Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Shadowing and hyperbolicity for infinite dimensional dynamical systems

Shadowing e hiperbolicidad para sistemas dinámicos de dimensión infinita

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Memoria para optar al grado de doctor presentada por

Carlos Roberto Takaessu Junior

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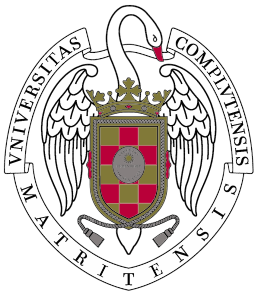
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Facultad de Ciencias Matemáticas

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UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

Shadowing and Hyperbolicity for Infinite Dimensional Dynamical Systems

Carlos Roberto Takaessu Junior

Tese de Doutorado do Programa de Pós-Graduação em Matemática (PPG-Mat)



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Shadowing and Hyperbolicity for Infinite Dimensional Dynamical Systems

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Abstract

In this work we extend results about Shadowing Lemma there are known on finite dimensional compact manifolds without border (or $\mathbb{R}^n$), to an infinite dimensional space. In fact, we proved that if $\{\mathcal{T}(t) : t \geq 0\}$ is a Morse-Smale semigroup defined on a Hilbert space with global attractor $\mathcal{A}$, then $\mathcal{T}(1)$ admits the Lipschitz Shadowing on $\mathcal{A}$ and there exists a neighborhood of $\mathcal{A}$ such that $\mathcal{T}(1)$ has the Holder-Shadowing property. Furthermore, we show some applications related to the robustness of Morse-Smale semigroups and the continuity of global attractors.

Resumen

En este trabajo extendemos los resultados sobre el Shadowing Lemma que se conocen en variedades compactas de dimensión finita sin borde (o $\mathbb{R}^n$), a un espacio de dimensión infinita. De hecho, demostramos que si $\{\mathcal{T}(t) : t \geq 0\}$ es un semigrupo de Morse-Smale definido en un espacio de Hilbert con atractor global $\mathcal{A}$, entonces $\mathcal{T}(1)$ admite Lipschitz Shadowing en $\mathcal{A}$ y existe un entorno de $\mathcal{A}$ tal que $\mathcal{T}(1)$ tiene la propriedad de Hölder-Shadowing. Además, mostramos algunas aplicaciones relacionadas con la robustez de semigrupos de Morse-Smale y la continuidad de atractores globales.

Resumo

Neste trabalho estendemos os resultados sobre Shadowing Lemma conhecidos em variedades compactas de dimensão finita sem borda (ou $\mathbb{R}^n$), para um espaço de dimensão infinita. De fato, provamos que se $\{\mathcal{T}(t) : t \geq 0\}$ é um semigrupo Morse-Smale definido em um espaço de Hilbert com atrator global $\mathcal{A}$, então $\mathcal{T}(1)$ admite Lipschitz-Shadowing em $\mathcal{A}$ e existe uma vizinhança de $\mathcal{A}$ tal que $\mathcal{T}(1)$ tem a propriedade de Holder-Shadowing. Além disso, mostramos algumas aplicações relacionadas à robusteza de semigrupos Morse-Smale e à continuidade de atratores globais.

Introduction

Since the 17th century, the theory of differential equations has become a powerful tool to model several phenomena in different areas of science. For instance, the motion of a planet around the sun is determined by the position of the planet at each time, measured in a certain coordinate system where the sun usually occupies the origin. The position is expressed by a function of time and this function satisfies certain equations involving the function and its derivatives (a differential equation) which encodes the laws of physics that govern this motion, that is the laws of Newtonian mechanics. Analyzing these equations is fundamental to predict the motion of the planet around the sun. In a similar way, the motion of a fluid is determined by a function that gives us the velocity of the fluid at each point in space and time. This function satisfies certain differential equations which encodes the laws of physics governing this process which turn out to be the Navier-Stokes equation. This process of expressing the behavior of an important quantity as a function which satisfy certain differential equation is very powerful and it is present in all areas of knowledge including biology (population models), economy and finance (evolution of stock markets), engineering (study of beams and structures), medicine (growth of tumors and bacteria), and so on.

It is clear that analyzing these differential equations is fundamental to understand the evolution of the physical quantity we are interested in. According to the type of equation obtained, a whole branch of mathematics with its own techniques and results has been developed: ordinary differential equations, partial differential equations, stochastic differential equations, delay differential equations, etc..

One important question we would like to address is the behavior of solutions of these differential equations for large times, that we refer to as the "asymptotic behavior" of solutions. In this respect, there is an important class of systems denoted by "dissipative systems" (the name is taken from the physical concept of "dissipation of energy") which are characterized by the fact that there exists a bounded set of states, say B, so that the evolution of each physical state enters into this bounded set at certain time and never leaves this set afterwards. This implies that for whatever initial condition that we consider, the system always lives in this bounded set B for large enough times. Therefore, analyzing the evolution of B as time goes to infinity will give us information on the asymptotic behavior of the whole system. The set of states which are obtained as limit for large times of the evolution of the set B, that we denote by $\mathcal{A}$, will attract each state as time goes to infinity. We refer to this set $\mathcal{A}$ as the "global attractor" of the system and it is an important concept for dissipative systems, as we are going to see. Studying the structure of the set $\mathcal{A}$ and analyzing the behavior of the solutions inside $\mathcal{A}$ will be fundamental in understanding the asymptotic behavior of the dissipative system.

In many instances the systems we are analyzing will depend on some parameters. These parameters may represent some physical quantities which in general are not completely determined but rather they are known with certain degree of precision. For instance in the planetary motion that we referred above, the mass of the sun or the universal gravity constant are only known up

to certain degree of accuracy; or for the behavior of fluids, the viscosity is only approximately known. In other situations the parameters in an equation may have other origin and will represent a perturbation that we make on a system. For instance we may consider the case where we have certain physical process like the motion of a fluid or a heat conduction problem in a very thin tube of thickness $\epsilon > 0$ small, and we are interested in understanding the behavior of the solutions when $\epsilon \to 0$. This is a particular case of a singular perturbation since the domain where the equation takes place undergoes a drastic change (the dimension goes from 3 to 1) as we let $\epsilon \to 0$. Hence, in general we will have a system depending on a parameter that we generically denote by ϵ and we want to analyze how the solutions depends on ϵ. If the system is dissipative and it has an attractor $\mathcal{A}_\epsilon$ for each value of ϵ it is important to analyze the dependence of the attractor on the parameter. Questions like:

- Is the set $\mathcal{A}_\epsilon$ continuous with respect to ϵ ?
- Is the dynamics inside the attractor $\mathcal{A}_\epsilon$ equivalent to the dynamics inside $\mathcal{A}_0$?
- How can we estimate the distance of the sets $\mathcal{A}_\epsilon$ to $\mathcal{A}_0$?

have been studied for a long time now but no complete and satisfactory answers have been obtained for a broad class of systems.

Answers to these questions have been obtained under some restrictions on the dissipative systems under consideration. One of them consists, roughly speaking, in requiring that any particular solution of the system converges to an equilibrium state, which is a solution of the system which do not vary with time. This class of systems are generally denoted as gradient systems (Definition 6) and are characterized by the existence of an energy which decreases along solutions. For these gradient systems the question of the continuity of the attractors can be addressed satisfactorily [28, 50, 51].

To address the other questions we need to impose more requirement to the class of gradient systems. One of them has to do with a subtle requirement that will guarantee some robustness to the attractor. This extra requirement will define the class of Morse-Smale systems, see Definition 11. Another important restriction is that the system is "essentially" finite dimensional, that is, we have a finite dimensional space or manifold which carries over all the asymptotic information of the system. This is satisfied if the system is expressed by an ordinary differential equation in $\mathbb{R}^N$ or in a smooth manifold of dimension N. But even if the system lives in an infinite dimensional space we may have finite dimensional submanifolds (usually denoted as "inertial manifolds") which carries all the asymptotic dynamics of the system, that is, the system can be reduced to a finite dimensional manifold. This finite dimensionality requirement together with the Morse-Smale gradient structure has been proved to be a natural framework to address the last two issues.

One fundamental technique to address the questions above is called Shadowing, as we will explain and define properly later on in the thesis, see Definition 12. This technique is suited and has been developed mainly for finite dimensional Morse-Smale systems and is the main reason why the questions above can be address for these type of systems. But the finite dimensionality reduction obtained via the existence of "inertial manifolds" is a very restrictive requirement and it is desired to be able to develop the technique of Shadowing for infinite dimensional Morse-Smale systems, as we are going to detail further ahead. This is the main aspect of this thesis.

In order to explain more in detail the framework and results of this thesis, we will need to introduce some notation and definitions.

Let us start with the fundamental concept of dynamical system or semigroup.

Definition 1 (Semigroup)**.** *Let (M, d) be a metric space, $\mathbb{P} = \mathbb{R}$ or $\mathbb{Z}$ and $\mathcal{C}(M)$ be the space of continuous functions from M into M. We say that the family $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$, where $\mathbb{P}^+ = \mathbb{P} \cap [0, +\infty)$, is a semigroup on the phase space M if it satisfies:*

1. $\mathcal{T}(0)x = x,\ \forall x \in M$;
2. $\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s),\ \forall t, s \in \mathbb{P}^+$;
3. *The map* $\mathbb{P}^+ \times M \ni (t, x) \mapsto \mathcal{T}(t)x$ *is continuous.*

When $\mathbb{P} = \mathbb{R}$ *we say that* $\mathcal{T}(\cdot)$ *is a continuous semigroup. In case* $\mathbb{P} = \mathbb{Z}$ *we say that* $\mathcal{T}(\cdot)$ *is a discrete semigroup.*

Remark 1. *From property (2) we have that*

$$\mathcal{T}(n) = \underbrace{\mathcal{T}(1) \circ \cdots \circ \mathcal{T}(1)}_{n \text{ times}} = (\mathcal{T}(1))^n, \text{ for all } n \in \mathbb{Z}^+.$$

Therefore, discrete semigroups are uniquely determined by the map $\mathcal{T}(1) : M \to M$*. With certain abuse of notation, we will identify a continuous map* $f : M \to M$ *with the discrete semigroup generated by* f*, that is,* $\mathcal{T} = \{f^n : n = 0, 1, 2, \dots\}$*. Note that if* $\mathcal{T}(\cdot)$ *is a continuous semigroup, then* $\{(\mathcal{T}(1))^n = \mathcal{T}(n) : n \in \mathbb{Z}^+\}$ *defines a discrete semigroup. In fact, this particular case of discrete semigroup will be used on Chapters 2 and 3.*

If $M = X$ *is a Banach space and the map* $\mathcal{T}(t) : X \to X$ *is differentiable for all* $t \in \mathbb{P}^+$*, we say that* $\{\mathcal{T}(t) : t \in \mathbb{P}^+\}$ *is a differentiable semigroup. Sometimes we will denote a semigroup* $\mathcal{T}(\cdot)$*, which is a family of continuous map, simply by* $\mathcal{T}$*.*

As we said before, the motivation behind continuous semigroups comes from differential equations. Briefly speaking, when we deal with an ordinary differential equation (ODE) on a Banach space X

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \in X, \end{cases} \tag{0.1}$$

and assume that (0.1) has a unique solution $x(\cdot, x_0) : \mathbb{R}^+ \to X$ through x_0, for all $x_0 \in X$, we can define $\mathcal{T}(t) := x(t, \cdot) : X \to X$ to obtain a continuous semigroup in X. The same argument works to partial differential equations (PDE's). Therefore, the theory of continuous semigroups are intimately connected with differential equations and is a powerful tool to study solutions of PDE's and ODE's, specially when the phase space is infinite dimensional [28, 33, 38, 23, 39, 9, 15, 21, 10, 16, 4, 18]. Very oftenly, the phase space is actually a Hilbert space [20, 39, 9, 15, 57, 61, 16, 30].

Regarding discrete semigroups, they are also often studied under different perspectives and are itself a large branch of mathematics, with many applications. For example, the theory of discrete semigroups can be used to obtain several results from number theory [64].

In both cases, continuous and discrete semigroups, we are concerned with understanding the behavior of the semigroup when time is large, that is, we want to study the asymptotic behavior of the semigroup. For example, in the case of continuous semigroup, we want to know how the solutions of a differential equation behaves when the time is large: do the solutions distance from each other? do they approach some set? do they behave chaotically? The same questions also hold for discrete semigroups. In fact, there are several open problems in mathematics regarding the asymptotic behavior of discrete semigroups, such as the famous Collatz conjecture [62].

Regarding semigroups where the phase space is infinite dimensional, the most important concept related to its dynamics and asymptotic behavior is called the global attractor $\mathcal{A}$. This compact set has a special property of attraction and dictates the behavior of each point x on the phase space M. In fact, if a semigroup $\mathcal{T}$ has a global attractor $\mathcal{A}$, then each point of the phase space will approach $\mathcal{A}$ as time goes to infinity. Let us introduce some concepts related to orbits and trajectories of subsets $B \subset M$ so we can define precisely what is a global attractor.

We start by defining what is a global solution/orbit. The nomenclature "global solution" comes from differential equations, since we already know that semigroups are connected with solutions of differential equations.

Definition 2 (Global Solution/Orbit)**.** *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup, discrete or continuous, on a metric space M. We say that a continuous function $\phi : \mathbb{P} \to M$ is a* ***global solution*** *(or global orbit) of $\mathcal{T}$ if satisfies $\mathcal{T}(s)\phi(t) = \phi(t+s)$ for all $s \in \mathbb{P}^+$ and $t \in \mathbb{P}$. If $\phi(t_0) = x \in M$ for some $t_0 \in \mathbb{P}$ we say that ϕ is a global solution through x. In this case, we usually assume (without lost of generality) that $\phi(0) = x$, since $\psi(t) = \phi(t+t_0)$ is a global solution with this property.*

Now we define some notions of trajectories, that describe the evolution of a certain subset.

Definition 3 (Trajectory/Orbit)**.** *Let $\mathcal{T} = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup on a metric space (M, d). We define*

1. *The* ***trajectory (or orbit)*** *of $S \subset M$ on the interval of time $[t_0, t_1] \subset \mathbb{R}^+$ as the set*
$$\gamma_{[t_0,t_1]}(S) := \bigcup_{s \in [t_0,t_1] \cap \mathbb{P}} \{\mathcal{T}(s)x : x \in S\}.$$
2. *The* ***positive trajectory (or positive orbit)*** *of $S \subset M$ as $\gamma^+(S) := \bigcup_{t \in \mathbb{P}^+} \gamma_{[0,t]}(S)$.*
3. *If $x \in M$ and ϕ is a global solution through x, the partial negative orbit of x at time t related to ϕ is the set $(\gamma_\phi)_t^-(x) := \{\phi(s) : s \in \mathbb{P}, s \leq t\}$. In particular, the* ***negative trajectory (or negative orbit)*** *of x related to ϕ is the set $(\gamma_\phi)_0^-(x)$.*

Let us present one final concept before define global attractors, which is invariance. Geometrically speaking, invariance describes if a subset $S \subset M$ will remain inside of itself (shrink), will become bigger or will stay the same as time evolves.

Definition 4 (Invariance and equilibrium point)**.** *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup on a metric space (M, d) and $S \subset M$. We say that*

1. *S is* ***positively invariant*** *if $\mathcal{T}(t)S \subset S$ for all $t \in \mathbb{P}^+$;*
2. *S is* ***negatively invariant*** *if $\mathcal{T}(t)S \supset S$ for all $t \in \mathbb{P}^+$;*
3. *S is* ***invariant*** *if $\mathcal{T}(t)S = S$ for all $t \in \mathbb{P}^+$. In particular, if $S = \{x^*\}$ is an invariant unitary set, we say that x^* is an* ***equilibrium point*** *of $\mathcal{T}$.*

Note that if $x^* \in M$ is an equilibrium point of $\mathcal{T}$, then the constant map $\phi(\cdot) \equiv x^*$ is a global solution of $\mathcal{T}$. Similarly, if $x_0 \in M$ and $\mathcal{T}(t_0)x_0 = x_0$ for some $t_0 > 0$, then

$$\phi(t) = \mathcal{T}(t - nt_0)x_0, \ \forall t \in [nt_0, (n+1)t_0] \cap \mathbb{P}, \ \forall n \in \mathbb{Z}$$

defines a global solution. In this case, we say that ϕ is a periodic solution. In particular, every equilibrium point x^* is a periodic solution.

Now we finally define properly what is a global attractor.

Definition 5 (Global Attractor)**.** *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup on a metric space (M, d). We say $\mathcal{A} \subset M$ is a* ***global attractor*** *if*

1. *$\mathcal{A}$ is compact;*
2. *$\mathcal{A}$ is invariant, i.e. $\mathcal{T}(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{P}^+$;*
3. *$\mathcal{A}$ attracts bounded subsets of M, that is, for each bounded subset B of M*
$$\lim_{t \to +\infty} dist_H(\mathcal{T}(t)B, \mathcal{A}) = 0, \tag{0.2}$$
where $dist_H(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} d(a_1, a_2)$ is the Hausdorff semi-distance between $A_1, A_2 \subset M$.

Note that if $\mathcal{T}$ has a global attractor $\mathcal{A}$, then all equilibrium points and periodic orbits are inside $\mathcal{A}$. In fact since periodic orbits are compact (consequently bounded), it follows from (0.2) that they must be attracted by $\mathcal{A}$ and therefore, from its invariance, they must be inside the global attractor $\mathcal{A}$. The same reasoning shows that the global attractor $\mathcal{A}$ is unique.

As we said, this special set dictates the whole dynamics of the semigroup, since the evolution of any point of the phase space will be close to $\mathcal{A}$. Therefore we are very interested in analyzing the behavior of solutions on $\mathcal{A}$ and on its neighborhood [30, 28, 34, 9, 16, 1, 17, 51, 56, 35]. In fact, a large part of the studies related to infinite-dimensional dynamical systems are based on studying the existence of the global attractor, to understand its structure and to analyze the behavior of the orbits inside $\mathcal{A}$, as we will see below.

The structure of the global attractor $\mathcal{A}$ can provide us several information about the dynamics of the semigroup. In fact, to study the global attractor is so important that there are classes of semigroups, namely Dynamically Gradient semigroups (Definition 8), that are characterized only by its dynamics inside the global attractor. Moreover, the dynamics inside the attractor $\mathcal{A}$ for this class of semigroups provides information about the whole phase space M. To be more precise, continuous Dynamically Gradient semigroups have an "energy function" $V : M \to \mathbb{R}$ on the entire phase space M, called Lyapunov function, which is extremely useful and provides a lot of properties. We will get into more details further on.

We are also very much interested in knowing what happens with the global attractor $\mathcal{A}$ if we make small perturbations on the semigroup. Could the structure of the global attractor remain the same if we perturb the semigroup? This kind of information is related to structural stability of semigroups and continuity of global attractors, as we will see further on, and is connected to some results of this thesis.

However, to understand well what happens inside the global attractor $\mathcal{A}$ and its behavior under perturbations is a very hard task for general nonlinear semigroups. Therefore, to be able to obtain more information regarding $\mathcal{A}$, we have to impose some conditions on the semigroup $\mathcal{T}$, that is, we have to restrict our study to special classes of semigroups. In our case, we will study Morse-Smale semigroups (Definition 11), as we are going to comment more later.

Before embarking on the definition and study of Morse-Smale semigroups, let us define first other types of semigroups. We start with the concept of Gradient semigroup, which is a semigroup that have a Lyapunov function.

Definition 6 (Gradient Semigroup)**.** *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{R}^+\}$ be a continuous semigroup on the metric space M with global attractor $\mathcal{A}$ and set of equilibrium points $\mathcal{E} = \{x_1^*, \dots, x_p^*\}$. We say that $\mathcal{T}(\cdot)$ is **Gradient** if there exists a function $V : M \to \mathbb{R}$ satisfying:*

1. *For each $x \in M$ the map $(0, +\infty) \ni t \mapsto V(\mathcal{T}(t)x)$ is non-increasing.*
2. *If $V(\mathcal{T}(t)x) = V(x)$ for all $t \geq 0$, then $x \in \mathcal{E}$.*

A function $V : M \to \mathbb{R}$ satisfying items 1 and 2 is called a Lyapunov function. Moreover, we will oftenly refer to the value of V at $x \in M$ by the energy of x.

We would like to highlight that our definition of Gradient semigroup is more particular than the usual one, since the original definition allows the set $\mathcal{E}$ to be infinite and does not demand the existence of a global attractor [28]. Moreover, there are alternative versions of Gradient semigroups that allows the set $\mathcal{E}$ to have another elements besides equilibrium points, such as periodic orbits for example [20]. Since in our case we will just have equilibria (and a finite amount of them), we include this hypothesis in the definition. We will proceed similarly with the concepts of Dynamically Gradient and Morse-Smale semigroups that will appear further on.

We now introduce the concept of homoclinic structure, which is based on the notion of "loops" produced by global solutions, before we announce the definition of Dynamically Gradient semigroups.

Definition 7 (Homoclinic structure)**.** *Assume that $\mathcal{T}(\cdot)$ is a semigroup on a metric space M with set of equilibria $\mathcal{E}$. An **homoclinic structure** is a subset $\{x^*_{l_1}, \dots, x^*_{l_k}\} \subset \mathcal{E}$ and a set of non constant global solution $\{\xi_i : \mathbb{P} \to M\}_{i=1,\dots,k}$, satisfying*

$$\lim_{t\to-\infty} \xi_i(t) = x^*_{l_i} \quad \text{and} \quad \lim_{t\to+\infty} \xi_i(t) = x^*_{l_{i+1}}, \quad i = 1, \dots, k$$

where we define $x^*_{l_{k+1}} = x^*_{l_1}$.

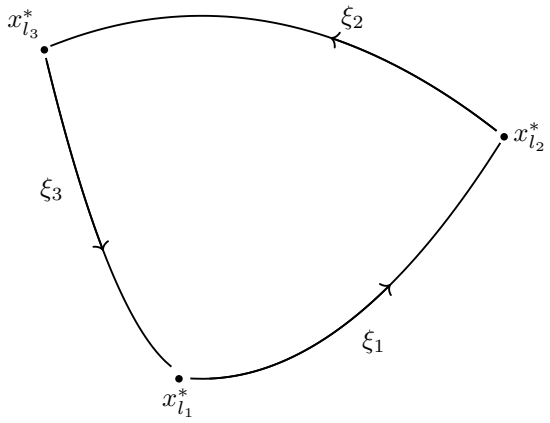


Figure 0.1: Example of homoclinic structure with $k = 3$

We are now ready to announce the definition of Dynamically Gradient semigroups. Instead of requiring the existence of a Lyapunov function, we will require certain dynamical behavior inside its global attractor.

Definition 8 (Dynamically Gradient Semigroup)**.** *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space (M, d), with global attractor $\mathcal{A}$ and a finite number of equilibria that we denote by $\mathcal{E} = \{x^*_1, \dots, x^*_p\}$. Then, we say that $\mathcal{T}(\cdot)$ is **Dynamically Gradient** if:*

(G1) For every bounded global solution $\xi : \mathbb{R} \to \mathcal{A}$ there exists $m, k \in \{1, \dots, p\}$ such that

$$\lim_{t\to-\infty} \xi(t) = x^*_m \text{ and } \lim_{t\to+\infty} \xi(t) = x^*_k.$$

(G2) There is no homoclinic structure in $\mathcal{A}$.

It is known that any Gradient semigroup is also Dynamically Gradient [9] (a sketch of the proof can be found in Proposition 1.6). As we can see, Dynamically Gradient semigroups are characterized only by its global attractor and as we will see later, in Section 1.1, every Dynamically Gradient semigroup is also Gradient, that is, both concepts are actually equivalent (see Corollary 1.11). Therefore, an information about the global attractor (which is a compact set) has provided us information about the whole phase space (that can be an infinite dimensional vector space). Moreover, because of its simple dynamics, it is easier to prove that Dynamically Gradient semigroups are robust under perturbations [20, 9, 1] instead of proving this property for Gradient Semigroups.

Now that we have established some of the importance of the global attractor, we will begin to address the more specific issues of this work. Our main goal is to prove that a certain property,

called Shadowing, occurs within the global attractor of Morse-Smale semigroups, which is a special class of semigroups. As we have mentioned above in this introduction, we are interest in obtain the Shadowing property because with it we can obtain results related to the robustness of the dynamics inside the global attractor and continuity of global attractors.

Before we present Morse-Smale semigroups, we will briefly discuss about the concepts of hyperbolic equilibria, non-wandering sets and unstable/stable manifolds, which are necessary concepts to define Morse-Smale semigroups. Since the definition of hyperbolicity is more technical and requires more mathematical background, we will just give an intuition about it and refer the reader to Definition 20 for the detailed description. Basically speaking, we say that an equilibrium is hyperbolic if it has some properties of exponential attraction and repulsion. Moreover, hyperbolic equilibria are preserved under small perturbations, that is, if $\mathcal{T}$ is a semigroup with a hyperbolic equilibrium x^*, then small perturbations of $\mathcal{T}$ will also have a hyperbolic equilibrium, which is close to x^*.

Let us now define the non-wandering points and non-wandering set. Geometrically, non-wandering points are points such that its orbit will always return close to where they started, even after a great amount of time.

Definition 9 (Non-wandering set)**.** *Let (M,d) be a metric space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup on M. We say that $x \in M$ is a **non-wandering** point of $\mathcal{T}(\cdot)$ if for any $t_0 \geq 0$ and neighborhood V_x of x there exists $t > t_0$, $t \in \mathbb{P}$, such that $\mathcal{T}(t)V_x \cap V_x \neq \emptyset$. We will denote the set of non-wandering points of $\mathcal{T}(\cdot)$ by Ω.*

Note that if x^* is a point in a periodic orbit then $\mathcal{T}(\tau)x^* = x^*$ for some $\tau > 0$ and therefore $\mathcal{T}(n\tau)x^* = x^*$ for all $n \in \mathbb{N}$. This implies that $x^* \in \Omega$. In particular, Ω contains all the periodic orbits and all the equilibrium points.

Let us talk about the unstable and stable manifolds before we proceed with the definition of Morse-Smale semigroups.

Definition 10 (Stable and unstable manifolds)**.** *Let $\mathcal{T} = \{\mathcal{T}(t) : t \in \mathbb{P}^+\} \subset \mathcal{C}(M)$ be a semigroup on a metric space (M,d), x^* be an equilibrium point and $B_M(x^*, r) := \{x \in M : d(x, x^*) < r\}$ be the open ball centered in x^* with radius $r > 0$. The **local stable manifold** $W^s_\delta(x^*)$ and the **local unstable manifold** $W^u_\delta(x^*)$ are given by the sets*

$$W^s_\delta(x^*) = \{x \in B_M(x^*, \delta) : \mathcal{T}(t)x \in B_M(x^*, \delta) \text{ for all } t \in \mathbb{P}^+ \text{ and } \lim_{t\to+\infty} \mathcal{T}(t)x = x^*\}$$

and

$$W^u_\delta(x^*) = \{x \in B_M(x^*, \delta) : \text{there exists a global solution } \phi \text{ through } x \text{ such that } \phi(t) \in B_M(x^*, \delta) \text{ for } t \in \mathbb{P} \cap (-\infty, 0] \text{ and } \lim_{t\to-\infty} \phi(t) = x^*\}.$$

We often denote the local stable and unstable manifolds as $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$ when $\delta > 0$ is understood.

*We also define the **stable manifold** $W^s(x^*)$ and **unstable manifold** $W^u(x^*)$ of x^* as*

$$W^s(x^*) = \{x \in M : \lim_{t\to+\infty} \mathcal{T}(t)x = x^*\}$$

and

$$W^u(x^*) = \{x \in M : \text{there exists a global solution } \phi \text{ through } x \text{ such that } \lim_{t\to-\infty} \phi(t) = x^*\}.$$

Under certain conditions (namely reversibility of the semigroup [9, Definitions 4.18 and 5.2]), it is known that if x^* is a hyperbolic equilibrium then its unstable manifold $W^u(x^*)$ is a $\mathcal{C}^1$-manifold [9,

Chapter 4]. This is the case for Morse-Smale semigroups, for example. A sketch of the proof is given in Theorem 1.12. Similarly, we can prove that $W^s_{loc}(x^*)$ is a $\mathcal{C}^1-$manifold when x^* is hyperbolic and $\mathcal{T}$ satisfies certain properties. Knowing that this special sets are manifolds allows us to use the concepts of transversality. Without getting into too many technical details, transversality can be understood as the opposite of tangency. That is, two submanifolds S_1, S_2 of a manifold S intersect transversely at the point $x \in S_1 \cap S_2$ if $T_xS_1 + T_xS_2 = T_xS$, where T_xP denotes the tangent space of the manifold P at the point x. Similar to hyperbolicity, transversality is an important property because of its robustness, that is, if we have transversality then small perturbation of the submanifolds will still intersect transversely [10, Lemma 5.8].

Now we introduce one of the key concepts of this work, the concept of Morse-Smale semigroup. In fact, Morse-Smale will be the class of semigroups that we are going to work with through almost the whole thesis. As we said before, we recommend the reader that is not familiar with the concepts of hyperbolicity to see Section 1.1.

Definition 11 (Morse-Smale)**.** *Let X be a Banach space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}^1(X)$ be a continuous semigroup with global attractor $\mathcal{A}$. We say that $\mathcal{T}(\cdot)$ is a **Morse-Smale** semigroup if it satisfies the following conditions:*

1. *$\mathcal{T}(t)|_{\mathcal{A}}$ is injective for all $t \geq 0$.*
2. *The Fréchet derivative $D_z\mathcal{T}(t) : X \to X$ of $\mathcal{T}(t)$ at z is an isomorphism onto its image for all $z \in \mathcal{A}$ and $t \geq 0$.*
3. *The non-wandering set of $\mathcal{T}(\cdot)$ is given by the set of equilibrium points $\Omega = \mathcal{E} = \{x_1^*, \cdots, x_p^*\}$, which are a finite number of them and each one is hyperbolic. In particular, there are no periodic orbits.*
4. *$\dim W^u_{loc}(x^*) < +\infty$ for every $x^* \in \Omega$, where dim is the dimension related to the differentiable manifold.*
5. *$W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ are transverse for all $i \neq j$, i.e., if $z \in W^u(x_i^*) \cap W^s_{loc}(x_j^*)$, then*

$$T_zW^u(x_i^*) + T_zW^s_{loc}(x_j^*) = X,$$

where $T_zW^u(x_i^*)$ and $T_zW^s_{loc}(x_j^*)$ are the tangent spaces of $W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ at z, respectively.

This class of semigroups have been on the literature for a while and is still extensively studied [30, 8, 9, 16, 13, 11]. On Section 1.1 we prove that Morse-Smale semigroups are also Dynamically Gradient (and Gradient) semigroups. In fact, those semigroups have a close relation [9, Corollary 5.12]. We would also like to recall that the usual definition of Morse-Smale semigroup just demand that $D_x\mathcal{T}(t)$ is a bounded injective operator, for all $t > 0$ and $x \in \mathcal{A}$ [9]. In order to simplify the text, we have redefined Morse-Smale semigroups as before, which still generalizes the concept of Morse-Smale dynamical systems on finite dimension, where the derivative is an isomorphism and the maps $\mathcal{T}(t)$ are diffeomorphisms.

Similarly to Definition 11, the notion of discrete Morse-Smale systems also exists [43]. Remembering that a map f can define a discrete semigroup (Remark 1), we can also define the Morse-Smale property for f. In fact, if M is a compact smooth manifold, Kupka and Smale showed that the set of maps $f : M \to M$ that satisfy the Morse-Smale property is generic (in the Whitney topology) in the set $\mathcal{C}^r(M)$ of C^r-differentiable maps from M into M [40, 60, 46, 43]. In the infinite dimensional context, which is our case, we also have a similar result that is obtained with the damped wave equation [13]. We will study this example with more details in Section 1.2.

Finally, we define the other key concept of this thesis, which is Shadowing, and then provide details about our results and the motivation behind this investigation.

Definition 12 (Shadowing)**.** *Let (M,d) be a metric space, $\mathcal{T}: M \to M$ be a map and $\mathbb{T} = \mathbb{Z}$ or $\mathbb{Z}^-$.*

1. *A sequence $\{x_n\}_{n\in\mathbb{T}}$ in M is said to be a δ-**pseudo orbit** of $\mathcal{T}$ for some $\delta > 0$ if*
$$d(\mathcal{T}x_n, x_{n+1}) \leq \delta, \ \forall\ n \in \mathbb{T}.$$

2. *Let $\{x_n\}_{n\in\mathbb{T}}$ and $\{z_n\}_{n\in\mathbb{T}}$ be sequences on M. We say that $\{z_n\}_{n\in\mathbb{T}}$ ϵ-**shadows** $\{x_n\}_{n\in\mathbb{T}}$, for some $\epsilon > 0$ if*
$$d(x_n, z_n) \leq \epsilon, \ \forall\ n \in \mathbb{T}.$$

3. *We say that $\mathcal{T}$ admits the **Shadowing** property if for any $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $\{x_n\}_{n\in\mathbb{T}}$ of $\mathcal{T}$ is ϵ-shadowed by an orbit of $\mathcal{T}$, that is, there exists $x \in M$ such that*
$$d(x_n, \mathcal{T}^n x) \leq \epsilon, \forall\ n \in \mathbb{T}.$$

4. *We say that $\mathcal{T}$ admits the α-**Hölder Shadowing** property if there exists $d_0, L > 0$ and $\alpha \in (0,1]$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is Ld^α-shadowed by an orbit of $\mathcal{T}$. If $\alpha = 1$ we say that $\mathcal{T}$ have the **Lipschitz Shadowing** property.*

5. *We say that $\mathcal{T}$ admits the **Logarithm Shadowing** property if there exists $d_0, L > 0$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is $Ld|\ln d|$-shadowed by an orbit of $\mathcal{T}$.*

Note that if $\alpha \in (0,1)$ then

$$\textit{Lipschitz Shadowing} \Rightarrow \textit{Logarithm Shadowing} \Rightarrow \alpha - \textit{Hölder Shadowing} \Rightarrow \textit{Shadowing}.$$

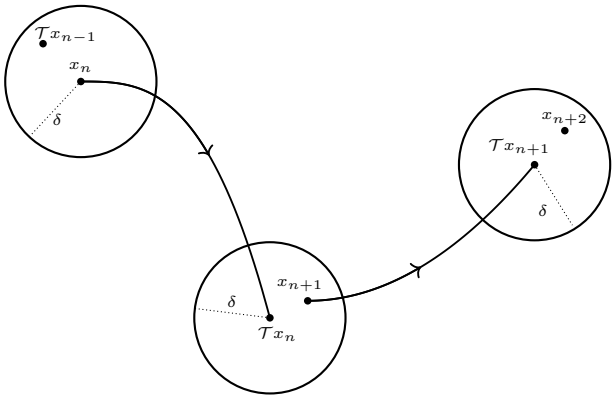


Figure 0.2: $\delta-$pseudo orbit of $\mathcal{T}$

The Shadowing property is a well known tool in the theory of dynamical systems and has been intensively studied over the years [47, 49, 41, 44, 48, 33, 27, 22, 58]. Such property has a lot of important consequences on finite dimensional dynamical systems and also on applied sciences. Moreover, Shadowing has also been studied on infinite dimensional phase spaces, see e.g. [47, 33, 22, 42].

It is known that if $\mathcal{T}(\cdot)$ is a continuous Morse-Smale semigroup defined on finite dimensional compact manifold without border, then $\mathcal{T}(1)$ has the Lipschitz Shadowing property [47, Chapter 2]. By compactification, the same holds if the phase space is $\mathbb{R}^n$ [58] and in particular we have Lipschitz Shadowing on a neighborhood of the global attractor $\mathcal{A} \subset \mathbb{R}^n$. On [58, 6] the authors used this property to obtain results about continuity of global attractors on $\mathbb{R}^n$ and on infinite

dimensional vector spaces. To be more specific, the authors assumed that the semigroup possess an inertial manifold $\mathcal{M}$ [25, 36, 55], that is a finite dimensional invariant manifold, to apply known results about Shadowing on compact manifolds. With this, they were able to obtain an optimal rate of convergence of the global attractors.

Unfortunately, inertial manifolds are not easily found in applications, since they usually request a strong gap condition on the eigenvalues of the linear operator associated to the equation [54]. Therefore, a natural question arises: is it possible to obtain Shadowing on infinite dimensional vector spaces without requesting the existence of inertial manifolds? As far as we know, the only result of Shadowing on a neighborhood of the global attractor in the infinite dimensional setting demands the existence of an inertial manifold (which is finite dimensional) and then appeal to the finite dimensional case, as we can see in [47, Section 3.4].

With this in mind, we were motivated to investigate Shadowing properties on semigroups, defined in infinite dimensional spaces, that does not (necessarily) have an inertial manifold $\mathcal{M}$. To be more precise, we want to prove that Morse-Smale semigroups, defined on infinite dimensional vector spaces, have the Shadowing property on a neighborhood of its global attractor $\mathcal{A}$, even without the existence of an inertial manifold. In fact, we have made great progress in this direction and the main result of this thesis states that we can remove the hypothesis related to the inertial manifold $\mathcal{M}$ and still preserve Shadowing, as we can see below.

Theorem 0.1. *Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup on a Hilbert space X with global attractor $\mathcal{A}$. Assume that the Fréchet derivative $D_x\mathcal{T}(t)$ of $\mathcal{T}(t)$ at x that satisfies:*

(H1) There exists $C_0, C_1 > 0$ such that

$$\| D_x\mathcal{T}(t) \|_{\mathcal{L}(X)} \leq C_1 e^{C_0 t}, \ \forall t \geq 0, \ \forall x \in \mathcal{A};$$

(H2) For each $x \in \mathcal{A}$ and $v \in X$ the map $[0, +\infty) \ni t \mapsto (D_x\mathcal{T}(t))v \in X$ is continuous.

Then,

1. *$\mathcal{T}(1)|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ has the Lipschitz Shadowing property (where $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}^-$);*
2. *For any positively invariant bounded neighborhood $\mathcal{U} \supset \mathcal{A}$, the map $\mathcal{T}(1)|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ has the α-Hölder-Shadowing property (where $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}^-$).*

Theoretically speaking, Theorem 0.1 is a great advance for the theory of Dynamical Systems, since Shadowing is a value property, with many applications, and as far as we know it is the first result of Shadowing on a neighborhood of the global attractor $\mathcal{A}$ that does not appeal to the result on finite dimension. We recall that the result of Lipschitz Shadowing for Morse-Smale semigroups defined in finite dimensional smooth compact manifolds (with no border) [47] also requires that items (H1) and (H2) hold, that is, Theorem 0.1 extends the known results of Lipschitz Shadowing on the global attractor to the infinite dimensional setting. Moreover, items (H1) and (H2) are also often satisfied in the infinite dimensional context [20, Theorem 6.33], especially when the semigroup $\mathcal{T}$ satisfies the variation of constants formula (see Definition 29). As an application of Theorem 0.1, we will obtain new information related to the geometrical structral stability of Morse-Smale semigroups [9, Chapter 5]. To be more precise, we will show that if $\mathcal{T}_0$ is a Morse-Smale semigroup satisfying items (H1) and (H2), then the bounded global orbits of a small perturbation $\mathcal{T}_\eta$ of $\mathcal{T}_0$ will be close to the bounded global orbits of $\mathcal{T}_0$, as we are going to comment further on.

We would like to highlight that the semigroup related to the damped wave operator, that will be commented in Section 1.2, does not satisfy the gap condition required to obtain an inertial manifold, see [54], and consequently it was not possible to obtain shadowing on a neighborhood of its global attractor by the already known results [47, Section 3.4]. Despite that, since the semigroup related

to the damped wave operator satisfies all assumptions from Theorem 0.1, when the nonlinearity is regular enough (see Corollary 1.34), we guarantee that the shadowing property still holds on a neighborhood of its attractor.

To better understand the difficulties in proving Theorem 0.1, we will now provide a quick sketch of our ideas and assure where the main difficulties lie when adapting results from finite dimensional dynamics to infinite dimensional ones. Our strategy was to study the (very beautiful) proof of Lipschitz Shadowing for Morse-Smale semigroups on compact smooth manifolds M (with no border), that is given in [47, Section 2.2], and then try to adapt it to infinite dimensional setting. Through his analysis, the author used the following (essential) facts:

(P1) The dimension of the compact manifold M is finite and consequently $\mathcal{T}(t) : M \to M$ is a diffeomorphism, that is, the Fréchet derivative $D_x\mathcal{T}(t) : M \to M$ of $\mathcal{T}(t)$ at x is an isomorphism for all $x \in M$ and $t \geq 0$. In particular, $\mathcal{T}(-t) := (\mathcal{T}(t))^{-1}$ is well defined and $\mathcal{T}$ defines a group in M, that is, $\mathcal{T}(t)$ is defined for all $t \in \mathbb{R}$;

(P2) M is a finite dimensional smooth manifold and therefore the concepts and tools from geometry are available, such as continuity of subbundles and the exponential map;

(P3) M is a smooth (finite dimensional) manifold and consequently, $W^s(x^*)$ and $W^u(x^*)$ are $\mathcal{C}^1$-manifolds for any hyperbolic equilibrium $x^* \in \Omega$;

(P4) $M = \bigcup_{x^* \in \mathcal{E}} W^u(x^*)$, where $\mathcal{E}$ denotes the set of equilibria.

Since (P1)-(P4) are essential to prove Lipschitz Shadowing on a smooth compact manifold M, the generalization to the infinite dimensional setting is not straightforward. To be more precise, assume that $\mathcal{T}$ is a Morse-Smale semigroup on the Banach space X and $\mathcal{U}$ is a positively invariant neighborhood of the global attractor $\mathcal{A}$. Defining the semigroup $\mathcal{T}|_{\mathcal{U}}$ we have:

(a) (P1) does not hold, since $D_x\mathcal{T}$ is not necessarily an isomorphism if X has infinite dimension. Consequently, $\mathcal{T}(t)$ is not a diffeomorphism and $\mathcal{T}|_{\mathcal{U}}$ does not define a group in $\mathcal{U}$;

(b) Regarding to (P2), if X is a Banach space and $\mathbb{S} = \{S(\lambda) \subset X\}_{\lambda \in \Lambda}$ is a family of subspaces of X, we have to provide a meaning to the statement "the family $\mathbb{S}$ is continuous" that gives us the properties expected, that were valid in the the context of manifolds;

(c) If X is a Banach space, we just know that $W^u(x^*)$ is a (finite dimensional) manifold. In relation to the stable manifold, we just know that $W^s_{loc}(x^*)$ is a $\mathcal{C}^1$-manifold (see Section 1.1). Consequently, (P3) is not necessarily satisfied;

(d) About (P4), we know that Morse-Smale semigroups are Gradient (Lemma 1.17) and in particular $\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} W^u(x^*)$ (see Theorem 1.3). Thus $\mathcal{U} \neq \bigcup_{x^* \in \mathcal{E}} W^u(x^*)$.

Putting all this difficulties together, we understand that our goal to extend Shadowing is challenging. With this in mind, we choose to proceed as follows: first we adapt the results about Lipschitz Shadowing only inside the global attractor $\mathcal{A}$ and then we try to extend this property obtained in $\mathcal{A}$ to the whole neighborhood $\mathcal{U}$. Indeed, if we consider the semigroup $\mathcal{T}|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ restricted to the global attractor, at least we know that $\mathcal{T}(t)$ is a group (Proposition 1.14), $D_x\mathcal{T}(t)$ is an isomorphism onto its range for all $x \in \mathcal{A}$, and $\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} W^u(x^*)$, which helps us a little bit with items (P1) and (P4). Let us first focus on the difficulties to prove Lipschitz Shadowing in the global attractor.

To understand the obstacles in proving that $\mathcal{T}|_{\mathcal{A}}$ has the Lipschitz Shadowing property when $\mathcal{T}$ is Morse-Smale and X is infinite dimensional, we can go over items (P1)-(P3) again. As we said before, the global attractor provides us a better structure then a neighborhood of itself, but items (P2) and (P3) are still a problem. Moreover, with respect to item (P1), $D_x\mathcal{T}(t)$ is not surjective,

which is one of the greatest issues in the proof. In fact, the main key to prove our result is to construct families of compatible subbundles (see Subsection 2.1.3), that appeared for the first time in [52]. Those subbundles consist on families of subspaces $\{S(x)\}_{x\in\Gamma}$ and $\{U(x)\}_{x\in\Gamma}$, where Γ is a set of parameters, that are defined through the tangent spaces $T_xW^s(x_i^*)$ and $T_xW^u(x_i^*)$ (that is the reason behind (P3)) and behaves continuously (that is why we need to solve (P2)). Moreover, to prove the existence of this subbundles, the surjectivity of the derivative $D_x\mathcal{T}(t)$ is essential (that is the reason behind (P1)). Let's go into a little more detail about each problem.

Regarding (P1) and the lackness of surjectivity of $D_x\mathcal{T}(t)$ when X is infinite dimensional: as we said above, the strategy to prove Lipschitz Shadowing in the global attractor $\mathcal{A}$ is to construct compatible subbundles. This construction is very technical and the main key to obtain this subbundles is Theorem 2.15, which is a generalization of [47, Lemma 2.2.9] to the infinite dimensional setting. To show [47, Lemma 2.2.9] in finite dimensional compact manifolds (with no border), the author use the Principle of Finite Induction, together with several technical lemmas, and appeals to the surjectivity of the derivative $D_x\mathcal{T}(t)$ to prove the induction hypothesis. Therefore, the surjectivity of the derivative is essential to prove [47, Lemma 2.2.9] and Lipschitz Shadowing on finite dimensional compact manifolds. To deal with the lackness of surjectivity in the infinite dimensional setting we need to explore more deeply the dynamics in $\mathcal{A}$ (Lemmas 2.16 and 2.17).

About (P2) and the lackness of smooth manifold structure: as we said above, Theorem 2.15 is the key to construct the compatible subbundles in $\mathcal{A}$ and we need several technical lemmas to prove it. In the context of finite dimensional compact smooth manifolds, the concepts and tools from geometry, such as continuity of subbundles (that are not available in our case) are used to prove several of this auxiliary lemmas. Consequently, the continuity of subbundles and the smooth manifold structure has a big role in the proof of [47, Lemma 2.2.9] and is an essential part of the construction of the compatible subbundles. That is the main reason behind the assumption of X being a Hilbert space in Theorem 0.1, since we still need to obtain geometric results in the infinite dimensional setting, where $\mathcal{A}$ is not a smooth manifold.

Concerning (P3) and $W^s(x^*)$: once again, the construction of the compatible subbundles is the reason behind the importance of the $\mathcal{C}^1$ manifold structure of $W^s(x^*)$ in the finite dimensional case. In fact, the proof of the finite dimensional version of Theorem 2.15 [47, Lemma 2.2.9] is very constructive and depends on the continuity of the subbundles $\{T_xW^s(x^*)\}_{x\in W^s(x^*)}$ and $\{T_xW^u(x^*)\}_{x\in W^u(x^*)}$, for each $x^*\in\Omega$. Hence, in the finite dimensional context it is important that $W^s(x^*)$ is a $\mathcal{C}^1$ manifold. To overcome this problem in the infinite dimensional setting, we will extend the family $\{T_xW^s(x^*)\}_{x\in W^s_{loc}(x^*)}$ in order to still maintain the properties that were hold in the finite dimensional case (where $T_xW^s(x^*)$ was defined for each $x\in W^s(x^*)$).

Once we have obtained the Lipschitz Shadowing property inside the global attractor, we can briefly discuss our strategy to extend the property of Shadowing on a neighborhood $\mathcal{U}\supset\mathcal{A}$. In fact, after showing Lipschitz Shadowing on $\mathcal{A}$ it is easier to extend the property on $\mathcal{U}$. The scheme of the proof is:

(N1) First we prove that δ-pseudo orbits $\{x_n\}_{n\in\mathbb{Z}}\subset\mathcal{U}$ of $\mathcal{T}(1)$ on $\mathcal{U}$ are close to $\mathcal{A}$, so that if $\delta\to 0$ then the $\delta-$pseudo orbit approaches $\mathcal{A}$;

(N2) We obtain a new sequence $\{y_n\}_{n\in\mathbb{Z}}\subset\mathcal{A}$ that is closed to $\{x_n\}_{n\in\mathbb{Z}}$ and is a $\delta'-$pseudo orbit;

(N3) We use the Lipschitz Shadowing on $\mathcal{A}$ to extract an orbit $\{\mathcal{T}(n)x\}_{n\in\mathbb{Z}}$, where $x\in\mathcal{A}$, that shadows $\{y_n\}_{n\in\mathbb{Z}}$;

(N4) We show that $\{\mathcal{T}(n)x\}_{n\in\mathbb{Z}}$ shadows $\{x_n\}_{n\in\mathbb{Z}}$ and finish the proof.

The hardest part of this proof is to show (N1) and after that, the proofs of items (N2), (N3) and (N4) are straightforward. As we will see on Chapter 3, we need to guarantee that the smaller the $\delta>0$, the closer the $\delta-$pseudo orbit is to the $\mathcal{A}$. Remember that if $\delta=0$ then $\{x_n\}_{n\in\mathbb{Z}}$ is a

bounded global solution and therefore $x_n \in \mathcal{A}$ (see Proposition 1.1), i.e., the distance of x_n to the global attractor is zero. Now we announce our main result related to the proof of (N1). It is in this step of the proof that we lose Lipschitz Shadowing and obtain only Hölder Shadowing on $\mathcal{U}$.

Lemma 0.2. *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space (M,d) with global attractor $\mathcal{A}$. Assume that there exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$ that is positively invariant, $\mathcal{T} := \mathcal{T}(1)|_{\mathcal{U}}$ is Lipschitz and*

$$dist_H(\mathcal{T}(n)\mathcal{U}, \mathcal{A}) \leq Ce^{-\gamma n}, \ \forall n \in \mathbb{N} \tag{0.3}$$

for some $C, \gamma > 0$. Then there exists $C_1 > 0$ and $\alpha \in (0,1)$ such that if $\{x_n\}_{n\in\mathbb{T}}$ is a d-pseudo-orbit of $\mathcal{T}$ in $\mathcal{U}$, $0 < d < 1$, then

$$d(x_n, \mathcal{A}) \leq C_1 \cdot d^{\alpha}, \ \forall n \in \mathbb{T}. \tag{0.4}$$

Note that Lemma 0.2 helps us to prove Hölder Shadowing on the neighborhood $\mathcal{U}$ but does not have any assumption related to Shadowing. In fact, Lemma 0.2 only requires that $\mathcal{A}$ attracts the neighborhood $\mathcal{U}$ exponentially, which is the case for any Gradient semigroup with hyperbolic equilibria [17]. In particular, the global attractor of Morse-Smale semigroups satisfies (0.3). We will also use Lemma 0.2 on Chapter 4 to guarantee general results about continuity of global attractors (see Section 4.2). We will comment more about the results of continuity of global attractors further on.

To finish this Introduction, we now comment about the applications related to the geometrical structural stability of Morse-Smale semigroups and continuity of global attractors. It is well known that Morse-Smale semigroups are topologically structurally stable, that is, small perturbations $\mathcal{T}_\eta$ of a Morse-Smale semigroup $\mathcal{T}_{\eta_0}$ are still Morse-Smale [9, Theorem 5.14]. In fact, Morse-Smale semigroups are even geometrically structural stable, that is, the global attractors $\mathcal{A}_\eta$ of the perturbed semigroups $\mathcal{T}_\eta$ have a phase-diagram isomorphism (as we are going to define below), with the global attractor $\mathcal{A}_{\eta_0}$ of $\mathcal{T}_{\eta_0}$.

Definition 13 (Phase-Diagram Isomorphism)**.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two dynamically gradient semigroups on a metric space (M,d), with global attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ and equilibria $\mathcal{E}_1 = \{x_1^*, \dots, x_p^*\}$ and $\mathcal{E}_2 = \{y_1^*, \dots, y_p^*\}$, respectively. We say that there exists a **phase diagram isomorphism** between $\mathcal{A}_1$ and $\mathcal{A}_2$ if there exists a bijection $D : \mathcal{E}_1 \to \mathcal{E}_2$ with the property: there exists $1 \leq i, j \leq p$ and a global solution $\xi : \mathbb{R} \to M$ of $\mathcal{T}_1$ satisfying*

$$\lim_{t\to-\infty} d(\xi(t), x_i^*) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t), x_j^*) = 0,$$

if and only if there is a global solution $\psi : \mathbb{R} \to M$ of $\mathcal{T}_2$ such that:

$$\lim_{t\to-\infty} d(\psi(t), D(x_i^*)) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\psi(t), D(x_j^*)) = 0.$$

However, the proof of the geometrical structural stability (Theorem 4.1) does not provide any information about the global solutions of the perturbed problem, i.e., we don't know if the bounded global solutions of the perturbed problem (that are contained in the perturbed global attractor) remain close to a bounded global solution of the limit problem. In order to prove that, in Section 4.1 we apply Theorem 0.1 to the Morse-Smale semigroup $\mathcal{T}_{\eta_0}$. To give the main idea behind the proof, let X be the Hilbert space where those semigroups are defined and $\mathcal{U}$ be the neighborhood of $\mathcal{A}_{\eta_0}$ with the properties stated in Theorem 0.1. Then for any $x \in \mathcal{A}_\eta$ we can define the sequence $x_n = \mathcal{T}_\eta(1)^n x$ for any $n \in \mathbb{Z}$, which is the orbit of x related to the semigroup $\mathcal{T}_\eta$. If we assume that $\mathcal{A}_\eta \subset \mathcal{U}$, then

$$d(\mathcal{T}_{\eta_0}(1)x_n, x_{n+1}) = d(\mathcal{T}_{\eta_0}(1)\mathcal{T}_\eta(1)^n x, \mathcal{T}_\eta(1)\mathcal{T}_\eta(1)^n x) \leq \| \mathcal{T}_\eta(1) - \mathcal{T}_{\eta_0}(1) \|_{L^\infty(\mathcal{U},X)} . \tag{0.5}$$

Therefore, the bounded orbits (or solutions) of the perturbed problem $\mathcal{T}_\eta$ are pseudo orbits of $\mathcal{T}_{\eta_0}$ and from Theorem 0.1 there exists a bounded orbit of $\mathcal{T}_{\eta_0}$ which is closed to the orbit x_n and conclude the proof. We have also considered non-autonomous perturbations (Corollary 4.4) of the semigroup $\mathcal{T}_{\eta_0}$.

Now we will comment about our results of continuity of global attractors. To provide more details, let us define our notion of continuity, which is based on the Hausdorff semi-distance given on Definition 5.

Definition 14. *Let (M,d) and Λ be metric spaces and $\mathcal{B} := \{B_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of M. We say that*

- *The family $\mathcal{B}$ is **upper semicontinuous** at $\lambda_0 \in \Lambda$ if*

$$dist_H(B_\lambda, B_{\lambda_0}) \to 0 \quad as \quad \lambda \to \lambda_0.$$

  If the family $\mathcal{B}$ is upper semicontinuous at λ_0 for any $\lambda_0 \in \Lambda$, we say that $\mathcal{B}$ is upper semicontinuous.

- *The family $\mathcal{B}$ is **lower semicontinuous** at $\lambda_0 \in \Lambda$ if*

$$dist_H(B_{\lambda_0}, B_\lambda) \to 0 \quad as \quad \lambda \to \lambda_0.$$

  If the family $\mathcal{B}$ is lower semicontinuous at λ_0 for any $\lambda_0 \in \Lambda$, we say that $\mathcal{B}$ is lower semicontinuous.

- *The family $\mathcal{B}$ is **continuous** on $\lambda_0 \in \Lambda$ if is upper and lower semicontinuous at λ_0. If the family $\mathcal{B}$ is continuous at λ_0 for any $\lambda_0 \in \Lambda$, we say that $\mathcal{B}$ is continuous.*

Sometimes, it is possible to dictate the distance of two global attractors $\mathcal{A}_0$ and $\mathcal{A}_\eta$ of the semigroups $\mathcal{T}_0$ and $\mathcal{T}_\eta$, respectively, through the distance of the semigroups $\mathcal{T}_0$ and $\mathcal{T}_\eta$. For example, if $\{\mathcal{T}_\eta(\cdot) : M \to M\}_{\eta\in[0,1]}$ is a family of semigroups defined in a metric space (M,d) and $\mathcal{U}$ is a neighborhood of $\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta$, there are conditions that guarantee the estimate

$$dist_H(\mathcal{A}_0, \mathcal{A}_\eta) + dist_H(\mathcal{A}_\eta, \mathcal{A}_0) < L \parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel^\alpha_{L^\infty(\mathcal{U},M)}, \ \forall \eta \in [0,1], \tag{0.6}$$

for some $L > 0$ and $\alpha \in (0,1]$. In this case, if $\parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel_{L^\infty(\mathcal{U},M)} \to 0$ as $\eta \to 0^+$, we have continuity of global attractors at 0.

With these notions of continuity, in Section 4.2 we prove that under certain (usual) conditions, such as equi-attraction and continuity of the family of semigroups, we have upper semicontinuity (Theorems 4.7 and 4.8) and continuity (Theorems 4.9 and 4.10) of global attractors. In fact, if the maps $\mathcal{T}_\eta(1)|_{\mathcal{U}}$ are Lipschitz, with Lipschitz constants uniformly bounded, we recover well known results of continuity of attractors [51, 24] using Lemma 0.2, with very short proofs and easy to follow. We will also provide a new result of continuity of global attractors, which allow us (under certain hypothesis) to approach the exponent α, related to the rate of convergence of attractors in equation (0.6), to 1 (Theorem 4.11).

To understand the idea behind the proofs, remember that we already know from (0.5) that the bounded orbits from a perturbed semigroup $\mathcal{T}_\eta$ are pseudo orbits of $\mathcal{T}_{\eta_0}$ and from Lemma 0.2 that the pseudo orbits of $\mathcal{T}_{\eta_0}$ are close to the global attractor $\mathcal{A}_{\eta_0}$. Hence, we know that the bounded orbits of $\mathcal{T}_\eta$ are close to $\mathcal{A}_{\eta_0}$ and thanks to the characterization of the global attractor (Proposition 1.1), we obtain that $\mathcal{A}_\eta$ is close to $\mathcal{A}_{\eta_0}$, which guarantees the upper semicontinuity. The proof of the lower semicontinuous is similar, but demands that the whole family of semigroup $\{\mathcal{T}_\eta\}_\eta$ have the same attraction rate. Note that in the sketch of the proof we did not use Shadowing in any moment, we have just used Lemma 0.2, that does not request that the semigroup is Morse-Smale (Lemma 3.8).

Finally, we describe how we have organized this work.

In Chapter 1 we define the main concepts of this work and provide a background on some topics from Dynamical Systems and Differential Equations. We have reserved Section 1.1 to show the relations between Gradient, Dynamically Gradient and Morse-Smale semigroups, as well as some properties of this classes of semigroups. In Section 1.2 we provide basic results related to the damped wave equation, such as the existence of global attractor for example. It is also in this Section that we give an example of Morse-Smale semigroups defined on a Hilbert space (Corollary 1.34).

In Chapter 2 we focus only on the dynamics inside of the global attractor $\mathcal{A}$ and proved item 1 of Theorem 0.1. In fact, in Section 2.1 we construct the compatible subbundles that were discussed before and deal with problems (P1)-(P3). In Section 2.2 we use the subbundles constructed on Section 2.1 to prove the Lipschitz Shadowing property on the global attractor.

In Chapter 3 we show general properties (without assuming that the semigroup is Morse-Smale) related to pseudo orbits that are valid on a neighborhood of $\mathcal{A}$, that is, we proved that if $\{x_n\}_{n\in\mathbb{Z}}$ is a δ-pseudo orbit on a neighborhood of $\mathcal{A}$, then the smaller is the $\delta > 0$ the closer x_n is to $\mathcal{A}$ (Lemmas 3.1 and 3.8). We have applied this results to Morse-Smale semigroups to prove Shadowing and Hölder-Shadowing on a neighborhood $\mathcal{U} \supset \mathcal{A}$ on Sections 3.1 and 3.2, respectively.

In Chapter 4 we mix results from Chapters 2 and 3 to obtain results about continuity of global attractors and structural stability of Morse-Smale semigroups, as we discussed before. To be more precise, in Section 4.1 we study the geometrical structural stability of Morse-Smale semigroups (that is well known) and prove that the orbits of perturbed Morse-Smale semigroups remain close to the orbits of the limit problem. Section 4.2 is dedicated to apply the theoretical results from Section 3 to guarantee continuity of global attractors.

Finally, we have an Appendix with basic results from Functional Analysis and more advanced results from Dynamical Systems.

Chapter 1
Fundamental Concepts

In this chapter we recall some of the fundamental concepts and results from infinite dimensional dynamical systems and differential equations. Some of these concepts will be present through the whole thesis and are essential for the reader's understanding.

We will stablish some notations through this work:

If X is a normed vector space and we did not explicitly state its norm, then we will denote it by $\|\cdot\|$.

If X, Y are normed vector spaces, we will denote the space of continuous linear operators from X into Y by $\mathcal{L}(X,Y)$. We also define $\mathcal{L}(X) := \mathcal{L}(X,X)$.

If $F : X \to X$ is a differentiable map on a Banach space X, then we will denote the Fréchet derivative of F at $x \in X$ by $D_xF \in \mathcal{L}(X)$.

If $F : X \to Y$ is a map and $C \subset X$ is a subset, then we will denote the range of F by $R(F)$ and the restriction of F on C by $F|_C : C \to Y$. Moreover, given a subset $D \subset Y$ we will denote the inverse image of D by $F^{-1}(D)$.

If (M,d) is a metric space and $x_0 \in M$, we will denote the open ball with radius $r > 0$ and centered at x_0 by $B_M(x_0,r) = \{x \in M : d(x,x_0) < r\}$. If the context is not ambiguous, we will denote the same open ball simply by $B(x_0,r)$.

If (M,d) is a metric space and $S \subset M$, we will denote the $r-$neighborhood of S by $\mathcal{O}_r(S) = \{x \in M : d(x,S) < r\}$.

If (M,d) is a metric space and $A, B \subset M$ are non-empty subsets of M, we will denote by $d(A,B) := \inf_{a\in A}\inf_{b\in B} d(a,b)$ the usual distance between A, B and by $dist_H(A,B) := \sup_{a\in A}\inf_{b\in B} d(a,b)$ the Hausdorff semi-distance between A, B.

If M is a metric space we denote the set of continuous functions from M into M by $\mathcal{C}(M)$. Similarly, if X is a Banach space we denote the set of maps from X into X that are r times continuously differentiable by $\mathcal{C}^r(X)$.

If X is a Hilbert space, we will denote its inner product by $\langle\cdot,\cdot\rangle$ (except in Section 1.2, where we define another notation).

1.1 Some Results from Dynamical Systems

In this section we define the basic concepts, related to the theory of nonlinear dynamical systems, that are necessary to understand the following chapters. Some of this concepts were already com-

mented in the Introduction and we will not repeat them here (semigroups, global attractor, global solution, etc).

Summarizing, in this section we present properties from Gradient, Dynamically Gradient and Morse-Smale semigroups, where the latter is one of the most fundamental concepts of this thesis, together with Shadowing. We will also provide a relation between these classes of semigroups (see Corollary 1.11 and Lemma 1.17). The great majority of concepts and results of this section can be found in [20, Chapter 5] and [9, Chapters 3-5]. Remember that we do not assume that the maps $\mathcal{T}(t)$, $t \geq 0$, on Definition 1 are linear. In fact, we have reserved Section 1.2 to stablish the theory of linear semigroups, in the sense of [45]. Since we are going through some results very briefly, we recommend the references given before for further details. In fact, since this section is the preliminaries of this thesis, we prove some results with details and for others we just give a sketch of the proof together with a reference that contains a detailed approach.

We recall that we will use $\mathcal{T}$ to denote a semigroup, that is, $\mathcal{T} := \{\mathcal{T}(t) : t \in \mathbb{P}^+\}$. Since the main goal of this thesis is to prove Theorem 0.1 (for continuous semigroups), in this section we will only consider continuous semigroups. Despite that, some results of this section still holds for discrete semigroups (Proposition 1.1, for example). Therefore, the reader must be aware that any semigroup $\mathcal{T}$ defined in this section will be continuous.

Now we provide the scheme of this Section: In Subsection 1.1.1 we introduce fundamental concepts from dynamical systems, such as ω and α limits, and provide results related to the dynamics of semigroups with global attractor. Some of theses results are dedicated to the class of Gradient and Dynamically Gradient semigroups, see Definitions 6 and 8. In this subsection we also prove that Gradient semigroups are Dynamically Gradient. Subsection 1.1.2 is dedicated to show that any Dynamically Gradient semigroup is Gradient, that is, both concepts are equivalent. Finally, in Subsection 1.1.3 we define properly some concepts related to Morse-Smale semigroups that were given in the Introduction, such as hyperbolic equilibrium point. We also prove several results related to the dynamics of Morse-Smale semigroups and show the relation between Gradient and Morse-Smale semigroups.

1.1.1 General Results of Semigroups with Global Attractor

In this subsection we introduce fundamental concepts from dynamical systems, such as α and ω limits (Definitions 15) and prove several results related to the dynamics of semigroups with global attractor $\mathcal{A}$. Great part of this technical results will be used through Chapter 2, which is the main theoretical contribution of this work. In fact, we highlight the following results of this subsection: Proposition 1.2, that will be used to prove several results of this thesis (including Lemma 2.16, that is our alternative to deal with one of the main theoretical obstacles of this work: the lackness of surjectivity of the derivative $D_x\mathcal{T}(t)$ of the semigroup), Theorem 1.3, which provides a characterization of the global attractor of Gradient semigroups, and Proposition 1.6, which shows that any Gradient semigroup is Dynamically Gradient.

We start this subsection by providing a characterization of the global attractor (Definition 5), based on global solutions (Definition 2). In particular, we show that the global attractor $\mathcal{A}$ is unique.

Proposition 1.1. *Let $\mathcal{T}$ be a semigroup on a metric space (M, d) with global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{x \in M : \text{there exists a bounded global solution } \phi : \mathbb{R} \to M \text{ through } x\}.$$

Consequently, any bounded global solution $\phi : \mathbb{R} \to M$ is contained in the global attractor $\mathcal{A}$.

Proof. If $\phi : M \to \mathbb{R}$ is a bounded global solution, then it must be attracted by $\mathcal{A}$ and therefore (by its invariance) $\phi(\mathbb{R}) \subset \mathcal{A}$. On the other hand, if $x \in \mathcal{A}$, then we can use the invariance of $\mathcal{A}$ to fix $x_{-1} \in \mathcal{A}$ such that $\mathcal{T}(1)x_{-1} = x$. We can also fix $x_{-2} \in \mathcal{A}$ such that $\mathcal{T}(1)x_{-2} = x_{-1}$ and proceeding like that we obtain a sequence $\{x_{-k}\}_{k\in\mathbb{N}} \subset \mathcal{A}$ with

$$\mathcal{T}(1)x_{-k} = x_{-k+1} \quad \text{and } \mathcal{T}(k)x_{-k} = x.$$

Now we can "glue" the solution such as

$$\phi(t) := \begin{cases} \mathcal{T}(t)x, \ t \geq 0; \\ \mathcal{T}(t+k)x_{-k}, \ t \in [-k, 0) \end{cases}$$

to obtain a bounded global solution through x. □

The next result allow us to extract a new global solution from a sequence of global solutions. This kind of result is very useful to prove some properties, especially when the proof is by contradiction (see Theorem 1.8 and Lemmas 1.4, 1.5 and 2.16, for example).

Proposition 1.2. *Suppose that $\mathcal{T} = \{\mathcal{T}(t) : t \geq 0\}$ is a semigroup on a metric space (M, d) with global attractor $\mathcal{A}$. Let $\{\sigma_k\}_{k\in\mathbb{N}}$ be a positive sequence with $\sigma_k \to \infty$ as $k \to \infty$, $\{u_k\}_{k\in\mathbb{N}}$ a bounded sequence in M, and let $J_k = \{s \in \mathbb{R} : -\sigma_k \leq s < \infty\}$. Define $\xi_k : J_k \to M$ by $\xi_k(s) = \mathcal{T}(s + \sigma_k)u_k$, $s \in J_k$. Then, there exists a global solution $\xi : \mathbb{R} \to M$ of $\mathcal{T}$ and a subsequence of $\{\xi_k\}_{k\in\mathbb{N}}$ (which we again denote by $\{\xi_k\}_{k\in\mathbb{N}}$) such that*

$$\lim_{k\to\infty} \xi_k(s) \to \xi(s), \quad \forall s \in \mathbb{R}. \tag{1.1}$$

Proof. Since $\mathcal{T}$ has a global attractor, then $\mathcal{T}$ is asymptotically compact [9, Theorem 1.23], that is, for any positive sequence $t_n \to +\infty$ and bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$, $\mathcal{T}(t_n)x_n$ has a convergent subsequence.

Therefore, there exists a subsequence, again denoted by ξ_k and $x_0 \in M$, such that

$$\xi_k(0) = \mathcal{T}(\sigma_k)\xi_k(-\sigma_k) \to x_0.$$

Let $\xi_0(\cdot) : \mathbb{R}^+ \to X$ be the solution of $\mathcal{T}$ defined by $\xi_0(t) = \mathcal{T}(t)x_0$, for all $t \geq 0$. Also,

$$\xi_k(s) = \mathcal{T}(s)\xi_k(0) \xrightarrow{k\to\infty} \mathcal{T}(s)x_0 = \xi_0(s), \quad \forall s \geq 0.$$

Proceeding analogously, $\{\xi_k(-1)\}_{k\in\mathbb{N}}$ has a convergent subsequence $\{\xi_k^1(-1)\}_{k\in\mathbb{N}}$ with limit x_{-1}. Redefining $\xi_0(s) := \mathcal{T}(s+1)x_{-1}$, for all $s \geq -1$, we have

$$\mathcal{T}(1)x_{-1} = \lim_{k\to\infty} \mathcal{T}(1)(\xi_k^1(-1)) = \lim_{k\to\infty} \xi_k^1(0) = x_0,$$

where ξ_0 is a solution of $\mathcal{T}$ with $\xi(-1) = x_{-1}$, $\xi(0) = x_0$, and

$$\xi_k^1(s) = \mathcal{T}(s+1)\xi_k^1(-1) \to \mathcal{T}(s+1)x_{-1} = \xi_0(s), \ \forall s \geq -1.$$

Proceeding like this we obtain subsequences $\{\xi_k^i\}_{k\in\mathbb{N}}, 1 \leq i \leq m-1$, such that $\{\xi_k^{i+1}\}_{k\in\mathbb{N}}$ is a subsequence of $\{\xi_k^i\}_{k\in\mathbb{N}}$ and

$$\xi_k^i(-i) \to x_{-i}, \ 1 \leq i \leq m-1 \ \text{ and } \ \mathcal{T}(1)x_{-i} = x_{-i+1}, 1 \leq i \leq m-1.$$

From our construction we also have a solution $\xi_0 : [-m, +\infty) \to M$ such that

$$\xi_k^i(s) \to \xi_0(s), \ \forall s \geq -i.$$

Proceeding like that we can construct a countable (infinite) family of global solutions $\{\xi_k^j\}_{k,j\in\mathbb{N}}$ and, using a Cantor diagonalization procedure, obtain a subsequence of ξ_k satisfying (1.1), which finishes the proof.

□

Now we define some standard concepts from Dynamical Systems: ω-limit and α-limit. These notions help us to understand where the points of the phase space "accumulate" forward and backwards (when it's possible).

Definition 15. *Let $\mathcal{T}$ be a semigroup on a metric space (M,d). Given a subset $S \subset M$, the ω-limit of S, denoted by $\omega(S)$, is the set*

$$\omega(S) = \bigcap_{t\geq 0} \overline{\{\mathcal{T}(s)S : s \geq t\}} = \bigcap_{t\geq 0} \overline{\gamma_{[s+\infty)}(S)},$$

where γ denotes the trajectory (Definition 3) In particular, if $S = \{x\}$ then $\omega(x) = \bigcap_{t\geq 0} \overline{\{\mathcal{T}(s)x : s \geq t\}}$.

If $x \in M$ and ϕ is a global solution through x, we define the $\alpha-$limit of x in relation to ϕ as

$$\alpha_\phi(x) = \bigcap_{t\leq 0} \overline{(\gamma_\phi)_t^-(x)},$$

where we are using the notation from Definition 3.

The following result gives us another characterization for the global attractor $\mathcal{A}$ in case $\mathcal{T}$ is Gradient (Definition 6). Such characterization will be used throughout Chapter 2.

Theorem 1.3. *Assume that $\mathcal{T}(\cdot)$ is a Gradient semigroup on a metric space M, with global attractor $\mathcal{A}$ and finite set of equilibria $\mathcal{E}$. Then*

$$\mathcal{A} = \bigcup_{x^*\in\mathcal{E}} W^u(x^*),$$

where $W^u(x^*)$ is the unstable manifold of x^* (see Definition 10)

Proof. Here we will give a sketch of the proof. We refer to [9, Theorem 3.8] and [20, Chapter 5] for more details.

We start by proving that $\mathcal{A} \subset W^u(\mathcal{E}) := \bigcup_{x^*\in\mathcal{E}} W^u(x^*)$. Since

$$\mathcal{A} = \{x \in M : \text{there exists a bounded global solution } \phi : \mathbb{R} \to M \text{ such that } \phi(0) = x\},$$

from Proposition 1.1, then for each $x \in \mathcal{A}$ there exists a global solution $\phi : \mathbb{R} \to X$ through x. Since $\phi(\mathbb{R}) \subset \mathcal{A}$ is relatively compact, it follows that $\alpha_\phi(x) \neq \varnothing$ (see Definition 3) and $\alpha_\phi(x) \subset \mathcal{E}$, which is the crucial point of the proof [28, Lemma 3.8.2]. This shows that $\mathcal{A} \subset W^u(\mathcal{E})$.

On the other hand, if $x \in W^u(x^*)$ for some $x^* \in \mathcal{E}$, then there is a global solution $\phi : \mathbb{R} \to M$ through x and $\lim_{t\to\pm\infty} \text{dist}_H(\phi(t), \mathcal{E}) = 0$, implying that $\phi(\mathbb{R})$ is bounded and $x \in \mathcal{A}$ (Proposition 1.1). This shows $W^u(\mathcal{E}) \subset \mathcal{A}$ and completes the proof. □

In the next result we introduce the Birkhoff number of a bounded set $B \subset M$. This concept will be used in Chapter 2 in order to construct the compatible subbundles, which are essential to prove Lipschitz Shadowing, as we discussed before in the introduction.

Lemma 1.4. *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space (M,d) with global attractor $\mathcal{A}$, set of equilibria $\mathcal{E}=\{x_1^*,\dots,x_p^*\}$ and that satisfies property (G1) from Definition 8. Then for each $\epsilon>0$ and bounded set $B\subset M$ there exists $t_0>0$, called Birkhoff number of B, such that*

$$\gamma_{[0,t_0]}(x)\cap\left(\bigcup_{j=1}^{p}\mathcal{O}_\epsilon(x_j^*)\right)\neq\emptyset,\ \forall x\in B.$$

Proof. In order to prove by contradiction, assume that there exists a bounded set $B\subset M$, an $\epsilon>0$ and sequences $t_n\to+\infty$ and $x_n\in B$ such that

$$\mathcal{T}(s)x_n\cap\left(\bigcup_{j=1}^{p}\mathcal{O}_\epsilon(x_j^*)\right)=\emptyset,\ \forall s\in[0,2t_n].$$

Then, from Proposition 1.2 there exists a global solution $\xi:\mathbb{R}\to M$ such that (up to a subsequence)

$$\mathcal{T}(s+t_n)x_n\xrightarrow{n\to+\infty}\xi(s),\ \forall s\in\mathbb{R}.$$

Since $d(\mathcal{T}(s)x_n,\mathcal{E})\geq\epsilon$ for all $s\in[-t_n,t_n]$, it follows that $d(\xi(s),\mathcal{E})\geq\epsilon$ for all $s\in\mathbb{R}$, which contradicts (G1).

□

Our next result is a technical lemma. Geometrically, it will stablish that if $\mathcal{T}(\cdot)$ is a dynamically gradient semigroup then for any equilibrium point x_i^* and neighborhood $B(x_i^*,\delta)$, there exists another neighborhood $B(x_i^*,\delta')\subset B(x_i^*,\delta)$ such that if a solution starts in $B(x_i^*,\delta')$ and leaves $B(x_i^*,\delta)$, then it will never return to $B(x_i^*,\delta')$ (see Figure 1.1).

Lemma 1.5. *Let $\mathcal{T}(\cdot)$ be a Dynamically Gradient semigroup relative to the set of equilibria $\mathcal{E}=\{x_1^*,\dots,x_p^*\}$ on a metric space (M,d). Then for any $i\in\{1,\dots,p\}$ and any $\delta>0$ there exists $0<\delta'<\delta$ such that if $x\in B(x_i^*,\delta')$ and $\mathcal{T}(t_0)x\notin B(x_i^*,\delta)$, for some $t_0>0$, we have*

$$\mathcal{T}(t)x\notin B(x_i^*,\delta'),\ \forall t\geq t_0.$$

Proof. Assuming by contradiction that the statement is false, then there exists $x^*\in\mathcal{E}$, $\delta>0$ small, $x_k\to x^*$ and $t_k,s_k\to+\infty$ such that

$$d(\mathcal{T}(t_k)x_k,x^*)=\delta\ \text{ and }\ d(\mathcal{T}(t_k+s)x_k,x^*)<\delta,\ \forall s\in[0,s_k].$$

Hence, for each $k\in\mathbb{N}$ we can define the solution $\xi_k=\mathcal{T}(t+t_k)x_k$ for all $t\in[-t_k,s_k]$ and from Proposition 1.2 there exists a global solution $\xi:\mathbb{R}\to M$ and a subsequence such that

$$\xi_k(s)\to\xi(s)\text{ as }k\to+\infty,\ \forall s\in\mathbb{R}.$$

Since $d(\xi_k(s),x^*)<\delta$ for all $s\in(-t_k,s_k)$ and $d(\xi_k(0),x^*)=\delta$, then $d(\xi(s),x^*)<\delta$ for all $s\in\mathbb{R}$ and $\xi\not\equiv x^*$. Assuming without lost of generality that $B(x^*,\delta)\cap\mathcal{E}=x^*$, it follows from (G1) that ξ satisfies

$$x^*\xleftarrow{t\to-\infty}\xi(t)\xrightarrow{t\to+\infty}x^*\ \text{ and }\ \xi\not\equiv x^*,$$

which contradicts (G2). □

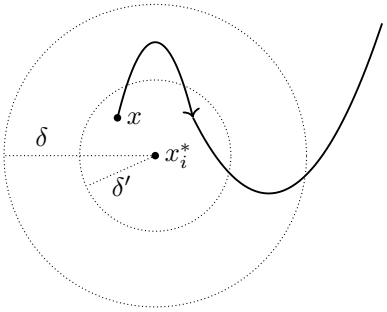


Figure 1.1: Dynamics of Lemma 1.5

In the next result we show that Gradient semigroups are also Dynamically Gradient. As we will see further on, both concepts are actually equivalent but for now we just prove this implication, since in the proof of the converse we need to construct a Lyapunov function, which makes the proof more complicated.

Proposition 1.6. *Let $\mathcal{T}(\cdot)$ be a Gradient semigroup on a metric space M, with global attractor $\mathcal{A}$ and set of equilibria $\mathcal{E} = \{x_1^*, \dots, x_p^*\}$. Then $\mathcal{T}(\cdot)$ is Dynamically Gradient.*

Proof. We will provide a sketch of the proof and recommend [9, 20] for a more detailed approach. Let $\mathcal{T}$ be a Gradient semigroup on M. We have to show that $\mathcal{T}$ satisfies properties (G1) and (G2) from Definition 8. To show (G1), observe that Theorem 1.3 already gives us that every bounded global solution ϕ satisfies $\phi(t) \xrightarrow{t\to-\infty} x^*$, for some $x^* \in \mathcal{E}$. It remains to show that for each $x \in \mathcal{A}$ there exists $y^* \in \mathcal{E}$ such that $\lim_{t\to+\infty} \mathcal{T}(t)x = y^*$. Since we are assuming the existence of a global attractor $\mathcal{A}$, then for each $x \in \mathcal{A}$ the omega limit

$$\omega(x) = \bigcap_{t\geq 0} \overline{\{\mathcal{T}(s)x : s \geq t\}}$$

is non-empty, invariant and attracts x [9, Lemma 1.20]. Moreover, using the Lyapunov function we can show that $\omega(x) \subset \mathcal{E}$ [9, Lemma 3.5]. Hence, $\omega(x) \subset \mathcal{E}$ attracts x and (G1) is satisfied. To see that (G2) is also satisfied, we assume that the semigroup $\mathcal{T}$ has a homoclinic structure that begins and finishes at the equilibrium x^*. Since the Lyapunov function is non-increasing through solutions and the homoclinic structure begins and finishes at x^*, we conclude that the energy of the orbits (in the homoclinal structure) is constant and therefore the solutions in the homoclinical structure must be on $\mathcal{E}$, which is a contradiction. This finishes the proof that Gradient semigroups are Dynamically Gradient. □

1.1.2 Dynamically Gradient semigroups are Gradient semigroups

This subsection is dedicated to prove that Dynamically Gradient semigroups are Gradient (Theorem 1.10). Consequently, both concepts are equivalent (Corollary 1.11). The main idea is to construct a Morse decomposition (see Definition 18) and from that construct a Lyapunov function (see Definition 6). To do that, we will have to introduce the concepts of local attractor and repeller (Definition 19). In order to maintain this section compact and organized, we will omit some technical results and maintain only the most important ones. For a more complete and detailed approach, we recommend the references [20, Chapter 5] and [9, Chapter 3]

Now we introduce the necessary concepts to define a Morse-decomposition.

Definition 16. *Let $\mathcal{T}$ be a semigroup in a metric space (M,d) and Ξ a invariant set, that is, $\mathcal{T}(t)\Xi = \Xi$ for all $t \geq 0$. Given an open neighborhood U of Ξ, we say that Ξ is the **maximal $\mathcal{T}$-invariant** set in U if for any $B \subset U$ invariant we have $B \subset \Xi$.*

We also say that Ξ is an isolated invariant set if there exists $\delta > 0$ such that Ξ is the maximal invariant set in $\mathcal{O}_\delta(\Xi)$ (see the notation established in the beginning of the chapter).

Definition 17. *We say that $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_p\}$ is a **disjoint collection of isolated invariant sets** if Ξ_i is an isolated invariant set for each $i \in \{1, \ldots, p\}$ and there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \varnothing, 1 \leqslant i < j \leqslant p$.*

If $\Xi = \{x^*\}$ is a unitary set we say that x^* is an isolated stationary solution (or isolated equilibrium).

Now we define the notion on Morse-decomposition, which will be used to construct a Lyapunov function for Dynamically Gradient semigroups.

Definition 18 (Morse-Decomposition)**.** *Let $\mathcal{T}$ be a semigroup on a metric space (M,d) with global attractor $\mathcal{A}$. An ordered n-tuple $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets is called a **Morse decomposition** of $\mathcal{A}$ if for any global solution $\phi : \mathbb{R} \to \mathcal{A}$ of $\mathcal{T}$ it holds*

$$\lim_{t\to-\infty} d(\phi(t), \Xi_j) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\phi(t), \Xi_i) = 0 \quad \text{with } i < j,$$

or $\phi(\mathbb{R}) \subset \Xi_i$ for some $1 \leqslant i \leqslant n$.

To be able to construct a Morse-decomposition for Dynamically Gradient semigroups, we need to "order" our equilibria in relation to its stability. For example, there will always be an equilibrium that attracts a neighborhood of itself [9, Lemma 3.34] and therefore is "stable". This kind of equilibrium can be also seen as a local attractor, which motivates our next definitions.

Definition 19 (Local Attractor and Repeller)**.** *Let $\mathcal{T}$ be a semigroup in a metric space M with a global attractor $\mathcal{A}$. We define:*

1. *A nonempty subset Ξ of $\mathcal{A}$ is a **local attractor** if there exists $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$;*
2. *The **repeller** Ξ^* associated to a local attractor Ξ is the set*

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \varnothing\}.$$

We call the duple (Ξ, Ξ^*) by attractor-repeller pair.

Now we can show that any Dynamically Gradient semigroup has a Morse-Decomposition.

Lemma 1.7. *Let $\mathcal{T}$ be a Dynamically Gradient semigroup with global attractor $\mathcal{A}$ and equilibria $\mathcal{E} = \{x_1^*, \cdots, x_p^*\}$. Then there is a reordering of $\mathcal{E}$ that it is a Morse decomposition of $\mathcal{A}$.*

Proof. Since this result requires more auxiliary lemmas, we will give a quick sketch of the proof and refer to [9, Theorem 3.35] for more details. We can assume without lost of generality that x_1^* a local attractor [9, Lemma 3.34] and

$$(x_1^*)^* = \{x \in \mathcal{A} : \omega(x) \cap x_1^* = \varnothing\}.$$

Consequently, $x_i^* \in (x_1^*)^*$ for each $i > 1$. Moreover, from [9, Lemma 3.33] it follows that for any $x \notin \mathcal{A} \setminus \{x_1^* \cup (x_1^*)^*\}$ and global solution $\phi : \mathbb{R} \to \mathcal{A}$ through x we have

$$\lim_{t\to-\infty} \operatorname{dist}_H\left(\phi(t),(x_1^*)^*\right)=0 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}_H\left(\phi(t),x_1^*\right)=0.$$

Since $(x_1^*)^*$ is invariant, we obtain a new semigroup $\mathcal{T}_1$ by restricting $\mathcal{T}(t)$ to $X_{1,0}:=(x_1^*)^*$. Note that $\mathcal{T}_1$ is a Dynamically Gradient semigroup in $X_{1,0}$, with disjoint collection of isolated invariant sets $\left\{x_2^*,\cdots,x_p^*\right\}$, and if $\phi_1:\mathbb{R}\to\mathcal{A}$ is a global solution of $\mathcal{T}_1$ such that $\phi_1(t)\to x_i^*$, then $i>1$.

Repeating the same process to $\mathcal{T}_1$, we assume without loss of generality that x_2^* is a local attractor for the semigroup $\mathcal{T}_1$ in $X_{1,0}$. If $X_{2,1}$ is the repeller associated to the isolated invariant set x_2^* for $\mathcal{T}_1$ in $X_{1,0}$, we may proceed similarly considering the restriction $\mathcal{T}_2$ of the semigroup $\mathcal{T}_1$ to $X_{2,1}$ and $\mathcal{T}_2$ is a Dynamically Gradient semigroup in $X_{2,1}$ with associated disjoint collection of isolated invariant sets $\left\{x_3^*,\cdots,x_p^*\right\}$. Note that in this time we have removed all global solutions $\phi:\mathbb{R}\to X$ of $\mathcal{T}$ such that $\phi(t)\to x_2^*$ as $t\to+\infty$ and therefore, if

$$x_j^* \overset{t\to-\infty}{\longleftarrow} \phi(t) \overset{t\to\infty}{\longrightarrow} x_2^*,$$

then $j\geqslant 2$.

We can proceed like this until all equilibria are exhausted to obtain a reordering of $\mathcal{E}$ such that x_j^* is a local attractor for the restriction of $\mathcal{T}$ to $X_{j-1,j-2}$ and $X_{0,-1}=\mathcal{A}$. Consequently, if $\phi:\mathbb{R}\to\mathcal{A}$ is a global solution of $\mathcal{T}$ and

$$x_j^* \overset{t\to-\infty}{\rightleftarrows} \xi(t) \overset{t\to\infty}{\longrightarrow} x_k^*,$$

then $j\geqslant k$. Therefore we have reordered $\mathcal{E}$ such that $\left\{x_1^*,\cdots,x_p^*\right\}$ is a Morse decomposition. □

In the next results we extract more information about the Morse-Decomposition so we can construct a Lyapunov function for any Dynamically Gradient semigroup.

Theorem 1.8. *Let $\mathcal{T}$ be a Dynamically Gradient semigroup on a metric space (M,d) with equilibria $\mathcal{E}=\left\{x_1^*,\cdots,x_p^*\right\}$, which is already reordered as a Morse decomposition. Define $A_0=\varnothing, A_1=x_1^*$ and*

$$A_j=\bigcup_{i=1}^{j} W^u\left(x_i^*\right),\ \forall j\in\{2,\ldots,p\}. \tag{1.2}$$

Then A_j is a local attractor for $\mathcal{T}$ in M and $x_j^*=A_j\cap A_{j-1}^*$ for all $j\in\{1,\ldots,p\}$.

Proof. In order to show that A_j is a local attractor, we will use [9, Lemma 3.31] and show that $A_j=A_{j-1}\cup W^u(x_j^*)$ is a local attractor of $\mathcal{T}|_{\mathcal{A}}$. Since A_j is compact [9, Lemma 3.36], we can fix $\delta_0>0$ such that $\mathcal{O}_{\delta_0}\left(A_j\right)\cap\left(\bigcup_{i=j+1}^{n} x_i^*\right)=\varnothing$.

If there are $\delta<\delta_0$ and $\delta'<\delta$ such that $\gamma^+\left(\mathcal{O}_{\delta'}\left(A_j\right)\right)\subset\mathcal{O}_{\delta}\left(A_j\right)$, then $\omega\left(\mathcal{O}_{\delta'}\left(A_j\right)\right)$ $\mathcal{T}$-attracts $\mathcal{O}_{\delta'}\left(A_j\right)$ and is contained in A_j ($\omega\left(\mathcal{O}_{\delta'}\left(A_j\right)\right)$ is $\mathcal{T}$-invariant), implying that A_j is a local attractor. Otherwise, for $\delta<\delta_0$, there exists a sequence of global solution $\xi_k:\mathbb{R}\to\mathcal{A}$ through $x_k\in\mathcal{A}\backslash A_j$, with $d(x_k,A_j)\to 0$, $d(\xi_k(t),A_j)\leqslant\delta$ for all $t\in[0,t_k]$ and $d(\xi_k(t_k),A_j)=\delta$. Using Proposition 1.2, we can construct a global solution $\xi:\mathbb{R}\to\mathcal{A}$ with

$$\xi(0)\notin A_j \text{ and } d(\xi(t),A_j)\leqslant\delta, \text{ for all } t\leqslant 0. \tag{1.3}$$

From (1.3) and the fact that $\mathcal{T}$ is a Dynamically Gradient semigroup, we must have $\xi(t)\to x_k^*$ as $t\to-\infty$ for some $1\leqslant k\leqslant j$, which implies that $\xi(0)\in A_j$ and contradicts (1.3). This finishes the proof that $\mathcal{A}_j$ is a local attractor.

Now we prove that $x_j^*=A_j\cap A_{j-1}^*$. Given $z\in A_j\cap A_{j-1}^*$ we know that the global solution $\xi:\mathbb{R}\to\mathcal{A}$ through z must satisfy that

$$\{x_i^* : 1 \leqslant i \leqslant j\} \xrightarrow{t\to-\infty} \xi(t) \xrightarrow{t\to\infty} \{x_i^* : j \leqslant i \leqslant n\}.$$

Since $\mathcal{T}$ is a Dynamically Gradient semigroup and $\mathcal{E}$ is already reordered to be a Morse-decomposition, we obtain that $z = x_j^*$. This shows that $A_j \cap A_{j-1}^* \subset x_j^*$. The other inclusion is obvious. □

Proposition 1.9. *Let $\mathcal{T}$ be a Dynamically Gradient semigroup on a metric space (M, d) with equilibria $\mathcal{E} = \{x_1^*, \ldots, x_p^*\}$ which is already reordered to be a Morse decomposition. Then,*

$$\bigcap_{j=0}^{p} (A_j \cup A_j^*) = \bigcup_{i=1}^{p} x_i^*$$

*Proof.*If $z \in \bigcup_{i=1}^{p} x_i^*$, let $k \in \{1, 2, \cdots, p\}$ be such that $z = x_k^* = A_k \cap A_{k-1}^*$ (Theorem 1.8). Hence $z \in A_k \subset A_{k+1} \subset \cdots \subset A_n$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \cdots \subset A_0^*$. Consequently

$$z \in \bigcap_{j=k}^{p} A_j \cap \bigcap_{j=1}^{k-1} A_j^* \subset \bigcap_{j=k}^{p} (A_j \cup A_j^*) \cap \bigcap_{j=0}^{k-1} (A_j \cup A_j^*) = \bigcap_{j=0}^{p} (A_j \cup A_j^*),$$

proving that $\bigcup_{i=1}^{p} x_i^* \subset \bigcap_{j=0}^{p} (A_j \cup A_j^*)$.

Now, let $z \in \bigcap_{j=0}^{p} \left(A_j \cup A_j^*\right)$ and $I := \{i_1, i_2, \cdots, i_k\}, J := \{j_1, j_2, \cdots, j_l\}$ such that $I \cup J = \{0, 1, \cdots, p\}$ with $I \cap J = \varnothing$ and $z \in A_i$ for all $i \in I$ and $z \in A_j^*$ for all $j \in J$. Note that if $i := \min I$, then $I = \{i, i+1, i+2, \cdots, n\}$ and $J = \{0, 1, \cdots, i-1\}$. Hence $z \in A_i \cap A_{i-1}^*$ and (using Theorem 1.8) the proof is completed. □

Now we can finally show that Dynamically Gradient semigroups are also Gradient, that is, it possess a Lyapunov function. Putting this together with Proposition 1.6, we will be able to conclude that the concepts of Gradient and Dynamically Gradient semigroups are equivalent.

Theorem 1.10. *Let $\mathcal{T}(\cdot)$ be a Dynamically Gradient semigroup on a metric space (M, d), with global attractor $\mathcal{A}$ and equilibria $\mathcal{E} = \{x_1^*, \ldots, x_p^*\}$, already reordered to be a Morse decomposition. Then, $\mathcal{T}(\cdot)$ is a Gradient semigroup and we can construct the Lyapunov function such that $V(x_i^*) = i$ for each $i \in \{1, \ldots, p\}$.*

Proof. We will provide a sketch of the proof, but the reader can find the detailed proof in [20, Chapter 5], [9, Chapter 3] and [1]. We will assume the following facts

1. The map $h : M \to \mathbb{R}$ given by

$$h(x) = \sup_{t\geqslant 0}\{d(T(t)x, \mathcal{A})\}, \forall x \in M$$

is well defined, continuous, nonincreasing along solutions of $\mathcal{T}$ and $h^{-1}(0) = \mathcal{A}$.

2. For each attractor-repeller pair (A_j, A_j^*) let $k_j : M \to \mathbb{R}$ be the function

$$k_j(z) = \sup_{t\geqslant 0} U_j(T(t)z), \ \forall z \in M,$$

where U_j is the canonical Urysohn function

$$U_j(z) = \frac{d(z, A_j)}{d(z, A_j) + d(z, A_j^*)}, \ \forall z \in M.$$

Then, k_j has the properties:
(i) k_j is continuous in M.
(ii) k_j is nonincreasing along solutions of $\mathcal{T}$.
(iii) $k_j^{-1}(0) = A_j$ and $k_j^{-1}(1) \cap \mathcal{A} = A_j^*$.
(iv) Given $z \in M$, if $k_j(\mathcal{T}(t)z) = k_j(z)$ for all $t \geqslant 0$, then $z \in A_j \cup A_j^*$.

Let $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_p = \mathcal{A}$ be the sequence of local attractors defined in (1.2) and $\varnothing = A_p^* \subset A_{p-1}^* \subset \cdots \subset A_0^* = \mathcal{A}$ be their respective repellers such that $x_j^* = A_j \cap A_{j-1}^*$, for each $j = 1, 2, \cdots, p$.

Let h be the function defined on item (1) and k_j be the functions defined in (2). Define the continuous function $V : M \to \mathbb{R}$ by

$$V(z) = h(z) + \sum_{j=1}^{p} k_j(z) \quad \text{for each } z \in M$$

We will show that V is a Lyapunov function with respect to $\mathcal{E}$. Indeed, since h and each k_j are nonincreasing along solutions of $\mathcal{T}$, so is V.

If $z \in M$ is such that $V(\mathcal{T}(t)z) = V(z)$ for all $t \geqslant 0$, we can use property (iv) of the functions k_j to guarantee that $z \in A_j \cup A_j^*$, for each $j = 0, 1, \cdots, p$ and consequently (from Proposition 1.9) $z \in \mathcal{E}$.

Finally, we can use property (iii) to guarantee that $k_j(x_k^*) = 0$ if $k \leqslant j \leqslant n$ and $k_j(x_k^*) = 1$ if $1 \leqslant j \leqslant k-1$. Therefore,

$$V(x_k^*) = \sum_{j=1}^{n} k_j(x_k^*) = \sum_{j=0}^{k-1} k_j(x_k^*) + \sum_{j=k}^{n} k_j(x_k^*) = \sum_{j=0}^{k-1} 1 + \sum_{j=k}^{n} 0 = k - 1$$

and we can define $\tilde{V} : M \to \mathbb{R}$ as $\tilde{V}(z) = V(z) + 1$ to obtain the property $\tilde{V}(x_k^*) = k$ for all $k \in \{1, \ldots, p\}$. □

Corollary 1.11. *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space M, with global attractor $\mathcal{A}$ and equilibria $\mathcal{E} = \{x_1^*, \ldots, x_p^*\}$. Then, $\mathcal{T}(\cdot)$ is Gradient if, and only if it is Dynamically Gradient.*

Proof. It follows imediatly from Proposition 1.6 and Theorem 1.10. □

1.1.3 Morse-Smale semigroups

This subsection is dedicated to one of the key concepts of this thesis: Morse-Smale semigroups. In fact, in this subsection we introduce fundamental concepts related to Morse-Smale semigroups, such as hyperbolicity (Definition 20) and recall the definitions of non-wandering set and Morse-Smale semigroups. Moreover, we prove several results related to the dynamics of Morse-Smale semigroups (e.g. Theorem 1.12 and Proposition 1.14).

As we just said, Morse-Smale semigroups are fundamental to this thesis, specially in Chapter 2, where we build and prove our theoretical results. Therefore, every result from this subsection is essential to properly understand this work. We highlight Theorem 1.12, which states that the unstable manifold $W^u(x^*)$ of a hyperbolic equilibrium point is a $\mathcal{C}^1$ manifold when $\mathcal{T}$ is Morse-Smale, Proposition 1.14, that allow us to work with a group instead of a semigroup, and Lemma 1.17, which shows that Morse-Smale semigroups are Gradient/Dynamically Gradient.

From now on, if $\{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}(X)$ is a differentiable semigroup on a Banach space X and $E \subset X$, we will denote the inverse image of the derivative by

$$(D_x\mathcal{T}(-t))E := (D_{\mathcal{T}(-t)x}\mathcal{T}(t))^{-1}E, \ \forall x \in X, \ \forall t \geq 0.$$

Moreover, if the operator $D_{\mathcal{T}(-t)x}\mathcal{T}(t)$ is injective and $\mathcal{T}(t)$ is injective, then we will also denote $D_x\mathcal{T}(-t) := (D_{\mathcal{T}(-t)x}\mathcal{T}(t))^{-1}$.

From now on we will focus on concepts related to Morse-Smale semigroups. We start by defining hyperbolicity, which is an essential property to the theory of Dynamical Systems [28, 20, 9, 33]. Our definition is based on [33].

Definition 20 (Hyperbolicity)**.** *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}(X)$ be a differentiable semigroup on a Banach space X and x^* be an equilibrium of $\mathcal{T}(\cdot)$, that is, $\mathcal{T}(t)x^* = x^*$ for all $t \geq 0$. We say that x^* is hyperbolic if satisfies:*

1. *There exists a continuous projection $P(x^*) : X \to X$ such that*

$$S_0(x^*) \oplus U_0(x^*) = X,$$

where $U_0(x^*) = R(P(x^*))$ and $S_0(x^*) = R(I - P(x^*))$;

2. *dim $U_0(x^*) < +\infty$;*

3. *$(D_{x^*}\mathcal{T}(t))P(x^*) = P(x^*)D_{x^*}\mathcal{T}(t)$ for all $t \geq 0$ and the operator*

$$(D_{x^*}\mathcal{T}(t))|_{U_0(x^*)} : U_0(x^*) \to U_0(x^*)$$

is an isomorphism. Consequently,

$$(D_{x^*}\mathcal{T}(t))U_0(x^*) = U_0(x^*) \ \forall t \in \mathbb{R} \quad \text{and} \quad (D_{x^*}\mathcal{T}(t))S_0(x^*) \subset S_0(x^*), \ \forall t \geq 0;$$

4. *There exists $C > 0, \lambda \in (0, 1)$ such that*

$$\| (D_{x^*}\mathcal{T}(t))v^s \| \leq C\lambda^t \| v^s \|, \ \forall v^s \in S_0(x^*), \ \forall t \geq 0,$$
$$\| (D_{x^*}\mathcal{T}(-t))v^u \| \leq C\lambda^t \| v^u \|, \ \forall v^u \in U_0(x^*), \ \forall t \geq 0.$$

The constants C and λ are called constants of hyperbolicity.

Remark 2. *On [33, Lemma 2.2] the author has proved that Definition 20 is equivalent to the definition of hyperbolicity defined via exponential dichotomy (Definition 41), given later on Chapter 4.*

Let us recall the notion of non-wandering set, already introduced before, that is necessary to define Morse-Smale semigroups.

Definition 21 (Non-wandering set)**.** *Let X be a Banach space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\}$ be semigroup on X. We say that $x \in X$ is a **non-wandering** point of $\mathcal{T}(\cdot)$ if for any $t_0 \geq 0$ and neighborhood V_x of x there exists $t > t_0$ such that $\mathcal{T}(t)V_x \cap V_x \neq \emptyset$. We will denote the set of non-wandering points of $\mathcal{T}(\cdot)$ by Ω.*

Now we are able to define, with all details, one of the key concepts of this thesis, which will be present on the great majority of our work.

Definition 22 (Morse-Smale)**.** *Let X be a Banach space and $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}^1(X)$ be a semigroup with global attractor $\mathcal{A}$. We say that $\mathcal{T}(\cdot)$ is a **Morse-Smale** semigroup if it satisfies the following conditions:*

1. *$\mathcal{T}(t)|_{\mathcal{A}}$ is injective for all $t \geq 0$.*

2. *The Fréchet derivative $D_z\mathcal{T}(t) : X \to X$ of $\mathcal{T}(t)$ at z is an isomorphism onto its image for all $z \in \mathcal{A}$ and $t \geq 0$.*
3. *The non-wandering set of $\mathcal{T}(\cdot)$ is given by the set of equilibrium points $\Omega = \mathcal{E} = \{x_1^*, \cdots, x_p^*\}$, which are a finite number of them and each one is hyperbolic. In particular, there are no periodic orbits.*
4. $\dim W^u_{loc}(x^*) < \infty$ *for every $x^* \in \Omega$, where dim is the dimension related to the differentiable manifold.*
5. *$W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ are transverse for all $i \neq j$, i.e., if $z \in W^u(x_i^*) \cap W^s_{loc}(x_j^*)$, then*

$$T_zW^u(x_i^*) + T_zW^s_{loc}(x_j^*) = X,$$

where $T_zW^u(x_i^*)$ and $T_zW^s_{loc}(x_j^*)$ are the tangent spaces of $W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$ at z, respectively.

Remark 3. *We would like to remember that usually the definition of Morse-Smale semigroup just demand that $D_x\mathcal{T}(t)$ is a bounded injective operator, for all $t > 0$ and $x \in \mathcal{A}$ (see e.g. [9]). In order to simplify the text, we have redefined what is a Morse-Smale semigroup as on Definition 22. The definition still generalizes the concept of Morse-Smale on finite dimensions (where the derivative is an isomorphism).*

We now proceed with another classical result from finite and infinite dimensional Dynamical Systems, which is the stable/unstable manifold theorem [9, 59].

Theorem 1.12. *Let $\{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}^1(X)$ be a Morse-Smale semigroup, x^* be a hyperbolic equilibrium point and denote by $B_X(x, s)$ the open ball in X centered in x with radius $s > 0$. Then, there exists $\delta > 0$ such that $W^u(x^*) \cap B_X(x^*, \delta)$ and $W^s(x^*) \cap B_X(x^*, \delta)$ are the graphs of $\mathcal{C}^1-$maps $\psi_u : B_{U_0(x^*)}(x^*, \delta) \to S_0(x^*)$ and $\psi_s : B_{S_0(x^*)}(x^*, \delta) \to U_0(x^*)$, respectively. Consequently, $W^u(x^*)$ and $W^s_{loc}(x^*)$ are $\mathcal{C}^1$-manifolds.*

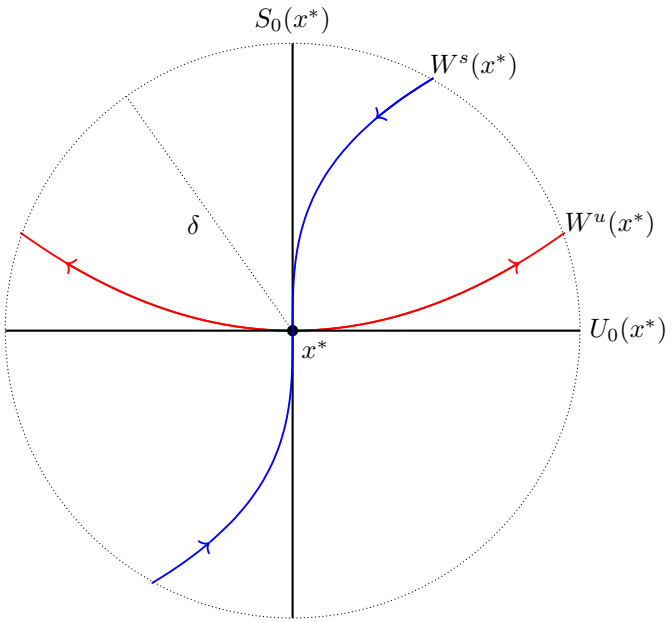


Figure 1.2: The local unstable and stable manifolds

Proof. We will provide a sketch of the proof for the unstable manifold and the proof for the local stable manifold will follow similarly. The reader can find a more rigorous proof in [9, Chapter 4].

The main idea is to prove that the local unstable manifold is the graph of a map obtained via the Banach fixed point Theorem and then conclude that the whole unstable manifold $W^u(x^*)$ is a $\mathcal{C}^1-$manifold.

- $W^u_{loc}(x^*)$ is the graph of a Lipschitz map

  For this part of the proof we recommend the reader to check the original reference [9, Theorem 4.4]. Since any continuous semigroup can be written a discrete semigroup, we will prove the statement for the discrete dynamical system $\{T^n : n \in \mathbb{N}\}$ where $T = \mathcal{T}(1)$. In fact, both semigroups have the same unstable manifold $W^u(x^*)$. Assuming that $x^* = 0$ without lost of generality, we write T as $T = L + N$, where $L : X \to X$ is a hyperbolic linear operator, that is, $\sigma(L) \cap S^1 = \emptyset$, where $\sigma(L)$ denote the spectrum of L (Definition 53) and $S^1 \subset \mathbb{C}$ denotes the unity sphere, and $N : X \to X$ is a $\mathcal{C}^1-$map that satisfies $N(0) = 0$ and $N'(0) = 0$. Since we are concerned with the behavior of the unstable manifold near 0, we can assume that N has Lipschitz constant $\gamma > 0$ that can be chosen to be sufficiently small. Considering the curve $\Gamma(t) = e^{it}$ for $t \in [0, 2\pi]$, we define the spectral projections

  $$\Pi_2 = \frac{1}{2\pi i}\int_\Gamma (\lambda - L)^{-1} d\lambda, \quad \text{and} \quad \Pi_1 = I - \Pi_2$$

  and the corresponding decomposition of X and L as $X = X_1 \oplus X_2$ and $L = L_1 \oplus L_2$, where $X_j = \Pi_j X, L_j = L|_{X_j}, j = 1, 2$, and

  $$\sigma(L_2) = \sigma(L) \cap \{\lambda \in \mathbb{C} : |\lambda| < 1\} \text{ and } \sigma(L_1) = \sigma(L) \cap \{\lambda \in \mathbb{C} : |\lambda| > 1\}.$$

  For $X = X_1 \oplus X_2$ consider the norm given by $\|x_1 + x_2\| := \max\{\|x_1\|, \|x_2\|\}$. Note that, with this norm in $X, \|\Pi_j\|_{\mathcal{L}(X)} = 1, j = 1, 2$.

  Then T can be written as

  $$\begin{aligned} T : X_1 \oplus X_2 &\to X_1 \oplus X_2 \\ (x_1, x_2) &\mapsto (L_1 x_1 + N_1(x_1 + x_2), L_2 x_2 + N_2(x_1 + x_2)) \end{aligned}$$

  where $N_j = \Pi_j \circ N$ has Lipschitz constant $Lip N_j \leqslant \gamma$ for $j = 1, 2$.
  Our goal is to show that $W^u(0) = \{\xi + \theta(\xi) : \xi \in X_1\}$, for some $\theta : X_1 \to X_2$ Lipschitz with $\theta(0) = 0$ and $\operatorname{Lip}\theta \leqslant 1$.

  Since $TW^u(0) \subset W^u(0)$, we expect that for all $\xi \in X_1$ there exists $\hat{\xi} \in X_1$ with $T(\xi + \theta(\xi)) = \hat{\xi} + \theta(\hat{\xi})$, that is,

  $$\begin{cases} \hat{\xi} = L_1\xi + N_1(\xi + \theta(\xi)) \\ \theta_*(\hat{\xi}) = L_2\theta(\xi) + N_2(\xi + \theta(\xi)) \end{cases} \tag{1.4}$$

  with $\theta_* = \theta$, where θ is the fixed point of the map $\theta \mapsto \theta_*$ defined by (1.4).
  In order to show that there exists such fixed point, let $\theta : X_1 \to X_2$ be a map satisfying $\theta(0) = 0$ and $\operatorname{Lip}\theta \leqslant 1$. First of all, to see that the map θ_* is well defined, note that $\xi \mapsto -L_1^{-1}N_1(\xi + \theta(\xi))$ is a contraction, and therefore for each $\hat{\xi} \in X_1$, there exists a unique $\xi \in X_1$ satisfying the first equation of (1.4).

  To find the fixed point, define the complete metric space $(\mathcal{F}, d)$ of Lipschitz maps $\theta : X_1 \to X_2$ with $\theta(0) = 0$, $\operatorname{Lip}\theta \leqslant 1$ and metric

  $$d(\theta, \tau) = \sup_{\xi \in X_1,} \frac{\|\theta(\xi) - \tau(\xi)\|}{\|\xi\|}.$$

  It is possible to show that the map $\theta \mapsto \theta_*$ takes $\mathcal{F}$ into itself and is a contraction.

It remains to show that the graph $G = \{\xi + \theta_*(\xi) : \xi \in X_1\}$ of the fixed point is the unstable manifold of 0. In fact, it is possible to show that $TG = G$, T is injective in G and $Lip\left((T|_G)^{-1}\right) < 1$, which will imply that each element of G converges exponentially to 0 backwards, that is, $G \subset W^u(0)$. To show that $W^u(0) \subset G$, we take $x \in W^u(0)$ and write $x = \xi + \theta_*(\xi) + \eta$, where $\xi \in X_1$ and $\eta \in X_2$. With some algebraic manipulation, it is possible to show, using the fact that $T^{-n}x$ converges to 0 exponentially, that $\eta = 0$ and therefore $G = W^u(0)$.

- $W^u(x^*)$ is a $\mathcal{C}^1$-manifold

  To prove the regularity of the manifold, the strategy is to use the following fact: If X, Y are Banach spaces, $Q \subset X$ is an open set and $f : Q \to Y$ is a locally Lipschitz map, then f is continuously differentiable in Q if, and only if for each $x_0 \in Q$ we have

$$\|f(x+h) - f(x) - f(x_0+h) + f(x_0)\| = o(\|h\|)$$

  as $(x, h) \to (x_0, 0)$.

  With this result in mind, it is possible to show that θ_* satisfy such property and consequently $W^u_{loc}(x^*)$ is a $\mathcal{C}^1$-manifold, since

$$W^u_{loc}(x^*) = \{x^* + \xi + \theta_*(\xi) : \xi \in X_1, \|\xi\| < r_0\},$$

  for some $r_0 > 0$.

  It remains to show that the whole unstable manifold $W^u(x^*)$ is a $\mathcal{C}^1-$manifold. Given $z \in W^u(x^*)$, fix the smallest nonnegative integer n_z such that $z \in T^{n_z}\left(W^u_{\text{loc}}\left(x^*\right)\right)$. Since T is Morse-Smale, $T^{n_z}\left(W^u_{\text{loc}}\left(x^*\right)\right)$ is open in $\mathcal{A}$ and we can fix a neighborhood $U_z \subset T^{n_z}\left(W^u_{loc}(x)\right)$ of z and let $W_z = T^{-n_z}U_z$ be a neighborhood in $W^u_{loc}(x)$. If $\pi_u : W^u_{loc}(x) \to X_1$ is the projection

$$\pi_u(x^* + \xi + \theta(\xi)) = \xi, \text{ for } \xi \in X_1, \|\xi\| < r_0,$$

  define $\varphi_z : \pi_u\left(W_z\right) \to U_z$ by $\varphi(\xi) = T^{n_z}(x + \xi + \theta(\xi))$. Then $\{(\pi_u\left(W_z\right), \varphi_z)\}$ is a $\mathcal{C}^1$ chart for $W^u(x)$, and since the inclusion $i : W^u(x) \to X$ has injective derivative with this chart, $W^u(x)$ is an injectively immersed $\mathcal{C}^1$ submanifold of X.

□

Theorem 1.13. *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\} \subset \mathcal{C}^1(X)$ be a Morse-Smale semigroup and Ω be the non-wandering set of $\mathcal{T}$. Then, for any $x^* \in \Omega$ it holds*

$$T_{x^*}W^u(x^*) = U_0(x^*) \text{ and } T_{x^*}W^s_{loc}(x^*) = S_0(x^*),$$

where S_0 and U_0 are the subspaces of X defined in Definition 20.

Proof. It is a consequence of the proof of Theorem 1.12 and the Implicit Function Theorem. The reader can find a more detailed proof in [9, Chapter 4]. □

The following result states that Morse-Smale semigroups behave like a group, which we define below, when restricted to the global attractor $\mathcal{A}$. In fact, Morse-Smale semigroups allow us to work with homeomorphisms instead of just continuous maps. This property will be often used in Chapter 2.

Proposition 1.14. *Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup on a Banach space X and define $\mathcal{T}(-t) := (\mathcal{T}(t)|_{\mathcal{A}})^{-1} : \mathcal{A} \to \mathcal{A}$ for all $t \geq 0$. Then:*

1. *$\{\mathcal{T}(t)|_{\mathcal{A}} : t \in \mathbb{R}\}$ is a group on $\mathcal{A}$, that is,*

a. $\mathcal{T}(0)|_{\mathcal{A}} = Id$;

b. $\mathcal{T}(t)|_{\mathcal{A}} \circ \mathcal{T}(s)|_{\mathcal{A}} = \mathcal{T}(t+s)|_{\mathcal{A}}$ *for all* $t, s \in \mathbb{R}$

c. The map $\mathbb{R} \times \mathcal{A} \ni (t, x) \mapsto \mathcal{T}(t)x$ *is continuous.*

2. For any $x^* \in \Omega$ *the map* $W^u(x^*) \ni x \mapsto \dim T_x W^u(x^*) < \infty$ *is constant.*

Proof.

1. The proof of item a) is obvious. To prove item b), firstly note that

$$\begin{aligned}\mathcal{T}(-t)\mathcal{T}(-s) &= (\mathcal{T}(t)|_{\mathcal{A}})^{-1}(\mathcal{T}(s)|_{\mathcal{A}})^{-1} = (\mathcal{T}(s)|_{\mathcal{A}} \circ \mathcal{T}(t)|_{\mathcal{A}})^{-1} \\ &= (\mathcal{T}(t+s)|_{\mathcal{A}})^{-1} = \mathcal{T}(-t-s) = \mathcal{T}(-s)\mathcal{T}(-t), \ \forall t, s \geq 0. \end{aligned} \tag{1.5}$$

Moreover,

$$\begin{aligned}\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(s)\mathcal{T}(t) &\Longrightarrow \mathcal{T}(s)(\mathcal{T}(t)|_{\mathcal{A}})^{-1} = (\mathcal{T}(t)|_{\mathcal{A}})^{-1}\mathcal{T}(s) \\ &\Longrightarrow \mathcal{T}(s)\mathcal{T}(-t) = \mathcal{T}(-t)\mathcal{T}(s), \ \forall t, s \geq 0. \end{aligned} \tag{1.6}$$

Hence, if $t > s > 0$ we have

$$\begin{aligned}&\mathcal{T}(t) \circ \mathcal{T}(-s) \circ (\mathcal{T}(t-s)|_{\mathcal{A}})^{-1} = \mathcal{T}(t)(\mathcal{T}(s)|_{\mathcal{A}})^{-1}(\mathcal{T}(t-s)|_{\mathcal{A}})^{-1} \\ =&\mathcal{T}(t)(\mathcal{T}(t-s)|_{\mathcal{A}} \circ \mathcal{T}(s)|_{\mathcal{A}})^{-1} = \mathcal{T}(t)(\mathcal{T}(t))^{-1} = Id\end{aligned}$$

and from (1.5) and (1.6)

$$(\mathcal{T}(t-s)|_{\mathcal{A}})^{-1}\mathcal{T}(t) \circ \mathcal{T}(-s) = Id = (\mathcal{T}(t-s)|_{\mathcal{A}})^{-1}\mathcal{T}(t) \circ \mathcal{T}(-s),$$

which implies

$$\mathcal{T}(t)|_{\mathcal{A}} \circ \mathcal{T}(-s) = \mathcal{T}(t-s)|_{\mathcal{A}}, \ \forall t > s > 0. \tag{1.7}$$

Similarly, if $s > t > 0$ we can use (1.7) to obtain

$$\mathcal{T}(t-s) = (\mathcal{T}(s-t)|_{\mathcal{A}})^{-1} = (\mathcal{T}(s)|_{\mathcal{A}} \circ \mathcal{T}(-t))^{-1} = \mathcal{T}(t)|_{\mathcal{A}} \circ \mathcal{T}(-s),$$

and conclude the proof of item b).

It remains to show the continuity of the map $\mathbb{R} \times \mathcal{A} \ni (t, x) \mapsto \mathcal{T}(t)x$ (specially when $t < 0$). From item (1) of Definition 22 and the compactness of $\mathcal{A}$ it follows that $\mathcal{T}(-t)$ is continuous on $\mathcal{A}$. Note that $(t, x) \mapsto \mathcal{T}(t)x$ is uniformly continuous on $[0, 1] \times \mathcal{A}$, that is, given $\epsilon > 0$ there exists $\delta > 0$ such that if

$$x, y \in \mathcal{A}, t, s \in [0, 1], d(x, y) < \delta, |t - s| < \delta \Rightarrow d(\mathcal{T}(t)x, \mathcal{T}(s)y) < \epsilon. \tag{1.8}$$

From (1.8) it follows that given $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$x, y \in \mathcal{A}, t \in [0, \delta], d(x, y) < \delta \Rightarrow d(\mathcal{T}(t)x, y) < \epsilon.$$

Then

$$d(x, \mathcal{T}(-t)x) = d(\mathcal{T}(t)\mathcal{T}(-t)x, \mathcal{T}(-t)x) < \epsilon, \ \forall x \in \mathcal{A}, |t| < \delta.$$

Therefore for any $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$|t| < \delta \Rightarrow d(\mathcal{T}(t)x, x) < \epsilon, \ \forall x \in \mathcal{A}. \tag{1.9}$$

Let $t_0 > 0$ and $x_0 \in \mathcal{A}$. Given $\epsilon > 0$ take $\delta > 0$ such that

$$d(x,x_0)<\delta \Rightarrow d(\mathcal{T}(-t_0)x,\mathcal{T}(-t_0)x_0)<\frac{\epsilon}{2}.$$

From (1.9) we can fix $\delta' \in (0,1)$ such that

$$|s|<\delta' \Rightarrow d(\mathcal{T}(s)x,x)<\frac{\epsilon}{2},\ \forall x\in\mathcal{A}.$$

Hence, if $d((-t,x),(-t_0,x_0))<\delta_1=\min\{\delta',\delta\}$ we have

$$\begin{aligned} d(\mathcal{T}(-t_0)x_0,\mathcal{T}(-t)x) &= d(\mathcal{T}(-t_0)x_0,\mathcal{T}(-t_0)x)+d(\mathcal{T}(-t_0)x,\mathcal{T}(-t)x)\\ &\leq d(\mathcal{T}(-t_0)x_0,\mathcal{T}(-t_0)x)+d(\mathcal{T}(-t_0)x,\mathcal{T}(t_0-t)\mathcal{T}(-t_0)x)\\ &<\epsilon. \end{aligned}$$

This finish the proof that $\{\mathcal{T}(t)|_{\mathcal{A}} : t\in\mathbb{R}\}$ is a group on $\mathcal{A}$.

2. It follows immediately from item (4) of Definition 22 and the fact that $W^u(x^*)$ is a continuous connected $\mathcal{C}^1$-manifold (see Theorem 1.12).

□

Now our goal is to show that Morse-Smale semigroups are Gradient/Dynamically Gradient (Lemma 1.17). In particular, the results from Subsections 1.1.1 and 1.1.2 hold for Morse-Smale semigroups.

We start with the following technical lemma.

Lemma 1.15. *Let $\mathcal{T}$ be a semigroup on a metric space (M,d) with global attractor $\mathcal{A}$ and non-wandering set Ω. Then Ω is compact, $\mathcal{T}(t)\Omega\subset\Omega$ for all $t\geqslant 0$ (Ω is positively invariant), $\omega(x)\subset\Omega$ for all $x\in M$ and $\alpha_\phi(x)\subset\Omega$ for all $x\in\mathcal{A}$.*

Proof. We start by showing the compactness of Ω. Since $\Omega\subset\mathcal{A}$, we just have to show that Ω is closed. Let $\{z_n\}\subset\Omega$ such that $z_n\to z\in\mathcal{A}, t_0\geqslant 0$ and V_z a neighborhood of z in M. Since $z_n\to z$, there exists $n_0\in\mathbb{N}$ such that V_z is a neighborhood of $z_{n_0}\in\Omega$, thus there exists $t\geqslant t_0$ such that $\mathcal{T}(t)V_z\cap V_z\neq\varnothing$, and consequently $z\in\Omega$.

To show that Ω is positively invariant, let $z\in\Omega, t, t_0\geqslant 0$ and V a neighborhood of $\mathcal{T}(t)z$. Then $\mathcal{T}(t)^{-1}(V)$ is a neighborhood of z and there exists $\tau\geqslant t_0$ such that $\mathcal{T}(\tau)\left(\mathcal{T}(t)^{-1}(V)\right)\cap\mathcal{T}(t)^{-1}(V)\neq\varnothing$, implying that $\mathcal{T}(\tau)V\cap V\neq\varnothing, \tau\geqslant t_0$. Therefore $\mathcal{T}(t)z\in\Omega$ and that Ω is positively invariant

Now we prove the last statement for the ω-limit and the proof for the α-limit will follow analogously. Let $z\in\omega(x)\subset\mathcal{A}$, for some $x\in M, t_0\geqslant 0$ and V a neighborhood of z in M. Since $z\in\omega(x)$, it follows from the characterization of the omega limit [9, Proposition 1.13] that there exists $t_n\geq 0$ such that $t_n\to\infty$ and $\mathcal{T}(t_n)x\xrightarrow{n\to+\infty} z$. Hence, there exists $n_0\in\mathbb{N}$ such that $\mathcal{T}(t_n)x\in V$ for all $n\geqslant n_0$ and we can choose $n_1\geqslant n_0$ such that $t:=t_{n_1}-t_{n_0}\geqslant t_0$. Thus $\mathcal{T}(t_{n_1})x\in V$ and $\mathcal{T}(t_{n_1})x=\mathcal{T}(t_{n_1}-t_{n_0})\mathcal{T}(t_{n_0})x=\mathcal{T}(t)\mathcal{T}(t_{n_0})x\in\mathcal{T}(t)V$, proving that $\mathcal{T}(t)V\cap V\neq\varnothing$ and $z\in\Omega$. □

The next result states that if we have an indirect connection between two equilbria x_i^* and x_k^* then we also have a direct connection between them. That is, if there exists (bounded) global solutions ϕ and ψ and another equilibrium x_j^* such that

$$x_i^*\xleftarrow{t\to-\infty}\phi(t)\xrightarrow{t\to+\infty}x_j^*\xleftarrow{t\to-\infty}\psi(t)\xrightarrow{t\to+\infty}x_k^*$$

then there exists another global solution ξ satisfying

$$x_i^*\xleftarrow{t\to-\infty}\xi(t)\xrightarrow{t\to+\infty}x_k^*.$$

This property is the final piece to show that Morse-Smale semigroups are Dynamically Gradient (and Gradient).

Proposition 1.16. *Let $\mathcal{T}$ be a Morse-Smale semigroup on a Banach space X with non-wandering set $\Omega = \{x_1^*, \cdots, x_p^*\}$. If $W^u(x_i^*) \cap W^s_{loc}(x_j^*) \neq \varnothing$ and $W^u(x_j^*) \cap W^s_{loc}(x_k^*) \neq \varnothing$, then $W^u(x_i^*) \cap W^s_{loc}(x_k^*) \neq \varnothing$.*

Proof. The proof will use the infinite dimensional version of the $\lambda-$lemma, which can be found in [9, Theorems 4.17 and 4.20] and we will not prove here. In fact, this result also depends on more auxilary lemmas, but we will replicate the proof here, since it's short. We recommend the reader to see the original proof in [9, Proposition 5.9] for more details.

Let x_{ij} be a point of intersection between $W^u(x_i^*)$ and $W^s_{loc}(x_j^*)$. Since $\mathcal{T}$ is Morse-Smale, those manifolds intersect transversely and from [9, Lemma 4.7] we can fix a submanifold $D^0 \subset X$ that satisfies

$$D^0 \cap W^s_{\text{loc}}(x_j^*) = \{x_{ij}\}, T_{x_{ij}} D^0 \oplus T_{x_{ij}} W^s_{\text{loc}}(x_j^*) = X \text{ and } D^0 \subset W^u(x_i^*).$$

If x_{jk} a point of transversal intersection between $W^u(x_j^*)$ and $W^s_{\text{loc}}(x_k^*)$, we can fix n_0 such that $x_{jk} \in \mathcal{T}^{n_0}\left(W^u_{\text{loc}}(x_j^*)\right)$. Hence, the global λ-lemma implies that there exist a neighborhood D^0_ρ of x_{ij} in D^0 and sets $R^n \subset \mathcal{T}^n\left(D^0_\rho\right) \subset W^u(x_i^*)$ such that

$$R^n \to \mathcal{T}^{n_0}\left(W^u_{loc}(x_j^*)\right),$$

in the $\mathcal{C}^1$ norm (see [9, Theorem 4.20]). It follows from the openness of the transversality [9, Lemma 5.8] that there exists a point $x_{ik} \in W^u(x_i^*) \cap W^s_{\text{loc}}(x_k^*)$. □

We can finally prove that Morse-Smale semigroups are a particular case of Dynamically Gradient (and Gradient) semigroups.

Lemma 1.17. *If $\mathcal{T}(\cdot)$ is a Morse-Smale semigroup on a Banach space X with non-wandering set $\Omega = \{x_1^*, \ldots, x_p^*\}$, then $\mathcal{T}(\cdot)$ is also a Gradient semigroup, that is, there exists a continuous function $V : X \to [0, +\infty)$, named Lyapunov function, such that we can order the non-wandering set Ω to satisfy*

1. *$V(x_i^*) = i$, for all $1 \leq i \leq p$.*
2. *If $x \notin \Omega$, then $\mathbb{R}^+ \ni t \mapsto V(\mathcal{T}(t)x)$ is strictly decreasing.*

Consequently, the global attractor is given by

$$\mathcal{A} = \bigcup_{i=1}^{p} W^u(x_i^*),$$

and for each $x \in X$ there exists $i \in \{i, \ldots, p\}$ such that $\mathcal{T}(t)x \to x_i^*$ as $t \to +\infty$.

Proof. From Theorem 1.10 we just have to prove that Morse-Smale semigroups are Dynamically Gradient. We will provide a sketch of the proof, which is again inspired by the proof found on [9, Theorem 5.10].

To show that $\mathcal{T}$ satisfies (G1), let $\xi : \mathbb{R} \to \mathcal{A}$ be a bounded global solution with $\xi(\mathbb{R}) \notin \Omega$ and let $x = \xi(0)$. It follows from Lemma 1.15 that $\omega(x), \alpha_\xi(x) \subset \Omega$. Since $\omega(x)$ and $\alpha_\xi(x)$ are connected, then (G1) must be satisfied.

It remains to show that $\mathcal{T}$ satisfies (G2). Assume by contradiction that this is not the case, that is, suppose that there exists a homoclinic structure related to $\mathcal{E} = \Omega$. Then, we can use Proposition 1.16 to obtain a homoclinic solution, that is, a global solution $\xi : \mathbb{R} \to X$ and $x^* \in \Omega$ such that

$$\xi(0) := x^0 \notin \Omega \quad \text{and} \quad x^* \xleftarrow{t \to -\infty} \xi(t) \xrightarrow{t \to +\infty} x^*$$

We can assume without lost of generality that $x^0 \in W^s_{\text{loc}}(x^*)$. We want to prove that x^0 is a non-wandering point to arive at a contradiction. Let V be a neighborhood of x^0 in X and $D^0 \subset V$ a submanifold transversal to $W^s_{loc}(x^*)$ in x^0. Then, the global λ-lemma [9, Theorem 4.20] implies that there exist a neighborhood D^0_ρ of x^0 in D^0 and subsets S_n of $\mathcal{T}^n\left(D^0_\rho\right) \subset \mathcal{T}^n(V)$ such that $S_n \to \mathcal{T}^{n_0}\left(W^u_{\text{loc}}(x^*)\right)$, where $n_0 \in \mathbb{N}$ is such that $x^0 \in \mathcal{T}^{n_0}\left(W^u_{\text{loc}}(x^*)\right)$, which proves that $x^0 \in \Omega$ and gives us a contradiction. Therefore $\mathcal{T}$ satisfies (G2).

The latter statement is a directly consequence of Theorem 1.3. □

We will not provide the proof of the next results, since they are straightforward. Despite that, they will be very useful on Chapter 2.

Lemma 1.18. *Let $\mathcal{T} = \{\mathcal{T}(t) : t \geq 0\}$ be a Gradient semigroup relative to the set $\{x^*_1, \dots, x^*_p\}$ on a metric space (M, d). Then, for any $n \in \mathbb{N}$ the semigroup $\mathcal{T}_n = \{\mathcal{T}(nt) : t \geq 0\}$ is Gradient relative to the set $\{x^*_1, \dots, x^*_p\}$, with the same Morse-decomposition as $\mathcal{T}$.*

Lemma 1.19. *Let $\mathcal{T} = \{\mathcal{T}(t) : t \geq 0\}$ be a semigroup on a metric space (M, d) and x^* be an equilibrium. Given $n \in \mathbb{N}$ define $\mathcal{T}_n$ as on Lemma 1.18 and denote by $W^u(\mathcal{T}, x^*)$ the unstable manifold of x^* related to the semigroup $\mathcal{T}$ and $W^u(\mathcal{T}^n, x^*)$ the unstable manifold of x^* related to $\mathcal{T}_n$. Then $W^u(\mathcal{T}, x^*) = W^u(\mathcal{T}^n, x^*)$. The same holds for the stable manifold.*

Corollary 1.20. *Let $\mathcal{T} = \{\mathcal{T}(t) : t \geq 0\}$ be a semigroup on a Banach space X. Given $n \in \mathbb{N}$ define $\mathcal{T}_n$ as on Lemma 1.18. If x^* is a hyperbolic equilibrium of $\mathcal{T}$ with constants C and λ, then it is also hyperbolic for $\mathcal{T}_n$ with constants C and λ^n and the unstable and (local) stable manifolds are $\mathcal{C}^1$ manifolds.*

Proof. The proof of the hyperbolicity is straighforward and the $\mathcal{C}^1$ manifolds follows from Lemma 1.19. □

We finish this section by recalling another key concepts of this thesis (that were already covered in the Introduction), which are related to the Shadowing property. In fact, our main goal is to prove that if $\{\mathcal{T}(t) : t \geq 0\}$ is a Morse-Smale semigroup defined on a Hilbert space X with global attractor $\mathcal{A}$, then $\mathcal{T}(1)|_{\mathcal{A}}$ has the Lipschitz Shadowing property and $\mathcal{T}(1)|_{\mathcal{U}}$ has the Hölder Shadowing property for any positively invariant bounded neighborhood $\mathcal{U} \supset \mathcal{A}$. We recall that the Lipschitz Shadowing property on the global attractor is known to be true when $\mathcal{A} = M$ is a (finite dimensional) compact manifold and $\mathcal{T}$ is Morse-Smale, as we can see in [47, Chapter 2]. Moreover, it is also known that the Lipschitz Shadowing property holds on a neighborhood of the global attractor $\mathcal{A}$ of a Morse-Smale semigroup $\mathcal{T}(\cdot)$ when X is either finite dimensional [58] or Hilbert with an inertial manifold [47, Chapter 4].

Definition 23 (Shadowing)**.** *Let (M, d) be a metric space, $\mathcal{T} : M \to M$ be a map and $\mathbb{T} = \mathbb{Z}, \mathbb{Z}^-$.*

1. *A sequence $\{x_n\}_{n\in\mathbb{T}}$ in M is said to be a δ-**pseudo orbit** of $\mathcal{T}$ (see figure 0.2) for some $\delta > 0$ if*
$$d(\mathcal{T}x_n, x_{n+1}) \leq \delta, \ \forall\ n \in \mathbb{T}.$$
2. *Let $\{x_n\}_{n\in\mathbb{T}}$ and $\{z_n\}_{n\in\mathbb{T}}$ be sequences on M. We say that $\{z_n\}_{n\in\mathbb{T}}$ ϵ-**shadows** $\{x_n\}_{n\in\mathbb{T}}$, for some $\epsilon > 0$ if*
$$d(x_n, z_n) \leq \epsilon, \ \forall\ n \in \mathbb{T}.$$
3. *We say that $\mathcal{T}$ admits the **Shadowing** property if for any $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $\{x_n\}_{n\in\mathbb{T}}$ of $\mathcal{T}$ is ϵ-shadowed by an orbit of $\mathcal{T}$, that is, there exists $x \in M$ such that*
$$d(x_n, \mathcal{T}^n x) \leq \epsilon, \forall\ n \in \mathbb{T}.$$

4. *We say that $\mathcal{T}$ admits the α-**Hölder Shadowing** property if there exists $d_0, L > 0$ and $\alpha \in (0, 1]$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is Ld^α-shadowed by a orbit of $\mathcal{T}$. If $\alpha = 1$ we say that $\mathcal{T}$ have the **Lipschitz Shadowing** property.*
5. *We say that $\mathcal{T}$ admits the **Logarithm Shadowing** property if there exists $d_0, L > 0$ such that any d-pseudo orbit of $\mathcal{T}$, with $d \in [0, d_0]$, is $Ld|\ln d|$-shadowed by an orbit of $\mathcal{T}$.*

Note that if $\alpha \in (0, 1)$ then

$$\textit{Lipschitz Shadowing} \Rightarrow \textit{Logarithm Shadowing} \Rightarrow \alpha - \textit{Hölder Shadowing} \Rightarrow \textit{Shadowing}.$$

1.2 The Damped Wave Equation

We dedicate this section to introduce the theory of linear semigroups/groups and show some results about the damped wave equation, such as well posedness, differentiability of the semigroup, existence of a global attractor and Gradient structure. We study the damped wave equation so we can find an example of a Morse-Smale semigroup defined on a Hilbert space that satisfies (H1) and (H2) from Theorem 0.1. In particular, the semigroup related to the damped wave operator satisfies the assumptions from Theorem 0.1 (see Corollary 1.34).

The scheme of this section is as follows: in Subsection 1.2.1 we define the basic concepts from linear semigroup theory, in the sense of [45]. In fact, we introduce the notion of semigroup, group, infinitesimal generator, etc. Subsection 1.2.2 is dedicated to study the damped wave equation (1.12), which is defined on a Hilbert space. We prove that problem (1.12) is well posed and define a semigroup which is $\mathcal{C}^1$ in relation to the initial conditions. Moreover, we finish this subsection with an example of a Morse-Smale semigroup defined on a Hilbert space, which is exactly the hypothesis used for our theoretical results stablished in Chapter 2.

As in Section 1.1, we will pass very briefly to some results, especially in Subsection 1.2.1, which is basically the preliminaries of Subsection 1.2.2. Most of the results of this section can be found in [45, 20].

1.2.1 Linear Semigroups and Groups

In this subsection we stablish the basics of the linear semigroup theory. The concepts and results of this subsection provide the theoretical background to the following subsection. This will be a very quickly introduction to the theory, and we recommend the books [45, 63, 20] to the reader that is not familiar with semigroup theory and looks for a complete and detailed approach.

Let us start with the fundamental definition.

Definition 24 (Linear Semigroup)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $\{L(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a family of bounded linear maps. This family is a **strongly continuous semigroup of bounded linear operators** (or simply continuous semigroup) if*

1. *$L(0) = Id$;*
2. *$L(t)L(s) = L(t + s)$, for all $t, s \geq 0$;*
3. *$\| L(t)x - x \| \xrightarrow{t \to 0^+} 0$, for all $x \in X$.*

Similarly, we can define a group of bounded linear operators, which allows the family of maps to "evolve" backwards. In fact, we will see further on that the solutions of the damped wave equation define a group.

Definition 25 (Linear Group)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $\{L(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$ be a family of bounded linear maps. This family is a* ***group of bounded linear operators*** *if*

1. $L(0) = Id$;
2. $L(t)L(s) = L(t+s)$, *for all* $t, s \in \mathbb{R}$;
3. $\| L(t)x - x \| \xrightarrow{t\to 0} 0$, *for all* $x \in X$.

The following result provide us an exponential bound to the norm of the linear operator $L(t)$, when $\{L(t) : t \geq 0\}$ is a continuous semigroup. The proof of this result can be found in [45, Corollary 1.4].

Theorem 1.21. *Let $\{L(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a strongly continuous semigroups of bounded linear operators. Then, there exists $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$\| L(t) \|_{\mathcal{L}(X)} \leq Me^{\omega t}, \ \forall t \geq 0.$$

Below we define the infinitesimal generator, which is one of the most fundamental concepts from Semigroup theory. We will go into a little more detail about its importance after the definition.

Definition 26 (Initesimal Generator)**.** *Let X be a Banach space and $\{L(t) : t \geq 0\} \subset \mathcal{L}(X)$ be a continuous semigroup. The* ***infinitesimal generator*** *of the semigroup $\{L(t) : t \geq 0\} \subset \mathcal{L}(X)$ is the operator $A : D(A) \subset X \to X$ defined as*

$$D(A) := \left\{ x \in X : \text{the limit } \lim_{h\to 0^+} \frac{L(h)x - x}{h} \text{ exists} \right\} \text{ and } Ax = \lim_{h\to 0^+} \frac{L(h)x - x}{h}, \ \forall x \in D(A).$$

It is a fact that the domain $D(A)$ is non-empty. In fact, $\bigcap_{n\in\mathbb{N}} D(A^n)$ is dense in X. We invite the reader to see [45, Chapter 1.1] for more details.

The infinitesimal generator has an important role on the theory of linear semigroups because we can associate A and the semigroup associated to it with the theory of differential equations. In fact, if $x_0 \in D(A)$, then the map $[0, \infty) \ni t \mapsto x(t) := L(t)x_0 \in X$ is continuously differentiable [45, Theorem 2.4] and satisfies

$$\begin{cases} \dot{x}(t) = Ax(t), & t > 0 \\ x(0) = x_0 \end{cases} \tag{1.10}$$

With this in mind, we define the concept of strong solution.

Definition 27. *A function $x \in \mathcal{C}([0,\infty), X) \cap \mathcal{C}^1((0,\infty), X)$ is said to be a* ***strong solution*** *of (1.10) if $x(0) = x_0$, $x(t) \in D(A)$ for $t > 0$, and (1.10) holds for $t > 0$.*

If $x_0 \in X$ does not belong to $D(A)$, it is still possible to give meaning to $x(t)$ as a solution of (1.10).

Definition 28. *A* ***weak solution*** *of (1.10) is a function $x \in \mathcal{C}([0,\infty), X)$ such that $x(0) = x_0$, and for every $x^* \in D(A^*)$, the function $[0,\infty) \ni t \mapsto \langle x(t), x^* \rangle$ is differentiable and*

$$\frac{d}{dt}\langle x(t), x^* \rangle = \langle x(t), A^* x^* \rangle, \quad t \geq 0,$$

where A^* is the dual operator of A (Definition 52).

Now we characterize the weak solutions of (1.10) and emphasize the potential of the theory of linear semigroups.

Theorem 1.22. *A function $x : [0, \infty) \to X$ is a weak solution of (1.10) if, and only if,*

$$x(t) = L(t)x_0, \quad t \geq 0. \tag{3.6}$$

In particular, there exists a unique weak solution of (1.10). Moreover, any strong solution of (1.10) is a weak solution. Converserly, if $x_0 \in D(A)$ then any weak solution of (1.10) is a strong solution.

Proof. See [45, Chapter 1]. □

Remark 4. *It is known that if $A : X \to X$ is bounded, then the solution of (1.10) is given by*

$$e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \in \mathcal{L}(X), \ \forall t \geq 0.$$

From this fact and Theorem 1.22, if $\{L(t)t \geq 0\}$ is a semigroup with infinitesimal generator A, we sometimes denote $L(t)$ by e^{At}.

Now we announce our last concepts before we study the damped wave equation. The next result is very well known in the literature and is oftenly used to prove the existence and uniqueness of solutions of linear differential equations.

Theorem 1.23 (Stone). *Let X be a Hilbert space and $A : D(A) \subset X \to X$ be a linear operator. Then, A is an infinitesimal generator of a group of unitary operators ($\| L(t) \|_{\mathcal{L}(X)} \leq 1$ for all $t \in \mathbb{R}$) if, and only if iA is self-adjoint (Definition 52).*

Proof. See [45, Theorem 10.8]. □

Theorem 1.23 will be helpful to study the damped wave equation, because the linear operator associate to it is a bounded perturbation of a linear operator that satisfies the assumptions from Theorem 1.23. With this, we will be able to prove that the linear operator associated to the damped wave equation also generates a group and therefore $L(t)$ is an isomorphism (because $L(t)$ has inverse $L(-t)$), that is, $0 \in \rho(L(t))$ for all $t \in \mathbb{R}$, whrere ρ denotes the resolvent set (Definition 53).

Now we introduce the notion of mild solution to semilinear equations, which we will adopt when dealing with the damped wave equation.

Definition 29. *Let X be a Banach space and consider the semilinear equation*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)), \ t > 0 \\ x(0) = x_0 \in X, \end{cases} \tag{1.11}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $A : D(A) \subset X \to X$ is the infinitesimal generator of the semigroup e^{At}. We say that $x : [0, \tau) \to X$ is a mild solution (or simply solution) of (1.11) if it satisfies the variation of constants formula

$$x(t) = e^{At}x_0 + \int_0^t e^{As} f(x(s))ds, \ \forall t \in [0, \tau).$$

Note that if $f \equiv 0$ then x is a mild solution of (1.11) if, and only if is a weak solution of (1.10).

1.2.2 Damped Wave Equation

From now on we will focus on the following autonomous wave equation posed on a smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u = f(u), \quad u|_{\partial\Omega=0} \\ u(x,0) = u_0 \in H_0^1(\Omega) \\ u_t(x,0) = v_0 \in L^2(\Omega) \end{cases} \tag{1.12}$$

We will follow the same approach as [20] to prove that problem (1.12) is well posed in the sense of Definition 29 and that the semigroup defined by its solutions is $\mathcal{C}^1$ in relation to the initial data and Gradient. Moreover, we will use these results to provide an example of a Morse-Smale semigroup defined on a Hilbert space (Corollary 1.34).

Let us start with our assumptions over the nonlinearity f. Assume that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies the growth estimate

$$|f'(s)| \leq k(1 + |s|^p) \quad \text{for all } s \in \mathbb{R}, \tag{1.13}$$

for some $p < 2$ and $k \geq 0$, and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq 0. \tag{1.14}$$

The case $p = 2$, known as the critical case, can also be studied but we will not approach here, since it is way more difficult and it is not the focus of our study. The reader can consult [5] for more details.

Note that (1.14) is equivalent to saying that for any $\gamma' > 0$ there exists a $c_{\gamma'}$ such that

$$sf(s) \leq \gamma' s^2 + c_{\gamma'} \quad \text{for all } s \in \mathbb{R}. \tag{1.15}$$

We set the phase space $X = H_0^1(\Omega) \times L^2(\Omega)$ for our problem. Denoting $v = u_t$, we can define

$$z = \begin{pmatrix} u \\ v \end{pmatrix},$$

and rewrite equation (1.12) as the abstract problem

$$z_t = \tilde{A}z + \tilde{F}(z), \tag{1.16}$$

where the operators $\tilde{A}$ and $\tilde{F}$ will be defined right away. The linear operator $\tilde{A}$ of equation (1.16) is given by

$$\tilde{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \tag{1.17}$$

where $A = -\Delta u$ is defined on the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$. The nonlinear term is

$$\tilde{F}(z) = \begin{pmatrix} 0 \\ F(z) \end{pmatrix},$$

where $F : H_0^1(\Omega) \times L^2(\Omega) \to L^2(\Omega)$ is defined as

$$F(z_1, z_2) = -\gamma z_2 + f^e(z_1), \tag{1.18}$$

where $f^e(z_1)(x) = f(z_1(x))$, for all $x \in \Omega$.

It is proved in [20, Proposition 6.21] that $\tilde{A}$ generates a strongly continuous semigroup of contractions in X. In order to prove the well posedness for the nonlinear case, we will show that f is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Omega)$ and the result will follow from [20, Theorem 6.27].

Lemma 1.24. *Suppose that*

$$|f(s_1) - f(s_2)| \leq c|s_1 - s_2|(1 + |s_1|^p + |s_2|^p), \quad s_1, s_2 \in \mathbb{R}, \tag{1.19}$$

with $p < 2$. Then for all $w_1, w_2 \in H^1(\Omega)$,

$$\|f(w_1) - f(w_2)\|_{L^2(\Omega)} \leq C\|w_1 - w_2\|_{H^1(\Omega)} \left(1 + \|w_1\|^p_{H^1(\Omega)} + \|w_2\|^p_{H^1(\Omega)}\right).$$

Consequently, for $z_1, z_2 \in X$,

$$\|\tilde{F}(z_1) - \tilde{F}(z_2)\|_X \leq C\|z_1 - z_2\|_X \left(1 + \|z_1\|^p_X + \|z_2\|^p_X\right). \tag{1.20}$$

Proof. From (1.19), Hölder's inequality, and the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, we obtain

$$\begin{aligned}
\|f(w_1) - f(w_2)\|_{L^2(\Omega)} &\leq c\left[\int_\Omega |w_1 - w_2|^2 (1 + |w_1|^p + |w_2|^p)^2\right]^{\frac{1}{2}} \\
&\leq c\left[\int_\Omega |w_1 - w_2|^6\right]^{\frac{1}{6}} \left[\int_\Omega (1 + |w_1|^p + |w_2|^p)^3\right]^{\frac{1}{3}} \\
&\leq C\|w_1 - w_2\|_{L^6(\Omega)} \left(1 + \|w_1\|^p_{L^{3p}(\Omega)} + \|w_2\|^p_{L^{3p}(\Omega)}\right) \\
&\leq C\|w_1 - w_2\|_{H^1(\Omega)} \left(1 + \|w_1\|^p_{H^1(\Omega)} + \|w_2\|^p_{H^1(\Omega)}\right).
\end{aligned}$$

The bound in (1.20) now follows in a straightforward way from the definition of F.

It follows from [20, Theorem 6.27] that for any $z_0 \in X$ there is a unique mild solution $(u(\cdot), v(\cdot))$ to (1.12) and (1.16) satisfying the initial condition $(u(0), v(0)) = z_0 \in X$ and defined on a maximal interval of existence $[0, \tau_{z_0})$. □

Remembering that Morse-Smale semigroups are $\mathcal{C}^1$ and we want find an example of Morse-Smale semigroups through the damped wave equation, now we focus on the differentiability of F so we can obtain that the solutions are $\mathcal{C}^1$ in relation to the initial conditions.

To prove the differentiability of F we just have to show the differentiability of f^e. To this end, we make the additional assumption

$$|f''(s)| \leq c(1 + |s|) \quad \text{for all } s \in \mathbb{R}. \tag{1.21}$$

As F is a map from X into $L^2(\Omega)$, its derivative DF is an element of $\mathcal{L}(X, L^2(\Omega))$ defined by

$$(h_1, h_2) \to (D_u f^e)h_1 - \gamma h_2.$$

Lemma 1.25. *If f satisfies (1.13) and (1.21), then $f^e(\cdot) : H^1_0(\Omega) \to L^2(\Omega)$ is continuously differentiable and $Df^e : H^1_0(\Omega) \to \mathcal{L}(H^1_0(\Omega), L^2(\Omega))$ is Lipschitz continuous in bounded subsets of $H^1_0(\Omega)$. Consequently, $DF : X \to \mathcal{L}(X, L^2(\Omega))$ is also Lipschitz continuous in bounded subsets of X.*

Proof. For $u \in H^1(\Omega)$ define $D_u f^e \in \mathcal{L}(H^1(\Omega), L^2(\Omega))$ by

$$(D_u f^e(h))(x) = f'(u(x))h(x), \ \forall x \in \Omega. \tag{1.22}$$

First we check that this is well defined. In fact, if $u, h \in H^1(\Omega)$, then $u, h \in L^6(\Omega)$, and using (1.13)

$$\int_\Omega |f'(u(x))|^2|h(x)|^2dx \le k\int_\Omega (1+u^p)^2h^2dx \le k\|(1+u^p)^2\|_{L^{3/2}(\Omega)}\|h^2\|_{L^3(\Omega)}$$
$$\le c(1+\|u\|^{2p}_{L^{3p}(\Omega)})\|h\|^2_{L^6(\Omega)} \le c(1+\|u\|^{2p}_{H^1(\Omega)})\|h\|^2_{H^1(\Omega)}.$$

It follows that $D_uf^e(h)\in L^2(\Omega)$ and that $D_uf^e\in\mathcal{L}(H^1(\Omega),L^2(\Omega))$. Now let us check that D_uf^e is indeed the Fréchet derivative of $f^e(\cdot)$ at u.

If $u,h\in H^1(\Omega)$, then

$$|f^e(u+h)-f^e(u)-D_uf^e(h)\|^2_{L^2(\Omega)} = \int_\Omega [f(u(x)+h(x))-f(u(x))-f'(u(x))h(x)]^2dx$$
$$\le \int_\Omega [f''(u(x)+\sigma(x)h(x))]h(x)^4dx,$$

where $\sigma(x)\in(0,1)$. Thus,

$$\|f^e(u+h)-f^e(u)-D_uf^e(h)\|^2_{L^2(\Omega)} \le \int_\Omega C(1+u(x)^2+h(x)^2)h(x)^4dx$$

$$\le C(1+\|u^2\|_{L^3(\Omega)}+\|h^2\|_{L^3(\Omega)})\|h^4\|_{L^{3/2}(\Omega)} \le C(1+\|u\|^2_{L^6(\Omega)}+\|h\|^2_{L^6(\Omega)})\|h\|^4_{L^6(\Omega)},$$

and consequently

$$\|f^e(u+h)-f^e(u)-D_uf^e(h)\|_{L^2(\Omega)} \le \tilde{C}(1+\|u\|_{H^1(\Omega)}+\|h\|_{H^1(\Omega)})\|h\|^2_{H^1(\Omega)}.$$

This proves the differentiability of $f^e(\cdot)$ in u for each $u\in H^1(\Omega)$ and shows (1.22).

Finally, we show that $u\to D_uf^e$ is Lipschitz continuous as a map from $H^1(\Omega)$ into $\mathcal{L}(H^1(\Omega),L^2(\Omega))$. Hence, if $u,v,h\in H^1(\Omega)$, then

$$\begin{aligned}\|D_uf^e(h)-D_vf^e(h)\|^2_{L^2(\Omega)} &= \int_\Omega (f'(u(x))-f'(v(x)))^2h(x)^2dx\\ &= \int_\Omega (f''(u-\theta(x)v(x)))^2(u(x)-v(x))^2h(x)^2dx \quad \text{(for some } \theta(x)\in(0,1))\\ &\le \int_\Omega c(1+u(x)^2+v(x)^2)(u(x)-v(x))^2h(x)^2dx\\ &\le c(1+\|u\|_{L^6(\Omega)}+\|v\|_{L^6(\Omega)})\|u-v\|^2_{L^6(\Omega)}\|h\|^2_{L^6(\Omega)}\\ &\le c(1+\|u\|_{H^1(\Omega)}+\|v\|_{H^1(\Omega)})\|u-v\|^2_{H^1(\Omega)}\|h\|^2_{H^1(\Omega)}.\end{aligned}$$

The continuous differentiability of F follows from that. □

Corollary 1.26. *Under the assumptions of Lemma 1.25 the solutions of (1.16) are continuously differentiable with respect to initial conditions.*

Proof. It follows from Lemma 1.25 and [20, Theorem 6.33]. □

Now we show that the solutions of (1.12) are globally defined and therefore we can define a nonlinear semigroup through its solution.

Proposition 1.27. *The solutions of (1.12) are globally defined, i.e., for each $z_0\in X$, $\tau_{z_0}=+\infty$.*

Proof. Consider the continuous functional $V_\delta: X\to\mathbb{R}$ defined by

$$V_\delta(u,v)=\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}+\frac{1}{2}\|v\|^2_{L^2(\Omega)}+\delta(u,v)-\int_\Omega F_0(u),\ \forall (u,v)\in X \tag{1.23}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$,

$$F_0(r) = \int_0^r f(s)ds,$$

and $\delta \geq 0$ will shortly be chosen appropriately.

It follows from (1.14) that for any $\gamma' > 0$ there exists $k_{\gamma'} > 0$ such that

$$F_0(r) \leq \int_0^r \gamma' \frac{s}{2} + k_{\gamma'} ds \leq \gamma' \frac{r^2}{4} + k_{\gamma'} r \leq \gamma' r^2 + c'_{\gamma'}. \tag{1.24}$$

Using this, it follows that

$$\begin{aligned}\frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|v\|^2 &= V_\delta(u,v) - \delta(u,v) + \int_\Omega F_0(u) \\ &\leq V_\delta(u,v) + \int_\Omega (\gamma' u^2 + c'_{\gamma'}) + \delta\lambda_1^{-1/2}\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)},\end{aligned}$$

where we have used Poincaré's inequality in the form $\|u\| \leq \lambda_1^{-1/2}\|\nabla u\|$. Thus,

$$\left(\frac{1}{2} - \gamma'\lambda_1^{-1} - \delta\frac{1}{2}\lambda_1^{-1}\right)\|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \delta\frac{1}{2}\right)\|v\|^2_{L^2(\Omega)} \leq V_\delta(u,v) + c'_{\gamma'}|\Omega|,$$

and provided that $\delta < \min\{1, \lambda_1\}$, one can choose γ' sufficiently small to show that

$$\|(u,v)\|^2_X \leq c_1 V_\delta(u,v) + c_2, \text{ for some } c_1, c_2 > 0. \tag{1.25}$$

It follows from [20, Theorem 6.29] that, if $z_0 = (u_0, v_0) \in D(\tilde{A})$ then $t \mapsto z(t, z_0)$ is continuously differentiable as a map from $[0, \tau_{z_0})$ into X, and while defined

$$\begin{aligned}\frac{d}{dt}V_\delta(z) &= (\nabla u, \nabla u_t) + (u_t, u_{tt}) - (f(u), u_t) + \delta\|u_t\|^2_{L^2(\Omega)} + \delta(u, u_{tt}) \\ &= -(\Delta u, u_t) + (u_t, -\gamma u_t + \Delta u + f(u)) - (f(u), u_t) + \delta\|u_t\|^2_{L^2(\Omega)} \\ &\qquad + \delta(u, -\gamma u_t + \Delta u + f(u)) \\ &= -(\gamma - \delta)\|u_t\|^2_{L^2(\Omega)} - \delta\|\nabla u\|^2_{L^2(\Omega)} - \gamma\delta(u, u_t) + \delta\int_\Omega u f(u).\end{aligned} \tag{1.26}$$

If $\delta = 0$, we know from (1.26) that $[t_0, \infty) \ni t \mapsto V_0(z(t, v_0)) \in \mathbb{R}$ is non-increasing. It follows from the fact that V_0 is continuous and bounded in bounded subsets of X and from (1.25) that, given $r > 0$, there is a constant $C(r) > 0$ such that

$$\sup\{\|z(t, z_0)\|_X : \|z_0\|_X \leq r \text{ and } t \in [0, \tau_{z_0})\} \leq C(r).$$

This implies that, for each $z_0 \in X$, the solution of (1.16), with $z(0, z_0) = z_0$, is defined for all $t \geq 0$; that is $\tau_{z_0} = +\infty$. □

From the proof of Proposition 1.27 we guarantee the following property.

Corollary 1.28. *Each bounded subset of X has a bounded orbit.*

Now that we have Proposition 1.27, we can define the semigroup $\{T(t) : t \geq 0\}$ in X by $T(t)z_0 = z(t, z_0)$, $t \geq 0$. The following results are dedicated to prove that this semigroup has a global attractor $\mathcal{A}$. In fact, we want to show that the semigroup $\{T(t) : t \geq 0\}$ is dissipative (defined on Proposition 1.29) and asymptotically compact (defined in the proof of Proposition 1.2) so we can use [20, Corollary 2.21] to guarantee the existence of the global attractor.

We start by proving that $\{T(t) : t \geq 0\}$ is dissipative.

Proposition 1.29. *The semigroup $\{T(t) : t \geq 0\}$ is dissipative, i.e., there exists $R > 0$ such that for any bounded set $B \subset X$ there exists $t_0(B) > 0$ satisfying*

$$T(t)B \subset B_X(0, R) \text{ for all } t \geq t_0(B). \tag{1.27}$$

Proof. Now one can use (1.15), (1.24), the Cauchy-Schwarz inequality, Young's inequality, and the Poincaré inequality to write, for any $0 < \epsilon < \gamma$,

$$\frac{d}{dt}V_\delta(z) \leq -(\gamma - \delta)\|u_t\|^2_{L^2(\Omega)} - \delta\|\nabla u\|^2_{L^2(\Omega)} - \gamma\delta(u, u_t) + \delta\int_\Omega(\gamma'|u|^2 + c_{\gamma'})$$

$$\leq -(\gamma - \delta)\|u_t\|^2_{L^2(\Omega)} - \left[\delta - \gamma'\delta\lambda_1^{-1}\right]\|\nabla u\|^2_{L^2(\Omega)} + \gamma\delta\|u\|_{L^2(\Omega)}\|u_t\|_{L^2(\Omega)} + \delta c_{\gamma'}|\Omega|$$

$$\leq -(\gamma - \delta - \frac{\delta}{2}\gamma^2\lambda_1^{-1})\|u_t\|^2_{L^2(\Omega)} - \left[\frac{\delta}{2} - \gamma'\delta\lambda_1^{-1}\right]\|\nabla u\|^2_{L^2(\Omega)} + \delta c_{\gamma'}|\Omega|.$$

Now choose $\delta \in (0, \min\{1, \lambda_1\})$ sufficiently small that

$$\gamma - \delta - \delta^2\gamma^2\lambda_1^{-1} > 0$$

and γ' sufficiently small that

$$\frac{\delta}{2} - \gamma'\delta\lambda_1^{-1} > 0.$$

Then, if $\vartheta := \min\{\gamma - \delta - \frac{\delta}{2}\gamma^2\lambda_1^{-1}, \frac{\delta}{2} - \gamma'\delta\lambda_1^{-1}\}$, then

$$\frac{d}{dt}V_\delta(z) \leq -\vartheta\|z\|^2_X + \delta c_{\gamma'}|\Omega|.$$

Choosing $r_0 = [\delta c_{\gamma'}|\Omega| + 1]/\vartheta$ we have $V_\delta(z(t, z_0)) \leq -t + V_\delta(z_0)$ as long as

$$\|z(t, z_0)\|_X \geq r_0.$$

It then follows from (1.25) that

$$\|z(t, z_0)\|_X \leq c_1[-t + V_\delta(z_0)] + c_2$$

as long as

$$\|z(t, z_0)\|_X \geq r_0.$$

Consequently, if $t \geq V_\delta(z_0)$, then $\|z(t, z_0)\|_X \leq \min\{c_2, r_0\}$. This proves that $\{T(t) : t \geq 0\}$ is dissipative. □

Now that we have shown that the process generated by our equation is dissipative, we need to show that $T(\cdot)$ is also asymptotically compact in order to apply [20, Theorem 2.23] and guarantee the existence of the global attractor.

We continue to work in the same phase space, $X = H_0^1(\Omega) \times L^2(\Omega)$, but now rewrite (1.16) as

$$w_t = Cw + F(w) \tag{1.28}$$

where

$$C = \begin{pmatrix} 0 & I \\ -A & -\gamma I \end{pmatrix} \quad \text{and} \quad F(w) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}. \tag{1.29}$$

We have proved that for each initial value $w_0 \in X$ and initial time $s \in \mathbb{R}$, the equation (1.28) possesses a unique solution, which we now write as the variation of constants formula (see Definition 29)

$$T(t)w_0 = L(t)w_0 + U(t)w_0, \tag{1.30}$$

where $L(t)$ is the solution operator (semigroup) for $w_t = Cw$, and

$$U(t)w_0 = \int_0^t L(t-\theta)F(T(\theta)w_0)\,d\theta. \tag{1.31}$$

To prove that $\{T(t) : t \geq 0\}$ is asymptotically compact, we will show that $L(t)$ decays exponentially and $U(t)$ is a compact operator (see [20, Theorem 2.37]).

Proposition 1.30. *There are constants $K, \alpha > 0$ such that*

$$\|L(t)\|_{\mathcal{L}(X)} \leq Ke^{-\alpha t} \quad \textit{for all} \quad t \geq 0,$$

and $U(t)$ is a compact operator from X into itself for all $t > 0$. In particular, applying [20, Theorem 2.37], the semigroup $T(\cdot)$ is asymptotically compact .

Proof. To prove the decay of $L(t)$, consider the functional

$$W_\delta(\varphi, \phi) = \frac{1}{2}\|\varphi\|^2_{H_0^1(\Omega)} + 2b(\varphi, \phi)_{L^2(\Omega)} + \frac{1}{2}\|\phi\|^2_{L^2(\Omega)}.$$

Proceeding as in Proposition 1.27 (without the nonlinear term) there exists $\alpha > 0$ such that

$$\dot{W}_\delta \leq -\alpha W_\delta \Longrightarrow W_\delta(L(t)(\varphi, \phi)) \leq W_\delta(\varphi, \phi)e^{-t\alpha}$$

and therefore

$$\|L(t)(\varphi, \phi)\|^2_X \leq Ke^{-t\alpha}\|(\varphi, \phi)\|_X.$$

To show that $U(t)$ is compact, we first show that f is bounded from $H_0^1(\Omega)$ into $W^{1,r}(\Omega)$, with $r = \frac{6}{3+p} > \frac{6}{5}$; indeed, it follows from (1.13) that $|f(s)| \leq k'(1 + |s|^{p+1})$, and so

$$\|f(u)(\cdot)\|^r_{W^{1,r}} = \int_\Omega \left(|f(u)|^r + |f'(u)|^r|\nabla u|^r\right)$$

$$\leq c\left(1 + \|u\|^{r(p+1)}_{L^{r(p+1)}} + \|\nabla u\|^r_{L^r} + \int_\Omega |u(x)|^{pr}|\nabla u|^r\right)$$

$$\leq c\left(1 + \|u\|^{r(p+1)}_{L^{r(p+1)}} + \|\nabla u\|^r_{L^r} + \|u\|^{pr}_{L^{\frac{2rp}{2-r}}}\|\nabla u\|^r_{L^2}\right).$$

Our choice of r implies that $r(p+1) < 6$ and $\frac{2pr}{2-r} = 6$. From the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ for $q \leq 6$ it follows that f^e is bounded from $H_0^1(\Omega)$ into $W^{1,r}(\Omega)$ and the latter is compactly embedded in L^2 (since $r > \frac{6}{5}$). Thus, F is bounded from X into $\{0\} \times W^{1,r}(\Omega)$ and the latter is compactly embedded in $H_0^1(\Omega) \times W^{1,r}(\Omega)$.

Now fix $t > 0$ and consider

$$U(t)w_0 = \int_0^t L(t-\theta)F(S(\theta)w_0)\,d\theta$$

for $w_0 \in B$, a bounded subset of X. Then, since orbits of bounded subsets of X under $\{T(t) : t \geq 0\}$ are bounded in X (Corollary 1.28), it follows that $U(t)$ is compact for each $t > 0$. □

Corollary 1.31. *The semigroup generated by equation (1.12) has a global attractor.*

Proof. It follows directly from Propositions 1.29,1.30 and [20, Corollary 2.21]. □

Remark 5. *Note that $C = \tilde{A} + A_0$, where C is given in (1.29), $\tilde{A}$ is given in (1.17) and A_0 is the bounded (in X) linear operator*

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma I \end{pmatrix}.$$

Since $i\tilde{A}$ is self-adjoint, it follows from Theorem 1.23 that $\tilde{A}$ generates a group of unitary linear operators. Moreover, since $A_0 \in \mathcal{L}(X)$ is bounded, the operator C also generates a group of bounded linear operators. In particular, $0 \in \rho(L(t))$ for all $t \geq 0$, where ρ is the resolvent set (Definition 53).

Now that we already know that $\{T(t) : t \geq 0\}$ is a semigroup with global attractor $\mathcal{A}$, we will show that it is also Gradient.

Proposition 1.32. *Assume that there are only finitely many solutions $\{u_1^*, \ldots, u_p^*\}$ of the elliptic problem*

$$\Delta u + f(u) = 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \quad (15.25)$$

i.e., only finitely many equilibria of (1.12). Write $E = \{e_1^*, \ldots, e_p^*\}$, where $e_i^* = \begin{pmatrix} u_i^* \\ 0 \end{pmatrix}$.

Then $T(\cdot)$ is Dynamically Gradient; that is, satisfies properties (G1) and (G2) from Definition 8. Consequently, from Corollary 1.11 and Theorem 1.3

$$\mathcal{A} = \bigcup_{i=1}^{n} W^u(e_i^*).$$

Proof. We first observe that the function $V_0(\cdot)$ in (1.23) (we take $\delta = 0$) is decreasing along solutions, i.e., if $\xi : [0, \infty) \to X$ is a solution of (1.12), then $V_0(\xi(t))$ is decreasing. In addition, it follows from (15.12) (making $\delta = 0$) that if $V_0(\xi(t))$ is constant in a non-trivial interval of $\mathbb{R}$, then ξ must be an equilibrium.

It follows that $V_0 : X \to \mathbb{R}$ is a Lyapunov function for (1.12) with respect to E, that is, the semigroup $T(\cdot)$ is Gradient and consequently Dynamically Gradient (see Corollary 1.11). □

Now we prove our last result before providing an example of a Morse-Smale semigroup.

Theorem 1.33. *Let X be a Banach space and A be the generator of a continuous group $\{e^{At} : t \in \mathbb{R}\}$ on X. Suppose that $f : X \to X$ is a function such that the semilinear equation*

$$\dot{x} = Ax + fx$$

generates a semigroup $\{T_f(t) : t \geq 0\}$ on X and the operator $K(t) : X \to X$ given by

$$X \ni x \mapsto \int_0^t e^{A(t-s)} f(T_f(s)x) ds$$

is compact and differentiable for all $t > 0$. Then, $T_f(t)$ is differentiable and $R(D_x T_f(t))$ is closed for each $t > 0$. In particular, $R(D_x T_f(t))$ is a Banach space and if $D_x T_f(t)$ is injective, then it is also an isomorphism onto its range [12, Corollary 2.7].

Proof. We know that for each $t > 0$ and $x \in X$ we have

$$D_x T_f(t) = e^{At} + D_x K(t) \Longrightarrow e^{-At} D_x T_f(t) = I + e^{-At} D_x K(t). \tag{1.32}$$

From Lemma 4.12 we know that $-D_x K(t)$ is compact and therefore $e^{-At} D_x K(t)$ is compact for each $t > 0$ and $x \in X$. Hence, from Fredholm alternative [12, Theorem 6.6], $R(e^{-At} D_x T_f(t))$ is closed and consequently $R(D_x T_f(t))$ is closed (e^{At} is an isomorphism). □

Now, we finally give an example of Morse-Smale semigroup. In fact, the following result shows that Morse-Smale semigroups are dense (it's actually generic) in some sense when we analyze the nonlinear term of the damped wave equation.

Corollary 1.34. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, $X = H_0^1(\Omega) \times L^2(\Omega)$ and $T_f(t)$ be the semigroup in X generated by the nonlinear damped wave equation, that is, $T_f(t)(u_0, v_0)$ is the solution on time t of the equation*

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u = f(u), \quad u|_{\partial\Omega = 0} \\ u(x,0) = u_0 \in H_0^1(\Omega) \\ u_t(x,0) = v_0 \in L^2(\Omega). \end{cases}.$$

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$ function satisfying conditions (1.13), (1.14) and (1.21). Then, items (H1) and (H2) from Theorem 0.1 holds. Moreover, for any neighborhood $\mathcal{O}$ of f in the $\mathcal{C}^2$-Whitney topology (see Chapter 3.4 of [47]) there exists $g \in \mathcal{O}$ such that T_g is a Morse-Smale semigroup.

Proof. First of all, note that in this section we already proved that $T_f(t)$ is well defined (Lemma 1.25), continuous differentiable (Corollary 1.26) and has a global attractor $\mathcal{A}$ (Corollary 1.31). Before we prove the Morse-Smale properties, let us show that T_f satisfies items (H1) and (H2) from Theorem 0.1. To prove (H1), we first recall that the linear semigroup $\{e^{\tilde{A}t} : t \geq 0\}$ related to the wave operator $\tilde{A}$, given in (1.17), is a contraction, that is, the constants of Theorem 1.21 related to the semigroup $\{e^{\tilde{A}t} : t \geq 0\}$ are given by $M = 1$ and $\omega = 0$. Now, it follows from the continuous differentiability of $\tilde{F}$ (Lemma 1.25) and [20, Theorem 6.33] that the derivative of the semigroup $D_x T_f(t)$ at $x \in \mathcal{A}$ is written as

$$D_x T_f(t) = e^{\tilde{A}t} + \int_0^t e^{\tilde{A}(t-s)} (D_{T_f(s)x} \tilde{F}) D_x T_f(s)\, ds \ \in \mathcal{L}(X), \ \forall t \geq 0, \ \forall x \in \mathcal{A}. \tag{1.33}$$

Fix $x_0 \in \mathcal{A}$ arbitrary. Now we can use Gronwall inequality and the fact that $D\tilde{F}$ is Lipschitz in bounded sets (Lemma 1.25), to obtain

$$\| D_x T_f(t) \|_{\mathcal{L}(X)} \leq 1 + \int_0^t \| (D_{T_f(s)x} \tilde{F}) \circ D_x T_f(s) \|_{\mathcal{L}(X)} \ ds, \ \forall t \geq 0, \ \forall x \in \mathcal{A}$$

$$\| D_x T_f(t) \|_{\mathcal{L}(X)} \leq 1 + \int_0^t \| (D_{T_f(s)x} \tilde{F} - D_{x_0} \tilde{F}) + D_{x_0} \tilde{F} \|_{\mathcal{L}(X)} \| D_x T_f(s) \|_{\mathcal{L}(X)} \ ds, \ \forall t \geq 0, \ \forall x \in \mathcal{A}$$

$$\Longrightarrow \| D_x T_f(t) \|_{\mathcal{L}(X)} \leq 1 + L \int_0^t \| D_x T_f(s) \|_{\mathcal{L}(X)} \ ds, \ \forall t \geq 0, \ \forall x \in \mathcal{A}$$

$$\Longrightarrow \| D_x T_f(t) \|_{\mathcal{L}(X)} \leq e^{Lt}, \ \forall t \geq 0, \ \forall x \in \mathcal{A}.$$

where $L = (2Lip(D\tilde{F}) + 1)M$, $Lip(D\tilde{F})$ is the Lipschitz constant of $D\tilde{F}|_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}(X)$ in the bounded subset $\mathcal{A}$ and $M = \max\{\max_{x\in\mathcal{A}} \| x \|, \max_{x\in\mathcal{A}} \| D_x\tilde{F} \|_{\mathcal{L}(X)}\}$. This finishes the proof of item (H1) from theorem 0.1. The proof of item (H2) follows immediately from the continuously differentiability of $\tilde{F}$ and [20, Theorem 6.33].

Now we prove the Morse-Smale properties required in Definition 22. We start by proving some properties for the semigroup T_f to then conclude all properties that are required to be a Morse-Smale semigroup for T_g, where g is in the $\mathcal{C}^2$- neighborhood (Whitney topology) of f. The injectiveness of $T_f(t)|_{\mathcal{A}}$ follows from the fact that $e^{\tilde{A}t}$ is a group (Remark 5) and $\tilde{F}$ is Lipschitz on bounded sets (since is its derivative is Lipschitz in bounded sets) and consequently the solution

$$T_f(t)x_0 = e^{\tilde{A}t}x_0 + \int_0^t e^{\tilde{A}(t-s)}\tilde{F}(T_f(s)x_0)ds$$

exists and is unique (Fixed Point Theorem) for all $t \in \mathbb{R}$ and $x_0 \in \mathcal{A}$, that is, $T_f(t)|_{\mathcal{A}}$ is injective. Similarly, since e^{At} is an isomorphism and the derivative of $T_f(t)$ at x also satisfies (1.33), then $D_xT_f(t)$ is injective.

Note that if $g \in \mathcal{O}$, where $\mathcal{O}$ is a neighborhood of f on the $\mathcal{C}^2$-Whitney topology, then g also satisfies (1.13), (1.14) and (1.21), implying that the map

$$x \mapsto \int_0^t e^{C(t-s)}G(T_g(s)x)ds$$

is also compact (Proposition 1.30), where C is linear operator related to the damped wave equation defined in (1.29) and G is defined analogously to F in (1.29). Hence, it follows from equation (1.30), Remark 5 and Theorem 1.33 that $D_xT_g(t)$ is an isomorphism onto its range.

To see that the non-wandering set Ω is the set of equilibria $\mathcal{E}$ just note that $T_g(t)$ is Dynamically Gradient in relation to $\mathcal{E}$. The conditions of hyperbolicity and transversality follows from the fact that the set of functions such that the semigroup $T_g(t)$ holds this properties are generic in the Whitney Topology [13, 9]. □

Remark 6. *We recall that the semigroup related to the damped wave operator does not satisfy the gap condition required to obtain an inertial manifold (see [54]) and consequently we can not prove shadowing on a neighborhood of its global attractor by the already known results [47, Section 3.4]. Besides that, because this semigroup is Morse-Smale and satisfy items (H1) and (H2) from Theorem 0.1, we will show that the shadowing property still holds on its neighborhood.*

Chapter 2

Lipschitz Shadowing on the Global Attractor

In this chapter we are going to extend the known theory of Lipschitz Shadowing on compact finite dimensional manifolds [47, Chapter 2] and $\mathbb{R}^n$ [58] to infinite dimensional Hilbert spaces. This chapter can be considered the essence of this thesis. We divide this chapter in two parts: in the first one (Section 2.1), we dedicate to adapt the results from [47, Chapter 2.2.1] to the context of infinite dimensional Hilbert spaces in order to construct a family of compatible subbundles (see Theorem 2.15). In fact, to prove this result we had to redefine some geometrical concepts, such as continuous subbundles (Definitions 30 and 31), to put in our context where $\mathcal{A}$ is not a manifold. Furthermore, we also have to deal with the lack of surjectivity of the derivative of the semigroup $\mathcal{T}(t)$ (Lemmas 2.16 and 2.17). In the second part of the chapter, that is Section 2.2, we use the compatible subbundles from Section 2.1 to prove that $\mathcal{T}(1)|_{\mathcal{A}}$ has the Lipschitz-Shadowing property. Hence, in this chapter we will focus only on the behavior inside the global attractor $\mathcal{A}$.

In this whole chapter we will assume that $\mathcal{T}(\cdot) = \{\mathcal{T}(t) : t \geq 0\}$ is a Morse-Smale semigroup on a Hilbert space X, with global attractor $\mathcal{A}$ and non-wandering set $\Omega = \{x_1^*, \dots, x_p^*\}$. We will also assume that $\mathcal{T}(\cdot)$ satisfies conditions (H1) and (H2) given in Theorem 0.1.

2.1 Compatible Subbundles

In this section we will construct families of compatible subbundles which we are going to use to prove the property of Lipschitz Shadowing in the global attractor $\mathcal{A}$. Most of the results were based on [47, Chapter 2.2.1]. In fact, although some proofs work exactly the same way (Lemmas 2.1 and 2.2), we had to adapt a large part of the results to make it work in our context, where $\mathcal{A}$ is not a manifold and the derivative is not surjective. In fact, the key to prove Lipschitz Shadowing on $\mathcal{A}$ is present in this section, that is Theorem 2.15.

Therefore, the main goal of this section is to prove the existence of compatible subbundles S, U that are defined in $\mathcal{A}$. Subbundles are defined in Definition 30 and are basically families of closed subspaces. To be able to do that, we will have to proceed with a very technical and constructive process, that we will try to summarize here.

1. In order to construct the subbundles S and U in $\mathcal{A}$, we will construct continuous subbundles S_i and U_i on a neighborhood $\mathcal{O}_i$ of x_i^*, for each $x_i^* \in \Omega$ (Theorem 2.15). Since $\mathcal{T}$ is Morse-Smale, therefore Gradient (Lemma 1.17), the trajectory of each point of $x \in \mathcal{A}$ will eventually enter on $\mathcal{O}_j$, for some j. Hence, for any point $x \in \mathcal{A}$ there will be a time $t > 0$ such that $\mathcal{T}(t)x \in \mathcal{O}_j$ for some $j \in \{1, \dots, p\}$. Since $S_j(\mathcal{T}(t)x)$ and $U_j(\mathcal{T}(t)x)$ are well defined, we will be able to define $S(x)$ and $U(x)$ through that. Now we have to construct the continuous subbundles S_i and U_i in $\mathcal{O}_i$ for each $i \in \{1, \dots, p\}$. We will focus on the construction of U_i and the construction of S_i will follow similarly.

2. To construct the subbundles U_i in $\mathcal{O}_i$, we will first construct subbundles $\mathcal{V}_i$ in a neighborhood Q of $D := V^{-1}(i+d) \cap W^s(x_i^*) \cap \mathcal{A}$, for some $d \in (0,1)$. We will construct this subbundle by induction. For example, note that $W^u(x_p^*)$ is a neighborhood of x_p^* and $U(x) = T_x W^u(x^*)$ is a continuous subbundle defined in $W^u(x_p^*)$. This will be the first step of our induction.
3. Defining the natural subbundle $T_x W^u(x_i^*)$ for $x \in W^u(x_i^*)$, we will prove that $\gamma^+(Q) \cup W^u(x_i^*)$ is a neighborhood of x_i^* (Lemma 2.1). Therefore we have already constructed a subbundle in $\mathcal{O}_i$.
4. We will reduce Q so this subbundle is continuous in $\mathcal{O}_i$ (Lemma 2.11).

This construction is complex and our results will not be proved in the same order exposed. In fact, to prove the existence of the subbundles S_i and U_i we first have to prove items 2, 3 and 4. For example, the first result of this Section will be Lemma 2.1, which is related to item 3.

Now we provide a scheme of this section: we start the section by proving some technical lemmas (that holds for any Gradient semigroup) that will be used to construct the subbundles S_i and U_i. One of these lemmas is related to item 3. In Subsection 1.2.1 we define continuity of subbundles using projections and prove several results related to continuity, such as Lemma 2.11, that is related to item 4. We recall that this notion of continuity was not used before to prove Shadowing, and that is why we have to show that our definition of continuity is appropriate and prove a lot of properties. In subsection 1.2.2 we use the results from the previous subsections to construct the subbundles S_i and U_i present on item 1, which finishes the section.

As we said before, in this section we will work only inside the global attractor $\mathcal{A}$. From now on we denote by V a Lyapunov function for the Morse-Smale semigroup $\mathcal{T}$ such that $V(x_i^*) = i$ for all $i \in \{1, \dots, p\}$ and V is decreasing. Such function exists by Theorem 1.17.

We now prove some technical lemmas, which are Lemmas 2.1 and 2.2. These lemmas give us information near the equilibrium points x_i* and will be used in several parts of this section, including Theorem 2.15. To be more precise, the following result allow us to obtain a neighborhood of the equilibrium point x_i^* through the unstable manifold and the evolution of a level curve.

Lemma 2.1. *Fix $i \in \{i, \dots, p\}$ and $d \in (0,1)$ and define the set*

$$D := V^{-1}(i+d) \cap W^s(x_i^*) \cap \mathcal{A}.$$

For any neighborhood Q of D in $H := V^{-1}(i+d) \cap \mathcal{A}$ the set $\gamma^+(Q) \cup W^u(x_i^*)$ contains a neighborhood (in $\mathcal{A}$) of x_i^*.

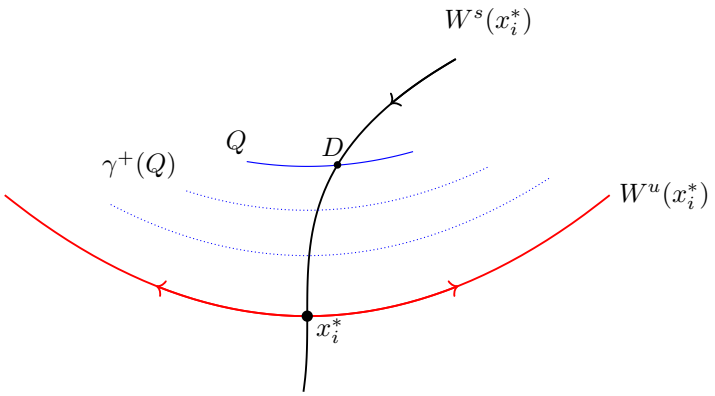


Figure 2.1: $\gamma^+(Q) \cup W^u(x_i^*)$ is a neighborhood of x_i^*

Proof.

Let Q be a neighborhood of D in H and suppose that the statement does not hold. Then there exists $\{x_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$ such that

$$x_m \to x_i^*,\ x_m \notin W^u(x_i^*) \cup \gamma^+(Q), \forall\ m \in \mathbb{N}. \tag{2.1}$$

We assume without lost of generality that

$$i - d < V(x_m), \forall\ m \in \mathbb{N}$$

and that there is a $j > i$ such that

$$\lim_{t\to-\infty} d(\mathcal{T}(t)x_m, x_j^*) = 0,\ \text{for every } m \in \mathbb{N}.$$

Since $x_m \in \mathcal{A}$, for every m there exists $t_m > 0$, with $t_m \to +\infty$, such that $y_m := \mathcal{T}(-t_m)x_m \in H$. Now, taking a subsequence if necessary, we suppose that there exists $y \in H$ such that $y_m \to y$, as $m \to +\infty$. Suppose that $y \notin W^s(x_i^*)$, then $y \in W^s(x_k^*)$ for some $k < i$. Thus there exists $t_0 > 0$ such that $V(\mathcal{T}(t_0)y) < i - d$. Since $t_0 - t_m < 0$ for large m we have $V(\mathcal{T}(t_0)y_m) < i - d$ for m chosen suitable large. Therefore, for large m, we have

$$i - d < V(x_m) \le V(\mathcal{T}(t_0 - t_m)x_m) < i - d,$$

which is a contradiction. Therefore, $y \in W^s(x_i^*) \cap H$ and $y_m \in Q$ for large m, implying that $x_m \in \gamma^+(Q)$. This contradicts (2.1) and completes the proof. □

The following technical lemma allows us to place the points (in the global attractor) of the stable manifold of x_i^* , which have energy close to i, as close to x_i^* as we want. We will use this property later, to construct the compatible subbundles. Note that this result does not follows immediately from the continuity of the semigroup and the Lyapunov function.

Lemma 2.2. *Let N be a neighborhood of x_i^* in $\mathcal{A}$, for some $i \in \{1, \dots, p\}$. Then, there exists $d' \in (0, 1)$ such that*

$$V^{-1}(i + d) \cap W^s(x_i^*) \cap \mathcal{A} \subset N, \forall\ d \in [0, d'].$$

Proof. Let N be a neighborhood of x_i^* on $\mathcal{A}$ and consider the set $\mathcal{H} = V^{-1}\left(i + \frac{1}{2}\right) \cap W^s(x_i^*) \cap \mathcal{A}$, which is compact and attracted to x_i^*. Take $T > 0$ such that

$$\mathcal{T}(t)\mathcal{H} \subset N, \forall\ t \ge T. \tag{2.2}$$

Since $\min\{V(s) : s \in \gamma_{[0,T]}\mathcal{H}\} > i$, there exists $d' \in \left(0, \frac{1}{2}\right)$ such that

$$\min\{V(s) : s \in \gamma_{[0,T]}\mathcal{H}\} > i + d'.$$

We will show that d' satisfies the properties that we expect. Let $x \in V^{-1}(i + d) \cap W^s(x_i^*) \cap \mathcal{A}$ for some $d \in [0, d']$. Then, there exists $y \in \mathcal{H}$ and $t > 0$ such that $x = \mathcal{T}(t)y$. Note that if $t < T$ we have $V(x) = V(\mathcal{T}(t)y) > i + d' > i + d$, which contradicts the fact that $x \in V^{-1}(i + d)$. Hence $t \ge T$ and by (2.2) we conclude that $x = \mathcal{T}(t)y \in N$. □

Remark 7. *Regarding Lemma 2.2, the same result holds for the unstable manifold. More precisely, given a neigborhood N of x_i^* on $\mathcal{A}$ there exists $d' \in (0, 1)$ such that*

$$V^{-1}(i + d) \cap W^u(x_i^*) \cap \mathcal{A} \subset N, \forall\ d \in [0, d'].$$

Analogously, Lemma 2.1 also holds for the unstable manifold of x_i^*.

2.1.1 Continuity of Subbundles

In this subsection we define continuity of subbundles, even when we do not have a manifold. Our definition of continuity will be via projections. We will prove a lot of results related to our sense of continuity that will be used through this chapter. Moreover, we introduce the neighborhoods $V_i \ni x_i^*$, defined in (2.5) and (2.6), that will be used to prove Lemma 2.11 (related to item 3) and Theorem 2.15, which is the main result of this section. Finally, in this subsection we define the notion of inclination of subspaces, present in the proof of the finite dimensional case, that will be used to prove several results of continuity in this subsection and Subsections 2.1.2 and 2.1.3. We recall that the main goal of this section is to obtain subbundles S_i and U_i that are continuous in a neighborhood $\mathcal{O}_i$ of x_i^*. This subsection is the key to obtain such continuity.

The next definition is an alternative, that we found very useful, to explore the notion of continuity of subbundles on infinite dimensional spaces using projections (see Definition 51). From now on it will be clear why the assumption of X being a Hilbert space and $D_x\mathcal{T}(t)$ being an isomorphism onto its range is necessary.

Definition 30. *Let X be a Banach space and (M,d) a metric space. For each $m \in M$ fix a closed subspace $E(m)$ of X. We say that the family $\{E(m) : m \in M\}$ is **continuous** (uniformly continuous) if there exists a family of projections $\{P(m) : m \in M\} \subset \mathcal{L}(X)$ such that $m \mapsto P(m)$ is continuous (uniformly continuous) in relation to the norm $\|\cdot\|_{\mathcal{L}(X)}$ and $R(P(m)) = E(m)$ for all $m \in M$.*

We will oftenly refer a (continuous) family $\{E(x) \subset X : x \in M\}$ as a (continuous) subbundle on the metric spaces M. Moreover, we denote by $TX|_M$ the subbundle $\{E(m) = X\}_{m\in M}$.

Now we define a "stronger" notion of continuity.

Definition 31. *Let X be a Hilbert space and (M,d) a metric space. For each $m \in M$ fix a closed subspace $E(m)$ of X and take $P(m) \in \mathcal{L}(X)$ as the orthogonal projection onto $E(m)$. We say that the family $\{E(m) : x \in M\}$ is **orthogonally continuous** if the family of projections $\{P(m) : m \in M\}$ is continuous in relation to the norm $\|\cdot\|_{\mathcal{L}(X)}$.*

Remark 8. *If $\{E(m) : m \in M\}$ is a uniformly continuous family of subspaces of X and the metric space M is connected, then it follows from Lemma 4.14 that $\dim E(m) = \dim E(m')$ for all $m, m' \in M$.*

The following notation will be used through this whole section.

Remark 9. *Let X be a Banach space. If we have the direct sum $V \oplus W = X$, with both subspaces V, W closed, we will denote by P_{VW} the projection $x_v + x_w \mapsto x_v$. Note that P_{VW} is continuous, since V, W are closed, and $P_{VW} + P_{WV} = Id$, where Id is the identity. In this notation, if X is a Hilbert space, the projection $P_{VV^\perp}$ is the orthogonal projection onto the closed subspace V and we simplify this notation by $P_V := P_{VV^\perp}$.*

The next lemma shows that Definitions 30 and 31 are equivalent if X is a Hilbert space.

Lemma 2.3. *Let X be a Hilbert space and $Q_n, Q \in \mathcal{L}(X)$ be continuous projections with $Q_n \to Q$ in $\mathcal{L}(X)$. Denoting $S_n := R(Q_n)$ and $S = R(Q)$, define the orthogonal projections*

$$P_n := P_{S_n} \text{ and } P := P_S.$$

Then $P_n \to P$ in $\mathcal{L}(X)$.

Proof. It is sufficient to show that $P(I-P_n) \to 0$ and $(I-P)P_n \to 0$ in $\mathcal{L}(X)$. We will prove that $(P-I)P_n \to 0$ and the other case will follow analogously by Lemma 4.18. Given $v \in X$, $\| v \| = 1$, it follows from the fact that P_n, P are orthogonal projections that

$$\begin{aligned}\| (P-I)P_n v \| =& \| PP_n v - P_n v \| \\ \leq& \| QP_n v - P_n v \| \\ =& \| QQ_n P_n v - Q_n P_n v \| \\ \leq& \| (Q-I)Q_n \|_{\mathcal{L}(X)} \to 0.\end{aligned}$$

□

Remembering that the unstable and local stable manifolds of a hyperbolic set are $\mathcal{C}^1$ manifolds if $\mathcal{T}$ is Morse-Smale (Theorem 1.12), for each $x \in W^s_{loc}(x_i^*) \cap W^u(x_j^*)$ we will denote by $\mathcal{S}(x)$ and $\mathcal{U}(x)$ the subspaces $T_x W^s_{loc}(x_i^*)$ and $T_x W^u(x_j^*)$ respectively. Note that $\mathcal{U}(x)$ is defined for any $x \in \mathcal{A}$ but $\mathcal{S}(x)$ is just defined if x is sufficiently close to an equilibrium x_i^*. Now we will extend the family $\mathcal{S}$ to the whole global attractor in a way that still holds its original properties (see Proposition 2.4). From this point on, the assumption of transversality, on condition 5 of Definition 22, will be frequently used.

We recall that if $E \subset X$, then we denote

$$(D_x\mathcal{T}(-t))E := (D_{\mathcal{T}(-t)x}\mathcal{T}(t))^{-1}E, \ \forall x \in \mathcal{A}, \ \forall t \geq 0.$$

Moreover, since $\mathcal{T}$ is Morse-Smale then the linear operator $D_{\mathcal{T}(-t)x}\mathcal{T}(t) \in \mathcal{L}(X)$ and the map $\mathcal{T}(t)|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ are injective with inverse continuous, and we denote

$$D_x\mathcal{T}(-t) := (D_{\mathcal{T}(-t)x}\mathcal{T}(t))^{-1}, \ \forall x \in \mathcal{A}, \ \forall t \geq 0.$$

Proposition 2.4. *The family $\mathcal{S}$ can be extend (not necessarily continuously) to the whole global attractor $\mathcal{A}$ in a way that still satisfies*

$$\mathcal{S}(x) + \mathcal{U}(x) = X, \ \forall x \in \mathcal{A} \tag{2.3}$$

and

$$(D_x\mathcal{T}(t))\mathcal{S}(x) \subset \mathcal{S}(\mathcal{T}(t)x), \ \forall x \in \mathcal{A}, \ \forall t \geq 0.$$

Proof. Fix $r > 0$ such that $W^s(x_i^*) \cap B[x_i^*, r]$ is part of the graph of a $\mathcal{C}^1$ map

$$\Psi : \mathcal{S}(x_i^*) \to \mathcal{U}(x_i^*)$$

and $B_i := B[x_i^*, r]$ (the closed ball in X of center x_i^* and radius r) satisfies

$$x \in W^s(x_i^*) \cap B_i \Rightarrow \mathcal{T}(t)x \in B_i, \ \forall t \geq 0.$$

Remember that $W^s(x_i^*) \cap B_i$ is a $\mathcal{C}^1$ manifold and $\mathcal{S}(x) = T_x W^s(x_i^*)$ is well defined for each $x \in B_i$. Given $x \in (\mathcal{A} \cap W^s(x_i^*)) \setminus B_i$ fix $t_0 = t_0(x) > 0$ as

$$t_0 = \min\{t > 0 : \mathcal{T}(t)x \in B_i\}. \tag{2.4}$$

Denoting $x_0 = \mathcal{T}(t_0)x \in B_i$ we define $\mathcal{S}(x)$ as below

$$\mathcal{S}(x) = (D_x\mathcal{T}(t_0))^{-1}\mathcal{S}(x_0) = (D_{x_0}\mathcal{T}(-t_0))\mathcal{S}(x_0).$$

In order to show (2.3), rembember that $\mathcal{U}(x) = T_x W^u(x_i^*)$ for each $x \in W^u(x_i^*)$, where $W^u(x_i^*)$ is a finite dimensional manifold and $D_x\mathcal{T}(t)$ is injective for any $x \in \mathcal{A}$ and $t \geq 0$, that is,

$$(D_x\mathcal{T}(t))\mathcal{U}(x) = \mathcal{U}(\mathcal{T}(t)x), \ \forall x \in \mathcal{A}, \ \forall t \in \mathbb{R}.$$

Therefore,

$$\mathcal{U}(x_0) \in R(D_x\mathcal{T}(t_0)) \ \text{ and } \ \mathcal{S}(x_0) + \mathcal{U}(x_0) = X$$

and (2.3) follows from Lemma 4.19.

Now we prove the positively invariance. Fix $x \in W^s(x_i*)$, $t, s \geq 0$ and consider the cases:

- If $x \in B_i$, then $\mathcal{T}(t)x \in B_i$ and $\mathcal{S}(x) = T_x W^s_{loc}(x_i^*)$ and $\mathcal{S}(\mathcal{T}(t)x) = T_{\mathcal{T}(t)x} W^s_{loc}(x_i^*)$. From the injectiveness of the derivative we conclude that $(D_x\mathcal{T}(t))\mathcal{S}(x) \subset \mathcal{S}(\mathcal{T}(t)x)$.
- If $x \notin B_i$ fix $t_0 > 0$ and $x_0 \in B_i$ as defined before. Then

$$(D_x\mathcal{T}(t_0))\mathcal{S}(x) = (D_x\mathcal{T}(t_0))(D_x\mathcal{T}(t_0))^{-1}\mathcal{S}(x_0) \subset \mathcal{S}(x_0).$$

- Let $x \notin B_i$ and fix $t_0 > 0$ and $x_0 \in B_i$ as defined before. Then, it follows from the previous items that for any $t \geq 0$ we have

$$(D_x\mathcal{T}(t_0+t))\mathcal{S}(x) = (D_{x_0}\mathcal{T}(t) \circ D_x\mathcal{T}(t_0))\mathcal{S}(x) \subset (D_{x_0}\mathcal{T}(t))\mathcal{S}(x_0) \subset \mathcal{S}(t_0+t).$$

- Let $x \notin B_i$ and fix $t_0 > 0$ and $x_0 \in B_i$ as defined before. Fix $t > 0$ such that $\mathcal{T}(t)x \notin B_i$. Then

$$\mathcal{S}(x) = (D_x\mathcal{T}(t_0))^{-1}\mathcal{S}(x_0) \quad \text{and} \quad \mathcal{S}(\mathcal{T}(t)x) = (D_{\mathcal{T}(t)x}\mathcal{T}(t_0-t))^{-1}\mathcal{S}(x_0).$$

  Therefore

$$\begin{aligned}(D_x\mathcal{T}(t))\mathcal{S}(x) &= (D_x\mathcal{T}(t))(D_x\mathcal{T}(t_0))^{-1}\mathcal{S}(x_0) = (D_x\mathcal{T}(t))(D_x\mathcal{T}(t_0-t+t))^{-1}\mathcal{S}(x_0)\\ &= (D_x\mathcal{T}(t))(D_{\mathcal{T}(t)x}\mathcal{T}(t_0-t) \circ D_x\mathcal{T}(t))^{-1}\mathcal{S}(x_0)\\ &\subset (D_{\mathcal{T}(t)x}\mathcal{T}(t_0-t))^{-1}\mathcal{S}(x_0) = \mathcal{S}(\mathcal{T}(t)x).\end{aligned}$$

  This concludes all cases and finishes the proof. □

Now we will construct a special neighborhood (in $\mathcal{A}$) $V_i \ni x_i^*$ for each $x_i^* \in \Omega$, where $V_i \subset B_i$, in the notation of Proposition 2.4. In fact, for each $x_i^* \in \Omega$ we want to find a neighborhood (in $\mathcal{A}$) V_i of x_i^* and continuous subbundles S_i', U_i' on V_i such that

$$\begin{aligned}U_i'(x) &= \mathcal{U}(x), \quad \text{for all} \quad x \in W^u(x_i^*) \cap V_i,\\ S_i'(x) &= \mathcal{S}(x), \quad \text{for all} \quad x \in W^s_{loc}(x_i^*) \cap V_i\end{aligned} \tag{2.5}$$

and

$$S_i'(x) \oplus U_i'(x) = X, \forall x \in V_i. \tag{2.6}$$

Those subbundles will be very helpful later, since they can be used as a "change of coordinates" on Lemma 2.14 and Theorem 2.15 (which is the essential key for the proof of Lipschitz Shadowing). Lemmas 2.5 and 2.6 below are dedicated to the proof of properties (2.5) and (2.6), respectively.

Lemma 2.5. *Let $\delta > 0$ such that $W^u_\delta(x^*)$ and $W^s_\delta(x^*)$ are graphs of the $\mathcal{C}^1-$maps (see Theorem 1.12)*

$$\Psi_u : B_{U_0(x^*)}(0,\delta) \to S_0(x^*) \quad \text{and} \quad \Psi_s : B_{S_0(x^*)}(0,\delta) \to U_0(x^*)$$

where $S_0(x^*) \oplus U_0(x^*) = X$ and $U_0(x^*)$ is finite dimensional. Consider the continuous family of projections $\{P_s(x) : x \in W^s_\delta(x^*)\}$ and $\{P_u(x) : x \in W^u_\delta(x^*)\}$, where

$$R(P_s(x)) = T_x W^s_\delta(x^*) \quad \text{and} \quad R(P_u(x)) = T_x W^u_\delta(x^*).$$

From Lemmas 4.25 and 4.26 those projections are well defined. Then, there exists a continuous extension of these families on $B_X(x^*,\delta)$.

Proof. Given $x \in B_X(x^*,\delta)$ we know that $x = x_s + x_u$, where $x_s \in S_0(x^*)$ and $x_u \in U_0(x^*)$. Defining $P_s(x) = P_s(x_s + \Psi_s(x_s))$ and $P_u(x) = P_u(x_u + \Psi_u(x_u))$ for each $x \in B_X(x^*,\delta)$, the extensions of P_s and P_u are well defined and continuous. Hence, we can define $S_i'(x) = R(P_s(x))$ and $U_i'(x) = R(P_u(x))$ for all $x \in B_X(x^*,\delta)$ to conclude the proof. □

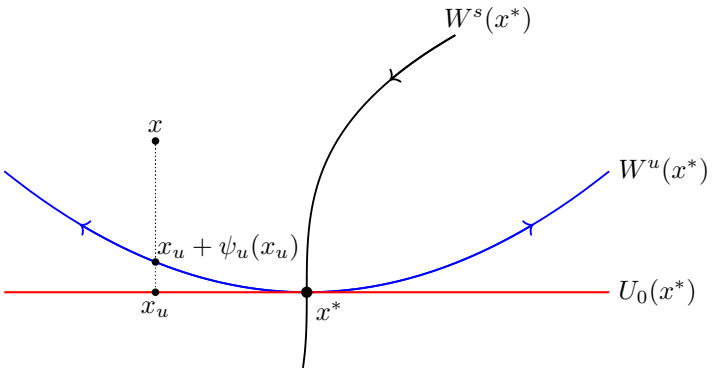


Figure 2.2: Construction of U_i' in $B_X(x^*, \delta)$

Lemma 2.6. *Let X be a Hilbert space and (M, d) a metric space. Assume that the families of closed subspaces $\{S(m) : m \in M\}$ and $\{U(m) : m \in M\}$ are continuous, dim $U(m) = n < +\infty$ for all $m \in M$ and*

$$S(m_0) \oplus U(m_0) = X$$

for some $m_0 \in M$. Then there exists $\epsilon > 0$ such that $S(m) \oplus U(m) = X$, for all $m \in B(m_0, \epsilon)$.

Proof.

Let $\{P(m) : m \in M\}$ and $\{Q(m) : m \in M\}$ be the orthogonal projections related to $\{S(m) : m \in M\}$ and $\{U(m) : m \in M\}$ respectively.

We will first prove that if $S(m_0) \oplus U(m_0) = X$ for some $m_0 \in M$, then there exists $\epsilon > 0$ such that $S(m) \cap U(m) = \{0\}$, for all $m \in B(m_0, \epsilon)$. Note that $S(m_0) \cap U(m_0) = \{0\}$ if and only if $P(m_0) - Q(m_0)$ is injective. Since the families of projections are continuous and the set of injective linear maps are open in $\mathcal{L}(X)$, it follows that there exists $\epsilon > 0$ such that $S(m) \cap U(m) = \{0\}$, for all $m \in B(m_0, \epsilon)$.

Now, we have to prove that if $S(m_0) + U(m_0) = X$ for some $m_0 \in M$, then there exists $\epsilon > 0$ such that $S(m) + U(m) = X$, for all $m \in B(m_0, \epsilon)$. First, we prove that $S(m_0) + U(m_0) = X$ if and only if $P(m_0) + Q(m_0)$ is surjective and the conclusion will follow from the continuity of the projections and the fact that the set of surjective maps is open in $\mathcal{L}(X)$. Note that it is obvious that if $P(m_0) + Q(m_0)$ is surjective then $S(m_0) + U(m_0) = X$, thus it remains to show the other way. Suppose that $S(m_0) + U(m_0) = X$ and take $v \in X$ arbitrary. Then, there exists $v_s \in S(m_0), v_u \in U(m_0)$ such that $v = v_s + v_u$. Since $S(m_0)^\perp + U(m_0)^\perp = X$, because $S(m_0) \cap U(m_0) = \{0\}$ and $U(m_0)$ is finite dimensional (see Lemma 4.15), we can take $w_s \in S(m_0)^\perp$ and $w_u \in U(m_0)^\perp$ such that

$$v_s - v_u = w_s + w_u.$$

Define $\tilde{v} = v_s - w_s = v_u + w_u$. Then $(P + Q)\tilde{v} = v$. □

Now we have concluded the existence of the neighborhoods V_i and the continuous subbundles S_i', U_i' stated before in (2.5) and (2.6). In order to prove Theorem 2.15 (construction of compatible subbundles), we have to construct one more continuous subbundle.

Let x_i^* fixed. Fix $d \in (0, 1)$, consider the set

$$D = V^{-1}(i + d) \cap W^s(x_i^*) \cap \mathcal{A}, \tag{2.7}$$

and let Q be a compact neighborhood of D in $V^{-1}(i + d) \cap \mathcal{A}$ with $Q \cap W^u(x_i^*) = \emptyset$. Assume that $\mathcal{V}$ is a continuous subbundle defined on Q such that

$$\mathcal{S}(x) \oplus \mathcal{V}(x) = X, \text{ for } x \in D. \tag{2.8}$$

Now, we extend the subbundle $\mathcal{V}$ to the set $\gamma(Q)^+ \cup W^u(x_i^*)$ as

$$\mathcal{V}(x) := \mathcal{U}(x),\ \forall x \in W^u(x^*),$$
$$\mathcal{V}(\mathcal{T}(t)x) := (D_x\mathcal{T}(t))\mathcal{V}(x),\ \forall x \in Q,\ \forall t > 0.$$

Our next goal is to show that we can reduce Q in a way that the subbundle $\mathcal{V}$ becomes continuous on $\gamma^+(Q) \cup W^u(x_i^*)$. Firstly, we will focus on results that guarantee the continuity on $\gamma^+(Q)$ (Lemma 2.8). We start with a technical lemma.

Lemma 2.7. *Let X be a Hilbert space and $T_n, T \in \mathcal{L}(X)$ be injective bounded linear maps such that $T_n^{-1} : R(T_n) \to X$ and $T^{-1} : R(T) \to X$ are bounded and $T_n \to T$ in $\mathcal{L}(X)$. Then*

$$\sup_{n\in\mathbb{N}} \| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)} < +\infty.$$

Proof. Assume that there exists a sequence $v_n \in R(T_n)$, with $\| v_n \| = 1$ for all n, such that

$$\| T_n^{-1} v_n \| \to +\infty.$$

Defining $w_n = T_n^{-1} v_n$, we have

$$T_n\left(\frac{w_n}{\| w_n \|}\right) = \frac{v_n}{\| w_n \|} \to 0,$$

which contradicts the fact that T is an isomorphism onto its image. □

The following result will guarantee that if $\mathcal{V} = \{\mathcal{V}(x) : x \in M\}$ is a continuous subbundle in the metric space M and T is an isomorphism onto its range, then $T\mathcal{V} = \{T\mathcal{V}(x) : x \in M\}$ is a (well defined) continuous subbundle.

Lemma 2.8. *Let X, Y be Hilbert spaces, $T_n, T \in \mathcal{L}(X, Y)$ be isomorphisms onto its images and $Q_n, Q \in \mathcal{L}(X)$ be continuous projections such that*

$$T_n \to T \quad \text{and} \quad Q_n \to Q \quad \text{in} \quad \mathcal{L}(X).$$

Define P_n the orthogonal projection onto $R(T_nQ_n)$ and P the orthogonal projection onto $R(TQ)$. Then $P_n \to P$ in $\mathcal{L}(X)$. In the particular case $Q_n = Q = Id$ we have $R(T_n) \to R(T)$, in the sense of Definition 31.

Proof. It is sufficient to show that $(I - P)P_n \to 0$ and $P(P_n - I) \to 0$ in $\mathcal{L}(X)$. We will only prove that $(I - P)P_n \to 0$ and the other case follows analogously by Lemma 4.18. Given $v \in X$, $\| v \| \leq 1$, we have

$$\| P_nv - PP_nv \| = d(P_nv, R(P)) = d(P_nv, R(TQ)).$$

Since $P_nv = T_nQ_nv_n$ for some v_n (we can consider $v_n = Q_nv_n$) with $\| v_n \| \leq \| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)}$, we have

$$\| P_nv - PP_nv \| \leq \| T_nQ_nv_n - TQv_n \| .$$

From Lemma 2.7 there exists $M > 0$ that bounds $\| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)}$ for all n and therefore

$$\sup_{\|v\|=1} \| P_nv - PP_nv \| \leq M \| T_nQ_n - TQ \|_{\mathcal{L}(X)} \to 0.$$

□

With this result, we are ready to prove the continuity of the subbundle $\mathcal{V}$ restricted to $\gamma^+(Q)$ (Lemma 2.11). Despite that, before we start the proof of the continuity on $\gamma(Q)^+ \cup W^u(x_i^*)$, we will show that, under certain conditions, the subbundle $\mathcal{V}$ can also be continuous on $\gamma^-(Q)$. We have to prove this "backwards continuity" to construct one of the compatible subbundles, as we will see further on.

Lemma 2.9. *Let X be a Hilbert space, $Q_n, Q \in \mathcal{L}(X)$ be projections and $T_n, T \in \mathcal{L}(X)$ be isomorphisms onto its images such that $Q_n \to Q$ and $T_n \to T$. If $R(Q_n) \subset R(T_n)$ for all n, and $R(Q) \subset R(T)$, then $T_n^{-1}Q_n \to T^{-1}Q$ in $\mathcal{L}(X)$. Moreover, if P_n and P are the orthogonal projections onto $R(T_n^{-1}Q_n)$ and $R(T^{-1}Q)$ respectively, then $P_n \to P$ in $\mathcal{L}(X)$.*

Proof. We know from Lemma 2.7 that $\sup_{n\in\mathbb{N}} \| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)}$ is bounded. Hence

$$\| T^{-1}Q - T_n^{-1}Q_n \|_{\mathcal{L}(X)} \leq \| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)} \| T_n T^{-1}Q - Q_n \|_{\mathcal{L}(X)} \to 0.$$

In order to prove that $P_n \to P$ in $\mathcal{L}(X)$, we will show that $(I-P)P_n \to 0$ and $P(P_n - I) \to 0$ in $\mathcal{L}(X)$. We will prove only that $(I-P)P_n \to 0$ and the other case will follow analogously by Lemma 4.18. Given $v \in X$ with $\| v \| = 1$ we have

$$\| PP_n v - P_n v \| = d(P_n v, R(P)) = d(P_n v, R(T^{-1}Q)).$$

Since $P_n v = T_n^{-1}Q_n v_n$, with $\| v_n \| = \| Q_n v_n \| \leq M := \sup_{n\in\mathbb{N}} \| T_n \|$ for all n, we have

$$\begin{aligned} \| PP_n v - P_n v \| &\leq \| T_n^{-1}Q_n v_n - T^{-1}Q v_n \| \leq M \| T_n^{-1}Q_n - T^{-1}Q \|_{\mathcal{L}(X)} \\ &\Rightarrow \sup_{\|v\|=1} \| PP_n v - P_n v \| \leq M \| T_n^{-1}Q_n - T^{-1}Q \|_{\mathcal{L}(X)} \to 0. \end{aligned}$$

□

Corollary 2.10. *Let X be a Hilbert space and $T_n, T \in \mathcal{L}(X)$ be isomorphisms onto its ranges such that $T_n \to T$ in $\mathcal{L}(X)$. Then*

$$\| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)} \to \| T^{-1} \|_{\mathcal{L}(R(T),X)}.$$

Proof. Just apply Lemma 2.9 with Q_n and Q being the orthogonal projections onto $R(T_n)$ and $R(T)$ respectively (they are continuous from Lemma 2.8) and note that

$$\| T_n^{-1}P_n \|_{\mathcal{L}(X)} = \| T_n^{-1} \|_{\mathcal{L}(R(T_n),X)} \quad \text{and} \quad \| T^{-1}P \|_{\mathcal{L}(X)} = \| T^{-1} \|_{\mathcal{L}(R(T),X)} \tag{2.9}$$

□

Now we prove the continuity of the subbundle $\mathcal{V}$ on $\gamma(Q)^+ \cup W^u(x_i^*)$.

Lemma 2.11. *Let V_i be the neighborhood of x_i^* mentioned in (2.5) and (2.6) and $d' > 0$ the number related to that neighborhood with the properties stated on Lemma 2.2. Fix $d \in (0, d')$, a neighborhood Q of D (see (2.7)) in $V^{-1}(i+d) \cap \mathcal{A}$ and a continuous subbundle $\mathcal{V}$ on Q satisfying (2.8). Then, we can reduce Q such that $\mathcal{V}$ is continuous on $(\gamma(Q)^+ \cup W^u(x_i^*)) \cap V_i$.*

Proof. First we prove the continuity of $\mathcal{V}$ on $\gamma^+(Q)$. We just have to show that the subbundles $\mathcal{V}_{t_0} := \{(D_x\mathcal{T}(t_0))\mathcal{V}(x)\}_{x\in Q}$ and $\mathcal{V}_{x_0} := \{\{(D_{x_0}\mathcal{T}(t))\mathcal{V}(x_0)\}_{t\geq 0}$ are continuous for each $t_0 > 0$ and $x_0 \in Q$. We start by proving the continuity of $\mathcal{V}_{t_0}$.

Assume that $x, x_n \in Q$ and $\mathcal{T}(t_0)x_n \to \mathcal{T}(t_0)x$ as $n \to +\infty$. Since $\mathcal{T}$ is a group in $\mathcal{A}$ (Proposition 1.14), then $x_n \to x$ and $\mathcal{V}(x_n) \to \mathcal{V}(x)$, in the sense of Definition 30. Since $\mathcal{T}(t_0) \in \mathcal{C}^1(X)$, can use Lemma 2.8 with $T_n = D_{x_n}\mathcal{T}(t_0)$,$T = D_x\mathcal{T}(t_0)$ and $Q_n = P_{\mathcal{V}(x_n)}, Q = P_{\mathcal{V}(x)}$ to guarantee that

$$\mathcal{V}(\mathcal{T}(t_0)x_n) = (D_{x_n}\mathcal{T}(t_0))\mathcal{V}(x_n) \to (D_x\mathcal{T}(t_0))\mathcal{V}(x) = \mathcal{V}(\mathcal{T}(t_0)x),$$

where the convergence is in the sense of projections (Definitions 30 and 31). Hence the subbundle $\mathcal{V}_{t_0}$ is continuous for each $t_0 \geq 0$.

Now we show the continuity of $\mathcal{V}_{x_0}$, for $x_0 \in D$ fixed. We recall that we are assuming item (H2) from Theorem 0.1. Consequently, for each $t \geq 0$ it holds

$$(0, +\infty) \ni t_n \to t \Longrightarrow (D_{x_0}\mathcal{T}(t_n))v \to (D_{x_0}\mathcal{T}(t))v, \text{ for all } v \in X. \tag{2.10}$$

Since $\mathcal{V}(x_0)$ is finite dimensional, with the same dimension of $U_0(x_i^*)$ given on Definition 20, it follows from (2.10) that

$$t_n \to t \Longrightarrow (D_{x_0}\mathcal{T}(t_n))|_{\mathcal{V}(x_0)} \to (D_{x_0}\mathcal{T}(t))|_{\mathcal{V}(x_0)} \text{ in } \mathcal{L}(\mathcal{V}(x_0), X).$$

Therefore we can use Lemma 2.8 once more to guarantee the continuity of the subbundle $\mathcal{V}_{x_0}$, for each $x_0 \in Q$.

Now we focus on the continuity of $\mathcal{V}$ on $\gamma^+(Q) \cup W^u(x_i^*)$. On Lemma 4.21 we replicate the proof of [47, Lemma 2.2.7] to show that the inclination of the subspaces $\mathcal{V}$ related to the decomposition $S_i'(x) \oplus U_i'(x) = X$ goes to 0. Here we will just show that this continuity (by inclination) implies our sense of continuity (Definitions 30 and 31).

Let (M, d) be a metric space and assume that $\{S'(x) : x \in M\}$, $\{U'(x) : x \in M\}$ and $\{\mathcal{V}(x) : x \in M\}$ are continuous families of closed subspaces of a Hilbert space X such that

$$S'(x) \oplus U'(x) = 0, \ \forall x \in M,$$
$$S'(x) \cap \mathcal{V}(x) = 0, \ \forall x \in M.$$

Given $v \in \mathcal{V}(x)$ non nulle, we can write $v = v^s + v^u$ with $v^s \in S'(x)$ and $v^u \in U'(x)$ in a unique way. We define the inclination of v in relation to $S'(x) \oplus U'(x) = X$ as $\alpha(v) := \dfrac{\| v^s \|}{\| v^u \|}$. We have to show that if $x_n \to x_0$ and every $v_n \in \mathcal{V}(x_n)$ satisfies $\alpha(v_n) \to 0$, then $\mathcal{V}(x_n)$ goes to $U'(x_0)$ in the sense of Definition 31.

Let $P_{U'(x)}$ and $P_{\mathcal{V}(x)}$ be the orthogonal projections onto $U'(x)$ and $\mathcal{V}(x)$ respectively, and $P_{U'(x)^\perp} = I - P_{U'(x)}$, $P_{\mathcal{V}(x)^\perp} = I - P_{\mathcal{V}(x)}$.

If $\mathcal{V}(x_n)$ does not converges to $U'(x_0)$, then (from Corollary 4.17) we have two possible cases:

$$P_{U'(x_0)^\perp} P_{\mathcal{V}(x_n)} \nrightarrow 0 \quad \text{or} \quad P_{U'(x_0)} P_{\mathcal{V}(x_n)^\perp} \nrightarrow 0,$$

where $P_{\mathcal{V}(x_n)}$ and $P_{U'(x_0)^\perp}$ are the orthogonal projections onto $\mathcal{V}(x_n)$ and $U'(x_0)^\perp$ respectively.

We will prove the first case by contradiction and the second case comes analogously. Assume that there exists $\epsilon \in (0, 1)$ and sequences $x_n \to x_0$ and $w_n \in X$, with $\| w_n \| \leq 1$, such that

$$\| P_{U'(x_0)^\perp} P_{\mathcal{V}(x_n)} w_n \| \geq \epsilon, \ \forall n \in \mathbb{N}. \tag{2.11}$$

Define $v_n = P_{\mathcal{V}(x_n)} w_n$. Take $\epsilon' > 0$ such that

$$\frac{\epsilon'}{1 - \epsilon'} < \epsilon$$

and fix large N such that

$$\frac{\| v_n^s \|}{\| v_n^u \|} < \epsilon', \ \forall n > N, \tag{2.12}$$

that is

$$-\epsilon' \| v_n^u \| < - \| v_n^s \|, \ \forall n > N.$$

From this we have

$$(1-\epsilon')\parallel v_n^u\parallel\leq\parallel v_n^u\parallel-\parallel v_n^s\parallel\leq\parallel v_n^s+v_n^u\parallel=\parallel v_n\parallel\ \ \forall n>N,$$

which implies

$$\parallel v_n^u\parallel\leq\frac{\parallel v_n\parallel}{1-\epsilon'},\ \forall n>N.$$

It follows from that and (2.12) that

$$\parallel v_n-v_n^u\parallel=\parallel v_n^s\parallel\leq\epsilon'\parallel v_n^u\parallel\leq\frac{\epsilon'}{1-\epsilon'}\parallel v_n\parallel\leq\frac{\epsilon'}{1-\epsilon'}<\epsilon\ \forall n>N.$$

Since $P_{U'(x_0)}$ is the orthogonal projection (minimize the distance between a point and $U'(x_0)$), then

$$\parallel P_{U(x_0)^\perp}P_{\mathcal{V}(x_n)}w_n\parallel=\parallel(I-P_{U(x_0)})v_n\parallel=\parallel v_n-P_{U(x_0)}v_n\parallel\leq\parallel v_n-v_n^u\parallel<\epsilon,\ \forall n>N,$$

which contradicts (2.11). □

The proposition below will be essential to prove Theorem 2.15, where we construct the compactible subbundles. The reason behind this result is as follows: in some moment we will have a continuous subbundle G on a metric space M and we will need to obtain a continuous subbundle using G and T^{-1}, where T is an isomorphism onto its image. It is not possible to simply define the subbundle $\{T^{-1}G(x)\}_{x\in M}$, because we do not know if this family is continuous (and this is false in general). The intuitively way to do that (see Lemma 2.9) is to work with the subbundle

$$L(x)=T^{-1}(G(x)\cap R(T)),\ \forall x\in M.$$

Unfortunately, we also do not know when the subbundle $G(x)\cap R(T)$ is continuous. For example, if we consider $X=\mathbb{R}^2$ and fix the continuous subbundles $\{F_m=\langle(1,1)\rangle\}_{m\in\mathbb{R}^+}$ and $\{E_m=\langle(1,m)\rangle\}_{m\in\mathbb{R}^+}$, where $\langle v\rangle$ denotes the subspace generated by $v\in\mathbb{R}^2$, then $F_m\cap E_m=\{0\}$ if $m\neq 1$ and $F_1=\langle(1,1)\rangle$. Therefore, the subbundle $\{F_m\cap E_m\}_{m\in\mathbb{R}^+}$ is not continuous (see Remark 8).

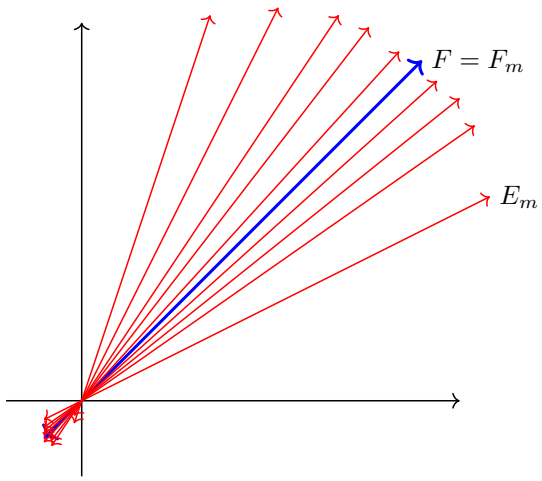


Figure 2.3: Discontinuity of the intersection of subbundles

Therefore we need a result that guarantees the continuity of a family of intersections.

Proposition 2.12. *Let (M,d) be a connected metric space, X be a Hilbert space and $\{E_m\}_{m\in M}$, $\{F_m\}_{m\in M}$ be continuous families of closed subspaces of X. If*

$$E_m+F_m=X\quad\text{and}\quad\dim(E_m)^\perp<+\infty,\ \text{for all } m\in M,$$

then the family $\{E_m\cap F_m\}_{m\in M}$ is continuous.

Proof. From $E_m + F_m = X$ we have $E_m^\perp \cap F_m^\perp = \{0\}$ and from $\dim(E_m)^\perp < +\infty$ we know that $E_m^\perp \oplus F_m^\perp$ is closed in X. Since $\{E_m \cap F_m\}_{m\in M}$ is continuous if, and only if $\{E_m^\perp \oplus F_m^\perp\}_{m\in M}$ is continuous, we just have to show that the last family is continuous. Take $m_n, m \in M$ such that $m_n \to m$ and define P_n, P the orthogonal projections on $E_{m_n}^\perp \oplus F_{m_n}^\perp$ and $E_m^\perp \oplus F_m^\perp$ respectively. We will show that $P_n \to P$ in $\mathcal{L}(X)$. Remember that E_x is isomorphic to E_y and F_x is isomorphic to F_y for every $x, y \in M$ (see Lemma 4.14). For each $n \in \mathbb{N}$ define the linear map

$$\begin{aligned} \psi_n : E_{m_n} \oplus F_{m_n} &\to E_m \oplus F_m \\ e + f &\mapsto (P_{E_m})e + (P_{F_m})f, \end{aligned}$$

where P_{E_m} and P_{F_m} are the orthogonal projections onto E_m and F_m, respectively. Note that

$$\| \psi_n - Id|_{E_{m_n}\oplus F_{m_n}} \|_{\mathcal{L}(E_{m_n}\oplus F_{m_n}, X)} \to 0.$$

In order to prove that $P_n - P \to 0$, we will prove that $(I - P)P_n \to 0$ and $P(I - P_n) \to 0$. It is sufficient to prove that $(I - P)P_n \to 0$ (the other case follows analogously by Lemma 4.18). Given $v \in X$, $\| v \| \leq 1$, we have

$$\| (I - P)P_n v \| = d(P_n v, E_m \oplus F_m) \leq \| P_n v - \psi_n(P_n v) \| \leq \| \psi_n - I|_{E_{m_n}\oplus F_{m_n}} \| \to 0.$$

□

Now that we proved Proposition 2.12, we can put it together with Lemma 2.9 to guarantee "backwards continuity" in some sense.

2.1.2 Lower semicontinuity and construction of new subbundles S_i and U_i

In this subsection we construct the subbundles S_i and U_i discussed in the beginning of the section. These subbundles will allow us to easily construct compatible subbundles S and U. In fact, this subsection is the most important one to construct the compatible subbundles and has one of the key results from this thesis, that is Theorem 2.15. Beyond its importance to construct the compatible subbundles, Theorem 2.15 is also one of the hardest results to adapt to the infinite dimensional context. Indeed, in the finite dimensional context the derivative of the semigroup $D_x\mathcal{T}(t)$ is an isomorphism, and the surjectivity is used very strongly to prove this result. Summarizing, the proof of Theorem 2.15 is via induction and the surjectivity is used to prove the induction hypothesis. To overcome this problem, we need two auxiliary results, that are Lemmas 2.16 and 2.17.

The notion of lower semicontinuity is also introduced in this subsection. This concept is more general then continuity (continuous implies lower semicontinuous) and will help us to prove Lemmas 2.16 and 2.17, discussed above. We will prove that the families $\mathcal{U}$ and $\mathcal{S}$ defined in Subsection 2.1.1 are lower semicontinuous (Lemma 2.14). In fact, the proof of Lemma 2.14 will be a guide to prove Theorem 2.15, since it has a similar construction to the proof of Theorem 2.15, but more simple.

Now we define the notion of lower semicontinuity to a family of vector subspaces, which is a weaker concept then continuity. Intuitively speaking, if X is a Banach space and (M, d) is a metric space, then a family of closed subspaces $\{E(x) \subset X : x \in M\}$ is lower semicontinuous at x_0 if there exists a neighborhood U of x_0 such that the subspaces $E(x)$ are "bigger" then $E(x_0)$, for $x \in U$. For example, the subbundle

$$E(x) = \begin{cases} \mathbb{R}^n, \text{ if } x \in \mathbb{R} \setminus \{0\}; \\ 0, \text{ if } x = 0 \end{cases}$$

is lower semicontinuous at $x = 0$ but it's not continuous $x = 0$. In fact, this subbundle is lower semicontinuous at x for all $x \in \mathbb{R}$.

The notion of lower semicontinuity has a important role on the proof of Theorem 2.15. In fact, lower semicontinuity can give us a property of robustness similar to Lemma 2.6 (see Remark 11). As we know, robustness is a very important in the theory of dynamical systems, e.g. hyperbolic equilibria and transversality are very important properties because of its robustness.

Definition 32. *Let (M,d) be a metric space and $\{E(x) : x \in M\}$ be a family of closed linear subspaces of a Hilbert space X. The family $\{E(x) : x \in M\}$ is called **lower semicontinuous** at $x_0 \in M$ if for any linear subspace $L \subset E(x_0)$ there exists a neighborhood $U \subset M$ of x_0 and a continuous subbundle F such that $F(x_0) = L$ and $F(x) \subset E(x)$, for $x \in U$.*

A family $\{E(x) : x \in M\}$ is called lower semicontinuous if it is lower semicontinuous at every $x \in M$.

Remark 10. *If a family $\{E(x) : x \in \mathcal{A}\}$ is continuous (Definition (31)) then it's lower semicontinuous, as we are going to see on the next lemma.*

Lemma 2.13. *Suppose that (M,d) is a metric space, X is a Hilbert space and $P(X) \subset \mathcal{L}(X)$ is the subset of orthogonal projections from X into X. If $f : M \to P(X)$ is continuous (the topology in $P(X)$ is the same as in $\mathcal{L}(X)$), then for any $x_0 \in M$ and closed subset $S(x) \subset R(f(x_0))$ there exists a map $g : M \to P(X)$ such that $R(g(x_0)) = S(x)$, $R(g(x)) \subset R(f(x))$ for all $x \in M$, and g is continuous on a neighborhood of x_0.*

Proof. Fix $x_0 \in M$ and a closed subspace $S \subset R(f(x_0))$. Let P_S be the orthogonal projection onto S and define $Q(x)$ as the orthogonal projection onto $R(P(x)P_S)$ for all $x \in M$. Note that $R(Q(x_0)) = S$ and $R(Q(x)) \subset R(P(x))$ for all $x \in M$. Lets prove that $\{Q(x)\}_{x\in M}$ is continuous on a neighborhood of x_0.

Let $r > 0$ such that if $x, y \in B(x_0, r)$ then $\| P(x) - P(y) \|_{\mathcal{L}(X)} < \frac{1}{2}$. Take $x, x_n \in B(x_0, r)$ with $x_n \to x$. We will show that $(I - Q(x))Q(x_n) \to 0$ and (from Lemma 4.18) $Q(x)(I - Q(x_n)) \to 0$ will follow analogously. If $v \in X$ with $\| v \| = 1$ we have

$$\| Q(x_n)v - Q(x)Q(x_n)v \| = d(Q(x_n)v, (P(x)P_S)X).$$

We can assume that $Q(x_n)v = P(x_n)P_S v_n$ for some $v_n \in X$. Note that $\| P(x_n)P_S v_n \| \leq 1$. Moreover,

$$\begin{aligned} \| P_S v_n \| = \| P(x_0)P_s v_n \| &\leq \| (P(x_n) - P(x_0))P_S v_n \| + \| P(x_n)P_S v_n \| \\ &\leq \frac{1}{2} \| P_S v_n \| + 1 \\ &\Longrightarrow \| P_S v_n \| \leq 2, \quad \forall n. \end{aligned}$$

Hence

$$\begin{aligned} \| Q(x_n)v - Q(x)Q(x_n)v \| &\leq \| P(x_n)P_S v_n - P(x)P_S v_n \| \\ &\leq 2 \| P(x_n) - P(x) \|_{\mathcal{L}(X)} \to 0. \end{aligned}$$

□

The following remark show us the that lower semicontinuity give us some robustness. To be more precise, if we have two subbundles that are lower semicontinuous and are transversal at a point x, then there exists a neighborhood of x that preserves the transversality.

Remark 11. *If E and L are a lower semicontinuous subbundle on $\mathcal{A}$ such that*

$$E(x) + L(x) = X,$$

for some $x \in \mathcal{A}$, it follows from Lemmas 2.6 and 4.13 that there exists a neighborhood V of x in $\mathcal{A}$ such that

$$E(y) + L(y) = X, \ \forall y \in V.$$

Now we prove that the families $\mathcal{U}$ and $\mathcal{S}$ defined in the beginning of this section are lower semicontinuous. This property will help us to prove the main result of this section, that is Theorem 2.15. Before we prove this property, let us first analyze if this result makes sense geometrically. We will argue to the family $\mathcal{U}$ and the same reasoning will work to $\mathcal{S}$.

Let $x \in \mathcal{A}$ and $x_i^* \in \Omega$ such that $x \in W^u(x_i^*)$. We will show that there exists a neighborhood $V_x \subset \mathcal{A}$ of x such that $V_x \subset \bigcup_{j=i}^{p} W^u(x_j^*)$, that is, there is a neighborhood (in $\mathcal{A}$) of x such that every element in the neighborhood is "equally or more unstable" then x, which induces us to believe that $\mathcal{U}$ is lower semicontinuous. In fact, if this is not the case then there exists a sequence $\mathcal{A} \ni x_n \to x$ such that $x_n \in W^u(x_k^*)$ with $k < i$. Since V is decreasing along solutions, then

$$V(\mathcal{T}(t)x_n) < k, \ \forall n \in \mathbb{N}, \ \forall t \geq 0. \tag{2.13}$$

Fix $t > 0$ such that $k < V(\mathcal{T}(-t)x)$. Since $x_n \to x$ and $\mathcal{T}$ is a group (Proposition 1.14), then

$$\mathcal{T}(-t)x_n \to \mathcal{T}(-t)x \Longrightarrow V(\mathcal{T}(-t)x_n) > k \ \text{ for large } n,$$

which contradicts (2.13). This proves our claim and concludes the intuition behind the lower semicontinuity of $\mathcal{U}$.

We recall the notation of positive orbit, that will be used in the proof: If $B \subset \mathcal{A}$, we define the positive orbit of B as the set $\gamma^+(B) := \bigcup_{t \geq 0} \mathcal{T}(t)B$.

Lemma 2.14. *The families $\{\mathcal{U}(x)\}_{x\in\mathcal{A}}$ and $\{\mathcal{S}(x)\}_{x\in\mathcal{A}}$ are lower semicontinuous.*

Proof. We will prove just the lower semicontinuity for the family $\mathcal{U}$. In fact, it is a little bit harder to prove for the family $\mathcal{S}$ (because the derivative is not surjective) but it is still possible. Since the strategy to prove semicontinuity for the family $\mathcal{S}$ is very similar to the proof that we will provide for the existence of the subbundles S_i on Theorem 2.15, we will not replicate the proof here.

Consider the following statement: for each $i \in \{x_1^*, \ldots, x_p^*\}$ there exists a neighborhood $Z_i \ni x_i^*$ in $\mathcal{A}$ and subbundles $\mathcal{G}_i$ of $TX|_{\gamma^+(Z_i)}$, continuous on Z_i, such that

$$\mathcal{G}_i(x) \subset \mathcal{U}(x), \ \forall x \in \gamma^+(Z_i) \tag{2.14}$$

and

$$\mathcal{G}_i(x) = \mathcal{U}(x), \ \forall x \in W^u(x_i^*). \tag{2.15}$$

Note that if we prove that statement we conclude that $\mathcal{U}$ is lower semicontinuous, since $\mathcal{G}_i$ is continuous (and consequently lower semicontinuous from Remark 10). The proof is by induction on i. Since $W^u(x_p^*)$ is a neighborhood of x_p^* in $\mathcal{A}$, the statement is true for $i = p$ with $Z_p = W^u(x_p^*)$ and $\mathcal{G}_p(x) = \mathcal{U}(x)$ for all $x \in W^u(x_p^*) = \gamma^+(W^u(x_p^*))$.

Suppose that the statement holds for $j \in \{i+1, \ldots, p\}$. We will show that it also holds for $j = i$. Using Lemma 2.2, fix $d \in (0,1)$ appropriate such that

$$D = V^{-1}(i + d') \cap W^s(x_i^*) \cap \mathcal{A} \subset V_i, \ \forall d' \in [0, d],$$

where V_i is given in (2.6). Fix $x \in D$ and $j > i$ such that $x \in W^u(x_j^*)$. Since $\mathcal{T}$ is Morse-Smale, we know that $\mathcal{S}(x) + \mathcal{U}(x) = X$ and therefore there exists a subspace $\mathcal{G}' \subset \mathcal{U}(x)$ such that

$$\mathcal{S}(x) \oplus \mathcal{G}' = X.$$

By the induction hypothesis there exists a neighborhood $Z(x)$ of x in $\mathcal{A} \cap V^{-1}(i + d)$ and continuous subbundles $G^{(x)}$ on $Z(x)$ satisfying

$$G^{(x)}(y) \subset \mathcal{U}(y),\ \forall y \in Z(x)$$

and

$$G^{(x)}(x) = \mathcal{G}'.$$

Remembering that $\mathcal{S}(x) = S_i(x)$, it follows from Lemma 2.6 that we can reduce $Z(x)$ to satisfy $Z(x) \subset V_i$ and

$$S_i'(y) \oplus G^{(x)}(y) = X,\ \forall y \in Z(x). \tag{2.16}$$

From (2.6), (2.16) and Lemma 4.22 it follows that $G^{(x)}(y)$ can be written as the graph of a continuous linear map

$$g_y^{(x)} : U_i'(y) \to S_i'(y)$$

for all $y \in Z(x)$ and $x \in D$. Let Q be a close neighborhood of D in $V^{-1}(i+d) \cap \mathcal{A}$ such that

$$Q \subset \bigcup_{x \in D} Z(x)$$

Since Q is compact there exists $x_1, \ldots, x_n \in D$ such that $Q \subset Z(x_1) \cup \cdots \cup Z(x_n)$.

Take a smooth partition of unity $\{\beta_k : Q \to \mathbb{R}\}_{k=1}^n$ subordinated to the covering $\{Z(x_k)\}_{k=1}^n$. Define the continuous map $g^Q : Q \to \mathcal{L}(X)$ as

$$g^Q(y) = \sum_{k=1}^{n} \beta_k(y) g_y^{(x_k)} P_{U_i'(y)},\ \forall y \in Q. \tag{2.17}$$

Denoting by $G^Q(y)$ the graph of $g^Q(y)|_{U_i'(y)}$ for all $y \in Q$, we obtain that $G^Q := \{G^Q(y)\}_{y \in Q}$ is a continuous subbundle on Q (Lemma 4.23). Given $v \in U_i'(y)$ we have

$$v + g^Q(y)v = \sum_{k=1}^{n} \beta_k(y)(v + g_y^{(x_k)} v).$$

Since $v + g_y^{(x_k)} v \in G^{(x_k)}(y) \subset \mathcal{U}(y)$ for any $k \in \{1, \ldots, p\}$, then

$$G^Q(y) \subset \mathcal{U}(y),\ \forall y \in Q. \tag{2.18}$$

Now we extend G^Q to the subbundle G_i defined on $\gamma^+(Q) \cup W^u(x_i^*)$ as below

$$G_i(\mathcal{T}(t)x) := (D_x\mathcal{T}(t))G^Q(x),\ \forall x \in Q,\ \forall t \geq 0, \tag{2.19}$$

$$G_i(x) := \mathcal{U}(x),\ \forall x \in W^u(x_i^*). \tag{2.20}$$

With that extension we guarantee that $G_i(x) \subset \mathcal{U}(x)$ for all $x \in \gamma(Q) \cup W^u(x_i^*)$ (remember that $\mathcal{U}$ is $D\mathcal{T}(t)$-invariant) and $G_i(x) = \mathcal{U}(x)$ for $x \in W^u(x_i^*)$. From Lemma 2.1 there exists a neighborhood $Z_i \subset \gamma^+(Q) \cup W^u(x_i^*)$ of x_i^* in $\mathcal{A}$. Thereofore Z_i and the subbundle $\mathcal{G}_i := G_i|_{\gamma^+(Z_i) \cup W^u(x_i^*)}$ satisfies our thesis and we have proved our claim by induction. □

The following theorem is one the main results of this work. Here we will construct a geometric structure that will allow us later, with tools from Functional Analysis, to obtain the Lipschitz Shadowing on the global attractor. The original idea of this Theorem can be found on the prestigious paper [52], where the author works with Morse-Smale diffeomorphims on a compact smooth manifold. The same result was also proved for vector fields on compact manifolds [53]. The reader can also consulte the proof on [47, Lemma 2.2.9]. The original proof is based on a induction which strongly uses the surjective of the derivative, which we do not have here. To overcomes this problem, we proved some auxilar results (Lemmas 2.16 and 2.17) about the dynamics inside the attractor $\mathcal{A}$. Moreover, if you have a diffeomorphism on a compact manifold you can use the Lyapunov norm,

see [47, Lemma 1.2.1], and assume without lost of generality that the constant of hiperbolicity C (see Definition 20) is equal to 1 and we do not have this tool in our context. To solve this little problem we have Remark 13.

The following proof can be a little difficult to follow, since it depends on many auxiliary lemmas. In general, those lemmas are easier to understand geometrically but are laborious to write rigorously.

We recall the following notation, that will be used through the proof: If $B \subset \mathcal{A}$ is a subset, we denote the negative and positive and trajectories of B by $\gamma^-(B) = \bigcup_{t\leq 0} \mathcal{T}(t)B$ and $\gamma^+(B) = \bigcup_{t\geq 0} \mathcal{T}(t)B$, respectively. We also denote the trajectory of B by $\gamma(B) = \gamma^-(B) \cup \gamma^+(B)$.

Theorem 2.15. *Let $\{\mathcal{T}(t) : t \geq 0\}$ be a Morse-Smale semigroup and $\Omega = \{x_1^*, \cdots, x_p^*\}$ be its non-wandering set. Then, for each $i \in \{1, \ldots, p\}$ there exists $\tilde{C} > 0$, $\lambda_1 \in (0, 1)$, neighborhoods $\mathcal{O}_i$ of x_i^* in $\mathcal{A}$ and subbundles S_i, U_i defined on $\overline{\mathcal{O}_i} \cup \gamma(\mathcal{O}_i)$ that are continuous on $\overline{\mathcal{O}_i}$ and satisfy:*

1. *For all $x \in \gamma(\mathcal{O}_i)$ it holds*

$$(D_x\mathcal{T}(t))S_i(x) \subset S_i(\mathcal{T}(t)x), \forall\ t \geq 0 \quad \text{and} \quad (D_x\mathcal{T}(t))U_i(x) = U_i(\mathcal{T}(t)x), \forall\ t \in \mathbb{R}.$$

2. *$S_i(x_i^*) = \mathcal{S}(x_i^*)$, and $U_i(x_i^*) = \mathcal{U}(x_i^*)$ for all $i \in \{1, \ldots, p\}$.*
3. *$S_j(x) \subset S_k(x)$, $U_k(x) \subset U_j(x)$, for every $x \in \gamma^+(\mathcal{O}_j) \cap \gamma^-(\mathcal{O}_k)$.*
4. *$S_i(x) \oplus U_i(x) = X$, for all $x \in \gamma^-(\mathcal{O}_i)$.*
5. *For every $x \in \mathcal{O}_i$ it holds*

$$\|(D_x\mathcal{T}(t))v^s\| \leq \tilde{C}\lambda_1^t\|v^s\|, \forall\ v^s \in S_i(x),\ \forall\ 0 \leq t \leq 1, \tag{2.21}$$
$$\|(D_x\mathcal{T}(-t))v^u\| \leq \tilde{C}\lambda_1^t\|v^u\|, \forall\ v^u \in U_i(x),\ \forall\ 0 \leq t \leq 1. \tag{2.22}$$

Proof. First of all, note that items (4) and (5) are consequences of items (1), (2), the hyperbolicity of x_i^* and the continuity of the subbundles (see Lemma 2.6). Therefore we will prove only itens (1), (2) and (3). We will first prove the existence of the subbundles U_i. If the derivative $D_x\mathcal{T}(t)$ is an isomorphism for all $x \in \mathcal{A}$ and $t \geq 0$ (like the finite dimensional case), then the proof will follow analogously for S_i. In our case the construction of the family S_i is harder since the derivative is not surjective and we still need to "walk" backwards continuously with the derivative. Hence, we will also provide a proof for the existence of the families S_i.

We will do the proof by induction, assuming that for each $j \in \{i+1, \ldots, p\}$ we have the neighborhoods $\mathcal{O}_j$ and the subbundles U_j on $\gamma(\mathcal{O}_j) \cup \overline{\mathcal{O}_j}$ satisfying itens (1)-(5) and the extra property below

$$\mathcal{S}(x) + U_j(x) = X,\ \forall x \in \gamma^+(\mathcal{O}_j). \tag{2.23}$$

On the finite dimensional case, property (2.23) was justified because the derivative $D_x\mathcal{T}(t)$ was surjective for all $t \geq 0$ and $x \in \mathcal{A}$. In our case, we have to prove this condition using another methods. To maintain the organization of this proof we will proceed as if (2.23) was true and justify later (Lemmas 2.16 and 2.17) why we can assume this condition without lost of generality.

Note that for $j = p$ the neighborhood $\mathcal{O}_p = W^u(x_p^*)$ and the subbundle $U_p(x) = \mathcal{U}(x)$, for all $x \in \gamma(\mathcal{O}_p) = \mathcal{O}_p$, satisfy our hypothesis. This proves our induction basis and therefore we can assume that our induction hypothesis holds for $j \in \{i+1, \ldots, p\}$. We will assume, without lost of generality, that the neighborhood $\mathcal{O}_j$ satisfies

$$\mathcal{O}_j \subset V^{-1}\left(j - \frac{1}{4}, j + \frac{1}{4}\right). \tag{2.24}$$

We can also assume that $\mathcal{O}_j$ is a closed (compact) neighborhood of x_j^* (we are considering the topology on $\mathcal{A}$, which is provided by X). Fix the neighborhood V_i of x_i^* in $\mathcal{A}$ such that

$$V_i \subset V^{-1}\left(i-\frac{1}{4}, i+\frac{1}{4}\right)$$

and we have the continuous subbundles S_i', U_i' on V_i as in (2.5) and (2.6). Take $d'>0$ as on Lemma 2.2 such that

$$W^s(x_i^*)\cap V^{-1}(i+d)\cap\mathcal{A}\subset V_i, \forall d\in[0,d']. \tag{2.25}$$

Fix $d\in(0,d']$ and define once more the set $D=W^s(x_i^*)\cap V^{-1}(i+d)\cap\mathcal{A}$. Note that

$$D\subset\bigcup_{j=i+1}^{p} W^u(x_j^*)\subset\bigcup_{j=i+1}^{p}\gamma^+(\text{int}\mathcal{O}_j)$$

and since D is compact there exists $q>0$ such that

$$D\subset\bigcup_{j=i+1}^{p}\gamma_{[0,q]}(\text{int}\mathcal{O}_j).$$

For each $x\in D$ define $j=j(x)$ such that

$$x\in\gamma_{[0,q]}(\mathcal{O}_j)\ \text{ and }\ x\notin\gamma_{[0,q]}(\mathcal{O}_k),\ \forall i<k<j.$$

This means that the neighborhood $\mathcal{O}_j$ is the last one that x visited (until q). Since the neighborhoods $\mathcal{O}_k$ are compact, there exists a neighborhood $Z(x)$ of x in $\mathcal{A}\cap V^{-1}(i+d)$ such that

$$Z(x)\subset\gamma_{[0,q]}\mathcal{O}_j\ \text{ and }\ Z(x)\cap\gamma_{[0,q]}\mathcal{O}_k=\emptyset,\ \forall i<k<j.$$

Claim: $Z(x)\cap\gamma^+(\mathcal{O}_k)=\emptyset,\ \forall i<k<j$. Indeed, suppose that there exists $y\in Z(x)\subset\gamma_{[0,q]}\mathcal{O}_j$ such that

$$y\in\gamma^+(\mathcal{O}_k)\setminus\gamma_{[0,q]}(\mathcal{O}_k),$$

for some $i<k<j$. Then $y=\mathcal{T}(s)y'$ for some $s>q$ and $y'\in\mathcal{O}_k$ satisfying

$$V(y')<k+\frac{1}{4}<j-\frac{1}{4}. \tag{2.26}$$

The last inequality follows from (2.24). From the decreasing property of the Lyapunov function we have

$$V(\mathcal{T}(t)y')<j-\frac{1}{4},\ \forall\ 0\le t\le s.$$

Hence $y\notin\gamma_{[0,q]}(\mathcal{O}_j)$, which is a contradiction. This proves the claim.

From the continuity of U_j (remember that j was chosen in a way that $x\in\gamma(\mathcal{O}_j)$), the fact that $S_i'(x)=\mathcal{S}(x)$ and from (2.23) there exists continuous subbundles $G^{(x)}$ of $TX|_{Z(x)}$ such that

$$G^{(x)}(y)\subset U_j(y)\ \text{ and }\ S_i'(y)\oplus G^{(x)}(y)=X,\ \forall y\in Z(x). \tag{2.27}$$

It follows from (2.6) and (2.27) that $G^{(x)}(y)$ can be written as the graph of a continuous linear map $g_y^{(x)}:U_i'(y)\to S_i'(y)$ for all $y\in Z(x), x\in D$ (Lemma 4.22). Let Q be a closed neighborhood of D in $V^{-1}(i+d)\cap\mathcal{A}$ and $x_1,\dots,x_n\in D$ such that $Q\subset\bigcup_{k=1}^{n}Z(x_k)$. Take a smooth partition of unity $\{\beta_k:Q\to\mathbb{R}\}_{k=1}^n$ subordinated to the covering $\{Z(x_k)\}_{k=1}^n$. Define the continuous map $g^Q:Q\to\mathcal{L}(X)$ as

$$g^Q(y) = \sum_{k=1}^{n} \beta_k(y) g_y^{(x_k)} P_{U_i'(y)}, \ \forall y \in Q. \tag{2.28}$$

Denoting by $G^Q(y)$ the graph of $g^Q(y)|_{U_i'(y)}$ for all $y \in Q$, we obtain that $G^Q := \{G^Q(y)\}_{y\in Q}$ is a continuous subbundle on Q (Lemma 4.23). We want to show that

$$G^Q(y) \subset U_{k'}(y), \ \forall y \in Q \cap \gamma^+(\mathcal{O}_k'), \ \forall i < k' \leq p. \tag{2.29}$$

Given $v \in U_i'(y)$ we have

$$v + g^Q(y)v = \sum_{k=1}^{n} \beta_k(y)(v + g_y^{(x_k)} v) \in U_j(y).$$

We know from our claim that $j \leq k'$ and therefore it follows from (2.27) and our induction hypothesis that

$$v + g^Q(y)v \in U_j(y) \subset U_{k'}(y),$$

which proves (2.29).

Now we extend our subbundle G^Q to a subbundle G_i on $\gamma(Q) \cup W^u(x_i^*)$ as

$$\begin{aligned} G_i(\mathcal{T}(t)x) &:= (D_x\mathcal{T}(t))G^Q(x), \ \forall x \in Q, \ \forall t \in \mathbb{R}, \\ G_i(x) &:= \mathcal{U}(x), \ \forall x \in W^u(x_i^*). \end{aligned}$$

Note that G^Q already satisfies itens (1), (2) and (3). If we reduce Q we obtain the continuity of the subbundle by Lemma 2.11. From Lemma 2.1 the set $\gamma(Q) \cup W^u(x_i^*)$ contains a neighborhood $\mathcal{O}_i$ of x_i^* that, together with the subbundle $U_i(x) := G_i(x)$ for all $x \in \gamma(\mathcal{O}_i) \cup \overline{\mathcal{O}_i}$, satisfy every condition from our thesis. This finishes the proof for the families U_i.

Before we proceed the proof for subbundles S_i, note that since U_i is continuous in $\mathcal{O}_i$ and $\mathcal{S}$ is lower semicontinuous (Lemma 2.14), we can use Remark 11 to guarantee that $\mathcal{S}(x)+U_i(x) = X$ for all $x \in \mathcal{O}_i$, reducing the neighborhood if it's necessary. Therefore, if the derivative of the semigroup is surjective it follows that

$$\mathcal{S}(x) + U_i(x) = X, \quad \forall x \in \gamma^+(\mathcal{O}_i)$$

and condition (2.23) is satisfied. This would finish the proof if X was finite dimensional or if $\mathcal{T}(t)$ was a diffeomorphism. Since this is not the case, we have to justify later why we can assume (2.23).

Now we proceed to construct the family S_i. Since the proof will follow the same reasoning as before, we will provide only the essential parts. Once more we will prove by induction. Note that the neighborhood $\mathcal{O}_1 = W^s(x_1^*)$ and the family $S_1(x) = X$ for all $x \in W^s(x_1^*)$ satisfy the properties required. Lets construct S_i and $\mathcal{O}_i$ assuming that S_j and $\mathcal{O}_j$ are construct for $j \in \{1, \ldots, i-1\}$, where S_j is continuous on $\overline{\mathcal{O}_j}$. We can proceed as before until the change of coordinates (2.27). Indeed, instead of doing this change of coordinates directly, we have to adapt the proof because the inverse of the derivative is not surjective. Indeed, the case U_i works directly because the subbundle $U_{j(x)}$ is continuous on $\gamma^+(\mathcal{O}_{j(x)})$ but for the familie $S_{j(x)}$ we do not have continuity on $\gamma^-(\mathcal{O}_{j(x)})$.

Arguing as in the proof of Proposition 2.4, we recall that for each $x \in \mathcal{A}$ and $t > 0$ we have

$$\mathcal{U}(x) = (D_{\mathcal{T}(-t)x}\mathcal{T}(t))\mathcal{U}(\mathcal{T}(-t)x),$$

that is,

$$\mathcal{U}(x) \subset R(D_{\mathcal{T}(-t)x}\mathcal{T}(t)), \ \forall t \geq 0, \ \forall x \in \mathcal{A}. \tag{2.30}$$

Let us remember that in this case we have

$$D = V^{-1}(i-d) \cap W^u(x_i^*)$$

and $U_i'(x) = \mathcal{U}(x)$ for each $x \in V_i \cap W^u(x_i^*)$. Fix $x \in D$ and $t_0 > 0$ such that $\mathcal{T}(t_0)x \in \mathcal{O}_{j(x)}$. Since the family $S_{j(x)}$ is continuous on $\mathcal{O}_{j(x)}$ and $\mathcal{U}$ is lower semicontinuous, we can use (2.23) to fix a neighborhood $Z(\mathcal{T}(t_0)x) \ni \mathcal{T}(t_0)x$ such that

$$S_{j(x)}(y) + \mathcal{U}(y) = X, \ \forall y \in Z(\mathcal{T}(t_0)x)$$

and

$$\dim(S_{j(x)}(y))^{\perp} = m < +\infty, \quad \forall y \in Z(\mathcal{T}(t_0)x).$$

Hence, from (2.30) we have

$$S_{j(x)}(y) + R(D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0)) = X, \quad \forall y \in Z(\mathcal{T}(t_0)x)$$

Therefore, we can use Proposition 2.12 to guarantee that the subbundle

$$\{L(y) := S_{j(x)}(y) \cap R(D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0))\}_{y \in Z(\mathcal{T}(t_0)x)}$$

is continuous. Now we can use Lemma 2.9 to define a continuous subbundle G_x' on $Z(x) := \mathcal{T}(-t_0)Z(\mathcal{T}(t_0)x)$ as

$$G_x'(\mathcal{T}(-t_0)y) := (D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0))^{-1}L(y), \ \forall y \in Z(\mathcal{T}(t_0)x).$$

Since $\mathcal{U}(y) \subset R(D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0))$ we have

$$L(y) + \mathcal{U}(y) = (S_j(y) + \mathcal{U}(y)) \cap R(D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0)) = R(D_{\mathcal{T}(-t_0)y}\mathcal{T}(t_0)), \ \forall y \in Z(\mathcal{T}(t_0)x)$$

and from Lemma 4.19

$$G_x'(y) + \mathcal{U}(y) = X, \ \forall y \in Z(x). \tag{2.31}$$

Therefore, we can obtain a continuous subbundle $\{G^{(x)}(y)\}_{y \in Z(x)}$ on $Z(x)$ (possibly reducing this neighborhood) satisfying

$$G^{(x)}(y) \subset S_j(y) \quad \text{and} \quad G^{(x)}(y) \oplus U_i'(y) = X, \quad \forall y \in Z(x).$$

After that we can proceed as before to construct the continuous subbundle G^Q on Q. After that we will extend this subbundle to $\gamma^-(Q)$ as usual. We will denote this extension by G_i, that is,

$$G_i(\mathcal{T}(-t)x) = (D_{\mathcal{T}(-t)x}\mathcal{T}(t))^{-1}G^Q(x), \ \forall x \in Q.$$

We know that G^Q is continuous on Q and $G^Q(y) + \mathcal{U}(y) = X$ for all $y \in Q$ (remember that $\mathcal{U}$ is lower semicontinuous). Hence $G^Q(y) + R(D_{\mathcal{T}(-t)y}\mathcal{T}(t)) = X$ and we can proceed as before to conclude that $\{G_i(y)\}_{y \in \gamma^-(Q)}$ is continuous. We now proceed as the other case to finish the proof. □

Remark 12. *Since the equilibria x_i^* are hyperbolic for each $i = 1, \dots, p$ then $U_0(x_i^*)$ (Definition 20) is finite dimensional. Consequently, the subbundles U_i constructed in Theorem 2.15 are also finite dimensional, for each $i = 1, \dots, p$ (see Remark 8). Moreover, from the construction made in Theorem 2.15 and from (2.30) each set $U_i(x)$ is contained in the range of $D_{\mathcal{T}(-t)x}\mathcal{T}(t)$. Therefore, we can use Lemmas 2.8 and 2.9 to guarantee that the subbundles U_i are continuous in $\gamma(\mathcal{O}_i) \cup \overline{\mathcal{O}_i}$, for each $i = 1, \dots, p$. This is not the case for the subbundles S_i, since they $S_i(x) = T_xW^s(x_i^*)$ for $x \in W^s_{loc}(x_i^*) \cap \mathcal{O}_i$ and we do not know if $T_xW^s(x_i^*) \subset R(D_{\mathcal{T}(-t)x}\mathcal{T}(t))$.*

The following lemmas justify why we can assume condition (2.23) without lost of generality.

Lemma 2.16. *Let $\mathcal{T}$ be a Gradient semigroup on X (in this Lemma X can be a metric space with metric d) with equilibria $\Omega = \{x_1^*, \dots, x_p^*\}$. Given $i \in \{1, \dots, p\}$, $k < i$, $d \in (0, 1)$ and $\epsilon, \epsilon' > 0$ there exists $\delta > 0$ such that if $x \in B(x_i^*, \delta) \cap \mathcal{A}$ satisfies*

1. There exists $t > 0$ such that $\mathcal{T}(t)x \in V^{-1}(k+d) \cap W^s(x_k^*)$.

2.

$$d(\mathcal{T}(t)x, x_j^*) \geq \epsilon', \ \forall t \in \mathbb{R}, \ \forall k < j < i. \tag{2.32}$$

Then, there exists a bounded global solution ξ, with $\xi(0) \in W^u(x_i^*)$, such that

$$d(\mathcal{T}(t)x, \xi(0)) < \epsilon.$$

Moreover, $d(\xi(t), x_j^*) \geq \epsilon', \ \forall t \in \mathbb{R}, \ \forall k < j < i.$

Proof. We will prove by contradiction. Suppose that there exists $\epsilon, \epsilon' > 0$, $\mathcal{A} \ni x_n \to x_i^*$ and $t_n \to +\infty$ such that

$$\mathcal{T}(t_n)x_n \in V^{-1}(k+d) \cap W^s(x_k^*),$$
$$d(\mathcal{T}(t)x_n, x_j^*) \geq \epsilon', \ \forall t \in \mathbb{R}, \ \forall k < j < i, \ \forall n \in \mathbb{N} \tag{2.33}$$

and

$$d(\mathcal{T}(t_n)x_n, W^u(x_i^*)) \geq \epsilon. \tag{2.34}$$

We can also assume without lost of generality that

$$i - \frac{1}{2} < V(x_n) < i + \frac{1}{2}. \tag{2.35}$$

For each n take $s_n > 0$ such that $V(\mathcal{T}(s_n)x_n) = i - \frac{1}{2}$. Despite that $s_n \to +\infty$ we will show that $t_n - s_n$ is bounded. Indeed, if $t_n - s_n \to +\infty$, then we can define a sequence of global solutions as below

$$\xi_n(t) = \mathcal{T}\left(t + \frac{t_n + s_n}{2}\right) x_n$$

which satisfies

$$k + d \leq V(\xi_n(t)) \leq i - \frac{1}{2}, \ \forall t \in \left[\frac{s_n - t_n}{2}, \frac{t_n - s_n}{2}\right].$$

Hence, we can use Proposition 1.2 and (considering a subsequence) there exists a bounded global solution $\xi(\cdot)$ such that

$$\xi_n(t) \to \xi(t) \text{ as } n \to +\infty, \ \forall t \in \mathbb{R} \tag{2.36}$$

and $k + d \leq V(\xi(t)) \leq i - \frac{1}{2}, \ \forall t \in \mathbb{R}$. Thus $\xi(0) \in W^s(x_j^*)$, with $k < j < i$ which contradicts (2.33). Thus $t_n - s_n$ is bounded, that is, there exists $M < 0$ such that

$$M \leq s_n - t_n, \ \forall n \in \mathbb{N}. \tag{2.37}$$

For each n define the global solution

$$\xi_n(t) = \mathcal{T}(t + t_n)x_n, \ \forall t \in \mathbb{R}$$

and (considering a subsequence) take $\xi(\cdot)$ a global solution such that

$$\xi_n(t) \to \xi(t), \text{ as } n \to +\infty, \ \forall t \in \mathbb{R}. \tag{2.38}$$

It follows from (2.35) that

$$i - \frac{1}{2} \leq V(\xi_n(t)) \leq i + \frac{1}{2}, \ \forall t \leq [-t_n, s_n - t_n], \ \forall n \in \mathbb{N}. \tag{2.39}$$

Putting (2.37) and (2.39) together we have (for n large satisfying $-t_n < M$)

$$i - \frac{1}{2} \leq V(\xi_n(t)) \leq i + \frac{1}{2}, \ \forall t \leq [-t_n, M], \ \forall n \in \mathbb{N}. \tag{2.40}$$

From (2.38) and (2.40) we conclude that

$$i - \frac{1}{2} \leq V(\xi(t)) \leq i + \frac{1}{2}, \ \forall t \in (-\infty, M).$$

Therefore

$$\xi_n(0) = \mathcal{T}(t_n)x_n \to \xi(0) \in W^u(x_i^*),$$

which contradicts (2.34). This concludes the proof. □

Lemma 2.17. *In the conditions of Lemma 2.15, we can reduce $\mathcal{O}_{i+1}, \ldots, \mathcal{O}_p$ such that if $x \in D$ and $x \in \gamma^+(\mathcal{O}_j)$, where $j = j(x)$ is defined as*

$$j(x) := \min\{i < k \leq p : x \in \gamma^+(\mathcal{O}_k)\},$$

then $\mathcal{S}(x) + U_j(x) = X$.

Proof. First we prove three auxiliary facts:

1. Fix d and $\mathcal{O}_i$ as on the proof of Lemma 2.15 and consider the set

$$D = V^{-1}(i + d) \cap W^s(x_i^*) \cap \mathcal{A}.$$

For each $x \in D$ fix $r(x)$ such that

$$y \in \mathcal{A}, d(y, x) < r(x) \Rightarrow V(y) > i + \frac{d}{2} \tag{2.41}$$

From the compactness of D there exists $\epsilon_1 > 0$ such that

$$\mathcal{O}_{\epsilon_1}(D) \subset \bigcup_{x \in D} B(x, r(x)), \text{ where the } \epsilon_1 - \text{neighborhood is taken in } \mathcal{A}. \tag{2.42}$$

2. Define the set $\mathcal{C} = W^u(x_{i+1}^*) \cap V^{-1}\left[i + \frac{d}{2}, i + 1 - \frac{d}{2}\right]$. We claim that $\mathcal{C}$ is compact. Indeed, assume that $\mathcal{C} \ni x_n \to x$ and $x \in W^u(x_j^*)$, for some $j > i + 1$. Then there exists $s < 0$ such that $V(\mathcal{T}(s)x) > i + 1 + \frac{1}{2}$ and $V(\mathcal{T}(s)x_n) > i + 1 + \frac{1}{2}$ for large n, which is a contradiction. Hence $\mathcal{C}$ is compact. Since

$$\mathcal{S}(x) + U_{i+1}(x) = \mathcal{S}(x) + \mathcal{U}(x) = X, \ \forall x \in \mathcal{C},$$

it follows from Lemma 2.14 and Remark 11 that for each $x \in \mathcal{C}$ there exists $\epsilon(x)$ such that

$$\mathcal{S}(y) + U_{i+1}(y) = X, \ \forall y \in B(x, \epsilon(x)). \tag{2.43}$$

From the compactness of $\mathcal{C}$ there exists $\epsilon_2 > 0$ such that

$$\mathcal{O}_{\epsilon_2}(\mathcal{C}) \subset \bigcup_{x \in \mathcal{C}} B(x, \epsilon(x)). \tag{2.44}$$

3. Given $\epsilon' > 0$ and $j > i$ define the set

$$\mathcal{H}_j := \{x \in W^u(x_j^*) : d(\gamma(x), x_k^*) \geq \epsilon', i < k < j\}.$$

We claim that the set

$$\mathcal{J}_j := \mathcal{H}_j \cap V^{-1}\left[i+\frac{d}{2}, j-\frac{d}{2}\right]$$

is compact. Indeed, let $\mathcal{J}_j \ni x_n \to x$. Take bounded global solutions ξ_n such that $\xi_n(0) = x_n$. We know from Proposition 1.2 that there exists a subsequence of ξ_n and a bounded global solution ξ such that

$$\xi_n(t) \to \xi(t), \text{ as } n \to +\infty, \ \forall t \in \mathbb{R}. \tag{2.45}$$

Note that $V(\xi(t)) \leq j$ for all $t \in \mathbb{R}$ and $V(x) \in \left[i+\frac{d}{2}, j-\frac{d}{2}\right]$, that is,

$$\xi(t) \to x_k^*, \text{ as } t \to -\infty, \text{ for some } i < k \leq j. \tag{2.46}$$

Since $d(\xi_n(t), x_k^*) \geq \epsilon'$ for all $i < k < j$ we conclude that $\xi(t) \to x_j^*$ as $t \to -\infty$, and $d(\xi(t), x_k^*) \geq \epsilon'$, that is, $x \in \mathcal{H}_j$. Obviously $x \in V^{-1}\left[i+\frac{d}{2}, j-\frac{d}{2}\right]$, which concludes the proof.

Now we are ready to prove our statement by induction.

1. For $j = i+1$: From Lemma 2.16 we can reduce $\mathcal{O}_{i+1}$ such that if $x \in \mathcal{O}_{i+1}$ and $\mathcal{T}(t)x \in W^s(x_i^*) \cap V^{-1}(i+d)$ then $d(\mathcal{T}(t)x, W^u(x_{i+1}^*)) < \frac{\epsilon}{2}$, where $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Hence, if $x \in D$ and $j(x) = i+1$, there exists $\tilde{x} \in W^u(x_{i+1}^*)$ with $d(x, \tilde{x}) < \epsilon$. Moreover, from (2.41) and (2.42) we know that $V(\tilde{x}) > i + \frac{d}{2}$, that is,
$$d(x, \mathcal{C}) < \epsilon \tag{2.47}$$
Finally, it follows from (2.43), (2.44) and (2.47) that $\mathcal{S}(x) + U_{i+1}(x) = X$.
2. Assume that for $i+1 \leq r < j$ it is valid that if $x \in \gamma^+(\mathcal{O}_r) \cap D$ and $x \notin \gamma^+(\mathcal{O}_k)$, $i < k < r$, then
$$\mathcal{S}(x) + U_r(x) = X.$$
Fix $\epsilon' > 0$ such that
$$B(x_k^*, \epsilon') \subset \mathcal{O}_k, i < k < j.$$
With that ϵ', consider the set $\mathcal{J}_j$ as on item 3. From the compactness of $\mathcal{J}_j$ we can argue as item 2 to guarantee the existence of an $\epsilon_3 > 0$ such that
$$x \in \mathcal{O}_{\epsilon_3}(\mathcal{J}_j) \Rightarrow \mathcal{S}(x) + U_j(x) = X. \tag{2.48}$$
From Lemma 2.16 we can reduce $\mathcal{O}_j$ such that if $x \in \mathcal{O}_j$,
$$d(\mathcal{T}(t)x, x_k^*) \geq \epsilon', \ \forall t \in \mathbb{R}, i < k < j$$
and $\mathcal{T}(t)x \in D$, then
$$d(\mathcal{T}(t)x, W^u(x_j^*)) < \frac{\epsilon}{2},$$
where $\epsilon = \min\{\epsilon_1, \epsilon_3\}$. Thus, if $x \in D$ and $j(x) = j$, there exists $\tilde{x} \in W^u(x_j^*)$ such that $d(x, \tilde{x}) < \epsilon$. Moreover, from (2.41) and (2.42) we know that $V(\tilde{x}) > i + \frac{d}{2}$, that is,
$$d(x, \mathcal{J}_j) < \epsilon \tag{2.49}$$
Finally, it follows from (2.48) that $\mathcal{S}(x) + U_j(x) = X$.

□

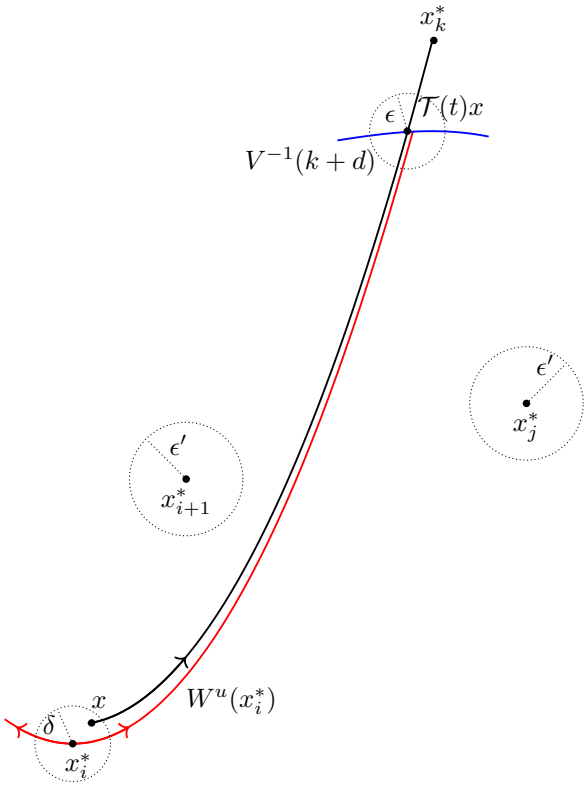


Figure 2.4: Dynamics of Lemma 2.16

Now that we have the subbundles S_i and U_i given in Lemma 2.15, we just need some auxiliary tools. In fact, Lemmas 2.18 and 2.19 will allow us to prove Lipschitz Shadowing for the operator $\mathcal{T}^N$, for some $N \in \mathbb{N}$, instead of the operator $\mathcal{T}$. Both results can be found in [47].

Lemma 2.18. *Let (M,d) be a metric space, $\mathcal{T} \in \mathcal{C}(X)$ be a Lipschitz map with Lipschitz constant L_1 and $\{x_n\}_{n\in\mathbb{T}}$ be a d-pseudo-orbit of $\mathcal{T}$. Then, for any $m, r \in \mathbb{N}$ we have*

$$d(\mathcal{T}^m(x_{rk}), x_{rk+m}) \leq \left(\frac{L_1^m - 1}{L_1 - 1}\right) d, \forall\, k \in \mathbb{T}. \tag{2.50}$$

In particular, if $z_n = x_{nN}$, for all $n \in \mathbb{T}$, the sequence $\{z_n\}_{n\in\mathbb{T}}$ is a $C_1 d$-pseudo-orbit of $\mathcal{T}^N$, with $C_1 = \dfrac{L_1^N - 1}{L_1 - 1}$, for $L_1 \neq 1$, and a $C_1 = Nd$ for $L_1 = 1$.

Proof. Fix $r \in \mathbb{N}$ arbitrary. Note that for $m = 0$ and $m = 1$ (2.50) is obvious. Suppose that (2.50) is valid for some $m \in \mathbb{N}$.

Then, for any $d-$pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ it holds

$$\begin{aligned} d(\mathcal{T}^{m+1}(x_{rk}), x_{rk+m+1}) &\leq d(\mathcal{T}^{m+1}(x_{rk}), \mathcal{T}x_{rk+m}) + d(\mathcal{T}x_{rk+m}, x_{rk+m+1}) \,\| \\ &\leq L_1 d(\mathcal{T}^m(x_{rk}), x_{rk+m}) + d \\ &\leq L_1 \left(\frac{L_1^m - 1}{L_1 - 1}\right) d + d \\ &= \left(\frac{L_1^{m+1} - 1}{L_1 - 1}\right) d, \ \forall\, k \in \mathbb{T}. \end{aligned}$$

□

Lemma 2.19. *Let (M,d) be a metric space, $\mathcal{T}\in\mathcal{C}(X)$ be a Lipschitz map with Lipschitz constant L_1 and $\{x_n\}_{n\in\mathbb{T}}$ be a d-pseudo-orbit of $\mathcal{T}$. Given $N\in\mathbb{N}$ define the sequence $z_n=x_{nN}$, for all $n\in\mathbb{T}$ and $\phi=\mathcal{T}^N$. Assume that there exists $L>1$ and $x\in X$ such that*

$$\| \phi^k x - z_k \| \leq Ld,\ \forall k\in\mathbb{T}. \tag{2.51}$$

Then we have

$$\| \mathcal{T}^k x - x_k \| \leq L^* d,\ \forall k\in\mathbb{T}, \tag{2.52}$$

where $L^*=(1+L_1+\cdots+L_1^N)L$ is independent of d.

Proof. Fix $N\in\mathbb{N}$. It is sufficient to show that for each $r\in\{0,\dots,N-1\}$ we have

$$d(\mathcal{T}^{Nk+r}x, x_{Nk+r}) \leq (1+L_1+\cdots+L_1^{r-1}+(L_1)^r)Ld,\ k\in\mathbb{T}. \tag{2.53}$$

It is clear that (2.53) holds for $r=0$. Now we assume that (2.53) holds for $r-1\in\{0,\dots,N-2\}$ in order to show this property for r. Indeed, since $L>1$ we have

$$\begin{aligned} d(\mathcal{T}^{Nk+r}x, x_{Nk+r}) &\leq d(\mathcal{T}^{Nk+r}x, \mathcal{T}x_{Nk+r-1}) + d(\mathcal{T}x_{Nk+r-1}, x_{Nk+r}) \\ &\leq L_1 d(\mathcal{T}^{Nk+r-1}x, x_{Nk+r-1}) + d \\ &\leq L_1(1+L_1+\cdots+L_1^{r-1})Ld + d \\ &\leq (1+L_1+\cdots+L_1^r)Ld \\ &\leq L'd. \end{aligned}$$

Since $L^*=L_1L+\cdots+L_1^NL\geq 1+L_1+\cdots+L_1^rL$ for all $r\in\{0,\cdots,N-1\}$, we obtain (2.52) and the proof is complete. □

With Lemmas 2.18 and 2.19 in hand, we can now manipulate our semigroup $\mathcal{T}$ in a way that item 5 of Theorem 2.15 becomes more powerful. To put more details, from item 5 we can deduce that if $x\in\mathcal{O}_i$ and $\gamma_{[0,t]}(x)\subset\mathcal{O}_i$ for some $t=n+r$, where $n\in\mathbb{N}$ and $r\in[0,1)$ then

$$\| (D_x\mathcal{T}(t)x)v^s \| \leq \tilde{C}^{n+1}\lambda_1^t,\ \forall v^s\in S_i(x).$$

Unfortunately, we don't know if $\tilde{C}^{n+1}\lambda_1^t$ goes to zero as $t\to+\infty$. The next remark states that, with some algebraic manipulation, we can obtain this information.

Remark 13. *We can assume from item (5) of Theorem 2.15 that there exists $\tilde{C}\geq 1$, $\lambda_0\in(0,1)$ such that if $t>0$ and $\mathcal{T}(s)x\in\mathcal{O}_i$ for all $s\in[0,t]$, then*

$$\| (D_x\mathcal{T}(t))v^s \| \leq \tilde{C}\lambda_0^t \| v^s \|,\ \forall v^s\in S_i(x). \tag{2.54}$$

Similarly, if $\mathcal{T}(s)x\in\mathcal{O}_i$ for all $s\in[-t,0]$ then

$$\| (D_x\mathcal{T}(-t))v^u \| \leq \tilde{C}\lambda_0^t \| v^u \|,\ \forall v^u\in U_i(x) \tag{2.55}$$

Indeed, fix $N\in\mathbb{N}$ such that $\lambda_0:=\tilde{C}\lambda_1^N<1$ and reduce the neighborhood $\mathcal{O}_i$ such that the inequalities 2.21 and 2.22 holds for $t\in[0,N]$ (we can reduce the neighborhood keeping the same $\lambda_1\in(0,1)$ and $\tilde{C}>0$). Consider the semigroup $\mathcal{T}_N$ (see Lemmas 1.18, 1.19 and Corollary 1.20). Take $x\in\mathcal{O}_i$ and $t>0$ satisfying $\mathcal{T}_N(s)x\in\mathcal{O}_i$ for all $s\in[0,t]$. Take $n\in\mathbb{N}$ and $r\in[0,1)$ such that $t=n+r$. Assuming without lost of generality that $\tilde{C}\geq 1$, we have

$$\begin{aligned} \| (D_x\mathcal{T}_N(t))v^s \| &= \| (D_x\mathcal{T}_N(n+r))v^s \| = \| (D_{\mathcal{T}_N(n)x}\mathcal{T}_N(r))\circ(D_x\mathcal{T}_N(n))v^s \| \\ &\leq \tilde{C}\lambda_1^{Nr} \| (D_x\mathcal{T}_N(n))v^s \| \leq \tilde{C}\lambda_1^{Nr}\lambda_0^n \| v^s \| \\ &= \tilde{C}^{1-r}\lambda_0^{n+r} \| v^s \| \leq \tilde{C}\lambda_0^t \| v^s \|,\ \forall v^s\in S_i(x). \end{aligned}$$

We can prove (2.55) analogously. Hence, properties 2.54 and 2.55 are valid for the semigroup $\mathcal{T}_N$. In fact, Lemmas 1.18, 1.19 and Corollary 1.20 show that we can work with the semigroup $\mathcal{T}_N$ instead of $\mathcal{T}$.

2.1.3 The Compatible Subbundles S and U

In this subsection we use the subbundles S_i and U_i of Subsection 2.1.2 to define the compatible subbundles S and U, independent of index i, on the global attractor $\mathcal{A}$. Furthermore, we will prove a lot of properties of the compatible subbundle, such as positively (negatively for U) invariance in relation to the derivative $D_x\mathcal{T}$ and boundness of the norms of the projections related to the direct sum $S(x) \oplus U(x) = X$. Summarizing, this subsection extract all properties of S and U that will be required later, on Section 2.2, to prove Lipschitz Shadowing on $\mathcal{A}$. We invite the reader to look Theorem 2.27 from Section 2.2, which is the Theorem that will connect the compatible subbundles with Lipschitz Shadowing. In fact, the properties that we are looking for in this subsection are basically the conditions present on the hypothesis of Theorem 2.27.

Now we proceed to construct the compatible subbundles S, U (independent of i) on $\mathcal{A}$ that will be the main key to prove shadowing on the attractor. In fact, using Theorem 2.15 the construction of these subbundles are very simple, as it follows: Fix neighborhoods $\mathcal{O}_i$ of x_i^* in $\mathcal{A}$ and continuous subbundles S_i, U_i on $\mathcal{O}_i$ as in Theorem 2.15. Define $W = \bigcup_{i=i}^{p} \mathcal{O}_i$ and take $T > 0$ the Birkhoff number related to W as in Lemma 1.4. Hence for each $x \in \mathcal{A}$ there exists $t \in [0, T]$ such that $\mathcal{T}(t)x \in W$.Take $t = t(x) \in [0, T]$ and $i \in \{1, \dots, p\}$ such that

$$\mathcal{T}(t)x \in \mathcal{O}_i \quad \text{and} \quad \mathcal{T}(s)x \cap \mathcal{O}_j = \emptyset, \ \forall s \in [0, t), \ \forall j \in \{1, \dots, p\}. \tag{2.56}$$

Define $S(x) = S_i(x)$ and $U(x) = U_i(x)$. Note that $S(x) \oplus U(x) = X$ for all $x \in \mathcal{A}$ and consequently we can define the projections $P_s(x) := P_{S(x)U(x)}$ and $P_u(x) := Id - P_s(x) = P_{U(x)S(x)}$ as for all $x \in \mathcal{A}$ (see the notation defined in Remark 9). We already know by Theorem 2.15 that the families of orthogonal projections $\{P_{S_i(x)} = P_{S(x)}\}_{x \in \mathcal{O}_i}$ and $\{P_{U_i(x)} = P_{U(x)}\}_{x \in \mathcal{O}_i}$ are continuous on $\mathcal{O}_i$, but its not obvious that the family $P_s(x)$ and $P_u(x)$ are continuous on $\mathcal{O}_i$. The next results are dedicated to show that we actually have such continuity (see Proposition 2.21).

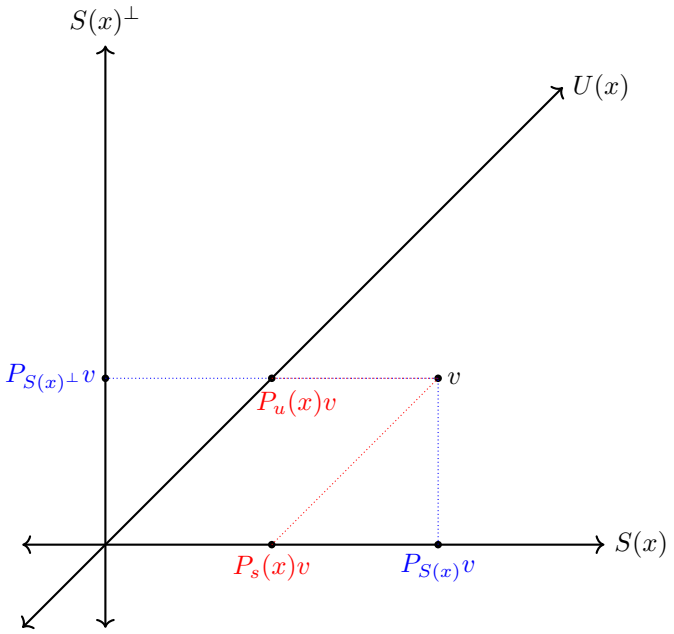


Figure 2.5: Projections $P_s(x)$ and $P_{S(x)}$

Lemma 2.20. *Let (M, d) be a metric space, X be a Banach space and $\{T(x) : x \in M\} \subset \mathcal{L}(X)$ be a family of bounded linear maps satisfying*

$$\| T(x) \|_{\mathcal{L}(X)} < 1, \ \forall x \in M.$$

If the map $x \mapsto T(x)$ is continuous (in $\mathcal{L}(X)$), then $x \mapsto (I - T(x))^{-1}$ is also continuous, where $I \in \mathcal{L}(X)$ is the identity.

Proof. First we will show that for any $x_0 \in M$ there exists a neighborhood $V \ni x_0$ and $K > 0$ such that

$$\| (I - T(x))^{-1} \|_{\mathcal{L}(X)} \leq K, \ \forall x \in V. \tag{2.57}$$

Indeed, note that

$$I - T(x) = (I - T(x_0))[I - (I - T(x_0))^{-1}(T(x) - T(x_0))].$$

Fixing a neighborhood W of x_0 with the property

$$\| (I - T(x_0))^{-1} \|_{\mathcal{L}(X)} \| T(x) - T(x_0) \|_{\mathcal{L}(X)} < 1, \ \forall x \in W,$$

we guarantee that

$$(I - T(x))^{-1} = \left(\sum_{n=0}^{\infty} (I - T(x_0))^{-n} (T(x) - T(x_0))^n \right) \circ (I - T(x_0))^{-1}, \ \forall x \in W. \tag{2.58}$$

Taking $V \subset W$ such that

$$\| (I - T(x_0))^{-1} \|_{\mathcal{L}(X)} \| T(x) - T(x_0) \|_{\mathcal{L}(X)} < \frac{1}{2}, \ \forall x \in V,$$

the series in (2.58) converges uniformly in V, that is, we can take $K = 2 \| (I - T(x_0))^{-1} \|$ to satisfy the condition

$$\| (I - T(x))^{-1} \|_{\mathcal{L}(X)} \leq K, \ \forall x \in V.$$

Now we prove the desired continuity. Note that from (2.57) it is sufficient to show the equality

$$(I - T(x))^{-1} - (I - T(x_0))^{-1} = (I - T(x))^{-1}(T(x) - T(x_0))(I - T(x_0))^{-1}. \tag{2.59}$$

In fact, we have

$$\begin{aligned} (I - T(x))^{-1} &= (I - T(x))^{-1}(I - T(x_0))(I - T(x_0))^{-1} \\ &= (I - T(x))^{-1}[(I - T(x)) + (T(x) - T(x_0))](I - T(x_0))^{-1} \\ &= (I - T(x_0))^{-1} + (I - T(x))^{-1}(T(x) - T(x_0))(I - T(x_0))^{-1}, \end{aligned}$$

which concludes the proof. □

Proposition 2.21. *Let (M, d) be a metric space, X be a Hilbert space and $\{S(x)\}_{x \in M}$, $\{U(x)\}_{x \in M}$ be two continuous subbundles in M, that is, the families of orthogonal projections $\{P_{S(x)} : x \in M\}$ and $\{P_{U(x)} : x \in M\}$ (in the notation of Remark 9) are continuous in $\mathcal{L}(X)$. Assume that $S(x) \oplus U(x) = X$, for all $x \in M$. Then, the family of projections $\{P_{SU}(x) : x \in M\}$ is continuous in $\mathcal{L}(X)$, where $P_{SU}(x) := P_{S(x)U(x)}$. Consequently, the family $\{P_{US}(x) : x \in M\}$ is continuous in $\mathcal{L}(X)$, since $P_{US}(x) = I - P_{SU}(x)$.*

Proof. First we will prove that following equality

$$P_{SU}(x) = (I - P_{S(x)}P_{U(x)})^{-1} \circ P_{S(x)} \circ (I - P_{S(x)}P_{U(x)}). \tag{2.60}$$

Note that from Lemma 4.20 the map $(I - P_{S(x)}P_{U(x)})^{-1}$ is well defined. To prove that (2.60) is valid, we will show that this equality holds in $S(x)$ and in $U(x)$. Indeed, if $v \in S(x)$ we have

$$\begin{aligned}[(I - P_{S(x)}P_{U(x)}) \circ P_{SU}(x)]v &= (I - P_{S(x)}P_{U(x)})v \\ &= v - P_{S(x)}P_{U(x)}v \\ &= P_{S(x)}v - P_{S(x)}P_{U(x)}v \\ &= P_{S(x)}(I - P_{S(x)}P_{U(x)})v.\end{aligned}$$

Hence (2.60) holds in $S(x)$. On other hand, if $v \in U(x)$ we have

$$\begin{aligned}[(I - P_{S(x)}P_{U(x)}) \circ P_{SU}(x)]v &= 0 \\ &= v - P_{U(x)}v \\ &= P_{S(x)}v - P_{S(x)}P_{U(x)}v \\ &= P_{S(x)} \circ (I - P_{S(x)}P_{U(x)})v\end{aligned}$$

Thus (2.60) also holds in $U(x)$, implying that holds in X. Therefore, the result follows from (2.60) and Lemma 2.20. □

Now we just have to prove two more properties for the subbundles S, U, estated in Lemmas 2.22 and 2.26, before we finish our section and conclude everything necessary to move to the next step. We start by proving the positively invariance for the subbundle S (and negatively invariance for U) and some kind of roubstness for that property.

Lemma 2.22. *Consider the subbundles S, U constructed before and take $x \in \mathcal{A}$. If $y = \mathcal{T}(t)x$, the following statement holds:*

1. *If $x \in \mathcal{A}$, then $(D_x\mathcal{T}(t))S(x) \subset S(y),\ \forall t \geq 0$ and $(D_x\mathcal{T}(t))U(x) \subset U(y),\ \forall t \leq 0$.*
2. *If $y_n \to y$, then there exists $n_0 > 0$ and isomorphisms $\Pi_n : X \to X$ for $n \geq n_0$, with $\| \Pi_n - Id \| \to 0$ such that*

$$\begin{aligned}\Pi_n((D_x\mathcal{T}(t))S(x)) \subset S(y_n), \forall t \geq T,\ \forall n \geq n_0, \\ \Pi_n((D_x\mathcal{T}(t))U(x)) \subset U(y_n), \forall t \leq -T,\ \forall n \geq n_0.\end{aligned}$$

Proof. We will divide the proof by the respective itens:

1. We start by proving the first item to S and for U will follows analogously. Let $x \in \mathcal{A}$, $t > 0$ and $i \in \{1, \dots, p\}$ such that $S(x) = S_i(x)$. Since $t > 0$ it follows from our construction that $S(y) = S_j(y)$ for some $j \leq i$ and from item (3) of Theorem 2.15 we conclude that

$$(D_x\mathcal{T}(t))S(x) = (D_x\mathcal{T}(t))S_i(x) \subset (D_x\mathcal{T}(t))S_j(x) = S_j(y) = S(y).$$

2. We will prove for S and for U will follow similarly. Suppose that $y_n \to y$, where $y = \mathcal{T}(t)x$, with $t \geq T$ (we choose T on the construction of S and U)). We know from the construction of S that there exists $t_0 \in [0, T]$ and $i \in \{1, \dots, p\}$ such that $\tilde{x} = \mathcal{T}(t_0)x$ satisfies (2.56). Define $t' = t - t_0 \geq 0$ and note that

$$x'_n := \mathcal{T}(-t')y_n \to \tilde{x}.$$

Note that $\tilde{x} \in \mathcal{O}_i$ and therefore we can assume that $x'_n \in \mathcal{O}_i$ too. From the continuity of the subbundle S_i on $\mathcal{O}_i$ we know that

$$S_i(x'_n) \to S_i(\tilde{x}),\ \text{as } n \to \infty.$$

We can assume that $\mathcal{T}(t)x \notin \mathcal{O}_i$ without lost of generality, otherwise $t_0 = t$. Then, we have

$$\begin{aligned}(D_x\mathcal{T}(t))S(x) &= (D_x\mathcal{T}(t))S_i(x) \subset S_i(\mathcal{T}(t)x) = S_i(\mathcal{T}(t')\mathcal{T}(t-t')x)\\ &= S_i(\mathcal{T}(t')\tilde{x}) = (D_{\tilde{x}}\mathcal{T}(t'))S_i(\tilde{x}) = \lim_{n\to\infty}(D_{x'_n}\mathcal{T}(t'))S_i(x'_n)\\ &= \lim_{n\to\infty}(D_{x'_n}\mathcal{T}(t'))S(x'_n).\end{aligned}$$

Denoting by I the identity operator and by $Q(x)$ and $Q(x'_n)$ the projections related to $(D_x\mathcal{T}(t))S(x)$ and $(D_{x'_n}\mathcal{T}(t'))S(x'_n)$ respectively, define

$$\Pi_n = I - Q(x) + Q(x'_n)Q(x).$$

Note that $\| \Pi_n - I \|_{\mathcal{L}(X)} \to 0$ and

$$\Pi_n((D_x\mathcal{T}(t))S(x)) = \Pi_n(Q(x)X) = Q(x'_n)(Q(x)X) \subset Q(x'_n)X \subset S(y_n),\ \forall n \in \mathbb{N}.$$

Taking $n_0 > 0$ such that $\| \Pi_n - I \|_{\mathcal{L}(X)} < 1$ for $n \geq n_0$ we conclude that Π_n is an isomorphism by Neumann series.

□

Corollary 2.23. *Given $\nu > 0$ and $t > T$ there exists $\epsilon > 0$ such that if $x, p \in \mathcal{A}$ and $y = \mathcal{T}(t)x$, $q = \mathcal{T}(t)p$ satisfies*

$$d(x,p) < \epsilon \quad \text{and} \quad d(y,q) < \epsilon$$

then there exists isomorphism $F(x,q) : X \to X$ and $G(x,q) : X \to X$ such that

$$\| F(x,q) - I \|_{\mathcal{L}(X)} \leq \nu \quad \text{and} \quad \| G(x,q) - I \|_{\mathcal{L}(X)} \leq \nu$$

and

$$F(x,q)((D_x\mathcal{T}(t))S(x)) \subset S(q) \quad \text{and} \quad G(x,q)((D_y\mathcal{T}(-t))U(y)) \subset U(p).$$

Proof. It follows from Lemma 2.22 and the compactness of $\mathcal{A}$. □

The following technical results will be used to prove the boundness of the projections P_s and P_u, defined after Remark 13, that is,

$$\sup_{x\in\mathcal{A}}(\| P_s(x) \|_{\mathcal{L}(X)} + \| P_u(x) \|_{\mathcal{L}(X)}) < +\infty.$$

Once again, we will use the concept of inclination (see Lemma 2.11). This time we will give a geometric intuition, since we will use and prove results related to inclination. To illustrate, let $X = \mathbb{R}^2$, $S = \langle(1,0)\rangle$, $U = \langle(0,1)\rangle$ and $\mathcal{V}(n) = \langle(1,n)\rangle$ for all $n \in \mathbb{N}^*$, where $\langle v\rangle$ denotes the subspace generated by $v \in X$. It is clear that $S \oplus U = X$, $\mathcal{V}(n) \cap S = \emptyset$ and $\mathcal{V}(n) \cap U = \emptyset$ for all $n \in \mathbb{N}^*$. Therefore, we can compute the inclination of $\mathcal{V}(n)$ in relation to both decompositions $S \oplus U = U \oplus S = X$. Geometrically, note that $\mathcal{V}(n)$ "approaches" the subspace U (Figure 2.6). Given $v = (c, nc) \in \mathcal{V}(n)$ non null ($c \neq 0$) we have

$$v = v^s + v^u = (c,0) + (0,nc),$$

where $(c,0) \in S$ and $(0,nc) \in U$. Hence,

$$\frac{|v^s|}{|v^u|} = \frac{1}{n} \quad \text{and} \quad \frac{|v^u|}{|v^s|} = n,$$

which implies that the inclination of $\mathcal{V}(n)$ in relation to $S \oplus U = X$ goes to 0 (the subspaces $\mathcal{V}(n)$ are leaning into U) and the inclination of $\mathcal{V}(n)$ in relation to $U \oplus S = X$ goes to infinity.

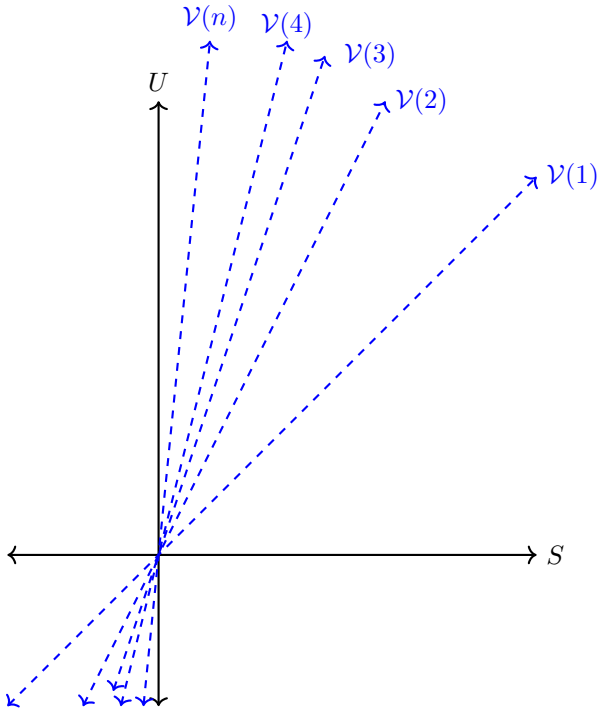


Figure 2.6: Inclination of $\mathcal{V}(n)$

As we said before, the inclination of $\mathcal{V}(n)$ went to infinity in relation to the decomposition $U \oplus S = X$ because $\mathcal{V}(n)$ was approaching U. Note that if we have defined a continuous family $\mathcal{V}(t) = \langle (1, t) \rangle$ and let the parameter t varies from $(0, +\infty)$ the same would happen. On the other hand, if the parameter t varies on a compact $K \subset (0, +\infty)$, then the continuous subbundle $\{\mathcal{V}(t) : t \in K\}$ would not approach U and its inclination would not explode. This is the idea behind the next lemma.

Lemma 2.24. *Let (M, d) be a compact metric space, X be a Hilbert space and $\{P(x) : x \in M\}$, $\{Q(x) : x \in M\}$, $\{Z(x) : x \in M\}$ be families of continuous projections such that*

$$S(x) \oplus U(x) = X, \text{ for all } x \in M,$$

and

$$S(x) \cap \mathcal{V}(x) = \{0\}, \text{ for all } x \in M,$$

where $S(x) = R(P(x))$, $U(x) = R(Q(x))$ and $\mathcal{V}(x) = R(Z(x))$. Given $v \in \mathcal{V}(x)$ there exists unique $v^s \in S(x)$ and $v^u \in U(x)$ such that

$$v = v^s + v^u.$$

with $v^u \neq 0$. Thus we can define the inclination of v in relation to the direct sum $S(x) \oplus U(x) = X$ as

$$\alpha(v) = \frac{\| v^s \|}{\| v^u \|}.$$

Now define the inclination of $\mathcal{V}(x)$ as

$$\alpha(\mathcal{V}(x)) := \sup_{\substack{v \in \mathcal{V}(x) \\ v \neq 0}} \alpha(v) = \sup_{\substack{v \in \mathcal{V}(x) \\ \|v\| = 1}} \alpha(v).$$

If either $S(x)$ or $\mathcal{V}(x)$ is finite dimensional for all $x \in X$ (with same dimension for each $x \in K$), then there exists $M' > 0$ such that

$$\sup_{x \in K} \alpha(\mathcal{V}(x)) \leq M'.$$

Proof. Given $x \in M$, define P_s the continuous projection below

$$\begin{aligned} P_s(x) : X =& S(x) \oplus U(x) \to S(x) \\ & v = v^s + v^u \mapsto v^s \end{aligned}$$

Denoting $P_u(x) = I - P_s(x)$, we have

$$\alpha(v) = \frac{\parallel P_s(x)v \parallel}{\parallel P_u(x)v \parallel} = \frac{\parallel (I - P_u(x))v \parallel}{\parallel P_u(x)v \parallel}, \ \forall x \in M, \ \forall v \in \mathcal{V}(x), v \neq 0.$$

Note that for any $x \in M$ and $v \in \mathcal{V}(x)$ non nulle it holds

$$\alpha(v) = \frac{\parallel (I - P_u(x))v \parallel}{\parallel P_u(x)v \parallel} \leq \frac{\parallel v \parallel}{\parallel P_u(x)v \parallel} + 1,$$

implying that

$$\alpha(\mathcal{V}(x)) \leq \sup_{\substack{v \in \mathcal{V}(x) \\ \|v\|=1}} \frac{1}{\parallel P_u(x)v \parallel} + 1 = \left(\inf_{\substack{v \in \mathcal{V}(x) \\ \|v\|=1}} \parallel P_u(x)v \parallel \right)^{-1} + 1. \tag{2.61}$$

We claim that there exists $\epsilon > 0$ satisfying

$$\inf_{\substack{v \in \mathcal{V}(x) \\ \|v\|=1}} \parallel P_u(x)v \parallel > \epsilon, \ \forall x \in M. \tag{2.62}$$

If that is not the case, there exists $x_n \in M$ and $v_n \in \mathcal{V}(x_n)$, $\parallel v_n \parallel = 1$ such that $P_u(x_n)v_n \to 0$. We can assume, since M is compact, that $x_n \to x \in M$ and consequently

$$d(v_n, S(x)) \to 0. \tag{2.63}$$

Now we have two possibilites:

1. If $\mathcal{V}(x)$ is finite dimensional: in this case we can assume without lost of generality that $v_n \to v$, where $v \in \mathcal{V}(x)$ (remember that $\mathcal{V}(x)$ is closed), $\parallel v \parallel = 1$. This implies that $v \in \mathcal{V}(x) \cap S(x)$, that is, $v = 0$ which contradicts $\parallel v \parallel = 1$.
2. If $S(x)$ is finite dimensional: It follows from (2.63) that there exists $s_n \in S(x)$ such that

   $$d(v_n, s_n) < \frac{1}{n}.$$

   Since $S(x)$ is finite dimensional and $s_n \in B_{S(x)}[0, 2]$ for large n, we can assume without lost of generality that $s_n \to s$, where $s \in S(x)$. Thus $v_n \to s$ and $s \in S(x) \cap \mathcal{V}(x)$, that is, $s = 0$, which contradicts

   $$\parallel v_n \parallel = 1, \text{ for all } n \in \mathbb{N}.$$

Putting (2.61) and (2.62) together we guarantee that for each $x \in M$ there exists $M_x > 0$ such that

$$\alpha(\mathcal{V}(x)) \leq \frac{M_x}{2}.$$

From Proposition 2.21 the maps $x \mapsto \parallel P_u(x) \parallel_{\mathcal{L}(X)}$ and $x \mapsto \parallel P_s(x) \parallel_{\mathcal{L}(X)}$ are continuous and therefore for each $x \in M$ there exists $r_x > 0$ such that

$$\alpha(\mathcal{V}(y)) \leq M_x, \ \forall y \in B(x, r_x).$$

Hence, it follows from the compactness of M that there exists $M' > 0$ such that

$$\alpha(\mathcal{V}(x)) \leq M', \ \forall x \in M.$$

□

In order to provide a geometric intuition of our next technical result, first note that in Figure 2.6 we had

$$S \oplus U = X, U = S^{\perp} \text{ and } U \oplus \mathcal{V}(n) = X \text{ for all } n.$$

Moreover, $\| P_{U\mathcal{V}(n)} \|_{\mathcal{L}(X)} \to +\infty$ and $\| P_{\mathcal{V}(n)U} \|_{\mathcal{L}(X)} \to +\infty$ as $n \to +\infty$, where we are using the notation from Remark 9. In fact, the norm of the projections grows as the subspaces are leaning into each other and minimize when the subspaces are orthogonal (orthogonal projections have norm 1). Therefore, our intuition leads us to the following: if we want to maintain the norm $\| P_{U\mathcal{V}(n)} \|_{\mathcal{L}(X)}$ of the projections bounded, we must control the inclination of the subspaces. This is the geometric ideia behind our next result.

Proposition 2.25. *Let X be a Hilbert space, Γ be a non-empty set and $\{S(x) : x \in \Gamma\}$ and $\{U(x) : x \in \Gamma\}$ be two families of closed subspaces of X (not necessarily continuous) with*

$$S(x) \oplus U(x) = X, \ \forall x \in \Gamma.$$

Since $U^{\perp}(x) \oplus U(x) = X$ we can calculate the inclination (see Lemma 2.24) of $S(x)$ related to $U(x)$ and $U^{\perp}(x)$. Assume that there exists $M > 0$ such that

$$\sup_{x \in \Gamma} \alpha(S(x)) \leq M,$$

that is,

$$\sup_{x \in \Gamma} \sup_{\substack{v \in S(x) \\ v \neq 0}} \frac{\| P_{UU^{\perp}}(x)v \|}{\| P_{U^{\perp}U}(x)v \|} \leq M. \tag{2.64}$$

Then, there exists $M' > 0$ such that

$$\sup_{x \in \Gamma} \{ \| P_{SU}(x) \|_{\mathcal{L}(X)} + \| P_{US}(x) \|_{\mathcal{L}(X)} \} \leq M',$$

where we are using the same notation as Proposition 2.21.

Proof. We just need to show

$$\sup_{x \in \Gamma} \{ \| P_{SU}(x) \|_{\mathcal{L}(X)} \} < +\infty$$

since

$$\| P_{US}(x) \|_{\mathcal{L}(X)} = \| I - P_{SU}(x) \|_{\mathcal{L}(X)} \leq 1 + \| P_{SU}(x) \|_{\mathcal{L}(X)} .$$

From Lemma 2.21 we know that

$$P_{SU}(x) = (I - P_{S(x)}P_{U(x)})^{-1} \circ P_{S(x)} \circ (I - P_{S(x)}P_{U(x)}), \ \forall x \in \Gamma.$$

Hence,

$$\| P_{SU}(x) \|_{\mathcal{L}(X)} \leq 2 \| (I - P_{S(x)}P_{U(x)})^{-1} \circ P_{S(x)} \|_{\mathcal{L}(X)}, \ \forall x \in \Gamma.$$

To estimate the term on the right side, we just need to find $\epsilon > 0$ satisfying

$$\| (I - P_{S(x)}P_{U(x)})v \| \geq \epsilon \| v \|, \ \forall v \in S(x), \forall x \in \Gamma,$$

that is,

$$\| (I - P_{S(x)}P_{U(x)})v \| \geq \epsilon, \ \forall v \in S(x), \| v \| = 1, \forall x \in \Gamma. \tag{2.65}$$

From (2.64) we have

$$\| P_{U^\perp U}(x)v \| \geq \frac{1}{M} \| P_{UU^\perp}(x)v \| \geq \frac{1}{M} \| v \| - \frac{1}{M} \| P_{U^\perp U}(x)v \|, \ \forall v \in S(x), \ \forall x \in \Gamma.$$

Thus

$$\| P_{U^\perp U}(x)v \| \geq \frac{1}{M+1} \| v \|, \forall v \in S(x), \ \forall x \in \Gamma. \tag{2.66}$$

For $v \in S(x)$, $\| v \| = 1$ we have

$$\begin{aligned} \| P_{U^\perp U}(x)P_{SS^\perp}(x)v \|^2 &= \langle P_{U^\perp U}(x)P_{SS^\perp}(x)v, P_{U^\perp U}(x)P_{SS^\perp}(x)v \rangle = \langle P_{SS^\perp}(x)v, P_{U^\perp U}(x)v \rangle \\ &= \langle v, P_{SS^\perp}(x)P_{U^\perp U}(x)v \rangle \leq \| P_{SS^\perp}(x)P_{U^\perp U}(x)v \| . \end{aligned}$$

Finally, putting the last inequality together with (2.66) we guarantee that for any $v \in S(x)$ with $\| v \| = 1$ we have

$$\begin{aligned} \| (I - P_{SS^\perp}(x)P_{UU^\perp}(x))v \| &= \| P_{SS^\perp}(x)(I - P_{UU^\perp}(x))v \| = \| P_{SS^\perp}(x)P_{U^\perp U}(x))v \| \\ &\geq \| P_{U^\perp U}(x)P_{SS^\perp}(x)v \|^2 \geq \frac{1}{(M+1)^2}. \end{aligned}$$

This proves (2.65) and concludes the proof. □

To finish this section, we obtain one final property from the subbundles S, U.

For each $i \in \{1, \dots, p\}$, take $\mathcal{O}_i' \subset \mathcal{O}_i$ as on Lemma 1.5 and define $T' > 0$ as the Birkhoff number (Lemma 1.4) related to the set $V' = \bigcup_{i=1}^{n} \mathcal{O}_i'$.

Remembering that we are assuming item (H1) from Theorem 0.1, we now fix $C_0 > 0$ and $C_1 > 1$ such that

$$\| D_x\mathcal{T}(t) \|_{\mathcal{L}(X)} \leq C_1 e^{C_0 t}, \ \forall t \in [0, T'], \ \forall x \in \mathcal{A}. \tag{2.67}$$

We recall that if our semigroup is defined through the variation of constants formula (see Definition 29) and $\mathcal{T}(t) \in \mathcal{C}^1(X)$ for $t \geq 0$, then it's derivative is also written through the variation of constants formula, as we can see in (1.33). Therefore, if the derivative of the nonlinearity f is Lipschitz in $\mathcal{A}$, we can use Gronwall's inequality to obtain (2.67), as we did in Corollary 1.34.

Lemma 2.26. *The subbundles S, U satisfy:*

1. *There exists $M > 0$ such that*

$$\| P_s(x) \|_{\mathcal{L}(X)}, \| P_u(x) \|_{\mathcal{L}(X)} \leq M, \ \forall x \in \mathcal{A},$$

   where P_s and P_u are the projections defined after Remark 13.

2. *There exists $C > 0$ such that for any $x \in \mathcal{A}$ it holds*

$$\| (D_x\mathcal{T}(t))v^s \| \leq Ce^{-\lambda_1 t} \| v^s \|, \ \forall v^s \in S(x), \ \forall t \geq 0 \tag{2.68}$$

   and

$$\| (D_x\mathcal{T}(-t))v^u \| \leq Ce^{-\lambda_1 t} \| v^u \|, \ \forall v^u \in U(x), \ \forall t \geq 0, \tag{2.69}$$

   where λ_1 is the constant on Remark 2.15.

Proof.

1. In order to prove the first item, note first that $\{\| P_s(x) \|\}_{x\in\mathcal{O}_i}$ is bounded by Lemma 2.15 and Proposition 2.21. It follows from the construction of S that, to estimate $P_s(x)$ it is sufficient to estimate $P_{S_i}(x)$ for $x \in \gamma^-_{[0,T]}(\mathcal{O}_i)$, which will be proved using Proposition 2.25. We already know from Lemma 2.24 that the inclination (defined on Lemma 2.24) of $S_i(x)$ related to $U_i(x)$ and $U_i^\perp(x)$ is bounded on $\overline{\mathcal{O}_i}$, that is, there exists $M > 0$ such that

$$\sup_{x\in\overline{\mathcal{O}_i}} \alpha(S_i(x)) = \sup_{x\in\overline{\mathcal{O}_i}} \sup_{\substack{v\in S_i(x)\\ \|v\|=1}} \frac{\| P_{U_iU_i^\perp}(x)v \|}{\| P_{U_i^\perp U_i}(x)v \|} \leq M.$$

We can estimate the inclination of $S_i(\mathcal{T}(-t)x)$, since $D_x\mathcal{T}(-t) : S_i(x) \cap R(D_{\mathcal{T}(-t)x}\mathcal{T}(t)) \to S_i(\mathcal{T}(-t)x)$ and $(D_x\mathcal{T}(-t))^{-1} : U_i(\mathcal{T}(-t)x) \to U_i(x)$ are isomorphisms. In fact, fix $x \in \mathcal{O}_i$, $v \in S_i(x) \cap R(D_{\mathcal{T}(-t)x}\mathcal{T}(t))$, $t \in [0,T]$ and define $x' = \mathcal{T}(-t)x$ and $v' = (D_x\mathcal{T}(-t))v \in S_i(x')$. Note that

$$\begin{aligned}\| P_{U_iU_i^\perp}(x)v - v \| &= \min_{u\in U_i(x)} \| v - u \| \\ &\leq \min_{u\in U_i(x)\cap R(D_{\mathcal{T}(-t)x}\mathcal{T}(t))} \| (D_{\mathcal{T}(-t)x}\mathcal{T}(t))v' - (D_{\mathcal{T}(-t)x}\mathcal{T}(t))D(\mathcal{T}(-t)x)u \| \\ &\leq \| D_{\mathcal{T}(-t)x}\mathcal{T}(t) \| \min_{u\in U_i(x')} \| v' - u \| = \| D_{\mathcal{T}(-t)x}\mathcal{T}(t) \| \| v' - P_{U_iU_i^\perp}(x')v' \| .\end{aligned}$$

Hence

$$\frac{1}{\| P_{U_i^\perp U_i}(x')v' \|} \leq \| D_{\mathcal{T}(-t)x}\mathcal{T}(t) \| \frac{1}{\| P_{U_i^\perp U_i}(x)v \|}.$$

Therefore for $v' \in S_i(x')$, $\| v' \| = 1$, we have

$$\frac{\| P_{U_iU_i^\perp}(x')v' \|}{\| P_{U_i^\perp U_i}(x')v' \|} \leq \| D_{\mathcal{T}(-t)x}\mathcal{T}(t) \| \frac{1}{\| P_{U_i^\perp U_i}(x)v \|} \leq M' \frac{1}{\| P_{U_i^\perp U_i}(x)v \|}.$$

and proceeding as the proof of Lemma 2.24 we conclude from (2.67) and Proposition 2.25.

2. Let C_0 be the constant in (2.67), λ_1 be the constant on Lemma 2.15 and p be the number of equilibria of our semigroup $\mathcal{T}(t)$. We will prove only (2.68) and (2.69) will follow similarly.

   Let $x \in \mathcal{A}$, $v \in S(x)$ and $t > 0$ ((2.68) is obvious for $t = 0$). Consider the set

$$\Theta = \{s \in [0,t] : \mathcal{T}(s)x \notin V\}$$

From the definition of T' and the fact that we have p equilibria, for any $\epsilon > 0$ there exists open segments (a_i, b_i) with $i = 1, \ldots, N$ such that

$$\Theta \subset \bigcup_{i=1}^{N} (a_i, b_i)$$

and

$$\sum_{i=1}^{n} (b_i - a_i) < pT' + \epsilon. \tag{2.70}$$

Define for each $= 1, \ldots, N$

$$v_i^- = (D_x\mathcal{T}(a_i))v, \quad v_i^+ = (D_x\mathcal{T}(b_i))v, \quad v' = (D_x\mathcal{T}(t))v.$$

Since $\mathcal{T}(s)x \in V$ for $s \in (0, a_1) \cup \cdots \cup (b_N, t)$ (we are assuming that x and $\mathcal{T}(t)x \in V$ bu the other cases follow analogously) it follows from item (5) of Lemma 2.15 and Remark 13 that

$$\| v_1^- \| \leq \tilde{C}\lambda_1^{a_1} \| v \|, \quad \| v_2^- \| \leq \tilde{C}\lambda_1^{a_2-b_1} \| v_1^+ \|, \ldots, \| v' \| \leq \tilde{C}\lambda_1^{t-b_N} \| v_n^+ \| . \tag{2.71}$$

and from 2.67 that

$$\| v_i^+ \| \leq C_1 \exp(C_0(b_i - a_i)) \| v_i^- \| . \tag{2.72}$$

Combining (2.70), (2.71) and (2.72) we have

$$\begin{aligned} \| v' \| &\leq \tilde{C}^N C_1^N \exp(C_0[(b_1 - a_1) + \cdots + (b_N - a_N)])\lambda_1^{a_1+(a_2-b_1)+\cdots+(t-b_N)} \| v \| \\ &\leq \tilde{C}^p C_1^p \exp(C_0(pT' + \epsilon))\lambda_1^{t-(T'+\epsilon)} \| v \| \\ &= (\exp(C_0\epsilon)\lambda_1^\epsilon)(\tilde{C}^p C_1^p \exp(C_0 pT')\lambda_1^{-T'})\lambda_1^t \| v \| . \end{aligned}$$

Since $\exp(C_0\epsilon)\lambda_1^\epsilon \to 1$ as $\epsilon \to 0$ we obtain that

$$\| (D_x\mathcal{T}(t))v \| \leq \tilde{C}^p C_1^p \exp(C_0 pT')\lambda_1^t \| v \|,$$

which concludes the proof of (2.68).

We can show (2.69) similarly. In fact, we can use Corollary 2.10 together with the fact that $D_x\mathcal{T}(t)$ is an isomorphism onto its range and the subbundles U_i are continuous (Remark 12) to also bound the norm of $(D_x\mathcal{T}(-t))|_{U(x)}$ uniformly for $t \in [0, T']$ and $x \in \mathcal{A}$. Hence, we can assume without lost of generality that C_0 and C_1 also satisfies

$$\| (D_x\mathcal{T}(-t))|_{U(x)} \|_{\mathcal{L}(U(x),X)} \leq C_1 e^{C_0|t|}, \ \forall t \in [0, T'], \ \forall x \in \mathcal{A}$$

and proceed exatcly like the first case, for the family S.

This concludes the proof.

□

2.2 Lipschitz Shadowing on the Global Attractor

Now that we construct the compatible subbundles S, U in $\mathcal{A}$ in Section 2.1, we are close to prove the main result of this chapter (Theorem 2.28). In fact, with the subbundles S, U we already have the geometric part of the proof done. Therefore, in this section we connect our geometrical results from Section 2.1 with an abstract result from Functional Analysis (Theorem 2.27) to finally prove the property of Lipschitz Shadowing in $\mathcal{A}$.

In this section $\{E_k\}_{k\in\mathbb{Z}}$ will be a sequence of Banach spaces and $\{\phi_k\}_{k\in\mathbb{Z}}$ a sequence of maps $\phi_k : E_k \to E_{k+1}$ such that

$$\phi_k = A_k + w_{k+1}$$

where $A_k : E_k \to E_{k+1}$ is linear. We will also denote $\mathcal{T} := \mathcal{T}(1)$ to simplify the notation. Note that in this case we have $\mathcal{T}^n = \mathcal{T}(n)$.

Let us announce the abstract result that will connects the compatible subbundles constructed in Section 2.1 with the notion of Lipschitz Shadowing. The following theorem is stated on $\mathbb{Z}$ but is also valid on $\mathbb{Z}^-$. The proof can be also seen in [47, Lemma 1.3.1] and [58, Lemma A.0.8]

Theorem 2.27. *Consider the following conditions:*

1. *For each $k \in \mathbb{Z}$ there exists continuous projections $P_k, Q_k : E_k \to E_k$ and constants $\lambda_1 \in (0, 1), \mathcal{C} > 0$ independent of k such that*

$$P_k + Q_k = Id, \qquad \| P_k \|_{\mathcal{L}(E_k)}, \| Q_k \|_{\mathcal{L}(E_k)} \leq M, \ \forall k \in \mathbb{Z} \tag{2.73}$$

and

$$\| A_k P_k \|_{\mathcal{L}(E_k, E_{k+1})} \leq \lambda_1, \quad A_k P_k(E_k) \subset P_{k+1} E_{k+1}, \ \forall k \in \mathbb{Z}. \tag{2.74}$$

2. There exists linear maps $B_k : Q_{k+1}E_{k+1} \to E_k$ satisfying

$$B_k Q_{k+1} E_{k+1} \subset Q_k E_k, \ \| B_k Q_{k+1} \|_{\mathcal{L}(E_{k+1}, E_k)} \leq \lambda_1, \ A_k B_k|_{Q_{k+1}E_{k+1}} = I|_{Q_{k+1}E_{k+1}}, \ \forall k \in \mathbb{Z}. \tag{2.75}$$

3. There exists $\Delta, D > 0$ with the property

$$\| w_{k+1} v - w_{k+1} v' \|_{E_{k+1}} \leq D \| v - v' \|_{E_k}, \ \forall \| v \|, \| v' \| \leq \Delta, \ \forall k \in \mathbb{Z}. \tag{2.76}$$

4. $DN_1 < 1$, where

$$N_1 = M \left(\frac{1 + \lambda_1}{1 - \lambda_1} \right).$$

If itens (1)-(4) are fullfilled, then there exists $d_0, L > 0$ with the property: if

$$\| \phi_k(0) \|_{E_{k+1}} \leq d \leq d_0, \ \forall k \in \mathbb{Z} \tag{2.77}$$

then there exists $v_k \in E_k$ such that

$$\| v_k \|_{E_k} \leq Ld \quad \textit{and} \quad \phi_k(v_k) = v_{k+1}, \ \forall k \in \mathbb{Z}. \tag{2.78}$$

Proof. Let Z be the Banach space of sequences $v = \{v_k\}_{k \geq 0}$ with $v_k \in E_k$ and the norm

$$\|v\|_\infty = \sup_{k \geq 0} \|v_k\|_{E_k}.$$

Define the operator H on Z by

$$H(w) = v = v_1 + v_2 + v_3, \quad w \in Z,$$

where $v_i = \{v_{ik}\}_{k \geq 0}$, with

$$v_k^1 = P_k w_k,$$

$$v_k^2 = \sum_{j=0}^{k-1} A_{k-1} \cdots A_j P_j w_j,$$

$$v_k^3 = -\sum_{j=k}^{\infty} B_k \cdots B_j Q_{j+1} w_{j+1}.$$

First we will prove that H is well defined, that is, that maps Z into itself, and estimate its norm. Fixing $w \in Z$, it follows from the above definition that

$$\|v_1\|_\infty \leq M \|w\|_\infty.$$

Moreover, since $P_k w_k \in S_k$, then

$$\|v_n^2\|_{E_n} \leq M \sum_{k=0}^{n-1} \lambda_1^{n-k} \|w_k\|_{E_k} \leq M \frac{\lambda_1}{1 - \lambda_1} \|w\|_\infty,$$

and similarly

$$\|v_n^3\|_{E_n} \leq M \frac{\lambda_1}{1 - \lambda_1} \|w\|_\infty.$$

Hence

$$\|H\|_{L(Z,Z)} \le M + M\frac{\lambda_1}{1-\lambda_1} + M\frac{\lambda_1}{1-\lambda_1} = M\frac{1+\lambda_1}{1-\lambda_1} = N_1. \tag{2.79}$$

Let $v = H(w)$. Calculating Av we obtain

$$A_n v_n^1 = A_n P_n w_n,$$

$$A_n v_n^2 = \sum_{k=0}^{n-1} A_n \cdots A_k P_k w_k,$$

$$A_n v_n^3 = -\left[A_n B_n Q_{n+1} w_{n+1} + \sum_{k=n+1}^{\infty} B_{n+1} \cdots B_k Q_{k+1} w_{k+1}\right]$$

and consequently

$$\begin{aligned} A_n v_n &= A_n P_n w_n + \sum_{k=0}^{n-1} A_n \cdots A_k P_k w_k - (I - P_{n+1}) w_{n+1} - \sum_{k=n+1}^{\infty} B_{n+1} \cdots B_k Q_{k+1} w_{k+1} \\ &= -w_{n+1} + v_{n+1}^1 + v_{n+1}^2 + v_{n+1}^3 \Longrightarrow v_{n+1} = A_n v_n + w_{n+1}, \quad n \ge 0. \end{aligned}$$

Define

$$L = \frac{N_1}{1 - DN_1}, \quad d_0 = \frac{\Delta}{L}. \tag{2.80}$$

Note that

$$\phi_k(v_k) = v_{k+1}, \quad k \ge 0,$$

is equivalent to

$$v_{k+1} = A_k v_k + w_{k+1}(v_k), \quad k \ge 0. \tag{2.81}$$

For $v \in Z$, we define

$$w(v) = \{w_k(v_{k-1})\}_{k \ge 0}, \quad \text{with} \quad w_0 = 0.$$

Then, a solution v of the equation

$$v = H(w(v)),$$

satisfies

$$\phi_k(v_k) = v_{k+1}.$$

Suppose

$$\|\phi_k(0)\|_{E_{k+1}} \le d.$$

We want to apply the Banach fixed point on the ball $B := \{\|v\|_\infty \le Ld\}$ in Z. Remember that $Ld \le Ld_0 = \Delta$. Using (2.76) and (2.79), for any $v, v' \in B$, $\|v\|_\infty, \|v'\|_\infty \le \Delta$ we have

$$\|H(w(v)) - H(w(v'))\|_\infty \le N_1 \|w(v) - w(v')\|_\infty \le DN_1 \|v - v'\|_\infty.$$

From (2.81), we obtain

$$\|w_{k+1}(0)\|_{E_{k+1}} = \|v_{k+1}\|_{E_{k+1}} = \|\phi_k(0)\|_{E_{k+1}} \le d.$$

I follows from (2.80) that for $v \in B$ it holds

$$\begin{aligned} \|H(w(v))\|_\infty &\le \|H(w(0))\|_\infty + \|H(w(v)) - H(w(0))\|_\infty \\ &\le N_1 d + N_1 D \|v\|_\infty \le N_1 d + N_1 DLd = d(N_1 + N_1 DL) = Ld, \end{aligned}$$

Therefore $H(w)$ maps B into itself and is a contraction (using (2.76)). Thus $H(w)$ has a fixed point and there exists a solution of the equation

$$v = H(w(v)),$$

which satisfies

$$\|v\|_\infty \leq Ld.$$

This concludes the proof. □

Despite the abstraction of Theorem 2.27, we can already understand how it connects the concept of compatible subbundles and shadowing. Indeed, the Banach spaces E_k will be X, for any k, and the projections P_k and Q_k on Theorem 2.27 will be the projections $P_s(x_k)$ and $P_u(x_k)$, where x_k is a pseudo orbit on $\mathcal{A}$. In this case $P_k E_k = S(x_k)$ and $Q_k E_k = U(x_k)$. Moreover, note that (2.77) and (2.78) are very similar to the Lipschitz Shadowing property.

Now we have all tools to proceed as the finite dimensional case [58] and prove the main theorem of this chapter.

Theorem 2.28 (Lipschitz Shadowing).

Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup with non-wandering set $\Omega = \{x_1^*, \dots, x_p^*\}$. If $\mathcal{T}(\cdot)$ satisfies (H1) and (H2), then $\mathcal{T} := \mathcal{T}(1)$ has the Lipschitz Shadowing property on the global attractor, that is, there exists $L, d_0 > 0$ such that for any $\{x_n\}_{n\in\mathbb{Z}}$ d-pseudo-orbit of $\mathcal{T}$ in $\mathcal{A}$, with $d \leq d_0$, there exists $x \in \mathcal{A}$ such that

$$d(\mathcal{T}^n x, x_{n+1}) \leq Ld, \ \forall k \in \mathbb{Z}.$$

Proof. This proof consists in fulfill the conditions of Theorem 2.27. Fix $\mu \in (0,1)$ arbitrary and $N \in \mathbb{N}$, $N > T$ such that

$$C\lambda_1^{N-1} \leq \mu, \tag{2.82}$$

where λ_1 and C are the constants on Lemma 2.26. Hence for each $x \in \mathcal{A}$ we have

$$\| (D_x\mathcal{T}^n)\ v^s \| \leq \mu \| v^s \|, \ \forall v^s \in S(x), \ \forall n \geq N-1, \tag{2.83}$$

$$\| (D_x\mathcal{T}^{-n})\ v^u \| \leq \mu \| v^u \|, \ \forall v^u \in U(x), \ \forall n \geq N-1. \tag{2.84}$$

Take M as on Lemma 2.26 such that

$$\| P_s(x) \|_{\mathcal{L}(X)}, \| P_u(x) \|_{\mathcal{L}(X)} \leq M, \ \forall x \in \mathcal{A}. \tag{2.85}$$

Take $K \geq M$ with

$$\| D_x\mathcal{T}^N \|_{\mathcal{L}(X)} \leq K, \ \forall x \in \mathcal{A}. \tag{2.86}$$

In order to apply condition (4) of Theorem 2.27 take $\nu_0 \in (0,1)$ satisfying

$$\lambda := (1+\nu_0)\mu < 1. \tag{2.87}$$

Define

$$N_1 = M\frac{1+\lambda}{1-\lambda}$$

and take constants $k, \nu \in (0, \nu_0)$ such that

$$kN_1 < 1 \quad \text{and} \quad K(2K+1)\nu < \frac{k}{2}. \tag{2.88}$$

We will prove the Lipschitz Shadowing property for $\psi := \mathcal{T}^N$ and the result for $\mathcal{T}$ will follow from Lemmas 2.18 and 2.19. Fix $d > 0$ and let $\{x_k\}_{k\in\mathbb{Z}}$ be a d-pseudo orbit of ψ. For each $k \in \mathbb{Z}$ denote $E_k = X$ and define

$$\phi_k : E_K \to E_{k+1} \tag{2.89}$$
$$v \mapsto \psi(x_k + v) - x_{k+1} \tag{2.90}$$

Note that $D_0\phi_k = D_{x_k}\psi$ and

$$\phi_k(v) = (D_0\phi_k)v + h_{k+1}(v), \forall v \in E_k, \tag{2.91}$$

where $h_{k+1} := \phi_k - D_0\phi_k$.

Since $D_0 h_{k+1} = 0$ and ψ is $\mathcal{C}^1$, there exists $d_0 > 0$ such that

$$\| h_{k+1}(v) - h_{k+1}(v') \| \leq \frac{k}{2} \| v - v' \|, \ \forall \| v \|, \| v' \| \leq d_0. \tag{2.92}$$

Fix $\epsilon > 0$ as on Lemma 2.23 (respectively no ν). From Corollary 2.23 (and the fact that $(\psi|_{\mathcal{A}})^{-1}$ is uniformly continuous) we can assume that $d_0 > 0$ satisfies

$$d(x_k, \psi^{-1}(x_{k+1})) \leq \epsilon \quad \text{and} \quad d(\psi(x_k), x_{k+1}) \leq \epsilon$$

and there exists isomorphisms $F(x_k, x_{k+1})$ and $G(x_k, x_{k+1})$ satisfying

$$\| F(x_k, x_{k+1}) - I \|, \| G(x_k, x_{k+1}) - I \|, \| G(x_k, x_{k+1})^{-1} - I \| \leq \nu, \tag{2.93}$$

where ν is the constant in (2.88) and

$$F(x_k, x_{k+1})(D_{x_k}\psi)P_s(x_k) \subset S(x_{k+1}), \ G(x_k, x_{k+1})(D_{x_{k+1}}\psi^{-1})U(x_{k+1}) \subset U(x_k).$$

Using the notation from Lemma 2.26 define the operators

$$A_k^s := F(x_k, x_{k+1})(D_{x_k}\psi)P_s(x_k) \qquad A_k^u := (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})P_u(x_k)$$

and

$$B_k := G(x_k, x_{k+1})D_{x_{k+1}}\psi^{-1} \qquad A_k := A_k^s + A_k^u. \tag{2.94}$$

Note that A_k^s relates the subspaces $S(x_k)$ and $S(x_{k+1})$ and B_k relates $U(x_{k+1})$ and $U(x_k)$.

Now we can write ϕ_k as

$$\phi_k = A_k + w_{k+1},$$

where

$$w_{k+1} = \phi_k - A_k = [D_{x_k}\psi - A_k] + h_{k+1}. \tag{2.95}$$

For now on we will prove that operators A_k, B_k and w_k satisfies all conditions from Theorem 2.27. We will start with the operator A_k^s. Given $v_s \in S(x_k) = P(x_k)E_k$ we know from Lemma 2.22 that $(D_{x_k}\psi)v_k \in S(\psi(x_k))$ and from (2.83) that

$$\| (D_{x_k}\psi)v^s \| \leq \mu \| v_s \| .$$

It follows from our choice of $F(x_k, x_{k+1})$ that

$$A_k^s S(x_k) \subset S(x_{k+1}) \tag{2.96}$$

and from (2.93), (2.87) and $\nu < \nu_0$ that

$$\| F(x_k, x_{k+1})(D_{x_k}\psi)v^s \| \leq (1+\nu)\mu \| v^s \| \leq \lambda \| v_s \| .$$

Hence

$$\| A_k^s|_{S(x_k)} \| \leq \lambda. \tag{2.97}$$

Analogously, for B_k we have

$$B_k(U(x_{k+1})) \subset U(x_k)$$

and for each $v^u \in U(x_{k+1})$ it holds

$$\| G(x_k, x_{k+1})(D_{x_{k+1}}\psi^{-1})v^u \| \leq (1+\nu)\mu \| v^u \| \leq \lambda \| v^u \| .$$

Therefore

$$\| B_k|_{U(x_k)} \| \leq \lambda. \tag{2.98}$$

Note that for any $v^u \in U(x_{k+1})$ we have

$$\begin{aligned} A_k^u B_k v^u &= (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})P_u(x_k)B_k v^u \\ &= (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})B_k v^u \\ &= (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})G(x_k, x_{k+1})(D_{x_{k+1}}\psi^{-1})v^u \\ &= v^u, \end{aligned}$$

that is,

$$A_k^u B_k|_{U(x_{k+1})} = Id_{U(x_{k+1})}. \tag{2.99}$$

Since $A_k|_{S(x_k)} = A_k^s$ and $A_k B_k|_{U(x_{k+1})} = A_k^u B_k|_{U(x_{k+1})}$, it follows from (2.97), (2.98) and (2.99) that A_k and B_k fulfill the conditions from itens (1) and (2) from Theorem 2.27, with $P_k = P_s(x_k)$ and $Q_k = P_u(x_k)$.

Now we show that w_k satisfies condition (3) from Theorem 2.27. First off all, note that $D\psi$ is uniformly continuous on $\mathcal{A}$ and we can reduce d_0 to satisfy

$$x, y \in \mathcal{A}, d(x,y) \leq d_0 \Longrightarrow \| D_x\psi - D_y\psi \| \leq \nu. \tag{2.100}$$

Remember that w_{k+1} is given in (2.95). First we will estimate $\| D_{x_k}\psi - A_k \|$. Given $d \leq d_0$ we have

$$\begin{aligned} \| D_{x_k}\psi - A_k \| =& \| (D_{x_k}\psi - A_k)I \| \\ =& \| (D_{x_k}\psi - A_k)(P_s(x_k) + P_u(x_k)) \| \\ \leq& \| (D_{x_k}\psi)P_s(x_k) - A_k^s \| + \| (D_{x_k}\psi)P_u(x_k) - A_k^u \| \\ & \| (D_{x_k}\psi)P_s(x_k) - F(x_k, x_{k+1})(D_{x_k}\psi)P_s(x_k) \| \\ & + \| (D_{x_k}\psi)P_u(x_k) - (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})P_u(x_k) \| . \end{aligned}$$

From (2.86) and (2.93) we can estimate the first term as

$$\| (D_{x_k}\psi)P_s(x_k) - F(x_k, x_{k+1})(D_{x_k}\psi)P_s(x_k) \| \leq K^2\nu.$$

To estimate the second term we use (2.100) as below

$$\begin{aligned} & \| (D_{x_k}\psi)P_u(x_k) - (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1})P_u(x_k) \| \\ & \leq K \| D_{x_k}\psi - (D_{\psi^{-1}(x_{k+1})}\psi)G^{-1}(x_k, x_{k+1}) \| \\ & \leq \| (D_{\psi^{-1}(x_{k+1})}\psi)(G^{-1}(x_k, x_{k+1}) - I) + D_{\psi^{-1}(x_{k+1})}\psi - D_{x_k}\psi \| \\ & \leq K^2\nu + K\nu \\ & \leq K\nu(K+1). \end{aligned}$$

Putting all these inequalities together with (2.88) we conclude that

$$\| D_{x_k}\psi - A_k \| \leq K^2\nu + K\nu(K+1) = \nu K(2K+1) \leq \frac{k}{2}. \tag{2.101}$$

It follows from (2.92) and (2.101) that

$$\| w_{k+1}(v) - w_{k+1}(v') \| \leq k \| v - v' \|, \forall \| v \|, \| v' \| \leq d_0. \tag{2.102}$$

Since $kN_1 < 1$ (see (2.88)), we conclude that w_k satisfies item (3) from Theorem 2.27 with $\Delta = d_0$.

Therefore ϕ_k satisfies all conditions from Theorem 2.27 and we can finally apply it to guarantee the existence of constants $d_1, L > 0$ satisfying (2.77) and (2.78).

Thus, if x_k is a d pseudo orbit of ψ with $d \leq d_2 := \min\{d_0, d_1\}$ we have

$$\| \phi_k(0) \| \leq d,$$

which implies the existence of $v_k \in X$ satisfying

$$\phi_k(v_k) = v_{k+1}, \tag{2.103}$$

and

$$\| v_k \| \leq Ld, \ \forall k \in \mathbb{Z}. \tag{2.104}$$

From (2.103) we have

$$\phi_k(v_k) = \psi(x_k + v_k) - x_{k+1} = \mathcal{T}^N(x_k + v_k) - x_{k+1} = v_{k+1} \Rightarrow \mathcal{T}^N(x_k + v_k) = x_{k+1} + v_{k+1}.$$

Hence $\{y_k := x_k + v_k\}_{k\in\mathbb{Z}}$ is an orbit of $\psi = \mathcal{T}^N$ and from (2.104)

$$d(\psi^k y_0, x_k) = d(y_k, x_k) \leq Ld.$$

Therefore $\mathcal{T}^N$ has the Lipschitz Shadowing property on $\mathcal{A}$ with constants L, d_2, which concludes the proof. □

Chapter 3

The Neighborhood of the Global Attractor

In this chapter we extend the results from Chapter 2, where we obtained Lipschitz Shadowing on the global attractor $\mathcal{A}$, to a neighborhood of $\mathcal{A}$ (see Theorems 3.2 and 3.10). As a corollary, we will prove item (2) from Theroem 0.1. To be more precise, in this chapter we prove that under some reasonable assumptions it holds

1. If $\mathcal{T}(1)$ has the Shadowing property on $\mathcal{A}$, then for any bounded and positively invariantneighborhood $\mathcal{U} \supset \mathcal{A}$ the map $\mathcal{T}(1)|_{\mathcal{U}}$ also has the Shadowing property.
2. If $\mathcal{T}(1)$ has the Lipschitz Shadowing property on $\mathcal{A}$, then for any bounded and positively invariant neighborhood $\mathcal{U} \supset \mathcal{A}$ that is exponentially attracted by $\mathcal{A}$, the map $\mathcal{T}(1)|_{\mathcal{U}}$ has the Hölder Shadowing property.

We prove items 1 and 2 in Sections 3.1 and 3.2, respectively. The results from this chapter will be used later, on Chapter 4, to obtain results related to the geometrical structural stability of Morse-Smale semigroups and continuity of global attractors.

In this chapter we proceed to use the notation $\mathcal{T} := \mathcal{T}(1)$, as in Section 2.2.

3.1 Shadowing on a Neighborhood of $\mathcal{A}$

In this section we will prove the following statement:

Let $\mathcal{T}$ be a semigroup in a metric space (M, d) with global attractor $\mathcal{A}$ and $\mathcal{U}$ be a positively invariant neighborhood of $\mathcal{A}$ such that $\mathcal{T}|_{\mathcal{U}}$ is Lipschitz, with Lipschitz constanst L_1. If $\mathcal{T}$ is a group in $\mathcal{A}$ and $\mathcal{T}|_{\mathcal{A}}$ has the Shadowing Property, then $\mathcal{T}|_{\mathcal{U}}$ also has the Shadowing property.

Moreover, we will also prove a result that does not depends on the Lipschitz Shadowing property, that is, we will estimate the distance from pseudo-orbits to the global attractor without using Shadowing (Lemma 3.1). This kind of argument will be used later to prove results related to continuity of global attractor on Chapter 4.

From now on we will denote by $\mathbb{T}$ the set $\mathbb{Z}$ or $\mathbb{Z}_- = \{m \in \mathbb{Z} : m \leq 0\}$.

Lemma 3.1. *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space (M, d) with global attractor $\mathcal{A}$ and $\mathcal{U}$ be a positively invariant neighborhood of $\mathcal{A}$ such that $\mathcal{T}$ restricted to $\mathcal{U}$ is Lipschitz, with Lipschitz constant L_1. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit $\{x_n\}_{n\in\mathbb{T}} \subset \mathcal{U}$ satisfies*

$$d(x_n, \mathcal{A}) < \epsilon, \ \forall\, n \in \mathbb{T}. \tag{3.1}$$

Proof. Let $\epsilon > 0$ and take $N \in \mathbb{N}$ such that

$$dist_H(\mathcal{T}^n\mathcal{U}, \mathcal{A}) < \frac{\epsilon}{4}, \ \forall n \geq N.$$

Now fix $\delta > 0$ such that $C_1\delta < \frac{\epsilon}{4}$, where C_1 is the constant on Lemma 2.18. Take $\{x_n\}_{n\in\mathbb{T}} \subset \mathcal{U}$ a δ-pseudo orbit and define $z_n = x_{nN}$. It follows from Lemma 2.18 that

$$d(\mathcal{T}^N z_n, z_{n+1}) < C_1\delta < \frac{\epsilon}{4}.$$

Therefore

$$d(z_{n+1}, \mathcal{A}) \leq d(z_{n+1}, \mathcal{T}^N z_n) + d(\mathcal{T}^N z_n, \mathcal{A}) < \frac{\epsilon}{2}.$$

Hence, for each $n \in \mathbb{T}$ there exists $\tilde{z}_n \in \mathcal{A}$ such that $d(x_{nN}, \tilde{z}_n) < \epsilon$. To conclude the proof apply the same argument above to $\{x_{n-r}\}_{n\in\mathbb{T}}$, for each $r \in \{1, \ldots, N-1\}$, instead of $\{x_n\}_{n\in\mathbb{T}}$. □

Now, we can put Theorem 2.28 and Lemma 3.1 together to prove Shadowing on a neighborhood of $\mathcal{A}$.

Theorem 3.2 (Shadowing on a neighborhood of $\mathcal{A}$). *Suppose that $\mathcal{T}(\cdot)$ is a semigroup on a metric space (M, d), with a global attractor $\mathcal{A}$ such that $\mathcal{T}(1)$ admits the Shadowing Property on $\mathcal{A}$. If $\mathcal{U}$ is a bounded positively invariant neighborhood of $\mathcal{A}$ such that $\mathcal{T} := \mathcal{T}(1)|_{\mathcal{U}}$ is Lipschitz, then $\mathcal{T}$ admits the Shadowing Property on $\mathcal{U}$.*

Proof. Fix $\epsilon > 0$ arbitrary and take $\delta' > 0$ such that for every δ'-pseudo orbit $\{x_n\}_{\in\mathbb{T}}$ in $\mathcal{A}$ there exists $x \in \mathcal{A}$ such that

$$d(\mathcal{T}x_n, x_{n+1}) < \frac{\epsilon}{2}.$$

Fix $\delta_1 > 0$ and $\beta \in (0, \frac{\epsilon}{2})$ satisfying

$$(1 + L_1)\beta + \delta_1 < \delta'. \tag{3.2}$$

From Lemma 3.1, we can fix $\delta \in (0, \delta_1)$ such that every δ-pseudo orbit on $\mathcal{U}$ is β close to the attractor.

Assume that $\{x_n\}_{n\in\mathbb{T}}$ is a $\delta-$pseudo orbit on $\mathcal{U}$. Then there exists $\{\tilde{x}_n\}_{n\in\mathbb{T}} \subset \mathcal{A}$ such that

$$d(x_n, \tilde{x_n}) < \beta, \ \forall n \in \mathbb{T}. \tag{3.3}$$

Now, we will show that $\{\tilde{x}_n\}_{n\in\mathbb{T}}$ is a δ'-pseudo orbit in $\mathcal{A}$. Indeed, note that

$$\begin{aligned} d(\mathcal{T}\tilde{x}_n, \tilde{x}_{n+1}) &\leq d(\mathcal{T}\tilde{x}_n, \mathcal{T}x_n) + d(\mathcal{T}x_n, x_{n+1}) + d(x_{n+1}, \tilde{x}_{n+1}) \\ &\leq L_1\beta + \delta + \beta \\ &< \delta', \end{aligned}$$

where L_1 is the Lipschitz constant of $\mathcal{T}|_{\mathcal{U}}$.

Hence $\{\tilde{x}_n\}_{n\in\mathbb{T}}$ is a δ'-pseudo orbit in $\mathcal{A}$. Then there exists $x \in \mathcal{A}$ such that $\{\mathcal{T}^n x\}_{n\in\mathbb{T}}$ $\frac{\epsilon}{2}$-shadows $\{\tilde{x}_n\}_{n\in\mathbb{T}}$. To complete the proof we will conclude that $\{\mathcal{T}^n x\}_{n\in\mathbb{T}}$ ϵ-shadows $\{x_n\}_{n\in\mathbb{T}}$. Indeed,

$$d(\mathcal{T}^n x, x_n) \leq d(\mathcal{T}^n x, \tilde{x}_n) + d(\tilde{x}_n, x_n) \leq \frac{\epsilon}{2} + \beta < \epsilon, \ \forall n \in \mathbb{T}.$$

□

Corollary 3.3. *Suppose that $\mathcal{T}(\cdot)$ is a Morse-Smale semigroup on a Hilbert space X with global attractor $\mathcal{A}$. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$ such that $\mathcal{T}(1)|_{\mathcal{U}}$ admits the Shadowing Property.*

3.2 α-Hölder Shadowing on a neighborhood of $\mathcal{A}$

In this section we will prove that if $\mathcal{T}$ is a Morse-Smale semigroup, then the α-Hölder Shadowing holds on any positively invariant bounded set $\mathcal{U} \supset \mathcal{A}$. We will see that the α depends on the Lipschitz constant of the map $\mathcal{T}$ (see Theorem 3.10). As in Section 3.1, we will also provide results that are independent of the Shadowing property (Lemma 3.8) and will help us to prove results related to continuity of global attactor on Chapter 4.

As we said, the Hölder exponent α depends on the Lipschitz constant L_1 of the map $\mathcal{T}$, and we will separate our study by cases: $L < 1$, $L_1 = 1$ and $L_1 > 1$. First we will start with contractions.

Lemma 3.4. *Let (M, d) be a complete metric space and $\mathcal{T} : M \to M$ be a contraction, that is, $L_1 := Lip(\mathcal{T}) < 1$. Assume that the fixed point x^* is the global attractor and let $\mathcal{U}$ be a bounded neighborhood of x^*. If $\{x_n\}_{n\in\mathbb{T}}$ is a $d-$ pseudo orbit of $\mathcal{T}$ on $\mathcal{U}$, then*

$$d(x_n, x^*) \leq \left(\frac{1}{1-L_1}\right) d, \ \forall n \in \mathbb{Z}.$$

Proof. Let $\{x_n\}_{n\in\mathbb{T}} \subset \mathcal{U}$ be a d-pseudo orbit of $\mathcal{T}$ and take $\epsilon > 0$ arbitrary. Fix $N \in \mathbb{N}$ such that

$$dist_H(\mathcal{T}^n\mathcal{U}, x^*) < \epsilon, \ \forall n \geq N.$$

From 2.18 we know that $z_n := x_{nN}$ is a $\left(\frac{1}{1-L_1}\right) d$ pseudo orbit of $\mathcal{T}^N$. Then

$$d(z_{n+1}, x^*) \leq d(z_{n+1}, \mathcal{T}^N z_n) + d(\mathcal{T}^N z_n, x^*) < \frac{1}{1-L_1} d + \epsilon, \ \forall n \in \mathbb{Z}.$$

Since $\epsilon > 0$ was arbitrary we guaratee that

$$d(z_{n+1}, x^*) \leq \left(\frac{1}{1-L_1}\right) d, \ \forall n \in \mathbb{Z}.$$

Just translating our original pseudo orbit we conclude the proof. □

Corollary 3.5. *Let X be a Banach space, $\mathcal{T} : X \to X$ be a bounded operator with $\| T \| < 1$ and $\mathcal{U}$ be a bounded neighborhood of the origin. Then, if $\{x_n\}_{n\in\mathbb{T}}$ is a $d-$pseudo orbit on $\mathcal{U}$ we have*

$$\| x_n \| \leq \left(\frac{1}{1-\| T \|}\right) d, \ \forall n \in \mathbb{Z}.$$

For our following results we will use the Lemma below.

Lemma 3.6. *Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup on a Banach Space X with global attractor $\mathcal{A}$ and $\mathcal{U} \subset X$ be a bounded set. Then there exists $\gamma, C > 0$ such that*

$$dist_H(\mathcal{T}^n(\mathcal{U}), \mathcal{A}) \leq Ce^{-\gamma n}, \ \forall n \in \mathbb{N}.$$

Proof. See [17]. □

Now we proceed with the case $L_1 = 1$.

Lemma 3.7. *Let (M, d) be a metric space, $\{\mathcal{T}(t) : M \to M : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and $\mathcal{U} \supset \mathcal{A}$. Assume that $\mathcal{A}$ attracts exponentially the neighborhood $\mathcal{U}$, that is, there exists $C \geq 1, \gamma > 0$ such that*

$$dist_H(T^n\mathcal{U},\mathcal{A}) \leq Ce^{-n\gamma},\ \forall n \in \mathbb{N}.$$

Moreover, suppose that $\mathcal{T} := \mathcal{T}(1)|_{\mathcal{U}}$ *is a Lipschitz map with* $Lip(\mathcal{T}) = 1$. *Then there exists* $d_0 > 0$ *and* $L > 0$ *such that if* $\{x_n\}_{n\in\mathbb{T}} \subset \mathcal{U}$ *is a* $d-$*pseudo orbit of* $\mathcal{T}$, *with* $0 \leq d \leq d_0$ *then*

$$d(x_n,\mathcal{A}) \leq Ld|\ln d|,\ \forall n \in \mathbb{Z}.$$

Proof. Let $\mathcal{U}$ be a neighborhood of $\mathcal{A}$ and $\{x_n\}_{n\in\mathbb{T}} \subset \mathcal{U}$ be a $d-$pseudo orbit. Fix $N \in \mathbb{N}$ such that

$$Ce^{-N\gamma} < d \leq Ce^{-(N-1)\gamma}.$$

From Lemma 2.18 we know that $z_n = x_{nN}$ is a Nd-pseudo orbit of $\mathcal{T}^N$. Note that

$$N < 1 + \frac{1}{\gamma}\ln\left(\frac{C}{d}\right)$$

and therefore

$$\begin{aligned} d(z_{n+1},\mathcal{A}) &\leq d(z_{n+1},T^Nz_n) + d(T^Nz_n,\mathcal{A}) \leq Nd + d \\ &\leq \left(2 + \frac{1}{\gamma}\ln C\right)d - \frac{1}{\gamma}d\ln d \\ &\leq \left(2 + \frac{1}{\gamma} + \frac{1}{\gamma}\ln C\right)d|\ln d|. \end{aligned}$$

From this we conclude our result. □

Now we study the general case $L_1 > 1$.

Lemma 3.8. *Let* $\mathcal{T}(\cdot)$ *be a semigroup with global attractor* $\mathcal{A}$. *Assume that there exists a neighborhood* $\mathcal{U}$ *of* $\mathcal{A}$ *that is positively invariant,* $\mathcal{T} := \mathcal{T}(1)|_{\mathcal{U}}$ *is Lipschitz with Lipschitz constant* L_1 *and there exists* $C, \gamma > 0$ *such that*

$$dist_H(\mathcal{T}^n\mathcal{U},\mathcal{A}) \leq Ce^{-\gamma n},\ \forall n \in \mathbb{N}.$$

Then there exists $C_2 > 0$ *and* $\alpha \in (0,1)$ *such that if* $\{x_n\}_{n\in\mathbb{T}}$ *is a d-pseudo-orbit of* $\mathcal{T}(1)$ *in* $\mathcal{U}$, $0 < d < 1$, *then*

$$d(x_n,\mathcal{A}) \leq C_2 \cdot d^{\alpha},\ \forall n \in \mathbb{T}. \tag{3.4}$$

Proof. Fix $\beta \in (0,1)$. For any $0 < d < 1$ arbitrary take $N \in \mathbb{N}$ such that

$$Ce^{-\gamma N} \leq K(d) \leq Ce^{-\gamma(N-1)}, \tag{3.5}$$

where C and γ are the constants on Lemma 3.6 and

$$K(d) := d^{\gamma(\log_{L_1} e)(1-\beta)}.$$

Define $z_n = x_{nN}$, for all $n \in \mathbb{T}$. From Lemma 2.18 we know that $\{z_n\}_{n\in\mathbb{T}}$ is a C_1d-pseudo-orbit of $\mathcal{T}^N$, where $C_1 = C_1(d) = \dfrac{L_1^N - 1}{L_1 - 1}$. Now , from (3.5), we estimate the value of L_1^N

$$L_1^N \leq L_1^{(\frac{\ln C/K(d)}{\gamma}+1)} = L_1^{(\frac{\ln C}{\gamma}+1)}L_1^{\frac{-\ln K(d)}{\gamma}} \leq \tilde{C}L_1^{\frac{-\ln K(d)}{\gamma}}, \tag{3.6}$$

where $\tilde{C} = L_1^{(\frac{\ln C}{\gamma}+1)}$. Now since $L_1^{\frac{-\ln K(d)}{\gamma}} = (L_1^{\log_{L_1} K(d)})^{\frac{-1}{\gamma\log_{L_1} e}} = K(d)^{\frac{-1}{\gamma\log_{L_1} e}}$, we obtain that

$$L_1^N \cdot d \leq \tilde{C}d^{\beta}.$$

Thus there exists $C' > 0$ independent of d such that

$$d(T^N z_n, z_{n+1}) \leq C_1(d)d \leq C'd^\beta.$$

From (3.5) we conclude that

$$\begin{aligned} d(z_{n+1}, \mathcal{A}) &\leq d(z_{n+1}, T^N z_n) + d(T^N z_n, \mathcal{A}) \\ &\leq C'd^\beta + d^{\gamma(\log_{L_1} e)(1-\beta)} \\ &\leq (C'+1)d^\alpha, \end{aligned}$$

where $\alpha = \min\{\beta, \frac{\gamma(1-\beta)}{\ln L_1}\}$. We can take $\beta = \frac{\gamma}{\gamma + \ln L_1}$ to maximize α. Thus, taking

$$C_2 = L_1^{\frac{\ln C}{\gamma}} \frac{L_1}{L_1 - 1} + 1 \tag{3.7}$$

we obtain (3.4) for the sequence $\{z_n\}_{n\in\mathbb{T}}$. To complete the proof apply the above arguments to the sequence $\{x_{n-r}\}_{n\in\mathbb{T}}$ for each $r \in \{1, \ldots, N-1\}$. □

We would like to highlight that Lemmas 3.1, 3.7 and 3.8 do not assume that $\mathcal{T}(1)$ has the Shadowing property. Those Lemmas will be very useful to obtain results, with very simple and short proofs, about continuity of global attractors.

Now, we prove the Hölder Shadowing property on a neighborhood of $\mathcal{A}$.

Theorem 3.9 (Shadowing is sufficient on $\mathcal{A}$). *Let $\mathcal{T}(\cdot)$ be a semigroup with global attractor $\mathcal{A}$. Assume that $\mathcal{T}$ has the property of Lipschitz Shadowing on $\mathcal{A}$. Moreover, assume that there exists $C > 0$, $\alpha \in (0,1)$ and a bounded neighborhood $\mathcal{U} \supset \mathcal{A}$ such that for any $d \in [0,1)$ and any $d-$pseudo orbit $\{x_n\}_{n\in\mathbb{Z}} \subset \mathcal{U}$ there exists $\{\tilde{x}_n\}_{n\in\mathbb{Z}} \subset \mathcal{A}$ such that*

$$d(x_n, \tilde{x}_n) \leq Cd^\alpha, \ \forall n \in \mathbb{Z}.$$

Then the α-Hölder Shadowing property holds on $\mathcal{U}$.

Proof. Assume that there exists $d_0', L > 0$ such that if a sequence $\{x_n\}_{n\in\mathbb{T}}$ in $\mathcal{A}$ satisfies

$$d(\mathcal{T}x_n, x_{n+1}) \leq d \leq d_0', \forall\ n \in \mathbb{T},$$

then there exists $x \in \mathcal{A}$ such that

$$d(\mathcal{T}^n x, x_n) \leq Ld, \forall\ n \in \mathbb{T}.$$

Suppose that $\{x_n\}_{n\in\mathbb{T}}$ is a d-pseudo-orbit of $\mathcal{T}$ on $\mathcal{U}$, with $0 \leq d \leq 1$. Then, from Lemma 3.8, for each $n \in \mathbb{T}$ there exists $\tilde{x_n}$ on $\mathcal{A}$ and $C > 0$ such that $d(x_n, \tilde{x_n}) \leq Cd^\alpha$. Note that

$$\begin{aligned} d(\mathcal{T}\tilde{x_n}, \tilde{x}_{n+1}) &\leq d(\mathcal{T}\tilde{x_n}, \mathcal{T}x_n) + d(\mathcal{T}x_n, x_{n+1}) + d(x_{n+1}, \tilde{x}_{n+1}) \\ &\leq L_1 Cd^\alpha + d^\alpha + Cd^\alpha \\ &= C_2 d^\alpha, \end{aligned}$$

where $C_2 = (L_1 C + 1 + C)$.

Fix $d_0 \in (0,1)$ satisfying

$$C_2 d_0^\alpha \leq d_0'$$

Hence, for $d \leq d_0$ we have that $\{\tilde{x}_n\}_{n\in\mathbb{T}}$ is a $C_2 d^\alpha$-psedo-orbit on $\mathcal{A}$ and consequently there exists $x \in \mathcal{A}$ such that

$$d(\mathcal{T}^n x, \tilde{x_n}) \leq C_2 Ld^\alpha, \forall\ n \in \mathbb{T}. \tag{3.8}$$

It follows from (3.8) that

$$\begin{aligned} d(\mathcal{T}^n x, x_n) &\leq d(\mathcal{T}^n x, \tilde{x_n}) + d(\tilde{x_n}, x_n) \\ &\leq C_2 L d^\alpha + C_2 d^\alpha \\ &\leq C_3 d^\alpha, \end{aligned} \tag{3.9}$$

where $C_3 = (L+1)C_2$. Hence, the constants d_0, C_3 satisfy our thesis. □

Note that Theorem 3.9 have used Lemma 3.8 in its proof. However, we can proceed as in the proof of Theorem 3.9 and use Lemmas 3.4 and 3.7, instead of Lemma 3.8, to obtain a more general result.

Theorem 3.10. *Let $\mathcal{T}(\cdot)$ be a semigroup on a metric space M with global attractor $\mathcal{A}$ and $\mathcal{U}$ be a positively invariant neighborhood of $\mathcal{A}$ such that $\mathcal{T}(1)|_{\mathcal{U}}$ is Lipschitz, with Lipschitz constant $L_1 \geq 0$, $\mathcal{T}(1)|_{\mathcal{A}}$ has the Lipschitz Shadowing property (on $\mathcal{A}$) and $\mathcal{U}$ is exponentially attracted by $\mathcal{A}$. Then, It holds:*

1. *If $L_1 < 1$ then $\mathcal{T}(1)$ has the Lipschitz Shadowing property on $\mathcal{U}$;*
2. *If $L_1 = 1$ then $\mathcal{T}(1)$ has the Logarithm Shadowing property on $\mathcal{U}$;*
3. *If $L_1 > 1$ then $\mathcal{T}(1)$ has the Hölder Shadowing property on $\mathcal{U}$.*

Proof. It follows imediatly from Lemmas 3.4, 3.7 and the proof of Theorem 3.9. □

The following result is an immediate consequence of Theorem 3.10.

Corollary 3.11. *Let $\mathcal{T}(\cdot)$ be a Morse-Smale semigroup on a Hilbert space X with global attractor $\mathcal{A}$ and that satisfies (H1) and (H2). Then, $\mathcal{T}(1)|_{\mathcal{A}}$ has the Lipschitz Shadowing property. Moreover, for any positively invariant bounded neighborhood $\mathcal{U} \supset \mathcal{A}$ the map $\mathcal{T}(1)|_{\mathcal{U}}$ has the Hölder-Shadowing property.*

Remark 14. *Note that we just proved Theorem 3.10 for negative and global pseudo orbits, that is, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{Z}^-$. In fact, this result is false for positive pseudo orbit. Indeed, suppose that there exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$ satisfying the conditions of Theorem 3.10 for positive pseudo orbits. Fix $x \in \mathcal{U} \setminus \mathcal{A}$ and note that the sequence $x_k = \mathcal{T}^k x$ for all $k \in \mathbb{N}$ is a $d-$ positive pseudo orbit, for all $d \geq 0$. This means that for each $d \in [0, d_0]$ there exists $x_d \in \mathcal{A}$ such that*

$$d(x_k, \mathcal{T}^k x_d) \leq L d^\alpha, \ \forall k \in \mathbb{N}$$

and in particular $d(x, x_d) < L d^\alpha$. Making $d \to 0$ it follows that $x \in \mathcal{A}$, which is a contradiction.

Although the result is not valid for positive pseudo orbits, we can obtain a weaker result for this class of sequences as we show it below.

Corollary 3.12 (Forward Shadowing). *Let $\mathcal{T}$ be a Morse-Smale semigroup on a Hilbert space X satisfying the hypothesis from Theorem 3.10. Then there exists $\alpha, d_0, L > 0$ such that the following statement holds: if $\{x_k\}_{k\in\mathbb{N}}$ is a d- pseudo orbit in $\mathcal{U}$, $0 \leq d \leq d_0$, and and $d(x_0, x_i^*) \leq d$, for some $i \in \{1, \ldots, p\}$ then there exists $x \in \mathcal{A}$ such that*

$$d(\mathcal{T}^k x, x_k) \leq L d^\alpha, \ \forall k \in \mathbb{N}.$$

Proof. Define $x_k = x_i^*$ for all $k \leq 0$. Then $\{x_k\}_{k\in\mathbb{Z}}$ is a d-pseudo orbit in $\mathcal{U}$ and the result follows from Theorem 3.10. □

Corollary 3.13. *Suppose that $\mathcal{T}$ satisfies the hypothesis of Theorem 3.9. Then, for each $0 \leq d \leq d_0$ and $x \in \mathcal{U}$ with $d(x, x_i^*) \leq d$, for some $i \in \{1, \ldots, p\}$ we have*

$$d(\mathcal{T}^n x, \mathcal{A}) \leq L d^\alpha, \ \forall n \in \mathbb{N}.$$

Proof. It follows from the fact that $\mathcal{T}^n x$ is a positive pseudo orbit and from Corollary 3.12. □

Corollary 3.14. *Assume that $\mathcal{T}(\cdot)$ is a semigroup on a metric space (M, d) that has the α-Hölder Shadowing property with parameters $\alpha \in (0, 1]$ and $d_0, L > 0$ on a neighborhood $\mathcal{U}$ of its global attractor $\mathcal{A}$. Then, if $x, \mathcal{T}(1)x \in \mathcal{U}$ and $d(\mathcal{T}(1)x, x) < d$, for some $d < d_0$, there exists a bounded global solution $\xi : \mathbb{R} \to M$ such that $d(x, \xi(n)) < Ld^\alpha$, for all $n \in \mathbb{Z}$.*

Proof. The sequence $\{x_n := x\}_{n\in\mathbb{N}}$ is a d-pseudo orbit of $\mathcal{T}(1)$. □

Chapter 4
Applications

In this chapter we will show applications related to the structural stability of Morse-Smale semigroups and to continuity of global attractors. About the structural stability, it is well known that Morse-Smale semigroups are robust under perturbations, i.e., small perturbations of Morse-Smale semigroups are also Morse-Smale (topological structural stability) and there is a phase-diagram isomorphism between the global attractors (geometrical structural stability). Despite that, it was not possible to know if the orbits from the perturbed global attractor remain close to the orbits from the original global attractor. In fact, we reserved Section 4.1 to show, using the Hölder Shadowing property, that these perturbed orbits are in fact close to the orbits from the original global attractor.

Regarding the applications related to the continuity of global attractors, on Section 4.2 we provide alternative proofs for some results from [51, 24]. In fact, in order to estimate the distance between to global attractors, our approach will be via pseudo-orbits and the proofs are really short and easy to follow. We will also provide a new result that allow us to obtain a better Hölder exponent, that is, we give sufficient hypothesis on the semigroup such that we can approximate α to 1.

4.1 Structural Stability of Morse-Smale Semigroups

In this section we are concerned in the behavior of the bounded global solutions from a perturbed Morse-Smale semigroup, that is, we want to show that the solutions on the global attractor of a perturbed Morse-Smale semigroup are close to the solutions of the limit global attractor (see Corollaries 4.2 and 4.4). We highlight that, if the semigroup is Morse-Smale, our results holds for autonomous and non-autonomous perturbations. The great majority of the results from this section can be found in [9].

We will start considering autonomous perturbations and introducing some concepts related to the robustness of Morse-Smale semigroups.

Definition 33. *Let (M,d) be a metric space and, for each $\eta \in [0,1]$, we consider a semigroup $\mathcal{T}_\eta$ in M. We say that the family of semigroups $\{\mathcal{T}_\eta\}_{\eta\in[0,1]}$ is:*

1. ***continuous*** *at $\eta_0 \in [0,1]$, if*

$$d\left(\mathcal{T}_\eta(t)x, \mathcal{T}_{\eta_0}(t)x\right) \to 0 \text{ as } \eta \to \eta_0,$$

   uniformly for (t,x) in compact subsets $\mathbb{R}^+ \times M$;

2. ***collectively asymptotically compact*** *in $\eta_0 \in [0,1]$ if for any sequences $t_k \geq 0$, with $t_k \to +\infty$, $\eta_k \in [0,1]$ with $\eta_k \to \eta_0$ and $\{x_k\}$ bounded in M, the sequence $\{\mathcal{T}_{\eta_k}(t_k)\, x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence in M.*

Definition 34 (Phase-Diagram Isomorphism)**.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two dynamically gradient semigroups on a metric space (M,d), with global attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ and equilibria $\mathcal{E}_1 = \{x_1^*, \dots, x_p^*\}$ and $\mathcal{E}_2 = \{y_1^*, \dots, y_p^*\}$, respectively. We say that there exists a* ***phase diagram isomorphism*** *between $\mathcal{A}_1$ and $\mathcal{A}_2$ if there exists a bijection $D : \mathcal{E}_1 \to \mathcal{E}_2$ with the property: there exists $1 \le i, j \le p$ and a global solution $\xi : \mathbb{R} \to M$ of $\mathcal{T}_1$ satisfying*

$$\lim_{t\to-\infty} d(\xi(t), x_i^*) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t), x_j^*) = 0,$$

if and only if there is a global solution $\psi : \mathbb{R} \to M$ of $\mathcal{T}_2$ such that:

$$\lim_{t\to-\infty} d(\psi(t), D(x_i^*)) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\psi(t), D(x_j^*)) = 0.$$

We now recall the result that assures the robustness of the Morse-Smale structure under small perturbations [9, Theorem 5.14].

Theorem 4.1. *Let $\{\mathcal{T}_\eta\}_{\eta\in[0,1]}$ be a family of semigroups defined on a Banach space X, which is continuous and collectively asymptotically compact at $\eta = 0$. Assume that:*

(a) $\mathcal{T}_\eta$ has a global attractor $\mathcal{A}_\eta$ for each $\eta \in [0,1]$ and $\overline{\bigcup_{\eta\in(0,1]} \mathcal{A}_\eta}$ is compact;

(b) $\{\mathcal{T}_\eta\}_{\eta\in[0,1]} \subset \mathcal{C}^1(X)$ and $\mathbb{R}^+ \times [0,1] \ni (t,\eta) \mapsto D_x\mathcal{T}_\eta(t) \in \mathcal{L}(X)$ is continuous for each $x \in X$;

(c) $\mathcal{T}_0$ is a Morse-Smale semigroup with non-wandering set $\Omega_0 = \mathcal{E}_0 = \{x_{1,0}^*, \cdots, x_{p,0}^*\}$, where $x_{i,0}^*$ is a hyperbolic equilibrium, $1 \le i \le p$.

Then, there exists an $\eta_1 \in (0,1]$ such that $\mathcal{T}_\eta$ is a Morse-Smale semigroup and there exists a phase diagram isomorphism between the attractors of $\mathcal{T}_\eta$ and $\mathcal{T}_0$, for all $\eta \in (0,\eta_1]$.

Proof. We will provide just a sketch of the proof, since it requires a lot of background. The reader can consult the complete proof in [9, Theorem 5.14].

We will write the proof by steps:

1. It is known that the permanence of hyperbolic equilibria and dynamically gradient structure under small perturbation holds. Hence, we can fix $\eta_1 > 0$ such that $\mathcal{T}_n$ is a dynamically gradient semigroup related to the set of hyperbolic equilibria $\mathcal{E}_\eta = \{x_{1,\eta}^*, \dots, x_{p,\eta}^*\}$, which it's already ordered to be a Morse-decomposition.
2. Since $\mathcal{T}_0$ is Morse-Smale and the unstable/stable manifolds behave continuously under perturbation, then we can reduce $\eta_1 > 0$ in a way that if $z \in W^u(x_{i,\eta}^*) \cap W^s_{loc}(x_{j,\eta})$, then $T_zW^u(x_{i,\eta}^*) + T_zW^s_{loc}(x_{j,\eta}) = X$ for all $\eta \in (0,\eta_1]$. Therefore, it follows from [9, Corollary 5.12] that $\mathcal{T}_\eta$ is Morse-Smale for all $\eta \in [0,\eta_1]$.
3. It remains to show the existence of a phase diagram isomorphism. Define the bijection $\mathcal{D} : \mathcal{E}_0 \to \mathcal{E}_\eta$ by $\mathcal{D}(x_i^*) = x_{i,\eta}$, for $i = 1, \cdots, p$. First we show that if there exists a connection between x_i^* and x_j^*, then there exists a connection between $x_{i,\eta}^*$ and $x_{j,\eta}^*$. Indeed, if $W^u(x_{i,0}^*) \cap W^s_{loc}(x_{j,0}^*) \neq \varnothing$, we can once more use the fact that $\mathcal{T}_0$ is Morse-Smale and the unstable and stable manifolds behave continuously in the $\mathcal{C}^1$ norm to guarantee that $W^u(x_{i,\eta}^*) \cap W^s_{loc}(x_{j,\eta}^*) \neq \varnothing$, for all $\eta \in (0,\eta_1]$, reducing η_1 if necessary.
4. Now we show the converse of item 3 to finish the proof. Assume that we have a sequence $\eta_k \to 0$ and a sequence of global solutions $\xi_k : \mathbb{R} \to X$ of $\mathcal{T}_{\eta_k}$ such that

$$x_{i,\eta_k}^* \xleftarrow{t\to-\infty} \xi_k(t) \xrightarrow{t\to+\infty} x_{j,\eta_k}^*.$$

Then, from [9, Lemma 3.44] there exists a finite collection of global solutions $\varphi_\ell : \mathbb{R} \to X$ of $\mathcal{T}_0$, with $\ell = 1, \cdots, m-1 (m \leqslant p)$, satsifying

$$x^*_{n_\ell} \xleftarrow{t\to-\infty} \varphi_\ell(t) \xrightarrow{t\to\infty} x^*_{n_{\ell+1}}, \text{ for all } \ell = 1, \cdots, m-1,$$

where $x^*_{n_1} = x^*_{i,0}, x^*_{n_m} = x^*_{j,0}$. Therefore $W^u(x^*_{i,0}) \cap W^s_{loc}(x^*_{j,0}) \neq \emptyset$ (see Proposition 1.16), which concludes the proof.

□

Note that the proof of Theorem 4.1 shows the phase diagram isomorphism but does not guarantee if the solutions on $\mathcal{A}_\eta$ are close to the solutions on $\mathcal{A}_0$. Now, we can use the results from Section 3 to obtain this kind of information.

Corollary 4.2. *Assume the same conditions from Theorem 4.1. Then, we already know that there exists $\eta_1 \in (0,1)$ such that $\mathcal{T}_\eta$ is a Morse-Smale semigroup and there exists a phase diagram isomorhispm between $\mathcal{A}_0$ and $\mathcal{A}_\eta$. Moreover, if X is a Hilbert space and $\mathcal{T}_{\eta_0}$ satisfies assumptions (H1) and (H2) from Theorem 0.1, then there exists $C > 0$ and $\alpha \in (0,1)$ such that if ξ_η is a solution of $\mathcal{T}_\eta$, with $\eta \in [0, \eta_1)$, satisfying*

$$x^*_{i,\eta} \xleftarrow{t\to-\infty} \xi_\eta \xrightarrow{t\to+\infty} x^*_{j,\eta},$$

then there exists a global solution ξ_0 of $\mathcal{T}_0$ such that

$$x^*_{i,0} \xleftarrow{t\to-\infty} \xi_0 \xrightarrow{t\to+\infty} x^*_{j,0} \tag{4.1}$$

and

$$d(\xi_\eta(t), \xi_0(t)) \leq C \parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel^\alpha_{L^\infty(\mathcal{A}_\eta, X)}, \ \forall t \in \mathbb{R}.$$

Proof. From assumption (a) we can fix a bounded neighborhood set $\mathcal{U} \supset \bigcup_{\eta\in[0,\eta_1]} \mathcal{A}_\eta$. Let ξ_η be a global solution of $\mathcal{T}_\eta$ as stated before and define $x_n = \xi_\eta(n)$ for all $n \in \mathbb{Z}$. Then,

$$d(\mathcal{T}_0(1)x_n, x_{n+1}) = d(\mathcal{T}_0(1)\xi_\eta(n), \mathcal{T}_\eta(1)\xi_\eta(n)) \leq \parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel_{L^\infty(\mathcal{A}_\eta, X)}$$

and x_n is a $\parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel_{L^\infty(\mathcal{A}_\eta,X)}$ −pseudo orbit of $\mathcal{T}_0(1)$ in $\mathcal{U}$. Therefore, it follows from Corollary 3.11 that there exists $C > 0$ and a bounded solution ξ_0 of $\mathcal{T}_0$ such that

$$d(\xi_0(n), \xi_\eta(n)) \leq C \parallel \mathcal{T}_0(1) - \mathcal{T}_\eta(1) \parallel^\alpha_{L^\infty(\mathcal{A}_\eta, X)}, \ \forall n \in \mathbb{Z}.$$

Since the equilibria $x^*_{k,\eta}$ are close to $x^*_{k,0}$ for η small [9, Theorem 4.24], and $\mathcal{T}_0$ is dynamically gradient from Lemma 1.17, then ξ_0 also satisfies 4.3 if η is small enough. □

In order to show a non-autonomous version of Corollary 4.2, we finally introduce, very briefly, concepts related to the non-autonomous theory.

Definition 35 (Evolution-Process)**.** *Let (M,d) be a metric space, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $\mathcal{P} := \{(x,y) \in \mathbb{T}^2 : x \geq y\}$ and $\mathcal{S} := \{\mathcal{S}(t,s) : (t,s) \in \mathcal{P}\} \subset \mathcal{C}(M)$ be a family of continuous maps. We say that $\mathcal{S}$ is an **evolution process** if*

1. *$\mathcal{S}(t,t) = Id$, for all $t \in \mathbb{R}$;*
2. *$\mathcal{S}(t,\tau)\mathcal{S}(\tau,s) = \mathcal{S}(t,s)$, for all $t \geq \tau \geq s$;*
3. *The map $\mathcal{P} \times M \ni ((t,s),x) \mapsto \mathcal{S}(t,s)x$ is continuous.*

When $\mathbb{T} = \mathbb{R}$ we say that $\mathcal{S}$ is a continuous evolution process. Analogously, if $\mathbb{T} = \mathbb{Z}$ we say that $\mathcal{S}$ is a discrete evolution process. We also say that $\mathcal{S}$ is a linear evolution process if M is a normed vector space X and $\{\mathcal{S}(t,s) : (t,s) \in \mathcal{P}\} \subset \mathcal{L}(X)$.

Note that if $\mathcal{T}$ is a semigroup, then $\mathcal{S}(t,s) := \mathcal{T}(t-s)$ for all $t \geq s$, is an evolution process. Hence, semigroups can be seen as particular cases of evolution processes. Now we generalize the concept of global attractor to the non-autonomous context.

Definition 36 (Pullback Attractor)**.** *Let (M,d) be a metric space and $\mathcal{S}$ be an continuous evolution process on M. The family $\mathbb{A} = \{A(t)\}_{t\in\mathbb{R}}$, with $A(t) \subset M$ non-empty, is a **pullback attractor** of $\mathcal{S}$ if*

1. *$A(t) \subset M$ is compact for each $t \in \mathbb{R}$;*
2. *The family $\mathbb{A}$ is $\mathcal{S}-$invariant, that is, $\mathcal{S}(t,s)A(s) = A(t)$, for all $t \geq s$;*
3. *For any bounded set $B \subset M$ it holds*

$$\lim_{s\to-\infty} dist_H(\mathcal{S}(t,s)B, A(t)) = 0, \ \forall t \in \mathbb{R}.$$

4. *$\mathbb{A}$ is the minimal family of closed subsets that satisfies item 3.*

Now we can put concepts related to dynamically gradient semigroups in the non-autonomous context.

Definition 37. *Let $\mathcal{S}$ be a continuous evolution process in a metric space (M,d) with a pullback attractor $\mathbb{A} = \{A(t)\}_{t\in\mathbb{R}}$. A family $\hat{E} = \{E(t)\}_{t\in\mathbb{R}}$, with $E(t) \subset A(t)$ for all $t \in \mathbb{R}$, is an **isolated $\mathcal{S}$-invariant family** if $\hat{E}$ is $\mathcal{S}$-invariant and there exists $r > 0$ such that if $\xi : \mathbb{R} \to M$ is a global solution of $\mathcal{S}$ with $\xi(t) \in \mathcal{O}_r(E(t))$ for all $t \in \mathbb{R}$ (the notation of $r-$neighborhood is given in Chapter 1), then $\xi(t) \in E(t)$ for all $t \in \mathbb{R}$.*

From now on we will denote a family $\{E(t)\}_{t\in\mathbb{R}}$ by $\hat{E}$.

Definition 38. *Let $\mathcal{S}$ be a continuous evolution process in a metric space (M,d) with a pullback attractor $\mathbb{A} = \{A(t)\}_{t\in\mathbb{R}}$. A collection $\{\hat{E}_1, \cdots, \hat{E}_p\}$ is called a **disjoint collection of isolated $\mathcal{S}$-invariant families** if $\hat{E}_i$ is an isolated $\mathcal{S}$-invariant family for $1 \leq i \leq p$, and there exists $\delta > 0$ such that*

$$\mathcal{O}_\delta(E_j(t)) \cap \mathcal{O}_\delta(E_k(t)) = \emptyset, \ \forall i \leq j < k \leq p, \ \forall t \in \mathbb{R}.$$

The following concepts are the non-autonomous versions of Definitions 7 and 8.

Definition 39. *Let $\mathcal{S}$ be an evolution process on a metric space (M,d) with a pullback attractor $\mathbb{A}$ and a disjoint collection of isolated $\mathcal{S}$-invariant families $\mathcal{E} = \{\hat{E}_1, \cdots, \hat{E}_p\}$. A **homoclinic structure** in $\mathbb{A}$ is a subcollection $\{\hat{E}_{i_1}, \cdots, \hat{E}_{i_k}\}$ of $\mathcal{E}$ together with global solutions $\{\xi_1, \cdots, \xi_k\}$ of $\mathcal{S}$ such that*

$$\lim_{t\to-\infty} d(\xi_j(t), E_{i_j}(t)) = 0 \quad \text{and} \quad \lim_{t\to+\infty} d(\xi_j(t), E_{i_{j+1}}(t)) = 0,$$

for $1 \leq j \leq k$, with $\hat{E}_{i_{k+1}} = \hat{E}_{i_1}$.

Definition 40. *Let $\mathcal{S}$ be an evolution process with a pullback attractor $\mathbb{A}$ in a metric space (M,d) and a disjoint collection of isolated $\mathcal{S}$-invariant families $\mathcal{E} = \{\hat{E}_1, \cdots, \hat{E}_p\}$. The family $\mathcal{S}$ is a **dynamically gradient evolution process** relative to $\mathcal{E}$ if satisfies:*

(G'1) given a bounded global solution $\xi : \mathbb{R} \to X$ of $\mathcal{S}$ there are $1 \leq i, j \leq p$ such that

$$\lim_{t\to-\infty} d(\xi(t), E_i(t)) = 0 \quad \text{and} \quad \lim_{t\to+\infty} d(\xi(t), E_j(t)) = 0;$$

(G'2) there are no homoclinic structures in $\mathbb{A}$.

Now we announce another notions of hyperbolicity (Definition 20), namely hyperbolic global solutions. In fact, the non-autonomous perturbation of an equilibrium point gives origin to another kind of sets, which is one of the motivations to define global hyperbolic solutions. To do that, we first need to recall the definition of exponential dichotomy, which was alredy mentioned before (Remark 2).

Definition 41. *We say that a linear evolution process $L(\cdot,\cdot)$, defined on a Banach space $(X,\|)$, has an **exponential dichotomy** with projection $P(\cdot)=\{P(t):t\in\mathbb{R}\}\subset\mathcal{L}(X)$, exponent ω, and constant M if*

(i) $P(t)L(t,\tau)=L(t,\tau)P(\tau)\quad$ for all $t\geq\tau$;

(ii) The restriction of $L(t,\tau)$ to $R(P(\tau))$ is an isomorphism from $R(P(\tau))$ onto $R(P(t))$, with inverse denoted by $L(\tau,t):R(P(t))\to R(P(\tau))$];

(iii) $P(\cdot)$ splits X into stable and unstable subspaces, so that

$$\|L(t,\tau)(I-P(\tau))\|_{\mathcal{L}(X)}\leq Me^{-\omega(t-\tau)},\quad t\geq\tau,$$
$$\|L(t,\tau)P(\tau)\|_{\mathcal{L}(X)}\leq Me^{\omega(t-\tau)},\quad t\leq\tau.$$

Note that the isomorphism between $R(P(t))$ and $R(P(\tau))$ implies that these subspaces have the same dimension for all $t,\tau\in\mathbb{R}$. Moreover, it is possible to show that the projections $P(t)$ depends continuously on t [20, Corollary 7.18].

From now on we write the process evolution $\mathcal{S}$, defined on a Banach space X, as the perturbation of a linear process, that is, we assume that $\mathcal{S}$ satisfies the variation of constants formula

$$\mathcal{S}(t,\tau)x=L(t,\tau)x+\int_\tau^t L(t,s)f(s,\mathcal{S}(s,\tau)x)ds,\ \forall x\in X, \tag{4.2}$$

where $f:\mathbb{R}\times X\to X$ is continuous differentiable and L is a linear evolution process.

Definition 42 (Hyperbolic Global Solution)**.** *Let $\mathcal{S}$ be a continuous evolution process, in a Banach space X, that satisfies the variation of constants formula (4.2). We say that a bounded global solution $\xi:\mathbb{R}\to X$ is a hyperbolic global solution of $\mathcal{S}$ if the linear evolution process*

$$T_\xi(t,\tau)=L(t,\tau)+\int_\tau^t L(t,\tau)D_xf(s,\xi(s))T(s,\tau)ds$$

has exponential dichotomy, where $D_xf(p,q)$ denote the derivative of f, related to the spatial coordinate, on $(p,q)\in\mathbb{R}\times X$.

Now we have everything we need to define Morse-Smale evolution processes.

Definition 43. *Let $\mathcal{S}$ be an evolution process in a Banach space X with a pullback attractor $\mathbb{A}$. We say that $\mathcal{S}$ is a Morse-Smale evolution process if:*

1. *$\mathcal{S}$ is a dynamically gradient evolution process with respect to a disjoint collection of isolated hyperbolic global solutions $\mathbf{E}=\{\xi_1,\cdots,\xi_p\}$;*
2. *$\mathcal{S}$ is a reversible evolution process, that is, $\mathcal{S}(t,s)|_{A(s)}:A(s)\to A(t)$ is injective, $\mathcal{S}(t,s)$ is differentiable in $A(s)$ and $D_z\mathcal{S}(t,s)\in\mathcal{L}(X)$ is an injectivelinear operator, for each $t\geqslant s\in\mathbb{R}$;*
3. *$W^u_{loc}(\xi_i)$ is finite dimensional, for all $i=1,\cdots,p$;*
4. *Given $\xi_i,\xi_j\in\mathbf{E}$ such that $W^u(\xi_i)(t_0)\cap W^s_{loc}(\xi_j)(t_0)\neq\varnothing$ for some $t_0\in\mathbb{R}$, there exists a point of transversal intersection in $W^u(\xi_i)(t_0)\cap W^s_{loc}(\xi_j)(t_0)$.*

Now we can define the last concepts, which are basically the non-autonomous version of Definitions 33 and 34, before enunciate our result related to the geometric structural stability.

Definition 44. *Let $\mathcal{R} = \{\mathcal{R}(t,s) : (t,s) \in \mathcal{P}\}$ and $\mathcal{S} = \{\mathcal{S}(t,s) : (t,s) \in \mathcal{P}\}$ be two dynamically gradient evolution processes (with pullback attractors) in a metric space (M,d) with respect to the families of isolated invariant bounded solutions $\mathbf{E}_{\mathcal{R}} = \{\xi_1, \cdots, \xi_p\}$ and $\mathbf{E}_{\mathcal{S}} = \{\psi_1, \cdots, \psi_p\}$, respectively. We say that there exists a **phase diagram isomorphism** between $\mathcal{R}$ and $\mathcal{S}$ if there exists a bijection $\mathcal{D} : \mathbf{E}_{\mathcal{R}} \to \mathbf{E}_{\mathcal{S}}$ with the property: there exists a global solution ξ of $\mathcal{R}$ with*

$$\lim_{t\to-\infty} d\left(\xi_i(t), \xi(t)\right) = 0 \quad \text{and} \quad \lim_{t\to\infty} d\left(\xi_j(t), \xi(t)\right) = 0$$

if and only if there exists a global solution ψ of $\mathcal{S}$ with

$$\lim_{t\to-\infty} d\left(\mathcal{B}\left(\xi_i\right)(t), \psi(t)\right) = 0 \quad \text{and} \quad \lim_{t\to\infty} d\left(\mathcal{B}\left(\xi_j\right)(t), \psi(t)\right) = 0.$$

The following result, that can be found in [9, Theorem 10.2], is the non-autonomous version of Theorem 4.1. We will ommit the proof here, since it is very similar to the autonomous case.

Theorem 4.3. *Let $\Lambda = [0,1]$ and let $\{\mathcal{S}_\lambda\}_{\lambda\in\Lambda}$ be a family of evolution processes such that:*

(a) $\mathcal{S}_\lambda$ has a pullback attractor $\mathbb{A}_\lambda = \{A_\lambda(t)\}_{t\in\mathbb{R}}$ and is reversible for each $\lambda \in \Lambda$;

(b) $\overline{\bigcup_{\lambda\in\Lambda}\bigcup_{t\in\mathbb{R}} A_\lambda(t)}$ is compact;

(c) for each compact set $K \subset \mathbb{R}^+ \times X$ we have

$$\sup_{s\in\mathbb{R}} \sup_{(t,x)\in K} \left\|\mathcal{S}_\lambda(t+s,s)x - \mathcal{S}_0(t+s,s)x\right\|_X \to 0 \quad \text{as } \lambda \to 0$$

(d) for each compact set $J \in \mathbb{R}^+$ we have

$$\sup_{s\in\mathbb{R}} \sup_{t\in J} \sup_{z\in A(s)} \left\|\mathcal{S}'_\lambda(t+s,s)(z) - \mathcal{S}'_0(t+s,s)(z)\right\|_{\mathcal{L}(X)} \to 0 \quad \text{as } \lambda \to 0$$

(e) $\mathcal{S}_0$ is a Morse-Smale semigroup (autonomous), that is, $\mathcal{S}_0(t,s) = \mathcal{T}(t-s)$ for all $t \geqslant s$, where $\mathcal{T} = \{\mathcal{T}(t) : t \geqslant 0\}$ is a Morse-Smale semigroup in X, with global attractor $\mathcal{A}$ (the pullback attractor is the family $A_0(t) = \mathcal{A}$, for all $t \in \mathbb{R}$) and set of isolated stationary equilibria $\mathbf{E}_0 = \mathcal{E} = \{e_1^*, \cdots, e_n^*\}$.

Then there exists $\lambda_1 \in (0,1)$ such that $\mathcal{S}_\lambda$ is a Morse-Smale evolution process with respect to $\mathbf{E}_\lambda = \{\xi_{1,\lambda}, \ldots, \xi_{p,\lambda}\}$ for all $\lambda \in [0,\lambda_1)$ and there exists a phase diagram isomorphism between $\mathcal{S}_\lambda$ and $\mathcal{T} = \mathcal{S}_0$.

As we said before, the proof of this result follows the same steps from Theorem 4.1 and once again we can not obtain information related to the distance of the orbits in $\mathbb{A}_\lambda$ in relation to the orbits in $\mathcal{A}_0$, that is, if the bounded global solutions from the perturbed problem remains close the bounded global solutions of the limit problem. With Shadowing, we can again guarantee this property, similar to Corollary 4.2, to non-autonomous perturbations.

Corollary 4.4. *Assume that the same conditions from Theorem 4.3 hold. Then there exists $\lambda_1 \in (0,1)$ such that $\mathcal{S}_\lambda$ is a Morse-Smale evolution process with respect to $\mathbf{E}_\lambda = \{\xi_{1,\lambda}, \ldots, \xi_{p,\lambda}\}$ for all $\lambda \in (0,\lambda_1)$ and there exists a phase diagram isomorphism between $\mathcal{S}_\lambda$ and $\mathcal{T} = \mathcal{S}_0$. Moreover, if X is a Hilbert space and $\mathcal{T} = \mathcal{S}_0$ satisfies conditions (H1) and (H2) from Theorem 0.1, then there exists $C > 0$ and $\alpha \in (0,1)$ such that if ψ_λ is a bounded global solution of $\mathcal{S}_\lambda$, with $\lambda \in [0,\lambda_1)$, satisfying*

$$\lim_{t\to-\infty} d(\psi_\lambda(t),\xi_{i,\lambda}(t))=0 \quad \textit{and} \quad \lim_{t\to+\infty} d(\psi_\lambda(t),\xi_{j,\lambda}(t))=0$$

for some $1\le j<i\le p$, *then there exists a global solution* ψ_0 *of* $\mathcal{S}_0=\mathcal{T}$ *such that*

$$\lim_{t\to-\infty} d(\psi_0(t),x_i^*)=0 \quad \textit{and} \quad \lim_{t\to+\infty} d(\psi_0(t),x_j^*)=0 \tag{4.3}$$

and

$$d(\psi_\lambda(t),\psi_0(t))\le C\sup_{t\in\mathbb{R}} \parallel \mathcal{S}_0(t+1,t)-\mathcal{S}_\lambda(t+1,t) \parallel_{L^\infty(\mathcal{A}_\lambda,X)},\ \forall t\in\mathbb{R}.$$

Proof. Given a global solution ψ_λ as stated before, define $x_n=\psi_\lambda(n)$ for all $n\in\mathbb{Z}$.

We just have to check if x_n is a pseudo-orbit of $\mathcal{T}(1)=\mathcal{S}_0$ so we can proceed as Theorem 4.2. In fact, if $n\in\mathbb{Z}$ then

$$\begin{aligned} d(\mathcal{T}(1)x_n,x_{n+1}) &= d(\mathcal{S}_0(n+1,n)x_n,\mathcal{S}_\lambda(n+1,n)x_n) \\ &\le \sup_{t\in\mathbb{R}} \parallel \mathcal{S}_0(t+1,t)-\mathcal{S}_\lambda(t+1,t) \parallel_{L^\infty(\mathcal{A}_\lambda,X)}. \end{aligned}$$

which concludes the proof. □

4.2 Continuity of Global Attractors

In this section we are concerned about getting results related to continuity of global attractors, using the theory developed on Chapter 3. First we will enunciate some concepts to understand what the authors proved in [51, 19]. Usually, it is hard to obtain the lower semicontinuity without knowing the structure of the global attractor, e.g. if is gradient [31]. We will also revisit the results from [24], which demands the assumption of equicontinuity (see Definition 49).

Definition 45 (Modulus of Continuity)**.** *Let* $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ *be a family of semigroups on a Banach space* X. *Assume that* $\mathcal{T}_\lambda$ *has a global attractor* $\mathcal{A}_\lambda$ *for each* $\lambda\in\Lambda$ *and let* $k:\Lambda\to[0,+\infty)$ *be a continuous function. We say that* $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ *has a* $k-$***modulus of continuity*** *at* λ_0 *if* $k(\lambda_0)=0$ *and there exist* $L,c,t_0>0$ *and a neighborhood* $V_0\ni\lambda_0$ *satisfying*

$$\parallel \mathcal{T}_\lambda(t)-\mathcal{T}_{\lambda_0}(t) \parallel_{L^\infty(U_0,X)}\le ce^{Lt}k(\lambda),\ \forall\lambda\in V_0,\ \forall t\ge t_0.$$

where

$$U_0=\bigcup_{\lambda\in V_0}\mathcal{A}_\lambda.$$

The following definition is an alternative to define continuity of a family of semigroups.

Definition 46 (Degenerated Modulus of Continuity)**.** *Let* $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ *be a family of semigroups on a Banach space* X. *Assume that* $\mathcal{T}_\lambda$ *has a global attractor* $\mathcal{A}_\lambda$ *for each* $\lambda\in\Lambda$. *We say that* $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ *has a* ***degenerated modulus of continuity*** *at* λ_0 *if there exist* $L,c>0$ *such that for each* $\delta>0$ *and* $t_0>0$ *there exists a neighborhood* $V_0\ni\lambda_0$ *such that*

$$\parallel \mathcal{T}_\lambda(t)-\mathcal{T}_{\lambda_0}(t) \parallel_{L^\infty(U_0,X)}\le ce^{Lt}\delta,\ \forall\lambda\in V_0,\ \forall t\in[0,t_0],$$

where $U_0=\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda$.

The following definitions give us the notion of a uniform attraction for the family of global attractor $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$.

Definition 47 (Equi-attraction)**.** *Let $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on a metric space M. Assume that $\mathcal{T}_\lambda$ has a global attractor $\mathcal{A}_\lambda$ for each $\lambda\in\Lambda$. We say that $\mathcal{A}_\lambda$ **equi-attracts** $B\subset X$ if for any $\epsilon>0$ there exists $T>0$ such that*

$$dist_H(B,\mathcal{A}_\lambda)<\epsilon,\ \forall t\geq T,\ \forall\lambda\in\Lambda.$$

Definition 48 (Exponentially Equi-attraction)**.** *Let $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on a metric space M. Assume that $\mathcal{T}_\lambda$ has a global attractor $\mathcal{A}_\lambda$ for each $\lambda\in\Lambda$ and there exists a bounded set $\mathcal{U}\supset\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda$. We say that $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ **exponentially equi-attracts** $\mathcal{U}$ if there exists $C,\gamma>0$ with the property*

$$dist_H(\mathcal{T}_\lambda(\mathcal{U}),\mathcal{A}_\lambda)\leq Ce^{-\gamma t}\ \forall\lambda\in\Lambda.$$

From now on we the concept of continuity given in Definition 14 will be present.

The following Theorem can be found in [19].

Theorem 4.5. *Let $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on a Banach space X. Assume that $\mathcal{T}_\lambda$ has a global attractor $\mathcal{A}_\lambda$ for each $\lambda\in\Lambda$ and this family of global attractors exponentially equi-attracts $\mathcal{U}\supset\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda$ with the same coefficients as in Definition 48. Then, it holds:*

1. *If $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ has a degenerated modulus of continuity at λ_0, then the family $\mathcal{A}_\lambda$ is continuous at λ_0;*
2. *If $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ has a k-modulus of continuity at λ_0, then the family $\mathcal{A}_\lambda$ is continuous at λ_0. Moreover, there exists $M>0$ such that*

$$dist_H(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})+dist_H(\mathcal{A}_{\lambda_0},\mathcal{A}_\lambda)\leq Mk(\lambda)^{\frac{\gamma}{\gamma+L}},$$

where $L>0$ is the constant related to the k-modulus of continuity (see Definition 45).

The next definition is present on [24].

Definition 49. *Let $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups on a metric space (M,d). We say that $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ is **equicontinuous** if for any $T,\epsilon>0$ and bounded set $B\subset M$ there exists $\delta>0$ such that*

$$\sup_{x\in B}d(\mathcal{T}_\lambda(t)x,\mathcal{T}_{\lambda'}(t)x)<\epsilon,\ \forall t\in[0,T],\ \forall\lambda,\lambda'\in\Lambda\text{ with }d(\lambda,\lambda')<\delta.$$

Theorem 4.6. *Let $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a equicontinuous family of semigroups on a Banach space X. Assume that $\mathcal{T}_\lambda$ has a global attractor $\mathcal{A}_\lambda$ for each $\lambda\in\Lambda$ and this family of global attractors equi-attracts any bounded set $B\subset X$. Then, the family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is continuous.*

Proof. See [24]. □

Now we will show our results related to continuity which has very simple proofs. This results were possible due to the relations between a pseudo orbit and the global attractor found on Chapter 3. We start showing our results of upper semicontinuity.

Theorem 4.7. *Let $\mathcal{T}_1$ be a semigroup on a metric space (M,d) with global attractor $\mathcal{A}_1$ and assume that there exists a neighborhood $\mathcal{U}$ of $\mathcal{A}_1$ such that $\mathcal{T}_1|_{\mathcal{U}}$ is Lipschitz. Then, for any $\epsilon>0$ there exists $\delta>0$ such that if $\mathcal{T}_2$ is semigroup with global attractor $\mathcal{A}_2\subset\mathcal{U}$ and $\|\mathcal{T}_1(1)-\mathcal{T}_2(1)\|_{L^\infty(\mathcal{U},M)}<\delta$ then*

$$dist_H(\mathcal{A}_2,\mathcal{A}_1)<\epsilon.$$

Proof. Fix $x \in \mathcal{A}_2$ arbitrary and define $x_n = \mathcal{T}_2^n x$ for all $n \in \mathbb{Z}$. Note that

$$d(\mathcal{T}_1(1)x_n, x_{n+1}) = d(\mathcal{T}_1(1)\mathcal{T}_2^n(1)x_n, \mathcal{T}_2(1)\mathcal{T}_2^n(1)x_n) \leq \| \mathcal{T}_1 - \mathcal{T}_2 \|_{L^\infty(\mathcal{U},M)}$$

and therefore $\{x_n\}_{n\in\mathbb{Z}}$ is a $\| \mathcal{T}_1 - \mathcal{T}_2 \|_{L^\infty(\mathcal{U},M)}$-pseudo-orbit of $\mathcal{T}_1$. Now the result follows from Lemma 3.1. □

Similarly to Theorem 4.7, we can use Lemma 3.8 instead of Lemma 3.1 to obtain a better estimate of the distance of the global attractors.

Theorem 4.8. *Let X be a metric space and $\mathcal{T}_1$ be a semigroup on X with global attractor $\mathcal{A}_1$. Assume that $\mathcal{A}_1 \subset \mathcal{U}$ and $\mathcal{T}_1|_{\mathcal{U}}$ is Lipschitz and $\mathcal{A}_1$ attracts $\mathcal{U}$ exponentially. Then there exists $L > 0$ and $\alpha \in (0,1)$ such that for any semigroup $\mathcal{T}_2$ with global attractor $\mathcal{A}_2 \subset \mathcal{U}$ and $\| \mathcal{T}_1 - \mathcal{T}_2 \|_{L^\infty(\mathcal{U},X)} < 1$ we have*

$$dist_H(\mathcal{A}_2, \mathcal{A}_1) < L \| \mathcal{T}_1(1) - \mathcal{T}_2(1) \|^\alpha_{L^\infty(\mathcal{U},X)} . \tag{4.4}$$

Proof. Fix $x \in \mathcal{A}_2$ and consider the sequence $x_n = \mathcal{T}_2^n x$ for all $n \in \mathbb{Z}$. Note that $x_n \in \mathcal{A}_2$ for all $n \in \mathbb{Z}$ and

$$\| \mathcal{T}_1(1)x_n - x_{n+1} \| = \| \mathcal{T}_1(1)x_n - \mathcal{T}_2(1)x_n \| \leq \| \mathcal{T}_1(1) - \mathcal{T}_2(1) \|_{L^\infty(\mathcal{U},X)}, \ \forall n \in \mathbb{Z},$$

that is, $\{x_n\}_{n\in\mathbb{Z}}$ is a $\| \mathcal{T}_1(1) - \mathcal{T}_2(1) \|_{L^\infty(\mathcal{U},X)}$-pseudo orbit of $\mathcal{T}_1$. Now the results follows from Lemma 3.8. □

Definition 50. *Let (M,d) be a metric space and $Y \subset X$. Define $Lip(Y)$ as the set of all Lipschitz map $f : Y \to X$. We will denote the Lipschitz constant of f as*

$$Lip_f(Y) = \sup_{y,y' \in Y, y \neq y'} \frac{d(fy, fy')}{d(y,y')}.$$

Now we show our results related to continuity of global attractors. As Theorems 4.5 and 4.6 we will require equiattraction of the global attractors.

Theorem 4.9. *Let Λ be a metric space, $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups with global attractors $\mathcal{A}_\lambda$ and $\mathcal{U} \subset X$ be a subset such that $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \subset \mathcal{U}$. Assume that there exists $r > 0$ such that $\mathcal{T}_\lambda(r) \in Lip(\mathcal{U})$ for all $\lambda \in \Lambda$ and*

1. *For each $\lambda \in \Lambda$ there exists a neighborhood $V \ni \lambda$ such that $\sup_{\lambda\in V} Lip_{T_\lambda(r)}(\mathcal{U}) < +\infty$;*
2. *For each $\lambda \in \Lambda$ there exists a neighborhood $V \ni \lambda$ such that the family $\{\mathcal{T}_\lambda\}_{\lambda\in V}$ equi-attracts $\mathcal{U}$;*
3. *$\| T_\lambda(r) - T_{\lambda_0}(r) \|_{L^\infty(\mathcal{U},X)} \to 0$ when $\lambda \to \lambda_0$.*

Then the family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is continuous.

Proof. From the proof of Lemma 3.1 we see that δ only dependes on ϵ, the Lipschitz constant and the number T given as in Definition 47. From this fact and the proof of Theorem 4.7 we complete the proof. □

Theorem 4.10. *Let Λ be a metric space, $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$ be a family of semigroups with exponential global attractors $\mathcal{A}_\lambda$ and $\mathcal{U} \subset X$ be a subset such that $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \subset \mathcal{U}$. Assume that there exists $r > 0$ such that $\mathcal{T}_\lambda(r) \in Lip(\mathcal{U})$ for all $\lambda \in \Lambda$ and assume that the following conditions holds:*

1. *$L_1 := \sup_{\lambda\in\Lambda} Lip_{T_\lambda(r)}(\mathcal{U}) < +\infty$;*

2. $\| \mathcal{T}_\lambda(r) - \mathcal{T}_{\lambda_0}(r) \|_{L^\infty(\mathcal{U},X)} \to 0$ *when* $\lambda \to \lambda_0,\ \forall \lambda_0 \in \Lambda$;

3. *The family* $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ *exponentially equi-attracts* $\mathcal{U}$ *with rate* γ.

Then, $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ *is continuous and there exists* $C > 0$ *such that*

$$dist_H(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) + dist_H(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) \leq C \| \mathcal{T}_\lambda(r) - \mathcal{T}_{\lambda_0}(r) \|_{L^\infty(\mathcal{U},X)}^{\frac{\gamma}{\gamma+\ln L_1}}.$$

Proof. We can reproduce the proofs of Lemma 3.8 and Theorem 4.8 replacing the Lipschitz contants by L_1 and using the coefficient of exponentially equi-atraction γ. □

Now we compare the Hölder exponents from Theorems 4.5 (item 2) and 4.10. It is clear that if we have a $k-$modulus of continuity then we already have

$$\| \mathcal{T}_\lambda(t_0) - \mathcal{T}_{\lambda_0}(t_0) \|_{L^\infty(\mathcal{U},X)} \to 0, \text{ for some } t_0 > 0$$

and we can apply Theorem 4.10 to obtain $\frac{\gamma}{\gamma+\ln L_1}$-Hölder continuity of the global attractors, where L_1 is given in the proof of Theorem 4.10 and γ is the constant of exponential equiattraction. To see the other implication, first note that

$$\begin{aligned}
&\| \mathcal{T}_\lambda(2) - \mathcal{T}_{\lambda_0}(2) \|_{L^\infty(\mathcal{U},X)} \\
&\quad = \| \mathcal{T}_\lambda(1)\mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1)\mathcal{T}_\lambda(1) + \mathcal{T}_{\lambda_0}(1)\mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1)\mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)} \\
&\quad \leq (1 + L_1) \| \mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)} \\
&\qquad\qquad \vdots \\
&\implies \| \mathcal{T}_\lambda(n) - \mathcal{T}_{\lambda_0}(n) \|_{L^\infty(\mathcal{U},X)} \leq (1 + L_1 + \cdots + L_1^n) \| \mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)}.
\end{aligned}$$

Assuming that $L_1 > 1$ and remembering that

$$\sum_{k=0}^{N} x^k \leq \left(\frac{x}{x-1}\right) x^N,\ \forall x > 1,\ \forall N \in \mathbb{N},$$

then

$$\| \mathcal{T}_\lambda(n) - \mathcal{T}_{\lambda_0}(n) \|_{L^\infty(\mathcal{U},X)} \leq \left(\frac{L_1}{L_1 - 1}\right) e^{n \ln L_1} \| \mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)}$$

and we obtain an estimate like Definition 45, where

$$c = \frac{L_1}{L_1 - 1},\ L = \ln L_1 \text{ and } k(\lambda) = \| \mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)}.$$

Therefore, if we asssume that item 2 from Theorem 4.5 holds, we have

$$dist_H(\mathcal{A}_1, \mathcal{A}_2) + dist_H(\mathcal{A}_2, \mathcal{A}_1) \leq M \| \mathcal{T}_\lambda(1) - \mathcal{T}_{\lambda_0}(1) \|_{L^\infty(\mathcal{U},X)}^{\frac{\gamma}{\gamma+\ln L_1}}.$$

Hence, Theorems 4.5 and 4.10 provide the same Hölder exponent, that is, our approach via pseudo-orbits keep the same exponent of Theorem 4.5 when we apply in the context of $k-$modulus of continuty.

Note that the main difference from Theorems 4.5 and 4.10 is that in the first one we just need to know the information on how distant the semigroups are on the fixed time $r > 0$. This actually makes a difference when we define the semigroup $\mathcal{T}_N$ (see Lemma 1.18), which global attractor attracts bounded sets with exponential rate $N\gamma$ when the exponential rate of $\mathcal{T}$ is γ. For example, if we look the approach via $k-$modulus of continuity, we will have

$$\| (\mathcal{T}_N)_\lambda(t) - (\mathcal{T}_N)_{\lambda_0}(t) \|_{L^\infty(U_0,X)} = \| \mathcal{T}_\lambda(Nt) - \mathcal{T}_{\lambda_0}(Nt) \|_{L^\infty(U_0,X)} \leq ce^{NLt}k(\lambda),\ \forall \lambda \in V_0,\ \forall t \geq t_0.$$

Hence, the Hölder exponent on Theorem 4.5 in this case is $\alpha = \frac{N\gamma}{N\gamma+NL} = \frac{\gamma}{\gamma+L}$, that is, to accelerate the semigroup does not help to obtain a better rate. On the other hand, when we look the proof of Lemma 3.8, to accelerate the semigroup makes difference of the Hölder expoenent. To be more precise, if L_1 is the Lipschitz constant of $\mathcal{T}(1)$ and L_n is the Lipschitz constant of $\mathcal{T}(n)$, then then the Hölder exponent relative to the semigroups $\mathcal{T}$ and $\mathcal{T}_N$ are $\alpha_1 = \frac{\gamma}{\gamma+\ln L_1}$ and $\alpha_N = \frac{N\gamma}{N\gamma+\ln L_n}$. Its obvious that we can estimate $L_n \leq L_1^N$, which gives us $\alpha_N \leq \alpha_1$. But if we can estimate L_n in a better way (not exponentially), this can help us to obtain a better Hölder exponent which is closer to one

With this in mind, we present our last result.

Theorem 4.11. *Assume that $\{\mathcal{T}(t) : t \geq 0\}$ is a semigroup with global attractor $\mathcal{A}$ and $\mathcal{U} \supset \mathcal{A}$ is exponentially attracted by $\mathcal{A}$ (as on Lemma 3.8). Denoting $L_n = Lip(\mathcal{T}(n))$ suppose without lost of generality that $L_1 > 1$. Denoting by $\mathcal{T} := \mathcal{T}(1)$, the following holds:*

1. *If*
$$\limsup_{n\to+\infty} \frac{n}{\ln L_n} = +\infty \tag{4.5}$$
then there exists $C_3 > 0$ such that given $\alpha \in (0,1)$ there exists $N \in \mathbb{N}$ such that if $\{S(t) : t \geq 0\}$ is a semigroup with global attractor $\mathcal{A}_s \subset \mathcal{U}$ then
$$dist_H(\mathcal{A}_s, \mathcal{A}) \leq C_3 \| \mathcal{T}^N - S^N \|^\alpha_{L^\infty(\mathcal{U},X)}.$$
In particular,
$$dist_H(\mathcal{A}_s, \mathcal{A}) \leq C_3 \limsup_{n\to+\infty} \| \mathcal{T}^n - S^n \|_{L^\infty(\mathcal{U},X)}.$$

2. *If*
$$\lim_{n\to+\infty} \frac{n}{\ln L_n} = +\infty \tag{4.6}$$
then there exists $C_3 > 0$ such that given $\alpha \in (0,1)$ there exists $N \in \mathbb{N}$ such that
$$dist_H(\mathcal{A}_s, \mathcal{A}) \leq C_3 \inf_{n\geq N} \| \mathcal{T}^n - S^n \|^\alpha_{L^\infty(\mathcal{U},X)}.$$
Consequently, we obtain
$$dist_H(\mathcal{A}_s, \mathcal{A}) \leq C_3 \liminf_{n\to+\infty} \| \mathcal{T}^n - S^n \|_{L^\infty(\mathcal{U},X)}.$$

3. *If (4.5) holds and $\mathcal{T}(1)$ has the Holder-Lipschitz Shadowing on $\mathcal{U}$, then given $\alpha \in (0,1)$ there exists $C_3 > 0$ such that*
$$dist_H(\mathcal{A}_s, \mathcal{A}) \leq C_3 \| \mathcal{T} - S \|^\alpha_{L^\infty(\mathcal{U},X)}.$$

Proof. From the proof of Lemma 3.8 and (3.7) we had

$$C_2 = L_1^{\frac{\ln C}{\gamma}} \frac{L_1}{L_1 - 1} + 1 \quad \text{and} \quad \alpha = \min\{(1-\beta), \frac{\beta\gamma}{\ln L_1}\}.$$

Given $n \in \mathbb{N}$ we can define the new semigroup $\mathcal{T}_n$ as in Lemma 1.18. In this case, $Lip(\mathcal{T}_n(1)) = L_n$, the global attractor of $\mathcal{T}_n$ is $\mathcal{A}$ and the constants of attraction are C and $n\gamma$. We can assume without lost of generality that

$$\sup_{n\in\mathbb{N}} L_n^{\frac{\ln C}{n\gamma}} \frac{L_n}{L_n - 1} \leq L_1^{\frac{\ln C}{\gamma}} \sup_{n\in\mathbb{N}} \frac{L_1^n}{L_1^n - 1} < +\infty$$

Hence there exists $C_3 > 0$ such that

$$\sup_{n\in\mathbb{N}} L_n^{\frac{\ln C}{n\gamma}} \frac{L_n}{L_n - 1} + 1 \leq C_3.$$

Given $\beta \in (0,1)$ take $N \in \mathbb{N}$ such that

$$\frac{\beta n\gamma}{\ln L_n} > 1 - \beta.$$

Note that if $\{x_n\}_{n\in\mathbb{Z}}$ is a $d-$pseudo orbit of $\mathcal{T}_N$ on $\mathcal{U}$ then

$$\sup_{n\in\mathbb{N}} d(x_n, \mathcal{A}) \leq C_3 d^{1-\beta}.$$

Itens 1,2 and 3 follows imediatly from this fact. □

Note that Theorem 4.11 requires that the semigroup has an exponential attractor but the Lipschitz constant of the semigroup doest not grow exponentially on a neighborhood of the global attractor. This means that we have to find a semigroup which has an exponential attractor but does not have any unstable hyperbolic equilibria (x^* is hyperbolic and $W^u(x^*) \neq x^*$), otherwise the Lipschitz constant of the semigroup would grow exponentially. We now provide some examples of semigroups that satisfy these conditions.

Consider the ODE

$$\begin{cases} \dot{x} = x^3(1-x)(1+x), \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{4.7}$$

We will show that the this ODE has a semigroup associated to it that satisfies the assumptions from Theorem 4.11. Note that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3(1-x)(1+x)$ is C^1, and consequently locally Lipschitz. Therefore, from Picard-Lindelöf Theorem [29] there exists $\epsilon > 0$ such that (4.7) has a unique solution $x(\cdot, x_0)$ defined in $[0, \tau_{x_0})$. Moreover, since

$$\langle x^3(1-x)(1+x), x\rangle = x^4 - x^6 = x^4(1-x^2) \leq 0 \quad \text{for} \quad |x| \geq 1, \tag{4.8}$$

then the solution $x(\cdot, x_0)$ does not explode in finite time and $\tau_{x_0} = +\infty$. Therefore we can define the semigroup $\mathcal{T}(t) : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{T}(t)x_0 = x(t, x_0)$ for all $t \geq 0$ and $x_0 \in \mathbb{R}$.

From (4.8) follows that the semigroup $\mathcal{T}$ is dissipative (see Lemma 1.29). Moreover, since bounded sets of $\mathbb{R}$ are relatively compact, then $\mathcal{T}$ is asymptotically compact (see Proposition 1.2) and from [9, Theorem 1.23] the semigroup $\mathcal{T}$ has a global attractor $\mathcal{A}$. Since the equilbria of $\mathcal{T}$ are given by the roots of $f(x) = x^3(1-x)(1+x)$, we have the set of equilibria $\mathcal{E} = \{-1, 0, 1\}$.

Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = \frac{x^4}{4} - \frac{x^6}{6}$ for all $x \in \mathbb{R}$. Note that ϕ is $\mathcal{C}^1$ and $\phi'(x) = x^3(1-x)(1+x)$. Therefore, it follows from [9, Example 3.2] that $\mathcal{T}$ is Gradient and $\mathcal{A} = \bigcup_{x*} W^u(x^*)$ (Theorem 1.3).

Note that

- $f(x) > 0$ for $x \in (-\infty, -1)$;
- $f(x) < 0$ for $x \in (-1, 0)$;
- $f(x) > 0$ for $x \in (0, 1)$;
- $f(x) < 0$ for $x \in (1, +\infty)$.

From that we conclude that the equilbria $\{-1, 1\}$ are stable, the equilibrium 0 is unstable and $\mathcal{A} = [-1, 1]$. Fix a neighborhood $\mathcal{U} \supset \mathcal{A}$. From our previous computations we already know that $\mathcal{T}(t)\mathcal{U} \subset \mathcal{U}$. Moreover, $\mathcal{T}(t) : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$ [20, Theorem 6.33] and consequently Lipscthiz on $\mathcal{U}$.

Now we check that the stable equilbria are hyperbolic (therefore attracts $\mathcal{U}$ at exponential rate) and the unstable equilibrium 0 is not hyperbolic (and therefore the Lipschitz constant L_n of $\mathcal{T}(n)$ does not grow exponentially).

Since the linearization at $x^* \in \{-1, 1\}$ is given by the ODE $\dot{x} = -2x$, then $\{-1, 1\}$ are hyperbolic equilibria. Finally, the linearization at $x^* = 0$ is given by $\dot{x} = 0$ and x^* is not hyperbolic. This shows that $\mathcal{T}$, $\mathcal{A}$ and $\mathcal{U}$ satisfy the assumptions from Theorem 4.11.

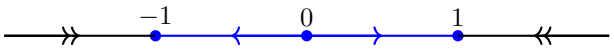


Figure 4.1: Dynamics near $\mathcal{A} = [-1, 1]$

From Example 4.2 it follows that the semigroup $\mathcal{T}$ generated by the ODE

$$\begin{cases} \dot{x} = x^3 - x^5, \\ \dot{y} = y^3 - y^5 \end{cases}$$

has global attractor $\mathcal{A} = [-1, 1] \times [-1, 1]$, which exponentially attracts any bounded set $\mathcal{U} \supset \mathcal{A}$, and the Lipschitz constant of $\mathcal{T}(n)$ does not grows exponentially.

The following example is based on [17]. In this paper the authors have showed the existence of a global attractor with exponentially attraction but that does not demand the hyperbolicity of the equilibria. We now adapt one of their examples.

Consider the ODE in $\mathbb{R}^n$

$$\dot{u} = -(1- \parallel u \parallel^2)^3(2- \parallel u \parallel)u.$$

We can argue as Example 4.2 to show that the this equation has solutions difined for all $t \geq 0$, which defines a semigroup $\mathcal{T}$ with global attractor $\mathcal{A}$.

Multipling both sides by u we have

$$\frac{1}{2}\frac{d}{dt} \parallel u \parallel^2 = -(1- \parallel u \parallel^2)^3(2- \parallel u \parallel) \parallel u \parallel^2$$

and denoting $\rho = \parallel u \parallel$ we have the scalar ODE

$$\frac{1}{2}\frac{d}{dt}\rho^2 = -(1-\rho^2)^3(2-\rho)\rho^2 \Longrightarrow \dot{\rho} = -(1-\rho^2)^3(2-\rho)\rho.$$

This equation also generates a semigroup $\mathcal{T}_\rho$ with global attractor $\mathcal{A}_\rho$. Moreover, we can argue as Example 4.2 to guarantee that $\mathcal{T}_\rho$ is Gradient, since $\rho \mapsto -(1-\rho^2)^3(2-\rho)\rho^2$ is $\mathcal{C}^1$. Therefore $\mathcal{A}_\rho = W^u(x^*)$, where $x^* \in \{0, 1, 2\}$. From a straightforward computation we obtain that the equilibria $\{0, 2\}$ are stable and $\{1\}$ is unstable and not hyperbolic. Therefore $\mathcal{A}_\rho = [0, 2]$ and $\mathcal{A} = B_{\mathbb{R}^n}[0, 2]$.

Note that if $\rho > 2$, then

$$\frac{d}{dt}(\rho - 2) \leq -54(\rho - 2)$$

and consequently

$$d(u(t), \mathcal{A}) \leq \max\{0, \parallel u(0) \parallel -2\}e^{-54t}.$$

Therefore $\mathcal{A}$ attracts point exponentially and from [17] we obtain that $\mathcal{A}$ attracts bounded sets of $\mathbb{R}^n$ exponentially. Moreover, since $\rho^* = 1$ is not hyperbolic, the Lipschitz constant $\tilde{L}_n$ of $\mathcal{T}(n)|\mathcal{A}$ does not grow exponentially. Therefore, for any positively invariant bounded neighborhood $\mathcal{U} \supset \mathcal{A}$, the sequence $L_n = Lip(\mathcal{T}|_\mathcal{U})$ does also not grows exponentially.

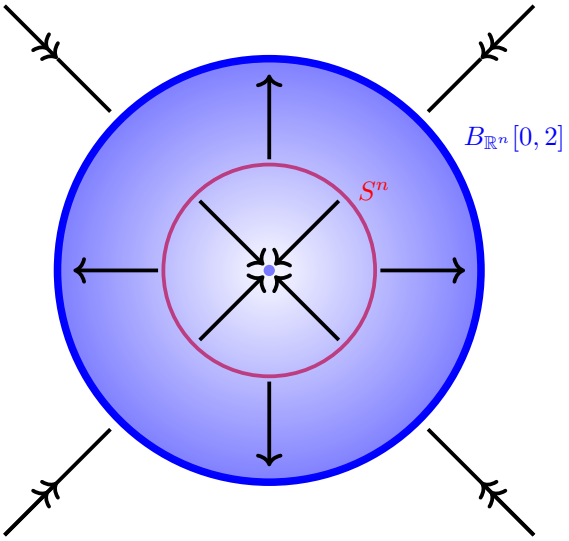


Figure 4.2: Dynamics near $\mathcal{A} = B_{\mathbb{R}^n}[0,2]$

Appendix

In this appendix we recall some concepts and results from Functional Analysis and Dynamical Systems that appear through the text. We will assume that the reader already has familiarity with some of the basics, such as compact operators, Banach spaces, Hilbert spaces, orthogonal projections and adjoint operators [12, 37].

Lemma 4.12. *Let $(X, \| \cdot \|)$ be a Banach space and $F : X \to X$ be a compact map which is differentiable at $x \in X$. Then the Fréchet derivative D_xF of F at x is a compact operator.*

Proof. Assume that D_xF is not a compact operator, that is, $D_xF(B_X[0,1])$ is not relatively compact (and therefore not totally bounded). Then there exists $\epsilon_0 > 0$ and a sequence $x_n \in X$, $\| x_n \| = 1$ such that

$$\| D_xF(x_k) - D_xF(x_l) \| > \epsilon_0, \ \forall l, k \in \mathbb{N}, l \neq k.$$

From the differenciability of F we can take $\delta > 0$ such that

$$\| F(x+v) - F(x) - D_xF(v) \| < \frac{\epsilon_0}{4} \| v \|, \ \forall v \in X, \| v \| < \delta.$$

Fixing $0 < \tau < \delta$ we have

$$\begin{aligned} &\| F(x+\tau x_k) - F(x+\tau x_l) \| \\ \geq &\| D_xF(\tau x_k) - D_xF(\tau x_l) \| - \| F(x+\tau x_k) - F(x) - D_xF(\tau x_k) \| \\ &\qquad - \| F(x+\tau x_l) - F(x) - D_xF(\tau x_l) \| \end{aligned}$$

and therefore

$$\| F(x+\tau x_k) - F(x+\tau x_l) \| > \tau \frac{\epsilon_0}{2} \quad \forall l, k \in \mathbb{N}, \ l \neq k. \tag{4.9}$$

Since F is compact the sequence $\{F(x+\tau x_n)\}_{n\in\mathbb{N}}$ should have a subsequence that converges, which contradicts (4.9). □

Lemma 4.13. *Let S, U be closed subspaces of a Hilbert space X such that $S + U = X$. Then there exists a closed (in X) subspace S' of S such that $S' \oplus U = X$.*

Proof. By hypothesis $S \cap U$ is a closed subspace of the Hilbert space S and therefore we can take the orthogonal subspace $S' = (S \cap U)^{\perp}$ in relation to S, that is,

$$(S \cap U) \oplus S' = S. \tag{4.10}$$

Note that $S' \subset S$ is closed on X. Moreover, if $x \in S' \cap U$ then $x \in (S \cap U)$ and $x \in (S \cap U)^{\perp}$, implying that $S' \cap U = \{0\}$. It remains to prove that $S' + U = X$. Take $x \in X$ arbitrary and fix

$x_u \in U$ and $x_s \in S$ such that $x = x_s + x_u$. From (4.10) there exists $v_u \in S \cap U$ and $x_m \in S'$ such that $x_s = v_u + x_m$. Hence, we have

$$x = x_s + x_u = x_m + (v_u + x_u),$$

which concludes the proof. □

Definition 51 (Projection)**.** *Let X be a normed vector space and $P \in \mathcal{L}(X)$. We say that $\mathcal{P}$ is a* ***continuous projection*** *if $P^2 = P$.*

Lemma 4.14. *Let P, Q be continuous projections on a normed vector space X with $\| P-Q \|_{\mathcal{L}(X)} < 1$. Then $R(P)$ is isomorphic to $R(Q)$.*

Proof. Let B and C be a basis of $R(P)$ and $R(Q)$ respectively. Define the map

$$\begin{aligned} F :& B \to Q(B) \\ & x \mapsto Qx \end{aligned}$$

Given $\{x_1, \ldots, x_n\} \subset B$ note that $\{Q(x_1), \ldots, Q(x_n)\}$ is linear independent, otherwise there exists non-null elements $\{\lambda_i\}_{i=1}^n \in \mathbb{K}$, where $\mathbb{K}$ is the field related to X, such that

$$\begin{aligned} \sum_{i=1}^n \lambda_i Q x_i = 0 = \sum_{i=1}^n \lambda_i (x_i - P x_i) \quad & (\text{because } x_i \in R(P)) \\ \Rightarrow \sum_{i=1}^n (P-Q)\lambda_i x_i = \sum_{i=1}^n \lambda_i x_i & \\ \Rightarrow \| \ (P-Q)(\sum_{i=1}^n \lambda_i x_i) \ \| = \| \sum_{i=1}^n \lambda_i x_i \ \| & \\ \Rightarrow \| \ (\sum_{i=1}^n \lambda_i x_i) \ \| > \| \ (\sum_{i=1}^n \lambda_i x_i) \ \|, & \end{aligned}$$

which is a contradiction. Note that this calculation also proofs that F is injective. Hence $Q(B)$ is linear independent and there exists an injection from $Q(B)$ to C and therefore an injection from B to C. By the same argument we conclude that there exists an injection from C to B and from Schröder–Bernstein Theorem [26, Theorem 0.8] we guarantee that there exists a bijection between B and C, that is, $R(P)$ and $R(Q)$ are isomorphic. □

Lemma 4.15. *Let X be a Hilbert space. If $V, W \subset X$ are vector subspaces where V is finite dimensional, W is closed and $V \oplus W = X$. Then $V^\perp \oplus W^\perp = X$.*

Proof. Assume that $x \in V^\perp \cap W^\perp$. Then,

$$\langle x, v + w \rangle = 0, \ \forall v \in V, \ \forall w \in W \Rightarrow x = 0.$$

To prove that $V^\perp + W^\perp = X$, just note that for every closed subspaces $A, B \subset X$ it holds

$$\overline{A^\perp + B^\perp} = (A \cap B)^\perp.$$

Since $V^\perp$ is closed and $W^\perp$ is finite dimensional (isomorphic to V) we know that $V + W$ is closed, which concludes the proof. □

Lemma 4.16. *Let X be a normed vector space and $P_n, P \in \mathcal{L}(X)$ be continuous projections for all $n \in \mathbb{N}$. Then $P_n \to P$ in $\mathcal{L}(X)$ if and only if $(I-P)P_n \to 0$ and $P(I-P_n) \to 0$ in $\mathcal{L}(X)$.*

Proof. If $P_n \to P$, then obviously $(I-P)P_n \to 0$ and $P(I-P_n) \to 0$. To prove the converse, note that

$$P_n - P = P_n - PP_n + PP_n - P = (I-P)P_n - P(I-P_n) \to 0.$$

□

Corollary 4.17. *Let P_n, P be orthogonal projections on a Hilbert space X. Then $P_n \to P$ if and only if $P_{P(X)^\perp}P_n \to 0$ and $PP_{P_n(X)^\perp} \to 0$, where we used the notation defined in Remark 9.*

Lemma 4.18. *If P, Q are continuous orthogonal projections on a Hilbert space X, then*

$\| PQ \|_{\mathcal{L}(X)} = \| QP \|_{\mathcal{L}(X)}$.

Proof. Just note that P, Q are self-adjoint [12, Chapter 6]. □

Lemma 4.19. *Let E be a normed vector space and V, W be subspaces of E. Let $T : E \to E$ be an injective linear map such that $W \subset T(E)$. Then*

$$T^{-1}(V+W) = T^{-1}(V) + T^{-1}(W),$$

where T^{-1} denotes the inverse image of T.

Proof. The inclusion $T^{-1}(V) + T^{-1}(W) \subset T^{-1}(V+W)$ is obvious. If $x \in T^{-1}(V+W)$ then $Tx = v + w$, with $v \in V$ and $w \in W$. Since $W \subset T(E)$ and T is injective we conclude that $v, w \in T(E)$ and therefore $x = T^{-1}v + T^{-1}w$. □

Lemma 4.20. *Let X be a Hilbert space and S, U be closed subspaces of X such that $S \oplus U = X$. Then, using the notation defined on Remark 9, we have*

$$\| P_{SS^\perp} \circ P_{UU^\perp} \| < 1.$$

Proof. If $S = \{0\}$ or $U = \{0\}$ it is obvious, so we will not consider this cases. Fix $v \in U$ with $\| v \| = 1$ and note that

$$v = P_{US}(v) \quad \text{and} \quad (P_{US} \circ P_{SS^\perp})v = 0.$$

Hence

$$1 = \| v \| = \| P_{US}(v) - P_{US}P_{SS^\perp}(v) \| \leq \| P_{US} \| \| v - P_{SS^\perp}v \| .$$

It follows that

$$\begin{aligned} \frac{1}{\| P_{US} \|^2} &\leq \langle v - P_{SS^\perp}v, v - P_{SS^\perp}v \rangle \\ &= \langle (I - P_{SS^\perp})v, v - P_{SS^\perp}v \rangle \\ &= \langle (I - P_{SS^\perp})v, v \rangle \\ &= \| v \|^2 - \langle P_{SS^\perp}v, (I - P_{SS^\perp})v + P_{SS^\perp}v) \rangle \\ &= 1 - \| P_{SS^\perp}v \| \end{aligned}$$

For any non null projection Q we have

$$\| Q \| = \| Q^2 \| \leq \| Q \|^2 \Rightarrow 1 \leq \| Q \| .$$

Therefore $\| P_{US} \| \geq 1$ and

$$\| P_{SS^\perp} v \| \leq \sqrt{1 - \frac{1}{\| P_{US} \|^2}}, \ \forall \| v \| = 1, v \in U.$$

Thus, we conclude

$$\| P_{SS^\perp} \circ P_{UU^\perp} \| \leq \sqrt{1 - \frac{1}{\| P_{US} \|^2}} < 1.$$

□

The following concepts are important to deal with the theory linear semigroups. We refer to [37, 45] for further details.

Definition 52. *Let X, Y be Banach spaces and X^*, Y^* be their dual spaces, respectively. Given a linear operator $A : D(A) \subset X \to Y$ which is densely defined ($\overline{D(A)} = X$), the **dual operator** $A^* : D(A^*) \subset Y^* \to X^*$ of A is defined as*

$$D(A^*) = \{y^* \in Y^* : \text{there exists } x^* \in X^* \text{ such that } y^*(Ax) = x^*(x), \ \forall x \in D(A)\} \tag{4.11}$$

and $A^* y^* = x^*$, where $x^* \in X^*$ is the (only) element of X^* satisfying (4.11).

*If X is Hilbert and $A = A^*$, we say that A is **self-adjoint** [45, Definition 10.7].*

Definition 53 (Resolvent and Spectrum)**.** *Let X be a Banach space over the field $\mathbb{K}$ and $A : D(A) \subset X \to X$ be a closed linear operator. We define the **resolvent** of A as the set*

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda - A \text{ is an isomorphism}\}.$$

*We also define the **spectrum** of A as $\sigma(A) = \mathbb{K} \setminus \rho(A)$.*

The following result can be found in [47, Lemma 2.2.7] and will help us in Lemma 2.11.

Lemma 4.21. *Let V_i be the neighborhood of x_i^*, mentioned in (2.5) and (2.6), and $d' > 0$ the number related to that neighborhood with the properties stated on Lemma 2.2. Fix $d \in (0, d')$, a neighborhood Q of D (see (2.7)) in $V^{-1}(i + d) \cap \mathcal{A}$ and a continuous subbundle $\mathcal{V}$ on Q satisfying (2.8). Then, we can reduce Q such that the inclination of $\mathcal{V}$ in relation to the direct sum $S_i'(x) \oplus U_i'(x) = X$ goes to 0.*

Proof. We will show that if $x_n \in \gamma^+(Q)$ goes to $x \in W^u(x_i^*)$ then the inclination of $\mathcal{V}(x_n)$ related to decomposition $\mathcal{S}_i'(x_n) \oplus \mathcal{U}_i'(x_n) = X$ goes to 0.

Given $x \in V_i$ and $v \in X$ we know from (2.6) that there exists uniques $v^s \in S_i'(x)$ and $v^u \in U_i'(x)$ such that $v = v^s + v^u$. If $v_u \neq 0$, we define the inclination of v by

$$\alpha(v) := \frac{\| v^s \|}{\| v^u \|}.$$

Note that from our choice of d we have $D \subset V_i$, so for $x \in D$ and $v \in \mathcal{V}(x)$, $v \neq 0$, we can computate the inclination $\alpha(v)$.

First we will show that we can reduce the neighborhood Q of D such that the inclination $\mathcal{V}(y)$, $y \in \gamma^+(Q) \cap V_i$, is bounded by a constant $\Delta > 0$.

Let $C > 0$ and $\lambda_0 \in (0, 1)$ be the constants associated to the hyperbolic set x_i^* and take $T > 0$ and $\Delta \in (0, 1)$ such that

$$C\lambda_0^T < \frac{1}{2} \text{ and } \rho = \frac{1}{R - \Delta} < 1, \quad \text{where} \quad R = \frac{\lambda_0^{-T}}{2C}. \tag{4.12}$$

For $x \in V_i$ define $F(x) : T_xX \to T_{\mathcal{T}(T)x}X$ by $F(x)v = ((D_x\mathcal{T}(T)))v$. Note that if $x_1 = \mathcal{T}(T)x \in V_i$ then $X = S_i'(x) \oplus U_i'(x) = S_i'(x_1) \oplus U_i'(x_1)$ and $F(x)$ can be represented by the matrix below

$$F(x) = \begin{pmatrix} F_{ss}(x) & F_{su}(x) \\ F_{us}(x) & F_{uu}(x) \end{pmatrix}$$

If $x \in x_i^*$ we know from from the hiperbolicity of x_i^* that $F_{us}(x_1^*) = F_{su}(x_1^*) = 0$ and

$$\| F_{ss}(x) \| \leq C\lambda_0^T \quad \text{and} \quad \| F_{uu}(x)^{-1} \| \leq C\lambda_0^T.$$

Thus, from (4.12) we know that for each $x \in x_i^*$ we have

$$\| F_{ss}(x) \| \leq \frac{1}{2} \| v^s \|, \forall v^s \in S(x), \tag{4.13}$$

$$\| F_{uu}(x)v^u \| \geq 2R \| v^u \|, \forall v^u \in U(x). \tag{4.14}$$

Take $\delta \in (0,1)$ such that

$$\delta < \frac{\Delta(R-\delta)(1-\rho)}{4} \quad \text{and} \quad \rho\Delta + \frac{\delta}{R-\delta} < \Delta. \tag{4.15}$$

It follows from estimatives (4.13) and (4.14) that we can reduce V_i so that the properties below are valid for each $x \in V_i$

$$\| F_{ss}(x) \| \leq 1, \| F_{us}(x) \|, \| F_{su}(x) \| \leq \delta, \| F_{uu}(x)v^u \| \geq R \| v^u \|, \forall v^u \in U'(x). \tag{4.16}$$

Define $D_1 = \gamma_{[0,T]}D$ and note that, from our choice of d' and the decreasing behavior of the Lyapunov function, D_1 is a subset of V_i and therefore it makes sense to discuss about the inclination of $\mathcal{V}(y)$ for $y \in D_1$. Since D_1 is compact and $\mathcal{V}$ is already continuous on $\gamma^+(Q) \supset D_1$, it follows from Lemma 2.24 that there exists $K > 0$ such that $\alpha(v) \leq K$, for all $v \in \mathcal{V}(y)$, $y \in D_1$.

Now we will show that there exists $m_0 > 0$ such that $\alpha(\mathcal{V}(y)) < \frac{\Delta}{2}$ for all $y \in D' := \mathcal{T}(m_0T)D_1 \subset V_i$. Indeed, take $x_0 \in D_1$ arbitrary and define the sequence $x_{k+1} = \mathcal{T}(T)x_k$, for all $k \in \mathbb{N}$. Let $v \in \mathcal{V}(x_0)$ non null and define $v_{k+1} = F(x_k)v_k \in \mathcal{V}(x_{k+1})$. We already know that each v_k is writen as $v_k = v_k^s + v_k^u$, where $v_k^s \in S_i'(x_k)$ and $v_k^u \in U_i'(x_k)$. From (4.16) we know that

$$\| v_1^s \| = \| F_{ss}(x_0)v_0^s + F_{su}(x_0)v_0^u \| \leq \| v_0^s \| + \delta \| v_0^u \| \tag{4.17}$$

and

$$\| v_1^u \| = \| F_{us}(x_0)v_0^s + F_{uu}(x_0)v_0^u \| \geq R \| v_0^u \| - \delta \| v_0^s \| = R \| v_0^u \|, \tag{4.18}$$

where the last inequality holds because $(D_x\mathcal{T}(t))S_i'(x) \subset S_i'(x)$ for all $x \in W^s(x_i^*) \cap V_i$ and consequently $F_{us}(x) = 0$ for $x \in W^s(x_i^*) \cap V_i$. It follows from (4.17) and (4.18) that

$$\alpha(v_1) \leq \frac{\| v_0^s \|}{R \| v_0^u \|} + \frac{\delta \| v_0^u \|}{R \| v_0^u \|} \leq \rho\alpha(v_0) + \frac{\delta}{R} \tag{4.19}$$

Proceeding with this process and using (4.15) we conclude that

$$\alpha(v_m) \leq \rho^m\alpha(v_0) + \frac{\delta}{R}(1 + \cdots + \rho^{m-1}) \leq \rho^m\alpha(v_0) + \frac{\delta}{R(1-\rho)} \leq \rho^m\alpha(v_0) + \frac{\Delta}{4} \tag{4.20}$$

Thus, we can find $m_0 > 0$ such that

$$\alpha(\mathcal{V}(y)) < \frac{\Delta}{2} \quad \text{for all} \quad y \in D' := \mathcal{T}(m_0T)D_1 \subset V_i.$$

Since $\mathcal{V}$ is continuous in $\gamma^+(Q)$ and $D' \subset \gamma^+(Q)$, there exists a neighborhood Q' of D' in $\mathcal{A}$ such that $Q' \subset \gamma^+(Q)$ and

$$\alpha(\mathcal{V}(y)) \leq \Delta, \forall\ y \in Q'. \tag{4.21}$$

Repeating the same reasoning above, we can show that for every $x_0 \in Q'$ and $v \in \mathcal{V}(x_1)$ non nulle, where $x_1 = \mathcal{T}(T)x_0$, we have $\alpha(v_1) \leq \Delta$. Indeed, given $x_0 \in Q'$ and $v \in \mathcal{V}(x_1)$ we have

$$\| v_1^s \| = \| F_{ss}(x_0)v_0^s + F_{su}(x_0)v_0^u \| \leq \| v_0^s \| + \delta \| v_0^u \|$$

and

$$\| v_1^u \| = \| F_{us}(x_0)v_0^s + F_{uu}(x_0)v_0^u \| \geq R \| v_0^u \| - \delta \| v_0^s \| .$$

It follows from (4.15) that

$$\alpha(v_1) \leq \frac{\| v_0^s \|}{R \| v_0^u \| - \delta \| v_0^s \|} + \frac{\delta \| v_0^u \|}{R \| v_0^u \| - \delta \| v_0^s \|} \leq \rho\Delta + \frac{\delta}{R - \delta} \leq \Delta.$$

Proceeding like this we conclude that

$$\alpha(v_m) \leq \rho\Delta + \frac{\delta}{R - \delta} \leq \Delta\ \forall m.$$

Now we can decrease Q so $\mathcal{T}(m_0T)Q \subset Q'$ and

$$V(x) > i + \frac{d}{2},\ \forall x \in Q. \tag{4.22}$$

With this argument, we guarantee that

$$\mathcal{V}(\mathcal{T}(t)x) \leq \Delta,\ \forall x \in Q,\ \forall t \geq m_0T. \tag{4.23}$$

This concludes the first part of the proof.

Now we have the necessary tools to prove the continuity of the subbundle $\mathcal{V}$ on $W^u(x_i^*)$. From Remark 7 there exists $d_1 \in (0, 1)$ that satisfies

$$V^{-1}(i - d) \cap W^u(x_i^*) \subset V_i, \forall\ d \in [0, d_1].$$

Define $\mathcal{H}_2 = V^{-1}(i - d_1) \cap W^u(x_i^*)$. To show that $\mathcal{V}$ is continuous on $W^u(x_i^*)$ it is sufficiently to prove the continuity on x_i^* and $\mathcal{H}_2$, since for every point x on $W^u(x_i^*) \setminus \{x_i^*\}$ there exists $t \in \mathbb{R}$ such that $\mathcal{T}(t)x \in \mathcal{H}_2$. We will show the continuity on $\mathcal{H}_2$ and the proof for x_i^* follows analagously.

Let $y \in \mathcal{H}_2$ and $y_n \in \gamma^+(Q)$ such that $y_n \to y$. We need to show that $\alpha(\mathcal{V}(y_n))$ goes to 0. Note that

$$i - d_1 \leq V(\mathcal{T}(t)y) \leq i, \forall t \in (-\infty, 0]. \tag{4.24}$$

We know that there exists $x_n \in Q$ and $t_n > 0$ such that $y_n = \mathcal{T}(t_nT)x_n$ and $V(y_n) \to i - d_1$. It follows from (4.22) and (4.24) that $t_n \to \infty$.

Note that $y \in W^u(x_i^*)$ implies $F_{us}(y) = 0$, since the family $\mathcal{U}$ is $D\mathcal{T}(t)$-invariant. Hence, for any $\epsilon > 0$ there exists a nighborhood $V(\epsilon)$ of $W^u(x_i^*)$ such that

$$\| F_{su}(x) \| \leq \epsilon_1 := \frac{\epsilon(R - \delta)(1 - \rho)}{2},\ \forall x \in V(\epsilon). \tag{4.25}$$

Since $y \in V(\epsilon)$ and $y_n \to y$, we can assume that for each n there exists s_n such that $\mathcal{T}(tT)x_n \in V(\epsilon)$, for $s_n \leq t \leq t_n$, and $\nu(n) = t_n - s_n \to +\infty$. Moreover, we can assume that $s_n > m_0$ for all n.

Denote $z_n = \mathcal{T}(s_nT)x_n \in V(\epsilon)$ and take $v_0 \in \mathcal{V}(z_n)$ non nulle. Once again, define $v_{k+1} = F(x_k)v_k$ for all $k > 0$. From (4.18) and (4.25) we obtain

$$\| v^s_{k+1} \| = \| F_{ss}(x_k)v^s_k + F_{su}(x_k)v^u_k \| \leq \| v^s_k \| + \epsilon_1 \| v^u_k \|$$

and

$$\| v^u_{k+1} \| = \| F_{us}(x_k)v^s_k + F_{uu}(x_k)v^u_k \| \geq R \| v^u_k \| - \delta \| v^s_k \|$$

which implies

$$\begin{aligned} \alpha(v_1) &\leq \rho\Delta + \frac{\epsilon_1}{R-\delta} \\ &\cdots \\ \alpha(v_k) &\leq \rho^k\Delta + \frac{\epsilon_1}{(R-\delta)(1-\rho)} \leq \rho^k\Delta + \epsilon, \ \forall\ 0 \leq k \leq \nu(n). \end{aligned}$$

Since $\nu(n) \to +\infty$ then $\alpha(v_{\nu(n)}) \to 0$ and we conclude that the subbundle $\mathcal{V}$ is continuous on $(\gamma(Q)^+ \cup W^u(x^*_i)) \cap V_i$. □

Lemma 4.22. *Let X be a normed vector space and $V, W, Z \subset X$ be closed subspaces. If*

$$V \oplus W = X \quad \text{and} \quad Z \oplus W = X$$

then there exists a bounded linear map $\psi : Z \to W$ such that $V = G(\psi)$, where $G(\psi)$ is the graph of ψ.

Proof. Define

$$\begin{aligned} \psi : & Z \to W \\ & z \mapsto -P_{WV}(z) \end{aligned}$$

Note that ψ is well defined and continuous (because the subspaces are closed). Moreover, $z + \psi(z) = P_{VW}(z) \in V$ and therefore $G(\psi) \subset V$. In order to prove that $V \subset G(\psi)$ fix $v \in V$ and define $z = P_{ZW}(v) \in Z$. Then

$$\begin{aligned} z + \psi(z) &= z - P_{WV}(z) = P_{ZW}(v) - P_{WV}P_{ZW}(v) = v - P_{WZ}(v) - P_{WV}P_{ZW}v \\ &= v - P_{WZ}(v) - P_{WV}(v - P_{WZ}(v)) = v - P_{WZ}(v) - P_{WV}(v) + P_{WZ}(v) = v - P_{WV}(v) = v, \end{aligned}$$

which concludes the proof. □

Lemma 4.23. *Let $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ be two normed vector spaces and $T_n, T : X \to Y$ be bounded linear maps such that $T_n \to T$ in $\mathcal{L}(X, Y)$. Denote by $G(T_n)$ and $G(T)$ the graphs of T_n and T, that are closed subspaces of $(X \times Y, |\cdot|_{X\times Y})$, where $|(x, y)| = \| x \|_X + \| y \|_Y$ for all $(x, y) \in X \times Y$. Then, $P_{G(T_n)} \to P_{G(T)}$, where we are using the notation estabilished in Remark 9.*

Proof. We have to show that $(I - P_{G(T)})P_{G(T_n)} \to 0$ and $P_{G(T)}(P_{G(T_n)} - I) \to 0$. We will show the first convergence and the second one will follow analogously from Lemma 4.18. Fix $v \in X \times Y$ with $|v| = 1$. Then, $P_{G(T_n)}v = (w_n, T_nw_n)$ for some $w_n \in X$ and $\| w_n \|_X \leq |P_{G(T_n)}v| \leq 1$. Hence

$$\begin{aligned} |P_{G(T_n)}v - P_{G(T)}P_{G(T_n)}v| &= d(P_{G(T_n)}v, G(T)) = d((w_n, T_nw_n), G(T)) \leq |(w_n, T_nw_n) - (w_n, Tw_n)| \\ &= \| T_nw_n - Tw_n \|_Y \leq \| T_n - T \|_{\mathcal{L}(X,Y)} \to 0 \ \text{ uniformly for } |v| \leq 1. \end{aligned}$$

□

Lemma 4.24. *Let x_i^* be a closed, maximal invariant set of a gradient semigroup $\mathcal{T}(t)$. Then there exists a neighborhood N of x_i^* such that $W^s(x_i^*) \cap N$ is closed in N. It $\mathcal{T}(t)$ is a group in $\mathcal{A}$, we have the same statement to the set $W^u(x_i^*) \cap N$.*

Proof. Take a neighborhood N of x_i^* such that $N \subset V^{-1}\left(i - \frac{1}{4}, i + \frac{1}{4}\right)$, where V is the Lyapunov function related to the semigroup $\mathcal{T}(t)$. Suppose that $x_n \in W^s(x_i^*) \cap N$ converges to $x \in N$. If $x \notin W^s(x_i^*)$, then $x \in W^s(x_j^*)$, for some $j < i$ and there exists $s > 0$ such that $\mathcal{T}(s)x < i - \frac{1}{2}$. Hence, for large n we have $V(\mathcal{T}(s)x_n) < i - \frac{1}{4}$, which is a contradiction. Therefore $x \in W^s(x_i^*)$ and $W^s(x_i^*) \cap N$ is closed as we claim. To prove the case $W^u(x_i^*) \cap N$ just use the same reasoning to the semigroup $\mathcal{T}(-t)$. □

Lemma 4.25. *Let $(X, \| \cdot \|)$ be a Banach space. For each $x \in W^s_{loc}(x_i^*)$ we have that $S(x) = T_x W^s(x_i^*)$ is closed.*

Proof. Since $W^s_{loc}(x_i^*)$ is the graph of a C^1 map $\Psi : S(x_i^*) \to U(x_i^*)$, it follows that $T_x W^s(x_i^*)$ is the graph of $D_x\Psi$. From the continuity of $D_x\Psi$ it follows that $S(x_i^*)$ is closed in $(X, |\cdot|)$, where $X = S(x_i^*) \oplus U(x_i^*)$ and

$$|v_s + v_u| = \| v_s \| + \| v_u \|, \ \forall v^s \in S(x_i^*), \ \forall v^u \in U(x_i^*).$$

From the open mapping theorem we have that $|\cdot|$ and $\|\cdot\|$ are equivalent norms in X, that is, the inclusion

$$i : (X, \| \cdot \|) \to (X, |\cdot|)$$

is continuous. Hence $i^{-1}(S(x)) = S(x)$ is closed in $(X, \| \cdot \|)$. □

Lemma 4.26. *The families of closed vector subspaces $\{\mathcal{S}(x) = T_x W^s_{loc}(x_i^*) : x \in W^s_{loc}(x_i^*)\}$ and $\{\mathcal{U}(x) = T_x W^u(x_i^*) : x \in W^u_{loc}(x_i^*)\}$ are continuous, in the sense of Definition 30.*

Proof. It follows from the fact that $W^s_{loc}(x_i^*)$ and $W^u_{loc}(x_i^*)$ are graphs of $\mathcal{C}^1$ maps (Theorem 1.12). □

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