(\boldsymbol{x}) = \bigoplus_{i=1}^{s} \mathcal{D}_i(\boldsymbol{x})\,. \tag{7.24}$$

with

$$\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^s)\,. \tag{7.25}$$

The variables $\boldsymbol{x}^i = (x^{i,1}, \ldots, x^{i,r_i})$ are defined over the integral leaves of the eigendistribution $\mathcal{D}_i$, whereas the remaining ones, namely

$$(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{i-1}, \boldsymbol{x}^{i+1}, \ldots, \boldsymbol{x}^s)$$

are coordinates of the leaves, i.e., are constant ($\boldsymbol{x}^1 = \boldsymbol{c}^1, \ldots, \boldsymbol{x}^{i-1} = \boldsymbol{c}^{i-1}, \boldsymbol{x}^{i+1} = \boldsymbol{c}^{i+1}, \ldots, \boldsymbol{x}^s = \boldsymbol{c}^s$) on each leaf $D_i(\boldsymbol{c})$ of the foliation. Here $\boldsymbol{c} := (\boldsymbol{c}^1, \ldots, \boldsymbol{c}^{i-1}, \boldsymbol{c}^{i+1}, \ldots, \boldsymbol{c}^s)$. Since all operators of the set $\mathcal{S}$ commute, every distribution $\mathcal{D}_i$ is invariant under the action of the operators

$\{\boldsymbol{K}_2, \dots, \boldsymbol{K}_w\}$. As a direct consequence of this property, all the operators $\boldsymbol{K}_\alpha \in \mathcal{S}$ in the local chart (7.25) take a block-diagonal form, where the ith $(r_i \times r_i)$ block matrix $[\boldsymbol{K}_\alpha^{(i)}]_{|_{D_i(\boldsymbol{c})}}$ coincides with the matrix $[\boldsymbol{K}_{\alpha|_{D_i(\boldsymbol{c})}}]$, $i = 1, \dots, s$, which represents the restricted operator $\boldsymbol{K}_\alpha$ to the leaf $D_i(\boldsymbol{c})$. □

Theorem 7.3.2. *Let $\mathcal{S} = \{\boldsymbol{K}_1, \dots, \boldsymbol{K}_w\}$, $\boldsymbol{K}_\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$, be a set of commuting operator fields. If*

(i) all the operators of the family have vanishing generalized torsion of level m:

$$\tau^{(m)}_{\boldsymbol{K}_\alpha}(X, Y) = \boldsymbol{0}\,, \qquad \alpha = 1, \dots, w, \qquad X, Y \in \mathfrak{X}(M) \tag{7.26}$$

(ii) their joint eigendistributions

$$\mathcal{V}_a(\boldsymbol{x}) := \bigoplus_{i_1, \dots, i_w}^{s_1, \dots, s_w} \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \bigcap \dots \bigcap \mathcal{D}^{(w)}_{i_w}(\boldsymbol{x})\,, \qquad a = 1, \dots, v \leq n \tag{7.27}$$

are pairwise integrable,

then there exist sets of local coordinates, adapted to the decomposition

$$T_{\boldsymbol{x}} M = \bigoplus_{a=1}^{v} \mathcal{V}_a(\boldsymbol{x}) \qquad \boldsymbol{x} \in M, \tag{7.28}$$

in which all operators $\boldsymbol{K}_\alpha$ admit simultaneously a block-diagonal form with possibly finer blocks with respect to those considered in Theorem 7.3.1.

Proof. According to Theorem 7.3.1, since by hypothesis $\tau^{(m)}_{\boldsymbol{K}_1}(X, Y) = \boldsymbol{0}$, $X, Y \in \mathfrak{X}(M)$, all the operators take simultaneously a block-diagonal form in a suitable local chart. Assume also that $\tau^{(m)}_{\boldsymbol{K}_2}(X, Y) = \boldsymbol{0}$. Then, the tangent space at any point $\boldsymbol{x}$ admits the finer decomposition

$$T_{\boldsymbol{x}} M = \bigoplus_{i_1, i_2}^{s_1, s_2} \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \cap \mathcal{D}^{(2)}_{i_2}(\boldsymbol{x})\,, \tag{7.29}$$

where $\mathcal{D}^{(2)}_{i_2}$ are the eigendistributions of $\boldsymbol{K}_2$. These eigendistributions are integrable by virtue of Theorem 7.1.1. Consequently, the generalized Haantjes Theorem 7.1.1 can also be applied to the restriction of $\boldsymbol{K}_2$ to $D^{(1)}_{i_1}(\boldsymbol{c}_1)$. Therefore, there exists a transformation of coordinates, acting only on the coordinates over the leaves of the foliation $D^{(1)}_{i_1}$

$$\Phi : M \to M, \qquad (\boldsymbol{x}^1, \dots, \boldsymbol{x}^{i_1}, \dots, \boldsymbol{x}^{s_1}) \mapsto (\boldsymbol{x}^1, \dots, \boldsymbol{y}^{i_1}, \dots, \boldsymbol{x}^{s_1})\,, \tag{7.30}$$

such that the new coordinates $\boldsymbol{y}^{i_1} = (y^{i_1,1}, \dots, y^{i_1, r_{i_1}}) = \boldsymbol{f}^{i_1}(\boldsymbol{x}^{i_1})$ are adapted to the decomposition

$$T_{\boldsymbol{x}} D^{(1)}_{i_1}(\boldsymbol{c}_1) = \bigoplus_{i_2}^{s_2} \mathcal{D}^{(1)}_{i_1}(\boldsymbol{x}) \cap \mathcal{D}^{(2)}_{i_2}(\boldsymbol{x}), \qquad \boldsymbol{x} \in D^{(1)}_{i_1}(\boldsymbol{c}_1)\,. \tag{7.31}$$

Thus, we have

$$[\boldsymbol{K}_\alpha^{(i_1)}] = \left[\begin{array}{c|c|c} \boldsymbol{K}_\alpha^{(i_1,1)} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \boldsymbol{K}_\alpha^{(i_1,s_2)} \end{array}\right], \qquad \alpha = 1, \dots, w \tag{7.32}$$

where $[\boldsymbol{K}_\alpha^{(i_1,j)}]_{|_{D_{i_1}^{(1)}(\boldsymbol{c}_1)\cap D_j^{(2)}(\boldsymbol{c}_2)}} = \left[\boldsymbol{K}_{\alpha|_{D_{i_1}^{(1)}(\boldsymbol{c}_1)\cap D_j^{(2)}(\boldsymbol{c}_2)}}\right]$, $j = 1, \dots, s_2$. Let us consider the decomposition

$$T_{\boldsymbol{x}}M = \bigoplus_{i_1,i_2}^{s_1,s_2} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \cap \mathcal{D}_{i_2}^{(2)}(\boldsymbol{x}) = \bigoplus_{\gamma=1}^{u} \mathcal{U}_\gamma(\boldsymbol{x}) \qquad \boldsymbol{x} \in M\,, \tag{7.33}$$

where in the direct sum (7.33) $\mathcal{U}_\gamma \neq \{\boldsymbol{0}\}$, $u \leq n$ and r_γ denotes the rank of $\mathcal{U}_\gamma$ ($\sum_{\gamma=1}^{u} r_\gamma = n$). Clearly, the distributions $\mathcal{U}_\gamma$ are invariant under the action of each operator $\boldsymbol{K}_\alpha \in \mathcal{S}$. Besides, these distributions are involutive as are realized as the intersection of involutive distributions. By assumption, their direct sums are also integrable. Therefore, there exist local charts in M of the form

$$\{U, (y^{1,1}, \dots, y^{1,r_1}; \dots; y^{i_1,1}, \dots, y^{i_1,r_{i_1}}; \dots; y^{s_1,1}, \dots, y^{s_1,r_{s_1}})\} \tag{7.34}$$

adapted to the decomposition (7.33), where all the operators $\boldsymbol{K}_\alpha \in \mathcal{S}$ admit simultaneously a (possibly) finer block-diagonal form. By extending the previous procedure to the Haantjes operators $\boldsymbol{K}_3, \dots, \boldsymbol{K}_w$, we obtain the decomposition

$$T_{\boldsymbol{x}}M = \bigoplus_{i_1,\dots,i_w}^{s_1,\dots,s_w} \mathcal{D}_{i_1}^{(1)}(\boldsymbol{x}) \bigcap \dots \bigcap \mathcal{D}_{i_w}^{(w)}(\boldsymbol{x}) = \bigoplus_{a=1}^{v} \mathcal{V}_a(\boldsymbol{x}), \tag{7.35}$$

where in the direct sum (7.35) $\mathcal{V}_a \neq \{\boldsymbol{0}\}$, $v \leq n$ and r_a denotes the rank of $\mathcal{V}_a$ ($\sum_{a=1}^{v} r_a = n$). Since the involutive distributions $\mathcal{V}_a$ are mutually integrable as a consequence of the previous reasoning, then there exist local charts

$$\{U, (\boldsymbol{y}^1, \dots, \boldsymbol{y}^a, \dots, \boldsymbol{y}^v)\} \tag{7.36}$$

adapted to the decomposition (7.35), such that

$$\mathcal{V}_a = \left\langle \frac{\partial}{\partial y^{a,1}}, \dots, \frac{\partial}{\partial y^{a,r_a}} \right\rangle \qquad a = 1, \dots, v, \tag{7.37}$$

where the natural frame $\left\{\frac{\partial}{\partial y^{a,1}}, \dots, \frac{\partial}{\partial y^{a,r_a}}\right\}$ over the leaves of $\mathcal{V}_a$ is formed by joint *generalized* eigenvector fields of the operators $\{\boldsymbol{K}_1, \dots, \boldsymbol{K}_w\}$. □

Remark 7.3.1. *Given a generalized Haantjes algebra $(M, \mathscr{H}^{(m)})$ with generator $\boldsymbol{L}$, the hypothesis (ii) of Theorem 7.3.2 is automatically satisfied. Indeed, the joint eigendistributions $\mathcal{V}_a$ either coincide with the eigendistributions of the generator $\boldsymbol{L}$ or are direct sums of them. In both cases, since (due to Theorem 7.1.1) the eigendistributions of $\boldsymbol{L}$ are mutually integrable, it turns out that the direct sums of the joint eigendistributions $\mathcal{V}_a$ are integrable as well.*

7.3.1 *The importance of the pairwise integrability*

In this section, we present a case that highlights the necessity of introducing the pairwise integrability condition for joint eigendistributions in hypothesis *ii)* of Theorem 7.3.2. Consider the operators $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, which in coordinates (x^1, x^2, x^3, x^4) take the form

$$\boldsymbol{K}_1(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{K}_2(\boldsymbol{x}) = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & (\lambda-\mu)\,x^1 \\ 0 & 0 & 0 & \mu \end{bmatrix}, \tag{7.38}$$

where λ and μ are two arbitrary constants.

The second operator becomes diagonal (and constant) in the coordinate system

$$\begin{aligned} y^1 &= x^1\,, \\ y^2 &= x^2\,, \\ y^3 &= x^3 - x^1 x^4\,, \\ y^4 &= x^4\,. \end{aligned} \tag{7.39}$$

The operators $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are indeed Nijenhuis. Consequently, they are also Haantjes operators (and of higher order as well). One of them is uniform (i.e., constant), while the other is non-trivial. Moreover, both operators are semisimple and commute.

Nevertheless, no coordinate system ensures the simultaneous diagonalization of both operators. However, this does not contradict either the present theory or the previous results concerning the simultaneous diagonalization of commuting Haantjes tensors.

The key point is that additional hypotheses are required for the theory to apply.

A) Regarding the possibility of simultaneous block-diagonalization, hypothesis *ii)* of Theorem 7.3.2 crucially requires the *pairwise integrability* of the intersections of the eigendistributions of the two operators. A direct analysis reveals that this hypothesis is not satisfied.

   Specifically, while the eigendistribution of each operator is Frobenius integrable, as are their intersections, not all direct sums of these intersections are integrable.

   In more detail, the eigendistributions of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are, respectively,

$$\begin{aligned} V_0 &= \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\rangle, & V_1 &= \left\langle \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4} \right\rangle; \\ W_\lambda &= \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3} \right\rangle, & W_\mu &= \left\langle \frac{\partial}{\partial x^2}, x^1 \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \right\rangle. \end{aligned}$$

   Their corresponding intersections are:

$$\begin{aligned} V_0 \cap W_\lambda &= \left\langle \frac{\partial}{\partial x^1} \right\rangle, & V_0 \cap W_\mu &= \left\langle \frac{\partial}{\partial x^2} \right\rangle; \\ V_1 \cap W_\lambda &= \left\langle \frac{\partial}{\partial x^3} \right\rangle, & V_1 \cap W_\mu &= \left\langle x^1 \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \right\rangle. \end{aligned}$$

Since $V_0 \cap W_\lambda \oplus V_1 \cap W_\mu$ is not integrable, hypothesis *ii)* of Theorem 7.3.2 does not hold, and thus the theorem is not applicable.

B) As discussed in Section 2.4.2, a necessary and sufficient condition for two Haantjes operators to be simultaneously diagonalized is that they form a commutative *Haantjes algebra* of operators. This condition, once again, is not met by $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$. Although both are Haantjes operators and commute, it is easy to verify that the linear combination $\alpha_1 \boldsymbol{K}_1 + \alpha_2 \boldsymbol{K}_2$, where α_1 and α_2 are arbitrary functions, is not a Haantjes operator. Likewise, their product $\boldsymbol{K}_1 \cdot \boldsymbol{K}_2$ is not a Haantjes operator.

In the following sections, we shall illustrate the results proposed. First, we will show some examples of generalized Haantjes algebras of operator fields. Then, we shall present the explicit construction of local charts of coordinates allowing their simultaneous block-diagonalization.

7.4 BLOCK-DIAGONALIZATION OF GENERALIZED HAANTJES ALGEBRAS

As stated in Theorems 7.3.1 and 7.3.2, we can block-diagonalize families of commuting generalized Nijenhuis operators by constructing a suitable coordinate chart. To this end, we propose the following procedure, analogous to the one in Section 3.3.6, except that the coordinate transformation is not required to be canonical.

1) Determine the joint eigendistributions $\mathcal{V}_a$ of the given family of operator fields.

2) Determine a basis of closed 1-forms for each of the the corresponding annihilators.

3) Integrate them to find the characteristic coordinates.

4) Compute the expression of the operators of the given family in these coordinates.

To illustrate this procedure, we shall first construct families of operator fields, depending on arbitrary functions and having a prescribed vanishing higher-level torsion. A direct approach consists in considering an operator field whose entries are all arbitrary functions, and imposing both that a certain higher-level torsion is vanishing, and simultaneously that the lower-level ones are not. In this way, one can obtain a set of differential constraints that can be solved (possibly with suitable ansätze) to obtain families of operator fields with a vanishing higher-level torsion, still depending on arbitrary functions. Generally speaking, there is a lot of freedom in this approach.

From this analysis it emerges that generalized Haantjes algebras are not rare. Thus, by specializing the arbitrary functions contained in the families of operators determined according to the procedure proposed, we can obtain infinitely many examples of new, higher-level Haantjes algebras.

7.4.1 *A third-level generalized Haantjes algebra*

Let M be a five-dimensional differentiable manifold. In local coordinates $\boldsymbol{x} = (x^1, x^2, x^3, x^4, x^5)$, a first example is given by the family of operators

$$\boldsymbol{L}(\boldsymbol{x}) = \begin{bmatrix} f_1 & 1 & 0 & 1 & 0 \\ f_1 - f_2 + 1 & f_1 + 1 & -f_3 & f_1 - f_2 + 1 & -f_3 \\ 1 & 0 & f_2 + f_3 & 1 & f_3 \\ f_2 - f_1 & -1 & f_3 & f_2 - 1 & f_3 \\ -1 & 0 & -f_3 - 1 & -1 & f_2 - f_3 - 1 \end{bmatrix}, \tag{7.40}$$

where $f_1 = f_1(x^1, x^2)$, $f_2 = f_2(x^3, x^4, x^5)$ and $f_3 = f_3(x^3)$ are arbitrary functions.

A direct computation allows us to prove the following result.

Proposition 7.4.1. *The family of operators $\boldsymbol{L}$ satisfies the following properties:*

- $\tau^{(3)}_{\boldsymbol{L}}(X, Y) = \boldsymbol{0}$
- $\mathcal{H}_{\boldsymbol{L}}(X, Y) \neq \boldsymbol{0}$ *if and only if* $f_3' \neq 0$.

The minimal polynomial of $\boldsymbol{L}$ is generically of fifth degree and reads

$$m_{\boldsymbol{L}}(\boldsymbol{x}, \lambda) = (\lambda - f_1 - 1)(\lambda - f_1 + 1)\,(\lambda - f_2 + 1)\,(\lambda - f_2)^2\,. \tag{7.41}$$

Thus, it coincides with the characteristic polynomial; therefore $\boldsymbol{L}$ is a cyclic operator and the Riesz indices ρ_i coincide with the rank of the eigendistributions $\mathcal{D}_i$. Also, notice that $\boldsymbol{L}$ is generically non-semisimple. The eigenvalues and generalized eigenvectors of $\boldsymbol{L}$ are

$$\begin{aligned} \lambda_1 &= f_1 + 1, \quad \rho_1 = 1, \quad \mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4} \right\rangle, \\ \lambda_2 &= f_1 - 1, \quad \rho_2 = 1, \quad \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4} \right\rangle, \\ \lambda_3 &= f_2 - 1, \quad \rho_3 = 1, \quad \mathcal{D}_3 = \left\langle f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} \right\rangle, \\ \lambda_4 &= f_2, \quad \rho_4 = 2, \quad \mathcal{D}_4 = \left\langle \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4} \right\rangle. \end{aligned} \tag{7.42}$$

From these eigendistributions, we can compute their characteristic distributions and consequently their annihilators:

$$\begin{aligned}
\mathcal{E}_1 = \oplus_{i\neq 1}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Rightarrow \mathcal{E}_1^\circ = \langle \mathrm{d}x^1 + \mathrm{d}x^2 + \mathrm{d}x^4 \rangle, \\
\mathcal{E}_2 = \oplus_{i\neq 2}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Rightarrow \mathcal{E}_2^\circ = \langle \mathrm{d}x^1 - \mathrm{d}x^2 - \mathrm{d}x^4 \rangle, \\
\mathcal{E}_3 = \oplus_{i\neq 3}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^5} \right\rangle \\
&\Rightarrow \mathcal{E}_3^\circ = \langle \mathrm{d}x^3 + \mathrm{d}x^5 \rangle, \\
\mathcal{E}_4 = \oplus_{i\neq 4}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} + 2\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^4}, f_3\frac{\partial}{\partial x^2} - f_3\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} \right\rangle \\
&\Rightarrow \mathcal{E}_4^\circ = \langle \mathrm{d}x^3, \mathrm{d}x^1 + \mathrm{d}x^4 + f_3\mathrm{d}x^5 \rangle.
\end{aligned} \tag{7.43}$$

Remark 7.4.1. *We can easily extract an Abelian cyclic algebra $\mathscr{H}^{(3)}$ from the class (7.40). In order to achieve that, for simplicity, we set $f_3 = x^3$ for all the operators of the algebra and vary the other arbitrary functions. Observe that all the operators of the family obtained in this way will share both the same eigendistributions and the degree of the minimal polynomial. Thus, any of them generate the whole algebra $\mathscr{H}^{(3)}$ with its independent powers. Notice that for this algebra the hypotheses of Theorem 7.3.2 are all satisfied, as discussed in Remark 7.3.1.*

By way of an example of simultaneous block-diagonalization, we can consider the three commuting operators

$$\boldsymbol{L}_1(\boldsymbol{x}) = \begin{bmatrix} x^1 & 1 & 0 & 1 & 0 \\ x^1+1 & x^1+1 & -x^3 & x^1+1 & -x^3 \\ 1 & 0 & x^3 & 1 & x^3 \\ -x^1 & -1 & x^3 & -1 & x^3 \\ -1 & 0 & -x^3-1 & -1 & -x^3-1 \end{bmatrix}, \tag{7.44}$$

$$\boldsymbol{L}_2(\boldsymbol{x}) = \begin{bmatrix} x^2 & 1 & 0 & 1 & 0 \\ x^2-x^5+1 & x^2+1 & -x^3 & x^2-x^5+1 & -x^3 \\ 1 & 0 & x^3+x^5 & 1 & x^3 \\ -x^2+x^5 & -1 & x^3 & x^5-1 & x^3 \\ -1 & 0 & -x^3-1 & -1 & -x^3+x^5-1 \end{bmatrix}, \tag{7.45}$$

$$\boldsymbol{L}_3(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ -x^4 - x^3x^5 + 1 & 1 & -x^3 & -x^4 - x^3x^5 + 1 & -x^3 \\ 1 & 0 & x^4 + x^3\,(x^5+1) & 1 & x^3 \\ x^4 + x^3x^5 & -1 & x^3 & x^4 + x^3x^5 - 1 & x^3 \\ -1 & 0 & -x^3 - 1 & -1 & x^4 + x^3\,(x^5-1) - 1 \end{bmatrix}. \tag{7.46}$$

These operators are $C^\infty(\mathbb{R})$-linearly independent. Any of these three operators can be chosen to be the generator of a level-three cyclic Haantjes algebra $\mathscr{H}^{(3)}$ of rank 5. A basis of this algebra is, for instance, $\mathscr{B} = \{\boldsymbol{I}, \boldsymbol{L}_1, \boldsymbol{L}_1^2, \boldsymbol{L}_1^3, \boldsymbol{L}_1^4\}$. In the following subsection, we shall determine a set of coordinates which block-diagonalize the algebra $\mathscr{H}^{(3)}$.

7.4.2 *Block-diagonalization*

By integrating the annihilators (7.43) of the eigendistributions of operators (7.44)-(7.46) and leaving $f_3 = f_3(x^3)$ arbitrary, we obtain a set of coordinates which block-diagonalize the algebra $\mathscr{H}^{(3)}$. We have explicitly:

$$\begin{aligned} y^1 &= x^1 + x^2 + x^4, \\ y^2 &= x^1 - x^2 - x^4, \\ y^3 &= x^3 + x^5, \\ y^4 &= x^3, \\ y^5 &= x^1 + x^4 + f_3(x^3)x^5. \end{aligned} \tag{7.47}$$

Precisely, in these block-separation coordinates, the operators $\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_3$ read:

$$\boldsymbol{L}_1(\boldsymbol{y}) = \left[\begin{array}{c|c|c|cc} \frac{1}{2}\,(y^1+y^2)+1 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2}\,(y^1+y^2)-1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & -y^3+2y^4 & 1 \\ 0 & 0 & 0 & -(y^3-2y^4)^2 & y^3-2y^4 \end{array}\right], \tag{7.48}$$

$$\boldsymbol{L}_2(\boldsymbol{y}) = \left[\begin{array}{c|c|c|cc} y^1+(y^3-y^4)y^4-y^5+1 & 0 & 0 & 0 & 0 \\ \hline 0 & y^1+(y^3-y^4)y^4-y^5-1 & 0 & 0 & 0 \\ \hline 0 & 0 & y^3-y^4-1 & 0 & 0 \\ \hline 0 & 0 & 0 & y^4 & 1 \\ 0 & 0 & 0 & -(y^3-2y^4)^2 & 2y^3-3y^4 \end{array}\right], \tag{7.49}$$

$$\boldsymbol{L}_3(\boldsymbol{y}) = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{2}(y^1+y^2)+y^5-1 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2}(y^1+y^2)-y^3+2y^4+y^5 & 1 \\ 0 & 0 & 0 & -(y^3-2y^4)^2 & -\frac{1}{2}(y^1+y^2)+y^3-2y^4+y^5 \end{array}\right]. \tag{7.50}$$

Observe that more generally, in coordinates (7.47) the whole class (7.40) takes the block-diagonal form

$$\boldsymbol{L}(\boldsymbol{y}) = \left[\begin{array}{c|c|c|c|c} f_1+1 & 0 & 0 & 0 & 0 \\ \hline 0 & f_1-1 & 0 & 0 & 0 \\ \hline 0 & 0 & f_2-1 & 0 & 0 \\ \hline 0 & 0 & 0 & f_2+f_3-(y^3-y^4)f_3' & 1 \\ 0 & 0 & 0 & -\left(f_3-(y^3-y^4)f_3'\right)^2 & f_2-f_3+(y^3-y^4)f_3' \end{array}\right], \tag{7.51}$$

where $f_1 = f_1\left(\frac{1}{2}(y^1+y^2), y^1+(y^3-y^4)f_3-y^5\right)$, $f_2 = f_2\left(y^4, -\frac{1}{2}(y^1+y^2)-(y^3-y^4)f_3+y^5, y^3-y^4\right)$ and $f_3 = f_3(y^4)$ are arbitrary functions (with $f_3' \neq 0$).

7.5 A FOURTH-LEVEL GENERALIZED HAANTJES ALGEBRA

Let us consider the following fourth-level generalized Nijenhuis family of operator fields defined over a seven-dimensional differentiable manifold M, which is obtained with the same procedure described in the previous section:

$$\boldsymbol{K}(\boldsymbol{x}) = \begin{bmatrix} g_1 & x^1 & x^1+g_2 & -x^1 & x^1 & -x^1 & x^1 \\ 1 & g_5 & -g_1+g_5+x^1 & g_4-g_5 & 0 & g_1-g_5-x^1 & -1-g_4+g_5 \\ 0 & 0 & g_1 & 0 & 0 & 0 & 0 \\ 1+g_1-g_4 & x^1 & x^1+g_2 & -1-x^1+g_4 & x^1 & -x^1 & x^1 \\ 0 & g_1-g_5 & g_1+g_3-g_5+\frac{1}{x^1} & -g_1+g_5 & g_1 & -g_1+g_5-\frac{1}{x^1} & g_1-g_5 \\ 0 & 0 & x^1 & 0 & 0 & g_1-x^1 & 0 \\ g_1-g_4 & x^1 & x^1+g_2 & -1-x^1 & x^1 & -x^1 & 1+x^1+g_4 \end{bmatrix}. \tag{7.52}$$

Here $g_1 = g_1(x^1,x^2,x^3,x^4)$, $g_2 = g_2(x^1,x^2,x^3,x^4)$, $g_3 = g_3(x^1,x^2,x^3,x^4)$, $g_4 = g_4(x^5,x^6)$, $g_5 = g_5(x^7)$.

As a result of a direct computation, one can prove the following

Proposition 7.5.1. *The operator family* (7.52) *satisfies the property:*

$$\tau^{(4)}_{\boldsymbol{K}}(X,Y) = 0\,.$$

In addition, its third-level torsion does not vanish whenever $g_3 \neq 0$.

7.5.1 *Spectral analysis*

Its minimal polynomial is generically of seventh degree,

$$m_{\boldsymbol{K}}(\boldsymbol{x},\lambda) = (\lambda - g_5)(\lambda - g_4 - 1)(\lambda - g_4 + 1)(\lambda - g_1 + x^1)(\lambda - g_1)^3, \tag{7.53}$$

and its eigenvalues and generalized eigenvectors are

$$\begin{aligned}
\lambda_1 = g_5, \quad \rho_1 = 1, \quad \mathcal{D}_1 &= \left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5} \right\rangle,\\
\lambda_2 = g_4 + 1, \quad \rho_2 = 1, \quad \mathcal{D}_2 &= \left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7} \right\rangle,\\
\lambda_3 = g_4 - 1, \quad \rho_3 = 1, \quad \mathcal{D}_3 &= \left\langle \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7} \right\rangle,\\
\lambda_4 = g_1 - x^1, \quad \rho_4 = 1, \quad \mathcal{D}_4 &= \left\langle \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7} \right\rangle,\\
\lambda_5 = g_1, \quad \rho_5 = 3, \quad \mathcal{D}_5 &= \left\langle \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6} \right\rangle.
\end{aligned} \tag{7.54}$$

The characteristic distributions and the corresponding annihilators are

$$\begin{aligned}
\mathcal{E}_1 = \oplus_{i\neq 1}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \right.\\
&\qquad \left. \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \right\rangle\\
\Rightarrow \mathcal{E}_1^\circ &= \langle \mathrm{d}x^2 + \mathrm{d}x^3 - \mathrm{d}x^4 - \mathrm{d}x^6 + \mathrm{d}x^7 \rangle,\\
\mathcal{E}_2 = \oplus_{i\neq 2}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \right.\\
&\qquad \left. \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \right\rangle\\
\Rightarrow \mathcal{E}_2^\circ &= \langle \mathrm{d}x^1 + \mathrm{d}x^4 - 2\mathrm{d}x^7 \rangle,\\
\mathcal{E}_3 = \oplus_{i\neq 3}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \right.\\
&\qquad \left. \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \right\rangle\\
\Rightarrow \mathcal{E}_3^\circ &= \langle \mathrm{d}x^1 - \mathrm{d}x^4 \rangle,\\
\mathcal{E}_4 = \oplus_{i\neq 4}\mathcal{D}_i &= \left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \right.\\
&\qquad \left. \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^6}, \frac{\partial}{\partial x^5} \right\rangle
\end{aligned} \tag{7.55}$$

$$
\begin{aligned}
&\Rightarrow \mathcal{E}_4^\circ = \langle \mathrm{d}x^3 - \mathrm{d}x^6 \rangle, \\
\mathcal{E}_5 = \oplus_{i\neq 5}\mathcal{D}_i = &\left\langle \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^7}, \frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^7}, \right. \\
&\left. \frac{\partial}{\partial x^1} - (x^1)^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^5} - (x^1)^2\frac{\partial}{\partial x^6} + \frac{\partial}{\partial x^7} \right\rangle \\
&\Rightarrow \mathcal{E}_5^\circ = \langle \mathrm{d}x^3, (x^1)^2\mathrm{d}x^1 + \mathrm{d}x^6, \mathrm{d}x^1 + \mathrm{d}x^2 - \mathrm{d}x^4 + \mathrm{d}x^5 - \mathrm{d}x^6 + \mathrm{d}x^7 \rangle \,.
\end{aligned}
$$

By specializing the arbitrary functions g_1, g_2, g_4, g_5, but keeping the same function g_3 in all operators to have commutativity, we easily obtain an Abelian, cyclic and fourth-level Haantjes algebra. As in the previous example, all the operators of the family obtained in this way will share both the same eigendistributions and the degree of the minimal polynomial. Thus, any of them generates the whole algebra.

To illustrate the simultaneous block-diagonalization procedure, by way of an example we can consider the three commuting operators

$$
\boldsymbol{K}_1(\boldsymbol{x}) = \begin{bmatrix}
x^2 & x^1 & x^1 & -x^1 & x^1 & -x^1 & x^1 \\
1 & 0 & x^1 - x^2 & 0 & 0 & -x^1 + x^2 & -1 \\
0 & 0 & x^2 & 0 & 0 & 0 & 0 \\
1 + x^2 & x^1 & x^1 & -1 - x^1 & x^1 & -x^1 & x^1 \\
0 & x^2 & x^2 & -x^2 & x^2 & -\frac{1}{x^1} - x^2 & x^2 \\
0 & 0 & x^1 & 0 & 0 & -x^1 + x^2 & 0 \\
x^2 & x^1 & x^1 & -1 - x^1 & x^1 & -x^1 & 1 + x^1
\end{bmatrix}, \tag{7.56}
$$

$$
\boldsymbol{K}_2(\boldsymbol{x}) = \begin{bmatrix}
x^3 & x^1 & -\frac{1}{x^1} & -x^1 & x^1 & -x^1 & x^1 \\
1 & x^7 & x^1 - x^3 + x^7 & x^6 - x^7 & 0 & -x^1 + x^3 - x^7 & -1 - x^6 + x^7 \\
0 & 0 & x^3 & 0 & 0 & 0 & 0 \\
1 + x^3 - x^6 & x^1 & -\frac{1}{x^1} & -1 - x^1 + x^6 & x^1 & -x^1 & x^1 \\
0 & x^3 - x^7 & x^3 - x^7 & -x^3 + x^7 & x^3 & -\frac{1}{x^1} - x^3 + x^7 & x^3 - x^7 \\
0 & 0 & x^1 & 0 & 0 & -x^1 + x^3 & 0 \\
x^3 - x^6 & x^1 & -\frac{1}{x^1} & -1 - x^1 & x^1 & -x^1 & 1 + x^1 + x^6
\end{bmatrix}, \tag{7.57}
$$

$$\boldsymbol{K}_3(\boldsymbol{x}) =$$

$$\begin{bmatrix} x^4 & x^1 & 0 & -x^1 & x^1 & -x^1 & x^1 \\ 1 & 1 & 1+x^1-x^4 & -1+x^5 & 0 & -1-x^1+x^4 & -x^5 \\ 0 & 0 & x^4 & 0 & 0 & 0 & 0 \\ 1+x^4-x^5 & x^1 & 0 & -1-x^1+x^5 & x^1 & -x^1 & x^1 \\ 0 & -1+x^4 & -1+x^4 & 1-x^4 & x^4 & 1-\frac{1}{x^1}-x^4 & -1+x^4 \\ 0 & 0 & x^1 & 0 & 0 & -x^1+x^4 & 0 \\ x^4-x^5 & x^1 & 0 & -1-x^1 & x^1 & -x^1 & 1+x^1+x^5 \end{bmatrix}. \tag{7.58}$$

These operators are $C^\infty(\mathbb{R})$-linearly independent. Each of them generates a fourth-level Haantjes algebra $\mathscr{H}^{(4)}$ of rank 7. A basis of this algebra is given, for instance, by $\mathscr{B} = \{\boldsymbol{I}, \boldsymbol{K}_1, \boldsymbol{K}_1^2, \boldsymbol{K}_1^3, \boldsymbol{K}_1^4, \boldsymbol{K}_1^5, \boldsymbol{K}_1^6\}$. In the following subsection, we shall determine a set of coordinates which block-diagonalize the whole algebra $\mathscr{H}^{(4)}$.

7.5.2 *Block-diagonalization*

By integrating the annihilators (7.55) of the eigendistributions of these operators, we get the block-separating coordinates:

$$\begin{aligned} y^1 &= x^2 + x^3 - x^4 - x^6 + x^7, \\ y^2 &= x^1 + x^4 - 2x^7, \\ y^3 &= x^1 - x^4, \\ y^4 &= x^3 - x^6, \\ y^5 &= x^3, \\ y^6 &= \frac{(x^1)^3}{3} + x^6, \\ y^7 &= x^1 + x^2 - x^4 + x^5 - x^6 + x^7. \end{aligned} \tag{7.59}$$

In these coordinates, all the operators of the algebra $\mathscr{H}^{(4)}$ block-diagonalize simultaneously. In particular, by defining $\chi = \sqrt[3]{3(y^4 - y^5 + y^6)}$, the operators $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3$ take the form:

$$\boldsymbol{K}_1(\boldsymbol{y}) =$$

$$\left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^1+\frac{1}{2}(y^2-y^3)-y^4-\chi & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & y^1+\frac{1}{2}(y^2-y^3)-y^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi+\chi^3 & y^1+\frac{1}{2}(y^2-y^3)-y^4-\chi & \chi^3 \\ 0 & 0 & 0 & 0 & \chi & -\chi^{-1} & y^1+\frac{1}{2}(y^2-y^3)-y^4+\chi \end{array}\right], \tag{7.60}$$

$$\boldsymbol{K}_2(\boldsymbol{y}) = \begin{bmatrix} -\frac{1}{2}(y^2+y^3)+\chi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-y^4+y^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-y^4+y^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^5-\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y^5-\chi & \chi^3 \\ 0 & 0 & 0 & 0 & -\chi^{-1} & -\chi^{-1} & y^5+\chi \end{bmatrix}, \tag{7.61}$$

$$\boldsymbol{K}_3(\boldsymbol{y}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-y^1+y^5-\chi+y^7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-y^1+y^5-\chi+y^7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y^3+\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi & -y^3 & \chi^3 \\ 0 & 0 & 0 & 0 & 0 & -\chi^{-1} & -y^3+2\chi \end{bmatrix}. \tag{7.62}$$

Also, in the block-separation coordinates, the family (7.52) takes a block-diagonal form:

$$\boldsymbol{K}(\boldsymbol{y}) = \begin{bmatrix} g_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_4+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_4-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1-\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_2\chi^2+\chi(1+\chi^2) & g_1-\chi & \chi^3 \\ 0 & 0 & 0 & 0 & g_3+g_2+\chi^{-1}(1+\chi^2) & -\chi^{-1} & g_1+\chi \end{bmatrix}. \tag{7.63}$$

Here $g_1 = g_1(\chi, y^1 + \frac{1}{2}(y^2-y^3) - y^4, y^5)$, $g_2 = g_2(\chi, y^1 + \frac{1}{2}(y^2-y^3) - y^4, y^5)$, $g_3 = g_3(\chi, y^1 + \frac{1}{2}(y^2-y^3) - y^4, y^5)$, $g_4 = g_4(-y^1-\chi+y^5+y^7, -y^4+y^5)$ and $g_5 = g_5(-\frac{1}{2}(y^2+y^3)+\chi)$ are arbitrary functions of their arguments.

7.6 POLARIZATIONS

Another interesting concept related to the torsions introduced in the previous sections is that of their polarization. As explained in [102], a *polarization* is defined as the process of going from a homogeneous to a multilinear form. Therefore, to conclude this chapter, we will give some hints of this idea by following the article [187], defining the polarization of generalized Nijenhuis torsions

and discussing one of its direct algebraic applications in the characterization of Haantjes $C^\infty(M)$-modules of operator fields.

7.6.1 *Defect and polarization of torsions*

A first example of this notion can be found in the fact that the FN bracket for $(1,1)$-tensor fields (see Eq. (2.4)) is the polarization of the Nijenhuis torsion. Since the Nijenhuis torsion $\tau_{\boldsymbol{A}}(X,Y)$ is a quadratic form depending on the entries of the operator $\boldsymbol{A}$, it can be polarized in a natural way. By doing so, one obtains that the bilinear form obtained coincides with the FN bracket:

$$[\boldsymbol{A},\boldsymbol{B}]_{\mathrm{FN}}(X,Y) = \tau_{\boldsymbol{A}+\boldsymbol{B}}(X,Y) - (\tau_{\boldsymbol{A}}(X,Y) + \tau_{\boldsymbol{B}}(X,Y))\,. \tag{7.64}$$

This result can be extended to derive the multilinear polarizations of generalized Nijenhuis torsions of arbitrary level m. To achieve this, the authors in [187] adopt the approach proposed in [48], introducing the concept of *defect of a torsion of level m* and defining the associated polarization 2-form based on this notion.

Definition 7.6.1. *Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ be a set of operator fields. The* defect of index *k of the generalized Nijenhuis torsion of level m is the vector-valued 2-form defined by*

$$\Delta_k^{(m)}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_k)(X,Y) := \sum_{\mathcal{I}\subseteq\{1,2,\ldots,k\}} (-1)^{k-|\mathcal{I}|}\,\tau^{(m)}_{\sum_{i\in\mathcal{I}}\boldsymbol{A}_i}(X,Y)\,, \tag{7.65}$$

where $k, m \in \mathbb{N}\setminus\{0\}$. Here, $\mathcal{I}$ denotes any non-empty subset extracted from $\{1,2,\ldots,k\}$, and $|\mathcal{I}|$ represents its cardinality.

By construction, this quantity is symmetric in its arguments $(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_k)$. Moreover, it satisfies the following properties.

Proposition 7.6.1. *The defect of index k is additive in each of its entries $\boldsymbol{A}_i$, $i = 1,\ldots,k$, if and only if it fulfils*

$$\Delta_{k+1}^{(m)}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_{k+1})(X,Y) = \boldsymbol{0}\,. \tag{7.66}$$

Proposition 7.6.2. *For $k > 2m$, it holds that*

$$\Delta_k^{(m)}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_k)(X,Y) = \boldsymbol{0}\,. \tag{7.67}$$

Building on this concept, we define the vector-valued 2-form representing the polarization form associated with the generalized Nijenhuis torsions using the polarization identity proposed in [102] (see Eq. (2.6.5), p. 43). This form corresponds to the defect of index k when $k = 2m$, where m represents the level of the generalized torsion.

Definition 7.6.2. *The* polarization *of the Nijenhuis torsion of level m is the vector-valued 2-form*

$$\begin{aligned}\mathcal{P}^{(m)}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_{2m})(X,Y) &:= \Delta_{2m}^{(m)}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_{2m})(X,Y)\\ &= \sum_{\mathcal{I}\subseteq\{1,2,\ldots,2m\}} (-1)^{2m-|\mathcal{I}|}\,\tau^{(m)}_{\sum_{i\in\mathcal{I}}\boldsymbol{A}_i}(X,Y)\,,\end{aligned} \tag{7.68}$$

where m is a non-zero natural number, $X, Y \in \mathfrak{X}(M)$ and $\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_{2m}$ are $(1,1)$-tensor fields.

Remark 7.6.1. *When defining polarizations, it is usual to include a factor* $\frac{1}{(2m)!}$*, as seen for instance in [102]. In this case, however, it was decided to omit it so that the case with* $m = 1$ *exactly recovers the FN bracket* (2.4)*:*

$$\mathcal{P}^{(1)}(\boldsymbol{A}_1, \boldsymbol{A}_2)(X, Y) \equiv [\boldsymbol{A}_1, \boldsymbol{A}_2]_{FN}(X, Y)\,. \tag{7.69}$$

It is also possible to introduce an equivalent alternative definition following the approach of [153], which is also useful:

$$\mathcal{P}^{(m)}(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_{2m})(X, Y) := \frac{\partial^{2m}}{\partial\lambda_1\,\partial\lambda_2\,\ldots\,\partial\lambda_{2m}}\,\tau^{(m)}_{\sum_{i=1}^{2m}\lambda_i\,\boldsymbol{A}_i}(X, Y)\,. \tag{7.70}$$

Here, the λ_i are distinct arbitrary real parameters.

Besides, when all its entries are equal, we recover the corresponding generalized Nijenhuis torsion:

$$\tau^{(m)}_{\boldsymbol{A}}(X, Y) = \Delta^{(m)}_1(\boldsymbol{A})(X, Y) = \frac{1}{(2m)!}\mathcal{P}^{(m)}(\underbrace{\boldsymbol{A}, \ldots, \boldsymbol{A}}_{2m\text{ times}})(X, Y)\,. \tag{7.71}$$

A study of the properties of the polarization form can be found in the article [187], among which we highlight the following.

Proposition 7.6.3. *If the operator fields* $\boldsymbol{A}_i$*,* $i = 1, 2, \ldots 2m$*, pairwise commute, then for* $m \geq 2$*, the polarization of the generalized Nijenhuis torsion is* $C^\infty(M)$*-multilinear.*

Since one instance where the commutativity condition is trivially satisfied occurs when all the operators are equal, this result is relevant to generalized Nijenhuis torsions. In fact, it aligns with the discussion found in Proposition 2.3.3 regarding the Haantjes torsion, as well as with Proposition 7.1.1 for generalized torsions.

7.6.2 *Generalized Haantjes modules and vector spaces*

The polarizations described above play an essential role in the study of generalized $C^\infty(M)$-modules of Nijenhuis operators. In [184] the notion of *Haantjes modules*, namely modules of Haantjes operators, was introduced; but the polarisations allow the characterisation of more general modules, defined in [186, 187].

Definition 7.6.3. *A* generalized Haantjes module *of level* m *is a pair* $(M, \mathscr{H}^{(m)}_M)$ *which satisfies the following conditions:*

(i) M *is a differentiable manifold of dimension* n*;*

(ii) $\mathscr{H}^{(m)}_M$ *is a set of operators* $\boldsymbol{K}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ *whose Nijenhuis torsion of level* m *vanishes, i.e.,* $\tau^{(m)}_{\boldsymbol{K}} = \boldsymbol{0}$*. Furthermore, they fulfil*

$$\begin{aligned}\tau^{(m)}_{(f\boldsymbol{K}_1 + g\boldsymbol{K}_2)}(X, Y) = \boldsymbol{0}\,, \qquad &\forall\, X, Y \in \mathfrak{X}(M)\,,\\ \forall\, f, g \in C^\infty(M)\,, \quad &\forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H}^{(m)}_M\,.\end{aligned} \tag{7.72}$$

Therefore, a generalized Haantjes module consists of a free module of generalized Nijenhuis operators of level m over the ring of smooth functions on M. In the specific case where the functions f and g in the property (7.72) are restricted to be real constants, we refer to it as a *generalized Haantjes vector space* of level m.

As the following result shows, the generalized Haantjes modules can be described in terms of polarizations.

Theorem 7.6.1. *Let M be a differentiable manifold, and let $\boldsymbol{A}, \boldsymbol{B}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ be two generalized Nijenhuis operators of level m. For $m \geq 1$, $\boldsymbol{A}$ and $\boldsymbol{B}$ generate a generalized Haantjes vector space of level m if and only if the $2m-1$ differential conditions*

$$\mathcal{P}^{(m)}(\underbrace{\boldsymbol{A},\dots,\boldsymbol{A}}_{2m-k \text{ times}},\underbrace{\boldsymbol{B},\dots,\boldsymbol{B}}_{k \text{ times}})(X,Y) = \mathbf{0}, \qquad k = 1,2,\dots,2m-1 \tag{7.73}$$

are satisfied. Moreover, for $m \geq 2$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, they generate a generalized Haantjes module of level m if and only if the same conditions (7.73) are satisfied.

Part IV

OTHER RELATED RESULTS ON DISCRETE INTEGRABILITY

Split the Lark – and you'll find the Music –
Bulb after Bulb, in Silver rolled –
Scantilly dealt to the Summer Morning,
Saved for your Ear when Lutes be old.

Emily Dickinson, *Split the Lark – and you'll find the Music*

8

A Frobenius-type theory for discrete systems

Consider a blade of steel, glinting in countless hands-shaping marble in a sculptor's grip, parting tissue beneath a surgeon's light, slicing wheat under a gilded moon. It is weapon, tool, artist; edges honed to infinite purposes, yet bound by a single truth: its sharpness divides the whole. The blade does not question what it cuts. It exists to sever, to dissect, to lay bare the hidden layers of things. Its essence is not in the act of division, but in the clarity it leaves behind.

Integrability, like the blade, adapts its edge to the task at hand. In classical mechanics, it is the Liouville-Arnold theorem–a sculptor's chisel, revealing symmetries in the grain of Hamiltonian chaos, carving order from motion with conserved quantities. For quantum realms, it becomes a surgeon's scalpel: observables must commute, entanglement unraveled into solvable threads. When nonlinear waves surge against the cliffs of PDEs, integrability swings like a scythe, reaping an infinite harvest of conserved currents to tame turbulence. Yet beneath these guises lies a shared core–the blade's unwavering purpose. Poincaré once reflected, *"A system of differential equations is only more or less integrable"*; so too does the blade's efficacy depend on the hand that guides it. Even in discrete landscapes, where difference equations replace differential flows, integrability endures–a crystalline fracture of order, unexpected yet precise, proving that division can forge unity.

This final chapter turns to integrability in discrete systems. A central challenge in modern mathematical physics lies in discretizing dynamical models while preserving their symmetries and integrability, as seen in formulations for ODEs and PDEs [18, 66, 110, 179]. Discrete approaches to

quantum mechanics, for instance, arise in quantum gravity frameworks [58, 62, 170], where spacetime itself is granular. Here, integrability becomes both a tool and a compass, guiding structure through the lattice.

A crucial difficulty when formulating these discretizations is presented by the no-go theorem proved in [90]. This theorem states that it is impossible to define a field theory on an infinite lattice equipped with a nontrivial product that simultaneously satisfies Leibniz's rule, translational invariance, and locality.

To circumvent this drawback, methods such as the computation of time-scales (see [72]) have been developed, which waive Leibniz's rule. This approach is based on unifying discrete and continuous calculus by defining a general domain that may be continuous, discrete, or mixed (time-scales or, more generally, measure chains) along with appropriate jump operators. It has been widely studied in recent years (see monograph [25]), yielding many results and applications.

Another possibility is offered by the *finite operator theory* (also called umbral calculus), developed by G. C. Rota and collaborators [165, 166, 169] as a language to unify the study of polynomial sequences and combinatorial problems. Its ideas have been employed as a novel theoretical framework for discretizing quantum-mechanical models [46, 106–109] and for multiscale analysis of dynamical systems on a lattice [108].

A consistent approach based on Roman-Rota theory was formulated more recently. The essential idea is to preserve Leibniz's rule by using discrete operators that act as derivatives with respect to a nonstandard multiplication rule. This idea was proposed from different perspectives and contexts (e.g., [28, 77, 195]), and later in [181, 182].

From a technical point of view, an appropriate nonlocal product of functions is defined that acts on the basic polynomials associated with a given discrete derivative (delta operator) in the same manner as the standard local product acts on monomials. In this way, the class of exact solutions expressed in terms of power series is preserved. Formally, the algebraic structure on which the discretization process is implemented is that of a Rota algebra [181], namely, a formal differential Galois algebra of power series equipped with a delta operator and a suitable product that ensures the derivation property.

The nonlocality issue of this approach is resolved by identifying the lattice points where the dynamics occurs with the set of zeros of the basic polynomials associated with the given delta operator. This fundamental requirement addresses the main problem encountered in previous "umbral" approaches to discretization, which were formulated in the language of (typically divergent) formal power series.

In [181, 182] the approach was developed within a categorical framework, and discrete analogs were obtained for a large class of linear and nonlinear dynamical systems whose coefficients are expressed in terms of polynomials. A discretization of the Euler equation was proposed in [163]. Moreover, a discrete analog of classical Frobenius theory for the analysis of ODE singularities was developed in [158]; this last example serves to illustrate the procedure.

Sections 8.1-8.3 present a brief review of some elementary notions and fundamental results of the Roman-Rota formulation of the theory of difference operators, which provides a natural language

for defining the theory of basic, Appell, and Sheffer sequences of polynomials. This approach can be regarded as the modern version of the umbral calculus of the 19th century introduced by Sylvester, Cayley, Blissard, and others [45]. After introducing these concepts, Sections 8.4 and 8.5 develop a Frobenius-type theory for analyzing regular and regular singular points, building on the framework proposed in [158]. In Section 8.6, this methodology is applied to the Bessel equation, demonstrating its practical utility.

8.1 BASIC NOTIONS OF FINITE OPERATOR CALCULUS

Let $\mathbb{N}$ denote the set of nonnegative integers and let $\mathbb{K}$ be a field of characteristic zero. Denote by $\mathcal{F}[[\mathbb{K}]]$ the algebra of formal power series in x, and by $\mathcal{P}$ the space of polynomials in the variable $x \in \mathbb{K}$. Throughout this discussion, $\mathbb{K} = \mathbb{R}$.

Definition 8.1.1. *An operator $T^h : \mathcal{P} \to \mathcal{P}$ satisfying*

$$T^h p(x) = p(x+h) \quad \textit{for } h > 0,$$

is called a shift operator.

For simplicity, we will adopt the notation $T := T^h$, with $h = 1$.

Definition 8.1.2. *An operator S is called* shift-invariant *if it commutes with every shift operator T^h.*

Let $\mathcal{A}$ denote the algebra of shift-invariant operators, equipped with the usual operations (sum and product of operators, and product of a scalar with an operator).

Definition 8.1.3. *A shift-invariant operator Q is called a* delta operator *if $Qx = \text{constant} \neq 0$.*

It immediately follows the following result.

Corollary 8.1.1. *For each constant $c \in \mathbb{R}$, $Q\,c = 0$.*

The concept of the delta operator has been extensively investigated in the literature (see, e.g., [169]). Typical examples include the standard derivative D, the forward discrete derivative $\Delta^+ = T - \mathbf{1}$, the backward derivative $\Delta^- = \mathbf{1} - T^{-1}$, and the symmetric derivative $\Delta^s = \frac{T - T^{-1}}{2}$.

Denote by t^k the operator D^k. Then its action on a polynomial is given by $t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}$. Extending this action by linearity, we can associate a differential operator with a formal power series:

$$F(t) = \sum_k \frac{a_k t^k}{k!} \quad \Rightarrow \quad F(D) = \sum_k \frac{a_k D^k}{k!}.$$

This correspondence permits delta operators to be represented in terms of a function $q(t)$, called the *representative* of Q. For instance, a non-standard example is the Gould operator $q^G(t) = e^{at}(e^{bt} - 1)$. Another interesting case is the Abel operator $q^A(t) = te^{\sigma t}$, whose basic sequence is the set of Abel polynomials $p_n^A(x) = x(x - n\sigma)^{n-1}$ [166].

Definition 8.1.4. *A sequence of polynomials* $\{p_n(x)\}_{n\in\mathbb{N}}$ *is called the sequence of* basic polynomials *associated with a delta operator* Q *if the following conditions hold:*

(i) $p_0(x) = 1$;

(ii) $p_n(0) = 0$ *for all* $n > 0$;

(iii) $Q\,p_n(x) = n\,p_{n-1}(x)$.

It is well known that there exists a unique sequence of basic polynomials associated with any delta operator [169]. For instance, if $Q = D$, the sequence is given by $p_n(x) = x^n$. For the forward and backward discrete derivatives $\Delta^{\pm}$, one obtains

$$p_0^{\pm}(x) = 1\,, \quad p_n^{\pm}(x) := x(x \mp 1)(x \mp 2)\cdots(x \mp (n-1))\,, \tag{8.1}$$

respectively. In the case of Δ^s, the basic polynomials are expressed in terms of Gould polynomials [108], which constitute the basic sequence for the operator q^G.

8.2 PINCHERLE DERIVATIVE AND BASIC POLYNOMIALS

The following technique, discussed in [107], permits the determination of the basic polynomials associated with an arbitrary delta operator through the algebraic notion of the Pincherle derivative. Let the operator of multiplication by x be denoted by x: $p(x) \mapsto x\,p(x)$.

Definition 8.2.1. *The* Pincherle derivative *of a delta operator* $Q \in \mathcal{A}$ *is defined by*

$$Q' := [Q, x]\,. \tag{8.2}$$

If Q is a delta operator, then Q' is invertible [169]. An operator $\beta \in \mathcal{A}$ is determined by requiring that the Heisenberg–Weyl algebra is satisfied [106]:

$$[Q,\, x\beta] = \mathbf{1}\,. \tag{8.3}$$

For a given delta operator Q, the operator β, called the conjugate operator of Q, exists uniquely and is given by $\beta = (Q')^{-1}$. In particular, for $Q = D$, one has $\beta = \mathbf{1}$; for $Q = \Delta^+$, $\beta = T^{-1}$; and for $Q = \Delta^-$, $\beta = T$.

Thus, the basic polynomials associated with a delta operator Q can be obtained via the relation

$$p_n(x) = (x\beta)^n \cdot 1. \tag{8.4}$$

8.3 DISCRETIZATION OF DYNAMICAL SYSTEMS: ROTA ALGEBRAS

For a given delta operator Q, its family of basic polynomials $\{p_n(x)\}_{n\in\mathbb{N}}$ forms a basis for the algebra of formal power series $\mathcal{F}$. Consequently, any function $f \in \mathcal{F}$ can be expanded as a formal series

$$f(x) = \sum_{n=0}^{\infty} a_n\,p_n(x)\,.$$

In $\mathcal{F}$, we define the $*$-product, an associative and commutative operation, by

$$p_n(x) * p_m(x) := p_{n+m}(x)\,. \tag{8.5}$$

This product, in the context of the forward difference operator Δ^+, was introduced in [195] and [77], and it extends naturally by linearity to spaces of formal power series. If functions $f, g \in \mathcal{F}$ are defined on a set of points $\mathcal{L} \subset \mathbb{R}$ (referred to as the lattice $\mathcal{L}$) and are expanded in the polynomial basis provided by the basic sequence $\{p_n(x)\}_{n\in\mathbb{N}}$, the following fundamental relation holds:

$$\Delta^+(f * g) = (\Delta^+ f) * g + f * (\Delta^+ g)\,. \tag{8.6}$$

In other words, Δ^+ acts as a derivation with respect to the $*$-product, thereby restoring the Leibniz rule on $\mathcal{F}$. Moreover, by identifying the lattice $\mathcal{L}$ with the set of zeros of the basic polynomials, all the formal series become truncated.

Based on these ideas, for any delta operator Q the notion of a *Rota algebra* can be defined (see [181]), providing a natural framework for the discretization procedure described above.

Definition 8.3.1. *A* Rota differential algebra *is a Galois differential algebra* $(\mathcal{F}, Q)$*, where* $(\mathcal{F}, +, \cdot, *_Q)$ *denotes an associative algebra of formal power series, the product* $*_Q$ *being the composition law defined by* (8.5)*, and* Q *is a delta operator acting as a derivation on* $\mathcal{F}$*:*

(i) $Q(a+b) = Q(a) + Q(b)\,, \qquad Q(\lambda a) = \lambda Q(a)\,, \qquad \lambda \in \mathbb{K}\,;$

(ii) $Q(a * b) = Q(a) * b + a * Q(b)\,.$

It was proved in [181] that there exists a unique Rota algebra $(\mathcal{F}, Q)$ associated with a given delta operator Q.

Hence, the discretization procedure consists in associating with a given ODE a formal equation defined within a Rota algebra by replacing the continuous derivative D with a delta operator Q (which acts as a Galois operator, as in Eq. (8.6)) and substituting the basic sequence x^n with the basic polynomials $p_n(x)$.

$$\begin{array}{ccc} D & \longrightarrow & Q \\ \uparrow\downarrow & & \uparrow\downarrow \\ x^n & \longrightarrow & p_n(x) \end{array}$$

This discretization scheme extends the *umbral correspondence* discussed in [46, 107, 109, 169] because it is no longer defined solely on formal power series but becomes *effective*. Due to the truncation phenomenon described above, it yields standard finite-difference dynamical models that admit convergent solutions in one-to-one correspondence with the analytic solutions of the original continuous model [181, 182, 195]. Different choices of Q (and, hence, of the associated Rota algebra) lead to different discrete models that share the same family of solutions, *mutatis mutandis*. In this technical sense, the discretization is said to preserve integrability.

8.4 GENERALIZING FROBENIUS ANALYSIS ON THE LATTICE: REGULAR POINTS

8.4.1 *Statement of the problem*

This section summarizes the main results presented in [158] concerning the discretization of the second-order linear differential equation

$$y'' + A(x)y' + B(x)y = 0\,. \tag{8.7}$$

More generally, our aim is to construct a theory on a discrete lattice that preserves the main results of the standard Frobenius approach based on the singularity analysis of Eq. (8.7).

According to standard terminology, if $A(x)$ and $B(x)$ are analytic functions in a neighborhood of a given point x_0, then x_0 is called a *regular point* of the equation; otherwise, it is *singular*. Assuming that $x = 0$ is a regular point, the functions $A(x)$ and $B(x)$ admit the power series expansions

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n x^n, \tag{8.8}$$

which converge in a neighborhood of $x = 0$. As is well known [176], there exists a basis for the solution space of Eq. (8.7) consisting of two analytic functions with convergence radius $R \geq R_0 := \min\{R_1, R_2\}$, where R_1 and R_2 denote the convergence radii of the series representing $A(x)$ and $B(x)$, respectively. The coefficients in the series solutions are uniquely determined once the initial conditions are fixed. In particular, the general form of the two independent series solutions is given by

$$y(x) = \sum_{k=0}^{\infty} \zeta_k x^k. \tag{8.9}$$

The discrete counterpart of Eq. (8.7) is introduced by means of the formal series

$$u(x) = \sum_{k=0}^{\infty} \zeta_k\, p_k(x)\,, \tag{8.10}$$

where $\{p_k(x)\}_{k\in\mathbb{N}}$ are the basic polynomials associated with a given delta operator Q. Also define the formal series

$$F(x) := \sum_{k=0}^{\infty} a_k\, p_k(x)\,, \qquad G(x) := \sum_{k=0}^{\infty} b_k\, p_k(x)\,, \tag{8.11}$$

where $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are the coefficients appearing in (8.8).

Definition 8.4.1. *Let Q be a delta operator and let $\{p_n(x)\}_{n\in\mathbb{N}}$ be the associated family of basic polynomials. We define the class of discrete dynamical systems by*

$$Q^2u(x) + F(x) * Qu(x) + G(x) * u(x) = 0\,, \tag{8.12}$$

where the operations are defined on the Rota algebra $(\mathcal{F}, +, \cdot, *_Q)$.

Lemma 8.4.1. *The dynamical model* (8.12) *admits the formal solution*

$$u(x) = \sum_{k=0}^{\infty} \zeta_k \, p_k(x) \,, \tag{8.13}$$

where the coefficients ζ_k *are determined by the recurrence relation*

$$\zeta_{k+2} = -\frac{1}{(k+2)(k+1)} \sum_{m=0}^{k} \left[(m+1) a_{k-m} \, \zeta_{m+1} + b_{k-m} \, \zeta_m \right] . \tag{8.14}$$

Remark 8.4.1. *As a direct consequence of Lemma 8.4.1, the class of analytic solutions of Eq.* (8.7) *is inherited by every discrete representative of Eq.* (8.12) *in the sense that every solution of the form* (8.9) *for Eq.* (8.7) *defines a corresponding solution of the form* (8.10) *for Eq.* (8.12).

The family of systems (8.12) can also be represented as a standard difference equation once a difference delta operator Q is chosen, with a dependent variable u_n defined on a suitable lattice $\mathcal{L}$ (namely, the set of zeros of the basic polynomials associated with Q). In the subsequent discussion the focus is on the case $Q = \Delta^+$. In this case, the lattice $\mathcal{L}^+$ consists of the points $x = n$, $n = 0, 1, 2, \ldots$. We define

$$u_n := u(n) = \sum_{k=0}^{n} \zeta_k \, p_k^+(n) = \sum_{k=0}^{n} \zeta_k \, \frac{n!}{(n-k)!} \,. \tag{8.15}$$

This relation can be inverted [181]:

$$\zeta_k = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} \, u_j \,. \tag{8.16}$$

Combining the inversion formula (8.16) with Lemma 8.4.1 yields the following result.

Theorem 8.4.1. *Consider the family of difference equations*

$$u_{n+2} - 2u_{n+1} + u_n + \sum_{l=0}^{n} \frac{n!}{(n-l)!} \left[a_l \, u_{n-l+1} - (a_l - b_l) \, u_{n-l} \right] = 0 \,, \tag{8.17}$$

where $n \in \mathbb{N}$ *and* $a_l, b_l \in \mathbb{R}$ *are the coefficients in the series expansions* (8.8). *Then, the function* $y(x) = \sum_{k=0}^{\infty} \zeta_k \, x^k$ *is a solution of the differential equation* (8.7) *if and only if the series* $u_n = \sum_{k=0}^{n} \zeta_k \, \frac{n!}{(n-k)!}$ *is a solution of the difference equation* (8.17).

Thus, Eq. (8.17) represents the discrete counterpart of Eq. (8.7) on the lattice $\mathcal{L}^+$; it can also be regarded as the realization of the abstract equation (8.12) on the Rota algebra $(\mathcal{F}, +, \cdot, *_{\Delta^+})$. All these results can be generalized to any other delta operators following the same procedure; indeed, an equivalent result to Theorem 8.4.1 for Δ^- is also established in [158]. Hereafter, for the sake of concreteness, we will list some basic examples of discrete models arising from Eq. (8.17).

8.4.2 *A constant coefficient model*

The constant coefficient equation

$$y'' + \alpha y' + \beta y = 0 \tag{8.18}$$

with $a_0 = \alpha$, $a_l = 0,\ l \geq 1$, $b_0 = \beta$, $b_l = 0,\ l \geq 1$ provides us with the difference equation

$$u_{n+2} + (\alpha - 2)u_{n+1} - (\alpha - \beta - 1)u_n = 0 \tag{8.19}$$

whose independent solutions, inherited from those of Eq. (8.18) are

$$u_n^{(1)} = \sum_{k=0}^{n} \frac{(\sqrt{\alpha^2 - 4\beta} - \alpha)^k}{2^k k!} \frac{n!}{(n-k)!}, \quad u_n^{(2)} = \sum_{k=0}^{n} \frac{(-1)^k (\sqrt{\alpha^2 - 4\beta} + \alpha)^k}{2^k k!} \frac{n!}{(n-k)!}.$$

8.4.3 *The Airy equation*

One of the most interesting examples of the application of the above theory is the Airy equation

$$y'' - xy = 0 \tag{8.20}$$

which, in quantum mechanics, represents the Schrödinger equation for a particle in a one-dimensional constant force field.

The discrete version of it is readily obtained from Eq. (8.17) by taking $a_l = 0,\ l \geq 0$, $b_0 = 0$, $b_1 = -1$, $b_l = 0,\ l \geq 2$. We immediately obtain an interesting discrete model.

Definition 8.4.2. *The difference equation*

$$u_{n+2} - 2u_{n+1} + u_n - nu_{n-1} = 0, \qquad n \in \mathbb{N} \tag{8.21}$$

will be said to be the discrete Airy equation on $\mathcal{L}^+$.

The following discrete Airy functions provide the two independent solutions of the Airy difference equation (8.21):

$$\mathrm{Ai}(n) = \sum_{k=0}^{n} \frac{1}{3^k k! 2 \cdot 5 \cdots (3k-1)} \frac{n!}{(n-3k)!},$$

$$\mathrm{Bi}(n) = \sum_{k=0}^{n} \frac{1}{3^k k! 4 \cdot 7 \cdots (3k+1)} \frac{n!}{(n-(3k+1))!}.$$

8.4.4 *The Hermite equation*

The Hermite equation

$$y'' - 2xy' + 2\lambda y = 0 \tag{8.22}$$

can be discretized by taking $a_0 = 0, a_1 = -2, a_l = 0,\ l \geq 2, b_0 = 2\lambda, b_l = 0,\ l \geq 1$.

Definition 8.4.3. *The Hermite difference equation is*

$$u_{n+2} - 2u_{n+1} + (1 + 2\lambda - 2n)u_n + 2nu_{n-1} = 0. \tag{8.23}$$

By way of an example, we shall consider the Hermite polynomials $H_N(x)$ which are exact solutions of the Hermite equation for $\lambda = N$, being N a nonnegative integer. For instance, for $N = 3$, the Hermite polynomial

$$H_3(x) = 8x^3 - 12x \tag{8.24}$$

corresponds to the function

$$u_n = 8p_3^+(n) - 12p_1^+(n) = 8n^3 - 24n^2 + 4n, \tag{8.25}$$

which is an exact solution of the Hermite difference equation (8.23) for $\lambda = 3$:

$$u_{n+2} - 2u_{n+1} + (7 - 2n)u_n + 2nu_{n-1} = 0. \tag{8.26}$$

8.4.5 *Continuum limit*

We shall discuss the continuum limits of the large family of discrete models introduced in this section. To this aim, we will introduce a step size h and define a regular mesh of points on $\mathbb{R}_{\geq 0}$, indexed by nh, $n \in \mathbb{N}$. We will consider the double limit $h \to 0$ and $n \to \infty$ while keeping nh finite. The basic polynomials associated with $\Delta_h^+ = \frac{T^h - 1}{h}$ are now $p_k^+(x) = \prod_{j=0}^{k-1}(x - jh)$. Thus,

$$p_k^+(nh) = h^k \prod_{j=0}^{k-1}(n - j) = \frac{n!h^k}{(n-k)!}. \tag{8.27}$$

Consequently, the coefficients ζ_k are given by:

$$\zeta_k = \frac{1}{h^k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} u_j, \tag{8.28}$$

with

$$u_n = u(nh) = \sum_{k=0}^{n} \zeta_k p_k^+(nh) = \sum_{k=0}^{n} \frac{n!h^k \zeta_k}{(n-k)!}. \tag{8.29}$$

The action of Δ_h^+ on a solution is expressed by

$$\begin{aligned}\Delta_h^+ u &= \frac{1}{h}(u_{n+1} - u_n) = \frac{1}{h}\left(\sum_{k=0}^{n+1} \frac{(n+1)!h^k\zeta_k}{(n+1-k)!} - \sum_{k=0}^{n} \frac{n!h^k\zeta_k}{(n-k)!}\right) \\ &= \sum_{k=0}^{n+1} h^{k-1}\zeta_k \left(\frac{(n+1)!}{(n+1-k)!} - \frac{n!}{(n-k)!}\right) = \sum_{k=0}^{n+1} k\zeta_k \frac{n!h^{k-1}}{(n-(k-1))!} = \sum_{k=0}^{n+1} k\zeta_k p_{k-1}^+(nh).\end{aligned} \tag{8.30}$$

Consequently,

$$(\Delta_h^+)^2 u = \frac{1}{h^2}(u_{n+2} - 2u_{n+1} + u_n) = \sum_{k=0}^{n+2} k(k-1)\zeta_k p_{k-2}^+(nh). \tag{8.31}$$

The equation (8.17) on our mesh takes the form

$$\frac{1}{h^2}(u_{n+2} - 2u_{n+1} + u_n) + \sum_{l=0}^{n} \frac{n!h^l}{(n-l)!}\left[\frac{a_l}{h}(u_{n-l+1} - u_{n-l}) + b_l\, u_{n-l}\right] = 0, \tag{8.32}$$

which reproduces Eq. (8.7) in the continuum limit. For instance, the difference Airy equation becomes

$$\frac{1}{h^2}(u_{n+2} - 2u_{n+1} + u_n) - nhu_{n-1} = 0, \tag{8.33}$$

whereas the difference Hermite equation reads

$$\frac{1}{h^2}(u_{n+2} - 2u_{n+1} + u_n) + 2\lambda u_n - 2n(u_n - u_{n-1}) = 0. \tag{8.34}$$

In the continuum limit, they coincide with the differential equations (8.20) and (8.22) respectively.

8.5 FROBENIUS THEORY FOR REGULAR SINGULAR POINTS

This section recapitulates the extension of the analysis from [158] to the more general case of differential equations of the form (8.7) that possess regular singular points. To this end, we rewrite Eq. (8.7) as

$$x^2 y'' + R(x)y' + S(x)y = 0\,, \tag{8.35}$$

where the functions $R(x) := x^2 A(x)$ and $S(x) := x^2 B(x)$, are analytic. In order to restrict the singularity of $A(x)$ to at most a simple pole, one imposes $R(0) = 0$. Thus, one may write

$$R(x) = \sum_{k=1}^{\infty} r_k x^k, \qquad S(x) = \sum_{k=0}^{\infty} s_k x^k. \tag{8.36}$$

According to classical Frobenius theory, there exists a solution of Eq. (8.35) of the form

$$y(x) = x^\lambda \sum_{k=0}^{\infty} \tilde{\zeta}_k x^k, \tag{8.37}$$

where λ is a root of the indicial polynomial

$$\lambda(\lambda-1)+r_1\lambda+s_0=0\,. \tag{8.38}$$

Restricting the analysis to the case of real roots, and assuming that the larger root is a non-negative integer, the solution can be written as

$$y(x)=\sum_{k=0}^{\infty}\tilde{\zeta}_k x^{\lambda+k}=\sum_{k=\lambda}^{\infty}\tilde{\zeta}_{k-\lambda}x^k=\sum_{k=0}^{\infty}\zeta_k x^k, \tag{8.39}$$

with the understanding that $\zeta_k=0$ for $k=0,\ldots,\lambda-1$. The subsequent procedure follows that of the regular case.

By analogy with the previous discussion, the formal series (8.10) is introduced (assuming $r_0=0$), and we define the dynamical system by

$$p_2(x)*Q^2u(x)+V(x)*Qu(x)+W(x)*u(x)=0, \tag{8.40}$$

on the abstract Rota algebra $(\mathcal{F},+,\cdot,*_Q)$, where

$$V(x)=\sum_{l=1}^{\infty}r_l\,p_l(x)\,,\qquad W(x)=\sum_{l=0}^{\infty}s_l\,p_l(x)\,.$$

Substituting $u_n=\sum_{k=0}^{n}\zeta_k\,p_k(n)$ and after suitable simplification, Eq. (8.40) takes the form

$$\sum_{k=0}^{\infty}\left\{k(k-1)\zeta_k+\sum_{l=0}^{k}\left[(k-l)r_{l+1}+s_l\right]\zeta_{k-l}\right\}p_k(x)=0\,. \tag{8.41}$$

Hence, we obtain the recurrence relation

$$\left[k(k-1)+kr_1+s_0\right]\zeta_k+\sum_{l=1}^{k}\left[(k-l)r_{l+1}+s_l\right]\zeta_{k-l}=0\,,\quad k=0,1,\ldots, \tag{8.42}$$

which implies in particular that

$$\begin{aligned}
s_0\,\zeta_0&=0\,,\\
(r_1+s_0)\,\zeta_1+s_1\,\zeta_0&=0\,,\\
(2+2r_1+s_0)\,\zeta_2+(r_2+s_1)\,\zeta_1+s_2\,\zeta_0&=0\,,\\
&\ldots
\end{aligned}$$

The difference equation for $u_n=\sum_{k=0}^{n}\zeta_k\,p_k(n)$ is obtained from Eq. (8.41). In the case $Q=\Delta^+$, by applying the inversion formula (8.16) and using the basic polynomials $p_k^+(n)$, the following result is deduced.

Theorem 8.5.1. *Consider the class of discrete dynamical systems*

$$n(n-1)(u_n - 2u_{n-1} + u_{n-2}) + \sum_{l=0}^{n} \frac{n!}{(n-l)!} \left[r_l \, u_{n-l+1} - (r_l - s_l) \, u_{n-l} \right] = 0 \,, \tag{8.43}$$

where $n \in \mathbb{N}$ *and* $r_l, s_l \in \mathbb{R}$ *are the coefficients of the series expansions* (8.36)*. Then, the function* $y(x) = \sum_{k=0}^{\infty} \zeta_k \, x^k$ *is a solution of the differential equation* (8.35) *if and only if the series* $u_n = \sum_{k=0}^{n} \zeta_k \, \frac{n!}{(n-k)!}$ *is a solution of the difference equation* (8.43)*.*

Thus, the family of difference equations (8.43) can be regarded as the discrete version of the ordinary differential equations (8.35). Both the continuum limit and the analysis for other delta operators can be performed analogously.

8.6 A DISCRETE BESSEL EQUATION

The theoretical framework developed above allows us to introduce, for example, a natural discretization for the Bessel equation. Due to the relevance of the Bessel equation in many application contexts, we shall discuss its discretization in detail here.

8.6.1 *The zero-order Bessel equation*

Let us discuss first the simplest example of the zero-order Bessel differential equation, which reads

$$x^2 y'' + xy' + x^2 y = 0 \tag{8.44}$$

and admits, as an independent solution, the zero-order Bessel function

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}} x^{2k} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots \tag{8.45}$$

In this case, $R(x) = x$, $S(x) = x^2$. For $n \geq 2$, we obtain a remarkable difference equation.

Definition 8.6.1. *The zero-order discrete Bessel equation is given by*

$$nu_n - (2n-1)u_{n-1} + 2(n-1)u_{n-2} = 0. \tag{8.46}$$

The discrete version of the Bessel function (8.45) $J_0(x)$ arising from our theory is

$$u_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}} p_{2k}^{+}(n) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{2^{2k}(n-2k)!(k!)^2}, \tag{8.47}$$

which represents a solution of the difference equation (8.46). Precisely, from Eq. (8.46) we obtain the relation

$$n \sum_{k=0}^{\infty} \frac{(-1)^k n!}{2^{2k}(n-2k)!(k!)^2} - (2n-1) \sum_{k=0}^{\infty} \frac{(-1)^k (n-1)!}{2^{2k}(n-1-2k)!(k!)^2}$$

$$+ 2(n-1)\sum_{k=0}^{\infty} \frac{(-1)^k (n-2)!}{2^{2k}(n-2-2k)!(k!)^2}. \tag{8.48}$$

It can be simplified, leading us to the expression

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} n (k!)^2 (n-2k)!}[n^2 - (4k+1)(n-2k)], \tag{8.49}$$

which vanishes for $n \geq 2$.

8.6.1.1 *Continuum limit*

With the same construction as in the case of regular points, one can easily derive, in a lattice of step h, the expression

$$(n+2)^2 u_{n+2} - 2(n+2)^2 u_{n+1} + (n+2)^2 u_n \\ + (n+2)u_{n+1} - (n+2)u_n + (n+1)(n+2)h^2 u_n = 0, \tag{8.50}$$

which can be written as:

$$((n+2)h)^2 \frac{u_{n+2} - 2u_{n+1} + u_n}{h^2} + (n+2)h\frac{u_{n+1} - u_n}{h} + (n+1)(n+2)h^2 u_n = 0. \tag{8.51}$$

In the limit ($h \to 0$, $n \to \infty$ and nh finite) we easily recover the zero-order Bessel equation.

8.6.2 *The Bessel equation of order ν*

Let us extend the previous discussion to the Bessel equation of order $\nu \in \mathbb{N}\backslash\{0\}$:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{8.52}$$

In this case, $R(x) = x$, $S(x) = x^2 - \nu^2$. Thus, the non-vanishing coefficients are

$$r_1 = 1, \quad s_0 = -\nu^2, \quad s_2 = 1. \tag{8.53}$$

Therefore, Eq. (8.43) converts into a simpler relation.

Definition 8.6.2. *The difference equation*

$$(n^2 - \nu^2)u_n - n(2n-1)u_{n-1} + 2n(n-1)u_{n-2} = 0 \tag{8.54}$$

is said to be the discrete Bessel equation of order $\nu \in \mathbb{N}\backslash\{0\}$.

8.6.2.1 *Exact solutions*

From the Bessel function of the first class and order ν

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k!(k+\nu)!} x^{2k+\nu} \tag{8.55}$$

we can define its discrete counterpart

$$u_n = \sum_{k=0}^{[(n-\nu)/2]} \frac{(-1)^k}{2^{2k+\nu} k!(k+\nu)!} p^{+}_{2k+\nu}(n), \tag{8.56}$$

which is an exact solution of (8.54).

Part V

CONCLUSIONS AND APPENDICES

Es que todo es muy raro, en cuanto te fijas un poco. Lo raro es vivir. Que estemos aquí sentados, que hablemos y se nos oiga, poner una frase detrás de otra sin mirar ningún libro, que no nos duela nada, que lo que bebemos entre por el camino que es y sepa cuándo tiene que torcer, que nos alimente el aire y a otros ya no, que según el antojo de las vísceras nos den ganas de hacer una cosa o la contraria y que de esas ganas dependa a lo mejor el destino, es mucho a la vez, tú, no se abarca, y lo más raro es que lo encontramos normal.

Carmen Martín Gaite, *Lo raro es vivir*

Conclusions and future perspectives

In the spirit of T.S. Eliot's words, *We shall not cease from exploration, and the end of all our exploring will be to arrive where we started and know the place for the first time,* this thesis embodies a journey of discovery through the intricate landscapes of Haantjes geometry, integrable systems, and discrete analysis. Our exploration has led us to revisit classical concepts with fresh insights, ultimately revealing the elegant structures that underpin both continuous and discrete models in Hamiltonian dynamics. As we return to the starting point enriched by our findings, we now perceive our original framework in a new light–deeper, more coherent, and brimming with possibilities for further inquiry.

The introductory chapters have set the stage by providing a comprehensive overview of geometric mechanics. We recalled the definition of symplectic manifolds and the associated Poisson and Jacobi structures, which play a pivotal role in modern Hamiltonian dynamics. The review of classical integrability via the Arnold-Liouville theorem and the Hamilton-Jacobi approach allowed us to motivate the search for suitable coordinate systems for separation of variables. In parallel, the theory of operator fields was revisited with a focus on Nijenhuis and Haantjes tensors. In particular, we detailed how the vanishing of the Nijenhuis (and, more generally, the Haantjes) torsion is intimately connected with the integrability of eigen-distributions. This framework provided the necessary background to understand how Haantjes algebras lead to the construction of local coordinate systems in which operator fields assume a block-diagonal form–a key feature that underpins the subsequent original contributions. The collective contributions of this thesis demonstrate that Haantjes geometry offers a unifying and versatile framework for addressing several central issues in the theory of integrable systems. The approach presented here is distinguished by its coordinate-

independent nature, which allows for the derivation of integrability conditions directly from the algebraic properties of tensor fields.

One of the main strengths of our work is its applicability to both semisimple and non-semisimple cases. In the superintegrable scenario, the identification of multiple Haantjes structures provides a deep insight into the hidden symmetries of the system. In contrast, the introduction of partial separability through non-semisimple $\omega\mathscr{H}$ manifolds makes it possible to deal with systems that do not admit a complete separation of variables. Such flexibility is crucial when confronting real-world problems where symmetries may be only partially realized or where the system exhibits inherent complexity.

In addition, our tensorial methods are well suited for implementation in computer algebra systems. The tests for vanishing torsion and the procedures for constructing Darboux–Haantjes coordinates can be algorithmically implemented, thus providing effective computational tools for researchers working on integrable systems. Such developments are expected to facilitate further progress in both the theoretical and applied aspects of integrability.

In Chapter 4, we addressed the problem of classical multiseparable Hamiltonian systems and superintegrability within the context of Haantjes geometry. Our work shows that a superintegrable system, which by definition possesses more independent integrals of motion than degrees of freedom, naturally admits several Haantjes structures. Each structure is associated with a distinct separable coordinate system, thereby providing alternative representations for the integrals of motion. In particular, we have demonstrated that the existence of multiple compatible Haantjes algebras guarantees that the Hamilton-Jacobi equation admits additive separation in different coordinate charts. This result not only extends the classical theory of separability but also has practical computational advantages when constructing explicit solutions for systems of physical interest, such as the Smorodinsky-Winternitz models. Our analysis establishes a clear correspondence between the spectral properties of the Haantjes operators and the separability structure of the underlying dynamical system.

In Chapter 5, a theory of partial separability is developed based on non-semisimple symplectic-Haantjes manifolds. In many physically relevant cases, complete separability is too restrictive; hence, it is essential to consider situations in which only a subset of variables can be separated. By relaxing the hypothesis of semisimplicity or maximal rank for the Haantjes algebras, we introduced the concept of partial separability. This framework leads to the construction of Darboux-Haantjes coordinates that block-diagonalize the operators of the Haantjes algebra. Such coordinates allow the Hamilton-Jacobi equations to be split into a mixture of ordinary and partial differential equations. The resulting theory generalizes classical approaches to separability and provides a unified geometric interpretation of integrable systems that admit only partial separation of variables. This contribution is particularly relevant in complex systems where complete separation may not be attainable, yet one can still exploit underlying symmetries to achieve significant simplification.

Chapter 6 is devoted to the study of Hamiltonian systems immersed in external magnetic fields from the perspective of symplectic-Haantjes geometry. In this setting, the Hamiltonian is modified by the presence of a vector potential, which in turn gives rise to a magnetic field. We have shown that even in the presence of such external interactions, one can construct a compatible $\omega\mathscr{H}$ man-

ifold. The corresponding Haantjes algebra naturally encodes the magnetic contributions, and the associated Darboux-Haantjes coordinates yield a separated form for the Hamilton-Jacobi equation. Moreover, by exploiting the underlying Stäckel geometry, our analysis has led to the derivation of new families of integrable Hamiltonian models. These models extend the class of known magnetic systems and provide fresh perspectives on the interplay between gauge freedom and separability. The results obtained in this chapter suggest that the Haantjes approach offers an intrinsic and unified framework for understanding integrability even when non-conservative forces are present. The incorporation of magnetic fields into the Haantjes framework underscores the robustness of our methods. The ability to construct compatible symplectic-Haantjes structures in the presence of gauge fields provides an elegant geometric explanation for the separability observed in many classical magnetic systems. Furthermore, by linking Haantjes geometry with the Stäckel approach, our work opens the possibility of discovering new integrable models that are amenable to explicit solution techniques.

In Chapter 7 we have studied a generalization of classical Nijenhuis torsions by defining higher-order torsions. This new perspective allows one to consider generalized Nijenhuis operators whose vanishing torsions imply integrability conditions for the eigen-distributions of operator fields. A key outcome of this analysis is the definition of generalized Haantjes algebras. These algebras consist of operators whose higher-level torsions vanish and are closed under the usual operations of addition and composition. The main theorem of this chapter guarantees that if a family of commuting operator fields has vanishing generalized torsion of a given level, then there exists a local coordinate chart in which all these operators are simultaneously block-diagonalizable. This result provides a powerful tensorial test for the block-diagonalizability of operator families, thereby bypassing the need for explicit spectral analysis. The generalized framework not only extends the classical theory but also opens new avenues for investigating normal forms of operator fields in various contexts.

In Chapter 8, a novel Frobenius-type framework for discrete dynamical systems is developed, providing a rigorous method to discretize second-order differential equations with analytic coefficients. By leveraging finite operator calculus and the theory of Rota algebras, the chapter establishes a discretization scheme that preserves key properties of the continuous models, such as the Leibniz rule and the exact correspondence of analytic solutions. This approach applies to prototypical equations–including the Airy, Hermite, and Bessel equations–yielding discrete analogues that maintain both integrability and the structure of their singularities. Thus, a bridge is built between continuous and discrete analysis, opening new perspectives for the study of integrable systems on lattices and offering a robust foundation for future research in numerical methods, quantum mechanics on discrete spaces, and the general theory of discrete integrable models.

The results presented in this work open several promising avenues for future research. A primary objective is to develop a comprehensive theory of normal forms for Haantjes operators. Indeed, unlike the semisimple case, where there exists a unique normal form (the diagonal one), in the non-semisimple case much less is known. Although current studies have largely focused on operators with real eigenvalues, there remains a significant need to investigate the full classification of Haantjes operators, including the case when these operators exhibit complex eigenvalues. Such an exten-

sion would not only enrich the theoretical framework but also offer new insights into the spectral properties and integrability conditions of operator fields in various dynamical systems.

Another compelling direction is the adaptation of the symplectic-Haantjes formalism to other geometric settings. In particular, it would be highly beneficial to extend the existing framework from symplectic manifolds to contact and cosymplectic manifolds. This generalization is expected to facilitate the study of systems with dissipation or explicit time dependence, where the traditional symplectic approach is often inadequate. By formulating a contact or cosymplectic version of Haantjes geometry, one could potentially uncover novel integrability properties and develop effective methods for the separation of variables in non-conservative systems.

The class of generalized Stäckel systems (5.26), which a priori possess less than n integrals of motion in involution, may admit a subfamily of completely integrable systems. Thus, it would be important to ascertain in full generality which conditions on the generalized Stäckel matrix (5.25) ensure the existence of further integrals necessary to complete the set of available Hamiltonians in involution. It would also be interesting to study the co-isotropic foliation associated with partially separable systems in the case of an incomplete family of Hamiltonians.

Another relevant question concerns the existence of non-semisimple Haantjes algebras. Given an integrable system, one can construct an associated semisimple Haantjes algebra due to the existence of action-angle variables defined on the Lagrangian foliation of invariant Liouville-Arnold tori. When we deal with superintegrable systems, additional integrals of motion are available. They are in involution with the Hamiltonian, but not necessarily with each other. Presently, it is an open problem whether one might associate with these additional integrals symplectic-Haantjes manifolds, in particular non-semisimple ones. This is the case, for instance, for the Jacobi-Calogero three-particle system discussed in Section 5.4.1. We believe that the solution to this problem is strictly related to the general study of normal forms for Haantjes operators.

There is also a need to deepen the understanding of the application of symplectic-Haantjes manifolds in magnetic systems. The interplay between magnetic fields and Haantjes structures has already shown promising results, yet many aspects remain unexplored. Future research should focus on elucidating how these geometric frameworks can be systematically applied to construct and analyze integrable models in the presence of magnetic interactions. Such studies could lead to the discovery of new magnetic models and provide a clearer interpretation of gauge invariance and separability in Hamiltonian dynamics.

In parallel, further investigations into generalized Haantjes algebras in the context of integrable systems are warranted. The generalization of classical Nijenhuis and Haantjes torsions to higher-order torsions introduces a rich algebraic structure that is still not completely understood. A deeper exploration of the properties, representations, and applications of these generalized algebras is likely to yield substantial advances in the classification of integrable systems, as well as in the construction of novel classes of Hamiltonian models. This line of inquiry could bridge existing gaps between algebraic and geometric methods in the study of integrability.

Additional perspectives arise from the work on superintegrable and partially separable systems. In the context of superintegrability, the existence of multiple Haantjes structures associated with distinct separable coordinate systems suggests that a more systematic classification of these struc-

tures may reveal new hidden symmetries. The development of algorithmic techniques for detecting and exploiting these Haantjes structures remains an open challenge with significant potential for advancing both theoretical and computational aspects of integrable systems. A particularly intriguing direction is the application of the Haantjes chains approach to the TTW potential, a family of superintegrable systems in the Euclidean plane characterized by higher-degree polynomial integrals of motion. To our knowledge, this methodology has not yet been applied to such systems, and its exploration could yield novel insights into their integrability properties. Similarly, the study of partial separability, particularly through the lens of non-maximal-rank Haantjes algebras, invites further inquiry. Refining the concept of partial separability and extending the generalized Stäckel theory to encompass such systems could provide novel tools for the analysis of complex dynamical behaviors that lie between full integrability and chaotic regimes.

Overall, these research directions highlight the vast potential of Haantjes geometry as a unifying framework in the theory of integrable systems. By bridging traditional symplectic methods with new geometric and algebraic approaches, future work in this area promises to deepen our understanding of integrability, separability, and superintegrability across a broad spectrum of physical and mathematical applications.

Moreover, further research on invariant discretizations promises to deepen our understanding of symmetry preservation in discrete models and opens several promising avenues. First, extending this framework to quantum-mechanical systems and quantum field theories could unify finite operator theory and Rota algebras with foundational physical models, enhancing computational and structural fidelity. Second, addressing the open problem of initial and boundary value problems within this theoretical context may deepen the understanding of stability and convergence properties in singularity-adapted discretizations. Finally, a critical future direction lies in rigorously analyzing the symmetry properties of these discrete systems through Lie group theory, which would not only elucidate their geometric invariants but also advance the broader goal of constructing discretizations that inherently preserve continuous symmetries.

I know not what tomorrow will bring.

A

Operators compatible with the symplectic form for two and three-dimensional systems

Let $\omega\mathscr{H}$ be a $2n$-dimensional symplectic-Haantjes manifold. In Section 3.4.4, we discussed that the most general operator compatible with the symplectic form is (3.84). However, it is useful to consider the explicit form of these operators for $n = 2$ and $n = 3$.

In the case where $n = 2$, the resulting operator can be writen in Darboux coordinates as

$$\begin{aligned}\boldsymbol{M} = \sum_{i,j=1}^{2} m_{ij}\left(\frac{\partial}{\partial q^i}\otimes \mathrm{d}q^j + \frac{\partial}{\partial p_j}\otimes \mathrm{d}p_i\right) + m_{14}\left(\frac{\partial}{\partial q^1}\otimes \mathrm{d}p_2 - \frac{\partial}{\partial q^2}\otimes \mathrm{d}p_1\right)\\ + m_{32}\left(\frac{\partial}{\partial p_1}\otimes \mathrm{d}q^2 - \frac{\partial}{\partial p_2}\otimes \mathrm{d}q^1\right),\end{aligned} \tag{A.1}$$

where m_{11}, m_{12}, m_{21}, m_{22}, m_{14}, m_{32} are 6 arbitrary functions that depend on the variables (q^1, q^2, p_1, p_2). To make the expressions easier to read, it is useful to represent these operators using the matrix notation

$$\boldsymbol{M} = \left[\begin{array}{cc|cc} m_{11} & m_{12} & 0 & m_{14} \\ m_{21} & m_{22} & -m_{14} & 0 \\ \hline 0 & m_{32} & m_{11} & m_{21} \\ -m_{32} & 0 & m_{12} & m_{22} \end{array}\right]. \tag{A.2}$$

Analogously, for $n = 3$ the most general operator compatible with the symplectic form is

$$\begin{aligned}\boldsymbol{M} = \sum_{i,j=1}^{3} m_{ij}\left(\frac{\partial}{\partial q^i}\otimes \mathrm{d}q^j + \frac{\partial}{\partial p_j}\otimes \mathrm{d}p_i\right) + m_{15}\left(\frac{\partial}{\partial q^1}\otimes \mathrm{d}p_2 - \frac{\partial}{\partial q^2}\otimes \mathrm{d}p_1\right)\\ + m_{16}\left(\frac{\partial}{\partial q^1}\otimes \mathrm{d}p_3 - \frac{\partial}{\partial q^3}\otimes \mathrm{d}p_1\right) + m_{26}\left(\frac{\partial}{\partial q^2}\otimes \mathrm{d}p_3 - \frac{\partial}{\partial q^3}\otimes \mathrm{d}p_2\right)\\ + m_{42}\left(\frac{\partial}{\partial p_1}\otimes \mathrm{d}q^2 - \frac{\partial}{\partial p_2}\otimes \mathrm{d}q^1\right) + m_{43}\left(\frac{\partial}{\partial p_1}\otimes \mathrm{d}q^3 - \frac{\partial}{\partial p_3}\otimes \mathrm{d}q^1\right)\\ + m_{53}\left(\frac{\partial}{\partial p_2}\otimes \mathrm{d}q^3 - \frac{\partial}{\partial p_3}\otimes \mathrm{d}q^2\right),\end{aligned} \tag{A.3}$$

where m_{11}, m_{12}, m_{13}, m_{21}, m_{22}, m_{23}, m_{31}, m_{32}, m_{33}, m_{15} m_{16}, m_{26}, m_{42}, m_{43}, m_{53} are 15 arbitrary functions that depend on the variables $(q^1, q^2, q^3, p_1, p_2, p_3)$. In the matrix notation, it reads

$$\boldsymbol{M} = \left[\begin{array}{ccc|ccc} m_{11} & m_{12} & m_{13} & 0 & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & -m_{15} & 0 & m_{26} \\ m_{31} & m_{32} & m_{33} & -m_{16} & -m_{26} & 0 \\ \hline 0 & m_{42} & m_{43} & m_{11} & m_{21} & m_{31} \\ -m_{42} & 0 & m_{53} & m_{12} & m_{22} & m_{32} \\ -m_{43} & -m_{53} & 0 & m_{13} & m_{23} & m_{33} \end{array}\right]. \tag{A.4}$$

It is straightforward to obtain equivalent expressions for $n \geq 4$, but these have not been necessary in this work.

B

On the Haantjes structure of the anisotropic oscillator

As an illustrative example of the form of the integral of motion $\mathfrak{R}(\Psi)$, Eq. (4.27), admitted by the anisotropic oscillator with Rosochatius terms (4.23), we shall consider the case $n_1 = 1, n_2 = 3$. The integral (4.27) reads explicitly

$$
\begin{aligned}
\mathfrak{R}(\Psi) = {} & 9\,\nu^8 x^6 y^2 + 27\,\frac{{c_1}^2 \nu^4 y^2}{x^2} - 27\,c_1\,x^2\nu^6 y^2 + \frac{{c_1}^3 c_2}{x^6 y^2} \\
& - 9\,\frac{{c_1}^3 \nu^2 y^2}{x^6} - \frac{\nu^6 x^6 c_2}{y^2} - 3\,\frac{{c_1}^2 \nu^2 c_2}{x^2 y^2} + 3\,\frac{c_1\,x^2 \nu^4 c_2}{y^2} \\
& + \left(162\,c_1\,\nu^4 y^2 - 135\,\nu^6 x^4 y^2 + 3\,\frac{{c_1}^2 c_2}{x^4 y^2} - 27\,\frac{{c_1}^2 \nu^2 y^2}{x^4} - 18\,\frac{c_1\,\nu^2 c_2}{y^2} + 15\,\frac{\nu^4 x^4 c_2}{y^2}\right) {p_x}^2 \\
& + \left(\frac{{c_1}^3}{x^6} - \nu^6 x^6 - 3\,\frac{{c_1}^2 \nu^2}{x^2} + 3\,c_1\,x^2\nu^4\right) {p_y}^2 + \left(3\,\frac{{c_1}^2}{x^4} - 18\,c_1\,\nu^2 + 15\,\nu^4 x^4\right) {p_x}^2 {p_y}^2 \\
& + {p_x}^6 {p_y}^2 + \left(36\,\frac{{c_1}^2 \nu^2 y}{x^3} - 72\,c_1\,x\nu^4 y + 36\,\nu^6 x^5 y\right) p_y\,p_x \\
& + \left(135\,\nu^4 x^2 y^2 + 3\,\frac{c_1\,c_2}{x^2 y^2} - 27\,\frac{c_1\,\nu^2 y^2}{x^2} - 15\,\frac{\nu^2 x^2 c_2}{y^2}\right) {p_x}^4 \\
& + \left(\frac{c_2}{y^2} - 9\,\nu^2 y^2\right) {p_x}^6 + \left(72\,\frac{c_1\,\nu^2 y}{x} - 120\,\nu^4 x^3 y\right) p_y\,{p_x}^3 + \left(3\,\frac{c_1}{x^2} - 15\,\nu^2 x^2\right) {p_x}^4 {p_y}^2 \\
& + 36\,\nu^2 {p_x}^5 x p_y\,y
\end{aligned}
\tag{B.1}
$$

and the elements of the Haantjes operator are

$$
\begin{aligned}
m_d = &-270\,\nu^6x^4y^2 + 324\,c_1\,\nu^4y^2 + 30\,\frac{\nu^4x^4c_2}{y^2} - 54\,\frac{{c_1}^2\nu^2y^2}{x^4} - 36\,\frac{c_1\,\nu^2c_2}{y^2} \\
&+ 6\,\frac{{c_1}^2c_2}{x^4y^2} + \left(540\,\nu^4x^2y^2 - 108\,\frac{c_1\,\nu^2y^2}{x^2} - 60\,\frac{\nu^2x^2c_2}{y^2} + 12\,\frac{c_1\,c_2}{x^2y^2}\right){p_x}^2 \\
&+ \left(30\,\nu^4x^4 - 36\,c_1\,\nu^2 + 6\,\frac{{c_1}^2}{x^4}\right){p_y}^2 + \left(-60\,\nu^2x^2 + 12\,\frac{c_1}{x^2}\right){p_x}^2{p_y}^2 \\
&+ \frac{p_y}{p_x}\left(36\,\frac{\nu^2{c_1}^2y}{x^3} - 72\,c_1\,x\nu^4y + 36\,\nu^6x^5y\right) + 180\,{p_x}^3xy\nu^2p_y \\
&+ \left(-360\,\nu^4x^3y + 216\,\frac{c_1\,\nu^2y}{x}\right)p_xp_y + \left(-54\,\nu^2y^2 + 6\,\frac{c_2}{y^2}\right){p_x}^4 + 6\,{p_x}^4{p_y}^2,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
m_{21} = &36\,\frac{\nu^2{c_1}^2y}{x^3} - 72\,c_1\,x\nu^4y + 36\,\nu^6x^5y + \left(72\,\frac{c_1\,\nu^2y}{x} - 120\,\nu^4x^3y\right){p_x}^2 \\
&+ \left(360\,\nu^4x^3y - 216\,\frac{c_1\,\nu^2y}{x}\right){p_y}^2 \\
&+ \frac{p_y}{p_x}\left(36\,\frac{c_1\,\nu^2c_2}{y^2} - 2\,\nu^6x^6 + 270\,\nu^6x^4y^2 + 2\,\frac{{c_1}^3}{x^6} + 6\,c_1\,x^2\nu^4\right. \\
&\qquad\left. -324\,c_1\,\nu^4y^2 - 30\,\frac{\nu^4x^4c_2}{y^2} - 6\,\frac{{c_1}^2\nu^2}{x^2} + 54\,\frac{{c_1}^2\nu^2y^2}{x^4} - 6\,\frac{{c_1}^2c_2}{x^4y^2}\right) \\
&+ \left(-30\,\nu^2x^2 + 54\,\nu^2y^2 + 6\,\frac{c_1}{x^2} - 6\,\frac{c_2}{y^2}\right)p_y{p_x}^3 + \left(60\,\nu^2x^2 - 12\,\frac{c_1}{x^2}\right){p_y}^3p_x \\
&- 6\,{p_y}^3{p_x}^3 + 2\,{p_x}^5p_y - 180\,{p_x}^2yx\nu^2{p_y}^2 \\
&+ \left(30\,\nu^4x^4 - 540\,\nu^4x^2y^2 + 6\,\frac{{c_1}^2}{x^4} - 36\,c_1\,\nu^2\right. \\
&\qquad\left. +108\,\frac{c_1\,\nu^2y^2}{x^2} + 60\,\frac{\nu^2x^2c_2}{y^2} - 12\,\frac{c_1\,c_2}{x^2y^2}\right)p_xp_y \\
&+ \frac{{p_y}^3}{p_x}\left(-30\,\nu^4x^4 - 6\,\frac{{c_1}^2}{x^4} + 36\,c_1\,\nu^2\right) + 36\,{p_x}^4yx\nu^2 \\
&+ \frac{{p_y}^2}{{p_x}^2}\left(-36\,\nu^6x^5y + 72\,c_1\,x\nu^4y - 36\,\frac{\nu^2{c_1}^2y}{x^3}\right),
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
m_{32} = &\left(324\,y\nu^4c_1 + 972\,\frac{y^3\nu^4c_1}{x^2} - 30\,\frac{\nu^4x^4c_2}{y^3}\right. \\
&\qquad\left. +1080\,\frac{\nu^4x^2c_2}{y} - 54\,\frac{y\nu^2{c_1}^2}{x^4} - 60\,\frac{\nu^2x^2{c_2}^2}{y^5} - 6\,\frac{{c_1}^2c_2}{x^4y^3}\right)p_x \\
&+ \left(12\,\frac{c_1\,{c_2}^2}{x^2y^5} + 36\,\frac{c_1\,\nu^2c_2}{y^3} - 216\,\frac{c_1\,\nu^2c_2}{x^2y} - 270\,x^4y\nu^6 - 4860\,x^2y^3\nu^6\right)p_x
\end{aligned}
$$

$$
\begin{aligned}
&+\left(216\,\frac{c_1\,\nu^2 c_2}{xy^2}+36\,\nu^6x^5+3240\,x^3y^2\nu^6\right.\\
&\qquad\left.+36\,\frac{\nu^2{c_1}^2}{x^3}-72\,c_1\,x\nu^4-1944\,\frac{c_1\,\nu^4y^2}{x}-360\,\frac{\nu^4x^3c_2}{y^2}\right)p_y\\
&+\left(-54\,\frac{c_1\,\nu^2y}{x^2}+30\,\frac{\nu^2x^2c_2}{y^3}-108\,\frac{c_2\,\nu^2}{y}\right.\\
&\qquad\left.-6\,\frac{c_1\,c_2}{x^2y^3}+270\,\nu^4x^2y+486\,y^3\nu^4+6\,\frac{{c_2}^2}{y^5}\right){p_x}^3\\
&+\frac{1}{p_x}\left(-54\,c_1\,x^2\nu^6y-2916\,y^3\nu^6c_1+2\,\frac{\nu^6x^6c_2}{y^3}-540\,\frac{\nu^6x^4c_2}{y}\right.\\
&\qquad\left.+54\,\frac{{c_1}^2\nu^4y}{x^2}+486\,\frac{y^3\nu^4{c_1}^2}{x^4}+30\,\frac{\nu^4x^4{c_2}^2}{y^5}-18\,\frac{{c_1}^3\nu^2y}{x^6}\right)\\
&+\frac{1}{p_x}\left(-2\,\frac{{c_1}^3c_2}{x^6y^3}+6\,\frac{{c_1}^2{c_2}^2}{x^4y^5}-6\,\frac{c_1\,x^2\nu^4c_2}{y^3}+648\,\frac{c_1\,\nu^4c_2}{y}+6\,\frac{\nu^2{c_1}^2c_2}{x^2y^3}\right.\\
&\qquad\left.-108\,\frac{\nu^2{c_1}^2c_2}{x^4y}-36\,\frac{c_1\,\nu^2{c_2}^2}{y^5}+18\,\nu^8x^6y+2430\,x^4y^3\nu^8\right)\\
&+\left(-18\,y\nu^2-2\,\frac{c_2}{y^3}\right){p_x}^5+\left(-54\,y\nu^2+6\,\frac{c_2}{y^3}\right){p_y}^2{p_x}^3\\
&+\left(540\,\nu^4x^2y-60\,\frac{\nu^2x^2c_2}{y^3}+12\,\frac{c_1\,c_2}{x^2y^3}-108\,\frac{c_1\,\nu^2y}{x^2}\right)p_x{p_y}^2\\
&+\left(-120\,\nu^4x^3-1620\,\nu^4xy^2+72\,\frac{c_1\,\nu^2}{x}+180\,\frac{\nu^2xc_2}{y^2}\right){p_x}^2p_y\\
&+\frac{p_y}{{p_x}^2}\left(-72\,\frac{c_1\,x\nu^4c_2}{y^2}+36\,\frac{\nu^2{c_1}^2c_2}{x^3y^2}-324\,\nu^8x^5y^2\right.\\
&\qquad\left.-324\,\frac{{c_1}^2\nu^4y^2}{x^3}+648\,c_1\,x\nu^6y^2+36\,\frac{\nu^6x^5c_2}{y^2}\right)\\
&+\frac{{p_y}^2}{px}\left(-36\,\frac{c_1\,\nu^2c_2}{y^3}-270\,x^4y\nu^6-54\,\frac{y\nu^2{c_1}^2}{x^4}\right.\\
&\qquad\left.+6\,\frac{{c_1}^2c_2}{x^4y^3}+324\,y\nu^4c_1+30\,\frac{\nu^4x^4c_2}{y^3}\right)+36\,{p_x}^4x\nu^2p_y. \qquad \text{(B.4)}
\end{aligned}
$$

C

Partial brackets in the proof of Theorem 5.2.1

We report here the explicit computation leading to Eq. (5.22):

$$
\begin{aligned}
\{H_\alpha, H_\beta\}\big|_a &= \sum_{j_a=1}^{\sigma_a} \left(\frac{\partial H_\alpha}{\partial q^{a,j_a}} \frac{\partial H_\beta}{\partial p_{a,j_a}} - \frac{\partial H_\beta}{\partial q^{a,j_a}} \frac{\partial H_\alpha}{\partial p_{a,j_a}} \right) \\
&= \sum_{j_a=1}^{\sigma_a} \left(\left(\sum_{i_a=1}^{\sigma_a} \frac{\partial H}{\partial q^{a,i_a}} (\boldsymbol{A}_a^{(\alpha)})_{j_a}^{i_a} + \frac{\partial H}{\partial p_{a,i_a}} (\boldsymbol{C}_a^{(\alpha)})_{j_a}^{i_a} \right) \right. \\
&\qquad \left. \left(\sum_{k_a=1}^{\sigma_a} \frac{\partial H}{\partial q^{a,k_a}} (\boldsymbol{B}_a^{(\beta)})_{j_a}^{k_a} + \frac{\partial H}{\partial p_{a,k_a}} ((\boldsymbol{A}_a^{(\beta)})^T)_{j_a}^{k_a} \right) \right. \\
&\quad - \left. \sum_{k_a=1}^{\sigma_a} \left(\frac{\partial H}{\partial q^{a,k_a}} (\boldsymbol{B}_a^{(\alpha)})_{j_a}^{k_a} + \frac{\partial H}{\partial p_{a,k_a}} ((\boldsymbol{A}_a^{(\alpha)})^T)_{j_a}^{k_a} \right) \right. \\
&\qquad \left. \left(\sum_{i_a=1}^{\sigma_a} \frac{\partial H}{\partial q^{a,k_a}} (\boldsymbol{A}_a^{(\beta)})_{j_a}^{i_a} + \frac{\partial H}{\partial p_{a,k_a}} ((\boldsymbol{C}_a^{(\beta)})^T)_{j_a}^{i_a} \right) \right) \\
&= \sum_{i_a,j_a,k_a=1}^{\sigma_a} \left(\frac{\partial H}{\partial q^{a,i_a}} \frac{\partial H}{\partial p_{a,k_a}} \left((\boldsymbol{A}_a^{(\alpha)})_{j_a}^{i_a} (\boldsymbol{A}_a^{(\beta)})_{k_a}^{j_a} - (\boldsymbol{A}_a^{(\alpha)})_{k_a}^{j_a} (\boldsymbol{A}_a^{(\beta)})_{j_a}^{i_a} \right) \right. \\
&\qquad \left. + \frac{\partial H}{\partial q^{a,k_a}} \frac{\partial H}{\partial p_{a,i_a}} \left((\boldsymbol{C}_a^{(\alpha)})_{j_a}^{i_a} (\boldsymbol{B}_a^{(\beta)})_{j_a}^{k_a} - (\boldsymbol{B}_a^{(\alpha)})_{j_a}^{k_a} (\boldsymbol{C}_a^{(\beta)})_{j_a}^{i_a} \right) \right.
\end{aligned}
$$

$$+\frac{\partial H}{\partial q^{a,i_a}}\frac{\partial H}{\partial q^{a,k_a}}\big((\boldsymbol{A}_a^{(\alpha)})_{j_a}^{i_a}(\boldsymbol{B}_a^{(\beta)})_{j_a}^{k_a}-(\boldsymbol{B}_a^{(\alpha)})_{j_a}^{k_a}(\boldsymbol{A}_a^{(\beta)})_{j_a}^{i_a}\big)$$
$$+\frac{\partial H}{\partial p_{a,i_a}}\frac{\partial H}{\partial p_{a,k_a}}\big((\boldsymbol{C}_a^{(\alpha)})_{j_a}^{i_a}(\boldsymbol{A}_a^{(\beta)})_{k_a}^{j_a}-(\boldsymbol{A}_a^{(\alpha)})_{k_a}^{j_a}(\boldsymbol{C}_a^{(\beta)})_{j_a}^{i_a}\big)\Bigg).$$

D

Partial and total separability of a completely integrable system

In Chapter 5 we showed that an integrable system may admit more than one $\omega\mathscr{H}$ structure. In particular, depending on whether one associates a semisimple or a non-semisimple Haantjes algebra, the corresponding HJ equations can be completely or partially separable. This example illustrates how, by choosing different Haantjes algebras, both partial and total separability can emerge within the same integrable framework, thereby highlighting the compatibility between these two scenarios.

To illustrate this point, we associate both a non-semisimple maximal Haantjes algebra (corresponding to Case (**I**) of Definition 5.1.1) and a semisimple one to a given integrable system, thereby obtaining charts in which the system is partially or totally separable, respectively. Consider the system defined by the following Hamiltonian functions:

$$H_1 = 4p_1^2 + 2p_2^2 + 4p_3^2 + q^1\left(q^1+2\right) + 4(q^2)^2 + q^3\left(q^3-2\right) + 4q^1q^2 + 2q^1q^3 + 4q^2q^3, \quad \text{(D.1)}$$

$$H_2 = p_1^2 + p_3^2 - 2p_1p_3 + q^1 - q^3, \quad \text{(D.2)}$$

$$H_3 = p_1 - p_2 + p_3. \quad \text{(D.3)}$$

This system is not totally separable in the original coordinates, since they do not satisfy Benenti's relations or the Levi-Civita test (see Section 1.3.4).

Let us start by considering the chain equation $\mathrm{d}H_2 = \boldsymbol{K}_2^T \mathrm{d}H_1$. The following Nijenhuis operator solves it:

$$\boldsymbol{K}_2 = \frac{1}{4}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]. \tag{D.4}$$

Its minimal polynomial is of second degree, $m_{\boldsymbol{K}_2}(\boldsymbol{q}, \boldsymbol{p}, \lambda) = \lambda\left(\lambda - \frac{1}{2}\right)$, and its eigenvalues and eigenvectors are

$$\begin{aligned} \lambda_1^{(2)} &= 0, \qquad \mathcal{D}_1^{(2)} = \operatorname{span}\left\{\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_2}\right\}, \\ \lambda_2^{(2)} &= \frac{1}{2}, \qquad \mathcal{D}_2^{(2)} = \operatorname{span}\left\{\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3}\right\}. \end{aligned} \tag{D.5}$$

The set $\mathscr{H}_0 = \operatorname{span}\{\boldsymbol{K}_1 = \boldsymbol{I}, \boldsymbol{K}_2\}$ represents an Abelian, semisimple Haantjes algebra of rank 2 (with $\boldsymbol{K}_2^2 = \frac{1}{2}\boldsymbol{K}_2$). In order to get an Abelian algebra of rank 3 associated to the whole system (D.1)-(D.3), we can extend it in two different ways. Each of them will provide us with a different separation coordinate system.

D.1 PARTIAL SEPARABILITY (CASE (I))

First, we look for a *non-semisimple* algebra of Haantjes operators. A Haantjes operator that solves the chain equation $\mathrm{d}H_3 = \boldsymbol{K}_3^T \mathrm{d}H_1$ reads

$$\boldsymbol{K}_3 = \frac{1}{8(p_1+p_3)^2}\left[\begin{array}{ccc|ccc} p_1+p_3 & 0 & p_1+p_3 & 0 & 0 & 0 \\ -p_1-p_2-p_3 & 2(p_1+p_3) & -p_1-p_2-p_3 & 0 & 0 & 0 \\ p_1+p_3 & 0 & p_1+p_3 & 0 & 0 & 0 \\ \hline 0 & -q^1-2q^2-q^3 & 0 & p_1+p_3 & -p_1-p_2-p_3 & p_1+p_3 \\ q^1+2q^2+q^3 & 0 & q^1+2q^2+q^3 & 0 & 2(p_1+p_3) & 0 \\ 0 & -q^1-2q^2-q^3 & 0 & p_1+p_3 & -p_1-p_2-p_3 & p_1+p_3 \end{array}\right] \tag{D.6}$$

in the coordinates $(q^1, q^2, q^3, p_1, p_2, p_3)$. Its minimal polynomial is of third degree: $m_{\boldsymbol{K}_3}(\boldsymbol{q}, \boldsymbol{p}, \lambda) = \lambda\left(\lambda - \frac{1}{4(p_1+p_3)}\right)^2$. Thus, the operator is non-semisimple. Its eigenvalues and generalized eigendistributions are

$$\lambda_1^{(3)} = \frac{1}{4(p_1+p_3)}, \quad \rho_1^{(3)} = 2,$$
$$\mathcal{D}_1^{(3)} = \operatorname{span}\left\{\frac{\partial}{\partial q^1}+\frac{\partial}{\partial q^3}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1}+\frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_2}\right\} = \mathcal{D}_1^{(2)} = \mathcal{V}_1, \quad \text{(D.7)}$$

$$\lambda_2^{(3)} = 0, \quad \rho_2^{(3)} = 1,$$
$$\mathcal{D}_2^{(3)} = \operatorname{span}\left\{\frac{\partial}{\partial q^1}-\frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1}-\frac{\partial}{\partial p_3}\right\} = \mathcal{D}_2^{(2)} = \mathcal{V}_2. \quad \text{(D.8)}$$

The set $\mathscr{H}_1 = \operatorname{span}\{\boldsymbol{K_1} = \boldsymbol{I}, \boldsymbol{K}_2, \boldsymbol{K}_3\}$ represents a non-semisimple Abelian Haantjes algebra of rank 3, with $\boldsymbol{K}_2\boldsymbol{K}_3 = \boldsymbol{0}$, $\boldsymbol{K}_3^2 = g_1\boldsymbol{I} + g_2\boldsymbol{K}_2 + g_3\boldsymbol{K}_3$, where

$$g_1 = -\frac{1}{16(p_1+p_3)^2}, \quad g_2 = -2g_1, \quad g_3 = \frac{1}{2(p_1+p_3)}.$$

These operators share the same generalized eigendistributions. Integrating the annihilators

$$\begin{aligned}\mathcal{V}_1^\circ &= \operatorname{span}\{\mathrm{d}q^1 - \mathrm{d}q^3, \mathrm{d}p_1 - \mathrm{d}p_3\},\\ \mathcal{V}_2^\circ &= \operatorname{span}\{\mathrm{d}q^1 + \mathrm{d}q^3, \mathrm{d}q^2, \mathrm{d}p_1 + \mathrm{d}p_3, \mathrm{d}p_2\},\end{aligned} \quad \text{(D.9)}$$

we obtain a new DH chart $(Q_1, Q_2, P_1, P_2, Q_3, P_3)$:

$$\begin{aligned}&Q^1 = \frac{1}{2}(q^1+q^3), \qquad Q^2 = q^2, \qquad Q^3 = \frac{1}{2}(q^1-q^3),\\ &P_1 = p_1+p_3, \qquad P_2 = p_2, \qquad P_3 = p_1-p_3.\end{aligned} \quad \text{(D.10)}$$

In this DH chart the Haantjes operators take the block-diagonal form

$$\boldsymbol{K}_2 = \frac{1}{2}\operatorname{diag}[0,0,0,0,1,1], \quad \text{(D.11)}$$

$$\boldsymbol{K}_3 = \frac{1}{4P_1^2}\left[\begin{array}{cccc|cc} P_1 & 0 & 0 & 0 & 0 & 0\\ -(P_1+P_2) & P_1 & 0 & 0 & 0 & 0\\ 0 & -2(Q_1+Q_2) & P_1 & -(P_1+P_2) & 0 & 0\\ 2(Q_1+Q_2) & 0 & 0 & P_1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]; \quad \text{(D.12)}$$

and the integrals of the motion read

$$H_1 = 2\left(P_1^2+P_2^2+P_3^2\right) + 4\,(Q^1+Q^2)^2 + 4Q^3, \quad \text{(D.13)}$$
$$H_2 = P_3^2 + 2Q^3, \quad \text{(D.14)}$$
$$H_3 = P_1 - P_2. \quad \text{(D.15)}$$

Since the system (D.13)-(D.15) fulfils the hypotheses of Theorem 5.2.1, it is partially separable in the new coordinates. In fact, H_1 and H_3 are in $\mathcal{P}$-involution:

$$\{H_1, H_3\}|_{(1,2)} := \frac{\partial H_1}{\partial Q^1}\frac{\partial H_3}{\partial P_1} - \frac{\partial H_1}{\partial P_1}\frac{\partial H_3}{\partial Q^1} + \frac{\partial H_1}{\partial Q^2}\frac{\partial H_3}{\partial P_2} - \frac{\partial H_1}{\partial P_2}\frac{\partial H_3}{\partial Q^2} = 0\,, \tag{D.16}$$

whereas (H_1, H_2) and (H_1, H_3) are in $\mathcal{T}$-involution.

The HJ equations associated with the system (D.13)-(D.15) read

$$2\left[\left(\frac{\partial W}{\partial Q^1}\right)^2 + \left(\frac{\partial W}{\partial Q^2}\right)^2 + \left(\frac{\partial W}{\partial Q^3}\right)^2\right] + 4\,(Q^1 + Q^2)^2 + 4Q^3 = h\,, \tag{D.17}$$

$$\left(\frac{\mathrm{d}W}{\mathrm{d}Q^3}\right)^2 + 2Q^3 = h_2\,, \tag{D.18}$$

$$\frac{\partial W}{\partial Q^1} - \frac{\partial W}{\partial Q^2} = h_3\,. \tag{D.19}$$

Seeking a complete integral of the form

$$W(Q^1, Q^2, Q^3; h, h_2, h_3) = W_1(Q^1, Q^2; h_1, h_2, h_3) + W_2(Q^3; h_1, h_2, h_3)\,,$$

we obtain a partially separated system of two PDEs and one ODE:

$$2\left[\left(\frac{\partial W_1}{\partial Q^1}\right)^2 + \left(\frac{\partial W_1}{\partial Q^2}\right)^2\right] + 4\,(Q^1 + Q^2)^2 = h_1, \tag{D.20}$$

$$\left(\frac{\mathrm{d}W_2}{\mathrm{d}Q^3}\right)^2 + 2Q^3 = h_2, \tag{D.21}$$

$$\frac{\partial W_1}{\partial Q^1} - \frac{\partial W_1}{\partial Q^2} = h_3 \tag{D.22}$$

where $h = h_1 + 2h_2$. The previous system can be integrated. The corresponding complete integral is given by

$$\begin{aligned} W &= W(Q^1, Q^2, Q^3; h_1, h_2, h_3) = W_1(Q^1, Q^2; h_1, h_2, h_3) + W_2(Q^3; h_1, h_2, h_3) \\ &= \frac{1}{2}\Bigg\{h_3(Q^1 - Q^2) + \frac{1}{2}(Q^1 + Q^2)\sqrt{h_1 - h_3^2 - 4(Q^1 + Q^2)^2} \\ &\quad - \frac{1}{4}\left(h_1 - h_3^2\right)\cdot \arctan\left[\frac{2(Q^1 + Q^2)\sqrt{h_1 - h_3^2 - 4(Q^1 + Q^2)^2}}{-h_1 + h_3^2 + 4(Q^1 + Q^2)^2}\right]\Bigg\} \\ &\quad + \frac{1}{3}(h_2 - 2Q^3)^{3/2}. \end{aligned} \tag{D.23}$$

D.2 TOTAL SEPARABILITY

We can also extend the algebra $\mathscr{H}_0$ in a different way, namely, by looking for an additional, *semisimple* Haantjes operator that solves the chain equation $\tilde{\boldsymbol{K}}_3^T \mathrm{d}H_1 = \mathrm{d}H_3$ and commutes with the generators of $\mathscr{H}_0$. A possible choice is

$$\tilde{\boldsymbol{K}}_3 = \frac{1}{8(p_1 - p_2 + p_3)} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right], \tag{D.24}$$

with minimal polynomial $m_{\tilde{\boldsymbol{K}}_3}(\boldsymbol{q}, \boldsymbol{p}, \lambda) = \lambda\left(\lambda - \frac{1}{2(p_1 - p_2 + p_3)}\right)$. This operator admits two distinct eigenvalues, whose proper eigendistributions are

$$\lambda_1^{(3)} = 0\,, \quad \rho_1^{(3)} = 1\,,$$
$$\mathcal{D}_1^{(3)} = \left\langle 2\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3} \right\rangle, \tag{D.25}$$

$$\lambda_2^{(3)} = \frac{1}{2(p_1 - p_2 + p_3)}\,, \quad \rho_2^{(3)} = 1\,,$$
$$\mathcal{D}_2^{(3)} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - 2\frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right\rangle. \tag{D.26}$$

In this way, we have obtained the semisimple algebra of rank 3, $\mathscr{H}_2 = \{\boldsymbol{I}, \boldsymbol{K}_2, \tilde{\boldsymbol{K}}_3\}$, with $\boldsymbol{K}_2\tilde{\boldsymbol{K}}_3 = \boldsymbol{0}$, $\tilde{\boldsymbol{K}}_3^2 = \lambda_2^{(3)}\tilde{\boldsymbol{K}}_3$. Thus, we can determine a chart where all of the operators diagonalize simultaneously. In turn, this will give us total separation variables for the HJ equations associated with the system (D.1)-(D.3). We obtain the non-trivial distributions

$$\tilde{\mathcal{V}}_1 = \mathcal{D}_1^{(2)} \cap \mathcal{D}_1^{(3)} = \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} + 2\frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right\rangle, \tag{D.27}$$

$$\tilde{\mathcal{V}}_2 = \mathcal{D}_1^{(2)} \cap \mathcal{D}_2^{(3)} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_3} \right\rangle, \tag{D.28}$$

$$\tilde{\mathcal{V}}_3 = \mathcal{D}_2^{(2)} \cap \mathcal{D}_1^{(3)} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1} - 2\frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right\rangle. \tag{D.29}$$

We can work out the corresponding annihilators

$$\begin{aligned} (\tilde{\mathcal{V}}_1 \oplus \tilde{\mathcal{V}}_2)^\circ &= \langle \mathrm{d}q^1 - 2\mathrm{d}q^2 + \mathrm{d}q^3, \mathrm{d}p_1 - \mathrm{d}p_2 + \mathrm{d}p_3 \rangle, \\ (\tilde{\mathcal{V}}_1 \oplus \tilde{\mathcal{V}}_3)^\circ &= \langle \mathrm{d}q^1 - \mathrm{d}q^3, \mathrm{d}p_1 - \mathrm{d}p_3 \rangle, \\ (\tilde{\mathcal{V}}_2 \oplus \tilde{\mathcal{V}}_3)^\circ &= \langle \mathrm{d}q^1 + 2\mathrm{d}q^2 + \mathrm{d}q^3, \mathrm{d}p_1 + \mathrm{d}p_2 + \mathrm{d}p_3 \rangle. \end{aligned} \tag{D.30}$$

Integrating them, we find the set of DH coordinates

$$\begin{aligned}
\tilde{Q}_1 &= \frac{1}{4}\left(q^1 + 2q^2 + q^3\right), & \tilde{P}_1 &= p_1 + p_2 + p_3,\\
\tilde{Q}_2 &= \frac{1}{4}\left(q^1 - 2q^2 + q^3\right), & \tilde{P}_2 &= p_1 - p_2 + p_3,\\
\tilde{Q}_3 &= \frac{1}{2}\left(q^1 - q^3\right), & \tilde{P}_3 &= p_1 - p_3\,,
\end{aligned} \tag{D.31}$$

which fulfil the canonical commutation relations. In this new coordinate chart, the operators of the algebra $\mathscr{H}_2$ read

$$\boldsymbol{K}_2 = \frac{1}{2}\,\mathrm{diag}\,[0,0,1,0,0,1]\,, \qquad \tilde{\boldsymbol{K}}_3 = \frac{1}{2\tilde{P}_2}\mathrm{diag}\,[0,1,0,0,1,0]\,; \tag{D.32}$$

and the Hamiltonian function and the integrals of the motion read

$$H_1 = \tilde{P}_1^2 + \tilde{P}_2^2 + 2\tilde{P}_3^2 + 16(\tilde{Q}^1)^2 + 4\tilde{Q}^3, \tag{D.33}$$

$$H_2 = \tilde{P}_3^2 + 2\tilde{Q}^3, \tag{D.34}$$

$$H_3 = \tilde{P}_2. \tag{D.35}$$

Searching for a Hamilton characteristic function of the totally separated form

$$W(\tilde{Q}^1, \tilde{Q}^2, \tilde{Q}^3; \tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = W_1(\tilde{Q}^1; \tilde{h}_1, \tilde{h}_2, \tilde{h}_3) + W_2(\tilde{Q}^2; \tilde{h}_1, \tilde{h}_2, \tilde{h}_3) + W_3(\tilde{Q}^3; \tilde{h}_1, \tilde{h}_2, \tilde{h}_3)\,,$$

the HJ equations read:

$$\left(\frac{\mathrm{d}W_1}{\mathrm{d}\tilde{Q}^1}\right)^2 + 16(\tilde{Q}^1)^2 = \tilde{h}_1, \quad \left(\frac{\mathrm{d}W_3}{\mathrm{d}\tilde{Q}^3}\right)^2 + 2\tilde{Q}^3 = \tilde{h}_3, \quad \frac{\mathrm{d}W_2}{\mathrm{d}\tilde{Q}^2} = \tilde{h}_2. \tag{D.36}$$

Therefore, a complete integral is:

$$W = \frac{1}{2}\tilde{Q}^1\sqrt{\tilde{h}_1 - 16\tilde{Q}_1^2} + \frac{1}{8}\tilde{h}_1 \arctan\frac{4\tilde{Q}^1}{\sqrt{\tilde{h}_1 - 16\tilde{Q}_1^2}} + \tilde{h}_2\,\tilde{Q}^2 + \frac{1}{3}(\tilde{h}_3 - 2\tilde{Q}^3)^{3/2}\,. \tag{D.37}$$

Observe that under the variable transformation

$$\begin{aligned}
\tilde{Q}^1 &= \frac{1}{2}(Q^1 + Q^2), & \tilde{Q}^2 &= \frac{1}{2}(Q^1 - Q^2), & \tilde{Q}^3 &= Q^3,\\
\tilde{P}_1 &= P_1 + P_2, & \tilde{P}_2 &= P_1 - P_2, & \tilde{P}_3 &= P_3,\\
\tilde{h}_1 &= h_1 - h_3^2, & \tilde{h}_2 &= h_3, & \tilde{h}_3 &= h_2;
\end{aligned} \tag{D.38}$$

the complete integral of Eq. (D.23) coincides with that of Eq. (D.37).

The previous approach allows us to construct total separation variables (D.31) for the system (D.1)-(D.3) in a systematic way. However, in this case they can also be obtained in a more direct way.

In the spirit of Eisenhart's approach [49], by combining algebraically the Hamiltonians (D.33)-(D.35), one can introduce the new Hamiltonian

$$\begin{aligned} \hat{H}_1 &= H_1 - 2H_3 - H_3^2 = (p_1 + p_2 + p_3)^2 + (q_1 + 2q_2 + q_3)^2 \\ H_2 &= p_1 - p_2 + p_3, \qquad H_3 = (p_1 - p_3)^2 + (q_1 - q_3). \end{aligned} \tag{D.39}$$

These relations suggest one to introduce by inspection the same coordinate chart (D.31), previously obtained via the Haantjes geometry. In these new coordinates, the Hamiltonian system

$$\hat{H}_1 = \hat{P}_1^2 + 16(\hat{Q}^1)^2, \quad \hat{H}_2 = H_2 = \tilde{P}_3^2 + 2\tilde{Q}^3, \quad \hat{H}_3 = H_3 = \tilde{P}_2 \tag{D.40}$$

is manifestly separated.

As discussed in Section 3.4.3, it has been shown that Killing tensors for the AKN systems can be naturally constructed within the theory of $\omega\mathscr{H}$ manifolds by projecting the Haantjes operators associated to the AKN systems from phase space into the configuration space. This, in turn, provides a direct connection between Haantjes geometry and Eisenhart's theory of Killing tensors for Stäckel systems.

The main advantage of Haantjes geometry is that it allows us to study the problem of separability in a systematic way and within a unified theoretical framework, also valid for systems outside the classical Stäckel geometry.

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