Universidad Autónoma de Madrid

Some topics in Parabolic and Elliptic PDEs

Mikel Ispizua Moreno

Supervisores: Matteo Bonforte y María del Mar González

Departamento de Matemáticas
Universidad Autónoma de Madrid

Marzo, 2024

Memoria para optar al título de
DOCTOR EN MATEMÁTICAS

Agradecimientos

Gracias Mar y Matteo por darme la oportunidad de trabajar en esto durante cuatro años, por enseñarme con paciencia y entusiasmo, por acogerme con cariño, por las "charlas a la pizarra" y, sobre todo, por no dejarme pagar un solo café en todo este tiempo. Gracias Peio e Irene, por compartir conmigo inquietudes, penas, viajes y piso. Molte grazie a Maria G. per avermi dato l'opportunità di ampliare i miei orizzonti matematici e per avermi coinvolto in un problema così bello. Grazie anche a Nicola e Lorenzo, il cui prezioso aiuto ha sbloccato problemi che da solo non sarei stato in grado di risolvere. Eskerrak eman nahi nizkieke ere txikitatik nire interes zientifikoa sustatzea lortu zuten irakasleei, eskerrik asko Amaia, Eneko, Javi... Gracias a esas personas que durante estos últimos cuatro años me han sostenido y me han soportado en los momentos en los que las cosas se han puesto más cuesta arriba. Y gracias Aita y Ama. Gracias Aita por darme lo que sea que hay que tener para llevar esto a cabo. Gracias Ama por enseñarme que uno tiene que lanzarse aun cuando no las tiene todas consigo. Gracias por el esfuerzo titánico que habéis hecho conmigo y con mis hermanos, a quienes también doy gracias, porque sí, porque son mis hermanos.

Introducción y conclusiones

Tipos de EDPs

Esta tesis es una recopilación de tres trabajos que el autor ha llevado a cabo en los últimos cuatro años. El área común de estos trabajos son las Ecuaciones en Derivadas Parciales (EDPs) parabólicas y elípticas. Sea $\Omega \subset \mathbb{R}^N$ un dominio con borde $\partial\Omega$, entonces, la EDP elíptica canónica es la *Ecuación de Laplace*:

$$-\Delta u = 0, \quad \text{en } \Omega,$$

con condiciones de contorno en $\partial\Omega$. Entre muchas interpretaciones, una típica de esta ecuación es que u representa una cantidad física (como la concentración de un químico, la temperatura, etc.) en equilibrio dentro del dominio Ω bajo ciertas condiciones en $\partial\Omega$. Para este tipo de ecuaciones un área de estudio significativa es el cálculo de autovalores y de cantidades asociadas a estos, como la rigidez torsional o la capacitancia. En particular, en el Capítulo 2, abordamos el problema de *optimización de forma*, buscando obtener estimaciones precisas basadas en la forma del dominio para una de estas cantidades mencionadas, la *δ-rigidez torsional en la frontera*.

En dicho capítulo presentamos los resultados del artículo *A Steklov version of torsional rigidity* ([31]), un trabajo conjunto con Lorenzo Brasco y María del Mar González.

Por otro lado, la ecuación diferencial parabólica estándar es la *Ecuación del Calor*:

$$u_t = \Delta u, \quad \text{en } (0,T) \times \Omega,$$

usualmente asociada a un estado inicial u_0. Pensando en u como la temperatura en el momento t y la posición x dentro de una región Ω, esta ecuación determina que la temperatura varía en el tiempo a la misma velocidad que sus segundas derivadas espaciales. Esta ecuación tiende a "suavizar" la función desconocida u, a regularizarla. Este tipo de ecuaciones se denominan *ecuaciones de difusión*.

Tanto la ecuación de Laplace como la ecuación del calor son lineales, es decir, que la suma de dos soluciones también es una solución. Sin embargo, la mayoría de las EDPs interesantes no lo son. Entre todas las ecuaciones *no lineales* la más cercana a la Ecuación del Calor es,

$$u_t = \Delta u^m.$$

Cuando $m > 1$ esta ecuación se conoce como la *Ecuación de Medios Porosos* (PME en inglés) y, cuando $m < 1$, se llama la *Ecuación de Difusión Rápida* (FDE en

inglés). El nombre "Ecuación de Medios Porosos" proviene de su uso para describir el flujo de un gas ideal en un medio poroso homogéneo.

Tanto la PME como la FDE pertenecen a una clase más amplia de ecuaciones de la forma

$$u_t = \nabla \cdot (a(u)\nabla u),$$

donde $a(u)$ se denomina coeficiente de difusión. En el caso de la PME y la FDE, este coeficiente toma la forma de $a(u) = mu^{m-1}$. Cuando $m > 1$ la ecuación degenera si $u = 0$, es decir, $a(0) = 0$ y la ecuación no difunde, por lo que las soluciones con datos iniciales de soporte compacto permanecen con soporte compacto, presentando una especie de frontera libre donde la solución se anula. Sin embargo, en el caso de FDE el coeficiente de difusión $a(u)$ tiende a infinito cuando $u \to 0$, por lo que incluso si los datos iniciales tienen soporte compacto, la solución será positiva para todo $t > 0$. Como puede observarse, un pequeño cambio en la "no linealidad" genera grandes cambios en el comportamiento de la ecuación.

Si ahora agregamos un término de *convección* a cualquier ecuación de difusión, la convertimos en una *ecuación de convección-difusión*:

$$u_t = \nabla \cdot (a(u)\nabla u) - \nabla \cdot (\mathbf{b}\, u),$$

donde $\mathbf{b}$ denota la velocidad del flujo. Continuando con el ejemplo anterior, esta ecuación modela la concentracion de un gas ideal en un medio poroso condicionado por un flujo determinado por $\mathbf{b}$. Una aplicación muy frecuente de las ecuaciones de convección-difusión la encontramos en la biología a la hora de describir movimientos colectivos de células, como por ejemplo, en el modelo de quimiotaxis de Keller-Segel. En este modelo las células están sujetas a un proceso de difusión y al mismo tiempo son atraídas por ciertas regiones ricas en nutrientes o por substancias químicas generadas por otras células.

El Capítulo 4 presenta los resultados del artículo *A quasilinear Keller-Segel model with saturated discontinuous advection* ([75]) en el cual el autor y sus colaboradores, Maria Gualdani y Nicola Zamponi, estudian una ecuación cuasilineal derivada de un modelo de quimiotaxis.

En los procesos de difusión descritos anteriormente se asume que las partículas que forman el sistema se mueven de manera continua o, alternativamente, en intervalos cortos que pueden representarse mediante una distribución gaussiana. Al depender únicamente del entorno más inmediato, es razonable que el movimiento de las partículas esté gobernado por un operador local, por ejemplo, por el laplaciano en el caso del movimiento browniano. Sin embargo, existen procesos de *difusión anómala* que requieren operadores no locales debido a que el movimiento de las partículas depende de la distribución de las mismas en todo el espacio. Entre estos operadores el más utilizado y estudiado es el laplaciano fraccionario $(-\Delta)^s$, con $s \in (0,1)$, que en $\mathbb{R}^N$ se puede definir, entre otras muchas formas equivalentes, a través de la integral singular

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \iint_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

donde $C_{N,s}$ es una constante de normalización y P.V. indica el valor principal. Como se puede observar, la integral es singular en $x = y$, lo que significa que para que la expresión sea calculable, se requiere cierta regularidad para u en x. Esta singularidad

es lo que, de alguna manera, representa la diferenciación de u, ya que necesitaremos cierta regularidad en el punto x para que la integral sea finita. Como se menciona anteriormente, en $\mathbb{R}^N$ hay varias maneras equivalentes de definir el laplaciano fraccionario (a través de la Transformada de Fourier, mediante el semigrupo de calor asociado con el operador $(-\Delta)^s$, etc.). Sin embargo, como veremos más adelante, en dominios acotados estas definiciones pueden no ser equivalentes.

El Capítulo 3 está dedicado a la ecuación de difusión rápida con operadores fraccionarios, presentando los resultados del artículo *The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains* ([23]) y de otro artículo en preparación que es la continuación natural de este. Ambos artículos son trabajos conjuntos con Matteo Bonforte y Peio Ibarrondo.

A continuación presentamos una introducción breve pero más específica de cada uno de los capítulos de forma independiente.

Rigidez torsional en la frontera

Aquí presentamos el problema abordado en [31],

A Steklov version of torsional rigidity,

un trabajo conjunto del autor con Lorenzo Brasco y María del Mar González, publicado en 2023 en *Communications in Contemporary Mathematics*.

Tomemos la ecuación de Laplace, agreguemos un término constante en el interior e impongamos condiciones de contorno de Dirichlet, es decir:

$$\begin{cases} -\Delta u &= 1, \quad \text{en } \Omega, \\ u &= 0, \quad \text{en } \partial\Omega. \end{cases}$$

La solución a esta ecuación se llama *función de tensión* y representa la distribución de la tensión generada en una viga cuando se aplica un momento sobre el eje longitudinal. Ω representa la sección de la viga y la tension total resultante se llama *rigidez torsional*, la cual puede expresarse como

$$\mathcal{T}(\Omega) = \int\!\!\int_\Omega u\,dx, \quad \text{o equivalentemente,} \quad \mathcal{T}(\Omega) = \sup_{\varphi|_{\partial\Omega}=0} \frac{\left(\int_\Omega \varphi\,dx\right)^2}{\int_\Omega |\nabla\varphi|^2\,dx}.$$

Sea $\lambda(\Omega)$ el primer autovalor de Dirichlet, es decir, la menor constante $\lambda > 0$ que satisface

$$\begin{cases} -\Delta u &= \lambda u, \quad \text{en } \Omega, \\ u &= 0, \quad \text{en } \partial\Omega. \end{cases}$$

Una desigualdad clásica de Pólya [122] establece que

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \leq 1.$$

Motivados por esta relación entre el primer autovalor de Dirichlet y la rigidez torsional, el objetivo del Capítulo 2 es encontrar un nuevo concepto de *rigidez torsional en la frontera* físicamente coherente y matemáticamente interesante relacionado con

un autovalor de Steklov modificado. Definimos este autovalor como la primera constante $\sigma > 0$ que satisface

$$\begin{cases} -\Delta u &= \delta u, \quad \text{en } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= \sigma\, u, \quad \text{en } \partial\Omega, \end{cases}$$

donde ν_Ω es el vector normal unitario exterior de Ω.

Ahora, definimos la *δ-rigidez torsional fronteriza* como

$$T(\Omega;\delta) = \sup \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}\right)^2}{\int_\Omega |\nabla\varphi|^2 \, dx + \delta^2 \int_\Omega \varphi^2 \, dx},$$

donde $\mathcal{H}^{N-1}$ representa la medida de Hausdorff (N − 1)-dimensional de la frontera $\partial\Omega$.

Desde un punto de vista variacional, cuando $\delta = 1$, dicho objeto corresponde a la constante óptima para el encaje de traza de $W^{1,2}(\Omega)$ en $L^1(\partial\Omega)$.

El problema de Cauchy-Dirichlet para difusiones no locales singulares

Esta sección está dedicada al problema abordado en [23],

The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains,

un trabajo conjunto del autor con Matteo Bonforte y Peio Ibarrondo, publicado en 2023 en *Discrete and Continuous Dynamical Systems*. En el Capítulo 3 también presentamos algunos resultados nuevos de otro artículo en preparación que es la continuación natural del mencionado anteriormente.

Estudiamos el Problema de Cauchy-Dirichlet (CDP en inglés) para una ecuación de difusión no lineal y no local de tipo singular de la forma

$$\begin{cases} \partial_t u(t,x) = -\mathcal{L} u^m(t,x) & \text{en } (0,\infty)\times\Omega\,, \\ u(t,\cdot) = 0 & \text{en la frontera lateral, } \forall t>0\,, \\ u(0,\cdot) = u_0 & \text{en } \Omega\,, \end{cases} \tag{1}$$

donde $0 < m < 1$ y $\Omega \subset \mathbb{R}^N$ es un dominio suave y acotado con $N \geq 1$.

El operador de difusión lineal $\mathcal{L}$ es un operador sub-Markoviano, perteneciente a una gran clase de operadores (locales y) no locales, que incluye las tres versiones diferentes del "laplaciano fraccionario", mientras que la no linealidad es de tipo singular, es decir, $u^m = |u|^{m-1}u$ con $0 < m < 1$. La ecuación prototipo es la Ecuación de Difusión Rápida, es decir, cuando $\mathcal{L} = -\Delta$ es el laplaciano de Dirichlet clásico. Centramos nuestra atención en modelos no locales cuyo prototipo es la llamada Ecuación de Difusión Rápida Fraccionaria, que corresponde al caso en que $\mathcal{L}$ es uno de los tres laplacianos fraccionarios de Dirichlet posibles en Ω: el laplaciano fraccionario restringido (RFL en inglés, también conocido como el "estándar" o simplemente "el" laplaciano fraccionario), el laplaciano fraccionario espectral (SFL en inglés) y el laplaciano fraccionario censurado (CFL en inglés, también conocido como laplaciano fraccional regional).

Dado que permitimos que el operador lineal $\mathcal{L}$ sea no local, la condición de frontera lateral es difícil de definir en general. Superamos esta dificultad utilizando la *formulación dual* del problema y *codificando las condiciones de contorno* a través de la *función de Green* del operador. Podemos reformular el (CDP) en su forma dual mediante el operador inverso $\mathcal{L}^{-1}$ como

$$\mathcal{L}^{-1}u_t(t,x) = -u^m(t,x).$$

Una EDP cuasilineal de quimiotaxis

A continuación, presentamos el problema abordado en el artículo [75],

A quasilinear Keller-Segel model with saturated discontinuous advection,

un trabajo conjunto del autor con Maria Gualdani y Nicola Zamponi.

La quimiotaxis es un fenómeno biológico que describe el movimiento colectivo de una población formada por individuos (como bacterias) que reaccionan a un estímulo químico externo. El modelo más simple para describir el movimiento quimiotáctico es el propuesto por Patlak y Keller-Segel, [85, 114]. En este modelo las células experimentan un proceso de difusión a la vez que son atraídas hacia regiones con nutrientes o ciertas sustancias químicas generadas por otras células.

En este trabajo nos centramos en un caso particular de un modelo de quimiotaxis con advección discontinua y en ausencia de nutriente establecido por Calvez-Hoffmann en [42]. Sea $d \geq 1$, este modelo pede expresarse como una versión del clásico Keller-Segel:

$$\begin{cases} \partial_t \rho - \Delta\rho + \chi \mathrm{div}\left(\rho \frac{\nabla S}{|\nabla S|}\right) &= 0, \qquad \text{en } \mathbb{R}^d \times (0,\infty), \\ -\Delta S + \alpha S &= \rho, \qquad \text{en } \mathbb{R}^d \times (0,\infty), \end{cases}$$

donde $\chi > 0$ es la sensibilidad de las células al químico y $\alpha \geq 0$ es el decaimiento natural del mismo. Tomando el límite singular $\alpha \to \infty$ (reescalando $S \mapsto S/\alpha$ e identificando el límite de viscosidad $S = \rho$) y escribiendo el resultado en forma de divergencia obtenemos, formalmente, la siguiente EDP cuasilineal:

$$\begin{cases} \partial_t \rho - \mathrm{div}\left[\left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)\nabla\rho\right] &= 0, \qquad \text{en } \mathbb{R}^d \times (0,\infty), \\ \rho(0) &= \rho_0, \qquad \text{en } \mathbb{R}^d. \end{cases}$$

Nuestro objetivo es demostrar la existencia de soluciones débiles para este problema. Sin embargo, vemos que el coeficiente de difusión es positivo cuando $|\nabla\rho| > \chi\rho$, negativo cuando $|\nabla\rho| < \chi\rho$ y se anula con la igualdad. Por lo tanto, como enfoque inicial, es natural demostrar la existencia del siguiente problema, modificación del anterior, que mantiene la dinámica principal del modelo (ambos comparten el mismo estado estacionario) pero que lo hace matemáticamente más abordable:

$$\begin{cases} \partial_t \rho - \mathrm{div}\left[\left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho\right] &= 0, \qquad \text{en } \mathbb{R}^d \times (0,\infty), \\ \rho(0) &= \rho_0, \qquad \text{en } \mathbb{R}^d, \end{cases} \tag{2}$$

donde $(\cdot)_+$ denota la parte positiva, es decir, $f_+ = \max\{f, 0\}$.

Conclusiones

En el Capítulo 2 se exponen algunas motivaciones físicas y matemáticas para estudiar la δ-rigidez torsional en la frontera. Además, presentamos algunas propiedades de la función de tensión y estudiamos el límite $\delta \to 0$. Para terminar, obtenemos estimaciones geométricas óptimas de la *δ-rigidez torsional fronteriza*, tanto para conjuntos simplemente conexos en el plano como para conjuntos convexos de $\mathbb{R}^N$.

En el Capítulo 3 proporcionamos una teoría básica completa para soluciones del problema (1): existencia y unicidad en la mayor clase de datos conocida hasta ahora, efectos clásicos de regularización (*smoothing effects* en inglés) $L^p - L^\infty$ y nuevas estimaciones "con pesos" que representan una novedad también en el caso local, es decir, $u_t = \Delta u^m$. A la hora de probar los *smoothing effects* comparamos dos estrategias diferentes: la iteración de Moser y el método de la función de Green. El carácter singular de la ecuación implica una propiedad peculiar de las soluciones: pueden extinguirse en tiempo finito. De hecho, mostramos tasas de extinción explícitas y precisas tanto por arriba como por abajo. A continuación, demostramos los llamados *Principios Globales de Harnack* (GHP en inglés), que proporcionan límites superiores e inferiores precisos de la solución en términos de una función (casi) explícita. Estos límites globales claramente implican los locales, las llamadas desigualdades de Harnack intrínsecas, que comparan el supremo y el ínfimo de las soluciones en "cilindros intrínsecos" adecuados, cuyo tamaño depende de la solución misma. Estas últimas desigualdades, a su vez, permiten deducir *regularidad de Hölder* en el interior de Ω y, aprovechando el GHP, extender cierta regularidad hasta la frontera. También demostramos una regularidad interior más alta, cuando el operador lo permite, mostrando que las soluciones acotadas son realmente clásicas. Esta última parte (frontera y regularidad interior más alta) está en proceso y forma parte de un artículo en sus etapas finales de preparación.

En el Capítulo 4, demostramos la existencia y la unicidad de las soluciones del problema (2). Para ello agregamos un término de viscosidad ($\varepsilon\Delta\rho_\varepsilon$) a la ecuación y demostramos la existencia de soluciones de este nuevo problema regularizado. Pare esto hacemos uso de la teoría para operadores monótonos y de un teorema de punto fijo. La dificultad principal al eliminar el término de viscosidad es identificar el límite del término no lineal $\left(1 - \frac{\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)\nabla\rho_\varepsilon$. Mediante un método de entropía, somos capaces de superar este problema y demostrar la existencia de soluciones. Finalmente, a través el mismo método de entropía, obtenemos la unicidad de soluciones.

Table of Contents

Acknowledgements **i**

Introducción y conclusiones **ii**

1 Introduction and conclusions **1**
1.1 Types of PDEs . . . 1
1.2 Boundary torsional rigidity . . . 3
1.3 The Cauchy-Dirichlet problem for singular nonlocal diffusions . . . 4
1.4 A quasilinear PDE from chemotaxis . . . 5
1.5 Conclusions . . . 6

2 A Steklov version of the torsional rigidity **7**
2.1 A Steklov version of torsional rigidity . . . 7
2.1.1 Background . . . 7
2.1.2 The boundary torsional rigidity . . . 9
2.1.3 Physical interpretation . . . 10
2.1.4 Main results: geometric estimates . . . 11
2.2 Boundary torsional rigidity . . . 12
2.2.1 Notation . . . 12
2.2.2 First properties . . . 13
2.2.3 Relation with a Steklov eigenvalue problem . . . 17
2.3 Some properties of the torsion function . . . 18
2.4 Exact solutions in some special sets . . . 26
2.5 Geometric estimates . . . 30
2.5.1 Planar simply connected sets . . . 30
2.5.2 Convex sets: lower bound . . . 34
2.5.3 Convex sets: upper bounds . . . 38
Appendix A . . . 42

3 The Cauchy-Dirichlet problem for FFDE **45**
3.1 Introduction . . . 45
3.1.1 Background . . . 45
3.1.2 The Cauchy-Dirichlet Problem for singular nonlocal diffusions. 47
3.1.3 About the lateral boundary condition and the dual formulation of the problem . . . 47
3.1.4 Main assumptions . . . 48

3.1.5 Different fractional Laplacian operators on domains and other examples . . . 50
3.1.6 The dual formulation of the problem . . . 52
3.1.7 Main results . . . 53
3.2 Smoothing effects. Moser iteration VS Green function method . . . 53
3.2.1 Moser iteration . . . 54
3.2.2 Green function method . . . 59
3.2.3 $H^* - L^\infty$ smoothing effects. Proof of Theorem 3.2.11 . . . 67
3.3 Existence and uniqueness . . . 69
3.3.1 Existence of GF solutions . . . 69
3.3.2 Time Monotonicity and $L^1_{\Phi_1}$-contractivity . . . 71
3.3.3 Nonnegative GF solutions are WDS . . . 75
3.3.4 Existence of nonnegative MWDS . . . 78
3.3.5 Existence of $L^1_{\Phi_1}$-strong solutions and uniqueness . . . 83
3.4 Energy estimates and Finite Extinction Time . . . 86
3.4.1 Nonlinear Rayleigh Quotients and extinction rates . . . 86
3.4.2 Norm estimates and bounds for the extinction time . . . 87
3.5 Boundary behaviour and Global Harnack Principles . . . 92
3.5.1 Upper boundary estimates . . . 93
3.5.2 Lower boundary estimates . . . 95
3.5.3 Global Harnack Principles . . . 97
3.6 Regularity . . . 98
Appendix B . . . 104

4 A quasilinear PDE from chemotaxis 109
4.1 Introduction . . . 109
4.2 A priori estimates and technical results . . . 113
4.3 Existence for the approximated equation . . . 120
4.3.1 Existence for the semi-linearized equation . . . 121
4.3.2 Existence for approximated solutions . . . 122
4.4 The limit $\varepsilon \to 0$ and proof of Theorem 4.1.1 . . . 125
4.4.1 Entropy method . . . 125
4.4.2 Identification of the limit of $\nabla \rho_\varepsilon$ . . . 132
4.4.3 Limit $\varepsilon \to 0$ in (4.3.5) . . . 135
4.4.4 Obtaining the initial data ρ_{in} for ρ . . . 135
4.4.5 Uniqueness . . . 136

Bibliography 139

Chapter 1

Introduction and conclusions

1.1 Types of PDEs

This thesis is a compilation of three works that the author has carried out over the past four years. The common area of these works is parabolic and elliptic Partial Differential Equations (PDEs). Let $\Omega \subset \mathbb{R}^N$ be any domain with boundary $\partial\Omega$, then, the canonical linear elliptic partial differential equation is the Laplace's Equation:

$$\begin{cases} -\Delta u &= 0, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega. \end{cases}$$

Among many interpretations, a typical one of this equation is that u represents a physical quantity (such as concentration of a chemical, temperature, etc.) in equilibrium within a domain Ω with some boundary conditions in $\partial\Omega$. For equations of this type, a significant area of study involves eigenvalues and associated quantities such as torsional rigidity or capacity. Particularly, in Chapter 2, we deal with the issue of shape optimization, aiming to derive sharp estimates for these quantities based on the domain's shape.

In the mentioned chapter we present the results of the paper *A Steklov version of torsional rigidity*, see [31], a joint work with Lorenzo Brasco and María del Mar González.

On the other hand, the standard parabolic equation is the Heat Equation:

$$u_t = \Delta u \quad \text{in } (0,T) \times \Omega \subset \mathbb{R}_+ \times \mathbb{R}^N,$$

usually associated with an initial state u_0. Thinking of u as the temperature at time t and position x within a region Ω, the equation tells us that the temperature will vary in time at the same rate as its second spatial derivatives. This equation tends to "smooth out" the unknown function, to regularize it. These types of equations are called diffusion equations.

Both the Laplace equation and the heat equation are linear, meaning that the sum of two solutions is also a solution. However, most interesting PDEs are nonlinear, and among them, the one closest to the Heat Equation is:

$$u_t = \Delta u^m.$$

When $m > 1$, this equation is known as the Porous Media Equation (PME), and when $m < 1$, it is called the Fast Diffusion Equation (FDE). The name "Porous Media Equation" stems from its use in describing the flow of an ideal gas in a homogeneous porous medium.

Both PME and FDE belong to a broader class of equations of the form

$$u_t = \nabla \cdot (a(u)\nabla u),$$

where $a(u)$ is often referred to as the diffusion coefficient, and in the case of PME and FDE, $a(u) = mu^{m-1}$. When $m > 1$, the equation degenerates when $u = 0$, meaning $a(0) = 0$, and the equation does not diffuse, so solutions with initial data of compact support remain with compact support, featuring a sort of free boundary where the solution vanishes. However, in the case of FDE, $a(u)$ tends to infinity as $u \to 0$, so even if the initial data has compact support, the solution will be positive for all $t > 0$. As can be seen, between these two equations, along with the Heat Equation, there are many similarities but also significant differences.

Now, if we add a *convection* term to any diffusion equation, we turn it into a convection-diffusion equation:

$$u_t = \nabla \cdot (a\nabla u) - \nabla \cdot (\mathbf{b}u),$$

where b is the velocity at which en external flux moves. Continuing with the previous example, this equation can model the concentration of an ideal gas in a porous medium conditioned by a specific flow with velocity $\mathbf{b}$. The convection-diffusion equation is also widely used in biology to describe the collective movement of cell groups, such as in Keller-Segel model for Chemotaxis. In such model, cells undergo a diffusion process while also being attracted to regions with food or certain chemicals generated by other cells (chemotaxis), meaning they have a bias towards certain directions.

Chapter 4 presents the results of the paper *A quasilinear Keller-Segel model with saturated discontinuous advection* [75], in which the author and collaborators, Maria Gualdani and Nicola Zamponi, study a quasilinear equation arising from a chemotaxis model.

In the diffusion processes described above, it is assumed that the particles forming the system move continuously or, alternatively, in short intervals that can be represented by a Gaussian distribution. However, this is not always the case; for instance, there are processes of *anomalous diffusion* that require non-local operators. Among these operators, the most used and studied is the fractional Laplacian $(-\Delta)^s$, with $s \in (0, 1)$, which in $\mathbb{R}^N$ has many equivalent definitions, such as through a singular integral:

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where $C_{N,s}$ is a normalization constant and P.V. indicates the principal value. As can be seen, the integral is singular at $x = y$, meaning that for the expression to be computable, a certain regularity for u at x is required. This singularity is what in some way differentiates u, so we will need a certain regularity for the integral to

be finite. As mentioned earlier, in $\mathbb{R}^N$, there are several equivalent ways to define the fractional Laplacian (via the Fourier Transform, through the heat semigroup associated with the operator $(-\Delta)^s$, etc.). However, as we will see, in bounded domains, these definitions may not be equivalent.

Chapter 3 is dedicated to the fast diffusion equation with fractional operators, presenting the results of the paper *The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains*, see [23], and of a forthcoming article which is the natural continuation of this one. Both papers are joint works with Matteo Bonforte and Peio Ibarrondo.

Next, we give a brief but more specific introduction of each of the chapters independently.

1.2 Boundary torsional rigidity

Here we present the problem addressed in [31],

A Steklov version of torsional rigidity,

a joint work of the author with Lorenzo Brasco and María del Mar González, published in 2023 in *Communications in Contemporary Mathematics*.

Let us take the Laplace's equation above, add a constant term in the interior and impose Dirichlet's boundary conditions, this is:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The solution to this equation is called the stress function and represents the distribution of the stress generated in a beam due to an applied torque. The total resultant torque due to this stress is called *torsional rigidity* and can be expressed as

$$\mathcal{T}(\Omega) = \iint_\Omega u \, \mathrm{d}x, \quad \text{or equivalently,} \quad \mathcal{T}(\Omega) = \sup_{\varphi|_{\partial\Omega}=0} \frac{\left(\iint_\Omega \varphi \, \mathrm{d}x\right)^2}{\iint_\Omega |\nabla\varphi|^2 \, \mathrm{d}x}.$$

Let $\lambda_1(\Omega)$ be the first Dirichlet's eigenvalue, this is, the smallest constant $\lambda_1 > 0$ satisfying

$$\begin{cases} -\Delta u = \lambda_1 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

A classic inequality by Pólya [122] states that

$$\frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|} \leq 1.$$

Motivated by this relation between the first Dirichlet's eigenvalue and torsional rigidity, the aim of Chapter 2 is to find a physically coherent and mathematically

interesting new concept for *boundary torsional rigidity* related to a modified Steklov eigenvalue. We define this eigenvalue as the first constant $\sigma > 0$ satisfying

$$\begin{cases} -\Delta u &= \delta u, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= \sigma\, u, \quad \text{on } \partial\Omega, \end{cases}$$

where ν_Ω is the unit outer normal vector of Ω.

Now, we define the *boundary δ-torsional rigidity* as

$$T(\Omega;\delta) = \sup \frac{\left(\int\!\!\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} \right)^2}{\int\!\!\int_\Omega |\nabla \varphi|^2 \, \mathrm{d}x + \delta^2 \int\!\!\int_\Omega \varphi^2 \, \mathrm{d}x}, \tag{1.2.1}$$

where $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure of the boundary $\partial\Omega$.

From a variational point of view, when $\delta = 1$, such a new object corresponds to the sharp constant for the trace embedding of $W^{1,2}(\Omega)$ into $L^1(\partial\Omega)$.

1.3 The Cauchy-Dirichlet problem for singular nonlocal diffusions

This section is devoted to the problem addressed in [23],

The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains,

a joint work of the author with Matteo Bonforte and Peio Ibarrondo, published in 2023 in *Discrete and Continuous Dynamical Systems*. In Chapter 3, the one dedicated to this problem, we also present some new results of a forthcoming paper which is the natural continuation of the one mentioned above.

We study the Cauchy-Dirichlet Problem (CDP) for a nonlinear and nonlocal diffusion equation of singular type of the form

$$\begin{cases} \partial_t u(t,x) = -\mathcal{L} u^m(t,x) & \text{on } (0,+\infty)\times\Omega\,, \\ u(t,\cdot) = 0 & \text{on the lateral boundary, } \forall t>0\,, \\ u(0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{CDP}$$

where $0 < m < 1$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, posed on a bounded Euclidean domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and $N \geq 1$.

The linear diffusion operator $\mathcal{L}$ is a sub-Markovian operator, belonging to a large class of (local and) nonlocal operators, which includes the three different versions of the Fractional Laplacian, while the nonlinearity is of singular type, namely $u^m = |u|^{m-1}u$ with $0 < m < 1$. The prototype equation is the Fast Diffusion Equation, when $\mathcal{L} = -\Delta$ is the classical Dirichlet Laplacian. We focus our attention on nonlocal models, whose prototype is the so-called Fractional Fast Diffusion Equation, which corresponds to the case when $\mathcal{L}$ is one of the three possible Dirichlet fractional Laplacians on Ω, namely, the restricted fractional Laplacian (RFL, also known as the

"standard" or simply "the" fractional Laplacian), the spectral fractional Laplacian (SFL) and the censored fractional Laplacian (CFL, also known as regional fractional Laplacian).

Since we are allowing the linear operator $\mathcal{L}$ to be nonlocal, the lateral boundary condition is tricky to define in general. We overcome this difficulty using the *a dual formulation* of the problem, *encoding the boundary conditions* through the *Green function* of the linear operator. We can reformulate (CDP) in its dual form by means of the inverse operator $\mathcal{L}^{-1}$ as

$$\mathcal{L}^{-1}u_t(t,x) = -u^m(t,x).$$

1.4 A quasilinear PDE from chemotaxis

Let us present the problem addressed in the preprint [75],

A quasilinear Keller-Segel model with saturated discontinuous advection,

a joint work of the author with Maria Gualdani and Nicola Zamponi.

Chemotaxis is a biological phenomenon describing the collective motion of a population formed of individuals (such as bacteria) that react to an external chemical stimulus. The simplest model for describing the chemotactic motion is the one proposed by Patlak and Keller-Segel, [85, 114]. In this model, cells undergo a diffusion process while also being attracted to regions with nutrients or certain chemicals generated by other cells, meaning they have a bias towards certain directions.

Let $d \geq 1$, in this work we focus in a particular case of the following minimal chemotactic model, stated by Calvez-Hoffmann in [42], with discontinuous advection and in the absence of nutrient, which can be expressed as a version of the classical Keller-Segel:

$$\begin{cases} \partial_t \rho - \Delta \rho + \chi \mathrm{div}\left(\rho \frac{\nabla S}{|\nabla S|}\right) = 0, & \text{in } \mathbb{R}^d \times (0,\infty), \\ -\Delta S + \alpha S = \rho, & \text{in } \mathbb{R}^d \times (0,\infty), \end{cases}$$

where $\chi > 0$ is the chemoatractant sensitivity and $\alpha \geq 0$ is the natural decay of the chemical. Taking the singular limit $\alpha \to \infty$ (if we rescale $S \mapsto S/\alpha$ and we identify the vanishing viscosity limit $S = \rho$) and writing the result in divergence form we formally get the following quasilinear PDE:

$$\begin{cases} \partial_t \rho - \mathrm{div}\left[\left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)\nabla\rho\right] = 0, & \text{in } \mathbb{R}^d \times (0,\infty), \\ \rho(0) = \rho_0, & \text{in } \mathbb{R}^d. \end{cases}$$

We aim to prove existence of weak solutions to this problem.

Here we see that the diffusion coefficient is positive when $|\nabla\rho| > \chi\rho$, negative when $|\nabla\rho| < \chi\rho$ and vanishes with equality. Therefore, as a first approach, it is natural to prove existence for the following problem, which is a modification of

(4.1.2) that maintains the main dynamics of the model (same stationary state) but makes it more mathematically tractable:

$$\begin{cases} \partial_t \rho - \operatorname{div}\left[\left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho\right] = 0, & \text{in } \mathbb{R}^d \times (0,\infty), \\ \rho(0) = \rho_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{1.4.1}$$

where $(\cdot)_+$ denotes the positive part, that is, $f_+ = \max\{f, 0\}$.

1.5 Conclusions

In Chapter 2 we give some physical and mathematical motivations to study the δ-torsional rigidity. We present some properties of the state function in (1.2.1) and we study the limit $\delta \to 0$. We obtain sharp geometric estimates of δ-torsional rigidity, both for planar simply connected sets and for convex sets in any dimension.

In Chapter 3 we provide a complete basic theory for solutions to problem (CDP): existence and uniqueness in the biggest class of data known so far, both for nonnegative and signed solutions; sharp smoothing estimates: classical $L^p - L^\infty$ smoothing effects, and new weighted estimates, which represent a novelty also in local case, i.e. $u_t = \Delta u^m$. We compare two strategies to prove smoothing effects: Moser iteration VS Green function method. The singular character of the equation implies a peculiar property of solutions: they may extinguish in finite time. Indeed we show explicit sharp extinction rates, both from above and from below. We next prove the so-called *Global Harnack Principles* (GHP), which gives precise pointwise upper and lower bounds of the solution in terms of a (almost) explicit function. These global bounds clearly imply the local ones, the so-called intrinsic Harnack inequalities, that compare the supremum and the infimum of solutions on suitable "intrinsic cylinders", whose size depends on the solution itself. These latter inequalities, in turn, allow to deduce *Hölder regularity* in the interior of Ω and, exploiting the GHP, we push the regularity up to the boundary. We shall also prove higher interior regularity, when the operator allows it, showing that bounded solutions are indeed classical. This last part (boundary and higher interior regularity) is still a work in progress (missing only minor technical details), and a paper is in the final stages of preparation.

In Chapter 4 we prove existence and uniqueness of solutions of the problem (1.4.1). This can be done by adding a viscosity term ($\varepsilon\Delta\rho_\varepsilon$) to the original equation. Then, we prove existence for this new regularized problem using theory of monotone operators and a fixed point argument. Once this is done, the main difficulty of the chapter is removing the viscosity term, particularly, identifying the limit of the nonlinear term $\left(1 - \frac{\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon$. We are able to overcome this issue by means of an entropy method. Finally, by the same entropy method we get uniqueness of solutions.

Chapter 2

A Steklov version of the torsional rigidity

In this chapter we present the results of the paper *A Steklov version of torsional rigidity*, [31], a joint work of the author of this thesis with Lorenzo Brasco and María del Mar González.

2.1 A Steklov version of torsional rigidity

2.1.1 Background

In this chapter we introduce a boundary version for the torsional rigidity which is modeled on the trace Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^1(\partial\Omega)$ for an open bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. This will be also closely related to the Steklov eigenvalue problem. Before we give the precise definitions in Section 2.1.2, let us briefly recall some facts about the usual *torsional rigidity* and the *Steklov eigenvalue problem*.

Let us consider an isotropic elastic cylindrical beam whose cross-section is represented by an open bounded simply connected set Ω in $\mathbb{R}^2$, with boundary $\partial\Omega$. The longitudinal axis of the beam is the z-axis, with the cross-sections perpendicular to it. The formulation of the so-called *torsion problem* began at the end of the 18th century, provided by Coulomb, for bars with circular cross section. In this case, it is assumed that the cross-sections rotate around z-axis as a rigid body under an applied torque. Nevertheless, because of the complex stress distribution in the bar, this does not happen when the cross-sections do not have circular symmetry and the problem must be reformulated.

The correct statement for beams of general shape was given (in 1856) by the French mechanician and mathematician A. B. de Saint-Venant, who proposed that the deformation in a twisted beam consists in two phenomena: the rotation of the projections in the x, y plane of the cross-sections as rigid bodies, as in the circular case; and the *warping*, equal for all the cross-sections, which do not remain plane after rotation. The distribution of stress generated in the beam due to an applied

torque is determined by the *stress function*, denoted by v_Ω, which satisfies

$$\begin{cases} -\Delta v_\Omega &= 1, \quad \text{in } \Omega, \\ v_\Omega &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

The total resultant torque due to this stress function is called *torsional rigidity* and can be expressed as

$$\mathcal{T}(\Omega) = \iint_\Omega v_\Omega \, \mathrm{d}x,$$

or equivalently,

$$\mathcal{T}(\Omega) = \sup_{v|_{\partial\Omega}=0} \frac{\left(\iint_\Omega v \, \mathrm{d}x \right)^2}{\iint_\Omega |\nabla v|^2 \, \mathrm{d}x}. \tag{2.1.2}$$

We refer the reader to [102] for a thorough description of this physical model.

De Saint-Venant himself stated in 1856 that the simply connected cross-section with maximal torsional rigidity is the circle. In other words, he conjectured the validity of the following inequality

$$\mathcal{T}(\Omega) \leq \mathcal{T}(\Omega^*), \tag{2.1.3}$$

where Ω^* is any circle having the same area as Ω. This statement was rigorously established, almost a hundred years later, by G. Pólya using a symmetrization method [120] and by E. Makai [103]. The latter also established the identification of equality cases, by means of a clever quantitative improvement of the inequality (2.1.3).

The torsional problem has been extensively studied for its mathematical interest, apart of its obvious physical importance: in addition to the classical book [122], we refer for example to [79, 89, 116, 118], for classical results, as well as to [8, 11, 12, 13, 36, 37], for more recent studies.

It is worth noting that formulation (2.1.2) for the torsional rigidity has a close connection with the variational characterization of the first eigenvalue of the Dirichlet-Laplacian and, in fact, similar techniques are used to obtain geometric estimates for both quantities (see the references above).

We now come to the Steklov eigenvalue problem: this was originally posed by V. A. Steklov at the turn of the 20th century as

$$\begin{cases} -\Delta u &= 0, \quad & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= \sigma \, u, \quad & \text{on } \partial\Omega, \end{cases}$$

where ν_Ω is the unit outer normal vector. It plays an important role in the study of the so called *sloshing problem*, in which the eigenvalues and eigenvectors of a mixed Steklov problem correspond to the fundamental frequencies and modes of vibration, respectively, of the surface of an inviscid, incompressible and heavy liquid in a container (see [93]). Apart from this interpretation, the Steklov problem has a large number of applications in physics and engineering. For instance, it describes stationary states of a heat distribution on a bounded domain Ω, where the heat flow at the boundary of Ω depends on the temperature at this boundary.

From a purely mathematical point of view, the Steklov problem has attracted a lot of interest, in particular, in the framework of *Spectral Shape Optimization*. In fact, the connection between the Steklov eigenvalues and the geometry of the domain presents some peculiarities that can not be observed in other eigenvalue problems. In 1954, R. Weinstock showed in [144] that, among all simply connected plane domains with given perimeter, the disc maximizes the first non-zero eigenvalue. Since then, many efforts have been devoted in order to give geometric estimates for the spectrum of the Steklov-Laplacian: without any attempt of completenes, we cite for example [30, 38, 40, 60, 61, 66, 69, 115].

From a different point of view, the Steklov eigenvalues can be seen as the eigenvalues of the *Dirichlet-to-Neumann operator*

$$\begin{array}{rccc} \mathcal{D} & : & W^{\frac{1}{2},2}(\partial\Omega) & \to & W^{-\frac{1}{2},2}(\partial\Omega), \\ & & f & \mapsto & \langle \nabla \mathrm{Har}(f), \nu_\Omega \rangle, \end{array}$$

where $\mathrm{Har}(f)$ is the harmonic extension of f to Ω. This concept is important in many fields such as electric impedance tomography, cloaking and so on (see [93, Section 5]).

2.1.2 The boundary torsional rigidity

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Given $\delta > 0$, we introduce a new object, the *boundary δ-torsional rigidity*, which is defined as

$$T(\Omega;\delta) = \sup_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int\!\!\!\!\!\!\diagup_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} \right)^2}{\int\!\!\!\!\!\!\diagup_{\Omega} |\nabla \varphi|^2 \, \mathrm{d}x + \delta^2 \int\!\!\!\!\!\!\diagup_{\Omega} \varphi^2 \, \mathrm{d}x}. \tag{2.1.4}$$

We use the symbol $\mathcal{H}^{N-1}$ to indicate the $(N-1)-$dimensional Hausdorff measure. One may check (see Proposition 2.2.3 below) that the supremum is attained by any non-zero multiple of a positive function $u_{\Omega,\delta} \in W^{1,2}(\Omega)$ which is the weak solution of the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u + \delta^2 u & = & 0, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle & = & 1, & \text{on } \partial\Omega. \end{array} \right. \tag{2.1.5}$$

The function $u_{\Omega,\delta}$ is known as the *boundary $\delta-$torsion function* of Ω. It is not difficult to see that we have

$$T(\Omega;\delta) = \int\!\!\!\!\!\!\diagup_{\partial\Omega} u_{\Omega,\delta} \, d\mathcal{H}^{N-1}. \tag{2.1.6}$$

We note here that, while interior regularity for equation (2.1.5) is not an issue, regularity up to the boundary is more delicate under our standing assumptions on Ω. Thus, the Neumann condition in (2.1.5) has to be interpreted in weak sense (we refer to Remark 2.3.2).

A strong motivation for the definition of $T(\Omega;\delta)$ comes from its relation to the ($\delta-$modified) Steklov problem. In particular, the relation to the first Steklov eigenvalue is given precisely in Proposition 2.2.6. The latter can be seen as the Steklov

version of the classical Pólya inequality relating torsional rigidity, volume and first eigenvalue of the Dirichlet-Laplacian, see [122, Chapter V, Section 4].

Additionally, we will discuss the asymptotic behavior of both $u_{\Omega,\delta}$ and $T(\Omega;\delta)$, as the positive parameter δ goes to 0. We will see in Theorem 2.3.4 that both quantities (once scaled by the dimensional factor δ^2) converge to a geometric quantity, which is a kind of *isoperimetric–type ratio*.

2.1.3 Physical interpretation

The *membrane analogy*, also known as *soap-film analogy*, was discovered by R. Prandtl in 1903. It states that the equations governing the stress distribution on a beam in torsion are the same as the ones describing the shape of a membrane deformed under a pressure applied across it (see [126, Section 9.3.4] and [135]). This fact establishes an analogy between the usual torsional rigidity and the volume under a forced membrane.

More precisely, let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set. For ease of presentation, we suppose in what follows that Ω has a smooth boundary, for example of class C^2. Let us consider a thin membrane stretched over the rigid frame represented by $\partial\Omega$, with uniform tension $\mathcal{N}$ and under an external pressure p. Then, by applying the equilibrium equations to this system, we get that the vertical displacement of the membrane $v(x,y)$ at a point $(x,y) \in \Omega$ solves

$$\begin{cases} -\Delta v &= p/\mathcal{N}, & \text{in } \Omega, \\ v &= 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.7}$$

When the pressure p is constant, we recognize the usual torsion problem (2.1.1), up to a multiplicative constant.

Here we are interested in a Neumann version of (2.1.7):

$$\begin{cases} -\Delta u + \delta^2 u &= f(x), & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.8}$$

In this case, the boundary of the membrane can move in the vertical direction but always keeps a horizontal angle. Moreover, the applied pressure depends on both the displacement of the membrane and the x position. For every $\varepsilon > 0$ small enough, let us define the $\varepsilon-$neighbourhood of the boundary

$$\Omega_\varepsilon = \Big\{ x \in \Omega \,:\, d_\Omega(x) \le \varepsilon \Big\},$$

and add a concentration effect near $\partial\Omega$ to problem (2.1.8), that is,

$$\begin{cases} -\Delta u + \delta^2 u &= \dfrac{C}{\varepsilon} 1_{\Omega_\varepsilon}, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.9}$$

where 1_E denotes the characteristic function of a set E. The precise limit behavior of this problem is proven by Arrieta, Jiménez-Casas, and Rodríguez-Bernal [6, Theorem 4.1]. Indeed, this shows that[1] by taking the limit as ε goes to 0, the unique

[1] With the notation of [6], our equation (2.1.9) corresponds to [6, formula (1.4)] with the choices

$$a(x) \equiv 1, \quad c(x) = b(x) = V_\varepsilon(x) = j_\varepsilon(x) = g_\varepsilon \equiv 0, \quad \lambda = \delta^2, \quad f_\varepsilon(x) \equiv C,$$

solution u_ε of (2.1.9) converges in $W^{1,2}(\Omega)$ to the solution of

$$\begin{cases} -\Delta u + \delta^2\, u &= 0, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= C, \quad \text{on } \partial\Omega, \end{cases}$$

which is our problem (2.1.5).

Another possible physical motivation is the following one. We consider the time evolution of the temperature u of a uniform heat conductor Ω, with an initial temperature equal to 0 and subject to a constant heat flux at the boundary. For simplicity, we normalize such a constant flux to be 1. In other words, the function u solves

$$\begin{cases} \Delta u &= u_t, \quad \text{in } \Omega \times (0, +\infty), \\ u &= 0, \quad \text{in } \Omega \times \{0\}, \\ \langle \nabla u, \nu_\Omega \rangle &= 1, \quad \text{on } \partial\Omega \in (0, +\infty). \end{cases}$$

If we introduce the family of probability measures $\{p_\delta\}_{\delta>0}$ on the time interval $(0, +\infty)$, given by

$$dp_\delta(t) = \delta^2 \, \exp(-\delta^2\, t)\, dt,$$

then it is not difficult to see that for every $\delta > 0$ the averaged (in time) temperature

$$U_\delta(x) := \int_0^{+\infty} u(x,t)\, dp_\delta(t) = \delta^2 \int_0^{+\infty} u(x,t)\, e^{-\delta^2\, t}\, dt,$$

solves (2.1.5), at least formally.

In this way, we can think of $u_{\Omega,\delta}$ as a sort of averaged temperature of the heat conductor Ω; correspondingly, the quantity $T(\Omega;\delta)$ can be seen as an *averaged heat content* of the boundary, thanks to (2.1.6). This interpretation is quite close in spirit to the probabilistic description of the usual torsional rigidity, in terms of Brownian motion. We refer for example to [8, 123].

2.1.4 Main results: geometric estimates

The exact expressions of $T(\Omega;\delta)$ and $u_{\Omega,\delta}$ can be computed explicitly only in a few special geometries (balls, spherical shells or the hyperrectangles, see Section 2.4). Thus, for a general set Ω, it is important to find (possibly sharp) bounds in terms of known geometric objects. This is the main objective of this chapter.

For open simply connected sets in two dimensions, it is possible to use the *conformal transplantation technique*: this amounts to use the Riemann mapping theorem to conformally transplant test functions form the disk to any bounded, simply connected $\Omega \subset \mathbb{R}^2$. This technique yields geometric estimates of eigenvalues for simply connected sets in the plane, see for example [119, 122, 132] and [144]. The key point here is that the Dirichlet energy is a conformal invariant. By carefully estimating the other norms contained in the definition of $T(\Omega;\delta)$, in Theorem 2.5.4 we are able to give a sharp lower bound for $T(\Omega;\delta)$ in terms of the corresponding quantity for the disk and what we call *boundary distortion radius* of Ω, defined precisely in Definition 2.5.1.

and $\Gamma = \partial\Omega$.

On the contrary, in higher dimensions conformal mappings are not available in general and we need to modify our approach. We restrict in this case our study to the class of convex sets. First, in Theorem 2.5.6, we prove a sharp lower bound on $T(\Omega;\delta)$ using the *method of interior parallels* introduced by Makai [104, 105] and Pólya [121]. We were inspired by an estimate of Pólya, which provides a lower bound on the usual torsional rigidity in terms of volume and perimeter, see [121]. We point out that the convexity assumption could be replaced by the requirement that the distance function d_Ω to $\partial\Omega$ is weakly superharmonic, which in general is a weaker property, see Remark 2.5.7 below.

Finally, a sharp upper bound for $T(\Omega;\delta)$ in the convex case is obtained in Theorem 2.5.11: this involves geometric quantities such as the inradius and the *proximal radius* of Ω. A nice by-product of our arguments is the sharp estimate of Corollary 2.5.12, which involves four different geometric quantities: the volume of Ω, the $(N-1)-$dimensional measure of its boundary, its proximal radius and its inradius.

Plan of the chapter. In Section 2.2 we set the main notations and discuss the first properties of the boundary torsional rigidity. We discuss well-posedness of the problem and present various equivalent variational characterizations, as in the classical case. We also obtain some basic upper and lower bounds on $T(\Omega;\delta)$. Then, in Section 2.3 we discuss some properties of the boundary $\delta-$torsion function $u_{\Omega,\delta}$ of a set and its asymptotic behavior as δ goes to 0. In Section 2.4 we compute the exact shape of the boundary $\delta-$torsion function, for some special sets: the ball, the spherical shell and the hyperrectangle. Section 2.5 is entirely devoted to prove sharp geometric estimates on the boundary $\delta-$torsional rigidity. Finally, the Appendix contains some technical results on the proximal radius of a convex set.

2.2 Boundary torsional rigidity

2.2.1 Notation

In what follows in this chapter, we will always consider $N \geq 2$. We will indicate by $B_R(x_0)$ the $N-$dimensional open ball with radius $R > 0$ and center $x_0 \in \mathbb{R}^N$. When the center coincides with the origin, we will simply write B_R. Finally, we will denote by B the ball with unit radius, centered at the origin. We also set

$$|B| = \omega_N.$$

For $a \in \mathbb{R}$, we will use the notation

$$a_+ = \max\{a, 0\}.$$

Let $\Omega \subset \mathbb{R}^N$ be a open bounded set, with Lipschitz boundary. We denote by ν_Ω its unit outer normal vector, which is well-defined $\mathcal{H}^{N-1}-$almost everywhere on $\partial\Omega$. By d_Ω we will indicate the *distance function* given by

$$d_\Omega(x) = \min_{y\in\partial\Omega} |x-y|, \qquad \text{for } x \in \Omega.$$

It is well-known that under the standing assumptions on Ω, we have the continuous trace embedding

$$W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega),$$

for every $1 \leq q \leq 2^{\#}$, where

$$2^{\#} = \begin{cases} \dfrac{2\,N-2}{N-2}, & \text{if } N \geq 3, \\ \\ \text{any finite exponent} \geq 1, & \text{if } N = 2, \end{cases}$$

see for example [67, Theorems 6.4.1 & 6.4.2]. Moreover, such an embedding is compact, whenever $1 \leq q < 2^{\#}$, see for example [67, Remark 6.10.5, points (i) & (ii)]. For every admissible q, we will set

$$\eta_q(\Omega) = \inf_{\varphi \in W^{1,2}(\Omega)} \left\{ \|\varphi\|^2_{W^{1,2}(\Omega)} \,:\, \|\varphi\|_{L^q(\partial\Omega)} = 1 \right\} > 0, \tag{2.2.1}$$

which is the sharp constant for such a trace embedding. Here we use the norm on $W^{1,2}(\Omega)$ defined by

$$\|\varphi\|_{W^{1,2}(\Omega)} = \left(\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \int_\Omega |\varphi|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \qquad \text{for every } \varphi \in W^{1,2}(\Omega).$$

2.2.2 First properties

Given $\delta > 0$, we have defined the boundary δ-torsional rigidity by (2.1.4). It is not difficult to see that for every $t > 0$ we have the scaling law

$$T\left(t\,\Omega; \frac{\delta}{t}\right) = t^N\, T(\Omega;\delta). \tag{2.2.2}$$

It is also easy to check that $T(\Omega;\delta)$ is related to the best Sobolev–type constant in (2.2.1). More precisely:

Lemma 2.2.1. *Under the previous assumptions, we have $T(\Omega;\delta) < +\infty$ and the supremum in* (2.1.4) *is attained. Moreover, the following estimates hold*

$$\frac{1}{\delta^2}\,\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq T(\Omega;\delta) \leq \frac{1}{\min\{1,\delta^2\}}\,\frac{1}{\eta_1(\Omega)}, \tag{2.2.3}$$

where $\eta_1(\Omega)$ is defined in (2.2.1).

Proof. We first observe that

$$T(\Omega;\delta) \leq \frac{1}{\min\{1,\delta^2\}} \sup_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} \right)^2}{\displaystyle\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \int_\Omega \varphi^2 \, \mathrm{d}x} = \frac{1}{\min\{1,\delta^2\}}\,\frac{1}{\eta_1(\Omega)}.$$

This shows that $T(\Omega;\delta) < +\infty$ and it also proves the upper bound in (2.2.3).

Existence of an extremal for $T(\Omega;\delta)$ can be proved by appealing to the Direct Method in the Calculus of Variations. It is sufficient to observe that the maximization in (2.1.4) is equivalently performed on the set $W^{1,2}(\Omega) \setminus W^{1,2}_0(\Omega)$. Thus, a maximizer of (2.1.4) can be obtained by solving

$$\inf_{u \in W^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega \varphi^2 \, \mathrm{d}x \,:\, \int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} = 1 \right\} \tag{2.2.4}$$

Any minimizing sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of functions in $W^{1,2}(\Omega)$ for this problem is such that

$$\int_\Omega |\nabla\varphi_n|^2 \,\mathrm{d}x + \fint_\Omega \varphi_n^2 \,\mathrm{d}x \le \frac{C}{\min\{1,\delta^2\}} \qquad \text{and} \qquad \fint_{\partial\Omega} \varphi_n \, d\mathcal{H}^{N-1} = 1,$$

for some $C > 0$ independent of n.

The first property, in conjuction with the Rellich-Kondrašov Theorem and the compactness of the trace embedding $W^{1,2}(\Omega) \hookrightarrow L^1(\partial\Omega)$, implies that there exists $u_0 \in W^{1,2}(\Omega)$ such that φ_n converges to u_0 (up to a subsequence), weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and strongly in $L^1(\partial\Omega)$. In particular, u_0 has still unit boundary integral. Thus the limit function u_0 is still admissible in (2.2.4) and, thanks to the lower semicontinuity of the functional, we get that u_0 is the desired solution.

Finally, in order to prove the lower bound (2.2.3), it is sufficient to use the characteristic function of Ω as a test function in the Rayleigh–type quotient which defines $T(\Omega;\delta)$. □

Remark 2.2.2. *As simple as it may appear, the lower bound in* (2.2.3) *is actually sharp in some sense. This will be shown in Remark 2.3.5, as a consequence of the asymptotic behavior of the quantity*

$$\delta^2\, T(\Omega;\delta), \qquad \text{as } \delta \searrow 0.$$

See Theorem 2.3.4 below. For the moment, we observe that from (2.2.3) *we get that*

$$\lim_{\delta\to 0^+} T(\Omega;\delta) = +\infty,$$

and $T(\Omega;\delta)$ diverges precisely at a rate δ^{-2}.

As for the usual torsional rigidity, $T(\Omega;\delta)$ can be equivalently defined through an unconstrained concave maximization problem:

Proposition 2.2.3. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with Lipschitz boundary. Then we have*

$$T(\Omega;\delta) = \sup_{\varphi\in W^{1,2}(\Omega)} \left\{ 2\fint_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} - \fint_\Omega |\nabla\varphi|^2 \,\mathrm{d}x - \delta^2 \fint_\Omega \varphi^2 \,\mathrm{d}x \right\} \tag{2.2.5}$$

Moreover, the supremum in (2.2.5) *is uniquely attained by a non-negative function $u_{\Omega,\delta} \in W^{1,2}(\Omega)$, which is the weak solution of the Neumann boundary value problem*

$$\begin{cases} -\Delta u + \delta^2 u &= 0, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega\rangle &= 1, \quad \text{on } \partial\Omega. \end{cases}$$

In other words, $u_{\Omega,\delta}$ satisfies

$$\fint_\Omega \langle \nabla u_{\Omega,\delta}, \nabla\varphi\rangle \,\mathrm{d}x + \delta^2 \fint_\Omega u_{\Omega,\delta}\,\varphi \,\mathrm{d}x = \fint_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}, \quad \text{for every } \varphi \in W^{1,2}(\Omega). \tag{2.2.6}$$

Finally, we also have

$$T(\Omega;\delta) = \fint_{\partial\Omega} u_{\Omega,\delta} \, d\mathcal{H}^{N-1}. \tag{2.2.7}$$

Proof. The existence of a solution for the maximization problem in (2.2.5) follows again from the Direct Method in the Calculus of Variations. Indeed, we first observe that by using the definition (2.2.1) of $\eta_1(\Omega)$ and Young's inequality we get, for every $L > 0$,

$$\begin{aligned}\mathfrak{F}(\varphi) := 2\int_{\partial\Omega}\varphi\,d\mathcal{H}^{N-1} - \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2\int_\Omega \varphi^2\,\mathrm{d}x \\ \le 2\int_{\partial\Omega}\varphi\,d\mathcal{H}^{N-1} - \min\{1,\delta^2\}\,\|\varphi\|^2_{W^{1,2}(\Omega)} \\ \le \frac{2}{\sqrt{\eta_1(\Omega)}}\,\|\varphi\|_{W^{1,2}(\Omega)} - \min\{1,\delta^2\}\,\|\varphi\|^2_{W^{1,2}(\Omega)} \\ \le \Big(L - \min\{1,\delta^2\}\Big)\|\varphi\|^2_{W^{1,2}(\Omega)} + \frac{1}{L\,\eta_1(\Omega)}.\end{aligned}$$

By choosing $L = \min\{1,\delta^2\}/2$, we obtain

$$\mathfrak{F}(\varphi) \le -\frac{1}{C}\,\|\varphi\|^2_{W^{1,2}(\Omega)} + C, \tag{2.2.8}$$

for some $C = C(N,\Omega,\delta) > 0$. This shows that

$$\sup_{\varphi\in W^{1,2}(\Omega)} \mathfrak{F}(\varphi) < +\infty.$$

Consequently, by the estimate (2.2.8), every maximizing sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is equi-bounded in $W^{1,2}(\Omega)$. Existence of a maximizer $u_{\Omega,\delta}$ can now be inferred as in the proof of Lemma 2.2.1.

Uniqueness of the solution follows from the strict concavity of the functional, while the definite sign property is a consequence of the uniqueness and the fact that

$$\mathfrak{F}(|\varphi|) \ge \mathfrak{F}(\varphi), \qquad \text{for every } \varphi \in W^{1,2}(\Omega).$$

Finally, the equation (2.2.6) is just the optimality condition for $u_{\Omega,\delta}$, which can be obtained by computing the first variation of the functional.

We now come to the proof of (2.2.5). We first notice that

$$\begin{aligned}\max_{\varphi\in W^{1,2}(\Omega)} \left\{2\int_{\partial\Omega}\varphi\,d\mathcal{H}^{N-1} - \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2\int_\Omega\varphi^2\,\mathrm{d}x\right\} \\ = \max_{\varphi\in W^{1,2}(\Omega)\setminus W^{1,2}_0(\Omega)} \left\{2\int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{N-1} - \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2\int_\Omega\varphi^2\,\mathrm{d}x\right\}\end{aligned}$$

Indeed, observe that for every $\varphi \in W^{1,2}_0(\Omega)$ we have

$$2\int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{N-1} - \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2\int_\Omega\varphi^2\,\mathrm{d}x \le 0,$$

while, if $\varphi \in W^{1,2}(\Omega)\setminus W^{1,2}_0(\Omega)$, then for every $t > 0$ small enough we have

$$2\,t\int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{N-1} - t^2\left(\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \delta^2\int_\Omega\varphi^2\,\mathrm{d}x\right) > 0.$$

This also gives in particular that

$$\max_{\varphi\in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)} \left\{ 2\int_{\partial\Omega} |\varphi|\, d\mathcal{H}^{N-1} - \fint_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2 \int_\Omega \varphi^2\,\mathrm{d}x \right\}$$
$$= \max_{\varphi\in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)} \sup_{t>0} \left\{ 2\,t \fint_{\partial\Omega} |\varphi|\, d\mathcal{H}^{N-1} - t^2 \left(\fint_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \delta^2 \int_\Omega \varphi^2\,\mathrm{d}x \right) \right\} \tag{2.2.9}$$

For every $\varphi \in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)$, the optimal choice of t is given by

$$t = \frac{\fint_{\partial\Omega} |\varphi|\, d\mathcal{H}^{N-1}}{\fint_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \delta^2 \fint_\Omega \varphi^2\,\mathrm{d}x}.$$

By substituting, we get (2.2.5).

Let us finally prove formula (2.2.7). From (2.2.5) it follows

$$T(\Omega;\delta) = 2\int_{\partial\Omega} u_{\Omega,\delta}\, d\mathcal{H}^{N-1} - \fint_\Omega |\nabla u_{\Omega,\delta}|^2\,\mathrm{d}x - \delta^2 \fint_\Omega u_{\Omega,\delta}^2\,\mathrm{d}x.$$

On the other hand, by taking $\varphi = u_{\Omega,\delta}$ as a test function in (2.2.6), we obtain

$$\fint_\Omega |\nabla u_{\Omega,\delta}|^2\,\mathrm{d}x + \delta^2 \int_\Omega u_{\Omega,\delta}^2\,\mathrm{d}x = \fint_\Omega u_{\Omega,\delta}\,\mathrm{d}x.$$

By joining the last two equations in display, we obtain (2.2.7). □

Definition 2.2.4. *The function $u_{\Omega,\delta}$ of Proposition 2.2.3 will be called boundary $\delta-$torsion function of Ω. We also observe that such a function enjoys the following scaling law for $t>0$:*

$$u_{t\,\Omega,\delta/t}(x) = t\,u_{\Omega,\delta}\left(\frac{x}{t}\right) \qquad \text{for every } x\in t\,\Omega. \tag{2.2.10}$$

It is not difficult to see that the concave maximization problem in (2.2.5) admits a dual formulation, in the sense of Convex Analysis. This in turn permits to equivalently define $T(\Omega;\delta)$ as a convex minimization problem. This fact will be useful in the sequel.

Lemma 2.2.5 (Dual formulation)**.** *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, with Lipschitz boundary. Let us set*

$$\mathcal{A}^+(\Omega) = \left\{ (\phi,g)\in L^2(\Omega;\mathbb{R}^N)\times L^2(\Omega) \,:\, \begin{array}{rl} -\mathrm{div}\,\phi + \delta^2 g \ge 0, & \text{in } \Omega \\ \langle \phi,\nu_\Omega\rangle \ge 1, & \text{on } \partial\Omega \end{array} \right\}$$

where the conditions have to be intended in weak sense, i.e.

$$\int_{\partial\Omega} \varphi\, d\mathcal{H}^{N-1} \le \fint_\Omega \langle\phi,\nabla\varphi\rangle\,\mathrm{d}x + \delta^2 \fint_\Omega g\,\varphi\,\mathrm{d}x, \qquad \text{for every } \varphi\in W^{1,2}(\Omega) \text{ non-negative.}$$

Then we have

$$T(\Omega;\delta) = \min_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{ \fint_\Omega |\phi|^2\,\mathrm{d}x + \delta^2 \int_\Omega g^2\,\mathrm{d}x \right\} \tag{2.2.11}$$

and the minimum is uniquely attained by the pair $(\nabla u_{\Omega,\delta}, u_{\Omega,\delta})$.

Proof. For every non-negative $\varphi \in W^{1,2}(\Omega)$ and every $(\phi, g) \in \mathcal{A}^+(\Omega)$, we have by Young's inequality

$$\begin{aligned}\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} &\leq \fint_\Omega \langle \phi, \nabla\varphi \rangle \, \mathrm{d}x + \delta^2 \fint_\Omega g\,\varphi \, \mathrm{d}x \\ &\leq \frac{1}{2}\int_\Omega |\phi|^2 \, \mathrm{d}x + \frac{1}{2}\fint_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \frac{\delta^2}{2}\int_\Omega g^2 \, \mathrm{d}x + \frac{\delta^2}{2}\fint_\Omega \varphi^2 \, \mathrm{d}x.\end{aligned}$$

This in particular gives

$$2\fint_\Omega \varphi \, d\mathcal{H}^{N-1} - \left(\fint_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega \varphi^2 \, \mathrm{d}x\right) \leq \fint_\Omega |\phi|^2 \, \mathrm{d}x + \delta^2 \fint_\Omega g^2 \, \mathrm{d}x.$$

and by arbitrariness of φ and (ϕ, g) we get

$$\max_{\varphi\in W^{1,2}(\Omega),\varphi\geq 0} \left\{2\int_{\partial\Omega}\varphi \, d\sigma - \fint_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - \delta^2\int_\Omega \varphi^2 \, \mathrm{d}x\right\} \leq \inf_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{\int_\Omega |\phi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega g^2 \, \mathrm{d}x\right\}\fint$$

Since the constraint $\varphi \geq 0$ can be dropped without affecting the maximum value, recalling (2.2.5) we get

$$T(\Omega;\delta) \leq \inf_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{\fint_\Omega |\phi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega g^2 \, \mathrm{d}x\right\}\fint$$

On the other hand, it is easy to see that the pair $(\phi_0, g_0) = (\nabla u_{\Omega,\delta}, u_{\Omega,\delta})$ is admissible and it gives

$$\begin{aligned}\inf_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{\int_\Omega |\phi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega g^2 \, \mathrm{d}x\right\} &\leq \fint_\Omega |\nabla u_{\Omega,\delta}|^2 \, \mathrm{d}x + \delta^2 \fint_\Omega u_{\Omega,\delta}^2 \, \mathrm{d}x \\ &= \fint_{\partial\Omega} u_{\Omega,\delta} \, d\mathcal{H}^{N-1} = T(\Omega;\delta).\end{aligned}$$

This finally proves (2.2.11) and the fact that $(\nabla u_{\Omega,\delta}, u_{\Omega,\delta})$ is a minimizer. Its uniqueness easily follows from the strict convexity of the variational problem in (2.2.11). □

2.2.3 Relation with a Steklov eigenvalue problem

Here we establish some relations between the boundary torsional rigidity and the first Steklov eigenvalue of the Schrödinger–type operator

$$u \mapsto -\Delta u + \delta^2 \, u.$$

The Steklov spectrum of this operator consists of the real numbers σ such that the following boundary value problem

$$\begin{cases} -\Delta u + \delta^2 \, u &= 0, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= \sigma \, u, \quad \text{on } \partial\Omega, \end{cases}$$

admits at least a non-trivial weak solution $u \in W^{1,2}(\Omega)$. It is well known that the whole spectrum consists of an increasing sequence of eigenvalues $\{\sigma_n(\Omega;\delta)\}_{n\in\mathbb{N}}$ diverging at infinity. Indeed, it is sufficient to observe that the resolvent operator

$$\begin{array}{rccc} \mathcal{R}_\delta & : & L^2(\partial\Omega) & \to & L^2(\partial\Omega) \\ & & f & \mapsto & u_f, \end{array}$$

is positive, self-adjoint and compact. Then we can apply the Spectral Theorem in order to prove the claimed structure of the spectrum. Here, by $u_f \in W^{1,2}(\Omega)$ we mean the unique weak solution to the Neumann boundary value problem

$$\begin{cases} -\Delta u + \delta^2 u &= 0, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= f, \quad \text{on } \partial\Omega. \end{cases}$$

Of course, compactness of $\mathcal{R}_\delta$ is again due to the compactness of the relevant trace embedding

$$W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega).$$

The first eigenvalue has the following variational characterization

$$\sigma_1(\Omega;\delta) = \min_{u \in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \delta^2 \iint_\Omega u^2 \, \mathrm{d}x}{\displaystyle\int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}}.$$

Observe that by choosing u to be the characteristic function of Ω, we obtain

$$\sigma_1(\Omega;\delta) \leq \delta^2 \frac{|\Omega|}{\mathcal{H}^{N-1}(\partial\Omega)} \qquad \text{and thus} \qquad \lim_{\delta \to 0^+} \sigma_1(\Omega;\delta) = 0.$$

This is consistent with the fact that the first eigenvalue of the Steklov-Laplacian is 0, associated to constant eigenfunctions (see for example [78, Chapter 7, Section 3]).

The relation between $\sigma_1(\Omega;\delta)$ and $T(\Omega;\delta)$ follows immediately by applying Hölder's inequality on the boundary integral

$$\left(\iint_{\partial\Omega} u \, d\mathcal{H}^{N-1} \right)^2 \leq \mathcal{H}^{N-1}(\partial\Omega) \iint_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}.$$

This yields the following estimate, which should be compared with the classical Pólya inequality relating torsional rigidity, volume and first eigenvalue of the Dirichlet-Laplacian, see [122, Chapter V, Section 4]:

Proposition 2.2.6. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with Lipschitz boundary. Then we have*

$$\frac{\sigma_1(\Omega;\delta)\, T(\Omega;\delta)}{\mathcal{H}^{N-1}(\partial\Omega)} \leq 1. \tag{2.2.12}$$

2.3 Some properties of the torsion function

In this section, we establish some quantitative properties of the function $u_{\Omega,\delta}$ and study its asymptotic behavior as δ goes to 0.

Proposition 2.3.1. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with Lipschitz boundary.*

(i) The $L^1(\Omega)$ norm of $u_{\Omega,\delta}$ is given by

$$\fint_{\Omega} u_{\Omega,\delta}\,\mathrm{d}x = \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\delta^2}; \tag{2.3.1}$$

(ii) its trace is in $L^\infty(\partial\Omega)$, with the following estimate: for every exponent $2 < q < 2^\#$

$$\|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)} \le C_q \left(\frac{T(\Omega;\delta)^{\frac{q-2}{q}}}{\min\{1,\delta^2\}\,\eta_q(\Omega)}\right)^{\frac{q}{2(q-1)}}, \tag{2.3.2}$$

where $\eta_q(\Omega)$ is the constant defined in (2.2.1) and $C_q > 0$ is a constant only depending on q, which blows-up as $q \searrow 2$;

(iii) we have $u_{\Omega,\delta} \in L^\infty(\Omega)$ and it holds

$$\|u_{\Omega,\delta}\|_{L^\infty(\Omega)} \le \|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)}. \tag{2.3.3}$$

Proof. Point *(i)* simply follows by testing (2.2.6) with the characteristic function of Ω.

We now prove that $u_{\Omega,\delta}$ has a bounded trace, by using a Moser's iteration. We first show that it is sufficient to prove the claimed estimate (2.3.2) for $\delta = 1$. Indeed, let us suppose that (2.3.2) holds true for $\delta = 1$, for every open bounded set, with Lipschitz boundary. By recalling (2.2.10), we would get

$$u_{\Omega,\delta}(x) = \frac{1}{\delta}\,u_{\delta\,\Omega,1}(\delta\,x), \qquad \text{for } x \in \Omega,$$

and thus

$$\|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)} = \frac{1}{\delta}\,\|u_{\delta\,\Omega,1}\|_{L^\infty(\partial(\delta\,\Omega))} \le \frac{C_q}{\delta}\left(\frac{T(\delta\,\Omega;1)^{\frac{q-2}{q}}}{\eta_q(\delta\,\Omega)}\right)^{\frac{q}{2(q-1)}}. \tag{2.3.4}$$

We now observe that by its definition and a change of variable, we have

$$\begin{aligned}
\eta_q(\delta\,\Omega) &= \inf_{\varphi\in W^{1,2}(\delta\Omega)\setminus W_0^{1,2}(\delta\Omega)} \frac{\displaystyle\int_{\delta\Omega} |\nabla\varphi|^2\,\mathrm{d}x + \fint_{\delta\Omega} \varphi^2\,\mathrm{d}x}{\left(\displaystyle\fint_{\partial\Omega} |\varphi|^q\,d\mathcal{H}^{N-1}\right)^{\frac{2}{q}}} \\
&= \inf_{\varphi\in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)} \frac{\delta^{N-2}\displaystyle\int_{\Omega} |\nabla\varphi|^2\,\mathrm{d}x + \delta^N \fint_{\Omega} \varphi^2\,\mathrm{d}x}{\delta^{(N-1)\frac{2}{q}}\left(\displaystyle\fint_{\partial\Omega} |\varphi|^q\,d\mathcal{H}^{N-1}\right)^{\frac{2}{q}}} \\
&\ge \delta^{N\frac{q-2}{q}-2\frac{q-1}{q}}\left(\min\{1,\delta^2\}\right)\eta_q(\Omega),
\end{aligned}$$

while by (2.2.2)

$$T(\delta\,\Omega;1) = \delta^N\,T(\Omega;\delta).$$

Going back to (2.3.4), we get the claimed upper bound in (2.3.2) for $\delta \neq 1$, as well.

In order to prove (2.3.2) with $\delta = 1$, for notational simplicity we will write u in place of $u_{\Omega,1}$. We fix $M > 0$ and $\beta \geq 1$, then we insert the test function

$$\varphi = u_M^\beta, \qquad \text{where } u_M = \min\{u, M\},$$

in the weak formulation (2.2.6). Observe that this is a feasible test function, thanks to the Chain Rule in Sobolev spaces. We then obtain

$$\beta \iint_{\{u\leq M\}} |\nabla u|^2 \, u^{\beta-1} \, \mathrm{d}x + \int_\Omega u \, u_M^\beta \, \mathrm{d}x = \iint_{\partial\Omega} u_M^\beta \, d\mathcal{H}^{N-1}.$$

By observing that

$$\beta \iint_{\{u\leq M\}} |\nabla u|^2 \, u^{\beta-1} \, \mathrm{d}x = \frac{4\,\beta}{(\beta+1)^2} \int_\Omega \nabla \left(u_M^{\frac{\beta+1}{2}} \right)^2 \, \mathrm{d}x,$$

and using that $u \geq u_M$, from the identity above we get

$$\frac{4\,\beta}{(\beta+1)^2} \int_\Omega \nabla \left(u_M^{\frac{\beta+1}{2}} \right)^2 \, \mathrm{d}x + \int_\Omega \left(u_M^{\frac{\beta+1}{2}} \right)^2 \, \mathrm{d}x \leq \iint_{\partial\Omega} u_M^\beta \, d\mathcal{H}^{N-1}.$$

This in particular entails that

$$\left\| u_M^{\frac{\beta+1}{2}} \right\|^2_{W^{1,2}(\Omega)} \leq \left(\frac{(\beta+1)^2}{4\,\beta} + 1 \right) \iint_{\partial\Omega} u_M^\beta \, d\mathcal{H}^{N-1}.$$

By recalling the notation (2.2.1), from the trace embedding with $2 < q < 2^\#$ we obtain

$$\eta_q(\Omega) \left(\int_{\partial\Omega} \left(u_M^{\frac{\beta+1}{2}} \right)^q \, d\mathcal{H}^{N-1} \right)^{\frac{2}{q}} \leq \left(\frac{(\beta+1)^2}{4\,\beta} + 1 \right) \iint_{\partial\Omega} u_M^\beta \, d\mathcal{H}^{N-1}, \tag{2.3.5}$$

which is an iterative scheme of reverse L^p inequalities. Before going further, we observe that for every $\beta \geq 1$ we have

$$\frac{(\beta+1)^2}{4\,\beta} + 1 \leq \frac{\beta+1}{2} + 1 \leq \beta + 1 \leq 2\,\beta,$$

and then define the sequence of exponents

$$\beta_0 = 1, \qquad \beta_{i+1} = \frac{q}{2}\,(\beta_i + 1), \qquad \text{for } i \in \mathbb{N}.$$

Then from (2.3.5) we get

$$\left(\iint_{\partial\Omega} (u_M)^{\beta_{i+1}} \, d\mathcal{H}^{N-1} \right)^{\frac{1}{\beta_{i+1}}} \leq \left(\frac{2}{\eta_q(\Omega)}\,\beta_i \right)^{\frac{q}{2}\frac{1}{\beta_{i+1}}} \left(\iint_{\partial\Omega} u_M^{\beta_i} \, d\mathcal{H}^{N-1} \right)^{\frac{q}{2}\frac{1}{\beta_{i+1}}}.$$

If we set for notational simplicity $Y_i = \|u_M\|_{L^{\beta_i}(\partial\Omega)}$, the previous scheme can be written as

$$Y_{i+1} \leq \left(\frac{2}{\eta_q(\Omega)}\,\beta_i\right)^{\frac{q}{2}\,\frac{1}{\beta_{i+1}}}\; Y_i^{\frac{q}{2}\,\frac{\beta_i}{\beta_{i+1}}}, \qquad \text{for every } i \in \mathbb{N}. \tag{2.3.6}$$

We start with $i = 0$ and iterate (2.3.6) n times. We then obtain

$$Y_n \leq \left(\frac{2}{\eta_q(\Omega)}\right)^{\frac{1}{\beta_n}\sum_{i=1}^{n}\left(\frac{q}{2}\right)^i} \left[\prod_{i=0}^{n-1}\beta_i^{\left(\frac{q}{2}\right)^{n-i}}\right]^{\frac{1}{\beta_n}} Y_0^{\left(\frac{q}{2}\right)^n\frac{\beta_0}{\beta_n}}. \tag{2.3.7}$$

We now wish to take the limit as n goes to ∞ in this estimate. For this, we observe that

$$\sum_{i=1}^{n}\left(\frac{q}{2}\right)^i \sim \frac{2}{q-2}\,\left(\frac{q}{2}\right)^{n+1}, \qquad \text{as } n \to \infty,$$

$$\beta_n = \left(\frac{q}{2}\right)^n\left(+\sum_{i=1}^{n}\left(\frac{q}{2}\right)^i\right) \sim \left(\frac{q}{2}\right)^n\left(\frac{2\,(q-1)}{q-2}, \qquad \text{as } n \to \infty,\right.$$

and

$$\lim_{n\to\infty}\left[\prod_{i=0}^{n-1}\beta_i^{\left(\frac{q}{2}\right)^{n-i}}\right]^{\frac{1}{\beta_n}} = \lim_{n\to\infty}\exp\;\frac{1}{\beta_n}\,\left(\frac{q}{2}\right)^n\left(\sum_{i=0}^{n-1}\left(\frac{2}{q}\right)^i\log\beta_i\right)\left(= C_q < +\infty.\right.$$

Thus from (2.3.7), we get

$$Y_\infty \leq C_q\,\left(\frac{2}{\eta_q(\Omega)}\right)^{\frac{q}{2(q-1)}}\,\left(Y_0\right)^{\frac{q-2}{2(q-1)}}.$$

Finally, by recalling the definition of Y_i and that of u_M, we get

$$\begin{aligned}
\|u_M\|_{L^\infty(\partial\Omega)} &\leq C_q\,\left(\frac{2}{\eta_q(\Omega)}\right)^{\frac{q}{2(q-1)}}\,\left(\int_{\partial\Omega} u_M\,d\mathcal{H}^{N-1}\right)^{\frac{q-2}{2\,(q-1)}} \\
&\leq C_q\,\left(\frac{2}{\eta_q(\Omega)}\right)^{\frac{q}{2(q-1)}}\,\left(\int_{\partial\Omega} u\,d\mathcal{H}^{N-1}\right)^{\frac{q-2}{2\,(q-1)}} \\
&= C_q\,\left(\frac{2}{\eta_q(\Omega)}\right)^{\frac{q}{2(q-1)}}\,\left(T(\Omega;\delta)\right)^{\frac{q-2}{2(q-1)}}.
\end{aligned}$$

We also used the identity (2.2.7) in the last equality. We eventually get the desired result from the previous estimate, by arbitrariness of $M > 0$. This concludes the proof of point *(ii)*.

Finally, we come to the proof of point *(iii)*, i.e. the maximum principle (2.3.3). Let us set for brevity $L = \|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)}$, then we have

$$u_{\Omega,\delta} \leq L, \qquad \text{on } \partial\Omega.$$

This implies that $\varphi = (u_{\Omega,\delta} - L)_+ \in W_0^{1,2}(\Omega)$. By inserting this test function in (2.2.6), we obtain

$$\int_\Omega \langle \nabla u_{\Omega,\delta}, \nabla (u_{\Omega,\delta} - L)_+ \rangle \, \mathrm{d}x + \delta^2 \int_\Omega u_{\Omega,\delta} \, (u_{\Omega,\delta} - L)_+ \, \mathrm{d}x = 0,$$

that is

$$\int_\Omega |\nabla (u_{\Omega,\delta} - L)_+|^2 \, \mathrm{d}x + \delta^2 \int_\Omega u_{\Omega,\delta} \, (u_{\Omega,\delta} - L)_+ \, \mathrm{d}x = 0.$$

Since both terms are non-negative, we get that they both must vanish. This in turn implies

$$\nabla (u_{\Omega,\delta} - L)_+ = 0, \qquad \text{a. e. in } \Omega,$$

and

$$u_{\Omega,\delta} \, (u_{\Omega,\delta} - L)_+ = 0, \qquad \text{a. e. in } \Omega.$$

The two informations entail that $(u_{\Omega,\delta} - L)_+$ must vanish almost everywhere in Ω, which means that

$$0 \leq u_{\Omega,\delta} \leq L = \|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)}, \qquad \text{a. e. in } \Omega.$$

Thus we get $\|u_{\Omega,\delta}\|_{L^\infty(\Omega)} \leq \|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)}$. □

Remark 2.3.2 (Regularity of $u_{\Omega,\delta}$)**.** *We remark that by classical Elliptic Regularity, the function $u_{\Omega,\delta}$ actually belongs to $C^\infty(\Omega)$, i.e. it is a classical solution of*

$$-\Delta u + \delta^2 \, u = 0, \qquad \text{in } \Omega.$$

However, the regularity up to the boundary is more delicate, under our standing assumptions on Ω. Indeed, well-known counterexamples show that global smoothness may fail in Lipschitz sets, already for harmonic functions and very simple boundary data, see for example [72]. Thus in general $u_{\Omega,\delta}$ has to be considered only as a weak solution of

$$\begin{cases} -\Delta u + \delta^2 \, u &= 0, \quad \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle &= 1, \quad \text{on } \partial\Omega, \end{cases}$$

i.e. the Neumann condition has to be interpreted only in weak sense.

Finally, we point out that the property $u_{\Omega,\delta} \geq 0$ can actually be improved to

$$u_{\Omega,\delta} > 0, \qquad \text{for every } x \in \Omega,$$

by virtue of the minimum principle, see [62, Chapter 6, Section 4, Theorem 4]. Observe that this holds true also if Ω is a disconnected set, by virtue of the non-homogeneous Neumann condition. In other words, if Ω is made of k connected components, we must have $u_{\Omega,\delta} > 0$ on each single component.

The boundary torsion functions are monotone with respect to the parameter δ. This is the content of the next result:

Lemma 2.3.3 (Monotonicity)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with Lipschitz boundary. For every $0 < \delta_0 < \delta_1$, we have*

$$u_{\Omega,\delta_0} > u_{\Omega,\delta_1}, \qquad \text{in } \Omega.$$

Proof. Let us set for simplicity

$$u_i := u_{\Omega,\delta_i}, \qquad i = 0, 1.$$

By subtracting the equations (2.2.6) for u_0 and u_1, we get

$$\fint_\Omega \langle \nabla(u_1 - u_0), \nabla\varphi \rangle \, \mathrm{d}x + \delta_1^2 \int_\Omega u_1 \, \varphi \, \mathrm{d}x - \delta_0^2 \fint_\Omega u_0 \, \varphi \, \mathrm{d}x = 0, \qquad \text{for every } \varphi \in W^{1,2}(\Omega). \tag{2.3.8}$$

We first observe that by choosing φ to be the characteristic function of Ω, we obtain

$$\delta_1^2 \int_\Omega u_1 \, \mathrm{d}x = \delta_0^2 \fint_\Omega u_0 \, \mathrm{d}x,$$

and thus we must have $u_0 \not\equiv u_1$.

We then choose the test function $\varphi = (u_1 - u_0)_+$ in (2.3.8), which implies

$$\int_\Omega |\nabla(u_1 - u_0)_+|^2 \, \mathrm{d}x + \fint_\Omega (\delta_1^2 \, u_1 - \delta_0^2 \, u_0) \, (u_1 - u_0)_+ \, \mathrm{d}x = 0.$$

This can be rearranged into

$$\int_\Omega |\nabla(u_1 - u_0)_+|^2 \, \mathrm{d}x + \delta_1^2 \fint_\Omega (u_1 - u_0) \, (u_1 - u_0)_+ \, \mathrm{d}x = (\delta_0^2 - \delta_1^2) \fint_\Omega u_0 \, (u_1 - u_0)_+ \, \mathrm{d}x.$$

The left-hand side is non-negative, thus by using that $\delta_1 > \delta_0 > 0$, we get that

$$\fint_\Omega u_0 \, (u_1 - u_0)_+ \, \mathrm{d}x \le 0.$$

Since $u_0 > 0$ by Remark 2.3.2, the previous estimate entails that $(u_1 - u_0)_+$ must vanish almost everywhere. This shows that

$$u_1 \le u_0, \qquad \text{a. e. in } \Omega.$$

By the interior smoothness of both functions, this property must actually hold everywhere.

Finally, in order to obtain the strict sign, it is sufficient to observe that the difference $v = u_0 - u_1$ is a non-negative smooth solution of

$$-\Delta v + \delta_0^2 \, v = (\delta_1^2 - \delta_0^2) \, u_1 \ge 0.$$

By using again the minimum principle, we get that

$$\text{either} \quad v \equiv 0 \quad \text{or} \quad v > 0.$$

However, the first fact has been already excluded at the beginning. This concludes the proof. □

Theorem 2.3.4 (Asymptotics for $\delta \to 0$)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, with Lipschitz boundary. Then we have*

$$\lim_{\delta\to 0^+} \|\nabla(\delta^2\, u_{\Omega,\delta})\|_{L^2(\Omega)} = 0, \tag{2.3.9}$$

and

$$\lim_{\delta\to 0^+} \left\| \delta^2\, u_{\Omega,\delta} - \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|} \right\|_{L^m(\Omega)} = 0, \qquad \text{for every } 2 \le m < \infty. \tag{2.3.10}$$

Moreover, it also holds

$$\lim_{\delta\to 0^+} \delta^2\, T(\Omega;\delta) = \frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|}.$$

Proof. From the identity

$$\int\!\!\!\int_{\Omega} |\nabla u_{\Omega,\delta}|^2 \, \mathrm{d}x + \delta^2 \int_\Omega (u_{\Omega,\delta})^2 \, \mathrm{d}x = \int\!\!\!\int_{\partial\Omega} u_{\Omega,\delta}\, d\mathcal{H}^{N-1}, \tag{2.3.11}$$

we get in particular, for every $0 < \delta \le 1$,

$$\begin{aligned} \int_\Omega |\nabla(\delta^2\, u_{\Omega,\delta})|^2 \, \mathrm{d}x \le \delta^4 \int\!\!\!\int_{\partial\Omega} u_{\Omega,\delta}\, d\mathcal{H}^{N-1} &= \delta^4\, T(\Omega;\delta) \\ &\le \frac{\delta^4}{\min\{1,\delta^2\}} \frac{1}{\eta_1(\Omega)} = \frac{\delta^2}{\eta_1(\Omega)}. \end{aligned} \tag{2.3.12}$$

Observe that we used formula (2.2.7) and the upper bound (2.2.3). This shows (2.3.9).

Moreover, still from (2.3.11), we also have

$$\int_\Omega (\delta^2\, u_{\Omega,\delta})^2 \, \mathrm{d}x \le \delta^2 \int\!\!\!\int_{\partial\Omega} u_{\Omega,\delta}\, d\mathcal{H}^{N-1} \le \frac{1}{\eta_1(\Omega)}.$$

These show that the family of functions $\{\delta^2\, u_{\Omega,\delta}\}_{0<\delta\le 1} \subset W^{1,2}(\Omega)$ is equi-bounded. By the Rellich-Kondrašov Theorem, for every infinitesimal decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ there exists a function $w \in W^{1,2}(\Omega)$ such that (up to a subsequence)

$$\lim_{n\to\infty} \|\delta_n^2\, u_{\Omega,\delta_n} - w\|_{L^2(\Omega)} = 0,$$

and

$$\lim_{n\to\infty} \int_\Omega \langle \nabla(\delta_n^2\, u_{\Omega,\delta_n}), \phi\rangle \, \mathrm{d}x = \int\!\!\!\int_{\Omega} \langle \nabla w, \phi\rangle \, \mathrm{d}x, \qquad \text{for every } \phi \in L^2(\Omega;\mathbb{R}^N).$$

Moreover, thanks to (2.3.12), we actually get that it must result $\nabla w = 0$ almost everywhere in Ω. The connectedness assumption entails that w is constant on Ω. In order to identify the value of such a constant, we recall that from (2.3.1) we have

$$\int\!\!\!\int_{\Omega} (\delta_n^2\, u_{\Omega,\delta_n}) \, \mathrm{d}x = \mathcal{H}^{N-1}(\partial\Omega).$$

By taking the limit as n goes to ∞ in this identity, we finally get

$$\iint_{\Omega} w \, \mathrm{d}x = \mathcal{H}^{N-1}(\partial\Omega).$$

By recalling that w is constant, we get

$$w = \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|}.$$

We now observe that the limit function w is uniquely determined and thus does not depend on the particular sequence $\{\delta_n\}_{n\in\mathbb{N}}$. Thus, we can finally infer convergence in L^2 of the whole family $\{\delta^2 u_{\Omega,\delta}\}_{0<\delta\leq 1}$, i.e. we showed (2.3.10) for $m = 2$.

In order to get (2.3.10) for an exponent $2 < m < \infty$, it is sufficient to observe that

$$\begin{aligned}\left\| \delta^2 u_{\Omega,\delta} - \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|} \right\|_{L^m(\Omega)} \leq & \left\| \delta^2 u_{\Omega,\delta} - \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|} \right\|_{L^\infty(\Omega)}^{1-\frac{2}{m}} \\ & \times \left\| \delta^2 u_{\Omega,\delta} - \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|} \right\|_{L^2(\Omega)}^{\frac{2}{m}}.\end{aligned}$$

The last term converges to 0 thanks to the proof above, while the L^∞ norm is uniformly bounded, thanks to the fact that by Proposition 2.3.1 we have for every $0 < \delta \leq 1$

$$\begin{aligned}\|\delta^2 u_{\Omega,\delta}\|_{L^\infty(\Omega)} \leq \|\delta^2 u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)} &\leq C_q \left(\frac{(\delta^2 T(\Omega;\delta))^{\frac{q-2}{q}}}{\eta_q(\Omega)} \right)^{\frac{q}{2(q-1)}} \\ &\leq C_q \left(\frac{\eta_1(\Omega)^{\frac{2-q}{q}}}{\eta_q(\Omega)} \right)^{\frac{q}{2(q-1)}},\end{aligned}$$

for a fixed $2 < q < 2^\#$. Observe that we also used the upper bound in (2.2.3).

Finally, the last part of the statement easily follows by taking the limit in formula (2.2.7) and using the compactness of the trace embedding. □

Remark 2.3.5. *With the previous result at hand, we can now prove that the lower bound*

$$\frac{1}{\delta^2} \frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq T(\Omega;\delta),$$

proved in (2.2.3) *is actually sharp, for every fixed $\delta > 0$. Indeed, let us take $\Omega \subset \mathbb{R}^N$ an open bounded connected set, with Lipschitz boundary. By the scaling law* (2.2.2)*, we have*

$$T(t\,\Omega,\delta) = t^N\, T(\Omega; t\,\delta), \qquad \text{for every } t > 0.$$

We thus obtain

$$\begin{aligned}\lim_{t\to 0^+} \delta^2\, T(t\,\Omega;\delta)\, \frac{|t\,\Omega|}{(\mathcal{H}^{N-1}(\partial(t\,\Omega)))^2} &= \lim_{t\to 0^+} \delta^2\, t^N\, T(\Omega; t\,\delta)\, \frac{|\Omega|\, t^N}{(\mathcal{H}^{N-1}(\partial\Omega)\, t^{N-1})^2} \\ &= \lim_{t\to 0^+} (\delta\, t)^2\, T(\Omega; t\,\delta)\, \frac{|\Omega|}{(\mathcal{H}^{N-1}(\partial\Omega))^2} = 1,\end{aligned}$$

thanks to Theorem 2.3.4.

2.4 Exact solutions in some special sets

In what follows, we will by indicate by I_α and K_α the *modified Bessel functions with index* α of first and second kind, respectively. We recall that these functions have the following asymptotic behavior for z converging to 0

$$I_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)} \left(\frac{z}{2}\right)^\alpha, \tag{2.4.1}$$

and

$$K_\alpha(z) \sim \begin{cases} -\log\left(\frac{z}{2}\right) & \text{for } \alpha = 0, \\ \frac{\Gamma(\alpha)}{2} \left(\frac{2}{z}\right)^\alpha, & \text{otherwise.} \end{cases} \tag{2.4.2}$$

Here Γ is the usual Gamma function. We refer to [107] or [97, Chapter 5] for the general properties of these functions.

Lemma 2.4.1 (Balls)**.** *Let $\delta > 0$ and let $B \subset \mathbb{R}^N$ be the ball of radius 1, centered at the origin. Then $u_{B,\delta}$ is a radially symmetric increasing function. This is explicitly given by*

$$u_{B,\delta}(x) = \mathcal{U}_\delta(|x|) \qquad \text{where} \quad \mathcal{U}_\delta(\varrho) = \frac{\varrho^{1-N/2}\, I_{N/2-1}(\delta\,\varrho)}{\delta\, I_{N/2}(\delta)}, \text{ for } 0 \le \varrho < 1. \tag{2.4.3}$$

Accordingly, we get

$$T(B;\delta) = \int_{\partial B} u_{B,\delta}\, d\mathcal{H}^{N-1} = \frac{N\,\omega_N\, I_{N/2-1}(\delta)}{\delta\, I_{N/2}(\delta)}. \tag{2.4.4}$$

Finally, the function $\varrho \mapsto \mathcal{U}_\delta'(\varrho)/\varrho$ is monotone non-decreasing.

Proof. Let O be an $N \times N$ rotation matrix, we set $v(x) = u_{B,\delta}(O\,x)$. Thanks to the symmetries of B and using the change of variable $y = O\,x$, we get that v is still a maximizer of (2.2.5). By uniqueness, we thus obtain

$$u_{B,\delta}(x) = u_{B,\delta}(O\,x), \qquad \text{for } x \in B.$$

By arbitrariness of the matrix O, this gives that u_B must be radially symmetric. Thus there exists $\mathcal{U}_\delta$ such that

$$u_{B,\delta}(x) = \mathcal{U}_\delta(|x|),$$

and the function $\mathcal{U}_\delta$ must be the unique solution of

$$\sup_{\varphi \in W^{1,2}(\mathrm{I})} \left\{ 2\,\varphi(1) - \int_{\mathrm{I}} |\varphi'(\varrho)|^2\, \varrho^{N-1}\, d\varrho - \delta^2 \int_{\mathrm{I}} \varphi(\varrho)^2\, \varrho^{N-1}\, d\varrho \right\} \tag{2.4.5}$$

where we set $\mathrm{I} = (0,1)$. We now construct the new admissible function

$$\widetilde{\mathcal{U}}_\delta(\varrho) = \left(\mathcal{U}_\delta(1) - \int_\varrho^1 (\mathcal{U}_\delta'(t))_+\, dt \right)_+. \tag{2.4.6}$$

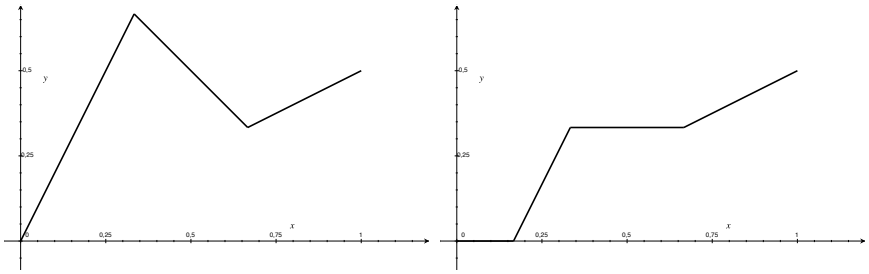


Figure 2.1: An example of function (on the left) and its rearrangement (on the right) given by (2.4.6).

Observe that this is non-negative and non-decreasing by construction, since

$$\frac{d}{d\varrho}\left(\mathcal{U}_\delta(1) - \int_\varrho^1 (\mathcal{U}_\delta'(t))_+\,dt\right) = (\mathcal{U}_\delta'(\varrho))_+ \geq 0,$$

and thus $\widetilde{\mathcal{U}}_\delta$ is the composition of two non-decreasing functions. Moreover, we have $\widetilde{\mathcal{U}}_\delta(1) = \mathcal{U}_\delta(1)$ and

$$\mathcal{U}_\delta(1) - \int_\varrho^1 (\mathcal{U}_\delta'(t))_+\,dt \leq \mathcal{U}_\delta(1) - \int_\varrho^1 \mathcal{U}_\delta'(t)\,dt = \mathcal{U}_\delta(\varrho).$$

By taking the positive part on both sides and using that $\mathcal{U}_\delta \geq 0$, we obtain

$$0 \leq \widetilde{\mathcal{U}}_\delta \leq \mathcal{U}_\delta.$$

Finally, by construction we also have

$$|\widetilde{\mathcal{U}}_\delta'(\varrho)| \leq |\mathcal{U}_\delta'(\varrho)|, \qquad \text{for a. e. } \varrho \in \mathrm{I}.$$

All these properties show that $\widetilde{\mathcal{U}}_\delta$ must be another maximizer of (2.4.5). By uniqueness, we obtain $\widetilde{\mathcal{U}}_\delta = \mathcal{U}_\delta$ and thus, $\mathcal{U}_\delta$ has the claimed monotonicity.

In order to explicitly determine $\mathcal{U}_\delta$, it is sufficient to write the optimality condition. This is given by the boundary value problem

$$\begin{cases} -\varrho^2\,\psi'' - \varrho\,(N-1)\,\psi' + \delta^2\,\varrho^2\,\psi &= 0, \quad \text{in I},\\ \psi'(0) &= 0,\\ \psi'(1) &= 1. \end{cases}$$

The general solution of the ODE is given by (see for example [97, Chapter 5, Section 7])

$$\psi(\varrho) = C\,\varrho^{1-\frac{N}{2}}\,I_{N/2-1}(\delta\,\varrho) + D\,\varrho^{1-\frac{N}{2}}\,K_{N/2-1}(\delta\,\varrho), \qquad C, D \in \mathbb{R}. \tag{2.4.7}$$

Thus, as K_α is singular at the origin by (2.4.2), the condition $w'(0) = 0$ implies that $D = 0$. This leads to

$$\mathcal{U}_\delta(\varrho) = C\,\varrho^{1-\frac{N}{2}}\,I_{N/2-1}(\delta\,\varrho).$$

We impose the other boundary condition $w'(1) = 1$ in order to fix the constant C. By recalling the relation (see [97, equation (5.7.9), page 110])

$$\frac{d}{d\varrho}(\varrho^{-\alpha}\, I_\alpha(\varrho)) = \varrho^{-\alpha}\, I_{\alpha+1}(\varrho), \tag{2.4.8}$$

and using this with $\alpha = N/2 - 1$, we finally get the expression (2.4.3).

We can now substitute this expression in (2.2.7) in order to find the explicit formula (2.4.4) for the torsional rigidity.

At last, we prove that

$$\mathcal{W}(\varrho) := \frac{\mathcal{U}'_\delta(\varrho)}{\varrho} = \frac{1}{\varrho}\,\frac{d}{d\varrho}\frac{\varrho^{1-N/2}\, I_{N/2-1}(\delta\,\varrho)}{\delta\, I_{N/2}(\delta)},$$

is monotone non-decreasing. By using (2.4.8), it is easy to see that this can be re-written as

$$\mathcal{W}(\varrho) = \frac{\varrho^{-N/2}\, I_{N/2}(\delta\,\varrho)}{I_{N/2}(\delta)}.$$

Thus $\mathcal{W}$ solves the ODE

$$-\varrho^2\,\mathcal{W}'' - \varrho\,(N+1)\,\mathcal{W}' + \delta^2\,\varrho^2\,\mathcal{W} = 0, \qquad \text{in I},$$

with boundary conditions

$$\mathcal{W}'(0) = 0 \qquad \text{and} \qquad \mathcal{W}'(1) = \delta\,\frac{I_{N/2+1}(\delta)}{I_{N/2}(\delta)} =: c_\delta.$$

We notice that these can be inferred by appealing again to (2.4.8) and (2.4.1). These properties and the first part of the proof imply that, if we denote by $B' \subset \mathbb{R}^{N+2}$ the $(N+2)-$dimensional unit ball centered at the origin, the radially symmetric function

$$w(x) := \mathcal{W}(|x|), \qquad \text{for } x \in B',$$

is the unique solution of

$$\begin{cases} -\Delta u + \delta^2\, u &= 0, \quad \text{in } B', \\ \langle \nabla u, \nu_\Omega \rangle &= c_\delta, \quad \text{on } \partial B'. \end{cases}$$

Thus, it is the unique maximizer of

$$\sup_{\varphi \in W^{1,2}(B')} \left\{ 2\, c_\delta \int_{\partial B'} \varphi\, d\mathcal{H}^{N+1} - \int_{B'} |\nabla \varphi|^2 \, \mathrm{d}x - \delta^2 \int_{B'} \varphi^2 \, \mathrm{d}x \right\}$$

By recalling that I_α is always positive for real arguments, we get that $c_\delta > 0$ and thus the claimed monotonicity of w now follows by using the same argument as for (2.4.5). □

Remark 2.4.2. *By recalling the scaling laws* (2.2.2) *and* (2.2.10)*, for a generic ball of radius* $R > 0$ *we have*

$$u_{B_R,\delta}(x) = R\, u_{B,\delta R}\left(\frac{x}{R}\right) = \frac{|x|^{1-\frac{N}{2}}\, I_{N/2-1}(\delta\,|x|)}{\delta\, R^{1-\frac{N}{2}}\, I_{N/2}(\delta\, R)},$$

and

$$T(B_R;\delta) = R^N\, T\,(B;\delta\, R) = R^{N-1}\, \frac{N\,\omega_N\, I_{N/2-1}(\delta\, R)}{\delta\, I_{N/2}(\delta\, R)}.$$

Lemma 2.4.3 (Spherical shells)**.** *Let* $\delta > 0$*. For* $0 < r < R$*, we consider the spherical shell*

$$\Omega = \Big\{x \in \mathbb{R}^N \,:\, r < |x| < R\Big\}.$$

Then $u_{\Omega,\delta}$ *is a radially symmetric function. This is explicitly given by*

$$u_{\Omega,\delta}(x) = \mathcal{V}_{r,R,\delta}(|x|), \qquad \text{for } x \in \Omega,$$

where

$$\mathcal{V}_{r,R,\delta}(\varrho) = C_0\, \varrho^{1-\frac{N}{2}}\, I_{N/2-1}(\delta\,\varrho) + D_0\, \varrho^{1-\frac{N}{2}}\, K_{N/2-1}(\delta\,\varrho),$$

and the constants $C_0 = C_0(r,R,\delta) \neq 0$ *and* $D_0 = D_0(r,R,\delta) \neq 0$ *are given by*

$$\begin{aligned} C_0 &= \frac{r^{1-\frac{N}{2}}\, K_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, K_{N/2}(\delta\, R)}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\Big(I_{N/2}(R)\, K_{N/2}(r) - I_{N/2}(r)\, K_{N/2}(R)\Big)},\\ D_0 &= \frac{r^{1-\frac{N}{2}}\, I_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, I_{N/2}(\delta\, R)}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\Big(I_{N/2}(r)\, K_{N/2}(R) - I_{N/2}(R)\, K_{N/2}(r)\Big)}. \end{aligned}$$

Accordingly, we get

$$\begin{aligned} T(\Omega;\delta) = {} & \frac{\Big[r^{1-\frac{N}{2}}\, K_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, K_{N/2}(\delta\, R)\Big]\,\Big[R^{1-\frac{N}{2}} I_{1-N/2}(\delta\, R) + r^{1-\frac{N}{2}}\, I_{1-N/2}(\delta\, r)\Big]}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\,\big[I_{N/2}(R)\, K_{N/2}(r) - I_{N/2}(r)\, K_{N/2}(R)\big]}\\ & + \frac{\Big[r^{1-\frac{N}{2}}\, I_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, I_{N/2}(\delta\, R)\Big]\,\Big[R^{1-\frac{N}{2}}\, K_{1-N/2}(\delta\, R) + r^{1-\frac{N}{2}}\, K_{1-N/2}(\delta\, r)\Big]}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\,\big[I_{N/2}(r)\, K_{N/2}(R) - I_{N/2}(R)\, K_{N/2}(r)\big]}. \end{aligned}$$

Proof. The radial symmetry of $u_{\Omega,\delta}$ can be obtained with the same proof of Lemma 2.4.1. In order to determine $u_{\Omega,\delta}$, we proceed as before, by seeking this time the solution of

$$\begin{cases} -\varrho^2\, w'' - \varrho\,(N-1)\, w' + \delta^2\, \varrho^2\, w &= 0, \qquad \text{in } (r,R),\\ w'(r) &= -1,\\ w'(R) &= 1. \end{cases}$$

By recalling the expression of the general solution (2.4.7), we can calculate the explicit form of both constants C and D by imposing the two boundary conditions. Once $u_{\Omega,\delta}$ is obtained, it is sufficient to use again (2.2.7) to get $T(\Omega;\delta)$, as well. We leave the details to the reader. □

Lemma 2.4.4 (Hyperrectangle). *Let $\delta > 0$ and let $\ell_1, \ell_2, \dots, \ell_N > 0$. If we set*

$$\Omega = \prod_{i=1}^{N} (-\ell_i, \ell_i),$$

then we have

$$u_{\Omega,\delta}(x) = \sum_{i=1}^{N} \frac{\cosh(\delta\, x_i)}{\delta\, \sinh(\delta\, \ell_i)}, \qquad \text{for every } x = (x_1, \dots, x_N) \in \Omega. \tag{2.4.9}$$

Its boundary torsional rigidity is given by

$$T(\Omega;\delta) = \sum_{k=1}^{N} \left[\frac{1}{\delta\, \tanh(\delta\, \ell_k)}\, \mathcal{H}^{N-1}(\Sigma_k) + \sum_{i \neq k} \frac{1}{\delta^2}\, \mathcal{H}^{N-2}(\Sigma_{k,i}) \right]$$

where

$$\Sigma_k = \Big\{ x \in \overline{\Omega} \,:\, |x_k| = \ell_k \Big\} \qquad \text{and} \qquad \Sigma_{k,i} = \Big\{ x \in \overline{\Omega} \,:\, |x_k| = \ell_k,\, |x_i| = \ell_i \Big\}.$$

Proof. One can directly verify that (2.4.9) is a classical solution of the optimality condition (2.2.6). In order to compute the boundary torsional rigidity, we use (2.4.9) in formula (2.2.7). We get

$$T(\Omega;\delta) = \int_{\partial\Omega} u_{\Omega,\delta}\, d\mathcal{H}^{N-1} = \sum_{k=1}^{N} \int_{\Sigma_k} \left[\frac{\cosh(\delta\, x_k)}{\delta\, \sinh(\delta\, \ell_k)} + \sum_{i \neq k} \frac{\cosh(\delta\, x_i)}{\delta\, \sinh(\delta\, \ell_i)} \right] d\mathcal{H}^{N-1}.$$

We then observe that for every $k \in \{1, \dots, N\}$, we have (observe that $\cosh(\delta\, x_k)$ is constant on Σ_k)

$$\int_{\Sigma_k} \frac{\cosh(\delta\, x_k)}{\delta\, \sinh(\delta\, \ell_k)}\, d\mathcal{H}^{N-1} = \frac{\cosh(\delta\, \ell_k)}{\delta\, \sinh(\delta\, \ell_k)}\, \mathcal{H}^{N-1}(\Sigma_k),$$

and for $i \neq k$ we get

$$\int_{\Sigma_k} \frac{\cosh(\delta\, x_i)}{\delta\, \sinh(\delta\, \ell_i)}\, d\mathcal{H}^{N-1} = \frac{4}{\delta^2} \prod_{j \neq i,k} (2\, \ell_j) = \frac{\mathcal{H}^{N-2}(\Sigma_{k,i})}{\delta^2}.$$

This concludes the proof. □

2.5 Geometric estimates

2.5.1 Planar simply connected sets

We start by proving a lower bound for the boundary torsional rigidity by using *conformal transplantation*, when $N = 2$. As we have mentioned in the introduction, this technique has been successfully employed in order to give geometric estimates for eigenvalues of planar sets, see for example [119, 122, 132, 144].

Roughly speaking, it consists in producing trial functions for a variational problem in an open simply connected set $\Omega \subset \mathbb{R}^2$, by simply composing functions defined on the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ with an holomorphic map $h : \Omega \to \mathbb{D}$. This is particularly useful when tackling problems involving the Dirichlet integral, due to its conformal invariance.

In order to state our lower bound, let us recall a few facts on conformal mappings. The celebrated *Riemann Mapping Theorem* (see [2, Chapter 6]) states that given any simply connected region $\Omega \subsetneq \mathbb{R}^2$ and a point $x_0 \in \Omega$, there exists a unique (up to a rotation) holomorphic isomorphism

$$f_{x_0} : \mathbb{D} \to \Omega, \qquad \text{with} \quad f_{x_0}(0) = x_0.$$

Furthermore, when $\partial\Omega$ is $C^{1,\alpha}$, we know that this mapping can be extended up to the boundary, it is C^1 in $\overline{\mathbb{D}}$ and we have

$$f'_{x_0}(x) \neq 0, \qquad \text{for every } x \in \partial\mathbb{D}.$$

We refer to [143, Theorem 1 & Theorem 2] for this result.

Definition 2.5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded simply connected set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. With the previous notation, we define the boundary distortion radius of Ω by*

$$\dot{\mathcal{R}}_\Omega := \inf_{x_0 \in \Omega} \left(\frac{1}{2\,\pi} \iint_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}.$$

It is readily seen from its definition that this quantity scales like a length.

Remark 2.5.2. *By using that $|f'_{x_0}|^2$ is a subharmonic function, we get that the map*

$$\varrho \mapsto \frac{1}{2\,\pi\,\varrho} \iint_{\{|x|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1,$$

is monotone non-decreasing, a result originally due to G. H. Hardy, see [77, Theorem III]. In particular, we have

$$|f'_{x_0}(0)| \leq \left(\frac{1}{2\,\pi\,\varrho} \iint_{\{|x|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq \left(\frac{1}{2\,\pi} \iint_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}.$$

Then the definition of boundary distortion radius is maybe better appreciated by recalling that

$$\dot{r}_\Omega := \sup_{x_0 \in \Omega} |f'_{x_0}(0)|,$$

is usually called conformal radius of Ω. This quantity naturally appears in many geometric estimates for the spectrum of the Laplacian of a planar simply connected set, see for example [119] and [122].

Finally, we may notice that the quantity

$$\left(\frac{1}{2\,\pi} \iint_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}},$$

can be recognized as the norm of the conformal map f'_{x_0} in the Hardy space $\mathrm{H}^2(\mathbb{D})$. We refer the reader to [91] for an introduction to these spaces.

Lemma 2.5.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. Then we have*

$$|\Omega| \leq \pi \, \dot{\mathcal{R}}_\Omega^2. \tag{2.5.1}$$

In particular, if Ω is a disk of radius R, we have $\dot{\mathcal{R}}_\Omega = R$ and the equality holds in (2.5.1).

Proof. Let $x_0 \in \Omega$, with the notation above we recall that $|f'_{x_0}|^2$ coincides with the Jacobian determinant of f_{x_0}, seen as a two-dimensional change of variables. Thus we have

$$|\Omega| = \int\!\!\!\int_{\mathbb{D}} |f'_{x_0}|^2 \, dw.$$

We can then write

$$\int\!\!\!\int_{\mathbb{D}} |f'_{x_0}|^2 \, dw = \int_0^1 \left(\int_{\{|x|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right) d\varrho.$$

By using Remark 2.5.2 and the arbitrariness of $x_0 \in \Omega$, we then get the desired estimate (2.5.1).

As for the second statement, we observe that for a disk $B_R(\xi_0)$, by choosing $x_0 = \xi_0$ and

$$f_{\xi_0}(x) = R\, x + \xi_0,$$

we have $|f'_{\xi_0}| = R$ and thus

$$\dot{\mathcal{R}}_{B_R(\xi_0)} \leq \left(\frac{1}{2\,\pi} \int\!\!\!\int_{\partial\mathbb{D}} |f'_{\xi_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} = R.$$

The reverse inequality follows from (2.5.1). This concludes the proof. □

We are now ready for the main result of this subsection.

Theorem 2.5.4. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. Then it holds*

$$\left(\frac{\mathcal{H}^1(\partial\Omega)}{2\,\pi} \right)^2 \frac{T\left(B_{\dot{\mathcal{R}}_\Omega}; \delta\right)}{\dot{\mathcal{R}}_\Omega^2} \leq T(\Omega; \delta). \tag{2.5.2}$$

Moreover, equality holds if and only if Ω is a disk.

Proof. We first give the estimate under the assumption that $\dot{\mathcal{R}}_\Omega = 1$. By definition, for every $\varepsilon > 0$, there exists $x_0 \in \Omega$ such that

$$\frac{1}{2\,\pi} \int\!\!\!\int_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 < 1 + \varepsilon. \tag{2.5.3}$$

For simplicity, we indicate by u the boundary $\delta-$torsion function of the unit disk $\mathbb{D}$, given by (2.4.3). With the notation above, we use the test function $\widetilde{u} = u \circ h_{x_0}$, where we have set

$$h_{x_0} := f_{x_0}^{-1} : \overline{\Omega} \to \overline{\mathbb{D}}.$$

Observe that $\widetilde{u} \in W^{1,2}(\Omega)$, thanks to the properties of f_{x_0}. This yields

$$\frac{1}{T(\Omega;\delta)} \leq \frac{\displaystyle\int_\Omega |\nabla \widetilde{u}|^2 \, \mathrm{d}x + \delta^2 \int_\Omega \widetilde{u}^2 \, \mathrm{d}x}{\left(\displaystyle\int_{\partial\Omega} \widetilde{u} \, d\mathcal{H}^1 \right)^2} = \left(\frac{\delta \, I_1(\delta)}{I_0(\delta)} \right)^2 \frac{\displaystyle\int_{\mathbb{D}} |\nabla u|^2 \, dw + \delta^2 \int_{\mathbb{D}} u^2 \, |f'_{x_0}(w)|^2 \, dw}{(\mathcal{H}^1(\partial\Omega))^2}, \tag{2.5.4}$$

where we used that $h_{x_0} : \partial\Omega \to \partial\mathbb{D}$ and that u is radially symmetric so that, by (2.4.3),

$$\widetilde{u} = \frac{I_0(\delta)}{\delta \, I_1(\delta)}, \qquad \text{on } \partial\Omega.$$

In order to estimate the integral

$$\int_{\mathbb{D}} u^2 \, |f'_{x_0}(w)|^2 \, dw = \int_0^1 u^2 \left(\int_{\{|w|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right) d\varrho,$$

we set

$$\Phi(\varrho) = \frac{1}{2\,\pi\,\varrho} \int_{\{|w|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1,$$

then we can rewrite

$$\int_{\mathbb{D}} u^2 \, |f'_{x_0}(w)|^2 \, dw = 2\,\pi \int_0^1 u^2 \, \Phi(\varrho) \, \varrho \, d\varrho.$$

From Remark 2.5.2 we know that $\varrho \mapsto \Phi(\varrho)$ is monotone non-decreasing. Thus we obtain

$$\int_{\mathbb{D}} u^2 \, |f'_{x_0}(w)|^2 \, dw \leq \left(2\,\pi \int_0^1 u^2 \, \varrho \, d\varrho \right) \Phi(1) = \Phi(1) \int_{\mathbb{D}} u^2 \, dw.$$

By recalling (2.5.3), this finally gives

$$\int_{\mathbb{D}} u^2 \, |f'_{x_0}(w)|^2 \, dw < (1+\varepsilon) \int_{\mathbb{D}} u^2 \, dw.$$

We insert this estimate into (2.5.4) and use that u is optimal for the disk. We get

$$\frac{1}{T(\Omega;\delta)} \leq \left(\frac{\delta \, I_1(\delta)}{I_0(\delta)} \right)^2 \frac{\displaystyle\int_{\partial\mathbb{D}} u \, d\mathcal{H}^1 + \delta^2 \, \varepsilon \int_{\mathbb{D}} u^2 \, dw}{(\mathcal{H}^1(\partial\Omega))^2}.$$

If we now let ε go to 0 and use (2.2.7), we obtain

$$\frac{1}{T(\Omega;\delta)} \leq \left(\frac{\delta \, I_1(\delta)}{I_0(\delta)} \right)^2 \frac{T(\mathbb{D};\delta)}{(\mathcal{H}^1(\partial\Omega))^2}.$$

Finally, from (2.4.4) we know that

$$\frac{\delta \, I_1(\delta)}{I_0(\delta)} = \frac{2\,\pi}{T(\mathbb{D};\delta)},$$

from which (2.5.2) follows, for the case $\dot{\mathcal{R}}_\Omega = 1$.

The general case can be obtained by scaling. Indeed, taking $t = 1/\dot{\mathcal{R}}_\Omega$, the scaled set $\widetilde{\Omega} = t\,\Omega$ has unit boundary distortion radius. Thus, by using the first part of the proof with δ/t and (2.2.2), we have

$$T(\Omega;\delta) = t^{-2}\,T\left(\widetilde{\Omega};\frac{\delta}{t}\right) \geq t^{-2}\quad \left(\frac{\mathcal{H}^1(\partial\widetilde{\Omega})}{2\,\pi}\right)^2 T\left(\mathbb{D};\frac{\delta}{t}\right) = \left(\frac{\mathcal{H}^1(\partial\Omega)}{2\,\pi}\right)^2 \frac{T\left(B_{\dot{\mathcal{R}}_\Omega};\delta\right)}{\dot{\mathcal{R}}_\Omega^2}$$

This concludes the proof of the inequality.

We now come to the equality cases. We first observe that disks achieve equality in (2.5.2), thanks to Lemma 2.5.3. Let us now suppose that Ω is such that equality holds in (2.5.2). For simplicity, we can assume again that $\dot{\mathcal{R}}_\Omega = 1$. This means that equality must hold everywhere in the proof above. In particular, there exists $x_0 \in \Omega$ such that $\widetilde{u} = u \circ h_{x_0}$ must be the boundary $\delta-$torsion function of Ω. This in particular implies that $\widetilde{u}$ must solve

$$-\Delta\widetilde{u} + \delta^2\,\widetilde{u} = 0, \qquad \text{in } \Omega.$$

By computing the Laplacian of $\widetilde{u}$, which is given by

$$\Delta\widetilde{u}(x) = |h'_{x_0}(x)|^2\,\Delta u(h_{x_0}(x)), \qquad \text{for } x \in \Omega,$$

and using that u solves

$$-\Delta u + \delta^2\,u = 0, \qquad \text{in } \mathbb{D},$$

we then obtain that

$$(1 - |h'_{x_0}|^2)\,\delta^2\,\widetilde{u} = 0, \qquad \text{in } \Omega.$$

Recalling that u is positive, the last identity entails that we must have $|h'_{x_0}| \equiv 1$ in Ω and thus $|f'_{x_0}| \equiv 1$ in $\mathbb{D}$. This implies that f'_{x_0} is constant, hence f_{x_0} is actually a linear isometry and thus Ω is a disk, with radius 1. □

2.5.2 Convex sets: lower bound

The following technical lemma will be used in a while. We will use again the notation $\mathrm{I} = (0,1)$.

Lemma 2.5.5. *For every $\delta > 0$, we have*

$$\alpha(\delta) := \sup_{\varphi\in W^{1,2}(\mathrm{I})} \frac{(\varphi(0))^2}{\displaystyle\int_{\mathrm{I}} |\varphi'|^2\,dt + \delta^2 \int_{\mathrm{I}} \varphi^2\,dt} = \frac{1}{\delta\,\tanh(\delta)}. \tag{2.5.5}$$

Moreover, the function

$$u_{\mathrm{I}}(t) = \frac{1}{\delta}\left(\frac{\cosh(\delta\,t)}{\tanh(\delta)} - \sinh(\delta\,t)\right)$$

attains the supremum in (2.5.5).

Proof. An argument as in (2.2.9) yields that the maximization problem (2.5.5) can be rephrased as

$$\alpha(\delta) = \sup_{\varphi \in W^{1,2}(\mathrm{I})} \left\{ 2\,\varphi(0) - \int_{\mathrm{I}} |\varphi'|^2\,dt - \delta^2 \int_{\mathrm{I}} \varphi^2\,dt \right\} \qquad (2.5.6)$$

The existence of a (unique) maximizer u_I for (2.5.6) is again easily proven by the Direct Method. Moreover, by concavity of the maximization problem, we have that φ solves (2.5.6) if and only if it verifies the optimality condition

$$\psi(0) = \int_{\mathrm{I}} \varphi'\,\psi'\,dt + \delta^2 \int_{\mathrm{I}} \varphi\,\psi\,dt, \qquad \text{for every } \psi \in W^{1,2}(\mathrm{I}).$$

This is the weak formulation of

$$\begin{cases} -\varphi'' + \delta^2\,\varphi = 0, & \text{in I}, \\ \varphi'(0) = -1, \\ \varphi'(1) = 0. \end{cases}$$

The latter is uniquely solved by the function

$$t \mapsto \frac{1}{\delta}\left(\frac{\cosh(\delta\,t)}{\tanh(\delta)} - \sinh(\delta\,t) \right)$$

which then coincides with the unique maximizer u_I. From (2.5.6) and the equation in weak form we get

$$\alpha(\delta) = 2\,u_I(0) - \int_{\mathrm{I}} |u_I'|^2\,dt - \delta^2 \int_{\mathrm{I}} u_I^2\,dt = u_{\mathrm{I}}(0).$$

By using the expression of u_{I}, we conclude. □

We recall the definition of *inradius* of Ω, i.e.

$$r_\Omega = \sup_{x \in \Omega} d_\Omega(x).$$

Observe that this is the radius of a largest ball inscribed in Ω. We can then prove the following sharp lower bound on $T(\Omega;\delta)$ for a convex set. This is somehow similar to an estimate by Pólya, which provides a lower bound on the usual torsional rigidity in terms of volume and perimeter, see [121]. We employ the same technique, i.e. the *method of interior parallels*, introduced by Makai [104, 105] and Pólya [121].

Theorem 2.5.6. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. We have the following estimate*

$$T(\Omega;\delta) > \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\delta\,\tanh\left(\delta\,r_\Omega\right)}.$$

Moreover, the estimate is sharp in the following sense: we have

$$\lim_{n\to\infty} \frac{T(\Omega_n;\delta)\,\tanh(\delta\,r_{\Omega_n})}{\mathcal{H}^{N-1}(\partial\Omega_n)} = \frac{1}{\delta}, \qquad \textit{for } \Omega_n := (-n,n)^{N-1} \times (-1,1).$$

Proof. We first prove the estimate under the assumption that $r_\Omega = 1$. We define a non-negative test function for the torsion functional (2.1.4) of the form

$$\varphi(x) = u_I(d_\Omega(x)), \qquad \text{for } x \in \Omega,$$

where u_I is the same function as in Lemma 2.5.5. We denote

$$\Omega_t = \Big\{ x \in \Omega \,:\, d_\Omega(x) = t \Big\}, \qquad \text{for } t \in [0,1].$$

Then, by using the Coarea Formula and the fact that $|\nabla d_\Omega| = 1$ almost everywhere in Ω, we get

$$\begin{aligned} T(\Omega;\delta) \geq \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} \right)^2}{\displaystyle\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \delta^2 \int_\Omega \varphi^2 \, \mathrm{d}x} &= \frac{(u_I(0))^2 \, (\mathcal{H}^{N-1}(\partial\Omega))^2}{\displaystyle\int_\Omega |u_I'(d_\Omega(x))|^2 \, \mathrm{d}x + \delta^2 \int_\Omega (u_I(d_\Omega(x))^2 \, \mathrm{d}x} \\ &= \frac{(u_I(0))^2 \, (\mathcal{H}^{N-1}(\partial\Omega))^2}{\displaystyle\int_0^1 \Big[|u_I'(t)|^2 + \delta^2 \, (u_I(t))^2 \Big] \mathcal{H}^{N-1}(\partial\Omega_t) \, dt} \\ &\geq \frac{(u_I(0))^2}{\displaystyle\int_0^1 \Big[|u_I'(t)|^2 + \delta^2 \, (u_I(t))^2 \Big] dt} \, \mathcal{H}^{N-1}(\partial\Omega). \end{aligned}$$

In the last inequality we used that

$$\mathcal{H}^{N-1}(\partial\Omega_t) \leq \mathcal{H}^{N-1}(\partial\Omega), \qquad \text{for } t \in (0,1), \tag{2.5.7}$$

which follows from convexity, see for example [39, Lemma 2.2.2]. Observe that this inequality is strict, for an open bounded convex set. If we now apply Lemma 2.5.5, we get the desired estimate in the case $r_\Omega = 1$.

The general case now follows by scaling: indeed, by taking $t = 1/r_\Omega$, the scaled set $\widetilde{\Omega} = t\,\Omega$ has unit inradius. Thus, by using the first part of the proof with δ/t and (2.2.2), we have

$$T(\Omega;\delta) = t^{-N} \, T\left(\widetilde{\Omega}; \frac{\delta}{t}\right) \geq t^{-N} \, \frac{\mathcal{H}^{N-1}(\partial\widetilde{\Omega})}{\dfrac{\delta}{t} \, \tanh\left(\dfrac{\delta}{t}\right)} = \frac{1}{t} \, \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\dfrac{\delta}{t} \, \tanh(\delta \, r_\Omega)} = \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\delta \, \tanh(\delta \, r_\Omega)},$$

as desired.

We now come to the proof of the sharpness. The idea is to look for equality in (2.5.7). For this, let us consider $\Omega_n := (-n,n)^{N-1} \times (-1,1) \subseteq \mathbb{R}^N$. By Lemma 2.4.4 with the choices

$$\ell_1 = \cdots = \ell_{N-1} = n \qquad \text{and} \qquad \ell_N = 1,$$

we know that

$$u_{\Omega_n,\delta}(x) = \frac{\cosh(\delta \, x_N)}{\delta \, \sinh(\delta)} + \frac{1}{\delta \, \sinh(\delta \, n)} \sum_{i=1}^{N-1} \cosh(\delta \, x_i),$$

and

$$T(\Omega_n;\delta) = \sum_{k=1}^{N-1} \left[\frac{1}{\delta\, \tanh(\delta\, n)}\, \mathcal{H}^{N-1}(\Sigma_k) + \sum_{i\neq k} \frac{1}{\delta^2}\, \mathcal{H}^{N-2}(\Sigma_{k,i}) \right] \Bigg(+ \left[\frac{1}{\delta\, \tanh(\delta)}\, \mathcal{H}^{N-1}(\Sigma_N) + \sum_{i=1}^{N-1} \frac{1}{\delta^2}\, \mathcal{H}^{N-2}(\Sigma_{N,i}) \right] \Bigg($$

where we used the notation Σ_k and $\Sigma_{k,i}$ from Lemma 2.4.4. We now observe that

$$\mathcal{H}^{N-1}(\Sigma_N) = 2^N\, n^{N-1}, \qquad \mathcal{H}^{N-1}(\Sigma_k) = 2^N\, n^{N-2}, \quad \text{for } k = 1,\dots,N-1,$$
$$\mathcal{H}^{N-2}(\Sigma_{N,i}) = 2^N\, n^{N-2}, \qquad \text{for } i = 1,\dots,N-1,$$

while (this occurs only for $N \geq 3$)

$$\mathcal{H}^{N-2}(\Sigma_{k,i}) = 2^N\, n^{N-3}, \quad \text{for } k = 1,\dots,N-1,\ i \in \{1,\dots,N-1\} \setminus \{k\},$$

and

$$\mathcal{H}^{N-2}(\Sigma_{k,N}) = 2^N\, n^{N-2}, \quad \text{for } k = 1,\dots,N-1.$$

This finally shows that

$$T(\Omega_n;\delta) \sim \frac{1}{\delta\, \tanh(\delta)}\, \mathcal{H}^{N-1}(\Sigma_N) = \frac{1}{\delta\, \tanh(\delta)}\, 2^N\, n^{N-1}, \qquad \text{as } n \to \infty.$$

As for the measure of the boundary, we have

$$\mathcal{H}^{N-1}(\partial\Omega_n) = 2\,(2\,n)^{N-1} + 2\,(N-1)\,(2\,n)^{N-2} \sim 2^N\, n^{N-1}, \qquad \text{as } n \to \infty,$$

while clearly by construction we have $r_{\Omega_n} = 1$. By gathering all these informations, we get

$$\frac{T(\Omega_n;\delta)\, \tanh(\delta\, r_{\Omega_n})}{\mathcal{H}^{N-1}(\partial\Omega_n)} \sim \frac{1}{\delta\, \tanh(\delta)}\, \frac{2^N\, n^{N-1}}{2^N\, n^{N-1}}\, \tanh(\delta) = \frac{1}{\delta}, \qquad \text{as } n \to \infty,$$

as claimed. □

Remark 2.5.7. *We observe that the convexity assumption in the previous result has been used only to guarantee the property* (2.5.7)*. Thus Theorem 2.5.6 continues to hold for every open bounded Lipschitz set* Ω *which enjoys this property. This is the case, for example, when* $N = 2$ *and* Ω *is simply connected or doubly connected, see [80, Section 2]. Other sets having property* (2.5.7) *are those for which the distance function* d_Ω *is weakly superharmonic in* Ω*, i.e. such that*

$$\int\!\!\int_\Omega \langle \nabla d_\Omega, \nabla\varphi \rangle\, dx \geq 0, \qquad \text{for every } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0,$$

see for example [29, Remark 3.2].

We recall that on a convex set Ω *the distance* d_Ω *is always weakly superharmonic, since it is a concave Lipschitz function. However, the weak superharmonicity of* d_Ω *is equivalent to the convexity of* Ω *only for dimension* $N = 2$ *(see [5, Theorem 2]), while in higher dimension this is a weaker condition. We refer to [5, Section 5] for an example of non-convex set with superharmonic distance.*

Finally, we also recall that if Ω *has a* C^2 *boundary, the superharmonicity of* d_Ω *is equivalent to the fact that the mean curvature of* $\partial\Omega$ *is non-negative (see [99]).*

2.5.3 Convex sets: upper bounds

By appealing to the dual formulation (2.2.11), we can obtain the following upper bound of geometric flavor, which can be seen as a sort of Steklov version of the so-called *Diaz-Weinstein inequality* (see [55]). For this, we first need to recall the definition of *high ridge set* of an open bounded set $\Omega \subset \mathbb{R}^N$ (see for example [50]). This is given by

$$M(\Omega) := \Big\{ x \in \Omega \,:\, B_{r_\Omega}(x) \subset \Omega \Big\},$$

i.e. this is the collection of centers of maximal balls inscribed in Ω.

Proposition 2.5.8. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then, we have the following upper bound*

$$T(\Omega;\delta) \leq \frac{1}{r_\Omega^2}\left(\mathcal{I}_{\#}(\Omega) + \frac{N^2}{\delta^2}\,|\Omega|\right) \qquad \text{where} \quad \mathcal{I}_{\#}(\Omega) = \min_{x_0 \in M(\Omega)} \int_\Omega |x - x_0|^2 \,\mathrm{d}x. \tag{2.5.8}$$

Proof. Let $x_0 \in M(\Omega)$, then accordingly we have $B_{r_\Omega}(x_0) \subset \Omega$. Recalling Lemma 2.2.5, we make the following choices

$$\phi_0 = \frac{x - x_0}{r_\Omega} \qquad \text{and} \qquad g_0 = \frac{N}{\delta^2\, r_\Omega}, \tag{2.5.9}$$

and observe that $(\phi_0, g_0) \in \mathcal{A}^+(\Omega)$. Indeed, we clearly have by construction

$$-\mathrm{div}\phi_0 + \delta^2\, g = 0, \qquad \text{in } \Omega.$$

As for the flux condition on the boundary, convexity of Ω entails that

$$\langle x - x_0, \nu_\Omega \rangle \geq r_\Omega, \qquad \text{for } \mathcal{H}^{N-1}\text{−a. e. } x \in \partial\Omega,$$

see for example [32, Lemma 2.1]. By appealing to Lemma 2.2.5 and to the arbitrariness of $x_0 \in M(\Omega)$, we finally get the claimed estimate. □

Remark 2.5.9. *By recalling the characterization for extremals of the dual problem* (2.2.11)*, it is not difficult to see that estimate* (2.5.8) *is not sharp. Indeed, the trial pair* (2.5.9) *is such that*

$$\phi_0 \neq \nabla g.$$

However, thanks to Theorem 2.3.4, from (2.5.8) *we recover in the limit the inequality*

$$\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} = \lim_{\delta \to 0^+} \delta^2\, T(\Omega;\delta) \leq \frac{N^2\,|\Omega|}{r_\Omega^2},$$

that is

$$\mathcal{H}^{N-1}(\partial\Omega) \leq \frac{N\,|\Omega|}{r_\Omega},$$

which is sharp. Equality in the latter is attained for balls, for examples.

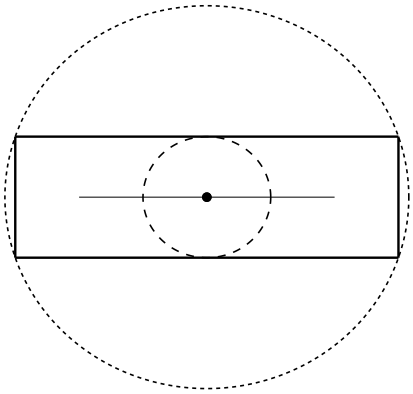


Figure 2.2: The construction of the proximal radius: in bold line, the boundary of the convex set Ω; the thin black line is the high ridge $M(\Omega)$. The black dot is the proximal center, which in this particular example coincides with the center of symmetry of the set.

Remark 2.5.10. *The geometric quantity $\mathcal{I}_{\#}(\Omega)$ defined above is quite similar to the more usual one*

$$\mathcal{I}(\Omega) = \min_{x_0 \in \mathbb{R}^N} \int\!\!\int_{\Omega} |x - x_0|^2 \, \mathrm{d}x,$$

which is called polar moment of inertia of Ω. However, in general we have

$$\mathcal{I}_{\#}(\Omega) \geq \mathcal{I}(\Omega),$$

and the inequality is strict, unless the barycenter of Ω, which is the unique minimizer for $\mathcal{I}(\Omega)$, belongs to $M(\Omega)$.

We now exploit once again the dual formulation (2.2.11), in order to give a sharp upper bound this time. At this aim, for an open bounded convex set $\Omega \subset \mathbb{R}^N$, we define its *proximal radius* by

$$L_\Omega := \inf \Big\{ R > 0 \,:\, \exists x_0 \in M(\Omega) \text{ such that } \Omega \subset B_R(x_0) \Big\}.$$

In other words, L_Ω is the radius of the smallest ball containing Ω and having center in the high ridge set $M(\Omega)$. We will call *proximal center* any point $x_\Omega \in M(\Omega)$, such that

$$\Omega \subset B_{L_\Omega}(x_\Omega).$$

Such a point exists and is unique by Lemma 2.5.13 below. Observe that by construction, for the proximal center we have

$$B_{r_\Omega}(x_\Omega) \subset \Omega \subset B_{L_\Omega}(x_\Omega). \tag{2.5.10}$$

This will be crucial for the proof of the following result.

Theorem 2.5.11. *Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. With the notation above, we have*

$$T(\Omega;\delta) \leq \left(\frac{r_\Omega}{L_\Omega}\right)^{N-2} \left(\frac{I_{N/2}(\delta\, L_\Omega)}{I_{N/2}(\delta\, r_\Omega)}\right)^2 T(B_{L_\Omega};\delta). \tag{2.5.11}$$

Moreover, equality holds if and only if Ω is a ball.

Proof. Without loss of generality, we can assume that the proximal center of Ω coincides with the origin. We set for brevity

$$C_{\Omega,\delta} = \mathcal{U}'_{\delta L_\Omega}\left(\frac{r_\Omega}{L_\Omega}\right)\Bigg(\tag{2.5.12}$$

where $\mathcal{U}_\delta$ is the function of Lemma 2.4.1. We then introduce the pair

$$\phi_0 = \frac{1}{C_{\Omega,\delta}}\,\nabla u_{B_{L_\Omega},\delta} \qquad \text{and} \qquad g_0 = \frac{1}{C_{\Omega,\delta}}\,u_{B_{L_\Omega},\delta},$$

where as always $u_{B_{L_\Omega},\delta}$ is the boundary $\delta-$torsional function of the ball B_{L_Ω}. We observe that by construction and using (2.5.10), we have

$$-\mathrm{div}\,\phi_0 + \delta^2\, g_0 = \frac{1}{C_{\Omega,\delta}}\,\left(-\Delta u_{B_{L_\Omega},\delta} + \delta^2\, u_{B_{L_\Omega},\delta}\right) = 0, \qquad \text{in } \Omega.$$

Moreover, since we have

$$u_{B_{L_\Omega},\delta}(x) = L_\Omega\, u_{B,\delta L_\Omega}\left(\frac{x}{L_\Omega}\right)\Bigg(= L_\Omega\,\mathcal{U}_{\delta L_\Omega}\left(\frac{|x|}{L_\Omega}\right)\Bigg($$

we get that on $\partial\Omega$ it holds

$$\langle \phi_0, \nu_{\partial\Omega}\rangle = \frac{1}{C_{\Omega,\delta}}\,\frac{1}{|x|}\,\mathcal{U}'_{\delta\, L_\Omega}\left(\frac{|x|}{L_\Omega}\right)\Bigg(\langle x, \nu_{\partial\Omega}\rangle \geq \frac{1}{C_{\Omega,\delta}}\,\frac{1}{|x|}\,\mathcal{U}'_{\delta\, L_\Omega}\left(\frac{|x|}{L_\Omega}\right)\Bigg(r_\Omega,$$

$\mathcal{H}^{N-1}-$almost everywhere, where we used again [32, Lemma 2.1]. We then recall that

$$\varrho \mapsto \frac{\mathcal{U}'_\delta(\varrho)}{\varrho},$$

is monotone non-decreasing by Lemma 2.4.1 and by construction

$$|x| \geq r_\Omega, \qquad \text{for every } x \in \partial\Omega.$$

This permits to estimate the flux of ϕ_0 on the boundary of Ω by

$$\langle \phi_0, \nu_{\partial\Omega}\rangle \geq \frac{1}{C_{\Omega,\delta}}\,\mathcal{U}'_{\delta\, L_\Omega}\left(\frac{r_\Omega}{L_\Omega}\right)\Bigg($$

Recalling the definition (2.5.12) of $C_{\Omega,\delta}$, we finally obtain that $(\phi_0, g_0) \in \mathcal{A}^+(\Omega)$. By Lemma 2.2.5, we thus obtain

$$\begin{aligned} T(\Omega;\delta) &\leq \frac{1}{(C_{\Omega,\delta})^2}\,\left[\int_\Omega\Bigg(|\nabla u_{B_{L_\Omega},\delta}|^2\,\mathrm{d}x + \delta^2\int_\Omega (u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x\right]\Bigg(\\ &\leq \frac{1}{(C_{\Omega,\delta})^2}\,\left[\int_{B_{L_\Omega}}\Bigg(|\nabla u_{B_{L_\Omega},\delta}|^2\,\mathrm{d}x + \delta^2\int_{B_{L_\Omega}}\Bigg((u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x\right] = \frac{T(B_{L_\Omega};\delta)}{(C_{\Omega,\delta})^2}. \end{aligned} \tag{2.5.13}$$

We now compute the constant $C_{\Omega,\delta}$

$$\begin{aligned} C_{\Omega,\delta} &= \mathcal{U}'_{\delta\, L_\Omega}\left(\frac{r_\Omega}{L_\Omega}\right)\left(= \frac{d}{d\varrho} \quad \frac{(\delta\, L_\Omega\, \varrho)^{1-N/2}\, I_{N/2-1}(\delta\, L_\Omega\, \varrho)}{(\delta\, L_\Omega)^{2-\frac{N}{2}}\, I_{N/2}(\delta\, L_\Omega)}\right)_{|\varrho=\frac{r_\Omega}{L_\Omega}} \\ &= \left(\frac{\varrho^{1-N/2}\, I_{N/2}(\delta\, L_\Omega\, \varrho)}{I_{N/2}(\delta\, L_\Omega)}\right)\Bigg(_{|\varrho=\frac{r_\Omega}{L_\Omega}} \\ &= \left(\frac{r_\Omega}{L_\Omega}\right)^{\frac{2-N}{2}} \frac{I_{N/2}(\delta\, r_\Omega)}{I_{N/2}(\delta\, L_\Omega)}, \end{aligned}$$

where we used (2.4.8) in order to compute the derivative. This concludes the proof of the inequality.

Let us now discuss equality cases. At first, it is quite clear that for any ball we get equality in (2.5.11). Indeed, in this case, the set $M(\Omega)$ just coincides with the center of the ball and $r_\Omega = L_\Omega$. We now assume that $\Omega \subset \mathbb{R}^N$ is an open bounded convex set for which (2.5.11) holds as an equality. As before, we can suppose that $x_\Omega = 0$. Then the equality sign must hold in all the inequalities above. In particular, equality in (2.5.13) entails that we must have

$$\int_\Omega (u_{B_{L_\Omega},\delta})^2 \, \mathrm{d}x = \iint_{B_{L_\Omega}} (u_{B_{L_\Omega},\delta})^2 \, \mathrm{d}x.$$

By virtue of (2.5.10) and the fact that $u_{B_{L_\Omega},\delta}$ does not vanish, we finally get that

$$|B_{L_\Omega} \setminus \Omega| = 0.$$

By convexity, this fact and (2.5.10) in turn imply that Ω must coincide with the ball B_{L_Ω}. □

From the previous result, we can get the following sharp geometric estimate, involving four geometric quantities.

Corollary 2.5.12. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. With the notation above, we have*

$$\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq N^2\, \omega_N \left(\frac{L_\Omega}{r_\Omega}\right)^2 L_\Omega^{N-2}. \tag{2.5.14}$$

Equality holds if and only if Ω is a ball.

Proof. We multiply inequality (2.5.11) by δ^2 and then take the limit as δ goes to 0. By using Theorem 2.3.4 and (2.4.1), we get

$$\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq \left(\frac{L_\Omega}{r_\Omega}\right)^2 \frac{(\mathcal{H}^{N-1}(\partial B_{L_\Omega}))^2}{|B_{L_\Omega}|}.$$

This gives the desired inequality (2.5.14).

It is straightforward to see that balls give equality in (2.5.14). On the other hand, let us suppose that Ω is an open bounded convex set which attains equality

in (2.5.14). In order to prove that Ω must be a ball, we go back to (2.5.13): with a slightly more careful estimate, we can obtain

$$\begin{aligned} T(\Omega;\delta) &\leq \frac{1}{(C_{\Omega,\delta})^2}\left[\int\!\!\!\int_{\Omega}|\nabla u_{B_{L_\Omega},\delta}|^2\,\mathrm{d}x+\delta^2\int_{\Omega}(u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x\right]\Bigg(\\ &\leq \frac{1}{(C_{\Omega,\delta})^2}\left[\int\!\!\!\int_{B_{L_\Omega}}|\nabla u_{B_{L_\Omega},\delta}|^2\,\mathrm{d}x+\delta^2\int_{B_{L_\Omega}}(u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x\right]\Bigg(\\ &- \frac{\delta^2}{(C_{\Omega,\delta})^2}\int\!\!\!\int_{B_{L_\Omega}\setminus\Omega}(u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x \\ &= \frac{T(B_{L_\Omega};\delta)}{(C_{\Omega,\delta})^2}-\frac{\delta^2}{(C_{\Omega,\delta})^2}\int\!\!\!\int_{B_{L_\Omega}\setminus\Omega}(u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x. \end{aligned}$$

By recalling the value of $C_{\Omega;\delta}$, we get the following enhanced version of (2.5.11)

$$T(\Omega;\delta)\leq\left(\frac{r_\Omega}{L_\Omega}\right)^{N-2}\left(\frac{I_{N/2}(\delta\,L_\Omega)}{I_{N/2}(\delta\,r_\Omega)}\right)^2\left[T(B_{L_\Omega};\delta)-\delta^2\int\!\!\!\int_{B_{L_\Omega}\setminus\Omega}(u_{B_{L_\Omega},\delta})^2\,\mathrm{d}x\right].$$

As in the first part of the proof, we multiply both sides by δ^2 and then take the limit as δ goes to 0. By Theorem 2.3.4, we now get the enhanced version of (2.5.14)

$$\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|}\leq\left(\frac{L_\Omega}{r_\Omega}\right)^2\left[\frac{(\mathcal{H}^{N-1}(\partial B_{L_\Omega}))^2}{|B_{L_\Omega}|}-\left(\frac{\mathcal{H}^{N-1}(\partial B_{L_\Omega})}{|B_{L_\Omega}|}\right)^2|B_{L_\Omega}\setminus\Omega|\right].$$

Thus, if Ω attains equality in (2.5.14), in particular it must result

$$|B_{L_\Omega}\setminus\Omega|=0,$$

which implies that Ω is a ball. □

Appendix A

We give in this section some elementary facts about the proximal radius of a convex set. We start with the following

Lemma 2.5.13. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded convex set. Then there exists a unique proximal center.*

Proof. We first observe that the high ridge set $M(\Omega)$ is a closed convex set. This follows from the fact that it coincides with the maximum level set $\{x\in\overline{\Omega}\,:\,d_\Omega(x)=r_\Omega\}$ and the distance function is continuous and concave on $\overline{\Omega}$. We now claim that

$$L_\Omega=\inf_{x\in M(\Omega)}\left(\max_{y\in\partial\Omega}|x-y|\right)\Bigg(\tag{2.5.15}$$

and that the infimum is attained by a proximal center. Indeed, first observe that the function

$$x\mapsto\max_{y\in\partial\Omega}|x-y|,$$

is 1−Lipschitz, as a supremum of a family of 1−Lipschitz uniformly bounded functions. By recalling that $M(\Omega)$ is compact, existence of a minimizer follows from Weierstrass' Theorem.

Let us now take $x_0 \in M(\Omega)$ to be one of these minimizers. By taking the ball $B_R(x_0)$ with $R = \max_{y\in\partial\Omega} |x_0 - y|$, we clearly have

$$\Omega \subset B_R(x_0),$$

and thus

$$L_\Omega \leq R = \max_{y\in\partial\Omega} |x_0 - y| = \min_{x\in M(\Omega)} \left(\max_{y\in\partial\Omega} |x - y|\right)$$

In order to prove the reverse inequality, let $r > 0$ be such that there exists $x_r \in M(\Omega)$ with the property

$$\Omega \subset B_r(x_r).$$

This implies that

$$|x_r - y| \leq r, \qquad \text{for every } y \in \partial\Omega,$$

and thus

$$\min_{x\in M(\Omega)} \left(\max_{y\in\partial\Omega} |x - y|\right) \leq r.$$

By taking the infimum over admissible $r > 0$, we conclude the proof of (2.5.15).

The previous part of the proof also shows the existence of at least a proximal center $x_0 \in M(\Omega)$. Let us suppose that $x_1 \in M(\Omega)$ is another proximal center. This implies that

$$\Omega \subset B_{L_\Omega}(x_0) \cap B_{L_\Omega}(x_1).$$

In particular, such an intersection is not empty and thus $L_\Omega > |x_0 - x_1|/2$. By convexity of $M(\Omega)$, the midpoint $\overline{x} = (x_0 + x_1)/2$ still belongs to $M(\Omega)$. If we define

$$\widetilde{R} = \sqrt{L_\Omega^2 - \frac{|x_0 - x_1|^2}{4}},$$

we have $0 < \widetilde{R} < L_\Omega$ and by construction

$$\Omega \subset B_{L_\Omega}(x_0) \cap B_{L_\Omega}(x_1) \subset B_{\widetilde{R}}(\overline{x}).$$

This violates the minimality of L_Ω, thus giving the desired contradiction. □

The proximal radius is actually comparable to more familiar geometric quantities, like the diameter $\operatorname{diam}(\Omega)$ and the *circumradius* R_Ω. The latter is defined by

$$R_\Omega = \inf\left\{R > 0 \,:\, \exists x \in \mathbb{R}^N \text{ such that } \Omega \subset B_R(x)\right\}.$$

This is the radius of the smallest ball entirely containing Ω.

Lemma 2.5.14. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then we have*

$$R_\Omega \leq L_\Omega < \operatorname{diam}(\Omega),$$

and both inequalities are sharp.

Proof. The first inequality $R_\Omega \leq L_\Omega$ is a trivial consequence of the definitions. We also observe that we have equality for every convex set such that R_Ω is attained by a ball centered on $M(\Omega)$: this happens for example for every convex sets having N orthogonal axis of symmetry.

For the second inequality, for every $y \in \partial\Omega$ and every $x \in M(\Omega)$ we have

$$|x - y| < \operatorname{diam}(\Omega),$$

by definition of diameter. Observe that the inequality is strict, since $M(\Omega)$ is at a distance r_Ω (i.e. the inradius) from the boundary $\partial\Omega$, thus its points can not be extremal for the diameter. By recalling (2.5.15), this gives the desired inequality.

In order to prove sharpness of this estimate, we confine ourselves for simplicity to $N = 2$. We take the sequence of right triangles $\{T_n\}_{n\in\mathbb{N}}$ having vertices at

$$P_1 = (-1, 0), \qquad P_2 = (1, 0), \qquad P_3 = \left(\cos\left(\frac{1}{n+1}\right), \sin\left(\frac{1}{n+1}\right)\right)\left(\right.$$

We clearly have that $\operatorname{diam}(T_n) = 2$. In order to compute L_{T_n}, we observe that the high ridge set is a singleton

$$M(T_n) = \{(\alpha_n, \beta_n)\}, \qquad \text{with } \lim_{n\to\infty} \alpha_n = 1, \quad \lim_{n\to\infty} \beta_n = 0.$$

Thus we have

$$\lim_{n\to\infty} L_{T_n} = \lim_{n\to\infty} |(\alpha_n, \beta_n) - P_1| = \lim_{n\to\infty} \sqrt{(\alpha_n + 1)^2 + \beta_n^2} = 2,$$

as desired. □

Chapter 3

The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains

In this chapter we present the results of the paper *The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains*, [23], a joint work of the author of this thesis with Matteo Bonforte and Peio Ibarrondo.

Some of the results in Section 3.4 and the totality of Section 3.5 are part of a work in progress also with Bonforte and Ibarrondo. This paper is the natural continuation of [23].

3.1 Introduction

In this chapter we study the homogeneous Cauchy Dirichlet Problem for a nonlinear and nonlocal diffusion equation of singular type of the form

$$\partial_t u = -\mathcal{L} u^m \tag{FFDE}$$

posed on a bounded Euclidean domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and $N \geq 1$. The linear diffusion operator $\mathcal{L}$ is a densely defined sub-Markovian operator, allowed to be of nonlocal type, while the nonlinearity is of singular type, namely $u^m = |u|^{m-1}u$ with $0 < m < 1$.

We shall now review some related results and bibliography.

3.1.1 Background

On the whole space, the prototype equation is the so-called Fractional Fast Diffusion Equation on the whole space, studied in [28, 109, 111, 112, 113],

$$u_t = -(-\Delta_{\mathbb{R}^N})^s u^m \tag{3.1.1}$$

where the Fractional Laplacian is commonly represented via the hypersingular kernel

$$(-\Delta_{\mathbb{R}^N})^s f(x) := P.V. \iint_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,,$$

but other equivalent definitions are possible on $\mathbb{R}^N$, see e.g. [41, 94].

These models have received a lot of attention in the last years, especially because of their natural interplay with probability and stochastic processes, but also for the numerous applications, for instance to anomalous diffusions in physics and biology, cf. [17, 98, 130]. These equations appear in boundary heat control problems [7], and also as hydrodynamic limits of interacting particle systems with long-range dynamics, cf. [81, 82]. We refer to [7, 26, 28, 43, 109, 130, 138, 139] for further details about possible applications of these useful nonlinear and nonlocal diffusion models.

A quite complete theory for the Cauchy problem on $\mathbb{R}^N$ for (3.1.1) has been developed for all $m > 0$, covering the Fractional Porous Medium Equation (FPME, $m > 1$) and the Fractional Fast Diffusion Equation (FFDE, $m < 1$) by Athanasopoulus, Bonforte, Caffarelli, De Pablo, Quirós, Rodriguez, Stan, Del Teso, Váquez, Volzone and many others, see for instance the fundamental contributions [7, 28, 71, 109, 110, 111, 112, 113, 129, 134, 137, 142], including numerical methods [51, 133]. Of course, when $m = 1$ we have the Fractional Heat Equation (FHE) which has a very rich literature, see [15, 25] and references therein.

These kind of problems have been extensively studied when $\mathcal{L} = -\Delta$, in which case the equation becomes the classical Porous Medium Equation ($m > 1$) or Fast Diffusion Equation ($m < 1$), see for instance the monographs by DiBenedetto [56, 57], Daskalopoulos and Kenig [52] and Vázquez [140, 141].

Even if *the results of this work apply also in the local case* ($s = 1$), and provide some *novelties also for the classical FDE* when $\mathcal{L} = -\Delta$ (and recover the classical results with a different proof), we shall focus our attention in nonlocal diffusion operators, in particular fractional Laplacian operators, emphasizing the differences with the classical (local) case. Since we are working on a bounded domain, the situation dramatically departs from the local case: to begin with, the very definition of "Fractional Laplacian" admits several non-equivalent versions[1]: the Restricted Fractional Laplacian (RFL), the Spectral Fractional Laplacian (SFL), and the Censored Fractional Laplacian (CFL); see Section 3.1.5 for more details.

The Cauchy-Dirichlet problem for the classical FDE, $u_t = \Delta u^m$, has attracted the attention of several researchers since the celebrated paper of Berryman and Holland [14]. The basic theory, local and global Harnack inequalities and optimal interior regularity are well understood [52, 57, 58]. The asymptotic behaviour is a delicate issue, started with the pioneering paper [14] and followed by progressive advances [3, 58, 63]. The sharp asymptotic results were found only recently by Bonforte and Figalli in [18]. Recent results appeared about optimal boundary regularity [84] and asymptotic behaviour [83].

[1] We adopt this terminology, nowadays accepted in the community, which has been introduced in previous works [20, 22, 21, 24, 26, 27] to avoid confusion in the abbreviations, but we point out that RFL is usually known as the Standard Fractional Laplacian, or plainly Fractional Laplacian, and CFL is often called Regional Fractional Laplacian.

3.1.2 The Cauchy-Dirichlet Problem for singular nonlocal diffusions.

Consider the problem

$$\begin{cases} \partial_t u(t,x) = -\mathcal{L}u^m(t,x) & \text{on } (0,+\infty)\times\Omega\,, \\ u(t,\cdot) = 0 & \text{on the lateral boundary, } \forall t>0\,, \\ u(0,\cdot) = u_0 & \text{in } \Omega\,, \end{cases} \tag{CDP}$$

where $0 < m < 1$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, which we assume at least of class $C^{2,\alpha}$, though most of the results indeed hold for $C^{1,1}$ domains, but we choose not to address delicate questions of boundary regularity here. Throughout the chapter we assume $N > 2s$, but most of the results can be easily adapted to the case when $N = 1$ and $s \in [1/2, 1)$.

When $m = 1$, we have the fractional heat equation, whose theory can be considered nowadays classical, since Fourier analysis allows to obtain a complete basic theory analogous to the case $s = 1$ of the local heat equation. However many basic questions have been solved only recently, like sharp bounds for the heat kernel by Zhang in [147], and more recently by many authors often by means of probabilistic techniques, we refer to Section 3.1.5 for more details and references. Many fine properties of solutions are based on the representation formula, and become quite simple once good (or better, sharp) bounds on the heat kernel are known.

3.1.3 About the lateral boundary condition and the dual formulation of the problem

Since we are allowing the linear operator $\mathcal{L}$ to be nonlocal, the lateral boundary condition is tricky to define in general, since it depends on the operator itself. Let us make an example with the three most common Dirichlet Fractional Laplacians on bounded domains, the Spectral (SFL), the Restricted (or the "standard", RFL) and the Censored (or Regional, CFL). The first one is defined as the spectral power of the Dirichlet Laplacian, hence the boundary condition needs only to be imposed on the topological boundary $\partial\Omega$. In the case of the RFL, since the operator needs to be defined on the whole $\mathbb{R}^N$, we need to impose that our solutions vanish on the complement of Ω, or we need to "restrict" the set of admissible functions to the ones supported in $\overline{\Omega}$, hence the name. As for the CFL the boundary condition is even more subtle: the Dirichlet boundary condition holds only for $s \in (1/2, 1)$, while when $s \in (0, 1/2]$ a kind of Neumann condition holds. This can be understood in terms of the underlying censored $\alpha-$stable process, see the original paper of Bogdan, Burdzy and Chen [16], for more details.

We overcome this difficulty by means of *a dual formulation of the problem*, through the *Green function* of the linear operator, that automatically *encodes the boundary conditions*.

Degenerate VS singular diffusions: a problem at the boundary. To better understand the issue, let us consider the local case,

$$u_t = \Delta u^m = m\nabla\cdot(u^{m-1}\nabla u)$$

it is clear that when $m > 1$ the (time dependent) diffusion matrix $u^{m-1}I$ becomes zero when u does, and in particular always at the boundary. This is the cause of the finite speed of propagation[2] and, in the nonlocal case, where the speed of propagation is infinite, of the appearance of the anomalous boundary behaviour, see [22]. On the other hand, when $m < 1$, the diffusion matrix becomes infinite when $u = 0$, in particular on the lateral boundary. This causes some regularity issues, for instance the restriction in the parameters $2s > \gamma$ in the smoothing effects, and other technical hypotheses. We shall explain in what follows all the subtle technicalities.

3.1.4 Main assumptions

In order to understand the main results, it is important to discuss our main hypotheses, their connections and basic implications.

Assumptions on $\mathcal{L}$. The linear operator $\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subseteq L^1(\Omega) \to L^1(\Omega)$ is densely defined and sub-Markovian, more precisely, it satisfies:

(A1) $\mathcal{L}$ is m-accretive in $L^1(\Omega)$

(A2) If $0 \le f \le 1$ then $0 \le e^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0), u \in \mathrm{dom}(\mathcal{L}), \mathcal{L}u \in L^q(\Omega)$, $1 \le q \le \infty$, $v \in L^{q/(q-1)}(\Omega), v(x) \in \beta(u(x))$ a.e., then

$$\iint_{\Omega} v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0\,.$$

We recall that the assumptions on $\mathcal{L}$ are the same as in [21], where detailed proofs of the claims below can be found as well as a rich set of examples of operators. See also [22, 27] and Section 3.1.5.

Assumptions on $\mathcal{L}^{-1}$. In order to develop the theory that follows in this paper we will need to deal with the dual form of (CDP). Therefore, we asume that $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : L^1(\Omega) \to L^1(\Omega)$ that can be represented by a kernel $\mathbb{G}_\Omega$ as

$$\mathcal{L}^{-1}[f](x) := \iint_{\Omega} \mathbb{G}_\Omega(x,y) f(y)\,\mathrm{d}y,$$

which satisfies at least one of the next assumptions, for some $s \in (0,1]$ and for a.e. $x, y \in \Omega$:

- There exists a constant $c_{1,\Omega} > 0$ such that

$$0 \le \mathbb{G}_\Omega(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\,. \tag{K1}$$

  According to Proposition 5.1 of [21], this assumption guarantees that $\mathcal{L}^{-1}$ is compact in $L^2(\Omega)$, hence that is has a discrete set of eigen-elements for $\mathcal{L}$,

[2] Finite speed of propagation can be understood as follows: in the whole space, compactly supported initial data produce compactly supported solutions at all times.

namely (λ_k, Φ_k), with $\Phi_k \in L^\infty(\Omega)$. In particular there exists a ground state $\Phi_1 \geq 0$, and the Poincaré inequality holds with $\lambda_1 > 0$

$$\lambda_1 \int_\Omega f\mathcal{L}^{-1} f \,\mathrm{d}x \leq \int_\Omega f^2 \,\mathrm{d}x \qquad \text{or equivalently} \qquad \lambda_1 \int_\Omega f^2 \,\mathrm{d}x \leq \int_\Omega f\mathcal{L} f \,\mathrm{d}x\,.$$

The eigenfunction are always Hölder continuous in the interior, see Theorem 7.1 of [21].

(K1) *implies Hardy-Littlewood-Sobolev and Sobolev type inequalities:* It has been shown in Theorem 7.5 of [24] that assumption (K1) implies the Hardy-Littlewood-Sobolev inequality:

$$\|f\|_{H^*(\Omega)} \leq \mathcal{H}_{\mathcal{L}} \|f\|_{(2^*)'} \qquad \text{with} \quad (2^*)' = \frac{2N}{N+2s}\,. \tag{3.1.2}$$

Moreover, following Lieb's duality argument [101], as in Proposition 7.4 of [24], we have that the above inequality is equivalent to the fractional Sobolev inequality:

$$\|f\|_{2^*} \leq \mathcal{S}_{\mathcal{L}} \|f\|_{H(\Omega)} \qquad \text{with } 2^* = \frac{2N}{N-2s}\,. \tag{3.1.3}$$

We refer to the Appendix of [24] for further details.

- The second assumption is needed when we want to take into account boundary behaviour. Let

$$\delta_\gamma(x) := \mathrm{dist}(x, \partial\Omega)^\gamma \qquad \text{for some } \gamma \in (0,1]\,.$$

There exist $c_{0,\Omega}, c_{1,\Omega} > 0$ such that

$$c_{0,\Omega}\delta_\gamma(x)\delta_\gamma(y) \leq \mathbb{G}_\Omega(x,y) \leq \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right). \tag{K2}$$

Proposition 5.3 of [21], guarantees that under this assumption, the boundary behaviour of eigenfunction is dictated by the power γ, namely

$$\underline{\kappa}\,\delta_\gamma(x) \leq \Phi_1(x) \leq \overline{\kappa}\,\delta_\gamma(x) \qquad \text{and} \qquad |\Phi_n(x)| \leq \overline{\kappa}_n\,\delta_\gamma(x)$$

In this case, eigenfunction are classical in the interior, i.e. $C^{2s+\alpha}$, whenever the operator allows it, and have a sharp boundary regularity $C^\gamma(\overline{\Omega})$, see Theorems 7.1 and 7.2 of [21] for more details.

It is therefore convenient to use the ground state as a smooth extension of the distance to the boundary (to the power γ), and this allows us to rephrase the assumption (K2) as follows:

$$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \leq \mathbb{G}_\Omega(x,y) \leq \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1\right). \tag{K3}$$

- In many examples, the Green function satisfies an even stronger estimate

$$\mathbb{G}_\Omega(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right) \tag{K4}$$

  This is the case for instance of the three most common Fractional Laplacians, namely for the SFL $\gamma = 1$, while for RFL $\gamma = s$, and CFL $\gamma = 2s-1$). See Section 3.1.5 for more details.

Notations. We will write the L^p norm in Ω as $\|f\|_p := (\int_\Omega |f|^p \,\mathrm{d}x)^{1/p}$, but for weighted norms with the first eigenvalue Φ_1 we use $\|f\|_{L^p_{\Phi_1}(\Omega)} := \left(\int_\Omega |f|^p \Phi_1 \,\mathrm{d}x\right)^{1/p}$ in the space $L^p_{\Phi_1}(\Omega) := \{f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{L^p_{\Phi_1}(\Omega)} < +\infty\}$. The Heaviside function will be denoted by $\mathrm{sign}_+(f) := \max\{0, \mathrm{sign} f\}$, and positive and negative part as $[f]_+ := \max\{0, f\}$ and $[f]_- = \max\{0, -f\}$, respectively. We also use the following notation for maxima and minima: $f \wedge g := \max\{f, g\}$, $f \vee g := \min\{f, g\}$.

3.1.5 Different fractional Laplacian operators on domains and other examples

We briefly exhibit a number of examples to which our results apply. These include a wide class of local and nonlocal operators. We just sketch the essential points, we shall give the values γ, s in each example, and specify the lateral boundary conditions. See [22, 27] for a more detailed exposition.

The Restricted Fractional Laplacian (RFL). We define the fractional Laplacian operator acting on a bounded domain by using the integral representation on the whole space in terms of a hypersingular kernel, namely

$$(-\Delta_{|\Omega})^s f(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+2s}} \,\mathrm{d}y$$

where we restrict the operator to functions that are zero outside Ω. The initial and boundary conditions associated to (CDP) are $u(t,x) = 0$ in $(0,\infty) \times \mathbb{R}^N \backslash \Omega$ and $u(0,\cdot) = u_0$. As explained in [24], such boundary condition can be understood via the Caffarelli-Silvestre extension [41]. The sharp expression of the boundary behaviour for the RFL was investigated in [124]. The sharp expression of the boundary behavior for the RFL was investigated in [125]. We refer to [24] for a careful construction of the RFL in the framework of fractional Sobolev spaces, and [15] for a probabilistic interpretation. See also [73]. For this operator, assumptions (A1), (A2) and (K4) are satisfied with $\gamma = s$, cf. [92].

Other RFL-type integral operators "with Hölder coefficients" can be considered

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x)-f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \,\mathrm{d}y\,,$$

where a is a measurable symmetric function, bounded between two positive constants, and satisfying

$$a(x,y) - a(x,x)\ \chi_{|x-y|<1} \le c|x-y|^\sigma\,, \qquad \text{with } 0 < s < \sigma \le 1\,,$$

for some $c > 0$. Actually, we can allow even more general kernels, cf. [26, 87]. Then, for all $s \in (0, 1]$, the Green function $\mathbb{G}_\Omega(x, y)$ of $\mathcal{L}$ satisfies (K4) with $\gamma = s$, cf. Corollary 1.4 of [87].

Spectral Fractional Laplacian (SFL). Consider the so-called *spectral definition* of the fractional power of the classical Dirichlet Laplacian Δ_Ω on Ω defined by a formula in terms of semigroups, namely

$$(-\Delta_\Omega)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_\Omega} f(x) - f(x)) \frac{\mathrm{d}t}{t^{1+s}} = \sum_{j=1}^\infty \lambda_j^s \hat{f}_j \phi_j,$$

where $(\lambda_j, \phi_j), j = 1, 2, \dots$ are the eigenvalues and eigenfunctions of the classical Dirichlet Laplacian on Ω, $\hat{f} = \int_\Omega f(x)\phi_j(x)\,\mathrm{d}x$, and $\|\phi_j\|_2 = 1$. The initial and boundary conditions associated to (CDP) are $u(t, x) = 0$ on $(0, \infty) \times \partial\Omega$ and $u(0, \cdot) = u_0$. For this operator, assumptions (A1), (A2), and (K2) are satisfied with $\gamma = 1$. Assumption (K2) and also (K4) can be obtained by the Heat kernel estimates valid for the case $s = 1$, cf. [53], as explained in [26, 27].

Spectral powers of linear operators in divergence form. Consider any operator $\mathcal{A} = -\sum_{i,j=1}^N \partial_i(a_{ij}\partial_j)$ with uniformly elliptic C^1 coefficients and with discrete spectrum (λ_j, ϕ_j) (consequence of uniform ellipticity). We can build the following spectral power of the operator: for any $s \in (0, 1]$

$$\mathcal{L}f(x) := \mathcal{A}^s f(x) := \sum_{j=1}^\infty \lambda_j^s \hat{f}_j \phi_j(x), \qquad \text{where} \qquad \hat{f}_j = \int_\Omega f(x)\phi_j \,\mathrm{d}x\,.$$

The Green function of this operator satisfies (K2) with $\gamma = 1$, see [53].

Censored Fractional Laplacian (CFL) with general kernels. This third kind of Fractional Laplacian was introduced in [16], in connection with censored stable processes. The operators takes the form:

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \, \frac{a(x, y)}{|x - y|^{N+2s}} \,\mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

where $a(x, y)$ is a symmetric function of class C^1 bounded between two positive constants. Actually, not only $\mathcal{L}$ satisfies (A1), (A2), but the Green function $\mathbb{G}_\Omega(x, y)$ satisfies (K4) with $\gamma = 2s - 1$, cf. Corollary 1.2 of [47]. The boundary condition is a bit mysterious here: notice that when $s \in (0, 1/2]$, the boundary condition transitions from Dirichlet to a Neumann type, as explained in [16]. In this case it is clear the advantage of the weak dual formulation (WDS) which encodes the boundary condition in the Green function. In literature this is also known as Regional Fractional Laplacian.

Other examples. As it is shown in [22, 27], this assumptions hold for many other examples:

(i) Sums of two fractional operators: $\mathcal{L} = (-\Delta_{|\Omega})^s + (-\Delta_{|\Omega})^\sigma$, with $0 < \sigma < s \le 1$, where $(-\Delta_{|\Omega})^s$ is the RFL. Here, (K4) holds with $s \in (0, 1)$ and $\gamma = s$, see [48].

(ii) Sum of the Laplacian and a nonlocal operator kernels

$$\mathcal{L} = a(-\Delta_{|\Omega}) + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

with

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \big(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y|\le 1}\big) \chi_{|y|\le 1} \mathrm{d}\nu(y)\,,$$

where the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies suitable integrability conditions at zero and infinity. Here, (K4) holds with $s = 1$, $\gamma = 1$.
(iii) Schrödinger equations for non-symmetric diffusions $\mathcal{L} = A + \mu \cdot \nabla + \nu$, where A is a uniformly elliptic operator with C^1 coefficients both in divergence and non-divergence form, more details can be found in [88]. Here, (K4) holds with $s = 1$, $\gamma = 1$.
(iv) Gradient perturbation of restricted fractional Laplacians, $\mathcal{L} = (-\Delta_{|\Omega})^s + b \cdot \nabla$, where b is a vector valued function belonging to a suitable Kato class. Here, (K4) holds with $\gamma = s$, see [45].
(v) Relativistic stable processes,

$$\mathcal{L} = \big(c^{1/s} - \Delta\big)^s - c\,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

Here, (K4) holds with $\gamma = s$, see [46].

3.1.6 The dual formulation of the problem

We can reformulate problem (CDP) in an equivalent dual form, by means of the inverse operator $\mathcal{L}^{-1}$:

$$\begin{cases} \mathcal{L}^{-1} u_t(t,x) = -u^m(t,x) & \text{on } (0,+\infty) \times \Omega\,, \\ u(0,\cdot) = u_0 & \text{in } \Omega\,. \end{cases} \tag{CDP*}$$

A clear advantage of this formulation is that the lateral boundary condition are encoded in the inverse $\mathcal{L}^{-1}$. Now we define a concept of weak solution suitable for the above formulation, firstly introduced in [26, 27].

Definition 3.1.1 (Weak dual solutions)**.** *Let $T > 0$, we say that $u \in C((0,T) : L^1_{\Phi_1}(\Omega))$ is a* weak dual solution *of* (CDP) *if $u^m \in L^1\big([0,T] : L^1_{\Phi_1}(\Omega)\big)$ and*

$$\int_0^T \int_\Omega \mathcal{L}^{-1} u \, \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega u^m \psi \, \mathrm{d}x \, \mathrm{d}t \qquad \forall \psi / \Phi_1 \in C^1_c((0,T) : L^\infty(\Omega))\,. \tag{WDS}$$

We say that u is a WDS of the Cauchy-Dirichlet Problem (CDP)*, corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$, if moreover $u \in C([0,T) : L^1_{\Phi_1}(\Omega))$, and*

$$\lim_{t \to 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}(\Omega)} = 0.$$

A WDS is called strong *if in addition $t\,\partial_t u \in L^\infty((0,T) : L^1_{\Phi_1}(\Omega))$. A WDS is called* Minimal Weak Dual Solution *(MWDS) if it is obtained as the non-decreasing limit of a sequence of semigroup (mild, gradient flow) solutions.*

3.1.7 Main results

The basic theory for the FFDE on bounded domains is only partially understood, [24, 106, 109], and not in the generality nor in the unified framework that we present here. We shall build *a basic theory,* which includes *existence and uniqueness in the largest (known so far) class of data,* namely H^* for signed solutions and $L^1_{\Phi_1}$ for nonnegative ones. We also prove different *energy estimates and $L^p - L^\infty$ smoothing effects.* For the proof of the smoothing estimates, we *compare the classical Moser iteration with the Green function method*, providing two different proofs of the smoothing effects, under different assumptions, in order to cover as many operators as possible. Once solutions are bounded we see that they extinguish in finite time and we show explicit sharp extinction rates, both above and below. Knowing this, we are able to prove *Global Harnack Principles*, which gives that the solution is bounded above and bellow globally both in space and time. GHP together with the theory of regularity for parabolic nonlocal operators given by Felsinger and Kassmann in [64] will provide *Hölder regularity* in the interior of Ω and at the boundary. *All the proofs are constructive,* meaning that all the constants in the inequalities are computable.

Plan of the chapter. In Section 3.2 we present two different strategies to obtain smoothing effects for solutions of CDP with diverse regularity of initial data. Section 3.3 is devoted to prove existence and uniqueness of MWDS as limit of GF solutions. In Section 3.4 we discuss the decay of the solutions and we see how this solutions vanish in finite time. In Section 3.5 we compute quantitative upper and lower bounds up to the boundary and we show GHP. Then, in Section 3.6 we show Hölder regularity of the solutions. Finally, in Appendix B. we compute some useful inequalities and a rigorous proof of the decay of $L^1_{\Phi_1}$ norm.

3.2 Smoothing effects. Moser iteration VS Green function method

The aim of this section is to present two different strategies to obtain Smoothing effects for solutions of (CDP) in order to be able to cover a larger class of operators. First, by a Moser iteration we show boundedness of weak dual solutions with initial data $u_0 \in L^p$ and p depending on $0 < m < 1$, the second is based on Green function method. For these two methods, we use different assumptions, as we shall explain carefully below. This first part will require some assumption on the operator together with the validity of functional inequalities of Sobolev and Stroock-Varopoulos type, which we call (M1). Sometime these ingredients are not at hand, but we can find results about the inverse of the operator. This is the key tool needed in order to use the Green function method, introduced in [26], based on the dual formulation of the problem, that allows to prove an "almost representation formula", i.e. pointwise bounds essential to prove the smoothing effects.

The advantage of the Green function method is that the proof is simpler and it allows to obtain weighted estimates, which are new also in the case of the classical FDE, i.e. $\mathcal{L} = -\Delta$. As we shall see, the smoothing effect does not always hold for merely L^1 or $L^1_{\Phi_1}$ functions, as in the degenerate case, cf. [22, 26, 27]. As it happens in the local case, more integrability is required to obtain bounded solutions

to FDE, and new exponents appear: m_c which can be characterized as the first m such that L^1 data do not produce bounded solutions, and $p_c = p_c(m)$ as the minimal integrability required to have bounded solutions. This is carefully explained in the monograph [140] and in the Appendix of [28]. In what follows, we present new critical exponents in the weighted and unweighted case.

Remark 3.2.1. ***Boundedness of signed solutions.*** All our smoothing effects are stated for nonnegative solutions, but they can easily be extended to signed solutions, since a closer inspection of the proof reveals that they hold for nonnegative subsolutions, hence they hold for the positive and negative part (both subsolutions) hence for $|u|$, just by summing the bounds (and possibly paying a 2 in $\overline{\kappa}$).

3.2.1 Moser iteration

One possible strategy to prove smoothing effects is to use a variant of the celebrated Moser iteration, following the main steps developed in [109] in the case of the fractional Porous medium equation. Moser iteration relies on two main ingredients: a suitable Sobolev inequality for the quadratic form associated to the operator, and a Stroock-Varopoulos type inequality. This was firstly used in [109], in the case of the Fractional Laplacian on $\mathbb{R}^N$ and of the standard Fractional Sobolev inequality on $H^s(\mathbb{R}^N)$, together with the relation $\int f\mathcal{L}f = \|f\|_H^2 \asymp \|f\|^2_{H^s(\mathbb{R}^N)}$. In our setting, we need the following Sobolev inequality: there exists $2^* > 2$ such that for all $f \in H(\Omega)$

$$\|f\|^2_{2^*} \le \mathcal{S}^2_{\mathcal{L}}\, \|f\|^2_{H(\Omega)} = \mathcal{S}^2_{\mathcal{L}} \iint_Q f\mathcal{L}f \,\mathrm{d}x = \mathcal{S}^2_{\mathcal{L}}\, \|\mathcal{L}^{1/2}u\|_2^2 \,. \tag{3.2.1}$$

We also deduce a family of Gagliardo-Nirenberg-Sobolev (GNS) inequality: let $p > q > 0$ and

$$\frac{2q}{q+m-1} \le 2^*, \qquad \text{and} \qquad \frac{q+m-1}{2q} = \frac{\theta}{2^*} + (1-\theta)\frac{q+m-1}{2p}\,,$$

then, by interpolation of L^p norms, we have

$$\|f\|_{\frac{2q}{q+m-1}} \le \|f\|^{\theta}_{2^*}\|f\|^{1-\theta}_{\frac{2p}{q+m-1}} \le \mathcal{S}^{\theta}_{\mathcal{L}}\, \|f\|^{\theta}_{H(\Omega)}\|f\|^{1-\theta}_{\frac{2p}{q+m-1}}\,. \tag{GNS}$$

However, GNS are not sufficient to run Moser iteration in the nonlocal setting. We need the Stroock-Varopoulos inequality, this is, that there exists a constant $c_{m,p} > 0$

$$\int_\Omega u^{q-1}\mathcal{L}u^m \,\mathrm{d}x \ge c_{m,q} \iint_Q u^{\frac{q+m-1}{2}}\mathcal{L}u^{\frac{q+m-1}{2}} \,\mathrm{d}x = c_{m,q} \left\| \mathcal{L}^{1/2}u^{\frac{q+m-1}{2}} \right\|_2^2 = c_{m,q} \left\| u^{\frac{q+m-1}{2}} \right\|^2_{H(\Omega)}\,. \tag{3.2.2}$$

So that, combining the two above inequalities, one gets

$$\iint_Q u^{q-1}\mathcal{L}u^m \,\mathrm{d}x \ge c_{m,q} \left\| u^{\frac{q+m-1}{2}} \right\|^2_{H(\Omega)} \ge c_{m,q}\mathcal{S}_{\mathcal{L}}^{-2} \frac{\|u\|_q^{\frac{q+m-1}{2\theta}}}{\|u\|_p^{\frac{1-\theta}{\theta}\frac{q+m-1}{2}}}\,. \tag{M1}$$

This implies the decay of the L^q-norm, i.e. $\|u(t)\|_q \le \|u(t_0)\|_q$ for all $t \ge t_0 \ge 0$ and all $q \in [1,\infty]$.

The latter inequality is the minimal assumption to run the Moser iteration. We shall discuss under which assumption this is true. Somehow, the Stroock-Varopoulos inequality is a quantitative version of (A2'), which only asserts that the above integral is nonnegative.

In the local case, the validity of this (M1) is usually a consequence of the classical ellipticity condition: there exist $0 < \lambda < \Lambda$ such that for all $\xi \in \mathbb{R}^N$

$$\text{if } \mathcal{L}[f] = -\sum_{i,j=1}^{N} \partial_i\big(a_{i,j}\partial_j f\big) \quad \text{then we need} \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{N} \xi_i a_{i,j}\xi_j \le \Lambda|\xi|^2\,, \tag{3.2.3}$$

and the Stroock-Varopoulos inequality follows by integration by parts and chain rule:

$$\int_\Omega u^{p-1}\mathcal{L}u^m \,\mathrm{d}x \ge \lambda \int_\Omega \nabla(u^{p-1})\cdot\nabla(u^m)\,\mathrm{d}x = c_{m,p}\lambda \int_\Omega \nabla u^{\frac{p+m-1}{2}}{}^2 \,\mathrm{d}x$$

and we obtain (M1) just by using the classical Sobolev inequality on $H^1_0(\Omega)$, with $2^* = \frac{2N}{N-2} > 2$.

In the nonlocal case, we need assumptions on the kernel of the operator: define the operator as

$$\text{if} \quad \mathcal{L}[f](x) = \text{P.V.} \int_{\mathbb{R}^N} [f(x) - f(y)]\, K(x,y)\,\mathrm{d}y \tag{3.2.4}$$

restricted to functions supported in $\overline{\Omega}$, and impose the ellipticity condition on the kernel:

$$\frac{\lambda}{|x-y|^{N+2s}} \le K(x,y) \le \frac{\Lambda}{|x-y|^{N+2s}}\,. \tag{3.2.5}$$

Under this assumptions, using the Stroock-Varopoulos inequality (3.2.2), and the Sobolev inequality on $H^s_0(\Omega)$, one can prove (M1) , as it has been done for the first time in [112]. See also [24] for more details about Sobolev spaces on bounded domains.

Unfortunately, the above assumptions do not cover the case of the Spectral Fractional Laplacian, and more generally "spectral type operators" , i.e. powers of elliptic operators of type (3.2.3) or even of type (3.2.4). This is because spectral operators can be represented with a kernel as follows:

$$\mathcal{L}^s[f] = \text{P.V.} \int_\Omega [f(x) - f(y)]\, K(x,y)\,\mathrm{d}y + B(x)f(x) \tag{3.2.6}$$

and there exist $0 < \lambda < \Lambda$ such that for a.e. $x, y \in \Omega$ we have

$$\lambda\frac{\delta_\gamma(x)\delta_\gamma(y)}{|x-y|^{N+2s}} \le K(x,y) \le \Lambda\frac{\delta_\gamma(x)\delta_\gamma(y)}{|x-y|^{N+2s}} \quad \text{and} \quad 0 \le B(x) \le \frac{\mathsf{C}}{\delta(x)^{2s}}\,, \tag{L1}$$

and having a kernel supported in Ω and zero at the boundary, they clearly do not satisfy the ellipticity condition (3.2.5). We will show how to deal with these operators more easily, with the Green function method in Section 3.2.2, but we can also take into account the following remark.

Assumption (K1) **and** (L1) **imply** (M1)**.** We have seen that assumption (K1) implies the validity of a Hardy-Littlewood (HLS) inequality: for all $f \in H^*(\Omega)$

$$\|f\|_{H^*}^2 = \|\mathcal{L}^{-1/2} f\|_2^2 \le \mathcal{H}_{\mathcal{L}} \|f\|_{(2^*)'} \qquad \text{with} \quad (2^*)' = \frac{2N}{N+2s}\,, \tag{3.2.7}$$

which is equivalent to the desired Sobolev inequality,

$$\|f\|_{2^*}^2 \le \mathcal{S}_{\mathcal{L}}^2 \, \|f\|_{H(\Omega)}^2 \qquad \text{with} \quad 2^* = \frac{2N}{N-2s}\,, \tag{3.2.8}$$

by using Legendre duality, as carefully explained in Appendix 7.8 of [24].

On the other hand, the Stroock-Varopoulos inequality follows by the fact that $K(x,y) \ge 0$, see Lemma 3.6.5 for a proof. Hence, (M1) holds under the assumption (K1), an upper bound on the Green function, the kernel of the inverse $\mathcal{L}^{-1}$.

We refer to [27, 21, 22] for more details and further examples of operators, briefly summarized in Section 3.1.5.

(K1) **and critical exponents for unweighted smoothings.** When dealing with the question of boundedness of solutions, it is convenient to introduce the following exponents

$$m_c := \frac{N-2s}{N} \qquad \text{and} \qquad p_c := \frac{N(1-m)}{2s}\,.$$

As in the local case $s = 1$, when $m < m_c$ we have that L^1 data do not necessarily produce bounded solutions, as firstly shown by Brezis and Friedman [35], see also a thorough discussion in [140, 28].

We begin by proving how (M1) easily implies smoothing effects from L^p to L^q with $p_c < p < q < +\infty$.

Lemma 3.2.2 (L^p - L^q smoothing)**.** *Let $\mathcal{L}$ satisfy* (A1), (A2)*, and* (M1) *with $2^* = 2N/(N-2s)$. Let u be a GF solution with $0 \le u_0 \in L^\infty(\Omega)$. Then, for all $\max\{1, p_c\} < p < q < +\infty$ there exists $\overline{\kappa}_{p,q} > 0$ such that for all $t > t_0 \ge 0$:*

$$\|u(t)\|_q \le \overline{\kappa}_{p,q} \, \frac{\|u(t_0)\|_p^{\frac{p\vartheta_p}{q\vartheta_q}}}{(t-t_0)^{\frac{N(q-p)}{q}\vartheta_p}}\,, \qquad \textit{with} \quad \vartheta_r = \frac{1}{2sr - N(1-m)} \tag{3.2.9}$$

where $\overline{\kappa}_{p,q}$ only depends on p, q, N, m, s, Ω and is given in (3.2.10)*.*

Proof. Let us begin by deriving the L^q-norm and using (M1), recalling that in this case the constant in the Stroock-Varopoulos inequality is given by

$$c_{m,q} = \frac{4(q-1)m}{(q+m-1)^2},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q \,\mathrm{d}x = q \iint_\Omega u^{q-1}\partial_t u \,\mathrm{d}x = -q \iint_\Omega u^{q-1}\mathcal{L}u^m \,\mathrm{d}x \le -\mathcal{S}_{\mathcal{L}}^{-2} \frac{4q(q-1)m}{(q+m-1)^2} \frac{\|u\|_q^{\frac{q+m-1}{2\theta}}}{\|u\|_p^{\frac{1-\theta}{\theta}\frac{q+m-1}{2}}}\,.$$

Few remarks are in order: first, we have to ensure that

$$\frac{2p}{p+m-1} \le 2^* = \frac{2N}{N-2s}\,, \qquad \text{that is} \qquad p \ge p_c = \frac{N(1-m)}{2s}\,.$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q\,\mathrm{d}x \le -\,\overline{\kappa}_0\,\left(\int_\Omega u^q\,\mathrm{d}x\right)^{\frac{q+m-1}{q\theta}} \quad \text{with} \quad \overline{\kappa}_0 = \frac{4q(q-1)m}{(q+m-1)^2}\,\frac{\mathcal{S}_{\mathcal{L}}^{-2}}{\|u_0\|_p^{\frac{(1-\theta)(q+m-1)}{\theta}}}\,.$$

Integrating the above inequality on $[t_0,t]$, we get

$$\left(\int_\Omega u^q\,\mathrm{d}x\right)^{-\frac{q+m-1}{q\theta}+1} \ge \overline{\kappa}_0\left(\frac{q+m-1}{q\theta}-1\right)(t-t_0)$$

recalling that

$$\frac{q+m-1}{q\theta}-1 = \frac{2sp-N(1-m)}{N(q-p)}$$

we obtain

$$\begin{aligned}\int_\Omega u(t)^q\,\mathrm{d}x &\le \left(\overline{\kappa}_0\,\frac{2sp-N(1-m)}{N(q-p)}\,(t-t_0)\right)^{-\frac{N(q-p)}{2sp-N(1-m)}}\\ &= \left(\frac{N\,\mathcal{S}_{\mathcal{L}}^2\,(q+m-1)^2(q-p)}{4\,q(q-1)\,m\,(2sp-N(1-m))}\;\cdot\;\frac{\|u_0\|_p^{p\,\frac{2sq-N(1-m)}{N(q-p)}}}{t-t_0}\right)^{\frac{N(q-p)}{2sp-N(1-m)}},\end{aligned}$$

which gives exactly (3.2.9) with the constant

$$\overline{\kappa}_{p,q} := \left(\frac{N\mathcal{S}_{\mathcal{L}}^2(q-p)(q+m-1)^2\vartheta_p}{4\,q\,(q-1)m}\right)^{\frac{N(q-p)\vartheta_p}{q}}\,. \tag{3.2.10}$$

□

Notice that we cannot let $q\to\infty$ in the Lemma above, since the constant blows up. In the next theorem we use Moser iteration to prove from Lemma 3.2.9 that solutions are bounded

Theorem 3.2.3 (L^p - L^∞ smoothing)**.** *Let $\mathcal{L}$ satisfy* (A1), (A2), *and* (M1) *with $2^* = 2N/(N-2s)$. Let u be a nonnegative WDS with $u_0\in L^p(\Omega)$ and $p>p_c$ if $m\in(0,m_c]$ or $p\ge 1$ if $m\in(m_c,1)$. Then, there exists a constant $\overline{\kappa}=\overline{\kappa}(N,m,s,p,\mathcal{S}_{\mathcal{L}},\Omega)>0$ such that for any $t>t_0\ge 0$*

$$\|u(t)\|_\infty \le \overline{\kappa}\,\frac{\|u(t_0)\|_p^{2sp\,\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \qquad \textit{with} \quad \vartheta_p = \frac{1}{2sp-N(1-m)}\,. \tag{3.2.11}$$

Proof. We split the proof in three steps: first we establish the bound (3.2.11) for nonnegative bounded GF solutions for $p>\max\{1,p_c\}$, then we establish it for $p=1$

when $m \in (m_c, 1)$ in Step 2, then we will deduce the final result by approximation in Step 3.

• STEP 1. *Smoothing estimates for bounded GF solutions.* We begin by establishing the bound (3.2.11) for nonnegative bounded GF solutions, more precisely we will prove the following

Claim: Let $\mathcal{L}$ satisfy (A1), (A2) and (M1) with $2^* = 2N/(N-2s)$ and let u be a GF solution with $0 \le u_0 \in L^\infty(\Omega)$. Then, for any $p > \max\{1, p_c\}$ there exists a constant $\overline{\kappa} = \overline{\kappa}(N, m, s, p, \mathcal{S}_\mathcal{L}, \Omega) > 0$ such that (3.2.11) holds for any $t > t_0 \ge 0$.

Proof of the claim. Let us rewrite (3.2.9) for each $k \ge 1$ with $p_k = 2^k p$ and t_k such that $t_k - t_{k-1} = \frac{t-t_0}{2^k}$,

$$\|u(t_k)\|_{p_k} \le \left(\frac{c_k}{t_k - t_{k-1}}\right)^{\frac{N(p_k - p_{k-1})}{p_k}\vartheta_{k-1}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\vartheta_{k-1}}{p_k\vartheta_k}} \tag{3.2.12}$$

where $\vartheta_k := \vartheta_{p_k} = (2sp_k - N(1-m))^{-1}$ and

$$c_k := \frac{N\mathcal{S}_\mathcal{L}^2 (p_k - p_{k-1})(p_k + m - 1)^2 \vartheta_{k-1}}{4\, p_k\, (p_k - 1) m}\,.$$

Then, let us bound c_k uniformly in k using the definition of p_k:

$$\begin{aligned} c_k &\le \frac{N\mathcal{S}_\mathcal{L}^2}{8m} \frac{(p_k + m - 1)^2}{(p_k - 1)(2sp_{k-1} - N(1-m))} = \frac{N\mathcal{S}_\mathcal{L}^2}{8m} \frac{(1 - \frac{1-m}{p_k})^2}{(1 - \frac{1}{p_k})(s - \frac{N(1-m)}{p_k})} \\ &\le \frac{N\mathcal{S}_\mathcal{L}^2}{2m} \frac{p^2}{(p-1)\,(2sp - N(1-m))} =: \overline{c} \end{aligned}$$

which is bounded since $p > p_c$. Hence, we can iterate in (3.2.12)

$$\begin{aligned} \|u(t_k)\|_{p_k} &\le \left(\frac{\overline{c}}{t_k - t_{k-1}}\right)^{\frac{N(p_k - p_{k-1})}{p_k}\vartheta_{k-1}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\vartheta_{k-1}}{p_k\vartheta_k}} \\ &= \left(\frac{\overline{c}}{t_k - t_{k-1}}\right)^{\frac{N(p_k - p_{k-1})}{p_k}\vartheta_{k-1}} \left(\frac{\overline{c}}{t_{k-1} - t_{k-2}}\right)^{\frac{N(p_{k-1} - p_{k-2})}{p_{k-1}}\vartheta_{k-2}\frac{p_{k-1}\vartheta_{k-1}}{p_k\vartheta_k}} \|u(t_{k-2})\|_{p_{k-2}}^{\frac{p_{k-2}\vartheta_{k-2}}{p_k\vartheta_k}} \\ &\;\;\vdots \\ &= \prod_{j=1}^{k} \left(\frac{\overline{c}}{t_j - t_{j-1}}\right)^{\frac{N(p_j - p_{j-1})}{p_k}\frac{\vartheta_j\vartheta_{j-1}}{\vartheta_k}} \|u(t_0)\|_p^{\frac{p\vartheta_p}{p_k\vartheta_k}} \\ &= \left[\prod_{j=1}^{k} \left(2^j \frac{\overline{c}}{t - t_0}\right)^{\frac{N(\vartheta_{j-1} - \vartheta_j)}{2s}}\right]^{\frac{1}{p_k\vartheta_k}} \|u(t_0)\|_p^{\frac{p\vartheta_p}{p_k\vartheta_k}}, \end{aligned}$$

where in the last inequality we have used that $(p_j - p_{j-1})\vartheta_{j-1}\vartheta_j = \frac{(\vartheta_{j-1} - \vartheta_j)}{2s}$.

Now, we see that the product is uniformly bounded with respect to k:

$$\begin{aligned}\prod_{j=1}^{k}\left(2^j\frac{\overline{c}}{t-t_0}\right)^{\frac{N(\vartheta_{j-1}-\vartheta_j)}{2s}} &= \exp\left[\frac{N\log(2)}{2s}\sum_{j=1}^{k} j(\vartheta_{j-1}-\vartheta_j)+\frac{N}{2s}\log\left(\frac{\overline{c}}{t-t_0}\right)\sum_{j=1}^{k}(\vartheta_{j-1}-\vartheta_j)\right]\\ &\le \exp\left[\frac{N\log(2)}{2s}\sum_{j=1}^{k}\frac{j}{2s\,p_{j-1}}+\frac{N}{2s}\log\left(\frac{\overline{c}}{t-t_0}\right)(\vartheta_p-\vartheta_k)\right]\\ &\le \exp\left[\frac{N\log(2)}{4s^2\,p}\sum_{j=1}^{k}\frac{j}{2^j}+\frac{N}{2s}\log\left(\frac{\overline{c}}{t-t_0}\right)(\vartheta_p-\vartheta_k)\right]\\ &= 2^{\frac{N}{2s^2p}}\left(\frac{\overline{c}}{t-t_0}\right)^{\frac{N(\vartheta_p-\vartheta_k)}{2s}}.\end{aligned}$$

Notice that $\lim_{k\to\infty}(p_k\vartheta_k)^{-1}=2s$ and $\lim_{k\to\infty}\vartheta_k=0$. Therefore, since $t\ge t_k$ for all $k\ge 1$ and decay of the L^p-norm holds, we conclude that

$$\begin{aligned}\|u(t)\|_\infty &= \lim_{k\to\infty}\|u(t)\|_{p_k}\le\lim_{k\to\infty}\|u(t_k)\|_{p_k}\\ &\le \lim_{k\to\infty}\left(2^{\frac{N}{2s^2p}}\left(\frac{\overline{c}}{t-t_0}\right)^{\frac{N(\vartheta_p-\vartheta_k)}{2s}}\right)^{\frac{1}{p_k\vartheta_k}}\|u(t_0)\|_p^{\frac{p\,\vartheta_p}{p_k\vartheta_k}}\le\overline{\kappa}\,\frac{\|u(t_0)\|_p^{2sp\,\vartheta_p}}{(t-t_0)^{N\vartheta_p}}.\end{aligned}$$

This is exactly (3.2.11) when $p>\max\{1,p_c\}$, and the proof of the claim is complete.

• Step 2. *The case $p=1$ and $m\in(m_c,1)$.* We extend the claim of Step 1 to the case $p=1$. Let u be a nonnegative gradient flow solution with $u_0\in L^\infty(\Omega)$ and $m\in(m_c,1)$. Take any $p_1>1>p_c$, hence $u(t)$ satisfies the claim for p_1,

$$\begin{aligned}\|u(t)\|_\infty &\le \frac{c_1}{(t-t_0)^{N\vartheta_{p_1}}}\|u(t_0)\|_{p_1}^{2s\,p_1\vartheta_{p_1}}=\frac{c_1}{(t-t_0)^{N\vartheta_{p_1}}}\left(\int_\Omega u(t_0)^{(p_1-1)+1}\,\mathrm{d}x\right)^{2s\vartheta_{p_1}}\\ &\le \frac{c_1\|u(t_0)\|_\infty^{2s\vartheta_{p_1}(p_1-1)}}{(t-t_0)^{N\vartheta_{p_1}}}\|u(t_0)\|_1^{2s\vartheta_{p_1}}\le\frac{1}{2}\|u(t)\|_\infty+2^{2s\vartheta_1(p_1-1)}\ \frac{c_1\,\|u(t_0)\|_1^{2s\vartheta_{p_1}}}{(t-t_0)^{N\vartheta_{p_1}}}\Bigg)^{\frac{\vartheta_1}{\vartheta_{p_1}}}\end{aligned}$$

where we have used Young inequality, $ab\le\frac{1}{2}a^{\frac{1}{\alpha}}+2^{\frac{\alpha}{1-\alpha}}\,b^{\frac{1}{1-\alpha}}$, with $\alpha=2s\,\vartheta_{p1}(p_1-1)<1$. We can eliminate the term $\frac{1}{2}\|u(t_0)\|_\infty$ by applying De Giorgi's Lemma 3.6.6 and conclude the proof of the claim also in this case.

• Step 3. *Approximation with GF solutions.* We approximate the initial data $0\le u_0\in L^p(\Omega)$ by truncation, $u_{0,n}=\min\{u_0,n\}$, so that $u_{0,n}\to u_0$ in L^p. Since $u_{0,n}\in L^\infty(\Omega)\subset H^*(\Omega)$, there exists a gradient flow solution $u_n(t)$ given by Theorem 3.3.2, that is also bounded and satisfies the Claim of Step 1. Hence, by lower semicontinuity of the L^∞ norm we obtain

$$\|u(t)\|_\infty\le\lim_{n\to\infty}\|u_n(t)\|_\infty\le\lim_{n\to\infty}\overline{\kappa}(t-t_0)^{-N\vartheta_p}\|u_n(t_0)\|_p^{2sp\vartheta_p}=\overline{\kappa}(t-t_0)^{-N\vartheta_p}\|u(t_0)\|_p^{2sp\vartheta_p}.$$

This concludes the proof. □

3.2.2 Green function method

In this section we will show the advantage of the Weak Dual Formulation (WDS) and the power of the Green function method: we shall obtain the fundamental pointwise

estimate (3.2.17) of Lemma 3.2.5, the "almost representation formula", the nonlinear analogous of the representation formula in the linear case, proven in [26, 27] in the case $m > 1$. Another advantage, is that we only need assumption (K1) to obtain the same results obtained with the Moser iteration in the previous section, with a much simpler proof. On the other hand, the Green function method allows also to prove weighted smoothing effects, and boundary estimates: since in the nonlocal setting it is quite rare to have weighted Sobolev type inequalities, essential to run a Moser Iteration, here we just need to know the boundary behaviour of the Green function, namely (K2).

Let us recall the assumptions on $\mathcal{L}^{-1}$ that we will use in this section:

- There exists a constant $c_{1,\Omega} > 0$ such that

$$0 \le \mathbb{G}_\Omega(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\,. \tag{K1}$$

- Let $\Phi_1 \ge 0$ be the first eigenfunction of $\mathcal{L}$. Then, $\Phi_1(x) \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$, and we assume

$$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{G}_\Omega(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1\right). \tag{K3}$$

As we have already explained in Section 3.1.4, assumption (K1) guarantees existence of a positive and bounded eigenfunction Φ_1, and the validity of HLS, Sobolev and Poincaré inequalities. If we assume (K2) then $\Phi_1 \asymp \mathrm{dist}(\cdot\,,\partial\Omega)^\gamma$, cf. [21] for further details. Also, (K2) implies (K3), which is more practical in the computations: Φ_1 turns out to be a smart and smooth extension of $\mathrm{dist}(\cdot\,,\partial\Omega)^\gamma$.

In what follows, we shall use the following useful Lemma of [21] that allows to estimate the L^q norm of the Green function, under (K1) and (K2) assumptions.

Lemma 3.2.4 (Green function estimates I, [21])**.** *Let $\mathbb{G}_\Omega$ be the kernel of $\mathcal{L}^{-1}$, and assume that* (K1) *holds. Then, for all $0 < q < N/(N-2s)$, there exists a constant $c_{2,\Omega}(q) > 0$ such that*

$$\sup_{x_0\in\Omega} \int_\Omega \mathbb{G}_\Omega^q(x,x_0)\,\mathrm{d}x \le c_{2,\Omega}(q)\,. \tag{3.2.13}$$

Moreover, if (K2) *holds, then for the same range of q there exists a constant $c_{3,\Omega}(q) > 0$ such that, for all $x_0 \in \Omega$,*

$$c_{3,\Omega}(q)\Phi_1(x_0) \le \left(\int_\Omega \mathbb{G}_\Omega^q(x,x_0)\,\mathrm{d}x\right)^{\frac{1}{q}} \le c_{4,\Omega}(q)\mathcal{B}_q(\Phi_1(x_0))\,, \tag{3.2.14}$$

where $\mathcal{B}_q : [0,\infty) \to [0,\infty)$ is defined as follows:

$$\mathcal{B}_q(\Phi_1(x_0)) := \begin{cases} \Phi_1(x_0)\,, & \text{for } 0 < q < \frac{N}{N-2s+\gamma}\,, \\ \Phi_1(x_0)\left(1 + \log\Phi_1(x_0)^{\frac{1}{q}}\right), & \text{for } q = \frac{N}{N-2s+\gamma}\,, \\ \Phi_1(x_0)^{\frac{N-q(N-2s)}{q\gamma}}\,, & \text{for } \frac{N}{N-2s+\gamma} < q < \frac{N}{N-2s}\,. \end{cases}$$

Finally, for all $0 \le f \in L^1_{\Phi_1}(\Omega)$, (K2) *implies that*

$$\int_\Omega f(x)\mathbb{G}_\Omega(x,x_0)\,\mathrm{d}x \ge c_{0,\Omega}\Phi_1(x_0)\|f\|_{L^1_{\Phi_1}(\Omega)} \qquad \text{for all } x_0 \in \Omega\,. \tag{3.2.15}$$

The constants $c_{i,\Omega}$, $i = 0,\dots,5$, depend only on s, N, γ, q, Ω, and have an explicit expression, cf. [21].

Lemma 3.2.5 (Fundamental pointwise estimates)**.** *Assume* (A1) *and* (A2)*. Let* u *be a bounded and nonnegative GF solution, then for all* $x_0 \in \Omega$ *and* $0 \le t_0 \le t_1$ *we have*

$$\frac{u^m(t_1,x)}{t_1^{\frac{m}{1-m}}} \le \frac{1}{1-m}\int_\Omega \frac{u(t_0,y)-u(t_1,y)}{t_1^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}\mathbb{G}_\Omega(x,y)\,\mathrm{d}y \le \frac{u^m(t_0,x)}{t_0^{\frac{m}{1-m}}}\,. \tag{3.2.16}$$

The proof presented here is an adaptation to the FDE of the proof of Proposition 4.2 of [26], or Proposition 5.1 of [27].

Proof. Let us give first a formal proof: test the weak formulation (WDS) of the equation with (the a priori not admissible test function) $\chi_{[t_0,t_1]}\,\delta_{x_0}$, in order to obtain on the left-hand side

$$\int_{t_0}^{t_1}\int_\Omega u_t(t,x)\,\mathcal{L}^{-1}[\delta_{x_0}]\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (u(t_1,x)-u(t_0,x))\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,,$$

and on the right-hand side

$$-\int_{t_0}^{t_1}\int_\Omega u^m(t,x)\delta_{x_0}(x)\,\mathrm{d}x\,\mathrm{d}t = -\int_{t_0}^{t_1} u^m(t,x_0)\,\mathrm{d}t\,.$$

Using the time monotonicity, namely that $\left(\frac{t}{t_1}\right)^{\frac{1}{1-m}} u(t_1) \le u(t) \le u(t_0)\left(\frac{t}{t_0}\right)^{\frac{1}{1-m}}$, we can estimate the time integral from above and below, and obtain (3.2.16).

For a rigorous proof it suffices to approximate δ_{x_0} with $\varphi_n = \frac{\chi_{B_{1/n}(x_0)}}{|B_{1/n}(x_0)|}$ which is admissible in the (WDS) formulation, and obtain that

$$\int_{t_0}^{t_1}\int_\Omega u^m(t,x)\varphi_n(x)\,\mathrm{d}x\,\mathrm{d}t \xrightarrow{n\to\infty} \int_{t_0}^{t_1} u^m(t,x_0)\,\mathrm{d}t$$

at every Lebesgue point of $u(t,\cdot)$. Also, we shall approximate $\chi_{[t_0,t_1]}$ with χ_n as in Lemma 3.6.7. □

In what follows we will only need the lower bound of (3.2.17), in the following form:

Proposition 3.2.6 (Fundamental upper bounds)**.** *Assume* (A1), (A2) *and Kato's inequality* (3.6.10)*. Let* u *be a bounded and nonnegative GF solution, then for all* $x_0 \in \Omega$ *and* $0 \le t_0 \le \tau < t \le t_1$ *we have*

$$u^{p+m-1}(t,x_0) \le \mathsf{c}_{p,m}\,\frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}}\int_\Omega u(\tau,x)^p\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,. \tag{3.2.17}$$

where $\mathsf{c}_{p,m} = \frac{p+m-1}{m(1-m)}$*. Note that when* $p = 1$ *the above inequality follows from* (3.2.16)*, and there is no need of assuming Kato's inequality* (3.6.10)*.*

Proof. First, we multiply the equation by $pu^{p-1}\mathbb{G}_\Omega$ and integrate on $[0,T]\times\Omega$. On one hand, we obtain

$$\int_\Omega \fint_{t_0}^{t_1} pu^{p-1}(t,x)\,\partial_t u(t,x)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}t\,\mathrm{d}x = \fint_\Omega (u(t_1,x)^p - u(t_0,x)^p)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x.$$

On the other hand, we use Kato's inequality with $v=u^m$ and $f(v)=\frac{m}{p+m-1}v^{\frac{p+m-1}{m}}$

$$\begin{aligned}
-p\fint_{t_0}^{t_1}\fint_\Omega u^{p-1}\mathcal{L}\left(u(t,x)^m\right)\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,\mathrm{d}t \leq \\
&\leq -\frac{pm}{p+m-1}\fint_{t_0}^{t_1}\fint_\Omega \mathcal{L}(u^{p+m-1}(t,x))\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\frac{pm}{p+m-1}\fint_{t_0}^{t_1} u^{p+m-1}(t,x_0)\,\mathrm{d}t\,.
\end{aligned}$$

Using the time monotonicity, namely that $\left(\frac{t}{t_1}\right)^{\frac{1}{1-m}}u(t_1)\leq u(t)$, we can estimate the time integral from above and obtain (3.2.6). □

Lemma 3.2.7 ($L^p_{\Phi_1}$-stability)**.** *Let u be a nonnegative GF solution with $u_0\in L^p_{\Phi_1}(\Omega)$ and $p\geq 1$, then*

$$\|u(t_1)\|_{L^p_{\Phi_1}(\Omega)} \leq \|u(t_0)\|_{L^p_{\Phi_1}(\Omega)} \qquad \text{for every} \quad 0<t_0<t_1<T\,.$$

Proof. Without loss of generality, we assume $u(t)\in L^\infty(\Omega)$. Let us multiply the equation pointwise with $pu^{p-1}\Phi_1\chi_{[t_0,t_1]}$ and integrate:

$$\begin{aligned}
\fint_{t_0}^{t_1}\int_\Omega pu^{p-1}\partial_t u\,\Phi_1\,\mathrm{d}x\,\mathrm{d}t &= \fint_0^T \fint_\Omega \partial_t\left(u^p\,\Phi_1\right)\chi_{[t_0,t_1]}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega u^p(t_1)\Phi_1\,\mathrm{d}x - \fint_\Omega u^p(t_0)\Phi_1\,\mathrm{d}x\,.
\end{aligned}$$

On the other hand, by Kato's inequality we get

$$\begin{aligned}
-\int_{t_0}^{t_1}\fint_\Omega pu^{p-1}\mathcal{L}u^m\,\Phi_1\,\mathrm{d}x\,\mathrm{d}t \leq -\frac{p(p+m-1)}{m}\int_{t_0}^{t_1}\int_\Omega \mathcal{L}u^{p+m-1}\Phi_1\,\mathrm{d}x\,\mathrm{d}t \\
= -\frac{\lambda_1\,p(p+m-1)}{m}\fint_{t_0}^{t_1}\fint_\Omega u^{p+m-1}\Phi_1\,\mathrm{d}x\,\mathrm{d}t \leq 0\,.
\end{aligned}$$

Finally, we shall approximate $\chi_{[t_0,t_1]}$ with χ_n as in Lemma 3.6.7. □

(K2) **and critical exponents for weighted smoothing effects.** Before presenting the main results of this section we shall introduce first two new exponents that naturally appear in this weighted setting

$$m_{c,\gamma} := \frac{N+\gamma-2s}{N} \qquad \text{and} \qquad p_{c,\gamma} := \frac{N(1-m)}{2s-\gamma}\,.$$

Now, we are in a position to prove the following two results:

Theorem 3.2.8 ($L^p - L^\infty$ smoothing)**.** *Let $N > 2s$, $m \in (0,1)$ and assume* (A1), (A2) *and either* (K1) *or* (M1)*. Let u be a nonnegative WDS of* (CDP) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, for every $t > t_0 \ge 0$ we have*

$$\|u(t)\|_\infty \le \overline{\kappa}\, \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \qquad \text{with} \quad \vartheta_p = \frac{1}{2sp - N(1-m)}\,, \tag{3.2.18}$$

where $0 < \overline{\kappa}$ only depends on N, m, s, p and Ω.

Theorem 3.2.9 ($L^p_{\Phi_1} - L^\infty$ smoothing)**.** *Let $N > 2s > \gamma$, $m \in (0,1)$ and assume* (A1), (A2) *and* (K2)*. Let u be a nonnegative WDS of* (CDP) *corresponding to the initial datum $u_0 \in L^p_{\Phi_1}(\Omega)$ with $p \ge 1$ if $m \in (m_{c,\gamma}, 1)$ or $p > p_{c,\gamma}$ if $m \in (0, m_{c,\gamma}]$. Then, for every $t > t_0 \ge 0$ we have*

$$\|u(t)\|_\infty \le \overline{\kappa}\, \frac{\|u(t_0)\|_{L^p_{\Phi_1}(\Omega)}^{(2s-\gamma)p\,\vartheta_{p,\gamma}}}{(t-t_0)^{N\vartheta_{p,\gamma}}} \qquad \text{with} \;\; \vartheta_{p,\gamma} = \frac{1}{(2s-\gamma)p - N(1-m)}\,. \tag{3.2.19}$$

Proof of Theorems 3.2.8 and 3.2.9

We shall prove simultaneously Theorems 3.2.8 and 3.2.9, indeed the only difference in their proof is one step (Step 2A and 2B below) that involve Green function estimates which hold under assumptions (K1) or (K2), respectively. This shows how the Green function method allows to deal with both weighted and unweighted smoothing effects essentially in the same way. It is enough to prove the result for bounded nonnegative GF solutions u and then approximate the WDS by means of GF solutions $u_n(t)$ starting at $u_{0,n}(t) = \min\{u_0, n\}$, see Step 3.

• Step 1. *Fundamental pointwise estimate and De Giorgi Lemma.* Let us assume that u is a nonnegative GF solution with $u_0 \in L^\infty(\Omega)$. Then the following estimate holds for all $0 \le t_0 < t_1$

$$\begin{aligned}\|u(t_1)\|_\infty^{p+m-1} \le 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \Bigg[\Bigg(\mathsf{c}_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau\in[t_0,t_1]\\ x_0\in\Omega}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \Bigg]^{\frac{p+m-1}{\varepsilon}} \Bigg(\\ + \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau\in[t_0,t_1]\\ x_0\in\Omega}} \int_{\Omega\setminus B_R(x_0)} u(\tau,x)^p\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,.\end{aligned} \tag{3.2.20}$$

To prove the above inequality, we apply the Fundamental upper bounds (3.2.17) of Proposition 3.2.6 to obtain that for all $x_0 \in \Omega$ and $0 \le t_0 \le \tau < t \le t_1$,

$$u^{p+m-1}(t,x_0) \le \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_\Omega u(\tau,x)^p\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\,. \tag{3.2.21}$$

Next, we split the last integral in two parts: fix $R > 0$ to be determined later and

let $\varepsilon \in (0, p+m-1)$

$$
\begin{aligned}
&u^{p+m-1}(t,x_0) \le \\
&\le \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}}\, \|u(\tau)\|_\infty^{p+m-1-\varepsilon} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \\
&+ \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{\Omega\setminus B_R(x_0)} u^{p}(\tau,x)\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \\
&\le \frac{1}{2}\|u(\tau)\|_\infty^{p+m-1} + 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[\mathsf{c}_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \right]^{\frac{p+m-1}{\varepsilon}} \\
&+ \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{\Omega\setminus B_R(x_0)} u(\tau,x)^{p}\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x
\end{aligned}
$$

by using Young's inequality, $ab \le \frac{1}{2}a^\sigma + 2^{\frac{1}{\sigma-1}}\, b^{\frac{\sigma}{\sigma-1}}$, with $\sigma = \frac{p+m-1}{p+m-1-\varepsilon} > 1$. Taking supremum on both sides we obtain for all $0 \le t_0 \le \tau < t \le t_1$,

$$
\begin{aligned}
&\|u(t)\|_\infty^{p+m-1} \le \\
&\le \frac{1}{2}\|u(\tau)\|_\infty^{p+m-1} \\
&+ 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[\mathsf{c}_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau\in[t_0,t_1]\\ x_0\in\Omega}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \right]^{\frac{p+m-1}{\varepsilon}} \\
&+ \mathsf{c}_{p,m}\, \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau\in[t_0,t_1]\\ x_0\in\Omega}} \int_{\Omega\setminus B_R(x_0)} u(\tau,x)^{p}\, \mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x.
\end{aligned}
$$

We can conclude by De Giorgi's Lemma 3.6.6, with the function $Z(t) := \|u(t)\|_\infty^{p+m-1}$ and obtain (3.2.20).

Important: The next step is the only point where we shall distinguish between weighted (Step 2B) and unweighted smoothing effects (Step 2A).

• Step 2.A. *$L^p - L^\infty$ smoothing effects via* (K1). Given any p as in Theorem 3.2.8, we shall prove the following estimate for bounded GF solutions: for every $t > t_0 \ge 0$ we have

$$
\|u(t)\|_\infty \le \overline{\kappa}\, \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \qquad \text{with} \quad \vartheta_p = \frac{1}{2sp - N(1-m)}\,, \tag{3.2.22}
$$

where $\overline{\kappa} > 0$ only depends on N, m, s, p and Ω.

The proof consists in carefully estimating the two terms in the right-hand side of (3.2.20) separately, using assumption (K1), namely $\mathbb{G}_\Omega(x_0,x) \le c_{1,\Omega}|x-x_0|^{-(N-2s)}$. First, we observe that for all exponent $0 < q < \frac{N}{N-2s}$ and $q' = \frac{q}{q-1} > \frac{N}{2s}$ we can use

Hölder's inequality

$$\begin{aligned}\int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x &\le c_{1,\Omega}\iint_{B_R(x_0)}\frac{u^{1-m+\varepsilon}(\tau,x)}{|x-x_0|^{N-2s}}\,\mathrm{d}x\\ &\le c_{1,\Omega}\|u(\tau)\|^{1-m+\varepsilon}_{q'(1-m+\varepsilon)}\left[\iint_{B_R(x_0)}\frac{1}{|x-x_0|^{q(N-2s)}}\,\mathrm{d}x\right]^{\frac{1}{q}}\\ &\le c_2\|u(t_0)\|^{1-m+\varepsilon}_{q'(1-m+\varepsilon)}\,R^{\frac{N-q(N-2s)}{q}}\\ &= c_2\|u(t_0)\|^{1-m+\varepsilon}_{q'(1-m+\varepsilon)}\,R^{2s-\frac{N}{q'}},\end{aligned}$$

where $c_2>0$ depends on q,N,s and $c_{1,\Omega}$. We have also used the L^p-norm decay since if we fix $\varepsilon\in\left(0,\frac{2sp-N(1-m)}{N}\right)\subset(0,p+m-1)$ we can choose $q':=\frac{p}{(1-m+\varepsilon)}$ as

$$\frac{N}{2s}<q'=\frac{p}{1-m+\varepsilon}\,.$$

On the other hand, using the L^p-norm decay we have

$$\int_{\Omega\setminus B_R(x_0)} u(\tau,x)^p\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x\le c_{1,\Omega}\iint_{\Omega\setminus B_R(x_0)}\frac{u(\tau,x)^p}{|x-x_0|^{N-2s}}\,\mathrm{d}x\le c_{1,\Omega}\frac{\|u(t_0)\|_p^p}{R^{N-2s}}\,.$$

Plugging the above estimates in (3.2.20) and choosing $t_0=0$, we obtain

$$\|u(t_1)\|_\infty^{p+m-1}\le 2^{\frac{p+m-1-\varepsilon}{\varepsilon}}\left[c_2\frac{\|u_0\|_p^{1-m+\varepsilon}}{t_1}R^{\frac{2sp-N(1-m+\varepsilon)}{p}}\right]^{\frac{p+m-1}{\varepsilon}}+c_3\,\frac{\|u_0\|_p^p}{t_1}\,\frac{1}{R^{N-2s}}\,.$$

Finally, we choose

$$R=\left(\frac{t_1}{\|u_0\|_p^{1-m}}\right)^{\frac{p}{2sp-N(1-m)}}$$

and, by the time-shift invariance of the equation, we obtain the desired smoothing effects (3.2.22) for every $p\ge1$ if $m\in(m_c,1)$ and $p>p_c$ if $m\in(0,m_c)$.

• Step 2.B. *$L^p_{\Phi_1}-L^\infty$ smoothing effects via* (K2). Under assumption (K2), we shall prove the following smoothing effect for bounded GF solutions and any p as in Theorem 3.2.9: for every $t>t_0\ge0$ we have

$$\|u(t)\|_\infty\le\overline{\kappa}\,\frac{\|u(t_0)\|^{(2s-\gamma)p\,\vartheta_{p,\gamma}}_{L^p_{\Phi_1}(\Omega)}}{(t-t_0)^{N\vartheta_{p,\gamma}}}\qquad\text{with }\ \vartheta_{p,\gamma}=\frac{1}{(2s-\gamma)p-N(1-m)}\,.\quad(3.2.23)$$

The proof follows analogously to Step 2.A, that is, we have to estimate properly the two terms of (3.2.20), but in this case we have to use assumption (K2), namely $\mathbb{G}_\Omega(x_0,x)\le c_{1,\Omega}|x-x_0|^{-(N+\gamma-2s)}\Phi_1(x)$.

First, observe that for every $0<q<\frac{N}{N+\gamma-2s}$ and $q'=\frac{q}{q-1}>\frac{N}{2s-\gamma}$ we can define

$p := q'(1-m+\varepsilon)$ to obtain by Hölder's inequality that

$$\begin{aligned}\int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x &\le c_{1,\Omega}\iint_{B_R(x_0)}\frac{u^{1-m+\varepsilon}(\tau,x)\Phi_1(x)}{|x-x_0|^{N+\gamma-2s}}\,\mathrm{d}x\\ &\le c_{1,\Omega}\|\Phi_1\|_\infty^{1-\frac{1}{q'}}\,\|u(\tau)\|^{1-m+\varepsilon}_{L^p_{\Phi_1}(\Omega)}\left[\iint_{B_R(x_0)}\frac{1}{|x-x_0|^{q(N+\gamma-2s)}}\,\mathrm{d}x\right]^{\frac{1}{q}}\\ &\le c_2\|u(t_0)\|^{1-m+\varepsilon}_{L^p_{\Phi_1}(\Omega)}\,R^{\frac{N-q(N+\gamma-2s)}{q}}\\ &= c_2\|u(t_0)\|^{1-m+\varepsilon}_{L^p_{\Phi_1}(\Omega)}\,R^{\frac{(2s-\gamma)p-N(1-m+\varepsilon)}{p}},\end{aligned}\tag{3.2.24}$$

where $c_2>0$ depends on q_0,N,s,Φ_1 and $c_{1,\Omega}$, and we have also used the $L^p_{\Phi_1}$-norm decay. Notice that we have to choose $\varepsilon\in\left(0,\frac{(2s-\gamma)p-N(1-m)}{N}\right)\subset(0,p+m-1)$ to ensure

$$\frac{N}{2s-\gamma}<q'=\frac{p}{1-m+\varepsilon}\,.$$

On the other hand, we use the $L^p_{\Phi_1}$-norm decay to obtain

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} u(\tau,x)^p\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x &\le c_{1,\Omega}\iint_{\Omega\setminus B_R(x_0)}\frac{u(\tau,x)^p\Phi_1(x)}{|x-x_0|^{N+\gamma-2s}}\,\mathrm{d}x\\ &\le c_{1,\Omega}\frac{\|u(t_0)\|^p_{L^p_{\Phi_1}(\Omega)}}{R^{N+\gamma-2s}}\,.\end{aligned}$$

Plugging the above estimates in (3.2.20) and choosing $t_0=0$, we obtain

$$\begin{aligned}\|u(t_1)\|_\infty^{p+m-1}\le 2^{\frac{p+m-1-\varepsilon}{\varepsilon}}&\left[c_2\,\frac{\|u_0\|^{1-m+\varepsilon}_{L^p_{\Phi_1}(\Omega)}}{t_1}R^{\frac{(2s-\gamma)p-N(1-m+\varepsilon)}{p}}\right]^{\frac{p+m-1}{\varepsilon}}\\ &+c_3\,\frac{\|u_0\|^p_{L^p_{\Phi_1}(\Omega)}}{t_1}\,\frac{1}{R^{N+\gamma-2s}}\,.\end{aligned}\tag{3.2.25}$$

with $c_2,c_3>0$ depending on p,N,s,γ,Φ_1 and $c_{1,\Omega}$. Finally, we choose in (3.2.25)

$$R=\left(\frac{t_1}{\|u_0\|_p^{1-m}}\right)^{\frac{p}{(2s-\gamma)p-N(1-m)}}$$

and, by the time-shift invariance of the equation, we obtain the desired smoothing effects (3.2.23) for every admissible p.

• Step 3. *Approximation with GF solutions.* We approximate the initial data $0\le u_0\in L^p(\Omega)$ by truncation, $u_{0,n}=\min\{u_0,n\}$, so that $u_{0,n}\to u_0$ in L^p. Since $u_{0,n}\in L^\infty(\Omega)\subset H^*(\Omega)$, there exists a gradient flow solution $u_n(t)$ given by Theorem 3.3.2, that is also bounded and satisfies the Claim of Step 1. Hence, by lower semicontinuity of the L^∞ norm we obtain

$$\begin{aligned}\|u(t)\|_\infty\le\lim_{n\to\infty}\|u_n(t)\|_\infty&\le\lim_{n\to\infty}\overline{\kappa}(t-t_0)^{-N\vartheta_p}\|u_n(t_0)\|_p^{2sp\vartheta_p}\\ &=\overline{\kappa}(t-t_0)^{-N\vartheta_p}\|u(t_0)\|_p^{2sp\vartheta_p}\,.\end{aligned}$$

This concludes the proof of the smoothing effects of Theorems 3.2.8 and 3.2.9 assuming Kato inequality.

• STEP 4. *Smoothing without using Kato inequality.* We notice that when $p = 1$, the fundamental upper bound (3.2.17) of Proposition 3.2.5 does not require the validity of Kato inequality, hence we can repeat Step 1 with $p = 1$, and obtain inequality (3.2.20) with $p = 1$ without using Kato inequality. We can then repeat Steps 2.A and 2.B respectively, and this is where the restriction $p < N/(2s-\gamma)$ comes from, as we shall see. The unweighted case corresponds to $\gamma = 0$ and follows with minor modifications.

The first integral of (3.2.20) can be estimated as in (3.2.24)

$$\begin{aligned}\int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau,x)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x &\le c_{1,\Omega}\iint_{B_R(x_0)}\frac{u^{1-m+\varepsilon}(\tau,x)\Phi_1(x)}{|x-x_0|^{N+\gamma-2s}}\,\mathrm{d}x\\ &\le c_2\|u(t_0)\|^{1-m+\varepsilon}_{L^{q'(1-m+\varepsilon)}_{\Phi_1}(\Omega)}\,R^{\frac{N-q(N+\gamma-2s)}{q}} = c_2\|u(t_0)\|^{1-m+\varepsilon}_{L^{p_0}_{\Phi_1}(\Omega)}\,R^{\frac{(2s-\gamma)p_0-N(1-m+\varepsilon)}{p_0}}\end{aligned} \tag{3.2.26}$$

where we have set $p_0 := q'(1-m+\varepsilon) > \frac{N(1-m+\varepsilon)}{2s-\gamma}$.

We next have to estimate the second integral of (3.2.20) as follows

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} u(\tau,x)\;\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x &\le c_{1,\Omega}\iint_{\Omega\setminus B_R(x_0)}\frac{u(\tau,x)\Phi_1(x)}{|x-x_0|^{N+\gamma-2s}}\,\mathrm{d}x\\ &\le c_{2,\Omega,p}\frac{\|u(t_0)\|_{L^q_{\Phi_1}(\Omega)}}{R^{q'(N+\gamma-2s)-N}}\,,\end{aligned} \tag{3.2.27}$$

with $q' > \frac{N}{N+\gamma-2s}$, so that $q < \frac{N}{2s-\gamma}$. This is where the restriction on p comes from. We would like to chose now $p_0 = q$, and this is possible only when $\frac{N(1-m+\varepsilon)}{2s-\gamma} < p_0 = q < \frac{N}{2s-\gamma}$ for all $\varepsilon \in (0,m)$, which means in fact for all $p_0 \in \left(\frac{N(1-m)}{2s-\gamma}, \frac{N}{2s-\gamma}\right)$. Hence, for all such p_0 we have

$$\|u(t_1)\|^m_\infty \le 2^{\frac{m-\varepsilon}{\varepsilon}}\left[c_2\,\frac{\|u_0\|^{1-m+\varepsilon}_{L^{p_0}_{\Phi_1}(\Omega)}}{t_1}R^{\frac{(2s-\gamma)p_0-N(1-m+\varepsilon)}{p_0}}\right]^{\frac{m}{\varepsilon}} + \frac{c_4}{t_1}\frac{\|u(t_0)\|_{L^{p_0}_{\Phi_1}(\Omega)}}{R^{\frac{p_0}{p_0-1}(N+\gamma-2s)-N}}\,,$$

with $c_2, c_4 > 0$ depending on $p_0, N, s, \gamma, \Phi_1$ and $c_{1,\Omega}$. Letting $R^{p_0\vartheta_{p_0}} = \|u_0\|^{m-1}_{L^{p_0}_{\Phi_1}(\Omega)}t_1$ gives the desired bounds. This concludes the proof of Theorems 3.2.8 and 3.2.9. □

3.2.3 $H^* - L^\infty$ smoothing effects. Proof of Theorem 3.2.11

It is possible to use the $L^p - L^\infty$ estimate obtained above to get $H^* - L^\infty$ smoothing effects, for which we have to introduce another critical exponent:

$$m_s = \frac{N-2s}{N+2s}\,.$$

For this estimate it is required to be in the regime $m \in (m_s, 1)$, since we need the energy estimates below.

Lemma 3.2.10 (GF solutions are Energy solutions)**.** *Assume* (A1) *and* (A2) *and let u be a GF solution with $u_0 \in H^*(\Omega)$. Then, for every $t_1 > t > t_0 \geq 0$ we have*

$$\|u^m(t_1)\|^2_{H(\Omega)} \;\leq\; \frac{1}{2m}\frac{\|u(t)\|^{1+m}_{1+m}}{(t_1-t)} \;\leq\; \frac{1}{2m(1+m)}\frac{\|u(t_0)\|^2_{H^*(\Omega)}}{(t_1-t)(t-t_0)}\,. \tag{3.2.28}$$

Proof. We provide a formal proof in order to explain the main ideas, for a rigorous proof see Proposition 11.9 of [4].
First inequality. Let us differentiate the H^*-norm as in (3.4.4) and use the decay of the L^{1+m}-norm

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x \leq -2m\,\frac{\left(\int_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x\right)^2}{\int_\Omega u^{1+m}\,\mathrm{d}x} \leq -2m\,\frac{\left(\int_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x\right)^2}{\int_\Omega u(t_0)^{1+m}\,\mathrm{d}x}\,.$$

The result folows by integrating on $[t,t_1]$.
Second inequality. We differentiate the L^{1+m}-norm (the energy) as in (3.4.2) to obtain the following ODE,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{1+m}\,\mathrm{d}x \leq -\frac{1+m}{\|u(t_0)\|^2_{H^*(\Omega)}}\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)^2.$$

Then, the upper estimate follows by integrating on $[t_0,t]$. □

From the above energy estimate easily follows:

Theorem 3.2.11 ($H^* - L^\infty$ smoothing)**.** *Let $N > 2s$, $m \in (m_s, 1)$ and assume* (A1)*,* (A2) *and* (K1)*. Let u be a nonnegative WDS of* (CDP) *corresponding to the initial datum $u_0 \in H^*(\Omega)$. Then, for every $t > t_0 \geq 0$ we have*

$$\|u(t)\|_\infty \leq \overline{\kappa}\,\frac{\|u(t_0)\|^{4s\,\vartheta_{1+m}}_{H^*(\Omega)}}{(t-t_0)^{(N+2s)\vartheta_{1+m}}} \qquad \textit{with} \quad \vartheta_{1+m} = \frac{1}{2s(1+m)-N(1-m)}\,, \tag{3.2.29}$$

where $0 < \overline{\kappa}$ only depends on N, m, s and Ω.

Proof of Theorem 3.2.11. We combine the energy estimate of Lemma 3.2.10 with $L^{1+m} - L^\infty$ smoothing effect. Given $t > t_0 \geq 0$, let us choose $\tilde{t} = \frac{t+t_0}{2}$ so that

$$\begin{aligned}\|u(t)\|_\infty \leq c\frac{\|u(\tilde{t})\|^{(1+m)2s\vartheta_{1+m}}_{1+m}}{(t-\tilde{t})^{N\vartheta_{1+m}}} &\leq \frac{c}{(1+m)}\frac{\|u(t_0)\|^{4s\vartheta_{1+m}}_{H^*(\Omega)}}{(t-\tilde{t})^{N\vartheta_{1+m}}(\tilde{t}-t_0)^{2s\vartheta_{1+m}}}\\ &\leq \overline{\kappa}\frac{\|u(t_0)\|^{4s\vartheta_{1+m}}_{H^*(\Omega)}}{(t-t_0)^{(N+2s)\vartheta_{1+m}}}\,.\end{aligned}$$

□

3.3 Existence and uniqueness

In this section, we prove the existence and uniqueness of MWDS. To do this, first, we prove the existence of GF solutions, adapting to our setting the celebrated theorem of Brezis-Komura [34, 90]. This class of solutions enjoys useful properties and we use them to build a monotone approximating sequence, needed to build the MWDS. In Section 3.3.2, we prove the time monotonicity of the norm and the fact that GF solutions are strong in $L^1_{\Phi_1}(\Omega)$, and we also give a explicit bound for $\|\partial_t u\|_{L^1_{\Phi_1}(\Omega)}$. Then, we use the strong formulation of the problem to prove the T-contractivity in $L^1_{\Phi_1}(\Omega)$. In Section 3.3.3, we prove that nonnegative GF solutions are indeed WDS. Finally, in Section 3.3.4 we give the proof of Theorem 3.3.15, and in Section 3.3.5 we prove Theorem 3.3.17.

3.3.1 Existence of GF solutions

We will construct our WDS, more precisely the MWDS, as the monotone limit of gradient flow solutions in a suitable Hilbert space, modeled ad hoc for the operator $\mathcal{L}$. The class of nonnegative WDS contains the nonnegative semigroup solutions that will form our approximating sequence, see Lemma 3.3.14. We shall construct such semigroup solutions using the celebrated theory of Brezis on Maximal Monotone Operators [34, 33], see also Komura [90] and the excellent lecture notes [4]: we define the free energy (or entropy functional) as follows

$$\mathcal{E}(u) = \begin{cases} \frac{1}{1+m}\int_\Omega |u|^{1+m}(x)\,\mathrm{d}x & \text{if} \quad u \in L^{1+m}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and show that $\mathcal{L}$ is the subdifferential of the convex and lower-semicontinuous energy functional $\mathcal{E}$ on the Hilbert space $H^*(\Omega)$. Note that $H^*(\Omega)$ is defined as the (topological) dual space of $H(\Omega)$, the domain of the quadratic form associated to $\mathcal{L}$:

$$H(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega u\,\mathcal{L}u\,\mathrm{d}x < \infty \right\}.$$

We can endow $H^*(\Omega)$ with the natural scalar product and the associated norm

$$\langle u, v\rangle_{H^*(\Omega)} = \int_\Omega u\,\mathcal{L}^{-1}v\,\mathrm{d}x \qquad \text{and} \qquad \|u\|^2_{H^*(\Omega)} = \langle u, u\rangle_{H^*(\Omega)} = \int_\Omega u\,\mathcal{L}^{-1}u\,\mathrm{d}x\,.$$

We refer to [24] for further details about the spaces $H(\Omega)$ and $H^*(\Omega)$ and their relation with the fractional Sobolev spaces $H^s_0(\Omega)$, $H^{1/2}_{00}(\Omega)$, $H^s(\Omega)$ and their duals $H^{-s}(\Omega)$. Following [4, 34, 33], we recall the definition of semigroup (or gradient flow) solutions adapted to our setting:

Definition 3.3.1 (Gradient flow)**.** *We say that $u : (0, +\infty) \to L^{1+m}(\Omega)$ is a* gradient flow (GF) *of the functional $\mathcal{E}$ defined above, if $u(t) \in AC_{loc}((0,\infty) : H^*(\Omega))$ and*

$$-\partial_t u(t) \in \partial\mathcal{E}(u(t)) \quad \text{for a.e. } t \in (0,\infty).$$

We say that $u(t)$ starts from $u_0 \in H^*(\Omega)$ if $\lim_{t\to 0^+} \|u(t) - u_0\|_{H^*(\Omega)} = 0$.

It is well-known that if $\mathcal{E}$ is convex and lower-semicontinuous then a locally absolutely continuous curve $u(t) \in H^*(\Omega)$ is a GF if and only if it is a EVI solution, see for instance Theorem 11.15 of [4]. We are now in the position to state the celebrated Brezis-Komura Theorem, the nonlinear analogous of the Hille-Yosida or Lumer-Phillips Theorem. The statement, adapted to our setting, reads:

Theorem 3.3.2 (Brezis-Komura [34, 90])**.** *For every $u_0 \in H^*(\Omega)$, there exists a unique gradient flow starting from u_0 that we denote by $u(t) = S_t u_0$. This defines a continuous semigroup $S_t : H^*(\Omega) \to L^{1+m}(\Omega)$ for $t > 0$ with the T-contraction property in $H^*(\Omega)$*

$$\|(S_t u_0 - S_t v_0)_\pm\|_{H^*(\Omega)} \le \|(u_0 - v_0)_\pm\|_{H^*(\Omega)}, \qquad \forall t > 0, \ \ \forall u_0, v_0 \in H^*(\Omega)\,. \quad (3.3.1)$$

Moreover, GF solutions are H^*-strong, namely for all $T > 0$ we have

$$t\, u^m,\, t\, \partial_t u \in L^\infty((0,T) : H^*(\Omega))\,.$$

Indeed, GF solutions are H-energy solution, i.e. the following energy estimate for every $t > t > t_1 \ge 0$ we have

$$\|u^m(t_1)\|^2_{H(\Omega)} \ \le\ \frac{(1+m)}{2m}\frac{\mathcal{E}(u(t))}{(t_1 - t)} \ \le\ \frac{1}{2m(1+m)}\frac{\|u(t_0)\|^2_{H^*(\Omega)}}{(t_1 - t)(t - t_0)}\,.$$

To apply Brezis-Komura Theorem, the functional $\mathcal{E}(u)$ must be convex and lower semicontinuous in $H^*(\Omega)$ as it is proven in the following lemma:

Lemma 3.3.3. *The functional $\mathcal{E}(u)$ is convex and lower semicontinuous in $H^*(\Omega)$.*

We prove the above lemma in Section 3.6. This is enough to prove Theorem 3.3.2, indeed:

Proof of Theorem 3.3.2. Once we know by Lemma 3.3.3 that $\mathcal{E}$ is convex and lower semicontinuous in $H^*(\Omega)$, the rest of the proof follows as in Theorem 2.2 of [24], which is the adaptation to our setting of the proof of Brezis [34] for the classical H^{-1} case. See also [4]. □

Remark 3.3.4. (i) Note that in this class of GF solutions, the comparison principle holds, namely by (3.3.1) it is clear that if $u_0 \le v_0$ a.e. in Ω, then $u \le v$ a.e. in $(0,\infty)\times\Omega$. A formal proof of (3.3.1) is the following: by Kato inequality we know that $\mathrm{sign}_+(u^m - v^m)\mathcal{L}(u^m - v^m) \ge \mathcal{L}(u^m - v^m)_+$ so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(u-v)_+\|^2_{H^*} &= -2\int_\Omega \mathrm{sign}_+(u^m - v^m)\mathcal{L}(u^m - v^m)\mathcal{L}^{-1}(u-v)_+\,\mathrm{d}x \\ &\le -2\int_\Omega (u^m - v^m)_+(u-v)_+\,\mathrm{d}x\,.\end{aligned}$$

Indeed, this is rigorous since solutions are H^*-strong, i.e. $\partial_t u(t,\cdot) \in H^*(\Omega)$.
(ii) Notice that uniqueness in H^* follows by (3.3.1), indeed we have the H^*-contraction (just summing the positive and negative part estimates)

$$\|u(t) - v(t)\|_{H^*(\Omega)} \le \|u_0 - v_0\|_{H^*(\Omega)} \qquad \text{for all } t \ge 0.$$

(iii) Note that here *we do not assume any sign condition on* u_0, and this is to the best of our knowledge *the largest class of data without sign restrictions, for which existence and uniqueness hold.* The above theorem generalizes the existence and uniqueness result of [24], Theorem 2.2, where the case of SFL and RFL were thoroughly investigated in the framework of fractional Sobolev Spaces. Most of that theory applies also in the present case, but we have preferred to simplify the setup aiming at a larger class of nonnegative solution, which are the WDS in the $L^1_{\Phi_1}$ framework.
(iv) This Theorem follows by adapting Brezis' proof [34] originally in the H^{-1} framework, to the present H^* setting, as it has been done in [24], where more general nonlinearities and further details are given; in particular, we recall that under assumption (K1) it is possible to identify H^* and H^{-s}, when $s \neq \frac{1}{2}$.

3.3.2 Time Monotonicity and $L^1_{\Phi_1}$-contractivity

In this section, we prove the Benilan-Crandall estimates, i.e. the time monotonicity of solutions. We shall also see that nonnegative GF solutions are strong in $L^1_{\Phi_1}(\Omega)$, and we show different $L^1_{\Phi_1}$-contractivity of solutions. All these properties hold for GF solutions, this makes them a very good choice to build the approximating sequence needed to construct the MWDS, that will inherit most of these properties.

In order to prove that GF solutions are strong in $L^1_{\Phi_1}(\Omega)$ we need the following theorem of Benilan and Gariepy [9], that we restate here in our notations:

Theorem 3.3.5 (Theorem 1.1 of [9])**.** *Let* $w = u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$, $p(r) = \frac{2}{1+m} r^{\frac{1-m}{1+m}}$ *and*

$$v = \int_0^w p(r)dr \in BV((0,T) : L^1_{\Phi_1}(\Omega)),$$

where BV is the class of functions of bounded variation. *Then* $v \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$ *and the chain rule holds for the time derivative:*

$$\partial_t v(t) = p(w(t))\partial_t w(t) = \partial_t u(t)\,.$$

We recall that $u \in L^1_{\Phi_1}(\Omega)$ is a BV function if its distributional derivative $\partial_t u$ can be represented by a finite Radon measure. In our case, the distributional derivative $\partial_t u$ must be a Radon measure, as the limit of incremental quotients of u that are uniformly bounded in $L^1_{\Phi_1}(\Omega)$. Our purpose is to prove that, indeed, this limit is a function of $L^1_{\Phi_1}(\Omega)$.

The following result holds for the whole class of nonnegative WDS, and will be essential later.

Lemma 3.3.6. *Let* $0 \le u \le v$ *be two ordered WDS of* (CDP) *with initial data* $0 \le u_0 \le v_0$. *Then, for all* $0 \le t_0 \le t_1 < \infty$, *the following property holds*

$$\int_\Omega [v(t_1) - u(t_1)]\Phi_1 \,\mathrm{d}x \le \int_\Omega [v(t_0) - u(t_0)]\Phi_1 \,\mathrm{d}x\,.$$

Proof. We apply the weak formulation (WDS) to the difference $v-u$, using the admisible test function $\psi_n = \Phi_1\,\chi_n$, where χ_n is the same as in Lemma 3.6.7. Then, the left-hand side of the equality (WDS) reads

$$\begin{aligned}\iint_0^T \int_\Omega \mathcal{L}^{-1}(v-u)\partial_t\psi_n \,\mathrm{d}x\,\mathrm{d}t &= \iint_0^T \int_\Omega (v-u)\partial_t\chi_n \mathcal{L}^{-1}\Phi_1 \,\mathrm{d}x\,\mathrm{d}t \\ &\xrightarrow{n\to\infty} \lambda_1^{-1}\int_\Omega \big[(v(t_0)-u(t_0)) - (v(t_1)-u(t_1))\big]\Phi_1\,\mathrm{d}x\,.\end{aligned}$$

As for the other side of the equality, we have

$$\begin{aligned}\iint_0^T \int_\Omega \mathcal{L}^{-1}(v-u)\partial_t\psi_n \,\mathrm{d}x\,\mathrm{d}t &= \iint_0^T \int_\Omega (v^m-u^m)\psi_n \,\mathrm{d}x\,\mathrm{d}t \\ &\xrightarrow{n\to\infty} \iint_{t_0}^{t_1} \int_\Omega (v^m-u^m)\Phi_1 \,\mathrm{d}x\,\mathrm{d}t \geq 0\,,\end{aligned}$$

since $u\leq v$ implies $u^m\leq v^m$ whenever $m>0$, and this proves the lemma. □

The proofs of the next results, Lemmata 3.3.7, 3.3.8 and 3.3.9, follow the ideas in [49, 9], adapted in [112] to the case of the RFL on $\mathbb{R}^N$ and on domains. However we offer few interesting twists and variants.

Lemma 3.3.7 (Time Monotonicity)**.** *Let u be a GF solution to* (CDP)*, then*

$$u_t \leq \frac{u}{(1-m)\,t}, \tag{3.3.2}$$

in the sense of distributions. This is the weak formulation of the fact that $t\mapsto t^{-\frac{1}{1-m}}\,u$ is non increasing on t for a.e. $x\in\Omega$.

We will adapt the classical proof of [49] for the L^1 case, see also [141].

Proof. First of all, notice that GF solutions satisfy T-contractivity property in H^*, hence, given two ordered solutions $u\leq v$, we have that for every $0\leq t_0\leq t_1\leq\infty$,

$$\begin{aligned}0\leq \|(u(t_1)-v(t_1))_+\|_{L^1_{\Phi_1}(\Omega)} &\leq \lambda_1^{1/2}\|(u(t_1)-v(t_1))_+\|_{H^*(\Omega)} \\ &\leq \lambda_1^{1/2}\|(u(t_0)-v(t_0))_+\|_{H^*(\Omega)} = 0,\end{aligned}$$

which implies comparison in $L^1_{\Phi_1}(\Omega)$ and, particularly, almost everywhere in Ω.

Consider the rescaled function

$$u_\lambda(t,x) = \lambda^{-\frac{1}{1-m}}u(\lambda t,x),$$

which is also a solution of (CDP). Now, for a fixed t, let us choose $\lambda = \frac{t+h}{t}$ with $h\geq 0$ and observe that, $\lambda\geq 1$ and $\lambda^{-\frac{1}{1-m}}\leq 1$, hence, $u_\lambda(0)\leq u_0$ and $u_\lambda(t)\leq u(t)$ by comparison. Then,

$$\begin{aligned}\frac{u(t+h,x)-u(t,x)}{h} &= \frac{1}{h}\left[\left(\frac{t+h}{t}\right)^{\frac{1}{1-m}} u_\lambda(t,x) - u_\lambda(t,x)\right] + \frac{u_\lambda(t,x)-u(t,x)}{h} \\ &\leq \frac{(t+h)^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}}{h}\,\frac{u_\lambda(t,x)}{t^{\frac{1}{1-m}}}.\end{aligned} \tag{3.3.3}$$

The above formula has to be understood in the distributional sense. Since we know by Theorem 3.3.2 that GF solutions are H^*-strong, namely $t\,\partial_t u \in L^\infty((0,T) : H^*(\Omega))$, this implies convergence of the right-hand side to the distributional time derivative $\partial_t u$ as $h \to 0^+$, and concludes the proof. □

Lemma 3.3.8. *Let u be a GF solution of* (CDP)*, then $\partial_t u^{\frac{1+m}{2}} \in L^2_{\mathrm{loc}}((0,T) : L^2(\Omega))$, more precisely, we have that for all $0 < t_0 < t_1 < T$*

$$\|\partial_t u^{\frac{1+m}{2}}\|^2_{L^2((t_0,t_1):\,L^2(\Omega))} \le \frac{c(1+m)\|u(t_0/2)\|^2_{H^*(\Omega)}}{4m^2}\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right). \tag{3.3.4}$$

Moreover, the bound is uniform in $t_0 \ge 0$, since $\|u(t_0/2)\|^2_{H^*(\Omega)} \le \|u_0\|^2_{H^*(\Omega)}$.

Proof. We follow the strategy of Lemma 8.1. of [109] who, in turn, gives a generalized result of the the original one in [9]. We would like to use $\xi(t)\partial_t u^m(t,x)$ as test function in the $H^*(\Omega)$ formulation of the problem (see Definition 3.3.11), where $\xi(t) \in C_c^\infty(0,\infty)$. However, a priori we do not know if $\partial_t u^m(t,x)$ is sufficiently regular for this purpose so, following the idea in [9], we will use the Steklov averages.

For any $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ we define the Steklov average as

$$g^h(t,x) = \frac{1}{h}\int_t^{t+h} g(\tau,x)\,\mathrm{d}\tau, \qquad \text{so that} \qquad \partial_t g^h(t,x) = \frac{g(t+h,x)-g(t+h,x)}{h}\,.$$

We already know that a GF solution satisfies the equation pointwise in the $H^*(\Omega)$ sense and, consequently, so does its Steklov average, i.e. for every $t \in (0,T)$

$$\partial_t u^h = -\mathcal{L}[(u^m)^h] \qquad \text{a.e. in } \Omega$$

with $\partial_t u^h, \mathcal{L}[(u^m)^h] \in H^*(\Omega)$. Now, we multiply the above equation by a suitable test function ψ and we integrate in time and space to get the following expression

$$\int_0^T \int_\Omega u_t^h \psi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega \mathcal{L}(u^m)^h \psi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \psi \in C_c^1([0,T], H(\Omega)).$$

At this point, notice that $\partial_t(u^m)^h$ has the same regularity as u^m, namely, by Lemma 3.2.10 we have that $u^m \in H(\Omega)$ so, therefore, we can take $\psi = \xi\partial_t(u^m)^h$ as test function in the equation above, where $\xi \in C_c^\infty((t_0/2,T))$, $0 \le \xi \le 1$ and $\xi = 1 \in [t_0,t_1]$ for $0 \le t_0 \le t_1 \le T$. Since everything is well defined now, we have

$$\begin{aligned}\int_0^T \int_\Omega \partial_t(u^h)\partial_t(u^m)^h \xi \,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{2}\int_0^T \int_\Omega \xi\frac{\partial}{\partial t}|\mathcal{L}^{1/2}[(u^m)^h]|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^T \int_\Omega \partial_t\xi\ |\mathcal{L}^{1/2}[(u^m)^h]|^2 \,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Then, we apply numerical inequality (3.6.12) on the left hand side to get

$$\begin{aligned}\partial_t(u^h)\partial_t(u^m)^h &= \frac{1}{h^2}\left[u(t+h)-u(t)\right]\left[u^m(t+h)-u(t)^m\right] \\ &\ge \frac{4m}{(1+m)^2}\frac{1}{h^2}\left[u^{\frac{1+m}{2}}(t+h)-u(t)^{\frac{1+m}{2}}\right]^2 = \frac{4m}{(1+m)^2}|\partial_t(u^{\frac{1+m}{2}})^h|^2.\end{aligned}$$

Now, notice that $\|\partial_t \xi\|_{L^\infty(\frac{t_0}{2},t_1)} \le c(\frac{2}{t_0} + \frac{1}{T-t_1})$ for every $t_0 < t_1$ in $(0,T)$, so that

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_\Omega |\partial_t (u^{\frac{1+m}{2}})^h|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \frac{(1+m)^2}{8m}\|\partial_t\xi\|_{L^\infty(\frac{t_0}{2},t_1)} \int_{t_0/2}^{t_1}\int_\Omega \partial_t |\mathcal{L}^{1/2}[(u^m)^h]|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \frac{(1+m)^2}{8m}\|\partial_t\xi\|_{L^\infty(\frac{t_0}{2},t_1)} \frac{1}{h^2}\int_{t_0/2}^{t_1}\int_t^{t+h}\int_t^{t+h}\int_\Omega u^m(\tau)\mathcal{L}u^m(\zeta)\,\mathrm{d}x\,\mathrm{d}\tau\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad\le \frac{(1+m)^2}{8m}\|\partial_t\xi\|_{L^\infty(\frac{t_0}{2},t_1)} \frac{1}{h^2}\int_{t_0/2}^{t_1}\int_t^{t+h}\int_t^{t+h} \|u(\tau)^m\|_{H(\Omega)}\|u(\zeta)^m\|_{H(\Omega)}\,\mathrm{d}\tau\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad\le \frac{(1+m)^2}{8m}\|\partial_t\xi\|_{L^\infty(\frac{t_0}{2},t_1)} \int_{t_0/2}^{t_1} \frac{2}{m(1+m)}\frac{\|u(t_0/2)\|^2_{H^*(\Omega)}}{t^2}\,\mathrm{d}t \\
&\quad\le \frac{c(1+m)\|u(t_0/2)\|^2_{H^*(\Omega)}}{4m^2}\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)
\end{aligned}$$

where we have used Cauchy-Schwarz inequality, in the third line, and Lemma 3.2.10. Finally, the result follows by taking the limit as $h \to 0$ on the left hand side and using Fatou's Lemma. □

Lemma 3.3.9. *Let $u \in H^*(\Omega)$ be a GF solution to the problem* (CDP) *with initial data $0 \le u_0 \in H^*(\Omega)$. Then, u is a strong solution in $L^1_{\Phi_1}(\Omega)$ and for every $t \in (0,T)$,*

$$\|\partial_t u\|_{L^1_{\Phi_1}(\Omega)} \le \frac{2\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)\,t} \le \frac{2\lambda_1^{1/2}\|u_0\|_{H^*(\Omega)}}{(1-m)\,t}.$$

We will adapt the classical proof of [49] for the L^1 case, see also [141]. The main difference in the $L^1_{\Phi_1}$ case, is that we will only use the "half" contractivity for ordered solutions of Lemma 3.3.6, instead of the full contractivity as in original proof of [49].

Proof. We follow the same idea of the proof of Lemma 3.3.7 to get

$$\frac{1}{h}\|u(t+h)-u(t)\|_{L^1_{\Phi_1}(\Omega)} \le \frac{1}{h}\left(\frac{t+h}{t}\right)^{\frac{1}{1-m}} - 1 \;\|u_\lambda(t)\|_{L^1_{\Phi_1}(\Omega)} + \frac{1}{h}\|u_\lambda(t)-u(t)\|_{L^1_{\Phi_1}(\Omega)}.$$

Here, using Lemma 3.3.6 and the decay of $L^1_{\Phi_1}(\Omega)$-norm we arrive to

$$\frac{1}{h}\|u(t+h)-u(t)\|_{L^1_{\Phi_1}(\Omega)} \le \frac{1}{h}\left(\frac{t+h}{t}\right)^{\frac{1}{1-m}} - 1 \;\|u_{\lambda,0}\|_{L^1_{\Phi_1}(\Omega)} + \frac{1}{h}\|u_{\lambda,0}-u_0\|_{L^1_{\Phi_1}(\Omega)}.$$

Then, passing to the limit as in Lemma 3.3.7 we get that the time increments are bounded in $L^1_{\Phi_1}(\Omega)$,

$$\lim_{h\to 0}\int_\Omega \frac{u(t+h,x)-u(t,x)}{h}\;\Phi_1\,\mathrm{d}x \le \frac{2\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)\,t} \le \frac{2\lambda_1^{1/2}\|u_0\|_{H^*(\Omega)}}{(1-m)\,t}, \tag{3.3.5}$$

by (3.3.12), thus, we have that $u \in BV((0,T),L^1_{\Phi_1}(\Omega))$. With this and Lemma 3.3.8, applying Theorem 3.3.5 with $v = u$, we conclude that $\partial_t u \in L^1_{\Phi_1}(\Omega)$. □

The above result implies that the equation is satisfied pointwise as functions of $L^1_{\Phi_1}(\Omega)$. We use this fact to prove the following proposition:

Proposition 3.3.10 (T-contractivity in $L^1_{\Phi_1}$)**.** *Let u and v be two GF solutions of* (CDP) *with initial data $u_0, v_0 \in H^*(\Omega)$. Then, for all $0 \le t_0 \le t_1 < \infty$ we have the following property*

$$\int_\Omega [u(t_1) - v(t_1)]_\pm \Phi_1 \,\mathrm{d}x \le \int\!\!\!\!\!\!\diagup_\Omega [u(t_0) - v(t_0)]_\pm \Phi_1 \,\mathrm{d}x\,.$$

Notice that T-contractivity implies usual contractivity,

$$\|u(t_1) - v(t_1)\|_{L^1_{\Phi_1}(\Omega)} \le \|u(t_0) - v(t_0)\|_{L^1_{\Phi_1}(\Omega)}\,.$$

Proof. It is sufficient to prove the result for $[\,\cdot\,]_+$, the case of $[\,\cdot\,]_-$ being completely analogous. We start from the strong formulation of (CDP). Let us multiply the equation by

$$\psi(t,x) = \mathrm{sign}_+(u-v)\Phi_1 \chi_{[t_0,t_1]}(t)$$

and integrate in time and space. Then, notice that on one hand

$$\begin{aligned}\int_0^T \int\!\!\!\!\!\!\diagup_\Omega (\partial_t u - \partial_t v)\,\psi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_\Omega (u-v)\,\partial_t\big(\mathrm{sign}_+(u-v)\chi_{[t_0,t_1]}\big)\Phi_1 \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Omega [u(t_0) - v(t_0)]_+ \Phi \,\mathrm{d}x + \int\!\!\!\!\!\!\diagup_\Omega [u(t_1) - v(t_1)]_+ \Phi_1 \,\mathrm{d}x\,.\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}&\int\!\!\!\!\!\!\diagup_0^T \int_\Omega (\partial_t u - \partial_t v)\,\psi \,\mathrm{d}x\,\mathrm{d}t = \int\!\!\!\!\!\!\diagup_0^T \int\!\!\!\!\!\!\diagup_\Omega -\mathcal{L}(u^m - v^m)\psi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int\!\!\!\!\!\!\diagup_{t_0}^{t_1} \int\!\!\!\!\!\!\diagup_\Omega \mathcal{L}(u^m - v^m)\mathrm{sign}_+(u-v)\Phi_1 \;\mathrm{d}x\,\mathrm{d}t \\ &\le -\int_{t_0}^{t_1} \int_\Omega \mathcal{L}\big([u^m - v^m]_+\big)\Phi_1 \,\mathrm{d}x\,\mathrm{d}t = -\lambda_1 \int_{t_0}^{t_1} \int_\Omega [u^m - v^m]_+ \Phi_1 \,\mathrm{d}x\,\mathrm{d}t \le 0\end{aligned}$$

where we have used Kato inequality (3.6.10) for the convex function $[\,\cdot\,]_+$. □

3.3.3 Nonnegative GF solutions are WDS

In this Section, we prove a technical fact needed in Section 3.3.4, in the proof of the existence of WDS. We will show in Lemma 3.3.12, that GF solutions, in the sense of Def. 3.3.1, satisfy a suitable weak formulation, that characterizes an "intermediate" class of solutions that we will call H^*-solutions. Finally, Proposition 3.3.14, shows that nonnegative H^*-solutions are WDS.

We need to introduce an intermediate concept of solution, a weak formulation suitable for GF solutions.

Definition 3.3.11 (H^*-solutions)**.** *$u \in C([0,T] : H^*(\Omega))$ is an* H^*-solution *of* (CDP) *if $u^m \in L^1([0,T] : H(\Omega))$ such that*

$$\int_0^T \int_\Omega u\psi_t \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega u^m \mathcal{L}\psi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \psi \in C_c^1([0,T], H(\Omega))\,,$$

or equivalently,

$$\int_0^T \int_\Omega u\partial_t(\mathcal{L}^{-1}\varphi) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega u^m \varphi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \varphi \in C_c^1([0,T], H^*(\Omega))\,.$$

Notice that both statements are equivalent due to the fact that $\mathcal{L}$ is an isomorphism between $H(\Omega)$ and $H^*(\Omega)$, as remarked in [24] where this concept of solutions has been previously introduced.

We show next that GF solutions satisfy the $H^*(\Omega)$ formulation, which constitutes the link between the two important class of solutions of this chapter, i.e. GF and WDS.

Lemma 3.3.12. *Let $u \in AC_{\mathrm{loc}}((0,\infty); H^*(\Omega))$ be a GF solution of* (CDP) *given by Theorem 3.3.2, then, it is an H^*-solution.*

Proof. Let $\epsilon \in \mathbb{R} \setminus \{0\}$ and $\varphi \in C_c^1([0,T], H^*(\Omega))$, then, from the definition of gradient flow and subdifferential we have that

$$\mathcal{E}(u+\epsilon\varphi) \geq \mathcal{E}(u) + \epsilon\langle -u_t, \varphi\rangle_{H^*\times H^*} \qquad \text{a.e. in } t \in (0,\infty), \quad \forall \varphi \in C_c^1([0,T], H^*(\Omega)).$$

Now, if we compute the Taylor expansion of $|u+\epsilon\varphi|^{m+1}$ around u we get

$$\epsilon \int_\Omega |u|^{m-1}u\varphi \,\mathrm{d}x + o(\epsilon) \geq -\epsilon \int_\Omega u_t \mathcal{L}^{-1}\varphi \,\mathrm{d}x.$$

Here, we divide by ϵ and noticing to the fact that ϵ can be either positive and negative, we find

$$\int_\Omega |u|^{m-1}u\varphi \,\mathrm{d}x = -\int_\Omega u_t \mathcal{L}^{-1}\varphi \,\mathrm{d}x \qquad \text{a.e. in } t \in (0,\infty), \quad \forall \varphi \in C_c^1([0,T], H^*(\Omega)).$$

with $u_t \in H^*(\Omega)$ and therefore $|u|^{m-1}u \in H(\Omega)$. As $x : (0,\infty) \to D(\mathcal{E}) \subset H^*$ is absolutely continuous we have that $u_t, |u|^{m-1}u \in L^1((0,T); H^*)$. Finally, since $\mathcal{L}$ is an isomorphism between H and H^* we have that $\varphi = \mathcal{L}\psi$ for $\psi \in H$, so integrating by parts in time we get

$$\int_0^T \int_\Omega |u|^{m-1}u\, \mathcal{L}\psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega u\psi_t \,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \psi \in C_c^1([0,T], H(\Omega)).$$

Hence u is a H^* solution. □

Lemma 3.3.13. *Let u be a nonnegative GF solution with $u_0 \in L^1_{\Phi_1}(\Omega)$. Then, for every $t_1 \geq t_0 \geq 0$ we have*

$$0 \leq \int_\Omega (u(t_0) - u(t_1))\Phi_1 \,\mathrm{d}x \leq c_m \|u_0\|^m_{L^1_{\Phi_1}(\Omega)}(t_1 - t_0)\,, \tag{3.3.6}$$

with $c_m = \lambda_1 \|\Phi_1\|_1^{1-m}$. Moreover, letting $T = T(u_0)$ be the extinction time, we get

$$\|u(t)\|_{L^1_{\Phi_1}(\Omega)} \leq \mathsf{c}_1 (T-t)^{\frac{1}{1-m}}\,. \tag{3.3.7}$$

Proof. Let us derive the $L^1_{\Phi_1}$-norm of the solution and use Hölder inequality,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\Phi_1\,\mathrm{d}x = -\iint_\Omega \mathcal{L}u^m\Phi_1\,\mathrm{d}x = -\lambda_1\iint_\Omega u^m\Phi_1\,\mathrm{d}x \\ \geq -\lambda_1\|\Phi_1\|_1^{1-m}\left(\iint_\Omega u(t)\Phi_1\,\mathrm{d}x\right)^m .\end{aligned} \tag{3.3.8}$$

We apply the decay of $L^1_{\Phi_1}(\Omega)$ on the right hand side and we integrate in (t_0,t_1) to get

$$\|u(t_1)\|_{L^1_{\Phi_1}(\Omega)} - \|u(t_0)\|_{L^1_{\Phi_1}(\Omega)} \geq -\lambda_1\|\Phi_1\|_1^{1-m}\|u_0\|^m_{L^1_{\Phi_1}(\Omega)}(t_1-t_0)$$

(3.3.6) follows by the nonnegativity of $u(t)$. For the extinction rate, we just integrate (3.3.8) on $[t,T]$. □

Proposition 3.3.14 (Nonnegative GF solutions are WDS)**.** *Let u be a GF solution of* (CDP) *with initial datum $0 \leq u_0 \in H^*(\Omega)$. Then, $u(t)$ is a WDS, according to Definition 3.1.1.*

Proof. By Lemma 3.3.12, we have that $u(t)$ is a H^* solution. Therefore, let us prove that the set of test functions in H^* solutions is bigger than in the case of weak dual solution. Since $\|\Phi_1\|_{L^2}=1$, we have

$$\|\psi\|^2_{H^*(\Omega)} = \iint_\Omega \psi\mathcal{L}^{-1}\psi\,\mathrm{d}x \leq \lambda_1^{-1}\|\psi/\Phi\|^2_\infty.$$

Moreover, by (3.3.12) we have that non negative solutions in H^* are indeed in the natural space for WDS:

$$\|u\|_{L^1_{\Phi_1}(\Omega)} \leq \lambda_1^{1/2}\|u\|_{H^*(\Omega)}.$$

Then, we have to verify that the next equality is true

$$\iint_0^T\int_\Omega u\partial_t(\mathcal{L}^{-1}\psi)\,\mathrm{d}x\,\mathrm{d}t = \iint_0^T\iint_\Omega \mathcal{L}^{-1}u\partial_t(\psi)\,\mathrm{d}x\,\mathrm{d}t \quad \forall\frac{\psi}{\Phi_1}\in C^1_c([0,T],L^\infty(\Omega)). \tag{3.3.9}$$

Indeed, we have

$$\|\partial_t\psi\|^2_{H^*(\Omega)} = \int_\Omega \partial_t\psi\mathcal{L}^{-1}(\partial\psi)\,\mathrm{d}x \leq \|\partial_t\psi/\Phi_1\|_\infty\iint_\Omega \Phi_1\mathcal{L}^{-1}(\partial_t\psi)\,\mathrm{d}x \leq \lambda_1^{-1}\|\partial_t\psi/\Phi_1\|^2_\infty$$

and (3.3.9) holds. It remains to prove that $u\in C((0,T]:L^1_{\Phi_1}(\Omega))$. From the time monotonicity of Lemma 3.3.7, we deduce that

$$u(t_1)-u(t_0) \leq \quad 1-\left(\frac{t_0}{t_1}\right)^{\frac{1}{1-m}} u(t_1) \qquad \forall t_1\geq t_0>0 \text{ and a.e. } x\in\Omega\,,$$

and hence we have the following estimate for the positive part,

$$[u(t_1)-u(t_0)]_+ \leq \frac{t_1^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}}\,u(t_1) \qquad \forall t_1\geq t_0>0 \text{ and a.e. } x\in\Omega\,. \tag{3.3.10}$$

Recalling that $|f| = 2[f]_+ - f$ for any function, let us bound $\|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\Omega)}$ as follows

$$\begin{aligned}\|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\Omega)} &= 2\int_\Omega [u(t_1) - u(t_0)]_+ \,\Phi_1 \,\mathrm{d}x - \iint_\Omega (u(t_1) - u(t_0))\,\Phi_1 \,\mathrm{d}x \\ &\le \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \int_\Omega u(t_1)\Phi_1 \,\mathrm{d}x + c_m \|u_0\|^m_{L^1_{\Phi_1}(\Omega)}(t_1 - t_0)\,,\end{aligned} \tag{3.3.11}$$

where we have used (3.3.10) and Lemma 3.3.13 to conclude. □

3.3.4 Existence of nonnegative MWDS

In this section we prove the existence of nonnegative MWDS by approximating WDS from below with bounded GF solutions, which reads:

Theorem 3.3.15 (Existence and uniqueness of nonnegative MWDS in $L^1_{\Phi_1}$)**.** *Let* (A1) *and* (A2) *hold. Then, for every* $0 \le u_0 \in L^1_{\Phi_1}(\Omega)$*, there exist a unique minimal weak dual solution* u *of* (CDP) *with*

$$\lim_{t\to 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}(\Omega)} = 0 \quad \textit{and} \quad \lim_{h\to 0^+} \left\| \frac{u(t+h) - u(t)}{h} \right\|_{L^1_{\Phi_1}(\Omega)} \le \frac{2\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)\,t}\,.$$

Moreover, the T-contraction estimates hold: let $0 \le u_0, v_0 \in L^1_{\Phi_1}(\Omega)$ *and* $u(t), v(t)$ *be the corresponding MWDS, then, for all* $t \ge 0$ *we have*

$$\|(u(t) - v(t))_\pm\|_{L^1_{\Phi_1}(\Omega)} \le \|(u_0 - v_0)_\pm\|_{L^1_{\Phi_1}(\Omega)}\,.$$

Remark 3.3.16. (i) As for the previous Theorem, the T-contraction implies comparison and uniqueness in $L^1_{\Phi_1}$ (via contractivity), but we stress on the fact that our proof only guarantees the validity of these properties for the MWDS, a priori we can not exclude that $u_0 \in L^1_{\Phi_1}$ produces other WDS.
(ii) A closer inspection of the proof reveals that the MWDS does not depend on the particular choice of the approximating sequence, we only require the approximating sequence to be monotone increasing, see Step 4 of the proof of Theorem 3.3.15 for more details.
(iii) The question of uniqueness is a delicate issue in this $L^1_{\Phi_1}$-setting, since the energy functional is

$$\mathcal{F}[u] = \iint_\Omega u^m \Phi_1 \,\mathrm{d}x$$

which is clearly *not convex* when $m \in (0,1)$. A full uniqueness result for WDS is still missing. This is an intriguing open problem and it would imply that all WDS are minimal (hence a priori more regular). This lack of convexity makes it impossible to apply Crandall-Liggett type Theorems, as done in the L^1-setting in [10]. See [71, 134] for related uniqueness results on $\mathbb{R}^N$.

(iv) *A larger "existence" class of nonnegative solutions.* When we deal with nonnegative functions, $H^*(\Omega) \subset L^1_{\Phi_1}(\Omega)$, indeed by Cauchy-Schwartz inequality and the normalization $\|\Phi_1\|_2 = 1$, we get

$$\begin{aligned} \|u\|_{L^1_{\Phi_1}(\Omega)} = \int_\Omega u\,\Phi_1\,\mathrm{d}x &= \iint_\Omega \mathcal{L}^{-\frac{1}{2}}(u)\,\mathcal{L}^{\frac{1}{2}}(\Phi_1)\,\mathrm{d}x \\ &\leq \left\| \mathcal{L}^{-\frac{1}{2}} u \right\|_2 \left\| \mathcal{L}^{\frac{1}{2}} \Phi_1 \right\|_2 = \lambda_1^{\frac{1}{2}} \|u\|_{H^*(\Omega)}\,. \end{aligned} \tag{3.3.12}$$

To the best of our knowledge, *the largest class of nonnegative data for which existence of solutions to the* (CDP) *is guaranteed, is precisely* $L^1_{\Phi_1}$.

Proof of Theorem 3.3.15. The proof is divided into several steps.

• Step 1. *Existence of the minimal limit solution..* We approximate $0 \leq u_0 \in L^1_{\Phi_1}(\Omega)$ from below with the non-decreasing and bounded sequence $0 \leq u_{0,n} = u_0 \wedge n \in L^\infty(\Omega)$. Recall that the integrability of $\mathbb{G}_\Omega$ implies that $L^\infty(\Omega) \subset H^*(\Omega)$, therefore $u_{0,n} \in H^*(\Omega)$ for all $n \geq 1$. By monotone convergence, we have that $u_{0,n} \to u_0$ in the strong $L^1_{\Phi_1}$ topology, namely

$$0 \leq \|u_0 - u_{0,n}\|_{L^1_{\Phi_1}(\Omega)} = \iint_\Omega (u_0 - u_{0,n})\Phi_1\,\mathrm{d}x \xrightarrow{n\to+\infty} 0\,. \tag{3.3.13}$$

Now, we are in the position to apply Theorem 3.3.2 which provides us the existence of the nonnegative GF solution $u_n(t)$ in $H^*(\Omega)$ starting at $u_{0,n}$. Since the L^∞ norm is non-increasing in time (just let $p \to \infty$ in (3.4.15)), we know that $u_n(t) \in L^\infty(\Omega)$. Next, Lemma 3.3.14 tells us that u_n is also a WDS corresponding to the initial datum $u_{0,n} \in L^\infty(\Omega)$. Finally, since GF solutions satisfy the comparison principle, we have that $u_n(t,x) \leq u_{n+1}(t,x)$ for a.e. $(t,x) \in (0,\infty)\times\Omega$.

Since for any fixed $t > 0$ and a.e. $x \in \Omega$ the sequence $u_n(t,x)$ is monotone increasing in n, the limit exists (can be $+\infty$) and we can define our candidate limit solution as follows

$$u(t,x) := \liminf_{n\to\infty} u_n(t,x) \qquad \text{for a.e. } (t,x) \in (0,\infty)\times\Omega\,.$$

We shall see that this limit solution defines a continuous path in $t \mapsto u(t) \in L^1_{\Phi_1}(\Omega)$. *Convergence in* $L^1_{\Phi_1}$. Notice that for all $t > 0$ we have that (take $k > n$ so that $u_k \geq u_n$)

$$\begin{aligned} \|u(t) - u_n(t)\|_{L^1_{\Phi_1}(\Omega)} &\leq \liminf_{k\to\infty} \|u_k(t) - u_n(t)\|_{L^1_{\Phi_1}(\Omega)} \leq \liminf_{k\to\infty} \|u_k(0) - u_n(0)\|_{L^1_{\Phi_1}(\Omega)} \\ &= \|u_0 - u_{0,n}\|_{L^1_{\Phi_1}(\Omega)} \xrightarrow{n\to+\infty} 0\,, \end{aligned} \tag{3.3.14}$$

where the first inequality follows by Fatou's lemma, in the second step we have used the $L^1_{\Phi_1}$ contraction for ordered WDS, and finally that $u_{0,n}$ and $u_k(0) = u_{0,k}$ tend to u_0 strongly in $L^1_{\Phi_1}(\Omega)$ by (3.3.13).

• Step 2. $u \in C((0,T], L^1_{\Phi_1}(\Omega))$. By Proposition 3.3.14, we know that $u_n \in C((0,T] : L^1_{\Phi_1}(\Omega))$, indeed, by (3.3.11) and the decay of $\|u_n(t)\|_{L^1_{\Phi_1}(\Omega)}$, we have that

for every $0 < t_0 \le t_1 \le T$

$$\|u_n(t_1) - u_n(t_0)\|_{L^1_{\Phi_1}(\Omega)} \le \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)} + c_m (t_1 - t_0)^{\frac{1}{1-m}} .$$

Then, since $u_{0,n} \to u_0$ strongly in $L^1_{\Phi_1}(\Omega)$ as explained in Step 1, we have that by Fatou's Lemma

$$\begin{aligned} \|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\Omega)} &\le \lim_{n\to\infty} \|u_n(t_1) - u_n(t_0)\|_{L^1_{\Phi_1}(\Omega)} \\ &\le \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \|u_0\|_{L^1_{\Phi_1}(\Omega)} + c_m (t_1 - t_0)^{\frac{1}{1-m}} , \end{aligned} \tag{3.3.15}$$

hence $u \in C((0,T], L^1_{\Phi_1}(\Omega))$ as desired. Notice that the above estimate just gives uniform boundedness of the limiting Radon measure when $t_0 = 0$, and it is not sufficient to conclude continuity on $[0,T]$.

• STEP 3. *The limit solution is a WDS.* Let us prove that the limit solution u constructed above is a WDS in the sense of Definition 3.1.1. Indeed, we shall show that it satisfies the weak formulation (WDS), which amounts to check that

$$\lim_{n\to\infty} \underbrace{\int_0^T \int_\Omega \mathcal{L}^{-1}(u - u_n)\, \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t}_{(I)} = \lim_{n\to\infty} \underbrace{\int_0^T \int_\Omega (u^m - u_n^m)\psi \, \mathrm{d}x \, \mathrm{d}t}_{(II)} = 0 .$$

First, notice that (I) vanishes as $n \to \infty$ as a consequence of (3.3.14)

$$\begin{aligned} (I) &\le \left\| \frac{\partial_t \psi}{\Phi_1} \right\|_\infty \int_0^T \int_\Omega \mathcal{L}^{-1}(u - u_n)\, \Phi_1 \, \mathrm{d}x \, \mathrm{d}t \\ &\le \lambda_1^{-1} \left\| \frac{\partial_t \psi}{\Phi_1} \right\|_\infty \int_0^T \int_\Omega (u(t) - u_n(t)) \Phi_1 \, \mathrm{d}x \, \mathrm{d}t \xrightarrow{n\to+\infty} 0 . \end{aligned}$$

On the other hand, by Proposition 3.3.10, we have that following the same argument as for (I),

$$\begin{aligned} (II) &\le \left\| \frac{\psi}{\Phi_1} \right\|_\infty \int_0^T \int_\Omega (u^m - u_n^m) \Phi_1 \, \mathrm{d}x \, \mathrm{d}t \le \left\| \frac{\psi}{\Phi_1} \right\|_\infty \int_0^T \int_\Omega (u - u_n)^m \Phi_1 \, \mathrm{d}x \, \mathrm{d}t \\ &\le \left\| \frac{\psi}{\Phi_1} \right\|_\infty \|\Phi_1\|_1^{1-m} \int_0^T \left(\int_\Omega (u(t) - u_n(t)) \Phi_1 \, \mathrm{d}x \right)^m \mathrm{d}t \xrightarrow{n\to+\infty} 0 . \end{aligned}$$

We conclude that $u \in C((0,T], L^1_{\Phi_1}(\Omega))$ is a MWDS.

• STEP 4. *The initial datum is taken strongly in $L^1_{\Phi_1}$.* Once we have shown that our limit solution is a WDS according to Definition 3.1.1, we need to show that it takes the initial datum strongly in $L^1_{\Phi_1}$, that is $u \in C([0,T], L^1_{\Phi_1}(\Omega))$ satisfying the limit $\lim_{t\to 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}} = 0$. Actually it is enough to check the latter which in turn implies the $L^1_{\Phi_1}$-continuity at zero. Let us add and substract in $\|u(t) - u_0\|_{L^1_{\Phi_1}(\Omega)}$ the truncation of the initial data $u_{0,n} \in L^\infty(\Omega) \cap H^*(\Omega)$ and the associated GF solutions $u_n(t) \in L^\infty(\Omega) \cap H^*(\Omega)$:

$$\begin{aligned}\iint_\Omega |u(t)-u_0|\Phi_1\,\mathrm{d}x &\\ &\le \int |u(t)-u_n(t)|\Phi_1\,\mathrm{d}x + \int |u_n(t)-u_{n,0}|\Phi_1\,\mathrm{d}x + \iint_\Omega |u_{0,n}-u_0|\Phi_1\,\mathrm{d}x\\ &= (I)+(II)+(III).\end{aligned}$$

Now let us analyse each addend in order to pass to the limit. Since $u_n(t)\le u(t)$, by Lemma 3.3.6 we have that

$$(I)=\int_\Omega (u(t)-u_n(t))\Phi_1\,\mathrm{d}x \le \iint_\Omega (u_0-u_{0,n})\Phi_1\,\mathrm{d}x.$$

Moreover, by construction it holds that $u_{0,n}\le u_0$, therefore

$$(III)=\iint_\Omega (u_0-u_{0,n})\Phi_1\,\mathrm{d}x\,.$$

On the other hand, let us estimate (II) using that $u_n(t)\rightharpoonup u_{n,0}$ in $H^*(\Omega)$ as $t\to 0^+$ by definition of GF solution and $\|u_n(t)\|_2\le\|u_{n,0}\|_2$ for all $t>0$.
First, we prove that weak convergence in $H^*(\Omega)$ implies weak convergence in $L^2(\Omega)$ using that $H(\Omega)$ is dense in $L^2(\Omega)$. Given any function $\psi\in L^2(\Omega)$ consider a sequence $\{\varphi_k\}_{k\ge1}\subset H(\Omega)$ satisfying $\varphi_k\to\psi$ in $L^2(\Omega)$. Then,

$$\begin{aligned}\int_\Omega (u_n(t)-u_{n,0})\psi\,\mathrm{d}x &= \int_\Omega (u_n(t)-u_{n,0})\varphi_k\,\mathrm{d}x + \iint_\Omega (u_n(t)-u_{n,0})(\psi-\varphi_k)\,\mathrm{d}x\\ &\le \iint_\Omega (u_n(t)-u_{n,0})\varphi_k\,\mathrm{d}x + \|u_n(t)-u_{n,0}\|_2\|\psi-\varphi_k\|_2\\ &\le \int_\Omega (u_n(t)-u_{n,0})\varphi_k\,\mathrm{d}x + 2\|u_{n,0}\|_2\,\|\psi-\varphi_k\|_2\,.\end{aligned}$$

If we take limit when $t\to0^+$ we obtain

$$\lim_{t\to0^+}\iint_\Omega (u_n(t)-u_{n,0})\psi\,\mathrm{d}x \le 2\|u_{n,0}\|_2\,\|\psi-\varphi_k\|_2 \qquad \forall k\ge1\,,$$

since $u_n(t)\rightharpoonup u_{n,0}$ in $H^*(\Omega)$. Letting $k\to\infty$ above gives the weak-$L^2(\Omega)$ convergence of $u_n(t)$ to $u_{n,0}$ as $t\to0^+$. Next we prove convergence of $L^2(\Omega)$-norms, namely that $\|u_n(t)\|_2\to\|u_{n,0}\|_2$ as $t\to0^+$, using the weak lower semicontinuity of the norm: since $u_n(t)\rightharpoonup u_{n,0}$ in $L^2(\Omega)$ and $\|u_n(t)\|_2\le\|u_{n,0}\|_2$, we have

$$\|u_{n,0}\|_2\le\liminf_{t\to0^+}\|u_n(t)\|_2\le\limsup_{t\to0^+}\|u_n(t)\|_2\le\|u_{n,0}\|_2\,.$$

This implies strong-$L^2(\Omega)$ convergence of $u_n(t)$ to $u_{n,0}$ as $t\to0^+$, since in any Hilbert space, weak convergence plus convergence of norms implies strong convergence. Then, we estimate (II) as follows

$$(II)=\iint_\Omega |u_n(t)-u_{n,0}|\Phi_1\,\mathrm{d}x\le\|u_n(t)-u_{n,0}\|_2\|\Phi_1\|_2=\|u_n(t)-u_{n,0}\|_2\xrightarrow{t\to0^+}0\,. \tag{3.3.16}$$

Finally, putting together the estimates of (I), (II) and (III) we have shown that

$$\int_\Omega |u(t)-u_0|\Phi_1\,\mathrm{d}x \le 2\iint_\Omega (u_0-u_{n,0})\Phi_1\,\mathrm{d}x + \|u_n(t)-u_{n,0}\|_2\,,$$

which implies by (3.3.16)

$$\lim_{t\to 0^+}\int_\Omega |u(t)-u_0|\Phi_1\,\mathrm{d}x \le 2\iint_\Omega (u_0-u_{n,0})\Phi_1\,\mathrm{d}x \qquad \forall n\ge 1\,.$$

We conclude taking limit when $n\to+\infty$ and applying (3.3.13).

• Step 5. *Independence of approximating sequence and uniqueness of MWDS.* Let us see that MWDS does not depend on the particular choice of the approximating sequence, we only require the approximating sequence to be monotone increasing. We adapt here the proof of Theorem 3.2 of [28] for reader's convenience. Let us choose another nonnegative monotone sequence $\{v_{0,k}\}\in L^\infty(\Omega)$ which converges to u_0 from below. Repeating the construction of the previous steps, there exists a weak dual solution $v(t,x)\in C\big([0,T]:L^1_{\Phi_1}(\Omega)\big)$ generated by the GF solutions starting from $v_{0,k}$. We want to prove that $u=v$, i.e. that the minimal weak dual solution is unique. For this, we start by proving that $u\ge v$ and then we prove the opposite inequality. First, let us prove that $u\ge v$. By Proposition 3.3.10, we have that

$$\lim_{n\to\infty}\iint_\Omega [v_k-u_n]_+\Phi_1 \le \lim_{n\to\infty}\int_\Omega [v_{0,k}-u_{0,n}]_+\Phi_1 \le \iint_\Omega [v_{0,k}-u_0]_+\Phi_1 = 0,$$

as $v_{0,k}\le u_0$ by construction. This gives that $v_k\le u$ for all k, so taking the limit we get that $v\le u$. The opposite inequality follows by exchanging the roles of u and v.

Remark. Notice that we have proven uniqueness of MWDS, which does not exclude a priori that there can be other WDS starting from u_0 which are not obtained as limit of a nondecreasing sequence of GF solutions. Strong uniqueness is known so far only for strong WDS solutions, c.f. Theorem 3.3.17.

• Step 6. *T-contractivity for MWDS. Comparison and Contractivity.* We approximate again $u_0\in L^1_{\Phi_1}(\Omega)$ by the same sequence as in the uniqueness proof u_n so by Lemma 3.3.10 we have that

$$\lim_{n\to\infty}\int_\Omega [u_n(t_1)-v_n(t_1)]_\pm\Phi_1 \le \lim_{n\to\infty}\iint_\Omega [u_{0,n}-v_{0,n}]_\pm\Phi_1.$$

The strong L^1 convergence of $u_{0,n},v_{0,n}$ implies the fact that $\iint_\Omega [u_{0,n}-v_{0,n}]_+\Phi_1\to\iint_\Omega [u_0-v_0]_+\Phi_1$, and Fatou's Lemma applied on the left-hand side allow to conclude. Consequently, we have contractivity for MWDS, hence, uniqueness in $L^1_{\Phi_1}$ of MWDS, as solutions get closer as t increases. Moreover, by T-contractivity we have that comparison holds for MWDS, i.e. if $u_0(x)\le v_0(x)$ a.e $x\in\Omega$ then

$$0\le\int_\Omega [u-v]_+\Phi_1 \le \iint_\Omega [u_0-v_0]_+\Phi_1 = 0,$$

which implies that $u(t)\le v(t)$ a.e $x\in\Omega$, as $\Phi_1>0$ in the interior of Ω.

• STEP 7. *Boundedness of incremental quotient of u* . Again, for the approximating sequence, notice that before passing to the limit $h \to 0$ in (3.3.5) we have

$$\iint_\Omega \frac{u_n(t+h,x)-u_n(t,x)}{h}\ \Phi_1\,\mathrm{d}x \le \frac{1}{h}[(t+h)^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}]\frac{2\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}}{t^{\frac{1}{1-m}}}\,.$$

Now, we pass to the limit as $n \to \infty$ on the right hand side and on the left by Fatou's lemma. The rest follows by the same argument as in the last part of the proof of Lemma 3.3.9. □

3.3.5 Existence of $L^1_{\Phi_1}$-strong solutions and uniqueness

We prove now in which cases MWDS are actually $L^1_{\Phi_1}$-strong solutions and satisfy the equation $u_t = -\mathcal{L}u^m$ pointwise as functions of $L^1_{\Phi_1}$. Indeed, this also provides T-contractivity hence strong uniqueness in $L^1_{\Phi_1}$. All of this is summarized in the following theorem:

Theorem 3.3.17 ($L^1_{\Phi_1}$-strong solutions)**.** *Let* (A1) *and* (A2) *hold. Then, the MWDS u corresponding to the initial datum* $0 \le u_0 \in L^1_{\Phi_1}(\Omega)$*, is moreover a* $L^1_{\Phi_1}$*-strong solution, with the bound*

$$\|\partial_t u(t)\|_{L^1_{\Phi_1}(\Omega)} \le \frac{2\,\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t} \qquad \forall t>0\,,$$

whenever one of the following additional conditions hold:

1. *Let* $N > 2s > \gamma$ *and either* (K1) *or* (M1) *hold. Moreover, assume* $u_0 \in L^p(\Omega)$ *with* $p \ge 1$ *if* $m \in (m_c, 1)$ *or* $p > p_c$ *if* $m \in (0, m_c]$*.*
2. *Let* $N, \gamma > 2s$*,* $m \in (0, \frac{2s}{\gamma})$ *and either* (K1) *or* (M1) *hold. Assume moreover that* $u_0 \in L^p(\Omega)$ *with* $p \ge 1$ *if* $m \in (m_c, 1)$ *or* $p > p_c$ *if* $m \in (0, m_c]$*.*
3. *Let* $N > 2s > \gamma$ *and* (K2) *hold. Assume moreover that* $u_0 \in L^p_{\Phi_1}(\Omega)$ *with* $p \ge 1$ *if* $m \in (m_{c,\gamma}, 1)$ *or* $p > p_{c,\gamma}$ *if* $m \in (0, m_{c,\gamma}]$*.*

Remark 3.3.18. *Essentially the above theorem says that MWDS are strong when they are bounded. Since the* $L^1 - L^\infty$ *smoothing effects are not true for* $L^1_{\Phi_1}$ *data when m is close to zero, it is therefore quite natural to expect that we need some extra* L^p *integrability on the initial datum to obtain bounded -hence strong- solutions, as it will be made precise below.*

Proof of Theorem 3.3.17. In the three cases we follow the same steps. We use Theorem 3.3.5 for the MWDS in order to prove that $\partial_t u$ is a integrable function, more precisely that $t\partial_t u \in L^\infty((0,T) : L^1_{\Phi_1}(\Omega)))$. We have to check if the assumptions of Theorem 3.3.5 are satisfied, namely, if the solution satisfies $u \in BV((0,T) : L^1_{\Phi_1}(\Omega))$ and $u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$.

• STEP 1. $u \in BV((0,T) : L^1_{\Phi_1}(\Omega))$: The former condition follows by applying Lemma 3.3.9 to the monotone increasing GF solutions $u_n(t)$ that approximate the MWDS $u(t)$ and passing to the limit,

$$\|\partial_t u\|_{L^1_{\Phi_1}(\Omega)} \le \lim_{n\to\infty} \|\partial_t u_n\|_{L^1_{\Phi_1}(\Omega)} \le \lim_{n\to\infty} \frac{2\,\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t} = \frac{2\,\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t}$$

since we use Fatou's Lemma and $u_{0,n} \to u_0$ in $L^1_{\Phi_1}(\Omega)$.

• STEP 2. $u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$: On the other hand, the latter condition will be satisfied by Lemma 3.3.8, but first we have to bound the $\|u_n(t_0/2)\|_{H^*(\Omega)}$ in terms of $\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}$ in order to pass to the limit. Thus, let study the three cases of Theorem 3.3.17:

1. Let $u_n(t)$ be a GF solution, then by Theorem 3.5.1, recalling the fact that $\mathcal{L}^{-1}\Phi_1 = \lambda_1^{-1}\Phi_1$, we have

$$\begin{aligned}
\|u_n(t)\|^2_{H^*(\Omega)} &= \int_\Omega u_n\,\mathcal{L}^{-1}u_n\,\mathrm{d}x \le \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}}\Big\|_\infty \int_\Omega (\mathcal{L}^{-1}u_n)\Phi_1\,\mathrm{d}x \\
&= \lambda_1^{-1}\,\frac{u_n(t)}{\Phi_1^{\frac{1}{m}}}\Big\|_\infty \|u_n(t)\|_{L^1_{\Phi_1}(\Omega)} \le \overline{\kappa}\,\lambda_1^{-1}\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}\frac{\|u_{0,n}\|_p^{\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}} \\
&\le \overline{\kappa}\,\lambda_1^{-1}\,\|\Phi_1\|_{\frac{p}{p-1}}\,\frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}}
\end{aligned}$$

where we have used the decay of the $L^1_{\Phi_1}$-norm and Hölder's inequality. Hence, combining the above estimate with inequality (3.3.4) of Lemma 3.3.8, we obtain

$$\begin{aligned}
\|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0,t_1):L^2(\Omega))} &\le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\|u_n(t_0/2)\|^2_{H^*(\Omega)} \\
&\le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\overline{\kappa}\lambda_1^{-1}\|\Phi_1\|_{\frac{p}{p-1}}\,\frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{(t_0/2)^{\frac{1+N\vartheta p}{m}}}\,.
\end{aligned}$$

We can let $n \to \infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n} \to u_0$ in $L^p(\Omega)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$.

2. Let $u_n(t)$ be a GF solution, then by Theorem 3.5.1 with $m < \frac{2s}{\gamma} < 1$ and $\mathcal{L}^{-1}\Phi_1 = \lambda_1^{-1}\Phi_1$, we have

$$\|u_n(t)\|^2_{H^*(\Omega)} = \int_\Omega u_n\,\mathcal{L}^{-1}u_n\,\mathrm{d}x \le \frac{u_n(t)}{\Phi_1^{\frac{2s}{\gamma m}}}\ {}_\infty\!\int\!\!\!\!\int_\Omega \Phi_1^{\frac{2s}{\gamma m}}\mathcal{L}^{-1}u_n\,\mathrm{d}x$$

$$\le \|\Phi_1\|_\infty^{\frac{2s}{\gamma m}-1}\ \frac{u_n(t)}{\Phi_1^{\frac{2s}{\gamma m}}}\ {}_\infty\!\int\!\!\!\!\int_\Omega \Phi_1\mathcal{L}^{-1}u_n\,\mathrm{d}x \le \lambda_1^{-1}\|\Phi_1\|_\infty^{\frac{2s}{\gamma m}-1}\|u_n(t)\|_{L^1_{\Phi_1}(\Omega)}\ \frac{u_n(t)}{\Phi_1^{\frac{2s}{\gamma m}}}\ {}_\infty$$

$$\le \overline{\kappa}\,\lambda_1^{-1}\|\Phi_1\|_\infty^{\frac{2s}{\gamma m}-1}\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}\frac{\|u_{0,n}\|_p^{\frac{2sp\vartheta_p}{m}}}{t^{\frac{1+N\vartheta_p}{m}}} \le \overline{\kappa}\lambda_1^{-1}\|\Phi_1\|_\infty^{\frac{2s}{\gamma m}-1}\|\Phi_1\|_{\frac{p}{p-1}}\ \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta_p}{m}}}{t^{\frac{1+N\vartheta_p}{m}}}$$

$$\le \overline{\kappa}\,\lambda_1^{-1}\ \|\Phi_1\|_\infty^{\frac{2s}{\gamma m}}|\Omega|^{\frac{p-1}{p}}\ \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta_p}{m}}}{t^{\frac{1+N\vartheta_p}{m}}},$$

where we have used the decay of the $L^1_{\Phi_1}$-norm and Hölder's inequality. Hence, combining the above estimate with inequality (3.3.4) of Lemma 3.3.8, we obtain

$$\|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0,t_1):L^2(\Omega))} \le$$

$$\le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\|u_n(t_0/2)\|^2_{H^*(\Omega)}$$

$$\le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\overline{\kappa}\,\lambda_1^{-1}\ \|\Phi_1\|_\infty^{\frac{2s}{\gamma m}}|\Omega|^{\frac{p-1}{p}}\ \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta_p}{m}}}{(t_0/2)^{\frac{1+N\vartheta_p}{m}}}.$$

We can let $n\to\infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n}\to u_0$ in $L^p(\Omega)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}}\in W^{1,1}((0,T):L^1_{\Phi_1}(\Omega))$.

3. Let $u_n(t)$ be a GF solution, then by Theorem 3.5.1 and recalling the fact that $\mathcal{L}^{-1}\Phi_1=\lambda_1^{-1}\Phi_1$, we have

$$\|u_n(t)\|^2_{H^*(\Omega)} = \int_\Omega u_n\,\mathcal{L}^{-1}u_n\,\mathrm{d}x \le \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}}\ {}_\infty\!\int\!\!\!\!\int_\Omega (\mathcal{L}^{-1}u_n)\Phi_1\,\mathrm{d}x$$

$$= \lambda_1^{-1}\ \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}}\ {}_\infty\|u_n(t)\|_{L^1_{\Phi_1}(\Omega)} \le \overline{\kappa}\,\lambda_1^{-1}\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}\ \frac{\|u_{0,n}\|_{L^p_{\Phi_1}(\Omega)}^{\frac{(2s-\gamma)p\vartheta_{p,\gamma}}{m}}}{t^{\frac{1+N\vartheta_{p,\gamma}}{m}}}$$

$$\le \overline{\kappa}\,\lambda_1^{-1}\ \|\Phi_1\|_1^{\frac{p}{p-1}}\ \frac{\|u_{0,n}\|_{L^p_{\Phi_1}(\Omega)}^{1+\frac{(2s-\gamma)p\vartheta_{p,\gamma}}{m}}}{t^{\frac{1+N\vartheta_{p,\gamma}}{m}}}$$

where we have used the decay of the $L^1_{\Phi_1}$-norm and Hölder's inequality. Notice that in the case $p=1$ we just stop at the second inequality and the Theorem is proven. In the rest of the cases, we combine the above estimate with inequality (3.3.4) of Lemma 3.3.8, to obtain

$$\|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0,t_1):L^2(\Omega))} \le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\|u_n(t_0/2)\|^2_{H^*(\Omega)}$$

$$\le c_m\left(\frac{2}{t_0}+\frac{1}{T-t_1}\right)\left(\frac{2}{t_0}-\frac{1}{t_1}\right)\overline{\kappa}\,\lambda_1^{-1}\ \|\Phi_1\|_1^{\frac{p}{p-1}}\ \frac{\|u_{0,n}\|_{L^p_{\Phi_1}(\Omega)}^{1+\frac{(2s-\gamma)p\vartheta_{p,\gamma}}{m}}}{(t_0/2)^{\frac{1+N\vartheta_{p,\gamma}}{m}}}.$$

We can let $n \to \infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n} \to u_0$ in $L^p(\Omega)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\Omega))$. □

3.4 Energy estimates and Finite Extinction Time

Once we have that solutions exist and are bounded for sufficiently smooth initial data, we will prove that they vanish in finite time, with estimates from above and below for the extinction time in different norms. Also we will show sharp time decay estimates as $t \to T^-$ that show how bounded solution extinguish in finite time, and constitute the first step towards the understanding of the asymptotic behaviour, which is a really delicate issue already in the local case, cf. for instance [18].

Some of the results from this section are presented in terms of nonlinear Rayleigh quotients and their decay along the flow:

3.4.1 Nonlinear Rayleigh Quotients and extinction rates

Let us consider the "Dual" Nonlinear Rayleigh Quotient

$$\mathcal{Q}^*[f] = \frac{\|f\|_{1+m}^{1+m}}{\|f\|_{H^*}^{1+m}} = \frac{\int_\Omega |f|^{1+m}\,\mathrm{d}x}{\left(\int_\Omega f\mathcal{L}^{-1}f\,\mathrm{d}x\right)^{\frac{1+m}{2}}}.$$

We shall show that $\mathcal{Q}^*[u(t)]$ is decreasing along the FFDE flow, namely that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}^*[u(t)]\\ &\le -(1+m)\frac{\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)^2}{\left(\int_\Omega u\mathcal{L}^{-1}u\,\mathrm{d}x\right)^{\frac{1+m}{2}+1}} + (1+m)\frac{\left(\int_\Omega u\mathcal{L}^{-1}u\,\mathrm{d}x\right)^{\frac{m-1}{2}}\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)}{\left(\int_\Omega u\mathcal{L}^{-1}u\,\mathrm{d}x\right)^{1+m}} = 0\,,\end{aligned} \tag{3.4.1}$$

since we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\mathcal{L}^{-1}u\,\mathrm{d}x\right)^{\frac{1+m}{2}} = -(1+m)\left(\int_\Omega u\mathcal{L}^{-1}u\,\mathrm{d}x\right)^{\frac{m-1}{2}}\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)($$

and also that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{u^{1+m}}{1+m}\,\mathrm{d}x = \int_\Omega u^m\,\partial_t u\,\mathrm{d}x = -\int_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x = -\int_\Omega \frac{(\mathcal{L}^{1/2}u^m)^2\,(\mathcal{L}^{-1/2}u)^2}{(\mathcal{L}^{-1/2}u)^2}\,\mathrm{d}x\\ \le -\frac{\|(\mathcal{L}^{1/2}u^m)(\mathcal{L}^{-1/2}u)\|_1^2}{\|(\mathcal{L}^{-1/2}u)^2\|_1} \le -\frac{1}{\|u(t_0)\|_{H^*}^2}\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)^2,\end{aligned} \tag{3.4.2}$$

where we have used the decay of the H^*-norm and the Cauchy-Schwarz inequality in the following form

$$\int_\Omega \frac{f^2}{g}\,\mathrm{d}x \ge \frac{\|f\|_1^2}{\|g\|_1}\,. \tag{3.4.3}$$

Let us consider the Nonlinear Rayleigh Quotient

$$\mathcal{Q}[f] = \frac{\|f^m\|^2_{H(\Omega)}}{\|f\|^{2m}_{1+m}} = \frac{\int_\Omega f^m \mathcal{L} f^m \,\mathrm{d}x}{\left(\int_\Omega |f|^{1+m}\,\mathrm{d}x\right)^{\frac{2m}{1+m}}}.$$

As before, $\mathcal{Q}[u(t)]$ decays along the FFDE flow,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}[u(t)] = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^m \mathcal{L} u^m \,\mathrm{d}x\right)}{\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)^{\frac{2m}{1+m}}} + 2m \frac{\left(\int_\Omega u^m \mathcal{L} u^m \,\mathrm{d}x\right)^2}{\left(\int_\Omega u^{1+m}\,\mathrm{d}x\right)^{\frac{2m}{1+m}+1}} \le 0$$

since (3.4.3) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^m \mathcal{L} u^m \,\mathrm{d}x = -2m \int_\Omega \frac{u^{2m}(\mathcal{L}u^m)^2}{u^{1+m}}\,\mathrm{d}x \le -2m\, \frac{\left(\int_\Omega u^m \mathcal{L} u^m \,\mathrm{d}x\right)^2}{\int_\Omega u^{1+m}\,\mathrm{d}x}\,. \tag{3.4.4}$$

The decay of the Nonlinear Rayleigh Quotient $\mathcal{Q}[u(t)]$ provides another proof of the upper estimate of the L^{1+m}-norm without using a Sobolev-type inequality.

3.4.2 Norm estimates and bounds for the extinction time

The estimates that we present about the L^p-norm requires Sobolev and Stroock-Varopoulos inequalities which can be deduced from assumption (K1) and (M1) as we show in Section 3.2.1. For the estimates involving smoothing effects with weighted norms we will require (K2).

Proposition 3.4.1 (H^* estimates and extinction time)**.** *Assume* (A1), (A2), (K1) *and let* $m \in (0,1)$. *Let* u *be a nonnegative WDS with* $u_0^\alpha \in H^*(\Omega)$, *where* $\alpha_c \le \alpha$ *with* $\alpha_c = \max\{1, \frac{(N+2s)(1-m)}{4s}\}$. *Then, there exists a finite extinction time* $T > 0$ *such that*

$$0 \le T = T(u_0) \le \mathsf{c}_\alpha \, \|u_0^\alpha\|^{\frac{1-m}{\alpha}}_{H^*(\Omega)} \qquad \forall \alpha > \alpha_c\,,$$

with c_α *depending on* α, m, s, N *and* $\mathcal{H}_\mathcal{L}$. *Moreover, for every* $0 \le t_0 \le t \le T$ *we have*

$$\mathsf{c}_\alpha(T-t) \le \|u^\alpha(t)\|^{\frac{1-m}{\alpha}}_{H^*(\Omega)} \le \|u^\alpha(t_0)\|^{\frac{1-m}{\alpha}}_{H^*(\Omega)} - \mathsf{c}_\alpha(t-t_0)\,. \tag{3.4.5}$$

Proof. Let us derive in time the H^*-norm of u^α for any $\alpha > \alpha_c$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^\alpha \mathcal{L}^{-1} u^\alpha \,\mathrm{d}x = 2\alpha \int_\Omega u^{\alpha-1}\,\partial_t u\, \mathcal{L}^{-1} u^\alpha \,\mathrm{d}x = -2\alpha \int_\Omega u^{\alpha-1}\,\mathcal{L}u^m\, \mathcal{L}^{-1} u^\alpha \,\mathrm{d}x \\
&\le -\frac{2\,\alpha\, m}{\alpha+m-1}\int_\Omega \mathcal{L}u^{\alpha+m-1}\mathcal{L}^{-1}u^\alpha \,\mathrm{d}x = -\frac{2\,\alpha\, m}{\alpha+m-1}\int_\Omega u^{2\alpha+m-1}\,\mathrm{d}x\,,
\end{aligned}$$

where we have applied Kato's inequality (3.6.10) with $f(v) = \frac{m}{\alpha+m-1} v^{\frac{\alpha+m-1}{m}}$ and $v = u^m$. Assumption (K1), ensures the validity of the HLS inequality for the H^*-norm (3.1.2), that applied to $f = u^\alpha$ with $q = \frac{2\alpha+m-1}{\alpha}$, gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^\alpha \mathcal{L}^{-1} u^\alpha \,\mathrm{d}x &\le -\frac{2\,\alpha\, m}{\alpha+m-1}\int_\Omega (u^\alpha)^{\frac{2\alpha+m-1}{\alpha}}\,\mathrm{d}x \\
&\le -\frac{2\,\alpha\, m\, \mathcal{H}_\mathcal{L}^{-\frac{2\alpha+m-1}{\alpha}}}{\alpha+m-1}\left(\int_\Omega u^\alpha \mathcal{L}^{-1} u^\alpha \,\mathrm{d}x\right)^{\frac{2\alpha+m-1}{2\alpha}}.
\end{aligned}$$

Integrating this ODE on the intervals $[t_0, t]$ and $[t, T]$ we conclude (3.4.5). The upper bound of the extinction time is provided by choosing $t = 0$ in the lower bound of (3.4.5). □

Once we have proven the extinction of the solution, we can study the rate of extinction. For this purpose, we provide the decay of the $L^1_{\Phi_1}$-norm in terms of the extinction time. This result is deduced directly from the formulation of the WDS without assumptions on $\mathcal{L}^{-1}$.

Proposition 3.4.2 ($L^1_{\Phi_1}$ estimate and extinction time)**.** *Assume* (A1), (A2), (K1) *and let* $m \in (0,1)$*. Let* u *be a nonnegative WDS with* $u_0 \in L^1_{\Phi_1}(\Omega)$*, and let* $T > 0$*, be its extinction time. Then*

$$\|u(t)\|_{L^1_{\Phi_1}(\Omega)} \le \mathsf{c}_1 (T-t)^{\frac{1}{1-m}} \qquad \forall 0 \le t < T\,,$$

with $\mathsf{c}_1 = \lambda_1^{\frac{1}{1-m}} \|\Phi_1\|_1$*. As a consequence, we have the following lower bound for the extinction time,*

$$T \ge \lambda_1 \left(\|u_0\|_{L^1_{\Phi_1}} \|\Phi_1\|_1 \right)^{1-m}.$$

Proof. Let us consider the test function $\psi(\tau, x) = \lambda_1 \Phi_1(x)\, \chi_{[t,T]}(\tau)$ in the weak dual formulation (we approximate $\chi_{[t,T]}$ as in Lemma 3.6.7) and use that Φ_1 satisfies $\mathcal{L}^{-1}\Phi = \lambda_1^{-1}\Phi_1$,

$$\begin{aligned}
\|u(t)\|_{L^1_{\Phi_1}} - \|u(T)\|_{L^1_{\Phi_1}} &= \int_0^T \!\!\int_\Omega u\, \partial_\tau \chi_{[t,T]}\, \Phi_1 \,\mathrm{d}x\,\mathrm{d}\tau = \lambda_1 \int_0^T \!\!\int_\Omega \mathcal{L}^{-1} u\, \partial_\tau \chi_{[t,T]}\, \Phi_1 \,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^T \!\!\int_\Omega \mathcal{L}^{-1} u\, \partial_\tau \psi \,\mathrm{d}x\,\mathrm{d}\tau = \int_0^T \!\!\int_\Omega u^m \psi \,\mathrm{d}x\,\mathrm{d}\tau = \lambda_1 \int_t^T \!\!\int_\Omega u^m \Phi_1 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \lambda_1 \|\Phi\|_1^{1-m} \int_t^T \|u(\tau)\|^m_{L^1_{\Phi_1}} \,\mathrm{d}\tau \le \lambda_1 \|\Phi_1\|_1^{1-m}\, \|u(t)\|^m_{L^1_{\Phi_1}}\, (T-t),
\end{aligned}$$

where we have use Hölder and the $L^1_{\Phi_1}$-decay in the last inequalities. The result follows from the fact that T is the extinction time, hence $\|u(T)\|_{L^1_{\Phi_1}} = 0$. □

Proposition 3.4.3 (H^* extinction rate)**.** *Let* $0 < m < 1$ *and* u *be a nonnegative WDS with initial data* $u_0 \in L^{1+m} \cap H^*(\Omega)$*. Assume* (A1) *and* (A2)*. If there exists an extinction time* $T > 0$ *and* (K1) *holds, then*

$$\|u(t)\|_{H^*(\Omega)} \le \mathsf{c}_1\, \mathcal{Q}^*[u_0]^{\frac{1}{1-m}} (T-t)^{\frac{1}{1-m}} \qquad \text{for every } \ 0 \le t < T, \tag{3.4.6}$$

with $\mathsf{c}_1 = (1-m)^{\frac{1}{1-m}}$*. In addition, if* $m \in (m_s, 1)$*, then we also have the lower estimate*

$$\mathsf{c}_0 (T-t)^{\frac{1}{1-m}} \le \|u(t)\|_{H^*(\Omega)} \le \mathsf{c}_1\, \mathcal{Q}^*[u_0]^{\frac{1}{1-m}} (T-t)^{\frac{1}{1-m}} \qquad \text{for every } \ 0 \le t < T, \tag{3.4.7}$$

with c_0 *depending on* m, s, N, Ω *and* $\mathcal{H}_{\mathcal{L}}$. *Moreover, let* $\epsilon \in (0,1)$ *and* $t_\epsilon = \epsilon T$, *then*

$$\|u(t)\|_{1+m}^{1+m} \le \mathsf{c}^* \mathcal{Q}^{*\,\frac{2}{1-m}}[u_0] \begin{cases} T^{\frac{2}{1-m}}\, t^{-1}, & \text{for } 0 < t \le t_\epsilon, \\ \mathsf{c}_\epsilon (T-t)^{\frac{1+m}{1-m}}, & \text{for } t_\epsilon < t \le T. \end{cases} \tag{3.4.8}$$

where $\mathsf{c}^* = \frac{(1-m)^{\frac{2}{1-m}}}{1+m}$ *and* $\mathsf{c}_\epsilon = \epsilon^{-1}(1-\epsilon)^{-\frac{1+m}{1-m}}$. *In addition,*

$$\|u^m(t)\|_{H(\Omega)}^2 \le \mathsf{c}^* \mathcal{Q}^{*\,\frac{2}{1-m}}[u_0] \begin{cases} T^{\frac{2}{1-m}}\, t^{-2}, & \text{for } 0 < t \le t_\epsilon, \\ \mathsf{c}_\epsilon (T-t)^{\frac{2m}{1-m}}, & \text{for } t_\epsilon < t \le T, \end{cases} \tag{3.4.9}$$

where $\mathsf{c}^* = \frac{(1-m)^{\frac{2}{1-m}}}{2m(1+m)}$ *and* $\mathsf{c}_\epsilon = \left[(1-\sqrt{1-\epsilon})(1-\epsilon)^{\frac{2+3m}{2(1+m)}}\right]^{-1}$.

Proof. Proof of (3.4.6). We derive the H^*-norm and use the monotonicity of $\mathcal{Q}^*[u(t)]$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \iint_\Omega u\mathcal{L}^{-1}u \,\mathrm{d}x = -2 \iint_\Omega u^{1+m} \,\mathrm{d}x &= -2\, \mathcal{Q}^*[u(t)] \left(\int_\Omega u\mathcal{L}^{-1}u \,\mathrm{d}x \right)^{\frac{1+m}{2}} \\ &\ge -2\, \mathcal{Q}^*[u_0] \left(\iint_\Omega u\mathcal{L}^{-1}u \,\mathrm{d}x \right)^{\frac{1+m}{2}}. \end{aligned}$$

The upper estimate follows from integrating in $[t,T]$ the ODE for $\|u(t)\|_{H^*}$.
Proof of lower bound in (3.4.7). Since $1+m > (2^*)' = \frac{2N}{N+2s}$, we have that (K1) implies HLS inequality. Therefore,

$$\|f\|_{H^*} \le \mathcal{H}_{\mathcal{L}} \|f\|_{\frac{2N}{N+2s}} \le \mathcal{H}_{\mathcal{L}} |\Omega|^{\frac{N+2s}{2N} - \frac{1}{1+m}} \|f\|_{1+m}.$$

Proof of (3.4.8). Now, we derive in time the H^*-norm of the solution $u(t)$ and we apply the estimate above,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_\Omega u\mathcal{L}^{-1}u \,\mathrm{d}x = -2 \iint_\Omega u^{1+m} \,\mathrm{d}x \le -2\, (c_\Omega \mathcal{H}_{\mathcal{L}})^{-(1+m)} \left(\int_\Omega u\mathcal{L}^{-1}u \,\mathrm{d}x \right)^{\frac{1+m}{2}}.$$

The lower estimate follows from integrating on [t,T] the ODE for $\|u(t)\|_{H^*}$. From (3.2.28) and (3.4.6) we have that for every $t > t_0 \ge 0$

$$\|u(t)\|_{1+m}^{1+m} \;\le\; \frac{1}{(1+m)} \frac{\|u(t_0)\|_{H^*(\Omega)}^2}{t-t_0} \le \frac{\mathsf{c}_1^2}{(1+m)} \mathcal{Q}^*[u_0]^{\frac{2}{1-m}} \frac{(T-t_0)^{\frac{2}{1-m}}}{t-t_0}. \tag{3.4.10}$$

Now, let $t \in [t_\epsilon, T)$ with $t_\epsilon = \epsilon\, T$ with $\epsilon \in (0,1)$ and $t_0 = \frac{t-\epsilon\, T}{1-\epsilon}$, and notice that

$$t_0 \ge 0 \iff t \ge t_\epsilon \qquad \text{and} \qquad t > t_0 \iff t < T.$$

Introducing this t_0 in (3.4.10) we arrive to

$$\|u(t)\|_{1+m}^{1+m} \;\le\; \frac{\mathsf{c}_1^2}{\epsilon\,(1-\epsilon)^{\frac{1+m}{1-m}}(1+m)} \mathcal{Q}^*[u_0]^{\frac{2}{1-m}} (T-t)^{\frac{1+m}{1-m}}, \tag{3.4.11}$$

so calling $\mathsf{c}^* = \frac{\mathsf{c}_1^2}{1+m}$ we arrive to the second part in (3.4.8). For the case $t < t_\epsilon$, we just take $t_0 = 0$ in 3.4.10 and the result follows.
Proof of (3.4.9) . For the case $t \geq t_\epsilon$ we use now the first part of estimate (3.2.28) combined with the previous decay (3.4.8). For this we choose t_{ϵ_0} defined as above such that for every $t \geq t_{\epsilon_0}$ we have

$$\|u^m(t_1)\|^2_{H(\Omega)} \;\leq\; \frac{\mathsf{c}^*\,\mathsf{c}_{\epsilon_0}}{2m}\mathcal{Q}^*[u_0]^{\frac{2}{1-m}}\frac{(T-t)^{\frac{1+m}{1-m}}}{(t_1-t)} \tag{3.4.12}$$

Now, set $t = \frac{t_1-\epsilon_0 T}{1-\epsilon_0}$ and notice that

$$t_0 = \frac{t_1-\epsilon_0 T}{1-\epsilon_0} \geq t_{\epsilon_0} \iff t_1 \geq (2\epsilon_0-\epsilon_0^2)\,T.$$

So, setting $\epsilon = (2\epsilon_0-\epsilon_0^2) \geq \epsilon_0$, i.e. $\epsilon_0 = 1-\sqrt{1-\epsilon}$, we have that for every $t_1 \in [t_\epsilon, T)$.

$$\|u^m(t_1)\|^2_{H(\Omega)} \;\leq\; \frac{\mathsf{c}^*\,\mathsf{c}_{\epsilon_0}}{\epsilon_0\,(1-\epsilon_0)^{\frac{2m}{1-m}}\,2m}\mathcal{Q}^*[u_0]^{\frac{2}{1-m}}(T-t_1)^{\frac{2m}{1-m}}. \tag{3.4.13}$$

Setting $t_1 = t$, $\mathsf{c}^* = \frac{\mathsf{c}_1^2}{2m(1+m)}$ and $\mathsf{c}_\epsilon = \frac{\mathsf{c}_{\epsilon_0}}{\epsilon_0\,(1-\epsilon_0)^{\frac{2m}{1-m}}}$ with $\epsilon_0 = 1-\sqrt{1-\epsilon}$ we get (3.4.9).

For $t < t_\epsilon$, from (3.2.28)

$$\|u^m(t_1)\|^2_{H(\Omega)} \;\leq\; \frac{1}{2m(1+m)}\frac{\|u(t_0)\|^2_{H^*(\Omega)}}{(t_1-t)(t-t_0)} \leq\; \frac{\mathsf{c}_1^2}{2m(1+m)}\mathcal{Q}^*[u_0]^{\frac{2}{1-m}}\frac{(T-t_0)^{\frac{2}{1-m}}}{(t_1-t)(t-t_0)}. \tag{3.4.14}$$

Here we just take $t_0 = 0$, $t = t_1/2$ and the result follows from the estimate above. □

Proposition 3.4.4 (L^p estimates and Extinction Time)**.** *Assume* (A1), (A2), (M1) *and let $m \in (0,1)$. Let u be a nonnegative WDS corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p > p_c$. Then, there exists a finite extinction time $T > 0$ such that*

$$0 \leq T = T(u_0) \leq c_p\|u_0\|_p^{1-m},$$

where $c_p > 0$ only depends on $p, m, s, N, \lambda_1, \mathcal{S}_\mathcal{L}, \Omega$. In addition, for every $0 \leq t_0 \leq t \leq T$ we have

$$c_p(T-t) \leq \|u(t)\|_p^{1-m} \leq \|u(t_0)\|_p^{1-m} - c_p(t-t_0)\,. \tag{3.4.15}$$

Moreover, for every $t \in [t_\epsilon, T)$, where $t_\epsilon = \epsilon T$ with $\epsilon \in (0,1)$, there exists $\mathsf{c}_{p,\epsilon}$, which blows up when $\epsilon \to 0$, such that

$$\|u(t)\|_p^p \leq \mathsf{c}_{p,\epsilon}\quad \overline{\kappa}_\epsilon\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\Bigg)^{[p-(1+m)]_+}\mathcal{Q}^*[u_0]^{\frac{2}{1-m}}(T-t)^{\frac{1+m}{1-m}}, \tag{3.4.16}$$

where

$$\mathsf{c}_{p,\epsilon} = \frac{p(1-m)^{\frac{3-m}{1-m}}}{2m(1+m)^2}\left[(1-\sqrt{1-\epsilon})(1-\epsilon)^{\frac{2+3m}{2(1+m)}}\right]^{-1} \qquad \textit{and} \qquad \overline{\kappa}_\epsilon = \overline{\kappa}/\epsilon^{N\vartheta_p},$$

with $\overline{\kappa}$ as in Theorem 3.2.8. When $p < 1+m$ the correct exponent comes from introducing $p = 1+m$ in (3.4.16) *and by Holder's inequality.*

Proof. Recall that (K1) and (L1) implies (M1) and Sobolev inequality, see Section 3.2.1. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\,\mathrm{d}x = -p\iint_\Omega u^{p-1}\mathcal{L}u^m\,\mathrm{d}x \le -c_{m,p}\|\mathcal{L}^{\frac12}u^{\frac{p+m-1}{2}}\|_2^2 \tag{3.4.17}$$
$$\le -c_{m,p,\Omega}\,\mathcal{S}_{\mathcal{L}}^{-1}\|u^{\frac{p+m-1}{2}}\|^2_{\frac{2p}{p+m-1}} = -c_{m,p,\Omega}\,\mathcal{S}_{\mathcal{L}}^{-1}\|u\|_p^{p+m-1},$$

where in the last inequality we have used Sobolev in the following form,

$$\|u\|_{\frac{2p}{p+m-1}} \le |\Omega|^{\frac{p+m-1}{2p}-\frac{1}{2^*}}\,\|u\|_{2^*} \le |\Omega|^{\frac{p+m-1}{2p}-\frac{1}{2^*}}\,\mathcal{S}_{\mathcal{L}}\,\|\mathcal{L}^{-1}u\|_2\,,$$

since $p > p_c$ implies $\frac{2p}{p+m-1} < 2^* = \frac{2N}{N-2s}$. Now, integrating the ODE that $\|u(t)\|_p^p$ satisfies on the intervals $[t_0, t]$ and $[t, T]$ we obtain (3.4.15). For the upper bound on T, just let $t = 0$ in (3.4.15).

Now, we prove (3.4.16). For $p > 1 + m$, the rest of the cases follow by Hölder, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\,\mathrm{d}x = -p\iint_\Omega u^{p-1}\mathcal{L}u^m\,\mathrm{d}x \ge -p\,\|u(t)\|_\infty^{p-(1+m)}\iint_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x$$
$$= -p\,\|u(t)\|_\infty^{p-(1+m)}\,\|u^m(t)\|^2_{H(\Omega)} \ge -p\quad\overline{\kappa}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{t_0^{N\vartheta_p}}\Bigg)^{p-(1+m)}\|u^m(t)\|^2_{H(\Omega)},$$

for every $t > t_0$, by Theorem 3.2.8. Now, by (3.4.9), for every $t \ge \epsilon T$ with $\epsilon \in (0,1)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Omega u^p\,\mathrm{d}x \ge -p\quad\frac{\overline{\kappa}}{\epsilon^{N\vartheta_p}}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{t_0^{N\vartheta_p}}\Bigg)^{p-(1+m)}\|u^m(t)\|^2_{H(\Omega)} \tag{3.4.18}$$
$$\ge -p\,\mathsf{c}^*\,\mathsf{c}_\epsilon\quad\frac{\overline{\kappa}}{\epsilon^{N\vartheta_p}}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{t_0^{N\vartheta_p}}\Bigg)^{p-(1+m)}\mathcal{Q}^{*\frac{2}{1-m}}[u_0](T-t)^{\frac{2m}{1-m}}$$

where $\overline{\kappa}_\epsilon = \overline{\kappa}/\epsilon^{N\vartheta_p}$, with c^* and c_ϵ as in (3.4.9); and $\overline{\kappa}$ as in Theorem 3.2.8. Integrating (3.4.18) over suitable $[t_0, t_1]$ and choosing $t_1 = T$ and $t_0 = t$ we arrive to (3.4.16). □

Proposition 3.4.5. *Let $N > 2s$, $m \in (0,1)$ and assume* (A1), (A2) *and either* (K1) *or* (M1). *Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, for all $t \in [t_\epsilon, T)$ with $\epsilon \in (0,1)$, there exists $\mathsf{c} = \mathsf{c}(N, m, s, p, \Omega, \epsilon)$ blowing up when $\epsilon \to 0$, such that*

$$\|u(t)\|_\infty \le \mathsf{c}_{\infty,\epsilon}\quad\overline{\kappa}_\epsilon\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\Bigg)^{2s\vartheta_p[p-(1+m)]_+}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}(T-t)^{\frac{\vartheta_p}{\vartheta_{1+m}}\frac{1}{1-m}}. \tag{3.4.19}$$

Proof. For every $t_0 > \epsilon_0 T$, by Theorem 3.2.8 and (3.4.16),

$$\|u(t)\|_\infty \le \overline{\kappa}\,\frac{\|u(t_0)\|_p^{2sp\,\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \le \overline{\kappa}\,\mathsf{c}_{p,\epsilon}^{2s\vartheta_p}\quad\overline{\kappa}_{\epsilon_0}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\Bigg)^{2s\vartheta_p[p-(1+m)]_+}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}\frac{(T-t_0)^{\frac{2s\vartheta_p(1+m)}{1-m}}}{(t-t_0)^{N\vartheta_p}}$$

with $\overline{\kappa}$ and $\mathsf{c}_{p,\epsilon}$ as in Theorem 3.2.8 and (3.4.16) respectively. Then, we set $t_0 = \frac{t-\epsilon_0 T}{1-\epsilon_0}$ and $\epsilon = 2\epsilon_0 - \epsilon_0^2$, and by the same argument as in (3.4.13) we have that for every $t \in [t_\epsilon, T)$

$$\|u(t)\|_\infty \le \mathsf{c}_\epsilon^\infty \quad \overline{\kappa}_{\epsilon_0} \left(\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}} \right)^{2s\vartheta_p[p-(1+m)]_+} \mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}} (T-t)^{\frac{2s\vartheta_p(1+m)-N\vartheta_p(1-m)}{1-m}}$$

where

$$\mathsf{c}_\epsilon^\infty = \frac{\overline{\kappa}\,\mathsf{c}_{p,\epsilon}^{2s\vartheta_p}}{\epsilon_0^{N\vartheta_p}(1-\epsilon_0)^{\frac{\vartheta_p}{\vartheta_{1+m}}\frac{1}{1-m}}} \qquad \text{with} \qquad \epsilon_0 = 1-\sqrt{1-\epsilon}.$$

We finish noticing that

$$\frac{2s\vartheta_p(1+m) - N\vartheta_p(1-m)}{1-m} = \frac{\vartheta_p}{\vartheta_{1+m}} \frac{1}{1-m}.$$

□

Proposition 3.4.6. *Let $N > 2s > \gamma$, $m \in (m_{c,\gamma}, 1)$ and assume* (A1), (A2) *and* (K2). *Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$ and let $\epsilon \in (0,1)$, then for every $t \in [t_\epsilon, T)$ with $t_\epsilon = \epsilon T$ we have*

$$\|u(t)\|_\infty \le \bar{\mathsf{c}}_{1,\epsilon}\,(T-t)^{\frac{1}{1-m}}, \tag{3.4.20}$$

with $\bar{\mathsf{c}}_{1,\epsilon} = \bar{\mathsf{c}}_{1,\epsilon}(N, m, s, p, \Omega, \lambda_1, \epsilon)$ blowing up when $\epsilon \to 0$.

Proof. By Theorem 3.2.9 and Proposition 3.4.2, for every $t > t_0 \ge 0$

$$\|u(t)\|_\infty \le \overline{\kappa}\,\frac{\|u(t_0)\|_{L^1_{\Phi_1}(\Omega)}^{(2s-\gamma)\,\vartheta_{1,\gamma}}}{(t-t_0)^{N\vartheta_{1,\gamma}}} \le \bar{\mathsf{c}}_1\,\frac{(T-t_0)^{\frac{(2s-\gamma)\,\vartheta_{1,\gamma}}{1-m}}}{(t-t_0)^{N\vartheta_{1,\gamma}}}, \tag{3.4.21}$$

where $\bar{\mathsf{c}}_1 = \overline{\kappa}\mathsf{c}_1^{(2s-\gamma)\,\vartheta_{1,\gamma}}$ and $\overline{\kappa}$ and c_1 are as in Theorem 3.2.9 and Proposition 3.4.2. Now, imposing $t > \epsilon\,T$ with $\epsilon \in (0,1)$ and introducing $t_0 = \frac{t-\epsilon T}{(1-\epsilon)}$ in (3.4.21) we get

$$\|u(t)\|_\infty \le \bar{\mathsf{c}}_{1,\epsilon}\,(T-t)^{\frac{1}{1-m}},$$

where $\bar{\mathsf{c}}_{1,\epsilon} = \frac{\bar{\mathsf{c}}_1}{\epsilon^{N\vartheta_{1,\gamma}}(1-\epsilon)^{\frac{1}{1-m}}}$. □

3.5 Boundary behaviour and Global Harnack Principles

By means of the estimates in the previous section, here we prove quantitative upper and lower bounds of the solutions in terms of the first eigenfunction of the operator $\mathcal{L}$, that we recall to satisfy

$$\Phi_1(x) \asymp \delta_\gamma(x) = \operatorname{dist}(x, \partial\Omega)^\gamma.$$

This provides the information necessary to state our Global Harnack Principles. In the classical setting, the first GHP for the fast diffusion equation where proven by Kwong [96, 95] and diBenedetto, Kwong and Vespri [58], in the range $m \in (m_s, 1)$. They introduce the name "Global Harnack Principle", because they provide a uniform Harnack inequality globally both in space and in time. In these works, the Harnack constants depend on the initial data u_0 by means of $\mathcal{Q}[u_0]$, that is, $\|\nabla u_0^m\|_2$ and $\|u_0\|_{1+m}$. In [19], Bonforte and Figalli provide a new proof which eliminates the dependence on $\mathcal{Q}[u_0]$, replacing it with $\mathcal{Q}^*[u_0]$. They also extend the result to the whole range $m \in (0, 1)$, paying the price of an extra time exponent in the bad range $m \in (0, m_s]$. By similar arguments to the ones in [19] we get the following results, which provide strong quantitative information of how the lateral boundary conditions are satisfied.

3.5.1 Upper boundary estimates

Theorem 3.5.1 (Upper boundary estimate)**.** *Let $N > 2s$, $m \in (0, 1)$ and assume* (A1), (A2) *and* (K2)*. Let u be a nonnegative WDS of* (CDP) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, $u(t)$ satisfies boundary condition and we have*

$$u^m(t, x_0) \leq \overline{\kappa}\, \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t - t_0)^{1+N\vartheta_p}}\, \mathcal{B}_1(\Phi_1(x_0)) \quad \text{for a.e. } x_0 \in \Omega \text{ and } \forall t > t_0 \geq 0, \tag{3.5.1}$$

with $\overline{\kappa}$ depending on p, m, s, γ, N and Ω, and $\mathcal{B}_1$ is defined as in Lemma 4.1 of [21]*,*

$$\mathcal{B}_1(\Phi_1(x_0)) := \begin{cases} \Phi_1(x_0), & \text{for } 2s > \gamma, \\ \Phi_1(x_0)(1 + |\log \Phi_1(x_0)|), & \text{for } 2s = \gamma, \\ \Phi_1(x_0)^{\frac{2s}{\gamma}}, & \text{for } 2s < \gamma. \end{cases} \tag{3.5.2}$$

In addition,

$$\frac{u^m}{\Phi_1} \leq \bar{\mathsf{c}}[u_0, t](T - t)^{\frac{m}{1-m} - \frac{2s\vartheta_p}{1-m}[p-(1+m)]}, \tag{3.5.3}$$

where

$$\bar{\mathsf{c}}[u_0, t] = \mathsf{c}_{n,p,s,N,\Omega} \left\{ \begin{array}{ll} \frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{\frac{2sp\vartheta_p}{1-m}}}, & \text{for } 0 < t \leq T/2, \\ \left(\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}} \right)^{2s\vartheta_p[p-(1+m)]_+} \mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}, & \text{for } T/2 < t \leq T. \end{array} \right.$$

Proof. Let us first prove (3.5.1). From Theorem 3.2.8 it follows that $u(t)$ is bounded for every $t > 0$. Indeed, for a.e. $x_0 \in \Omega$ and $t > 0$ we have (recall that $\mathcal{B}_1$ is defined in (3.5.2))

$$\int_\Omega u(t/2, x)\mathbb{G}_\Omega(x_0, x)\, \mathrm{d}x \leq \|u(t/2)\|_\infty \|\mathbb{G}_\Omega(x_0, \cdot)\|_{L^1(\Omega)} \leq c_1 \|u(t/2)\|_\infty \mathcal{B}_1(\Phi_1(x_0))\,, \tag{3.5.4}$$

where in the last step we have used Lemma 4.2 of [21] which requires (K2). We remark that constant c_1 depends on m, s, γ and N. Then, we use (3.5.4) in estimate (3.2.17) with $p = 1$, $t_1 = t$ and $\tau = t/2$,

$$u^m(t,x_0) \le \frac{2^{\frac{1}{1-m}}}{(1-m)\,t} \iint_\Omega u(t/2,x)\,\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \le \frac{2^{\frac{1}{1-m}}c_1}{(1-m)\,t}\|u(t/2)\|_\infty \mathcal{B}_1(\Phi_1(x_0))\,.$$

Now, let us apply the smoothing estimate of Theorem 3.2.8,

$$u^m(t,x_0) \le \frac{2^{\frac{1}{1-m}}c_1}{(1-m)\,t}\,\|u(t/2)\|_\infty\,\mathcal{B}_1(\Phi_1(x_0)) \le \frac{2^{\frac{2sp\vartheta_p}{1-m}}c_1}{(1-m)}\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{1+N\vartheta_p}}\mathcal{B}_1(\Phi_1(x_0))\,.$$

Estimate (3.5.1) follows from the property of semigroups (time-shift invariance). For the weighted result with $2s > \gamma$, we repeat this last step with the smoothing estimate of Theorem 3.2.9,

$$u^m(t,x_0) \le \frac{2^{\frac{1}{1-m}}c_1}{(1-m)\,t}\|u(t/2)\|_\infty\Phi_1(x_0) \le \frac{2^{\frac{(2s-\gamma)p\vartheta_{p,\gamma}}{1-m}}c_1}{(1-m)}\frac{\|u_0\|_{L^p_{\Phi_1}(\Omega)}^{(2s-\gamma)p\vartheta_{p,\gamma}}}{t^{1+N\vartheta_{p,\gamma}}}\Phi_1(x_0)\,. \quad \square$$

Now, to prove (3.5.3) we take the pointwise estimate (3.2.16) and compute

$$\frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \frac{\mathsf{C}}{1-m}\frac{t^{\frac{m}{1-m}}}{t^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}\|u(t_0)\|_\infty \le \mathsf{C}\left(\frac{T}{t_0}\right)^{\frac{m}{1-m}}\frac{\|u(t_0)\|_\infty}{t-t_0}, \tag{3.5.5}$$

using that $t^{\frac{1}{1-m}}$ is convex and so $\frac{t_0^{\frac{m}{1-m}}}{1-m} \le t^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}$. Now, we use (3.4.19) and we get, for any t_0 such that $t_0 \ge t_{\epsilon_0} > 0$,

$$\frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \mathsf{C}\cdot\mathsf{C}^\infty_{\epsilon_0}\quad \overline{\kappa}_{\epsilon_0}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\Bigg)^{2s\vartheta_p[p-(1+m)]_+}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}\frac{(T-t_0)^{\frac{\vartheta_p}{\vartheta_{1+m}}\frac{1}{1-m}}}{t-t_0}.$$

Then, we set $t_0 = \frac{t-\epsilon_0 T}{1-\epsilon_0}$ and $\epsilon = 2\epsilon_0 - \epsilon_0^2$, as in (3.4.13). Choosing ϵ_0 such that $\epsilon = 1/2$, we have that for every $t \in [T/2, T)$

$$\frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \mathsf{C}_{m,p,s,N,\Omega}\quad \frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\Bigg)^{2s\vartheta_p[p-(1+m)]_+}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}(T-t)^{\frac{m}{1-m}-\frac{2s\vartheta_p}{1-m}[p-(1+m)]}.$$

In the case $t < T/2$, we just use Theorem 3.2.8 in the first inequality in (3.5.5) and we get

$$\frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \frac{\mathsf{C}}{1-m}\frac{t^{\frac{m}{1-m}}}{t^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}\overline{\kappa}\,\frac{\|u_0\|_p^{2sp\vartheta_p}}{t_0^{N\vartheta_p}} \le C\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{N\vartheta_p+1}} \le C\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{\frac{2sp\vartheta_p}{1-m}}}(T-t)^{\frac{m}{1-m}},$$

where C depends on m, p, s, N and Ω. The expression above implies (3.5.3) when $t \le T/2$. $\square$

Theorem 3.5.2. *Let $N > 2s > \gamma$, $m \in (m_{c,\gamma}, 1)$ and assume* (A1), (A2) *and* (K3). *Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$. Then, for every $t \in [T/2, T)$ we have*

$$\frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \mathsf{c}_\gamma \, (T-t)^{\frac{m}{1-m}}, \tag{3.5.6}$$

where

$$\mathsf{c}_\gamma = \mathsf{c}_{n,p,s,N,\Omega} \begin{cases} \frac{1}{\frac{m}{1-m}}, & \text{for } 0 < t \le T/2, \\ 1, & \text{for } T/2 < t \le T. \end{cases}$$

Proof. We follow the same proof of Theorem 3.5.1 until (3.5.5), where we just apply the L^∞ estimate (3.4.20). We choose ϵ_0 such that the result holds for all $T/2 \le t$. □

3.5.2 Lower boundary estimates

Theorem 3.5.3 ($L^1_{\Phi_1}$ norm lower estimates)**.** *Let $N > 2s > \gamma$, $m \in (0,1)$ and assume* (A1), (A2) *and* (K3). *Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, for every $t \in [T/2, T)$ we have*

$$\|u(t)\|_{L^1_{\Phi_1}} \ge \mathsf{c}(T-t)^{\frac{1}{1-m} + \frac{2s\vartheta_p[p-(1+m)]_+}{1-m}}, \tag{3.5.7}$$

where

$$\mathsf{c} = \begin{cases} \dfrac{\mathsf{c}_2}{\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}(2-m)}{1-m}}}, & \text{when } m \in (m_s, m_{\gamma,c}], \\ \dfrac{\mathsf{c}_3}{\left[\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}} \left(\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+}\right]^{p-2m+1}}, & \text{when } m \in (0, m_s]. \end{cases}$$

In addition, if $u_0 \in L^1_{\Phi_1}(\Omega)$ and $m \in (m_{c,\gamma}, 1)$, then

$$\|u(t)\|_{L^1_{\Phi_1}} \ge \mathsf{c}_1 (T-t)^{\frac{1}{1-m}}. \tag{3.5.8}$$

The constants $\mathsf{c}_1, \mathsf{c}_2$ and c_1 depend only on m, p, s, N, Ω.

Proof. We follow the proof in [19] for (3.5.7). First, differentiating the $L^1_{\Phi_1}$ norm and setting $p \ge 2m$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\Phi_1 \,\mathrm{d}x &= -\int_\Omega \mathcal{L}(u^m)\Phi_1 \,\mathrm{d}x = -\lambda_1 \int_\Omega u^m \Phi_1 \,\mathrm{d}x \\ &= -\lambda_1 \int_\Omega \frac{u^p\,\Phi_1}{u^{p-2m}\,u^m}\,\mathrm{d}x \le -\frac{\lambda_1\,\|u\|_p^p}{\|u\|_\infty^{p-2m}\|\frac{u^m}{\Phi_1}\|_\infty} \le -\frac{\lambda_1\,\mathsf{c}_p^{\frac{p}{1-m}}\,(T-t)^{\frac{p}{1-m}}}{\|u\|_\infty^{p-2m}\|\frac{u^m}{\Phi_1}\|_\infty} \end{aligned} \tag{3.5.9}$$

where we have used the lower estimate for the L^p norm in (3.4.16). Now, we split the proof in three parts depending of the range of $m \in (0,1)$.

• $m \in (m_s, 1)$. Here, we take $p = 1 + m$ in the estimates (3.4.19) and (3.5.3) and choose ϵ such that

$$\|u(t)\|_\infty \le \mathsf{c}_{\infty,1/2}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}}{1-m}}(T-t)^{\frac{1}{1-m}} \quad \text{and} \quad \left\|\frac{u^m(t)}{\Phi_1}\right\|_\infty \le \mathsf{c}_{b,1/2}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}}{1-m}}(T-t)^{\frac{m}{1-m}} \tag{3.5.10}$$

for every $t \in [T/2, T)$, with both c and c' depending only on n, p, s, n and Ω.

Introducing (3.5.10) in (3.5.9) with $p = 1 + m$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\Phi_1\,\mathrm{d}x \le -\frac{\lambda_1\,\mathsf{c}_{1+m}^{\frac{1+m}{1-m}}\,(T-t)^{\frac{m}{1-m}}}{\mathsf{c}_{\infty,1/2}^{1-m}\,\mathsf{c}_{b,1/2}\,\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}(2-m)}{1-m}}}. \tag{3.5.11}$$

Then, integrating over $[t, T]$ we get

$$\|u(t)\|_{L^1_{\Phi_1}} \ge \frac{\lambda_1\,(1-m)\,\mathsf{c}_{1+m}^{\frac{1+m}{1-m}}\,(T-t)^{\frac{1}{1-m}}}{\mathsf{c}_{\infty,1/2}^{1-m}\,\mathsf{c}_{b,1/2}\,\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}(2-m)}{1-m}}} = \underline{\mathsf{c}}(u_0)\,(T-t)^{\frac{1}{1-m}}$$

for all $t \in [T/2, T)$.

• $m \in (m_{c,\gamma}, 1)$. In this case, we are allowed to use estimates (3.4.20) and (3.5.6) so that, setting again $p = 1 + m$ in (3.5.9) and $\epsilon = 1/2$, we get for all $t \in [T/2, T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\Phi_1\,\mathrm{d}x \le -\frac{\lambda_1\,\mathsf{c}_{1+m}^{\frac{1+m}{1-m}}\,(T-t)^{\frac{1+m}{1-m}}}{\|u\|_\infty^{1-m}\|\frac{u^m}{\Phi_1}\|_\infty} \le -\frac{\lambda_1\,\mathsf{c}_{1+m}^{\frac{1+m}{1-m}}\,(T-t)^{\frac{m}{1-m}}}{\bar{\mathsf{c}}_{1,1/2}^{1-m}\,\mathsf{c}_{m_c,1/2}}$$

which integrated over $[t, T]$ gives

$$\|u(t)\|_{L^1_{\Phi_1}} \ge \frac{\lambda_1\,(1-m)\,\mathsf{c}_{1+m}^{\frac{1+m}{1-m}}\,(T-t)^{\frac{1}{1-m}}}{\bar{\mathsf{c}}_{1,1/2}^{1-m}\,\mathsf{c}_{m_c,1/2}} = \underline{\mathsf{c}}\,(T-t)^{\frac{1}{1-m}}. \tag{3.5.12}$$

• $m \in [0, m_s)$. For the remaining case, in which we have that $p > 1+m > 2m$, we introduce estimates (3.4.19) and (3.5.3) in (3.5.9) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\Phi_1\,\mathrm{d}x &\le -\frac{\lambda_1\,\mathsf{c}_p^{\frac{p}{1-m}}\,(T-t)^{\frac{p}{1-m}}}{\|u\|_\infty^{p-2m}\|\frac{u^m}{\Phi_1}\|_\infty} \\ &\le -\frac{\lambda_1\,(T-t)^{\frac{m}{1-m}+\frac{2s\vartheta_p[p-(1+m)]}{1-m}}}{\mathsf{c}_p^{\frac{p}{1-m}}\,\mathsf{c}_{\infty,\epsilon}^{p-2m}\,\mathsf{c}_{b,\epsilon}\left(\overline{\kappa}_{\epsilon_0}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+(p-2m+1)}\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}(p-2m+1)}},\end{aligned}$$

which integrated over [t,T] gives

$$\begin{aligned}\|u(t)\|_{L^1_{\Phi_1}} &\ge \frac{\lambda_1\,(1-m)\,(1+2s\vartheta_p(p-2m+1)[p-(1+m)])^{-1}\,(T-t)^{\frac{1}{1-m}+\frac{2s\vartheta_p[p-(1+m)]}{1-m}}}{\mathsf{c}_p^{\frac{p}{1-m}}\,\mathsf{c}_{\infty,\epsilon}^{p-2m}\,\mathsf{c}_{b,\epsilon}\left[\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}\left(\overline{\kappa}_{\epsilon_0}\,\frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+}\right]^{p-2m+1}} \\ &= \underline{\mathsf{c}}_p(u_0)(T-t)^{\frac{1}{1-m}+\frac{2s\vartheta_p[p-(1+m)]}{1-m}}.\end{aligned}$$

Setting ϵ and ϵ_0 such that the expression above holds for $t \in [T/2, T]$ we get the result. □

Theorem 3.5.4 (**Global lower bounds**). *Let $N > 2s$, $m \in (0,1)$ and assume* (A1), (A2) *and* (K3). *Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ and extinction time $T = T(u_0)$. Then, for all $t \in (0,T)$:*

$$\frac{u^m(t,x_0)}{\Phi_1} \geq \underline{\mathsf{c}}[u_0] \left(\frac{t}{T}\right)^{\frac{m}{1-m}} (T-t)^{\frac{m}{1-m}+\frac{2s\vartheta_p[p-(1+m)]_+}{1-m}}, \tag{3.5.13}$$

where

$$\underline{\mathsf{c}}[u_0] = \begin{cases} \dfrac{\underline{\mathsf{c}}_2}{\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_{1+m}(2-m)}{1-m}}}, & \text{when } m \in (m_s, m_{\gamma,c}]\,, \\ \dfrac{\underline{\mathsf{c}}_3}{\left(\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}} \quad \frac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+}\Big]^{p-2m+1}}, & \text{when } m \in (0, m_s]\,. \end{cases}$$

In addition, if $u_0 \in L^1_{\Phi_1}(\Omega)$ and $m \in (m_{\gamma,c}, 1)$, then

$$\frac{u^m(t,x_0)}{\Phi_1} \geq \underline{\mathsf{c}}_1 \left(\frac{t}{T}\right)^{\frac{m}{1-m}} (T-t)^{\frac{m}{1-m}}. \tag{3.5.14}$$

Constants $\underline{\mathsf{c}}_1$, $\underline{\mathsf{c}}_2$ and $\underline{\mathsf{c}}_3$ depend only on m, p, s, N, Ω.

Proof. When $t \in [T/2, T]$, to prove estimates (3.5.13) and (3.5.14) we compute

$$\begin{aligned} u^m(t,x_0) &\geq \frac{1}{1-m} \frac{t^{\frac{m}{1-m}}}{T^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}} \int_\Omega u(t,x)\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \\ &\geq \frac{\mathsf{c}}{1-m} \frac{t^{\frac{m}{1-m}}}{T^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}} \|u(t)\|_{L^1_{\Phi_1}(\Omega)}\, \Phi_1(x_0) \\ &\geq \mathsf{c} \left(\frac{t}{T}\right)^{\frac{m}{1-m}} \frac{\|u(t)\|_{L^1_{\Phi_1}(\Omega)}}{T-t}\, \Phi_1(x_0) \end{aligned}$$

where we have used that $T^{1/(1-m)} - t^{1/(1-m)} \leq \frac{1}{1-m}\, T^{m/(1-m)}\,(T-t)$, as $t^{1/(1-m)}$ is convex and Lemma 3.2.4. Choosing the right lower estimates for $L^1_{\Phi_1}$-norm in 3.5.3 both results follow.

When $0 < t < T/2$ we notice that by (3.2.16)

$$\frac{u^m(t_0)}{\Phi_1} \geq \left(\frac{t_0}{t}\right)^{\frac{m}{1-m}} \frac{u^m(t)}{\Phi_1}.$$

Using $0 < t_0 < T/2 \leq t \leq T$ and the estimate for $\frac{u^m(t)}{\Phi_1}$ when $t \in [T/2, T]$ we get the result for $t_0 \in (0, T/2)$. □

3.5.3 Global Harnack Principles

Here we combine upper and lower estimates above to get the following Global Harnack Principles:

Theorem 3.5.5 (Global Harnack Principle I)**.** *Let $N > 2s > \gamma$, $m \in (0,1)$ and assume* (A1)*,* (A2) *and* (K3)*. Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$ with extinction time $T = T(u_0)$. Then, for all $t \in (0,T)$, we have*

$$\underline{\mathsf{c}}[u_0] \left(\frac{t}{T}\right)^{\frac{m}{1-m}} (T-t)^{\frac{m}{1-m}+\frac{2s\vartheta_p[p-(1+m)]_+}{1-m}} \le \frac{u^m}{\Phi_1} \le \bar{\mathsf{c}}[u_0,t](T-t)^{\frac{m}{1-m}-\frac{2s\vartheta_p}{1-m}[p-(1+m)]_+}, \tag{3.5.15}$$

where

$$\bar{\mathsf{c}}[u_0,t] = \mathsf{c}_{n,p,s,N,\Omega} \begin{cases} \dfrac{\|u_0\|_p^{2sp\vartheta_p}}{t^{\frac{2sp\vartheta_p}{1-m}}}, & \text{for } 0 < t \le T/2, \\ \left(\dfrac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+} \mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}}, & \text{for } T/2 < t \le T. \end{cases}$$

and

$$\underline{\mathsf{c}}[u_0] = \frac{\mathsf{c}_{m,p,s,N,\Omega}}{\left[\mathcal{Q}^*[u_0]^{\frac{4s\vartheta_p}{1-m}} \left(\dfrac{\|u_0\|_p^{2sp\,\vartheta_p}}{T^{N\vartheta_p}}\right)^{2s\vartheta_p[p-(1+m)]_+}\right]^{p-2m+1}}.$$

Theorem 3.5.6 (Global Harnack Principle II)**.** *Let $N > 2s > \gamma$, $m \in (m_{c,\gamma}, 1)$ and assume* (A1)*,* (A2) *and* (K3)*. Let u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$ with extinction time $T = T(u_0)$. Then, for all $t \in (0,T)$, we have*

$$\underline{\mathsf{c}} \left(\frac{t}{T}\right)^{\frac{m}{1-m}} (T-t)^{\frac{m}{1-m}} \le \frac{u^m(t,x_0)}{\Phi_1(x_0)} \le \bar{\mathsf{c}}[t]\,(T-t)^{\frac{m}{1-m}} \tag{3.5.16}$$

$$\bar{\mathsf{c}}[t] = \bar{\mathsf{c}} \cdot \begin{cases} \frac{1}{t^{\frac{m}{1-m}}}, & \text{for } 0 < t \le T/2, \\ 1, & \text{for } T/2 < t \le T, \end{cases}$$

where $\underline{\mathsf{c}}$ and $\bar{\mathsf{c}}$ depend only on m, p, s, N, Ω.

Remark 3.5.7. (i) Notice that, when $m \in (m_s, 1)$ we can choose $p = 1 + m$ in (3.5.15) and get matching exponents in the GHP:

$$\underline{\mathsf{c}}[u_0] \left(\frac{t}{T}\right)^{\frac{m}{1-m}} (T-t)^{\frac{m}{1-m}} \le \frac{u^m}{\Phi_1} \le \bar{\mathsf{c}}[u_0,t](T-t)^{\frac{m}{1-m}},$$

(ii) When $m \in (m_{c,\gamma}, 1)$, we remove the dependence of the constant of the GHP in the initial data, see definition of $\underline{\mathsf{c}}$ and $\bar{\mathsf{c}}(t)$ in (3.5.16).

3.6 Regularity

Once we know that solutions to (CDP) are globally positive and bounded, with precise pointwise bounds given by the GHP, the next natural question is whether or not they are continuous. Indeed, we show that they are Hölder continuous in

space and time and up to the boundary, cf. Theorem 3.6.1. For the general class of operators that we consider, this is the best result that one can expect, as the coefficients of the linear operator are, in general, only measurable and bounded. This is proven in Theorem 3.6.1, by means of a localization argument and Hölder estimates for linear parabolic operators with measurable coefficients. An intriguing open problem is represented by the optimality of the Hölder exponent. This part is still a work in progress, as we shall comment further on. On the one hand, we shall show that the solution can be classical in the interior, namely $C_{t,x}^{\beta/2s,\beta}$ for some $\beta \in (0,1)$, when the operator $\mathcal{L}$ allows it. We shall see how to use standard Schauder estimates for linear operators to obtain such improvement, namely Theorem 3.6.3. On the other hand, the GHP show that the optimal boundary regularity may differ from the one in the interior. We conjecture that the optimal exponent is determined by the power of the distance to the boundary in the GHP, namely $\alpha = \gamma/m > 0$. We notice that γ/m can be bigger than one, in which case the C^α regularity has to be understood in the sense of $C^{[\alpha],\alpha-[\alpha]}$, where $[\alpha]$ is the integer part of α. This would follow by exploiting the interior higher regularity and delicate localization and scaling arguments, in the spirit of [20, 22]. From now on, we will assume the following form of the operator:

$$\mathcal{L}f(x) = \iint_{\mathbb{R}^N} \frac{[f(x)-f(y)]a(x,y)}{|x-y|^{N+2s}}\,\mathrm{d}y + B(x)u(x), \tag{3.6.1}$$

where

$$\lambda \le a(x,y) \le \Lambda, \qquad \text{and} \quad 0 \le B \in L^\infty(\Omega).$$

for some $0 < \lambda, \Lambda < \infty$. In what follows, for the sake of simplicity, we will ignore the lower order term $B(x)u(x)$, but all the calculations work also with this term.

We are now ready to state the first main result of this section, concerning Hölder regularity for solutions.

Theorem 3.6.1 (Interior and boundary Hölder regularity)**.** *Let $N > 2s > \gamma$, $m \in (0,1)$ and assume* (A1)*,* (A2) *and* (K3)*. Let $\mathcal{L}$ be as in* (3.6.1) *and u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$. Then, for any $0 < t_0 < t_1 < T$*
Interior regularity: There exists $\alpha \in (0,1)$ such that

$$\|u\|_{C^{\alpha/2s,\alpha}((t_0,t_1)\times\Omega_\epsilon)} \le C_\epsilon, \tag{3.6.2}$$

where, for fixed $0 < \epsilon$, $\Omega_\epsilon \subset \Omega$ is such that, if $x_0 \in \Omega_\epsilon$ then $\epsilon < \mathrm{dist}(x_0, \partial\Omega)$.
Boundary regularity: For any $\alpha' \in (0, \frac{2s\gamma}{2sm+\gamma(1-m)} \wedge \alpha)$ we have that

$$\|u\|_{C^{\alpha'/2s,\alpha'}((t_0,t_1)\times\bar{\Omega})} \le C. \tag{3.6.3}$$

The proof follows the main lines of [20, 22]. We first localize the problem in a domain inside $(0,T)\times\Omega$. Next we rewrite the nonlinear problem as a linear one with "good" coefficients $a(t,x,y)$, to be able to exploit the linear theory for nonlocal parabolic operators, in our case represented by a celebrated result by Felsinger and Kassmann [64], recalled below. We can use the GHP of Theorem 3.5.5 to ensure

that "the coefficients $a(t,x,y)$ are uniformly parabolic", that is there exist constants $0 < \underline{\lambda} < \bar{\lambda}$ such that $\underline{\lambda} \le a(t,x,y) \le \bar{\lambda}$. We finally extend the C^α regularity *up to the boundary.*

We recall here the key result needed in the proof of Hölder regularity of solutions, which we have adapted from [64] to our notations and we have modified by adding a non homogeneous term f.

Theorem 3.6.2 (Felsinger-Kassmann [64])**.** *Let $\mathcal{L}$ be like.*

$$\mathcal{L}_a f(x) = P.V. \iint_{\mathbb{R}^N} [f(x) - f(y)] \frac{a(t,x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y$$

with $\Lambda^{-1} \le a(t,x,y) \le \Lambda$ for some $1 \le \Lambda < \infty$. Then, there exists $\beta \in (0,1)$ depending on N, s and Λ such that for every solution to $\partial_t u + \mathcal{L}u = f$ in $(0,1) \times B_1$ with $\|f\|_{L^\infty((0,1)\times B_1)} < C$ satisfies

$$\|u\|_{C^{\beta/2s,\beta}(1/2,1;B_{1/2})} \le C \left(\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)} + \|f\|_{L^\infty((0,1)\times B_1)} \right)$$

for some C that depends only on N, θ, m, s and Λ.

Proof of 3.5.6. We begin by proving the interior Hölder estimates. Given $(t_0, x_0) \in (0,T) \times \Omega$, let $R \ge 4$ such that

$$\frac{\mathrm{dist}(x_0, \partial\Omega)}{R} \le \left(\frac{T - t_0}{2} \right)^{1/\theta}$$

and we define

$$r = \frac{\mathrm{dist}(x_0, \partial\Omega)}{R}.$$

Now, define $\theta := 2s + \gamma(\frac{1}{m} - 1)$ and rescale the equation

$$\partial_t u + \mathcal{L}u^m = 0 \qquad \text{in} \quad (t_0, t_0 + r^\theta) \times B_r(x_0) \subset (0,T) \times \Omega \,,$$

so that

$$u_r(t,x) = r^{-\gamma/m} u(t_0 + r^\theta t, x_0 + rx) \,,$$

is solution of

$$\partial_t u_r + \mathcal{L}u_r^m = 0 \qquad \text{in} \quad (0,1) \times B_3(0) \,. \tag{3.6.4}$$

By Theorem 3.5.5, we have that for all $(t,x) \in (0,1) \times B_3(0)$

$$r^{-\gamma/m} \underline{c}[u_0] (T - t_0 - r^\theta t)^{\frac{1}{1-m}+\sigma^+} \le \frac{u_r(t,x)}{\mathrm{dist}(x_0 + rx, \partial\Omega)^{\gamma/m}} \le r^{-\gamma/m} \underline{c}[u_0] (T - t_0 - r^\theta t)^{\frac{1}{1-m}-\sigma} \,. \tag{3.6.5}$$

Notice that

i) $$1 \le r^{-\gamma/m} \mathrm{dist}(x_0 + rx, \partial\Omega)^{\gamma/m} \le 2R^{\gamma/m} \qquad \forall x \in B_3(0) \,,$$

ii) $$\frac{T - t_0}{2} \;\le\; (T - t_0 - r^\theta t) \;\le T - t_0 \qquad \forall t \in (0,1) \,.$$

Hence, from (3.6.5) we deduce the following bounds for $u_r(t,x)$ with $(t,x) \in (0,1) \times B_3(0)$,

$$\underline{c}[u_0](T-t_0)^{\frac{1}{1-m}+\sigma} \leq u_r(t,x) \leq \overline{c}[u_0](T-t_0)^{\frac{1}{1-m}-\sigma}\,. \tag{3.6.6}$$

Now, let $\eta \in C_c^\infty(B_2)$ be such that $\eta = 1$ in B_1. Then, we can write

$$u_r^m(t,x) - u_r^m(t,y) = [u_r(t,x) - u_r(t,y)]a(t,x,y) + h(t,x,y),$$

where

$$a(t,x,y) = \frac{u_r^m(t,x) - u_r^m(t,y)}{u_r(t,x) - u_r(t,y)}\eta(x-y) + [1-\eta(x-y)]$$

and

$$h(t,x,y) = [u_r^m(t,x) - u_r^m(t,y) - u_r(t,x) + u_r^m(t,y)](1-\eta(x-y)).$$

So we can write (3.6.4) as

$$\partial_t u_r + \mathcal{L}_{a,r} u_r = f_r \qquad \text{in} \quad (0,1) \times B_3(0)\,,$$

where

$$\mathcal{L}_{a,r} u_r(t,x) = \text{P.V.} \iint_{\mathbb{R}^N} (u_r(t,x) - u_r(t,y))a(t,x,y)K_r(x,y)\,\mathrm{d}y$$

and

$$f_r(t,x) = \text{P.V.} \iint_{\mathbb{R}^N} h(t,x,y)K_r(x,y)\,\mathrm{d}y\,.$$

Notice that, as $m \in (0,1)$ we have that

$$\frac{m}{\max\{u_r^{1-m}(t,x), u_r^{1-m}(t,y)\}} \leq \frac{u_r^m(t,x) - u_r^m(t,y)}{u_r(t,x) - u_r(t,y)} \leq \frac{m}{\min\{u_r^{1-m}(t,x), u_r^{1-m}(t,y)\}}.$$

Then, for $(t,x,y) \in (0,1) \times B_1 \times \mathbb{R}^N$ we have the following estimate

$$\lambda \leq a(t,x,y) \leq \Lambda,$$

where

$$\lambda = \frac{m}{\overline{c}[u_0]^{1-m}(T-t_0)^{1-\sigma(1-m)}}\eta(x-y) + [1-\eta(x-y)]$$

and

$$\Lambda = \frac{m}{\underline{c}[u_0]^{1-m}(T-t_0)^{1+\sigma(1-m)}}\eta(x-y) + [1-\eta(x-y)].$$

On the other hand, is easy to see that $h(t,x,y) = 0$ when $|x-y| \leq 1$. This, with the fact that u_r is bounded, leads to

$$\|f_r\|_{L^\infty(0,1)\times B_1} \leq C.$$

So we satisfy all conditions in Theorem 3.6.2 and we have that

$$\|u_r\|_{C^{\alpha/2s,\alpha}((1/2,1)\times B_{1/2})} \le C\left(\|u_r\|_{L^\infty((0,1)\times\mathbb{R}^N)} + \|f_r\|_{L^\infty((0,1)\times B_1)}\right)$$

Recalling the definition of u_r, the expression above implies

$$\sup_{\substack{\tau,\tau'\in(1/2,1)\\ x,x'\in B_{1/2}}} \frac{r^{-\gamma/m}|u(t_0+r^\theta\tau, x_0+rx)| - u(t_0+r^\theta\tau', x_0+rx')}{|\tau-\tau'|^{\frac{\alpha}{2s}} + |x-x'|^\alpha} \le C.$$

Now, we can write

$$\begin{aligned}
&\frac{r^{-\gamma/m}|u(t_0+r^\theta\tau, x_0+rx) - u(t_0+r^\theta\tau', x_0+rx')|}{|\tau-\tau'|^{\frac{\alpha}{2s}} + |x-x'|^\alpha} \\
&\qquad = \frac{r^{-\gamma/m}|u(t_0+r^{\theta\tau}, x_0+rx)| - u(t_0+r^{\theta\tau'}, x_0+rx')}{\frac{1}{r^{\frac{\alpha\theta}{2s}}}|r^\theta\tau - r^\theta\tau'|^{\frac{\alpha}{2s}} + \frac{1}{r^\alpha}|rx-rx'|^\alpha} \\
&\qquad \ge \frac{r^{-\frac{\gamma}{m}+\frac{\alpha\theta}{2s}}|u(t_0+r^{\theta\tau}, x_0+rx)| - u(t_0+r^{\theta\tau'}, x_0+rx')}{|r^\theta\tau - r^\theta\tau'|^{\frac{\alpha}{2s}} + |rx-rx'|^\alpha}
\end{aligned}$$

This gives

$$\|u\|_{C^{\alpha/2s,\alpha}((t_0+r^\theta/2,t_0+r^\theta)\times B_{r/2}(x_0))} \le C r^{\frac{\gamma}{m}-\frac{\alpha\theta}{2s}},$$

which, by a covering argument, yields the interior regularity result.

The Hölder regularity can be extended up to the boundary by noticing that if $\alpha \le \frac{\gamma 2s}{\theta m}$, the right hand side in the above formula does not depend on r. We can therefore let $r \to 0$ we get the regularity up to the boundary in Ω, namely inequality (3.6.3). □

The last main result of this section consists in showing that solutions are classical when the operator allows it. Let us be more precise: we shall provide a bound for the $C^{1+2s,\beta+2s}$ norm, where β is the Hölder regularity exponent of the coefficients of the operator $\mathcal{L}$. This implies that weak dual solutions of (CDP) are classical when bounded.

Theorem 3.6.3 (Classical solutions in the interior)**.** *Let $N > 2s > \gamma$, $m \in (0,1)$ and assume* (A1), (A2) *and* (K3)*. Let $\mathcal{L}$ be as in* (3.6.1) *and u be a nonnegative WDS of* (FFDE) *corresponding to the initial datum $u_0 \in L^p(\Omega)$. Assume in addition*

$$|a(x,y) - a(x',y')| \le c(|x-x'|^\beta + |y-y'|^\beta), \quad \textit{with} \quad \beta \in (0, 1\wedge 2s),$$

such that $\beta + 2s$ is not an integer. Then,

$$\|u\|_{C^{1+\beta/2s,2s+\beta}((t_0,t_1)\times\Omega_\epsilon)} \le C_\epsilon, \tag{3.6.7}$$

where C_ϵ depends only on the norm $\|u\|_{C^{\beta_1/2s,\beta_1}((t_0,t_1)\times\Omega_\epsilon)}$, where $\beta_1 = (\alpha\wedge\beta)$.

The proof relies on an improvement of the arguments used in the proof of Theorem 3.6.1, but we replace the linear result with a Schauder estimate for uniformly parabolic equations given by Dong and Zhang in [76, Theorem 1.1]. Unfortunately, their result only holds for "a priori" smooth solutions, hence it does not directly apply to weak (dual) solutions as the result by Felsinger and Kassmann. This difficulty can be overcome by an approximation argument, by means of mollifiers, incremental quotients, or simply by regularizing the initial data. As of now, this Theorem only has a "formal proof", which we have decided to include because it allows to appreciate the main ideas. Making it rigorous is a work in progress (a joint work with Bonforte and Ibarrondo, in final stages of preparation), and we sketch the approximation argument at the end of the proof.

Proof. Let u_r, $\mathcal{L}_{a,r}$ and f_r be as in the proof of Theorem 3.6.1. Once we know that $u_r \in C^{\frac{\alpha}{2s},\alpha}((1/2,1)\times B_{1/2})$, we just apply Theorem 1.1. by Dong and Zhang in [76]. For this, let us define $v = \rho U_r$, where $\rho \in C_c^\infty(B_{1/2})$ is a cut-off function such that $\rho \equiv 1$ in $B_{1/4}$. Then, for any $(t,x) \in (1/2)\times B_{1/4}$ we have

$$\partial_t u_r + \mathcal{L}_{a,r} u_r = \partial v_t + \mathcal{L}_{a,r}[v_r] + \mathcal{L}_{a,r}[u_r - v_r] = f_r,$$

this is,

$$\partial v_t + \mathcal{L}_{a,r} v_r = f_r + g_r, \quad \text{in } (1/2,1)\times B_{1/4},$$

where

$$g_r(t,x) = \mathcal{L}_{a,r}[v_r - u_r] = \text{P.V.} \iint_{\mathbb{R}^N\setminus B_{1/4}} u_r(t,y)(1-\rho(t,y))a(t,x,y)K_r(x,y)\,\mathrm{d}y\,.$$

Notice that $u_r(t,y)(1-\rho(t,y)) = 0$ in $B_{1/4}$. Let us define $\beta_1 = (\alpha \wedge \beta)$, where β is the one in the regularity assumption of $K(x,y)$. It is easy to see that

$$\|g_r + f_r\|_{C^{\beta_1/2s,\beta_1}((1/2,1)\times B_{1/4})} \le C_0 \quad \text{and} \quad v \in C^{\beta_1/2s,\beta}((1/2,1)\times\mathbb{R}^N),$$

and that

$$|a(t_1,x_1,y)K_r(x_1,y) - a(t_2,x_2,y)K_r(x_2,y)| \le \frac{C\,\left(|t_1-t_2|^{\beta_1/2s} + |x_1-x_2|^{\beta_1}\right)}{y^{N+2s}}$$

Then, formally, by Theorem 1.1. in [76] we have that

$$\|v\|_{C^{1+\beta_1/2s,2s+\beta_1}((3/4,1)\times B_{1/8})} \le C\left(\|v\|_{C^{\beta_1/2s,\beta_1}((1/2,1)\times B_{1/4})} + C_0\right)$$

At this point, if $\alpha \ge \beta$, and thus $\beta_1 = \beta$, we are done. Otherwise, we repeat the whole argument with $\beta_1 = \beta$ and we get

$$\|u_r\|_{C^{1+\beta/2s,2s+\beta}((7/8,1)\times B_{1/16})} \le C\left(\|u_r\|_{C^{\beta/2s,\beta}((3/4,1)\times B_{1/8})} + C_0\right) \tag{3.6.8}$$

Recalling the definition of u_r, by a covering argument we get (3.6.7).

This concludes the proof for smooth solutions, hence it is still formal for general bounded (hence Hölder continuous) WDS. This is due to the fact that Theorem

1.1. in [76] only applies to $C^{1+\beta/2s,2s+\beta}$ solutions, for some $\beta \in (0,1)$. In order to make the previous proof rigorous, it is sufficient to approximate the initial data by a sequence of $u_{0,n}$ of smooth functions, such that $u_{0,n} \xrightarrow{n\to\infty} 0$. Next, we need to prove that smooth data generate sufficiently smooth solutions, namely $u_{0,n} \in C_c^\infty$ implies (at least) $u_n \in C^{1+\beta/2s,2s+\beta}$, for which the above proof is rigorous and estimate (3.6.8) is valid for such u_n. Since the constants involved in estimate (3.6.8) do not depend on u_n, we are allowed to take the limit as $n \to \infty$ and conclude the proof. □

Appendix B

Useful inequalities

Operators with nonnegative kernels. Let $\mathcal{L}$ be an operator of the form

$$\mathcal{L}[v](x) := \text{P.V.} \int_\Omega [v(x) - v(y)]\, \mathbb{K}(x,y)\, \mathrm{d}y + B(x)v(x) \qquad \text{with} \qquad \mathbb{K}, B \geq 0\,. \tag{3.6.9}$$

We show next a simple formal proof of Kato and Stroock-Varopoulos ([131, 136]) inequalities, without any claim of originality, indeed different proofs appear in literature for operators like $\mathcal{L}$. Our proof of Stroock-Varopoulos is inspired from [109], and from personal communications with the authors of that paper. Of course, justifying that all the integrals involved are finite would make this proofs rigorous. Throughout the paper, we pay attention to this fact and all the involved quantities are finite.

Lemma 3.6.4. (Kato's inequality). *Let $\mathcal{L}$ be an operator as in* (3.6.9)*, and let $f \in C^0(\mathbb{R})$ be a convex function with $f(0) \leq 0$. Then, if $\mathcal{L}[v] \in L^1_{\mathrm{loc}(\Omega)}$, Kato inequality holds in the sense of distributions:*

$$\mathcal{L}f(v) \leq f'(v)\, \mathcal{L}v\,. \tag{3.6.10}$$

Proof. Recall that a convex function lies above its tangent:

$$f(v) - f(w) \leq f'(v)(v-w) \qquad \forall v, w \in \mathbb{R}.$$

Then, since $\mathbb{K}(x,y) > 0$ and $B(x) \geq 0$, it holds that

$$\begin{aligned}
\mathcal{L}f(v(x)) &= \int_{\mathbb{R}^N} [f(v(x)) - f(v(y))]\, \mathbb{K}(x,y)\, \mathrm{d}y + B(x) f(v(x)) \\
&\leq \int_{\mathbb{R}^N} [f(v(x)) - f(v(y))]\, \mathbb{K}(x,y)\, \mathrm{d}y + B(x)\, [f(v(x)) - f(0)] \\
&\leq \int_{\mathbb{R}^N} f'(v(x))\, [v(x) - v(y)]\, \mathbb{K}(x,y)\, \mathrm{d}y + B(x) f'(v(x)) v(x) \\
&= f'(v(x)) \mathcal{L}f(v(x))\,.
\end{aligned}$$

□

Lemma 3.6.5. (Stroock-Varopoulos inequality). *Let $\mathcal{L}$ be an operator as in* (3.6.9) *with $s \in (0,1)$. Then for any $q > 1$*

$$\int_\Omega v^{q-1}\mathcal{L}v\,\mathrm{d}x \geq \frac{4(q-1)}{q^2}\iint_{\mathbb{R}^N} \mathcal{L}^{1/2}v^{q/2}{}^2\,\mathrm{d}x \tag{3.6.11}$$

for all $v \in L^q(\Omega)$ such that $\mathcal{L}^{s/2}v \in L^q(\Omega)$.

Proof. In this proof we will use the following inequality: let $0 \leq \alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 2$, then

$$(a^\alpha - b^\alpha)(a^\beta - b^\beta) \geq \alpha\beta(a-b)^2, \qquad \forall a, b \in \mathbb{R}, \tag{3.6.12}$$

which can be proved just by Cauchy-Schwarz inequality as follows

$$\begin{aligned}(a-b)^2 = \Big(\int_b^a \mathrm{d}t\Big)^2 = \Big(\int_b^a t^{\frac{\alpha-1}{2}}\,t^{\frac{\beta-1}{2}}\,\mathrm{d}t\Big)^2 &\leq \Big(\int_b^a t^{\alpha-1}dt\Big)\Big(\int_b^a t^{\beta-1}dt\Big)\\ &= \frac{(a^\alpha - b^\alpha)}{\alpha}\frac{(a^\beta - b^\beta)}{\beta}.\end{aligned}$$

Using the symmetry of the kernel, Fubini's theorem and inequality (3.6.12) we get

$$\begin{aligned}\iint_{\mathbb{R}^N} v^{q-1}\mathcal{L}v\,\mathrm{d}x &= \int_\Omega \Big[\int_\Omega v^{q-1}(x)\big(v(x)-v(y)\big)\mathbb{K}(x,y)\,\mathrm{d}y + B(x)v^q(x)\Big]\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega\int_\Omega \big(v^{q-1}(x)-v^{q-1}(y)\big)\big(v(x)-v(y)\big)\mathbb{K}(x,y)\,\mathrm{d}y\,\mathrm{d}x + \iint_{\mathbb{R}^N} B(x)v^q(x)\,\mathrm{d}x\\ &\geq \frac{1}{2}\frac{4(q-1)}{q^2}\int_\Omega\int_\Omega \big(v^{q/2}(x)-v^{q/2}(y)\big)^2\mathbb{K}(x,y)\,\mathrm{d}y\,\mathrm{d}x + \iint_{\mathbb{R}^N} B(x)v^q(x)\,\mathrm{d}x\\ &= \frac{4(q-1)}{q^2}\iint_{\mathbb{R}^N} v^{q/2}\mathcal{L}v^{q/2}.\end{aligned}$$

Now, we can use the assumption (L1) to get

$$\iint_{\mathbb{R}^N} v^{q/2}\mathcal{L}v^{q/2} \asymp \int_\Omega\int_\Omega \frac{\big(v^{q/2}(x)-v^{q/2}(y)\big)^2}{|x-y|^{N+2s}}\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega B(x)v^q(x)\,\mathrm{d}x,$$

which tells us that $\iint_{\mathbb{R}^N} v^{q/2}\mathcal{L}v^{q/2}$ is a Dirichlet form (symmetric and closed). This and the fact that $-\mathcal{L}$ is a non-positive self-adjoint operator allows us to write

$$\int_\Omega v^{q/2}\mathcal{L}v^{q/2} = \iint_{\mathbb{R}^N} \mathcal{L}^{1/2}v^{q/2}{}^2\,\mathrm{d}x\,.$$

□

We recall a variant of a brilliant Lemma due to De Giorgi, often used in Elliptic equations that we adapt here for our purposes.

Lemma 3.6.6. (De Giorgi). *Let $Z(t)$ be a bounded nonnegative function in the interval $[t_0, t_1]$. Assume that for $t_0 \leq \tau < t \leq t_1$ we have*

$$Z(t) \leq \theta Z(\tau) + \frac{A}{(t-\tau)^\alpha} + \frac{B}{(t-s)^\beta} + C \tag{3.6.13}$$

with $A, B, C \geq 0$, $\alpha > 0$ *and* $0 \leq \theta \leq 1$. *Then*

$$Z(t_1) \leq \frac{Ac(\alpha,\lambda,\theta)}{(t_1-t_0)^\alpha} + \frac{Bc(\beta,\lambda,\theta)}{(t_1-t_0)^\beta} + \frac{C}{(1-\theta)} \tag{3.6.14}$$

where

$$c(\alpha,\lambda,\theta) = \frac{1}{(1-\lambda)^\alpha\left(1-\frac{\theta}{\lambda^\alpha}\right)} \qquad \textit{for any} \qquad \lambda \in (\theta^{\frac{1}{\alpha}}, 1)\,.$$

Proof. We define the sequence $\{\tau_i\}$ as follows

$$\tau_0 = t_1, \quad \tau_{i+1} = \tau_i - (1-\lambda)\lambda^i(t_1-t_0)$$

so $\tau_k = t_1 - \sum_{i=0}^{k-1}(1-\lambda)\lambda^i(t_1-t_0)$ and $\lim_{k\to\infty}\tau_k = t_0$. Now, we can iteratively write (3.6.13) as

$$\begin{aligned}
Z(\tau_0) \leq& \theta Z(\tau_1) + \frac{A}{(\tau_0-\tau_1)^\alpha} + \frac{B}{(\tau_0-\tau_1)^\beta} + C \\
\leq& \theta^2 Z(\tau_2) + \frac{\theta A}{(\tau_1-\tau_2)^\alpha} + \frac{\theta B}{(\tau_1-\tau_2)^\beta} + \theta C + \frac{A}{(\tau_0-\tau_1)^\alpha} + \frac{B}{(\tau_0-\tau_1)^\beta} + C \\
&\vdots \\
\leq& \theta^k Z(\tau_k) + \frac{A}{(1-\lambda)^\alpha(t_1-t_0)^\alpha}\sum_{i=0}^{k-1}\left(\frac{\theta}{\lambda^\alpha}\right)^i + \frac{B}{(1-\lambda)^\beta(t_1-t_0)^\beta}\sum_{i=0}^{k-1}\left(\frac{\theta}{\lambda^\beta}\right)^i + C\sum_{i=0}^{k-1}\theta^i.
\end{aligned}$$

Finally, as the series on the right-hand side converges, we take the limit as $k \to \infty$ and we get (3.6.14). □

Technical proofs

Lemma 3.6.7. *(Decay of* $L^1_{\Phi_1}$*-norm). Let* $u(t)$ *be a nonnegative WDS, then*

$$\|u(t_1)\|_{L^1_{\Phi_1}} \leq \|u(t_0)\|_{L^1_{\Phi_1}} \qquad \textit{for every} \quad 0 < t_0 < t_1 < T\,.$$

Proof. We want to use the inadmissible test function $\psi(t,x) = \chi_{[t_0,t_1]}(t)\Phi_1(x)$ in the equation, therefore we consider the C^1_c approximation in time $\psi_n(t,x) = \chi_n(t)\Phi_1(x)$ where

$$\chi_n(t) = \begin{cases} \frac{n^2}{2}(t-t_0+\frac{1}{n})^2 & \text{if} \quad t \in [t_0-\frac{1}{n}, t_0), \\ 1-\frac{n^2}{2}(t-t_0-\frac{1}{n})^2 & \text{if} \quad t \in [t_0, t_0+\frac{1}{n}), \\ 1 & \text{if} \quad t \in [t_0+\frac{1}{n}, t_1-\frac{1}{n}) \\ 1-\frac{n^2}{2}(t-t_1+\frac{1}{n})^2 & \text{if} \quad t \in [t_1-\frac{1}{n}, t_1), \\ \frac{n^2}{2}(t-t_1-\frac{1}{n})^2 & \text{if} \quad t \in [t_1, t_1+\frac{1}{n}), \\ 0 & \text{otherwise}\,. \end{cases}$$

Using the LHS of (WDS), we obtain

$$\int_0^T\int_\Omega \mathcal{L}^{-1}u\;\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega u\;\partial_t\chi_n\,\mathcal{L}^{-1}\Phi_1\,\mathrm{d}x\,\mathrm{d}t = \lambda_1^{-1}\int_0^T \|u(t)\|_{L^1_{\Phi_1}}\;\partial_t\chi_n(t)\,\mathrm{d}t\,.$$

Let us show that $\partial_t \chi_n$ is an approximation of $\delta_{t_0} - \delta_{t_1}$. We know that

$$\partial_t \chi_n = \begin{cases} n^2(t - t_0 + \frac{1}{n}) & \text{if} \quad t \in [t_0 - \frac{1}{n}, t_0), \\ -n^2(t - t_0 - \frac{1}{n}) & \text{if} \quad t \in [t_0, t_0 + \frac{1}{n}), \\ -n^2(t - t_1 + \frac{1}{n}) & \text{if} \quad t \in [t_1 - \frac{1}{n}, t_1), \\ n^2(t - t_1 - \frac{1}{n}) & \text{if} \quad t \in [t_1, t_1 + \frac{1}{n}), \\ 0 & \text{otherwise}. \end{cases}$$

Now, let us consider the following approximation of the identity:

$$h_n(t) = \begin{cases} n^2(t + \frac{1}{n}) & \text{if} \quad t \in [-\frac{1}{n}, 0), \\ -n^2(t - \frac{1}{n}) & \text{if} \quad t \in [0, \frac{1}{n}), \\ 0 & \text{otherwise}. \end{cases}$$

It can be seen that for any $f \in C(\mathbb{R})$ it holds that $\lim_{n\to\infty} \int_{\mathbb{R}} f(t) h_n(t)\,\mathrm{d}t = f(0)$, since we can take limits in

$$\inf_{t\in[-\frac{1}{n},\frac{1}{n}]} \{f(t)\} \leq \int_{\mathbb{R}} f(t) h_n(t)\,\mathrm{d}t \leq \sup_{t\in[-\frac{1}{n},\frac{1}{n}]} \{f(t)\} \qquad \forall n > 0\,.$$

Notice that the $\inf\{f\}$ and the $\sup\{f\}$ above are attained due to Weierstrass Theorem.
Then, as we can rewrite $\partial\chi_n(t) = h_n(t - t_0) - h_n(t - t_1)$, we conclude that

$$\begin{aligned} \|u(t_0)\|_{L^1_{\Phi_1}} - \|u(t_1)\|_{L^1_{\Phi_1}} &= \lim_{n\to\infty} \int_0^T \|u(t)\|_{L^1_{\Phi_1}} \big(h_n(t - t_0) - h_n(t - t_1)\big)\,\mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^T \|u(t)\|_{L^1_{\Phi_1}} \partial_t \chi_n(t)\,\mathrm{d}t \\ &= \lim_{n\to\infty} \lambda_1 \int_0^T \int_\Omega u^m \Phi_1 \chi_n\,\mathrm{d}x\,\mathrm{d}t \geq 0\,, \end{aligned}$$

since using ψ_n as a test function in (WDS), it follows that

$$\int_0^T \|u(t)\|_{L^1_{\Phi_1}} \partial_t \chi_n(t)\,\mathrm{d}t = \lambda_1 \int_0^T \int_\Omega u^m\, \Phi_1\, \chi_n\,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \forall n > 0\,.$$

□

Proof of Lemma 3.3.3. We define $j(r) = \frac{1}{m+1}|r|^{m+1}$ and we notice that it satisfies $\lim_{|r|\to+\infty} \frac{j(r)}{|r|} = +\infty$. Convexity condition follows from the convexity of j, thus, let us check the l.s.c condition. We have two different convergence settings:

- Let $u_n \to u$ in $H^*(\Omega)$ and assume $\forall \lambda > 0 \;\; \exists n \in \mathbb{N}$ such that $\int_\Omega j(u_n) > \lambda$. Then, it holds that
$$\liminf_{n\to\infty} \mathcal{E}(u_n) = +\infty \geq \mathcal{E}(u)\,.$$
- Let $u_n \to u$ in $H^*(\Omega)$ and suppose $\exists \lambda > 0$ such that $\forall n \in \mathbb{N} \;\; \int_\Omega j(u_n) \leq \lambda$.

In this second case, we follow the same strategy as Brezis does for $H^{-1}(\Omega)$ in [34]. First, we show that $(u_n)_{n\in\mathbb{N}}$ is uniformly integrable, i.e.

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\text{ such that if }\; |E| < \delta \;\text{ then }\; \iint_E |u_n| < \epsilon \;\; \forall n \in \mathbb{N}.$$

Let $A > \frac{2\lambda}{\epsilon}$ and R be such that $\frac{j(r)}{|r|} \geq A$ for $|r| > R$. If we choose $\delta < \frac{\epsilon}{2R}$, then

$$\begin{aligned}\int_E |u_n| dx &= \iint_{\{x\in E: u_n(x)\geq R\}} |u_n| \,\mathrm{d}x + \iint_{\{x\in E: u_n(x)<R\}} |u_n| \,\mathrm{d}x \\ &\leq \int_\Omega \frac{j(u_n)}{A} \,\mathrm{d}x + \iint_E R \,\mathrm{d}x \leq \frac{\lambda}{A} + R\delta < \varepsilon\,.\end{aligned}$$

Then, we apply Dunford-Pettis theorem, which states that in bounded domains uniformly integrable implies weakly precompact in $L^1(\Omega)$. Hence, $(u_n)_{n\in\mathbb{N}}$ has a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges weakly in $L^1(\Omega)$, that is,

$$\exists \tilde{u} \in L^1(\Omega) \;\; \text{ such that } \;\; u_{n_k} \rightharpoonup \tilde{u} \;\text{ in }\; L^1(\Omega).$$

Further, $u = \tilde{u}$ in $H^*(\Omega)$, as $\{u_n\}$ already converges weakly to u in $H^*(\Omega)$ and $C_c^\infty(\Omega) \subset L^\infty(\Omega) \cap H(\Omega)$, which is dense in H.

Now, to conclude the proof, we just have to show that $\mathcal{E}(u)$ is weakly lower semicontinuous in $L^1(\Omega)$. So, let $u_n \to u$ in $L^1(\Omega)$, then, there exists a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ which converges almost everywhere to u. Since j is non negative, we can apply Fatou's Lemma,

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega j(u_n) \,\mathrm{d}x &\geq \iint_\Omega \liminf_{n\to\infty} j(u_n) \,\mathrm{d}x = \iint_\Omega \lim_{n\to\infty} \inf_{m\geq n} \{j(u_n)\} \,\mathrm{d}x \\ &\geq \iint_\Omega \lim_{l\to\infty} \inf_{m\geq n_l} \{j(u_{n_k})\} \,\mathrm{d}x = \int_\Omega \liminf_{l\to\infty} j(u_{n_l}) \,\mathrm{d}x = \iint_\Omega j(u) \,\mathrm{d}x,\end{aligned}$$

where we have used in the last equality that j is continuous (notice that it suffices to be l.s.c.). Finally, since $\mathcal{E}(u)$ is convex and l.s.c. in $L^1(\Omega)$, it holds that $\mathcal{E}(u)$ is weakly l.s.c. in $L^1(\Omega)$ by Mazur's Lemma.

□

Chapter 4

A quasilinear Keller-Segel model with saturated discontinuous advection

In this chapter we present the results of the preprint *A quasilinear Keller-Segel model with saturated discontinuous advection*, [75], a joint work of the author of this thesis with Maria Gualdani and Nicola Zamponi.

4.1 Introduction

Since the classical experiment of J. Adler on *Escherichia coli* [1] many experiments have confirmed the formation and propagation of traveling pulses of bacteria. The Keller-Segel model has been widely used as a basis to explain this phenomena in bacterial colonies driven by chemotaxis, see [86, 100]: it describes how the cell density moves towards a nutrient N and a chemical molecule S (*chemoatractant*) which is generated by the cells themselves. The velocity of this movement is proportional to the gradient of these two quantities. The Keller-Segel model has been the subject of intense mathematical investigation.

In recent works ([44, 59, 65, 108, 117, 127]), a novel class of models has been formulated, originating from a mesoscopic description of the *run-and-tumble* phenomenon at the individual cell level. Mathematically, these models deviate from the classical Keller-Segel framework in that the cells' velocity (or flux) is saturated and is proportional to $\frac{\nabla S}{|\nabla S|}$ and $\frac{\nabla N}{|\nabla N|}$. The model, in its more general form, reads as

$$\begin{aligned}\partial_t \rho &= D_\rho \Delta \rho - \operatorname{div}(\rho u_S + \rho u_N),\\ \partial_t S &= D_S \Delta S - \alpha S + \beta \rho,\\ \partial_t N &= D_N \Delta N - \gamma \rho N,\end{aligned}$$

where

$$u_S = \chi_S J(\partial_t S, |\nabla S|)\frac{\nabla S}{|\nabla S|}, \quad u_N = \chi_N J(\partial_t N, |\nabla N|)\frac{\nabla N}{|\nabla N|}.$$

In [127], the authors present an explicit formula for traveling pulses and establish the existence of a steady state, mirroring the physical occurrence of cluster formation

at the bacteria level (e.g., in the absence of nutrients). They also demonstrate the alignment of numerical experiments with actual data. The bounded fluxes, distinct from the flux in Keller-Segel, offer a significant advantage in investigating global well-posedness and, notably, in exploring traveling pulses and long-term asymptotics. Overall, this model has demonstrated greater robustness in replicating the observed traveling bands in experiments, see [127].

More recently, Calvez and Hoffmann [42] considered a simplified version of the above model (absence of nutrient and quasi-stationary chemoattractant signal)

$$\begin{cases} \partial_t \rho - \Delta \rho + \chi \mathrm{div}\left(\rho \frac{\nabla S}{|\nabla S|}\right) = 0, \\ -\Delta S + \alpha S = \rho, \end{cases} \tag{4.1.1}$$

and perform nonlinear stability analysis. In particular they show that if the initial data is near, in some norm, to the equilibrium function $\rho_\infty = \frac{\chi}{2} e^{-\chi|x-x_0|}$, then the corresponding solution converges to ρ_∞ exponentially fast.

In this chapter, we deal with the model obtained from (4.1.1) by performing the singular limit in α. After rescaling $S \mapsto S/\alpha$, the limit $\alpha \to \infty$ yields the following quasi-linear problem:

$$\partial_t \rho - \Delta \rho + \chi \mathrm{div}\left(\rho \frac{\nabla \rho}{|\nabla \rho|}\right) = 0. \tag{4.1.2}$$

Unlike (4.1.1), the advection depends on the gradient of the density, making it a competing term with the diffusion. Note, in fact, that

$$\Delta \rho - \chi \mathrm{div}\left(\rho \frac{\nabla \rho}{|\nabla \rho|}\right) = \mathrm{div}\left(\nabla \rho \left(1 - \frac{\chi \rho}{|\nabla \rho|}\right)\right)$$

which shows negative diffusion for profiles that have $\chi\rho \geq |\nabla\rho|$. For example, Gaussian functions $e^{-|x|^2}$ will make the diffusion coefficient negative around the zero. The system is diffusive as long as $|\nabla\rho| > \chi\rho$, and the steady-state is governed by the equation

$$|\nabla \rho_\infty| = \chi \rho_\infty. \tag{4.1.3}$$

One solution to (4.1.3) is

$$\rho_\infty(x) = c e^{-\chi|x-x_0|}, \tag{4.1.4}$$

for some c_1, x_0 determined by the conservation of mass and the property of invariance by translation of the equation. Other admissible (i.e. belonging to $L^1(\mathbb{R}^d)$) solutions to (4.1.3) exist; for example, for $d = 1$ another solution is the "multi-peak profile"

$$\rho_\infty(x) = \max\{c_0 \exp(-\chi|x-x_0|), c_1 \exp(-\chi|x-x_1|), \ldots c_k \exp(-\chi|x-x_k|)\}, \tag{4.1.5}$$

for a suitable choice of $k \geq 0$, $c_0 \ldots c_k > 0$, and $x_0 \ldots x_{k+1} \in \mathbb{R}$, while for $d \geq 2$ another solution is the "factorized profile"

$$\rho_\infty(x) = c \exp\left(-\frac{\chi}{\sqrt{d}} \sum_{i=1}^{d} |x_i - x_{0,i}|\right) \tag{4.1.6}$$

which equals the product of (suitably rescaled) single variable steady states. The proof of existence of solution to (4.1.2) is still out of reach. The main difficulty is the control on the diffusion coefficient, which can change along the flow and destroy the parabolic nature of the equation. We believe that if the initial data makes the diffusion coefficient positive, the coefficient will remain non-negative throughout the evolution. This is, however, still a claim. Problem (4.1.2) has a more tractable form in one dimension

$$\partial_t \rho - \partial_{xx}\rho + \chi \partial_x \left(\rho \, \mathrm{sign}(\rho_x)\right) = 0,$$

and is the subject of current investigation [74].

As a first attempt to understand (4.1.2), we introduce an alternative model that retains some of the essential characteristics of (4.1.2) while being mathematically more tractable. Specifically, we consider

$$\begin{cases} \partial_t \rho - \mathrm{div}\left[\left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho\right] = 0, & \text{in } \mathbb{R}^d \times (0,\infty), \\ \rho(0) = \rho_{in}, & \text{in } \mathbb{R}^d, \end{cases} \tag{4.1.7}$$

where $(\cdot)_+$ denotes the positive part. Here we use the convention $f = f_+ - f_-$. Equation (4.1.7) is quasilinear. The type of nonlinearity poses major challenges in the construction of solutions. Let us, however, quickly summarize why global well-posedness for (4.1.7) is more in-reach. First, observe that the operator

$$\left(1 - \frac{\chi}{|\nabla \log \rho|}\right)_+ \nabla \log \rho$$

is monotone. More precisely,

$$\left[\left(1 - \frac{\chi}{|\nabla \log \rho|}\right)_+ \nabla \log \rho - \left(1 - \frac{\chi}{|\nabla \log z|}\right)_+ \nabla \log z\right](\nabla \log \rho - \nabla \log z) \leq 0.$$

This monotonicity property, coupled with the theory of monotone operators and a fixed-point argument, proves fundamental in the proof of existence of solutions to (4.1.7) equipped with extra viscosity terms $\varepsilon\Delta\rho$ and $\varepsilon\rho$. The crux of the analysis is the vanishing viscosity limit $\varepsilon \to 0$ and the corresponding identification of the limit of the nonlinear term

$$\left(1 - \frac{\chi}{|\nabla \log \rho_\varepsilon|}\right)_+ \nabla \rho_\varepsilon.$$

A common strategy would involve showing strong convergence of one of the two terms. However, standard regularity techniques, usually based on obtaining estimates for the first derivative through second-order test functions, prove unsuitable here due to the discontinuity of the advection term and the discontinuity of $\nabla\rho$ itself. We overcome this problem using the Boltzmann relative entropy between two viscosity solutions. Let $H[\rho_\varepsilon, \rho_\delta]$ be the nonnegative functional defined as

$$H[\rho_\varepsilon, \rho_\delta] = \int_{\mathbb{R}^d} \rho_\delta \left[\frac{\rho_\varepsilon}{\rho_\delta} \log \frac{\rho_\varepsilon}{\rho_\delta} - \frac{\rho_\varepsilon}{\rho_\delta} + 1\right] \mathrm{d}x.$$

This functional measures the distance, in L^1 norm, between the functions ρ_ε and ρ_δ. Direct computations show that

$$\begin{aligned}\frac{d}{dt}H[\rho_\varepsilon,\rho_\delta] =& -\frac{1}{2}\int_{\mathbb{R}^d}\rho_\varepsilon\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\right]^2\mathrm{d}x\\ &-\frac{1}{2}\int_{\mathbb{R}^d}\rho_\varepsilon\ \nabla\log\left(\frac{\rho_\varepsilon}{\rho_\delta}\right)^2\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ + \left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\right]\mathrm{d}x\\ &+\text{error order terms},\end{aligned}$$

where the error terms derive from the viscosities $\varepsilon\Delta\rho_\varepsilon$, $\varepsilon\rho_\varepsilon$, $\delta\Delta\rho_\delta$ and $\delta\rho_\delta$. As ε and δ approach zero, the error terms vanish, and consequently

$$-\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\right]^2\mathrm{d}x\,\mathrm{d}t\to 0,$$

and

$$-\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\ \nabla\log\left(\frac{\rho_\varepsilon}{\rho_\delta}\right)^2\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ + \left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\right]\mathrm{d}x\,\mathrm{d}t\to 0.$$

Note that $H[\rho_\varepsilon,\rho_\delta](0)=0$ because $\rho_\varepsilon$and ρ_δ have same initial data. The two limits above will imply that, as ε and δ approach zero,

$$\left(1-\frac{\chi}{|\nabla\log\rho_\varepsilon|}\right)_+\nabla\rho_\varepsilon \rightharpoonup \left(1-\frac{\chi}{|\nabla\log\rho|}\right)_+\nabla\rho,$$

weakly in L^2. The identification of this limit is the content of Section 4.4.1. Almost automatically, the relative entropy argument will also provide uniqueness of the solution for the system without viscosity. In conclusion, our analysis establishes the existence of a unique solution in $L^p(t,T;C^\alpha(\mathbb{R}^d))$ for any $t>0$. We obtain regularity for $\partial_t\rho$ in Sobolev spaces with negative coefficients. The question of whether one can obtain more regularity is left for future investigation. Potential insights might be drawn from existing results on the p-Laplacian equation [54, 68]. However, the structure of our degenerate set is slightly more complicated than the one of the p-Laplacian, namely $|\nabla\rho|=0$. A De Giorgi argument for Hölder regularity might pose challenges, since our diffusion coefficient is degenerate, probably in a set of positive measure, and depends in a nonlinear way on $\nabla\log\rho$.

Here is our main result:

Theorem 4.1.1. *Let $\rho_{\mathrm{in}}\in L^p(\mathbb{R}^d)\cap L^1_2(\mathbb{R}^d)$, with $p>d$, $\rho_{\mathrm{in}}>0$ a.e. in $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d}\rho_{\mathrm{in}}|\log\rho_{\mathrm{in}}|<\infty$. Then, for every $T>0$ there exists an unique solution $\rho:[0,T)\times\mathbb{R}^d\to\mathbb{R}$ to (4.1.7) such that*

$$\int_0^T\int_{\mathbb{R}^d}\partial_t\rho\,\varphi\ \mathrm{d}x\,\mathrm{d}t=-\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi\rho}{|\nabla\rho|}\right)_+\nabla\rho\nabla\varphi\ \mathrm{d}x\,\mathrm{d}t,\quad\forall\varphi\in L^2(0,T;H^1(\mathbb{R}^d)),\tag{4.1.8}$$

with $\rho\to\rho_{\mathrm{in}}$ strongly in $L^p(\mathbb{R}^d)$ as $t\to 0$. This solution belongs to $C(0,T;L^p(\mathbb{R}^d))\cap L^p(t,T,C^\alpha(\mathbb{R}^d))$ for all $t>0$ and has the following properties:

(i) Mass Conservation:

$$\|\rho(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_{\text{in}}\|_{L^1(\mathbb{R}^d)}.$$

(ii) L^p norm bound:

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leq \|\rho_{\text{in}}\|_{L^p(\mathbb{R}^d)}.$$

(iii) Moment evolution:

$$\int_{\mathbb{R}^d} \rho(t)(1+|x|^2)\,\mathrm{d}x \leq e^{C\,t} \iint_{\mathbb{R}^d} \rho_{\text{in}}(1+|x|^2)\,\mathrm{d}x.$$

(iv) Entropy bound:

$$\left(\iint_{\mathbb{R}^d} \rho|\log\rho|\,\mathrm{d}x\right)(T) + \int_0^T \iint_{\mathbb{R}^d} \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \leq \bar{C}[\rho_{\text{in}}].$$

(iv) L^∞ smoothing:

$$\sup_{t>0} \|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C\|\rho_{\text{in}}\|_{L^p(\mathbb{R}^d)} \left(1 + \frac{1}{t^{\frac{d}{2p}}}\right)($$

(v) L^p bound of the gradient:

$$\|\nabla\rho_\varepsilon\|_{L^p((t,T)\times\mathbb{R}^d)} < C(\rho_{\text{in}}), \quad \forall\, t > 0.$$

The proof contains two approximation steps: initially we introduce viscosity and regularize the initial data f_{in} with a sequence of very smooth functions that converge (strongly in L^p) to f_{in}. We establish the existence of solutions to the approximated problem and subsequently remove the viscosity. In the last step we show how the solution attains the initial data strongly in L^p sense.

Plan of the chapter. The following section provides a concise summary of the most relevant a-priori estimates. Section 4.3 delves into the global well-posedness of the model with viscosity. Finally, Section 4.4 focuses on the vanishing viscosity limit and establishes uniqueness.

4.2 A priori estimates and technical results

In this section we give some a priori estimates of the solutions to

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \partial_t\rho\varphi\,\mathrm{d}x\,\mathrm{d}t = &-\int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)\nabla\rho_\varepsilon\nabla\varphi\;\mathrm{d}x\,\mathrm{d}t \\ &-\varepsilon\int_0^T \int_{\mathbb{R}^d} \nabla\rho_\varepsilon\nabla\varphi\;\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi\;\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.2.1}$$

for any $\varphi \in L^2(0,T;H^1(\mathbb{R}^d))$ with initial smooth data ρ_0. The function ρ_0 belongs to $W^{1,p} \cap L^\infty \cap L^1_2(\mathbb{R}^d)$, with $p > d$.

The following a priori estimates hold:

Lemma 4.2.1 (Mass evolution)**.** *Let ρ_ε be as stated, then*

$$\int_{\mathbb{R}^d} \rho_\varepsilon(t) = e^{-\varepsilon t} \int_{\mathbb{R}^d} \rho_0.$$

Proof. Testing (4.2.1) with the test function $\varphi = \frac{1}{\left(1+\frac{|x|^2}{R^2}\right)^{\alpha/2}}$, for some $\alpha > d$, we get

$$\begin{aligned}\left(\int_{\mathbb{R}^d} \rho_\varepsilon \varphi \,\mathrm{d}x\right)(T) = \ & (1+\varepsilon)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \Delta\varphi \,\mathrm{d}x\,\mathrm{d}t + \chi \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \frac{\nabla\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ & - \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_- \nabla\rho_\varepsilon\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x.\end{aligned} \tag{4.2.2}$$

But notice that

$$\begin{aligned}-\int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_- \nabla\rho_\varepsilon\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^d\cap\{|\nabla\rho_\varepsilon|\le\chi\rho_\varepsilon\}} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right) \nabla\rho_\varepsilon\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \int_{\mathbb{R}^d\cap\{|\nabla\rho_\varepsilon|\le\chi\rho_\varepsilon\}} \nabla\rho_\varepsilon\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \chi\int_0^T \int_{\mathbb{R}^d\cap\{|\nabla\rho_\varepsilon|\le\chi\rho_\varepsilon\}} \rho_\varepsilon \frac{\nabla\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\le (1+\chi)\int_0^T \int_{\mathbb{R}^d\cap\{|\nabla\rho_\varepsilon|\le\chi\rho_\varepsilon\}} \rho_\varepsilon|\nabla\varphi| \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

So, introducing the estimate above in (4.2.2) we get

$$\begin{aligned}\left(\int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\right)(T) \le & (1+\varepsilon)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t + (2\chi+1)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\,|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x \\ = & (1+\varepsilon)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \left[\frac{\alpha(\frac{\alpha}{2}+1)\frac{|x|^2}{R^4}}{\left(1+\frac{|x|^2}{R^2}\right)^{\frac{\alpha}{2}+2}} - \frac{\alpha d\frac{1}{R^2}}{\left(1+\frac{|x|^2}{R^2}\right)^{\frac{\alpha}{2}+1}}\right] \mathrm{d}x\,\mathrm{d}t \\ & + 3\chi\int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon\alpha\frac{|x|}{R^2}}{\left(1+\frac{|x|^2}{R^2}\right)^{\frac{\alpha}{2}+1}} \mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x \\ \le & (1+\varepsilon)\frac{\alpha}{2R}\int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon(\frac{\alpha}{2}+1-d)}{\left(1+\frac{|x|^2}{R^2}\right)^{\frac{\alpha}{2}+1}} \mathrm{d}x\,\mathrm{d}t + 3\chi\frac{\alpha}{R}\int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon}{\left(1+\frac{|x|^2}{R^2}\right)^{\frac{\alpha}{2}}} \\ & - \varepsilon\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x.\end{aligned}$$

Now, we choose α such that $\frac{\alpha}{2} + 1 - d > 0$ and get

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\right)(T) \le \frac{C}{R}\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x, \tag{4.2.3}$$

where $C > 0$. Gronwall's Lemma yields

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\right)(T) \le e^{\frac{CT}{R}} \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x \le e^{\frac{CT}{R}} \int_{\mathbb{R}^d} \rho_0 \,\mathrm{d}x.$$

By monotone convergence we can take the limit $R \to \infty$ and conclude that $\int \rho_\varepsilon \leq \int \rho_0$.

Now that we know that the mass of ρ_ε is bounded, we test again (4.2.1) with φ, but this time we pass directly to the limit $R \to \infty$ in (4.2.2) to find

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon \,\mathrm{d}x\right)(T) = -\varepsilon \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \,\mathrm{d}x + \int_{\mathbb{R}^d} \rho_0 \,\mathrm{d}x. \tag{4.2.4}$$

The above identity can be rewritten as an initial value problem for an ODE, which can be easily solved, yielding

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon \,\mathrm{d}x\right)(T) = e^{-\varepsilon T} \int_{\mathbb{R}^d} \rho_0 \,\mathrm{d}x,$$

which finishes the proof. □

Proposition 4.2.2 (L^p estimates)**.** *Let ρ_ε be as stated, then*

$$\|\rho_\varepsilon\|_{L^\infty((0,T);L^p(\mathbb{R}^d))} \leq \|\rho_0\|_{L^p(\mathbb{R}^d)}, \qquad \|\nabla \rho_\varepsilon^{\frac{p}{2}}\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C,$$

where C only depends on $\|\rho_0\|_{L^p}, p$ and χ.

Proof. First, notice that testing (4.2.1) with ρ_ε^{p-1}we get that $\int_{\mathbb{R}^d} \rho_\varepsilon^p(t) \leq \int_{\mathbb{R}^d} \rho_0^p$. Then, we compute

$$\begin{aligned}
&\left(\frac{1}{p}\int_{\mathbb{R}^d} \rho_\varepsilon^p \,\mathrm{d}x\right)(T) \\
&\leq -(p-1)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon^{p-2}|\nabla \rho_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \chi(p-1)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon^{p-1}|\nabla \rho_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_{\mathbb{R}^d} \rho_0^p \,\mathrm{d}x \\
&\leq -\frac{(p-1)}{2}\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon^{p-2}|\nabla \rho_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t + \chi^2(p-1)\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon^p \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_{\mathbb{R}^d} \rho_0^p \,\mathrm{d}x \\
&\leq -\frac{2(p-1)}{p^2}\int_0^T \int_{\mathbb{R}^d} |\nabla \rho_\varepsilon^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}t + \chi^2(p-1)\int_0^T \int_{\mathbb{R}^d} \rho_0^p \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_{\mathbb{R}^d} \rho_0^p \,\mathrm{d}x,
\end{aligned}$$

which yields

$$\|\rho_\varepsilon\|^p_{L^\infty(0,T;L^p(\mathbb{R}^d))} + \|\nabla \rho_\varepsilon^{\frac{p}{2}}\|_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C,$$

with the right hand side independent of ε. □

Proposition 4.2.3 (Moment Evolution)**.** *Let ρ_ε be as stated, then*

$$\int_{\mathbb{R}^d} \rho_\varepsilon(t)(1+|x|^2) \,\mathrm{d}x \leq e^{C\,t} \int_{\mathbb{R}^d} \rho_0(1+|x|^2) \,\mathrm{d}x,$$

where C is independent of ε.

Proof. We test the equation with $(1+|x|^2)\varphi$, where $\varphi = \frac{1}{\left(1+\frac{|x|^2}{R^2}\right)^{\alpha/2}}$ as in Lemma 4.2.1. Then, by Young's inequality, noticing that $|\nabla\varphi|, |\Delta\varphi| \leq \varphi$ and $0 \leq \left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ |\nabla\rho_\varepsilon| \leq 2\chi\rho_\varepsilon$, we get

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon(1+|x|^2)\varphi \,\mathrm{d}x\right)(T) \leq C\int_0^T\int_{\mathbb{R}^d} \rho_\varepsilon(1+|x|^2)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_0(1+|x|^2)\varphi \,\mathrm{d}x.$$

Gronwall's lemma gives

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon(1+|x|^2)\varphi \,\mathrm{d}x\right)(T) \leq e^{C\,t}\int_{\mathbb{R}^d} \rho_0(1+|x|^2)\varphi \,\mathrm{d}x.$$

By dominated convergence we pass to the limit in R and we conclude the proof of the proposition. □

Proposition 4.2.4 (Estimate for $\|\nabla\sqrt{\rho_\varepsilon}\|_{L^2((0,T)\times\mathbb{R}^d)}$)**.** *Let ρ_ε be as stated, then*

$$\left(\int_{\mathbb{R}^d} \rho_\varepsilon \log\rho_\varepsilon \,\mathrm{d}x\right)(T) + \int_0^T\int_{\mathbb{R}^d} \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \leq \bar{C}(\rho_0)$$

where $\bar{C}$ depends on the moment, entropy and L^2 norm of the initial data.

Proof. Let us test (4.2.1) with the test function $\varphi = \log(\rho_\varepsilon+\sigma) - \log\sigma$, with $0 < \sigma << 1$. After some computations we get

$$\begin{aligned}\left(\int_{\mathbb{R}^d} \left[(\rho_\varepsilon+\sigma)\log(\rho_\varepsilon+\sigma) - \rho_\varepsilon\log\sigma\right]\,\mathrm{d}x\right)(T) + \varepsilon\int_0^T\int_{\mathbb{R}^d} \rho_\varepsilon\left[\log(\rho_\varepsilon+\sigma) - \log\sigma\right]\,\mathrm{d}x\,\mathrm{d}t \\ \leq \int_{\mathbb{R}^d}\left[(\rho_0+\sigma)\log(\rho_0+\sigma) - \rho_0\log\sigma\right]\,\mathrm{d}x,\end{aligned}$$

which implies

$$\begin{aligned}\left(\int_{\mathbb{R}^d} (\rho_\varepsilon+\sigma)\log(\rho_\varepsilon+\sigma)\,\mathrm{d}x\right)(T) \leq \int_{\mathbb{R}^d}(\rho_0+\sigma)\log(\rho_0+\sigma)\,\mathrm{d}x \\ - \varepsilon\int_0^T\int_{\mathbb{R}^d} \rho_\varepsilon\log(\rho_\varepsilon+\sigma)\,\mathrm{d}x\,\mathrm{d}t \\ + \log\sigma\left(\int_{\mathbb{R}^d}\rho_\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^d}\rho_0\,\mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}$$

The last term in the previous inequality is zero by (4.2.4). Taking $\sigma \to 0$, by monotone and dominated convergence, we get

$$\left(\int_{\mathbb{R}^d}\rho_\varepsilon\log\rho_\varepsilon\,\mathrm{d}x\right)(T) \leq \int_{\mathbb{R}^d}\rho_0\log\rho_0\,\mathrm{d}x - \varepsilon\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\log\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

In particular,

$$\left(\int_{\mathbb{R}^d}\rho_\varepsilon|\log\rho_\varepsilon|\,\mathrm{d}x\right)(T) \leq e^{\varepsilon T}C(\rho_0),$$

where $C(\rho_0)$ is a constant that depends on $\int_{\mathbb{R}^d} \rho_0 |\log \rho_0| \, \mathrm{d}x$, the mass and the second moment of ρ_0, that is, $\int \rho_0(1+|x|^2)$.

To bound $\int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon}$, we define $h[\rho_\varepsilon](t) := \left(\int_{\mathbb{R}^d} \rho_\varepsilon \log(\rho_\varepsilon) \, \mathrm{d}x\right)(t)$. Using the fact that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right)_+ = \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right) + \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right)_-,$$

and repeating the same computations as before, we obtain

$$h[\rho_\varepsilon](T) + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon + \sigma} \, \mathrm{d}x \, \mathrm{d}t \leq \frac{\chi^2}{2} \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \, \mathrm{d}x \, \mathrm{d}t - \varepsilon \int_0^T h[\rho_\varepsilon](t) \, \mathrm{d}t + h[\rho_0],$$

which yields

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon + \sigma} \, \mathrm{d}x \, \mathrm{d}t \leq \frac{\chi^2}{2} \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + e^{\varepsilon T} C[\rho_0].$$

□

Lemma 4.2.5 ($\|\nabla \rho_\varepsilon\|_{L^p}$ estimate)**.** *Let ρ_ε be as stated, then*

$$\|\nabla \rho_\varepsilon\|_{L^p((t,T)\times\mathbb{R}^d)} < \frac{1}{t^{1/2}} C(\|\rho_0\|_{L^p}), \quad \forall \, t > 0.$$

Proof. Define f as

$$f := \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right)_+ \nabla \rho_\varepsilon - \nabla \rho_\varepsilon.$$

Note that each component of f belongs to $L^\infty(0,T;L^q(\mathbb{R}^d))$ for any $q \in [1,+\infty)$. In fact, in the domain where $|\nabla \rho_\varepsilon| \leq \chi \rho_\varepsilon$, the vector function f coincides with $\nabla \rho_\varepsilon$, which is bounded by $\chi \rho_\varepsilon$. In the complementary domain, we have that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right)_+ = \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla \rho_\varepsilon|}\right)$$

and $|f| \leq \chi \rho_\varepsilon$. With this in mind, consider the function u_i, solution to

$$u_{it} - (1+\varepsilon)\Delta u_i + \varepsilon u_i = f_i, \quad u_i(x,0) = 0, \quad i = 1, ..., d.$$

Parabolic regularity theory tells us that $u_i \in L^q(0,T;W^{2,q})$, with uniform bounds in ε. Direct computations show that ρ_ε can be written as the sum of $\mathrm{div}(u) + g$, where g solves

$$g_t = (1+\varepsilon)\Delta g - \varepsilon g, \quad g(x,0) = \rho_0.$$

Since $\rho_0 \in L^p(\mathbb{R}^d)$, we have that

$$\sup_{t>0} \|\nabla g\|_{L^p} \leq \frac{c}{t^{1/2}} \|\rho_0\|_{L^p},$$

with c independent of ε. Summarizing, $\mathrm{div}(u) \in L^p(0,T;W^{1,p})$, $g \in L^\infty(t,T;W^{1,p})$, and the thesis follows.

□

Lemma 4.2.6. *Let ρ_ε be as stated, then for every $\tau > 0$ it holds*

$$\|\rho_\varepsilon\|_{L^\infty(\tau,T;\mathbb{R}^d)} \leq CT^{\frac{1}{p}} \left(1 + \tau^{-\frac{d+2}{2p}}\right) \|\rho_0\|_{L^p(\mathbb{R}^d)},$$

where C only depends on p, χ and other universal constants.

Proof. Let us consider the following sequence of times

$$T_k = \tau(1 - 2^{-k}),$$

and the following sequence of energies

$$U_k = \int_{T_k}^{T} \int_{\mathbb{R}^d} \rho_{\varepsilon,k}^p \,\mathrm{d}x\,\mathrm{d}t, \quad \text{where} \quad \rho_{\varepsilon,k} = (\rho_\varepsilon - C_k)_+, \quad \text{and} \quad C_k = M(1 - 2^{-k}),$$

with $M > 0$ to be fixed later. Then, we test (4.2.1) with $\rho_{\varepsilon,k+1}^{p-1}$ and we integrate in time in (t_1, t_2), where $0 \leq T_k \leq t_1 \leq T_{k+1} \leq t_2 \leq T$. We find

$$\begin{aligned}\frac{1}{p} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} &\partial_t(\rho_{\varepsilon,k+1}^p) \,\mathrm{d}x\,\mathrm{d}t \\ &\leq -\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \nabla\rho_\varepsilon \nabla(\rho_{\varepsilon,k+1}^{p-1}) \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{\chi\rho}{|\nabla\rho_\varepsilon|} \nabla(\rho_{\varepsilon,k+1}^{p-1}) \nabla\rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

After some computations, from the expression above we get

$$\begin{aligned}\frac{1}{p} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \partial_t(\rho_{\varepsilon,k+1}^p) \,\mathrm{d}x\,\mathrm{d}t + \frac{(p-1)}{p^2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla \rho_{\varepsilon,k+1}^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{(p-1)}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x\,\mathrm{d}t \\ + 2\chi^2 C_{k+1}^2 (p-1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^{p-2} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.2.5}$$

Now, we notice that, in the set $\rho_{\varepsilon,k+1} > 0$, $\rho_{\varepsilon,k} = \rho_{\varepsilon,k+1} + C_{k+1} - C_k$, i.e. that $\frac{\rho_{\varepsilon,k}}{C_{k+1}-C_k} = \frac{\rho_{\varepsilon,k+1}}{C_{k+1}-C_k} + 1 > 1$. Then, for any $\alpha > 0$ we have $\left(\frac{\rho_{\varepsilon,k}}{C_{k+1}-C_k}\right)^\alpha > 1$ and, consequently $\left(\frac{1}{C_{k+1}-C_k}\right)^\alpha \rho_{\varepsilon,k}^{1+\alpha} > \rho_{\varepsilon,k+1}$. Now, let us choose $\alpha = \frac{2}{p-2}$ to get

$$\rho_{\varepsilon,k+1} \leq \left(\frac{1}{C_{k+1} - C_k}\right)^{\frac{2}{p-2}} \rho_{\varepsilon,k}^{\frac{p}{p-2}} \mathbb{1}_{\{\rho_{\varepsilon,k+1}>0\}}.$$

Introducing this expression in (4.2.5) we get

$$\begin{aligned}\frac{1}{p} \left(\int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x\right)(t_2) + \frac{(p-1)}{p^2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla \rho_{\varepsilon,k+1}^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{(p-1)}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x\,\mathrm{d}t \\ + 2\chi^2(p-1) \left(\frac{C_{k+1}}{C_{k+1} - C_k}\right)^2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \rho_{\varepsilon,k}^p \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{p} \left(\int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x\right)(t_1).\end{aligned}$$

Then, averaging t_1 in (T_k, T_{k+1}) and taking the supremum in $t_2 \in (T_{k+1}, T)$ we have

$$\begin{aligned} E_{k+1} &:= \sup_{t\in[T_{k+1},T]} \left(\int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x \right) + \int_{T_{k+1}}^T \int_{\mathbb{R}^d} |\nabla \rho_{\varepsilon,k+1}^{\frac{p}{2}}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C \left(1 + 2^k + \frac{2^k}{\tau} \right) \int_{T_k}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k}^p \,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where C only depends on p and χ. From now on, this C will change from line to line for convenience, always depending only on p and χ. Now, notice that E_{k+1} controls $\rho_{\varepsilon,k+1}$ in $L^\infty(T_{k+1}, T; L^p(\mathbb{R}^d))$ and, thanks to Sobolev, $L^p(T_{k+1}, T; L^{\frac{dp}{d-2}}(\mathbb{R}^d))$, so, by interpolation

$$\|\rho_{\varepsilon,k+1}\|^p_{L^{\frac{p(d+2)}{d}}(\mathbb{R}^d\times(T_{k+1},T))} \leq CE_{k+1} \leq C' \left(1 + 2^k + \frac{2^k}{\tau} \right) \int_{T_k}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k}^p \,\mathrm{d}x\,\mathrm{d}t. \quad (4.2.6)$$

Then, Chebyshev's inequality yields:

$$\begin{aligned} U_{k+1} &:= \int_{T_{k+1}}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^p \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \left(\int_{T_{k+1}}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^{\frac{p(d+2)}{d}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{d}{d+2}} \left(\int_{T_{k+1}}^T \int_{\mathbb{R}^d} \mathbb{1}_{\{\rho_{\varepsilon,k+1}>0\}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{d+2}} \\ &= \left(\int_{T_{k+1}}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k+1}^{\frac{p(d+2)}{d}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{d}{d+2}} \left(\int_{T_{k+1}}^T \int_{\mathbb{R}^d} \mathbb{1}_{\{\rho_{\varepsilon,k}>\frac{M}{2^{k+1}}\}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{d+2}} \\ &\leq CE_{k+1} \left(\frac{2^{k+1}}{M} \right)^{\frac{2p}{d+2}} \left(\int_{T_k}^T \int_{\mathbb{R}^d} \rho_{\varepsilon,k}^p \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{d+2}} \\ &\leq \left(1 + \frac{1}{\tau} \right) \frac{C^k}{M^{\frac{2p}{d+2}}} U_k^\beta, \end{aligned}$$

where $\beta = 1 + \frac{2}{d+2}$, using (4.2.6) to obtain the second and the third inequality. From [70, Lemma 7.1] it follows that $U_k \to 0$ as $k \to \infty$, i.e. that $\rho_\varepsilon \leq M$, if

$$\left(1 + \frac{1}{\tau} \right)^{\frac{d+2}{2}} \frac{C}{M^p} U_0 < 1.$$

So, we fix M in order to have

$$C \left(1 + \frac{1}{\tau} \right)^{\frac{d+2}{2p}} U_0^{1/p} \sim M.$$

Finally, we conclude from Proposition 4.2.2

$$\|\rho_\varepsilon\|_{L^\infty(0,T;\mathbb{R}^d)} \leq M \sim C \left(1 + \frac{1}{\tau} \right)^{\frac{d+2}{2p}} \|\rho_\varepsilon\|_{L^p(0,T;\mathbb{R}^d)} \leq CT^{\frac{1}{p}} \left(1 + \frac{1}{\tau^{\frac{d+2}{2p}}} \right) \|\rho_0\|_{L^p(\mathbb{R}^d)},$$

as desired. □

Lemma 4.2.7 (Compactness Lemma)**.** *Let $\{f_n\}$ be a bounded sequence in $L^2(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;\mathbb{R}^d, \sqrt{1+|x|^2}\,\mathrm{d}x)$ such that $\partial_t f_n$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^d))$, then*

$$f_n \to f \qquad \textit{strongly in } \ L^2(0,T;\mathbb{R}^d), \qquad \textit{as } n \to \infty.$$

Proof. As $\{f_n\}$ is uniformly bounded in $L^2(0,T;H^1(\mathbb{R}^d))$ we have that

$$f_n \rightharpoonup f \quad \text{weakly in } L^2(0,T;L^{\frac{2d}{d-2}}(\mathbb{R}^d)).$$

Then, for any $R>0$, as $\partial_t f_n \in L^2(0,T;H^{-1}(B_R))$ and $H^1(B_R) \subset\subset L^q(B_R)$ with $2 \le q < \frac{2d}{d-2}$, by Aubin-Lions Theorem (Theorem 5 in [128]), we have that a subsequence, $\{f_n^R\}$, converges strongly to f as follows

$$f_n^R \to f \quad \text{strongly in } L^2(0,T;L^q(B_R)).$$

By a Cantor diagonal argument, we have a new subsequence f_n, independent of R, such that

$$f_n \to f \quad \text{strongly in } L^2(0,T;L^q(B_R)),$$

and consequently,

$$f_n \to f \quad \text{a.e. in } [0,T]\times\mathbb{R}^d.$$

By Fatou's Lemma, we also have

$$\int_0^T \iint_{\mathbb{R}^d} f^2\sqrt{1+|x|^2}\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d} f_n^2\sqrt{1+|x|^2}\,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Now, setting $q=2$, for any $\delta>0$, there exist sufficiently large n and R such that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}|f_n-f|^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\iint_{B_R}|f_n-f|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d\setminus B_R}|f_n-f|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{\delta}{2} + \frac{1}{\sqrt{1+R^2}}\int_0^T\iint_{\mathbb{R}^d\setminus B_R}|f_n-f|^2\sqrt{1+|x|^2}\,\mathrm{d}x\,\mathrm{d}t \le \delta,\end{aligned}$$

and this completes the proof. □

4.3 Existence for the approximated equation

This section is devoted to show existence of solutions to (4.2.1) with smooth initial data. First, we semi-linearize (4.2.1) and obtain a monotone functional inside the divergence operator. Then, by standard theory for monotone operators we prove existence of this new problem. A Leray-Schauder fixed point will yield existence of solutions to (4.2.1).

4.3.1 Existence for the semi-linearized equation

For given ε, $\sigma \in [0,1]$ and $z \in L^2(0,T;L^2(\mathbb{R}^d))$, let us present the following regularized weak formulation of (4.1.7)

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \rho_\varepsilon \, \varphi \, \mathrm{d}x \, \mathrm{d}t = -\sigma \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi z_+}{|\nabla \rho_\varepsilon|}\right)_+ \nabla \rho_\varepsilon \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \varepsilon \int_0^T \int_{\mathbb{R}^d} \nabla \rho_\varepsilon \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \varepsilon \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t, \tag{4.3.1}$$

$\forall \varphi \in L^2(0,T;H^1(\mathbb{R}^d))$, where $\rho_\varepsilon(0,\cdot) = \rho_{\mathrm{in},\varepsilon}$ and $\rho_{\mathrm{in},\varepsilon}$ is a $W^{1,p} \cap L^\infty(\mathbb{R}^d)$ approximation of ρ_{in} such that $\rho_{\mathrm{in},\varepsilon} \to \rho_{\mathrm{in}}$ strongly in $L^p(\mathbb{R}^d)$ when $\varepsilon \to 0$. In order to prove the existence of the above equation, we use the general theory of monotone operators, concretely, [146, Theorem 30.A]. We define the following operator A as

$$\langle A[u](t), \varphi \rangle = \sigma \int_{\mathbb{R}^d} \left(1 - \frac{\chi z_+}{|\nabla u(t,x)|}\right)_+ \nabla u(t,x) \nabla \varphi \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}^d} \nabla u(t,x) \nabla \varphi \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}^d} u(t,x) \varphi \, \mathrm{d}x,$$

which must satisfy the following properties:

1. Monotonicity. The operator A is monotone, since $\forall\, u, v \in H^1(\mathbb{R}^d)$ we have

$$\langle A[u] - A[v], u - v \rangle = \sigma \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi z_+}{|\nabla u|}\right)_+ \nabla u - \left(1 - \frac{\chi z_+}{|\nabla v|}\right)_+ \nabla v \right] \nabla (u - v) \, \mathrm{d}x + \varepsilon \|u - v\|^2_{H^1(\mathbb{R}^d)} \geq 0.$$

2. Hemicontinuity. Let us check that the map

$$t \mapsto \langle A[u + tv], \varphi \rangle$$

is continuous in $t \in [0,1]$ for all $u, v, \varphi \in H^1(\mathbb{R}^d)$. Notice that

$$\left(1 - \frac{\chi z_+}{|\nabla (u + tv)|}\right)_+ \nabla (u + tv) \nabla \varphi \ \leq |\nabla u||\nabla \varphi| + |\nabla v||\nabla \varphi|$$

so

$$\lim_{t \to t_0} \langle A[u + tv], \varphi \rangle = \sigma \int_{\mathbb{R}^d} \left(1 - \frac{\chi z_+}{|\nabla (u + t_0 v)|}\right)_+ \nabla (u + t_0 v) \nabla \varphi \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}^d} \nabla (u + t_0 v) \nabla \varphi \, \mathrm{d}x + \varepsilon \int_{\mathbb{R}^d} (u + t_0 v) \varphi \, \mathrm{d}x = \langle A(u + t_0 v), \varphi \rangle,$$

by dominated convergence.

3. Coerciveness. We just compute

$$\langle A[u], u \rangle = \sigma \int_{\mathbb{R}^d} \left(1 - \frac{\chi z_+}{|\nabla u|}\right)_+ |\nabla u|^2 \, \mathrm{d}x + \varepsilon \|u\|^2_{H^1(\mathbb{R}^d)} \geq \varepsilon \|u\|^2_{H^1(\mathbb{R}^d)},$$

so A is coercive.

4. Growth condition. For this we have to prove that $\|A[u]\|_{H^*(\mathbb{R}^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)}$, but, by Hölder,

$$\begin{aligned}\|A[u]\|_{H^*(\mathbb{R}^d)} &= \sup_{\substack{\varphi\in H^1(\mathbb{R}^d)\\ \|\varphi\|\leq 1}} \langle A[u],\varphi\rangle \\ &= \sup_{\substack{\varphi\in H^1(\mathbb{R}^d)\\ \|\varphi\|\leq 1}} \int_{\mathbb{R}^d}\left[\sigma\left(1-\frac{\chi z_+}{|\nabla u|}\right)_+\Big(\nabla u\nabla\varphi+\varepsilon\nabla u\nabla\varphi+\varepsilon u\varphi\right]\Big(\mathrm{d}x \\ &\leq C\|u\|_{H^1(\mathbb{R}^d)}.\end{aligned}$$

5. Measurability. By assumption.

Then, by [146, Theorem 30.A], there exists a unique weak solution $\rho_\varepsilon \in L^2(0,T;H^1(\mathbb{R}^d))$ to (4.3.1) with $\partial_t\rho \in L^2(0,T;H^{-1}(\mathbb{R}^d))$ satisfying the initial condition in the sense of the strong $L^2(\mathbb{R}^d)$ convergence:

$$\rho_\varepsilon(t) \to \rho_{in,\varepsilon} \quad \text{strongly in } L^2(\mathbb{R}^d) \text{ as } t\to 0. \tag{4.3.2}$$

Therefore we can define the mapping

$$\mathcal{T} : (z,\sigma) \in L^2((0,T)\times\mathbb{R}^d)\times[0,1] \mapsto \rho_\varepsilon \in L^2((0,T)\times\mathbb{R}^d),$$

where ρ_ε is the unique solution to (4.3.1).

4.3.2 Existence for approximated solutions

Now, we are going to use the Leray-Schauder fixed point theorem to show the existence of a solution to (4.2.1). To use this theorem we require $\mathcal{T}$ to be compact. First, using ρ_ε as test function in (4.3.1) we get

$$\begin{aligned}\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\partial_t(\rho_\varepsilon^2)\,\mathrm{d}x\,\mathrm{d}t = &-\sigma\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi(z_+ +\varepsilon)}{|\nabla\rho(t,x)|}\right)_+|\nabla\rho_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\ &-\varepsilon\int_0^T\int_{\mathbb{R}^d}|\nabla\rho_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which implies

$$\frac{1}{2}\|\rho_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))}+\varepsilon\|\rho_\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^d))} \leq \int_{\mathbb{R}^d}\rho^2_{\mathrm{in},\varepsilon}\ \mathrm{d}x.$$

Thus, the $L^2(0,T;H^1(\mathbb{R}^d))$ norm of any solution ρ_ε has a uniform bound. Now, we test (4.3.1) with $\varphi = \rho_\varepsilon\sqrt{1+|x|^2}\phi_R$, where $\phi_R \in C_c^\infty(B_{2R})$ is such that $\phi_R \equiv 1$ in B_R and $|\nabla\phi_R| \leq \frac{C}{R}$ for some $C>0$. Doing this we get

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d}\rho_\varepsilon^2\sqrt{1+|x|^2}\phi_R\,\mathrm{d}x \leq\ &\frac{1}{2}\int_{\mathbb{R}^d}\rho^2_{\mathrm{in},\varepsilon}\sqrt{1+|x|^2}\,\mathrm{d}x+(\sigma+\varepsilon)\int_0^T\int_{B_{2R}}\frac{\rho_\varepsilon|\nabla\rho_\varepsilon|\cdot|x|}{\sqrt{1+|x|^2}}\,\mathrm{d}x\,\mathrm{d}t \\ &+(\sigma+\varepsilon)\int_0^T\int_{B_{2R}}|\nabla\rho_\varepsilon|\rho_\varepsilon\sqrt{1+|x|^2}|\nabla\phi_R|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then, using $\frac{|x|}{\sqrt{1+|x|^2}} \leq 1$, Young's inequality and that $|\nabla \phi_R| \leq \frac{C}{R}$, we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon^2 \sqrt{1+|x|^2}\phi_R \,\mathrm{d}x\,\mathrm{d}t \leq & \int_0^T \int_{\mathbb{R}^d} \rho_{\mathrm{in},\varepsilon}^2 \sqrt{1+|x|^2}\,\mathrm{d}x\,\mathrm{d}t + c\|\rho_\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^d))} \\ & + \frac{C\sqrt{1+4|R|^2}}{R} \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon |\nabla \rho_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t \\ \leq & C(\rho_{\mathrm{in},\varepsilon}).\end{aligned}$$

Taking the limit $R \to \infty$ we get that solutions to (4.3.1) are bounded in $L^2(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;\mathbb{R}^d, \sqrt{1+|x|^2}\,\mathrm{d}x)$. Also, for any $\varphi \in L^2(0,T;H^1(\mathbb{R}^d))$, by weak formulation (4.3.1), we have

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \rho_\varepsilon(t)\varphi \,\mathrm{d}x\,\mathrm{d}t \ \leq C\|\rho_\varepsilon(t)\|_{H^1(\mathbb{R}^d)}\|\varphi\|_{L^2(0,T;H^1(\mathbb{R}^d))} \leq C\|\varphi\|_{L^2(0,T;H^1(\mathbb{R}^d))}. \tag{4.3.3}$$

By Lemma 4.2.7, $\mathcal{T}$ maps bounded sets to precompact sets.

Now, we have to prove that the map is continuous, i.e. that $\lim_{n\to\infty} \mathcal{T}(\sigma_n, z_n) = \mathcal{T}(\sigma, z)$ when $z_n \to z$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ and $\sigma_n \to \sigma$ in $[0,1]$. Is easy to see that $\lim_{n\to\infty} \mathcal{T}(\sigma_n, z_n) = \lim_{n\to\infty} \mathcal{T}(\sigma, z_n)$. Then, as the map is compact, it only remains to prove

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi z_+}{|\nabla \rho_\varepsilon|}\right)_+ \nabla \rho_\varepsilon \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t,$$

but as $\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{R}^d))$, it has a weakly convergent subsequence and we only need $\nabla \rho_{\varepsilon,n} \to \nabla \rho_\varepsilon$ a.e. $x \in \mathbb{R}^d$. To show this convergence, we prove that $\{\nabla \rho_{\varepsilon,k}\}$ is a Cauchy sequence in $L^2(0,T;L^2(\mathbb{R}^d))$: let $\rho_{\varepsilon,n}$ and $\rho_{\varepsilon,k}$ solutions to (4.3.1) with z_n and z_k respectively. We subtract the equation for $\rho_{\varepsilon,k}$ from the one of $\rho_{\varepsilon,n}$ and get, after some manipulations:

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \partial_t(\rho_{\varepsilon,n} - \rho_{\varepsilon,k})\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k} - \left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \varepsilon \int_0^T \int_{\mathbb{R}^d} (\nabla \rho_{\varepsilon,n} - \nabla \rho_{\varepsilon,k}) \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t - \varepsilon \int_0^T \int_{\mathbb{R}^d} (\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+\right] \nabla \rho_{\varepsilon,k} \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \varepsilon \int_0^T \int_{\mathbb{R}^d} (\nabla \rho_{\varepsilon,n} - \nabla \rho_{\varepsilon,k})_+ \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t - \varepsilon \int_0^T \int_{\mathbb{R}^d} (\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \varphi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We use the test function $\varphi = \rho_{\varepsilon,n} - \rho_{\varepsilon,k}$ and obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} |\rho_{\varepsilon,n} - \rho_{\varepsilon,k}|^2 \,\mathrm{d}x\right)(t) \\
&\leq -\sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} - \left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+\right] \nabla \rho_{\varepsilon,k} \cdot \nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \varepsilon \int_0^T \int_{\mathbb{R}^d} |\nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k})|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leq -\sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} - \left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \frac{\sigma^2}{2\varepsilon} \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+\right]^2 |\nabla \rho_{\varepsilon,k}|^2 \\
&\quad - \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^d} |\nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k})|^2 \,\mathrm{d}x \,\mathrm{d}t, \qquad (4.3.4)
\end{aligned}$$

using the Young's inequality in the second estimate. Now come two key points in the argument. The first one is that, due to monotonicity, we have

$$-\sigma \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,n}|}\right)_+ \nabla \rho_{\varepsilon,n} - \left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \nabla \rho_{\varepsilon,k}\right] \nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k}) \,\mathrm{d}x \,\mathrm{d}t \leq 0.$$

The second one is that $\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+$ is Lipschitz continuous with respect to $(z_n)_+$. In fact, we have the following

$$\left(1 - \frac{\chi(z_n)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ - \left(1 - \frac{\chi(z_k)_+}{|\nabla \rho_{\varepsilon,k}|}\right)_+ \leq \frac{\chi}{|\nabla \rho_{\varepsilon,k}|} |z_n - z_k|.$$

Introducing this in (4.3.4) yields:

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} |\rho_{\varepsilon,n} - \rho_{\varepsilon,k}|^2 \,\mathrm{d}x\right)(t) \leq \frac{\chi \sigma^2}{2\varepsilon} \int_0^T \int_{\mathbb{R}^d} |(z_n)_+ - (z_k)_+|^2 \,\mathrm{d}x \,\mathrm{d}t \\
- \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^d} |\nabla(\rho_{\varepsilon,n} - \rho_{\varepsilon,k})|^2 \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

We conclude that $\{\nabla \rho_{\varepsilon,n}\}$ is a Cauchy sequence in $L^2(0,T;L^2(\mathbb{R}^d))$, and therefore, $\nabla \rho_{\varepsilon,n} \to \nabla \rho_\varepsilon$ a.e. $x \in [0,T] \times \mathbb{R}^d$. The map $\mathcal{T}$ is continuous, compact and constant when $\sigma = 0$. Moreover, fixed points are uniformly bounded in σ.

Finally, we have to prove that fixed points of $\mathcal{T}$ are nonnegative so we can remove the + sign. Using Stampacchia's truncation argument, we introduce $(\rho_\varepsilon)_-$ in (4.2.1) as test function and we get

$$\left(\int_{\mathbb{R}^d} (\rho_\varepsilon)_-^2 \,\mathrm{d}x\right)(t) \leq \int_{\mathbb{R}^d} (\rho_{\mathrm{in},\varepsilon})_-^2 \,\mathrm{d}x.$$

If $\rho_{\text{in},\varepsilon} \geq 0$, solutions to $\mathcal{T}(\sigma, \rho_\varepsilon) = \rho_\varepsilon$ are nonnegative.

Then, Leray-Schauder fixed-point yields a solution to

$$\iint_0^T \int_{\mathbb{R}^d} \partial_t \rho_\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t = -\iint_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t - \varepsilon \iint_0^T \int_{\mathbb{R}^d} \nabla\rho_\varepsilon \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t - \varepsilon \iint_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t, \tag{4.3.5}$$

for $\forall\ \varphi \in L^2(0,T; H^1(\mathbb{R}^d))$.

4.4 The limit $\varepsilon \to 0$ and proof of Theorem 4.1.1

The main difficulty in removing the viscosity terms $\varepsilon\Delta\rho_\varepsilon$ and $-\varepsilon\rho_\varepsilon$ is to pass to the limit in the nonlinear term $\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon$. We need to show that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{in } L^2(0,T;\mathbb{R}^d).$$

We already know that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \Big\|_{L^2(0,T;\mathbb{R}^d)} \leq C,$$

which implies,

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup G \quad \text{in } L^2(0,T;\mathbb{R}^d).$$

The identification $G = \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho$ will follow by almost everywhere convergence of ρ_ε (given by Lemma (4.2.7)) and of $\nabla\rho_\varepsilon$. The next two subsections will be devoted to show that

$$\nabla\rho_\varepsilon \to \nabla\rho \quad \text{a.e. in } \ ((0,T)\times\mathbb{R}^d)\backslash\{|\nabla\rho| \leq \chi\rho\}.$$

Then, in the last two subsections, we will prove the L^p convergence of ρ to the initial data ρ_{in} and the uniqueness of the solutions.

4.4.1 Entropy method

The major tool of this argument is the *relative entropy* between two functions u and v, defined as:

$$H_{\text{rel}}[u|v] \equiv \int_{\mathbb{R}^d} \left(u \log\frac{u}{v} - u + v\right) \mathrm{d}x. \tag{4.4.1}$$

Notice that, for $u, v \geq 0$

$$H_{\text{rel}}[u|v](T) \geq 0, \quad \text{as} \quad r\log(r) - r + 1 \geq 0, \quad \text{for all} \quad r \geq 0,$$

and $H[u|v] = 0$ if and only if $u = v$. Let us outline how we use (4.4.1) to show that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{in } L^2(0,T;\mathbb{R}^d).$$

Consider ρ_ε and ρ_δ two viscosity solutions to (4.3.5) with viscosity constants ε and δ respectively. A formal computation shows that

$$H[\rho_\varepsilon|\rho_\delta](T) - H[\rho_{\text{in},\varepsilon}|\rho_{\text{in},\delta}] + \int_0^T \mathcal{D}[\rho_\varepsilon|\rho_\delta]\,\mathrm{d}t = \text{error terms}. \tag{4.4.2}$$

The relative entropy production $\mathcal{D}[\rho_\varepsilon|\rho_\delta]$ is nonnegative as a direct consequence of the monotonicity of the operator $\left(1 - \frac{\chi}{|\nabla\log\rho|}\right)_+ \nabla\log\rho$. The error terms originate from the viscosities $\varepsilon\Delta\rho_\varepsilon$, $\delta\Delta\rho_\delta$, $\varepsilon\rho_\varepsilon$ and $\delta\rho_\delta$. As $\varepsilon, \delta \to 0$, $H[\rho_{\text{in},\varepsilon}|\rho_{\text{in},\delta}] \to 0$ by construction, and the error terms also vanish. Since both H and $\mathcal{D}$ are nonnegative, as $\delta, \varepsilon \to 0$, we must have

$$H[\rho_\varepsilon|\rho_\delta](T) \to 0 \quad \text{and} \quad \int_0^T \mathcal{D}[\rho_\varepsilon|\rho_\delta]\,\mathrm{d}t \to 0.$$

From the convergence

$$\int_0^T \mathcal{D}[\rho_\varepsilon|\rho_\delta]\,\mathrm{d}t \to 0,$$

we obtain the almost everywhere convergence of $\nabla\rho_\varepsilon$ in the nondegenerate region $\{|\nabla\rho| > \chi\rho\}$. Consequently, we can identify the limit $\left(1 - \frac{\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon$.

One can make this argument rigorous and obtain (4.4.2) by testing (4.3.5) for ρ_ε and for ρ_δ respectively with test functions $\varphi_1 = \log\left(\frac{\rho_\varepsilon}{\rho_\delta}\right)$ and $\varphi_2 = \frac{\rho_\varepsilon}{\rho_\delta}$. Since neither φ_1 nor φ_2 are admissible test functions, we modify them accordingly, at the expense of some error terms. Specifically, our test functions are of the form

$$\varphi_1 = \log\left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma}\right)\phi_R, \quad \text{and} \quad \varphi_2 = \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma}\phi_R,$$

with $\sigma > 0$ a small constant and ϕ_R a cut-off function supported in B_R. More precisely, we chose $R = \sigma^{-\frac{1}{\alpha d}}$, with $\alpha > 0$ to be fixed, and $\phi_R \in C_c^\infty(B_{2R})$ such that $\phi_R \equiv 1$ in B_R and $|\nabla\phi_R| \le \frac{C}{R}$ for some $C > 0$. With this choice of φ_1 and φ_2 we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} &[\varphi_1\partial_t\rho_\varepsilon - \varphi_2\partial_t\rho_\delta]\,\mathrm{d}x = \\ &= \frac{d}{dt}\int_{\mathbb{R}^d}\left[(\rho_\varepsilon + \sigma)\log\left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma}\right) - \rho_\varepsilon + \rho_\delta\right]\phi_R\,\mathrm{d}x - \frac{d}{dt}\int_{\mathbb{R}^d}\rho_\delta\phi_R\,\mathrm{d}x.\end{aligned} \tag{4.4.3}$$

By Lemma 4.2.7, we know that

$$\rho_\varepsilon \to \rho \quad \text{strongly in } L^2((0,T)\times\mathbb{R}^d).$$

Hence, we can chose $\varepsilon, \delta > 0$ small enough such that

$$\int_0^T \int_{\mathbb{R}^d} |\rho_\varepsilon - \rho_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t < \sigma^2. \tag{4.4.4}$$

We do not write $\varepsilon(\sigma), \delta(\sigma)$ and $R(\sigma)$ on σ for readability purposes, but notice that

$$\lim_{\sigma\to 0} \varepsilon, \delta = 0, \quad \text{and} \quad \lim_{\sigma\to 0} R(\sigma) = \infty.$$

Next, we subtract equation (4.3.5) for ρ_δ, tested with φ_2, to the equation (4.3.5) for ρ_ε, tested with φ_1. This yields:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \left[\left(\log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \partial_t \rho_\varepsilon - \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma}\, \partial_t \rho_\delta \right] \phi_R \,\mathrm{d}x\,\mathrm{d}t \qquad (4.4.5)\\
&= -\int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi \rho_\varepsilon}{|\nabla \rho_\varepsilon|} \right)_+ \nabla \rho_\varepsilon \left[\nabla \left(\log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \phi_R + \left(\log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \nabla \phi_R \right] \mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi \rho_\delta}{|\nabla \rho_\delta|} \right)_+ \nabla \rho_\delta \left[\nabla \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \phi_R + \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \nabla \phi_R \right] \mathrm{d}x\,\mathrm{d}t + E_{\varepsilon,\delta},
\end{aligned}$$

where $E_{\varepsilon,\delta}$ are error terms, which must vanish when $\sigma \to 0$.

Note that (4.4.5) is an approximation of $\int_0^T \frac{d}{\mathrm{d}t} H_{\mathrm{rel}}[\rho_\varepsilon|\rho_\delta]$. We will show

$$\lim_{\varepsilon,\delta\to 0} \int_0^T \frac{d}{\mathrm{d}t} H_{\mathrm{rel}}[\rho_\varepsilon|\rho_\delta] \le 0,$$

to find that, in fact, $\lim_{\varepsilon,\delta\to 0} H_{\mathrm{rel}}[\rho_\varepsilon|\rho_\delta](T) = 0$. This limit will provide the necessary information about the convergence of $\nabla \rho_\varepsilon$.

To do this, let us analyze the error terms first:

$$\begin{aligned}
E_{\varepsilon,\delta} = &-\varepsilon \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \phi_R \,\mathrm{d}x\,\mathrm{d}t + \delta \int_0^T \int_{\mathbb{R}^d} \rho_\delta \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \phi_R \,\mathrm{d}x\,\mathrm{d}t && (4.4.6a)\\
&- \int_0^T \int_{\mathbb{R}^d} \phi_R \left[\varepsilon \nabla \rho_\varepsilon \nabla \left(\log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) - \delta \nabla \rho_\delta \nabla \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \right] \mathrm{d}x\,\mathrm{d}t && (4.4.6b)\\
&- \int_0^T \int_{\mathbb{R}^d} \nabla \phi_R \left[\varepsilon \nabla \rho_\varepsilon \left(\log \frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) - \delta \nabla \rho_\delta \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right) \right] \mathrm{d}x\,\mathrm{d}t. && (4.4.6c)
\end{aligned}$$

For the first term above, assuming without loss of generality $\varepsilon < \delta$ and recalling

that we can always choose δ such that $\delta < \sigma << 1$ we compute

$$\begin{aligned}
(4.4.6a) =& -\varepsilon H[\rho_\varepsilon|\rho_\delta] + \varepsilon\sigma \int_0^T\!\!\int_{\mathbb{R}^d} \log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\phi_R \,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}(\rho_\varepsilon-\rho_\delta)\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
&+ \delta\int_0^T\!\!\int_{\mathbb{R}^d}\rho_\delta\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
\leq& -\varepsilon H[\rho_\varepsilon|\rho_\delta] + \sigma^2\int_0^T\!\!\int_{\mathbb{R}^d}\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t \qquad (4.4.7)\\
&+ \sigma\int_0^T\!\!\int_{\mathbb{R}^d}\rho_\delta\phi_R\,\mathrm{d}x\,\mathrm{d}t + \sigma\int_0^T\!\!\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
\leq& -\varepsilon H[\rho_\varepsilon|\rho_\delta] + \sigma\int_0^T\!\!\int_{\mathbb{R}^d}(2\rho_\varepsilon+\rho_\delta+\phi_R)\,\mathrm{d}x\,\mathrm{d}t \\
\leq& -\varepsilon H[\rho_\varepsilon|\rho_\delta] + C\sigma^{\frac{\alpha-1}{\alpha}}, \qquad (4.4.8)
\end{aligned}$$

where $C>0$ only depends on the L^1 norm of the initial data. In the last inequality we have used Lemma 4.2.1 and the fact that $\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x = C(d)\sigma^{\frac{\alpha-1}{\alpha}}$. From now on, C will be a positive universal constant that can change from line to line.

For expression (4.4.6b), we compute

$$\begin{aligned}
(4.4.6b) =& -\varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t + (\varepsilon+\delta)\int_0^T\!\!\int_{\mathbb{R}^d}\frac{\nabla\rho_\varepsilon\nabla\rho_\delta}{\rho_\delta+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
&- \delta\int_0^T\!\!\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\left(\frac{\nabla\rho_\delta}{\rho_\delta+\sigma}\right)^2\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
\leq& \frac{(\varepsilon+\delta)}{2}\int_0^T\!\!\int_{\mathbb{R}^d}\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t + \frac{(\varepsilon+\delta)}{2}\int_0^T\!\!\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\left(\frac{\nabla\rho_\delta}{\rho_\delta+\sigma}\right)^2\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
&- \delta\int_0^T\!\!\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\left(\frac{\nabla\rho_\delta}{\rho_\delta+\sigma}\right)^2\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
\leq& \frac{(\varepsilon+\delta)}{2}\int_0^T\!\!\int_{\mathbb{R}^d}\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon}\phi_R\,\mathrm{d}x\,\mathrm{d}t \\
\leq& \sigma C, \qquad (4.4.9)
\end{aligned}$$

using Lemma 4.2.4 in the last inequality. For the first term in (4.4.6c) we get

$$\begin{aligned}
\varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}\nabla\phi_R\nabla\rho_\varepsilon\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\,\mathrm{d}x\,\mathrm{d}t \leq& \varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}|\log(\rho_\varepsilon+\sigma)||\nabla\phi_R||\nabla\rho_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \\
&+ \varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}|\log(\rho_\delta+\sigma)||\nabla\phi_R||\nabla\rho_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $|\log(\rho+\sigma)| \leq |\log(\sigma)| + (\rho+\sigma)$ and $|\nabla\phi_R| \leq C\sigma^{\frac{1}{\alpha d}}$, we estimate the two

integrals above as

$$\begin{aligned}
\varepsilon \int_0^T \int_{\mathbb{R}^d} & |\log(\rho_\varepsilon+\sigma)\nabla\phi_R\nabla\rho_\varepsilon| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\sigma^{\frac{1}{\alpha d}+1} \int_0^T \int_{B_{2R}} (|\log(\sigma)||\nabla\rho_\varepsilon| + |\nabla\rho_\varepsilon|(\rho_\varepsilon+\sigma)) \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\sigma^{\frac{1}{\alpha d}+1}|\log(\sigma)| \int_0^T \int_{B_{2R}} \left(\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} + \rho_\varepsilon\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{C}{2}\sigma^{\frac{1}{\alpha d}+1} \int_0^T \int_{B_{2R}} \left(\sigma\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} + \sigma\rho_\varepsilon + |\nabla\rho_\varepsilon|^2 + \rho_\varepsilon^2\right) \\
&\leq C\sigma, \qquad (4.4.10)
\end{aligned}$$

for sufficiently small σ, by Lemma 4.2.1 and Proposition 4.2.4 in the last inequality. Similar computations can be done for the integral involving ρ_δ.

For the second term in (4.4.6c), we compute

$$\begin{aligned}
\delta \int_0^T \int_{\mathbb{R}^d} \nabla\phi_R\nabla\rho_\delta \left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right) \mathrm{d}x\,\mathrm{d}t \;&\leq \sigma^{\frac{1}{\alpha d}} \int_0^T \int_{\mathbb{R}^d} |\nabla\rho_\delta|(\rho_\varepsilon+\sigma)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{\sigma^{\frac{1}{\alpha d}}}{2} \int_0^T \int_{\mathbb{R}^d} \left(|\nabla\rho_\delta|^2 + \rho_\varepsilon^2 + \sigma\frac{|\nabla\rho_\delta|^2}{\rho_\delta} + \sigma\rho_\delta\right) \mathrm{d}x\,\mathrm{d}t \\
&\leq \sigma^{\frac{1}{\alpha d}} C, \qquad (4.4.11)
\end{aligned}$$

using Proposition 4.2.2 in the last inequality. Summarizing, by (4.4.7), (4.4.9), (4.4.10) and (4.4.11), for a sufficiently small σ, we have the following estimate for the error term

$$E_{\varepsilon,\delta} \leq -\varepsilon H[\rho_\varepsilon|\rho_\delta] + C\,\sigma^{\frac{1}{\alpha d}}. \qquad (4.4.12)$$

Now, we have to estimate the following two terms from (4.4.5) involving $\nabla\phi_R$:

$$I_1 := -\int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right) \nabla\rho_\varepsilon\nabla\phi_R \log\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right) \mathrm{d}x\,\mathrm{d}t, \qquad (4.4.13a)$$

$$I_2 := \int_0^T \int_{\mathbb{R}^d} \left(1 - \frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right) \nabla\rho_\delta\nabla\phi_R \left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right) \mathrm{d}x\,\mathrm{d}t. \qquad (4.4.13b)$$

Pursuing the same strategy as in (4.4.10), for (4.4.13a) we get that

$$|I_1| \lesssim C\sigma^{\frac{1}{\alpha d}}|\log(\sigma)|.$$

Similarly, (4.4.13b) can be estimated as

$$\begin{aligned}
|I_2| &\leq \sigma^{\frac{1}{\alpha d}} \int_0^T \int_{\mathbb{R}^d} |\nabla\rho_\delta| \left(1 + \frac{|\rho_\varepsilon-\rho_\delta|}{\rho_\delta+\sigma}\right) \mathrm{d}x\,\mathrm{d}t \\
&\leq \sigma^{\frac{1}{\alpha d}} \int_0^T \int_{\mathbb{R}^d} \left(\frac{|\nabla\rho_\delta|^2}{\rho_\delta} + \rho_\delta + |\nabla\rho_\delta|^2 + \frac{1}{\sigma^2}|\rho_\varepsilon-\rho_\delta|^2\right) \mathrm{d}x\,\mathrm{d}t \\
&\leq \sigma^{\frac{1}{\alpha d}} C,
\end{aligned}$$

by a priori estimates and (4.4.4).

Now, let us see what happens with the remaining term in (4.4.5). These are

$$I_3 := -\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+\nabla\rho_\varepsilon\nabla\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t \tag{4.4.14}$$

$$I_4 := \int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\nabla\rho_\delta\nabla\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4.15}$$

We can reformulate I_3 as

$$\begin{aligned}I_3 = &-\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\nabla\rho_\varepsilon\nabla\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\right]\nabla\rho_\varepsilon\nabla\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We rewrite I_4 in a similar way. It follows

$$\begin{aligned}I_3+I_4 = &-\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\nabla\rho_\varepsilon\nabla\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\nabla\rho_\delta\nabla\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t+R_1+R_2,\end{aligned}$$

where

$$R_1 := -\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\right]\nabla\rho_\varepsilon\nabla\left(\log\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t,$$

$$R_2 := \int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+-\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\right]\nabla\rho_\delta\nabla\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)\phi_R\,\mathrm{d}x\,\mathrm{d}t.$$

We recall that

$$\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\le\frac{\chi\sigma}{|\nabla\rho_\varepsilon|}, \tag{4.4.16}$$

and also that

$$0\le\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\right]$$

Similar bounds hold in R_2 for the term with ρ_δ. Using this, we estimate R_1 as follows:

$$\begin{aligned}R_1\le&\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\right]\nabla\rho_\varepsilon\frac{\nabla\rho_\delta}{\rho_\delta+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t\\ \le&\chi\sigma\int_0^T\int_{\mathbb{R}^d}\frac{|\nabla\rho_\delta|}{\rho_\delta+\sigma}\phi_R\,\mathrm{d}x\,\mathrm{d}t\le\chi\sigma^{1/2}\int_0^T\int_{\mathbb{R}^d}\frac{|\nabla\rho_\delta|}{\sqrt{\rho_\delta+\sigma}}\phi_R\,\mathrm{d}x\,\mathrm{d}t\\ \le&\chi\sigma^{\frac{1}{2}-\frac{1}{2\alpha}}C,\end{aligned}$$

which vanishes when $\sigma \to 0$ if $\alpha > 1$. We estimate R_2 in a similar way:

$$
\begin{aligned}
R_2 \leq & \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi \rho_\delta}{|\nabla \rho_\delta|} \right)_+ - \left(1 - \frac{\chi(\rho_\delta + \sigma)}{|\nabla \rho_\delta|} \right)_+ \right] \frac{\nabla \rho_\delta \nabla \rho_\varepsilon}{\rho_\delta + \sigma} \phi_R \, \mathrm{d}x \, \mathrm{d}t \\
\leq & \chi \sigma \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon|}{\rho_\delta + \sigma} \phi_R \, \mathrm{d}x \, \mathrm{d}t \leq \chi \sigma^{1/2} \int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon| \sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon} \sqrt{\rho_\delta + \sigma}} \phi_R \, \mathrm{d}x \, \mathrm{d}t \\
\leq & \chi \sigma^{1/2} \left(\int_0^T \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon|}{\rho_\varepsilon} \phi_R \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon}{\rho_\delta + \sigma} \phi_R \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \\
\leq & C \sigma^{1/2} \left(\int_0^T \int_{\mathbb{R}^d} \frac{|\rho_\varepsilon - \rho_\delta|}{\rho_\varepsilon + \sigma} \phi_R \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \frac{\rho_\delta}{\rho_\delta + \sigma} \phi_R \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \\
\leq & C \sigma^{1/2} \left(\int_0^T \int_{\mathbb{R}^d} \frac{|\rho_\varepsilon - \rho_\delta|^2}{\sigma^2} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \phi_R^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \phi_R \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \\
\leq & C \sigma^{1/2} \left(1 + \int_0^T \int_{\mathbb{R}^d} (\phi_R + \phi_R^2) \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \\
\leq & C \sigma^{\frac{1}{2} - \frac{1}{2\alpha}}.
\end{aligned}
$$

We conclude that, if $\alpha > 1$, $R_1, R_2 \to 0$ when $\sigma \to 0$.

Straightforward computations lead to

$$
\begin{aligned}
I_3 + I_4 = & - \int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon + \sigma}{2} \left[\left(1 - \frac{\chi(\rho_\varepsilon + \sigma)}{|\nabla \rho_\varepsilon|} \right)_+ - \left(1 - \frac{\chi(\rho_\delta + \sigma)}{|\nabla \rho_\delta|} \right)_+ \right] \times \\
& \times \left(|\nabla \log(\rho_\varepsilon + \sigma)|^2 - |\nabla \log(\rho_\delta + \sigma)|^2 \right) \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon + \sigma}{2} \nabla \log \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right)^2 \left[\left(1 - \frac{\chi(\rho_\varepsilon + \sigma)}{|\nabla \rho_\varepsilon|} \right)_+ + \left(1 - \frac{\chi(\rho_\delta + \sigma)}{|\nabla \rho_\delta|} \right)_+ \right] \mathrm{d}x \, \mathrm{d}t \\
& + R_1 + R_2 \\
\leq & - \int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon + \sigma}{2} \left[\left(1 - \frac{\chi(\rho_\varepsilon + \sigma)}{|\nabla \rho_\varepsilon|} \right)_+ - \left(1 - \frac{\chi(\rho_\delta + \sigma)}{|\nabla \rho_\delta|} \right)_+ \right]^2 \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^T \int_{\mathbb{R}^d} \frac{\rho_\varepsilon + \sigma}{2} \nabla \log \left(\frac{\rho_\varepsilon + \sigma}{\rho_\delta + \sigma} \right)^2 \left[\left(1 - \frac{\chi(\rho_\varepsilon + \sigma)}{|\nabla \rho_\varepsilon|} \right)_+ + \left(1 - \frac{\chi(\rho_\delta + \sigma)}{|\nabla \rho_\delta|} \right)_+ \right] \mathrm{d}x \, \mathrm{d}t \\
& + R_1 + R_2,
\end{aligned}
$$

where, in the second inequality, we have used that

$$
\left[\left(1 - \frac{\chi}{r} \right)_+ - \left(1 - \frac{\chi}{s} \right)_+ \right]^2 \leq C \left[\left(1 - \frac{\chi}{r} \right)_+ - \left(1 - \frac{\chi}{s} \right)_+ \right] \left(r^2 - s^2 \right), \quad \text{for any} \quad r, s > 0.
$$

Therefore, (4.4.3) and (4.4.5) give

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d}\left[(\rho_\varepsilon+\sigma)\log\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)-\rho_\varepsilon+\rho_\delta\right]\mathrm{d}x\right)(T)\\
&\quad-\int_{\mathbb{R}^d}\left[(\rho_{in,\varepsilon}+\sigma)\log\left(\frac{\rho_{in,\varepsilon}+\sigma}{\rho_{in,\delta}+\sigma}\right)-\rho_{in,\varepsilon}+\rho_{in,\delta}\right]\mathrm{d}x\\
&=\int_{\mathbb{R}^d}(\rho_\delta(T)-\rho_{in,\delta})\phi_R\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\left[\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\right]^2\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\ \nabla\log\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)^2\left[\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\right]\mathrm{d}x\,\mathrm{d}t\\
&\quad+R_1+R_2+I_1+I_2+E_{\varepsilon,\delta},
\end{aligned}\tag{4.4.17}$$

where, R_1, R_2, I_1, I_2 and $E_{\varepsilon,\delta}$ vanish when $\sigma,\varepsilon,\delta\to 0$. Now, we need to prove $\int_{\mathbb{R}^d}(\rho_\delta(T)-\rho_{in,\delta})\phi_R\,\mathrm{d}x\to 0$ when $\sigma,\varepsilon,\delta\to 0$. For this, we test (4.3.5) with ϕ_R to get

$$\begin{aligned}
\left(\int_{\mathbb{R}^d}\rho_\delta\phi_R\,\mathrm{d}x\right)(T)=\int_{\mathbb{R}^d}\rho_{\mathrm{in},\delta}\phi_R\,\mathrm{d}x-\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\nabla\rho_\delta\nabla\phi_R\,\mathrm{d}x\,\mathrm{d}t\\
-\varepsilon\int_0^T\int_{\mathbb{R}^d}\nabla\rho_\delta\nabla\phi_R\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\int_0^T\int_{\mathbb{R}^d}\rho_\delta\phi_R\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{4.4.18}$$

Since $|\nabla\phi_R|\le\frac{C}{R(\sigma)}\to 0$ and $\delta\to 0$ when $\sigma\to 0$, we have that

$$\lim_{\sigma\to 0}\int_0^T\int_{\mathbb{R}^d}\left(1-\frac{\chi\rho_\delta}{|\nabla\rho_\delta|}\right)_+\nabla\rho_\delta\nabla\phi_R\,\mathrm{d}x\,\mathrm{d}t\ \le\lim_{\sigma\to 0}\frac{C}{R(\sigma)}\|\nabla\sqrt{\rho_\delta}\|_2\|\rho_\delta\|_1=0.$$

The viscosity terms in (4.4.18) also vanish when $\sigma\to 0$, and therefore we get that

$$\lim_{\sigma\to 0}\left(\int_{\mathbb{R}^d}\rho_\delta\varphi\,\mathrm{d}x\right)(T)=\lim_{\sigma\to 0}\int_{\mathbb{R}^d}\rho_{\mathrm{in},\delta}\varphi\,\mathrm{d}x.$$

Finally, taking the limit $\sigma,\varepsilon,\delta\to 0$ in (4.4.17) yields

$$\lim_{\epsilon,\delta\to 0}\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\left[\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+-\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\right]^2\phi_R\,\mathrm{d}x\,\mathrm{d}t=0,\tag{4.4.19a}$$

$$\begin{aligned}
\lim_{\epsilon,\delta\to 0}\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}(\rho_\varepsilon+\sigma)\ \nabla\log\left(\frac{\rho_\varepsilon+\sigma}{\rho_\delta+\sigma}\right)^2\times\\
\times\left[\left(1-\frac{\chi(\rho_\varepsilon+\sigma)}{|\nabla\rho_\varepsilon|}\right)_+\left(1-\frac{\chi(\rho_\delta+\sigma)}{|\nabla\rho_\delta|}\right)_+\right]\phi_R\,\mathrm{d}x\,\mathrm{d}t=0.
\end{aligned}\tag{4.4.19b}$$

4.4.2 Identification of the limit of $\nabla\rho_\varepsilon$

Now, from (4.4.19a) is easy to see that

$$\left(1-\frac{\chi(\rho_\varepsilon+\varepsilon)}{|\nabla\rho_\varepsilon|}\right)_+\to\overline{G}\quad\text{a.e. } x\in[0,T]\times\left(\mathbb{R}^d\setminus\{\rho=0\}\right),$$

And on the other hand, from (4.4.19b)

$$\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon+\varepsilon}\to w \quad \text{a.e. in } [0,T]\times\left(\mathbb{R}^d\setminus\left(\{\rho=0\}\cup\{\overline{G}=0\}\right)\right),$$

Notice, moreover, that if $\left(1-\frac{\chi(\rho_\varepsilon+\varepsilon)}{|\nabla\rho_\varepsilon|}\right)_+$ converges almost everywhere in $[0,T]\times\mathbb{R}^d\setminus\{\rho=0\}$ then

$$\frac{|\nabla\rho_\varepsilon|}{\rho_\varepsilon+\varepsilon}<\infty \quad \text{a.e. in } [0,T]\times\left(\mathbb{R}^d\setminus\{\rho=0\}\right),$$

and we have

$$(\rho_\varepsilon+\varepsilon)^3\frac{|\nabla\rho_\varepsilon|}{\rho_\varepsilon+\varepsilon}\left(\frac{|\nabla\rho_\varepsilon|}{\rho_\varepsilon+\varepsilon}+\chi\right)\left(1-\frac{\chi(\rho_\varepsilon+\varepsilon)}{|\nabla\rho_\varepsilon|}\right)_+\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon+\varepsilon}\to\overline{w},$$

almost everywhere in $[0,T]\times\left(\mathbb{R}^d\setminus\{\rho=0\}\right)$, or, which is equivalent,

$$\left(|\nabla\rho_\varepsilon|^2-(\rho_\varepsilon+\varepsilon)^2\chi^2\right)_+\nabla\rho_\varepsilon\to\overline{w},$$

almost everywhere in $[0,T]\times\left(\mathbb{R}^d\setminus\{\rho=0\}\right)$. But as $\rho_\varepsilon\to\rho$ strongly in $L^2(0,T;\mathbb{R}^d)$, $\rho_\varepsilon\to\rho$ a.e. in $(0,T)\times\mathbb{R}^d$, and

$$\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon\to\overline{w},$$

almost everywhere in $[0,T]\times\left(\mathbb{R}^d\setminus\{\rho=0\}\right)$.

Define $Q_{R,\tau,T}=[\tau,T]\times(B_R\cap\{\rho>0\})$. Egorov's and Lusin's Theorems imply that, for every $\eta>0$, there exists $Q^\eta_{R,\tau,T}\subset Q_{R,\tau,T}$ compact such that $|Q_{R,\tau,T}\setminus Q^\eta_{R,\tau,T}|<\eta$,

$$\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon\in C^0(Q^\eta_{R,\tau,T}),$$
$$\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon\to\overline{w} \quad \text{uniformly in } Q^\eta_{R,\tau,T}.$$

Then, as $\nabla\rho_\varepsilon\rightharpoonup\nabla\rho$ weakly in $L^p(\mathbb{R}^d\times(0,T))$ we have that

$$\begin{aligned}
&\iint_{Q^\eta_{R,\tau,T}}\left[\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon-\left(|\nabla\rho|^2-\rho^2\chi^2\right)_+\nabla\rho\right](\nabla\rho_\varepsilon-\nabla\rho)\,\mathrm{d}x\,\mathrm{d}t\\
&=\iint_{Q^\eta_{R,\tau,T}}\left[\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon-\overline{w}\right](\nabla\rho_\varepsilon-\nabla\rho)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{Q^\eta_{R,\tau,T}}\left[\overline{w}-\left(|\nabla\rho|^2-\rho^2\chi^2\right)_+\nabla\rho\right](\nabla\rho_\varepsilon-\nabla\rho)\,\mathrm{d}x\,\mathrm{d}t\to 0.
\end{aligned}$$

But, notice that

$$\begin{aligned}
&\int_{Q^\eta_{R,\tau,T}}\left[\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+\nabla\rho_\varepsilon-\left(|\nabla\rho|^2-\rho^2\chi^2\right)_+\nabla\rho\right](\nabla\rho_\varepsilon-\nabla\rho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
=&\frac{1}{2}\iint_{Q^\eta_{R,\tau,T}}\left[\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+-\left(|\nabla\rho|^2-\rho^2\chi^2\right)_+\right]\times|\nabla(\rho_\varepsilon-\rho)|^2\,\mathrm{d}x\,\mathrm{d}t \qquad (4.4.20a)\\
&+\frac{1}{2}\iint_{Q^\eta_{R,\tau,T}}\left[\left(|\nabla\rho_\varepsilon|^2-\rho^2\chi^2\right)_+-\left(|\nabla\rho|^2-\rho^2\chi^2\right)_+\right]\times\left(|\nabla\rho_\varepsilon|^2-|\nabla\rho_\varepsilon|^2\right)\mathrm{d}x\,\mathrm{d}t\to 0. \qquad (4.4.20b)
\end{aligned}$$

Since both integrals in (4.4.20a), (4.4.20b) are nonnegative, the convergence to 0 of (4.4.20a) implies that we can find a subsequence (not relabeled) such that $\nabla\rho_\varepsilon \to \nabla\rho$ a.e. in $Q^\eta_{R,\tau,T} \setminus \{\rho\chi \geq |\nabla\rho|\}$. W.l.o.g. we can choose $\eta = 1/N$, $N \in \mathbb{N}$, and assume that $Q^{1/N_1}_{R,\tau,T} \subset Q^{1/N_2}_{R,\tau,T}$ for $N_1 < N_2$. Since $|Q_{R,\tau,T} \setminus Q^\eta_{R,\tau,T}| < \eta$, a Cantor diagonal argument yields the existence of a subsequence (not relabeled) such that $\nabla\rho_\varepsilon \to \nabla\rho$ a.e. in $Q_{R,\tau,T} \setminus \{\rho\chi \geq |\nabla\rho|\}$. Via another Cantor diagonal argument we conclude that (up to a subsequence)

$$\nabla\rho_\varepsilon \to \nabla\rho \quad \text{a.e. in } (0,T) \times \left(\mathbb{R}^d \setminus \{\rho\chi \geq |\nabla\rho|\}\right) \tag{4.4.21}$$

Thus

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{in } L^2\left((0,T) \times \left(\mathbb{R}^d \setminus \{\rho\chi \geq |\nabla\rho|\}\right)\right)$$

Furthermore

$$\begin{aligned}
&\left[\left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_+ - \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+\right] \times \left(|\nabla\rho_\varepsilon|^2 - |\nabla\rho_\varepsilon|^2\right) \\
&= \left[\left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_+ - \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+\right]^2 \\
&\quad - \left[\left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_+ - \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+\right] \left[\left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_- - \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_-\right] \\
&= \left[\left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_+ - \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+\right]^2 \\
&\quad + \left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_+ \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+ + \left(|\nabla\rho_\varepsilon|^2 - \rho^2\chi^2\right)_- \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_-.
\end{aligned}$$

So, (4.4.20b) gives

$$\left(|\nabla\rho_\varepsilon|^2 - \rho_\varepsilon^2\chi^2\right)_+ \to \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+ \quad \text{a.e. in } L^2\left((0,T) \times \left(\mathbb{R}^d \cap \{\rho > 0\}\right)\right)$$

Moreover, notice that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \ (|\nabla\rho_\varepsilon| + \rho_\varepsilon\chi)_+ &\leq \lim_{\varepsilon\to 0} \left(|\nabla\rho_\varepsilon|^2 - \rho_\varepsilon^2\chi^2\right)_+ \\
&= \left(|\nabla\rho|^2 - \rho^2\chi^2\right)_+ = 0,
\end{aligned}$$

a.e. in $(0,T) \times \left(\mathbb{R}^d \cap \{\rho\chi \geq |\nabla\rho, |\ \rho > 0\}\right)$. Finally, it holds

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \ \leq 2\sqrt{\rho_\varepsilon}|\nabla\sqrt{\rho_\varepsilon}| \to 0 = \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{a.e. in } \{\rho > 0\}.$$

This means that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \to \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{a.e. in } (0,T) \times \mathbb{R}^d,$$

and we finally get

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{weakly in } L^2((0,T) \times \mathbb{R}^d).$$

4.4.3 Limit $\varepsilon \to 0$ in (4.3.5)

As in (4.3.3), we have that

$$\|\partial_t \rho_\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \leq C\|\rho_{\mathrm{in},\varepsilon}\|_{L^2(\mathbb{R}^d)},$$

and, as $\rho_{\mathrm{in},\varepsilon}$ is uniformly bounded in $L^2(0,T;\mathbb{R}^d)$, this yields

$$\partial_t \rho_\varepsilon \rightharpoonup \partial_t \rho \quad \text{weakly in} \quad L^2(0,T;H^{-1}).$$

On the other hand, due to what is stated in the previous section, we have that

$$\left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+ \nabla\rho_\varepsilon \rightharpoonup \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+ \nabla\rho \quad \text{weakly in} \quad L^2(0,T;\mathbb{R}^d).$$

For the regularizing terms we have the following:

$$\begin{aligned}\lim_{\varepsilon\to 0}\ \varepsilon \int_0^T \int_{\mathbb{R}^d} \nabla\rho_\varepsilon \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t \ &\leq \lim_{\varepsilon\to 0} \varepsilon C\|\rho_{\mathrm{in},\varepsilon}\|_{L^2(0,T;\mathbb{R}^d)}\|\varphi\|_{L^2(0,T;\mathbb{R}^d)} \\ &= 0.\end{aligned}$$

Finally we have that ρ is the weak solution to (4.1.8).

4.4.4 Obtaining the initial data ρ_{in} for ρ

In what follows, we will show that $\rho(t) \to \rho_{\mathrm{in}}$ strongly in $L^p(\mathbb{R}^d)$ as $t \to 0$.

We start by reminding that ρ_ε is bounded in the space

$$\left\{u \in L^2(0,T;H^1(\mathbb{R}^d)) \quad : \quad \partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^d))\right.$$

which embeds continuously into $C^0([0,T];L^2(\mathbb{R}^d))$ [145, Prop. 23.23]. In particular it follows that $\rho_\varepsilon(t) \rightharpoonup \rho(t)$ weakly in $H^{-1}(\mathbb{R}^d)$ for every $t \in [0,T]$, hence

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \rho_\varepsilon(t)\varphi dx = \int_{\mathbb{R}^d} \rho(t)\varphi dx \quad \forall\varphi \in H^1(\mathbb{R}^d). \tag{4.4.22}$$

Also remember that the initial condition for the approximated problem (4.3.5) is satisfied in the sense of (4.3.2).

Define for brevity $A_\varepsilon := \varepsilon + \left(1 - \frac{\chi\rho_\varepsilon}{|\nabla\rho_\varepsilon|}\right)_+$, $A := \left(1 - \frac{\chi\rho}{|\nabla\rho|}\right)_+$. Choose $\varphi \in H^1(\mathbb{R}^d)$ independent of time in (4.3.5). It follows

$$\begin{aligned}&\int_{\mathbb{R}^d} \rho_\varepsilon(\tau)\varphi \,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_{\mathrm{in},\varepsilon}\varphi \,\mathrm{d}x \\ &= \int_0^\tau \int_{\mathbb{R}^d} A_\varepsilon \nabla\rho_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_0^\tau \int_{\mathbb{R}^d} \rho_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\leq (1+\varepsilon)\left(\|\nabla\rho_\varepsilon\|_{L^2(0,\tau;L^2(\mathbb{R}^d))}\|\nabla\varphi\|_{L^2(0,\tau;L^2(\mathbb{R}^d))} + \|\rho_\varepsilon\|_{L^2(0,\tau;L^2(\mathbb{R}^d))}\|\varphi\|_{L^2(0,\tau;L^2(\mathbb{R}^d))}\right) \\ &\leq C\tau^{1/2}\|\varphi\|_{H^1(\mathbb{R}^d)}.\end{aligned}$$

Relation (4.4.22) allows us to take the limit $\varepsilon \to 0$ in the above estimate and obtain

$$\int_{\mathbb{R}^d} \rho(\tau)\varphi \,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_0\varphi \,\mathrm{d}x \ \le C\tau^{1/2}\|\varphi\|_{H^1(\mathbb{R}^d)},$$

implying

$$\rho(t) \to \rho_0 \quad \text{strongly in } H^{-1}(\mathbb{R}^d) \text{ as } t \to 0. \tag{4.4.23}$$

Define now, for $M > 0$ arbitrary,

$$F_M(s) = \int_0^s f_M(\sigma)d\sigma, \quad f_M(s) = \begin{cases} ps^{p-1} & 0 \le s \le M, \\ p(p-1)M^{p-2}(s-M) + pM^{p-1} & s > M. \end{cases}.$$

Choose $\varphi = f_M(\rho)\chi_{[t_1,t_2]}(t) \in L^2(0,T;H^1(\mathbb{R}^d))$ in (4.1.8). It follows

$$\begin{aligned}\int_{\mathbb{R}^d} F_M(\rho(t_2))\,\mathrm{d}x - \int_{\mathbb{R}^d} F_M(\rho(t_1))\,\mathrm{d}x &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d} Af'_M(\rho)|\nabla\rho|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{4}{p^2}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} Af'_M(\rho)\rho^{2-p}|\nabla\rho^{p/2}|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $f'_M(\rho)\rho^{2-p} = p(p-1)\min(\rho^{p-2}, M^{p-2})\rho^{2-p} \le p(p-1)$, $F_M(\rho) \le \rho^p$, Proposition 4.2.2 and the dominated convergence theorem allow us to take the limit $M \to \infty$ in the above identity and deduce

$$\int_{\mathbb{R}^d} \rho(t_2)^p\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho(t_1)^p\,\mathrm{d}x = \frac{4(p-1)}{p}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} A|\nabla\rho^{p/2}|^2\,\mathrm{d}x\,\mathrm{d}t$$

for a.e. $t_1, t_2 \in [0,T]$. It follows that the function $t \mapsto \|\rho(t)\|^p_{L^p(\mathbb{R}^d)}$ is in $W^{1,1}(0,T)$, hence, up to changing the values of $\rho : [0,T] \to L^p(\mathbb{R}^d)$ in a zero measure set, it holds that $t \mapsto \|\rho(t)\|^p_{L^p(\mathbb{R}^d)}$ is continuous in $[0,T]$. As a consequence, the bound $\|\rho(t)\|_{L^p(\mathbb{R}^d)} \le \|\rho_0\|_{L^p(\mathbb{R}^d)}$ holds for every $t \in [0,T]$. In particular $\rho(t)$ is weakly convergent (up to subsequences) in $L^p(\mathbb{R}^d)$ as $t \to 0$; however, its weak limit has to be ρ_0 due to (4.4.23), hence $\rho(t) \rightharpoonup \rho_0$ weakly in $L^p(\mathbb{R}^d)$ as $t \to 0$ (no longer up to subsequences). It follows

$$\|\rho_0\|_{L^p(\mathbb{R}^d)} \le \liminf_{t\to 0} \|\rho(t)\|_{L^p(\mathbb{R}^d)} \le \limsup_{t\to 0} \|\rho(t)\|_{L^p(\mathbb{R}^d)} \le \|\rho_0\|_{L^p(\mathbb{R}^d)},$$

implying that $\|\rho(t)\|_{L^p(\mathbb{R}^d)} \to \|\rho_0\|_{L^p(\mathbb{R}^d)}$ as $t \to 0$. This fact and the weak convergence $\rho(t) \rightharpoonup \rho_0$ in $L^p(\mathbb{R}^d)$ imply that $\rho(t) \to \rho_{\text{in}}$ strongly in $L^p(\mathbb{R}^d)$ as $t \to 0$.

4.4.5 Uniqueness

Let ρ_1 and ρ_2 be two solutions to (4.1.8) with the same initial data. As in Section 4.4, we subtract equation (4.1.8) for ρ_2 tested with $\varphi_2 = \frac{\rho_1+\sigma}{\rho_2+\sigma} - 1 = \frac{\rho_1-\rho_2}{\rho_2+\sigma}$ to the equation for ρ_1 tested with $\varphi_1 = \log\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)$. Following the same computations as

in (4.4.3), we get

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} [\varphi_1 \partial_t \rho_1 - \varphi_2 \partial_t \rho_2] \,\mathrm{d}x\, dt &= \\ &= \int_0^T \frac{d}{dt} \int_{\mathbb{R}^d} (\rho_2+\sigma)\left[\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)\log\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right) - \left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)+1\right]\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} (\rho_2+\sigma)\left[\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)\log\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right) - \left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)+1\right](T)\,\mathrm{d}x.\end{aligned}$$

After some manipulations, this yields

$$\begin{aligned}&\int_{\mathbb{R}^d} (\rho_2+\sigma)\left[\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)\log\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right) - \left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)+1\right](T)\,\mathrm{d}x \qquad (4.4.24)\\ &= -\frac{1}{2}\int_0^T\int_{\mathbb{R}^d} (\rho_1+\sigma)\left[\left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+ - \left(1-\frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+\right]\\ &\qquad\cdot\left(|\nabla\log(\rho_1+\sigma)|^2 - |\nabla\log(\rho_2+\sigma)|^2\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad -\frac{1}{2}\int_0^T\int_{\mathbb{R}^d} (\rho_1+\sigma)\ \nabla\log\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)^2\left[\left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+ + \left(1-\frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+\right]\mathrm{d}x\,\mathrm{d}t\\ &\quad +\xi_1+\xi_2\\ &\le \xi_1+\xi_2\end{aligned}$$

where

$$\xi_1 := -\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_1}{|\nabla\rho_1|}\right)_+ - \left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+\right]\nabla\rho_1\nabla\left(\log\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)\mathrm{d}x\,\mathrm{d}t,$$

and

$$\xi_2 := \int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1-\frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+\right]\nabla\rho_2\nabla\left(\frac{\rho_1+\sigma}{\rho_2+\sigma}\right)\mathrm{d}x\,\mathrm{d}t.$$

We want to show that $\limsup_{\sigma\to 0}(\xi_1+\xi_2)\le 0$. We estimate first ξ_1 and then ξ_2.

$$\begin{aligned}\xi_1 &\le \int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_1}{|\nabla\rho_1|}\right)_+ - \left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+\right]\nabla\rho_1\frac{\nabla\rho_2}{\rho_2+\sigma}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \underbrace{\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_1}{|\nabla\rho_1|}\right)_+ - \left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+\right]\frac{|\nabla\rho_1|^2}{\rho_2+\sigma}\,\mathrm{d}x\,\mathrm{d}t}_{\text{(I)}}\\ &\quad + \underbrace{\frac{1}{2}\int_0^T\int_{\mathbb{R}^d}\left[\left(1-\frac{\chi\rho_1}{|\nabla\rho_1|}\right)_+ - \left(1-\frac{\chi(\rho_1+\sigma)}{|\nabla\rho_1|}\right)_+\right]\frac{|\nabla\rho_2|^2}{\rho_2+\sigma}\,\mathrm{d}x\,\mathrm{d}t}_{\text{(II)}}.\end{aligned}$$

It is easy to see that the integrand in (II) is controlled by $|\nabla\sqrt{\rho_2}|^2 \in L^1(\mathbb{R}^d\times(0,T))$. For (I) we use the Lipschitz condition (4.4.16) and we get

$$|\text{Integrand in (I)}| \le \frac{1}{2}|\nabla\rho_1| \le |\nabla\sqrt{\rho_1}|\sqrt{\rho_1} \in L^1(\mathbb{R}^d\times(0,T)).$$

This shows that ξ_1 is uniformly bounded with respect to σ. Let us estimate ξ_2:

$$\begin{aligned}
\xi_2 &\le \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1 - \frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+ \right] \frac{\nabla\rho_2 \nabla\rho_1}{\rho_2+\sigma} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1 - \frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+ \right] \frac{(\rho_1+\sigma)|\nabla\rho_2|^2}{(\rho_2+\sigma)^2} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1 - \frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+ \right] \frac{|\nabla\rho_1|^2}{\rho_1+\sigma} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1 - \frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+ \right] \frac{(\rho_1+\sigma)|\nabla\rho_2|^2}{(\rho_2+\sigma)^2} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \left[\left(1 - \frac{\chi\rho_2}{|\nabla\rho_2|}\right)_+ - \left(1 - \frac{\chi(\rho_2+\sigma)}{|\nabla\rho_2|}\right)_+ \right] \frac{|\nabla\rho_1|^2}{\rho_1} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

The integrand in the last expression is controlled by $|\nabla\sqrt{\rho_1}| \in L^1(\mathbb{R}^d \times (0,T))$. Therefore by dominated convergence we can take the limit $\sigma \to 0$ in (4.4.24) to obtain

$$\int_{\mathbb{R}^d} \left[\rho_1 \log\left(\frac{\rho_1}{\rho_2}\right) - \rho_1 + \rho_2 \right](T)\,\mathrm{d}x \le \limsup_{\sigma\to 0} (\xi_1 + \xi_2) \le 0.$$

This implies that $\rho_1 = \rho_2$.

Bibliography

[1] J. Adler. "Chemotaxis in bacteria". In: *Annual Review of Biochemistry* 44.1 (1975), pp. 341–356.

[2] L. V. Ahlfors. *Complex Analysis*. 2. McGraw-Hill Book Company, 1966.

[3] G. Akagi. "Stability of non-isolated asymptotic profiles for fast diffusion". In: *Communications in Mathematical Physics* 345 (2016), pp. 77–100.

[4] L. Ambrosio, E. Brué, D. Semola, et al. *Lectures on optimal transport*. Springer, 2021.

[5] D. H. Armitage and Ü. Kuran. "The convexity of a domain and the superharmonicity of the signed distance function". In: *Proceedings of the American Mathematical Society* 93.4 (1985), pp. 598–600.

[6] J. M. Arrieta, A. Jiménez-Casas, and A. Rodríguez-Bernal. "Flux terms and Robin boundary conditions as limit of reactions and potentials concentrating at the boundary". In: *Rev. Mat. Iberoam* 24.1 (2008), pp. 183–211.

[7] I. Athanasopoulos and L. A. Caffarelli. "Continuity of the temperature in boundary heat control problems". In: *Advances in Mathematics* 224.1 (2010), pp. 293–315.

[8] R. Bañuelos, M. Van den Berg, and T. Carroll. "Torsional rigidity and expected lifetime of Brownian motion". In: *Journal of the London Mathematical Society* 66.2 (2002), pp. 499–512.

[9] P. Bénilan and R. Gariepy. "Strong solutions in L1 of degenerate parabolic equations". In: *Journal of Differential Equations* 119.2 (1995), pp. 473–502.

[10] E. Berchio, M. Bonforte, G. Grillo, and M. Muratori. "The fractional porous medium equation on noncompact Riemannian manifolds". In: *Mathematische Annalen* (2023), pp. 1–49.

[11] M. van den Berg, G. Buttazzo, and A. Pratelli. "On relations between principal eigenvalue and torsional rigidity". In: *Communications in Contemporary Mathematics* 23.8 (2021), p. 2050093.

[12] M. van den Berg, V. Ferone, C. Nitsch, and C. Trombetti. "On Pólya's inequality for torsional rigidity and first Dirichlet eigenvalue". In: *Integral Equations and Operator Theory* 86 (2016), pp. 579–600.

[13] M. van den Berg and A. Malchiodi. "On some variational problems involving capacity, torsional rigidity, perimeter and measure". In: *Advances in Calculus of Variations* 16.4 (2023), pp. 961–974.

[14] J. G. Berryman and C. J. Holland. “Stability of the separable solution for fast diffusion”. In: *Archive for Rational Mechanics and Analysis* 74.4 (1980), pp. 379–388.

[15] R. M. Blumenthal and R. K. Getoor. “Some theorems on stable processes”. In: *Transactions of the American Mathematical Society* 95.2 (1960), pp. 263–273.

[16] K. Bogdan, K. Burdzy, and Z. Q. Chen. “Censored stable processes”. In: *Probability Theory and Related Fields* 127 (2003), pp. 89–152.

[17] M. Bologna, C. Tsallis, and P. Grigolini. “Anomalous diffusion associated with nonlinear fractional derivative Fokker-Planck-like equation: Exact time-dependent solutions”. In: *Physical Review E* 62.2 (2000), pp. 2213–2218.

[18] M. Bonforte and A. Figalli. “Sharp extinction rates for fast diffusion equations on generic bounded domains”. In: *Communications on Pure and Applied Mathematics* 74.4 (2021), pp. 744–789.

[19] M. Bonforte and A. Figalli. “The Cauchy–Dirichlet problem for the fast diffusion equation on bounded domains”. In: *Nonlinear Analysis* 239 (2024), p. 113394.

[20] M. Bonforte, A. Figalli, and X. Ros-Oton. “Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains”. In: *Communications on Pure and Applied Mathematics* 70.8 (2017), pp. 1472–1508.

[21] M. Bonforte, A. Figalli, and J. L. Vázquez. “Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations”. In: *Calculus of Variations and Partial Differential Equations* 57 (2018), pp. 1–34.

[22] M. Bonforte, A. Figalli, and J. L. Vázquez. “Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains”. In: *Analysis & PDE* 11.4 (2018), pp. 945–982.

[23] M. Bonforte, P. Ibarrondo, and M. Ispizua. “The Cauchy-Dirichlet problem for singular nonlocal diffusions on bounded domains”. In: *Discrete and Continuous Dynamical Systems* 43.3 (2023), pp. 1090–1142.

[24] M. Bonforte, Y. Sire, and J. L. Vázquez. “Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains”. In: *Discrete and Continuous Dynamical Systems* 35.12 (2014).

[25] M. Bonforte, Y. Sire, and J. L. Vázquez. “Optimal existence and uniqueness theory for the fractional heat equation”. In: *Nonlinear Analysis: Theory, Methods & Applications* 153 (2017), pp. 142–168.

[26] M. Bonforte and J. L. Vázquez. “A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains”. In: *Archive for Rational Mechanics and Analysis* 218 (2015), pp. 317–362.

[27] M. Bonforte and J. L. Vázquez. “Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds”. In: *Nonlinear Analysis* 131 (2016), pp. 363–398.

[28] M. Bonforte and J. L. Vázquez. "Quantitative local and global a priori estimates for fractional nonlinear diffusion equations". In: *Advances in Mathematics* 250 (2014), pp. 242–284.

[29] L. Brasco. "On principal frequencies and isoperimetric ratios in convex sets". In: *Annales de la Faculté des Sciences de Toulouse: Mathématiques*. Vol. 29. 4. 2020, pp. 977–1005.

[30] L. Brasco, G. De Philippis, and B. Ruffini. "Spectral optimization for the Stekloff–Laplacian: the stability issue". In: *Journal of Functional Analysis* 262.11 (2012), pp. 4675–4710.

[31] L. Brasco, M. González, and M. Ispizua. "A Steklov version of the torsional rigidity". In: *Communications in Contemporary Mathematics* (2023), p. 2350037.

[32] L. Brasco and D. Mazzoleni. "On principal frequencies, volume and inradius in convex sets". In: *Nonlinear Differential Equations and Applications NoDEA* 27.2 (2020), p. 12.

[33] H. Brezis. *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. Elsevier, 1973.

[34] H. Brézis. "Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations". In: *Contributions to Nonlinear Functional Analysis*. Elsevier, 1971, pp. 101–156.

[35] H. Brézis and A. Friedman. *Nonlinear parabolic equations involving measures as initial conditions*. Mathematics Research Center, University of Wisconsin-Madison, 1981.

[36] L. Briani, G. Buttazzo, and F. Prinari. "Inequalities between torsional rigidity and principal eigenvalue of the p-Laplacian". In: *Calculus of Variations and Partial Differential Equations* 61.78 (2022), p. 78.

[37] L. Briani, G. Buttazzo, and F. Prinari. "Some inequalities involving perimeter and torsional rigidity". In: *Applied Mathematics & Optimization* 84 (2021), pp. 2727–2741.

[38] F. Brock. "An isoperimetric inequality for eigenvalues of the Stekloff problem". In: *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* 81.1 (2001), pp. 69–71.

[39] D. Bucur and G. Buttazzo. *Variational methods in shape optimization problems*. Birkhäuser Boston, Inc., 2005.

[40] D. Bucur, V. Ferone, C. Nitsch, and C. Trombetti. "Weinstock inequality in higher dimensions". In: *Journal of Differential Geometry* 118.1 (2021), pp. 1–21.

[41] L. Caffarelli and L. Silvestre. "An extension problem related to the fractional Laplacian". In: *Communications in Partial Differential Equations* 32.8 (2007), pp. 1245–1260.

[42] V. Calvez and F. Hoffmann. "Nonlinear stability of chemotactic clustering with discontinuous advection". In: *Annales de la Faculté des Sciences de Toulouse : Mathématiques* 2 (2023).

[43] J. A. Carrillo, M. del Pino, A. Figalli, G. Mingione, and J. L. Vázquez. "The mathematical theories of diffusion: nonlinear and fractional diffusion". In: *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions: Cetraro, Italy 2016* (2017), pp. 205–278.

[44] F. A. C. C. Chalub, P. A. Markowich, B. Perthame, and C. Schmeiser. *Kinetic models for chemotaxis and their drift-diffusion limits*. Springer, 2004.

[45] Z. Q Chen, P. Kim, and R. Song. "Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation". In: (2012).

[46] Z. Q Chen, P. Kim, and R. Song. "Green function estimates for relativistic stable processes in half-space-like open sets". In: *Stochastic Processes and Their Applications* 121.5 (2011), pp. 1148–1172.

[47] Z. Q Chen, P. Kim, and R. Song. "Two-sided heat kernel estimates for censored stable-like processes". In: *Probability Theory and Related Fields* 146.3-4 (2010), pp. 361–399.

[48] Z. Q. Chen, P. Kim, and R. Song. "Dirichlet heat kernel estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$". In: *Illinois Journal of Mathematics* 54.4 (2010), pp. 1357–1392.

[49] M. G Crandall and P. Benilan. *Regularizing effects of homogeneous evolution equations*. University of Wisconsin–Madison, Mathematics Research Center, 1980.

[50] G. Crasta and I. Fragalà. "On the characterization of some classes of proximally smooth sets". In: *ESAIM: Control, Optimisation and Calculus of Variations* 22.3 (2016), pp. 710–727.

[51] N. Cusimano, F. del Teso, and L. Gerardo-Giorda. "Numerical approximations for fractional elliptic equations via the method of semigroups". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 54.3 (2020), pp. 751–774.

[52] P. Daskalopoulos and C. E. Kenig. *Degenerate diffusions: initial value problems and local regularity theory*. Vol. 1. European Mathematical Society, 2007.

[53] E. B. Davies. *Heat kernels and spectral theory*. 92. Cambridge University Press, 1989.

[54] E. Di Benedetto. "On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients". In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 13.3 (1986), pp. 487–535.

[55] J. B. Díaz and A. Weinstein. "The torsional rigidity and variational methods". In: *American Journal of Mathematics* 70.1 (1948), pp. 107–116.

[56] E. DiBenedetto. *Degenerate parabolic equations*. Springer Science & Business Media, 1993.

[57] E. DiBenedetto, U. P. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular parabolic equations*. Springer Science & Business Media, 2011.

[58] E. DiBenedetto, Y. Kwong, and V. Vespri. "Local Space–analiticity of Solutions of Certain Singular Parabolic Equations". In: *Indiana University Mathematics Journal* (1991), pp. 741–765.

[59] R. Erban and H. G. Othmer. "Taxis equations for amoeboid cells". In: *Journal of Mathematical Biology* 54 (2007), pp. 847–885.

[60] J. F. Escobar. "A comparison theorem for the first non-zero Steklov eigenvalue". In: *Journal of Functional Analysis* 178.1 (2000), pp. 143–155.

[61] J. F. Escobar. "An isoperimetric inequality and the first Steklov eigenvalue". In: *Journal of functional analysis* 165.1 (1999), pp. 101–116.

[62] L. C. Evans. *Partial differential equations*. Vol. 19. American Mathematical Society, 2022.

[63] E. Feireisl and F. Simondon. "Convergence for semilinear degenerate parabolic equations in several space dimensions". In: *Journal of Dynamics and Differential Equations* 12 (2000), pp. 647–673.

[64] M. Felsinger and M. Kassmann. "Local regularity for parabolic nonlocal operators". In: *Communications in Partial Differential Equations* 39.9 (2013), pp. 1539–1573.

[65] F. Filbet, P. Laurencot, and B. Perthame. "Derivation of hyperbolic models for chemosensitive movement". In: *Journal of Mathematical Biology* 50.2 (2005), pp. 189–207.

[66] A. Fraser and R. Schoen. "The first Steklov eigenvalue, conformal geometry, and minimal surfaces". In: *Advances in Mathematics* 226.5 (2011), pp. 4011–4030.

[67] S. Fucik, O. John, and A. Kufner. *Function spaces*. Vol. 3. Springer Science & Business Media, 1977.

[68] U. Gianazza, M. Surnachev, and V. Vespri. "A new proof of the Hölder continuity of solutions to p-Laplace type parabolic equations". In: *Advances in Calculus of Variations* 3.3 (2010), pp. 263–278.

[69] A. Girouard and I. Polterovich. "Spectral geometry of the Steklov problem (survey article)". In: *Journal of Spectral Theory* 7.2 (2017), pp. 321–359.

[70] E. Giusti. *Direct methods in the calculus of variations*. World Scientific, 2003.

[71] G. Grillo, M. Muratori, and F. Punzo. "Fractional porous media equations: existence and uniqueness of weak solutions with measure data". In: *Calculus of Variations and Partial Differential Equations* 54 (2015), pp. 3303–3335.

[72] P. Grisvard. *Elliptic problems in nonsmooth domains*. SIAM, 2011.

[73] G. Grubb. "Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators". In: *Advances in Mathematics* 268 (2015), pp. 478–528.

[74] M. Gualdani, M. Ispizua, I. Kim, and N. Zamponi. "A chemotaxis model with saturated discontinuous flux: the symmetric case". In: *In Progress* (2024).

[75] M. Gualdani, M. Ispizua, and N. Zamponi. "A quasilinear Keller-Segel model with saturated discontinuous advection". In: *Preprint in arXiv:2403.06820* (2024).

[76] H. Zhang H. Dong. "On Schauder estimates for a class of nonlocal fully nonlinear parabolic equations". In: *Calculus of Variations and Partial Differential Equations* (2019).

[77] G. H. Hardy. "The mean value of the modulus of an analytic function". In: *Proceedings of the London Mathemathical Society* 2.1 (1915), pp. 269–277.

[78] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Springer Science & Business Media, 2006.

[79] A. Henrot and M. Pierre. *Variation et optimisation de formes: une analyse géométrique*. Vol. 48. Springer Science & Business Media, 2006.

[80] J. Hersch. "Contribution to the method of interior parallels applied to vibrating membranes". In: *Studies in Mathematical Analysis and Related Topics, Stanford University Press* (1962), pp. 132–139.

[81] M. Jara, T. Komorowski, and S. Olla. "Limit theorems for additive functionals of a Markov chain". In: *The Annals of Applied Probability* 19.6 (2009), pp. 2270–2300.

[82] M. D. Jara. "Non-equilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps". In: *Communications pn Pure and Applied Mathematics* 62.2 (2009), pp. 198–214.

[83] T. Jin and J. Xiong. "Bubbling and extinction for some fast diffusion equations in bounded domains". In: *Transactions of the American Mathematical Society, Series B* 10.36 (2023), pp. 1287–1332.

[84] T. Jin and J. Xiong. "Optimal boundary regularity for fast diffusion equations in bounded domains". In: *American Journal of Mathematics* 145.1 (2023), pp. 151–219.

[85] E. F. Keller and L. A. Segel. "Initiation of slime mold aggregation viewed as an instability". In: *Journal of Theoretical Biology* 26.3 (1970), pp. 399–415.

[86] E. F. Keller and L. A. Segel. "Traveling bands of chemotactic bacteria: a theoretical analysis". In: *Journal of Theoretical Biology* 30.2 (1971), pp. 235–248.

[87] K. Y. Kim and P. Kim. "Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in C1, η open sets". In: *Stochastic Processes and their Applications* 124.9 (2014), pp. 3055–3083.

[88] P. Kim and R. Song. "Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts". In: *Journal of Mathematical Analysis and Applications* 332.1 (2007), pp. 57–80.

[89] M. T. Kohler-Jobin. “Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique I. Première partie: une démonstration de la conjecture isopérimétrique $P\lambda^2 \geq \pi j_0^{4/2}$ de Pólya et Szegó”. In: *Zeitschrift fúr Angewandte Mathematik und Physik ZAMP* 29 (1978), pp. 757–766.

[90] Y. Komura. “Nonlinear semi-groups in Hilbert space”. In: *Journal of the Mathematical Society of Japan* 19.4 (1967), pp. 493–507.

[91] P. Koosis. *Introduction to Hp spaces*. 115. Cambridge University Press, 1998.

[92] T. Kulczycki. “Properties of Green function of symmetric stable processes”. In: *Probability and Mathematical Statistics* 17.2 (1997), pp. 339–364.

[93] N. Kuznetsov, T. Kulczycki, M. Kwaśnicki, A. Nazarov, S. Poborchi, I. Polterovich, and B. Siudeja. “The legacy of Vladimir Andreevich Steklov”. In: *Notices of the AMS* 61.1 (2014), p. 190.

[94] M. Kwaśnicki. “Ten equivalent definitions of the fractional Laplace operator”. In: *Fractional Calculus and Applied Analysis* 20.1 (2017), pp. 7–51.

[95] Y. C. Kwong. “Boundary behavior of the fast diffusion equation”. In: *Transactions of the American Mathematical Society* 322.1 (1990), pp. 263–283.

[96] Y. C. Kwong. “Interior and boundary regularity of solutions to a plasma type equation”. In: *Proceedings of the American Mathematical Society* 104.2 (1988), pp. 472–478.

[97] N. N. Lebedev. *Special functions and their applications*. Ed. by R. A. Silverman. Dover Publications, Inc., 1972.

[98] E. K. Lenzi, R. S. Mendes, and C. Tsallis. “Crossover in diffusion equation: Anomalous and normal behaviors”. In: *Physical Review E* 67.3 (2003), p. 031104.

[99] R. T. Lewis, J. Li, and Y. Li. “A geometric characterization of a sharp Hardy inequality”. In: *Journal of Functional Analysis* 262.7 (2012), pp. 3159–3185.

[100] T. Li and Z.-A. Wang. “Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis”. In: *SIAM Journal on Applied Mathematics* 70.5 (2010), pp. 1522–1541.

[101] E. H. Lieb. “Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities”. In: *Inequalities: Selecta of Elliott H. Lieb*. Springer, 1983, pp. 529–554.

[102] A. I. Lurie. *Theory of elasticity*. Springer Science & Business Media, 2010.

[103] E. Makai. “A proof of Saint-Venant’s theorem on torsional rigidity”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 17 (1966), pp. 419–422.

[104] E. Makai. “Bounds for the principal frequency of a membrane and the torsional rigidity of a beam”. In: *Acta Sci. Math. Szeged* 20 (1959), pp. 33–35.

[105] E. Makai. “On the principal frequency of a convex membrane and related problems”. In: *Czechoslovak Mathematical Journal* 9.1 (1959), pp. 66–70.

[106] Q. H. Nguyen and J. L. Vázquez. "Porous medium equation with nonlocal pressure in a bounded domain". In: *Communications in Partial Differential Equations* 43.10 (2018), pp. 1502–1539.

[107] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.1.12 of 2023-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: `https://dlmf.nist.gov/`.

[108] H. G. Othmer and T. Hillen. "The diffusion limit of transport equations derived from velocity-jump processes". In: *SIAM Journal on Applied Mathematics* 61.3 (2000), pp. 751–775.

[109] A de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. "A general fractional porous medium equation". In: *Communications on Pure and Applied Mathematics* 65.9 (2012), pp. 1242–1284.

[110] A. de Pablo, F. Quirós, and A. Rodríguez. "Nonlocal filtration equations with rough kernels". In: *Nonlinear Analysis* 137 (2016), pp. 402–425.

[111] A. de Pablo, F. Quirós, and A. Rodríguez. "Regularity theory for singular nonlocal diffusion equations". In: *Calculus of Variations and Partial Differential Equations* 57.5 (2018), p. 136.

[112] A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. "A fractional porous medium equation". In: *Advances in Mathematics* 226.2 (2011), pp. 1378–1409.

[113] A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. "Classical solutions and higher regularity for nonlinear fractional diffusion equations". In: *Journal of the European Mathematical Society* 19.7 (2017), pp. 1949–1975.

[114] C. S. Patlak. "Random walk with persistence and external bias". In: *The Bulletin of Mathematical Biophysics* 15 (1953), pp. 311–338.

[115] L. E. Payne. "New isoperimetric inequalities for eigenvalues and other physical quantities". In: *Communications on Pure and Applied Mathematics* 9.3 (1956), pp. 531–542.

[116] L. E. Payne. "Some isoperimetric inequalities in the torsion problem for multiply connected regions". In: *Studies in Math. Analysis and Related Topics* (1962), pp. 270–280.

[117] B. Perthame. *Transport equations in biology*. Springer Science & Business Media, 2006.

[118] G. Polya and A. Weinstein. "On the torsional rigidity of multiply connected cross-sections". In: *Annals of Mathematics* (1950), pp. 154–163.

[119] G. Pólya. "On the characteristic frequencies of a symmetric membrane". In: *Mathematische Zeitschrift* 63.1 (1955), pp. 331–337.

[120] G. Pólya. "Torsional rigidity, principal frequency, electrostatic capacity and symmetrization". In: *Quarterly of Applied Mathematics* 6.3 (1948), pp. 267–277.

[121] G. Pólya. "Two more inequalities between physical and geometrical quantities". In: *J. Indian Math. Soc.(NS)* 24.3-4 (1960), pp. 413–419.

[122] G. Pólya and G. Szegö. *Isoperimetric inequalities in mathematical physics*. 27. Princeton University Press, 1951.

[123] S. C. Port and C. J. Stone. *Brownian motion and classical potential theory*. Probability and Mathematical Statistics. Academic Press [Harcourt Brace Jovanovich, Publishers], 1978.

[124] X. Ros-Oton. "Nonlocal elliptic equations in bounded domains: a survey". In: *Publicacions matematiques* (2016), pp. 3–26.

[125] X. Ros-Oton and J. Serra. "The Dirichlet problem for the fractional Laplacian: regularity up to the boundary". In: *Journal de Mathématiques Pures et Appliquées* 101.3 (2014), pp. 275–302.

[126] M. H. Sadd. *Elasticity: theory, applications, and numerics*. Academic Press, 2009.

[127] J. Saragosti, V. Calvez, N. Bournaveas, A. Buguin, P. Silberzan, and B. Perthame. "Mathematical description of bacterial traveling pulses". In: *PLoS computational biology* 6.8 (2010), e1000890.

[128] J. Simon. "Compact sets in the space $L^p(O,T;B)$". In: *Annali di Matematica Pura ed Applicata* 146 (1986), pp. 65–96.

[129] D. Stan, F. del Teso, and J. L. Vázquez. "Transformations of self-similar solutions for porous medium equations of fractional type". In: *Nonlinear Analysis: Theory, Methods & Applications* 119 (2015), pp. 62–73.

[130] D. Stan and J. L. Vázquez. "The Fisher-KPP equation with nonlinear fractional diffusion". In: *SIAM Journal on Mathematical Analysis* 46.5 (2014), pp. 3241–3276.

[131] Daniel W Stroock. *An introduction to the theory of large deviations*. Springer Science & Business Media, 2012.

[132] G. Szegö. "Inequalities for certain eigenvalues of a membrane of given area". In: *Journal of Rational Mechanics and Analysis* 3 (1954), pp. 343–356.

[133] F. del Teso, J. Endal, and E. R. Jakobsen. "Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: Schemes and experiments". In: *SIAM Journal on Numerical Analysis* 56.6 (2018), pp. 3611–3647.

[134] F. del Teso, J. Endal, and E. R. Jakobsen. "Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type". In: *Advances in Mathematics* 305 (2017), pp. 78–143.

[135] S. Timoschenko. "A membrane analogy to flexure". In: *Proceedings of the London Mathematical Society* 2.1 (1922), pp. 398–407.

[136] N.T. Varopoulos. "Hardy-Littlewood theory for semigroups". In: *Journal of Functional Analysis* 63.2 (1985), pp. 240–260.

[137] J. L. Vázquez. “Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type”. In: *Journal of the European Mathematical Society* 16.4 (2014), pp. 769–803.

[138] J. L. Vázquez. “Nonlinear diffusion with fractional Laplacian operators”. In: (2012), pp. 271–298.

[139] J. L. Vázquez. “Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators”. In: *Discrete and Contiunous Dynamical Systems* 7.4 (2014).

[140] J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type*. Vol. 33. OUP Oxford, 2006.

[141] J. L. Vázquez. *The porous medium equation: mathematical theory*. Oxford University Press, 2007.

[142] J. L. Vázquez and B. Volzone. “Optimal estimates for fractional fast diffusion equations”. In: *Journal de Mathématiques Pures et Appliquées* 103.2 (2015), pp. 535–556.

[143] S. E. Warschawski. “On differentiability at the boundary in conformal mapping”. In: *Proceedings of the American Mathematical Society* 12.4 (1961), pp. 614–620.

[144] R. Weinstock. “Inequalities for a classical eigenvalue problem”. In: *Journal of Rational Mechanics and Analysis* 3 (1954), pp. 745–753.

[145] E. Zeidler. *Nonlinear functional analysis and its applications: II/A: linear monotone operators*. Springer Science & Business Media, 2013.

[146] E. Zeidler. *Nonlinear functional analysis and its applications: II/B: nonlinear monotone operators*. Springer Science & Business Media, 2013.

[147] Q. S. Zhang. “The global behavior of heat kernels in exterior domains”. In: *Journal of Functional Analysis* 200.1 (2003), pp. 160–176.