Tesis Doctoral

On Some Isotopy Questions in 3-Dimensional Contact Topology

Dahyana Eugenia Farias Uncovich

Programa de Doctorado en Matemáticas

Facultad de Ciencias - UAM.

Dirección:

Director Dr. Eduardo Fernández Fuertes

Director Dr. Francisco Presas Mata

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A todos los que confiaron en mi

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Abstract in English

First Part: We provide a topological characterization for a family of bypasses with a fixed attaching arc to be contractible. This characterization is formulated in terms of the existence of a bypass that is disjoint from the given family away from the attaching region. As an application, we provide new proofs of several h-principles in overtwisted contact 3-manifolds.

Second Part: We classify convex disks with a fixed characteristic foliation and Legendrian boundary, up to contact isotopy relative to the boundary, in every closed overtwisted contact 3-manifold. This classification covers cases where the neighborhood of such a disk is tight or where the boundary violates the Bennequin-Eliashberg inequality. We show that this classification coincides with the formal one, establishing a h-principle for these disks. As a corollary, we deduce that the space of Legendrian unknots with $\mathrm{tb} + |\mathrm{rot}| \leq -1$ in every closed overtwisted contact 3-manifold satisfies the h-principle at the level of fundamental groups. Finally, we determine the contact mapping class group of the complement of every Legendrian unknot in an overtwisted 3-sphere with a non-positive tb invariant.

Abstract en castellano

Primera Parte: Proporcionamos una caracterización topológica para que una familia de bypasses con un arco de unión fijo sea contráctil. Esta caracterización se formula en términos de la existencia de un bypass que sea disjunto de la familia dada fuera de la región de unión. Como aplicación, ofrecemos nuevas demostraciones de varios h-principios en variedades de contacto 3-dimensionales overtwisted.

Segunda Parte: Clasificamos discos convexos con una foliación característica fija y borde Legendriano, salvo isotopía de contacto relativa al borde, en toda variedad de contacto 3-dimensional cerrada y overtwisted. Esta clasificación abarca casos en los que el entorno de dicho disco es tight o en los que le borde viola la desigualdad de Bennequin-Eliashberg. Demostramos que esta clasificación coincide con la clasificación formal, estableciendo un h-principio para estos discos. Como corolario, deducimos que el espacio de nudos Legendrianos triviales con $\mathrm{tb} + |\mathrm{rot}| \leq -1$ en toda variedad de contacto 3-dimensional cerrada y overtwisted satisface el h-principio a nivel de grupos fundamentales. Finalmente, determinamos el grupo de clases de isotopía de contacto del complemento de todo nudo Legendriano trivial en una 3-esfera overtwisted con un invariante tb no positivo.

Introduction

The original results presented in this thesis are based on the following preprints:

- [27] Dahyana Farias and Eduardo Fernández. *A bypass in the middle*, https://arxiv.org/abs/2411.06467.
- [28] Dahyana Farias, Eduardo Fernández, Francisco Presas and Guillermo Sánchez-Arellano. *Convex disks with Legendrian boundary in overtwisted contact 3-manifolds*, https://arxiv.org/abs/2501.09232.

First part: Bypasses

The contents of this part are based on [27] and appear in Chapter 2. Since their inception by Ko Honda [47], bypass disks have been a central object in the classification problem for both contact structures in a given smooth 3-manifold and Legendrians in a given contact 3-manifold. Determining the existence or non-existence of certain bypasses is a crucial aspect of the general strategies employed in these classifications (e.g., [24, 47, 48]).

One of the recent questions that has garnered attention is the classification of families of contact structures and Legendrian embeddings. This interest arises, among other things, from the fact that classifying 1-parameter families of contact structures is closely related to classifying contactomorphisms up to contact isotopy, which is key to understanding the Legendrian isotopy problem outside the standard contact 3-sphere, and that Legendrian loops can be used to construct interesting Lagrangian fillings or higher-dimensional Legendrians (see, for example, [6, 12, 16, 21, 31, 32, 33, 34, 41, 43, 54, 56, 60]). As a consequence of the Fuchs-Tabachnikov theorem [5, 35, 57], this classification can be reduced to studying the homotopy groups of certain bypass embedding spaces, i.e., (families of) bypass embeddings up to homotopy among (families of) bypass embeddings. For Legendrians, this reduction follows from the fact that bypasses can be used to destabilize Legendrians. For contact structures, this follows by attempting to mimic Giroux's construction of adapted open books [42] in a parametric fashion combined with [31].

Motivated by these, we study bypass embedding spaces. Our main result shows that the only potentially interesting behavior in the contact isotopy problem for bypass embeddings arises when

intersections between bypasses are present. For experts, this parallels the contact isotopy problem for overtwisted disks: it is topological in nature when there are no intersections [14], but can become highly non-trivial when intersections are involved [60].

Summary of the results in this part:

A *bypass embedding* is defined as a smooth embedding $b : \mathbb{D}^2_+ \to (M, \xi)$, of the half-disk $\mathbb{D}^2_+ = \mathbb{D}^2 \cap \{y \geq 0\} \subseteq \mathbb{R}^2$ with Legendrian boundary where the characteristic foliation matches a predefined bypass foliation $\mathcal{F}_B$ depicted in Figure 1. The *attaching arc* of the bypass is the Legendrian arc $\gamma = \{b(x, 0) : -1 \leq x \leq 1\}$.

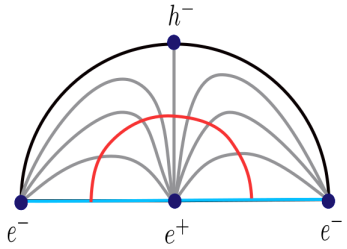


Figure 1: A bypass

- A bypass embedding b is associated with a convex surface $\Sigma \subseteq (M, \xi)$ if $b(\mathbb{D}^2_+) \cap \Sigma = \gamma$.
- Two bypass embeddings b and $\hat{b}$ coincide near the attaching arc if they agree in a neighborhood of γ. If their images outside this neighborhood are disjoint, $\hat{b}$ is disjoint from b away from γ.

The space of bypass embeddings for a surface Σ with a fixed attaching arc γ is denoted by $\mathcal{B}_\gamma(\Sigma, (M, \xi))$, while $\mathcal{B}^b_\gamma(\Sigma, (M, \xi))$ denotes the subspace disjoint from b away from γ.

The main result of this chapter provides a necessary and sufficient condition for a family of bypass embeddings with fixed attaching arc to be contractible

Theorem 0.0.1 (Bypass in the Middle)**.** *The inclusion $\mathcal{B}^b_\gamma(\Sigma, (M, \xi)) \hookrightarrow \mathcal{B}_\gamma(\Sigma, (M, \xi))$ is null-homotopic.*

Then, we will find several applications of our main result to the study of overtwisted contact 3-manifolds, introduced by Eliashberg in [17]. This are contact 3-manifolds chracterized by the existence of the so-called overtwisted disk, and satisfy an h-principle as proved in [17]. The novelty is that we will not make use of the overtwisted h-principle to prove this results.

Corollary 0.0.2. *Let (M, ξ) be an overtwisted contact 3-manifold, $\Delta_{OT} \subseteq (M, \xi)$ an overtwisted disk, and $\Sigma \subseteq (M \setminus \Delta_{OT}, \xi)$ a convex surface. Then, for any given attaching arc $\gamma \subseteq \Sigma$, the space of bypass embeddings $\mathcal{B}_\gamma(\Sigma, (M \setminus \Delta_{OT}, \xi))$ is contractible.*

In the next result we will denote by $\mathrm{Emb}^{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$ the space of *overtwisted disk embeddings* into a contact manifold (M, ξ).

Corollary 0.0.3. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk. Then, the 1-jet evaluation map at the origin*

$$\mathrm{ev}_0 : \mathrm{Emb}^{\mathrm{OT}}(\mathbb{D}^2, (M \setminus \Delta_{\mathrm{OT}}, \xi)) \to \mathrm{CFr}(M \setminus \Delta_{\mathrm{OT}}, \xi)$$

is a weak homotopy equivalence.

This result implies that two overtwisted disks are contact isotopic if a third disjoint overtwisted disk exists. However, the space of overtwisted disk embeddings may be disconnected as shown by Vogel [60]. In the class of strongly overtwisted contact 3-manifold introduced by Fernández the result can be improved to the whole manifold [30].

We will end this Chapter by studying Legendrian embedding spaces in overtwisted contact 3-manifolds. We will focus on *loose* Legendrians, i.e. with overtwisted complement. The space of Legendrian embeddings in a contact 3-manifold (M, ξ) will be denoted by $\mathcal{L}(M, \xi)$. Similarly, the space of formal Legendrians will be denoted by $F\mathcal{L}(M, \xi)$.

In chapter 2 we will explain that bypass embedding allow to destabilize Legendrians, hence the Bypass in the Middle Theorem will allow to study stabilization operators acting on Legendrian embedding spaces:

Theorem 0.0.4. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk. Then, for every $t \in \mathbb{S}^1$, the stabilization at time t operators*

$$S^t_+ : \mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi) \to \mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi)$$

and

$$S^t_- : \mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi) \to \mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi)$$

are weak homotopy equivalences.

From here we will deduce:

Corollary 0.0.5. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk. Then, the natural inclusion*

$$\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi) \hookrightarrow F\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi)$$

is a weak homotopy equivalence.

This result cannot be improved in general, as some loose Legendrians are formally isotopic but not Legendrian isotopic [60]. However, in strongly overtwisted manifolds, formal isotopy of loose Legendrians implies Legendrian isotopy [30].

Second part: Convex Disk

The contents of this part are based on [28] and appear in Chapter 3. Overtwisted contact structures were introduced by Eliashberg in 3 dimensions, and by Borman-Eliashberg-Murphy in all dimensions in the seminal works [2, 17]. As mentioned before these structures are governed by an h-principle, the results in the literature regarding classification problems have been scarce. Some results regarding the coarse classification of submanifolds, i.e., up to contactomorphisms smoothly isotopic to the identity, have been obtained by Chatterjee, Eliashberg-Fraser, Etnyre, Etnyre-Min-Mukherjee, and Geiges-Onaran [7, 20, 23, 25, 37, 38]. On the other hand, contact isotopy questions in overtwisted contact manifolds can be intricate even for loose objects, i.e., those with an overtwisted complement, as shown independently by Chekanov and Vogel [60]. Nevertheless, there have been few works dealing with classifications up to contact isotopy, such as the aforementioned [60], the works of Cahn-Chernov, Cardona-Presas, Ding-Geiges, Dymara, and Farias-Fernández [3, 4, 11, 13, 14, 27] for the classification of loose objects avoiding a fixed overtwisted disk, or the work of Fernández on strongly overtwisted contact 3-manifolds [30].

In this chapter, we study convex disks with fixed characteristic foliations and fixed Legendrian boundaries, up to contact isotopy, in overtwisted contact 3-manifolds. In most cases, we achieve a full classification by establishing a 1-parameter h-principle result. The analogous question for tight contact manifolds was studied by Colin in [10], with its parametrized version analyzed by Fernández-Martínez–Aguinaga-Presas in [31]. The methods employed throughout this chapter rely on a version of Eliashberg's h-principle [17] with non-fixed overtwisted disk, combined with a careful study of isotopies of overtwisted disks.

Summary of the results in this part:

Let (M,ξ) be an overtwisted contact 3-manifold and $\mathbb{D}^2 \subseteq (M,\xi)$ an embedded convex 2-disk with Legendrian boundary and characteristic foliation $\mathcal{F}$ determined by $T\mathbb{D}^2 \cap \xi$. We define the *space of $\mathcal{F}$-embeddings* as the space of all smooth embeddings of $\mathbb{D}^2 \hookrightarrow (M,\xi)$ that induce the characteristic foliation $\mathcal{F}$ on $\mathbb{D}^2$, we also require all the embeddings to agree in a neighborhood of the Legendrian boundary $\partial\mathbb{D}^2$. This space is denoted by $\mathrm{Emb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)$. The space of *formal $\mathcal{F}$-embeddings,* that will be precisely defined in 1.2.2 for the sake of readability, but informally can be thought as pairs (e,P_s), where $e:\mathbb{D}^2\hookrightarrow M$ is a smooth embedding and P_s, $s\in[0,1]$, is a homotopy of plane fields over $e(\mathbb{D}^2)$ so that $P_0 = de(T\mathbb{D}^2)$ and $P_1\cap\xi$ determines the charcteristic foliation $\mathcal{F}$. As before, we require our formal embeddings to coincide near the boundary. The space of formal $\mathcal{F}$-embeddings is denoted by $\mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)$.

Theorem 0.0.6. *Let (M,ξ) be an overtwisted contact 3-manifold and $\mathbb{D}^2\subseteq(M,\xi)$ an embedded convex disk with Legendrian boundary and characteristic foliation $\mathcal{F}$. If either $\mathcal{F}$ is tight or $\mathrm{tb}(\partial\mathbb{D}^2)+|\mathrm{rot}(\partial\mathbb{D}^2)|>-1$, the inclusion*

$$\mathrm{Emb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)\hookrightarrow\mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)$$

induces an isomorphism at the level of path-connected components. Moreover, the first relative homotopy

group is trivial:

$$\pi_1\left(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M,\xi), \mathrm{rel}\,\partial), \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M,\xi), \mathrm{rel}\,\partial)\right) = 0.$$

The motivation for studying spaces of convex disks with Legendrian boundary is to understand the spaces of Legendrian unknots in overtwisted contact 3-manifolds (M,ξ). Let $\Lambda \subseteq (M,\xi)$ be an embedded Legendrian unknot. We denote by $\mathcal{L}(\Lambda,(M,\xi))$ the path-connected component in the space of Legendrian embeddings containing Λ, and by $F\mathcal{L}(\Lambda,(M,\xi))$ the corresponding component within the space of formal Legendrians.

Our first application establishes an h-principle at the level of fundamental groups for Legendrian unknots that lie in a Darboux ball in overtwisted contact 3-manifolds.

Corollary 0.0.7. *Let $\Lambda \subseteq (M,\xi)$ be an embedded Legendrian unknot in an overtwisted contact 3-manifold. If Λ lies in a Darboux ball, then the inclusion*

$$\mathcal{L}(\Lambda,(M,\xi)) \hookrightarrow F\mathcal{L}(\Lambda,(M,\xi))$$

induces an isomorphism at the level of fundamental groups.

In the case where the underlying 3-manifold is the 3-sphere, the fundamental group of the space of Legendrian embeddings is described precisely.

Corollary 0.0.8. *Let $\Lambda \subseteq (\mathbb{S}^3,\xi)$ be an embedded Legendrian unknot in some overtwisted contact 3-sphere. Then, there is an isomorphism:*

$$\pi_1(\mathcal{L}(\Lambda,(\mathbb{S}^3,\xi))) \cong \mathbb{Z}\oplus\mathbb{Z},$$

where one $\mathbb{Z}$-factor is generated by a full rotation of Λ in a Darboux ball, and the other $\mathbb{Z}$-factor is generated by a loop of long Legendrian unknots, which are contractible as loops of smooth long unknots.

Remark 0.0.9. The situation stands in sharp contrast with the one for the tight contact 3-sphere described in [31]. For instance, there does not exists a non-contractible loop of long Legendrian unknots with $\mathrm{tb} = -1$ which is contractible as a loop of long smooth unknots in the standard tight 3-sphere.

To label them in $\mathbb{S}^3$ we fix the trivialization of $T\mathbb{S}^3$ given by the quaternions. Then, every contact structure ξ on $\mathbb{S}^3$ defines a Gauss map $f_\xi : \mathbb{S}^3 \to \mathbb{S}^2$, yielding to a map

$$\mathcal{C}_{\mathrm{OT}}(\mathbb{S}^3) \to \mathrm{Maps}(\mathbb{S}^3, \mathbb{S}^2)$$

that induces an isomorphism at the level of path-connected components. In particular, $\pi_0(\mathcal{C}_{\mathrm{OT}}(\mathbb{S}^3)) \cong \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, where the isomorphism is given by associating to each overtwisted contact structure ξ the Hopf invariant $h(f_\xi) \in \mathbb{Z}$ of the associated Gauss map. We fix a list of representatives $\{\xi_k : k \in \mathbb{Z}, h(f_{\xi_k}) = k\} \subseteq \mathcal{C}_{\mathrm{OT}}(\mathbb{S}^3)$ of these path-connected components.

Let $\mathrm{CStr}_{\mathrm{OT}}(M)$ denote the space of overtwisted contact structures on M. By Eliashberg's h-principle, the isotopy classes of overtwisted contact structures on M correspond to homotopy classes of

plane fields on M. To label them in $\mathbb{S}^3$ we fix the trivialization of $T\mathbb{S}^3$ given by the quaternions, so every contact structure ξ has an associated Gauss map $f_\xi : \mathbb{S}^3 \to \mathbb{S}^2$. In particular, $\pi_0(\mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3)) \cong \mathbb{Z}$, with the isomorphism given by the Hopf invariant $h(f_\xi) \in \mathbb{Z}$ of the Gauss map. We fix a list of representatives $\{\xi_k : k \in \mathbb{Z}, h(f_{\xi_k}) = k\} \subseteq \mathcal{C}_{\mathrm{OT}}(\mathbb{S}^3)$ of these path-connected components.

Given a Legendrian $\Lambda \subseteq (\mathbb{S}^3, \xi)$, we denote by $(\mathcal{O}p(\Lambda), \xi)$ a standard neighborhood of Λ in $(\mathbb{S}^3, \xi)$, and by $(C(\Lambda), \xi)$ its complement. We will denote by $\mathrm{Cont}(C(\Lambda), \xi)$ the group of contactomorphisms of $(C(\Lambda), \xi)$, and by $\mathrm{FCont}(C(\Lambda), \xi)$ the group of formal contactomorphisms. Our (formal) contactomorphisms are assumed to be the identity near the boundary.

Corollary 0.0.10. *Let $\xi \in \mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3)$ be some overtwisted contact structure on $\mathbb{S}^3$ and $\Lambda \subseteq (\mathbb{S}^3, \xi)$ a Legendrian unknot with $\mathrm{tb}(\Lambda) \leq 0$. Consider the inclusion*

$$i : \mathrm{Cont}(C(\Lambda), \xi) \hookrightarrow \mathrm{FCont}(C(\Lambda), \xi). \tag{1}$$

Then,

(a) If $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| > -1$ then the induced map $\pi_0(i)$ is an isomorphism. In particular, we have that $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

(b) If $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| \leq -1$ then

(i) If $\xi \ncong \xi_{-1}$ then the induced map $\pi_0(i)$ is an isomorphism. In particular, $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

(ii) If $\xi \cong \xi_{-1}$ then the induced map $\pi_0(i)$ is surjective but has order 2 kernel. In particular, $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Case (a) in this result also follows from the work of Fernández on strongly overtwisted contact 3-manifolds [30]. Our proof is independent of this. On the other hand, case (b) are the only examples of overtwisted contact 3-manifolds, besides the 3-sphere studied by Chekanov and Vogel, independently [60], for which there is a description of its contact mapping class group.

Conclusion

Conclusion I

In the first part of this thesis, we have provided a topological characterization to determine when a family of bypasses with a fixed attaching arc is contractible. This characterization was formulated in terms of the existence of a bypass that is disjoint from the given family outside the attaching region. Based on this characterization, we established new proofs of several h-principles in overtwisted 3-dimensional contact manifolds. These results not only reinforce the understanding of flexibility in contact geometry but also provide alternative tools for studying contact structures

in low dimensions. Overall, the findings of this research expand the existing theoretical framework and open new avenues for exploration in the theory of bypasses and their relationship with h-principles in contact topology.

Future Research: In recent years, the study of convex hypersurface theory and bypass attachments in higher dimensions has gained attention following the foundational work of Honda and Huang ([51] and [49]). One difficulty in adapting three-dimensional convex surface theory methods to the higher-dimensional setting is that they often rely on results that are not available in higher dimensions, such as Eliashberg's uniqueness result for tight contact structures in the 3-sphere ([18]). The entire proof of the Bypass in the Middle Theorem has been designed to avoid the use of such technology. A natural line of research would be to attempt a full adaptation of the proof to the higher-dimensional case and explore potential applications in this setting.

Conclusión I (español)

En la primera parte de esta tesis, hemos proporcionado una caracterización topológica para determinar cuándo una familia de bypasses con un arco de fijación dado es contráctil. Dicha caracterización se formuló en términos de la existencia de un bypass que sea disjunto de la familia dada fuera de la región de fijación. Con base en esta caracterización, establecimos nuevas demostraciones de varios h-principios en variedades de contacto sobreenrolladas de dimensión 3. Estos resultados no solo refuerzan la comprensión de la flexibilidad en la geometría de contacto, sino que también proporcionan herramientas alternativas para el estudio de estructuras de contacto en bajas dimensiones. En general, los hallazgos de esta investigación amplían el marco teórico existente y abren nuevas vías de exploración en la teoría de bypasses y su relación con los h-principios en topología de contacto.

Investigación Futura En los últimos años, el estudio de la teoría de hipersuperficies convexas y la adhesión de bypasses en dimensiones superiores ha ganado relevancia, en particular, tras los trabajos fundamentales de Honda y Huang ([51] y [49]). Una de las dificultades para adaptar los métodos de la teoría de superficies convexas en dimensión tres a dimensiones superiores radica en que estos dependen de resultados que no están disponibles en dimensiones más altas, como el teorema de unicidad de Eliashberg para estructuras de contacto tensas en la 3-esfera ([18]). Toda la demostración del teorema del Bypass en el Medio ha sido diseñada para evitar el uso de dicha tecnología. Una línea natural de investigación sería intentar una adaptación completa de la demostración al caso de dimensión superior y explorar posibles aplicaciones en este contexto.

Conclusion II

In the second part of this thesis, we have classified convex discs with a fixed characteristic foliation and Legendrian boundary, up to contact isotopy relative to the boundary, in every closed overtwisted 3-dimensional contact manifold. This classification includes both cases where the neighborhood of the disc is tight and those where the boundary violates the Bennequin-Eliashberg inequality. Furthermore, we proved that this classification coincides with the formal classification, thereby establishing an h-principle for these discs.

As a consequence of these results, we deduced that the space of trivial Legendrian knots with $\mathrm{tb} + |\mathrm{rot}| \leq -1$ in any closed overtwisted 3-dimensional contact manifold satisfies the h-principle at the level of fundamental groups. Finally, we determined the group of contact isotopy classes of the complement of any trivial Legendrian knot in an overtwisted 3-sphere with a non-positive tb invariant. These results not only strengthen the understanding of the structure and classification of convex discs and Legendrian knots in the context of contact topology but also extend the scope of h-principles in this setting, providing new tools for studying flexibility in contact geometry.

Future Research: The natural follow-up project to the problem studied in this part would be to determine whether the inclusion

$$\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial) \hookrightarrow \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$$

is a weak homotopy equivalence, rather than merely an isomorphism on path-connected components. This does not immediately follow from the strategy used in this document to prove Theorem 3.1.1, as it relies on a method for constructing isotopies of overtwisted disks that does not clearly extend to higher-dimensional families of embeddings.

Conclusión II (español)

En la segunda parte de esta tesis, hemos clasificado los discos convexos con una foliación característica fija y frontera Legendriana, hasta isotopía de contacto relativa a la frontera, en cualquier variedad de contacto sobreenrollada cerrada de dimensión 3. Esta clasificación incluye tanto los casos en que el entorno del disco es tenso como aquellos en los que la frontera viola la desigualdad de Bennequin-Eliashberg. Además, demostramos que esta clasificación coincide con la clasificación formal, estableciendo así un h-principio para estos discos.

Como consecuencia de estos resultados, dedujimos que el espacio de nudos Legendrianos triviales con $\mathrm{tb} + |\mathrm{rot}| \leq -1$ en cualquier variedad de contacto sobreenrollada cerrada de dimensión 3 satisface el h-principio a nivel de grupos fundamentales. Finalmente, determinamos el grupo de clases de isotopía de contacto del complemento de cualquier nudo Legendriano trivial en una 3-esfera sobreenrollada con un invariante tb no positivo. Estos resultados no solo fortalecen la comprensión de la estructura y clasificación de discos convexos y nudos Legendrianos en el contexto de la topología de contacto, sino que también amplían el alcance de los h-principios en este ámbito, proporcionando nuevas herramientas para el estudio de la flexibilidad en la geometría de contacto.

Investigación Futura El proyecto de seguimiento natural del problema estudiado en esta parte sería determinar si la inclusión

$$\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial) \hookrightarrow \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$$

es una equivalencia homotópica débil, en lugar de ser simplemente un isomorfismo en los componentes conexos por caminos. Esto no se sigue inmediatamente de la estrategia utilizada en este documento para demostrar el Teorema 3.1.1, ya que se basa en un método para construir isotopías de discos sobreenrollados que no se extiende de manera evidente a familias de embebimientos de mayor dimensión.

Outline:

This thesis is organized into three chapters. The first chapter, titled Preliminaries 1, introduces all the necessary concepts in contact topology and Legendrian knots necessary to understand the two parts of this work.

The second chapter of this thesis is organized as follows. In Section 2.2, we review the basic concepts of bypass that will be used both, in the first part as a main topic and as a tool in the second. In Section 2.3, we prove that the space of trivial bypasses is contractible (Theorem 2.3.1) and use this result to establish the main theorem, Theorem 2.1.1. Finally, in Section 2.4, we prove all the applications of Theorem 2.1.1 for overtwisted contact 3-manifolds.

The third chapter is organized as follows. In Section 3.4, we quickly review of the isotopy discretization theorem 3.4.1, triangle bypasses and the formal class of a bypass trival. In Section 3.5 reviews Eliashberg's h-principle from [17] and a variation of it with non-fixed overtwisted disk, which will be essential throughout the Chapter (Theorem 3.5.5 and Corollary 3.5.6). Section 3.6, the most technical part of the work, is devoted to constructing isotopies of overtwisted disks in the complement of isotopies of smooth disks with fixed boundary. Section 3.7 presents the proof of Theorem 3.1.1. Finally, in Section 3.8, we complete the proofs of Corollaries 3.2.1, 3.2.2, and 3.3.1.

Chapter 1

Preliminaries

In this chapter we will give all the basic concepts of contact topology, most of it has been obtained from Geiges' book [36], so the reader can go there for more information.

1.1 Contact Basics

Let's start by considering M a soft manifold of odd dimension, to which we will give a structure with ξ a sub- bundle of TM of codimension one. We will call this pair (M, ξ) a *contact manifold* and ξ a *contact structure*. This structure locally we can always understood as $\xi = \ker \alpha$, where α is a smooth 1-form in M, which satisfy

$$\alpha \wedge (d\alpha)^n \neq 0,$$

this condition is called *maximally non-integrable* and α is called *contact form*. Also if ξ is co-oriented hyperplane field admits a global 1-form α that defines it, this is $TM \cong \xi \oplus \xi^{\perp}$, equivalently TM/ξ is trivial.

On the other hand if $(M, \xi = \ker \alpha)$ is a contact manifold we have a very distinguished vector field associated with a contact form called *Reeb vector field* uniquely defined by the equations

$$(1) d\alpha(R_\alpha, \cdot) \equiv 0, \qquad (2) \alpha(R_\alpha) \equiv 1. \tag{1.1}$$

Until now we thought of a manifold with a structure, but it could have more than one contact structure that defines it or we could want to relate two different manifolds, for this we define the following concept: Let (M_1, ξ_1) and (M_2, ξ_2) be two contact manifold , we will say that they are *contactomorphic* if there exists a diffeomorphism $f : M_1 \to M_2$ with $df(\xi_1) = \xi_2$, where $df : TM_1 \to TM_2$ is the differential of f. Also if $\xi_i = \ker \alpha_i$, $i = 1, 2$, we can say that α_1 and $f^*\alpha_2$ determine the same hyperplane field, then there is a never null function $\lambda : M_1 \to \mathbb{R} \setminus \{0\}$ such that $f^*\alpha_2 = \lambda\alpha_1$.

The following example will be helpful to fully understand this way of relating two contact structures:

Example 1.1.1. Consider first $\mathbb{R}^3$ with Cartesian coordinates given by (x, y, z), and we will provide it with a contact structure with $\xi_{\text{std}} = \ker \alpha_{\text{std}}$, where $\alpha_{\text{std}} = dz - ydx$ is the *standard contact form*.

For the other case we will take $\mathbb{R}^3$ with cylindrical coordinates given by (r, φ, z), these define the 1-form $\alpha_2 = dz + r^2 d\varphi = dz + xdy - ydx$. Therefore if we consider a map smooth $f : (\mathbb{R}^3, \xi_{\text{std}}) \to (\mathbb{R}^3, \xi_2)$ such that

$$f(x, y, z) = \left(\frac{y - x}{2}, \frac{y + x}{2}, z - \frac{xy}{2}\right),$$

then easily checking $f^*\alpha_2 = \alpha_{\text{std}}$ we obtain that f is a contactmorphism.

On the other hand if we consider in $\mathbb{R}^3$ the contact structure $\xi_{\text{OT}} = \ker \alpha_{\text{OT}}$ called the *standard 'overtwisted' contact structure*, where the 1–form $\alpha_{\text{OT}} = \cos r dz + r \sin r d\varphi$, such that

$$r \mapsto \begin{cases} \frac{\sin r}{r} \text{ for } r \neq 0, \\ 1 \text{ for } r = 0. \end{cases}$$

We can notice that both the standard contact structure and the overtwisted are horizontal along the z-axis However, while the rotation angle of ξ_{std} approaches but never reaches $pi/2$, i.e. the contact planes never become vertical, the contact planes of ξ_{OT} form infinities complete turns as one moves in a radial direction. Bennequin [1] proved that $(\mathbb{R}^3, \xi_{std})$ and $(\mathbb{R}^3, \xi_{\text{OT}})$ are in fact not contactomorphic. This is one of the major results of contact topology. We have just seen that contact structures are divided into two types, those that are *overtwisted* or *tight*. Later we will tell you a way to distinguish them.

Another example of contactactomorphism would be to think of a smooth submanifolds, this concept is called *contact submanifold* . If N is a submanifold of (M, ξ) and N has a contact structure ξ' we say that this is a *contact submanifold* of (M, ξ) if $TN \cap \xi_{|N} = \xi'$.

Also if $\xi = \ker \alpha$ and $i : N \to M$ denotes the inclusion map, then the condition of being a contact submanifols is equivalent to $\xi' = \ker(i^*\alpha)$.

1.1.1 Contact Hamiltonian

Let $(M, \xi = \ker \alpha)$ be a contact manifold we can talk about another important vector field X called *contact vector field* this satisfy that $d\psi_t(\xi) = \xi$ for all $t \in \mathbb{R}$, where ψ_t its flow of X vector field on M. We will also say that X is an *strict contact vector field* of α if $\psi_t^*\alpha = \alpha$ for all t.

So if we think about the tangent space of the contactmorphism group $\text{Cont}(M, \xi)$ in the Identity, that is, its Lie Algebra, it is precisely the space of the contact vector fields. After fixing a contact form $\alpha \in \Omega^1(M)$; i.e. $\xi = \ker \alpha$, the latter can be identified with the smooth function space $C^\infty(M)$ as follows.

Given a contact vector field X we can assign it a smooth function $H_X := \alpha(X)$ known as the *contact Hamiltonian.* This mapping identifies the space of contact vector fields with the space of smooth functions $C^\infty(M)$. In fact, given a function $H \in C^\infty(M)$ there is a unique contact vector field X_H

determined by the relations $\alpha(X_H) = H$ and $i_{X_H} d\alpha = dH(R)\alpha - dH$, where R is the Reeb vector field of α.

These concepts will be very useful for the proofs of the following theorems, which are the main ones that relate all contact structures locally. On the other hand, we will also use these vector fields in the demonstrations of this thesis.

Theorem 1.1.2 (Gray stability theorem)**.**
Let ξ_t , $t \in [0,1]$, be a smooth family of contact structures on a closed manifold M. Then there is an isotopy $(\Psi_t)_{t\in[0,1]}$ of M such that

$$d\Psi_t(\xi_0) = \xi_t \text{ for each } t \in [0,1].$$

Theorem 1.1.3 (Darboux's theorem)**.**
Let α be a contact form on the $(2n+1)$–dimensional manifold M and p a point on M. Then there are coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$ on a neighbourhood $U \subset M$ of p such that $p = (0, \dots, 0)$ and

$$\alpha_{|U} = dz - \sum_{j=1}^{n} x_j dy_j.$$

1.1.2 Convex surface Theory

Recall that a *contact vector field* in a contact 3-manifold (M, ξ) is a vector field whose flow preserves the contact structure; i.e. is given by contactomorphisms. An embedded surface $\Sigma \subseteq (M, \xi)$ is said to be *convex* if there exists a contact vector field X transverse to it. Similarly, an embedding $e : \Sigma \to (M, \xi)$ is said to be *convex* if $e(\Sigma)$ is convex.

There exists a one-dimensional, oriented, singular foliation that characterizes the behavior of the surface Σ, referred to as the *characteristic foliation*. This foliation is determined by the intersection $T\Sigma \cap \xi$ and is denoted by $\mathcal{F}$. Similarly, for embeddings, we can define $\mathcal{F}$ as the pullback of the foliation given by $Te(\Sigma) \cap \xi$.

Another key component in this theory is the *dividing set* of a convex surface $\Sigma \subseteq (M, \xi)$ associated with a contact vector field X. This set is represented by a family of embedded transverse curves $\Gamma_X := \{p \in \Sigma : X(p) \in \xi_p\}$. The isotopy class of Γ_X is independent of the choice of X, as the space of contact vector fields transverse to Σ is contractible. Therefore, we will simply denote it by Γ.

Due to the convexity of $\Sigma \subseteq (M, \xi)$, there exists an *I-invariant neighborhood* of Σ, which is contactomorphic to $(\Sigma \times I, \xi_{\text{inv}} = \ker(f\,dt + \beta))$, $t \in I := [0,1]$, for some $f \in C^\infty(\Sigma)$ and $\beta \in \Omega^1(\Sigma)$. With these coordinates, the dividing set of Σ is then determined by $\Gamma = f^{-1}(0)$.

The following result highlights the feasibility of approximating a given surface by an arbitrarily close convex one, initially proven by Giroux [39] and later for the case of the Legendriand boundary by Honda [47].

Theorem 1.1.4 (Giroux, Honda)**.** *Let (M, ξ) be a contact manifold and $\Sigma \subset M$ a compact smoothly embedded surface with possibly empty Legendrian boundary with non-positive* tb*. Then, there exists a C^∞-small perturbation of Σ (C^0 near the boundary), which makes it convex.*

The following theorem establishes the relationship between the contact condition and the characteristic foliation of a surface, showing that it is fully determined by a contact germ near the surface:

Theorem 1.1.5 (Giroux). *Let $\Sigma_i \subseteq (M_i, \xi_i)$ be an embedded compact surface in contact 3–manifold with characteristic foliation $\mathcal{F}_i$, $i \in \{0,1\}$. Assume that there is a diffeomorphism $\phi : \Sigma_0 \to \Sigma_1$ such that $\phi_* \mathcal{F}_0 = \mathcal{F}_1$. Then, there is a contactomorphism $\Phi : (\mathcal{O}p(\Sigma_0), \xi_0) \to (\mathcal{O}p(\Sigma_1), \xi_1)$ with $\Phi(\Sigma_0) = \Sigma_1$ and such that $\Phi_{|\Sigma_0}$ is isotopic to ϕ via an isotopy preserving the characteristic foliation.*

Moreover, when the surface is also convex the contact germ is determined, after a small isotopy, by the dividing set, this is known as *Giroux Realization Theorem.* To state this result let us first introduce some notation. We write $\mathrm{Emb}(\Sigma, (M, \xi), \mathcal{F})$ to denote the space of embeddings $e : \Sigma \to M$ with characteristic foliation $\mathcal{F}$. Similarly, we will denote by $\mathrm{Emb}(\Sigma, (M, \xi), \Gamma)$ is the space of convex embeddings $e : \Sigma \to M$ with dividing set Γ.

Theorem 1.1.6 (Giroux Realization Theorem [41]). *Let $e : \Sigma \to (M, \xi)$ be a convex embedding with characteristic foliation $\mathcal{F}$ and dividing set Γ. Then, the natural inclusion*

$$i : \mathrm{Emb}(\Sigma, (M, \xi), \mathcal{F}) \to \mathrm{Emb}(\Sigma, (M, \xi), \Gamma)$$

is a weak homotopy equivalence.

The following Fraser result will be useful later, which is consequence from the Realization theorem:

Lemma 1.1.7 (Pivot Lemma). *Let $e : \mathbb{D} \to (M, \xi)$ be a embedding with a characteristic foliation $\mathcal{F}$, which consists only of one positive elliptic singularity p and unstable orbits from p which exit transversely from $e(\partial\mathbb{D})$. If δ_1, δ_2 are two unstable orbits meeting at p, and $\delta_i \cap e(\partial\mathbb{D}) = p_i$, then, after a C^∞–small perturbation of e fixing $\partial\mathbb{D}$, we obtain e' whose characteristic foliation has exactly one positive elliptic singularity p' and unstable orbits from p' exiting transversely from $\partial\mathbb{D}$, and for which the orbits passing through p_1, p_2 meet tangentially at p.*

Finally we will state a fundamental result of Giroux that rules out several configurations of dividing sets on tight contact 3-manifolds:

Theorem 1.1.8 (Giroux Tightness Criteria[40]). *Let $\Sigma \subseteq (M, \xi)$ be a convex surface with possibly empty Legendrian boundary. Then, an I-invariant neighborhood of Σ is tight if and only if one of the following two conditions holds*

(i) $\Sigma = \mathbb{S}^2$ *and the dividing set Γ is connected, or*

(ii) $\Sigma \neq \mathbb{S}^2$ *and the dividing set does not contain homotopically trivial curves.*

Lemma 1.1.9 (Edge-Rounding, [47]). *Let Σ_1 and Σ_2 be convex surface with Legendrian boundary, which intersects transversely along a common Legendrian curve γ in (M, ξ) a contact manifold. Assume that a neighborhood $N(\gamma)$ of γ is locally contactomorphic to $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq \varepsilon\}$ with the contact*

structure given by $\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$ *for some* $n \in \mathbb{N}$ *and* $\Sigma_1 \cap N(\gamma) = \{x = 0, 0 \leq y \leq \varepsilon\}$, $\Sigma_2 \cap N(\gamma) = \{y = 0, 0 \leq x \leq \varepsilon\}$. *If we join* Σ_1 *and* Σ_2 *along* $x = y = 0$ *and round the common edge, the resulting surface is convex, and the dividing curve* $z = k/2n$ *on* Σ_1 *will connect to the dividing curve* $z = (k/2n) - (1/4n)$ *on* Σ_2, *where* $k = 0, \ldots, 2n-1$. *Here we are assuming that* Σ_1 *and* Σ_2 *have been oriented so that the orientations agree after rounding, see Figure 1.1.*

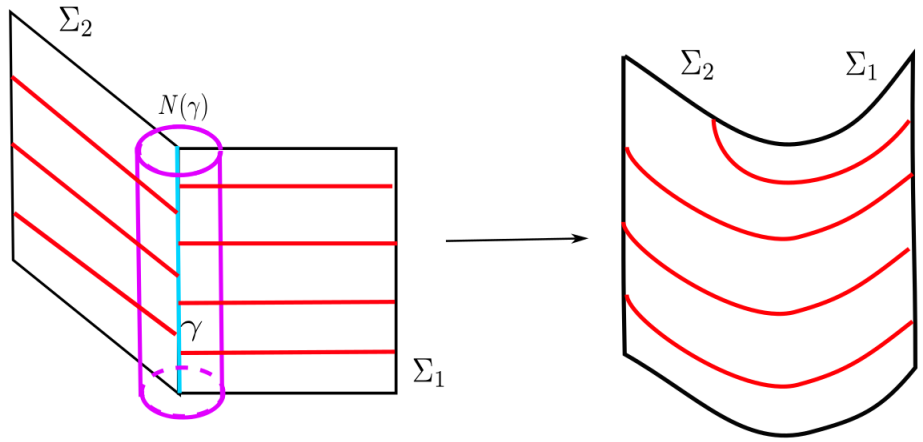


Figure 1.1:

1.2 Legendrian basics

In this section we will give the basic concepts of Legendrian knots, for more detailed information on this topic you can see [22] and [36].

Let (M, ξ) be a contact manifold. Recall that a submanifold L of (M, ξ) is called an *isotropic submanifold* if $T_pL \subset \xi_p$ for all points $p \in L$. Furthermore, if $2n + 1 = dim(M)$ and the isotropic submanifold L is of dimension n, we will call it the *Legendrian submanifold*.

In this work we will use both the concept of Legendrian knot in a contact 3-manifold (M, ξ), where the latter is a Legendrian embedding $\gamma : \mathbb{S}^1 \to M$, that is, an embedding satisfying

$$\gamma'(\theta) \in \xi_\gamma(\theta) \text{ for all } \theta \in \mathbb{S}^1. \tag{1.2}$$

We denote by $\mathcal{L}(M, \xi)$ the *space of Legendrian embeddings* .

In order to graph a knot L in $(\mathbb{R}^3, \xi_{\text{std}})$ there are two way, *front projection* and *Lagrangian projection.* We will concentrate on the front projection.

If γ is a parametrized of L given by $\gamma(\theta) = (x(\theta), y(\theta), z(\theta))$ in $(\mathbb{R}^3, \xi_{\text{std}})$, the frontal projection of L is the image of the coordinates $(x(\theta), z(\theta))$. This projection is characterized by the following conditions obtained from the equation 1.2:

- does not have vertical tangencies
- is an immersion except at finite points called *cusps.*
- at each crossing, the branch with the lowest slope will be above the other.

In the Figure 1.2 we can see tree examples of frontal projections:

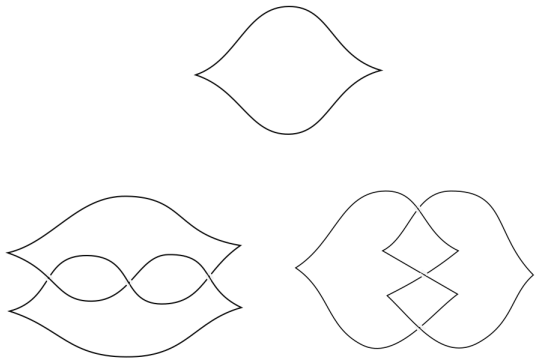


Figure 1.2: First the frontal projection of unknot and down the frontal projections of the left trefoil knot and right trefoil knot

Every Legendrian $L \subseteq (M, \xi)$ has an standard neighborhood $(\mathcal{O}p(L), \xi)$ contactomorphic to $(\mathbb{S}^1 \times \mathbb{R}^2, \xi_{\text{std}} = \ker(dz - yd\theta))$. Here, there is also a well-defined front projection onto $\mathbb{S}^1 \times \mathbb{R}(\theta, z)$ that satisfies similar rules as the ones above.

As a first important result we have the relation between smooth and Legendrian knots:

Theorem 1.2.1. *Let $\gamma : \mathbb{S}^1 \to (M, \xi)$ be a knot in a contact 3–manifold. Then γ can be C^0–approximated by a Legendrian knot isotopic to γ.*

The following result tells us how to relate the Legendrian knots through their frontal projections.

Theorem 1.2.2 ([58])**.** *Two front diagrams represent the Legendrian isotopic Legendrian knots if and only if they are related by regular homotopy and a sequence of Legendrian Reidemeister moves as Figure 1.3.*

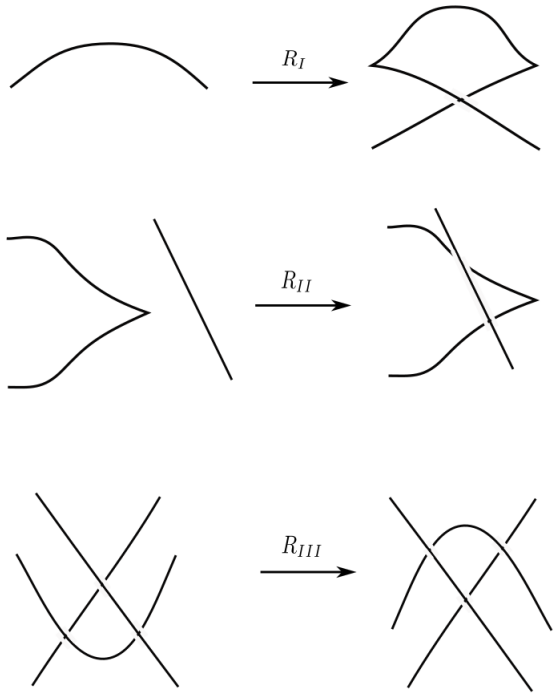


Figure 1.3: Legendrian Reidemeister moves

1.2.1 Invariants of Legendrian knots

We will begin by giving the formal definitions of two of the most important Legendrian knot invariants, *Thurston Bennequin* and *rotation number,* and then focus on how to calculate them from a Seifert surface. We consider in this section (M, ξ) a 3-contact manifold with ξ a cooriented contact structure and L a nullhomologous oriented Legendrian knot in M. We will denote by $\mathcal{N}_L$ the *normal bundle* of L. An identification of $\mathcal{N}_L$ with $L \times \mathbb{R}^2$ is called *trivialization.* The normal bundle is canonically trivialized by a Seifert surface Σ of L, i.e. $\partial\Sigma = L$.

On the other hand, a Legendrian knot has a *contact trivialization* induced by the contact structure, which is obtained from the intersection of the structure with the normal bundle. Then the turns that the contact framing makes with respect to the canonical trivialization is the *Thurston Bennequin invariant* of L, denoted by $tb(L) \in \mathbb{Z}$.

We can define the *rotation number* of L is defined as follows. Consider a Seifert surface Σ of L. It follows that $\xi_{|\Sigma}$ is canonically trivializable. The rotation number of L with respect to Σ is the number of turns that the oriented line $TL \subseteq \xi$ makes with respect to this trivialization, and it is denoted by $\text{rot}(L, [\Sigma]) \in \mathbb{Z}$. This numerical invariant depends, in general, on the homology class $[\Sigma] \in H_2(M, L)$. However, in the case that the Euler class $e(\xi) = 0 \in H_2(M)$ is also independent of this (see [36]) for details.

Theorem 1.2.3 (Kanda [55])**.** *Let L be a Legendrian, Σ a convex Seifert surface of L, and Γ a dividing set*

of Σ. *Then*

1. $\mathrm{tb}(\gamma) = -\frac{1}{2}\#(\Gamma \cap \gamma)$,
2. $\mathrm{rot}(\gamma) = \chi(\Sigma_+) - \chi(\Sigma_-)$,

where Σ_+ *and* Σ_- *are positive and negative component of* $\Sigma \setminus \Gamma$, *finally* $\chi(\Sigma)$ *is the Euler characteristic.*

We continue to define the object that we mentioned earlier that will separate the contact structures into two types, overtwisted and tight.

Recall the contact manifold $(\mathbb{R}^3, \xi_{\mathrm{OT}} = \ker(\cos r dz + r \sin r d\varphi))$, which was described in the example 1.1.1. Let Δ_{OT} be the disk given by $\{z = 0, r \leq \pi\}$, where its boundary $\partial\Delta_{OT}$ is a Legendrian curve for ξ_{OT}. The characteristic foliation of this consists of all radial lines, with singular points at the origin and at all boundary points, see Figure 1.4. We say that a contact manifold (M, ξ) is

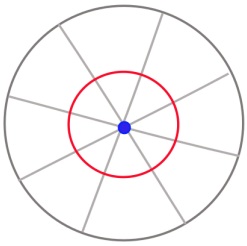


Figure 1.4: Overtwisted disk. The dividing set is depicted in red and the characteristic foliation in gray.

overtwisted if it contains a overtwisted disk Δ_{OT}. Otherwise we will say that the contact manifold is *tight*.

Also an overtwisted disc have as its boundary $\partial\Delta$ a Legendrian curve for ξ whose surface framing coincides with its contact framing, i.e. $\mathrm{tb}(\partial\Delta) = 0$.

On the other hand we can define a convex surface Σ to be *tight* if there exists a neighborhood of Σ that is tight. Conversely, we say that a convex surface is *overtwisted* if every neighborhood of Σ is overtwisted.

The following theorem by Colin establishes a fundamental connection between the tightness property in contact manifolds and convex embeddings of spheres, showing how local tightness in a region influences the global tightness of the manifold.

Theorem 1.2.4 (Colin [10])**.**
Let (M, ξ) *be a contact manifold and* $f : \mathbb{S}^2 \to (M, \xi)$ *a convex embedding. If* $(M \setminus f(\mathbb{S}^2), \xi)$ *is tight, then* (M, ξ) *is tight.*

This celebrated result provides a bound on the possible formal invariants of a Legendrian knot in the tight case.

Theorem 1.2.5 (Bennequin-Eliashberg Inequality [1, 18]). *Let $L \subseteq (M, \xi)$ be an oriented Legendrian knot in a tight contact 3–manifold (M, ξ), and Σ a Seifert surface of L. Then, the following holds*

$$\mathrm{tb}(L) + |\,\mathrm{rot}(L, [\Sigma])| \leq -\chi(\Sigma).$$

Note that when $\Sigma = \mathbb{D}^2$, so L is an unknot, then this yields the restriction $\mathrm{tb}(L) \leq -1$.

1.2.2 Formal $\mathcal{F}$-embeddings

All the contents presented in this section are based on [30], to which the reader is referred for a detailed and rigorous exposition of the relevant material.

Let Σ be a compact, connected, and oriented surface. Consider a contact manifold $(\Sigma \times \mathbb{R}, \eta)$, where η is an $\mathbb{R}$-invariant contact structure. The surface $\Sigma = \Sigma \times \{0\}$ is convex with respect to η and has characteristic foliation $\mathcal{F}$. From now on we will write $\eta =: \xi_{\mathcal{F}}$.

Given a contact 3-manifold (M, ξ), we define the *space of $\mathcal{F}$-embeddings* as the set of all embeddings of Σ into (M, ξ) that induce the characteristic foliation $\mathcal{F}$ on Σ. This space is denoted by $\mathrm{Emb}(\Sigma, \mathcal{F}, (M, \xi))$. In the case where $\partial\Sigma \neq \emptyset$, we impose the additional condition that all embeddings must agree in a neighborhood of the boundary $\partial\Sigma$.

In order to state the definition of a formal $\mathcal{F}$-embedding, it is necessary to consider germs of formal isocontact embeddings in a neighborhood of the surface. Let $\varepsilon > 0$ be a small positive parameter. We define an abstract *formal embedding* from $\Sigma \times (-\varepsilon, \varepsilon)$ into (M, ξ) as a pair (E, F_s) satisfying the following conditions:

1. $E : \Sigma \times (-\varepsilon, \varepsilon) \to M$ is a smooth embedding, and
2. $F_s : T(\Sigma \times (-\varepsilon, \varepsilon)) \to TM$, for $s \in [0, 1]$, is a homotopy of bundle isomorphisms covering E, such that $F_0 = dE$.

Under these conditions we can define the following equivalence relation: Let $(E, F_s) : \Sigma \times (-\varepsilon, \varepsilon) \to M$ and $(E', F'_s) : \Sigma \times (-\varepsilon', \varepsilon') \to M$ be two formal embeddings, they are equivalent if there exists $0 < \delta < min(\varepsilon, \varepsilon')$ such that $(E, F_s)|_{\Sigma\times(-\delta,\delta)} = (E', F'_s)|_{\Sigma\times(-\delta,\delta)}$. Then, we refer to the equivalence class of (E, F_s) under this relation as a *germ of a formal embedding* over Σ. With a slight abuse of notation, this equivalence class will also be denoted by (E, F_s). The space of germs of formal embeddings is denoted by $\mathrm{FEmb}^{germ}(\Sigma, M)$

Given a contact structure ξ in $\Sigma \times (-\varepsilon, \varepsilon)$, we will say that a formal embedding (E, F_s) is a *formal isocontact embedding* if $F_1(\xi) = \xi|_{E(\Sigma\times(-\varepsilon,\varepsilon))}$. A formal $\mathcal{F}$-embedding of Σ into (M, ξ) is simply a germ of a formal isocontact embedding $(E, F_s) : (\Sigma \times (-\varepsilon, \varepsilon), \xi_{\mathcal{F}}) \to (M, \xi)$. The space of formal $\mathcal{F}$-embeddings is denoted by $\mathrm{FEmb}(\Sigma, \mathcal{F}, (M, \xi))$. As explained in [30], since the space of germs of contact structures inducing the same characteristic foliation over a surface is contractible, the spaces of $\mathcal{F}$-embeddings of surfaces into a given contact 3-manifold (M, ξ) and germs of isocontact

embeddings $(E, dE) : (\Sigma \times (-\varepsilon, \varepsilon), \xi_{\mathcal{F}}) \to (M, \xi)$ are weakly homotopy equivalent. From now on, we will use both interchangeably.

Chapter 2

Bypasses

2.1 Main result

Before presenting our main result, we introduce some notation. Let (M, ξ) be a contact 3-manifold, and let $\Sigma \subseteq (M, \xi)$ be a convex surface. A *bypass embedding* is an embedding $b : \mathbb{D}^2_+ \to (M, \xi)$ of the half-disk $\mathbb{D}^2_+ = \mathbb{D}^2 \cap \{y \geq 0\} \subseteq \mathbb{R}^2$ with Legendrian boundary, whose characteristic foliation coincides with the bypass characteristic foliation $\mathcal{F}_B$ shown in Figure 1. More specifically, the foliation on $\mathbb{D}^2_+$ determined by $db^{-1}(db(T\mathbb{D}^2_+) \cap \xi)$ is precisely $\mathcal{F}_B$. The *attaching arc* of the bypass is the Legendrian arc $\gamma = \{b(x, 0) : -1 \leq x \leq 1\}$.

The bypass embedding b is said to be a bypass embedding *for* Σ if $b(\mathbb{D}^2_+) \cap \Sigma = \gamma$. Given another bypass embedding $\hat{b}$ with the same attaching arc γ, we say that $\hat{b}$ coincides with b near the attaching arc if there exists an open neighborhood $U \subseteq \mathbb{D}^2_+$ of $\{(x, 0) : -1 \leq x \leq 1\}$ such that $b|_{\overline{U}} = \hat{b}|_{\overline{U}}$. If, in addition, this neighborhood can be chosen so that $b(\mathbb{D}^2_+ \setminus \overline{U}) \cap \hat{b}(\mathbb{D}^2_+ \setminus \overline{U}) = \emptyset$, we say that $\hat{b}$ is disjoint from b away from the attaching arc. Note that this terminology involves a slight abuse of language: b is considered disjoint from itself away from the attaching arc. A schematic representation of two bypasses disjoint away from the attaching arc is shown in Figure 2.6.

We denote by $\mathcal{B}_\gamma(\Sigma, (M, \xi))$ the space of bypass embeddings for Σ with a fixed attaching arc γ that coincide with b near the attaching arc. Similarly, we denote by $\mathcal{B}^b_\gamma(\Sigma, (M, \xi)) \subseteq \mathcal{B}_\gamma(\Sigma, (M, \xi))$ the subspace of bypass embeddings that are disjoint from b away from the attaching arc.

Our main result establishes a necessary and sufficient condition for a family of bypass embeddings with a fixed attaching arc to be contractible:

Theorem 2.1.1 (Bypass in the Middle)**.** *The inclusion $\mathcal{B}^b_\gamma(\Sigma, (M, \xi)) \hookrightarrow \mathcal{B}_\gamma(\Sigma, (M, \xi))$ is null-homotopic.*

Remark 2.1.2. Let $b_0, b_1 \in \mathcal{B}_\gamma(\Sigma, (M, \xi))$ be two bypass embeddings for Σ that coincide near the attaching arc γ. The result above implies that if there exists a third bypass embedding $b \in \mathcal{B}_\gamma(\Sigma, (M, \xi))$ that coincides with both b_0 and b_1 near the attaching arc, but becomes disjoint from both b_0 and b_1 away from the attaching arc—hereafter referred to as a *bypass in the middle* -then there exists a homotopy of bypass embeddings connecting b_0 and b_1.

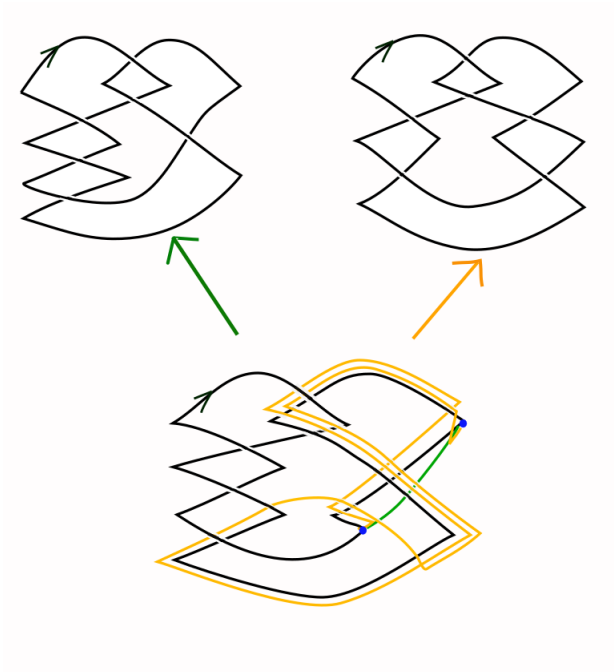


Figure 2.1: In the second row, a Legendrian knot is depicted along with the boundaries of two bypasses (shown in green and orange) attached to its standard neighborhood. These bypasses are distinct because they destabilize the Legendrian knot into the two Chekanov-Eliashberg $m(5_2)$ knots shown in the first row, which are not Legendrian isotopic.

In particular, if b_0 and b_1 are disjoint away from γ, they are isotopic as bypass embeddings. Therefore, any pair of bypass embeddings that are potentially of interest *must* intersect. This situation is entirely analogous to the case of overtwisted disk embeddings: two overtwisted disk embeddings that are disjoint from a third overtwisted disk are contact isotopic, whereas two overtwisted disks that intersect may fail to be contact isotopic [14, 60].

Remark 2.1.3. The existence of the bypass in the middle b in Theorem 2.1.1 is crucial: the space $\mathcal{B}_\gamma(\Sigma, (M, \xi))$ is not necessarily connected in general. For example, the standard neighborhood of the stabilization of one of the Chekanov-Eliashberg knots [8] admits two bypasses with the same attaching arc that are not contact isotopic. The boundaries of these bypasses are shown in Figure 2.1. These bypasses can be distinguished using Legendrian contact homology [8].

Our proof is elementary in the sense that it does not rely on any parametric statements from 3-dimensional contact topology, such as those in [21, 31]. Instead, it critically leverages the fact that a trivial bypass is indeed trivial [48, 50], a result that Honda-Huang proved in all dimensions [50], therefore without invoking Eliashberg's uniqueness theorem for tight contact structures on the 3-sphere [18]. Additionally, we use a parametric extension of this result: the space of trivial bypasses is contractible (Theorem 2.3.1). We decided to avoid using [21, 31] in the proof so it could be potentially upgraded to the higher-dimensional case.

2.1.1 Applications to overtwisted contact 3-manifolds

To demonstrate the broad applicability of the Bypass in the Middle Theorem (Theorem 2.1.1), we will use it to (re)prove several well-known h-principle-type results for overtwisted contact 3-manifolds. The novelty lies in the fact that our proofs rely solely on Theorem 2.1.1 and do not invoke Eliashberg's general overtwisted h-principle [17].

Our first result addresses the contractibility of the space of bypasses in the complement of an overtwisted disk:

Corollary 2.1.4. *Let (M, ξ) be an overtwisted contact 3-manifold, $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk and $\Sigma \subseteq (M \backslash \Delta_{\mathrm{OT}}, \xi)$ a convex surface. Then, for any given attaching arc $\gamma \subseteq \Sigma$ the space of bypass embeddings $\mathcal{B}_\gamma(\Sigma, (M \setminus \Delta_{\mathrm{OT}}, \xi))$ is contractible.*

An *overtwisted disk embedding* is a smooth embedding $e : \mathbb{D}^2 \to (M, \xi)$ of an overtwisted disk; that is, its induced characteristic foliation coincides with that of the overtwisted disk depicted in Figure 1.4. The space of overtwisted disk embeddings, denoted by $\mathrm{Emb}^{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$, is particularly relevant for the study of the homotopy type of the contactomorphism group of (M, ξ), as discussed in [14, 30, 60]. Let $\mathrm{CFr}(M, \xi)$ denote the total space of the bundle of contact frames of a contact 3-manifold (M, ξ).

Since every overtwisted disk can be viewed as a pair of disjoint bypasses smoothly glued along their attaching arc, we will derive the following parametric h-principle for overtwisted disks in the complement of a fixed overtwisted disk, based on the Bypass in the Middle Theorem:

Corollary 2.1.5. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk. Then, the 1-jet evaluation map at the origin*

$$\mathrm{ev}_0 : \mathrm{Emb}^{\mathrm{OT}}(\mathbb{D}^2, (M \backslash \Delta_{\mathrm{OT}}, \xi)) \to \mathrm{CFr}(M \backslash \Delta_{\mathrm{OT}}, \xi)$$

is a weak homotopy equivalence.

In particular, the previous corollary implies that two overtwisted disks are contact isotopic if there exists a third overtwisted disk that is disjoint from both. This result was previously established by Dymara [14] for any overtwisted 3-sphere, and for a general overtwisted contact 3-manifold, it follows easily from the ideas in [30].

Analogous to the situation in Theorem 2.1.1, the condition of avoiding a fixed overtwisted disk cannot, in general, be removed: the space of overtwisted disk embeddings could be disconnected, as shown independently by Chekanov and Vogel [60]. On the other hand, [30] a parametric h-principle for the entire space of overtwisted disks holds for the subclass of strongly overtwisted contact 3-manifolds and also for the subspace of overtwisted disks with fixed boundary [30].

Finally, we will use the Bypass in the Middle Theorem (Theorem 2.1.1) to study families of loose Legendrians in overtwisted contact 3-manifolds, that is, Legendrians with an overtwisted complement. The space of Legendrian embeddings into a contact 3-manifold (M, ξ) is denoted by $\mathcal{L}(M, \xi)$.

Theorem 2.1.6. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{OT} \subseteq (M, \xi)$ an overtwisted disk. Then, for every $t \in \mathbb{S}^1$ the stabilization at time t operators*

$$S^t_+ : \mathcal{L}(M \backslash \Delta_{OT}, \xi) \to \mathcal{L}(M \backslash \Delta_{OT}, \xi)$$

and

$$S^t_- : \mathcal{L}(M \backslash \Delta_{OT}, \xi) \to \mathcal{L}(M \backslash \Delta_{OT}, \xi)$$

are weak homotopy equivalences.

Here, the stabilization operators are defined as Legendrian satellite maps, following [32]. These operators, introduced in Section 2.2.1, act on each Legendrian by adding a positive (or negative) stabilization at time t, up to Legendrian isotopy.

Recall that a *formal Legendrian embedding* into a contact 3-manifold (M, ξ) is a pair (β, F_s), where

- $\beta : \mathbb{S}^1 \hookrightarrow M$ is a smooth embedding,
- $F_s : T\mathbb{S}^1 \to \beta^* TM$, $s \in [0, 1]$, is a homotopy of bundle monomorphisms such that $F_0 = d\beta$ and $\text{Image}(F_1) \subseteq \beta^* \xi$.

The space of formal Legendrian embeddings into (M, ξ) is denoted by $F\mathcal{L}(M, \xi)$. We will use Theorem 2.1.6 in conjunction with the parametric Fuchs-Tabachnikov Theorem [5, 35, 57] to show that.

Corollary 2.1.7. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{OT} \subseteq (M, \xi)$ an overtwisted disk. Then, the natural inclusion*

$$\mathcal{L}(M \backslash \Delta_{OT}, \xi) \hookrightarrow F\mathcal{L}(M \backslash \Delta_{OT}, \xi)$$

is a weak homotopy equivalence.

This result can be proved (in all dimensions) using the h-principle for overtwisted contact structures [2, 17], as well as results such as those in [4, 9]. It is important to note that this result cannot be improved in general: there exist loose Legendrians that are formally isotopic but not Legendrian isotopic [60]. On the other hand, if the underlying 3-manifold is strongly overtwisted, two formally isotopic loose Legendrians are Legendrian isotopic, and there is a complete h-principle for the subclass of strongly loose Legendrians [30].

2.2 Bypass basics

We will begin this chapter by providing the definition of a bypass, which was introduced by K. Honda in [47].

A *bypass* is a convex half-disk D with Legendrian boundary, generated by two Legendrian curves γ and β, which we will refer to as the *attaching arc* and the *Honda arc* , respectively. This half-disk

has four singularities: three elliptic singularities over γ, where two of the same sign are shared with β, and a hyperbolic singularity in β of the same sign as the elliptic ones, as shown in Figure 1.

There are two types of bypasses: *positive* and *negative* , which are determined by the sign of the central elliptic singularity. We work with positive bypasses unless otherwise indicated. However, the same arguments can be applied to negative bypasses.

Bypasses have proven to be extremely useful for describing the convexity defect in 1-parameter families of surfaces. Additionally, they allow us to construct a series of unions with the surface in order to isotope it to another surface. This union is known as a *Bypass Attachment* , which we now describe:

Let Σ be a convex surface and D a bypass disk, intersecting Σ only at the attaching arc γ. Then, a neighborhood of $\Sigma \cup D$ in (M, ξ), which we denote by $\mathcal{O}p(\Sigma \cup D)$, is diffeomorphic to $(\Sigma \times I, \xi)$. Here, $\Sigma = \Sigma \times \{\varepsilon\}$ for $\varepsilon > 0$, and $\Sigma \times \{1\}$ is convex. Its dividing set can be obtained from that of Σ by following the modification shown in Figure 2.2, where the movement applied to the dividing set of the surface is called *bypass attachment* . To achieve this smooth change in the dividing set, we will need to apply the *Edge Rounding Lemma* (Lemma 1.1.9) and the *Pivot Lemma* (Lemma 1.1.7).

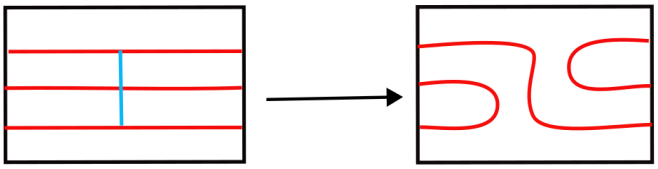


Figure 2.2: Result of attaching a bypass

We denote by $\mathcal{B}_\gamma(\Sigma, (M, \xi))$ the space of bypasses embeddings for Σ, where the attaching arc γ intersects the dividing set of Σ at exactly three points. These embeddings coincide over an open neighborhood of γ and have a fixed characteristic foliation.

It is possible that this space is empty, but there are two important cases in which it is not. The first case is as follows:

Lemma 2.2.1 ([59]). *If $M \setminus \Sigma$ is overtwisted and $\gamma \subset \Sigma$ is a possible attaching arc then exists bypass with attaching arc γ to Σ.*

We can observe that to obtain the bypass in the Lemma, we only need an overtwisted disk $\Delta_{OT} \subseteq (M \setminus \Sigma, \xi)$ and a Legendrian curve $\delta : [0,1] \to (M, \xi)$ such that $\delta(0) \in \gamma \subseteq \Sigma$, $\delta(1) \in \Delta_{OT}$, and $\delta(0,1) \cap \Sigma = \delta(0,1) \cap \Delta_{OT} = \emptyset$. Hence, the bypass is the half-disk cobounded by γ and the Legendrian connected sum of γ and $\partial\Delta_{OT}$ following δ.

The second case is *trivial bypasses* . These are known as the *Right to Life Principle* in [52, 47]. A bypass is called *trivial* for Σ if, when performing the bypass attachment, it does not change the

isotopy class of the dividing set. On the other hand, a Legendrian arc $\gamma \subseteq \Sigma$ intersecting the dividing set at precisely three points is said to be a *trivial attaching arc* if every (abstract) bypass for Σ along γ is trivial. In the following Figure 2.3, we depict a trivial bypass.

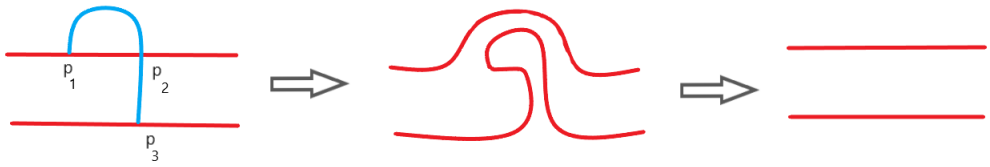


Figure 2.3: Trivial bypass attachment

Lemma 2.2.2 (Honda). *Let (M, ξ) be a contact 3-manifold, Σ be a convex surface with possibly empty Legendrian boundary and $\gamma \subset \Sigma$ be a trivial attaching arc. Then, there exists a trivial bypass D with attaching arc γ. Moreover, the neighborhood $(\overline{Op}(\Sigma \cup D), \xi)$ is contactomorphic to the standard I-invariant neighborhood $(\Sigma \times I, \xi_{inv})$.*

2.2.1 (De)stabilizations, bypasses and Fuchs-Tabachnikov Theorem

We introduce Legendrian stabilizations as a Legendrian satellite operation, e.g. [26], so it can be carried out parametrically [32].

Consider the contact 3-manifold $(\mathbb{S}^1 \times \mathbb{D}^2, \xi_{\text{std}}) \subseteq (J^1\mathbb{S}^1, \xi_{\text{std}} = \ker(dy - xd\theta))$, where $(x, y) \in \mathbb{D}^2$ and $\theta \in \mathbb{S}^1$; and the *Lengendrian 0-section* $s_0 : \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times \mathbb{D}^2$. Fix a small $\varepsilon > 0$, we define the positive and negative stabilizations of s_0 at time t as the Legendrian curves

$$s^t_{\pm} : \mathbb{S}^1 \hookrightarrow (\mathbb{S}^1 \times \mathbb{D}^2, \xi_{\text{std}})$$

whose fronts are depicted in Figure 2.4. These satisfy that $s^t_{\pm}(z) = s_0(z)$ for all $z \in \mathbb{S}^1 \backslash (t - \varepsilon, t + \varepsilon)$, whereas for $z \in (t - \varepsilon, t + \varepsilon)$ their front traces a zigzag as depicted in the figure. This definition is independent of $\varepsilon > 0$ up to Legendrian isotopy.

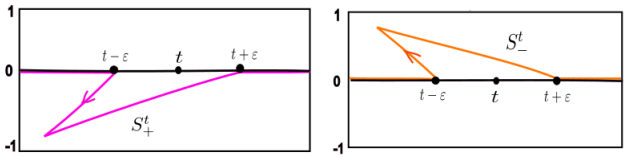


Figure 2.4: Stabilizations of s_0

Let us recall the Legendrian satellite construction. Consider a Legendrian embedding γ into a contact 3-manifold (M, ξ) together with a standard contact neighborhood of it, $T_\gamma : (\mathbb{S}^1 \times \mathbb{D}^2, \xi_{std}) \to$

(M, ξ), where $T_\gamma(\theta, 0, 0) = \gamma(\theta)$. Let $\beta : \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{D}^2, \xi_{std})$ be another Legendrian embedding. Associated with this data we define *Legendrian satellite* embedding:

$$T_\gamma \circ \beta : \mathbb{S}^1 \to (M, \xi).$$

The Legendrian γ is called *companion embedding* and β the *pattern embedding*. The uniqueness, up to contact isotopy, of the standard neighborhood of γ implies that this definition is independent of T_γ up to Legendrian isotopy.

Given a Legendrian embedding $\gamma \in \mathcal{L}(M, \xi)$ we define the positive, resp. negative, *stabilization* of γ at time $t \in \mathbb{S}^1$ as the Legendrian satellite embedding with companion γ and pattern the curve $s^t_+ : \mathbb{S}^1 \hookrightarrow (\mathbb{S}^1 \times \mathbb{D}^2, \xi_{\text{std}})$, resp. s^t_-, defined above. Following [32] there exist well-defined stabilization maps at time $t \in \mathbb{S}^1$

$$S^t_+ : \mathcal{L}(M, \xi) \to \mathcal{L}(M, \xi) \text{ and } S^t_- : \mathcal{L}(M, \xi) \to \mathcal{L}(M, \xi). \tag{2.1}$$

We say that β' is a *destabilization* of β if $S^t_\pm(\beta') = \beta$ for some t. It follows from the local construction at $(\mathbb{S}^1 \times \mathbb{D}^2, \xi_{\text{std}})$ that s_0 is a destabilization of both s^t_+ and s^t_-. Moreover, there is a bypass disk b_+, resp. b_-, naturally co-bounded by the Honda arc $-s_0([t - \varepsilon, t + \varepsilon])$ and by the attaching arc $s^t_+([t - \varepsilon, t + \varepsilon])$, resp. $s^t_-([t - \varepsilon, t + \varepsilon])$. The bypass b_+ is positive whereas the b_- is negative. We depicted the bypass b_+ in Figure 2.5.

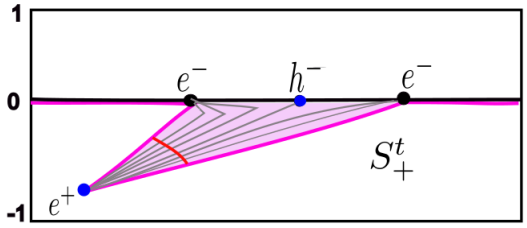


Figure 2.5: Bypass disk resulting from a stabilization

In general, given β a Legendrian embedding with $\gamma = \beta_{|I_0}$, where $I_0 \subset \mathbb{S}^1$ a subinterval and $b : D \to (M, \xi)$ a bypass disk embedding such that the attaching arc of b coincides with γ and $b(D) \cap \beta(\mathbb{S}^1) = \gamma(I_0)$, we can find a new Legendrian embedding β' such that $S^t_\pm(\beta') = \beta$, i.e. β' is a destabilization of β. To do this we replace γ by the Honda arc of b with the opposite orientation after using the pivot Lemma 1.1.7 twice to smooth the corners. Since Giroux realization is parametric all this process can be done parametrically meaning that given a family of Legendrian curves and bypasses as before one can simultaneously destabilize the Legendrians.

The main motivation behind this work rests on the previous destabilization procedure by means of bypasses embeddings together with the Fuchs-Tabachnikov Theorem [35] that states that two formally isotopic Legendrians became Legendrian isotopic after adding sufficiently many stabilizations (of both signs). In this Chapter, we will make use of the parametric version of this result proved by Casals-del Pino [5], see also Murphy [57].

Theorem 2.2.3 (Fuchs-Tabachnikov [5]).
Let (M, ξ) be a contact 3-manifold and $\gamma_z : \mathbb{S}^1 \to (M, \xi)$ a parameterized family of formal Legendrian knots $z \in \mathbb{D}^k$, such that γ_z Legendrian knots for $z \in Op(\partial\mathbb{D}^k)$. Then, there exists a parameterized family $\tilde{\gamma}_z$ of Legendrian knots such that $\tilde{\gamma}_z$, $z \in Op(\partial\mathbb{D}^k)$, is obtained from γ_z by stabilization.

Remark 2.2.4. The previous result states that multiple applications of the operators S^t_+ and S^t_-, defined in (2.1), on the space of Legendrian embeddings allow for gaining flexibility. This is analogous to the situation in higher dimensions, with the crucial difference being that in dimension 3, the stabilization map changes the Legendrian isotopy class, whereas, as observed by Murphy [57], this is not the case in higher dimensions for stabilizations along circles, as defined in [15], for Legendrians with at least one stabilization, i.e., Loose Legendrians, since these can be self-replicated. Nevertheless, one can still attempt to use this result to prove flexibility results for Legendrian embeddings in dimension 3 by trying to invert (in homotopy) the operators S^t_+ and S^t_-. This, of course, is not generally possible [8]. However, it follows from our previous discussion that the obstruction to doing this is measured by the homotopy groups of a certain space of bypass embeddings. Therefore, Theorem 2.1.1 can be applied to show that this indeed can be achieved in some special situations (see Theorem 2.1.6).

2.3 Bypass in the Middle Theorem

In this section we will prove the Bypass in the Middle Theorem 2.1.1. The main ingredient in the proof is Theorem 2.3.1, that could be thought as a parametric version of Honda's triviality Lemma 2.2.2.

2.3.1 The space of trivial bypasses

Theorem 2.3.1. *Let $\Sigma \subset (M, \xi)$ be a convex surface with possibly empty Legendrian boundary and γ attaching arc for Σ of a trivial bypass. Then, the space $\mathcal{B}_\gamma(\Sigma, (M, \xi))$ is contractible.*

Proof. Let $b \in \mathcal{B}_\gamma(\Sigma, (M, \xi))$ and $D = \text{Image}(b)$. The map

$$\text{Cont}_0(M, \xi, \text{rel}\,\Sigma) \to \mathcal{B}_\gamma(\Sigma, (M, \xi)), \varphi \mapsto \varphi \circ b;$$

defines a fibration

$$\text{Cont}_0(M, \xi, \text{rel}\,\Sigma \cup D) \hookrightarrow \text{Cont}_0(M, \xi, \text{rel}\,\Sigma) \to \mathcal{B}_\gamma(\Sigma, (M, \xi)).$$

Here $\text{Cont}_0(M, \xi, \text{rel}\,\Sigma)$ stands for the path connected components of the identity map and $\text{Cont}_0(M, \xi, \text{rel}\,\Sigma \cup D) = \text{Cont}_0(M, \xi, \text{rel}\,\Sigma) \cap \text{Cont}(M, \xi, \text{rel}\,\Sigma \cup D)$. It follows from Lemma 2.3.2, proved below, that the fibration map is surjective. Moreover, by Lemma 2.3.3, also proved below; the inclusion of the fiber in the total space is a weak homotopy equivalence. Therefore, the base is contractible and the result follows. □

It remains to prove Lemmas 2.3.2 and 2.3.3.

Lemma 2.3.2. *The map* $\mathrm{Cont}_0(M,\xi,\mathrm{rel}\,\Sigma) \to \mathcal{B}_\gamma(\Sigma,(M,\xi)), \varphi \mapsto \varphi \circ b$ *is surjective.*

Proof. Let $\tilde{b} \in \mathcal{B}_\gamma(\Sigma,(M,\xi))$ be a trivial bypass. We must find a contactomorphism $\tilde{\varphi}$, contact isotopic to the identity such that $\tilde{\varphi} \circ b = \tilde{b}$.

Write $D = \mathrm{Image}(b)$, $\tilde{D} = \mathrm{Image}(\tilde{b})$ and $I = [0,1]$. Consider an I-invariant neighborhood $(\Sigma \times I, \xi_{inv} = \ker(f\,dt + \beta))$ of Σ, where $f \in C^\infty(\Sigma)$ and $\beta \in \Omega^1(\Sigma)$. Since b and $\tilde{b}$ are trivial bypasses, Lemma 2.2.2 implies the existence of contactomorphisms

$$F : (\Sigma \times I, \xi_{inv}) \to (\overline{\mathcal{O}p(\Sigma \cup D)}, \xi),$$

$$\tilde{F} : (\Sigma \times I, \xi_{inv}) \to (\overline{\mathcal{O}p(\Sigma \cup \tilde{D})}, \xi).$$

We can assume that

(i) $F_{|\Sigma\times[0,2\varepsilon]} = \tilde{F}_{|\Sigma\times[0,2\varepsilon]}$ for some small $\varepsilon > 0$,

(ii) $F(\Sigma \times \{\varepsilon\}) = \tilde{F}(\Sigma \times \{\varepsilon\}) = \Sigma \subseteq M$.

Fix $\delta \in (0,\varepsilon)$ small such that $\tilde{D} \subset \tilde{F}(\Sigma \times [0, 1-\delta])$. Consider the Hamiltonian

$$H(p,t) = -\rho(t) f(p),$$

in $\Sigma \times I$, where $\rho : [0,1] \to [0,1]$ is a smooth bump function such that

(a) $\rho(t) = 0$ for $t \in \mathcal{O}p([0,\varepsilon] \cup \{1\})$ and

(b) $\rho(t) = 1$ for $t \in \mathcal{O}p([\varepsilon+\delta, 1-\delta])$.

Let X_H be the associated contact vector field in $(\Sigma \times I, \xi_{\mathrm{inv}})$ with respect to the contact form $\alpha = f\,dt + \beta$. Then, the contact vector field $\tilde{F}_* X_H$ generates a contact flow $g_u \in \mathrm{Cont}_0(M,\xi,\mathrm{rel}\,\Sigma)$, $u \in \mathbb{R}$, with support in $\tilde{F}(\Sigma \times I)$. Fix $U > 0$ big enough so that $g_U(\tilde{D}) \subset F(\Sigma \times I)$ and consider $\hat{b} := g_U \circ \tilde{b}$.

Let $\hat{\gamma} \times \{\varepsilon\} = F^{-1}(\gamma) \subseteq \Sigma \times \{\varepsilon\}$ be the pre-image under F of the attaching arc of both D and $\hat{D} := \mathrm{Image}(\hat{b}) = g_U(\tilde{D})$. Again Lemma 2.2.2 implies that

$$(\Sigma \times I \setminus \mathcal{O}p(F^{-1}(D) \cup \hat{\gamma} \times [0,\varepsilon]), \xi_{inv}) \cong (\Sigma \times I \setminus \mathcal{O}p(F^{-1}(\hat{D}) \cup \hat{\gamma} \times [0,\varepsilon]), \xi_{inv}) \cong (\Sigma \times I, \xi_{inv}).$$

This, together with Theorem 1.1.5, allow us to find $\hat{\varphi} \in \mathrm{Cont}(\Sigma \times I, \xi_{inv})$ such that $\hat{\varphi} \circ F^{-1} \circ b = F^{-1} \circ \hat{b}$. We claim that we can assume that $\hat{\varphi}$ is contact isotopic to the identity. Indeed, if this is not the case we can use a contact Hamiltonian as before to isotope $\hat{\varphi}^{-1}$ to a contactomorphism $G \in \mathrm{Cont}(\Sigma \times I, \xi_{inv})$ with support in $\mathcal{O}p(\Sigma \times \{1\})$. Hence, $G \circ \hat{\varphi} \in \mathrm{Cont}_0(\Sigma \times I, \xi_{inv})$ would be contact isotopic to the identity and still satisfy $G \circ \hat{\varphi} \circ F^{-1} \circ b = F^{-1} \circ \hat{b}$.

Therefore, the contactomorphism

$$\tilde{\varphi} = g_U^{-1} \circ F \circ \hat{\varphi} \circ F^{-1} \in \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma)$$

is contact isotopic to the identity, here $F \circ \hat{\varphi} \circ F^{-1}$ is defined to be the identity outside Image(F), and satisfies that $\tilde{\varphi} \circ b = \tilde{b}$ as required. This concludes the argument. □

Lemma 2.3.3. *The inclusion of the fiber*

$$\mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma \cup D) \hookrightarrow \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma)$$

is a weak homotopy equivalence.

Proof. Let (K, G) be a compact CW-pair and

$$\Phi : (K, G) \to (\mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma), \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma \cup D))$$

a continuous map. It is enough to find a homotopy

$$\Phi_s : (K, G) \to (\mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma), \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma \cup D)), s \in [0, 1],$$

such that

(a) $\Phi_0 = \Phi$,

(b) $\Phi_s(g) = \Phi(g)$, for $(s, g) \in [0, 1] \times G$,

(c) $\mathrm{Image}(\Phi_1) \subseteq \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma \cup D)$.

Lemma 2.2.2 implies the existence of a contactomorphism

$$F : (\Sigma \times I, \xi_{inv}) \to (\overline{Op(\Sigma \cup D)}, \xi)),$$

where $\xi_{inv} = \ker(f\, dt + \beta)$ and $F(\Sigma \times \{\varepsilon\}) = \Sigma \subseteq M$ for some $\varepsilon > 0$ small. We may further assume that $\Phi(k)_{|\,\mathrm{Image}(F)} = \mathrm{Id}$ for all $k \in G$.

Fix $\delta > 0$ small enough such that

(i) $D \subseteq F(\Sigma \times [0, 1 - \delta])$,

(ii) $\Phi(k)_{|Op(F(\Sigma \times [0, \varepsilon + \delta])} = \mathrm{Id}$ for all $k \in K$.

Let X_H be the contact vector field in $(\Sigma \times I, \xi_{inv})$ defined in the proof of Lemma 2.3.2. Consider the contact flow $g_u \in \mathrm{Cont}_0(M, \xi, \mathrm{rel}\, \Sigma)$, $u \in \mathbb{R}$, with support in $F(\Sigma \times I)$, of the contact vector field $F_* X_H$. Fix $U > 0$ big enough such that $g_U(F(\Sigma \times [0, 1 - \delta]))$ is contained in $F(\Sigma \times [0, \varepsilon + \delta])$. The required homotopy is given by

$$\Phi_s = g_{sU}^{-1} \circ \Phi \circ g_{sU}, s \in [0, 1].$$

□

2.3.2 Proof of Theorem 2.1.1

Let us explain the intuition behind the proof of Theorem 2.1.1, where the goal is to show that a family of bypasses admitting a bypass in the middle is contractible. The existence of a bypass in the middle allows us to reduce the problem to a parametric family of tight 3-balls, where the problem can be solved in each ball individually (Theorem 2.3.1). To piece together the different solutions corresponding to the various balls, we use a microfibration argument, where the contractibility of the fibers is ensured once more by Theorem 2.3.1.

Proof of Theorem 2.1.1. Let K be a compact parameter space and $b^z \in \mathcal{B}^b_\gamma(\Sigma, (M, \xi))$, $z \in K$, a continuous family of bypasses, all of them disjoint from b away from the attaching arc γ.

It is enough to find a homotopy $b^{z,t} \in \mathcal{B}_\gamma(\Sigma, (M, \xi))$, $t \in [0, 1]$, such that

(a) $b^{z,0} = b^z$,

(b) $b^{z,1} = b$.

We will reformulate the existence of such a homotopy as the existence of a section of a (micro)fibration with non-empty contractible fiber so the result will follow.

Write $D^z = \text{Image}(b^z)$, $z \in K$, and $D = \text{Image}(b)$. For each $z \in K$, the half-disks D^z and D are disjoint away from γ; hence, we can find an embedded closed 3-disk C^z such that

(i) $D^z \cup D \subseteq C^z$,

(ii) $\Sigma \cap C^z = \Sigma \cap \partial C^z = \overline{\mathcal{O}p_\Sigma(\gamma)}$ is independent of $z \in K$, and

(iii) (C^z, ξ) is tight.

Conditions (i) and (ii) are obvious. To check condition (iii) observe that, as a consequence of Theorem 1.1.5, $(\mathcal{O}p(D^z \cup D), \xi)$ contactly embeds in an I-invariant neighborhood of D, which is tight by Giroux Criteria Theorem 1.1.8. Therefore, taking C^z small enough we can ensure tightness.

Denote by $\mathcal{X}$ the space of pairs $(z, \hat{b}^{z,t})$ where

(a) $z \in K$,

(b) $\hat{b}^{z,t} \in \mathcal{B}_\gamma(\partial C^z, (C^z, \xi))$, $t \in [0, 1]$, is a homotopy of bypasses between $b^z = \hat{b}^{z,0}$ and $b = \hat{b}^{z,1}$.

The projection

$$\pi : \mathcal{X} \to K, \quad (z, \hat{b}^{z,t}) \mapsto z;$$

is naturally a microfibration. Moreover, since each (C^z, ξ) is tight, every bypass attached to ∂C^z inside C^z is trivial. Therefore, every fiber is contractible because of Theorem 2.3.1. By Weiss

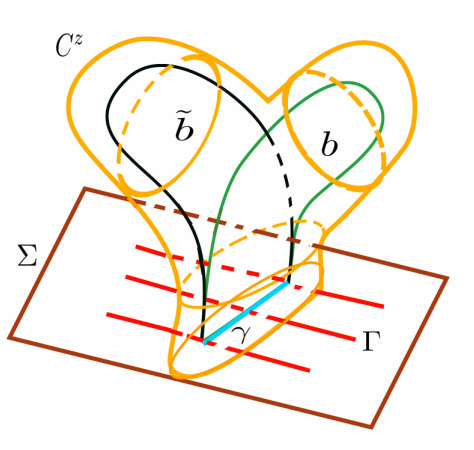


Figure 2.6: Schematic depiction of the 3-disk C^z.

Microfibration Lemma [61] we conclude that π is a Serre fibration with contractible fiber. [1] In particular, there exists a section $s : K \to \mathcal{X}$. The second component of $s(k)$ is the desired homotopy. □

Remark 2.3.4. The previous argument can be easily used to prove the following more general property. Let K be a compact parameter space and $B_0, B_0', B_1 : K \to \mathcal{B}_\gamma(\Sigma, (M, \xi))$ two families of bypass embeddings such that

$$B_0(k), B_0'(k) \in \mathcal{B}_\gamma^{B_1(k)}(\Sigma, (M, \xi))$$

for all $k \in K$. Then, B_0 and B_0' are homotopic through families of bypass embeddings. In particular, taking $B_0' = B_1$ we see that if $B_0(k) \in \mathcal{B}_\gamma^{B_1(k)}(\Sigma, (M, \xi))$ for all $k \in K$, then B_0 and B_1 are homotopic families of bypass embeddings. The details are left to the reader.

2.4 Applications to overtwisted contact 3-manifolds

In this Section we will prove all the corollaries of the Bypass in the Middle Theorem 2.1.1 stated in 2.1.1.

2.4.1 Proof of Corollary 2.1.4

We will use a parametric version of the argument used in the proof of Lemma 2.2.1, to then be able to apply the Bypass the middle Theorem 2.1.1.

We begin by considering $b^z \in \mathcal{B}_\gamma(\Sigma, (M \backslash \Delta_{OT}, \xi))$, $z \in \mathbb{S}^k$, a family of bypasses. We explain how to homotope this family to a constant bypass.

[1] In fact, choosing C^z carefully enough one can ensure that it is a Serre fibration, without appealing to the Microfibration Lemma.

Since b^z is contractible, relative to γ, as a family of smooth half-disk embeddings, we can find a family of smooth arcs $\delta^{z,r} : [0,1] \to M$, $(z,r) \in \mathbb{S}^k \times [0,1]$, such that

- $\delta := \delta^{z,0}$ is independent of $z \in \mathbb{S}^k$.
- $\delta^{z,r}_{|\mathcal{O}p(\{0,1\})} = \delta_{|\mathcal{O}p(\{0,1\})}$ are Legendrian arcs, for all $(z,r) \in \mathbb{S}^k \times [0,1]$.
- $\text{Image}(\delta^{z,r}) \cap \Delta_{\text{OT}} = \delta^{z,r}(1) = \delta(1)$.
- $\text{Image}(\delta^{z,r}) \cap \Sigma = \delta^{z,r}(0)$.
- $\delta^{z,1}(t) = b^z(0,t)$ for $t \in [0,\varepsilon]$, where $\varepsilon > 0$ is a small constant.
- $\text{Image}(\delta^{z,1}_{|(\varepsilon,1]}) \cap \text{Image}(b^z) = \emptyset$.

The first condition implies that the parameter space can be thought as a disk. Therefore, we can apply Fuchs-Tabachnikov Theorem 2.2.3 to further ensure that

- $\delta^{z,r}$ is Legendrian.

Apply the Contact Isotopy Extension Theorem to the family of Legendrian arcs $\delta^{z,r}$ to find a family of contactomorphisms $\varphi^{z,r} \in \text{Cont}(M,\xi;\text{rel}\,\Sigma \cup \Delta_{\text{OT}})$, $(z,r) \in \mathbb{S}^k \times [0,1]$, such that

- $\varphi^{z,0} = \text{Id}$,
- $\varphi^{z,r} \circ \delta = \delta^{z,r}$.

We use this family of contactomorphisms to define a homotopy of bypasses,

$$b^{z,r} = (\varphi^{z,r})^{-1} \circ b^z;$$

between $b^{z,0} = b^z$ and a new family $b^{z,1}$. Observe that the Legendrian arc δ is disjoint from all the bypasses $b^{z,1}$, $z \in \mathbb{S}^k$, away from the attaching region. Therefore, we can use the arc δ together with Lemma 2.2.1 to find a bypass disk embedding B disjoint from the whole family $b^{z,1}$. Apply the Bypass in the Middle Theorem 2.1.1 to conclude that $b^{z,1}$ is a contractible family of bypasses. □

2.4.2 Proof of Corollary 2.1.5

The map ev_0 is a surjective Serre fibration, so we must show that the fiber is weakly contractible.

Fix a base is point $(p,v) \in \text{CFr}(M \setminus \Delta_{\text{OT}}, \xi)$ the fiber is $\text{Emb}^{(p,v)}_{\text{OT}}(\mathbb{D}^2, (M \setminus \Delta_{\text{OT}}, \xi))$, which is the space of embeddings $e \in \text{Emb}_{\text{OT}}(\mathbb{D}^2, (M \setminus \Delta_{\text{OT}}, \xi))$ such that $ev_0(e) = (e(0), d_0e(e_1,e_2)) = (p,v)$.

By Giroux realization Theorem 1.1.6 the space $\text{Emb}^{(p,v)}_{\text{OT}}(\mathbb{D}^2, (M \setminus \Delta_{\text{OT}}, \xi))$ is weakly homotopy equivalent to the space of convex embeddings $e : \mathbb{D}^2 \subset \mathbb{R}^2(x,y) \to (M \setminus \Delta_{\text{OT}}, \xi)$ such that

(i) $ev_0(e) = (p, v)$,

(ii) $e_{|\mathbb{D}^2\cap\{x\leq 0\}}$ and $e_{|\mathbb{D}^2\cap\{x\geq 0\}}$ are bypass disk embeddings.

Denote this space as $\mathrm{DB}_{(p,v)}(M \setminus \Delta_{\mathrm{OT}}, \xi)$.

We will show that $\mathrm{DB}_{(p,v)}(M \setminus \Delta_{\mathrm{OT}}, \xi)$ is contractible. Let $E : \mathbb{S}^k \to \mathrm{DB}_{(p,v)}(M \setminus \Delta_{\mathrm{OT}}, \xi)$ be any sphere. Denote the Legendrian embeddings given by the restriction of each $E(z)$ to the Y-axis by $\gamma^z = E(z)_{|\mathbb{D}^2\cap\{x=0\}} : [-1, 1] \to (M \setminus \Delta_{\mathrm{OT}}, \xi)$, $y \mapsto \gamma^z(y) = E(z)(0, y)$.

Condition (i) above implies that, after a possible contact isotopy, we can assume that

(a) $\gamma^z \equiv \gamma$ is constant map and,

(b) $E(z)_{|\mathbb{D}^2\cap\{-\varepsilon\leq x\leq\varepsilon\}}$ is constant for some small $\varepsilon > 0$.

Our goal is to apply the Bypass in the Middle Theorem 2.1.1 to both families $E(z)_{|\mathbb{D}^2\cap\{x\leq 0\}}$ and $E(z)_{|\mathbb{D}^2\cap\{x\geq 0\}}$ of bypasses to conclude.

First, we observe that both families of bypasses are attached to the same convex rectangle with Legendrian boundary. This rectangle is obtained by slightly enlarging γ and pushing it forward and backwards in the direction of the contact vector field transverse to $E(z)$. We denote such a rectangle embedding by $R : [-1, 1] \times [-2, 2] \to (M \setminus \Delta_{\mathrm{OT}}, \xi)$. It satisfies the following properties such that

- $\mathrm{Image}(R) \cap \mathrm{Image}(E(z)) = \mathrm{Image}(\gamma)$ and $R(0, y) = \gamma(y)$ for $-1 \leq y \leq 1$,
- For each $z_0 \in [-1, 1]$, the horizontal segment $H_{z_0} = \{R(z_0, y) : -2 \leq y \leq 2\}$ is a Legendrian ruling.
- For each $y_0 \in \{-2, -\frac{1}{2}, \frac{1}{2}, 2\}$, the vertical segment $V_{y_0} = \{R(z, y_0) : -1 \leq z \leq 1\}$ is a Legendrian divide.
- The dividing set Γ_R of R is given by the vertical segments $V_{y_0} = \{R(z, y_0) : -1 \leq z \leq 1\}$, where $y_0 \in \{-1, 0, 1\}$.

Observe that $E(z)_{|\mathbb{D}^2\cap\{x\geq 0\}}$, $z \in \mathbb{S}^k$, is a family of bypasses for R with attaching arc γ attached from above. On the other hand, $E(z)_{|\mathbb{D}^2\cap\{x\leq 0\}}$, $z \in \mathbb{S}^k$, is also a family of bypasses for R with attaching arc γ but attached from below.

We apply Corollary 2.1.4 to the family of bypass embeddings $E(z)_{|\mathbb{D}^2\cap\{x\geq 0\}}$ combined with the Contact Isotopy Extension Theorem to find a homotopy $E^r : \mathbb{S}^k \to \mathrm{DB}_{(p,v)}(M \setminus \Delta_{\mathrm{OT}}, \xi)$, $r \in [0, 1]$, so that $E^0 = E$ and

$$E^1(z)_{|\mathbb{D}^2\cap\{x\geq 0\}} =: \tilde{b}$$

is a constant bypass.

It remains to fix the family $E^1(z)_{|\mathbb{D}^2\cap\{x\leq 0\}}$, $z \in \mathbb{S}^k$. The argument is completely analogous to the one just explained since the process can be done relative to $\tilde{b}$. This concludes the proof. □

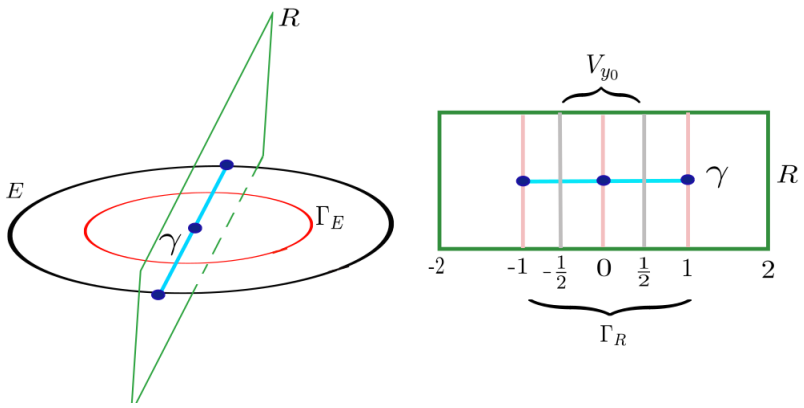


Figure 2.7: Bypasses $E(z)_{|\mathbb{D}^2\cap\{x\geq 0\}}$ and $E(z)_{|\mathbb{D}^2\cap\{x\leq 0\}}$ with attaching arc γ.

2.4.3 Proof of Theorem 2.1.6

We prove the result for the positive stabilization map S^t_+ since for the negative stabilization map S^t_- the proof is the same.

The surjectivity of the map $\pi_0(S^t_+)$ follows from Lemma 2.2.1. Indeed, given $\gamma \in \mathcal{L}(M \setminus \Delta_{\text{OT}}, \xi)$ we can use the Lemma to find a bypass attached to γ to destabilize γ. This is just the well-known fact that the connected sum of a Legendrian and the boundary of an overtwisted disk disjoint form it is a destabilization of the initial Legendrian.

To conclude we will check that for all $k \geq 1$ and every pair of maps $\Phi : \mathbb{D}^k \to \mathcal{L}(M \setminus \Delta_{\text{OT}})$ and $\partial\tilde{\Phi} : \partial\mathbb{D}^k \to \mathcal{L}(M \setminus \Delta_{\text{OT}})$, such that $\Phi_{|\partial\mathbb{D}^k} = S^t_+ \circ \partial\tilde{\Phi}$; there exists, after a possible homotopy of Φ relative to $\partial\mathbb{D}^k$, a map $\tilde{\Phi} : \mathbb{D}^k \to \mathcal{L}(M \setminus \Delta_{\text{OT}})$ such that

(i) $\tilde{\Phi}_{|\partial\mathbb{D}^k} = \partial\tilde{\Phi}$,

(ii) $S^t_+ \circ \tilde{\Phi} = \Phi$.

The assumption $\Phi_{|\partial\mathbb{D}^k} = S^t_+ \circ \partial\tilde{\Phi}$ is equivalent to the existence of a family of bypass embeddings $B : \partial\mathbb{D}^k \to \mathcal{B}(M \setminus \Delta_{\text{OT}}, \xi)$ such that

1. The attaching arc of $B(z)$ lies in $\Phi(z)$ for all $z \in \partial\mathbb{D}^k$
2. The bypass $B(z)$ and the Legendrian $\Phi(z)$ intersectect only at the attaching arc of $B(z)$ for all $z \in \partial\mathbb{D}^k$.
3. Replacing the attaching arc of $B(z)$ in $\Phi(z)$ by the Honda arc of $B(z)$, gives (after smoothing with the Lemma 1.1.7) the Legendrian $\tilde{\Phi}(z)$.

Moreover, the existence of the map $\tilde{\Phi}$ will follow from the existence of an extension $\tilde{B} : \mathbb{D}^k \to \mathcal{B}(M \setminus \Delta_{\text{OT}}, \xi)$ of the map B, i.e. $\tilde{B}_{|\partial\mathbb{D}^k} = B$; that satisfies properties (1) and (2) above for all $z \in \mathbb{D}^k$.

Fix coordinates $(z, r) \in \mathbb{D}^k = \partial\mathbb{D}^k \times [0,1]/\simeq$ in the disk $\mathbb{D}^k$. The Contact Isotopy Extension Theorem allows to find a family of contactomorphisms $\varphi : \mathbb{D}^k \to \mathrm{Cont}(M \setminus \Delta_{\mathrm{OT}}, \xi)$ such that

- $\varphi(z, 0) = \mathrm{Id}$,
- $\varphi(z, r) \circ \Phi(z, 0) = \Phi(z, r)$ for $(z, r) \in \partial\mathbb{D}^k \times [0, 1]$.

Define the family of bypasses $\hat{B} : \partial\mathbb{D}^k \times [0,1] \to \mathcal{B}(M \setminus \Delta_{\mathrm{OT}}, \xi)$ via the expression

$$\hat{B}(z, r) = \varphi(z, r) \circ \varphi^{-1}(z, 1) \circ B(z, 1).$$

This family extends the initial family of bypasses B, i.e. $\hat{B}_{|\partial\mathbb{D}^k \times \{1\}} = B$; and satisfies properties (1) and (2) above. However, in general, the map $\hat{B}_{|\partial\mathbb{D}^k \times \{0\}}$ is not constant as required and defines a $\partial\mathbb{D}^k$-family of bypasses, with fixed attaching arc over the Legendrian $\Phi(z, 0)$. Nevertheless, since all these bypasses avoid a fixed overtiwsted disk, we can invoke Corollary 2.1.4 to contract this family to a fixed bypass and, therefore, define the required map $\tilde{B} : \mathbb{D}^k \to \mathcal{B}(M \setminus \Delta_{\mathrm{OT}}, \xi)$. This concludes the argument. □

2.4.4 Proof of Corollary 2.1.7

This essentially follows from Theorem 2.1.6 combined with the Fuchs-Tabachnikov theorem 2.2.3. We must show that the maps

$$\pi_k(i) : \pi_k(\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi)) \to \pi_k(F\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}))$$

induced by the natural inclusion $i : \mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi) \hookrightarrow F\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi)$ are injective and surjective for all k.

We check first the injectivity property. Let $A \in \ker(\pi_k(i))$ be an element of the kernel. Then, Theorem 2.2.3 implies that there exists times $t_1^+, t_1^-, t_2^+, t_2^-, \ldots, t_n^+, t_n^- \in \mathbb{S}^1$ so that

$$A \in \ker \pi_k(S_-^{t_n^-} \circ S_+^{t_n^+} \circ \cdots S_-^{t_1^-} \circ S_+^{t_1^+}).$$

By Theorem 2.1.6 we conclude $A = 0$.

Let us prove that the map is surjective. Let $A \in \pi_k(F\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi))$ be a formal class. Then, by Theorem 2.2.3 there exists another formal class $\hat{A} \in \pi_k(F\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi))$, where the homotopy group has a different base point, obtained from A by several stabilizations such that

$$\hat{A} = \pi_k(i)(\hat{B})$$

for some $\hat{B} \in \pi_k(\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi))$. Let $B \in \pi_k(\mathcal{L}(M \setminus \Delta_{\mathrm{OT}}, \xi))$ be the unique Legendrian class satisfying

$$\hat{B} = \pi_k(S_-^{t_n^-} \circ S_+^{t_n^+} \circ \cdots S_-^{t_1^-} \circ S_+^{t_1^+})(B),$$

that exists because of Theorem 2.1.6. It remains to check that $\pi_k(i)(B) = A$. The only subtlety is that, a priori, we could lose control of the formal class during the previous process. However, we claim that this is not the case. Following [5], the class $\hat{A}$ is obtained first by making the formal Legendrians (of a representative) of A Legendrian near many points of the domain and adding several Legendrian stabilizations. Hence the claim follows from the following observation.

Lemma 2.4.1. *Let K be a compact parameter space, $\varepsilon > 0$ and $\delta^k : [-1-\varepsilon, 1+\varepsilon] \to (M, \xi)$, $k \in K$, a family of Legendrian arcs. Assume that there exists two families of bypass embeddings $b_i^k \in \mathcal{B}(M, \xi)$, $(i, k) \in \{0, 1\} \times K$, so that*

- *The attaching arc of b_i^k is $\delta^k([-1, 1])$ for all $(i, k) \in \{0, 1\} \times K$,*
- *b_0^k and b_1^k coincide near the attaching arc for all $k \in K$, and*
- *$\text{Image}(b_i^k) \cap \text{Image}(\delta^k) = \delta^k([-1, 1])$ for all $(i, k) \in \{0, 1\} \times K$.*

Then, the families of Legendrian arcs β_0^k and β_1^k obtained as a destabilization of the family δ^k by means of b_0^k and b_1^k, respectively, are formally Legendrian isotopic.

Proof. The existence of the two families of bypass embeddings allows to reduce the statement to the non-parametric case and to a precise local model. In the latter the result is obvious. □

After this Lemma $\pi_k(i)(B) = A$ and, therefore, the map $\pi_k(i)$ is surjective. This concludes the argument. □

Chapter 3

Convex disks with Legendrian boundary in overtwisted contact 3-manifolds.

3.1 Main result

Let (M, ξ) be a cooriented overtwisted contact 3-manifold and $\mathbb{D}^2 \subseteq (M, \xi)$ an embedded convex 2-disk with Legendrian boundary and characteristic foliation $\mathcal{F}$ determined by $T\mathbb{D}^2 \cap \xi$. We will say that the characteristic foliation $\mathcal{F}$ is tight, resp. overtwisted, whether the germ of contact structures in the neighborhoods of the disk is tight, resp. overtwisted. We consider the *space of $\mathcal{F}$-embeddings* as the space of all smooth embeddings of $\mathbb{D}^2 \hookrightarrow (M, \xi)$ that induce the characteristic foliation $\mathcal{F}$ on $\mathbb{D}^2$, we also require all the embeddings to agree in a neighborhood of the Legendrian boundary $\partial\mathbb{D}^2$. This space is denoted by $\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$. The space of *formal $\mathcal{F}$-embeddings,* as defined in 1.2.2, is denoted by $\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$. As before, we require our formal embeddings to coincide near the boundary.

Recall that for a null-homologous oriented Legendrian knot $\Lambda \subseteq (M, \xi)$ with Seifer surface Σ there are two classical invariants, namely the Thurston-Bennequin number $\mathrm{tb}(\Lambda)$ and the rotation number $\mathrm{rot}(\Lambda, [\Sigma])$. Although the Thurston-Bennequin invariant is independent of choices the rotation number could depend on the homology class of the Seifert surface $[\Sigma] \in H_2(M, \Lambda)$ whenever the Euler class $e(\xi) \in H_2(M)$ is non-zero. We refer the reader to [36] for a careful discussion on this. In the cases in which the choice of Seifert surface is clear from the context, such as when considering the Legendrian boundary $\partial\Sigma$ of a surface Σ, we will simply write $\mathrm{rot}(\Lambda)$.

Theorem 3.1.1. *Let (M, ξ) be an overtwisted contact 3-manifold and $\mathbb{D}^2 \subseteq (M, \xi)$ an embedded convex disk with Legendrian boundary and characteristic foliation $\mathcal{F}$. Assume that either $\mathcal{F}$ is tight or $\mathrm{tb}(\partial\mathbb{D}^2) + |\mathrm{rot}(\partial\mathbb{D}^2)| > -1$. Then, the inclusion*

$$\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial) \hookrightarrow \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial) \tag{3.1}$$

induces an isomorphism at the level of path-connected components. Moreover, the first relative homotopy

group is trivial:

$$\pi_1(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial), \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)) = 0.$$

Remark 3.1.2. Theorem 3.1.1 cannot be proved directly from Eliashberg's h-principle [17] as the following example shows. Consider two disk embeddings $e_0, e_1 \in \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi_{\mathrm{std}}), \mathrm{rel}\,\partial)$ in the standard tight 3-sphere that cobound an embedded pinched 3-disk (D, ξ_{std}). Perform contact connected sum $(\mathbb{S}^3, \xi_{\mathrm{std}})\#(\mathbb{S}^3, \xi_0) := (\mathbb{S}^3, \xi)$, where the connected sum region lies inside (D, ξ_{std}), and $(\mathbb{S}^3, \xi_0)$ is an overtwisted contact structure in the formal class of the standard one.

We think of e_0 and e_1 as embeddings in $(\mathbb{S}^3, \xi)$. Notice that now they cobound an overtwisted pinched 3-disk (D, ξ) with tight complement. The schematic of the situation is described in Figure 3.1. It is easy to see that there exists a formal isotopy e_t, $t \in [0, 1]$, between both disks. However, there is no overtwisted disk $\Delta \subseteq (\mathbb{S}^3, \xi)$ so that e_0 and e_1 are even smoothly isotopic relative to it. Therefore, Eliashberg's result cannot be directly applied. Nevertheless, Theorem 3.1.1 implies that there exists a contactomorphism $\varphi : (\mathbb{S}^3, \xi) \to (\mathbb{S}^3, \xi)$, contact isotopic to the identity relative to the boundary of the disks so that $\varphi \circ e_0 = e_1$, leading to the non-obvious observation that the pinched 3-disks (D, ξ) and $(\varphi(D), \xi)$, schematically depicted in Figure 3.1, are contact isotopic relative to the equator.

To overcome the described situation we will use a version of Eliashberg's h-principle in which the overtwisted disk is allowed to vary (Theorem 3.5.5 and Corollary 3.5.6). This will require the creation of an isotopy of overtwisted disks Δ_t, $t \in [0, 1]$, so that $\mathrm{Image}(e_t) \cap \Delta_t = \emptyset$ for all $t \in [0, 1]$. The existence of such an isotopy in general will be the heart of the proof.

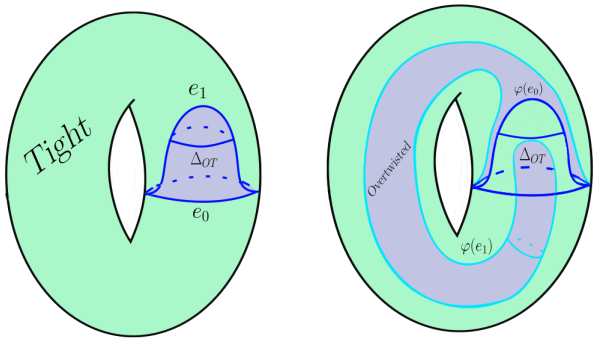


Figure 3.1: (a). Schematic of the two disks e_0 and e_1. The situation is described in the solid torus complementary to the boundary of the disks. The pinched 3-disk (D, ξ) is depicted in gray, the tight region is depicted in green. (b) Effect of applying the contactomorphism φ to the pinched 3-disk (D, ξ).

Remark 3.1.3. The work of the second author on strongly overtwisted contact 3-manifolds addresses the cases in which $\mathrm{tb}(\partial\mathbb{D}^2) + |\mathrm{rot}(\partial\mathbb{D}^2)| > -1$ or (M, ξ) is strongly overtwisted [30]. Even more, in these cases the inclusion (3.1) is a weak homotopy equivalence. In this chapter we will

address the case in which $\mathrm{tb}(\partial\mathbb{D}^2) + |\mathrm{rot}(\partial\mathbb{D}^2)| \leq -1$ and the characteristic foliation $\mathcal{F}$ is tight. However, our method works in both cases and is independent of [30].

Remark 3.1.4. In the case in which $\mathrm{tb}(\partial\mathbb{D}^2) + |\mathrm{rot}(\partial\mathbb{D}^2)| \leq -1$ and $\mathcal{F}$ is overtwisted the result is not true in general. For instance, in an overtwisted contact 3-manifold with disconnected space of overtwisted disks one could take a disk with tight characteristic foliation and use two transverse arcs to connect sum it with two non-isotopic overtwisted disks. Hence obtaining two disks with overtwisted characteristic foliation that are not contact isotopic and can be assumed formally isotopic. However, under the stronger assumption that the space of overtwisted disks is connected the same conclusion of Theorem 3.1.1 holds also in this case. Notice, that this assumption holds for every overtwisted contact 3-sphere but one [60] and every strongly overtwisted contact 3-manifold [30]. It is an open question if there exists an overtwisted contact 3-manifold (M, ξ) with $M \neq \mathbb{S}^3$ for which the space of overtwisted disks is disconnected.

3.2 Legendrian unknots

Our main motivation to study spaces of convex disks with Legendrian boundary was to understand spaces of Legendrian unknots in overtwisted contact 3-manifolds. Let $\Lambda \subseteq (M, \xi)$ be an embedded Legendrian unknot, and fix an inclusion $i : \mathbb{S}^1 \cong \Lambda \hookrightarrow (M, \xi)$. We will denote by $\mathcal{L}(\Lambda, (M, \xi))$ the path-connected component of the inclusion $\Lambda \hookrightarrow (M, \xi)$ in the space of Legendrian embeddings. Similarly, we will denote by $\mathcal{L}_{(p,v)}(\Lambda, (M, \xi)) \subseteq \mathcal{L}(\Lambda, (M, \xi))$ the subspace of *long* Legendrian unknots, i.e. Legendrian embeddings $\gamma \in \mathcal{L}(\Lambda, (M, \xi))$ so that $(\gamma(0), \gamma'(0)) = (i(0), i'(0)) =: (p, v)$.

Assume that $\Lambda \subseteq (\mathbb{D}^3, \xi_{\mathrm{std}}) \subseteq (M, \xi)$ lies in a Darboux ball so we may find an embedded convex disk $\mathbb{D}^2 \subseteq (\mathbb{D}^3, \xi_{\mathrm{std}}) \subseteq (M, \xi)$ with tight characeristic foliation $\mathcal{F}$ such that $\partial\mathbb{D}^2 = \Lambda$. Then, it was observed by Fernández, Martínez-Aguinaga and Presas in [31] that there is a weak homotopy equivalence

$$\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial) \cong \Omega\mathcal{L}_{(p,v)}(\Lambda, (M, \xi)).$$

In particular, Theorem 3.1.1 allows us to compute the fundamental group of the space of Legendrian unknots that lie in a Darboux ball in every overtwisted contact 3-manifold.

Recall that a formal Legendrian embedding is a pair (γ, F_s) such that

- $\gamma : \mathbb{S}^1 \hookrightarrow M$ is a smooth embedding and
- $F_s : T\mathbb{S}^1 \to \gamma^* TM$, $s \in [0, 1]$, is a homotopy of bundle monomorphisms such that $F_0 = d\gamma$ and $\mathrm{Image}(F_1) \subseteq \gamma^*\xi$.

Notice that every Legendrian γ is naturally a formal Legendrian just by taking $F_s \equiv d\gamma$ the constant homotopy. We will denote by $F\mathcal{L}(\Lambda, (M, \xi))$ the path-connected component of Λ in the space of formal Legendrians.

Corollary 3.2.1. *Let $\Lambda \subseteq (M, \xi)$ be an embedded Legendrian unknot in an overtwisted contact 3-manifold. Assume that Λ lies in some Darboux ball. Then, the inclusion*

$$\mathcal{L}(\Lambda, (M, \xi)) \hookrightarrow F\mathcal{L}(\Lambda, (M, \xi)) \tag{3.2}$$

induces an isomorphism at the level of fundamental groups.

In the case in which the underlying 3-manifold is the 3-sphere we precisely describe the fundamental group of these Legendrian embedding spaces.

Corollary 3.2.2. *Let $\Lambda \subseteq (\mathbb{S}^3, \xi)$ be an embedded Legendrian unknot is some overtwisted contact 3-sphere that lies in some Darboux ball. Then, there is an isomorphism*

$$\pi_1(\mathcal{L}(\Lambda, (\mathbb{S}^3, \xi)) \cong \mathbb{Z} \oplus \mathbb{Z},$$

in which one $\mathbb{Z}$-factor is generated by a full-rotations of Λ in a Darboux ball and the other $\mathbb{Z}$-factor is generated by a loop of long Legendrian unknots which are contractible as loops of smooth long unknots.

Remark 3.2.3. The situation stands in sharp contrast with the one for the tight contact 3-sphere described in [31]. For instance, there does not exists a non-contractible loop of long Legendrian unknots with tb = -1 which is contractible as a loop of long smooth unknots in the standard tight 3-sphere.

3.3 The contact mapping class group of a Legendrian unknot with negative tb in an overtwisted 3-sphere

Denote by $\mathrm{CStr}_{\mathrm{OT}}(M)$ the space of overtwisted contact structures on a given 3-manifold M. Recall that as a consequence of Eliashberg's h-principle there are isotopy classes of overtwisted contact structures on M are in bijection with homotopy classes of plane fields on M [17]. To label them in $\mathbb{S}^3$ we fix the trivialization of $T\mathbb{S}^3$ given by the quaternions. Then, every contact structure ξ on $\mathbb{S}^3$ defines a Gauss map $f_\xi : \mathbb{S}^3 \to \mathbb{S}^2$, yielding to a map

$$\mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3) \to \mathrm{Maps}(\mathbb{S}^3, \mathbb{S}^2)$$

that induces an isomorphism at the level of path-connected components. In particular, $\pi_0(\mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3)) \cong \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, where the isomorphism is given by associating to each overtwisted contact structure ξ the Hopf invariant $h(f_\xi) \in \mathbb{Z}$ of the associated Gauss map. We fix a list of representatives $\{\xi_k : k \in \mathbb{Z}, h(f_{\xi_k}) = k\} \subseteq \mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3)$ of these path-connected components.

Finally, given a Legendrian $\Lambda \subseteq (\mathbb{S}^3, \xi)$ we will always denote by $(\mathcal{O}p(\Lambda), \xi) := (\mathcal{O}p(\Lambda), \xi_{|\mathcal{O}p(\Lambda)}) \subseteq (\mathbb{S}^3, \xi)$ a standard neighborhood of Λ, which is unique up to contact isotopy fixing Λ. We will define the Legendrian complement of Λ as $(C(\Lambda), \xi) := (\mathbb{S}^3 \backslash \mathcal{O}p(\Lambda), \xi_{|\mathbb{S}^3 \backslash \mathcal{O}p(\Lambda)})$.

Recall that a *formal contactomorphism* of a contact manifold (M, ξ) is a pair (φ, F_s), where $\varphi \in \mathrm{Diff}(M)$ and $F_s \in \mathrm{Iso}(TM, \varphi^* TM)$, $s \in [0, 1]$, is a homotopy of bundle isomorphisms such that

$F_0 = d\varphi$ and $F_1(\xi) = \varphi^*\xi$. The group of formal contactomorphisms is denoted by $\mathrm{FCont}(M, \xi)$. A contactomorphism is a formal contactomorphism (φ, F_s) such that $F_s \equiv d\varphi$. We will denote the group of contactomorphisms by $\mathrm{Cont}(M, \xi) \subseteq \mathrm{FCont}(M, \xi)$ We will always assume that our formal contactomorphisms are the indentity near the possibly empty boundary ∂M.

Combining the Theorem 3.1.1 with Chekanov and Vogel theorem [60] about the contact mapping class group of an overtwisted sphere we conclude that

Corollary 3.3.1. *Let $\xi \in \mathrm{CStr}_{\mathrm{OT}}(\mathbb{S}^3)$ be some overtwisted contact structure on $\mathbb{S}^3$ and $\Lambda \subseteq (\mathbb{S}^3, \xi)$ a Legendrian unknot with $\mathrm{tb}(\Lambda) \leq 0$. Consider the inclusion*

$$i : \mathrm{Cont}(C(\Lambda), \xi) \hookrightarrow \mathrm{FCont}(C(\Lambda), \xi). \tag{3.3}$$

Then,

(a) If $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| > -1$ then the induced map $\pi_0(i)$ is an isomorphism. In particular, we have that $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

(b) If $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| \leq -1$ then

(i) If $\xi \ncong \xi_{-1}$ then the induced map $\pi_0(i)$ is an isomorphism. In particular, $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

(ii) If $\xi \cong \xi_{-1}$ then the induced map $\pi_0(i)$ is surjective but has order 2 kernel. In particular, $\pi_0(\mathrm{Cont}(C(\Lambda), \xi)) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Remark 3.3.2. Case (a) in the Theorem above also follows from the work [30] of the second author, in which is shown that the inclusion i is a weak homotopy equivalence in this case. Our proof is independent of this.

Remark 3.3.3. The "exotic" $\mathbb{Z}_2$-factor appearing in case (b)(ii) is the one found by Chekanov and Vogel, independently [60]. The fact that this exotic factor should appear is obvious since every contactomorphism of a connected contact 3-manifold can be assumed to fix a Darboux ball and the Legendrian unknot Λ lies in some Darboux ball by the assumption (b). The actual content of (b)(ii) is that this is the only exotic factor that appears in the complement of such a Legendrian.

3.4 Basics

In this second project, it will be necessary to define more specific concepts related to bypasses. Additionally, we will present the results required to address the objectives outlined in this chapter.

The following theorem on isotopy discretization provides a method for connecting two isotopic convex surfaces in a tight contact manifold through a sequence of intermediate surfaces, each related by a simple operation, known as bypass attachment. This result is crucial for understanding the behavior of convex surfaces under isotopies and for constructing explicit paths between them.

Theorem 3.4.1 (Isotopy Discretization [10, 48]). *Let (M, ξ) be a tight contact manifold, and let Σ and Σ' be convex surfaces in M with the same (posibly empty) Legendrian boundary. If they are isotopic (relative to the boundary) then there is a sequence of convex surfaces $\Sigma_0 = \Sigma, \Sigma_1, \ldots, \Sigma_n = \Sigma'$ such that*

- *Σ_i and Σ_{i+1} cobounder a $\Sigma \times I$ in M, and*
- *Σ_i and Σ_{i+1} are related by a single bypass attachment.*

Definition 3.4.2. Let $(\Sigma \times I, \xi_{\text{inv}})$ be an I-invariant contact structure and $\gamma \subseteq \Sigma$ be a Legendrian arc intersecting the dividing set in 3 points. We call a *bypass triangle attachment along γ* the composition of three bypasses with attaching arcs given by γ, γ_1 and γ_2, as in Figure 3.2.

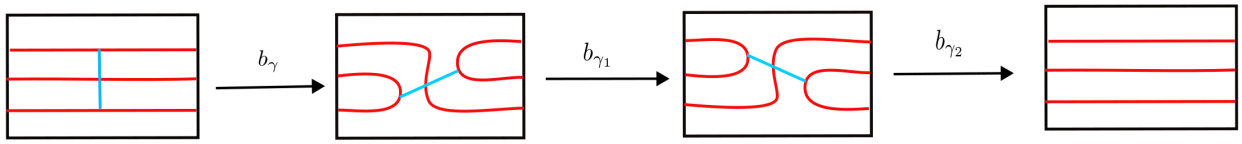


Figure 3.2: Bypass triangle

We will denote by $\xi_{BT}(\gamma)$ the contact structure on $\Sigma \times I$ obtained after attaching a bypass triangle along γ from above at $\Sigma \times \{1\}$. In the cases in which the choice of attaching arc is clear from the context or not relevant we will drop γ from the notation.

3.4.1 The formal class of a bypass triangle

Consider the 3-contact manifold $(\mathbb{D}^2 \times I, \xi_{\text{inv}})$ with $\xi_{\text{inv}} = \ker(f\,dt + \beta)$, where $t \in I = [0, 1]$, $f \in C^\infty(\mathbb{D}^2)$ and $\beta \in \Omega^1(\mathbb{D}^2)$. This contact structure induces a trivialization of $T(\mathbb{D}^2 \times I)$, which defines a constant Gauss function $G_{\xi_{\text{inv}}} : \mathbb{D}^2 \times I \to \mathbb{S}^2$.

Consider another plane field ξ on $\mathbb{D}^2 \times I$ such that $\xi(p) = \xi_{\text{inv}}(p)$ for all $p \in \mathcal{O}p(\partial(\mathbb{D}^2 \times I))$ and the associated Gauss map $G_\xi : \mathbb{D}^2 \times I \to \mathbb{S}^2$. We will denote by $h(\xi) \in \mathbb{Z}$ the *Hopf invariant* of the map G_ξ. If we denote by $\text{FCStr}(\mathbb{D}^2 \times I, \partial)$ the space of formal contact structures (plane fields) on $\mathbb{D}^2 \times I$ that coincide with ξ_{inv} near the boundary, then there is a bijection

$$\pi_0(\text{FCStr}(\mathbb{D}^2 \times I, \partial)) \to \mathbb{Z}, \xi \mapsto h(\xi);$$

such that $h(\xi_{\text{inv}}) = 0$.

A key result that we will use is the following

Theorem 3.4.3 (Huang [53]). *Consider the contact manifold $(\mathbb{D}^2 \times I, \xi_{BT})$ obtained by attaching a bypass triangle from above, then $h(\xi_{BT}) = -1$.*

3.5 Overtwisted h-principles

We will begin this section by recalling the overtwisted h-principle proved by Eliashberg in [17] and several consequences of it. We will then derive the h-principle with non-fixed overtwisted disk, that we call loose overtwisted h-principle. This is the content of Theorem 3.5.5 and Corollary 3.5.6 and will be crucial in our proof of Theorem 3.1.1.

3.5.1 Overtwisted h-principle

Given a 3-manifold we will denote by $\mathrm{FCStr}(M)$ the space of formal contact structures on M and by $\mathrm{CStr}(M) \subseteq \mathrm{FCStr}(M)$ the subspace of contact structures. In the case that ∂M is non-empty we will assume that all our formal contact structures agree near ∂M. Given a subset $G \subseteq M$ equipped with a germ of contact structure we will denote by $\mathrm{FCStr}(M, \mathrm{rel}\, G) \subseteq \mathrm{FCStr}(M)$ the subspace of formal contact structures that induced the given germ over G, and we define the corresponding space of contact structures as $\mathrm{CStr}(M, \mathrm{rel}\, G) = \mathrm{FCStr}(M, \mathrm{rel}\, G) \cap \mathrm{CStr}(M)$.

Theorem 3.5.1 (Eliashberg [17])**.** *Let M be a closed 3-manifold and $\mathbb{D}^2_{\mathrm{OT}} \subset M$ an embedded 2-disk equipped with the germ of an overtwisted disk. Then, the natural inclusion*

$$i : \mathrm{CStr}(M, \mathrm{rel}\, \mathbb{D}^2_{OT}) \hookrightarrow \mathrm{FCStr}(M, \mathrm{rel}\, \mathbb{D}^2_{OT})$$

is a weak homotopy equivalence.

For contactomorphisms we have the analogous result as observed by Dymara in [14] for the case of the 3-sphere and in general by Fernández in [30]

Theorem 3.5.2. *Let (M, ξ) be an overtwisted contact 3-manifold and $\Delta_{\mathrm{OT}} \subseteq (M, \xi)$ an overtwisted disk. Then, the inclusion* $\mathrm{Cont}(M, \xi, \mathrm{rel}\, \Delta_{\mathrm{OT}}) \hookrightarrow \mathrm{FCont}(M, \xi, \mathrm{rel}\, \Delta_{\mathrm{OT}})$ *is a weak homotopy equivalence.*

A smooth embedding $e : \mathbb{D}^2 \to (M, \xi)$ of an overtwisted disk is referred to as an *overtwisted disk embedding* if the characteristic foliation it induces coincides with that of the overtwisted disk illustrated in Figure 1.4. The *space of overtwisted disk embeddings*, denoted by $\mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$, plays a particularly significant role in studying the homotopy type of the contactomorphism group of (M, ξ); see [14, 30, 60] for further details.

Let $\mathrm{CFr}(M, \xi)$ denote the total space of the bundle of contact frames over the contact 3-manifold (M, ξ).

The following result can be proved by using Theorem 3.5.1. At the level of path-connected components in $\mathbb{S}^3$ this was observed by Dymara in [14]. A proof in full generality, that is independent from Theorem 3.5.1, was recently obtained by Farias-Fernández in [27] and is contained in the first part of the thesis.

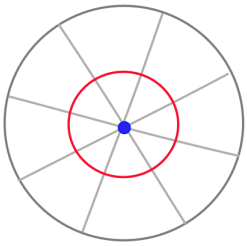


Figure 3.3: Overtwisted disk. The dividing set is depicted in red and the characteristic foliation in gray.

Theorem 3.5.3. *Let (M,ξ) be an overtwisted contact 3-manifold and $\Delta_{OT} \subseteq (M,\xi)$ an overtwisted disk. Then, the 1-jet evaluation map at the origin*

$$ev_0 : \mathrm{Emb}_{OT}(\mathbb{D}^2, (M\backslash\Delta_{OT}, \xi)) \to \mathrm{CFr}(M\backslash\Delta_{OT}, \xi)$$

is a weak homotopy equivalence. In particular, two disjoint overtwisted disks are contact isotopic.

3.5.2 Loose Overtwisted h-principle

In the following discussion we will restrict ourselves to embedding spaces for simplicity. However, the whole discussion trivially applies to germs of embeddings as the ones treated in Section 1.2.2. A *formal embedding* of a manifold N into a manifold M is a pair (e, F_s) where $e : N \to M$ is a smooth embedding and $F_s : TN \to TM$, $s \in [0,1]$, is a homotopy of bundle monomorphisms covering e such that $F_0 = de$. We denote the space of formal embeddings by $\mathrm{FEmb}(N,M)$. We will regard the space of embeddings $\mathrm{Emb}(N,M)$ as a subspace of $\mathrm{FEmb}(N,M)$. Since for a formal embedding (e, F_s), there is no restriction on F_1 (besides being a monomorphism) the natural inclusion

$$i : \mathrm{Emb}(N,M) \hookrightarrow \mathrm{FEmb}(N,M) \tag{3.4}$$

is a weak homotopy equivalence.

Given a subgroup $G \subseteq \mathrm{Diff}(M)$, there is a natural action of G on $\mathrm{FEmb}(N,M)$ defined via post-composition, that preserves $\mathrm{Emb}(N,M)$ so it induces an action on this subspace. Specifically, the action is defined as $G \times \mathrm{FEmb}(N,M) \to \mathrm{FEmb}(N,M), (g,(e,F_s)) \mapsto g\cdot(e,F_s) := (g\circ e, dg\circ F_s)$. A subspace $X \subseteq \mathrm{FEmb}(N,M)$ is said to be *G-invariant* if $G\cdot X = X$.

Definition 3.5.4. Let (M,ξ) be an overtwisted contact 3-manifold and K a parameter space. A family $(e^k, F_s^k) \in \mathrm{FEmb}(N,M)$, $k \in K$, is said to be a *loose family* if there exists a family $d^k \in \mathrm{Emb}_{OT}(\mathbb{D}^2,(M,\xi))$ such that $e^k(N) \cap d^k(\mathbb{D}^2) = \emptyset$. In this case, we also say that (e^k, F_s^k) is *loose with respect to d^k*.

Let (M,ξ) be an overtwisted contact 3-manifold with a fixed overtwisted disk embedding $d \in \mathrm{Emb}_{OT}(\mathbb{D}^2,(M,\xi))$. For a subspace $X \subseteq \mathrm{FEmb}(N,M)$ we denote by $X_d \subseteq X$ the subspace consisting of embeddings that are loose with respect to d.

Finally, given a subspace $X \subseteq \mathrm{FEmb}(N, M)$, the space of *loose pairs* is defined as

$$\mathrm{Loose}\, X = \{(e, d) : d \in \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi)), e \in X_d\}.$$

Notice that if $X \subseteq Y \subseteq \mathrm{FEmb}(N, M)$, then there is a natural inclusion

$$i : \mathrm{Loose}\, X \hookrightarrow \mathrm{Loose}\, Y.$$

Theorem 3.5.5 (Loose Overtwisted h-principle)**.**
Let (M, ξ) be an overtwisted contact 3-manifold. Consider two subspaces $X \subseteq Y \subseteq \mathrm{FEmb}(N, M)$ such that

(i) For every overtwisted disk $d \in \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$ the inclusion $i : X_d \hookrightarrow Y_d$ is a weak homotopy equivalence and

(ii) X and Y are $\mathrm{Cont}(M, \xi)$-invariant.

Then, the inclusion

$$i : \mathrm{Loose}\, X \hookrightarrow \mathrm{Loose}\, Y$$

is a weak homotopy equivalence.

Proof. Applying the Contact Isotopy Extension Theorem, together with condition (ii), we obtain that the projection $\mathrm{Loose}\, X \to \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$ is a fibration with fiber X_d, where $d \in \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$. Similarly, the projection $\mathrm{Loose}\, Y \to \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi))$ is a fibration with fiber Y_d.

Therefore, consider the following commutative diagram

$$\begin{array}{ccccc} X_d & \hookrightarrow & \mathrm{Loose}\, X & \longrightarrow & \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi)) \\ \downarrow & & \downarrow & & \downarrow \\ Y_d & \hookrightarrow & \mathrm{Loose}\, Y & \longrightarrow & \mathrm{Emb}_{\mathrm{OT}}(\mathbb{D}^2, (M, \xi)) \end{array}$$

The inclusion between the fibers is a homotopy equivalence due to condition (i) of the hypothesis, while the inclusion between the bases is simply the identity map. In particular, the latter is also a homotopy equivalence. Therefore, by the Five Lemma on the long exact sequence of homotopy groups, it follows that the inclusion $\mathrm{Loose}\, X \hookrightarrow \mathrm{Loose}\, Y$ is a weak homotopy equivalence. □

3.5.3 Convex disks with overtwisted disks in the complement

We will apply Theorem 3.5.5 to the space of embeddings of convex disks with Legendrian boundary and fixed characteristic foliation $\mathcal{F}$ into an overtwisted contact 3-manifold (M, ξ) and its formal counterpart. Namely, the spaces

$$\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, \partial) \subseteq \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, \partial).$$

The following holds

Corollary 3.5.6. *The inclusion*

$$\text{Loose Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial) \hookrightarrow \text{Loose FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial)$$

is a weak homotopy equivalence.

Proof. Note that $\text{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial)$ and $\text{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial)$ are subspaces of the space of germs of formal embeddings $\text{FEmb}^{germ}(\mathbb{D}^2, M, \text{rel}\,\partial)$, see Section 1.2.2. Let $\Lambda \subseteq (M, \xi)$ be the Legendrian boundary of the disks. Our goal is to apply Theorem 3.5.5 relative to a small open set $\mathcal{O}p(\Lambda)$, where $X = \text{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial)$ and $Y = \text{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \text{rel}\,\partial)$.

Clearly X and Y are $\text{Cont}(M, \xi, \text{rel}\,\Lambda)$-invariant. Therefore, it remains to check condition (i) in Theorem 3.5.5. That is, that for every overtwisted disk embedding $d \in \text{Emb}_{\text{OT}}(\mathbb{D}^2, M)$ the inclusion $X_d \hookrightarrow Y_d$ is a weak homotopy equivalence. The proof of the latter is analogous to the one given in [30, Proposition 3.6] and the details are left to the reader. □

3.6 Creating isotopies of overtwisted disks

Let (M, ξ) be a closed overtwisted contact 3-manifold. We will consider an isotopy of smooth disks $e^t : \mathbb{D}^2 \hookrightarrow M$, $t \in [0, 1]$, all of them coinciding near the Legendrian boundary, that we assume to be discretized. That is, there exists a finite sequence of times $0 = t_0 < t_1 < \ldots < t_n = 1$ such that the image of

$$E_i : \mathbb{D}^2 \times [t_i, t_{i+1}] \to (M, \xi), \quad (p, t) \mapsto e^t(p),$$

is an embedded pinched 3-disk and E_i is a smooth embedding away from $\mathcal{O}p(\partial\mathbb{D}^2) \times [t_i, t_{i+1}]$. We define $D_i := \text{Image}(E_i)$. We will say that D_i and E_i are going upwards, resp. downwards, if E_i is orientation preserving, resp. orientation reversing. A pinched 3-disk going upwards is illustrated in Figure 3.4. We will further assume that e^{t_i} is convex for every $i \in \{0, \ldots, n\}$ with characteristic foliation $\mathcal{F}_i$.

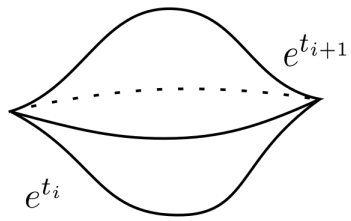


Figure 3.4: A pinched 3-disk

The goal of this Section is to find an isotopy of overtwisted disks Δ^t, $t \in [0, 1]$, such that $\Delta^t \cap \text{Image}(e^t) = \emptyset$ for all $t \in [0, 1]$. This will be key to apply Corollary 3.5.6 to prove Theorem 3.1.1

latter. We divide the argument into two cases, one in which the disks (D_i, ξ) are tight and other in which they are overtwisted. We need also to interpolate between both cases, therefore we need to find isotopies of overtwisted disks with prescribed initial conditions.

3.6.1 Tight case

First, we deal with the tight case:

Proposition 3.6.1 (Tight case)**.** *Let Δ_{OT} be an overtwisted disk in the complement of e^0. Assume that the pinched 3-disks (D_i, ξ) are tight for every $i \in \{0, \dots, n\}$. Then, there exists an isotopy Δ^t, $t \in [0,1]$, of overtwisted disks such that*

- $\Delta^0 = \Delta_{\mathrm{OT}}$ *and*
- $\Delta^t \cap \mathrm{Image}(e^t) = \emptyset$ *for all* $t \in [0,1]$.

Proof. Consider a 1-parameter family of smooth 3-disks $B^t = \mathcal{O}p(\mathrm{Image}(e^t))$, $t \in [0,1]$. We take B^0 small enough so that $B^0 \cap \Delta_{\mathrm{OT}} = \emptyset$. By the hypothesis we can assume that (B^t, ξ) is tight for every $t \in [0,1]$. In particular, the 2-spheres $S^t := \partial B^t$ are tight for every $t \in [0,1]$. Write $\Lambda = e^t(\partial \mathbb{D}^2)$ and notice that $S^t \cap \mathrm{Image}(e^t) = \emptyset$ for every $t \in [0,1]$ and in particular $S^t \cap \Lambda = \emptyset$.

By [29], or a small adaptation of [10], we can ensure the existence of a 1-parameter family of convex spheres with the same characteristic foliation $\hat{S}^t$, $t \in [0,1]$, which is C^0-close to S^t. We can also achieve that $\hat{S}^0 \cap \Delta_{\mathrm{OT}} = \emptyset$, $\hat{S}^t \cap \mathrm{Image}(e^t) = \emptyset$ and $\hat{S}^t \cap \Lambda = \emptyset$ for every $t \in [0,1]$.

To conclude apply the Contact Isotopy extension Theorem to find a contact isotopy $\varphi^t \in \mathrm{Cont}(M, \xi)$, $t \in [0,1]$, such that $\varphi^0 = \mathrm{Id}$, $\varphi^t(\hat{S}^0) = \hat{S}^t$ and $\varphi^t_{|\mathcal{O}p(\Lambda)} = \mathrm{Id}_{|\mathcal{O}p(\Lambda)}$. The isotopy of overtwisted disks is then given by $\Delta^t := \varphi^t(\Delta_{\mathrm{OT}})$. □

3.6.2 Overtwisted case

Finally we address the overtwisted case. In order to do so we need some preparatory lemmas. The first one addresses the problem without prescribing the initial condition.

Lemma 3.6.2. *Assume that the pinched 3-disks (D_i, ξ) are overtwisted for every $i \in \{0, \dots, n\}$. If $n > 1$ further assume that the embedding e^{t_i} is overtwisted for every $i \in \{1, \dots, n-1\}$. Then, there exists an isotopy of overtwisted disks Δ^t,$t \in [0,1]$, such that* $\mathrm{Image}(e^t) \cap \Delta^t = \emptyset$.

Proof. We will consider several overtwisted disks near each disk e^{t_i}, $i \in \{1, \dots, n-1\}$, and explain how to interpolate appropriately between them. We will do this inductively over the overtwisted pinched 3-disks (D_i, ξ).

Since e^{t_i} is convex for every $i \in \{0,1,\dots,n\}$ and, by hypothesis, every small neighborhood $(\mathcal{O}p(\text{Image}(e^{t_i})), \xi)$ is overtwisted for every $i \in \{1,\dots,n-1\}$, we can find a sufficiently small $\varepsilon > 0$ and overtwisted disks $\Delta_j^{i,\pm}$, for $(i,j) \in \{1,\dots,n-1\} \times \{1,3\}$, such that

(a) $\Delta_j^{i,\pm} \subseteq \mathcal{O}p(\text{Image}(e^{t_i \pm j\varepsilon}))$. In particular, $\Delta_j^{i,-} \subseteq D_{i-1}$ and $\Delta_j^{i,+} \subseteq D_i$.

(b) $\Delta_j^{i,\pm} \cap \text{Image}(e^{t_i}) = \emptyset$.

(c) $\Delta_j^{i,\pm} \cap \text{Image}(e^{t_i \pm 2\varepsilon}) = \emptyset$.

(d) $\Delta_1^{i,+}, \Delta_3^{i,+}, \Delta_1^{i,-}$ and $\Delta_3^{i,-}$ are pairwise disjoint for every $i \in \{1,\dots,n-1\}$.

(e) $\Delta_1^{i,+}, \Delta_3^{i,+}, \Delta_1^{i+1,-}$ and $\Delta_3^{i+1,-}$ are pairwise disjoint for every $i \in \{1,\dots,n-2\}$.

The way of finding these overtwisted disks is straightforward: we can assume that there exists a positive $\varepsilon > 0$ such that the isotopies $e^t, t \in [t_i - 4\varepsilon, t_i + 4\varepsilon]$, are composed of disks with fixed characteristic foliation; hence we can consider an I-invariant neighborhood of e^{t_i} and an overtwisted disk in a σ–small neighborhood of e^{t_i} with $\sigma \ll \varepsilon$ and several parallel copies of it. The last property can be ensured since all those overtwisted disks lie in the same pinched 3-disk D_i.

Now we explain how to define the required isotopy of overtwisted disks. We proceed inductively over the pinched 3-disks D_i, $i \in \{0,\dots,n-1\}$. We start with D_0, that is, for $t \in [0, t_1]$.

Observe that $\Delta_1^{1,-}$ and $\Delta_3^{1,-}$ are disjoint by (d) above. Moreover, they are disjoint from $\text{Image}(e^{t_1-2\varepsilon})$ by (c). Hence, since two disjoint overtwisted disks in the connected overtwisted contact 3-manifold $M \setminus \text{Image}(e^{t_1-2\varepsilon})$ are contact isotopic, by Theorem 3.5.3 we can find an isotopy of overtwisted disks O_t, $t \in [0,1]$, and $\delta > 0$ very small such that $O_0 = \Delta_1^{1,-}$, $O_1 = \Delta_3^{1,-}$ and $O_t \cap \text{Image}(e^{t_1-2\varepsilon+s}) = \emptyset$ for $s \in [-\delta, \delta]$. See Figure 3.5.

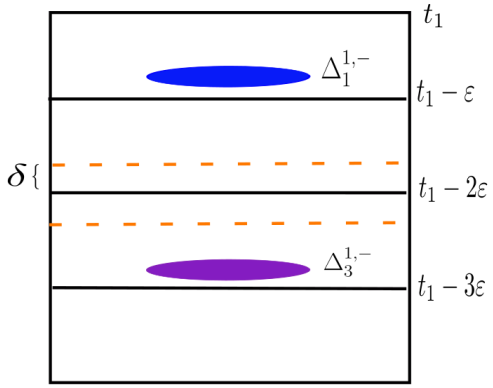


Figure 3.5: Schematic of the construction over D_0.

Therefore, for $t \in [0, t_1]$, we can define:

$$\Delta^t = \begin{cases} \Delta_1^{1,-} & t \in [0, t_1 - 2\varepsilon], \\ O_{\frac{t-(t_1-2\varepsilon)}{\delta}} & t \in [t_1 - 2\varepsilon, t_1 - 2\varepsilon + \delta], \\ \Delta_3^{1,-} & t \in [t_1 - 2\varepsilon + \delta, t_1]. \end{cases}$$

This defines the required isotopy over D_0. To extend it over D_1, that is for $t \in [t_1, t_2]$, proceed as follows. First for $t \in [t_1, t_1 + \delta]$ interpolate between $\Delta_3^{1,-}$ and $\Delta_3^{1,+}$. This can be done because of property (b) and (d) above. Secondly, for $t \in [t_1+\delta, t_1+2\delta]$ interpolate between $\Delta_3^{1,+}$ and $\Delta_1^{2,-}$. This can be done because of property (e). In both cases we are using Theorem 3.5.3. Finally, proceed exactly as before to define the isotopy over D_1. Continue this process inductively to conclude. □

The following key lemma will allow us to assume that if $\mathcal{F}_0$ and $\mathcal{F}_n$ are tight characteristic foliations then we can assume that both coincide after possibly adding several *tight* bypasses, which causes no problem due to Proposition 3.6.1.

Lemma 3.6.3. *Let $(\mathbb{D}_i^2, \Gamma_i)$, $i \in \{0, 1\}$, be two convex disks with the same Legendrian boundary with tight dividing sets Γ_i such that*

(i) *Γ_0 and Γ_1 coincide near the boundary.*

(ii) *$-\frac{1}{2}\#(\partial\mathbb{D}_1^2 \cap \Gamma_1) = -\frac{1}{2}\#(\partial\mathbb{D}_0^2 \cap \Gamma_0)$ and*

(iii) *$\chi(\mathbb{D}_{1,+}^2) - \chi(\mathbb{D}_{1,-}^2) = \chi(\mathbb{D}_{0,+}^2) - \chi(\mathbb{D}_{0,-}^2)$.*

Then, there exists a sequence of tight abstract bypasses such that after attaching it to $\mathbb{D}_0^2$ from above turns Γ_0 into Γ_1, up to isotopy. There also exists a sequence of tight abstract bypasses turning Γ_0 into Γ_1 when attached to $\mathbb{D}_0^2$ from below.

Remark 3.6.4. This result is likely well-known, although we were unable to find a direct reference. The existence of a sequence of bypasses from both above and below, which transforms one dividing set into the other, follows immediately, after Theorem 1.2.3, from the Eliashberg-Fraser classification of Legendrian unknots [20], and the Colin-Honda discretized isotopy (Theorem 3.4.1) between convex surfaces with Legendrian boundary. This is based on the observation that each such disk contactly embeds into $(\mathbb{R}^3, \xi_{\text{std}})$, as a consequence of Eliashberg's classification of tight contact structures on $\mathbb{R}^3$ [19].

Proof. We prove the existence of the sequence of bypasses attached from above. The other case is analogous. We proceed by induction on the number of curves in the dividing set. If there is only one curve, the disk is standard and there is nothing to prove.

Now, assume that there are n curves in the dividing set, with $n > 1$. If all the curves are ∂-parallel (the disk is multi-standard in the terminology of [31]), there is nothing to prove, since there is only one possible configuration. Otherwise, let $\mathcal{P} = \{p_1, \dots, p_{2k}\}$ denote the set of points ordered

cyclically and counterclockwise such that $\partial\mathbb{D}^2_1 \cap \Gamma_1 = \partial\mathbb{D}^2_0 \cap \Gamma_0 = \mathcal{P}$. Since $n > 1$, there exists a component $\Gamma_{1,1}$ of Γ_1 that intersects the boundary into two consecutive points of $\mathcal{P}$, i.e. $\Gamma_{1,1}$ is ∂-parallel. Without loss of generality, suppose that these points are p_1 and p_2.

Let $\Gamma_{0,0}$ denote the component of the dividing set Γ_0 that intersects the point p_1. This component must also intersect another point of the form p_{2+2j}, where $j \in \mathbb{Z}$. If $j = 0$ there is nothing to do. There are other two cases to consider:

First, if $p_{2+2j} \neq p_{2k}$, then we consider the two components of Γ_0 that contain p_{2k} and p_2. Notice that they must be different. Attach from above the obvious bypass whose attaching arc intersects these two components and $\Gamma_{0,0}$. After this, $\Gamma_{0,0}$ is transformed into a component that connects p_1 to p_2.

Second, if $p_{2+2j} = p_{2k}$. By the hypothesis, there exists a component $\Gamma_{0,2}$ that is not ∂-parallel and that can be connected with $\Gamma_{0,0}$ by an embedded arc whose interior does not intersect the dividing set. We can then find another component $\Gamma_{0,k}$ in the half-disk delimited by $\Gamma_{0,2}$ that does not contain $\Gamma_{0,0}$, for which the previous arc can be extended to intersect this component in such a way that it only intersects the dividing curves at three points. Attach a bypass, with attaching arc at this previous arc. After this, $\Gamma_{0,0}$ no longer connects p_1 to p_{2k}, as in the previous case.

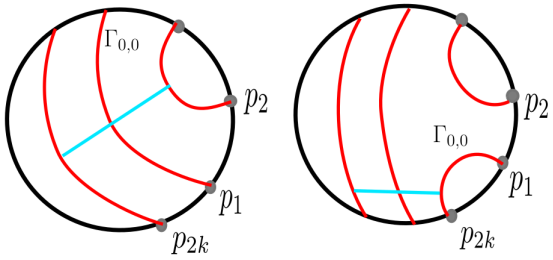


Figure 3.6: Possible configurations of dividing sets described in the proof. At the left it is depicted the case $p_{2+2j} \neq p_{2k}$ and at the right the case $p_{2+2j} = p_{2k}$

Hence, after attaching one or two tight bypasses from above we can arrange that both dividing sets coincide on the curve connecting p_1 to p_2. Since $n > 1$ we can apply Theorem 1.1.6 (or the Legendrian Realization Principle from [47]) to find a parallel Legendrian arc to $\Gamma_{0,0}$, which delimits a subdisk with Legendrian boundary. This subdisk has only one component of the dividing set, namely $\Gamma_{0,0}$. Therefore, the other subdisk associated with this Legendrian curve has $n - 1 < n$ components in both dividing sets. Finally, we can use the inductive hypothesis to conclude. □

Now we are ready to address the case in which the pinched 3-disks are overtwisted. In this case, we cannot find the isotopy of overtwisted disks for the given smooth isotopy e^t and a priori given starting overtwisted disk, in general, but we will be able to deform our isotopy e^t into a new one for which we can solve the problem. That is the content of the following

Proposition 3.6.5 (Overtwisted case). *Let Δ_{OT} be an overtwisted disk in the complement of e^0. Assume that the pinched 3-disks (D_i, ξ) are overtwisted for $i \in \{0, \dots, n-1\}$, and that the embeddings e^0 and e^1*

are tight. If $n > 1$ further assume that the embedding e^{t_i} is overtwisted for $i \in \{1, \dots, n-1\}$. Then, there exists a homotopy of isotopies $e^{t,u} : \mathbb{D}^2 \hookrightarrow M$, $(t,u) \in [0,1]^2$, relative to the boundary, such that

(i) $e^{t,u} = e^t$ *for all* $(t,u) \in [0,1] \times \{0\} \cup \{0,1\} \times [0,1]$,

(ii) The characteristic foliation of $e^{t,1}$ *is constant for* $0 \leq t \leq 1/2$,

(iii) $e^{t,1}$, $t \in [\frac{1}{2}, 1]$, *is a discretized isotopy composed of tight pinched 3-disks; and*

(iv) There exists an isotopy of overtwisted disks Δ^t, $t \in [0,1]$, *such that* $\Delta^0 = \Delta_{\mathrm{OT}}$ *and* $\Delta^t \cap \mathrm{Image}(e^{t,1}) = \emptyset$.

Proof. Consider the overtwisted pinched 3-disk (D_{n-1}, ξ). If this pinched 3-disk is going downwards (resp. upwards) we can use Lemma 3.6.3 to find a sequence of tight bypasses that after attaching them to e^1 from above (resp. below) turns the dividing set of e^1 into the one of e^0. Notice that we can apply this Lemma because of Theorem 1.2.3. Since the pinched 3-disk (D_{n-1}, ξ) is overtwisted we can realize such a sequence inside this pinched 3-disk by Lemma 2.2.1. Moreover, by Giroux Realization Theorem 1.1.6 we may further assume that the disk obtained from e^1 after attaching such a sequence of bypasses has the same characteristic foliation as e^0.

In conclusion, after a possible smooth isotopy supported in the pinched 3-disk (D_{n-1}, ξ) we may assume that there exists a time $\hat{t} \in (t_{n-1}, 1]$ such that $e^{\hat{t}}$ is convex with the same characteristic foliation than e^0 and such that the isotopy e^t, $t \in [\hat{t}, 1]$, decomposes as a sequence of tight pinched 3-disks.

From now on, we will focus on the isotopy e^t, $t \in [0, \hat{t}]$. Denote by $\mathcal{F} := \mathcal{F}_0$ the characteristic foliation of e^0 and $e^{\hat{t}}$. Our goal is to use the h-principle for $\mathcal{F}$-embeddings, Corollary 3.5.6, with this isotopy. However, there may be a formal obstruction to equip the smooth isotopy e^t, $t \in [0, \hat{t}]$, with a formal structure and apply the h-principle. In the next lines we will explain how to compute such a obstruction and how to sort it out by possibly adding more tight pinched 3-disks.

To compute the formal obstruction we proceed as follows. First, use use the smooth isotopy e^t, $t \in [0, \hat{t}]$, to extend the formal $\mathcal{F}$-embedding data of e_0 (which is actually genuine) up to $e^{\hat{t}}$; thereby obtaining a formal isotopy (e^t, F^t_s), $(t,s) \in [0, \hat{t}] \times [0,1]$. We need to describe the homotopy obstruction to making $F^{\hat{t}}_s$ constant. Define a plane field $\eta = \eta(p, \tau)$ in $\mathbb{D}^2 \times [0,1]$ as follows:

$$\eta(p, \tau) = F^{\hat{t}}_\tau(\xi(p)).$$

This plane field coincides with the I-invariant contact structure $(\mathbb{D}^2 \times [0,1], \xi_{\mathrm{inv}})$ defined by the convex characteristic foliation $\mathcal{F}$ over $\partial(\mathbb{D}^2 \times [0,1])$. The obstruction to making $F^{\hat{t}}_s$ constant is precisely the obstruction to homotoping the plane field η into ξ_{inv}, relative to the boundary. To analyze this, we trivialize the tangent bundle of $\mathbb{D}^2 \times [0,1]$ using ξ_{inv}, in such a way that ξ_{inv} has constant Gauss map, which allows us to identify this obstruction with the Hopf invariant of the associated Gauss map $G_\eta : (\mathbb{D}^2 \times [0,1], \partial) \to (\mathbb{S}^2, p_N)$. We denote this invariant by $h(\eta) \in \mathbb{Z}$.

If $h(\eta) = 0$ then there is nothing to do to obtain this formal structure. Let us assume that $h(\eta) \neq 0$. We suppose that obvious pinched 3-disk $\hat{D}$ bounded by $e^{t_{n-1}}$ and $e^{\hat{t}}$ is going upwards, the other case is analogous.

If $h(\eta) < 0$: Since the pinched 3-disk $(\hat{D}, \xi)$ is overtwisted near Image$(e^{t_{n-1}})$, we may add $|h(\eta)|$ bypass triangles *downwards* to $e^{\hat{t}}$ (Lemma 2.2.1). This allows us to find a new time $t^* \in (t^{n-1}, \hat{t})$ such that, after a possible isotopy, e^{t^*} is an $\mathcal{F}$-embedding. The obvious pinched 3-disk between e^{t^*} and $e^{\hat{t}}$ is given by a sequence of bypass triangles, in particular, it decomposes as a sequence of tight pinched 3-disks. Finally, adding a bypass triangle downwards increases the Hopf invariant by 1 as stated in Theorem 3.4.3. Therefore, there is no obstruction to equip the isotopy e^t, $t \in [0, t^*]$, with a formal structure.

If $h(\eta) > 0$: We proceed similarly to the previous case. First, create a bypass triangle downwards to obtain a new $\mathcal{F}$-embedding $e^{\tilde{t}}$, for some $\tilde{t} \in (t_{n-1}, \hat{t})$. Note that by Theorem 3.4.3 the obstruction to equip the isotopy e^t, $t \in [0, \tilde{t}]$, is now $h(\eta) + 1$. However, the pinched 3-disk enclosed by $e^{\tilde{t}}$ and $e^{\hat{t}}$ is overtwisted, since it is described as an embedded bypass triangle. Therefore, we may find a total of $h(G_\eta) + 1$ bypass triangles attached to $e^{\tilde{t}}$ from above inside this pinched 3-disk. After this, it is possible to find a time $t^* \in (\tilde{t}, \hat{t})$ such that, after a possible isotopy, e^{t^*} is an $\mathcal{F}$-embedding and the isotopy e^t, $t \in [0, t^*]$, admits a formal structure. Moreover, the isotopy e^t, $t \in [t^*, \hat{t}]$, is isotopic, relative to the boundary, to the isotopy that first interpolates back from e^{t^*} to $e^{\tilde{t}}$, following the sequence of bypass triangles backwards, and then realizes obvious isotopy from $e^{\tilde{t}}$ to $e^{\hat{t}}$ given by the initial bypass triangle upwards. In particular, we may assume that it decomposes as a sequence of tight pinched 3-disks.

In all the cases, we found an $\mathcal{F}$-embedding e^{t^*}, with $t^* \in (t_{n-1}, \hat{t}]$, such that

- The smooth isotopy e^t, $t \in [0, t^*]$, admits a formal structure and is discretized by overtwisted pinched 3-disks.
- The isotopy between e^t, $t \in [t^*, 1]$ is discretized by a sequence of *tight* pinched 3-disks.

We are now ready to conclude the proof. First, we apply Lemma 3.6.2 to construct an isotopy of overtwisted disks Δ^t, $t \in [0, t^*]$, such that Image$(e^t) \cap \Delta^t = \emptyset$ for all $t \in [0, t^*]$. We apply the h-principle given in Corollary 3.5.6 to (e^t, Δ^t), $t \in [0, t^*]$, and perform an obvious reparametrization of the t-variable, to conclude the existence of a homotopy $e^{t,u}$, $(t, u) \in [0, 1]^2$, satisfying properties (i), (ii) and (iii) stated above. To achieve property (iv), we simply use the Contact Isotopy Extension theorem for $t \in [0, \frac{1}{2}]$ in combination with Proposition 3.6.1 for $t \in [\frac{1}{2}, 1]$. This concludes the argument. □

3.6.3 The general case

We conclude this Section with the general existence statement of the isotopy of overtwisted disks that we will need.

Theorem 3.6.6. *Let (M,ξ) be a closed overtwisted contact 3-manifold. Let $e^t : \mathbb{D}^2 \hookrightarrow M$, $t \in [0,1]$, be an isotopy of smooth embeddings fixed near the Legendrian boundary $\Lambda = e^t(\partial\mathbb{D}^2)$ between two convex disks e^0 and e^1 with the same characteristic foliation $\mathcal{F}$. Assume that either $\mathcal{F}$ is tight or $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| > -1$. Then, there exists a homotopy of isotopies $e^{t,u} : \mathbb{D}^2 \hookrightarrow M$, $(t,u) \in [0,1]^2$, relative to the boundary, and an isotopy of overtwisted disks Δ^t, $t \in [0,1]$, such that*

(i) $e^{t,u} = e^t$ *for* $(t,u) \in [0,1] \times \{0\} \cup \{0,1\} \times [0,1]$

(ii) $\Delta^t \cap \mathrm{Image}(e^{t,1}) = \emptyset$.

Proof. We may assume that the isotopy is discretized as at the beginning of the section. If $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| > -1$, where the rotation is measured with respect to one the disks of the given isotopy, then, by the Bennequin-Eliashberg inequality (Theorem 1.2.5), $(\mathcal{O}p(\mathrm{Image}(e^t)), \xi)$ is overtwisted for all $t \in [0,1]$. Hence, we can directly apply Lemma 3.6.2 to find the required isotopy of overtwisted disks Δ^t, $t \in [0,1]$, and simply set $e^{t,u} \equiv e^t$.

Assume now that $\mathcal{F}$ is tight, and in particular $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| \leq -1$. Since $\mathcal{F}$ is tight then the discretized isotopy decomposes in smaller discretized isotopies as the ones treated in Propositions 3.6.1 and 3.6.5. By Colin's Theorem 1.2.4 we may fix an overtwisted disk Δ_{OT} in the complement of the tight disk e^0. Declare $\Delta^0 = \Delta_{\mathrm{OT}}$ and apply Propositions 3.6.1 and 3.6.5 iteratively to conclude. □

Remark 3.6.7. This Theorem is the key result that does not hold, in general, in the case in which $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| \leq -1$ and $\mathcal{F}$ is overtwisted. In this case, it is not true that every discretized isotopy decomposes in smaller discretized isotopies as in Propositions 3.6.1 and 3.6.5. The case that could appear and is not covered by these results is when there is a discretized isotopy $e^t : \mathbb{D}^2 \to M$, $t \in [0,1]$, composed by two overtwisted pinched 3-disks (D_0,ξ) and (D_1,ξ) for which $D_0 \cap D_1$ has a tight neighborhood while the embeddings e_0 and e_1 are overtwisted. In this case one cannot, in general, interpolate between the overtwisted disks for (D_0,ξ) and the ones for (D_1,ξ) since they could intersect. However, under the stronger assumption that the space of overtwisted disks is connected, we could also sort out this case. Notice that this assumption does not always hold [60] but is true for a wide class of overtwisted contact 3-manifolds [30].

3.7 Proof of Theorem 3.1.1

We start by addressing the surjectivity statement in Theorem 3.1.1:

Lemma 3.7.1 (Surjectivity)**.**
Given $(e,F_s) \in \mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)$ there is a homotopy $(e^t,F_s^t) \in \mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)$, with $t \in [0,1]$ such that

(i) $(e^0,F_s^0) = (e,F_s)$,

(ii) $(e^1, F_s^1) = (e^1, de^1) \in \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$, *for all* $s \in [0,1]$.

Proof. Let $D = e(\mathbb{D}^2 \times \{0\})$ and $\gamma = \partial D$. By hypothesis, we have $\mathrm{tb}(\gamma) \leq 0$. Using Proposition 1.1.4, we can assume that D is convex. We can ensure the existence of an overtwisted disk Δ_{OT} in the complement of a small neighborhood of D. Indeed, let Γ be a dividing set of D. Then, if Γ contain a closed curve, by Giroux's Criterion (Theorem 1.1.8), an I-invariant neighborhood of D is overtwisted and there exists an overtwisted disk in the complement of D. On the other hand if Γ does not contains closed curves, by Giroux's Criterion (Theorem 1.1.8) we obtain that $(e(\mathbb{D}^2 \times I_\varepsilon), \xi)$ is tight. Consequently, by Colin's Theorem 1.2.4, $M \setminus e(\mathbb{D}^2 \times I_\varepsilon)$ is overtwisted. In both cases, we can assume the existence of an overtwisted disk $\Delta_{\mathrm{OT}} \subseteq M \setminus e(\mathbb{D}^2 \times I_\varepsilon)$. Now the proof follows from Corollary 3.5.6. □

It remains to verify the injectivity of the Theorem 3.1.1, which reduces to proving the following:

Lemma 3.7.2 (Injectivity)**.**

$$\pi_1(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial), \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)) = 0.$$

Proof. Consider a formal isotopy $(e^t, F_s^t) \in \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$, $t \in [0,1]$, such that $(e^i, F_s^i) \in \mathrm{Emb}(\mathbb{D}^2, (M, \xi), \mathcal{F}, \mathrm{rel}\,\partial)$ for $i \in \{0,1\}$. Apply Theorem 3.6.6 to define $(e^{t,u}, F_s^{t,u}) \in \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$, $(t,u) \in [0,1] \times [0, \frac{1}{2}]$, such that

(i) $(e^{t,u}, F_s^{t,u}) = (e^t, F_s^t)$ for $(t,u,s) \in [0,1] \times \{0\} \times [0,1] \cup \{0,1\} \times [0, \frac{1}{2}] \times [0,1]$.

(ii) There exists an isotopy of overtwisted disks Δ^t, $t \in [0,1]$, such that $\Delta^t \cap \mathrm{Image}(e^{t,\frac{1}{2}}) = \emptyset$ for all $t \in [0,1]$.

Apply the h-principle from Corollary 3.5.6 to the isotopy $((e^{t,\frac{1}{2}}, F_s^{t,\frac{1}{2}}), \Delta^t) \in \mathrm{Loose\,FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$ to extend the previous homotopy over $u \in [\frac{1}{2}, 1]$ in such a way that

(i) $(e^{t,u}, F_s^{t,u}) = (e^t, F_s^t)$ for $(t,u,s) \in \{0,1\} \times [\frac{1}{2}, 1] \times [0,1]$.

(ii) $(e^{t,1}, F_s^{t,1}) \in \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)$ for all $t \in [0,1]$.

This proves that $\pi_1(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial), \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\,\partial)) = 0$ and, therefore concludes the argument. □

3.8 Applications

The purpose of this Section is to prove Corollaries 3.2.1, 3.2.2 and 3.3.1. We will first prove and recall some results for overtwisted contact 3-spheres that we will need for Corollaries 3.2.2 and 3.3.1. Finally, we will prove the mentioned Corollaries.

3.8.1 Convex disks with Legendrian boundary in an overtwisted 3-sphere

We begin by recalling the following

Theorem 3.8.1 (Hatcher [44, 46]). *The space* $\mathrm{Emb}(\mathbb{D}^2, \mathbb{S}^3, \mathrm{rel}\,\partial)$ *is contractible.*

This result together with Theorem 3.1.1 allow us to conclude the following

Corollary 3.8.2. *Let* $(\mathbb{S}^3, \xi)$ *be an overtwisted contact 3-sphere and* $\mathbb{D}^2 \subseteq (\mathbb{S}^3, \xi)$ *an embedded convex disk with Legendrian boundary and characteristic foliation* $\mathcal{F}$*. Assume that either* $\mathcal{F}$ *is tight or* $\mathrm{tb}(\partial\mathbb{D}^2) + |\mathrm{rot}(\partial\mathbb{D}^2)| > -1$*. Then,*

$$\pi_0(\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial)) \cong \mathbb{Z}.$$

Proof. By Theorem 3.1.1 it is enough to check that $\pi_0(\mathrm{FEmb}(\mathbb{D}^2, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial)) \cong \mathbb{Z}$. Consider the following fibration:

$$\mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Iso}(T\mathbb{S}^3|_{\mathbb{D}^2}, \mathrm{rel}\,\partial) \hookrightarrow \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial) \to \mathrm{Emb}(\mathbb{D}^2, \mathbb{S}^3, \mathrm{rel}\,\partial), \tag{3.5}$$

where the projection map is the natural forgetful map and $\mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Iso}(T\mathbb{S}^3|_{\mathbb{D}^2}, \mathrm{rel}\,\partial)$ denotes the space of paths F_s, $s \in [0,1]$, of vector bundle isomorphisms that start at a point $F_0 = *$ and end at $F_1 \in \mathrm{Maps}(\mathbb{D}^2, \mathrm{U}(1))$, that are fixed at the boundary of $\mathbb{D}^2$, where we see $\mathrm{U}(1)$ as a subset of $\mathrm{SO}(3) \subset \mathrm{GL}^+(3, \mathbb{R})$.

We can express the fiber as follows:

$$\begin{aligned}\mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Iso}(T\mathbb{S}^3|_{\mathbb{D}^2}, \mathrm{rel}\,\partial) &= \mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Maps}(\mathbb{D}^2, \mathrm{GL}^+(3,\mathbb{R}), \mathrm{rel}\,\partial)\\ &\cong \mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Maps}(\mathbb{S}^2, \mathrm{GL}^+(3,\mathbb{R}), \mathrm{rel}\,p)\\ &\cong \mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Maps}(\mathbb{S}^2, \mathrm{SO}(3), \mathrm{rel}\,p),\end{aligned}$$

where $p \in \mathbb{S}^2$.

On the other hand, it is known that $\mathrm{Path}_{(*,\mathrm{U}(1))}\,\mathrm{Maps}(\mathbb{S}^2, \mathrm{SO}(3), \mathrm{rel}\,p) = \mathrm{Path}_{(*,\mathrm{U}(1))}\,\Omega^2(\mathrm{SO}(3))$ projects to $\mathrm{Maps}(\mathbb{S}^2, \mathrm{U}(1), \mathrm{rel}\,p) = \Omega^2(\mathrm{U}(1))$. Thus, we obtain the fibration

$$\Omega^3(\mathrm{SO}(3)) \hookrightarrow \mathrm{Path}_{(*,\mathrm{U}(1))}\,\Omega^2(\mathrm{SO}(3)) \to \Omega^2(\mathrm{U}(1)),$$

where the projection map consists on taking the endpoint of the path, and $\Omega^2(\mathrm{U}(1)) \cong \Omega^2(\mathbb{S}^1)$, which is contractible. Consequently, $\Omega^3(\mathrm{SO}(3))$ is homotopically equivalent to the fiber of fibration (3.5), and therefore we have the following long exact sequence:

$$\cdots \longrightarrow \pi_1(\mathrm{Emb}(\mathbb{D}^2, \mathbb{S}^3, \mathrm{rel}\,\partial))$$
$$\pi_0(\Omega^3(\mathrm{SO}(3))) \longrightarrow \pi_0(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial)) \longrightarrow \pi_0(\mathrm{Emb}(\mathbb{D}^2, \mathbb{S}^3, \mathrm{rel}\,\partial)).$$

Since $\mathrm{Emb}(\mathbb{D}^2, \mathbb{S}^3, \mathrm{rel}\,\partial)$ is contractible by Theorem 3.8.1, and $\pi_0(\Omega^3(\mathrm{SO}(3))) = \pi_3(\mathrm{SO}(3)) = \mathbb{Z}$, we conclude that:

$$\pi_0(\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial)) \cong \pi_0(\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\,\partial)) \cong \mathbb{Z}.$$

□

3.8.2 Contactomorphisms of an overtwisted 3-sphere

We start with the following observation

Lemma 3.8.3. *Let $(\mathbb{S}^3, \xi)$ be an overtwisted contact 3-sphere. Then, the natural 1-jet evaluation map $\mathrm{Cont}(\mathbb{S}^3, \xi) \to \mathrm{CFr}(\mathbb{S}^3, \xi)$ is surjective at the level of fundamental groups.*

Proof. This follows from the fact that we can choose ξ to be invariant under rotations around one complex plane in $\mathbb{C}^2 \supseteq \mathbb{S}^3$. □

Given a point $p \in \mathbb{S}^3$ we will denote by $\mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, p) \subseteq \mathrm{Cont}(\mathbb{S}^3, \xi)$ the subspace of contactomorphisms that are fixed near a small Darboux disk containing p. We will make use of the following

Theorem 3.8.4 (Chekanov, Vogel [60])**.** *Let $(\mathbb{S}^3, \xi)$ be an overtwisted contact 3-sphere and $p \in \mathbb{S}^3$ a given point. Then,*

$$\pi_0(\mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, p) \cong \pi_0(\mathrm{Cont}(\mathbb{S}^3, \xi)) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } \xi \cong \xi_{-1}, \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

Proof. Without fixing the Darboux ball this is [60, Theorem 4.5]. The case in which the Darboux ball is fixed follows from this case by studying the fibration

$$\mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, p) \hookrightarrow \mathrm{Cont}(\mathbb{S}^3, \xi) \to \mathrm{CFr}(\mathbb{S}^3, \xi)$$

and using Lemma 3.8.3. □

3.8.3 Proof of Corollary 3.2.1

Let $\Lambda \subseteq (\mathbb{D}^3, \xi_{\mathrm{std}}) \subseteq (M, \xi)$ be a Legendrian unknot that lies in a Darboux ball. Fix an embedded convex disk $\mathbb{D}^2 \subseteq (M, \xi)$ with $\partial\mathbb{D}^2 = \Lambda$ and tight characteristic foliation $\mathcal{F}$. We will consider the auxiliary spaces $\mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, p)$ and $\mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, p)$ of embeddings and formal embeddings of disks with characteristic foliation $\mathcal{F}$ that are fixed near a point $p \in \Lambda = \partial\mathbb{D}^2$.

Consider the following commutative diagram in which the rows are Serre fibrations and the columns are the natural inclusions:

$$\begin{array}{ccccc} \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, \partial) & \longrightarrow & \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, p) & \longrightarrow & \mathcal{L}_{(p,v)}(\Lambda, (M, \xi)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, \partial) & \longrightarrow & \mathrm{FEmb}(\mathbb{D}^2, \mathcal{F}, (M, \xi), \mathrm{rel}\, p) & \longrightarrow & F\mathcal{L}_{(p,v)}(\Lambda, (M, \xi)) \end{array}$$

Here, $F\mathcal{L}_{(p,v)}(\Lambda,(M,\xi)) = \{(\gamma,F_s) \in F\mathcal{L}(\Lambda,(M,\xi)) : \gamma(0) = p, F_s(0) = v, s \in [0,1]\}$. It was observed in [31] that $\mathrm{Emb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\, p)$ is contractible. On the other hand, it is straightforward to check that $\mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\, p)$ is also contractible. Therefore, there is a commutative square

$$\begin{array}{ccc} \Omega\mathcal{L}_{(p,v)}(\Lambda,(M,\xi)) & \longrightarrow & \mathrm{Emb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial) \\ \downarrow & & \downarrow \\ \Omega F\mathcal{L}_{(p,v)}(\Lambda,(M,\xi)) & \longrightarrow & \mathrm{FEmb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial) \end{array}$$

in which the horizontal maps are weak homotopy equivalences.

Therefore, by Theorem 3.1.1, we conclude that the inclusion $\mathcal{L}_{(p,v)}(\Lambda,(M,\xi)) \hookrightarrow F\mathcal{L}_{(p,v)}(\Lambda,(M,\xi))$ induces an isomorphism at the level of fundamental groups.

To conclude the proof we will consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{L}_{(p,v)}(\Lambda,(M,\xi)) & \longrightarrow & \mathcal{L}(\Lambda,(M,\xi)) & \longrightarrow & \mathrm{CFr}(M,\xi)) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{F\mathcal{L}}_{(p,v)}(\Lambda,(M,\xi)) & \longrightarrow & F\mathcal{L}(\Lambda,(M,\xi)) & \longrightarrow & \mathrm{CFr}(M,\xi)) \end{array}$$

induced by the evaluation map $F\mathcal{L}(\Lambda,(M,\xi)) \to \mathrm{CFr}(M,\xi), (\gamma,F_s) \mapsto (\gamma(0),F_1(0))$. Here, the rows are Serre fibrations and the colums are inclusions. Notice that the fiber $\widetilde{F\mathcal{L}}_{(p,v)}(\Lambda,(M,\xi))$ is weakly homotopy equivalent to $F\mathcal{L}_{(p,v)}(\Lambda,(M,\xi))$. The result now follows by analyzing the long exact sequences in homotopy and the Five Lemma since the homomorphism

$$\pi_1(\mathcal{L}_{(p,v)}(\Lambda,(M,\xi))) \to \pi_1(F\mathcal{L}_{(p,v)}(\Lambda,(M,\xi)))$$

is an isomorphism as we just proved. This concludes the argument. □

3.8.4 Proof of Corollary 3.2.2

Consider the fibration

$$\mathcal{L}_{(p,v)}(\Lambda,(\mathbb{S}^3,\xi)) \hookrightarrow \mathcal{L}(\mathbb{S}^3,\xi) \to \mathrm{CFr}(\mathbb{S}^3,\xi) \tag{3.6}$$

Notice that $\mathrm{CFr}(\mathbb{S}^3,\xi) \cong \mathbb{S}^3 \times \mathrm{U}(1)$ since every oriented plane bundle over $\mathbb{S}^3$ is trivializable. In particular, $\pi_1(\mathrm{CFr}(\mathbb{S}^3,\xi)) \cong \mathbb{Z}$ and $\pi_2(\mathrm{CFr}(\mathbb{S}^3,\xi)) = 0$. Moreover, as explained in the proof of Corollary 3.2.1 there is an isomorphism

$$\pi_1(\mathcal{L}_{(p,v)}(\Lambda,(\mathbb{S}^3,\xi))) \cong \pi_0(\mathrm{Emb}(\mathbb{D}^2,\mathcal{F},(M,\xi),\mathrm{rel}\,\partial)) \cong \mathbb{Z},$$

where in the last isomorphism we have used Corollary 3.8.2. Using all this information in the long exact sequence in homotopy associated to the fibration (3.6) we conclude that there is a short exact sequence

$$0 \mapsto \mathbb{Z} \cong \pi_1(\mathcal{L}_{(p,v)}(\Lambda, (\mathbb{S}^3, \xi))) \to \pi_1(\mathcal{L}(\Lambda, (\mathbb{S}^3, \xi))) \to \mathbb{Z} \to 0$$

From here it follows that $\pi_1(\mathcal{L}(\Lambda, (\mathbb{S}^3, \xi)))$ is generated by two independent loops of Legendrian unknots as described in Corollary 3.2.2. One loop A represented by a full-rotation of Λ in a Darboux chart and the other loop B represented by a loop of long Legendrian unknots, which is contractible as a loop of long smooth unknots by a result of Hatcher [45]. Moreover, it follows that $\pi_1(\mathcal{L}(\Lambda, (\mathbb{S}^3, \xi)))$ is isomorphic to either $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$[1]. Since the later group is non abelian it enough to check that the loops A and B commute to conclude the proof.

By Lemma 3.8.3 the map $\pi_1(\mathrm{Cont}(\mathbb{S}^3, \xi)) \to \pi_1(\mathrm{CFr}(\mathbb{S}^3, \xi))$ is surjective. Therefore, there exists a loop of contactomorphisms $\varphi^t \in \mathrm{Cont}(\mathbb{S}^3, \xi)$, $t \in [0,1]$, $\varphi^0 = \varphi^1 = \mathrm{Id}$, such that $A = \varphi^t(\Lambda)$. We will further assume that $\varphi^t = \mathrm{Id}$ for $t \in [0, \frac{1}{2}]$. On the other hand, we may represent $B = \Lambda^t$, $t \in [0,1]$, by a loop of long Legendrians such that $\Lambda^t = \Lambda$ for all $t \in [\frac{1}{2}, 1]$. It then follows that the concatenation $A * B$ can be represented by the loop $\varphi^t(\Lambda^t)$, $t \in [0,1]$, it is now straightforward to check that $A * B = B * A$ as elements in $\pi_1(\mathcal{L}(\Lambda, (\mathbb{S}^3, \xi)))$. This concludes the argument. □

3.8.5 Proof of Corollary 3.3.1

Let $\Lambda \subseteq (\mathbb{S}^3, \xi)$ be some Legendrian unknot with non-positive tb in an overtwisted contact 3-sphere. We fix an embedded convex disk $\mathbb{D}^2 \subseteq (\mathbb{S}^3, \xi)$ with boundary $\partial \mathbb{D}^2 = \Lambda$. In the case in which $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| \leq -1$ we will assume that the characteristic foliation $\mathcal{F}$ of $\mathbb{D}^2$ is tight. We will make the identification $\mathrm{Cont}(C(\Lambda), \xi) = \mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, \Lambda)$.

We consider first the case in which $\mathrm{tb}(\Lambda) + |\mathrm{rot}(\Lambda)| > -1$. Consider the natural fibration

$$\mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, \mathbb{D}^2) \hookrightarrow \mathrm{Cont}(\mathbb{S}^3, \xi, \mathrm{rel}\, \Lambda) \to \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\, \partial). \tag{3.7}$$

given by post-composition of the inclusion $\mathbb{D}^2 \hookrightarrow (\mathbb{S}^3, \xi)$ with a contactomorphism.

We claim that the fibration map is surjective in homotopy groups. Indeed, this is an application of Eliashberg's h-principle (Theorem 3.5.1). We explain the non-parametric case since the general case is analogous. Given two elements $e, f \in \mathrm{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \mathrm{rel}\, \partial)$ it is not hard to find an orientation preserving diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{S}^3)$ such that $f = \varphi \circ e$, since the space of smooth embeddings is connected by Schoenflies Theorem (in the parametric case, we use that this space is contractible by Theorem 3.8.1). Since the embeddings e and f have the same characteristic foliation we may assume that φ is an actual contactomorphism over $(Op(\mathrm{Image}(e)), \xi)$. The contact structures $\varphi_* \xi$ and ξ are homotopic as formal contact structures relative $Op(\mathrm{Image}(f))$ (in the parametric case, we use that the obtained spheres of formal contact structures are contractible because the group $\mathrm{Diff}(\mathbb{S}^3, \mathrm{rel}\, \Lambda)$ is contractible by a combination of Theorem 3.8.1 and the Hatcher's proof of the Smale conjecture [45][2], so it acts trivially in homotopy on the space of formal contact

[1] Recall that, since $\mathrm{Aut}(\mathbb{Z}) = \{\mathrm{Id}, -\mathrm{Id}\}$, there are exactly two semidirect products of $\mathbb{Z}$ with itself, and only one of the two is commutative, which is precisely the trivial product.

[2] This appears as one of the equivalent forms of the Smale conjecture at the end of [45]. To prove this equivalence Theorem 3.8.1 is needed.

structures). Since $(\mathcal{O}p(\text{Image}(f)), \xi)$ is overtwisted, by the assumption $\text{tb}(\Lambda) + |\text{rot}(\Lambda)| > -1$, we can use Theorem 3.5.1 to find a homotopy of contact structures $\xi^t \in \text{CStr}(\mathbb{S}^3)$, $t \in [0,1]$, such that $\xi^0 = \varphi_* \xi$, $\xi^1 = \xi$ and $\xi^t_{|\mathcal{O}p(\text{Image}(f))} = \xi_{|\mathcal{O}p(\text{Image}(f))}$ for all $t \in [0,1]$. By Gray stability there exists an isotopy $g^t \in \text{Diff}(\mathbb{S}^3, \text{rel Image}(f))$, $t \in [0,1]$, such that $g^t_* \xi^0 = \xi^t$. The diffeomorphism $G = g^1 \circ \varphi \in \text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \Lambda)$ and satisfies $G_* \xi = \xi$, as required.

On the other hand, it was observed by Dymara in [14] that there is a weak homotopy equivalence $\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \mathbb{D}^2) \cong \text{FCont}(\mathbb{S}^3, \xi, \text{rel}\, \mathbb{D}^2) \cong \Omega^4 \mathbb{S}^2$. Here, we are using again that $(\mathcal{O}p(\mathbb{D}^2), \xi)$ is overtwisted so $(\mathbb{S}^3 \backslash \mathbb{D}^2, \xi)$ is overtwisted at infinity (see Theorem 3.5.2). In particular,

$$\pi_0(\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \mathbb{D}^2)) \cong \pi_0(\text{FCont}(\mathbb{S}^3, \xi, \text{rel}\, \mathbb{D}^2)) \cong \pi_4(\mathbb{S}^2) \cong \mathbb{Z}_2.$$

On the other hand, by Theorem 3.1.1 and Corollary 3.8.2 there is an isomorphism

$$\pi_0(\text{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \text{rel}\, \partial)) \cong \pi_0(\text{FEmb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \text{rel}\, \partial)) \cong \mathbb{Z}.$$

Hence, it follows from the short exact sequence in homotopy associated to the fibration (3.7) that there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to \pi_0(\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \Lambda)) \to \mathbb{Z} \to 0$$

from which it follows that

$$\pi_0(\text{Cont}(C(\Lambda), \xi)) = \pi_0(\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \Lambda)) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

This concludes the argument in the case that $\text{tb}(\Lambda) + |\text{rot}(\Lambda)| > -1$.

We now deal with the case $\text{tb}(\Lambda) + |\text{rot}(\Lambda)| \leq -1$. Consider the fibration

$$\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \Lambda) \hookrightarrow \text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, p) \to \mathcal{L}_{(p,v)}(\Lambda, (\mathbb{S}^3, \xi))$$

given by post-composition of the inclusion $\Lambda \hookrightarrow (\mathbb{S}^3, \xi)$ with a given contactomorphism. Observe that the fibration map factors as

$$\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, p) \to \text{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \text{rel}\, p) \to \mathcal{L}_{(p,v)}(\Lambda, (\mathbb{S}^3, \xi)).$$

Therefore, since the space $\text{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \text{rel}\, p)$ is contractible because $\mathcal{F}$ is tight by [31] we conclude that the fibration map is null-homotopic. In particular, there are weak homotopy equivalences

$$\text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, \Lambda) \cong \text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, p) \times \Omega \mathcal{L}_{(p,v)}(\Lambda, (\mathbb{S}^3, \xi)) \cong \text{Cont}(\mathbb{S}^3, \xi, \text{rel}\, p) \times \text{Emb}(\mathbb{D}^2, \mathcal{F}, (\mathbb{S}^3, \xi), \text{rel}\, \partial),$$

where the last one was explained in the proof of Corollary 3.2.1. The result now follows by combining Corollary 3.8.2 and Theorem 3.8.4. □

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