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TESIS DOCTORAL

Reduction by fiberwise actions andhigher-order reduction in Field Theory

Reducción por acción fibrada y de orden superior en Teoría de Campos

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Madrid

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Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



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Abstract

The aim of this thesis is to develop two new types of reduction by symmetries for covariant Lagrangian field theories; namely, reduction by fiberwise actions, also known as local gauge reduction or, simply, gauge reduction, and higher-order Euler–Poincaré reduction. The technique utilized in both cases consists of studying the geometry of the reduced configuration bundle, i.e., the quotient of the original configuration bundle by the corresponding action, and next transferring the principle of stationary action or Hamilton's principle to that reduced bundle. The reduced variations are not free any more, but they are restricted, and when applied to the reduced Lagrangian they yield the reduced field equations. In addition, the reconstruction problem, which consists of finding the compatibility condition that guarantee that the solutions of the reduced equations come from solutions of the original equations, and the relation of the reduced equations (or some of them) with the Noether conservation law are addressed.

In the gauge reduction case, a Lie group fiber bundle acts fiberwisely in the configuration bundle of the theory and the Lagrangian is invariant by an affine Lie group subbundle of the first jet lift of this action. The geometry of the quotient is studied by means of an Ehresmann connection that is equivariant for the fibered action. For this reason, we include a chapter devoted to the study of the properties and the curvature of this type of connections, which generalize the notions of principal connection and affine connection. For higher-order Euler–Poincaré reduction, the Lagrangian is defined in the higher-order jet bundle of a principal bundle and it is invariant by the action of the structure group of the principal bundle. The geometry of the reduced bundle is studied canonically, i.e., without using a connection.

Lastly, the theoretical results are illustrated through several examples. More specifically, gauge reduction is applied to the classical case of global symmetries (Lagrange–Poincaré reduction), to the invariance by the full jet, to Electromagnetism (including Electromagnetism with higher-order forms as well as symmetry breaking) and to non-Abelian gauge theories (in particular, to the Yang–Mills theory, where we obtain the well-known Utiyama theorem). For this last instance, the Lie groupoid structure of the 2-jet bundle of functions from a smooth manifold to a Lie group has to be trivialized with the aid of a linear connection. Hence, a chapter investigating this jet bundles is included. To conclude, higher-order Euler–Poincaré reduction is applied to higher-order multivariate splines on Riemannian Lie groups. This type of splines is used to interpolate data on Lie groups while minimizing higher-order derivatives, such as the acceleration.

Resumen

El objetivo de esta tesis es desarrollar dos nuevos tipos de reducción por simetrías para teorías de campos lagrangianas covariantes, a saber, reducción por acción fibrada, también conocida como reducción *gauge* local o, simplemente, reducción *gauge*, y reducción de Euler–Poincaré de orden superior. En ambos casos, la técnica empleada consiste en estudiar la geometría del fibrado de configuración reducido, esto es, el cociente del fibrado de configuración original por la acción correspondiente, y a continuación transferir el principio de acción estacionaria o principio de Hamilton a dicho fibrado reducido. Las variaciones reducidas ya no son libres, sino que están restringidas, y al aplicarlas a la lagrangiana reducida se obtienen las ecuaciones de campo reducidas. Además de lo anterior, se abordan el problema de reconstrucción, dando las condiciones de compatibilidad que garantizan que las soluciones de las ecuaciones reducidas proceden de soluciones de las ecuaciones originales, y la relación de las ecuaciones reducidas (o parte de ellas) con las leyes de conservación dadas por el teorema de Noether.

En el caso de reducción *gauge*, un fibrado de grupos de Lie actúa fibra a fibra en el fibrado de configuración de la teoría y la lagrangiana es invariante por un subfibrado de grupos de Lie afín de la prolongación al primer jet de esta acción. La geometría del cociente se estudia mediante una conexión de Ehresmann equivariante por la acción fibrada, por lo que se incluye un capítulo estudiando las propiedades de este tipo de conexiones y su curvatura, las cuales generalizan las nociones de conexión principal y de conexión afín. En el caso de reducción de Euler–Poincaré de orden superior, la lagrangiana está definida en el fibrado de jets de orden superior de un fibrado principal y es invariante por la acción del grupo de estructura del fibrado principal. La geometría del fibrado reducido se estudia de forma canónica, es decir, sin necesidad de utilizar una conexión.

Por último, los resultados teóricos se ilustran mediante diferentes ejemplos. Específicamente, la reducción *gauge* se aplica al caso clásico de simetrías globales (reducción de Lagrange–Poincaré), a la invarianza por el jet completo, al electromagnetismo (incluyendo electromagnetismo con formas de orden superior y ruptura espontánea de simetría) y a teorías *gauge* no-abelianas (en particular, a la teoría de Yang–Mills, obteniendo el conocido teorema de Utiyama). Para este último ejemplo es preciso trivializar la estructura de grupoide de Lie del fibrado de 2-jets de funciones de una variedad diferenciable a un grupo de Lie mediante una conexión lineal, por lo que se incluye un capítulo estudiando estos fibrados. Finalmente, la reducción de Euler–Poincaré de orden superior se aplica a splines de orden superior multivariables definidos en grupos de Lie riemannianos. Este tipo de splines son útiles para interpolar datos en grupos de Lie minimizando derivadas de orden superior, como la aceleración.

Introduction

Reduction in geometric Mechanics exploits the symmetries of many physical systems to formulate the dynamical equations in lower dimensional spaces and obtain conserved quantities. In the mathematical model, symmetry means the invariance of the functions describing the system (either the Lagrangian or the Hamiltonian, the constraints, the external forces, etc.) by the action of a Lie group. This invariance allows for transferring the variational and geometric structures to a lower dimensional space: the space of orbits of the group action, leading to the reduced equations of the system. The modern formulation of reduction by symmetries utilizes differential geometric tools and was introduced in the seminal works of Arnold [2], Smale [88], Meyer [75], and Marsden and Weinstein [73], although these ideas were present in earlier works of Lagrange, Poisson, Jacobi, and Noether. Thereafter, this method has been investigated in many other situations including Lagrangian reduction [22, 23, 71, 72], where the reduced equations come from a reduced variational principle.

This philosophy has been extended successfully to the realm of Classical Field Theory [15, 43] and, in particular, to covariant field theories [3, 11, 12, 13, 14, 15, 16, 17, 32, 43], among others. Here, the configuration space is a fiber bundle, the objects under study are sections of this bundle (the fields) and the phase space is the corresponding jet bundle. The base manifold can be the spacetime or, for non-dynamical problems, any manifold. One of the key differences between reduction in Mechanics and Field Theory is that, in the former, the solutions of the reduced equations always come from solutions of the original equations (at least locally), whereas this is not true for the latter. In order for a solution of the reduced field equations to come from a solution of the original equations an additional requirement, known as the *compatibility condition*, has to be satisfied. This is due to the fact that every Ehresmann connection on a bundle over a 1-dimensional base space is (locally) integrable, but this does not hold for higher dimensional base manifolds. Therefore, the compatibility condition is given by the vanishing of the curvature of a certain connection. The problem of finding the appropriate connection in order to ensure that solutions of the original equations can be recovered, which is somehow the inverse problem to reduction, is known as the *reconstruction problem*. For instance, for covariant Euler–Poincaré reduction, the reduced bundle is nothing but the bundle of connections of a principal bundle. Therefore, the reduced sections are connections themselves and the compatibility condition establishes that their own curvature must vanish. For other types of reduction the compatibility condition is not that immediate, but it still depends on some connection build from the reduced section.

The first attempts to carry out reduction by symmetries in the realm of Lagrangian field theories were given in [12, 17], and dealt with theories with a principal bundle as configuration bundle of theory that are invariant by the structure group of that bundle. The corresponding reduced equations are known as the *covariant Euler–Poincaré equations*. This was later generalized to other situations, leading to the *covariant Lagrange–Poincaré equations* [13, 16, 32], where the group of symmetries is given by a Lie group acting vertically on the total space of the configuration bundle of the theory. This type of symmetries are known as *global symmetries*. Nevertheless, there is a

wide variety of field theories where symmetries are not global but *local*, that is, the action of the symmetry group depends on the point of the base manifold. This is also known as *local gauge symmetry* (or simply gauge symmetry) and its geometric model is written in terms of a Lie group fiber bundle acting fiberwisely on the configuration bundle. The main instances of this framework are pure gauge theories such as Electromagnetism or Yang–Mills, where gauge transformations are sections of the cotangent bundle of the base manifold or the jet space of an adjoint bundle, respectively [37, 66]. Interestingly, these gauge theories possess a particular kind of reduction result: the Utiyama's theorem [39, 89], also generalized to the setting of interaction [4]. This theorem proves that gauge invariant Lagrangian densities only depend on the curvature (force), instead of depending on the principal connection itself (potential). Nevertheless, it is frustrating to see that the standard field theory reduction scheme (the aforementioned Lagrange–Poincaré reduction) stemmed from geometric Mechanics does not cover gauge symmetries, a fact that has prevented the gauge case from taking advantage of the rich geometric interpretations and constructions that the powerful reduction program has provided so far (for example, geometric integrators just to mention one of them). On the other hand, Lagrangian reduction in Mechanics (and, in particular, Euler–Poincaré reduction) has been extended to higher-order systems, i.e., those systems whose Lagrangian density depends on higher-order derivatives, such as the acceleration (see for example [40, 41, 42]). However, there is not a counterpart in field theory yet, not even for the Euler–Poincaré case, despite the fact that there are interesting systems and situations that fit in this context.

The purpose of the present thesis is to address the aforementioned situations by providing two new reduction schemes for covariant field theories: reduction by (local) gauge symmetries, and higher-order Euler–Poincaré reduction. More specifically, the mathematical model for these two types of reduction is the following:

(i) **Gauge reduction** or **reduction by fiberwise actions**. The configuration bundle of the system is a (smooth) fiber bundle $Y \to X$, and a Lie group fiber bundle $\mathcal{G} \to X$ acts fiberwisely on it. Unlike the Lagrange–Poincaré case, the Lie group bundle does not act on the first jet bundle of the configuration bundle, $J^1Y \to X$, but its jet bundle has to be considered, $J^1\mathcal{G} \to X$. In addition, we consider the fibered action restricted to an affine Lie group subbundle, $H \subset J^1\mathcal{G}$, since most physical systems are not invariant by the full jet bundle. The reduced configuration bundle is thus the 'fibered' quotient $\left(J^1Y\right)/H \to Y/\mathcal{G}$. This reduction theory is presented in Chapter 4 and some applications are discussed in Chapter 5.

(ii) **Higher-order Euler–Poincaré reduction**. The configuration bundle is given by a principal G-bundle, $P \to X$, and the Lagrangian density depends on higher-order derivatives, i.e., it is defined on J^kP, the k-th order jet bundle of the configuration bundle. Unlike the gauge reduction case, here we only consider the global action of G on J^kP. The reduced configuration bundle is thus $\left(J^kP\right)/G \to X$. This type of reduction is developed in Chapter 6 and it is also applied to multivariate splines on Lie groups.

Although there exist different approaches to reduction, such as dynamical reduction [15, 43] or Hamiltonian reduction [3, 14], we confine ourselves to the covariant Lagrangian formulation and, thus, the reduction technique involves the transfer of the variational principle to the reduced spaces. More specifically, the steps performed are the following:

(i) **Identification of the reduced bundle**. The first step after posing the problem is to study the geometry of the reduced configuration bundle, i.e., the quotient of the bundle where the Lagrangian density is defined by the group of symmetries. For reduction by fiberwise actions, the reduced bundle is $\left(J^1Y\right)/H \to Y/\mathcal{G}$ and, in order to study its geometry, we need to fix an Ehresmann connection on $Y \to Y/\mathcal{G}$ that is equivariant by the fibered action. Since this kind of connections were not studied in the previous literature, we introduce them and study their main properties in Chapter 2. We call them *generalized principal connections* as they extend the

notion of principal connection on a principal bundle. On the other hand, for higher-order Euler–Poincaré reduction the reduced bundle is $\left(J^k P\right)/G \to X$, which turns out to be isomorphic to the subbundle of $J^{k-1}C(P)$, where $C(P) = \left(J^1 P\right)/G$ is the bundle of connections, given by (pointwise) connections with vanishing curvature (up to certain order). This identification is canonical, i.e., it does not involve any connections.

(ii) **Reduction of the variational principle**. Once the reduced bundle is properly identified, the original variational principle —the principle of stationary action, also known as the Hamilton's principle—, has to be transferred to the reduced bundle by means of the corresponding isomorphism. Furthermore, thanks to the symmetry invariance, the *reduced Lagrangian*, also known as the *dropped Lagrangian*, is well-defined on the reduced space. By applying the reduced variational principle to the reduced Lagrangian, we arrive at the main result of the theory, the *reduced field equations*.

(iii) **Reconstruction problem**. As explained above, the solutions of the reduced field equations do not need to come from solutions of the original equations (not even locally). For this reason, it is convenient to find conditions that guarantee the equivalence between solutions of the original and the reduced equations. For reduction by fiberwise actions, this condition is given the vanishing curvature of some generalized principal connection built from the reduced solution. However, for higher-order Euler–Poincaré reduction, the reconstruction condition is surprisingly given a priori and, thus, every solution of the reduced equations on the reduced space is originated by a solution of the original equations.

(iv) **Noether's theorem**. It is usually possible to reinterpret some of the reduced equations as a Noether conservation law, meaning that these equations are satisfied if and only if the Noether current is preserved. We show that this is the case for the reduction problems considered in this work.

After performing the aforementioned reduction procedures, some illustrative examples are presented. In particular, for the last application of the gauge reduction theory, non-Abelian gauge theories, the Lie group bundle of symmetries is $\mathcal{G} = J^1\tilde{G}$, where $\tilde{G} = (P \times G)/G$ is the Lie group bundle associated to a given principal G-bundle, $P \to X$, via the conjugacy. The affine Lie group subbundle of symmetries is $H = J^2\tilde{G}$, which locally reads as $J^2(X, G)$, the bundle of 2-jets of functions from X to G. In order to obtain the reduced equations, the Lie group bundle structure of this bundle is needed. For this reason, we include Chapter 3, where the bundles $J^k(X, G) \to X$, $k \in \mathbb{Z}^+$, together with their Lie group bundle structure, are trivialized by using a linear connection on the tangent bundle of X. On the other hand, the higher-order Euler–Poincaré reduction procedure is applied to the analysis of multivariate k-splines on Lie groups, which are utilized to interpolate data while minimizing some cost functional.

The research carried out during the doctoral period has led to the following articles:

1. M. Castrillón López and Á. Rodríguez Abella, *Gauge reduction in covariant field theory*, 2023, https://arxiv.org/abs/2202.04578.
2. M. Castrillón López and Á. Rodríguez Abella, *Higher order jet bundles of Lie group-valued functions*, Journal of Lie Theory **33** (2023), no. 3, 831–844, https://www.heldermann.de/JLT/JLT33/JLT333/jlt33037.htm.
3. M. Castrillón López and Á. Rodríguez Abella, *Higher-order Euler–Poincaré field equations*, 2023, https://arxiv.org/abs/2311.18462.
4. M. Castrillón López and Á. Rodríguez Abella, *Principal bundles and connections modelled by Lie group bundles*, Geometriae Dedicata **217** (2023), no. 32, 1-26, https://link.springer.com/article/10.1007/s10711-023-00766-2.

5. Á. Rodríguez Abella, *Interconnection of Lagrange–Dirac systems through nonstandard interaction structures*, International Journal of Geometric Methods in Modern Physics **20** (2023), no. 07, 2350124, https://www.worldscientific.com/doi/10.1142/S0219887823501244.
6. Á. Rodríguez Abella and L.J. Colombo, *Nonholonomic reduction for mechanical systems with collisions*, 2023, https://arxiv.org/abs/2311.07707.
7. Á. Rodríguez Abella and M. Leok, *Discrete Dirac reduction of implicit Lagrangian systems with abelian symmetry groups*, Journal of Geometric Mechanics **15** (2023), no. 1, 319–356, https://www.aimspress.com/article/doi/10.3934/jgm.2023013.

The contents of this dissertation are organized as follows:

§ Chapter 1 is devoted to recall the geometric formulation of Lagrangian field theories as well as the classical Lagrange–Poincaré reduction scheme developed in [32] for systems invariant by the action of a Lie group bundle. We recover the Euler–Poincaré field equations as a particular case when the configuration bundle of the theory is a principal bundle and the group of symmetries is the corresponding structure group.

§ In Chapter 2 we investigate the geometry of principal bundles when the structure group is substituted by a Lie group fiber bundle acting fiberwisely. This is the situation that appears naturally when performing gauge reduction. Moreover, we study some properties of the Ehresmann connections that are equivariant by the fiberwise action and, in particular, we conclude that they always exist and are associated to a connection on the corresponding Lie group bundle. We also present several examples including standard principal connections and affine connections.

§ Chapter 3 is devoted to study the bundle of k-th order jets of functions from a smooth manifold to a Lie group. More specifically, the Lie groupoid structure of this bundle is trivialized by means of a linear connection. The geometric results presented here are utilized to apply the gauge reduction theory to non-Abelian gauge theories such as Yang–Mills theory.

Chapters 4, 5 and 6 constitute the core of this work, as they present the aforementioned reduction results: gauge reduction and higher-order covariant Euler–Poincaré reduction.

§ In Chapter 4 we start by studying the geometry of the reduced configuration bundle with the aid of a generalized principal connection. After that, we transfer the variational principle to the reduced bundle and apply it to the reduced Lagrangian, thus obtaining the reduced field equations. We conclude the chapter by addressing the reconstruction problem and the relation of the reduced equations with the Noether theorem.

§ Chapter 5 presents some applications of the gauge reduction theory; namely, classical Lagrange–Poincaré reduction, invariance by the full jet, Electromagnetism (including k-form Electromagnetism and symmetry breaking by product groups) and non-Abelian gauge theories (recovering the well known Utiyama theorem and the Yang–Mills equations as a particular case).

§ Chapter 6 starts by extending the Lagrangian formulation of field theories to the higher-order setting, i.e., for the case where the Lagrangian density depends on higher-order derivatives of the fields. After that, the geometry of the reduced configuration bundle is studied. Unlike the case of gauge symmetries, here we do not require a connection to define the identification, i.e., it is canonical. This identification allows for transferring the higher-order Hamilton's principle to the reduced configuration bundle and, thus, to compute the higher-order Euler–Poincaré field equations after applying it to the reduced Lagrangian. As for gauge reduction, we address the reconstruction problem, which turns out to be trivial, i.e., all sections of the reduced bundle satisfy the compatibility condition, and we relate the reduced equations with the Noether theorem. At last, the higher-order Euler–Poincaré reduction theory is applied to multivariate splines on Riemannian Lie groups.

Notations and conventions

Every manifold or map is assumed to be smooth, meaning C^∞, unless otherwise stated. Let X and Y be smooth manifolds with $\dim X = n$ and $\dim Y = n+m$, and $\pi_{Y,X} : Y \to X$ be a (locally trivial) fiber bundle. The fiber over each $x \in X$ is denoted by $Y_x = \pi_{Y,X}^{-1}(\{x\})$ and the family of (global) sections of $\pi_{Y,X} : Y \to X$ is denoted either by $\Gamma(\pi_{Y,X})$ or by $\Gamma(Y)$ when there is no possible confusion about the base space. In particular, the family of vector fields on X is denoted by $\mathfrak{X}(X) = \Gamma(TX)$, where TX is the tangent bundle of X, and the family of k-forms on X is denoted by $\Omega^k(X) = \Gamma\left(\bigwedge^k T^*X\right)$, where T^*X is the cotangent bundle of X. Likewise, the space of local sections on an open set $\mathcal{U} \subset X$ is denoted by $\Gamma(\mathcal{U}, \pi_{Y,X})$. Given a vector field $U \in \mathfrak{X}(X)$, the left interior multiplication by U is denoted by $\iota_U : \Omega^k(X) \to \Omega^{k-1}(X)$. The tangent map of $f \in C^\infty(X, X')$, where X' is another manifold, is denoted either by $df : TX \to TX'$ or $f_* : TX \to TX'$. In the same vein, the pull-back of $\alpha \in \Omega^k(X')$ is denoted by $f^*\alpha \in \Omega^k(X)$ and its exterior derivative is denoted by $\mathrm{d}\alpha \in \Omega^{k+1}(X')$. A compact interval will be denoted by $I = [a, b] \subset \mathbb{R}$. When working in local coordinates, the Einstein summation convention for repeated indices and multi-indices will be assumed. In addition, indices will be denoted by lowercase letters ($\mu \in \mathbb{N}$, etc.), and multi-indices by capital letters ($J \in \mathbb{N}^n$, etc.). Given two multi-indices $J = (J_1, \dots, J_n), I = (I_1, \dots, I_n) \in \mathbb{N}^n$, we write

$$J! = J_1! \dots J_n!, \qquad \binom{J}{I} = \binom{J_1}{I_1} \dots \binom{J_n}{I_n}.$$

Given $k \in \mathbb{Z}^+$, the k-th order jet bundle of $\pi_{Y,X}$ (cf., for instance, [87]) is denoted by $\pi_{J^kY,X} : J^kY \to X$ and its elements by $j^k_x s$. Similarly, its fibers are denoted by $J^k_x Y$, for each $x \in X$. The k-th jet lift of a section $s \in \Gamma(\pi_{Y,X})$ is denoted by $j^k s \in \Gamma\left(\pi_{J^kY,X}\right)$ and it is called holonomic section. Recall that, for $0 \le l < k$, the maps $\pi_{k,l} : J^kY \to J^lY$, $\pi_{k,l}(j^k_x s) = j^l_x s$, are fiber bundles, where we denote $J^0Y = Y$. In particular, the first jet bundle of $\pi_{Y,X}$ may be regarded as

$$J^1Y = \{\gamma_y : T_xX \to T_yY \mid \gamma_y \text{ linear},\ x = \pi_{Y,X}(y),\ (d\pi_{Y,X})_y \circ \gamma_y = \mathrm{id}_{T_xX}\},$$

where id_{T_xX} denotes the identity map on T_xX. Subsequently, 1-jets may be regarded as tangent maps, $j^1_x s \equiv (ds)_x : T_xX \to T_{s(x)}Y$. Recall that $\pi_{J^1Y,Y}$ is an affine bundle modelled on $\pi_{T^*X \otimes VY,Y}$, where $VY = \ker(\pi_{Y,X})_*$ is the so-called vertical bundle. Given a manifold F, the k-th order jet bundle of the trivial bundle, $\pi_{X\times F,X} : X \times F \to X$, is known as the k-th order jet of functions from X to F. Since any section $s \in \Gamma(\pi_{X\times F,X})$ is of the form $s = (\mathrm{id}_X, f)$ for some function $f : X \to F$, elements of $J^k(X, F) = J^k(X \times F)$ are denoted by $j^k_x f$.

An Ehresmann connection on $\pi_{Y,X} : Y \to X$ (see, for example, [76, Ch.IV]) is a fiber map $\mathfrak{I} : TY \to VY$ such that $\mathfrak{I}|_{VY} = \mathrm{id}_{VY}$. Similarly, we can understand an Ehresmann connection as a horizontal distribution, i.e., a vector subbundle $HY \subset TY$ complementary to VY. Finally, an Ehresmann connection can also be regarded as a section of the first jet bundle, $\nu \in \Gamma(\pi_{J^1Y,Y})$. In addition, a linear connection on a vector bundle $\pi_{V,X} : V \to X$ is denoted by $\nabla : \Gamma(\pi_{V,X}) \to \Omega^1(X, V) = \Gamma(\pi_{T^*X\otimes V,X})$, and the corresponding covariant derivative is denoted by ∇/dt.

Given a (connected) Lie group, G, its Lie algebra is denoted by $\mathfrak{g} = T_eG$. The adjoint representations of G and $\mathfrak{g}$ are denoted by $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$ and $\mathrm{ad}(\xi) = \mathrm{ad}_\xi \in \mathrm{End}(\mathfrak{g})$, respectively, for each $g \in G$ and $\xi \in \mathfrak{g}$. The dual of $\mathfrak{g}$ will be denoted by $\mathfrak{g}^*$ and the coadjoint representation will be denoted by $\mathrm{ad}^*(\xi) = \mathrm{ad}^*_\xi \in \mathrm{End}(\mathfrak{g}^*)$. Let P be a smooth manifold and $\Psi : P \times G \to P$ be a free and proper right action of G on P. Then the corresponding quotient projection, $\pi_{P,X} : P \to X$ is a principal G-bundle (cf., for example, [55, Ch.II] or [56, Ch.III]), where we denote $X = P/G$. Recall that the vertical bundle of a principal bundle is trivial, $VP \simeq P \times \mathfrak{g}$, where $\mathfrak{g} = T_eG$ is the Lie algebra of G. Hence, an Ehresmann connection on $\pi_{P,X}$ may be regarded as a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that $A(\xi^*) = \xi$ for each $\xi \in \mathfrak{g}$, where $\xi^* \in \mathfrak{X}(P)$ is the fundamental field or infinitesimal generator of the action. Moreover, A is said to be a principal connection if it is G-equivariant, i.e.,

$(\Psi_g)^* A = \mathrm{Ad}_{g^{-1}} \circ A$ for each $g \in G$, where $\Psi_g(y) = \Psi(y, g)$ for each $y \in P$. The curvature of the principal connection, $F^A \in \Omega^2(P, \mathfrak{g})$, is a basic form of the adjoint type, thus inducing the reduced curvature, $\tilde{F}^A \in \Omega^2(X, \tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}} = (P \times \mathfrak{g})/G$ is the adjoint bundle of $\pi_{P,X}$. Lastly, recall that $\pi_{\tilde{\mathfrak{g}},X}$ is the Lie algebra bundle of the Lie group bundle $\pi_{\tilde{G},X} : \tilde{G} \to X$, where $\tilde{G} = (X \times G)/G$ with the action on G on itself given by conjugacy (this will be further developed in Chapter 2). The coadjoint bundle of $\pi_{P,X}$ is dual of the adjoint bundle and it will be denoted by $\pi_{\tilde{\mathfrak{g}}^*,X} : \tilde{\mathfrak{g}}^* \to X$, where $\tilde{\mathfrak{g}} = (P \times \mathfrak{g}^*)/G$. The (fiberwise) adjoint and coadjoint representations are denoted by the same symbols for brevity, i.e., $\mathrm{Ad}_{\tilde{g}} \in \mathrm{Aut}(\tilde{\mathfrak{g}}_x)$, $\mathrm{ad}(\tilde{\xi}) = \mathrm{ad}_{\tilde{\xi}} \in \mathrm{End}(\tilde{\mathfrak{g}}_x)$ and $\mathrm{ad}^*(\tilde{\xi}) = \mathrm{ad}^*_{\tilde{\xi}} \in \mathrm{End}(\tilde{\mathfrak{g}}_x^*)$ for each $\tilde{g} \in \tilde{G}_x$, $\tilde{\xi} \in \tilde{\mathfrak{g}}_x$ and $x \in X$.

Chapter 1

Preliminaries on covariant Lagrange–Poincaré reduction

This thesis is framed in the context of the geometric formulation of Lagrangian field theories, which relies on a comprehensive use of differential geometric tools and calculus of variations to obtain the dynamical equations for the systems under study. As explained in the Introduction, our main goal is to generalize the Lagrange–Poincaré reduction scheme to systems satisfying a gauge invariance, i.e., those whose symmetries are given by a fiberwise action (see Chapter 4). A particular case of the Lagrange–Poincaré equations are the Euler–Poincaré equations, which are obtained when the configuration bundle of the theory is a principal bundle. Our second main goal is to generalize the Euler–Poincaré reduction to account for higher-order field theories, i.e., those theories whose Lagrangian densities depend on higher-order derivatives of the fields (see Chapter 6). For this reason, in this introductory chapter we briefly recall the geometric construction of Lagrangian field theories (see [47, 48, 66] for a detailed exposition of these contents), as well as the Lagrange–Poincaré reduction procedure for such theories (cf. [16, 32]).

1.1 Geometric formulation of Lagrangian field theories

A *(first order) Lagrangian density* on a fiber bundle $\pi_{Y,X} : Y \to X$ is a bundle morphism $\mathfrak{L} : J^1Y \to \bigwedge^n T^*X$ covering the identity on X, where $n = \dim X$. Assuming that X is orientable and $v \in \Omega^n(X)$ is a volume form, we can write $\mathfrak{L} = Lv$ for certain $L : J^1Y \to \mathbb{R}$ called *Lagrangian function* or simply *Lagrangian*. Henceforth, we suppose that X is compact for simplicity. The *action functional* defined by $\mathfrak{L}$ is the map

$$\mathbb{S} : \Gamma(\pi_{Y,X}) \to \mathbb{R}, \quad s \mapsto \int_X \mathfrak{L}\left(j^1 s\right).$$

A *variation* of a section $s \in \Gamma(\pi_{Y,X})$ is a smooth 1-parameter family $\{s_t \in \Gamma(\pi_{Y,X}) \mid t \in (-\epsilon, \epsilon)\}$ such that $s_0 = s$. The vector field $\delta s = ds_t/dt|_{t=0} \in \Gamma(\pi_{s^*TY,X})$ along s is called the *infinitesimal variation*. In what follows, we consider only $\pi_{Y,X}$-vertical variations, that is, $d\pi_{Y,X} \circ \delta s = 0$.

Definition 1.1 *A section $s \in \Gamma(\pi_{Y,X})$ is* critical for the variational problem defined by $\mathfrak{L}$ *if the variation of the corresponding action functional*[1] *vanishes for every vertical variation of s, that is,*

$$\delta\mathbb{S}(s) = \frac{d}{dt}\Big|_{t=0} \mathbb{S}(s_t) = \frac{d}{dt}\Big|_{t=0} \int_X \mathfrak{L}\left(j^1 s_t\right) = 0, \qquad \forall \delta s \in \Gamma(\pi_{s^*TY,X}).$$

In addition, the *Euler–Lagrange operator* associated to $\mathfrak{L}$ is the bundle morphism

$$\mathcal{EL}(\mathfrak{L}) : J^2Y \to V^*Y \otimes \bigwedge^n T^*X,$$

[1] The variation of $\mathbb{S}$ only depends on the infinitesimal variation. This means that if $\{s_t\}$ and $\{s'_t\}$ are two variations of s such that $\delta s = \delta s'$, then $d\mathbb{S}(s_t)/dt|_{t=0} = d\mathbb{S}(s'_t)/dt|_{t=0}$

over J^1Y, where we omit the necessary pull-backs for simplicity (see [16, §2.4] for details). If (x^μ, y^α) are bundle coordinates for $\pi_{Y,X}$, $v = d^n x$, and $(x^\mu, y^\alpha, y^\alpha_\mu, y^\alpha_{\mu\nu})$ are the induced bundle coordinates on $\pi_{J^2Y,X}$, then the Euler–Lagrange operator locally reads

$$\mathcal{EL}(\mathfrak{L}) = -\left(\frac{\partial L}{\partial y^\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^\alpha_\mu}\right)\right) dy^\alpha \otimes d^n x.$$

The following theorem states that, as in Classical Mechanics, a section is stationary if and only if it satisfies the Euler–Lagrange equations.

Theorem 1.1 ([48, Theorem 3.1]). *Let $\mathfrak{L}\colon J^1Y \to \bigwedge^n T^*X$ be a Lagrangian density and $s \in \Gamma(\pi_{Y,X})$. Then the following statements are equivalent:*

(i) The section s is critical for the variational problem defined by $\mathfrak{L}$.

(ii) The section s satisfies the Euler–Lagrange field equations, *$\mathcal{EL}(\mathfrak{L})\left(j^2 s\right) = 0$.*

(iii) In local coordinates such that $\mathfrak{L} = Lv$, $v = d^n x$, the section s satisfies the following equations:

$$\frac{\partial L}{\partial y^\alpha}(j^1 s) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^\alpha_\mu}(j^1 s)\right) = 0, \qquad 1 \le \alpha \le m.$$

Remark 1.1 (Noncompact base space) *The calculus of variations described above is straightforwardly extended to a non-compact base manifold X by considering compactly supported variations. In other words, given a section $s \in \Gamma(\pi_{Y,X})$, the only variations δs allowed are those for which $\{x \in X \mid \delta s(x) \neq 0\}$ is contained in some open subset $\mathcal{U} \subset X$ with compact closure. Observe that, in particular, δs vanishes on the boundary of $\mathcal{U}$.*

1.2 Covariant Lagrange–Poincaré reduction

Let $\mathfrak{L} = Lv : J^1Y \to \bigwedge^n T^*X$ be a Lagrangian density, as in the previous section. Let G be a Lie group and $\Psi : Y \times G \to Y$ be a free and proper (right) action that preserves the fibers, i.e.,

$$\pi_{Y,X} \circ \Psi_g = \pi_{Y,X}, \qquad g \in G, \tag{1.1}$$

where we denote by $\Psi_g : Y \to Y$ the map defined as $\Psi_g(y) = \Psi(y, g) = y \cdot g$ for each $y \in Y$. This action may be lifted to the (first) jet bundle of $\pi_{Y,X}$ as follows:

$$\Psi^{(1)} : J^1Y \times G \to J^1Y, \quad (j^1_x s, g) \mapsto j^1_x(\Psi_g \circ s) = j^1_x(s \cdot g). \tag{1.2}$$

Henceforth, we assume that our Lagrangian is *invariant* by the lifted action, i.e., $L \circ \Psi^{(1)}_g = L$. In this situation, G is said to be a *group of symmetries* of the system, and the *reduced* or *dropped Lagrangian* is well defined as

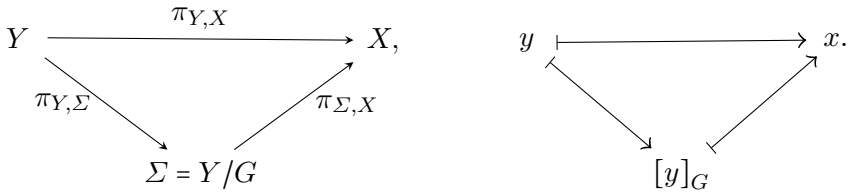
$$l : \left(J^1Y\right)/G \to \mathbb{R}, \quad [j^1_x s]_G \mapsto l\left([j^1_x s]_G\right) = L(j^1_x s),$$

where $[\cdot]_G$ denotes the equivalence class given by the G-action.

1.2.1 Reduced configuration space

As the G-action is free and proper, the quotient $\Sigma = Y/G$ is a smooth manifold. Moreover, the quotient projection $\pi_{Y,\Sigma} : Y \to \Sigma$ is a principal G-bundle, and the map $\pi_{\Sigma,X} : \Sigma \to X$ given by

$\pi_{\Sigma,X}([y]_G) = \pi_{Y,X}(y)$ for each $[y]_G \in \Sigma$ (which is well defined thanks to (1.1)) is a fiber bundle. In summary, we have the following commutative diagram:



In order to study the geometry of the *reduced configuration bundle*, i.e., of the quotient $\left(J^1Y\right)/G$, we fix a principal connection $A \in \Omega^1(Y,\mathfrak{g})$ on $\pi_{Y,\Sigma}$. Recall that it induces a linear connection ∇^A on the adjoint bundle $\pi_{\tilde{\mathfrak{g}},\Sigma}$ (cf. [32, Equation (2.6)]). We have the following isomorphism of affine bundles over the identity $\mathrm{id}_\Sigma : \Sigma \to \Sigma$:

$$\left(J^1Y\right)/G \to J^1\Sigma \times_\Sigma \left(T^*X \otimes \tilde{\mathfrak{g}}\right), \quad \left[j^1_x s\right]_G \mapsto \left(j^1_x\sigma_s, [s(x),(s^*A)_x]_G\right), \tag{1.3}$$

where $\times_X$ denotes the fibered product. Note that the elements of $T^*X \otimes \tilde{\mathfrak{g}}$ may be regarded as covectors taking values in the adjoint bundle. Therefore, the reduced Lagrangian may be regarded as defined on $J^1\Sigma \times_\Sigma (T^*X \otimes \tilde{\mathfrak{g}})$ and given a section $s \in \Gamma(\pi_{Y,X})$, the *reduced section* is $\overline{s} = [s, s^*A]_G \in \Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X})$. Observe that both the original and the reduced sections project to the same section of $\pi_{\Sigma,X}$; namely, $\sigma_s = \pi_{Y,\Sigma}\circ s = \pi_{T^*X\otimes\tilde{\mathfrak{g}},\Sigma}\circ\overline{s}$. The infinitesimal variation of the reduced section induced by an infinitesimal variation of the original section may be computed with the aid of the connections A and ∇^A (see Theorem 1.2 below).

1.2.2 Lagrange–Poincaré field equations

Let ∇^Σ be a linear connection on the tangent bundle of Σ projectable onto a linear connection ∇^X on the tangent bundle of X. Then the connections A and ∇^Σ induce an affine connection $\nabla^\times$ on the reduced bundle, $\pi_{J^1\Sigma\times_\Sigma(T^*X\otimes\tilde{\mathfrak{g}}),\Sigma}$, which enables us to compute the horizontal derivative of the reduced Lagrangian, $\delta l/\delta\sigma_s \in \Gamma(\pi_{T^*\Sigma,X})$:

$$\left\langle \frac{\delta l}{\delta\sigma_s}(x), U_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(\gamma_U(t)^h\right), \qquad U_x \in T_{\sigma_s(x)}\Sigma,$$

where $\gamma_U : (-\epsilon,\epsilon) \to \Sigma$ is such that $\gamma'(0) = U_x$ and $\gamma_U(t)^h \in J^1\Sigma\times_\Sigma(T^*X\otimes\tilde{\mathfrak{g}})$ is the horizontal lift of γ at $(j^1_x\sigma_s, \overline{s}(x))$ given by $\nabla^\times$. As usual, $\langle\cdot,\cdot\rangle$ denotes the corresponding dual pairing. Furthermore, the fiber derivatives of the reduced Lagrangian are the sections

$$\frac{\delta l}{\delta j^1\sigma_s} \in \Gamma(\pi_{TX\otimes V^*\Sigma,X}), \qquad \frac{\delta l}{\delta\overline{s}} \in \Gamma(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}),$$

are defined as

$$\left\langle \frac{\delta l}{\delta j^1\sigma_s}(x), V_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s + t\,V_x, \overline{s}(x)\right), \qquad V_x \in T^*_xX \otimes V_{\sigma_s(x)}\Sigma,$$
$$\left\langle \frac{\delta l}{\delta\overline{s}}(x), W_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s, \overline{s}(x) + t\,W_x\right), \qquad W_x \in T^*_xX \otimes \tilde{\mathfrak{g}}_{\sigma_s(x)}.$$

Lastly, in order to write the equations we define the following operators (cf. [32, Equation (3.7)]):

$$\overline{\nabla}^A : \mathfrak{X}(X) \times \Gamma(\pi_{\tilde{\mathfrak{g}},X}) \to \Gamma(\pi_{\tilde{\mathfrak{g}},X}), \qquad (U,\xi) \mapsto \left(\overline{\nabla}^A_U \xi\right)(x) = \left.\frac{\nabla^A(\xi\circ\gamma)(t)}{dt}\right|_{t=0}, \tag{1.4}$$

$$\overline{\nabla}^\Sigma : \mathfrak{X}(X) \times \Gamma(\pi_{T\Sigma,X}) \to \Gamma(\pi_{T\Sigma,X}), \qquad (U,\alpha) \mapsto \left(\overline{\nabla}^\Sigma_U \alpha\right)(x) = \left.\frac{\nabla^\Sigma(\alpha\circ\gamma)(t)}{dt}\right|_{t=0}, \tag{1.5}$$

for each $x \in X$, where $\gamma : (-\epsilon,\epsilon) \to X$ is such that $\gamma'(0) = U(x)$. The corresponding *divergences*, i.e., minus the adjoints of the previous operators, are denoted by

$$\operatorname{div}^A : \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right) \to \Gamma\left(\pi_{\tilde{\mathfrak{g}}^*,X}\right), \tag{1.6}$$

$$\operatorname{div}^\Sigma : \Gamma\left(\pi_{TX\otimes V^*\Sigma,X}\right) \to \Gamma\left(\pi_{V^*\Sigma,X}\right). \tag{1.7}$$

Note that for the latter we restrict ourselves to vertical sections, which is possible thanks to ∇^Σ being projectable. More explicitly, they are defined by the following conditions:

$$\int_X \left\langle \zeta, \overline{\nabla}^A \xi \right\rangle v = -\int_X \left\langle \operatorname{div}^A \zeta, \xi \right\rangle v, \qquad \int_X \left\langle \beta, \overline{\nabla}^\Sigma \alpha \right\rangle v = -\int_X \left\langle \operatorname{div}^\Sigma \beta, \alpha \right\rangle v,$$

for each $\zeta \in \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right)$, $\xi \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$, $\beta \in \Gamma(\pi_{TX\otimes V^*\Sigma,X})$ and $\alpha \in \Gamma(\pi_{V\Sigma,X})$.

Given a reduced section $\overline{s} \in \Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X})$, the coadjoint representation of $\mathfrak{g}^*$ induces a map:

$$\operatorname{ad}^*_{\overline{s}} : \Gamma(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}) \to \Gamma(\pi_{\tilde{\mathfrak{g}}^*,X}),$$

which is well-defined for every $\overline{\zeta} \in \Gamma(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X})$ such that $\pi_{TX\otimes\tilde{\mathfrak{g}}^*,\Sigma} \circ \overline{\zeta} = \pi_{T^*X\otimes\tilde{\mathfrak{g}},\Sigma} \circ \overline{s}$.

Theorem 1.2 ([32, Theorem 3.5]). *Let X be a oriented manifold with volume form $v \in \Omega^n(X)$, $\pi : Y \to X$ be a fiber bundle, G be a Lie group and $\Psi : Y \times G \to Y$ be a free and proper action satisfying (1.1). Let $\mathfrak{L} = Lv : J^1Y \to \bigwedge^n T^*X$ be a Lagrangian density invariant by the first jet lift of the G-action.*

Given a principal connection $A \in \Omega^1(Y,\mathfrak{g})$ on $\pi_{Y,\Sigma}$, where $\Sigma = Y/G$, the reduced Lagrangian is denoted by $l : \left(J^1Y\right)/G \simeq J^1\Sigma \times_\Sigma (T^*X \otimes \tilde{\mathfrak{g}}) \to \mathbb{R}$. Let ∇^Σ be a torsionless linear connection on the tangent bundle of Σ projectable onto a linear connection ∇^X on the tangent bundle of X, and consider the induced affine connection on the reduced bundle.

Let $s \in \Gamma(\pi_{Y,X})$, $\overline{s} = [s, s^*A]_G$ be the reduced section and $\sigma_s = \pi_{Y,\Sigma} \circ s$. Then the following statements are equivalent:

(i) The variational principle $\delta \int_X \mathfrak{L}(j^1 s) = 0$ holds for arbitrary vertical variations $\delta s \in \Gamma(\pi_{s^*TY,X})$.

(ii) The section s satisfies the Euler–Lagrange field equations for L, $\mathcal{EL}(\mathfrak{L})\left(j^2 s\right) = 0$.

(iii) The variational principle $\delta \int_X l\left(j^1\sigma_s, \overline{s}\right) v$ holds for variations of the form

$$\delta^A \overline{s} = \overline{\nabla}^A \overline{\eta} - \operatorname{ad}_{\overline{s}}(\overline{\eta}) + \tilde{F}^A(\delta\sigma_s, d\sigma_s) \in \Gamma(\pi_{\sigma_s^*\tilde{\mathfrak{g}},X}),$$

where $\delta^A\overline{s}$ denotes the vertical part of $\delta\overline{s}$, $\tilde{F}^A \in \Omega^2(\Sigma,\tilde{\mathfrak{g}})$ denotes the reduced curvature of A, and $\delta\sigma_s \in \Gamma(\pi_{\sigma_s^*T\Sigma,X})$ and $\overline{\eta} \in \Gamma(\pi_{\sigma_s^*\tilde{\mathfrak{g}},X})$ are arbitrary.

(iv) The reduced section $\overline{s}$ satisfies the Lagrange–Poincaré field equations*:*

$$\begin{cases} \dfrac{\delta l}{\delta\sigma_s} - \operatorname{div}^\Sigma\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta\overline{s}}, \iota_{d\sigma_s}\tilde{F}^A \right\rangle, \\ \left(\operatorname{div}^A - \operatorname{ad}^*_{\overline{s}}\right)\left(\dfrac{\delta l}{\delta\overline{s}}\right) = 0. \end{cases}$$

Note that the first equation, which is known as the *horizontal equation*, holds on $\pi_{\sigma_s^*V^*\Sigma,X}$, whereas the second equation, which is known as the *vertical equation*, holds in $\pi_{\sigma_s^*\tilde{\mathfrak{g}}^*,X}$.

Example 1.1 (Euler–Poincaré reduction) *For the case $\Sigma \simeq X$, i.e., when $\pi_{Y,X}$ is a principal G-bundle, the horizontal reduced equation does not appear and the quotient*

$$C(Y) = (J^1Y)/G \to X,$$

is nothing but the bundle of connections of $\pi_{Y,X}$, which is an affine bundle modelled on $T^*X \otimes \tilde{\mathfrak{g}} \to X$. Hence, the quotient of a section $s \in \Gamma(\pi_{Y,X})$ defines a principal connection $[s]_G \in \Gamma(\pi_{C(Y),X})$ of $\pi_{Y,X}$. It is easy to check that the reduced section satisfies $[s]_G = A - \overline{s}$ when the principal connection A is regarded as a section of the bundle of connections. In this case, the vertical reduced equation is known as the Euler–Poincaré equation *(cf. [12, 16]).*

1.2.3 Reconstruction

Unlike Mechanics (i.e., $X = \mathbb{R}$), where the solutions of the reduced equations always come from solutions of the original equations, in field theories there are differential and topological obstructions to this situation and, thus, certain compatibility conditions are to be imposed in order to achieve reconstruction. For any $\overline{s} \in \Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X})$, we define

$$Y^\sigma = \pi_{Y,\Sigma}^{-1}(\sigma(X)) = \{y \in Y : \pi_{Y,\Sigma}(y) = (\sigma \circ \pi_{Y,X})(y)\} \subset Y.$$

where $\sigma = \pi_{T^*X\otimes\tilde{\mathfrak{g}},\Sigma} \circ \overline{s} \in \Gamma(\pi_{\Sigma,X})$. The G-action restricts to Y^σ and $Y^\sigma/G \simeq X$. Hence, $\pi_{Y^\sigma,X} : Y^\sigma \to X$ is a principal G-bundle and the principal connection $A \in \Omega^1(Y,\mathfrak{g})$ may be restricted to a principal connection $A^\sigma \in \Omega^1(Y^\sigma,\mathfrak{g})$ on $\pi_{Y^\sigma,X}$. In addition, the section $\overline{s}$ may be regarded as section of $T^*X \otimes \tilde{\mathfrak{g}}^\sigma \to X$, where $\tilde{\mathfrak{g}}^\sigma = (Y^\sigma \times \mathfrak{g})/G$ is the adjoint bundle of $\pi_{Y^\sigma,X}$, that is, as a 1-form $\overline{s} \in \Omega^1(X,\tilde{\mathfrak{g}}^\sigma)$. Recall that there is a bijection $\Omega^1(X,\tilde{\mathfrak{g}}^\sigma) \ni \overline{s} \leftrightarrow \omega^{\overline{s}} \in \overline{\Omega}^1(Y^\sigma,\mathfrak{g})$, where the latter denotes the family of basic forms of the adjoint type, and that the family of principal connections on $\pi_{Y^\sigma,X}$ is an affine space modelled on $\overline{\Omega}^1(Y^\sigma,\mathfrak{g})$ [55, Chapter 2]. Therefore, we have the principal connection $A^{\overline{s}} = A^\sigma - \omega^{\overline{s}} \in \Omega^1(Y^\sigma,\mathfrak{g})$ on $\pi_{Y^\sigma,X}$.

Theorem 1.3 ([32, Theorem 3.6]). *Let $s \in \Gamma(\pi_{Y,X})$ be a solution of the Euler–Lagrange equations. Then the reduced section $\overline{s} = [s, s^*A]_G$ is a solution of the Lagrange–Poincaré equations and the principal connection $A^{\overline{s}} = A^{\sigma_s} - \omega^{\overline{s}} \in \Omega^1(Y^\sigma,\mathfrak{g})$ on $\pi_{Y^\sigma,X}$ is flat, where $\sigma_s = \pi_{Y,\Sigma} \circ s$.*

Conversely, let $\overline{s} \in \Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X})$ be a solution of the Lagrange–Poincaré equations and $\sigma = \pi_{T^*X\otimes\tilde{\mathfrak{g}},\Sigma} \circ \overline{s}$. If the principal connection $A^{\overline{s}} = A^\sigma - \omega^{\overline{s}} \in \Omega^1(Y^\sigma,\mathfrak{g})$ on $\pi_{Y^\sigma,X}$ is flat and has trivial holonomy, then the integral leaves of $A^{\overline{s}}$ (which define sections of $\pi_{Y^\sigma,X}$ and, thus, sections of $\pi_{Y,X}$) are solutions of the Euler–Lagrange equations. Furthermore, if $L \subset Y$ is one of such integral leaves, any other solution of the Euler–Lagrange equations is obtained as $\Psi_g(L)$ for some $g \in G$.

Observe that if $A^{\overline{s}}$ is flat, we can always restrict ourselves to an open, simply connected subset $\mathcal{U} \subset X$. Hence the holonomy of $A^{\overline{s}}|_{\mathcal{U}}$ vanishes and the reconstruction condition holds locally. However, if the base manifold is not simply connected, the flatness of $A^{\overline{s}}$ is not enough to ensure that the reduced section can be reconstructed, as illustrated in the following example.

Example 1.2 ([11, §5.2]) *Let S^1 be the unit circle and $Y = S^1 \times \mathbb{R}$. Then the projection onto the first component, $\pi_1 : Y \to S^1$, is a principal bundle with structure group $\mathbb{R}$ acting by translation. Since it is a trivial bundle, $J^1Y = T^*S^1 \otimes T\mathbb{R}$ and a section $s \in \Gamma(\pi_1)$ may be written as $s(\phi) = (\phi, \hat{s}(\phi))$ for each $\phi \in S^1$, where $\hat{s} \in C^\infty(S^1)$. Hence, we can consider the following Lagrangian,*

$$L : T^*S^1 \otimes T\mathbb{R} \to \mathbb{R}, \quad \left(\hat{s}, \hat{s}'\right) \mapsto \left(\hat{s}'\right)^2.$$

The corresponding Euler–Lagrange equation is readily seen to be $\hat{s}'' = 0$*, whose solutions are constant functions on* S^1 *(due to the periodicity).*

On the other hand, L *is clearly* $\mathbb{R}$*-invariant. The reduced configuration bundle is given by* $\left(T^*S^1 \otimes T\mathbb{R}\right)/\mathbb{R} = T^*S^1$*. Similarly, the reduced Lagrangian reads* $l(\sigma) = |\sigma|^2$ *for each* $\sigma = f d\phi \in \Gamma(T^*S^1) = \Omega^1(S^1)$*, where* $f \in C^\infty(S^1)$*. The corresponding Euler–Poincaré equations read* $df = 0$*. In this case, there are solutions of the reduced equations that do not come from solutions of the original equations. For instance,* $\sigma = d\phi$ *is a flat connection and locally defines* $\hat{s}$ *as a determination of the angle, which is not globally defined as it has a discontinuity. Geometrically, the horizontal bundle of the principal connection defined by* σ *is given by* $HY = \operatorname{span}\{\partial_\phi - \partial_t\}$*, where* t *denotes the global coordinate of* $\mathbb{R}$*. Note that* HY *is integrable and its integral leaves are helices, which do not define sections of* π_1*.*

1.2.4 Noether theorem

The *Noether theorem* takes advantage of the symmetries of a system to provide conserved quantities along the solutions. More specifically, a vector field $U \in \mathfrak{X}(Y)$ is said to be an *infinitesimal symmetry* of the Lagrangian $L : J^1Y \to \mathbb{R}$ when $\pounds_{U^{(1)}}\mathfrak{L} = 0$ or, equivalently, when $\pounds_{U^{(1)}}\Theta_{\mathfrak{L}} = 0$, where $\pounds$ denotes the Lie derivative, $U^{(1)} \in \mathfrak{X}(J^1Y)$ is the first jet prolongation of U (cf. [87, Definition 6.4.16]) and $\Theta_{\mathfrak{L}} \in \Omega^n(J^1Y)$ is the (covariant) Cartan form of the Lagrangian density (cf. [87, §5]). In such case, given a solution of the Euler–Lagrange equations $s \in \Gamma(\pi_{Y,X})$, the Noether theorem gives the following conservation law (cf. [48, §4D]):

$$\mathrm{d}\left(\left(j^1 s\right)^* \iota_{U^{(1)}}\Theta_{\mathfrak{L}}\right) = 0. \tag{1.8}$$

In our case, the G-invariance of the Lagrangian ensures that the fundamental field $\xi^* \in \mathfrak{X}(Y)$ is an infinitesimal symmetry of $\mathfrak{L}$ for each $\xi \in \mathfrak{g}$. In order to gather these infinitesimal symmetries, we define the *Noether current* (also known as the *covariant momentum map* or *multimomentum map*) as the $\mathfrak{g}^*$-valued form $\mathcal{J} \in \Omega^{n-1}(J^1P, \mathfrak{g}^*)$ given by

$$\left\langle \mathcal{J}\left(j^1_x s\right), \xi \right\rangle = \iota_{(\xi^*)^{(1)}}\Theta_{\mathfrak{L}}\left(j^1_x s\right), \qquad j^1_x s \in J^1P,\ \xi \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Hence, the Noether theorem ensures that $\mathrm{d}\left(\left(j^1 s\right)^* \mathcal{J}\right) = 0$ for each solution of the Euler–Lagrange equations $s \in \Gamma(\pi_{Y,X})$.

In addition, the Noether current is a tensorial form of the adjoint type (cf. [55, §II.5]) on $J^1Y \to \left(J^1Y\right)/G \simeq J^1\Sigma \times_\Sigma (T^*X \otimes \tilde{\mathfrak{g}})$. Subsequently, it can be projected to the reduced configuration bundle, i.e., it defines a $\tilde{\mathfrak{G}}^*$-valued form $\tilde{\mathcal{J}} \in \Omega^{n-1}\left(J^1\Sigma \times_\Sigma (T^*X \otimes \tilde{\mathfrak{g}}), \tilde{\mathfrak{G}}^*\right)$, where $\tilde{\mathfrak{G}}^* = \left(J^1Y \times \mathfrak{g}\right)/G$ is the adjoint bundle of $J^1Y \to \left(J^1Y\right)/G$. Note that $\tilde{\mathfrak{G}}^* = \pi^*\tilde{\mathfrak{g}}^*$, where $\pi : J^1\Sigma \times_\Sigma (T^*X \otimes \mathfrak{g}) \to \Sigma$ is the natural projection. As a result, if $\overline{s} = [s, s^*A]_G$ is the reduced section of a solution of the Euler–Lagrange equations $s \in \Gamma(\pi_{Y,X})$ and $\sigma_s = \pi_{Y,\Sigma} \circ s$, then the Noether conservation law reads $\mathrm{d}^A\left(\left(j^1\sigma_s, \overline{s}\right)^* \tilde{\mathcal{J}}\right) = 0$, where $\mathrm{d}^A : \Omega^k(X, \tilde{\mathfrak{g}}^*) \to \Omega^{k+1}(X, \tilde{\mathfrak{g}}^*)$ is the covariant exterior derivative induced by A.

Theorem 1.4 (Noether theorem). *Let* $\overline{s} \in \Gamma(\pi_{T^*X \otimes \tilde{\mathfrak{g}}, X})$ *and* $\sigma = \pi_{T^*X \otimes \tilde{\mathfrak{g}}, \Sigma} \circ \overline{s} \in \Gamma(\pi_{\Sigma,X})$*. Then the Noether conservation law* $\mathrm{d}^A\left(\left(j^1\sigma, \overline{s}\right)^* \tilde{\mathcal{J}}\right) = 0$ *holds if and only if the vertical Lagrange–Poincaré equation,* $\left(\operatorname{div}^A - \operatorname{ad}^*_{\overline{s}}\right)\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0$*, holds.*

Chapter 2

Principal bundles and connections modelled by Lie group bundles

This chapter is based on our first work [21], where we provide a new geometric tool needed to perform the gauge reduction procedure in Chapter 4 below, namely: generalized principal bundles and connections (see also [36]). This type of bundles arise from the quotient of a smooth bundle by a Lie group bundle and they extend the standard notion of principal bundle. Lie group bundles, along with their infinitesimal counterparts, Lie algebra bundles, have been extensively studied in the literature (cf. [31]) as they naturally emerge in various geometric contexts and find applications in different areas. It is noteworthy that Lie group bundles may have non-isomorphic algebraic structures among different fibers. This situation was addressed in the seminal work by Douady and Lazard [31] (see [1] for a recent approach). While this scenario is interesting, our focus is confined to the case where the fibers are algebraically isomorphic and, thus, the bundle is locally trivial from an algebraic viewpoint. This decision is primarily motivated by the prevalence of such bundles in applications, particularly when dealing with reduction by local symmetries, as detailed in Chapter 4 (see also [37, 81]). It is worth noting that fibered actions can also be extended to the Lie groupoid setting (as seen in [26, 29, 64]), although this aspect is not discussed in this work.

Furthermore, we study the main properties and the curvature of Ehresmann connections that are equivariant with respect to the fibered action, which we call generalized principal connections. This type of connections require a careful approach given that, unlike standard principal connections, they are always associated to a Lie group bundle connection, i.e., a connection that respects the multiplicative structure of the Lie group bundle. This situation yields an additional term in the equivariance formula and does not have a counterpart in the standard case. From the perspective of fibered actions, standard principal connections may be regarded as a particular case of generalized principal connections where the Lie group bundle and the corresponding Lie group bundle connection are both trivial. Hence, the setting here presented is also useful for classical reduction –i.e., reduction by global symmetries– within Mechanics [22, 23, 72] and Field Theory [12, 16, 32].

2.1 Generalized principal bundles

2.1.1 Actions of Lie group bundles

A *Lie group fiber bundle* with typical fiber a Lie group G is a fiber bundle $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ such that for any point $x \in X$ the fiber $\mathcal{G}_x$ is equipped with a Lie group structure and there is a neighborhood $\mathcal{U} \subset X$ and a diffeomorphism $x \in \mathcal{U} \times G \to \pi^{-1}_{\mathcal{G},X}(\mathcal{U})$ preserving the Lie group structure fiberwisely.

Note that the multiplication map $M : \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ and the inversion map $\cdot^{-1} : \mathcal{G} \to \mathcal{G}$ are bundle morphisms covering the identity $\mathrm{id}_X : X \to X$. Likewise, the map $1 : X \to \mathcal{G}$ that assigns the identity element $1_x \in \mathcal{G}_x$ to each $x \in X$ is a global section (called the *unit section*) of $\pi_{\mathcal{G},X}$. Any Lie group

bundle defines a *Lie algebra bundle* $\pi_{\mathfrak{g},X}:\mathfrak{g}\to X$ as the vector bundle whose fiber $\mathfrak{g}_x$ at each $x\in X$ is the Lie algebra of $\mathcal{G}_x$. That is, the Lie algebra bundle is the pull-back bundle $\mathfrak{g}=1^*(V\mathcal{G})$, where $V\mathcal{G}\subset T\mathcal{G}$ is the vertical bundle of $\pi_{\mathcal{G},X}$, i.e., the kernel of $(\pi_{\mathcal{G},X})_*$.

Remark 2.1 (Jets of Lie group fiber bundles) *Let $\pi_{\mathcal{G},X}:\mathcal{G}\to X$ be a Lie group fiber bundle and $r\geq 0$ be an integer. Then the r-th jet bundle of $\pi_{\mathcal{G},X}$, $J^r\mathcal{G}\to X$, is again a Lie group fiber bundle (see, for example, [37, §3, Th. 1]). The multiplication is inherited from the Lie group bundle structure of $\pi_{\mathcal{G},X}$, that is,*

$$M\left(j^r_x\gamma_1,j^r_x\gamma_2\right)=j^r_x\left(M\circ(\gamma_1,\gamma_2)\right),\qquad j^r_x\gamma_1,j^r_x\gamma_2\in J^r\mathcal{G}.$$

We consider subgroups of Lie group bundles in the following sense.

Definition 2.1 *A* Lie group subbundle *of a Lie group bundle $\pi_{\mathcal{G},X}:\mathcal{G}\to X$ is a Lie group bundle $\pi_{\mathcal{H},X}:\mathcal{H}\to X$ such that $\mathcal{H}$ is a submanifold of $\mathcal{G}$ and $\mathcal{H}_x$ is a Lie subgroup of $\mathcal{G}_x$ for each $x\in X$. It is said to be* closed *if $\mathcal{H}_x$ is a closed Lie subgroup of $\mathcal{G}_x$ for every $x\in X$.*

Let $\pi_{Y,X}$ be a fiber bundle and $\pi_{\mathcal{G},X}$ be a Lie group fiber bundle.

Definition 2.2 *A* (right) fibered action *of $\pi_{\mathcal{G},X}$ on $\pi_{Y,X}$ is a bundle morphism*

$$\Phi:Y\times_X\mathcal{G}\to Y$$

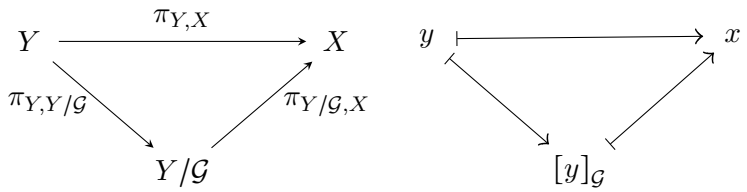
covering the identity $\mathrm{id}_X:X\to X$ *such that $\Phi(y,hg)=\Phi(\Phi(y,h),g)$ and $\Phi(y,1_x)=y$, for all $(y,g),(y,h)\in Y\times_X\mathcal{G}$, $\pi_{\mathcal{G},X}(y)=x$.*

For the sake of simplicity, we will denote $\Phi(y,g)=y\cdot g$ and we will say that $\pi_{\mathcal{G},X}$ acts fiberwisely on the right on $\pi_{Y,X}$. Note that Φ induces a right action on each fiber, $\Phi_x=\Phi|_{Y_x\times\mathcal{G}_x}:Y_x\times\mathcal{G}_x\to Y_x$. The fibered action is said to be *free* if $y\cdot g=y$ for some $(y,g)\in Y\times_X\mathcal{G}$ implies that $g=1_x$, $x=\pi_{Y,X}(y)$. In the same way, it is said to be *proper* if the bundle morphism $Y\times_X\mathcal{G}\ni(y,g)\mapsto(y,y\cdot g)\in Y\times_X Y$ is proper. If Φ is free and proper, so is each action Φ_x, since the fibers of a bundle are closed.

As the fibered action is vertical (i.e., it covers the identity id_X), we may regard the quotient space $Y/\mathcal{G}$ as the disjoint union of the quotients of the fibers by the induced actions, that is,

$$Y/\mathcal{G}=\bigsqcup_{x\in X}Y_x/\mathcal{G}_x=\{[y]_{\mathcal{G}}=(x,[y]_{\mathcal{G}_x}):x\in X,y\in Y_x\}.$$

Obviously, the following diagram is commutative:



(2.1)

Example 2.1 (Jet lift of fibered actions) *Let $\Phi:Y\times_X\mathcal{G}\to Y$ be a (right) fibered action of a Lie group bundle $\pi_{\mathcal{G},X}$ on a fiber bundle $\pi_{Y,X}$. The first jet extension of Φ turns out to be a (right) fibered action of $\pi_{J^1\mathcal{G},X}$ on $\pi_{J^1Y,X}$,*

$$\Phi^{(1)}:J^1Y\times_X J^1\mathcal{G}\to J^1Y,\quad \left(j^1_xs,j^1_x\gamma\right)\mapsto j^1_x(\Phi\circ(s,\gamma)).$$

If we regard 1-jets as tangent maps of sections at a point, that is, $j^1_xs\equiv(ds)_x$ and $j^1_x\gamma\equiv(d\gamma)_x$ for certain local sections s and γ, then $\Phi^{(1)}$ may be seen as

$$\Phi^{(1)}(j^1_xs,j^1_x\gamma)\equiv(d\Phi)_{(s(x),\gamma(x))}\circ((ds)_x,(d\gamma)_x):T_xX\to T_{s(x)\cdot\gamma(x)}Y.$$

2.1.2 Smooth structure of fibered quotients

Theorem 2.1. *If $\pi_{\mathcal{G},X}$ acts on $\pi_{Y,X}$ freely and properly, then $Y/\mathcal{G}$ admits a unique smooth structure such that*

(i) $\pi_{Y,Y/\mathcal{G}}$ is a fiber bundle with typical fiber G, called generalized principal bundle.

(ii) $\pi_{Y/\mathcal{G},X}$ is a fibered manifold, i.e., a surjective submersion.

Proof. Taking trivializations of $\pi_{Y,X}$ and $\pi_{\mathcal{G},X}$ on the same neighborhood $\mathcal{U} \subset X$, the local expression of the actions is:

$$\mathcal{U} \times \hat{Y} \times G \to \mathcal{U} \times \hat{Y},$$

where $\hat{Y}$ is the typical fiber of Y. This can be seen as a standard Lie group action acting trivially on $\mathcal{U}$. The classical results on Lie group actions (for example, see [60, Theorem 7.10]) provide a unique smooth structure on $(\mathcal{U} \times \hat{Y})/G$ that can be used as a chart on $Y/\mathcal{G}$. In these trivializations, the projections $\pi_{Y,Y/\mathcal{G}}$ and $\pi_{Y/\mathcal{G},X}$ are $\mathcal{U} \times \hat{Y} \to (\mathcal{U} \times \hat{Y})/G$ and $(\mathcal{U} \times \hat{Y})/G \to \mathcal{U}$ which gives *(i)* and *(ii)*.

Remark 2.2 *If a Lie group G acts freely, properly and fiberwisely on a bundle $\pi_{Y,X} : Y \to X$, then $\pi_{Y/G,X} : Y/G \to X$ is a bundle with typical fiber $\hat{Y}/G$. However, in the case of action of Lie group bundle, since the action depends on X, the notion of typical fiber is more delicate. For example, if X is not connected, the topology of the typical fiber may differ on each component. This is the case of the action of $X \times \mathbb{Z}$ on $X \times (S^1 \times \mathbb{R})$ with $X = \mathbb{R} - \{0\}$ defined as*

$$\Phi : Y \times_X \mathcal{G} \to Y, \quad \big((t, (\cos\theta, \sin\theta), z), (t, n)\big) \mapsto \left(t, \left(\operatorname{sgn}(t)^n \cos\theta, \sin\theta\right), z + n\right).$$

The typical fiber is a Klein bottle for $t < 0$ and a torus for $t > 0$.

For a connected base manifold X, the quotient $\hat{Y}/G$ is well-defined (up to diffeomorphisms). Indeed, given any two points $p, q \in X$ connected by a path, $p = \gamma(0), q = \gamma(1)$, the action on $Y_t = \pi^{-1}_{Y,X}(\gamma(t))$ can be regarded (from the local trivializations of $\mathcal{G}$ and Y) as a homotopy of diffeomorphisms of G on $\hat{Y}$. This gives a diffeomorphism of the fibers of $Y/\mathcal{G}$ on p and q.

We can build a trivializing atlas for $\pi_{Y,Y/\mathcal{G}}$ starting from a trivializing atlas $\left\{\left(\mathcal{U}_\alpha, \psi^\alpha_{\mathcal{G}}\right) : \alpha \in \Lambda\right\}$ of $\pi_{\mathcal{G},X}$. For $\alpha \in \Lambda$, let $\mathcal{V}_\alpha = \pi^{-1}_{Y/\mathcal{G},X}(\mathcal{U}_\alpha)$ and pick a local section $\hat{s} \in \Gamma(\mathcal{V}_\alpha, \pi_{Y,Y/\mathcal{G}})$ (the existence of that local section may require the choice of a smaller $\mathcal{U}_\alpha$). We define

$$\psi^\alpha_Y : Y|_{\mathcal{V}_\alpha} = \pi^{-1}_{Y,Y/\mathcal{G}}(\mathcal{V}_\alpha) \to \mathcal{V}_\alpha \times G, \quad y \mapsto \left([y]_{\mathcal{G}}, \hat{h}\right), \tag{2.2}$$

where $\hat{h} \in G$ is such that $y = \hat{s}([y]_{\mathcal{G}}) \cdot g$, with $g = (\psi^\alpha_{\mathcal{G}})^{-1}\left(\pi_{Y/\mathcal{G},X}([y]_{\mathcal{G}}), \hat{h}\right) \in \mathcal{G}$. The element $\hat{h}$ exists since $Orb(y) = \pi^{-1}_{Y,Y/\mathcal{G}}\left([y]_{\mathcal{G}}\right)$ and it is unique because the fibered action is free. It turns out that $(\mathcal{V}_\alpha, \psi^\alpha_Y)$ is a trivialization for $\pi_{Y,Y/\mathcal{G}}$. Indeed, its inverse is given by

$$(\psi^\alpha_Y)^{-1} : \mathcal{V}_\alpha \times G \to Y|_{\mathcal{V}_\alpha}, \quad \left([y]_{\mathcal{G}}, \hat{h}\right) \mapsto \hat{s}([y]_{\mathcal{G}}) \cdot (\psi^\alpha_{\mathcal{G}})^{-1}\left(\pi_{Y/\mathcal{G},X}([y]_{\mathcal{G}}), \hat{h}\right).$$

As a result, $\{(\mathcal{V}_\alpha, \psi^\alpha_Y) \mid \alpha \in \Lambda\}$ is an atlas for $\pi_{Y,Y/\mathcal{G}}$.

Observe that, by definition of Lie group bundle, the maps $\psi^\alpha_{\mathcal{G}}|_{\mathcal{G}_x} : \mathcal{G}_x \to \{x\} \times G$, $x \in \mathcal{U}_\alpha$, are Lie group homomorphisms. It is thus straightforward that the fibered action is locally given by the right multiplication on the Lie group G.

Corollary 2.2. *Let $x \in \mathcal{U}_\alpha$, $y = (\psi^\alpha_Y)^{-1}\left([y]_{\mathcal{G}}, \hat{h}\right) \in Y_x$ and $g = (\psi^\alpha_{\mathcal{G}})^{-1}(x, \hat{g}) \in \mathcal{G}_x$, then*

$$y \cdot g = (\psi^\alpha_Y)^{-1}\left([y]_{\mathcal{G}}, \hat{h}\hat{g}\right). \tag{2.3}$$

By construction, $y = \hat{s}([y]_{\mathcal{G}}) \cdot (\psi^{\alpha}_{\mathcal{G}})^{-1}\left(x, \hat{h}\right)$. Hence,

$$\Phi(y,g) = \left(\hat{s}([y]_{\mathcal{G}}) \cdot (\psi^{\alpha}_{\mathcal{G}})^{-1}\left(x,\hat{h}\right)\right) \cdot (\psi^{\alpha}_{\mathcal{G}})^{-1}(x,\hat{g}) = \hat{s}([y]_{\mathcal{G}}) \cdot (\psi^{\alpha}_{\mathcal{G}})^{-1}\left(x, \hat{h}\hat{g}\right)$$

Moreover, for each $x \in \mathcal{U}_\alpha$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}_x & \xrightarrow{\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x}} & \{x\} \times G \\ \exp \uparrow & & \uparrow \left(\mathrm{id}_{\{x\}}, \exp\right) \\ \mathfrak{G}_x & \xrightarrow{(d\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x})_{1_x}} & \{x\} \times \mathfrak{g} \end{array}$$

where $\pi_{\mathfrak{G},X}$ is the Lie algebra bundle of $\pi_{\mathcal{G},X}$, $\mathfrak{g}$ is the Lie algebra of G and exp is the exponential map between the Lie algebra and its corresponding Lie group. Furthermore, note that $(d\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x})_{1_x}$ is a linear isomorphism, since $\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x}$ is a diffeomorphism. In other words, for each $\xi = (d\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x})^{-1}_{1_x}\left(x, \hat{\xi}\right) \in \mathfrak{G}_x$ we have

$$\exp(\xi) = (\psi^{\alpha}_{\mathcal{G}})^{-1}\left(x, \exp\hat{\xi}\right). \tag{2.4}$$

In fact, we can define a linear trivialization $(\mathcal{U}_\alpha, \psi^{\alpha}_{\mathfrak{G}})$ for $\pi_{\mathfrak{G},X}$ from $(\mathcal{U}_\alpha, \psi^{\alpha}_{\mathcal{G}})$. Namely,

$$\psi^{\alpha}_{\mathfrak{G}} : \mathfrak{G}|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathfrak{g}, \quad \xi \mapsto (d\psi^{\alpha}_{\mathcal{G}}|_{\mathcal{G}_x})_{1_x}(\xi), \qquad x = \pi_{\mathfrak{G},X}(\xi). \tag{2.5}$$

2.1.3 Infinitesimal generators

If we fix $x \in X$, $y_0 \in Y_x$ and $g_0 \in \mathcal{G}_x$, we can consider the maps

$$\Phi_{y_0} : \mathcal{G}_x \to Y_x, \quad g \mapsto y_0 \cdot g; \qquad \Phi_{g_0} : Y_x \to Y_x, \quad y \mapsto y \cdot g_0.$$

In the same way, we denote by $L_{g_0} : \mathcal{G}_x \to \mathcal{G}_x$ and $R_{g_0} : \mathcal{G}_x \to \mathcal{G}_x$ the left and right multiplication by $g_0 \in \mathcal{G}_x$, respectively. *Infinitesimal generators* (or *fundamental fields*) are defined in the same fashion as in classical actions of Lie groups. Namely, for each $\xi \in \mathfrak{G}_x$, then $\xi^* \in \mathfrak{X}(Y_x)$ is defined as

$$\xi^*_y = \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\xi) = (d\Phi_y)_{1_x}(\xi), \qquad y \in Y_x. \tag{2.6}$$

Fundamental vector fields are $\pi_{Y,Y/\mathcal{G}}$-vertical, i.e., $\xi^*_y \in V_yY = \ker(d\pi_{Y,Y/\mathcal{G}})_y$ for each $y \in Y_x$. Of course, they are also $\pi_{Y,X}$-vertical.

Proposition 2.3. *Let $\pi_{\mathfrak{G},X}$ be the Lie algebra bundle of $\pi_{\mathcal{G},X}$. The following map is a vertical isomorphism of vector bundles over Y:*

$$Y \times_X \mathfrak{G} \to VY, \qquad (y,\xi) \mapsto \xi^*_y. \tag{2.7}$$

In addition, for any $(g,\xi) \in \mathcal{G} \times_X \mathfrak{G}$, we have:

$$(\Phi_g)_*(\xi^*) = \left(\mathrm{Ad}_{g^{-1}}(\xi)\right)^*. \tag{2.8}$$

Proof. It is clear that the morphism is vertical by definition. Fix $y \in Y$ with $\pi_{Y,X}(y) = x$. Let us see that the map $Y \times_X \mathfrak{G}_y = \mathfrak{G}_x \ni \xi \mapsto \xi^*_y \in V_yY$ is a linear isomorphism. The linearity is clear from the second equality of (2.6). Since $\dim \mathfrak{G}_x = \dim \mathcal{G}_x = \dim V_yY$ (the last equality is for being $\pi_{Y,Y/\mathcal{G}}$ a submersion), we just need to prove the injectivity to conclude. Let $\xi \in \mathfrak{G}_x$ be such that $\xi^*_y = 0$. Then, $y \cdot \exp t\xi = y$ for every $t \in (-\epsilon, \epsilon)$. Since the action is free, this says that $\exp t\xi = 1_x$ for every $t \in (-\epsilon, \epsilon)$ and, hence, $\xi = 0$. The second part is a straightforward computation.

2.1.4 Generalized associated bundles

Let $\pi_{F,X} : F \to X$ be a fiber bundle on which $\pi_{\mathcal{G},X}$ acts fiberwisely on the left. This yields a right fibered action of $\pi_{\mathcal{G},X}$ on product bundle $\pi_{Y\times_X F,X}$. Namely,

$$(y,f)\cdot g = (y\cdot g, g^{-1}\cdot f), \qquad (y,f,g)\in Y\times_X F\times_X \mathcal{G}.$$

If the fibered action of $\pi_{\mathcal{G},X}$ on $\pi_{Y,X}$ is free and proper, so is the induced action on $\pi_{Y\times_X F,X}$. In such case, we have the smooth manifold

$$Y\times_{\mathcal{G}} F = (Y\times_X F)/\mathcal{G}.$$

We denote the equivalence classes by $[y,f]_{\mathcal{G}}\in Y\times_{\mathcal{G}} F$.

Proposition 2.4. *In the above conditions, if $\hat{F}$ is the typical fiber of $\pi_{F,X}$, then $\pi_{Y\times_{\mathcal{G}}F,Y/\mathcal{G}}$ is a fiber bundle with typical fiber $\hat{F}$, called* generalized associated bundle*, where*

$$\pi_{Y\times_{\mathcal{G}}F,Y/\mathcal{G}} : Y\times_{\mathcal{G}} F\to Y/\mathcal{G}, \quad [y,f]_{\mathcal{G}}\mapsto [y]_{\mathcal{G}}.$$

Proof. It is clear that the projection is well-defined and surjective. For each $[y_0,f_0]_{\mathcal{G}}\in Y\times_{\mathcal{G}}F$, let us find a trivialization through that point. Let $\mathcal{U}\subset X$ be a trivializing set of both $\pi_{\mathcal{G},X}$ and $\pi_{F,X}$, such that $\pi_{Y,X}(y_0)=\pi_{F,X}(f_0)\in\mathcal{U}$. Let $\mathcal{V}=\pi^{-1}_{Y/\mathcal{G},X}(\mathcal{U})$ and suppose that it is a trivializing set of $\pi_{Y,Y/\mathcal{G}}$ (maybe we need to choose a smaller $\mathcal{U}$ in order to achieve this). Using the above trivializations, we define

$$\phi : (Y\times_{\mathcal{G}} F)|_{\mathcal{V}}\to \mathcal{V}\times\hat{F}, \quad [(\sigma,g),(x,f)]_{\mathcal{G}}\mapsto (\sigma, g\cdot f),$$

where the action $G\times\hat{F}\to\hat{F}$ is induced by the fibered action of $\pi_{\mathcal{G},X}$ on $\pi_{F,X}$. Observe that this action may depend on the base point x. Nevertheless, the condition of $\Phi : Y\times_X\mathcal{G}\to Y$ is smooth ensures that ϕ is also smooth.

Of course, $[y_0,f_0]_{\mathcal{G}}\in(Y\times_{\mathcal{G}}F)|_{\mathcal{V}}$, since $\pi_{Y,X}(y_0)=\pi_{F,X}(f_0)\in\mathcal{U}$. In fact, the pair $(\mathcal{V},\phi)$ is a trivialization of $\pi_{Y\times_{\mathcal{G}}F,Y/\mathcal{G}}$. Indeed, the inverse map is given by

$$\phi^{-1} : \mathcal{V}\times\hat{F}\to (Y\times_{\mathcal{G}}F)|_{\mathcal{V}}, \quad (\sigma,f)\mapsto [(\sigma,1),(x,f)]_{\mathcal{G}},$$

where again $x=\pi_{Y/\mathcal{G},X}(\sigma)$.

Example 2.2 (Conjugacy bundle) *A Lie group bundle $\pi_{\mathcal{G},X}$ acts fiberwisely on the left on itself by conjugation: $g\cdot h = c_g(h) = ghg^{-1}$, $(g,h)\in\mathcal{G}\times_X\mathcal{G}$. We can thus consider the* generalized conjugacy bundle*, $Y\times_{\mathcal{G}}\mathcal{G}$, which inherits the Lie group bundle structure from $\pi_{\mathcal{G},X}$.*

Example 2.3 (Adjoint bundle) *In the same vein, $\pi_{\mathcal{G},X}$ acts fiberwisely on the left on $\pi_{\mathfrak{g},X}$ via the adjoint map: $g\cdot\xi = \mathrm{Ad}_g(\xi) = (dc_g)_{1_x}(\xi)$, $(g,\xi)\in\mathcal{G}\times_X\mathfrak{g}$. The corresponding quotient is the* generalized adjoint bundle*, which we will denote by $\tilde{\mathfrak{g}} = Y\times_{\mathcal{G}}\mathfrak{g}$. It is a vector bundle equipped with a Lie algebra bundle structure.*

Example 2.4 (Coadjoint bundle) *Let $\mathfrak{g}^*$ be the dual vector bundle of $\pi_{\mathfrak{g},X}$, where $\pi_{\mathcal{G},X}$ acts fiberwisely on the left via the coadjoint representation, $g\cdot\eta = \mathrm{Ad}^*_{g^{-1}}(\eta)$, $(g,\eta)\in\mathcal{G}\times_X\mathfrak{g}^*$. Using this action we get the* generalized coadjoint bundle*, $\tilde{\mathfrak{g}}^* = Y\times_{\mathcal{G}}\mathfrak{g}^*$, which is a vector bundle.*

2.2 Generalized principal connections

2.2.1 Lie group bundle connections

If $\pi_{\mathcal{G},X}$ is a Lie group bundle, an Ehresmann connection

$$\nu : T\mathcal{G} \to V\mathcal{G}, \qquad U_g \mapsto U_g^v,$$

can be also regarded as a vertical bundle map (denoted by the same letter for the sake of simplicity)

$$\nu : T\mathcal{G} \to \mathfrak{g}, \qquad U_g \mapsto \left(dR_{g^{-1}}\right)_g (U_g^v).$$

Furthermore, it is natural to impose a compatibility of ν with the algebraic structure of $\mathcal{G}$.

Definition 2.3 *A* Lie group bundle connection *on $\pi_{\mathcal{G},X}$ is an Ehresmann connection $\nu : T\mathcal{G} \to V\mathcal{G}$ satisfying*

(i) $\ker \nu_{1_x} = (d1)_x(T_xX)$ *for each* $x \in X$.

(ii) For every $(g,h) \in \mathcal{G} \times_X \mathcal{G}$ *and* $(U_g, U_h) \in T_g\mathcal{G} \times_{T_xX} T_h\mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$, *then:*

$$\nu_{gh}\left((dM)_{(g,h)}(U_g,U_h)\right) = (dR_h)_g\left(\nu_g(U_g)\right) + (dL_g)_h\left(\nu_h(U_h)\right),$$

where $M : \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ *is the fiber multiplication map.*

The corresponding conditions when regarded as a map $\nu : T\mathcal{G} \to \mathfrak{g}$ *are*

(i) $\ker \nu_{1_x} = (d1)_x(T_xX)$ *for each* $x \in X$.

(ii) For every $(g,h) \in \mathcal{G} \times_X \mathcal{G}$ *and* $(U_g, U_h) \in T_g\mathcal{G} \times_{T_xX} T_h\mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$, *then:*

$$\nu_{gh}\left((dM)_{(g,h)}(U_g,U_h)\right) = \nu_g(U_g) + \mathrm{Ad}_g\left(\nu_h(U_h)\right).$$

Geometric interpretations of Lie group bundle connections are provided by the following results. We denote by ${}^{\nu}\big\|$ the parallel transport of ν and by $Hor_g^{\nu} : T_xX \to T_g\mathcal{G}$ its horizontal lift at any $g \in \mathcal{G}$, $x = \pi_{\mathcal{G},X}(g)$.

Proposition 2.5. *Let ν be an Ehresmann connection on $\pi_{\mathcal{G},X}$ such that* $\ker \nu_{1_x} = (d1)_x(T_xX)$ *for each $x \in X$. Then ν is a Lie group connection if and only if for any curve $x : I \to X$ we have*

$${}^{\nu}\big\|_{x(a)}^{x(b)}(gh) = \left({}^{\nu}\big\|_{x(a)}^{x(b)} g\right)\left({}^{\nu}\big\|_{x(a)}^{x(b)} h\right), \qquad g,h \in \mathcal{G}_{x(a)}. \tag{2.9}$$

Consequently,

$${}^{\nu}\big\|_{x(a)}^{x(b)} g^{-1} = \left({}^{\nu}\big\|_{x(a)}^{x(b)} g\right)^{-1}, \qquad {}^{\nu}\big\|_{x(a)}^{x(b)} 1_{x(a)} = 1_{x(b)}. \tag{2.10}$$

Proof. Suppose that ν is a Lie group connection. Let $x : I \to X$ be a curve and $g,h \in \mathcal{G}_{x(a)}$. Denote $\alpha_1(t) = {}^{\nu}\big\|_{x(a)}^{x(t)} g$, $\alpha_2(t) = {}^{\nu}\big\|_{x(a)}^{x(t)} h$, $t \in I$, and $\alpha = M \circ (\alpha_1, \alpha_2)$. To show that $\alpha(t) = {}^{\nu}\big\|_{x(a)}^{x(t)}(gh)$, $t \in I$, we use the uniqueness of the parallel transport. It is clear that $\pi_{\mathcal{G},X} \circ \alpha = x$ and $\alpha(a) = gh$, so we only need to check that it is horizontal,

$$\begin{aligned}\nu_{\alpha(t)}\left(\alpha'(t)\right) &= \nu_{\alpha(t)}\left((dM)_{(\alpha_1(t),\alpha_2(t))}\left(\alpha_1'(t),\alpha_2'(t)\right)\right)\\ &= \left(dR_{\alpha_2(t)}\right)_{\alpha_1(t)}\left(\nu_{\alpha_1(t)}\left(\alpha_1'(t)\right)\right) + \left(dL_{\alpha_1(t)}\right)_{\alpha_2(t)}\left(\nu_{\alpha_2(t)}\left(\alpha_2'(t)\right)\right) = 0,\end{aligned}$$

since α_1 and α_2 are horizontal.

Conversely, suppose that ν satisfies (2.9). Let $x \in X$, $g,h \in \mathcal{G}_x$ and $(U_g,U_h) \in T_g\mathcal{G} \times_{T_xX} T_h\mathcal{G}$. Then there exists $\alpha = (\alpha_1,\alpha_2) : (-\epsilon,\epsilon) \to \mathcal{G} \times_X \mathcal{G}$ such that $\alpha'(0) = (\alpha_1'(0),\alpha_2'(0)) = (U_g,U_h)$. Denote $x = \pi_{\mathcal{G},X} \circ \alpha_1 = \pi_{\mathcal{G},X} \circ \alpha_2$. We conclude that ν is a Lie group connection as a straightforward consequence of (2.9) and the definition of covariant derivative,

$$\begin{aligned}\nu_{gh}\left((dM)_{(g,h)}(U_g,U_h)\right) &= \nu_{gh}\left((M\circ\alpha)'(0)\right)\\ &= \frac{D^\nu(M\circ\alpha)(0)}{Dt}\\ &= (dR_h)_g\left(\frac{D^\nu\alpha_1(0)}{Dt}\right) + (dL_g)_h\left(\frac{D^\nu\alpha_2(0)}{Dt}\right)\\ &= (dR_h)_g\left(\nu_g(U_g)\right) + (dL_g)_h\left(\nu_h(U_h)\right).\end{aligned}$$

Proposition 2.6. *Let ν be an Ehresmann connection on $\pi_{\mathcal{G},X}$ and consider the corresponding jet section $\hat{\nu} \in \Gamma(\pi_{J^1\mathcal{G},\mathcal{G}})$. Then ν is a Lie group bundle connection if and only if*

(i) $\hat{\nu} \circ 1 = j^1 1 = d1$,

(ii) for each $(g,h) \in \mathcal{G} \times_X \mathcal{G}$, we have that $\hat{\nu}(gh) = \hat{\nu}(g)\,\hat{\nu}(h)$ with respect to the Lie group bundle structure in $J^1\mathcal{G}$.

Proof. It is clear that (i) is equivalent to the condition: $\ker \nu_{1_x} = (d1)_x(T_xX)$ for each $x \in X$. Thanks to the previous Proposition, it is enough to show that (ii) is equivalent to (2.9). Let $x \in X$, $U_x \in T_xX$ and $g,h \in \mathcal{G}_x$ and pick a curve $\alpha : (-\epsilon,\epsilon) \to X$ such that $\alpha'(0) = U_x$. We define the following curve:

$$\beta_g(t) = {}^{\nu}\Big\|_{\alpha(0)}^{\alpha(t)} g, \qquad t \in (-\epsilon,\epsilon).$$

We have $\beta_g'(0) = Hor_g^\nu(U_x) = \hat{\nu}(g)(U_x)$, where $Hor_g^\nu : T_xX \to T_g\mathcal{G}$ is the horizontal lifting given by ν. Performing the same construction with h and gh we obtain curves $\beta_h, \beta_{gh} : (-\epsilon,\epsilon) \to T\mathcal{G}$.

Suppose that (2.9) is satisfied, then $\beta_{gh} = M \circ (\beta_g,\beta_h)$. Therefore

$$\begin{aligned}\hat{\nu}(gh)(U_x) &= \beta_{gh}'(0)\\ &= (dM)_{(g,h)}\left(\beta_g'(0),\beta_h'(0)\right)\\ &= (dM)_{(g,h)}\left(\hat{\nu}(g)(U_x),\hat{\nu}(h)(U_x)\right).\end{aligned}$$

Since the equality is valid for every $U_x \in T_xX$, we conclude that $\hat{\nu}(gh) = \hat{\nu}(g)\,\hat{\nu}(h)$.

Conversely, if (ii) holds, then $\beta_{gh}(0) = gh = \beta_g(0)\,\beta_h(0)$ and $\beta_{gh}'(0) = (dM)_{(g,h)}(\beta_g'(0),\beta_h'(0)) = (M\circ(\beta_g,\beta_h))'(0)$. By uniqueness of the parallel transport we obtain that $\beta_{gh} = M\circ(\beta_g,\beta_h)$ for each local curve $\alpha : (-\epsilon,\epsilon) \to X$ such that $\alpha(0) = x$.

From the proof above we also deduce that the condition

$$Hor_{gh}^\nu = (dM)_{(g,h)} \circ \left(Hor_g^\nu, Hor_h^\nu\right), \qquad (g,h) \in \mathcal{G} \times_X \mathcal{G},$$

together with (i) of Definition 2.3 characterize Lie group connections. More generally, we have the following property.

Proposition 2.7. *Let ν be a Lie group connection on $\pi_{\mathcal{G},X}$. Then for each $(g,h) \in \mathcal{G}\times_X\mathcal{G}$, $U_x \in T_xX$, $x = \pi_{\mathcal{G},X}(g)$, and $U_h \in T_h\mathcal{G}$ such that $(d\pi_{\mathcal{G},X})_h(U_h) = U_x$ we have*

$$(dM)_{(g,h)}\left(Hor_g^\nu(U_x),U_h\right) = Hor_{gh}^\nu(U_x) + (dL_g)_h\left(\nu_h(U_h)\right).$$

Proof. Thanks to the uniqueness of the horizontal lifting, it is enough to show that

$$U = (dM)_{(g,h)}\left(Hor^{\nu}_{g}(U_x), U_h\right) - (dL_g)_h\left(\nu_h(U_h)\right)$$

is horizontal and projects to U_x. For horizontality we have that

$$\begin{aligned}\nu_{gh}(U) &= (dR_h)_g\left(\nu_g\left(Hor^{\nu}_{g}(U_x)\right)\right) + (dL_g)_h\left(\nu_h(U_h)\right) - \nu_{gh}\left((dL_g)_h\left(\nu_h(U_h)\right)\right)\\ &= (dL_g)_h\left(\nu_h(U_h)\right) - \nu_{gh}\left((dL_g)_h\left(\nu_h(U_h)\right)\right) \;=\; 0.\end{aligned}$$

The last equality is a particular case of property (ii) of the definition of Lie group connection with $U_g = 0$. In such case, we have $(dM)_{(g,h)}(0, U_h) = (dL_g)(U_h)$ and, thus,

$$\nu_{gh}\left((dL_g)(U_h)\right) = \nu_{gh}\left((dM)_{(g,h)}(0, U_h)\right) = (dL_g)_h\left(\nu_h(U_h)\right).$$

At last, we check that it projects to U_x. In order to do so, let $\alpha = (\alpha_1, \alpha_2) : (-\epsilon, \epsilon) \to \mathcal{G} \times_X \mathcal{G}$ such that $\alpha(0) = (g, h)$ and $\alpha'(0) = \left(Hor^{\nu}_{g}(U_x), U_h\right)$,

$$\begin{aligned}(d\pi_{\mathcal{G},X})_{gh}(U) &= (d\pi_{\mathcal{G},X})_{gh}\left((dM)_{(g,h)}\left(Hor^{\nu}_{g}(U_x), U_h\right)\right) - (d\pi_{\mathcal{G},X})_{gh}\left((dL_g)_h\left(\nu_h(U_h)\right)\right)\\ &= d(\pi_{\mathcal{G},X}\circ M)_{(g,h)}\left(Hor^{\nu}_{g}(U_x), U_h\right)\\ &= \left.\frac{d}{dt}\right|_{t=0}(\pi_{\mathcal{G},X}\circ M\circ\alpha)(t)\\ &= \left.\frac{d}{dt}\right|_{t=0}(\pi_{\mathcal{G},X}\circ\alpha_1)(t)\\ &= (d\pi_{\mathcal{G},X})_g\left(Hor^{\nu}_{g}(U_x)\right) \;=\; U_x.\end{aligned}$$

With respect to this last Proposition, if the vector U_h is horizontal, then

$$(dM)_{(g,h)}(H_g, U_h) = H_{gh}, \qquad U_h \in T_h\mathcal{G},\ \pi_{\mathcal{G},X}(g) = \pi_{\mathcal{G},X}(h),$$

an identity that is similar to the corresponding property of connections on standard principal bundles.

2.2.2 Induced connection on the Lie algebra bundle

A Lie group connection induces a linear connection on the corresponding Lie algebra bundle.

Proposition 2.8. *Let $x : I \to X$ be a smooth curve. Then the map $\mathfrak{g}\big\|_{x(a)}^{x(b)} : \mathfrak{g}_{x(a)} \to \mathfrak{g}_{x(b)}$ defined as*

$$\mathfrak{g}\big\|_{x(a)}^{x(b)}\xi = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon\,\xi), \quad \xi \in \mathfrak{g}_{x(a)}$$

is a linear parallel transport on $\pi_{\mathfrak{g},X}$.

Proof. The map is well-defined, since ${}^{\nu}\big\|_{x(a)}^{x(b)} 1_{x(a)} = 1_{x(b)}$. Likewise, it is easy to check that $\mathfrak{g}\big\|_{x(a)}^{x(b)}\lambda\xi = \lambda\,\mathfrak{g}\big\|_{x(a)}^{x(b)}\xi$ for each $\xi \in \mathfrak{g}_{x(a)}$ and $\lambda \in \mathbb{R}$. To conclude, let $\xi, \eta \in \mathfrak{g}_{x(a)}$ and $\epsilon \in \mathbb{R}$ "small enough". We apply the Zassenhaus formula (for example, cf. [10]):

$$\exp(\epsilon(\xi+\eta)) = \exp(\epsilon\,\xi)\exp(\epsilon\,\eta)\prod_{n=2}^{\infty}\exp(C_n(\epsilon\,\xi, \epsilon\,\eta)),$$

where $C_n(\epsilon\xi, \epsilon\eta)$ is a homogeneous Lie polynomial in $\{\epsilon\xi, \epsilon\eta\}$ for each $n \geq 2$. Thus, $\exp(\epsilon(\xi+\eta)) = \exp(\epsilon\xi)\exp(\epsilon\eta)\exp\left(o(\epsilon^2)\right)$ and are done:

$$\begin{aligned}
{}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}(\xi+\eta) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon(\xi+\eta)) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \left(\exp(\epsilon\xi)\exp(\epsilon\eta)\exp\left(o\left(\epsilon^2\right)\right)\right) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left({}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon\xi)\right)\left({}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon\eta)\right)\left({}^{\nu}\big\|_{x(a)}^{x(b)} \exp\left(o\left(\epsilon^2\right)\right)\right) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon\xi) + \frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \exp(\epsilon\eta) + \underbrace{\frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(b)} \exp\left(o\left(\epsilon^2\right)\right)}_{0} \\
&= {}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\xi + {}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\eta.
\end{aligned}$$

This linear parallel transport naturally respects the adjoint representation of $\mathfrak{g}$.

Proposition 2.9. *Let $x : I \to X$. Then*

$$ {}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)} \mathrm{Ad}_g(\xi) = \mathrm{Ad}_{{}^{\nu}\|_{x(a)}^{x(b)} g}\left({}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\xi\right), \qquad g \in \mathcal{G}_{x(a)},\ \xi \in \mathfrak{g}_{x(a)}. $$

Proof. Consider the curves $\alpha(t) = \mathrm{Ad}_{{}^{\nu}\|_{x(a)}^{x(t)} g}\left({}^{\mathfrak{g}}\big\|_{x(a)}^{x(t)}\xi\right)$ and $\beta(t) = {}^{\mathfrak{g}}\big\|_{x(a)}^{x(t)}\mathrm{Ad}_g(\xi)$, $t \in I$, which satisfy $\alpha(a) = \beta(a) = \mathrm{Ad}_g(\xi)$ and $\pi_{\mathfrak{g},X} \circ \alpha = \pi_{\mathfrak{g},X} \circ \beta = x$. Thanks to the uniqueness of parallel transport, to show that $\alpha = \beta$ it is enough to check that $\alpha' = \beta'$. For each $t \in I$, using the definition of ${}^{\mathfrak{g}}\big\|$ we have

$$\begin{aligned}
\alpha'(t) &= \frac{d}{dt}\frac{d}{d\epsilon}\Big|_{\epsilon=0} c_{{}^{\nu}\|_{x(a)}^{x(t)} g}\left({}^{\nu}\big\|_{x(a)}^{x(t)} \exp(\epsilon\xi)\right) \\
&\overset{(\star)}{=} \frac{d}{dt}\frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(t)} c_g\left(\exp(\epsilon\xi)\right) \\
&\overset{(\dagger)}{=} \frac{d}{dt}\frac{d}{d\epsilon}\Big|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(t)} \exp\left((dc_g)_{1_{x(a)}}(\epsilon\xi)\right) \\
&= \beta'(t),
\end{aligned}$$

where $(\star)$ is due to ν being a Lie group connection and $(\dagger)$ comes from the fact that $c_g : \mathcal{G}_{x(a)} \to \mathcal{G}_{x(a)}$ is a Lie group homomorphism.

We denote by $\nabla^{\mathfrak{g}}$ and $\nabla^{\mathfrak{g}}/dt$ the linear connection and the covariant derivative on $\pi_{\mathfrak{g},X}$ corresponding to this parallel transport ${}^{\mathfrak{g}}\big\|$, respectively. A more practical way of regarding $\nabla^{\mathfrak{g}}$ may be the following. Consider the Lie group bundle connection as a map $\nu : T\mathcal{G} \to \mathfrak{g}$. Then, for any section $\xi \in \Gamma(\pi_{\mathfrak{g},X})$, we have

$$ \nabla^{\mathfrak{g}}\xi = \frac{d}{dt}\Big|_{t=0} \nu \circ d\exp(t\xi). \tag{2.11} $$

The next result is the infinitesimal version of Proposition 2.9.

Proposition 2.10. *Let $g : I \to \mathcal{G}$ and $\xi : I \to \mathfrak{g}$ be such that $\pi_{\mathcal{G},X} \circ g = \pi_{\mathfrak{g},X} \circ \xi = x$. Then for all $t \in I$ we have*

$$ \frac{\nabla^{\mathfrak{g}}\left(\mathrm{Ad}_g \circ \xi\right)(t)}{dt} = \mathrm{Ad}_{g(t)}\left(\frac{\nabla^{\mathfrak{g}}\xi(t)}{dt}\right) + \left[\left(dR_{g(t)^{-1}}\right)_{g(t)}\left(\frac{\nabla^{\nu} g(t)}{dt}\right), \mathrm{Ad}_{g(t)}(\xi(t))\right]_{\mathcal{G}}. $$

Proof. Fix $t \in I$ and denote $x = \pi_{\mathcal{G},X} \circ g$. Let $\alpha : (-\epsilon,\epsilon) \to \mathcal{G}_{x(t)}$ be the curve given by $\alpha(s) = {}^{\nu}\|_{x(t+s)}^{x(t)} g(t+s)$ for each $s \in (-\epsilon,\epsilon)$. We have $\alpha(0) = g(t)$ and $\alpha'(0) = \nabla^{\nu} g(t)/dt$. Denoting $\beta(s) = \alpha(s)\, g(t)^{-1} = R_{g(t)^{-1}}(\alpha(s))$, $s \in (-\epsilon,\epsilon)$, we have that $\beta(0) = 1_{x(t)}$ and $\beta'(0) = \left(dR_{g(t)^{-1}}\right)_{g(t)} (\nabla^{\nu} g(t)/dt)$. Hence,

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{\alpha(s)}(\xi(t)) &= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{\beta(s)}\left(\mathrm{Ad}_{g(t)}(\xi(t))\right) \\
&= \mathrm{ad}(\beta'(0))\left(\mathrm{Ad}_{g(t)}(\xi(t))\right) \\
&= \left[\left(dR_{g(t)^{-1}}\right)_{g(t)}\left(\frac{\nabla^{\nu} g(t)}{dt}\right), \mathrm{Ad}_{g(t)}(\xi(t))\right]_{\mathcal{G}}.
\end{aligned}$$

Using this, we conclude

$$\begin{aligned}
\frac{\nabla^{\mathfrak{g}}\left(\mathrm{Ad}_g \circ \xi\right)(t)}{dt} &= \left.\frac{d}{ds}\right|_{s=0} {}^{\nabla^{\mathfrak{g}}}\|_{x(t+s)}^{x(t)} \mathrm{Ad}_{g(t+s)}(\xi(t+s)) \\
&= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{{}^{\nu}\|_{x(t+s)}^{x(t)} g(t+s)}\left({}^{\nabla^{\mathfrak{g}}}\|_{x(t+s)}^{x(t)} \xi(t+s)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{g(t)}\left({}^{\nabla^{\mathfrak{g}}}\|_{x(t+s)}^{x(t)} \xi(t+s)\right) + \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}_{{}^{\nu}\|_{x(t+s)}^{x(t)} g(t+s)}(\xi(t)) \\
&= \mathrm{Ad}_{g(t)}\left(\frac{\nabla^{\mathfrak{g}} \xi(t)}{dt}\right) + \left[\left(dR_{g(t)^{-1}}\right)_{g(t)}\left(\frac{\nabla^{\nu} g(t)}{dt}\right), \mathrm{Ad}_{g(t)}(\xi(t))\right]_{\mathcal{G}}.
\end{aligned}$$

2.2.3 Generalized principal connections

Let $\pi_{Y,X} : Y \to X$ be a fiber bundle on which a Lie group bundle $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ acts freely and properly on the right, and denote by $\Phi : Y \times_X \mathcal{G} \to Y$ the fibered action.

Lemma 2.11. *For every $(y,g) \in Y \times_X \mathcal{G}$ and $\xi, \eta \in \mathfrak{g}_x$, $x = \pi_{\mathcal{G},X}(g)$, we have*

$$(d\Phi)_{(y,g)}\left(\xi_y^*, \eta_g^*\right) = \left(\mathrm{Ad}_{g^{-1}}(\xi + \eta)\right)_{y\cdot g}^*,$$

where $\eta_g^* = d/dt|_{t=0} \exp(t\eta) g$ is the infinitesimal generator of η at g.

Proof. Observe that

$$\begin{aligned}
\left(\mathrm{Ad}_{g^{-1}}(\eta)\right)_{y\cdot g}^* &= \left.\frac{d}{dt}\right|_{t=0} y \cdot g \exp\left(t\, \mathrm{Ad}_{g^{-1}}(\eta)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} y \cdot g \exp\left(t \left.\frac{d}{ds}\right|_{s=0} g^{-1} \exp(s\eta)\, g\right)\Big) \\
&= \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\eta)\, g \\
&= \left.\frac{d}{dt}\right|_{t=0} \Phi_y\left(\exp(t\eta)\, g\right).
\end{aligned}$$

The result is a straightforward computation using the previous equation and (2.8),

$$
\begin{aligned}
(d\Phi)_{(y,g)}(\xi^*_y, \eta^*_g) &= \left.\frac{d}{dt}\right|_{t=0} \Phi\left(y \cdot \exp(t\xi), \exp(t\eta)\, g\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \Phi\left(y \cdot \exp(t\xi), g\right) + \left.\frac{d}{dt}\right|_{t=0} \Phi\left(y, \exp(t\eta)\, g\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \Phi_g\left(y \cdot \exp(t\xi)\right) + \left.\frac{d}{dt}\right|_{t=0} \Phi_y\left(\exp(t\eta)\, g\right) \\
&= (d\Phi_g)_y(\xi^*_y) + \left(\mathrm{Ad}_{g^{-1}}(\eta)\right)^*_{y\cdot g} \\
&= \left(\mathrm{Ad}_{g^{-1}}(\xi)\right)^*_{y\cdot g} + \mathrm{Ad}_{g^{-1}}(\eta)^*_{y\cdot g} \\
&= \mathrm{Ad}_{g^{-1}}(\xi+\eta)^*_{y\cdot g}\,.
\end{aligned}
$$

Ehresmann connections on $\pi_{Y,Y/\mathcal{G}}$ are identified with forms as follows.

Proposition 2.12. *There is a bijective correspondence between Ehresmann connections on $\pi_{Y,Y/\mathcal{G}}$ and 1-forms*[1] *$\omega \in \Omega^1(Y, \mathfrak{g})$ such that*

$$\omega_y(\xi^*_y) = \xi, \qquad (y, \xi) \in Y \times_X \mathfrak{g}.$$

Furthermore, such forms satisfy

$$\omega_{y\cdot g}\left((d\Phi_g)_y(\xi^*_y)\right) = \mathrm{Ad}_{g^{-1}}\left(\omega_y(\xi^*_y)\right), \qquad (y, g, \xi) \in Y \times_X \mathcal{G} \times_X \mathfrak{g}.$$

Proof. The equivalence between Ehresmann connections on $\pi_{Y,Y/\mathcal{G}}$ and the 1-forms as in the statement is a consequence of the isomorphism (2.7).

For the second part, let $(y, g, \xi) \in Y \times_X \mathcal{G} \times_X \mathfrak{g}$, then:

$$\omega_{y\cdot g}\left((d\Phi_g)_y(\xi^*_y)\right) = \omega_{y\cdot g}\left((\mathrm{Ad}_{g^{-1}}\xi)^*_{y\cdot g}\right) = \mathrm{Ad}_{g^{-1}}(\xi) = \mathrm{Ad}_{g^{-1}}\left(\omega_y(\xi^*_y)\right),$$

where equation (2.8) has been taken into account.

Observe that $U^v_y = \omega_y(U_y)^*_y$ for each $U_y \in T_yY$, $y \in Y$, and, thus U_y is horizontal if and only if $\omega_y(U_y) = 0$. We are ready to introduce generalized principal connections.

Definition 2.4 *Let ν be an Ehresmann connection on the Lie group bundle $\pi_{\mathcal{G},X}$. A* generalized principal connection *on $\pi_{Y,Y/\mathcal{G}}$ associated to ν is an Ehresmann connection $H \subset TY$ on $\pi_{Y,Y/\mathcal{G}}$ satisfying:*

$$\left[(d\Phi)_{(y,g)}(U_y, U_g)\right]^v = (d\Phi_g)_y(U^v_y) + (dL_{g^{-1}})_g(\nu_g(U_g))^*_{y\cdot g},$$

for every $(y, g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, where $x = \pi_{Y,X}(y)$.

Thanks to Proposition 2.12, it is easy to check that the following is an equivalent way of defining a generalized principal connection.

Definition 2.5 *Let ν be an Ehresmann connection on the Lie group bundle $\pi_{\mathcal{G},X}$. A* generalized principal connection *on $\pi_{Y,Y/\mathcal{G}}$ associated to ν is a form $\omega \in \Omega^1(Y, \mathfrak{g})$ satisfying:*

(i) (Complementarity) $\omega_y(\xi^*_y) = \xi$ for every $(y, \xi) \in Y \times_X \mathfrak{g}$.

*(ii) (*Ad*-equivariance) For each $(y, g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $x = \pi_{Y,X}(x)$, then*

$$\omega_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y, U_g)\right) = \mathrm{Ad}_{g^{-1}}\left(\omega_y(U_y) + \nu_g(U_g)\right).$$

[1] In fact, we are considering forms with values in $(\pi_{Y,X})^*(\mathfrak{g}) = Y \times_X \mathfrak{g}$ but, for the sake of simplicity, we do not write it. In the same fashion, abusing the notation we suppose that $\omega_y \in T^*_yY \otimes \mathfrak{g}_x$ for each $y \in Y$, $x = \pi_{Y,X}(y)$.

Roughly speaking, generalized principal connections extend the property given in Lemma 2.11 and Proposition 2.12 to non-necessarily vertical vectors U_g with respect to ν.

The next result gives a geometric interpretation of the above Definitions in terms of the parallel transports ${}^{\nu}\|$ and ${}^{\omega}\|$, in the same vein as Proposition 2.5.

Proposition 2.13. *Let $\nu : T\mathcal{G} \to \mathfrak{g}$ and $\omega \in \Omega^1(Y, \mathfrak{g})$ be Ehresmann connections on $\pi_{\mathcal{G},X}$ and $\pi_{Y,Y/\mathcal{G}}$, respectively. Then ω is a generalized principal connection associated to ν if and only if for any curve $\gamma : I \to Y/\mathcal{G}$, the corresponding parallel transports satisfy*

$$ {}^{\omega}\|_{\gamma(a)}^{\gamma(t)}(y \cdot g) = \left({}^{\omega}\|_{\gamma(a)}^{\gamma(t)} y\right) \cdot \left({}^{\nu}\|_{x(a)}^{x(t)} g\right), \qquad g \in \mathcal{G}_{x(a)},\ y \in Y_{\gamma(a)},\ t \in I, \tag{2.12} $$

where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$.

Proof. Suppose that ω is a generalized principal connection associated to ν. Let $\gamma : I \to Y/\mathcal{G}$ be a curve, $x = \pi_{Y/\mathcal{G},X} \circ \gamma$, $y \in Y_{\gamma(a)}$ and $g \in \mathcal{G}_{x(a)}$. Denote $\alpha_1(\cdot) = {}^{\omega}\|_{\gamma(a)}^{\gamma(\cdot)} y$, $\alpha_2(\cdot) = {}^{\nu}\|_{x(a)}^{x(\cdot)} g$ and $\alpha = \Phi \circ (\alpha_1, \alpha_2)$. To show that $\alpha(\cdot) = {}^{\omega}\|_{\gamma(a)}^{\gamma(\cdot)}(y \cdot g)$ we use the uniqueness of the parallel transport. It is clear that $\pi_{Y,Y/\mathcal{G}} \circ \alpha = \gamma$ and $\alpha(a) = y \cdot g$, so we only need to check that α is horizontal. For each $t \in I$ we have

$$ \begin{aligned} \alpha'(t)^v &= (d\Phi)_{(\alpha_1(t),\alpha_2(t))}\left(\alpha_1'(t), \alpha_2'(t)\right)^v \\ &= \left(d\Phi_{\alpha_2(t)}\right)_{\alpha_1(t)}\left(\alpha_1'(t)^v\right) + \left(dL_{\alpha_2(t)^{-1}}\right)_{\alpha_2(t)}\left(\nu_{\alpha_2(t)}\left(\alpha_2'(t)\right)\right)^*_{\alpha(t)} \ = \ 0, \end{aligned} $$

since α_1 and α_2 are horizontal for the corresponding connections.

Conversely, suppose that ω and ν satisfy (2.12). Let $(y, g) \in Y \times_X \mathcal{G}$ and $(U_y, U_g) \in T_yY \times_{T_xX} T_g$. We consider $\alpha = (\alpha_1, \alpha_2) : (-\epsilon, \epsilon) \to Y \times_X \mathcal{G}$ such that $\alpha'(0) = (\alpha_1'(0), \alpha_2'(0)) = (U_y, U_g)$ and we denote $\gamma = \pi_{Y,Y/\mathcal{G}} \circ \alpha_1$ and $x = \pi_{Y,X} \circ \alpha_1 = \pi_{\mathcal{G},X} \circ \alpha_2$. A straightforward consequence of (2.12) and the definition of covariant derivative is the following:

$$ \begin{aligned} (\Phi \circ \alpha)'(0)^v &= \left.\frac{D^{\omega}(\Phi \circ \alpha)}{Dt}\right|_{t=0} \\ &= (d\Phi_g)_y\left(\left.\frac{D^{\omega}\alpha_1}{Dt}\right|_{t=0}\right) + (d\Phi_y)_g\left(\left.\frac{D^{\nu}\alpha_2}{Dt}\right|_{t=0}\right) \\ &= (d\Phi_g)_y\left(U_y^v\right) + (d\Phi_y)_g\left(U_g^v\right). \end{aligned} $$

Furthermore, note that $(d\Phi_y)_g\left(U_g^v\right) = (d\Phi_{y\cdot g})_{1_x}\left((dL_{g^{-1}})_g\left(U_g^v\right)\right) = (dL_{g^{-1}})_g\left(U_g^v\right)^*_{y\cdot g}$. Thus, we conclude that ω is a generalized principal connection associated to ν:

$$ (d\Phi)_{(y,g)}\left(U_y, U_g\right)^v = (\Phi \circ \alpha)'(0)^v = (d\Phi_g)_y\left(U_y^v\right) + (dL_{g^{-1}})_g\left(U_g^v\right)^*_{y\cdot g}. $$

Proposition 2.14. *Let $\nu : T\mathcal{G} \to \mathfrak{g}$ and $\omega \in \Omega^1(Y, \mathfrak{g})$ be Ehresmann connections on $\pi_{\mathcal{G},X}$ and $\pi_{Y,Y/\mathcal{G}}$, respectively, and suppose that $Y/\mathcal{G} = X$. Let $\hat{\omega} \in \Gamma(\pi_{J^1Y,Y})$ and $\hat{\nu} \in \Gamma(\pi_{J^1\mathcal{G},G})$ be the corresponding jet sections. Then ω is a generalized principal connection associated to ν if and only if*

$$ \hat{\omega}(y \cdot g) = \hat{\omega}(y) \cdot \hat{\nu}(g), \qquad (y, g) \in Y \times_X \mathfrak{g}, \tag{2.13} $$

where the fibered action of $\pi_{J^1\mathcal{G},X}$ on $\pi_{J^1Y,X}$ is given by the first jet extension of Φ (recall Example 2.1).

Proof. Thanks to Proposition 2.13 we only need to show that (2.13) is equivalent to (2.12). Let $(y, g) \in Y \times_X \mathcal{G}$, $x = \pi_{Y,X}(y)$ and $U_x \in T_xX$, and pick a curve $\alpha : (-\epsilon, \epsilon) \to X$ with $\alpha'(0) = U_x$. We define the curves

$$\beta_y(t) = {}^{\omega}\big\|_{\alpha(0)}^{\alpha(t)} y, \quad \beta_{y\cdot g}(t) = {}^{\omega}\big\|_{\alpha(0)}^{\alpha(t)} (y\cdot g), \quad \beta_y(t) = {}^{\nu}\big\|_{\alpha(0)}^{\alpha(t)} g,$$

This way, we have $\beta_g'(0) = Hor_g^{\nu}(U_x) = \hat{\nu}(g)(U_x)$ and $\beta_y'(0) = Hor_y^{\omega}(U_x) = \hat{\omega}(y)(U_x)$, and analogous for $\beta_{y\cdot g}$.

Suppose that (2.12) is satisfied. Then $\beta_{y\cdot g} = \Phi \circ (\beta_y, \beta_g)$. Hence

$$\begin{aligned}\hat{\omega}(y\cdot g)(U_x) &= \beta_{y\cdot g}'(0)\\ &= (d\Phi)_{(y,g)}\left(\beta_y'(0), \beta_g'(0)\right)\\ &= (d\Phi)_{(y,g)}\left(\hat{\omega}(y)(U_x), \hat{\nu}(g)(U_x)\right).\end{aligned}$$

This equation is valid for every $U_x \in T_xX$, whence $\hat{\omega}(y\cdot g) = \hat{\omega}(y)\cdot\hat{\nu}(g)$.

Conversely, if (2.13) is satisfied, then $\beta_{y\cdot g}(0) = y\cdot g = \beta_y(0)\cdot\beta_g(0) = (\Phi\circ(\beta_y,\beta_g))(0)$ and

$$\begin{aligned}\beta_{y\cdot g}'(0) &= \hat{\omega}(y\cdot g)(U_x)\\ &= \left(\hat{\omega}(y)\cdot\hat{\nu}(g)\right)(U_x)\\ &= (d\Phi)_{(y,g)}\left(\hat{\omega}(y)(U_x), \hat{\nu}(g)(U_x)\right)\\ &= (d\Phi)_{(y,g)}\left(\beta_y'(0), \beta_g'(0)\right)\\ &= (\Phi\circ(\beta_y,\beta_g))'(0).\end{aligned}$$

By uniqueness of the parallel transport we conclude that $\beta_{y\cdot g} = \Phi\circ(\beta_y,\beta_g)$.

We denote by $Hor_y^{\omega} : T_{[y]_{\mathcal{G}}}(Y/\mathcal{G}) \to T_yY$ the *horizontal lift* given by ω at $y \in Y$. A different geometric interpretation of generalized principal connections is the following.

Proposition 2.15. *Let $(y,g) \in Y \times_X \mathcal{G}$ and $(U_{[y]_{\mathcal{G}}}, U_g) \in T_{[y]_{\mathcal{G}}}(Y/\mathcal{G}) \times_{T_xX} T_g\mathcal{G}$, where $x = \pi_{Y,X}(y)$. Then*

$$(d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right) = Hor_{y\cdot g}^{\omega}(U_{[y]_{\mathcal{G}}}) + (dL_{g^{-1}})_g(\nu_g(U_g))_{y\cdot g}^{*}.$$

Proof. We just need to prove that $U = (d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right) - (dL_{g^{-1}})_g(\nu_g(U_g))_{y\cdot g}^{*}$ is horizontal and projects to $U_{[y]_{\mathcal{G}}}$. With respect to the former

$$\begin{aligned}U^v &= (d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right)^v - (dL_{g^{-1}})_g(\nu_g(U_g))_{y\cdot g}^{*}\\ &= (d\Phi_g)_y\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}})^v\right) \;=\; 0.\end{aligned}$$

For the latter let $\alpha = (\alpha_1,\alpha_2) : (-\epsilon,\epsilon) \to Y\times_X\mathcal{G}$ be such that $\alpha'(0) = \left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right)$. We then have

$$\begin{aligned}(d\pi_{Y,Y/\mathcal{G}})_{y\cdot g}(U) &= (d\pi_{Y,Y/\mathcal{G}})_{y\cdot g}\left((d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right)\right)\\ &= d(\pi_{Y,Y/\mathcal{G}}\circ\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}}), U_g\right)\\ &= \left.\frac{d}{dt}\right|_{t=0}(\pi_{Y,Y/\mathcal{G}}\circ\Phi\circ\alpha)(t)\\ &= \left.\frac{d}{dt}\right|_{t=0}(\pi_{Y,Y/\mathcal{G}}\circ\alpha_1)(t)\\ &= (d\pi_{Y,Y/\mathcal{G}})_y\left(Hor_y^{\omega}(U_{[y]_{\mathcal{G}}})\right)\\ &= U_{[y]_{\mathcal{G}}}.\end{aligned}$$

Theorem 2.16 (Existence of generalized principal connections). *If X is a paracompact smooth manifold, then there exists an Ehresmann connection ν on $\pi_{\mathcal{G},X}$ and a generalized principal connection ω on $\pi_{Y,Y/\mathcal{G}}$ associated to ν.*

Proof. Let $\{(\mathcal{U}_\alpha, \psi^\alpha_{\mathcal{G}}) \mid \alpha \in \Lambda\}$ be a trivializing atlas for $\pi_{\mathcal{G},X}$, $\{(\mathcal{V}_\alpha = \pi^{-1}_{Y/\mathcal{G},X}(\mathcal{U}_\alpha), \psi^\alpha_Y) \mid \alpha \in \Lambda\}$ be the induced trivializing atlas for $\pi_{Y,Y/\mathcal{G}}$, as in (2.2), and $\{(\mathcal{U}_\alpha, \psi^\alpha_{\mathfrak{g}}) \mid \alpha \in \Lambda\}$ be the induced trivializing atlas for $\pi_{\mathfrak{g},X}$, as in (2.5). For $\hat{g} \in G$, we denote by $\hat{L}_{\hat{g}} : G \to G$ and $\hat{R}_{\hat{g}} : G \to G$ the left and right multiplication by $\hat{g}$, respectively. In the same way, we denote by $\widehat{Ad}$ the adjoint representation of G.

Fixed $\alpha \in \Lambda$, we define the local Ehresmann connection $\nu_\alpha : T\mathcal{G}|_{\mathcal{U}_\alpha} \to V\mathcal{G}|_{\mathcal{U}_\alpha}$ on $\pi_{\mathcal{G},X}|_{\mathcal{U}_\alpha} : \mathcal{G}|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha$ as follows

$$(\nu_\alpha)_g \left((d\psi^\alpha_{\mathcal{G}})^{-1}_g \left(U_x, \left(d\hat{L}_{\hat{g}} \right)_1 (\hat{\eta}) \right) \right) = (d\psi^\alpha_{\mathcal{G}})^{-1}_g \left(0_x, \left(d\hat{L}_{\hat{g}} \right)_1 (\hat{\eta}) \right), \qquad U_x \in T_xX,\ \hat{\eta} \in \mathfrak{g},$$

where $x \in \mathcal{U}_\alpha$, $\hat{g} \in G$ and $g = (\psi^\alpha_{\mathcal{G}})^{-1}(x, \hat{g}) \in \mathcal{G}|_{\mathcal{U}_\alpha}$. Likewise, we define the local 1-form $\omega_\alpha \in \Omega^1 \left(Y|_{\mathcal{V}_\alpha}, \mathfrak{g}|_{\mathcal{U}_\alpha} \right)$ as

$$(\omega_\alpha)_y \left((d\psi^\alpha_Y)^{-1}_y \left(U_{[y]_\mathcal{G}}, \left(d\hat{L}_{\hat{h}} \right)_1 (\hat{\xi}) \right) \right) = (\psi^\alpha_{\mathfrak{g}})^{-1} \left(x, \hat{\xi} \right), \qquad U_{[y]_\mathcal{G}} \in T_{[y]_\mathcal{G}}(Y/\mathcal{G}),\ \hat{\xi} \in \mathfrak{g},$$

where $[y]_\mathcal{G} \in \mathcal{V}_\alpha$, $\hat{h} \in G$ and $y = (\psi^\alpha_Y)^{-1} \left([y]_\mathcal{G}, \hat{h} \right) \in Y|_{\mathcal{V}_\alpha}$. We show that ν_α and ω_α satisfy both properties of Definition 2.5:

(i) (Complementarity) Let $\xi = (\psi^\alpha_{\mathfrak{g}})^{-1} \left(x, \hat{\xi} \right) \in \mathfrak{g}|_{\mathcal{U}_\alpha}$ and $y = (\psi^\alpha_Y)^{-1} \left([y]_\mathcal{G}, \hat{h} \right) \in Y|_{\mathcal{V}_\alpha}$. Using Corollary 2.2 and equation (2.4) we get

$$\xi^*_y = (d\psi^\alpha_Y)^{-1}_y \left(0_{[y]_\mathcal{G}}, \left(d\hat{L}_{\hat{h}} \right)_1 (\hat{\xi}) \right). \tag{2.14}$$

Thence,

$$(\omega_\alpha)_y \left(\xi^*_y \right) = (\omega_\alpha)_y \left((d\psi^\alpha_Y)^{-1}_y \left(0_{[y]_\mathcal{G}}, \left(d\hat{L}_{\hat{h}} \right)_1 (\hat{\xi}) \right) \right) = (\psi^\alpha_{\mathfrak{g}})^{-1} \left(x, \hat{\xi} \right) = \xi.$$

(ii) (Ad-equivariance) Let $\hat{h}, \hat{g} \in G$ and $[y]_\mathcal{G} \in \mathcal{V}_\alpha$, and consider $y = (\psi^\alpha_Y)^{-1} \left([y]_\mathcal{G}, \hat{h} \right)$ and $g = (\psi^\alpha_{\mathcal{G}})^{-1}(x, \hat{g})$, where $x = \pi_{Y/\mathcal{G},X}([y]_\mathcal{G}) \in \mathcal{U}_\alpha$. Likewise, let $\hat{\xi}, \hat{\eta} \in \mathfrak{g}$ and $U_{[y]_\mathcal{G}} \in T_{[y]_\mathcal{G}}(Y/\mathcal{G})$, and consider

$$U_y = (d\psi^\alpha_Y)^{-1}_y \left(U_{[y]_\mathcal{G}}, \left(d\hat{L}_{\hat{h}} \right)_1 (\hat{\xi}) \right) \in T_yY, \qquad U_g = (d\psi^\alpha_{\mathcal{G}})^{-1}_g \left(U_x, \left(d\hat{L}_{\hat{g}} \right)_1 (\hat{\eta}) \right) \in T_g\mathcal{G},$$

where $U_x = (d\pi_{Y/\mathcal{G},X})_{[y]_\mathcal{G}} \left(U_{[y]_\mathcal{G}} \right) \in T_xX$. Consider curves $\gamma : (-\epsilon, \epsilon) \to Y|_{\mathcal{V}_\alpha}$ and $\beta : (-\epsilon, \epsilon) \to \mathcal{G}|_{\mathcal{U}_\alpha}$ such that $\gamma'(0) = U_y$ and $\beta'(0) = U_g$. Then we have that $\psi^\alpha_Y \circ \gamma = (\gamma_1, \gamma_2)$ and $\psi^\alpha_{\mathcal{G}} \circ \beta = (\beta_1, \beta_2)$ for certain curves $\gamma_1 : (-\epsilon, \epsilon) \to \mathcal{V}_\alpha$, $\gamma_2 : (-\epsilon, \epsilon) \to G$, $\beta_1 : (-\epsilon, \epsilon) \to \mathcal{U}_\alpha$ and $\beta_2 : (-\epsilon, \epsilon) \to G$ such that $\gamma_1'(0) = U_{[y]_\mathcal{G}}$, $\gamma_2'(0) = \left(d\hat{L}_{\hat{h}} \right)_1 (\hat{\xi})$, $\beta_1'(0) = U_x$ and $\beta_2'(0) = \left(d\hat{L}_{\hat{g}} \right)_1 (\hat{\eta})$. Note that we can always choose the curves satisfying $\pi_{Y/\mathcal{G},X} \circ \gamma_1 = \beta_1$. Using these curves, Corollary 2.2 and Equation (2.14) it can be seen that

$$\begin{aligned} (d\Phi)_{(y,g)} (U_y, U_g) &= \left. \frac{d}{dt} \right|_{t=0} \Phi(\alpha(t), \beta(t)) \\ &= (d\psi^\alpha_Y)^{-1}_{y \cdot g} \left(U_{[y]_\mathcal{G}}, \left(d\hat{L}_{\hat{h}\hat{g}} \right)_1 \left(\widehat{Ad}_{\hat{g}^{-1}} (\hat{\xi}) \right) \right) + (\psi^\alpha_{\mathfrak{g}})^{-1} (x, \hat{\eta})^*_{y \cdot g}. \end{aligned}$$

This, together with property (i), lead to

$$\begin{aligned} (\omega_\alpha)_{y \cdot g} \left((d\Phi)_{y \cdot g} (U_y, U_g) \right) &= (\psi^\alpha_{\mathfrak{g}})^{-1} \left(x, \widehat{Ad}_{\hat{g}^{-1}} (\hat{\xi}) \right) + (\psi^\alpha_{\mathfrak{g}})^{-1} (x, \hat{\eta}) \\ &= \mathrm{Ad}_{g^{-1}} \left((\psi^\alpha_{\mathfrak{g}})^{-1} \left(x, \hat{\xi} \right) \right) + (d\psi^\alpha_{\mathcal{G}})^{-1}_{1_x} (0_x, \hat{\eta}) \\ &= \mathrm{Ad}_{g^{-1}} \left((\omega_\alpha)_y (U_y) \right) + \mathrm{Ad}_{g^{-1}} \left(\left(dR_{g^{-1}} \right)_g \left((\nu_\alpha)_g (U_g) \right) \right) \\ &= \mathrm{Ad}_{g^{-1}} \left((\omega_\alpha)_y (U_y) + (\nu_\alpha)_g (U_g) \right), \end{aligned}$$

where we have used (2.5) and the fact that $(d\psi^\alpha_{\mathcal{G}})^{-1}_{1_x} (0_x, \hat{\eta}) = (d\psi^\alpha_{\mathcal{G}}|_{\mathcal{G}_x})^{-1}_{1_x} (x, \hat{\eta})$.

Therefore, ω_α is a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}|_{\mathcal{V}_\alpha} : Y|_{\mathcal{V}_\alpha} \to \mathcal{V}_\alpha$ associated to ν_α. At last, we take a smooth partition of unity $\{\theta_\alpha \mid \alpha \in \Lambda\}$ on X subordinated to $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$. It is easy to check that $\{\Theta_\alpha = \theta_\alpha \circ \pi_{Y/\mathcal{G},X} \mid \alpha \in \Lambda\}$ is a smooth partition of unity on $Y/\mathcal{G}$ subordinated to $\{\mathcal{V}_\alpha \mid \alpha \in \Lambda\}$. Now set $\omega = \sum_{\alpha\in\Lambda}(\Theta_\alpha \circ \pi_{Y,Y/\mathcal{G}})\,\omega_\alpha$ and $\nu = \sum_{\alpha\in\Lambda}(\theta_\alpha \circ \pi_{\mathcal{G},X})\,\nu_\alpha$. They are a well defined 1-form $\omega \in \Omega^1(Y,\mathfrak{g})$ and a well defined Ehresmann connection $\nu : T\mathcal{G} \to V\mathcal{G}$. It is straightforward that ν and ω satisfy both properties of Definition 2.5. Thus, ω is a generalized principal connection associated to ν.

A 1-form $\alpha \in \Omega^1(Y,\mathfrak{g})$ is said to be *tensorial of the adjoint type* if it is *horizontal*, i.e., $\alpha_y(U_y) = 0$ for each $U_y \in V_yY = \ker(d\pi_{Y,Y/\mathcal{G}})_y$, $y \in Y$, and Ad-*equivariant*, i.e.,

$$\alpha_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y,U_g)\right) = \mathrm{Ad}_{g^{-1}}\left(\alpha_y(U_y)\right),$$

for each $(U_y,U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $(y,g) \in Y \times_X \mathcal{G}$, $x = \pi_{Y,X}(y)$. The family of tensorial 1-forms of the adjoint type is denoted by $\overline{\Omega}^1(Y,\mathfrak{g})$. These forms can be reduced to $Y/\mathcal{G}$ in the sense that we have a bijection

$$\overline{\Omega}^1(Y,\mathfrak{g}) \to \Omega^1(Y/\mathcal{G},\tilde{\mathfrak{g}}), \quad \alpha \mapsto \tilde{\alpha},$$

given by $\tilde{\alpha}_{[y]_\mathcal{G}}(U_{[y]_\mathcal{G}}) = [y,\alpha_y(U_y)]_\mathcal{G}$, where $U_y \in T_yY$ projects to $U_{[y]_\mathcal{G}} \in T_{[y]_\mathcal{G}}(Y/\mathcal{G})$, $y \in Y$. By an abuse of notation, we identify $\alpha \equiv \tilde{\alpha}$. This construction can be straightforwardly generalized to k-forms.

Proposition 2.17. *Let $\nu : T\mathcal{G} \to \mathfrak{g}$ be an Ehresmann connection on $\pi_{\mathcal{G},X}$. The family of generalized principal connections on $\pi_{Y,Y/\mathcal{G}}$ associated to ν is an affine space modelled on $\overline{\Omega}^1(Y,\mathfrak{g})$.*

Proof. On the one hand, if $\omega_1,\omega_2 \in \Omega^1(Y,\mathfrak{g})$ are generalized principal connections on $\pi_{Y,Y/\mathcal{G}}$ associated to ν, then $\omega_1 - \omega_2 \in \overline{\Omega}^1(Y,\mathfrak{g})$. The Ad-equivariance comes from the Ad-equivariance of ω_1 and ω_2 and the fact that the adjoint map is linear. For the horizontality, let $y \in Y$ and $V_y \in V_yY = \ker\left(d\pi_{Y,Y/\mathcal{G}}\right)_y$, we have

$$(\omega_1-\omega_2)_y(V_y)^*_y = (\omega_1)_y(V_y)^*_y - (\omega_2)_y(V_y)^*_y = V_y - V_y = 0.$$

As the map (2.7) is an isomorphism, we deduce that $(\omega_1-\omega_2)_y(V_y) = 0$.

On the other hand, given a generalized principal connection $\omega \in \Omega^1(Y,\mathfrak{g})$ associated to ν and a 1-form $\overline{\omega} \in \overline{\Omega}^1(Y,\mathfrak{g})$, a straightforward computation shows that $\omega + \overline{\omega}$ is a generalized principal connection, since $\overline{\omega}$ vanish on vertical vectors.

Now we give another interpretation of a generalized principal connection as a connection on $\pi_{Y\times_X\mathcal{G},X}$ equivariant under the fibered action.

Proposition 2.18. *Let $\omega \in \Omega^1(Y,\mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to an Ehresmann connection $\nu : T\mathcal{G} \to V\mathcal{G}$ on $\pi_{\mathcal{G},X}$. Then the map*

$$\varpi : T(Y\times_X\mathcal{G}) \to V(Y\times_X\mathcal{G}), \quad (U_y,U_g) \mapsto \left(\omega_y(U_y)^*_y, \nu_g(U_g)\right),$$

where $V(Y\times_X\mathcal{G}) = \ker\left(\pi_{Y\times_X\mathcal{G},X}\right)_*$, is an Ehresmann connection on $\pi_{Y\times_X\mathcal{G},X}$ equivariant under the map $\mathbf{\Phi} : Y\times_X\mathcal{G} \to Y\times_X\mathcal{G}$ defined as $\mathbf{\Phi}(y,g) = (y\cdot g, g)$, i.e.,

$$(d\mathbf{\Phi})_{(y,g)}\left(\varpi_{(y,g)}(U_y,U_g)\right) = \varpi_{\mathbf{\Phi}(y,g)}\left((d\mathbf{\Phi})_{y,g}(U_y,U_g)\right),$$

for each $(U_y,U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$ and $(y,g) \in Y\times_X\mathcal{G}$.

Proof. To begin with, observe that ϖ is well defined, i.e., $\varpi_{(y,g)}(U_y, U_g) \in V_{(y,g)}(Y \times_X \mathcal{G})$ for every $(U_y, U_g) \in T_yY \times_{T_xX} T_g\mathcal{G}$, $(y,g) \in Y \times_X \mathcal{G}$. This is a straightforward consequence of equality $V_{(y,g)}(Y \times_X \mathcal{G}) = V_yY \times V_g\mathcal{G}$, where $V_yY = \ker(\pi_{Y,X})_*$, and the fact that infinitesimal generators are $\pi_{Y,X}$-vertical. Likewise, it is clear that ϖ is a vertical vector bundle morphism over $Y \times_X \mathcal{G}$, since ω_y, ν_g and the infinitesimal generator are linear maps. To conclude, let us check the $\mathbf{\Phi}$-equivariance. Note that $(d\mathbf{\Phi})_{(y,g)}(U_y, U_g) = \big((d\Phi)_{(y,g)}(U_y, U_g), U_g\big)$. Hence,

$$\begin{aligned}
(d\mathbf{\Phi})_{(y,g)}\left(\varpi_{(y,g)}(U_y, U_g)\right) &= (d\mathbf{\Phi})_{(y,g)}\left(\omega_y(U_y)^*_y, \nu_g(U_g)\right) \\
&= \left((d\Phi)_{(y,g)}\left(\omega_y(U_y)^*_y, \nu_g(U_g)\right), \nu_g(U_g)\right) \\
&= \left(\mathrm{Ad}_{g^{-1}}\left(\omega_y(U_y) + \nu_g(U_g)\right)^*_{\Phi(y,g)}, \nu_g(U_g)\right) \\
&= \left(\omega_{\Phi(y,g)}\left((d\Phi)_{(y,g)}(U_y, U_g)\right)^*_{\Phi(y,g)}, \nu_g(U_g)\right) \\
&= \varpi_{(\Phi(y,g),g)}\left((d\Phi)_{(y,g)}(U_y, U_g), U_g\right) \\
&= \varpi_{\mathbf{\Phi}(y,g)}\left((d\mathbf{\Phi})_{(y,g)}(U_y, U_g)\right).
\end{aligned}$$

So far, we have considered generalized principal connections associated to arbitrary Ehresmann connections on $\mathcal{G} \to X$. We now prove that this Ehresmann connections must be a Lie group bundle connection, that is, they must respect the algebraic structure of $\pi_{\mathcal{G},X}$.

Proposition 2.19. *Let $\omega \in \Omega^1(Y, \mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to an Ehresmann connection $\nu : T\mathcal{G} \to \mathfrak{g}$ on $\pi_{\mathcal{G},X}$. Then ν is a Lie group bundle connection.*

Proof. Let $\gamma : I \to Y/\mathcal{G}$ projecting onto $x : I \to X$. Thanks to (2.12) for each $y \in Y_{\gamma(a)}$ and $g, h \in \mathcal{G}_{x(a)}$ we have

$$\begin{aligned}
\left({}^{\omega}\big\|^{\gamma(b)}_{\gamma(a)} y\right) \cdot \left({}^{\nu}\big\|^{x(b)}_{x(a)} g\right)\left({}^{\nu}\big\|^{x(b)}_{x(a)} h\right) &= \left({}^{\omega}\big\|^{\gamma(b)}_{\gamma(a)}(y \cdot g)\right) \cdot \left({}^{\nu}\big\|^{x(b)}_{x(a)} h\right) \\
&= {}^{\omega}\big\|^{\gamma(b)}_{\gamma(a)}(y \cdot gh) \\
&= \left({}^{\omega}\big\|^{\gamma(b)}_{\gamma(a)} y\right) \cdot \left({}^{\nu}\big\|^{x(b)}_{x(a)}(gh)\right).
\end{aligned}$$

Since the action is free, we conclude that property (2.9) holds.

On the other hand, applying (2.12) with $g = 1_{x(a)}$ and using again the fact that the action is free, we get

$${}^{\nu}\big\|^{x(t)}_{x(a)} 1_{x(a)} = 1_{x(b)}.$$

This gives that $\ker \nu_{1_x} = (d1)_x(T_xX)$ for each $x \in X$ and we conclude thanks to Proposition 2.5. Indeed, let $U_x \in \ker \nu_{1_x} = H_{1_x}$ and denote $u_x = (d\pi_{\mathcal{G},X})_{1_x}(U_x) \in T_xX$. Let $x : (-\epsilon, \epsilon) \to X$ be such that $x'(0) = u_x$. Then we have that

$$U_x = \frac{d}{dt}\Big|_{t=0} {}^{\nu}\big\|^{x(t)}_{x(0)} 1_x = \frac{d}{dt}\Big|_{t=0} 1_{x(t)} = (d1)_x(u_x).$$

Therefore, $\ker \nu_{1_x} \subset (d1)_x(T_xX)$. The other inclusion is due to the fact that both are vector spaces of the same dimension.

2.2.4 Curvature

The *curvature* ω as an Ehresmann connection (see, for example [56, §9.4]) is the VY-valued 2-form $\Omega \in \Omega^2(Y, VY)$ defined as

$$\Omega(U_1,U_2) = -\left[U_1 - \omega(U_1)^*, U_2 - \omega(U_2)^*\right], \qquad U_1,U_2 \in \mathfrak{X}(Y).$$

This is equivalent, by the identification (2.7), to the $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(Y,\mathfrak{g})$

$$\Omega(U_1,U_2) = -\omega\left(\left[U_1 - \omega(U_1)^*, U_2 - \omega(U_2)^*\right]\right),$$

which will be denoted the same.

The linear connection $\nabla^{\mathfrak{g}}$ induced on $\pi_{\mathfrak{g},X}$ by ν enables us to express the curvature as follows.

Proposition 2.20. *Let* $\mathrm{d}^{\mathfrak{g}}$ *be the exterior covariant derivative*[2] *associated to* $\nabla^{\mathfrak{g}}$. *Then*[3]

$$\Omega\left(U_1,U_2\right) = \mathrm{d}^{\mathfrak{g}}\omega\left(U_1^h, U_2^h\right), \qquad U_1,U_2 \in \mathfrak{X}(Y).$$

As in the case of (standard) principal connections, it is possible to regard the curvature as a 2-form on the base space $Y/\mathcal{G}$ with values in $\tilde{\mathfrak{g}}$.

Definition 2.6 *The* reduced curvature *of* ω *is the 2-form* $\tilde{\Omega} \in \Omega^2\left(Y/\mathcal{G},\tilde{\mathfrak{g}}\right)$ *taking values in the adjoint bundle given by*

$$\tilde{\Omega}_{[y]_{\mathcal{G}}}\left(U_1,U_2\right) = \left[y, \Omega_y\left(Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right)\right]_{\mathcal{G}}$$

for each $[y]_{\mathcal{G}} \in Y/\mathcal{G}$ *and* $U_1,U_2 \in T_{[y]_{\mathcal{G}}}(Y/\mathcal{G})$, *where* $y \in Y$ *is such that* $\pi_{Y,Y/\mathcal{G}}(y) = [y]_{\mathcal{G}}$.

The reduced curvature is well-defined, i.e., it does not depend on the choice of $y \in Y$. Indeed, let $g \in \mathcal{G}_x$, where $x = \pi_{Y,X}(y)$, $u_i = (d\pi_{Y/\mathcal{G},X})_{[y]_{\mathcal{G}}}(U_i) \in T_xX$ for $i = 1,2$ and $\gamma \in \Gamma(\pi_{\mathcal{G},X})$ be such that $\gamma(x) = g$ and $\nu_{\gamma(x)} \circ (d\gamma)_x = 0$. Proposition 2.15 gives

$$Hor_{y\cdot g}^{\omega}(U_i) = (d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_i), (d\gamma)_x(u_i)\right), \qquad i = 1,2.$$

Hence, we have

$$\begin{aligned}
\left[y\cdot g, \Omega_{y\cdot g}\left(Hor_{y\cdot g}^{\omega}(U_1), Hor_{y\cdot g}^{\omega}(U_2)\right)\right]_{\mathcal{G}} &= \left[y\cdot g, -\omega_{y\cdot g}\left(\left[Hor_{y\cdot g}^{\omega}(U_1), Hor_{y\cdot g}^{\omega}(U_2)\right]\right)\right]_{\mathcal{G}} \\
&= \left[y\cdot g, -\omega_{y\cdot g}\left(\left[(d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_1),(d\gamma)_x(u_1)\right), (d\Phi)_{(y,g)}\left(Hor_y^{\omega}(U_2),(d\gamma)_x(u_2)\right)\right]\right)\right]_{\mathcal{G}} \\
&\overset{(\star)}{=} \left[y\cdot g, -\omega_{y\cdot g}\left((d\Phi)_{(y,g)}\left(\left[Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right], \left[(d\gamma)_x(u_1),(d\gamma)_x(u_2)\right]\right)\right)\right]_{\mathcal{G}} \\
&= \left[y\cdot g, -\omega_{y\cdot g}\left((d\Phi)_{(y,g)}\left(\left[Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right], (d\gamma)_x\left([u_1,u_2]\right)\right)\right)\right]_{\mathcal{G}} \\
&= \left[y\cdot g, -\mathrm{Ad}_{g^{-1}}\left(\omega_y\left(\left[Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right]\right)\right)\right]_{\mathcal{G}} \\
&= \left[y, -\omega_y\left(\left[Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right]\right)\right]_{\mathcal{G}} \\
&= \left[y, \Omega_y\left(Hor_y^{\omega}(U_1), Hor_y^{\omega}(U_2)\right)\right]_{\mathcal{G}},
\end{aligned} \tag{2.15}$$

where we have used that $\left[(d\gamma)_x(u_1),(d\gamma)_x(u_2)\right] = (d\gamma)_x\left([u_1,u_2]\right)$.

[2] The *exterior covariant derivative* of a linear connection ∇^E on a vector bundle $\pi_{E,X}$ is an operator on the family of E-valued forms on X, $\mathrm{d}^E : \Omega^k(X,E) \to \Omega^{k+1}(X,E)$. For a 1-form $\alpha \in \Omega^1(X,E)$ it is given by

$$\mathrm{d}^E\alpha(U_1,U_2) = \nabla^E_{U_1}(\alpha(U_2)) - \nabla^E_{U_2}(\alpha(U_1)) - \alpha([U_1,U_2]), \qquad U_1,U_2 \in \mathfrak{X}(X).$$

[3] The generalized connection ω takes values on the vector bundle $\pi_{Y\times_X\mathfrak{g},Y}$, which is the pull-back of $\pi_{\mathfrak{g},X}$ by $\pi_{Y,X}$. Hence, we also need to pull-back the linear connection $\nabla^{\mathfrak{g}}$ to compute the exterior covariant derivative of ω. Abusing the notation, we denote the pull-back connection by the same symbol: $\pi^*_{Y,X}\left(\nabla^{\mathfrak{g}}\right) \equiv \nabla^{\mathfrak{g}}$.

2.3 Examples

2.3.1 Standard principal bundles and connections

Generalized principal connections reduce to usual principal connections on (standard) principal bundles. Let $\pi_{P,X}$ be a principal G-bundle and denote by $\Psi : P \times G \to P$ the corresponding right action. We define a fibered action of the trivial Lie group bundle $\mathcal{G} = X \times G$ on $\pi_{P,X}$ as

$$\Phi : P \times_X \mathcal{G} \to P, \quad (y, (x, \hat{g})) \mapsto \Psi_{\hat{g}}(y) = y \cdot \hat{g}.$$

Note that $P/\mathcal{G} \simeq X$, so we can regard $\pi_{P,X}$ as a generalized principal bundle with respect to this fibered action.

Let ν_0 be the trivial connection on $\pi_{\mathcal{G},X}$, that is, the one given by

$$(\nu_0)_{(x,\hat{g})}\left(U_x, U_{\hat{g}}\right) = (0_x, U_{\hat{g}}), \qquad (U_x, U_{\hat{g}}) \in T_{(x,\hat{g})}\mathcal{G} = T_xX \oplus T_{\hat{g}}G.$$

In addition, note that the Lie algebra bundle of $\pi_{\mathcal{G},X}$ is $\mathfrak{g} = X \times \mathfrak{g}$.

Proposition 2.21. *Let $A \in \Omega^1(P, \mathfrak{g})$ and $\omega \in \Omega^1(P, \mathfrak{g})$ be such that*

$$\omega_y(U_y) = (x, A_y(U_y)), \qquad y \in P,\ U_y \in T_yP,\ x = \pi_{P,X}(y).$$

Then ω is a generalized principal connection associated to ν_0 if and only if A is a (standard) principal connection.

Proof. Let $(y, \xi) \in P \times_X \mathfrak{g}$ and denote $\xi = (x, \hat{\xi})$, $x = \pi_{P,X}(y)$, for some $\hat{\xi} \in \mathfrak{g}$. Observe that

$$\xi^*_y = \frac{d}{dt}\Big|_{t=0} y \cdot \exp(t\,\xi) = \frac{d}{dt}\Big|_{t=0} y \cdot \left(x, \exp(t\,\hat{\xi})\right) = \frac{d}{dt}\Big|_{t=0} y \cdot \exp\left(t\,\hat{\xi}\right) = \hat{\xi}^\star_y,$$

where $\hat{\xi}^\star_y$ is the infinitesimal generator of $\hat{\xi} \in \mathfrak{g}$ at y in the sense of principal bundles. Hence,

$$\omega_y(\xi^*_y) = \xi \quad \Longleftrightarrow \quad \left(x, A_y\left(\hat{\xi}^\star_y\right)\right) = \left(x, \hat{\xi}\right) \quad \Longleftrightarrow \quad A_y\left(\hat{\xi}^\star_y\right) = \hat{\xi}.$$

Subsequently, we only need to check Ad-equivariance to conclude.

($\Rightarrow$) Suppose that ω is a generalized principal connection on $\pi_{P,X}$ associated to ν_0. Let $y \in P$, $U_y \in T_yP$ and $\hat{g} \in G$, and denote $g = (x, \hat{g})$, $x = \pi_{P,X}(y)$. Pick $\alpha : (-\epsilon, \epsilon) \to P$ such that $\alpha(0) = y$ and $\alpha'(0) = U_y$, we have

$$(d\Psi_{\hat{g}})_y(U_y) = \frac{d}{dt}\Big|_{t=0} \Psi_{\hat{g}}\left(\alpha(t)\right) = \frac{d}{dt}\Big|_{t=0} \Phi\left(\alpha(t), \beta(t)\right) = (d\Phi)_{(\alpha(0),\beta(0))}\left(\alpha'(0), \beta'(0)\right),$$

where $\beta : (-\epsilon, \epsilon) \to \mathcal{G}$ is given by $\beta(t) = ((\pi_{P,X} \circ \alpha)(t), \hat{g})$. Note that $\beta(0) = (x, \hat{g}) = g$ and $\beta'(0) = (U_x, 0_{\hat{g}})$ for $U_x = (d\pi_{P,X})_y(U_y) \in T_xX$. As a result,

$$(d\Psi_{\hat{g}})_y(U_y) = (d\Phi)_{(y,g)}\left(U_y, U_g\right), \qquad U_g = (U_x, 0_{\hat{g}}).$$

Now, property (ii) of Definition 2.5 ensures that A is a principal connection on $\pi_{P,X}$:

$$\begin{aligned}
\left(x, A_{y\cdot\hat{g}}\left((d\Psi_{\hat{g}})_y(U_y)\right)\right) &= \omega_{y\cdot g}\left((d\Phi)_{(y,g)}\left(U_y, U_g\right)\right) \\
&= \mathrm{Ad}_{g^{-1}}\left(\omega_y(U_y) + (\nu_0)_g(U_x, 0_{\hat{g}})\right) \\
&= \mathrm{Ad}_{g^{-1}}\left(x, A_y(U_y)\right) \\
&= \left(x, \mathrm{Ad}_{\hat{g}^{-1}}\left(A_y(U_y)\right)\right).
\end{aligned}$$

($\Leftarrow$) Suppose that A is a principal connection on $\pi_{P,X}$. Let $(y,g) \in P \times_X \mathcal{G}$ with $g = (x,\hat{g})$, and $(U_y, U_g) \in T_yP \times_{T_xX} T_g\mathcal{G}$ with $U_g = (U_x, U_{\hat{g}}) \in T_xX \oplus T_{\hat{g}}G$. Observe that

$$\begin{aligned}(d\Phi)_{(y,g)}\,(U_y,U_g) &= (d\Psi)_{(y,\hat{g})}\left(U_y,U_{\hat{g}}\right)\\ &= (d\Psi_{\hat{g}})_y(U_y) + (d\Psi_y)_{\hat{g}}(U_{\hat{g}})\\ &= (d\Psi_{\hat{g}})_y(U_y) + (d\Psi_{y\cdot\hat{g}})_1\left((dL_{\hat{g}^{-1}})_{\hat{g}}(U_{\hat{g}})\right)\\ &= (d\Psi_{\hat{g}})_y(U_y) + (dL_{\hat{g}^{-1}})_{\hat{g}}(U_{\hat{g}})^{\star}_{y\cdot\hat{g}},\end{aligned}$$

where $\Psi_y : G \to P$ is given by $\Psi_y(\hat{g}) = y\cdot\hat{g}$ and $L_{\hat{g}} : G \to G$ denotes the left multiplication by $\hat{g}$. By using this, we conclude

$$\begin{aligned}\omega_{y\cdot g}\left((d\Phi)_{(y,g)}(U_y,U_g)\right) &= \left(x, A_{y\cdot\hat{g}}\left((d\Psi_{\hat{g}})_y(U_y) + (dL_{\hat{g}^{-1}})_{\hat{g}}(U_{\hat{g}})^{\star}_{y\cdot\hat{g}}\right)\right)\\ &= \left(x, \mathrm{Ad}_{\hat{g}^{-1}}\left(A_y(U_y)\right) + (dL_{\hat{g}^{-1}})_{\hat{g}}(U_{\hat{g}})\right)\\ &= \mathrm{Ad}_{g^{-1}}\left(\omega_y(U_y)\right) + (dL_{g^{-1}})_g\left((\nu_0)_g(U_g)\right)\\ &= \mathrm{Ad}_{g^{-1}}\left(\omega_y(U_y) + (\nu_0)_g(U_g)\right).\end{aligned}$$

2.3.2 Affine bundles and connections

Generalized principal connections on an affine bundle are just affine connections (see [55, 86]) associated to linear connections on the modelling vector bundle. A vector bundle $\pi_{\overline{E},X}$ is an abelian Lie group bundle with the additive structure. A Lie group connection $\nu : T\overline{E} \to \overline{E}$ is just a linear connection, since it respects this additive structure. If we consider linear bundle coordinates (x^μ, v^α) corresponding to a basis of local sections $\{e_\alpha : 1 \le \alpha \le m\}$ of $\pi_{\overline{E},X}$, then we may write

$$\nu(x^\mu, v^\alpha) = \nu^\alpha_{\mu,\beta}(x^\mu)\, v^\beta\, dx^\mu \otimes e_\alpha + dv^\alpha \otimes e_\alpha,$$

for some (local) functions $\nu^\alpha_{\mu,\beta} \in C^\infty(X)$, $1 \le \mu \le n$, $1 \le \alpha, \beta \le m$.

Now consider an affine bundle $\pi_{E,X}$ modelled on $\pi_{\overline{E},X}$. We have the following fibered action

$$E \times_X \overline{E} \to E, \quad (y,v) \mapsto y + v.$$

This action is clearly free and proper, and it satisfies that $E/\overline{E} \simeq X$, so we can regard $\pi_{E,X}$ as a generalized principal bundle. In the following result we see that generalized principal connections for this action are just affine connections.

Proposition 2.22. *Let $\omega \in \Omega^1\left(E, \overline{E}\right)$ and ν be a linear connection on $\pi_{\overline{E},X}$. Then ω is an affine connection on $\pi_{E,X}$ with ν as underlying linear connection if and only if ω is a generalized principal connection on $\pi_{E,X}$ associated to ν.*

Proof. Let (x^μ, y^α) be affine bundle coordinates for $\pi_{E,X}$ associated to (x^μ, v^α). First, we suppose that ω is an affine connection with ν as underlying linear connection. The local expression of ω is

$$\omega(x^\mu, y^\alpha) = \left(\nu^\alpha_{\mu,\beta}(x^\mu)y^\beta + \Gamma^\alpha_\mu(x^\mu)\right) dx^\mu \otimes e_\alpha + dy^\alpha \otimes e_\alpha$$

for some (local) functions $\Gamma^\alpha_\mu \in C^\infty(X)$, $1 \le \mu \le n$, $1 \le \alpha \le m$. It is straightforward to check the equivariance condition from Definition 2.5, that for this case reads $\omega_{y+v} = \omega_y + \nu_v$ for each $(y,v) \in E \times_X \overline{E}$ (observe that the adjoint representation is trivial, since the group is abelian).

Conversely, suppose that ω is a generalized principal connection associated to ν. Locally, ω will be given by

$$\omega(x^\mu, y^\alpha) = \omega^\alpha_\mu(x^\mu, y^\alpha)\, dx^\mu \otimes e_\alpha + dy^\alpha \otimes e_\alpha$$

for some (local) functions $\omega^\alpha_\mu \in C^\infty(E)$, $1 \le \mu \le n$, $1 \le \alpha \le m$. It must satisfy the equivariance condition, that is, $\omega(x^\mu, y^\alpha + v^\alpha) = \omega(x^\mu, y^\alpha) + \nu(x^\mu, v^\alpha)$ for every $(x^\mu, y^\alpha, v^\alpha) \in E \times_X \overline{E}$. This gives

$$\omega^\alpha_\mu(x^\mu, y^\alpha + v^\alpha) = \omega^\alpha_\mu(x^\mu, y^\alpha) + \nu^\alpha_{\mu,\beta}(x^\mu)v^\beta, \qquad 1 \le \mu \le n,\ 1 \le \alpha \le m.$$

Thence, $\omega^\alpha_\mu(x^\mu, y^\alpha) = \Gamma^\alpha_\mu(x^\mu) + \nu^\alpha_{\mu,\beta}(x^\mu)y^\beta$ for $1 \le \mu \le n$ and $1 \le \alpha \le m$, where $\Gamma^\alpha_\mu(x^\mu) = \omega^\alpha_\mu(x^\mu, 0)$. This is the local expression of an affine connection with ν as underlying linear connection.

2.3.3 Gauge transformations

Let $\pi_{P,X} : P \to X$ be a (standard) principal bundle with structure group G. Recall that gauge transformations on $\pi_{P,X}$ are in a bijective correspondence with sections of the Lie group bundle

$$\tilde{G} = (P \times G)/G \to X,$$

where the right action of G on $P \times G$ is given by conjugacy, i.e., $(p, g)\cdot h = (p\cdot h, c_{h^{-1}}(g)) = (p\cdot h, h^{-1}gh)$ for each $p \in P$ and $g, h \in G$. This provides a fibered action of $\pi_{\tilde{G},X}$ on $\pi_{P,X}$. This bundle is a main instance of a Lie group bundle and, in particular, the question of when an arbitrary Lie group bundle $\mathcal{G} \to X$ is the adjoint bundle of a (standard) principal bundle has interesting topological consequences (for example, see [63]).

Principal connections on $\pi_{P,X}$ are in a bijective correspondence with sections of the bundle of connections, $\pi_{C(P),X}$. The group of gauge transformations acts on connections; namely, this action is fiberwisely expressed as

$$J^1\tilde{G} \times_X C(P) \to C(P), \quad (j^1_x\gamma, [j^1_x s]) \mapsto [j^1_x(\gamma \circ s)].$$

The 1-jet lift of this action is

$$J^1\left(J^1\tilde{G}\right) \times_X J^1C(P) \to J^1C(P).$$

This is again a (left) fibered action, but it is not free. Nevertheless, we may consider the following Lie group subbundle of $J^1\left(J^1\tilde{G}\right)$,

$$J^2_0\tilde{G} = \left\{j^2_x\gamma \in J^2\tilde{G} \mid \gamma(x) = 1_{\tilde{G}_x}\right\} \subset J^2\tilde{G} \subset J^1\left(J^1\tilde{G}\right).$$

The restriction of the action to this Lie group subbundle is free and proper, making

$$J^1C(P) \to J^1C(P)/J^2_0\tilde{G}$$

a generalized principal bundle that is not a (standard) principal bundle. Furthermore, the quotient is isomorphic to the *curvature bundle* of $\pi_{P,X}$,

$$J^1C(P)/J^2_0\tilde{G} \tilde{\to} \wedge^2 T^*X \otimes \tilde{\mathfrak{g}}, \quad \left[j^1_x A\right]_{J^2_0\tilde{G}} \mapsto \tilde{F}^A_x,$$

where we denote by $\tilde{F}^A \in \Omega^2(X, \tilde{\mathfrak{g}})$ the reduced curvature corresponding to a principal connection $A \in \Omega^1(P, \mathfrak{g})$. This is (the main part of) the geometric statement of the well-known Utiyama theorem (for example, see [39, §5]).

2.3.3.1 An example of generalized principal connection

Within the framework of gauge field theories analyzed above, we now present a simple example of a generalized principal connection. Even though one initially would like to have such connection on $C(P) = (J^1P)/G$, the action of $J^1\tilde{G}$ on $C(P)$ is not free. To remedy this, we may consider the action on J^1P instead,

$$\Phi : J^1\tilde{G} \times_X J^1P \to J^1P, \quad (j^1_x\varphi, j^1_x s) \mapsto j^1_x(\varphi \circ s),$$

that is free and and proper. Actually, this construction is conceptually close to the case of affine connections (see §2.3.2 above), since $J^1\tilde{G}$ acts transitively on J^1P.

For the sake of brevity, in the following we work in a trivializing chart of $\pi_{P,X}$, i.e., we pick $\mathcal{U} \subset X$ with $\mathcal{U} \simeq \mathbb{R}^n$, so that we can write $P|_{\mathcal{U}} = \mathcal{U} \times G$, $\tilde{G}|_{\mathcal{U}} = \mathcal{U} \times G$ and $\tilde{\mathfrak{g}}|_{\mathcal{U}} = \mathcal{U} \times \mathfrak{g}$. For the sake of simplicity, we write $\mathcal{U} = X$. Furthermore, consider the right trivialization

$$TG \simeq G \times \mathfrak{g}, \quad U_g \mapsto \left(g, (dR_{g^{-1}})_g(U_g)\right).$$

By using this, we have

$$J^1P = J^1\tilde{G} = G \ltimes (T^*X \otimes \mathfrak{g}), \qquad J^1\tilde{\mathfrak{g}} = \mathfrak{g} \oplus (T^*X \otimes \mathfrak{g}),$$

where the (fiberwise) semidirect product $\ltimes$ is given by the adjoint representation, that is,

$$(g, \xi_x) \cdot \left(g', \xi'_x\right) = \left(gg', \xi_x + \mathrm{Ad}_g \circ \xi'_x\right), \qquad (g, \xi_x), \left(g', \xi'_x\right) \in J^1_x\tilde{G} = G \ltimes (T^*_xX \otimes \mathfrak{g}).$$

The same expression holds for the trivialized (left) fibered action, i.e.,

$$(g, \xi_x) \cdot (h, A_x) = (gh, \xi_x + \mathrm{Ad}_g \circ A_x), \qquad (g, \xi_x) \in J^1_x\tilde{G} = J^1_xP \ni (h, A_x). \tag{2.16}$$

Remark 2.3 *It is easy to check that* $(g, \xi_x)^{-1} = \left(g^{-1}, -\mathrm{Ad}_{g^{-1}} \circ \xi_x\right)$ *for each* $(g, \xi_x) \in J^1\tilde{G}$. *In addition, the adjoint representation of* $J^1\tilde{G}$ *is given by*

$$\mathrm{Ad}_{(g,\xi_x)}(\eta, \phi_x) = (\mathrm{Ad}_g(\eta), \mathrm{Ad}_g \circ \phi_x - [\mathrm{Ad}_g(\eta), \xi_x])$$

for each $(g, \xi_x) \in J^1\tilde{G}$ *and* $(\eta, \phi_x) \in J^1\tilde{\mathfrak{g}}$, *where* $[\cdot, \cdot]$ *is the Lie bracket of* $\mathfrak{g}$.

On the other hand, the trivialization $P = X \times G$ also allows us to identify

$$J^1(J^1P) = J^1(J^1\tilde{G}) = G \ltimes ((T^*X \otimes \mathfrak{g}) \oplus (T^*X \otimes \mathfrak{g}) \oplus (T^*X \otimes T^*X \otimes \mathfrak{g})).$$

Moreover, the jet extension of the action (2.16) is given by

$$\begin{aligned}(g, \xi_x, \eta_x, \phi_x) \cdot (h, A_x, \chi_x, \alpha_x) \\ = (gh, \mathrm{Ad}_g \circ A_x + \xi_x; \mathrm{Ad}_g \circ \chi_x + \eta_x, \mathrm{Ad}_g \circ \alpha_x + \phi_x + [\eta_x, \mathrm{Ad}_g \circ A_x])\end{aligned} \tag{2.17}$$

for each $(g, \xi_x, \eta_x, \phi_x) \in J^1(J^1\tilde{G})$ and $(h, A_x, \chi_x, \alpha_x) \in J^1(J^1P)$.

Lemma 2.23. *The jet section* $\hat{\nu} \in \Gamma\left(\pi_{J^1(J^1\tilde{G}), J^1\tilde{G}}\right)$ *defined as*

$$\hat{\nu}(g, \xi_x) = (g, \xi_x, 0_x, 0_x), \qquad (g, \xi_x) \in J^1\tilde{G},$$

is a Lie group bundle connection on $\pi_{J^1\tilde{G}, X}$.

Proof. It is clear that $(\hat{\nu} \circ 1)(x) = \hat{\nu}(1, 0_x) = (1, 0_x, 0_x, 0_x) = (d1)_x$ for each $x \in X$. Besides, we have

$$\begin{aligned}\hat{\nu}\left((g, \xi_x) \cdot \left(g', \xi'_x\right)\right) &= \hat{\nu}\left(gg', \xi_x + \mathrm{Ad}_g \circ \xi'_x\right) \\ &= \left(gg', \xi_x + \mathrm{Ad}_g \circ \xi'_x, 0_x, 0_x\right) \\ &= (g, \xi_x, 0_x, 0_x) \cdot \left(g', \xi'_x, 0_x, 0_x\right) \\ &= \hat{\nu}(g, \xi_x) \cdot \hat{\nu}\left(g', \xi'_x\right)\end{aligned}$$

and we conclude by Proposition 2.6.

Observe that the Lie algebra bundle of $J^1\tilde{G}$ is $J^1\tilde{\mathfrak{g}}$. Theorem 2.16 ensures that there exists a generalized principal connection $\omega \in \Omega^1(J^1P, J^1\tilde{\mathfrak{g}})$ associated to ν.

Proposition 2.24. *An Ehresmann connection $\omega \in \Omega^1(J^1P, J^1\tilde{\mathfrak{g}})$ with corresponding jet section $\hat{\omega} \in \Gamma(\pi_{J^1(J^1P),J^1P})$ is a generalized principal connection on $\pi_{J^1P,X}$ associated to ν if and only if*

$$\hat{\omega}(h, A_x) = (h, A_x, \mathrm{Ad}_h \circ f(x), \mathrm{Ad}_h \circ g(x)), \qquad (h, A_x) \in J^1P,$$

for some sections $f \in \Gamma(\pi_{T^*X\otimes\mathfrak{g},X})$ and $g \in \Gamma(\pi_{T^*X\otimes T^*X\otimes\mathfrak{g},X})$.

Proof. Observe that we may write

$$\hat{\omega}(h, A_x) = (h, A_x, \varpi(h, A_x), \tilde{\varpi}(h, A_x)), \qquad (h, A_x) \in J^1P,$$

for some $\varpi : J^1P \to T^*X \otimes \mathfrak{g}$ and $\tilde{\varpi} : J^1P \to T^*X \otimes T^*X \otimes \mathfrak{g}$. By choosing $(h, A_x) = (1, 0_x)$ we get

$$\hat{\omega}\left((g, \xi_x) \cdot (1, 0_x)\right) = \hat{\omega}(g, \xi_x) = (g, \xi_x, \varpi(g, \xi_x), \tilde{\varpi}(g, \xi_x)).$$

Likewise, from (2.17) we have

$$\begin{aligned}\hat{\nu}(g, \xi_x) \cdot \hat{\omega}(1, 0_x) &= (g, \xi_x, 0_x, 0_x) \cdot (1, 0_x, \varpi(1, 0_x), \tilde{\varpi}(1, 0_x)) \\ &= (g, \xi_x, \mathrm{Ad}_g \circ \varpi(1, 0_x), \mathrm{Ad}_g \circ \tilde{\varpi}(1, 0_x)).\end{aligned}$$

Proposition 2.14 ensures that ω is a generalized principal connection on $\pi_{J^1P,X}$ associated to ν if and only if

$$\varpi(g, \xi_x) = \mathrm{Ad}_g \circ \varpi(1, 0_x), \qquad \tilde{\varpi}(g, \xi_x) = \mathrm{Ad}_g \circ \tilde{\varpi}(1, 0_x),$$

for each $(g, \xi_x) \in J^1P$. We conclude by denoting $f(x) = \varpi(1, 0_x)$ and $g(x) = \tilde{\varpi}(1, 0_x)$ for each $x \in X$.

Chapter 3

Higher-order jet bundles of Lie group-valued functions

This short chapter corresponds to our paper [20], where the bundle of k-jets of functions from a smooth manifold, X, to a Lie group, G, is investigated, for any positive integer k. Our goal is to trivialize this bundle by using a linear connection on the tangent bundle of X, and to obtain the explicit expression of the groupoid multiplication on the trivialized space, as well as the formula for the inverse element. Furthermore, in the last section we give the local expression of the trivialization on a coordinated chart of X.

Recall, for each $k \in \mathbb{Z}^+$, the family of k-jets of smooth curves, $c : \mathbb{R} \to G$, where G is a Lie group, is naturally a Lie group fiber bundle over $\mathbb{R}$. We denote it by $J^k(\mathbb{R}, G)$, and its fiber over each $t \in \mathbb{R}$ by $J^k_t(\mathbb{R}, G)$. In fact, these spaces are frequently known as higher-order tangent spaces and denoted by T^kG. In [93], the Lie group structure of $J^k_0(\mathbb{R}, G)$ was investigated. Namely, the space was trivialized by the right, yielding an isomorphism of bundles over G,

$$J^k_0(\mathbb{R}, G) \simeq G \times (\mathfrak{g} \oplus \overset{k}{\cdots} \oplus \mathfrak{g}).$$

Following that, the explicit expression of the multiplication, as well as the inverse element, were studied under this identification.

We aim to generalize the results in [93] when an arbitrary manifold, X, is considered instead of $\mathbb{R}$. As we will see, a linear connection on the cotangent bundle of X is necessary in order to trivialize the jet bundle, $J^k(X, G)$. It is important to point out that the generalization analyzed here goes beyond a mere complication of the computations occurring in higher-order tangent bundles of Lie groups. The situation for $\dim X \geq 2$ implies qualitative and important differences that entail the structure jet bundles in their full complexity. In particular, in this case the trivialization is not an isomorphism any more, but only an injective morphism. Notwithstanding, we take advantage of this injection to compute explicit formulas for the fibered multiplication and the inverse element. At last, the local expression of the trivialization is given after choosing a coordinates chart of X. In particular, this enables us to compute the image of the injective morphism for $k = 2$.

Jets of curves on Lie groups are essential in the analysis of higher-order Lagrangian mechanical systems over Lie groups [33], as well as in the higher-order Euler–Poincaré reduction procedure [40] and its applications to optimal control theory [25]. In turn, our generalization to jets of functions defined on arbitrary manifolds is of great relevance in the field theoretical counterpart of such systems and, in particular, in the analysis of non-Abelian gauge theories from the gauge reduction perspective, as we will see in Chapter 5.

3.1 Partly ordered and anti-lexicographically ordered partitions

Let $j \in \mathbb{Z}^+$. A partition of $\{1,\dots,j\}$ of length $l \in \{1,\dots,j\}$ is a tuple $\lambda = (\lambda_1,\dots,\lambda_l)$ where $\lambda_1,\dots,\lambda_l \subset \{1,\dots,j\}$ are disjoint subsets such that $\lambda_1 \cup \cdots \cup \lambda_l = \{1,\dots,j\}$. We denote by $\mathcal{P}(j)$ the family of all partitions of $\{1,\dots,j\}$. Likewise, we denote $j_i = |\lambda_i|$, the cardinality of λ_i, $1 \le i \le l$. Of course, $j_1 + \cdots + j_l = j$ for each partition $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}(j)$.

Definition 3.1 *A partition $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}(j)$ is* partly ordered *if $\alpha^i_1 < \cdots < \alpha^i_{j_i}$ for each $1 \le i \le l$, where we denote $\lambda_i = \{\alpha^i_1,\dots,\alpha^i_{j_i}\}$.*

Observe that we ask for the integers to be ordered only within each subset λ_i, $1 \le i \le l$. The set of all partly ordered partitions of $\{1,\dots,j\}$ is denoted by $\mathcal{P}^+(j)$. At last, we consider the family of partly ordered partitions with $\alpha^1_1 = 1$, i.e.,

$$\mathcal{P}^+_1(j) = \{\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}^+(j) \mid 1 \in \lambda_1\}.$$

Proposition 3.1. *Let $j \in \mathbb{Z}^+$ and $1 \le l \le j$. Fixed $j_1,\dots,j_l \in \{1,\dots,j\}$ such that $j_1 + \cdots + j_l = j$, there are exactly*

$$c(j_1,\dots,j_l) = \binom{j-1}{j_1-1}\prod_{i=2}^{l}\binom{j_i+\cdots+j_l}{j_i}$$

different partitions $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}^+_1(j)$ with $|\lambda_i| = j_i$, $1 \le i \le l$. In particular, the cardinality of $\mathcal{P}^+_1(j)$ is

$$|\mathcal{P}^+_1(j)| = \sum_{l=1}^{j}\sum_{j_1+\cdots+j_l=j} c(j_1,\dots,j_l).$$

Proof. For the first part, we write $\lambda_1 = \{\alpha^1_1 = 1 < \alpha^1_2 < \cdots < \alpha^1_{j_1}\}$ for each $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}^+_1(j)$. Thus, there are $\binom{j-1}{j_1-1}$ ways of choosing the elements $\alpha^1_2,\dots,\alpha^1_{j_1}$. We have $n-j_1$ remaining elements, whence there are $\binom{j-j_1}{j_2}$ choices for the elements of λ_2. Generally, there are $\binom{j-j_1\cdots-j_{i-1}}{j_i}$ choices for the elements of λ_i, $2 \le i \le l$. In short, the number of partitions in $\mathcal{P}^+_1(j)$ with $|\lambda_i| = j_i$, $1 \le i \le l$, is exactly

$$c(j_1,\dots,j_l) = \binom{j-1}{j_1-1}\prod_{i=2}^{l}\binom{j-j_1\cdots-j_{i-1}}{j_i}.$$

We conclude by recalling that $j_1 + \cdots + j_l = j$. The second part is a straightforward consequence of the first one.

Given a partition $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}(j)$, we consider two ways to derive a new partition of $\{1,\dots,j+1\}$. Namely, fixed $1 \le s \le l$ we define

(i) $\lambda^+_{[s]} = (\lambda_1,\dots,\lambda_{s-1},\lambda_s \cup \{j+1\},\lambda_{s+1},\dots,\lambda_l)$.

(ii) $\lambda^-_{[s]} = (\lambda_1,\dots,\lambda_s,\{j+1\},\lambda_{s+1},\dots,\lambda_l)$.

For instance, given $\lambda = (\{23\},\{1\}) \in \mathcal{P}(3)$, we have $\lambda^+_{[1]} = (\{234\},\{1\})$ and $\lambda^-_{[1]} = (\{23\},\{4\},\{1\})$. We have the following lemmas.

Lemma 3.2. *Let $j \in \mathbb{Z}^+$, $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}^+_1(j)$ and $1 \le s \le l$. Then $\lambda^+_{[s]}, \lambda^-_{[s]} \in \mathcal{P}^+_1(j+1)$.*

Lemma 3.3. *Let $j \in \mathbb{Z}^+$ and $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathcal{P}^+_1(j+1)$. Then there exists a unique $\hat{\lambda} \in \mathcal{P}^+_1(j)$ and $1 \le s \le l$ such that $\lambda = \hat{\lambda}^+_{[s]}$ or $\lambda = \hat{\lambda}^-_{[s]}$.*

Proof. Let $1 \leq s \leq l$ be such that $j+1 \in \lambda_s$. If $|\lambda_s| > 1$, we pick

$$\hat{\lambda} = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s - \{j+1\}, \lambda_{s+1}, \ldots, \lambda_l) \in \mathcal{P}_1^+(j).$$

It is thus clear that $\hat{\lambda}^+_{[s]} = \lambda$. Analogously, if $|\lambda_s| = 1$, then $s \geq 2$ and we choose

$$\hat{\lambda} = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_{s+1}, \ldots, \lambda_l) \in \mathcal{P}_1^+(j).$$

It is straightforward that $\hat{\lambda}^-_{[s-1]} = \lambda$.

Lastly, observe that if we have two different partitions $\hat{\lambda}, \tilde{\lambda} \in \mathcal{P}_1^+(j)$, then $\hat{\lambda}^+_{[s]}$, $\hat{\lambda}^-_{[s]}$, $\tilde{\lambda}^+_{[s]}$ and $\tilde{\lambda}^-_{[s]}$ are all distinct, whenever they are defined. Hence, the partition $\hat{\lambda}$ yielding λ is unique.

The previous lemmas have the following straightforward result.

Proposition 3.4. *Let $j \in \mathbb{Z}^+$ and denote by $\sqcup$ the disjoint union. Then*

$$\mathcal{P}_1^+(j+1) = \bigsqcup_{\lambda \in \mathcal{P}_1^+(j)} \bigsqcup_{s=1}^{l} \left(\left\{ \lambda^+_{[s]} \right\} \sqcup \left\{ \lambda^-_{[s]} \right\} \right).$$

Additionally, we are interested in anti-lexicographically ordered partitions.

Definition 3.2 *A partition $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(j)$ is* anti-lexicographically ordered *if $\max \lambda_1 < \cdots < \max \lambda_l$, that is to say, if each subset contains the highest available number when going from right to left.*

We denote by $\mathcal{P}^a(j)$ the set of all anti-lexicographically ordered partitions. In order to illustrate both definitions, consider $\mathcal{P}(3)$. Then $\lambda = (\{12\}, \{3\})$ is both partly and anti-lexicographically ordered, $\lambda = (\{3\}, \{12\})$ is partly ordered but not anti-lexicographically ordered, $\lambda = (\{21\}, \{3\})$ is anti-lexicographically ordered, but not partly ordered, and $\lambda = (\{3\}, \{21\})$ is not partly nor anti-lexicographically ordered.

Proposition 3.5 ([93]). *Let $j \in \mathbb{Z}^+$ and $1 \leq l \leq j$. Fixed $j_1, \ldots, j_l \in \{1, \ldots, j\}$ such that $j_1 + \cdots + j_l = j$, there are exactly*

$$N(j_1, \ldots, j_l) = \prod_{i=2}^{l} \binom{j_1 + \cdots + j_i - 1}{j_i - 1}$$

different partitions $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}^a(j)$ with $|\lambda_i| = j_i$, $1 \leq i \leq l$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}(j)$, we derive a partition of $\{1, \ldots, j+1\}$ as follows. Fix $1 \leq s \leq l$ and define

$$\lambda_{[s]} = \left(\hat{\lambda}_1, \ldots, \hat{\lambda}_s \cup \{1\}, \ldots \hat{\lambda}_l \right),$$

where $\hat{\lambda}_i = \{\alpha_1^i + 1, \ldots, \alpha_{j_i}^i + 1\}$ for each $1 \leq i \leq l$. Similarly, we define

$$\lambda_{[0]} = \left(\{1\}, \hat{\lambda}_1, \ldots, \hat{\lambda}_l \right).$$

Lemma 3.6. *Let $j \in \mathbb{Z}^+$, $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}^a(j)$ and $0 \leq s \leq l$. Then $\lambda_{[s]} \in \mathcal{P}^a(j+1)$.*

Lemma 3.7 ([93]). *Let $j \in \mathbb{Z}^+$ and $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P}^a(j+1)$. Then there exists a unique $\hat{\lambda} \in \mathcal{P}^a(j)$ and $0 \leq s \leq l$ such that $\lambda = \hat{\lambda}_{[s]}$.*

Proposition 3.8. *For each $j \in \mathbb{Z}^+$ we have*

$$\mathcal{P}^a(j+1) = \bigsqcup_{\lambda \in \mathcal{P}^a(j)} \bigsqcup_{s=0}^{l} \left\{ \lambda_{[s]} \right\}.$$

3.2 Trivialization for the higher-order jet bundle

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra and X be a smooth manifold. In the following, we regard the jet defined by a (local) function $g : X \to G$ as an element of the vector bundle $j^1_x g \equiv (dg)_x \in T^*X \otimes TG$ over X. Let $\nabla^X : \Gamma(TX) \to \Gamma(T^*X \otimes TX)$ be a linear connection on the tangent bundle of X, and consider the dual connection, $\nabla : \Gamma(T^*X) \to \Gamma(T^*X \otimes T^*X)$, on the cotangent bundle. For each $k \in \mathbb{Z}^+$, let

$$\nabla^{(k)} : \Gamma\left(\bigotimes^k T^*X \to X\right) \to \Gamma\left(\bigotimes^{k+1} T^*X \to X\right)$$

be the corresponding tensor product connection on $\bigotimes^k T^*X \to X$, which is again a linear connection [65, §2.4]. Note that $\nabla^{(1)} = \nabla$. At last, we extend our linear connections to maps,

$$\tilde{\nabla}^{(k)} : \Gamma\left(\bigotimes^k T^*X \otimes \mathfrak{g} \to X\right) \to \Gamma\left(\bigotimes^{k+1} T^*X \otimes \mathfrak{g} \to X\right),$$

In the previous expression, the tensor product only concerns the cotangent bundles, but the parenthesis is omitted for brevity, i.e.,

$$\bigotimes^k T^*X \otimes \mathfrak{g} = \left(\bigotimes^k T^*X\right) \otimes \mathfrak{g}.$$

Observe that this works although $\nabla^{(k)}$ is not tensorial, since $\mathfrak{g}$ is not a vector bundle, but a vector space. Indeed, if $(\mathcal{U}, x)$ is a coordinated chart of X, where we write $x = (x^\mu) = (x^1, \dots, x^n)$, and $\{B_1, \dots, B_m\}$ is a basis of $\mathfrak{g}$, we set

$$\tilde{\nabla}^{(k)}\left(\xi^\alpha_{\mu_1,\dots,\mu_k}\, dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k} \otimes B_\alpha\right) = \nabla^{(k)}\left(\xi^\alpha_{\mu_1,\dots,\mu_k}\, dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k}\right) \otimes B_\alpha,$$

for each $\xi^\alpha_{\mu_1,\dots,\mu_k} : \mathcal{U} \to \mathbb{R}$, $1 \le \mu_1, \dots, \mu_k \le n$, $1 \le \alpha \le m$.

Theorem 3.9. *For each $k \in \mathbb{Z}^+$, the following is an injective morphism of bundles over $X \times G$,*

$$J^k(X,G) \hookrightarrow G \times \oplus_{j=1}^k \left(\bigotimes^j T^*X \otimes \mathfrak{g}\right), \qquad j^k_x g \mapsto \left(g(x), \xi^{(1)}(x), \dots, \xi^{(k)}(x)\right), \tag{3.1}$$

where

$$\xi^{(1)}(x) = \left(dR_{g(x)^{-1}}\right)_{g(x)} \circ (dg)_x \in T^*_x X \otimes \mathfrak{g},$$

and

$$\xi^{(j)}(x) = (\tilde{\nabla}^{(j-1)}\xi^{(j-1)})(x) \in \bigotimes^j T^*_x X \otimes \mathfrak{g}, \quad 2 \le j \le k.$$

Proof. We show the result by induction in k. For $k = 1$, the first jet defined by a (local) function $g : X \to G$ is given by $j^1_x g = (dg)_x : T_x X \to T_{g(x)} G$. Subsequently, the map is given by composing this jet with the right trivialization $T_g G \ni U_g \mapsto (g, (dR_{g^{-1}})_g(U_g)) \in G \times \mathfrak{g}$. Namely,

$$J^1(X,G) \hookrightarrow G \times (T^*X \otimes \mathfrak{g}), \qquad j^1_x g \mapsto \left(g(x), (dR_{g(x)^{-1}}) \circ (dg)_x\right),$$

which is injective since it is an isomorphism of fiber bundles.

Now, given $k > 1$, we assume that (3.1) is injective for $k-1$. Let $j^k_x g, j^k_x h \in J^k(X,G)$ be such that

$$\left(g(x), \xi^{(1)}(x), \dots, \xi^{(k)}(x)\right) \neq \left(h(x), \eta^{(1)}(x), \dots, \eta^{(k)}(x)\right),$$

where $\xi^{(1)} = \left(dR_{g^{-1}}\right)_g \circ dg$, $\eta^{(1)} = (dR_{h^{-1}})_h \circ dh$, $\xi^{(j)} = \tilde{\nabla}^{(j-1)}\xi^{(j-1)}$ and $\eta^{(j)} = \tilde{\nabla}^{(j-1)}\eta^{(j-1)}$, $2 \le j \le k$. If either $g(x) \neq h(x)$ or $\xi^{(j)}(x) \neq \eta^{(j)}(x)$ for some $1 \le j \le k-1$, then $j^k_x g \neq j^k_x h$ by the induction hypothesis. At last, suppose that $g(x) = h(x)$, $\xi^{(j)}(x) = \eta^{(j)}(x)$, $1 \le j \le k-1$ and $\xi^{(k)}(x) \neq \eta^{(k)}(x)$. Recall that, in coordinates, we may write (cf. [65, §2.4]) $\nabla^{(k-1)} = \mathrm{d} + \Gamma^{(k-1)}$, where $\mathrm{d} : \Gamma\left(\bigotimes^{k-1} T^*X\right) \to \Gamma\left(\bigotimes^k T^*X\right)$ is the exterior derivative (which is well-defined, since we are

working on a coordinated chart) and $\Gamma^{(k-1)} \in \Gamma\left(T^*X \otimes \mathrm{End}\left(\bigotimes^{k-1} T^*X\right)\right)$, where $\mathrm{End}\left(\bigotimes^{k-1} T^*X\right)$ is the bundle of endomorphisms of $\bigotimes^{k-1} T^*X$. We have

$$\Gamma^{(k-1)}\left(\xi^{(k-1)}(x)\right) = \Gamma^{(k-1)}\left(\eta^{(k-1)}(x)\right)$$

and

$$\tilde{\nabla}^{(k-1)}\xi^{(k-1)}(x) \neq \tilde{\nabla}^{(k-1)}\eta^{(k-1)}(x).$$

Thence $\left(\mathrm{d}\xi^{(k-1)}\right)_x \neq \left(\mathrm{d}\eta^{(k-1)}\right)_x$. Subsequently, $j^k_x g \neq j^k_x h$, and we conclude.

By means of this map, we denote elements of the k-th order jet bundle by

$$\left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right) \in J^k(X, G),$$

for some $g \in G$ and $\xi^{(j)} \in \bigotimes^j T^*X \otimes \mathfrak{g}$, $1 \leq j \leq k$, all of them projecting to certain $x \in X$.

Remark 3.1 *Except for $k = 1$, the morphism (3.1) does not need to be surjective, that is, there are elements in $G \times \bigoplus_{j=1}^k \left(\bigotimes^j T^*X \otimes \mathfrak{g}\right)$, $k \geq 2$, that do not represent an element of $J^k(X, G)$. However, Example 3.2 below will show that (3.1) is always bijective for the particular case $X = \mathbb{R}$.*

3.3 Lie groupoid structure of the higher-order jet bundle

Since G is a Lie group, for each $k \in \mathbb{Z}^+$ the jet bundle $J^k(X, G) \to X$ is endowed with a natural fibered multiplication. More specifically, the following map is well-defined:

$$J^k(X, G) \times_X J^k(X, G) \to J^k(X, G), \qquad \left(j^k_x g, j^k_x h\right) \mapsto j^k_x(gh),$$

where $\times_X$ denotes the fibered product of bundles over X. Therefore, $J^k(X, G) \rightrightarrows X$ is a Lie groupoid with both the source and the target maps being $\pi_{J^k(X,G),X}$. The aim of this section is to compute the expression of the fibered multiplication under the identification (3.1). We start with the following technical lemma.

Lemma 3.10. *Let $j \in \mathbb{Z}^+$, $g : X \to G$ and $\chi^{(j)} \in \Gamma\left(\bigotimes^j T^*X \otimes \mathfrak{g} \to X\right)$, and denote $\chi = \left(dR_{g^{-1}}\right)_g \circ dg \in \Gamma\left(T^*X \otimes \mathfrak{g}\right)$. Then for each $U \in \mathfrak{X}(X)$ we have*

$$\tilde{\nabla}^{(j)}_U\left(\mathrm{Ad}_g \circ \chi^{(j)}\right) = \mathrm{ad}(\chi(U))\left(\mathrm{Ad}_g \circ \chi^{(j)}\right) + \mathrm{Ad}_g \circ \tilde{\nabla}^{(j)}_U \chi^{(j)}.$$

Proof. Let $U_1, \ldots, U_j \in \mathfrak{X}(X)$, and recall that we denote by ∇^X the linear connection on the tangent bundle, TX. We pick a curve $\gamma : (-\epsilon, \epsilon) \to X$ such that $\gamma(0) = x$ and $\gamma'(0) = U(x)$. For the sake of brevity, we denote by $\varphi : X \to \mathfrak{g}$ the map defined as $x \mapsto \chi^{(j)}(x)(U_1(x), \ldots, U_j(x))$. Therefore,

$$\begin{aligned} U\left(\mathrm{Ad}_g(\varphi)\right)(x) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{(g\circ\gamma)}\left(\varphi \circ \gamma\right)(t) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{(g\circ\gamma)g(x)^{-1}}\left(\mathrm{Ad}_{g(x)}\left(\varphi \circ \gamma\right)\right)(t) \\ &= \mathrm{ad}\left(\chi(x)\left(U(x)\right)\right)\left(\mathrm{Ad}_{g(x)}\left(\varphi(x)\right)\right) + \mathrm{Ad}_{g(x)}\left((d\varphi)_x(U(x))\right) \\ &= \mathrm{ad}\left(\chi(x)\left(U(x)\right)\right)\left(\mathrm{Ad}_{g(x)}\left(\varphi(x)\right)\right) + \mathrm{Ad}_{g(x)}\left(U(x)\left(\varphi(x)\right)\right). \end{aligned}$$

As a result, we get

$$U\left(\mathrm{Ad}_g(\varphi)\right) = \mathrm{ad}\left(\chi(U)\right)\left(\mathrm{Ad}_g(\varphi)\right) + \mathrm{Ad}_g(U(\varphi)).$$

By using this, we conclude

$$
\begin{aligned}
&\tilde{\nabla}^{(j)}_U \left(\mathrm{Ad}_g \circ \chi^{(j)}\right)(U_1,\dots,U_j)\\
&\quad = U\left(\left(\mathrm{Ad}_g \circ \chi^{(j)}\right)(U_1,\dots,U_j)\right) - \sum_{i=1}^{j}\left(\mathrm{Ad}_g \circ \chi^{(j)}\right)\left(U_1,\dots,\nabla^X_U U_i,\dots,U_j\right)\\
&\quad = \mathrm{ad}\left(\chi(U)\right)\left(\mathrm{Ad}_g(\varphi)\right) + \mathrm{Ad}_g(U(\varphi)) - \mathrm{Ad}_g\left(\sum_{i=1}^{j}\chi^{(j)}\left(U_1,\dots,\nabla^X_U U_i,\dots,U_j\right)\right)\\
&\quad = \mathrm{ad}\left(\chi(U)\right)\left(\mathrm{Ad}_g(\varphi)\right) + \mathrm{Ad}_g\left(U(\varphi) - \sum_{i=1}^{j}\chi^{(j)}\left(U_1,\dots,\nabla^X_U U_i,\dots,U_j\right)\right)\\
&\quad = \mathrm{ad}(\chi(U))\left(\left(\mathrm{Ad}_g \circ \chi^{(j)}\right)(U_1,\dots,U_j)\right) + \mathrm{Ad}_g\left(\tilde{\nabla}^{(j)}_U \chi^{(j)}(U_1,\dots,U_j)\right).
\end{aligned}
$$

Recall that for each $k, r \in \mathbb{Z}^+$ there is an injective map $J^{k+r}(X,G) \hookrightarrow J^k\left(J^r(X,G)\right), j^{k+r}_x g \mapsto j^k_x\left(j^r g\right)$. This allows us to identify

$$
J^k(X,G) \ni j^k_x g \simeq j^1_x(j^{k-1}g) \in J^1\left(J^{k-1}(X,G)\right). \tag{3.2}
$$

Theorem 3.11. *Let $x \in X$ and $\left(g,\xi^{(1)},\dots,\xi^{(k)}\right),\left(h,\eta^{(1)},\dots,\eta^{(k)}\right) \in J^k_x(X,G)$. Then their fibered product is given by*

$$
\left(g,\xi^{(1)},\dots,\xi^{(k)}\right)\left(h,\eta^{(1)},\dots,\eta^{(k)}\right) = \left(gh,\zeta^{(1)},\dots,\zeta^{(k)}\right),
$$

where

$$
\zeta^{(j)} = \xi^{(j)} + \sum_{\lambda\in\mathcal{P}^a(j)} \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\xi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \eta^{(j_l)}\right), \qquad 1 \le j \le k. \tag{3.3}
$$

Proof. We prove it by induction in k. For $k = 1$, consider two (local) functions $\tilde{g},\tilde{h} : X \to G$ such that $\tilde{g}(x) = g$, $\tilde{h}(x) = h$, $\left(dR_{g^{-1}}\right)_g \circ (d\tilde{g})_x = \xi^{(1)}$ and $\left(dR_{h^{-1}}\right)_h \circ (d\tilde{h})_x = \eta^{(1)}$. Recall that $\left(g,\xi^{(1)}\right)\left(h,\eta^{(1)}\right) \simeq j^1_x(\tilde{g}\tilde{h})$. By using this, (3.1) and the Leibniz rule, it can be checked that

$$
\left(g,\xi^{(1)}\right)\left(h,\eta^{(1)}\right) = \left(gh,\left(dR_{(gh)^{-1}}\right)_{gh} \circ d\left(\tilde{g}\tilde{h}\right)_x\right) = \left(gh,\xi^{(1)} + \mathrm{Ad}_g \circ \eta^{(1)}\right) = \left(gh,\zeta^{(1)}\right).
$$

Now, given $k > 1$, we assume that the result holds for $k-1$ and we pick a (local) function $\tilde{g} : X \to G$ such that $j^k_x\tilde{g} \simeq \left(g,\xi^{(1)},\dots,\xi^{(k)}\right)$ via (3.1). Moreover, we have $j^{k-1}\tilde{g} \simeq \left(\tilde{g},\tilde{\xi}^{(1)},\dots,\tilde{\xi}^{(k-1)}\right)$. In the same vein, we pick a (local) function $\tilde{h} : X \to G$ such that $j^k_x\tilde{h} \simeq \left(h,\eta^{(1)},\dots,\eta^{(k)}\right)$. By the induction hypothesis, $\left(j^{k-1}\tilde{g}\right)\left(j^{k-1}\tilde{h}\right) \simeq \left(\tilde{g}\tilde{h},\tilde{\zeta}^{(1)},\dots,\tilde{\zeta}^{(k-1)}\right)$. Thanks to this and (3.2), we may write

$$
\begin{aligned}
\left(g,\xi^{(1)},\dots,\xi^{(k)}\right)\left(h,\eta^{(1)},\dots,\eta^{(k)}\right) &\simeq j^1_x\left(\left(j^{k-1}\tilde{g}\right)\left(j^{k-1}\tilde{h}\right)\right)\\
&\simeq \left(gh,\zeta^{(1)},\dots,\zeta^{(k-1)},\tilde{\nabla}^{(k-1)}\tilde{\zeta}^{(k-1)}(x)\right).
\end{aligned}
$$

By recalling that $\tilde{\nabla}^{(j)}\tilde{\xi}^{(j)}(x) = \xi^{(j+1)}$, $1 \le j \le k-1$, and analogous for $\tilde{\eta}^{(j)}$, and by using Proposition 3.8, we finish

$$
\begin{aligned}
\tilde{\nabla}^{(k-1)}\tilde{\zeta}^{(k-1)}(x) &= \tilde{\nabla}^{(k-1)}\left(\tilde{\xi}^{(k-1)} + \sum_{\lambda\in\mathcal{P}^a(k-1)} \mathrm{ad}\left(\tilde{\xi}^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\tilde{\xi}^{(j_1)}\right)\left(\mathrm{Ad}_{\tilde{g}} \circ \tilde{\eta}^{(j_l)}\right)\right)(x)\\
&= \xi^{(k)} + \sum_{\lambda\in\mathcal{P}^a(k-1)}\sum_{s=1}^{l-1} \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\xi^{(j_s+1)}\right)\dots\mathrm{ad}\left(\xi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \eta^{(j_l)}\right)\\
&\quad + \sum_{\lambda\in\mathcal{P}^a(k-1)} \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\xi^{(j_1)}\right)\left(\mathrm{ad}\left(\xi^{(1)}\right)\left(\mathrm{Ad}_g \circ \eta^{(j_l)}\right) + \mathrm{Ad}_g \circ \eta^{(j_l+1)}\right)\\
&= \xi^{(k)} + \sum_{\lambda\in\mathcal{P}^a(k)} \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\xi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \eta^{(j_l)}\right) = \zeta^{(k)}.
\end{aligned}
$$

For the second equation, we have used that for each $j \in \mathbb{Z}^+$ and $j_1, j_2 \in \mathbb{Z}^+$ such that $j_1 + j_2 = j$ we have

$$\begin{aligned}\tilde{\nabla}_U^{(j)} \mathrm{ad}\left(\chi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \chi^{(j_2)}\right) &= \mathrm{ad}\left(\tilde{\nabla}_U^{(j_1)}\chi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \chi^{(j_2)}\right) \\ &\quad + \mathrm{ad}\left(\chi^{(j_1)}\right)\left(\mathrm{ad}\left(\chi(U)\right)\left(\mathrm{Ad}_g \circ \chi^{(j_2)}\right) + \mathrm{Ad}_g \circ \tilde{\nabla}_U^{(j_2)}\chi^{(j_2)}\right)\end{aligned} \tag{3.4}$$

for each (local) vector field $U \in \mathfrak{X}(X)$, each (local) section

$$\chi^{(j_i)} \in \Gamma\left(\otimes^{j_i} T^*X \otimes \mathfrak{g} \to X\right), \qquad i = 1, 2,$$

and each (local) function $g : X \to G$, where $\chi = \left(dR_{g^{-1}}\right)_g \circ dg \in \Gamma\left(T^*X \otimes \mathfrak{g} \to X\right)$. This is a straightforward consequence of the Leibniz rule, Lemma 3.10 and the associativity of the tensor product of linear connections, i.e., $\tilde{\nabla}^{(j)} = \tilde{\nabla}^{(j_1)} \otimes \tilde{\nabla}^{(j_2)}$.

Observe that the identity element is given by $(e, 0, \ldots, 0) \in J^k(X, G)$, where $e \in G$ is the identity element.

Corollary 3.12. *Let* $\left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right) \in J^k(X, G)$. *Then its inverse is given by* $\left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right)^{-1} = \left(g^{-1}, \omega^{(1)}, \ldots, \omega^{(k)}\right)$, *where*

$$\omega^{(j)} = \sum_{\lambda \in \mathcal{P}^a(j)} (-1)^l \ \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{ad}\left(\xi^{(j_1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l)}\right)\right), \qquad 1 \le j \le k. \tag{3.5}$$

Proof. For $k = 1$, given $\left(g, \xi^{(1)}\right) \in J^1(X, G)$, it is easy to check using (3.3) that

$$\left(g, \xi^{(1)}\right)\left(g^{-1}, -\mathrm{Ad}_{g^{-1}} \circ \xi^{(1)}\right) = (e, 0),$$

whence $\left(g, \xi^{(1)}\right)^{-1} = \left(g^{-1}, -\mathrm{Ad}_{g^{-1}} \circ \xi^{(1)}\right)$.

Now, given $k > 1$, we assume that the result holds for $k - 1$. Let $\tilde{g} : X \to G$ be a (local) function such that $j_x^k \tilde{g} \simeq \left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right)$ via (3.1). Thanks to (3.2) and the induction hypothesis, we may write

$$\left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right)^{-1} \simeq \left(j_x^k \tilde{g}\right)^{-1} \simeq j_x^1 \left(j^{k-1}\tilde{g}\right)^{-1} = j_x^1 \left(\tilde{g}^{-1}, \tilde{\omega}^{(1)}, \ldots, \tilde{\omega}^{(k-1)}\right).$$

Hence,

$$\left(g, \xi^{(1)}, \ldots, \xi^{(k)}\right)^{-1} = \left(g^{-1}, \omega^{(1)}, \ldots, \omega^{(k-1)}, \left(\tilde{\nabla}^{(k-1)}\tilde{\omega}^{(k-1)}\right)(x)\right).$$

By recalling that $\left(\tilde{\nabla}^{(j)}\tilde{\xi}^{(j)}\right)(x) = \xi^{(j+1)}$, $1 \le j \le k-1$, and by using Proposition 3.8, we conclude:

$$\begin{aligned}\left(\tilde{\nabla}^{(k-1)}\tilde{\omega}^{(k-1)}\right)(x) &= \tilde{\nabla}^{(k-1)}\left(\sum_{\lambda \in \mathcal{P}^a(k-1)} (-1)^l \ \mathrm{Ad}_{\tilde{g}^{-1}} \circ \left(\mathrm{ad}\left(\tilde{\xi}^{(j_1)}\right) \ldots \mathrm{ad}\left(\tilde{\xi}^{(j_{l-1})}\right)\left(\tilde{\xi}^{(j_l)}\right)\right)\right)(x) \\ &= \sum_{\lambda \in \mathcal{P}^a(k-1)} (-1)^{l+1} \ \mathrm{ad}\left(\mathrm{Ad}_{g^{-1}} \circ \xi^{(1)}\right)\left(\mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{ad}\left(\xi^{(j_1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l)}\right)\right)\right) \\ &\quad + \sum_{\lambda \in \mathcal{P}^a(k-1)} (-1)^l \ \mathrm{Ad}_{g^{-1}} \circ \sum_{s=1}^{l-1} \left(\mathrm{ad}\left(\xi^{(j_1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_s+1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l)}\right)\right) \\ &\quad + \sum_{\lambda \in \mathcal{P}^a(k-1)} (-1)^l \ \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{ad}\left(\xi^{(j_1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l+1)}\right)\right) \\ &= \sum_{\lambda \in \mathcal{P}^a(k)} (-1)^l \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{ad}\left(\xi^{(j_1)}\right) \ldots \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l)}\right)\right) = \omega^{(k)},\end{aligned}$$

where we have used (3.4) and the fact that $\mathrm{ad}\left(\mathrm{Ad}_h(\xi)\right)\left(\mathrm{Ad}_h(\eta)\right) = \mathrm{Ad}_h\left(\mathrm{ad}(\xi)(\eta)\right)$ for each $h \in G$ and $\xi, \eta \in \mathfrak{g}$.

Thanks to Proposition 3.5, for $1 \le j \le k$, we may rewrite (3.3) as

$$\zeta^{(j)} = \xi^{(j)} + \sum_{l=1}^{j} \sum_{j_1+\cdots+j_l=j} N(j_1,\dots,j_l)\ \mathrm{ad}\left(\xi^{(j_{l-1})}\right)\dots\mathrm{ad}\left(\xi^{(j_1)}\right)\left(\mathrm{Ad}_g \circ \eta^{(j_l)}\right).$$

Similarly, (3.5) may be expressed as

$$\omega^{(j)} = \sum_{l=1}^{j} \sum_{j_1+\cdots+j_l=j} (-1)^l\ N(j_1,\dots,j_l)\ \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{ad}\left(\xi^{(j_1)}\right)\dots\mathrm{ad}\left(\xi^{(j_{l-1})}\right)\left(\xi^{(j_l)}\right)\right).$$

Example 3.1 (Second order jet) *Let us compute the explicit expressions for the case $k = 2$. Let $\left(g,\xi^{(1)},\xi^{(2)}\right),\left(h,\eta^{(1)},\eta^{(2)}\right) \in J^2_x(X,G)$ and denote by $[\cdot,\cdot]$ the Lie bracket on $\mathfrak{g}$. An easy check from (3.3) yields*

$$\left(g,\xi^{(1)},\xi^{(2)}\right)\left(h,\eta^{(1)},\eta^{(2)}\right) = \left(gh,\xi^{(1)}+\mathrm{Ad}_g\circ\eta^{(1)},\xi^{(2)}+\mathrm{Ad}_g\circ\eta^{(2)}+\left[\xi^{(1)},\mathrm{Ad}_g\circ\eta^{(1)}\right]\right).$$

In the same vein, from (3.5) we obtain

$$\left(g,\xi^{(1)},\xi^{(2)}\right)^{-1} = \left(g^{-1},-\mathrm{Ad}_{g^{-1}}\circ\xi^{(1)},-\mathrm{Ad}_{g^{-1}}\circ\xi^{(2)}+\mathrm{Ad}_{g^{-1}}\circ\left[\xi^{(1)},\xi^{(1)}\right]\right).$$

3.4 Local expression of the trivialization

In this section, we work in a coordinated chart $(\mathcal{U},x)$ of X, where we write $x = (x^\mu) = (x^1,\dots,x^n)$. This enables us to (locally) identify $X = \mathbb{R}^n$ and, hence, $\bigotimes^k T^*X = X \times \bigotimes^k \mathbb{R}^n$. Note that sections of that bundle are of the form $\xi^{(k)} = \left(\mathrm{id}_X,\hat{\xi}^{(k)}\right)$ for some function $\hat{\xi}^{(k)} : X \to \bigotimes^k \mathbb{R}^n$. By abusing the notation, we identify $\xi^{(k)} = \hat{\xi}^{(k)}$. This way, if we pick the canonical flat connection on the trivial bundle $T^*X = X \times \mathbb{R}^n \to X$, then we have $\nabla^{(k)}\alpha = d\alpha$ for each $\alpha \in \Gamma\left(X \times \bigotimes^k \mathbb{R}^n \to X\right)$.

Example 3.2 *Let $X = \mathbb{R}$ and denote by $t \in \mathbb{R}$ its global coordinate. In this case, a function $g : \mathbb{R} \to G$ is a just curve on G and, thus, its derivative, $dg : T\mathbb{R} \to TG$, is uniquely determined by its velocity, $\dot{g} : \mathbb{R} \to g^*(TG)$. Namely, for each $t \in \mathbb{R}$ we have*

$$(dg)_t : T_t\mathbb{R} \simeq \mathbb{R} \to T_{g(t)}G, \qquad v \mapsto (dg)_t(v) = \dot{g}(t)v.$$

By using the velocity instead of the derivative, we get a map analogous to (3.1),

$$J^k(\mathbb{R},G) \to G \times k\mathfrak{g}, \qquad j^k_t g \mapsto \left(g(t),\xi^{(1)}(t),\dots,\xi^{(k)}(t)\right),$$

where $k\mathfrak{g} = \mathfrak{g} \oplus \overset{k}{\dots} \oplus \mathfrak{g}$ and

$$\begin{cases} \xi^{(1)}(t) = \left(dR_{g(t)^{-1}}\right)_{g(t)}\left(\dot{g}(t)\right) \in \mathfrak{g}, \\ \xi^{(j)}(t) = \dot{\xi}^{(j-1)}(t) \in \mathfrak{g}, \quad 2 \le j \le k. \end{cases}$$

Note that the information about the base point $t \in \mathbb{R}$ is lost and, subsequently, the new map is not injective anymore. Nevertheless, if we restrict ourselves to a single fiber —for instance, the fiber over $t = 0$— we obtain an isomorphism of bundles over G: $J^k_0(\mathbb{R},G) \simeq G \times k\mathfrak{g}$. The group structure of $J^k_0(\mathbb{R},G)$ under this identification was originally investigated in [93].

Additionally, we now assume that we are working with a matrix group, i.e., $G \subset GL(N,\mathbb{R})$ and $N \in \mathbb{Z}^+$. Hence, $(dR_h)_g(U_g) = U_g h$, is the matrix multiplication for each $g,h \in G$ and $U_g \in T_gG$.

Let $g = (g_{ab})_{a,b=1}^N : X \to G$ be a (local) function, where $g_{ab} : X \to \mathbb{R}$, $1 \le a, b \le N$. Then, for each multi-index J, we may consider the partial derivative of g as the following (local) function

$$\frac{\partial^{|J|} g}{\partial x^J} = \left(\frac{\partial^{|J|} g_{ab}}{\partial x^J} \right)_{a,b=1}^N : X \to \mathfrak{gl}(N, \mathbb{R}).$$

Subsequently, the tangent map of g is given by $dg = (\partial g / \partial x^\mu) dx^\mu$, and analogous for higher-order tangent maps: $d^2 g \equiv d(dg) = (\partial^2 g / \partial x^\mu \partial x^\nu) dx^\nu \otimes dx^\mu$, and so on. In order to keep the notation simple, it will be useful to define the maps

$$\xi_{\mu_1 \dots \mu_j} = \frac{\partial^j g}{\partial x^{\mu_1} \dots \partial x^{\mu_j}} g^{-1} : X \to \mathfrak{gl}(N, \mathbb{R}), \qquad 1 \le \mu_1, \dots, \mu_j \le n, \ j \in \mathbb{Z}^+.$$

Observe that $\xi_{\mu_{\sigma_1(1)} \dots \mu_{\sigma_j(j)}} = \xi_{\mu_1 \dots \mu_j}$ for any permutation $\sigma = (\sigma_1, \dots, \sigma_j) \in \mathcal{S}_j$. Hence, we can always rearrange the indices to be in ascending order. By the Leibniz's rule, it is easily checked that

$$\frac{\partial \xi_{\mu_1 \dots \mu_j}}{\partial x^\nu} = \xi_{\mu_1 \dots \mu_j \nu} - \xi_{\mu_1 \dots \mu_j} \xi_\nu, \qquad 1 \le \mu_1, \dots, \mu_j, \nu \le n, \ j \in \mathbb{Z}^+. \tag{3.6}$$

Lastly, given $1 \le \mu_1, \dots, \mu_j \le n$ and a subset $\kappa = \{\kappa_1, \dots, \kappa_r\} \subset \{1, \dots, j\}$, $1 \le r \le j$, we denote

$$\xi_{\kappa(\mu_1 \dots \mu_j)} = \xi_{\mu_{\kappa_1} \dots \mu_{\kappa_r}}.$$

This way, given a partition $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{P}(j)$ we denote

$$\xi_{\lambda(\mu_1 \dots \mu_j)} = \prod_{i=1}^l \xi_{\lambda_i(\mu_1 \dots \mu_j)} = \xi_{\lambda_1(\mu_1 \dots \mu_j)} \cdots \xi_{\lambda_l(\mu_1 \dots \mu_j)}. \tag{3.7}$$

Lemma 3.13. *For each $k \in \mathbb{Z}^+$, $k \ge 2$, $1 \le \mu_1, \dots, \mu_k \le n$ and $\lambda \in \mathcal{P}_1^+(k-1)$ we have*

$$\frac{\partial}{\partial x^{\mu_k}} \xi_{\lambda(\mu_1 \dots \mu_{k-1})} = \sum_{s=1}^l \left(\xi_{\lambda^+_{[s]}(\mu_1 \dots \mu_k)} - \xi_{\lambda^-_{[s]}(\mu_1 \dots \mu_k)} \right).$$

Proof. It is a computation using (3.6), as well as the definitions of $\lambda^+_{[s]}$ and $\lambda^-_{[s]}$. Indeed,

$$\begin{aligned}
\frac{\partial}{\partial x^{\mu_k}} \xi_{\lambda(\mu_1 \dots \mu_{k-1})} &= \frac{\partial}{\partial x^{\mu_k}} \prod_{i=1}^l \xi_{\lambda_i(\mu_1 \dots \mu_{k-1})} \\
&= \sum_{s=1}^l \left(\left(\prod_{i=1}^{s-1} \xi_{\lambda_i(\mu_1 \dots \mu_{k-1})} \right) \left(\frac{\partial}{\partial x^{\mu_k}} \xi_{\lambda_s(\mu_1 \dots \mu_{k-1})} \right) \left(\prod_{i=s+1}^l \xi_{\lambda_i(\mu_1 \dots \mu_{k-1})} \right) \right) \\
&= \sum_{s=1}^l \left(\left(\prod_{i=1}^{s-1} \xi_{\lambda_i(\mu_1 \dots \mu_{k-1})} \right) \left(\xi_{(\lambda_s \cup \{k\})(\mu_1 \dots \mu_k)} - \xi_{\lambda_s(\mu_1 \dots \mu_{k-1})} \xi_{\mu_k} \right) \left(\prod_{i=s+1}^l \xi_{\lambda_i(\mu_1 \dots \mu_{k-1})} \right) \right) \\
&= \sum_{s=1}^l \left(\xi_{\lambda^+_{[s]}(\mu_1 \dots \mu_k)} - \xi_{\lambda^-_{[s]}(\mu_1 \dots \mu_k)} \right).
\end{aligned}$$

We are ready to compute the coordinated expression of the injective morphism $J^k(X, G) \hookrightarrow G \times \bigoplus_{j=1}^k \left(\bigotimes^j T^* X \otimes \mathfrak{g} \right)$.

Theorem 3.14. *Let $k, N \in \mathbb{Z}^+$, ∇ be the canonical flat connection on the trivial bundle $T^* X = \mathbb{R}^n \times \mathbb{R}^n$ and $G \subset GL(N, \mathbb{R})$ (and, thus, $\mathfrak{g} \subset \mathfrak{gl}(N, \mathbb{R})$). Then the components of the map (3.1) are given by*

$$\xi^{(j)}(x) = \sum_{\lambda \in \mathcal{P}_1^+(j)} \epsilon(\lambda) \ \xi_{\lambda(\mu_1 \dots \mu_j)}(x) \ dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_j}, \qquad 1 \le j \le k, \tag{3.8}$$

where $\epsilon : \mathcal{P}_1^+(j) \to \{-1, 1\}$ is defined by recurrence as follows: $\epsilon(\{1\}) = 1$, $\epsilon(\lambda^+_{[s]}) = \epsilon(\lambda)$ and $\epsilon(\lambda^-_{[s]}) = -\epsilon(\lambda)$ for each $j \in \mathbb{Z}^+$, $\lambda \in \mathcal{P}_1^+(j)$ and $1 \le s \le j$.

Proof. Again, we show the result by induction in k. For $k = 1$, we have

$$\xi^{(1)}(x) = (dg)_x g(x)^{-1} = (\partial g/\partial x^\mu)(x) g(x)^{-1} dx^\mu = \xi_\mu(x) dx^\mu.$$

Note that this expression agrees with (3.8) since $\mathcal{P}_1^+(1) = \{(\{1\})\}$ and $\epsilon(\{1\}) = 1$.

To conclude, given $k > 1$, we assume that the result holds for $k-1$. Hence,

$$\begin{aligned}
\xi^{(k)}(x) = \left(d\xi^{(k-1)}\right)_x &= d\left(\sum_{\lambda\in\mathcal{P}_1^+(k-1)} \epsilon(\lambda)\xi_{\lambda(\mu_1\dots\mu_{k-1})} dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_{k-1}}\right)_x \\
&= \sum_{\lambda\in\mathcal{P}_1^+(k-1)} \epsilon(\lambda)\left(\frac{\partial}{\partial x^{\mu_k}}\xi_{\lambda(\mu_1\dots\mu_{k-1})}\right)(x) dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_k} \\
&= \sum_{\lambda\in\mathcal{P}_1^+(k-1)} \sum_{s=1}^{l} \epsilon(\lambda)\left(\xi_{\lambda^+_{[s]}(\mu_1\dots\mu_k)}(x) - \xi_{\lambda^-_{[s]}(\mu_1\dots\mu_k)}(x)\right) dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_k} \\
&= \sum_{\lambda\in\mathcal{P}_1^+(k)} \epsilon(\lambda)\xi_{\lambda(\mu_1\dots\mu_k)}(x) dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_k},
\end{aligned}$$

where we have used Proposition 3.4 and Lemma 3.13.

Recall that given a Lie algebra $\mathfrak{g}$ there exists an associative algebra with identity, which is called *universal enveloping algebra* of $\mathfrak{g}$ (cf. [49, 92]) and is denoted by $U(\mathfrak{g})$, together with a linear map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ such that

(i) $\iota([\xi,\eta]) = \iota(\xi)\iota(\eta) - \iota(\eta)\iota(\xi)$ for each $\xi, \eta \in \mathfrak{g}$.

(ii) $U(\mathfrak{g})$ is generated, as an algebra, by $\{\iota(\xi) \mid \xi \in \mathfrak{g}\}$.

(iii) If $\mathcal{A}$ is an associative algebra with identity and $i : \mathfrak{g} \to \mathcal{A}$ is linear and $i([\xi,\eta]) = i(\xi)i(\eta) - i(\eta)i(\xi)$ for each $\xi, \eta \in \mathfrak{g}$, then there exists a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $i = \phi \circ \iota$.

The following result ensures that we may identify $\mathfrak{g} \ni \xi \equiv \iota(\xi) \in U(\mathfrak{g})$ and we regard $\mathfrak{g} \subset U(\mathfrak{g})$ as a vector subspace.

Theorem 3.15 (Poincaré–Birkhoff–Witt [92, Theorem 3.2.2]). *Let $\{B_1, \dots, B_m\}$ be a basis of $\mathfrak{g}$, then*

$$\operatorname{span}\left\{\iota(\xi_1)^{\alpha_1}\dots\iota(\xi_m)^{\alpha_m} \mid \alpha_i \in \mathbb{Z},\ \alpha_i \geq 0,\ 1 \leq i \leq m\right\},$$

where $\iota(\xi)^0 = 1$, the identity of $U(\mathfrak{g})$, is a basis of $U(\mathfrak{g})$ as a vector space. In particular, the map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective.

In the vein of future generalization of Theorem 3.14 to non-matrix groups we note that the maps $\xi_{\mu_1\dots\mu_j}$ take values in the universal enveloping algebra of $\mathfrak{g}$. This can be checked by induction; namely, for each $j \in \mathbb{Z}^+$, $\xi_{\mu_1\dots\mu_j} : X \to U_j(\mathfrak{g})$, where we define

$$U_j(\mathfrak{g}) = \mathfrak{g} + \operatorname{span}\left\{B_{i_1}\dots B_{i_j} \mid (i_1,\dots,i_j) \in \mathcal{S}_j\right\} \subset U(\mathfrak{g}),$$

the sum being the one of vector subspaces. Note that the sum becomes direct sum, $\oplus$, for $j > 1$, whereas $U_1(\mathfrak{g}) = \mathfrak{g} + \mathfrak{g} = \mathfrak{g}$. In any case, despite expression (3.8), the elements $\xi^{(j)}$ take values in $\mathfrak{g}$ by construction.

Lastly, we use the previous theorem to compute the image of the map (3.1) for $k = 2$. We denote by

$$\operatorname{Sym} : \otimes^k T^*X \to \vee^k T^*X, \qquad \operatorname{Skew} : \otimes^k T^*X \to \wedge^k T^*X$$

the symmetrization and skew-symmetrization maps, respectively. Recall that for $k = 2$ these maps induce the decomposition $T^*X \otimes T^*X = (T^*X \vee T^*X) \oplus (T^*X \wedge T^*X)$. Similarly, for each $\xi_1, \xi_2 \in U(\mathfrak{g})$ we denote by $[\xi_1, \xi_2] = \xi_1\xi_2 - \xi_2\xi_1$ the commutator, and by $\{\xi_1, \xi_2\} = \xi_1\xi_2 + \xi_2\xi_1$ the anti-commutator.

Corollary 3.16. *Let $X = \mathbb{R}^n$ and $G \subset GL(N, \mathbb{R})$ (and, thus, $\mathfrak{g} \subset \mathfrak{gl}(N, \mathbb{R})$). Then*

$$J^2(X, G) \simeq G \times V_2 \subset G \times (T^*X \otimes \mathfrak{g}) \oplus (T^*X \otimes T^*X \otimes \mathfrak{g}),$$

where V_2 is the affine sub-bundle defined as

$$V_2 = \left\{ (\xi_x, \alpha_x) \in (T^*X \otimes \mathfrak{g}) \oplus (T^*X \otimes T^*X \otimes \mathfrak{g}) \mid \mathrm{Skew}(\alpha_x) = -\frac{1}{2}[\xi_x, \xi_x] \right\}.$$

In particular,

$$\mathrm{Sym}(\alpha_x) = \xi_{\mu\nu}(x) dx^\mu \otimes dx^\nu - \frac{1}{2}\{\xi_x, \xi_x\}.$$

Proof. Since the map (3.1) is an injective morphism, it is enough to show that its image is, precisely, $G \times ((T^*X \otimes \mathfrak{g}) \oplus V_2)$. Thank to the previous theorem, we know that (3.1) is given by

$$j^2_x g \mapsto \left(g(x), \xi^{(1)}(x) = \xi_\mu(x) dx^\mu, \xi^{(2)}(x) = (\xi_{\mu\nu}(x) - \xi_\mu(x)\xi_\nu(x)) dx^\mu \otimes dx^\nu \right),$$

where $\xi_\mu(x) = (\partial g / \partial x^\mu)(x) g(x)^{-1}$ and $\xi_{\mu\nu}(x) = (\partial^2 g / \partial x^\mu \partial x^\nu)(x) g(x)^{-1}$. It is clear that the first and second components of the map are surjective onto G and $T^*X \otimes \mathfrak{g}$, respectively. Additionally, it is easy to check that

$$\mathrm{Skew}\left(\xi^{(2)}(x)\right) = -\frac{1}{2}\left[\xi^{(1)}(x), \xi^{(1)}(x)\right],$$

Analogously,

$$\mathrm{Sym}\left(\xi^{(2)}(x)\right) = \xi_{\mu\nu}(x) dx^\mu \otimes dx^\nu - \frac{1}{2}\left\{\xi^{(1)}(x), \xi^{(1)}(x)\right\},$$

and we conclude.

Chapter 4

Gauge reduction in covariant field theory

This chapter is based on our work [18], where we develop a Lagrangian reduction theory for covariant field theories with gauge symmetries. Some preliminary results were presented in [81]. These symmetries are modeled by a Lie group fiber bundle acting fiberwisely on a configuration bundle. In order to reduce the variational principle, we utilize generalized principal connections. After obtaining the reduced equations, we give the reconstruction condition and we relate the vertical reduced equation with the Noether theorem. Some applications to illustrate the theory are given in Chapter 5. The construction of a general Lagrangian reduction procedure for (covariant) field theories with local symmetries here presented extends the classical case investigated in [32] for global symmetries. In some sense, we can say that this is the culmination of the reduction program in its generality.

As explained in the introduction, for this type of reduction we consider a fibered action of a Lie group bundle $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ on the configuration bundle of the theory $\pi_{Y,X} : Y \to X$. Since the Lagrangian density is defined on the first jet bundle J^1Y, we lift the action to the first jets, thus yielding a fibered action of $\pi_{J^1\mathcal{G},X} : J^1\mathcal{G} \to X$ on $\pi_{J^1Y,X} : J^1Y \to X$. In most examples found in the literature the Lagrangian density is not invariant by the whole jet bundle, but only by an affine Lie group subbundle $H \subset J^1\mathcal{G}$. This bundle $H \to X$ can be thought as a non-holonomic constraint in the nature of the derivatives of the symmetries. The existence of this constraint is the main source of complexity of the gauge reduction theory, since the invariance by the full jet is actually rather trivial as we show in the examples of the next chapter. Therefore, the reduced configuration bundle is $(J^1Y)/H \to Y/\mathcal{G}$. We start the chapter by analyzing the geometry of $(J^1Y)/H$. with the aid of a generalized principal connection. Next, we drop the Hamilton's variational principle to the reduced bundle and we apply it to the reduced Lagrangian in order to obtain the reduced field equations. We finish by investigating the reconstruction problem and the relation of the reduced equations with the Noether theorem.

4.1 Geometry of the reduced configuration space

Let $\pi_{\mathcal{G},X}$ be a Lie group bundle. A Lie group bundle connection ν provides an identification of the affine bundle $J^1\mathcal{G} \to \mathcal{G}$ with its model vector bundle $T^*X \otimes V\mathcal{G} \simeq \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$. If the Lie group bundle connection is regarded as a 1-form ν taking values in $\mathfrak{g}$, the identification is given explicitly as

$$\Theta_\nu : J^1\mathcal{G} \to \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}), \quad j^1_x\eta \mapsto (\eta(x), \nu \circ (d\eta)_x)\,. \tag{4.1}$$

We can transfer the Lie group bundle structure of $J^1\mathcal{G} \to X$ (for example, cf. [37, §3, Theorem 1]) via the above identification, as

$$(g, \xi_x)\,(h, \eta_x) = (gh, \xi_x + \mathrm{Ad}_g \circ \eta_x), \qquad (g, \xi_x), (h, \eta_x) \in \mathcal{G}_x \times (T^*_x X \otimes \mathfrak{g}_x),\ x \in X.$$

Note that the identity element is $(1_x, 0_x) \in \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$.

Assumption. Henceforth, let $\pi_{Y,X}$ be a fiber bundle on which $\pi_{\mathcal{G},X}$ acts fiberwisely, freely and properly, and $\pi_{H,X}$ be a closed Lie group subbundle $H \subset J^1\mathcal{G}$ of $\pi_{J^1\mathcal{G},X}$ such that $\pi_{J^1\mathcal{G},\mathcal{G}}(H) = \mathcal{G}$. Besides, we suppose that $\pi_{H,\mathcal{G}}$ is an affine subbundle of $\pi_{J^1\mathcal{G},\mathcal{G}}$. Identification (4.1) enables us to regard

$$\Theta_\nu(H) \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}),$$

We denote $\Theta_\nu(H)$ by the same symbol, H, since the context will clarify the distinction between the two objects. Then $H \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$ is both a closed Lie group subbundle over X and an affine subbundle over $\mathcal{G}$.

Proposition 4.1. *Let $H \subset J^1\mathcal{G}$ be a closed Lie group subbundle of $\pi_{J^1\mathcal{G},X}$ and an affine subbundle of $\pi_{J^1\mathcal{G},\mathcal{G}}$. Let $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ be a bundle morphism such that $(g, \varphi(g)) \in H$ for each $g \in \mathcal{G}$ and[1] $\varphi(1_x) = 0_x$ for each $x \in X$. Taking into account the identification (4.1) there exists a vector subbundle $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$ such that*

$$H = \{(g, \hat{\eta}_x) \in \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}) \mid \hat{\eta}_x = \varphi(g) + \eta_x,\ \eta_x \in \mathfrak{H}_x\}\,.$$

Furthermore, $\mathfrak{H}$ is Ad*-invariant, i.e., $\mathrm{Ad}_g(\mathfrak{H}_x) \subset \mathfrak{H}_x$ for each $x \in X$ and $g \in \mathcal{G}_x$.*

Proof. Let $g \in \mathcal{G}$ and $x = \pi_{\mathcal{G},X}(g) \in X$. Since $H_g \subset \{g\} \times (T^*_x X \otimes \mathfrak{g}_x)$ is an affine subspace, we may write $H_g = \{g\} \times (\varphi(g) + \mathfrak{H}_g)$ for certain vector subspace $\mathfrak{H}_g \subset T^*_x X \otimes \mathfrak{g}_x$. In addition, $H_x \subset \mathcal{G}_x \times (T^*_x X \otimes \mathfrak{g}_x)$ is a Lie subgroup, whence

$$(1_x, \eta_x) \cdot (g, \varphi(g)) = (g, \varphi(g) + \eta_x) \in H_g, \qquad \eta_x \in \mathfrak{H}_{1_x}.$$

Hence, $\mathfrak{H}_g = \mathfrak{H}_{1_x}$ and we denote $\mathfrak{H}_x \equiv \mathfrak{H}_{1_x}$. We thus define $\mathfrak{H} = \bigsqcup_{x \in X} \mathfrak{H}_x \to X$.

For the second part, observe that for each $\eta_x \in \mathfrak{H}_x$ and $g \in \mathcal{G}_x$ we have

$$(g, \varphi(g))(1_x, \eta_x) = (g, \varphi(g) + \mathrm{Ad}_g \circ \eta_x) \in H_g.$$

Note that, as a vector subbundle, we may consider the quotient $(T^*X \otimes \mathfrak{g})/\mathfrak{H}$, which is again a vector bundle over X. We denote the corresponding elements by $[\xi_x]_{\mathfrak{H}}$.

Lemma 4.2. *The map $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ satisfies*

$$[\varphi(gh)]_{\mathfrak{H}} = [\varphi(g) + \mathrm{Ad}_g \circ \varphi(h)]_{\mathfrak{H}}\,, \qquad g, h \in \mathcal{G}_x,\ x \in X.$$

Proof. We have that

$$(g, \varphi(g)) \cdot (h, \varphi(h)) = (gh, \varphi(g) + \mathrm{Ad}_g \circ \varphi(h)) \in H_{gh}.$$

Hence, $\varphi(g) + \mathrm{Ad}_d \circ \varphi(h) - \varphi(gh) \in \mathfrak{H}_x$ and we conclude.

Now we define the following map

$$\begin{array}{rl} \Phi_\varphi : (Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}) \times_X \mathcal{G} \to & Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H} \\ ((y, [\xi_x]_{\mathfrak{H}}), g) \mapsto & \left(y \cdot g, [\mathrm{Ad}_{g^{-1}} \circ (\xi_x + \varphi(g))]_{\mathfrak{H}}\right). \end{array} \tag{4.2}$$

Remark 4.1 *If the action of $\mathcal{G}$ on Y is left, then the map (4.2) should be*

$$\begin{array}{rl} \Phi_\varphi : \mathcal{G} \times_X (Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}) \to & Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H} \\ (g, (y, [\xi_x]_{\mathfrak{H}})) \mapsto & (g \cdot y, [\mathrm{Ad}_g \circ \xi_x + \varphi(g)]_{\mathfrak{H}}) \end{array} \tag{4.3}$$

[1] Observe that $(1_x, 0_x) \in H_{1_x}$, since $H \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$ is a Lie group subbundle (over X).

Thanks to the Ad-invariance of $\mathfrak{H}$, the map (4.2) is well-defined. Moreover, a straightforward computation shows that it is a (free and proper) right fibered action. As a result, we may consider the corresponding quotient, which is a fiber bundle over $Y/\mathcal{G}$:

$$(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} = \frac{Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}}{\Phi_\varphi}. \tag{4.4}$$

The elements of this quotient are denoted by $[\![y, \xi_x]\!]_{\mathfrak{H},\varphi}$, where $y \in Y_x$, $\xi_x \in (T^*X \otimes \mathfrak{g})_x$ and $x \in X$. Analogously, we have the fibered action given by (4.2) with $\varphi = 0$, which yields the quotient

$$(T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}} = \frac{Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}}{\Phi_0}$$

Proposition 4.3. *In the above conditions, the following statements are true:*

(i) $(T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}} \to Y/\mathcal{G}$ *is a vector bundle.*

(ii) $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to Y/\mathcal{G}$ *is an affine bundle modelled on* $(T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}} \to Y/\mathcal{G}$.

Proof. For (i), it is easy to check that the following operations are well-defined

$$[\![y, \xi_x]\!]_{\mathfrak{H},0} + [\![y, \zeta_x]\!]_{\mathfrak{H},0} = [\![y, \xi_x + \zeta_x]\!]_{\mathfrak{H},0}, \quad \lambda[\![y, \xi_x]\!]_{\mathfrak{H},0} = [\![y, \lambda\xi_x]\!]_{\mathfrak{H},0},$$

for each $[\![y, \xi_x]\!]_{\mathfrak{H},0}, [\![y, \zeta_x]\!]_{\mathfrak{H},0} \in (T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}$ and $\lambda \in \mathbb{R}$. Likewise, for (ii), the following map is well-defined

$$[\![y, \xi_x]\!]_{\mathfrak{H},\varphi} + [\![y, \zeta_x]\!]_{\mathfrak{H},0} = [\![y, \xi_x + \zeta_x]\!]_{\mathfrak{H},\varphi},$$

for each $[\![y, \xi_x]\!]_{\mathfrak{H},\varphi} \in (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$ and $[\![y, \zeta_x]\!]_{\mathfrak{H},0} \in (T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}$.

Remark 4.2 *Observe that, given a Lie group subbundle $H \subset J^1\mathcal{G}$, the map $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ is not unique. Furthermore, $H \subset \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$ is a vector subbundle over $\mathcal{G}$ (that is, $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} = (T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}$) if and only we can choose the map $\varphi = 0$. This is equivalent to the condition $\hat{\nu}(\mathcal{G}) \subset H$ when we regard H as a Lie group subbundle of $J^1\mathcal{G}$.*

Even though the case $\varphi = 0$ is very relevant in examples, in the following we will assume that φ does not necessarily vanish.

Analogous to the adjoint bundle defined in Example 2.3, we can consider the tensor product $T^*X \otimes \tilde{\mathfrak{g}}$ as the quotient

$$T^*X \otimes \tilde{\mathfrak{g}} = \frac{Y \times_X (T^*X \otimes \mathfrak{g})}{\mathcal{G}} \to Y/\mathcal{G}$$

by letting $\mathcal{G}$ act trivially on T^*X. As the vector subbundle $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$ obtained in Proposition 4.1 is Ad-equivariant, the action of $\mathcal{G}$ on $Y \times_X (T^*X \otimes \mathfrak{g})$ restricts to $Y \times_X \mathfrak{H}$ and we may consider the corresponding quotient

$$\tilde{\mathfrak{H}} = \frac{Y \times_X \mathfrak{H}}{\mathcal{G}} \subset T^*X \otimes \tilde{\mathfrak{g}} \to Y/\mathcal{G} \tag{4.5}$$

In addition, $\pi_{\tilde{\mathfrak{H}},Y/\mathcal{G}}$ is a vector subbundle of $\pi_{T^*X\otimes\tilde{\mathfrak{g}},Y/\mathcal{G}}$, so it makes sense to consider the corresponding quotient vector bundle, $(T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}$, whose elements are denoted by $[\![y, \xi_x]\!]_{\tilde{\mathfrak{H}}}$, where $y \in Y_x$, $\xi_x \in (T^*X \otimes \mathfrak{g})_x$ and $x \in X$. Moreover, it is easily seen to be isomorphic to $(T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}$.

Lemma 4.4. *In the above conditions, we have the following isomorphism of vector bundles over $Y/\mathcal{G}$,*

$$\frac{T^*X \otimes \tilde{\mathfrak{g}}}{\tilde{\mathfrak{H}}} \simeq \frac{Y \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}}{\Phi_0} = (T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}.$$

On the other hand, the first jet extension of the fibered action Φ is a right fibered action of $\pi_{J^1\mathcal{G},X}$ on $\pi_{J^1Y,X}$

$$\Phi^{(1)} : J^1Y \times_X J^1\mathcal{G} \to J^1Y, \quad \left(j^1_x s, j^1_x\gamma\right) \mapsto j^1_x(\Phi(s,\gamma)). \tag{4.6}$$

where, for (local) sections $s \in \Gamma(\pi_{Y,X})$ and $\gamma \in \Gamma(\pi_{\mathcal{G},X})$, we have the (local) section $\Phi(s,\gamma)(x) = \Phi(s(x),\gamma(x))$, $x \in X$. If we regard 1-jets as tangent maps of sections at a point, that is, $j^1_x s \equiv (ds)_x$ and $j^1_x\gamma \equiv (d\gamma)_x$, we may rewrite Φ^1 as

$$\Phi^{(1)}(j^1_x s, j^1_x\gamma) = (d\Phi)_{(s(x),\gamma(x))} \circ (ds_x, d\gamma_x) : T_xX \to T_{s(x)\cdot\gamma(x)}Y.$$

The following result studies the geometry of the quotient of $\pi_{J^1Y,X}$ by the fibered action (4.6) when restricted to the Lie group subbundle $H \subset J^1\mathcal{G}$.

Theorem 4.5. *Let $\omega \in \Omega^1(Y,\mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$ and consider a Lie group subbundle $H \subset J^1\mathcal{G}$ as in Proposition 4.1. Then the following map is a bundle isomorphism over $Y/\mathcal{G}$:*

$$\begin{aligned} \left(J^1Y\right)/H \to \quad & J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \\ \left[j^1_x s\right]_H \quad & \mapsto \left(j^1_x\sigma_s, \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x\right]\!\right]_{\mathfrak{H},\varphi}\right), \end{aligned} \tag{4.7}$$

where $\sigma_s = [s]_{\mathcal{G}} = \pi_{Y,Y/\mathcal{G}} \circ s \in \Gamma(\pi_{Y/\mathcal{G},X})$.

Proof. For $j^1_x s \in J^1Y$ and $j^1_x\eta \in H$, let s and η be (local) sections defining those jet elements. On one hand, it is clear that $\sigma_{\Phi(s,\eta)} = \sigma_s$. On the other, by (4.1) we have $j^1_x\eta \simeq (\eta(x), \nu \circ (d\eta)_x) \in H$, whence $\nu \circ (d\eta)_x = \varphi(\eta(x)) + \eta_x$ for some $\eta_x \in \mathfrak{H}_x$, by Proposition 4.1. Therefore,

$$\begin{aligned} &\left[\!\left[\Phi(s,\eta)(x), \omega_{\Phi(s,\eta)(x)} \circ (d\Phi(s,\eta))_x\right]\!\right]_{\mathfrak{H},\varphi} \\ &\quad = \left[\!\left[s(x)\cdot\eta(x), \omega_{s(x)\cdot\eta(x)} \circ (d\Phi)_{(s(x),\eta(x))} \circ ((ds)_x, (d\eta)_x)\right]\!\right]_{\mathfrak{H},\varphi} \\ &\quad = \left[\!\left[s(x)\cdot\eta(x), \mathrm{Ad}_{\eta(x)^{-1}}\left(\omega_{s(x)} \circ (ds)_x + \nu \circ (d\eta)_x\right)\right]\!\right]_{\mathfrak{H},\varphi} \\ &\quad = \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x + \eta_x\right]\!\right]_{\mathfrak{H},\varphi} \\ &\quad = \left[\!\left[s(x), \omega_{s(x)} \circ (ds)_x\right]\!\right]_{\mathfrak{H},\varphi}. \end{aligned}$$

Subsequently, the map is well-defined. A straightforward computation shows that the inverse map is

$$J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to \left(J^1Y\right)/H, \quad \left(j^1_x\sigma, \left[\!\left[y, \xi_x\right]\!\right]_{\mathfrak{H},\varphi}\right) \mapsto \left[Hor^\omega_y \circ (d\sigma)_x + (\xi_x)^*_y\right]_H,$$

where $Hor^\omega_y : T_{[y]_{\mathcal{G}}}(Y/\mathcal{G}) \to T_yY$ is the horizontal lifting given by ω at $y \in Y$. Observe that it is well-defined thanks to Lemma 2.11 and Proposition 2.15.

4.1.1 Connections on the reduced spaces

Let ω be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$ and ∇^X be a linear connection on the tangent bundle of X. These connections induce a linear connection on the vector bundle $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$, and an affine connection on the affine bundle $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}$ as follows. Consider the linear connection $\nabla^{\mathfrak{g}}$ on $\pi_{\mathfrak{g},X}$ induced by ν (cf. Proposition 2.8). As above, denote by ${}^{\nu}\|$, ${}^{\mathfrak{g}}\|$ and ${}^{\omega}\|$ the corresponding parallel transports. It is easy to check from the definition that the connections ν and $\nabla^{\mathfrak{g}}$ satisfy the following compatibility relation:

$$ {}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\mathrm{Ad}_g(\xi) = \mathrm{Ad}_{{}^{\nu}\|_{x(a)}^{x(b)}g}\left({}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\xi\right), \qquad g \in \mathcal{G}_{x(a)},\ \xi \in \mathfrak{g}_{x(a)} \tag{4.8} $$

for every curve $x : I \to X$. Indeed, consider the curves $\alpha : I \to \mathfrak{g}$ and $\beta : I \to \mathfrak{g}$ given by

$$ \alpha(t) = \mathrm{Ad}_{{}^{\nu}\|_{x(a)}^{x(t)}g}\left({}^{\mathfrak{g}}\big\|_{x(a)}^{x(t)}\xi\right), \qquad \beta(t) = {}^{\mathfrak{g}}\big\|_{x(a)}^{x(t)}\mathrm{Ad}_g(\xi), \qquad t \in I, $$

Note that $\alpha(a) = \beta(a) = \mathrm{Ad}_g(X)$ and $\pi_{\mathfrak{g},X} \circ \alpha = x$. Thanks to the uniqueness of parallel transport, to show that $\alpha = \beta$ it is enough to check that $\alpha' = \beta'$. This holds, as for each $t \in I$ we have

$$ \begin{aligned} \alpha'(t) &= \frac{d}{dt}\left(dc_{{}^{\nu}\|_{x(a)}^{x(t)}g}\right)_{1_x}\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(t)}\exp(\epsilon\xi)\right) \\ &= \frac{d}{dt}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} c_{{}^{\nu}\|_{x(a)}^{x(t)}g}\left({}^{\nu}\big\|_{x(a)}^{x(t)}\exp(\epsilon\xi)\right) \\ &= \frac{d}{dt}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(t)}c_g\left(\exp(\epsilon\xi)\right) \\ &= \frac{d}{dt}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} {}^{\nu}\big\|_{x(a)}^{x(t)}\exp\left((dc_g)_{1_{x(a)}}(\epsilon\xi)\right) = \beta'(t), \end{aligned} $$

where we have used that ν is a Lie group connection and $c_g : \mathcal{G}_{x(a)} \to \mathcal{G}_{x(a)}$ is a Lie group homomorphism. Equation (4.8) and Proposition 2.13 ensure that the following parallel transport is well-defined.

Proposition 4.6. *The assignment sending any curve $\gamma : I \to Y/\mathcal{G}$ to the map*

$$ \tilde{\mathfrak{g}}\big\|_{\gamma(a)}^{\gamma(b)} : \tilde{\mathfrak{g}}_{\gamma(a)} \to \tilde{\mathfrak{g}}_{\gamma(b)}, \quad [y,\xi]_{\mathcal{G}} \mapsto \left[{}^{\omega}\big\|_{\gamma(a)}^{\gamma(b)}y, {}^{\mathfrak{g}}\big\|_{x(a)}^{x(b)}\xi\right]_{\mathcal{G}}, $$

is a linear parallel transport on $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$, where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$.

We denote by $\nabla^{\tilde{\mathfrak{g}}}$ the corresponding linear connection on $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}$ and by $\nabla^{\tilde{\mathfrak{g}}}/dt$ the corresponding covariant derivative.

Lemma 4.7. *Let $\nabla^{\mathfrak{H}}$ be a linear connection on $\pi_{\mathfrak{H},X}$. The assignment sending a curve $x : I \to X$ to the map*

$$ {}^{H}\big\|_{x(a)}^{x(b)} : H_{x(a)} \to H_{x(b)}, \quad {}^{H}\big\|_{x(a)}^{x(b)}(g,\varphi(g)+\eta_{x(a)}) = \left({}^{\nu}\big\|_{x(a)}^{x(b)}g, \varphi\left({}^{\nu}\big\|_{x(a)}^{x(b)}g\right) + {}^{\mathfrak{H}}\big\|_{x(a)}^{x(b)}\eta_{x(a)}\right) $$

for each $(g,\varphi(g)+\eta_{x(a)}) \in H_{x(a)}$ is a parallel transport on $\pi_{H,X}$.

In the same vein, a parallel transport may be defined on the vector bundle $\overline{H} = \mathcal{G} \times_X \mathfrak{H} \to X$ as follows,

$$ {}^{\overline{H}}\big\|_{x(a)}^{x(b)} : \overline{H}_{x(a)} \to \overline{H}_{x(b)}, \quad {}^{\overline{H}}\big\|_{x(a)}^{x(b)}(g,\eta_{x(a)}) = \left({}^{\nu}\big\|_{x(a)}^{x(b)}g, {}^{\mathfrak{H}}\big\|_{x(a)}^{x(b)}\eta_{x(a)}\right). $$

Now we extend $\nabla^{\mathfrak{H}}$ to a linear connection $\nabla^{\otimes}$ on $T^*X \otimes \mathfrak{g}$ that is *compatible with* ν, i.e., the relation (4.8) holds for ${}^{\otimes}\|$ instead of ${}^{\mathfrak{g}}\|$. Similarly, $\nabla^{\otimes}$ is said to be *compatible with* φ if it satisfies

$$ {}^{\otimes}\big\|_{x(a)}^{x(b)}\varphi(g) = \varphi\left({}^{\nu}\big\|_{x(a)}^{x(b)}g\right), \qquad g \in \mathcal{G}_{x(a)}, \tag{4.9} $$

for each curve $x : I \to X$. Henceforth, we assume that $\nabla^{\otimes}$ is compatible with ν and φ. We are ready to define a linear connection on the reduced bundle.

Proposition 4.8. *The assignment that each curve $\gamma : I \to Y/\mathcal{G}$ corresponds to the map*

$$\Big\|_{\gamma(a)}^{\gamma(b)} : \big((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}\big)_{\gamma(a)} \to \big((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}\big)_{\gamma(b)}\,, \quad [\![y, \xi_x]\!]_{\mathfrak{H},\varphi} \mapsto \Big[\!\!\Big[{}^{\omega}\Big\|_{\gamma(a)}^{\gamma(b)} y,\, {}^{\otimes}\Big\|_{x(a)}^{x(b)} \xi_x \Big]\!\!\Big]_{\mathfrak{H},\varphi}\,,$$

where $x = \pi_{Y/\mathcal{G},X} \circ \gamma$, is an affine parallel transport on $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}$.

Proposition 2.13, (4.8), together with the Ad-invariance of $\mathfrak{H}$ and (4.9), guarantee that it is well-defined. Besides, it is affine thanks to the linearity of $\nabla^{\otimes}$. We denote by ∇ the affine connection on $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}$ associated to $\big\|$ and by ∇/dt the corresponding covariant derivative.

4.2 Reduced variations on $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$

Given a section $s \in \Gamma(\pi_{Y,X})$, the corresponding *reduced section* is defined to be

$$\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi} \in \Gamma\left(\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},X}\right).$$

Observe that the projection $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}$ is nothing but the quotient section $\sigma_s = [s]_{\mathcal{G}} = \pi_{Y,Y/\mathcal{G}} \circ s \in \Gamma\left(\pi_{Y/\mathcal{G},X}\right)$. We aim to compute the variation $\delta\overline{s}$ of the reduced section induced by a variation δs of the unreduced section. More particularly, we are first interested in the vertical part

$$\delta^{\nabla}\overline{s}(x) = \delta\overline{s}(x)^v = \left.\frac{\nabla\overline{s}_t(x)}{dt}\right|_{t=0}, \qquad x \in X,$$

of that reduced variation with respect to the connection ∇ on $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}$ built in Proposition 4.8. For that, we analyze below the corresponding expression of $\delta^{\nabla}\overline{s}$ when the variation $\delta s = ds_t/dt|_{t=0}$ is vertical or horizontal with respect to ω. By linearity, the expression of $\delta^{\nabla}\overline{s}$ will be the combination of both terms. In addition, since $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to Y/\mathcal{G}$ is an affine bundle, we will identify the tangent space of each fiber with the associated vector space, i.e.,

$$V_{\overline{y}}\big((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}\big) = T_{\overline{y}}\big(\big((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}\big)_{\sigma}\big) = \big((T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}}\big)_{\sigma}\,,$$

for each $\overline{y} \in (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$, where $\sigma = \pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}(\overline{y}) \in Y/\mathcal{G}$. Analogously, the restriction of the tangent map of $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ to $\mathfrak{g} = 1^*(V\mathcal{G})$ yields the following morphism of vector bundles,

$$\varphi_* = d\varphi|_{\mathfrak{g}} : \mathfrak{g} \to T^*X \otimes \mathfrak{g}.$$

Lemma 4.9. *If $\{s_t\}$ is a $\pi_{Y,Y/\mathcal{G}}$-vertical variation, that is, $\delta s = \xi_s^*$ for some $\xi \in \Gamma(\pi_{\mathfrak{g},X})$, then we can suppose that it is of the form $s_t = s \cdot \exp(t\,\xi)$. In that case,*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\,\xi)} \circ (s_t^*\omega + \varphi(\exp(-t\,\xi))) = \nabla^{\mathfrak{g}}\xi - \varphi_*\xi.$$

Proof. The first statement is because the variation of the functional only depends on the infinitesimal variation. For the second part, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\,\xi)} \circ s_t^*\omega &= \left.\frac{d}{dt}\right|_{t=0} s_t^*\omega + \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\,\xi)} \circ s^*\omega \\
&= \left.\frac{d}{dt}\right|_{t=0} \omega_{\Phi(s,\exp(t\,\xi))}\left((d\Phi)_{(s,\exp(t\,\xi))}\left(ds, d\exp(t\,\xi)\right)\right) + \mathrm{ad}(\xi)(s^*\omega) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(-t\,\xi)}\left(s^*\omega + \nu \circ d\exp(t\,\xi)\right) + [\xi, s^*\omega] \\
&= \left.\frac{d}{dt}\right|_{t=0} \nu \circ d\exp(t\,\xi) + \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(-t\,\xi)} \circ \nu \circ d1 \\
&\overset{(\star)}{=} \left.\frac{d}{dt}\right|_{t=0} \nu \circ d\exp(t\,\xi) \\
&= \nabla^{\mathfrak{g}}\xi.
\end{aligned}$$

Equality $(\star)$ comes from property (i) in the definition of Lie group connection. Similarly,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi)} \circ \varphi(\exp(-t\xi)) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(-t\xi)) + \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi)} \circ \varphi(\exp(0)) \\ &= -\varphi_* \left((d\exp)_0(\xi)\right) + \mathrm{ad}(\xi)(\varphi(1)) \\ &= -\varphi_* \xi,\end{aligned}$$

where we have used that $\varphi(1) = 0$ and $(d\exp)_0 = \mathrm{id}_{\mathfrak{g}}$.

Proposition 4.10 ($\delta^\nabla \overline{s}$ for vertical variations). *If $\{s_t\}$ is $\pi_{Y,Y/\mathcal{G}}$-vertical, that is, $\delta s = \xi_s^*$ for some $\xi \in \Gamma(\pi_{\mathfrak{g},X})$, then:*

$$\delta^\nabla \overline{s}(x) = \left[\!\left[s(x), \left(\nabla^{\mathfrak{g}}\xi\right)(x) - (\varphi_*\xi)(x) \right]\!\right]_{\mathfrak{H},0}, \qquad x \in X.$$

Proof. The reduced variations are

$$\overline{s}_t(x) = \left[\!\left[s(x) \cdot \exp(t\xi(x)), (s_t^*\omega)_x \right]\!\right]_{\mathfrak{H},\varphi} = \left[\!\left[s(x), \mathrm{Ad}_{\exp(t\xi(x))} \circ ((s_t^*\omega)_x + \varphi(\exp(-t\xi(x)))) \right]\!\right]_{\mathfrak{H},\varphi}.$$

Since $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}_t = \sigma_s$ for all $t \in (-\epsilon,\epsilon)$, we have that $\{\overline{s}_t\}$ is a $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}}$-vertical variation and $\nabla\overline{s}_t(x)/dt|_{t=0} = d\overline{s}_t(x)/dt|_{t=0}$. Subsequently, the previous Lemma results in

$$\begin{aligned}\delta^\nabla \overline{s}(x) &= \left[\!\left[s(x), \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi(x))} \circ ((s_t^*\omega)_x + \varphi(\exp(-t\,\xi_x))) \right]\!\right]_{\mathfrak{H},0} \\ &= \left[\!\left[s(x), (\nabla^{\mathfrak{g}}\xi)(x) - (\varphi_*\xi)(x) \right]\!\right]_{\mathfrak{H},0}.\end{aligned}$$

Lemma 4.11. *If $\{s_t\}$ is ω-horizontal, i.e., $\omega(\delta s) = 0$, then the horizontal component of $\delta\overline{s}$ with respect to ∇ is*

$$\delta\overline{s}(x)^h = \left.\frac{d}{dt}\right|_{t=0} \left[\!\left[s_t(x), (s^*\omega)_x \right]\!\right]_{\mathfrak{H},\varphi}, \qquad x \in X.$$

Proof. Let $Hor^\nabla_{\overline{s}(x)} : T_{\sigma_s(x)}(Y/\mathcal{G}) \to T_{\overline{s}(x)}((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}})$ be the horizontal lift given by ∇ at $\overline{s}(x)$. Then $\delta\overline{s}(x)^h = Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x))$, since $(d\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}})_{\overline{s}(x)} \circ \delta\overline{s}(x) = \delta\sigma_s(x)$. By construction of ∇, we have[2]:

$$Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x)) = (dq)_{(s(x),(s^*\omega)_x)} \left(Hor^\omega_{s(x)}(\delta\sigma_s(x)), Hor^{\nabla^\otimes}_{(s^*\omega)_x}(0_x) \right),$$

where $q : Y \times_X (T^*X \otimes \mathfrak{g}) \to (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$ is the quotient projection. The horizontality of $\delta s(x)$ yields $Hor^\omega_{s(x)}(\delta\sigma_s(x)) = \delta s(x)$, and it is clear that $Hor^{\nabla^\otimes}_{(s^*\omega)_x}(0_x) = 0_{(s^*\omega)_x}$. Hence,

$$\delta\overline{s}(x)^h = Hor^\nabla_{\overline{s}(x)}(\delta\sigma_s(x)) = \left.\frac{d}{dt}\right|_{t=0} (q \circ \gamma)(t) = \left.\frac{d}{dt}\right|_{t=0} \left[\!\left[s_t(x), (s^*\omega)_x \right]\!\right]_{\mathfrak{H},\varphi},$$

where for the computation of the tangent map dq, we consider the curve $\gamma : (-\epsilon,\epsilon) \to Y \times_X (T^*X \otimes \mathfrak{g})$, $\gamma(t) = (s_t(x), (s^*\omega)_x)$, since $\gamma(0) = (s(x), (s^*\omega)_x)$ and $\gamma'(0) = (\delta s(x), 0_x)$.

Proposition 4.12 ($\delta^\nabla \overline{s}$ for horizontal variations). *If $\{s_t\}$ is ω-horizontal, i.e., $\omega(\delta s) = 0$, then*

$$\delta^\nabla \overline{s}(x) = \left[\!\left[s(x), \Omega_{s(x)}(\delta s(x), (ds)_x) \right]\!\right]_{\mathfrak{H},0}, \qquad x \in X.$$

[2] Recall that $(d\pi_{Y/\mathcal{G},X})_{\sigma_s(x)}(\delta\sigma_s(x)) = 0$, since the variation $\{s_t\}$ is $\pi_{Y,X}$-vertical.

Proof. Thanks to the previous Lemma:

$$\begin{aligned}\delta^{\nabla}\overline{s}(x) &= \delta\overline{s}(x) - \delta\overline{s}(x)^h \\ &= \left.\frac{d}{dt}\right|_{t=0} [\![s_t(x), (s_t^*\omega)_x]\!]_{\mathfrak{H},\varphi} - \left.\frac{d}{dt}\right|_{t=0} [\![s_t(x), (s^*\omega)_x]\!]_{\mathfrak{H},\varphi} \\ &= \left.\frac{d}{dt}\right|_{t=0} [\![s_t(x), (s_t^*\omega)_x - (s^*\omega)_x]\!]_{\mathfrak{H},0} \\ &= \left[\!\!\left[s(x), \left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x \right]\!\!\right]_{\mathfrak{H},0}.\end{aligned}$$

Since the formula that we are proving is local, we can suppose that our bundles are trivial, that is, $\mathcal{G} = X \times G$, $\mathfrak{g} = X \times \mathfrak{g}$ and $Y = Y/\mathcal{G} \times G$, where $\mathfrak{g}$ is the Lie algebra of G. We can thus regard ω as a 1-form on Y with values in $\mathfrak{g}$. Then

$$\left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x = s^*\pounds_{\delta s}\omega = s^*(i_{\delta s}d\omega + d(i_{\delta s}\omega)) = d\omega_{s(x)}(\delta s, ds),$$

where $\pounds$ is the Lie derivative and $\omega(\delta s) = 0$. Then

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} (s_t^*\omega)_x &= d\omega_{s(x)}(\delta s, (ds)^h) + d\omega_{s(x)}(\delta s, (ds)^v) \\ &= d\omega_{s(x)}(\delta s, (ds)^h) \\ &= \mathrm{d}^{\mathfrak{G}}\omega(\delta s, ds) = \Omega_{s(x)}(\delta s, ds),\end{aligned}$$

since $d\omega_{s(x)}(\delta s, (ds)^v) = \delta s(\omega((ds)^v)) - \omega_{s(x)}([\delta s, (ds)^v]) = 0$. Indeed, we are working on a trivialization, so we can write $(ds)^v = \hat{\xi}^*$ for some $\hat{\xi} \in \mathfrak{g}$. Thus, $\omega((ds)^v) = \omega\left(\hat{\xi}^*\right) = \hat{\xi}$ and $\delta s(\hat{\xi}) = 0$. In addition, we have

$$\begin{aligned}[\delta s, (ds)^v] &= \left(\pounds_{(ds)^v}\delta s\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} (d\Phi)_{(s,g(t))}\left(\delta s, 0_{g(t)}\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} U_t = U,\end{aligned}$$

where $g : (-\epsilon, \epsilon) \to G$ is defined as $g(t) = \exp\left(t\hat{\xi}\right)$. Note that $U \in T_sY$ is $\pi_{Y,Y/\mathcal{G}}$-horizontal since so is each $Z_t \in T_{s\cdot g(t)}Y$:

$$\omega_{s\cdot g(t)}\left((d\Phi)_{s,g(t)}\left(\delta s, 0_{g(t)}\right)\right) = \mathrm{Ad}_{g(t)^{-1}}\left(\omega_s(\delta s(x)) + \nu(0_{g(t)})\right) = 0.$$

Since every arbitrary variation δs can be split into its $\pi_{Y,Y/\mathcal{G}}$-vertical and horizontal parts, we obtain the following result. We also make use of Lemma 4.4.

Corollary 4.13. *Let δs be a variation of a section $s \in \Gamma(\pi_{Y,X})$ and consider the induced variation $\delta^{\nabla}\overline{s}$ of the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi} \in \Gamma(\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},X})$. Then:*

$$\delta^{\nabla}\overline{s} = \left[\!\!\left[s, \nabla^{\mathfrak{G}}\xi - \varphi_*\xi + \Omega\left(\delta s, ds\right) \right]\!\!\right]_{\tilde{\mathfrak{H}}}, \qquad \xi = \omega(\delta s).$$

At last, we express the reduced variation in terms of the reduced section. To that end, we define the operator $\overline{\nabla}^{\tilde{\mathfrak{G}}} : \Gamma(\pi_{\tilde{\mathfrak{G}},X}) \to \Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{G}},X})$ as

$$\left(\overline{\nabla}^{\tilde{\mathfrak{G}}}_U\tilde{\xi}\right)(x_0) = \left.\frac{\nabla^{\tilde{\mathfrak{G}}}\left(\tilde{\xi}\circ x\right)(t)}{dt}\right|_{t=0}, \qquad \tilde{\xi} \in \Gamma(\pi_{\tilde{\mathfrak{G}},X}),\ U \in \mathfrak{X}(X),\ x_0 \in X, \tag{4.10}$$

where $x : (-\epsilon, \epsilon) \to X$ is a curve such that $x'(0) = U(x_0)$.

Lemma 4.14. *For each $\tilde{\xi} = [s,\xi]_{\mathcal{G}} \in \Gamma(\pi_{\tilde{\mathfrak{g}},X})$, we have*

$$\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} = [s, \nabla^{\mathfrak{g}}\xi + \mathrm{ad}_{s^*\omega}(\xi)]_{\mathcal{G}}.$$

Proof. Given $U \in \mathfrak{X}(X)$ and $x_0 \in X$, let $x : (-\epsilon,\epsilon) \to X$ be such that $x'(0) = U(x_0)$. Let $\gamma = \pi_{Y,Y/\mathcal{G}} \circ s \circ x : (-\epsilon,\epsilon) \to Y/\mathcal{G}$ and write

$$(s \circ x)(t) = \left({}^{\omega}\|_{\gamma(0)}^{\gamma(t)} s(x_0)\right) \cdot g(t) \tag{4.11}$$

for some $g : (-\epsilon,\epsilon) \to \mathcal{G}$. Observe that $\pi_{\mathcal{G},X} \circ g = x$ and $g(0) = 1_{x_0}$. From Proposition 4.6 and (4.8), we get

$$\begin{aligned}
\left(\overline{\nabla}^{\tilde{\mathfrak{g}}}_U \tilde{\xi}\right)(x_0) &= \left.\frac{\nabla^{\tilde{\mathfrak{g}}}\left(\tilde{\xi} \circ x\right)(t)}{dt}\right|_{t=0} \\
&= \left.\frac{d}{dt}\right|_{t=0} {}^{\tilde{\mathfrak{g}}}\|_{\gamma(t)}^{\gamma(0)} \left[(s \circ x)(t), (\xi \circ x)(t)\right]_{\mathcal{G}} \\
&= \left.\frac{d}{dt}\right|_{t=0} {}^{\tilde{\mathfrak{g}}}\|_{\gamma(t)}^{\gamma(0)} \left[{}^{\omega}\|_{\gamma(0)}^{\gamma(t)} s(x_0), \mathrm{Ad}_{g(t)} \circ (\xi \circ x)(t)\right]_{\mathcal{G}} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[{}^{\omega}\|_{\gamma(t)}^{\gamma(0)} \left({}^{\omega}\|_{\gamma(0)}^{\gamma(t)} s(x_0)\right), {}^{\mathfrak{g}}\|_{x(t)}^{x(0)} \left(\mathrm{Ad}_{g(t)} \circ (\xi \circ x)(t)\right)\right]_{\mathcal{G}} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[s(x_0), \mathrm{Ad}_{{}^{\nu}\|_{x(t)}^{x(0)} g(t)} \circ {}^{\mathfrak{g}}\|_{x(t)}^{x(0)} (\xi \circ x)(t)\right]_{\mathcal{G}} \\
&= \left[s(x_0), \left(\nabla^{\mathfrak{g}}_U \xi\right)(x_0) + \mathrm{ad}_{\zeta}(\xi(x_0))\right]_{\mathcal{G}},
\end{aligned}$$

where we denote $\zeta = \left.\frac{d}{dt}\right|_{t=0} {}^{\nu}\|_{x(t)}^{x(0)} g(t) \in \mathfrak{g}_{x_0}$. By taking the covariant derivative in (4.11), we obtain

$$\begin{aligned}
\left.\frac{D^{\omega}}{Dt}\right|_{t=0} (s \circ x)(t) &= \left.\frac{d}{dt}\right|_{t=0} {}^{\omega}\|_{\gamma(t)}^{\gamma(0)} (s \circ x)(t) \\
&= \left.\frac{d}{dt}\right|_{t=0} {}^{\omega}\|_{\gamma(t)}^{\gamma(0)} \left(\left({}^{\omega}\|_{\gamma(0)}^{\gamma(t)} s(x_0)\right) \cdot g(t)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(s(x_0) \cdot {}^{\nu}\|_{x(t)}^{x(0)} g(t)\right) \\
&= \zeta^*_{s(x_0)},
\end{aligned}$$

where Proposition 2.13 has been used. We conclude by applying ω at both sides of the previous equation:

$$\begin{aligned}
(s^*\omega)_{x_0}(U(x_0)) &= \omega_{s(x_0)}\left((ds)_{x_0}(U(x_0))\right) \\
&= \omega_{s(x_0)}\left(\left.\frac{d}{dt}\right|_{t=0} (s \circ x)(t)\right) \\
&= \omega_{s(x_0)}\left(\left.\frac{D^{\omega}}{Dt}\right|_{t=0} (s \circ x)(t)\right) \\
&= \omega_{s(x_0)}\left(\zeta^*_{s(x_0)}\right) \\
&= \zeta,
\end{aligned}$$

where (i) of Definition 2.5 has been used.

By linearity, the following map is well defined

$$\tilde{\varphi}_* : \tilde{\mathfrak{g}} \to T^*X \otimes \tilde{\mathfrak{g}}, \quad [y,\xi_x]_{\mathcal{G}} \mapsto [y, \varphi_* \xi_x]_{\mathcal{G}}. \tag{4.12}$$

Moreover, given $\alpha\otimes\tilde{\zeta}\in T^*X\otimes\tilde{\mathfrak{g}}$, we define the vector bundle morphism

$$\mathrm{ad}_{\alpha\otimes\tilde{\zeta}}:\tilde{\mathfrak{g}}\to T^*X\otimes\tilde{\mathfrak{g}},\quad \tilde{\xi}\mapsto\alpha\otimes\mathrm{ad}_{\tilde{\zeta}}(\tilde{\xi}),\tag{4.13}$$

which is well defined provided $\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}(\tilde{\xi})=\pi_{\tilde{\mathfrak{g}},Y/\mathcal{G}}(\tilde{\zeta})$, where $\mathrm{ad}:\tilde{\mathfrak{g}}\times_{Y/\mathcal{G}}\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}$ is the (fibered) adjoint representation on the (generalized) adjoint bundle. Note that we keep the same symbol for both maps for simplicity. By denoting $\tilde{s}=[s,s^*\omega]_{\mathcal{G}}\in\Gamma(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X})$ and $\tilde{\xi}=[s,\xi]_{\mathcal{G}}\in\Gamma(\pi_{\tilde{\mathfrak{g}},X})$, it is clear that $[s,\mathrm{ad}_{s^*\omega}(\xi)]_{\mathcal{G}}=\mathrm{ad}_{\tilde{s}}(\tilde{\xi})$. Thence, the following result is now straightforward.

Corollary 4.15. *Let δs be a variation of a section $s\in\Gamma(\pi_{Y,X})$ and consider the induced variation of the reduced section $\overline{s}=[\![s,s^*\omega]\!]_{\tilde{\mathfrak{H}},\varphi}$. Then for each $x\in X$ we have*

$$\delta^{\nabla}\overline{s}(x)=\left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi}-\mathrm{ad}_{\tilde{s}}(\tilde{\xi})-\tilde{\varphi}_*\tilde{\xi}+\tilde{\Omega}(\delta\sigma_s,d\sigma_s)\right]_{\tilde{\mathfrak{H}}},$$

where $\tilde{\xi}(x)=[s(x),\omega_{s(x)}(\delta s(x))]_{\mathcal{G}}$ and $\tilde{s}(x)=[s(x),(s^*\omega)_x]_{\mathcal{G}}$ for each $x\in X$.

4.3 Variations on $J^1(Y/\mathcal{G})$

The induced variations $\delta\sigma_s$ of $\sigma_s=\pi_{Y,Y/\mathcal{G}}\circ s$ are just the projection of the variations of s, that is,

$$\delta\sigma_s=d\pi_{Y,Y/\mathcal{G}}(\delta s).$$

In particular, these reduced variations are free, with no particular constraints (in contrast with the constraints for $\delta\overline{s}$ analyzed in the previous section). For later convenience, we now analyze the vertical part of the 1-jet lift of $\delta\sigma_s$ to $J^1(Y/\mathcal{G})$ with respect to a suitable connection.

Let $\nabla^{Y/\mathcal{G}}$ be a linear connection on the tangent bundle $T(Y/\mathcal{G})\to Y/\mathcal{G}$ and consider the operator $\overline{\nabla}^{Y/\mathcal{G}}:\Gamma(\pi_{T(Y/\mathcal{G}),X})\to\Gamma(\pi_{T^*X\otimes T(Y/\mathcal{G}),X})$ defined as

$$\left(\overline{\nabla}^{Y/\mathcal{G}}_U\alpha\right)(x)=\left.\frac{\nabla^{Y/\mathcal{G}}(\alpha\circ\gamma)(t)}{dt}\right|_{t=0},\qquad \alpha\in\Gamma(\pi_{T(Y/\mathcal{G}),X}),\ U\in\mathfrak{X}(X),\ x\in X,\tag{4.14}$$

where $\gamma:(-\epsilon,\epsilon)\to X$ is such that $\gamma'(0)=U(x)$. The following Lemma is an adaptation of [32, Corollary 3.4].

Lemma 4.16. *Let $\{s_t\}$ be a variation of a section $s\in\Gamma(\pi_{Y,X})$ and consider the induced variation of $\sigma_s\in\Gamma(\pi_{Y/\mathcal{G},X})$. Then:*

$$\left.\frac{\nabla^{Y/\mathcal{G}}d(\sigma_s)_t(U_x)}{dt}\right|_{t=0}=\left(\overline{\nabla}^{Y/\mathcal{G}}_U\delta\sigma_s\right)(x)+T^{Y/\mathcal{G}}\left(\delta\sigma_s(x),j^1_x\sigma_s(U_x)\right),\qquad x\in X,\ U_x\in T_xX,$$

where $U\in\mathfrak{X}(X)$ is such that $U(x)=U_x$, and $T^{Y/\mathcal{G}}\in\mathcal{T}^1_2(Y/\mathcal{G})$ is the torsion tensor *of $\nabla^{Y/\mathcal{G}}$.*

The connection $\nabla^{Y/\mathcal{G}}$ is said to be *projectable* on a linear connection ∇^X on $\pi_{TX,X}$ if the following diagram is commutative:

$$\begin{array}{ccc} T\left(T(Y/\mathcal{G})\right) & \xrightarrow{\nu^{Y/\mathcal{G}}} & T(Y/\mathcal{G}) \\ {\scriptstyle d(d\pi_{Y/\mathcal{G},X})}\downarrow & & \downarrow{\scriptstyle d\pi_{Y/\mathcal{G},X}} \\ T(TX) & \xrightarrow{\nu^X} & TX \end{array}$$

where $\nu^{Y/\mathcal{G}}$ and ν^X are the vertical projections of the connections $\nabla^{Y/\mathcal{G}}$ and ∇^X, respectively. Consider the vector bundle

$$\pi_{V,Y/\mathcal{G}} : V = T^*X \otimes T(Y/\mathcal{G}) \to Y/\mathcal{G}.$$

A section $\rho \in \Gamma(\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}})$ can be regarded as a section $\rho \in \Gamma(\pi_{V,Y/\mathcal{G}})$ since $J^1(Y/\mathcal{G}) \subset V$. Likewise, ρ can be regarded as a connection on $\pi_{Y/\mathcal{G},X}$. The next Proposition is an adaptation of [54, Theorem 3.1, Lemma 3.1] to our case. See also [32, Equations (3.13), (3.14)].

Proposition 4.17. *If $\nabla^{Y/\mathcal{G}}$ is projectable onto ∇^X, then it induces an affine connection $\nabla^{J^1(Y/\mathcal{G})}$ on $\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}$ given by*

$$\nabla^{J^1(Y/\mathcal{G})}_Z \rho = (id \otimes \nu^\rho) \circ \nabla^V_Z \rho, \qquad Z \in \mathfrak{X}(Y/\mathcal{G}),\ \rho \in \Gamma(\pi_{J^1(Y/\mathcal{G}),Y/\mathcal{G}}),$$

where $\mathrm{id} : T^*X \to T^*X$ is the identity map, ν^ρ is the vertical projection associated to ρ, and ∇^V is the linear connection induced on $V = T^*X \otimes T(Y/\mathcal{G}) \to Y/\mathcal{G}$ by the tensor product of the connections $\nabla^{Y/\mathcal{G}}$ and ∇^X.

Corollary 4.18. *For any variation $\delta\sigma_s = d/dt|_{t=0}(\sigma_s)_t$ of a reduced variation σ_s, the vertical part with respect to the connection $\nabla^{J^1(Y/\mathcal{G})}$ of its 1-jet lift is given by*

$$\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s = \left.\frac{\nabla^{J^1(Y/\mathcal{G})} j^1(\sigma_s)_t}{dt}\right|_{t=0} = \overline{\nabla}^{Y/\mathcal{G}} \delta\sigma_s + T^{Y/\mathcal{G}}(\delta\sigma_s, d\sigma_s).$$

Proof. Given $x \in X$ and $U_x \in T_xX$, the tensor product connection ∇^V satisfies

$$\frac{\nabla^V j^1_x(\sigma_s)_t}{dt}(U_x) = \frac{\nabla^{Y/\mathcal{G}} j^1_x(\sigma_s)_t(U_x)}{dt} - (j^1_x\sigma_s)\left(\frac{\nabla^X U_x}{dt}\right) = \frac{\nabla^{Y/\mathcal{G}} j^1_x(\sigma_s)_t(U_x)}{dt}.$$

Since the projection of $j^1_x(\sigma_s)_t(U)_x$ by $d\pi_{Y/\mathcal{G},X}$ is constantly U_x, we have that $\nabla^{Y/\mathcal{G}} j^1_x(\sigma_s)_t(U_x)/dt$ is $\pi_{Y/\mathcal{G},X}$ vertical so that

$$\frac{\nabla^{Y/\mathcal{G}} j^1_x(\sigma_s)_t}{dt} = (id \otimes \nu^{j^1\sigma_s}) \circ \frac{\nabla^V j^1_x(\sigma_s)_t}{dt} = \frac{\nabla^V j^1_x(\sigma_s)_t}{dt},$$

and the proof is complete by Lemma 4.16.

If the connection ∇^X is torsionless (and that will be our choice from now on), the formula above simply reads

$$\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s = \overline{\nabla}^{Y/\mathcal{G}} \delta\sigma_s.$$

4.4 Reduced field equations

Let $\mathfrak{L} : J^1Y \to \bigwedge^n T^*X$ be a (first order) Lagrangian density on a fiber bundle $\pi_{Y,X} : Y \to X$ with compact base manifold X. Assuming that X is orientable and $v \in \Omega^n(X)$ is a volume form, we write $\mathfrak{L} = Lv$ for certain $L : J^1Y \to \mathbb{R}$. Let $\pi_{\mathcal{G},X} : \mathcal{G} \to X$ be a Lie group bundle and $H \subset J^1\mathcal{G}$ be a closed, affine, Lie group subbundle, as in Proposition 4.1. If L is H-invariant, i.e., if

$$L\left(\Phi^{(1)}\left(j^1_x s, j^1_x \eta\right)\right) = L\left(j^1_x s\right), \qquad \forall \left(j^1_x s, j^1_x \eta\right) \in J^1Y \times_X H,$$

then the *dropped* or *reduced Lagrangian* is well defined as

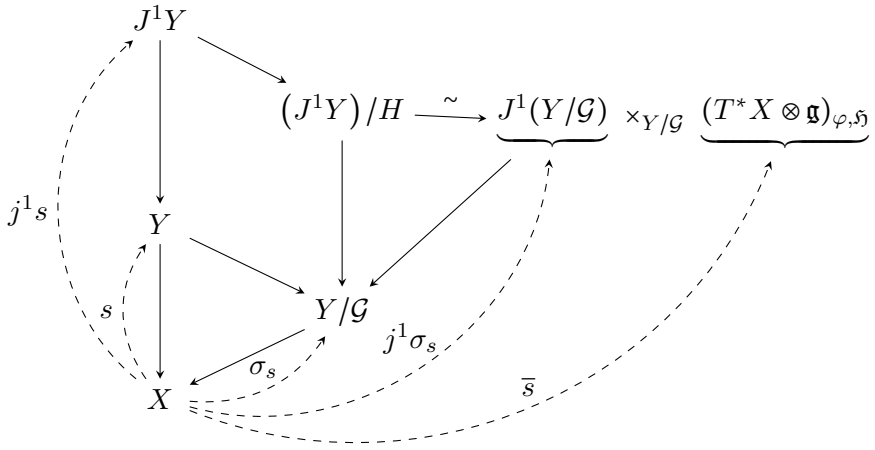
$$l : \left(J^1Y\right)/H \to \mathbb{R}, \quad [j^1_x s]_H \mapsto l\left(\left[j^1_x s\right]_H\right) = L\left(j^1_x s\right).$$

Let $\omega \in \Omega^1(Y, \mathfrak{g})$ be a generalized principal connection on $\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$, and choose a map $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ for the Lie group subbundle H as in Proposition 4.1. Thanks to Theorem 4.5, we may regard the reduced Lagrangian as defined on $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$. Recall that, given $s \in \Gamma(\pi_{Y,X})$, the corresponding *reduced section* is

$$\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi} \in \Gamma\left(\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},X}\right).$$

A variation $\{s_t\}$ of s induces a variation $\left\{\overline{s}_t = [\![s_t, s_t^*\omega]\!]_{\mathfrak{H},\varphi}\right\}$ of the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi}$. By construction, $L\left(j^1 s_t\right) = l\left(j^1(\sigma_s)_t, \overline{s}_t\right)$ for every $t \in (-\epsilon, \epsilon)$. Therefore:

$$\left.\frac{d}{dt}\right|_{t=0} \int_X L\left(j^1 s_t\right) v = \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v \tag{4.15}$$



Let $\mathfrak{H}^\circ \subset TX \otimes \mathfrak{g}^*$ be the annihilator of $\mathfrak{H} \subset T^*X \otimes \mathfrak{g}$. Then, $\tilde{\mathfrak{H}}^\circ = (Y \times_X \mathfrak{H}^\circ)/\mathcal{G} \subset TX \otimes \tilde{\mathfrak{g}}^*$ is the annihilator of $\tilde{\mathfrak{H}} \subset T^*X \otimes \tilde{\mathfrak{g}}$. This space is canonically isomorphic to the dual vector bundle

$$\tilde{\mathfrak{H}}^\circ \simeq \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)^*,$$

where the dual pairing is given by $\left\langle [y, \zeta_x]_{\mathcal{G}}, [\![y, \xi_x]\!]_{\tilde{\mathfrak{H}}} \right\rangle = \langle \zeta_x, \xi_x \rangle$ for each $[y, \zeta_x]_{\mathcal{G}} \in \tilde{\mathfrak{H}}^\circ$ and $[\![y, \xi_x]\!]_{\tilde{\mathfrak{H}}} \in (T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}$.

On the other hand, $J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to Y/\mathcal{G}$ is an affine bundle, since both $J^1(Y/\mathcal{G})$ and $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}$ are affine bundles over $Y/\mathcal{G}$. Let $\nabla^\times$ be the affine connection on this bundle induced by the affine connections $\nabla^{J^1(Y/\mathcal{G})}$ and ∇ introduced above.

Definition 4.1 *Let $s \in \Gamma(\pi_{Y,X})$ and consider the reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi}$, as well as $\sigma_s = \pi_{Y,Y/\mathcal{G}} \circ s$. The* partial derivatives *of the reduced Lagrangian*

$$l : J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to \mathbb{R}$$

are the sections

$$\frac{\delta l}{\delta \sigma_s} \in \Gamma\left(\pi_{T^*(Y/\mathcal{G}),X}\right), \quad \frac{\delta l}{\delta j^1\sigma_s} \in \Gamma\left(\pi_{TX\otimes V^*(Y/\mathcal{G}),X}\right), \quad \frac{\delta l}{\delta \overline{s}} \in \Gamma\left(\pi_{\tilde{\mathfrak{H}}^\circ,X}\right)$$

defined as

$$\left\langle \frac{\delta l}{\delta \sigma_s}(x), U_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(\gamma_U(t)^h(t)\right), \qquad \forall U_x \in T_{\sigma_s(x)}(Y/\mathcal{G}),$$
$$\left\langle \frac{\delta l}{\delta j^1\sigma_s}(x), V_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s + t\,V_x, \overline{s}(x)\right), \qquad \forall V_x \in T^*_x X \otimes V_{\sigma_s(x)}(Y/\mathcal{G}),$$
$$\left\langle \frac{\delta l}{\delta \overline{s}}(x), W_x \right\rangle = \left.\frac{d}{dt}\right|_{t=0} l\left(j^1_x\sigma_s, \overline{s}(x) + t\,W_x\right), \qquad \forall W_x \in= \left((T^*X \otimes \tilde{\mathfrak{g}})/\tilde{\mathfrak{H}}\right)_{\sigma_s(x)},$$

for each $x \in X$, where $\gamma_U(t)^h$ is the horizontal lift with respect to the connection $\nabla^\times$ of a curve $\gamma_U : (-\epsilon, \epsilon) \to Y/\mathcal{G}$ such that $\gamma'(0) = U_x$. As usual, $\langle \cdot, \cdot \rangle$ denotes the corresponding dual pairings.

We remark that, whereas the two latter derivatives are (intrinsic) fiber derivatives, the partial derivative $\delta l/\delta\sigma_s$ depends on the choice of the connections. Anyway, all of them are sections projecting onto σ_s.

Since $\delta l/\delta\overline{s}$ lies in the annihilator of $\tilde{\mathfrak{H}}$, then for each $[\![s, \xi]\!]_{\tilde{\mathfrak{H}}} \in \Gamma\left(\pi_{(T^*X\otimes\tilde{\mathfrak{g}})/\tilde{\mathfrak{H}},X}\right)$ the dual pairing satisfies

$$\left\langle \frac{\delta l}{\delta \overline{s}}, [\![s, \xi]\!]_{\tilde{\mathfrak{H}}} \right\rangle = \left\langle \frac{\delta l}{\delta \overline{s}}, [s, \xi]_{\mathcal{G}} \right\rangle \tag{4.16}$$

Indeed, let $g \in \Gamma(\pi_{\mathcal{G},X})$ and $\eta \in \Gamma(\pi_{\mathfrak{H},X})$. Then $\left[s \cdot g, \mathrm{Ad}_{g^{-1}} \circ (\xi + \eta)\right]_{\mathcal{G}} = [s, \xi]_{\mathcal{G}} + [s, \eta]_{\mathcal{G}}$ and $\left\langle \delta l/\delta\overline{s}, [s, \eta]_{\mathcal{G}} \right\rangle = 0$, since $[s, \eta]_{\mathcal{G}} \in \tilde{\mathfrak{H}}$.

Definition 4.2 *The* divergence *of the operator $\overline{\nabla}^{\tilde{\mathfrak{g}}}$ defined in (4.10) is minus the adjoint of $\overline{\nabla}^{\tilde{\mathfrak{g}}}$, i.e., the operator* $\mathrm{div}^{\tilde{\mathfrak{g}}} : \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right) \to \Gamma\left(\pi_{\tilde{\mathfrak{g}}^*,X}\right)$ *given by:*

$$\int_X \left\langle \zeta, \overline{\nabla}^{\tilde{\mathfrak{g}}} \xi \right\rangle v = -\int_X \left\langle \mathrm{div}^{\tilde{\mathfrak{g}}} \zeta, \xi \right\rangle v$$

for every $\zeta \in \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right)$ and $\xi \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$.

Analogously, the divergence of $\overline{\nabla}^{Y/\mathcal{G}}$ *is minus the adjoint of (4.14) restricted to vertical sections (the restriction is allowed since the linear connection $\nabla^{Y/\mathcal{G}}$ is projectable), that is,* $\mathrm{div}^{Y/\mathcal{G}} : \Gamma\left(\pi_{TX\otimes V^*(Y/\mathcal{G}),X}\right) \to \Gamma\left(\pi_{V^*(Y/\mathcal{G}),X}\right)$.

In the same vein, the dual operator of $\tilde{\varphi}_* : \tilde{\mathfrak{g}} \to T^*X \otimes \tilde{\mathfrak{g}}$ given in (4.12) is denoted by $\tilde{\varphi}^\dagger_* : TX \otimes \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$.

The (pointwise) *coadjoint representation* of the morphism $\mathrm{ad}_{\tilde{s}} : \tilde{\mathfrak{g}} \to T^*X \otimes \tilde{\mathfrak{g}}$ introduced in (4.13), i.e., minus its dual morphism, is denoted by $\mathrm{ad}^*_{\tilde{s}} : TX \otimes \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$.

Lemma 4.19. *In the previous conditions, the coadjoint representation can be restricted to $\tilde{\mathfrak{H}}^\circ$ yielding the following vector bundle morphism:*

$$\mathrm{ad}^*_{\tilde{s}} : \tilde{\mathfrak{H}}^\circ \to \tilde{\mathfrak{g}}^*, \quad \tilde{\zeta} \mapsto \mathrm{ad}^*_{\tilde{s}}(\tilde{\zeta}).$$

Proof. Let $x \in X$. As a consequence of the Ad-invariance of $\mathfrak{H}$, we have that $\mathrm{ad}_{\tilde{\eta}}(\tilde{\xi}) \in \tilde{\mathfrak{H}}$ for each $\tilde{\eta} \in \tilde{\mathfrak{H}}_{\sigma_s(x)}$ and $\tilde{\xi} \in \tilde{\mathfrak{g}}_{\sigma_s(x)}$. Indeed, $\mathrm{ad}_{\tilde{\eta}}(\tilde{\xi}) = -[\tilde{\xi}, \tilde{\eta}] = d/dt|_{t=0}\mathrm{Ad}_{\exp(t\,\tilde{\xi})} \circ \tilde{\eta} \in \tilde{\mathfrak{H}}$. Hence, the map $\mathrm{ad}^*_{\tilde{s}} : \tilde{\mathfrak{H}}^\circ \to \tilde{\mathfrak{g}}^*$ is well defined. Indeed, for each $\tilde{\zeta} \in \tilde{\mathfrak{H}}^\circ$ we have $\langle \mathrm{ad}^*_{\tilde{s}(x)+\tilde{\eta}}(\tilde{\zeta}), \tilde{\xi}\rangle = \langle \tilde{\zeta}, \mathrm{ad}_{\tilde{s}(x)+\tilde{\eta}}(\tilde{\xi})\rangle = \langle \tilde{\zeta}, \mathrm{ad}_{\tilde{s}(x)}(\tilde{\xi})\rangle = \langle \mathrm{ad}^*_{\tilde{s}(x)}(\tilde{\zeta}), \tilde{\xi}\rangle$, where Proposition 4.3 and Lemma 4.4 have been taken into account.

Theorem 4.20 (Reduced field equations). *Let $\pi_{Y,X}$ be a fiber bundle over a compact manifold X, $\pi_{\mathcal{G},X}$ be a Lie group bundle and $\pi_{\mathfrak{g},X}$ be its Lie algebra bundle. Suppose that $\pi_{\mathcal{G},X}$ acts fiberwisely, freely and properly on the right on $\pi_{Y,X}$. Let $\omega \in \Omega^1(Y, \mathfrak{g})$ be a generalized principal connection on*

$\pi_{Y,Y/\mathcal{G}}$ associated to a Lie group connection ν on $\pi_{\mathcal{G},X}$. Let $H \subset J^1\mathcal{G}$ be a Lie group subbundle projecting surjectively onto $\mathcal{G}$ and such that, with the identification $J^1\mathcal{G} \simeq \mathcal{G} \times_X T^*X \otimes \mathfrak{g}$ given by ν, it is an affine subbundle $\varphi + \mathfrak{H}$ as in Proposition 4.1.

For an H-invariant Lagrangian density $\mathfrak{L} = Lv : J^1Y \to \bigwedge^n T^*X$ we consider the corresponding reduced Lagrangian $l : J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to \mathbb{R}$. Then, for any section $s \in \Gamma(\pi_{Y,X})$ and its reduced section $\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi}$ and $\sigma = \pi_{Y,Y/\mathcal{G}} \circ s$, the following assertions are equivalent:

(i) The variational principle $\delta \int_X \mathfrak{L}(j^1 s) = 0$ holds for arbitrary (vertical) variations of s.

(ii) The section s satisfies the Euler–Lagrange equations for L, i.e., $\mathcal{EL}(L)\left(j^2 s\right) = 0$.

(iii) The variational principle $\delta \int_X l(j^1\sigma_s, \overline{s})v = 0$ holds for variations of the form:

$$\delta^\nabla \overline{s} = \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} - \mathrm{ad}_{\tilde{s}}\left(\tilde{\xi}\right) - \tilde{\varphi}_*\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{H}}},$$

where $\tilde{\xi} \in \Gamma(\pi_{\tilde{\mathfrak{g}},X})$ is an arbitrary section and $\delta\sigma_s$ is an arbitrary variation of σ_s.

(iv) The reduced section $\overline{s}$ satisfies the reduced field equations*:*

$$\begin{cases} \dfrac{\delta l}{\delta\sigma_s} - \mathrm{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta\overline{s}}, \iota_{d\sigma_s}\tilde{\Omega}\right\rangle, \\ \left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\overline{s}} + \tilde{\varphi}^\dagger_*\right)\left(\dfrac{\delta l}{\delta\overline{s}}\right) = 0. \end{cases}$$

The expressions above take into account the connections described in §4.1.1.

Proof. The equivalence of (i) and (ii) is a well-known fact as stated above. The equivalence of (i) and (iii) is a straightforward consequence of equation (4.15) and Corollary 4.15. To complete the proof, we show the equivalence of (iii) and (iv).

Let δs be a $\pi_{Y,X}$-vertical variation of s and consider induced variations of σ_s and $\overline{s}$. Then:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v &= \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)\right) v \\ &= \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)^h\right) v + \int_X dl\left(\delta\left(j^1\sigma_s, \overline{s}\right)^v\right) v \\ &= \int_X \left\langle \frac{\delta l}{\delta\sigma_s}, \delta\sigma_s\right\rangle v + \int_X \left\langle \frac{\delta l}{\delta j^1\sigma_s}, \delta^{J^1(Y/\mathcal{G})} j^1\sigma_s\right\rangle v + \int_X \left\langle \frac{\delta l}{\delta\overline{s}}, \delta^\nabla\overline{s}\right\rangle v, \end{aligned}$$

where the vertical parts $\delta^{J^1(Y/\mathcal{G})} j^1\sigma_s$ and $\delta^\nabla\overline{s}$ are defined with the connections $\nabla^{J^1(Y/\mathcal{G})}$ and ∇ respectively. Making use of Corollaries 4.15 and 4.18 (with vanishing torsion) we get

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right) v = &\int_X \left\langle \frac{\delta l}{\delta\sigma_s}, \delta\sigma_s\right\rangle v + \int_X \left\langle \frac{\delta l}{\delta j^1\sigma_s}, \overline{\nabla}^{Y/\mathcal{G}}\delta\sigma_s\right\rangle v \\ &+ \int_X \left\langle \frac{\delta l}{\delta\overline{s}}, \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} - \mathrm{ad}_{\tilde{s}}\left(\tilde{\xi}\right) - \tilde{\varphi}_*\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{H}}}\right\rangle v. \end{aligned}$$

Thanks to (4.16), we may write:

$$\left\langle \frac{\delta l}{\delta\overline{s}}, \left[\overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} - \mathrm{ad}_{\tilde{s}}\left(\tilde{\xi}\right) - \tilde{\varphi}_*\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right]_{\tilde{\mathfrak{H}}}\right\rangle = \left\langle \frac{\delta l}{\delta\overline{s}}, \overline{\nabla}^{\tilde{\mathfrak{g}}}\tilde{\xi} - \mathrm{ad}_{\tilde{s}}\left(\tilde{\xi}\right) - \tilde{\varphi}_*\tilde{\xi} + \tilde{\Omega}(\delta\sigma_s, d\sigma_s)\right\rangle.$$

Analogously, thanks to Lemma 4.19, we have

$$\left\langle \frac{\delta l}{\delta \overline{s}}, \mathrm{ad}_{\overline{s}}\left(\tilde{\xi}\right)\right\rangle = \left\langle \mathrm{ad}_{\overline{s}}\left(\frac{\delta l}{\delta \overline{s}}\right), \tilde{\xi}\right\rangle.$$

Using this and the divergence operators defined above we get:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\int_X l\left(j^1(\sigma_s)_t, \overline{s}_t\right)v = &\int_X \left\langle \frac{\delta l}{\delta \sigma_s}, \delta\sigma_s\right\rangle v + \int_X \left\langle -\mathrm{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right), \delta\sigma_s\right\rangle v\\ &+ \int_X \left\langle \left(-\mathrm{div}^{\tilde{\mathfrak{g}}} + \mathrm{ad}^*_{\overline{s}} - \tilde{\varphi}^\dagger_*\right)\left(\frac{\delta l}{\delta \overline{s}}\right), \tilde{\xi}\right\rangle v\\ &+ \int_X \left\langle -\left\langle \frac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega}\right\rangle, \delta\sigma_s\right\rangle v.\end{aligned}$$

As a result, the variational principle of *(iii)* reads

$$\begin{aligned}&\int_X \left\langle \frac{\delta l}{\delta \sigma_s} - \mathrm{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1\sigma_s}\right) - \left\langle \frac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega}\right\rangle, \delta\sigma_s\right\rangle v\\ &- \int_X \left\langle \left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\overline{s}} + \tilde{\varphi}^\dagger_*\right)\left(\frac{\delta l}{\delta \overline{s}}\right), \tilde{\xi}\right\rangle v = 0\end{aligned}$$

for every section $\tilde{\xi} \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$ and every variation $\delta\sigma_s$ of σ_s.

The previous Theorem is the main result of this Chapter. In §5.1 we will see that this result is a generalization of Theorem 1.2.

Remark 4.3 *The first equation holds on $\pi_{\sigma_s^* V^*(Y/\mathcal{G}),X}$, and the second one on $\pi_{\sigma_s^*\tilde{\mathfrak{g}}^*,X}$. On the other hand, if the connection $\nabla^{Y/\mathcal{G}}$ had non-vanishing torsion, then the reduced equations would read*

$$\begin{cases}\dfrac{\delta l}{\delta \sigma_s} - \mathrm{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1\sigma_s}\right) + \left\langle \dfrac{\delta l}{\delta j^1\sigma_s}, \iota_{d\sigma_s}T^{Y/\mathcal{G}}\right\rangle = \left\langle \dfrac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega}\right\rangle,\\ \left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\overline{s}} + \tilde{\varphi}^\dagger_*\right)\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0.\end{cases}$$

4.5 Reconstruction

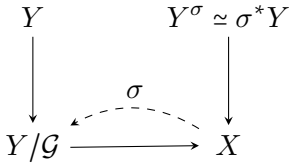
Let $\overline{s}$ be a section of the reduced bundle $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to X$ and let

$$\sigma = \pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s}$$

be the induced section of $Y/\mathcal{G} \to X$. We consider the subset

$$Y^\sigma = \pi^{-1}_{Y,Y/\mathcal{G}}\left(\sigma(X)\right) = \{y \in Y : \pi_{Y,Y/\mathcal{G}}(y) = \sigma(\pi_{Y,X}(y))\} \subset Y.$$

The action of $\mathcal{G}$ on Y restricts to Y^σ and $Y^\sigma/\mathcal{G} \simeq X$. In fact, we can regard Y^σ as a pull-back bundle $Y^\sigma \simeq \sigma^*Y \to X$ on which the Lie group bundle $\mathcal{G}$ acts transitively along the fibers:



In particular, the adjoint bundle of $Y^\sigma \to X$ is the pull-back $\sigma^*\tilde{\mathfrak{g}} \to X$. From this point of view, the section $\overline{s}$ can be also considered as a section of the pull-backed bundle

$$\sigma^*((T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}}) = \frac{Y^\sigma \times_X (T^*X \otimes \mathfrak{g})/\mathfrak{H}}{\Phi_\varphi} \to X,$$

Moreover, the restriction of the connection form ω to Y^σ is a generalized principal connection on $Y^\sigma \to X$ associated to the Lie group connection ν, which we denote by $\sigma^*\omega$. From Proposition 2.17 we know that, for any section $\tilde{\xi}$ of $T^*X \otimes \sigma^*\tilde{\mathfrak{g}} \to X$, the form $\sigma^*\omega - \tilde{\xi}$ is a generalized principal connection on $Y^\sigma \to X$.

Proposition 4.21. *Let $s \in \Gamma(\pi_{Y,X})$. We regard s as a section of $Y^\sigma \to X$, where $\sigma = \pi_{Y,Y/\mathcal{G}} \circ s$. Then the generalized principal connection defined as*

$$\omega^{\overline{s}} = \sigma^*\omega - \tilde{s}$$

is flat, where $\tilde{s} = [s, s^*\omega]_{\mathcal{G}} \in \Gamma(\pi_{T^*X\otimes\sigma^*\tilde{\mathfrak{g}},X})$.

Proof. To begin with, we have that $s(X) \subset Y^\sigma$ is an integral leaf of $\sigma^*\omega - \tilde{s}$. Indeed, it is easy to check that $(ds)_x(T_xX)$ is $\omega^{\overline{s}}$-horizontal for each $x \in X$. This means that $\mathrm{Curv}\left(\omega^{\overline{s}}\right)_{s(x)} = 0$ for each $x \in X$, where $\mathrm{Curv}\left(\omega^{\overline{s}}\right)$ is the curvature of $\omega^{\overline{s}}$.

An analogous computation to (2.15) yields

$$\left[s(x)\cdot g, \mathrm{Curv}\left(\omega^{\overline{s}}\right)_{s(x)\cdot g}\right]_{\mathcal{G}} = \left[s(x), \mathrm{Curv}\left(\omega^{\overline{s}}\right)_{s(x)}\right]_{\mathcal{G}} = 0, \qquad x \in X,\ g \in \mathcal{G}_x,$$

whence $\mathrm{Curv}\left(\omega^{\overline{s}}\right)_y = 0$ for every $y \in Y^\sigma$.

Theorem 4.22 (Reconstruction). *Let $\overline{s} \in \Gamma\left(\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},X}\right)$ be a critical section for the variational problem defined in (iii) of Theorem 4.20, and $\sigma = \pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},Y/\mathcal{G}} \circ \overline{s} \in \Gamma(\pi_{Y/\mathcal{G},X})$. Let $\tilde{s}$ be a section of $T^*X \otimes \sigma^*\tilde{\mathfrak{g}} \to X$ such that $\overline{s}(x) = [\![y, \xi_x]\!]_{\mathfrak{H},\varphi}$ for each $x \in X$, where $\tilde{s}(x) = [y, \xi_x]_{\mathcal{G}}$. If the connection*

$$\omega^{\overline{s}} = \sigma^*\omega - \tilde{s}$$

is flat and has trivial holonomy, then the integral leaves of the $\omega^{\overline{s}}$ are critical sections of the variational problem defined by L. Furthermore, any critical section of L is obtained in this way.

Proof. The trivial holonomy of $\omega^{\overline{s}}$ implies that its integral leaves are sections of $\sigma^*Y \to X$ that project to the reduced section $\overline{s}$. According to Theorem 4.20, these sections are critical.

Conversely, if s is a critical section of L, then $\omega^{\overline{s}}$ is flat and s is an integral leaf of $\omega^{\overline{s}}$, by Proposition 4.21.

Corollary 4.23. *If X is simply connected, then any connection has trivial holonomy and we have the following equivalence of equations*

$$\mathcal{EL}(L)(j^1 s) = 0 \iff \begin{cases} \dfrac{\delta l}{\delta \sigma_s} - \mathrm{div}^{Y/\mathcal{G}}\left(\dfrac{\delta l}{\delta j^1 \sigma_s}\right) = \left\langle \dfrac{\delta l}{\delta \overline{s}}, \iota_{d\sigma_s}\tilde{\Omega}\right\rangle \\ \left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\overline{s}} + \tilde{\varphi}^\dagger_*\right)\left(\dfrac{\delta l}{\delta \overline{s}}\right) = 0 \\ \mathrm{Curv}\left(\omega^{\overline{s}}\right) = 0 \end{cases}$$

For non-simply connected manifolds, the equivalence above holds locally only. There are topological obstructions and examples of reduced sections that do not admit global unreduced sections (cf. [11, §5.2]).

Remark 4.4 *Given a reduced section $\overline{s}$, there may be many sections $\tilde{s}$ satisfying the conditions of Theorem 4.22. Each choice will induce different solution s, and the transitions between them are governed by the symmetries of the system, that is, sections of $\mathcal{G} \to X$ with 1-jets lying on $H \to X$.*

4.6 Noether theorem

The well known Noether's theorem establishes that infinitesimal symmetries of the Lagrangian density yield preserved quantities for the dynamics of the system. The aim of this section is to show that the vertical part of the reduced equation is equivalent to the Noether's conservation law defined by the action of a Lie group bundle. As before, let $\mathfrak{L} = Lv$ be an H-invariant Lagrangian density for some $H \subset J^1\mathcal{G} \simeq \mathcal{G} \times_X (T^*X \otimes \mathfrak{g})$ as in Proposition 4.1

Definition 4.3 *An* infinitesimal symmetry *of $\mathfrak{L}$ is a vector field $U \in \mathfrak{X}(J^1Y)$ such that $\pounds_U \mathfrak{L} = 0$ or, equivalently, $\pounds_U \Theta_{\mathfrak{L}} = 0$, where $\pounds$ denotes the Lie derivative and $\Theta_{\mathfrak{L}} \in \Omega^n(J^1Y)$ is the (covariant) Cartan form of $\mathfrak{L}$ (cf., for instance, [87, §5]).*

If $s \in \Gamma(\pi_{Y,X})$ is a critical section for $\mathfrak{L}$ and $U \in \mathfrak{X}(J^1Y)$ is an infinitesimal symmetry of the Lagrangian density, the Noether's theorem gives (see, for example, [38]) the following conservation law

$$\mathrm{d}\left(\left(j^1 s\right)^* \iota_U \Theta_{\mathfrak{L}}\right) = 0.$$

In particular, if $\xi \in \Gamma(\pi_{\mathfrak{g},X})$ is such that its 1-jet extension falls in the Lie algebra bundle $\mathrm{Lie}(H)$ of H, it is clear that $(\xi^*)^{(1)} \in \mathfrak{X}(J^1Y)$ is an infinitesimal symmetry of $\mathfrak{L}$, thanks to the H-invariance. Recall that, from Proposition 4.1, we know that

$$\mathrm{Lie}(H) = \{(\xi, \hat{\eta}_x) \in \mathfrak{g} \times_X (T^*X \otimes \mathfrak{g}) \mid \hat{\eta}_x = \varphi_*(\xi) + \eta_x,\ \eta_x \in \mathfrak{H}_x\} \subset J^1\mathfrak{g}. \tag{4.17}$$

Theorem 4.24. *Let $L : J^1Y \to \mathbb{R}$ be an H-invariant Lagrangian for some subgroup bundle $H \subset J^1\mathcal{G}$ as in Proposition 4.1 and let s be a section of $\pi_{Y,X}$. If $\overline{s} = [\![s, s^*\omega]\!]_{\mathfrak{H},\varphi}$ is the induced section of $\pi_{(T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}},X}$ and $l : J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} \to \mathbb{R}$ is the reduced Lagrangian, then the Noether conservation law*

$$\mathrm{d}((j^1 s)^* i_{(\xi^*)^{(1)}} \Theta_{\mathfrak{L}}) = 0$$

holds for any section $\xi \in \Gamma(\pi_{\mathfrak{g},X})$ such that $\xi^{(1)} \in \Gamma(\pi_{\mathrm{Lie}(H),X})$ if and only if the vertical reduced equation

$$\left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\overline{s}} + \tilde{\varphi}^\dagger_*\right)\left(\frac{\delta l}{\delta \overline{s}}\right) = 0$$

is satisfied for $\overline{s}$.

Proof. The result being local enables us to work in coordinates. Some conditions on these coordinates will be imposed along the proof.

Let (x^μ, y^i, y^α) be bundle coordinates for $\pi_{Y,X}$ and consider the corresponding bundle coordinates $(x^\mu, y^i, y^\alpha, v^i_\mu, v^\alpha_\mu)$ for $\pi_{J^1Y,X}$. Suppose that they are chosen so that (x^μ, y^i) are bundle coordinates for $\pi_{Y/\mathcal{G},X}$ and, thus, (x^μ, y^i, v^i_μ) are bundle coordinates for $\pi_{J^1(Y/\mathcal{G}),X}$. Let $\{B_\alpha : 1 \le \alpha \le m\}$ be a basis of local sections of $\pi_{\mathfrak{g},X}$ and $\{B^\alpha : 1 \le \alpha \le m\}$ be its dual basis. For a fixed $x_0 = (x^\mu_0) \in X$, we suppose that, using these coordinates, we have

$$\mathrm{Lie}(H)_{x_0} = \mathrm{span}\{(dx^\mu)_{x_0} \otimes B_\alpha(x_0) : 1 \le \mu \le r, 1 \le \alpha \le s\} \subset T^*_{x_0}X \otimes \mathfrak{g}_{x_0} \tag{4.18}$$

for some r, $1 \le r \le n$, and s, $1 \le s \le m$. We start by finding the local expression of the infinitesimal symmetries of our Lagrangian density.

We also suppose that (y^α) are normal coordinates of G on a neighbourhood $\mathcal{U} \subset G$ of the identity element, 1. This means that there exist $g^\alpha_{\beta\gamma} \in C^\infty(\mathcal{U}\times\mathcal{U})$, $1 \le \alpha,\beta,\gamma \le m$, such that

$$y^\alpha(\hat g_1\hat g_2) = y^\alpha(\hat g_1) + y^\alpha(\hat g_2) + g^\alpha_{\beta\gamma}(\hat g_1,\hat g_2)\, y^\beta(\hat g_1)\, y^\gamma(\hat g_2), \qquad 1 \le \alpha \le m,$$

for each $\hat g_1, \hat g_2 \in \mathcal{U}$ such that $\hat g_1\hat g_2 \in \mathcal{U}$. Hence, the infinitesimal generators are given by

$$\big(B_\beta\big)^*_y = \big(\delta^\alpha_\beta + g^\alpha_{\gamma\beta}(\hat g, 1) y^\gamma(\hat g)\big)\,(\partial_\alpha)_y, \qquad y = (\sigma,\hat g) \in Y/\mathcal{G}\times\mathcal{U},\ 1 \le \beta \le m.$$

Denote by $\hat g : \mathbb{R}^m \to G$ the inverse of $(y^\alpha) : G \to \mathbb{R}^m$, and define $f^\gamma_{\alpha\beta} \in C^\infty(\mathbb{R}^m)$, $1 \le \alpha,\beta,\gamma \le m$, as

$$f^\gamma_{\alpha\beta}(y^\alpha) = g^\gamma_{\alpha\beta}(\hat g(y^\alpha), 1)$$

Therefore, given $\xi = \xi^\beta B_\beta \in \Gamma(\pi_{\mathfrak{g},X})$, where $\xi^\beta \in C^\infty(X)$, $1 \le \beta \le m$, the previous expressions yield

$$\xi^*(x^\mu, y^i, y^\alpha) = \xi^\beta(x^\mu)\left(\delta^\alpha_\beta + f^\alpha_{\gamma\beta}(y^\alpha) y^\gamma\right)\partial_\alpha.$$

From the formula of 1-jet lift of vector fields (for example, see, [87]), we get

$$(\xi^*)^{(1)} = \xi^\beta\left(\delta^\alpha_\beta + f^\alpha_{\gamma\beta} y^\gamma\right)\partial_\alpha + \left(\partial_\mu\xi^\beta\left(\delta^\alpha_\beta + f^\alpha_{\gamma\beta} y^\gamma\right) + v^\delta_\mu \xi^\beta\left(\partial_\delta f^\alpha_{\gamma\beta} y^\gamma + f^\alpha_{\delta\beta}\right)\right)\partial^\mu_\alpha. \tag{4.19}$$

On the other hand, in coordinates the map $\varphi : \mathcal{G} \to T^*X \otimes \mathfrak{g}$ reads

$$\varphi(x^\mu, y^\alpha) = \varphi^\alpha_\mu(x^\mu, y^\alpha) dx^\mu \otimes B_\alpha,$$

for some $\varphi^\alpha_\mu \in C^\infty(\mathcal{G})$. By denoting $\hat\varphi^\alpha_{\mu,\beta}(x^\mu) = (\partial\varphi^\alpha_\mu/\partial y^\beta)(x^\mu, y^\alpha = 0)$, we have

$$\varphi_*(\xi) = \xi^\beta \hat\varphi^\alpha_{\mu,\beta} dx^\mu \otimes B_\alpha, \qquad \xi = \xi^\alpha B_\alpha \in \Gamma(\pi_{\mathfrak{g},X}). \tag{4.20}$$

Similarly, our generalized principal connection, $\omega \in \Omega^1(Y,\mathfrak{g})$, locally reads

$$\omega(x^\mu, y^i, y^\alpha) = \left(\omega^\alpha_\mu(x^\mu, y^i) dx^\mu + \omega^\alpha_i(x^\mu, y^i) dy^i + dy^\alpha\right)\otimes B_\alpha,$$

for some functions $\omega^\alpha_\mu, \omega^\alpha_i \in C^\infty(Y/\mathcal{G})$, $1 \le \mu \le n$, $1 \le i \le k$, $1 \le \alpha \le m$, being $k \in \mathbb{Z}^+$ the dimension of the fiber of $\pi_{Y/\mathcal{G},X}$. Furthermore, there exist bundle coordinates $(x^\mu, y^i, v^i_\mu; w^\alpha_\mu)$ for the reduced space, where the indices of w^α_μ go through $\mu \in \{r+1,\dots,n\}$ and $\alpha \in \{s+1,\dots,m\}$, such that identification of Theorem 4.5 is given by

$$\begin{array}{ccc} \big(J^1Y\big)/H & \to & J^1(Y/\mathcal{G}) \times_{Y/\mathcal{G}} (T^*X\otimes\mathfrak{g})_{\varphi,\mathfrak{H}} \\ \big[x^\mu, y^i, y^\alpha, v^i_\mu, v^\alpha_\mu\big]_H & \mapsto & \big(x^\mu, y^i, v^i_\mu; w^\alpha_\mu\big) \end{array}$$

and, in the fiber over x_0, they satisfy

$$w^\alpha_\mu = \varphi^\alpha_\mu(x^\mu_0, 0) + \omega^\alpha_\mu(x^\mu_0, y^i) + \omega^\alpha_i(x^\mu_0, y^i) v^i_\mu + v^\alpha_\mu, \qquad r+1 \le \mu \le n,\ s+1 \le \alpha \le m.$$

Likewise, using the definition of reduced Lagrangian, i.e.,

$$l(x^\mu, y^i, v^i_\mu; w^\alpha_\mu) = L(x^\mu, y^i, y^\alpha, v^i_\mu, v^\alpha_\mu),$$

it is easy to find the local expression of the vertical equation. Namely, the local expression of the partial derivative of l is

$$\frac{\delta l}{\delta \bar s}(x^\mu) = \frac{\partial l}{\partial w^\alpha_\mu}\left(x^\mu, s^i(x^\mu), \frac{\partial s^i(x^\mu)}{\partial x^\mu}; w^\alpha_\mu(x^\mu)\right)\partial_\mu \otimes B^\alpha,$$

and then, the vertical reduced equation becomes

$$\left(\mathrm{div}^{\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\tilde{s}} + \tilde{\varphi}^\dagger_*\right)\left(\frac{\partial l}{\partial w^\alpha_\mu}\partial_\mu \otimes B^\alpha\right) = 0. \tag{4.21}$$

From Lemma 4.14, we may locally write $\mathrm{div}^{\tilde{\mathfrak{g}}} = \mathrm{div}^{\mathfrak{g}} + \mathrm{ad}^*_{\tilde{s}}$, where $\mathrm{div}^{\mathfrak{g}}$ is the divergence of $\nabla^{\mathfrak{g}}$. Observe that $\sigma_s^*(\tilde{\mathfrak{g}}) \simeq \mathfrak{g}$, since we are working locally. By definition, $\mathrm{div}\langle\zeta,\xi\rangle = \left\langle \mathrm{div}^{\mathfrak{g}}\zeta, \xi\right\rangle + \left\langle \zeta, \nabla^{\mathfrak{g}}\xi\right\rangle$, for each $\zeta = \zeta^\mu_\alpha \partial_\mu \otimes B^\alpha \in \Gamma(\pi_{TX\otimes\mathfrak{g}^*,X})$ and $\xi = \xi^\alpha B_\alpha \in \Gamma(\pi_{\mathfrak{g},X})$. An easy computation using this expression shows that

$$\mathrm{div}^{\mathfrak{g}}\zeta = \left(\partial_\mu \zeta^\mu_\alpha - \Gamma^\beta_{\mu,\alpha}\zeta^\mu_\beta\right) B^\alpha$$

where $\Gamma^\beta_{\mu,\alpha} \in C^\infty(X)$, $1 \le \mu \le n$, $1 \le \alpha, \beta \le m$, are the Christoffel symbols of $\nabla^{\mathfrak{g}}$, i.e.,

$$\nabla^{\mathfrak{g}}\xi = \left(\partial_\mu \xi^\alpha + \Gamma^\alpha_{\mu,\beta}\xi^\beta\right) dx^\mu \otimes B_\alpha, \qquad \xi = \xi^\alpha B_\alpha \in \Gamma(\pi_{\mathfrak{g},X}).$$

We choose the bundle coordinates on $\pi_{\mathfrak{g},X}$ so that $\nabla^{\mathfrak{g}}$ is flat at $x_0 = (x^\mu_0)$, i.e., $\Gamma^\beta_{\mu,\alpha}(x^\mu_0) = 0$ for each $1 \le \mu \le n$ and $1 \le \alpha, \beta \le m$. Analogously, $\langle \varphi_*(\xi), \zeta\rangle = \left\langle \xi, \varphi^\dagger_*(\zeta)\right\rangle$. Thus, from (4.20) we get

$$\tilde{\varphi}^\dagger_*(\zeta) = \hat{\varphi}^\beta_{\mu,\alpha}\zeta^\mu_\beta B^\alpha, \qquad \zeta = \zeta^\mu_\alpha \partial_\mu \otimes B^\alpha \in \Gamma(\pi_{TX\otimes\mathfrak{g}^*,X}).$$

As a result, (4.21) reads

$$\left(\partial_\mu\left(\frac{\partial l}{\partial w^\alpha_\mu}\right) + \hat{\varphi}^\beta_{\mu,\alpha}\frac{\partial l}{\partial w^\beta_\mu}\right)(x_0)B^\alpha(x_0) = 0, \tag{4.22}$$

where we recall that the summation is for $r+1 \le \mu \le n$ and $s+1 \le \alpha, \beta \le m$.

Using the expression for the coordinates w^α_μ at x_0 we have

- $\dfrac{\partial L}{\partial v^\alpha_\mu}(x_0) = \begin{cases} 0, & 1 \le \mu \le r,\ 1 \le \alpha \le s, \\ \dfrac{\partial l}{\partial w^\alpha_\mu}(x_0), & r+1 \le \mu \le n, \quad s+1 \le \alpha \le m. \end{cases}$
- $\dfrac{\partial L}{\partial v^i_\mu}(x_0) = \begin{cases} \dfrac{\partial l}{\partial v^i_\mu}(x_0), & 1 \le \mu \le r, \\ \dfrac{\partial l}{\partial v^i_\mu}(x_0) + \displaystyle\sum_{\alpha=s+1}^{m} \frac{\partial l}{\partial w^\alpha_\mu}(x_0)\omega^\alpha_i, & r+1 \le \mu \le n. \end{cases}$

In addition, if the volume form is given by $v = d^n x = dx^1 \wedge \cdots \wedge dx^n$, then the local expression of the Poincaré-Cartan form is

$$\Theta_{\mathfrak{L}} = \frac{\partial L}{\partial v^i_\mu}\left(dy^i - v^i_\mu dx^\mu\right) \wedge d^{n-1}x_\mu + \frac{\partial L}{\partial v^\alpha_\mu}\left(dy^\alpha - v^\alpha_\mu dx^\mu\right) \wedge d^{n-1}x_\mu + L\, d^n x,$$

where $d^{n-1}x_\mu = \iota_{\partial_\mu} d^n x$. By using (4.19), we obtain

$$\begin{aligned} \iota_{(\xi^*)^{(1)}}\Theta_{\mathfrak{L}}\left(x^\mu, y^i, y^\alpha, v^i_\mu, v^\alpha_\mu\right) &= \frac{\partial L}{\partial v^\alpha_\mu}\left(x^\mu, y^i, y^\alpha, v^i_\mu, v^\alpha_\mu\right)\xi^\beta(x^\mu)\left(\delta^\alpha_\beta + f^\alpha_{\gamma\beta}(y^\alpha)y^\gamma\right) d^{n-1}x_\mu \\ &= \frac{\partial l}{\partial w^\alpha_\mu}\left(x^\mu, y^i, v^i_\mu; w^\alpha_\mu\right)\xi^\beta(x^\mu)\left(\delta^\alpha_\beta + f^\alpha_{\gamma\beta}(y^\alpha)y^\gamma\right) d^{n-1}x_\mu. \end{aligned}$$

The last condition on the coordinates is the requirement that $s(x_0) = (\sigma_s(x_0), 1) \in Y/\mathcal{G} \times G$, i.e., $s^\alpha(x^\mu_0) = 0$, $1 \le \alpha \le m$. Therefore,

$$\left(j^1 s\right)^*\left(\iota_{(\xi^*)^{(1)}}\Theta_{\mathfrak{L}}\right)(x^\mu_0) = \frac{\partial l}{\partial w^\alpha_\mu}\left(x^\mu_0, s^i(x^\mu_0), \partial_\mu s^i(x^\mu_0); w^\alpha_\mu(x^\mu_0)\right)\xi^\alpha(x^\mu_0) d^{n-1}x_\mu.$$

Observe that the jet extension of ξ is given by $j^1\xi = (\xi, d\xi) \in \Gamma(\pi_{\mathfrak{g}\times X(T^*X\otimes\mathfrak{g}),X})$. Subsequently, from (4.17), (4.18) and (4.20) we deduce that the condition $\xi^{(1)} \in \Gamma(\pi_{\mathrm{Lie}(H),X})$ at x_0 reads

$$(\partial_\mu \xi^\alpha)(x_0) = \xi^\beta(x_0)\hat{\varphi}^\alpha_{\mu,\beta}(x_0), \qquad r+1 \le \mu \le n,\ s+1 \le \alpha \le m.$$

Therefore, the Noether conservation law reads

$$\begin{aligned}\mathrm{d}\left(\frac{\partial l}{\partial w^\alpha_\mu}\xi^\alpha d^{n-1}x_\mu\right)(x_0) &= \left[\partial_\mu\left(\frac{\partial l}{\partial w^\alpha_\mu}\right)\xi^\alpha + \frac{\partial l}{\partial w^\alpha_\mu}\partial_\mu\xi^\alpha\right](x_0)(d^n x)_{x_0}\\ &= \left[\partial_\mu\left(\frac{\partial l}{\partial w^\alpha_\mu}\right) + \frac{\partial l}{\partial w^\beta_\mu}\hat{\varphi}^\beta_{\mu,\alpha}\right](x_0)\xi^\alpha(x_0)(d^n x)_{x_0} = 0.\end{aligned}$$

As $\xi(x_0) = \xi^\alpha(x_0)B_\alpha(x_0) \in \mathfrak{g}_{x_0}$ is arbitrary, this is equivalent to (4.22), and we conclude.

Chapter 5

Applications of the gauge reduction theory

This chapter is devoted to provide some applications to illustrate the significance of the gauge reduction theory developed in the previous chapter. We begin with classical reduction in field theories (global symmetries), followed by reduction under the full jet symmetry. We continue with Electromagnetism in vacuum, as well as k-form Electromagnetism. Moreover, the case of symmetry breaking when there is a product of groups is analyzed for Electromagnetism. The last application is to non-Abelian gauge theories, where the Lie group bundle of symmetries is $\mathcal{G} = J^1\tilde{G}$, being $\tilde{G} = (P \times G)/G$ the Lie group bundle associated to a given principal G-bundle, $P \to X$, via the conjugacy. The affine Lie group subbundle corresponding to *gauge transformations* is $H = J^2\tilde{G}$, which locally reads as $J^2(X, G)$, the bundle of 2-jets of functions from X to G. In order to obtain the reduced equations, the trivialized Lie group bundle structure considered in Chapter 3 is utilized. In particular, we get a new proof of Utiyama theorem from a reduction point of view. In addition, the reduced equations, together with the reconstruction condition, are the celebrated Yang–Mills equations for the appropriated Lagrangian density. In fact, the reduction technique of this work sheds light to an uncomfortable situation that Utiyama reduction does not fully explain: even though the reduced Lagrangian depends on curvature only, the equations still depend on both curvature and connection. Surprisingly, we prove that this is connected with the fact that the non-Abelian gauge symmetry is not a free action and the reduction is, in principle, singular. There is not, even in Mechanics, a well defined theory of Lagrangian singular reduction. The way to overcome this difficulty is to enlarge the phase bundle (from the bundle of connections $C(P) = \left(J^1P\right)/G$ to its 1-jet bundle J^1P) before reduction. It turns out that the reduced phase bundle comprises both connections and curvature and thus the equations as well. But the elimination of that augmented geometry after reduction (in a sort of unreduction-reduction process) gives that the Lagrangian does not depend on the connection, without eliminating that dependence in the equations.

5.1 Classical case: global symmetries

The Lagrange–Poincaré reduction scheme recalled in §1.2 may be recovered as a particular case of our theory. Namely, the action considered in that section may be regarded as a fibered action of the trivial Lie group bundle,

$$\mathcal{G} = X \times G,$$

by setting $\Phi(y, g) = \Psi_{\hat{g}}(y) = y \cdot \hat{g}$ for each $(y, g) \in Y \times_X \mathcal{G}$, with $g = (x, \hat{g})$. It is clear that the fibered and the standard quotients agree, i.e., $Y/\mathcal{G} \simeq Y/G = \Sigma$.

Recall that $\pi_{Y,\Sigma}$ is a principal bundle and the Lie algebra bundle of $\pi_{\mathcal{G},X}$ is $\mathfrak{g} = X \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. From Proposition 2.21 we know that a generalized principal connection $\omega \in \Omega^1(Y, \mathfrak{g})$ on $\pi_{Y/\mathcal{G}}$ associated to the trivial connection ν_0 on $\pi_{\mathcal{G},X}$ is simply a principal connection $A \in \Omega^1(Y, \mathfrak{g})$. Thanks to the bijective correspondence between (local) sections of $\pi_{\mathcal{G},X}$ and (local)

functions $X \to G$, the identification (4.1) for ν_0 may be written as

$$J^1\mathcal{G} \simeq G \times T^*X \otimes \mathfrak{g}, \qquad j^1_x\gamma \mapsto \big(\hat{\gamma}(x), d(R_{\hat{\gamma}(x)^{-1}} \circ \hat{\gamma})_x\big),$$

where $\gamma = (\mathrm{id}_X, \hat{\gamma})$ and $R_g : G \to G$ denotes the right multiplication by $g \in G$. Let H be the Lie group subbundle of $\pi_{J^1\mathcal{G},X}$ corresponding to locally constant functions, i.e.,

$$H \simeq G \times T^*X \otimes \{0\} \simeq X \times G = \mathcal{G}.$$

Note that $\mathfrak{H}^\perp = TX \otimes \mathfrak{g}^*$ and, hence, $Y \times_{\mathcal{G}} \mathfrak{H}^\perp \simeq TX \otimes \tilde{\mathfrak{g}}^*$. The first jet extension of the fibered action (4.6) restricted to $\pi_{H,X}$ and first jet extension of the Lie group action (1.2) yield the same quotient:

$$\big(J^1Y\big)/H \simeq \big(J^1Y\big)/G.$$

Since $\varphi = 0$ and $\mathfrak{H} = \{0\}$, the quotient (4.4) is isomorphic to

$$(T^*X \otimes \mathfrak{g})_{0,\mathfrak{H}} \tilde{\to} T^*X \otimes \tilde{\mathfrak{g}}, \quad \Big[\!\Big[y, (x, \hat{X}_x)\Big]\!\Big]_{\mathfrak{H},0} \mapsto \big[y, \hat{X}_x\big]_G,$$

where $\tilde{\mathfrak{g}} = (Y \times \mathfrak{g})/G$ is the adjoint bundle of $\pi_{Y,\Sigma}$.

Theorem 5.1. *Let $A \in \Omega^1(Y, \mathfrak{g})$ be a principal connection on $\pi_{Y,\Sigma}$ and consider the corresponding generalized principal connection $\omega \in \Omega^1(Y, \mathfrak{g})$ on $\pi_{Y,Y/\mathcal{G}}$ associated to the canonical connection ν_0 on $\pi_{\mathcal{G},X}$. Then the fiber diffeomorphism (4.7) reduces to (1.3) under the previous identifications.*

Let $A \in \Omega^1(Y, \mathfrak{g})$ be a principal connection on $\pi_{Y,\Sigma}$, $\nabla^{\mathfrak{g}}_0$ be the canonical connection on $\pi_{\mathfrak{g},X}$ and ∇^X be a linear connection on $\pi_{TX,X}$, and consider the linear connection $\nabla^{\otimes} = \nabla^* \otimes \nabla^{\mathfrak{g}}_0$ on $\pi_{T^*X\otimes\mathfrak{g},X}$, where ∇^* is the dual of ∇^X. They yield a linear connection ∇ on $\pi_{(T^*X\otimes\mathfrak{g})_{0,\mathfrak{H}},Y/\mathcal{G}} \simeq \pi_{T^*X\otimes\tilde{\mathfrak{g}},\Sigma}$, as we have seen in Proposition 4.8. By using bases of local sections of these vector bundles, it can be seen that ∇ agrees with the linear connection defined in [32, Equation (3.19)]. Recall that the linear connection ∇^A on $\pi_{\tilde{\mathfrak{g}},\Sigma}$ associated to A descends to (1.4), whose divergence is denoted by (1.6). Lastly, given a torsion free linear connection ∇^Σ on $\pi_{T\Sigma,\Sigma}$ projectable onto ∇^X, consider the induced affine connection $\nabla^{J^1\Sigma}$ on $\pi_{J^1\Sigma,\Sigma}$, as well as the operator (1.5) and its divergence (1.7).

Theorem 5.2. *Let $L : J^1Y \to \mathbb{R}$ be a G-invariant Lagrangian and $l : J^1\Sigma \times_\Sigma (T^*X \otimes \tilde{\mathfrak{g}}) \to \mathbb{R}$ be the reduced Lagrangian. Let $s \in \Gamma(\pi_{Y,X})$ and consider the reduced section $\overline{s} = [s, s^*A]_G \in \Gamma\big(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\big)$. Then the reduced equations for $\overline{s}$ given in Theorem 4.20 are equivalent to the Lagrange–Poincaré field equations given in Theorem 1.2.*

5.2 Full jet symmetry

Let $\pi_{\mathcal{G},X}$ be a Lie group bundle endowed with a Lie group connection ν, and suppose that it acts (on the right) on a fiber bundle $\pi_{Y,X}$ freely and properly. In this example we consider a Lagrangian $L : J^1Y \to \mathbb{R}$ which is invariant by the whole jet bundle, that is,

$$H = J^1\mathcal{G} \overset{\nu}{\simeq} \mathcal{G} \times_X (T^*X \otimes \mathfrak{g}).$$

It is clear that $(T^*X \otimes \mathfrak{g})_{\varphi,\mathfrak{H}} = X$. Consequently, the identification of Theorem 4.5 can be performed without fixing a generalized principal connection:

$$\big(J^1Y\big)/J^1\mathcal{G} \to J^1(Y/\mathcal{G}), \quad \big[j^1_x s\big]_{J^1\mathcal{G}} \mapsto j^1_x\sigma_s. \tag{5.1}$$

In the same way, since the reduced section $\overline{s}$ vanishes, the second equation of *(iv)* in Theorem 4.20 does not appear. Therefore, the reduced equations are the usual *Euler–Lagrange equations* for the reduced Lagrangian $l : J^1(Y/\mathcal{G}) \to \mathbb{R}$:

$$\frac{\delta l}{\delta \sigma_s} - \operatorname{div}^{Y/\mathcal{G}}\left(\frac{\delta l}{\delta j^1 \sigma_s}\right) = 0.$$

Of course, in order to write them we have fixed a linear connection $\nabla^{Y/\mathcal{G}}$ on $\pi_{T(Y/\mathcal{G}),Y/\mathcal{G}}$ projectable onto a linear connection ∇^X on $\pi_{TX,X}$.

5.3 Electromagnetism in vacuum

To describe Electromagnetism in vacuum as an Abelian geometric Yang–Mills theory, let (X, g) be a 4-dimensional, compact, oriented pseudo-Riemannian manifold with volume form $v_g \in \Omega^4(X)$, and let $\pi_{P,X} : P \to X$ be a principal U(1)-bundle. The configuration bundle of this theory is the bundle of connections of $\pi_{P,X}$, i.e.,

$$C(P) = \left(J^1 P\right)/\mathrm{U}(1) \to X.$$

Recall that it is an affine bundle modelled on $T^*X \to X$.

Definition 5.1 *The Maxwell Lagrangian density for Electromagnetism in vacuum is* $\mathfrak{L} = L\, v_g$ *with:*

$$L\left(j^1 A\right) = g\left(\tilde{F}^A, \tilde{F}^A\right), \qquad A \in \Gamma(\pi_{C(P),X}),$$

where $\tilde{F}^A \in \Omega^2(X)$ *is the* reduced curvature *of the principal connection* A.

Gauge transformations are defined by maps $g : X \to \mathrm{U}(1)$, that is, sections of the trivial bundle $X \times \mathrm{U}(1) \to X$. The Maxwell Lagrangian is invariant with respect to the transformation

$$A_x \mapsto A_x + (dg)_x g(x)^{-1}, \qquad x \in X,$$

which can be understood as a symmetry by the fibered action $C(P) \times J^1(X, \mathrm{U}(1)) \to C(P)$ defined above. Note that the value $g(x) \in \mathrm{U}(1)$ does not play an essential role and that $(dg)_x\, g(x)^{-1} \in T^*_x X \otimes \mathfrak{u}(1) \simeq T^*_x X$. Subsequently, this fibered action induces another one,

$$C(P) \times_X T^*X \to C(P), \quad (A_x, \alpha_x) \mapsto A_x + \alpha_x, \tag{5.2}$$

which can be straightforwardly extended to the first jets, $J^1C(P) \times_X J^1(T^*X) \to J^1C(P)$. By virtue of this, the Lie group bundle of symmetries for Electromagnetism is the cotangent bundle of X with the additive structure,

$$\mathcal{G} = T^*X \to X.$$

Second derivatives of gauge transformations correspond to the Lie group subbundle $J^2(X, \mathrm{U}(1)) \subset J^1\left(J^1(X, \mathrm{U}(1))\right)$. One can readily obtain the corresponding subbundle of jet symmetries as

$$H = \left\{j^1_x \alpha \in J^1(T^*X) : (\mathrm{d}\alpha)_x = 0\right\} \subset J^1(T^*X).$$

Note that $C(P)/T^*X = X$. In order to study the quotient $J^1C(P)/H$ we need to consider a generalized principal connection on $\pi_{C(P),X}$. The following lemma can be easily proven using local coordinates.

Lemma 5.3. *Let ν be a linear connection on $\pi_{T^*X,X}$, then it is Lie group bundle connection and any affine connection ω on $\pi_{C(P),X}$ modelled on ν is a generalized principal connection associated to ν. Furthermore, if the connection ν is torsionless, then $\hat{\nu}(T^*X) \subset H$, where $\hat{\nu} \in \Gamma\left(\pi_{J^1(T^*X),T^*X}\right)$ is the section induced by ν.*

The identification (4.1) reads $J^1(T^*X) \overset{\nu}{\simeq} T^*X \times_X (T^*X \otimes T^*X)$ and for a torsionless connection ν we have

$$H \overset{\nu}{\simeq} T^*X \times_X \bigvee^2 T^*X. \tag{5.3}$$

Subsequently, in Proposition 4.1 we may choose $\varphi = 0$ and $\mathfrak{H} = \bigvee^2 T^*X$.

Proposition 5.4. *Let ω be the affine connection on $\pi_{C(P),X}$ induced by a torsionless linear connection ν on $\pi_{T^*X,X}$. Then the isomorphism of Theorem 4.5 is the curvature mapping (up to a minus sign), that is,*

$$J^1C(P)/H \to \bigwedge^2 T^*X, \quad \left[j^1_x A\right]_H \mapsto \mathrm{Skew}\left(A^*\omega\right)_x = -\tilde{F}^A_x,$$

where $\mathrm{Skew} : T^*X \otimes T^*X \to \bigwedge^2 T^*X$ *is the* skew-symmetrization.

Proof. Firstly, note that $(T^*X \otimes T^*X)/\bigvee^2 T^*X \simeq \bigwedge^2 T^*X$ via the skew-symmetrization. Likewise, as $\varphi = 0$ and the Lie group is Abelian, the action (4.2) reads

$$\left(C(P) \times_X \bigvee^2 T^*X\right) \times_X T^*X \to C(P) \times_X \bigvee^2 T^*X, \quad \left((A_x, \xi_x), \eta_x\right) \mapsto (A_x + \eta_x, \xi_x).$$

Subsequently, $(T^*X \otimes T^*X)_{0,\mathfrak{H}} \simeq \bigwedge^2 T^*X$, whence the isomorphism of Theorem 4.5 becomes $J^1C(P)/H \simeq \bigwedge^2 T^*X$, since $C(P)/T^*X \simeq X$. To conclude, by using local coordinates it can be checked that $\mathrm{Skew}(A^*\omega) = -\tilde{F}^A$ for each $A \in \Gamma(\pi_{C(P),X})$.

Therefore, for each $\tilde{F}^A \in \Gamma\left(\pi_{\bigwedge^2 T^*X,X}\right)$, the reduced Lagrangian is given by

$$l\left(\tilde{F}^A\right) = g\left(\tilde{F}^A, \tilde{F}^A\right),$$

for which the partial derivative is[1]

$$\frac{\delta l}{\delta \tilde{F}^A} = 2\, \iota_{\tilde{F}^A} g \in \Gamma\left(\pi_{\bigwedge^2 TX,X}\right).$$

Let $\nabla^* : \Gamma(\pi_{T^*X,X}) \to \Gamma(\pi_{T^*X\otimes T^*X,X})$ be the covariant derivative on $\pi_{T^*X,X}$ corresponding to the linear connection ν and $\mathrm{div}^* : \Gamma(\pi_{TX\otimes TX,X}) \to \Gamma(\pi_{TX,X})$ be its divergence. From Theorem 4.20 we know that if $A \in \Gamma(\pi_{C(P),X})$ is a solution of the Euler–Lagrange equations for L, then the reduced section $\tilde{F}^A \in \Gamma\left(\pi_{\bigwedge^2 T^*X,X}\right)$ satisfies the following reduced equation,

$$\mathrm{div}^*\left(\iota_{\tilde{F}^A} g\right) = 0. \tag{5.4}$$

Recall that the *Hodge star operator* $\star : \Omega^k(X) \to \Omega^{4-k}(X)$ is defined implicitly as:

$$\alpha \wedge \star\beta = g(\alpha, \beta)\, v_g, \qquad \alpha, \beta \in \Omega^k(X), \tag{5.5}$$

and it satisfies $\star\star = (-1)^{k(4-k)}\epsilon(g)$ on $\Omega^k(X)$, being $\epsilon(g)$ the parity of the signature of g.

[1] Observe that $\left(\bigwedge^2 T^*X\right)^* = \bigwedge^2 TX = \left(\bigvee^2 T^*X\right)^\perp = \left((T^*X \otimes T^*X)/\bigvee^2 T^*X\right)^*$

Theorem 5.5 (Maxwell equations). *In the above conditions, the reduced equation* (5.4) *is equivalent to the* Maxwell equation in vacuum*, that is:*

$$\mathrm{d}^\star \tilde{F}^A = 0,$$

where $\mathrm{d}^\star = \star \circ \mathrm{d} \circ \star : \Omega^k(X) \to \Omega^{k-1}(X)$ *denotes the* codifferential.

The equivalence is proved in local charts, making use of the isomorphisms $\sharp : T^*X \to TX$ and $\flat : TX \to T^*X$ implicitly defined as $g\left(\alpha^\sharp, U\right) = g\left(\alpha, U_\flat\right) = \alpha(U), \qquad (\alpha, U) \in T^*X \times_X TX.$

Theorem 5.6 (Reconstruction). *Let* $\mathcal{U} \subset X$ *be a simply connected domain and let*

$$F \in \Gamma\left(\mathcal{U}, \pi_{\bigwedge^2 T^*X, X}\right)$$

be a solution of the Maxwell's equations in vacuum. Then there exists a solution $A \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right)$ *of the Euler–Lagrange equations for* L *such that* $F = \tilde{F}^A$ *if and only if the following compatibility condition holds:*

$$\mathrm{d}F = 0.$$

In short, we have the following local equivalence for sections $A \in \Gamma(\pi_{C(P),X})$:

$$\mathcal{EL}(L)\left(j^1 A\right) = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \mathrm{d}^\star \tilde{F}^A = 0, \\ \mathrm{d}\tilde{F}^A = 0. \end{cases}$$

5.3.1 k-form Electromagnetism

When the principal bundle P is trivial $P = X \times \mathrm{U}(1)$, the bundle of connections is T^*X and the Maxwell formulation of the previous section is defined on 1-forms. This can be generalized to k-form Electromagnetism, $k \in \mathbb{N}$ (see for example [78]). In this case, both the configuration and Lie group bundle are the same so that the fiberwise actions is

$$\bigwedge^k T^*X \times_X \bigwedge^k T^*X \to \bigwedge^k T^*X, \quad (A_x, \alpha_x) \mapsto A_x + \alpha_x.$$

A generalized principal connection $\omega \in \Omega^1\left(\bigwedge^k T^*X, \bigwedge^k T^*X\right)$ on $\pi_{\bigwedge^k T^*X, X}$ is just a linear connection on that vector bundle, and it is associated to itself. The corresponding isomorphism (4.1) is

$$J^1\left(\bigwedge^k T^*X\right) \to \bigwedge^k T^*X \oplus \left(T^*X \otimes \bigwedge^k T^*X\right), \quad j^1_x\alpha \mapsto \left(\alpha(x), (\alpha^*\omega)_x\right).$$

Analogous to classical Electromagnetism, we pick the Lie group subbundle of closed forms and we restrict the previous isomorphism to it, i.e.,

$$H = \left\{j^1_x\alpha \in J^1\left(\bigwedge^k T^*X\right) : (\mathrm{d}\alpha)_x = 0\right\} \overset{\omega}{\cong} \bigwedge^k T^*X \oplus \left(T^*X \vee \bigwedge^k T^*X\right).$$

By using local coordinates, it can be shown that the identification of Theorem 4.5 reads (compare to Proposition 5.4)

$$J^1\left(\bigwedge^k T^*X\right) \Big/ H \to \bigwedge^{k+1} T^*X, \quad \left[j^1_x A\right]_H \mapsto \mathrm{Skew}(A^*\omega)_x = -(\mathrm{d}A)_x.$$

The Yang–Mills Lagrangian $L : J^1\left(\bigwedge^k T^*X\right) \to \mathbb{R}$ is defined as

$$L\left(j^1 A\right) = g(\mathrm{d}A, \mathrm{d}A), \qquad A \in \Gamma\left(\pi_{\bigwedge^{k+1} T^*X, X}\right).$$

It is H-invariant, so we may consider the reduced Lagrangian $l : \bigwedge^{k+1} T^*X \to \mathbb{R}$. Namely, it is given by

$$l(C) = g(C, C), \qquad C \in \Gamma\left(\pi_{\bigwedge^{k+1} T^*X, X}\right).$$

Fixed $C \in \Gamma\left(\pi_{\bigwedge^{k+1} T^*X, X}\right)$, the partial derivative is $\delta l/\delta C = 2\,\iota_C g \in \Gamma\left(\pi_{\bigwedge^{k+1} TX, X}\right)$ and, hence, the reduced equation is

$$\operatorname{div}^*(\iota_C g) = 0.$$

In the previous expression, $\operatorname{div}^* : \Gamma\left(\pi_{TX \otimes \bigwedge^k TX}\right) \to \Gamma\left(\pi_{\bigwedge^k TX, X}\right)$ is the divergence of the linear connection ω. Similarly to classical Electromagnetism, this reduced equation is equivalent to the Maxwell equations,

$$\mathrm{d}^\star C = 0.$$

5.4 Symmetry breaking by product groups

In this example, we consider a gauge theory whose structure group is a direct product and we suppose that the gauge symmetry is broken to the subgroup given by one of the factors. More specifically, let $\pi_{P,X} : P \to X$ be a principal G-bundle with

$$G = N \times \mathrm{U}(1),$$

being N a semisimple Lie group, and denote by $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}$ the corresponding Lie algebra. Consider the Lie subgroup $G_0 = \{1\} \times \mathrm{U}(1) \simeq \mathrm{U}(1)$. Since action of G on G_0 by conjugation is trivial, we have $\tilde{G} = (P \times G)/G \simeq (P \times N)/G \times \mathrm{U}(1)$. Hence, we may write:

$$J^1\tilde{G} \simeq J^1\left(\frac{P \times N}{G}\right) \times_X J^1(X, \mathrm{U}(1)).$$

As a usual gauge theory, the configuration bundle is the bundle of connections of $\pi_{P,X}$, that is, $\pi_{C(P),X}$, but we suppose that the symmetry is broken to G_0, i.e., we only consider gauge transformations coming from elements of this subgroup. In other words, we restrict the right fibered action $C(P) \times_X J^1\tilde{G} \to C(P)$ to the Lie group subbundle $J^1(X, \mathrm{U}(1)) \simeq \{1\} \times_X J^1(X, \mathrm{U}(1)) \subset J^1\tilde{G}$. A quick computation shows that it is given by:

$$C(P) \times_X J^1(X, \mathrm{U}(1)) \to C(P), \quad \left(A_x, j^1_x g\right) \mapsto A_x + (dg)_x g(x)^{-1}.$$

Analogous to Electromagnetism in vacuum, T^*X may be taken as the Lie group bundle of symmetries,

$$C(P) \times_X T^*X \to C(P), \quad (A_x, \alpha_x) \mapsto A_x + \alpha_x. \tag{5.6}$$

A trivialization of $\pi_{P,X}$ enables us to prove the following result.

Lemma 5.7. *Let $P_0 = P/G_0$, which is a principal N-bundle over X, and consider the corresponding bundle of connections, $C(P_0) \to X$. Then there exists a bundle isomorphism given by*

$$C(P)/T^*X \to C(P_0), \quad [A_x]_{T^*X} \mapsto (A_0)_x.$$

On the other hand, from the jet extension of the fibered action (5.6), we are only interested in elements coming from $J^2(X, \mathrm{U}(1)) \subset J^1\left(J^1(X, \mathrm{U}(1))\right)$, which correspond to the Lie group subbundle

$$H = \left\{j^1_x \alpha \in J^1(T^*X) : (\mathrm{d}\alpha)_x = 0\right\} \subset J^1(T^*X).$$

Let ν be a linear connection on $\pi_{T^*X,X}$ such that $\hat{\nu}(T^*X) \subset H$. Note that equation (5.3) is also valid for this case, so we may choose $\varphi = 0$ and, hence, $\mathfrak{H} = \bigvee^2 T^*X$. Similarly, we have

$$(T^*X \otimes T^*X)_{0,\mathfrak{H}} \simeq C(P_0) \times_X \bigwedge^2 T^*X.$$

A slight modification of the proof of Proposition 5.4 leads to the following result.

Proposition 5.8. *Let $\omega \in \Omega^1(C(P), T^*X)$ be a generalized principal connection on $\pi_{C(P),C(P_0)}$ associated to ν. Then the identification of Theorem 4.5 reads*

$$J^1C(P)/H \to J^1C(P_0) \times_X C(P_0) \times_X \bigwedge^2 T^*X, \quad \left[j^1_x A\right]_H \mapsto \left(j^1_x A_0, A_0(x), -\tilde{F}^A_x\right).$$

Thanks to this identification, for each section $A \in \Gamma(\pi_{C(P),X})$ we define the reduced section as

$$\overline{A} = \left(A_0, \tilde{F}^A\right) \in \Gamma\left(\pi_{C(P_0)\times_X \bigwedge^2 T^*X,X}\right).$$

Let $L : J^1C(P) \to \mathbb{R}$ be an H-invariant Lagrangian density and consider the reduced Lagrangian, $l : J^1C(P_0) \times_X C(P_0) \times_X \bigwedge^2 T^*X \to \mathbb{R}$. Let ∇^0 be a torsion free linear connection on $T(C(P_0))$ projectable onto a linear connection ∇^X on TX. We know that it induces an affine connection $\nabla^{(1)}$ on $\pi_{J^1C(P_0),C(P_0)}$. In addition, we assume that ν is the dual connection of ∇^X. These connections induce an affine connection on the reduced space, as described in Proposition 4.8. The partial derivatives of the reduced Lagrangian are

$$\frac{\delta l}{\delta A_0} \in \Gamma\left(\pi_{T^*(C(P_0)),X}\right), \quad \frac{\delta l}{\delta j^1 A_0} \in \Gamma\left(\pi_{TX\otimes V^*(C(P_0)),X}\right), \quad \frac{\delta l}{\delta \tilde{F}^A} \in \Gamma\left(\pi_{\bigwedge^2 TX,X}\right).$$

Let $\nabla^* : \Gamma(\pi_{T^*X,X}) \to \Gamma(\pi_{T^*X\otimes T^*X,X})$ be the covariant derivative associated to ν and denote by $\mathrm{div}^* : \Gamma(\pi_{TX\otimes TX,X}) \to \Gamma(\pi_{TX,X})$ its divergence. Likewise, let $\mathrm{div}^0 : \Gamma(\pi_{TX\otimes V^*(C(P_0)),X}) \to \Gamma(\pi_{V^*(C(P_0)),X})$ be the divergence of the operator $\overline{\nabla}^0$ defined from ∇^0 in (4.14). The reduced equations are straightforwardly obtained from *(iv)* of Theorem 4.20.

Theorem 5.9. *Let $\omega \in \Omega^1(C(P), T^*X)$ be a generalized principal connection on $\pi_{C(P),C(P_0)}$ associated to ν, and $A \in \Gamma(\pi_{C(P),X})$ be a solution of the Euler–Lagrange equations for L. Then the reduced section $\overline{A} = \left(A_0, \tilde{F}^A\right) \in \Gamma(\pi_{C(P_0)\times_X \bigwedge^2 T^*X,X})$ satisfies the following reduced equations:*

$$\begin{cases} \dfrac{\delta l}{\delta A_0} - \mathrm{div}^0\left(\dfrac{\delta l}{\delta j^1 A_0}\right) = \left\langle \dfrac{\delta l}{\delta \tilde{F}^A}, \iota_{dA_0}\tilde{\Omega} \right\rangle, \\ \mathrm{div}^*\left(\dfrac{\delta l}{\delta \tilde{F}^A}\right) = 0. \end{cases}$$

5.5 Non-Abelian gauge theories

In §5.3 we presented Electromagnetism, which constitutes an instance of an Abelian gauge theory. In this last example, those results are extended to non-Abelian gauge theories. In particular, we show that the Utiyama theorem (see [89] for the original version, and [39] for the geometric version) and the Yang–Mills equations may be obtained through the gauge reduction process described in the previous chapter.

5.5.1 Geometric formulation

Let $\pi_{P,X} : P \to X$ be a (standard) principal bundle with a semisimple structure group G. The configuration bundle of gauge theories is the bundle of connections:

$$\pi_{C(P),X} : C(P) = \left(J^1P\right)/G \to X,$$

whose sections define principal connections on $\pi_{P,X}$, and which is an affine bundle modelled on $\pi_{T^*X\otimes\mathfrak{g},X} : T^*X \otimes \tilde{\mathfrak{g}} \to X$, the bundle of covectors taking values in the adjoint bundle $\tilde{\mathfrak{g}} = (P \times \mathfrak{g})/G$, where $\mathfrak{g}$ is the Lie algebra of G and the action of G on $\mathfrak{g}$ is given by the adjoint representation.

The main instance in this framework is the *Yang–Mills theory*. In this case, given a pseudo-Riemannian metric, g, on X and the Killing metric, $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, on the Lie algebra, we first define a fibered inner product:

$$\langle \cdot, \cdot \rangle_g : \left(\wedge^2 T^*X \otimes \tilde{\mathfrak{g}}\right) \times_X \left(\wedge^2 T^*X \otimes \tilde{\mathfrak{g}}\right) \to \mathbb{R}, \quad (\alpha_1 \otimes [p, \xi_1]_G, \alpha_2 \otimes [p, \xi_2]_G) \mapsto g(\alpha_1, \alpha_2)\, K(\xi_1, \xi_2).$$

Then the Yang–Mills Lagrangian density is $\mathfrak{L} = L\, v_g$, where $v_g \in \Omega^n(X)$ is the pseudo-Riemannian volume form and

$$L : J^1C(P) \to \mathbb{R}, \quad j^1_x A \mapsto \left\langle \tilde{F}^A_x, \tilde{F}^A_x \right\rangle_g . \tag{5.7}$$

The Yang–Mills Lagrangian density is a particular case of a gauge invariant Lagrangian. From the perspective of fibered actions, gauge transformations can be regarded as sections of the associated bundle $\pi_{\tilde{G},X} : \tilde{G} = (P \times G)/G \to X$, where the action of G on itself is by conjugation, which is a Lie group bundle with the fiberwise group structure inherited from G. This Lie group bundle acts on the left on $\pi_{P,X}$. In order to have a fiber action on $C(P)$, we need first derivatives of the sections of $\pi_{\tilde{G},X}$, i.e., we consider its first jet bundle. More precisely, we define

$$J^1\tilde{G} \times_X C(P) \to C(P), \quad \left(j^1_x\gamma, [j^1_x s]_G\right) \mapsto \left[j^1_x(\gamma \cdot s)\right]_G . \tag{5.8}$$

Following the notation in the previous chapter, $\mathcal{G} = J^1\tilde{G}$ is the Lie group bundle of the reduction. Furthermore, the gauge invariance of a Lagrangian is understood with respect to the Lie group subbundle $H = J^2\tilde{G} \subset J^1\left(J^1\tilde{G}\right) = J^1\mathcal{G}$.

Unfortunately, since the fibered action (5.8) is not free, the corresponding quotients fail to be manifolds and our results cannot be applied as they are. However, this singular situation can be fixed by extending the configuration bundle of the theory; namely, we take $\pi_{J^1P,X}$ instead of $\pi_{C(P),X}$ as the configuration bundle. The action of $J^1\tilde{G}$ is now free and, in fact, fiberwisely transitive, i.e., $\left(J^1P\right)/J^1\tilde{G} \simeq X$.

Observe that any Lagrangian $L : J^1C(P) \to \mathbb{R}$ may be lifted to a new (unreduced) Lagrangian

$$\hat{L} : J^1\left(J^1P\right) \to \mathbb{R}, \quad j^1_x\hat{s} \mapsto \hat{L}\left(j^1_x\hat{s}\right) = L\left(j^1_x[\hat{s}]_G\right). \tag{5.9}$$

As a straightforward consequence of (5.1), a function $\hat{L} \in C^\infty\left(J^1\left(J^1P\right)\right)$ comes from a function $L \in C^\infty\left(J^1C(P)\right)$ if and only if it is invariant with respect to the following (right) fibered action:

$$J^1\left(J^1P\right) \times_X J^1(X, G) \to J^1\left(J^1P\right), \quad \left(j^1_x\hat{s}, j^1_x\tilde{h}\right) \mapsto j^1_x\left(\hat{s} \cdot \tilde{h}\right). \tag{5.10}$$

5.5.2 Trivialized spaces and actions

Since all the results concerning reduction are local, for simplicity of the exposition, we will confine ourselves to trivializing charts. More precisely, we may assume that $P = X \times G$ with $X = \mathbb{R}^n$. Hence,

$\tilde{G} = X \times G$ and $\tilde{\mathfrak{g}} = X \times \mathfrak{g}$. Moreover, sections $s = (\mathrm{id}_X, \tilde{g}) \in \Gamma(\pi_{P,X})$ may be identified with functions $\tilde{g} : X \to G$, as in Chapter 3. Recall from Theorem 3.11 that by means of the right trivialization of the tangent bundle of G,

$$TG \simeq G \times \mathfrak{g}, \qquad U_g \mapsto \left(g, \xi = (dR_{g^{-1}})_g(U_g)\right),$$

we have the following identifications:

$$J^1 P = J^1 \tilde{G} = G \ltimes (T^*X \otimes \mathfrak{g}), \qquad J^1 \tilde{\mathfrak{g}} = \mathfrak{g} \times (T^*X \otimes \mathfrak{g}), \tag{5.11}$$

where the (fibered) semidirect product is given with respect to the adjoint representation, i.e.,

$$(g, \xi_x) \cdot (h, A_x) = (gh, \xi_x + \mathrm{Ad}_g \circ A_x), \qquad (g, \xi_x), (h, A_x) \in J^1 \tilde{G}, \tag{5.12}$$

This expression also holds for the fibered action of $\pi_{J^1\tilde{G},X}$ on $\pi_{J^1P,X}$. The inverse element is given by

$$(g, \xi_x)^{-1} = (g^{-1}, -\mathrm{Ad}_{g^{-1}} \circ \xi_x), \qquad (g, \xi_x) \in G \ltimes (T^*X \otimes \mathfrak{g}),$$

and the adjoint representation reads

$$\mathrm{Ad}_{(g,\xi_x)}(\eta, \phi_x) = (\mathrm{Ad}_g(\eta), \mathrm{Ad}_g \circ \phi_x + [\xi_x, \mathrm{Ad}_g(\eta)]) \tag{5.13}$$

for every $(g, \xi_x) \in G \ltimes (T^*X \otimes \mathfrak{g})$ and $(\eta, \phi_x) \in \mathfrak{g} \times (T^*X \otimes \mathfrak{g})$, where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. With the aid of a linear connection ∇^X on the tangent bundle of X, Theorem 3.9 yields the following identification of the iterated jet:

$$\begin{aligned} J^1\left(J^1 P\right) = J^1\left(J^1 \tilde{G}\right) &\simeq G \times (T^*X \otimes \mathfrak{g}) \times_X (T^*X \otimes \mathfrak{g}) \times_X (T^*X \otimes T^*X \otimes \mathfrak{g}) \\ j^1_x\left(\tilde{g}, \tilde{\xi}\right) &\mapsto \left(g = \tilde{g}(x), \tilde{\xi}(x); \left(dR_{g^{-1}}\right)_g \circ (d\tilde{g})_x, \tilde{\nabla}\tilde{\xi}(x)\right), \end{aligned} \tag{5.14}$$

where ∇ is the dual connection of ∇^X and $\tilde{\nabla}$ is the extension of ∇ to $\mathfrak{g}$-valued forms on X.

Lastly, Corollary 3.16 ensures that the second jet bundle takes the form

$$H = J^2\tilde{G} \simeq \left\{(g, \xi_x; \eta_x, \phi_x) \in J^1\left(J^1\tilde{G}\right) : \xi_x = \eta_x,\ \mathrm{Skew}(\phi_x) = -\frac{1}{2}[\xi_x, \xi_x]\right\}, \tag{5.15}$$

where $\mathrm{Skew} : T^*X \otimes T^*X \to \bigwedge^2 T^*X$ is the skew-symmetrization map.

Remark 5.1 *The previous identification hold provided we are dealing with a matrix group, as in §3.4. Therefore, we assume this is the case in the following. Furthermore, recall that the bracket of 1-forms is understood as follows:*

$$[\xi_x, \eta_x](U_x, V_x) = [\xi_x(U_x), \eta_x(V_x)], \qquad \xi_x, \eta_x \in T^*_x X \otimes \mathfrak{g},\ U_x, V_x \in T_x X,\ x \in X.$$

5.5.3 Reduced configuration space

Consider the Lie group bundle connection ν on $J^1\tilde{G} = G \ltimes T^*X \otimes \mathfrak{g} \to X$ given in Lemma 2.23, as well as the generalized principal connection ω associated to ν given in Proposition 2.24 with $f = 0$ and $g = 0$, that is,

$$\hat{\omega} : J^1 P \to J^1\left(J^1 P\right), \quad (h, A_x) \mapsto (h, A_x; 0_x, 0_x). \tag{5.16}$$

Clearly, the isomorphism (4.1) reduces to the identity map and, from (5.15), the affine map defining the subbundle H is

$$\varphi : J^1\tilde{G} \to T^*X \times_X J^1\tilde{\mathfrak{g}}, \quad (g,\xi_x) \mapsto \left(\xi_x, -\frac{1}{2}[\xi_x,\xi_x]\right). \tag{5.17}$$

Moreover, the vector subbundle from Proposition 4.1 is given by

$$\mathfrak{H} = \{0\} \times_X \left(\bigvee^2 T^*X \otimes \mathfrak{g}\right) \subset (T^*X \otimes \mathfrak{g}) \times_X (T^*X \otimes T^*X \otimes \mathfrak{g}) = T^*X \otimes J^1\tilde{\mathfrak{g}},$$

From (5.13), we know that the quotient (4.5) is given by $\tilde{\mathfrak{H}} = \mathfrak{H}$.

Proposition 5.10. *Let $x \in X$ and $\hat{s} = (\tilde{g},\tilde{\xi}) \in \Gamma\left(\pi_{J^1P,X}\right)$, and write $j^1_x\hat{s} = (g,\xi_x;\eta_x,\phi_x) \in J^1\left(J^1P\right)$ under the identification (5.14). Then the pull-back by $\hat{s}$ of the connection 1-form corresponding to (5.16) is given by*

$$\left(\hat{s}^*\omega\right)_x = (\eta_x, \phi_x + [\xi_x,\eta_x]).$$

Proof. Recall that (5.16) is understood through identification (5.14). More specifically, if $(\gamma,\rho) \in \Gamma\left(\pi_{J^1P,X}\right)$ is such that $j^1_x(\gamma,\rho) \simeq \hat{\omega}(g,\xi_x)$ via (5.14), then $\gamma(x) = g$, $\rho(x) = \xi_x$, $(d\gamma)_x = 0$ and $\tilde{\nabla}\rho(x) = 0$. Observe that $\rho : X \to T^*X \otimes \mathfrak{g}$ can always be chosen so that $(d\rho)^h_x = \left(d\tilde{\xi}\right)^h_x$, where the superscript h denotes the horizontal part given by $\tilde{\nabla}$. As a result,

$$(d\hat{s})^v_x = (d\hat{s})_x - d(\gamma,\rho)_x = \left((d\tilde{g})_x, \tilde{\nabla}\tilde{\xi}(x)\right) = ((dR_g)_1 \circ \eta_x, \phi_x), \tag{5.18}$$

where the superscript v denotes the vertical part given by ω.

On the other hand, we compute the infinitesimal generator of $(\alpha,\beta_x) \in J^1\tilde{\mathfrak{g}}$ for the fibered action (5.12). Namely, (2.6) leads to

$$\begin{aligned}
(\alpha,\beta_x)^*_{(g,\xi_x)} &= \left.\frac{d}{dt}\right|_{t=0} (\exp(t\alpha), t\beta_x) \cdot (g,\xi_x) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\exp(t\alpha)g, t\beta_x + \mathrm{Ad}_{\exp(t\alpha)} \circ \xi_x\right) \\
&= ((dR_g)_1(\alpha), \beta_x - [\xi_x,\alpha]).
\end{aligned}$$

Lastly, by using this expression with $\alpha = \eta_x$ and $\beta_x = \phi_x + [\xi_x,\eta_x]$, together with (5.18) and (i) of Definition 2.5, we conclude:

$$\begin{aligned}
\left(\hat{s}^*\omega\right)_x &= \omega_{\hat{s}(x)} \circ (d\hat{s})_x \\
&= \omega_{(g,\xi_x)} \circ (d\hat{s})^v_x \\
&= \omega_{(g,\xi_x)} \circ ((dR_g)_1 \circ \eta_x, \phi_x) \\
&= \omega_{(g,\xi_x)} \circ (\eta_x, \phi_x + [\xi_x,\eta_x])^*_{(g,\xi_x)} \\
&= (\eta_x, \phi_x + [\xi_x,\eta_x]).
\end{aligned}$$

Proposition 5.11 (Reduced configuration space). *In the above conditions, the identification (4.7) reads*

$$\begin{aligned}
\left(J^1\left(J^1P\right)\right)/J^2\tilde{G} &\simeq (T^*X \otimes \mathfrak{g}) \oplus \left(\bigwedge^2 T^*X \otimes \mathfrak{g}\right) \\
[g,\xi_x;\eta_x,\phi_x]_{J^2\tilde{G}} &\mapsto \left(\mathrm{Ad}_{g^{-1}} \circ (\eta_x - \xi_x), \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{Skew} \circ \phi_x - \frac{1}{2}[\xi_x,\xi_x]\right)\right).
\end{aligned} \tag{5.19}$$

Proof. Observe that

$$\frac{T^*X \otimes J^1\tilde{\mathfrak{g}}}{\mathfrak{H}} \simeq (T^*X \otimes \mathfrak{g}) \times_X \left(\bigwedge^2 T^*X \otimes \mathfrak{g}\right), \qquad [\eta_x,\phi_x]_{\mathfrak{H}} \mapsto \left(\eta_x, \hat{\phi}_x = \mathrm{Skew} \circ \phi_x\right). \tag{5.20}$$

By using this, (5.12) and (5.13), it can be checked that the action (4.3) is given by

$$(h, A_x)\cdot\big((g,\xi_x),(\eta_x,\hat{\phi}_x)\big) = \Big(\big(hg, A_x + \mathrm{Ad}_h\circ\xi_x\big), \big(\mathrm{Ad}_h\circ\eta_x + A_x, \mathrm{Ad}_h\circ\hat{\phi}_x + \mathrm{Skew}\circ[A_x, \mathrm{Ad}_h\circ\eta_x] - \frac{1}{2}[A_x,A_x]\big)\Big),$$

for each $(g,\xi_x),(h,A_x)\in J^1P = J^1\tilde{G}$ and $(\eta_x,\hat{\phi}_x)\in(T^*X\otimes\mathfrak{g})\times_X\left(\bigwedge^2 T^*X\otimes\mathfrak{g}\right)$. Subsequently,

$$[\![(g,\xi_x),(\eta_x,\phi_x)]\!]_{\mathfrak{H},\varphi} = \left[\!\!\left[(1,0_x),\left(\mathrm{Ad}_{g^{-1}}\circ(\eta_x-\xi_x), \mathrm{Ad}_{g^{-1}}\circ\left(\mathrm{Skew}\circ(\phi_x - [\xi_x,\eta_x]) - \frac{1}{2}[\xi_x,\xi_x]\right)\right)\right]\!\!\right]_{\mathfrak{H},\varphi},$$

for each $(g,\xi_x)\in J^1P$ and $(\eta_x,\phi_x)\in T^*X\otimes J^1\tilde{\mathfrak{g}}$. Observe that, for each $U_x,V_x\in T_xX$, we have

$$\begin{aligned}\mathrm{Skew}\circ[\xi_x,\eta_x](U_x,V_x) &= \frac{1}{2}\left([\xi_x(U_x),\eta_x(V_x)] - [\xi_x(V_x),\eta_x(U_x)]\right)\\ &= \frac{1}{2}([\eta_x(U_x),\xi_x(V_x)] - [\eta_x(V_x),\xi_x(U_x)])\\ &= \mathrm{Skew}\circ[\eta_x,\xi_x](U_x,V_x).\end{aligned}$$

By using this and Proposition 5.10, we conclude:

$$\begin{aligned}[g,\xi_x;\eta_x,\phi_x]_{J^2\tilde{G}} &\overset{(4.7)}{\simeq} [\![(g,\xi_x),(\eta_x,\phi_x+[\xi_x,\eta_x])]\!]_{\mathfrak{H},\varphi}\\ &\overset{(5.20)}{\simeq}\left(\mathrm{Ad}_{g^{-1}}\circ(\eta_x-\xi_x), \mathrm{Ad}_{g^{-1}}\circ\left(\mathrm{Skew}\circ(\phi_x) - \frac{1}{2}[\xi_x,\xi_x]\right)\right).\end{aligned}$$

5.5.4 Utiyama's theorem

In order to show Utiyama's theorem we now present two auxiliary results that allow for reformulate the identification of the reduced configuration space.

Lemma 5.12. *The (right) fibered action (5.10) induces the following (right) fibered action on the reduced space:*

$$(A_x,F_x)\cdot(h,\mu_x) = \left(\mathrm{Ad}_{h^{-1}}\circ(A_x+\mu_x), \mathrm{Ad}_{h^{-1}}\circ F_x\right),$$

for each $(A_x,F_x)\in(T^*X\otimes\mathfrak{g})\times_X\left(\bigwedge^2 T^*X\otimes\mathfrak{g}\right)$ and $(h,\mu_x)\in G\ltimes(T^*X\otimes\mathfrak{g})$.

Proof. Firstly, we transfer the right action

$$J^1P\times G\to J^1P,\quad \left(j^1_x\tilde{g},h\right)\mapsto j^1_x\left(\tilde{g}h\right),$$

to the trivialized space (5.11), yielding

$$J^1P\times G\to J^1P,\quad ((g,\xi_x),h)\mapsto(gh,\xi_x).$$

As a result, the right fibered action (5.10) reads

$$J^1\left(J^1P\right)\times_X J^1(X,G)\to J^1\left(J^1P\right),\qquad \left(j^1_x\left(\tilde{g},\tilde{\xi}\right), j^1_x\tilde{h}\right)\mapsto j^1_x\left(\tilde{g}\tilde{h},\tilde{\xi}\right).$$

By taking the identification (5.14) into account, we arrive at

$$(g,\xi_x;\eta_x,\phi_x)\cdot(h,\mu_x) = (gh,\xi_x;\eta_x+\mathrm{Ad}_g\circ\mu_x,\phi_x).$$

At last, we transfer this fibered action through identification (5.19), i.e,

$$A_x = \mathrm{Ad}_{g^{-1}} \circ (\eta_x - \xi_x), \qquad F_x = \mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{Skew} \circ \phi_x - \frac{1}{2}[\xi_x, \xi_x] \right),$$

which gives

$$(A_x, F_x) \cdot (h, \mu_x) = (\mathrm{Ad}_{h^{-1}} \circ (A_x + \mu_x), \mathrm{Ad}_{h^{-1}} \circ F_x).$$

Lemma 5.13. *Under the above trivializations, the projection of the bundle of connections reads*

$$J^1 P \to C(P), \quad (g, \xi_x) \mapsto A_x = -\mathrm{Ad}_{g^{-1}} \circ \xi_x.$$

Furthermore, by identifying

$$J^1 C(P) \simeq (T^*X \otimes \mathfrak{g}) \times_X (T^*X \otimes T^*X \otimes \mathfrak{g}), \quad j^1_x A \mapsto (A_x = A(x),\ B_x = \tilde{\nabla} A(x)),$$

the jet lift of the projection reads

$$J^1 \left(J^1 P \right) \to J^1 C(P), \quad (g, \xi_x; \eta_x, \phi_x) \mapsto \left(A_x = -\mathrm{Ad}_{g^{-1}} \circ \xi_x,\ B_x = -\mathrm{Ad}_{g^{-1}} \circ (\phi_x - [\eta_x, \xi_x]) \right).$$

Proof. A connection form $A_x \in T^*_x X \otimes \mathfrak{g}$ defines a principal connection on $\pi_{P,X}$ as follows,

$$T_x X \oplus T_g G \mapsto \mathfrak{g}, \quad (U_x, U_g) \mapsto \left(dL_{g^{-1}} \right)_g (U_g) + A_x(U_x).$$

On the other hand, let $\tilde{g} : X \to G$ be a local function such that $\tilde{g}(x) = g$ and $(dR_{g^{-1}})_g \circ (d\tilde{g})_x = \xi_x$. Thus, the principal connection defined by (g, ξ_x) reads

$$T_x X \oplus T_g G \to \mathfrak{g}, \quad (U_x, U_g) \mapsto (dL_{g^{-1}})_g (U_g) - \mathrm{Ad}_{g^{-1}} (\xi_x(U_x)),$$

where we have used that $(U_x, U_g)^v = (U_x, U_g) - d\left(\mathrm{id}_X, \tilde{g}\right)_x (U_x) = (0_x, U_g - (d\tilde{g})_x (U_x))$. By equating the two previous expression we conclude that $A_x = -\mathrm{Ad}_{g^{-1}} \circ \xi_x$.

For the second part, let $\tilde{g} : X \to G$ and $\tilde{\xi} \in \Gamma(\pi_{T^*X \otimes \mathfrak{g}, X})$ be such that $\tilde{g}(x) = g$, $(dR_{g^{-1}})_g \circ (d\tilde{g})_x = \eta_x$, $\tilde{\xi}(x) = \xi_x$ and $\tilde{\nabla}\tilde{\xi}(x) = \phi_x$. By using Lemma 3.10 and the fact that $(dR_g)_{g^{-1}} \circ \left(d\tilde{g}^{-1} \right)_x = -\mathrm{Ad}_{g^{-1}} \circ \eta_x$, we conclude:

$$\begin{aligned} B_x &= -\tilde{\nabla} \left(\mathrm{Ad}_{\tilde{g}^{-1}} \circ \tilde{\xi} \right)_x \\ &= -\left(-[\mathrm{Ad}_{g^{-1}} \circ \eta_x, \mathrm{Ad}_{g^{-1}} \circ \xi_x] + \mathrm{Ad}_{g^{-1}} \circ \phi_x \right) \\ &= -\mathrm{Ad}_{g^{-1}} \circ (\phi_x - [\eta_x, \xi_x]). \end{aligned}$$

A straightforward consequence of the previous Lemma is that the curvature of the connection $A_x = -\mathrm{Ad}_{g^{-1}} \circ \xi_x$ reads

$$F^A_{(x,g)} = -\mathrm{Ad}_{g^{-1}} \circ \left(\mathrm{Skew} \circ (\phi_x - [\eta_x, \xi_x]) - \frac{1}{2}[\xi_x, \xi_x] \right) \in \bigwedge^2 T^*X \otimes \mathfrak{g}. \tag{5.21}$$

Locally, the reduced curvature is obtained when the previous expression is evaluated at the identity element $g = 1$, i.e., $\tilde{F}^A_x = F^A_{(x,1)} \in \bigwedge^2 T^*X \otimes \mathfrak{g}$.

Corollary 5.14 (Reduced configuration space). *The identification (5.19) may be written as*

$$\left(J^1 \left(J^1 P \right) \right) / J^2 \tilde{G} \simeq (T^*X \otimes \mathfrak{g}) \oplus \left(\bigwedge^2 T^*X \otimes \mathfrak{g} \right), \quad [1, \xi_x; 0_x, \phi_x]_{J^2\tilde{G}} \mapsto \left(A_x, -\tilde{F}^A_x \right).$$

Proof. Given $[1,\xi_x;0_x,\phi_x]_{J^2\tilde{G}}$ it is a straightforward consequence of Lemma 5.13 and equation (5.21). To conclude, let us check that we can always choose representatives of the form $(1,\xi_x;0_x,\phi_x)\in J^1\left(J^1P\right)$. Indeed, given $(g,\xi_x;\eta_x,\phi_x)\in J^1\left(J^1P\right)$, let

$$\left(g^{-1},-\mathrm{Ad}_{g^{-1}}\circ\eta_x;-\mathrm{Ad}_{g^{-1}}\circ\eta_x,-\frac{1}{2}\mathrm{Ad}_{g^{-1}}\circ[\eta_x,\eta_x]\right)\in J^2\tilde{G}.$$

Hence,

$$\begin{aligned}\left(g^{-1},-\mathrm{Ad}_{g^{-1}}\circ\eta_x;-\mathrm{Ad}_{g^{-1}}\circ\eta_x,-\frac{1}{2}\mathrm{Ad}_{g^{-1}}\circ[\eta_x,\eta_x]\right)\cdot(g,\xi_x;\eta_x,\phi_x)\\ =\left(1,\mathrm{Ad}_{g^{-1}}\circ(\xi_x-\eta_x);0_x,\mathrm{Ad}_{g^{-1}}\circ\left(\phi_x-[\eta_x,\xi_x]-\frac{1}{2}[\eta_x,\eta_x]\right)\right).\end{aligned}$$

Therefore, $[g,\xi_x;\eta_x,\phi_x]_{J^2\tilde{G}}=\left[1,\tilde{\xi}_x;0_x,\tilde{\phi}_x\right]_{J^2\tilde{G}}$, where

$$\tilde{\xi}_x=\mathrm{Ad}_{g^{-1}}\circ(\xi_x-\eta_x),\qquad \tilde{\phi}_x=\mathrm{Ad}_{g^{-1}}\circ\left(\phi_x-[\eta_x,\xi_x]-\frac{1}{2}[\eta_x,\eta_x]\right).$$

The Utiyama's theorem is now straightforward.

Corollary 5.15 (Utiyama's theorem). *A Lagrangian $L:J^1C(P)\to\mathbb{R}$ is gauge invariant if and only if $L=l\circ\tilde{F}$, where*

$$\tilde{F}:J^1C(P)\to\textstyle\bigwedge^2T^*X\otimes\tilde{\mathfrak{g}},\quad j^1_xA\mapsto\tilde{F}^A_x,$$

is the reduced curvature map*, and $l:\bigwedge^2T^*X\otimes\tilde{\mathfrak{g}}\to\mathbb{R}$ is an* Ad*-invariant function.*

Proof. As above, we work locally. Let $\hat{L}:J^1\left(J^1P\right)\to\mathbb{R}$ be the lifting of L, as in (5.9), which is $J^1(X,G)$-invariant. Let $(1,\xi_x;0_x,\phi_x)\in J^1\left(J^1P\right)$ projecting to $j^1_xA\in J^1C(P)$ (recall Lemma 5.13). Then Corollary 5.14 ensures the existence of a reduced Lagrangian, $\hat{l}:(T^*X\otimes\mathfrak{g})\oplus\left(\bigwedge^2T^*X\otimes\mathfrak{g}\right)\to\mathbb{R}$, such that $L\left(j^1_xA\right)=\hat{L}(1,\xi_x;0_x,\phi_x)=\hat{l}\left(A_x,-\tilde{F}^A_x\right)$. Lastly, from Lemma 5.12 we know that $\hat{l}$ do not depend on its first variable and it is Ad-invariant on its second variable, that is, there exists an Ad-invariant function $l:\bigwedge^2T^*X\otimes\mathfrak{g}\to\mathbb{R}$ such that $\hat{l}\left(A_x,-\tilde{F}^A_x\right)=l\left(\tilde{F}^A_x\right)$, and we conclude.

This represents an alternative proof of the Utiyama's theorem that relies on the gauge reduction theory developed in the previous chapter. Roughly speaking, this new approach takes advantage of the gauge invariance of the Lagrangian density to transfer it to the corresponding fibered quotient and, by means of the identification of Corollary 5.14, this quotient space is seen to be nothing but the curvature bundle of the theory. Of course, the Ad-invariance of the reduced Lagrangian is also taken into account from our perspective.

5.5.5 Reduced field equations

Lastly, we present two lemmas in order to compute the reduced equations given in (iv) of Theorem 4.20 for non-Abelian gauge theories.

Lemma 5.16. *Let $\varphi:J^1\tilde{G}\to T^*X\times_XJ^1\tilde{\mathfrak{g}}$ be the bundle morphism defined in (5.17). Then the map (4.12) reads*

$$\varphi_*:J^1\tilde{\mathfrak{g}}\to T^*X\otimes J^1\tilde{\mathfrak{g}},\quad(\eta,\phi_x)\mapsto(\phi_x,0_x).$$

Therefore, its adjoint is given by

$$\varphi_*^\dagger:TX\otimes\left(J^1\tilde{\mathfrak{g}}\right)^*\to\left(J^1\tilde{\mathfrak{g}}\right)^*,\quad(\alpha_x,\beta_x)\mapsto(0,\alpha_x).$$

Proof. The vertical space of $\pi_{J^1\tilde{G},X}$ at each $(g,\xi_x) \in J^1\tilde{G}$ reads

$$V_{(g,\xi_x)}\left(G \ltimes (T^*X \otimes \mathfrak{g})\right) \simeq T_gG \times (T_x^*X \otimes \mathfrak{g}),$$

where we identify

$$T_x^*X \otimes \mathfrak{g} \simeq V_{\xi_x}(T^*X \otimes \mathfrak{g}), \quad \phi_x \mapsto \left.\frac{d}{dt}\right|_{t=0} \xi_x + t\phi_x.$$

In particular, for $(g,\xi_x) = (1,0_x)$ we have

$$J_x^1\tilde{\mathfrak{g}} = V_{(1,0_x)}\left(G \ltimes (T^*X \otimes \mathfrak{g})\right) \simeq \mathfrak{g} \times (T_x^*X \otimes \mathfrak{g}).$$

As a result, for each $(\eta,\phi_x) \in J_x^1\tilde{\mathfrak{g}}$ we get

$$\begin{aligned}
\varphi_*(\eta,\phi_x) &= (d\varphi)_{(1,0_x)}(\eta,\phi_x) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(t\eta), 0_x + t\phi_x) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(t\phi_x, -\frac{t^2}{2}[\phi_x,\phi_x]\right) \\
&= (\phi_x, 0_x) \in T_x^*X \times J_x^1\tilde{\mathfrak{g}}.
\end{aligned}$$

For the second part, let $(\alpha_x,\beta_x) \in T_xX \otimes \left(J_x^1\tilde{\mathfrak{g}}\right)^*$. Thence,

$$\langle \phi_x, \alpha_x\rangle = \langle (\phi_x,0_x), (\alpha_x,\beta_x)\rangle = \langle \varphi_*(\eta,\phi_x), (\alpha_x,\beta_x)\rangle = \left\langle (\eta,\phi_x), \varphi_*^\dagger(\alpha_x,\beta_x)\right\rangle.$$

Since the previous expression is valid for every $(\eta,\phi_x) \in J_x^1\tilde{\mathfrak{g}}$, we conclude.

The exponential map on the Lie group G is denoted by $\exp : \mathfrak{g} \to G$. Note that the exponential map on $T^*X \otimes \mathfrak{g}$ is the identity as it is a vector bundle. The exponential map on the fibered semidirect product is denoted by

$$\tilde{\exp} : J^1\tilde{\mathfrak{g}} = \mathfrak{g} \ltimes (T^*X \otimes \mathfrak{g}) \to J^1\tilde{G} = G \ltimes (T^*X \otimes \mathfrak{g}).$$

Lemma 5.17. *The covariant derivative on $\pi_{J^1\tilde{\mathfrak{g}},X}$ induced by the Lie group connection ν is given by $\nabla^{J^1\tilde{\mathfrak{g}}}(\eta,\phi) = \left(d\eta, \tilde{\nabla}\phi\right)$ for each $(\eta,\phi) \in \Gamma(\pi_{J^1\tilde{\mathfrak{g}},X})$.*

Proof. To begin with, from [80, Theorem 3.1] we know that

$$\tilde{\exp}(t\eta, t\phi) = \left(\exp(t\eta), \mathrm{Ad}_{\exp(t\eta)} \circ t\phi + o(t^2)\right). \tag{5.22}$$

Furthermore, recall from the proof of Proposition 5.10 that the infinitesimal generator of $(\alpha,\beta_x) \in J^1\tilde{\mathfrak{g}}$ for the fibered action (5.12) is given by $(\alpha,\beta_x)^*_{(g,\xi_x)} = ((dR_g)_1(\alpha), \beta_x - [\xi_x,\alpha])$. Subsequently, ν reads

$$\nu : T\left(J^1\tilde{G}\right) \to J^1\tilde{\mathfrak{g}}, \quad (U_g, U_{\xi_x}) \mapsto \left((dR_{g^{-1}})_g(U_g), U_{\xi_x}^v - \left[\xi_x, (dR_{g^{-1}})_g(U_g)\right]\right), \tag{5.23}$$

where the superscript v denotes the vertical part given by $\tilde{\nabla}$ and the identification $V_{\xi_x}(T^*X \otimes \mathfrak{g}) \simeq T_x^*X \otimes \mathfrak{g}$ has been used.

On the other hand, recall that (cf., for instance, [92, Theorem 2.14.3])

$$(d\exp)_{t\eta} = (dL_{\exp(t\eta)})_e \circ \frac{1 - \exp(-\mathrm{ad}_{t\eta})}{\mathrm{ad}_{t\eta}} = (dL_{\exp(t\eta)})_e + o(t) : \mathfrak{g} \to T_{\exp(t\eta)}G,$$

being

$$\frac{1-\exp(-\mathrm{ad}_{t\eta})}{\mathrm{ad}_{t\eta}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!}\mathrm{ad}_{t\eta}^i \in \mathrm{End}(\mathfrak{g}).$$

From this and Lemma 3.10, we get

$$\begin{aligned} &\bullet\quad d\exp(t\eta) = (d\exp)_{t\eta}\circ td\eta = (dL_{\exp(t\eta)})_e \circ td\eta + o(t^2),\\ &\bullet\quad \tilde{\nabla}\left(\mathrm{Ad}_{\exp(t\eta)}\circ t\phi\right) = \mathrm{Ad}_{\exp(t\eta)}\circ t\tilde{\nabla}\phi + o(t^2). \end{aligned} \tag{5.24}$$

By gathering all, we conclude:

$$\begin{aligned} \nabla^{J^1\tilde{\mathfrak{g}}}(\eta,\phi) &\overset{(2.11)}{=} \left.\frac{d}{dt}\right|_{t=0} \nu\circ d\tilde{\exp}(t\eta, t\phi)\\ &\overset{(5.22)}{=} \left.\frac{d}{dt}\right|_{t=0} \nu\circ\left(d\exp(t\eta), d\left(\mathrm{Ad}_{\exp(t\eta)}\circ t\phi\right) + o(t^2)\right)\\ &\overset{(5.23)}{=} \left.\frac{d}{dt}\right|_{t=0} \left((dR_{\exp(-t\eta)})_{\exp(t\eta)}\circ d\exp(t\eta), \tilde{\nabla}\left(\mathrm{Ad}_{\exp(t\eta)}\circ t\phi\right) + o(t^2)\right)\\ &\overset{(5.24)}{=} \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{\exp(t\eta)}\circ td\eta + o(t^2), \mathrm{Ad}_{\exp(t\eta)}\circ t\tilde{\nabla}\phi + o(t^2)\right)\\ &= \left(d\eta, \tilde{\nabla}\phi\right). \end{aligned}$$

Lemma 5.18. *Let $j^1_x\hat{s} = (1,\xi_x;0_x,\phi_x) \in J^1\left(J^1P\right)$ under the identification (5.14). From Corollary 5.14, we may write $\overline{s}(x) = (A_x, -\tilde{F}_x) \in (T^*X\otimes\mathfrak{g})\times\left(\bigwedge^2 T^*X\otimes\mathfrak{g}\right)$. Then the coadjoint representation introduced in Lemma 4.19 is given by*

$$\mathrm{ad}^*_{\overline{s}} : \mathfrak{H}^\circ \to \left(J^1\tilde{\mathfrak{g}}\right)^*, \quad (\varpi_x, \Pi_x) \mapsto \left(\mathrm{ad}^*_{A_x}\circ\varpi_x - \mathrm{ad}^*_{\tilde{F}_x}\circ\Pi_x,\ \mathrm{ad}^*_{A_x}\circ\Pi_x\right).$$

where $\tilde{\mathfrak{H}}^\circ = (TX\otimes\mathfrak{g}^*)\times_X\left(\bigwedge^2 TX\otimes\mathfrak{g}^*\right)$.

Proof. Firstly, we compute the adjoint representation corresponding to (5.13). Let $(\alpha,\beta_x),(\eta,\phi'_x)\in\mathfrak{g}\times(T^*_xX\otimes\mathfrak{g})$, then

$$\begin{aligned} \mathrm{ad}_{(\eta,\phi'_x)}(\alpha,\beta_x) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\tilde{\exp}(t\eta,t\phi'_x)}(\alpha,\beta_x)\\ &\overset{(5.22)}{=} \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\left(\exp(t\eta),\mathrm{Ad}_{\exp(t\eta)}\circ t\phi'_x + o(t^2)\right)}(\alpha,\beta_x)\\ &\overset{(5.13)}{=} \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{\exp(t\eta)}(\alpha), \mathrm{Ad}_{\exp(t\eta)}\circ\beta_x + t\left(\mathrm{Ad}_{\exp(t\eta)}\circ[\phi'_x,\alpha]\right) + o(t^2)\right)\\ &= \left(\mathrm{ad}_\eta(\alpha), \mathrm{ad}_\eta\circ\beta_x + \mathrm{ad}_{\phi'_x}(\alpha)\right). \end{aligned}$$

Subsequently, for each $(\alpha,\beta_x)\in\mathfrak{g}\times(T^*_xX\otimes\mathfrak{g})$ and $(\varpi_x,\Pi_x)\in(TX\otimes\mathfrak{g}^*)\times(TX\otimes TX\otimes\mathfrak{g}^*)$, we have

$$\begin{aligned} \left\langle \mathrm{ad}^*_{\overline{s}(x)}(\varpi_x,\Pi_x),(\alpha,\beta_x)\right\rangle &= -\left\langle(\varpi_x,\Pi_x), \mathrm{ad}_{(A_x,-\tilde{F}_x)}(\alpha,\beta_x)\right\rangle\\ &= -\left\langle(\varpi_x,\Pi_x), \left(\mathrm{ad}_{A_x}\circ\alpha, \mathrm{ad}_{A_x}\circ\beta_x - \mathrm{ad}_{\tilde{F}_x}(\alpha)\right)\right\rangle\\ &= \left\langle\mathrm{ad}^*_{A_x}\circ\varpi_x,\alpha\right\rangle + \left\langle\mathrm{ad}^*_{A_x}\circ\Pi_x,\beta_x\right\rangle - \left\langle\mathrm{ad}^*_{\tilde{F}_x}\circ\Pi_x,\alpha\right\rangle. \end{aligned}$$

Since the previous expression is valid for every $(\alpha,\beta_x)\in\mathfrak{g}\times(T^*_xX\otimes\mathfrak{g})$, we conclude.

Theorem 5.19 (Reduced field equations). *Let $\tilde{\mathrm{div}} : \Gamma(TX\otimes TX\otimes\mathfrak{g}^*)\to\Gamma(TX\otimes\mathfrak{g}^*)$ be the divergence of $\tilde{\nabla}$, i.e., minus its adjoint. A section $(A,-\tilde{F})\in\Gamma\left(T^*X\otimes\mathfrak{g}\right)\oplus\left(\bigwedge^2T^*X\otimes\mathfrak{g}\right))$ is critical for $\hat{l}$ if and only if*

$$\mathrm{ad}^*_{\tilde{F}}\left(\frac{\delta l}{\delta \tilde{F}}\right) = 0, \tag{5.25}$$

$$\left(\tilde{\mathrm{div}} - \mathrm{ad}^*_A\right)\left(\frac{\delta l}{\delta \tilde{F}}\right) = 0. \tag{5.26}$$

Proof. From (iv) of Theorem 4.20, it is clear that the horizontal equation vanishes since

$$\left(J^1 P\right)\big/ J^1\tilde{G} \simeq X.$$

Similarly, the vertical equation reads

$$\left(\mathrm{div}^{J^1\tilde{\mathfrak{g}}} - \mathrm{ad}^*_{\tilde{s}} + \varphi^\dagger_*\right)\left(\frac{\delta \hat{l}}{\delta A}, \frac{\delta \hat{l}}{\delta \tilde{F}}\right) = 0,$$

where $\mathrm{div}^{J^1\tilde{\mathfrak{g}}}$ is the divergence of $\nabla^{J^1\tilde{\mathfrak{g}}}$. By using Lemmas 5.16, 5.17 and 5.18, we get

$$d^*\frac{\delta \hat{l}}{\delta A} - \mathrm{ad}^*_A \circ \frac{\delta \hat{l}}{\delta A} + \mathrm{ad}^*_{\tilde{F}} \circ \frac{\delta \hat{l}}{\delta \tilde{F}^A} = 0,$$

$$\tilde{\mathrm{div}}\frac{\delta \hat{l}}{\delta \tilde{F}} - \mathrm{ad}^*_A \circ \frac{\delta \hat{l}}{\delta \tilde{F}} + \frac{\delta \hat{l}}{\delta A} = 0.$$

We conclude by using Corollary 5.15, which yields

$$\frac{\delta \hat{l}}{\delta A} = \frac{\delta l}{\delta A} = 0, \qquad \frac{\delta \hat{l}}{\delta \tilde{F}} = -\frac{\delta l}{\delta \tilde{F}}.$$

In fact, the next result ensures that the first equation is nothing but the already known Ad-invariance of l.

Proposition 5.20. *Equation* (5.25) *is equivalent to the* Ad*-invariance of the reduced Lagrangian* $l : \bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}} \to \mathbb{R}$*, i.e.,* $l \circ \mathrm{Ad}_g = l$ *for each* $g \in G$*.*

Proof. Given $\tilde{F} \in \bigwedge^2 T^*X \otimes \mathfrak{g}$, the Ad-equivariance of l reads

$$\left.\frac{d}{dt}\right|_{t=0} l\left(\mathrm{Ad}_{\exp(t\xi)} \circ \tilde{F}\right) = \left.\frac{d}{dt}\right|_{t=0} l\left(\tilde{F}\right) = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Additionally, we have

$$\left.\frac{d}{dt}\right|_{t=0} l\left(\mathrm{Ad}_{\exp(t\xi)} \circ \tilde{F}\right) = \left\langle \frac{\delta l}{\delta \tilde{F}}, \mathrm{ad}_\xi \circ \tilde{F}\right\rangle = \left\langle \frac{\delta l}{\delta \tilde{F}}, -\mathrm{ad}_{\tilde{F}}(\xi)\right\rangle = -\left\langle \mathrm{ad}^*_{\tilde{F}}\left(\frac{\delta l}{\delta \tilde{F}}\right), \xi\right\rangle.$$

Since the previous expression is valid for every $\xi \in \mathfrak{g}$, we conclude.

We have thus the single equation (5.26). Note that, even though the reduced Lagrangian l does not depend on A, this variable is still a variational field of the reduced problem and it appears in the equations.

In particular, for the Yang–Mills reduced Lagrangian, $l(\tilde{F}) = \langle \tilde{F}, \tilde{F}\rangle_g$, we recover half of the Yang–Mills equations. In order to prove that, recall that the Hodge star operator (5.5) is may be extended to vector bundle valued forms, $\star : \Omega^k(X, \tilde{\mathfrak{g}}) \to \Omega^{n-k}(X, \tilde{\mathfrak{g}})$. Additionally, a principal connection $A \in \Gamma(\pi_{C(P),X})$ yields a covariant exterior derivative $\mathrm{d}^A : \Omega^k(X, \tilde{\mathfrak{g}}) \to \Omega^{k+1}(X, \tilde{\mathfrak{g}})$. Hence, we can define the covariant codifferential as

$$\left(\mathrm{d}^A\right)^\star = \star \circ \mathrm{d}^A \circ \star : \Omega^k(X, \tilde{\mathfrak{g}}) \to \Omega^{k-1}(X, \tilde{\mathfrak{g}})$$

Theorem 5.21. *Equation* (5.26) *for the Yang–Mills Lagrangian* (5.7) *is equivalent to*

$$\left(\mathrm{d}^A\right)^\star \tilde{F} = 0. \tag{5.27}$$

Proof. Let (x^μ) be local coordinates for X, $\{B_\alpha \mid 1 \le \alpha \le m\}$ be a basis of $\mathfrak{g}$ and $\{B^\alpha \mid 1 \le \alpha \le m\}$ be its dual basis. The Riemannian metric and the Killing form are locally given by

$$g = g_{\mu\sigma}\, dx^\mu \otimes dx^\sigma, \qquad K = K_{\alpha\beta} B^\alpha \otimes B^\beta,$$

for some local functions $g_{\mu\sigma} \in C^\infty(X)$ such that $g_{\mu\sigma} = g_{\sigma\mu}$, $1 \le \mu, \sigma \le n$, and $K_{\alpha\beta} \in \mathbb{R}$ such that $K_{\alpha\beta} = K_{\beta\alpha}$, $1 \le \alpha, \beta \le m$. The induced metric on 1-forms reads $g = g^{\mu\sigma}\, \partial_\mu \otimes \partial_\sigma$, where $(g^{\mu\sigma}) = (g_{\mu\sigma})^{-1}$ as matrices. Moreover, let $\eta = \eta^\alpha_\mu\, dx^\mu \otimes B_\alpha \in \Gamma(T^*X \otimes \mathfrak{g})$. The linear connection $\tilde{\nabla} : \Gamma(T^*X \otimes \mathfrak{g}) \to \Gamma(T^*X \otimes T^*X \otimes \mathfrak{g})$ locally reads

$$\tilde{\nabla}\eta = \left(\frac{\partial \eta^\alpha_\mu}{\partial x^\sigma} + \omega^\rho_{\mu\sigma} \eta^\alpha_\rho \right) dx^\mu \otimes dx^\sigma \otimes B_\alpha,$$

for certain functions $\omega^\rho_{\mu\sigma} \in C^\infty(X)$, $1 \le \mu, \sigma, \rho \le n$, which can be assumed to satisfy $\omega^\rho_{\mu\sigma} = \omega^\rho_{\sigma\mu}$ for each $1 \le \mu, \sigma, \rho \le n$, that is, $\tilde{\nabla}$ is assumed to be torsionless. In the same fashion, given $A = A^\alpha_\mu\, dx^\mu \otimes B_\alpha \in \Gamma(T^*X \otimes \mathfrak{g})$, the adjoint representation $\mathrm{ad}_A : \Gamma(T^*X \otimes \mathfrak{g}) \to \Gamma(T^*X \otimes T^*X \otimes \mathfrak{g})$ reads

$$\mathrm{ad}_A(\eta) = c^\alpha_{\beta\gamma}\, A^\beta_\mu\, \eta^\gamma_\sigma\, dx^\mu \otimes dx^\sigma \otimes B_\alpha,$$

where $c^\alpha_{\beta\gamma}$, $1 \le \alpha, \beta, \gamma \le m$, are the structure constants of the basis $\{B_\alpha \mid 1 \le \alpha \le m\}$. In addition, recall that the covariant exterior derivative for 1-forms is locally given by

$$\mathrm{d}^A\eta = \mathrm{d}\eta + [A, \eta] = \left(\frac{\partial \eta^\alpha_\mu}{\partial x^\sigma} + c^\alpha_{\beta\gamma}\, A^\beta_\mu\, \eta^\gamma_\sigma \right) dx^\mu \otimes dx^\sigma \otimes B_\alpha.$$

Let $\xi = \xi^{\mu\sigma}_\alpha\, \partial_\mu \otimes \partial_\sigma \otimes B^\alpha \in \Gamma\left(\bigwedge^2 TX \otimes \mathfrak{g}^*\right)$, that is, $\xi^{\mu\sigma}_\alpha + \xi^{\sigma\mu}_\alpha = 0$, $1 \le \mu, \sigma \le n$, $1 \le \alpha \le m$, and denote $\xi_\flat = g_{\mu\rho}\, g_{\sigma\tau}\, K^{\alpha\beta}\, \xi^{\mu\sigma}_\alpha\, dx^\rho \otimes dx^\sigma \otimes B_\beta \in \Gamma\left(\bigwedge^2 T^*X \otimes \mathfrak{g}\right)$, where $(K^{\alpha\beta}) = (K_{\alpha\beta})^{-1}$ as matrices. On the one hand, by using the previous expressions, we get

$$\begin{aligned}
\int_X \left\langle \eta, \left(\tilde{\mathrm{div}} - \mathrm{ad}^*_A\right)(\xi) \right\rangle v_g &= \int_X \left\langle \left(\tilde{\nabla} + \mathrm{ad}_A\right)(\eta), \xi \right\rangle v_g \\
&= \int_X \left\langle \left(\frac{\partial \eta^\alpha_\mu}{\partial x^\sigma} + \omega^\rho_{\mu\sigma} \eta^\alpha_\rho + c^\alpha_{\beta\gamma}\, A^\beta_\mu\, \eta^\gamma_\sigma \right) dx^\mu \otimes dx^\sigma \otimes B_\alpha, \xi^{\mu\sigma}_\alpha\, \partial_\mu \otimes \partial_\sigma \otimes B^\alpha \right\rangle v_g \\
&\overset{(\dagger)}{=} \int_X \left\langle \mathrm{d}^A\eta, \xi \right\rangle v_g \\
&\overset{(\star)}{=} \int_X \left\langle \eta, \left(\mathrm{d}^A\right)^\star \xi_\flat \right\rangle v_g.
\end{aligned}$$

For equality $(\dagger)$ we have used that $\omega^\rho_{\mu\sigma}\, \eta^\alpha_\rho\, \xi^{\mu\sigma}_\alpha = 0$ due to the symmetry properties, and for equality $(\star)$ we have used that the codifferential is the adjoint of the exterior derivative. Since this is valid for every $\eta \in \Gamma(T^*X \otimes \mathfrak{g})$, we conclude that $\left(\tilde{\mathrm{div}} - \mathrm{ad}^*_A\right)(\xi) = \left(\mathrm{d}^A\right)^\star \xi_\flat$.

Observe that the Yang–Mills Lagrangian (5.7) is locally given by $l(\tilde{F}) = K_{\alpha\beta}\, g^{\mu\rho}\, g^{\sigma\tau}\, F^\alpha_{\mu\sigma}\, F^\beta_{\rho\tau}$ for each $\tilde{F} = F^\alpha_{\mu\sigma}\, dx^\mu \otimes dx^\sigma \otimes B_\alpha \in \Gamma\left(\bigwedge^2 T^*X \otimes \mathfrak{g}\right)$, whence we conclude:

$$\frac{\delta l}{\delta \tilde{F}} = K_{\alpha\beta}\, g^{\mu\rho}\, g^{\sigma\tau}\, F^\alpha_{\mu\sigma}\, \partial_\rho \otimes \partial_\tau \otimes B^\beta, \qquad \left(\frac{\delta l}{\delta \tilde{F}}\right)_\flat = F^\beta_{\rho\tau}\, dx^\rho \otimes dx^\sigma \otimes B_\beta = \tilde{F}.$$

Equation (5.27) may be rewritten as $\star\left(\mathrm{d} \star \tilde{F} + [A, \star\tilde{F}]\right) = 0$ and, together with the reconstruction equation, $\mathrm{d}^A\tilde{F} = \mathrm{d}\tilde{F} + [A, \tilde{F}] = 0$, are the Yang–Mills equations. In conclusion, the reduced equations consists of the Ad-invariance condition for the reduced Lagrangian $l : \bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}} \to \mathbb{R}$, as well as the dynamical equations. When the original Lagrangian is the Yang–Mills Lagrangian, then the dynamical equations, together with the reconstruction condition, are the Yang–Mills equations.

Chapter 6

Higher-order Euler–Poincaré reduction

This chapter is based on our work [19], where the Euler–Poincaré reduction theory for covariant field theory investigated in [12] is extended to higher-order theories, i.e., those theories whose Lagrangian densities not only depend on the values of the field and its velocities, but also on higher-order derivatives.

Firstly, some essential facts about higher-order jet bundles are recalled in Section 6.1, including the total partial derivatives and the jet prolongation of vector fields. The formulation of higher-order Lagrangian field theories is recalled in 6.2. Next, in Section 6.3 the geometry of the reduced configuration space $(J^kP)/G$ for a k-th order Lagrangian theory on a principal bundle is investigated. The understanding of this reduced phase bundle is essential for a correct analysis of the reduced variational principle. For first order theories, this reduced bundle is the bundle of connections $C(P)$, the sections of which are principal connections on P. However, even though one might suspect that $(J^kP)/G \simeq J^{k-1}C(P)$, $k \geq 2$, it turns out that this reduced phase bundle is the space of $(k-1)$-th order pointwise principal connections with vanishing curvature up to $k-2$ order. This fact is already shown for particular cases in [53] and is analyzed in depth in [34]. We follow the strategy of the latter. The main result of the paper is established in Section 6.4, where we compute the reduced equations, i.e., the higher-order Euler–Poincaré field equations. For $k \geq 2$ the reconstruction process providing solutions of the original problem from solution of the reduced equations is given in detail (at least locally). In this case, the situation $k = 1$ and $k \geq 2$ show a different conceptual behaviour, since the first order case requires an auxiliary compatibility condition, whereas that condition is implicit in the set of admissible solution for higher-order Lagrangians. Furthermore, the reduced equations are reinterpreted as a Noether conservation law in Section 6.5. At last, Section 6.6 is devoted to illustrate the previous theory by computing the reduced equations for multivariate k-splines (brane splines) on Lie groups.

6.1 Higher-order jet bundles

We summarize the main results about higher-order jet bundles that we will need in the following (see, for example, [87, Chapter 6]). Let $\pi_{Y,X} : Y \to X$ be a fiber bundle and $k \in \mathbb{Z}^+$. The k-th order jet bundle of $\pi_{Y,X}$ is denoted by

$$\pi_{J^kY,X} : J^kY \longrightarrow X,$$

and its elements by $j^k_x s$. The k-th jet lift of a section $s \in \Gamma(\pi_{Y,X})$ is denoted by $j^k s \in \Gamma\left(\pi_{J^kY,X}\right)$. Recall that the maps $\pi_{k,l} : J^kY \to J^lY$, $j^k_x s \mapsto j^l_x s$, $0 \leq l < k$, are fiber bundles, where we denote $J^0Y = Y$. In addition, $\pi_{k,k-1}$ is an affine bundle modelled on

$$\pi^*_{J^{k-1}Y,X}\left(\bigvee^k T^*X\right) \otimes \pi^*_{k-1,0}(VY),$$

where $VY = \ker(\pi_{Y,X})_*$ is the vertical bundle of $\pi_{Y,X}$.

Let (x^μ, y^α) be bundle coordinates for $\pi_{Y,X}$. The induced coordinates on J^kY are $(x^\mu, y^\alpha; y^\alpha_J)$, $1 \le |J| \le k$, where $|J| = J_1 + \cdots + J_n$ is the length of J. If, locally, $s(x^\mu) = (x^\mu, s^\alpha(x^\mu))$ for some (local) functions $s^\alpha \in C^\infty(X)$, $1 \le \alpha \le m$, then $j^k s(x^\mu) = (x^\mu, s^\alpha(x^\mu); s^\alpha_J(x^\mu))$ with

$$s^\alpha_J(x^\mu) = \frac{\partial^{|J|}}{\partial x^J}\bigg|_{x=(x^\mu)} s^\alpha(x) = \left(\frac{\partial}{\partial x^1}\right)^{J_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{J_n}\bigg|_{x=(x^\mu)} s^\alpha(x), \qquad 1 \le |J| \le k.$$

Definition 6.1 *Let $\pi_{Y,X}$ and $\pi_{Y',X'}$ be fiber bundles, $k \in \mathbb{Z}^+$, and $F : Y \to Y'$ be a bundle morphism covering a diffeomorphism $f : X \to X'$. The* k-th order jet lift of F *is the map $j^kF : J^kY \to J^kY'$ defined as $j^k_x s \mapsto j^k_{f(x)}\left(F \circ s \circ f^{-1}\right)$.*

If $\pi_{Y'',X''} : Y'' \to X''$ is another fiber bundle and $G : Y' \to Y''$ is bundle morphism covering a diffeomorphism $g : X' \to X''$, then $j^k(G \circ F) = j^kG \circ j^kF$. In other words, the following diagram is commutative.

$$\begin{array}{ccccc}
 & & j^k(F\circ G) & & \\
 & \overset{j^kF}{\nearrow} & & \overset{j^kG}{\nearrow} & \\
J^kY & \longrightarrow & J^kY' & \longrightarrow & J^kY'' \\
\downarrow{\scriptstyle \pi_{k,0}} & & \downarrow{\scriptstyle \pi'_{k,0}} & & \downarrow{\scriptstyle \pi''_{k,0}} \\
Y & \xrightarrow{F} & Y' & \xrightarrow{G} & Y'' \\
\downarrow{\scriptstyle \pi_{Y,X}} & & \downarrow{\scriptstyle \pi_{Y',X'}} & & \downarrow{\scriptstyle \pi_{Y'',X''}} \\
X & \xrightarrow{f} & X' & \xrightarrow{g} & X''
\end{array} \tag{6.1}$$

Proposition 6.1. *The following is a canonical decomposition of vector bundles over J^kY,*

$$\pi^*_{k+1,k}\left(T(J^kY)\right) = \pi^*_{k+1,k}\left(V(J^kY)\right) \oplus H(\pi_{k+1,k}),$$

where $V(J^kY) = \ker(\pi_{J^kY,X})_*$ is the vertical bundle of $\pi_{J^kY,X}$ and $H(\pi_{k+1,k})_{j^{k+1}_x s} = d(j^k s)_x(T_xX)$ for each $j^{k+1}_x s \in J^{k+1}Y$.

Given $j^{k+1}_x s \in J^{k+1}Y$ and $U_x \in T_xX$, the vector $U^k_{j^{k+1}_x s} = d(j^k s)_x(U_x) \in H(\pi_{k+1,k})$ is called *k-th holonomic lift of U_x by $j^{k+1}_x s$*. In coordinates, if we write $j^{k+1}_x s = (x^\mu, y^\alpha; y^\alpha_J)$ and $U_x = U^\mu \partial_\mu$, then

$$U^k_{j^{k+1}_x s} = U^\mu\left(\partial_\mu + y^\alpha_{1_\mu}\partial_\alpha + \sum_{|J|=1}^{k} y^\alpha_{J+1_\mu}\partial^J_\alpha\right),$$

where 1_μ is the multi-index given by $(1_\mu)_\nu = \delta_{\mu\nu}$, and ∂_μ, ∂_α and ∂^J_α are the partial derivatives of the coordinates x^μ, y^α, y^α_J, respectively, $1 \le \mu, \nu \le n$, $1 \le \alpha \le m$, $1 \le |J| \le k$. In particular, the k-th holonomic lifts of the partial vector fields, ∂_μ, $1 \le \mu \le n$, are called *coordinate total derivatives* and are given by

$$\frac{d}{dx^\mu} = (\partial_\mu)^k_{j^{k+1}_x s} = \partial_\mu + y^\alpha_{1_\mu}\partial_\alpha + y^\alpha_{J+1_\mu}\partial^J_\alpha.$$

Example 6.1 (Total time derivative) *Let Q be a smooth manifold and consider the trivial bundle $\pi_{\mathbb{R}\times Q,\mathbb{R}}$ with (local) coordinates (t, q^α). Given $q(t) = (t, q^\alpha(t))$, the* total time derivative *is given by*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial q^\alpha}{\partial t}\partial_\alpha + \sum_{j=1}^{k} \frac{\partial^{j+1} q^\alpha}{\partial t^{j+1}}\partial^j_\alpha$$

Note that given a (local) function $f \in C^\infty(J^k Y)$ and a multi-index $J = (J^1, \ldots, J^n)$, then

$$\frac{d^{|J|} f}{dx^J} \in C^\infty(J^{k+|J|} Y).$$

Furthermore, for a section $s \in \Gamma(\pi_{Y,X})$ we have

$$\frac{d^{|J|} f}{dx^J} \circ j^{k+|J|} s = \frac{\partial^{|J|} \left(f \circ j^k s \right)}{\partial x^J}. \tag{6.2}$$

Next, we define the prolongation of vector fields.

Definition 6.2 *Let $l \in \mathbb{Z}^+$. A* generalized vector field *on $J^l Y$ is a section $U \in \Gamma\left(\pi_{l,0}^* TY \to J^l Y\right)$. Furthermore, it is said to be vertical if $U(j_x^l s) \in V_{s(x)} Y$ for each $j_x^l s \in J^l Y$.*

Note that (standard) vector fields on Y would be regarded as a generalized vector fields for $l = 0$. Locally, generalized vector fields are given by

$$U = U^\mu \partial_\mu + U^\alpha \partial_\alpha, \qquad U^\mu, U^\alpha : J^l Y \to \mathbb{R},\ 1 \leq \mu \leq n,\ 1 \leq \alpha \leq m.$$

This way, vertical generalized vector fields are those ones such that $U^\mu = 0$ for $1 \leq \mu \leq n$.

Given a generalized vector field, $U \in \Gamma\left(\pi_{l,0}^* TY \to J^l Y\right)$, and $k \in \mathbb{Z}^+$, it may be defined the *k-th order prolongation of U*, which is a section $U^{(k)} \in \Gamma\left(\pi_{k+l,k}^* \left(T\left(J^k Y\right)\right) \to J^{k+l} Y\right)$ (cf. [87, Definition 6.4.16]). In particular, k-the prolongations of (standard) vector fields on Y are (standard) vector fields on $J^k Y$.

Proposition 6.2. *Let $k \in \mathbb{Z}^+$, $(x^\mu, y^\alpha; y^\alpha_J)$ be bundle coordinates for $J^k Y$ and $V = V^\alpha \partial_\alpha \in \mathfrak{X}(Y)$ be a vertical vector field on Y, where $V^\alpha : Y \to \mathbb{R}$, $1 \leq \alpha \leq m$. Then k-th order prolongation of V is given by*

$$V^{(k)} = V^\alpha \partial_\alpha + \frac{d^{|J|} V^\alpha}{dx^J} \partial^J_\alpha \in \mathfrak{X}\left(J^k Y\right).$$

Proposition 6.3. *Let $k \in \mathbb{Z}^+$ and $U \in \mathfrak{X}(Y)$ be a $\pi_{Y,X}$-projectable vector field on Y. Then the flow of its k-th order prolongation, $U^{(k)} \in \mathfrak{X}\left(J^k Y\right)$, is $\left\{j^k \phi_t \mid t \in (-\epsilon, \epsilon)\right\}$, where $\{\phi_t \mid t \in (-\epsilon, \epsilon)\}$ is the flow of U.*

To conclude this brief overview, we present the Leibniz rule for higher-order, multivariable calculus, which can be straightforwardly proven by induction in the number of factors (see [50, Proposition 6] for two factors).

Lemma 6.4 (Higher-order Leibniz rule). *Let $m \in \mathbb{Z}^+$ and $f_1, \ldots, f_m \in C^\infty(\mathbb{R}^n)$. Then*

$$\frac{\partial^{|I|}}{\partial x^I} \prod_{\alpha=1}^m f_\alpha = \sum_{I^{(1)} + \cdots + I^{(m)} = I} \frac{I!}{I^{(1)}! \ldots I^{(m)}!} \prod_{\alpha=1}^m \frac{\partial^{|I^{(\alpha)}|} f_\alpha}{\partial x^{I^{(\alpha)}}}, \qquad |I| \geq 0.$$

6.2 Calculus of variations for higher-order Lagrangian densities

We now recall higher-order calculus of variations (for a comprehensive exposition see, for example, [7, 35, 77]). Let X be a compact, $\pi_{Y,X} : Y \to X$ be a fiber bundle, and $k \in \mathbb{Z}^+$. A *k-th order Lagrangian density* on $\pi_{Y,X}$ is a bundle morphism

$$\mathfrak{L} : J^k Y \longrightarrow \bigwedge^n T^* X$$

covering the identity on X. If X is oriented by a volume form $v \in \Omega^n(X)$, we will write $\mathfrak{L} = Lv$ for a function $L \in C^\infty\left(J^k Y\right)$ known as *Lagrangian*. The *action functional* defined by $\mathfrak{L}$ is

$$\mathbb{S} : \Gamma(\pi_{Y,X}) \to \mathbb{R}, \qquad s \mapsto \int_X \mathfrak{L}\left(j^k s\right).$$

A *variation* of $s \in \Gamma\left(\pi_{Y,X}\right)$ is a 1-parameter family of sections $\{s_t \in \Gamma\left(\pi_{Y,X}\right) \mid t \in (-\epsilon, \epsilon),\ s_0 = s\}$. The corresponding *infinitesimal variation* is

$$\delta s = \left.\frac{d}{dt}\right|_{t=0} s_t \in \Gamma\left(\pi_{s^* TY,X}\right).$$

Henceforth, only $\pi_{Y,X}$-vertical variations are considered, that is, those ones satisfying $\delta s(x) \in V_{s(x)} Y$ for each $x \in X$. The variation of the action functional induced by $\{s_t\}$ is defined as

$$\delta \mathbb{S}[s] = \left.\frac{d\mathbb{S}[s_t]}{dt}\right|_{t=0},$$

and it only depends on the infinitesimal variation. That is to say, if $\{s_t\}$ and $\{s'_t\}$ are two variations of s such that $\delta s = \delta s'$, then $d\mathbb{S}[s_t]/dt|_{t=0} = d\mathbb{S}[s'_t]/dt|_{t=0}$.

Definition 6.3 *A section $s \in \Gamma\left(\pi_{Y,X}\right)$ is* critical *or* stationary for $\mathbb{S}$ *if the variation of the corresponding action functional vanishes for every vertical variation of s, i.e.,*

$$\delta \mathbb{S}[s] = 0, \qquad \delta s \in \Gamma\left(\pi_{s^* VY,X}\right).$$

Unlike the first order case, the *covariant Cartan form*,

$$\Theta_{\mathfrak{L}} \in \Omega^n\left(J^{2k-1} Y\right),$$

is not uniquely defined for higher-order Lagrangians if $\dim X = n > 1$. The main reason is that, although there is a canonical embedding $J^k Y \hookrightarrow J^1\left(J^{k-1} Y\right)$, there are many different choices for the corresponding projection. Nevertheless, it is always possible to construct a globally defined projection by means of tubular neighborhoods, thus yielding a globally defined covariant Cartan form on $J^k Y$ (cf. [87, Theorem 6.5.13]). In any case, there is a unique choice for the *Euler–Lagrange form* associated to $\mathfrak{L}$,

$$\mathcal{EL}(\mathfrak{L}) \in \Omega^n\left(J^{2k} Y, \pi^*_{2k,0}\left(V^* Y\right)\right),$$

where $\pi_{V^* Y,Y}$ is the dual of the vertical bundle $\pi_{VY,Y}$. There exist adapted coordinates $(x^\mu, y^\alpha; y^\alpha_J)$ for $J^{2k} Y$ such that the covariant Cartan form is locally given by[1]

$$\Theta_{\mathfrak{L}} = \sum_{|I|=0}^{k-1} \sum_{|J|=0}^{k-|I|-1} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\right) \left(dy^\alpha_I - y^\alpha_{I+1_\nu} dx^\nu\right) \wedge \left(\iota_{\partial_\mu} v\right) + Lv, \tag{6.3}$$

where $\iota : \mathfrak{X}(X) \times \Omega^{k+1}(X) \to \Omega^k(X)$ denotes the left interior product. Similarly, the Euler–Lagrange form is given by

$$\mathcal{EL}(\mathfrak{L}) = \sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \left(\frac{\partial L}{\partial y^\alpha_J}\right) v \otimes dy^\alpha. \tag{6.4}$$

[1] Since the higher-order covariant Cartan form is not uniquely determined, its local expression depends on the choice of bundle coordinates (cf. [48, §3B], [87, §5.5] for first order, and [28, §5], [87, §6.5] for higher orders).

Proposition 6.5. *Let $\delta s \in \Gamma(\pi_{s^*VY,X})$ be a variation of a section $s \in \Gamma(\pi_{Y,X})$. Then the variation of the action functional is given by*

$$\delta\mathbb{S}[s] = \int_X \left\langle \left(j^{2k}s\right)^* \mathcal{EL}(\mathfrak{L}), \delta s \right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing within the vector bundle $VY \to Y$.

Proof. Since the variation is vertical, we have $s_t = \phi_t \circ s$, $t \in (-\epsilon, \epsilon)$, where $\{\phi_t : Y \to Y \mid t \in (-\epsilon,\epsilon)\}$ is the flow of a vertical, $\pi_{Y,X}$-projectable vector field $V \in \mathfrak{X}(Y)$. Subsequently,

$$\left.\frac{d}{dt}\right|_{t=0} \mathfrak{L}\left(j^k s_t\right) = \left.\frac{d}{dt}\right|_{t=0} \left(j^k s\right)^* \left(\mathfrak{L}(j^k\phi_t)\right) = \left(j^k s\right)^* \left(\left.\frac{d}{dt}\right|_{t=0} \mathfrak{L}\left(j^k\phi_t\right)\right) = \left(j^k s\right)^* \left(\pounds_{V^{(k)}}\mathfrak{L}\right), \tag{6.5}$$

where £ denotes the Lie derivative and we have used the commutativity of (6.1) and Proposition 6.3. Since X is boundaryless, the Stokes theorem leads to

$$\int_X \left(j^k s\right)^* \mathrm{d}\left(\iota_{V^{(k)}}\mathfrak{L}\right) = \int_X \mathrm{d}\left(\left(j^k s\right)^* \left(\iota_{V^{(k)}}\mathfrak{L}\right)\right) = \int_{\partial X} \left(j^k s\right)^* \left(\iota_{V^{(k)}}\mathfrak{L}\right) = 0. \tag{6.6}$$

Similarly, we pick bundle coordinates $(x^\mu, y^\alpha; y^\alpha_J)$ for J^kY, which allows us to write $V = V^\alpha\partial_\alpha$ for certain (local) functions $V^\alpha : X \to \mathbb{R}$, $1 \le \alpha \le m$. Thus, Proposition 6.2 yields

$$\iota_{V^{(k)}}\mathrm{d}\mathfrak{L} = \left(\frac{\partial L}{\partial y^\alpha}V^\alpha + \frac{\partial L}{\partial y^\alpha_J}\frac{d^{|J|}V^\alpha}{dx^J}\right)v. \tag{6.7}$$

Lastly, the higher-order integration by parts formula for boundaryless manifolds (see, for example, [9, Lemma 4.5]), together with (6.2), give

$$\begin{aligned}\int_X \left(j^k s\right)^* \left(\frac{\partial L}{\partial y^\alpha_J}\frac{d^{|J|}V^\alpha}{dx^J}\,v\right) &= \int_X \frac{\partial}{\partial y^\alpha_J}\left(L\circ j^k s\right)\frac{\partial^{|J|}V^\alpha}{\partial x^J}\,v \\ &= (-1)^{|J|}\int_X \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial}{\partial y^\alpha_J}\left(L\circ j^k s\right)\right)V^\alpha\,v \\ &= (-1)^{|J|}\int_X \left(j^{2k}s\right)^*\left(\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_J}\right)V^\alpha\,v\right).\end{aligned} \tag{6.8}$$

On the other hand, it is clear that $\delta s = (d/dt)|_{t=0}(\phi_t \circ s) = V \circ s = V^\alpha\partial_\alpha \in \Gamma(\pi_{s^*VY,X})$. Hence, form the local expression of the Euler–Lagrange form (6.4) we get

$$\left\langle \left(j^{2k}s\right)^* \mathcal{EL}(\mathfrak{L}), \delta s\right\rangle = \left(j^{2k}s\right)^*\left(\sum_{|J|=0}^{k}(-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_J}\right)\right)V^\alpha\,v. \tag{6.9}$$

By using Cartan's formula, i.e., $\pounds = d\circ\iota + \iota\circ d$, and by gathering the previous expressions, we finish

$$\begin{aligned}\delta\mathbb{S}[s] &= \left.\frac{d}{dt}\right|_{t=0}\int_X \mathfrak{L}\left(j^k s_t\right) \overset{(6.5)}{=} \int_X \left(j^k s\right)^*\left(\pounds_{V^{(k)}}\mathfrak{L}\right) \\ &= \int_X \left(j^k s\right)^*\left(\iota_{V^{(k)}}\mathrm{d}\mathfrak{L}\right) + \int_X \left(j^k s\right)^*\mathrm{d}\left(\iota_{V^{(k)}}\mathfrak{L}\right) \overset{(6.6),(6.7)}{=} \int_X \left(j^k s\right)^*\left(\frac{\partial L}{\partial y^\alpha}V^\alpha + \frac{\partial L}{\partial y^\alpha_J}\frac{d^{|J|}V^\alpha}{dx^J}\right)v \\ &\overset{(6.8)}{=} \int_X \left(j^{2k}s\right)^*\left(\frac{\partial L}{\partial y^\alpha} + (-1)^{|J|}\frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_J}\right)\right)V^\alpha\,v \overset{(6.9)}{=} \int_X \left\langle \left(j^{2k}s\right)^*\mathcal{EL}(\mathfrak{L}), \delta s\right\rangle.\end{aligned}$$

The following result is a straightforward consequence of the previous Proposition and it gives the higher-order version of the well-known Euler–Lagrange equations.

Theorem 6.6. *Let $\mathfrak{L} = Lv : J^kY \to \bigwedge^n T^*X$ be a k-th order Lagrangian density, and consider the corresponding Euler–Lagrange form, $\mathcal{EL}(\mathfrak{L})$. Then the following statements for a section $s \in \Gamma(\pi_{Y,X})$ are equivalent:*

(i) The variational principle $\delta\mathbb{S}[s] = 0$ holds for arbitrary variations $\delta s \in \Gamma(\pi_{s^*VY,X})$.

(ii) s satisfies the k-th order Euler–Lagrange field equations, i.e., $(j^{2k}s)^*\mathcal{EL}(\mathfrak{L}) = 0$.

From (6.2) and (6.4), we obtain the local expression of the higher-order Euler–Lagrange equations:

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\partial^{|J|}}{\partial x^J} \left(\frac{\partial L}{\partial y^\alpha_J} \left(j^k s \right) \right) = 0, \qquad 1 \le \alpha \le m.$$

Remark 6.1 *The calculus of variations described above is straightforwardly extended to a non-compact base manifold X by considering compactly supported variations. In other words, given a section $s \in \Gamma(\pi_{Y,X})$, the only variations δs allowed are those satisfying*

$$j^{k-1}_x \, \delta s = 0, \qquad x \in \partial\mathcal{U},$$

for some open subset $\mathcal{U} \subset X$ with compact closure, $\overline{\mathcal{U}}$. Locally, this condition ensures that δs as well as its partial derivatives up to order $k-1$ vanish on the boundary, $\partial\mathcal{U}$. As a result, no boundary terms appear when integrating by parts and, thus, Proposition 6.5 is still valid.

6.3 Geometry of the reduced configuration space

Let G be a Lie group, $\mathfrak{g} = T_eG$ be its Lie algebra, $e \in G$ being the identity element, and $\mathfrak{g}^*$ be the dual of the Lie algebra. The corresponding exponential map is denoted by $\exp : \mathfrak{g} \to G$, and the adjoint representation of G is denoted by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ for each $g \in G$. In the same vein, the adjoint and coadjoint representations of $\mathfrak{g}$ are denoted by $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$, respectively, for each $\xi \in \mathfrak{g}$.

Let $\pi_{P,X} : P \to X$ be a principal G-bundle. The corresponding right action is denoted by

$$\Psi : P \times G \to P, \qquad (y, g) \mapsto \Psi_y(g) = \Psi_g(y) = y \cdot g.$$

Recall that the infinitesimal generator of $\xi \in \mathfrak{g}$ is the vertical vector field $\xi^* \in \mathfrak{X}(P)$ given by

$$\xi^*_y = \left. \frac{d}{dt} \right|_{t=0} y \cdot \exp(t\xi) = (d\Psi_y)_e (\xi), \qquad y \in P.$$

For each $k \in \mathbb{Z}^+$, the k-th jet extension of the action,

$$\Psi^{(k)} : J^kP \times G \to J^kP, \qquad \left(j^k_x s, g \right) \mapsto j^k_x (s \cdot g),$$

is again free and proper, thus yielding a principal G-bundle, $J^kP \to J^kP/G$.

We denote by $C(P) = \left(J^1P\right)/G \to X$ the bundle of connections of $\pi_{P,X}$, which is an affine bundle modelled on $T^*X \otimes \tilde{\mathfrak{g}} \to X$. Recall that there is a bijective correspondence between (local) sections of $\pi_{C(P),X}$ and (local) principal connections on $\pi_{P,X}$, which we denote by

$$\Omega^1(P, \mathfrak{g}) \ni A \overset{1:1}{\longleftrightarrow} \sigma_A \in \Gamma(\pi_{C(P),X}).$$

The following result ensures that holonomic sections of the jet bundle yield flat principal connections.

Lemma 6.7. *Let $s \in \Gamma_{loc}(\pi_{P,X})$, where $\Gamma_{loc}(\pi_{P,X})$ denotes the family of local sections of $\pi_{P,X}$, and define $\sigma_A = \left[j^1 s\right]_G \in \Gamma_{loc}(\pi_{C(P),X})$. Then the local connection $A \in \Omega^1_{loc}(P, \mathfrak{g})$ is flat.*

Proof. The local section s induces a trivialization of $\pi_{P,X}$ on its domain $\mathcal{U} \subset X$,

$$\varphi : \mathcal{U} \times G \xrightarrow{\sim} P|_{\mathcal{U}}, \qquad (x, g) \mapsto \varphi(x, g) = s(x) \cdot g.$$

Recall that any connection is uniquely determined by its horizontal lift at each point. Given $y = s(x) \cdot g \in P|_{\mathcal{U}}$, we denote by $Hor^A_y : T_x X \to T_y P$ the horizontal lifting of A at y. Since $\sigma_A = \left[j^1 s\right]_G$, we have

$$Hor^A_y(U_x) = d(\Psi_g \circ s)_x(U_x) \in T_y P, \qquad U_x \in T_x X. \tag{6.10}$$

On the other hand, consider the canonical flat connection $A_0 \in \Omega^1(X \times G, \mathfrak{g})$ on the trivial bundle $\pi_{X\times G,X}$. For each $(x, g) \in X \times G$ it is defined as

$$Hor^{A_0}_{(x,g)}(U_x) = (U_x, 0_g) \in T_{(x,g)}(X \times G), \qquad U_x \in T_x X.$$

It turns out that $\varphi^* A = A_0|_{\mathcal{U}\times G}$, so A is a flat connection. Indeed, let $(x, g) \in \mathcal{U} \times G$ and $U_x \in T_x X$. Since φ covers the identity on $\mathcal{U}$, we have

$$\begin{aligned} Hor^{\varphi^* A}_{(x,g)}(U_x) &= \left(d\varphi^{-1}\right)_{s(x)\cdot g}\left(Hor^A_{\varphi(x,g)}(U_x)\right) = \left(d\varphi^{-1}\right)_{s(x)\cdot g}\left(d(\Psi_g \circ s)_x(U_x)\right) \\ &= d\left(\varphi^{-1} \circ \Psi_g \circ s\right)_x(U_x) = (d\iota_g)_x(U_x) = (U_x, 0_g) = Hor^{A_0}_{(x,g)}(U_x), \end{aligned}$$

where $X \ni x \mapsto \iota_g(x) = (x, g) \in X \times G$.

The main goal of this section is to study the geometry of the quotient $\left(J^k P\right)/G$, which was first investigated in [34]. For the convenience of the reader, we present a detailed proof of the main result (see Corollary 6.14 below). To begin with, we introduce the following bundle morphisms.

Proposition 6.8. *For each $k \geq 2$, the bundle morphism $\Theta_k : J^k P \to J^{k-1} C(P)$ defined as $j^k_x s \mapsto j^{k-1}_x [j^1 s]_G$ is well-defined and it descends to a map on the quotient, $\Xi_k : \left(J^k P\right)/G \to J^{k-1} C(P)$.*

Proof. To begin with, note that Θ_k is the composition of the natural immersion $J^k P \hookrightarrow J^{k-1}\left(J^1 P\right)$ with the $(k-1)$-th jet lift of the canonical projection $\pi_1 : J^1 P \to C(P)$. Thus, it is well-defined.

For the second part, let $g \in G$. Then

$$\Theta_k\left((j^k_x s) \cdot g\right) = \Theta_k\left(j^k_x (s \cdot g)\right) = j^{k-1}_x [j^1(s \cdot g)]_G = j^{k-1}_x [(j^1 s) \cdot g]_G = j^{k-1}_x [j^1 s]_G = \Theta_k\left(j^k_x s\right).$$

The previous result may be summarized in the following commutative diagram.

$$\begin{array}{ccc} J^k P & \xrightarrow{\Theta_k} & J^{k-1} C(P) \\ {\scriptstyle \pi_k}\downarrow & \nearrow_{\Xi_k} & \\ \left(J^k P\right)/G & & \end{array} \qquad \begin{array}{ccc} j^k_x s & \longmapsto & j^{k-1}_x [j^1 s]_G \\ \downarrow & \nearrow & \\ \left[j^k_x s\right]_G & & \end{array} \tag{6.11}$$

Consider the *curvature map*, i.e.,

$$\tilde{F} : J^1 C(P) \to \textstyle\bigwedge^2 T^* X \otimes \tilde{\mathfrak{g}}, \qquad j^1_x \sigma_A \mapsto \tilde{F}^A(x),$$

where $\tilde{F}^A \in \Omega^2(X, \tilde{\mathfrak{g}})$ is curvature of the (local) principal connection $A \in \Omega^1(P, \mathfrak{g})$ regarded as a 2-from on M with values in the adjoint bundle. By lifting it to the $(k-2)$-jet and by composing it with the natural immersion of $J^{k-1}C(P) \hookrightarrow J^{k-2}\big(J^1C(P)\big)$ we obtain

$$j^{k-2}\tilde{F} : J^{k-1}C(P) \to J^{k-2}\big(\textstyle\bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}}\big), \qquad j^{k-1}_x\sigma_A \mapsto j^{k-2}_x\tilde{F}^A.$$

Let $\hat{0} : X \to \bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}}$ be the zero section, and consider its $(k-2)$-jet lift, $j^{k-2}\hat{0} : X \to J^{k-2}\big(\bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}}\big)$. We define the kernel of $j^{k-2}\tilde{F}$ as

$$\ker j^{k-2}\tilde{F} = \big\{ j^{k-1}_x\sigma_A \in J^{k-1}C(P) \mid j^{k-2}_x\tilde{F}^A = j^{k-2}_x\hat{0} \big\}.$$

Proposition 6.9. *For each $k \geq 2$, we have*

(i) Ξ_k is an injective bundle morphism over X.

(ii) $\operatorname{im}\Xi_k \subset \ker j^{k-2}\tilde{F}$.

Proof. To check *(i)*, let $\big[j^k_x s\big]_G \neq \big[j^k_x s'\big]_G$ be two different elements in $\big(J^kP\big)/G$. In other words, $j^k_x s \neq (j^k_x s')\cdot g = j^k_x(s'\cdot g)$ for each $g \in G$. Subsequently, $[j^1_x s]_G \neq [j^1_x s']_G$ and, thus, $\Xi_k\big(\big[j^k_x s\big]_G\big) \neq \Xi_k\big(\big[j^k_x s'\big]_G\big)$. On the other hand, *(ii)* is a straightforward consequence of Lemma 6.7.

Let us introduce a trivializing chart of $\pi_{P,X}$, i.e., we pick an open subset $\mathcal{U} \subset X$ with $\mathcal{U} \simeq \mathbb{R}^n$ via the local coordinates $x = (x^\mu)$ and such that $P|_{\mathcal{U}} \simeq \mathcal{U} \times G$. To keep the notation simple, we write $\mathcal{U} = X$. Likewise, let $\{B_\alpha \in \mathfrak{g} \mid 1 \leq \alpha \leq m\}$ be a basis of $\mathfrak{g}$ and consider normal coordinates (y^α) in a neighborhood of the identity element $e \in G$, i.e., $g = \exp(y^\alpha B_\alpha)$. This way, we have normal bundle coordinates (x^μ, y^α) for $\pi_{P,X}$. Note that for each $g = \exp(y^\alpha B_\alpha), h = \exp(z^\alpha B_\alpha) \in G$ close enough to the identity, we have $gh = \exp\big(f^\alpha_{CD} y^C z^D B_\alpha\big)$, where $C = (C_1, \ldots, C_m)$ is a multi-index, $y^C = (y^1)^{C_1} \ldots (y^m)^{C_m}$ and f^α_{CD}, $1 \leq \alpha \leq m$, $|C|, |D| \geq 0$, are the constants of the Baker–Campbell–Hausdorff formula (cf. [92, §2.15] and [34, Theorem 4.1]). By using this, it can be checked that

$$P \times \mathfrak{g} \simeq VP, \qquad ((x^\mu, y^\alpha), \xi^\alpha B_\alpha) \mapsto a^\alpha_\beta \xi^\beta \partial_\alpha, \tag{6.12}$$

where

$$a^\alpha_\beta(y^1, \ldots, y^m) = f^\alpha_{C1_\beta} y^C.$$

Note that for $y = (x, e)$ we have $a^\alpha_\beta(0, \ldots, 0) = \delta^\alpha_\beta$, $1 \leq \alpha \leq m$. Hence, the inverse of the matrix $(a^\alpha_\beta(y^\alpha))$ is well defined in a neighbourhood of $y = (x, e)$, and we denote it by $(b^\alpha_\beta(y^1, \ldots, y^m))$. In such case, the inverse of the previous isomorphism is given by

$$VP \simeq P \times \mathfrak{g}, \qquad (y, U^\alpha \partial_\alpha) \mapsto (y, b^\alpha_\beta U^\beta B_\alpha). \tag{6.13}$$

We pick bundle coordinates (x^μ, A^α_μ) and $(x^\mu, F^\alpha_{\mu\nu})$ for $\pi_{C(P),X}$ and $\pi_{\bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}}, X}$, respectively. Similarly, we consider the corresponding bundle coordinates $(x^\mu, y^\alpha; y^\alpha_J)$, $(x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J})$ and $(x^\mu, F^\alpha_{\mu\nu}; F^\alpha_{\mu\nu,J})$ for J^kP, $J^{k-1}C(P)$ and $J^{k-2}\big(\bigwedge^2 T^*X \otimes \tilde{\mathfrak{g}}\big)$, respectively. In order to keep the notation simple, we will write $y^\alpha_{J_0} = y^\alpha$, $A^\alpha_{\mu,J_0} = A^\alpha_\mu$ and $F^\alpha_{\mu\nu,J_0} = F^\alpha_{\mu\nu}$ for $J_0 = (0, \ldots, 0)$.

Remark 6.2 *Given two multi-indices I, J, the inequality $I \leq J$ means that $I_\mu \leq J_\mu$ for $1 \leq \mu \leq n$. Likewise, we denote $I < J$ when $I \leq J$ and $I \neq J$, that is, $I_\mu \leq J_\mu$ for $1 \leq \mu \leq n$ and there exists $\mu_0 \in \{1, \ldots, n\}$ such that $I_{\mu_0} < J_{\mu_0}$.*

Let us compute the coordinated expression of Θ_k.

Proposition 6.10. *For each $k \geq 2$ and each $(x^\mu, y^\alpha; y^\alpha_J) \in J^k P$ with $|y^\alpha|$ small enough, $1 \leq \alpha \leq m$, we have $\Theta_k(x^\mu, y^\alpha; y^\alpha_J) = (x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J})$, where*[2]

$$A^\alpha_\mu = b^\alpha_\beta(y^\alpha) y^\beta_{1_\mu}, \quad A^\alpha_{\mu,J} = \frac{\partial^{|J|}}{\partial x^J}\left(b^\alpha_\beta(y^\alpha) y^\beta_{1_\mu}\right)\Big|_x, \quad 1 \leq \mu \leq n,\ 1 \leq \alpha \leq m,\ 1 \leq |J| \leq k-1.$$

Proof. We denote $j^1_x s = (x^\mu, y^\alpha; y^\alpha_J)$. Recall from (6.10) that the horizontal lift given by $[j^1 s]_G \in C(P)$ reads $Hor^A_{s(x)} = dx^\mu \otimes \partial_\mu + y^\alpha_{1_\mu} dx^\mu \otimes \partial_\alpha$. From this and (6.13), we get

$$A_{s(x)} = b^\alpha_\beta(y^\alpha)\left(-y^\beta_{1_\mu} dx^\mu + dy^\alpha\right) \otimes B_\alpha \in T^*_{s(x)} P \otimes \mathfrak{g}.$$

Then $[j^1 s]_G = (x^\mu, A^\alpha_\mu = b^\alpha_\beta(y^\alpha) y^\beta_{1_\mu})$. By taking partial derivatives, we conclude.

Now some technical lemmas are presented.

Lemma 6.11. *Let $k \geq 2$ and $j^{k-1}_x \sigma_A = (x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J}) \in J^{k-1} C(P)$. Then $j^{k-2}_x \tilde{F}^A = \left(x^\mu, F^\alpha_{\mu\nu}; F^\alpha_{\mu\nu,J}\right) \in J^{k-2}\left(\bigwedge^2 T^* X \otimes \tilde{\mathfrak{g}}\right)$ is given by*

$$F^\alpha_{\mu\nu,J} = \frac{1}{2}\left(A^\alpha_{\nu,J+1_\mu} - A^\alpha_{\mu,J+1_\nu} + \sum_{I \leq J} \binom{J}{I} c^\alpha_{\beta\gamma} A^\beta_{\mu,I} A^\gamma_{\nu,J-I}\right), \tag{6.14}$$

for each $1 \leq \mu, \nu \leq n$, $0 \leq |J| \leq k-2$ and $1 \leq \alpha \leq m$, where we denote by $c^\alpha_{\beta\gamma} = 2 f^\alpha_{1_\beta 1_\gamma}$, $1 \leq \alpha, \beta, \gamma \leq m$, the structure constants of $\mathfrak{g}$.

Proof. We show it by induction in $0 \leq |J| \leq k-2$. The case $|J| = 0$ is straightforward, since the curvature of $j^1_x \sigma_A = (x^\mu, A^\alpha_\mu; A^\alpha_{\mu,1_\nu})$ is given by (cf. [86, Chapter 1, §4])

$$F^\alpha_{\mu\nu} = \frac{1}{2}\left(A^\alpha_{\nu,1_\mu} - A^\alpha_{\mu,1_\nu} + c^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\right).$$

Now assume that (6.14) holds for $1 \leq |J| (< k-2)$. To conclude, we need to show that it holds for $J' = J + 1_\sigma$ for each $1 \leq \sigma \leq n$. Note that

$$\binom{J'}{I} = \binom{J + 1_\sigma}{I} = \binom{J}{I} + \binom{J}{I - 1_\sigma}, \qquad 1_\sigma \leq I \leq J.$$

Furthermore, note that

$$\sum_{\substack{I \leq J,\\ I_\sigma = 0}} \binom{J'}{I} = \sum_{\substack{I \leq J,\\ I_\sigma = 0}} \binom{J}{I}, \qquad \sum_{\substack{I \leq J',\\ I_\sigma = J_\sigma + 1}} \binom{J'}{I} = \sum_{\substack{I \leq J',\\ I_\sigma = J_\sigma + 1}} \binom{J}{I - 1_\sigma}.$$

Hence,

$$\sum_{I \leq J'} \binom{J'}{I} = \sum_{\substack{I \leq J,\\ I_\sigma = 0}} \binom{J}{I} + \sum_{1_\sigma \leq I \leq J} \left(\binom{J}{I} + \binom{J}{I - 1_\sigma}\right) + \sum_{\substack{I \leq J',\\ I_\sigma = J_\sigma + 1}} \binom{J}{I - 1_\sigma} = \sum_{I \leq J} \binom{J}{I} + \sum_{1_\sigma \leq I \leq J'} \binom{J}{I - 1_\sigma}.$$

By using this and by taking the partial derivative of (6.14) with respect to x^σ, we finish

[2] By abusing the notation, for each multi-index J we denote

$$\frac{\partial^J b^\alpha_\beta}{\partial x^J}(y)\Big|_x = \frac{\partial b^\alpha_\beta}{\partial y^\gamma}(y)\Big|_{y^\alpha = 0} y^\gamma_J, \qquad 1 \leq \alpha, \beta \leq m.$$

$$
\begin{aligned}
F^{\alpha}_{\mu\nu,J'} &= \frac{1}{2}\left(A^{\alpha}_{\nu,J'+1_\mu} - A^{\alpha}_{\mu,J'+1_\nu} + \sum_{I\leq J}\binom{J}{I}c^{\alpha}_{\beta\gamma}\left(A^{\beta}_{\mu,I+1_\sigma}A^{\gamma}_{\nu,J-I} + A^{\beta}_{\mu,I}A^{\gamma}_{\nu,J-I+1_\sigma}\right)\right)\\
&= \frac{1}{2}\left(A^{\alpha}_{\nu,J'+1_\mu} - A^{\alpha}_{\mu,J'+1_\nu} + \sum_{1_\sigma\leq I\leq J'}\binom{J}{I-1_\sigma}c^{\alpha}_{\beta\gamma}A^{\beta}_{\mu,I}A^{\gamma}_{\nu,J'-I} + \sum_{I\leq J}\binom{J}{I}c^{\alpha}_{\beta\gamma}A^{\beta}_{\mu,I}A^{\gamma}_{\nu,J'-I}\right)\\
&= \frac{1}{2}\left(A^{\alpha}_{\nu,J'+1_\mu} - A^{\alpha}_{\mu,J'+1_\nu} + \sum_{I\leq J'}\binom{J'}{I}c^{\alpha}_{\beta\gamma}A^{\beta}_{\mu,I}A^{\gamma}_{\nu,J'-I}\right).
\end{aligned}
$$

Lemma 6.12. *Let $k\geq 2$ and $g=\exp(y^\alpha B_\alpha)\in G$ with $|y^\alpha|$ small enough, $1\leq\alpha\leq m$. Then*

$$
c^{\lambda}_{\beta\gamma}b^{\beta}_{\alpha}(y)b^{\gamma}_{\epsilon}(y) = \frac{\partial b^{\lambda}_{\alpha}}{\partial y^{\epsilon}}(y) - \frac{\partial b^{\lambda}_{\epsilon}}{\partial y^{\alpha}}(y), \qquad 1\leq\alpha,\epsilon,\lambda\leq m.
$$

Proof. For each $1\leq\beta,\gamma\leq m$ and $y=(x^\mu,y^\alpha)\in P$, the infinitesimal generator is given in coordinates by $(B_\beta)^*_y = a^{\alpha}_{\beta}(y)\partial_\alpha$ and, from the equality $[B_\beta,B_\gamma]=c^{\alpha}_{\beta\gamma}B_\alpha$, they satisfy

$$
c^{\alpha}_{\beta\gamma}a^{\delta}_{\alpha}(t)\partial_\delta = \left[a^{\alpha}_{\beta}(y)\partial_\alpha, a^{\delta}_{\gamma}(y)\partial_\delta\right] = \left(a^{\alpha}_{\beta}(y)\frac{\partial a^{\delta}_{\gamma}}{\partial y^{\alpha}}(y) - a^{\alpha}_{\gamma}\frac{\partial a^{\delta}_{\beta}(y^\alpha)}{\partial y^{\alpha}}(y)\right)\partial_\delta. \tag{6.15}
$$

Since $\left(a^{\alpha}_{\beta}(y)\right)\left(b^{\alpha}_{\beta}(y)\right)=\mathbb{I}$, the identity matrix, we have $a^{\delta}_{\gamma}(y)b^{\beta}_{\delta}(y)=\delta^{\beta}_{\gamma}$ and, thus,

$$
\frac{\partial a^{\delta}_{\gamma}}{\partial y^{\alpha}}(y)b^{\beta}_{\delta}(y) + a^{\delta}_{\gamma}(y)\frac{\partial b^{\beta}_{\delta}(y^\alpha)}{\partial y^{\alpha}}(y) = 0, \qquad 1\leq\alpha,\beta,\gamma\leq m.
$$

As a result,

$$
\frac{\partial a^{\delta}_{\gamma}}{\partial y^{\alpha}}(y) = -a^{\delta}_{\beta}(y)a^{\epsilon}_{\gamma}(y)\frac{\partial b^{\beta}_{\epsilon}}{\partial y^{\alpha}}(y), \qquad 1\leq\alpha,\gamma,\delta\leq m.
$$

By substituting this into (6.15) and rearranging indices, we get

$$
c^{\kappa}_{\beta\gamma}a^{\delta}_{\kappa}(y) = a^{\alpha}_{\beta}(y)a^{\delta}_{\lambda}(y)a^{\epsilon}_{\gamma}(y)\left(-\frac{\partial b^{\lambda}_{\epsilon}}{\partial y^{\alpha}}(y) + \frac{\partial b^{\lambda}_{\alpha}}{\partial y^{\epsilon}}(y)\right), \qquad 1\leq\beta,\gamma,\delta\leq m.
$$

Lastly, we multiply each side by $b^{\beta'}_{\alpha}(y)b^{\lambda'}_{\delta}(y)b^{\gamma'}_{\epsilon}(y)$ and we sum over α, δ and ϵ:

$$
c^{\lambda}_{\beta\gamma}b^{\beta}_{\alpha}(y)b^{\gamma}_{\epsilon}(y) = \frac{\partial b^{\lambda}_{\alpha}}{\partial y^{\epsilon}}(y) - \frac{\partial b^{\lambda}_{\epsilon}}{\partial y^{\alpha}}(y), \qquad 1\leq\alpha,\epsilon,\lambda\leq m.
$$

We are ready to show the main result about the map Ξ_k.

Theorem 6.13. *For each $k\geq 2$, we have that:*

(i) $\operatorname{im}\Xi_k = \ker j^{k-2}\tilde{F}\subset J^{k-1}C(P)$ *and,*

(ii) Ξ_k *is a proper map.*

Proof. To begin with, we show *(i)*. Thanks to Proposition 6.9*(ii)*, we only need to prove that $\ker j^{k-2}\tilde{F}\subset\operatorname{im}\Xi_k$. Similarly, the commutativity of (6.11) yields $\operatorname{im}\Xi_k=\operatorname{im}\Theta_k$.

The fact that the maps are local enables us to work locally. More specifically, we employ the bundle coordinates introduced above. Let $j^{k-1}_x\sigma_A=(x^\mu,A^{\alpha}_{\mu};A^{\alpha}_{\mu,J})\in\ker j^{k-2}\tilde{F}$. We need to find $j^k_x s=(x^\mu,y^\alpha;y^\alpha_J)\in J^kP$, such that $\Theta_k(j^k_x s)=j^{k-1}_x\sigma_A$. For that matter, we set $y^\alpha=0$, $1\leq\alpha\leq m$. Besides, we define y^α_J, $1\leq|J|\leq k$, by recurrence. Namely, fix $0\leq r\leq k-1$, and assume that y^α_J has been defined for $1\leq\alpha\leq m$, $0\leq|J|\leq r-1$ in such a way that they satisfy

$$\frac{\partial^{|J|}}{\partial x^J}\left(b^\alpha_\beta(y)y^\beta_{1_\mu}\right)\Big|_x = A^\alpha_{\mu,J}, \qquad 0\le|J|\le r-2,\ 1\le\mu\le n,\ 1\le\alpha\le m. \tag{6.16}$$

Now pick $J\in\mathbb{N}^n$ with $|J|=r$ and consider $1\le\mu\le n$ such that $J_\mu>0$. Then we set

$$y^\alpha_J(\mu) = A^\alpha_{\mu,J-1_\mu} - \sum_{I<J-1_\mu}\binom{J-1_\mu}{I}\frac{\partial^{|J-1_\mu-I|}b^\alpha_\beta(y)}{\partial x^{J-1_\mu-I}}\Big|_x y^\beta_{I+1_\mu}, \qquad 1\le\alpha\le m.$$

Observe that it makes sense, since the elements y^α_J considered in the RHS are only up to order $r-1$. In addition, for each $1\le\alpha\le m$ we have

$$\frac{\partial^{|J-1_\mu|}}{\partial x^{J-1_\mu}}\left(b^\alpha_\beta(y)y^\beta_{1_\mu}\right)\Big|_x = y^\alpha_J(\mu) + \sum_{I<J-1_\mu}\binom{J-1_\mu}{I}\frac{\partial^{|J-1_\mu-I|}b^\alpha_\beta(y)}{\partial x^{J-1_\mu-I}}\Big|_x y^\beta_{I+1_\mu} = A^\alpha_{\mu,J-1_\mu},$$

since $b^\alpha_\beta(y^\alpha=0)=\delta^\alpha_\beta$. Hence, $y^\alpha_J(\mu)$ keeps satisfying (6.16).

Let us check that the definition do not depend on the μ chosen, so we can write $y^\alpha_J=y^\alpha_J(\mu)$. Let $1\le\nu\le n$ be another index such that $J_\nu>0$. This enables us to write $J=J'+1_\mu+1_\nu$ for some multi-index J' with $|J'|=r-2$. Hence, $J-1_\mu=J'+1_\nu$ and $J-1_\nu=J'+1_\mu$. Therefore,

$$\begin{aligned}
&\sum_{I<J'+1_\nu}\binom{J'+1_\nu}{I}\frac{\partial^{|J'+1_\nu-I|}b^\alpha_\beta}{\partial x^{J'+1_\nu-I}}(y)\Big|_x y^\beta_{I+1_\mu} - \sum_{I<J'+1_\mu}\binom{J'+1_\mu}{I}\frac{\partial^{|J'+1_\mu-I|}b^\alpha_\beta}{\partial x^{J'+1_\mu-I}}(y)\Big|_x y^\beta_{I+1_\nu} \\
&= \frac{\partial^{|J'+1_\nu|}}{\partial x^{J'+1_\nu}}\left(b^\alpha_\beta(y)y^\beta_{1_\mu}\right)\Big|_x - \frac{\partial^{|J'+1_\mu|}}{\partial x^{J'+1_\mu}}\left(b^\alpha_\beta(y)y^\beta_{1_\nu}\right)\Big|_x \\
&= \frac{\partial^{|J'|}}{\partial(y)x^{J'}}\left(\frac{\partial b^\alpha_\beta}{\partial y^\gamma}\Big|_x(y)\,y^\gamma_{1_\nu}y^\beta_{1_\mu} + b^\alpha_\beta(y)y^\beta_{1_\mu+1_\nu} - \frac{\partial b^\alpha_\beta}{\partial y^\gamma}\Big|_x(y)\,y^\gamma_{1_\mu}y^\beta_{1_\nu} - b^\alpha_\beta(y)y^\beta_{1_\mu+1_\nu}\right) \\
&= \frac{\partial^{|J'|}}{\partial x^{J'}}\left(\left(\frac{\partial b^\alpha_\gamma}{\partial y^\beta}(y) - \frac{\partial b^\alpha_\beta}{\partial y^\gamma}(y)\right)y^\gamma_{1_\mu}y^\beta_{1_\nu}\right)\Big|_x
\end{aligned} \tag{6.17}$$

Similarly, by inserting the condition $j^{k-1}_x\sigma_A\in\ker j^{k-2}\tilde F$, i.e., $j^{k-2}_x\tilde F^A = j^{k-2}_x\hat 0$, in Lemma 6.11, and by using (6.16), we obtain

$$\begin{aligned}
A^\alpha_{\mu,J'+1_\nu} - A^\alpha_{\nu,J'+1_\mu} &= \sum_{I\le J'}\binom{J'}{I}c^\alpha_{\lambda\epsilon}A^\lambda_{\mu,I}A^\epsilon_{\nu,J'-I} \\
&= \sum_{I\le J'}\binom{J'}{I}c^\alpha_{\lambda\epsilon}\frac{\partial^{|I|}}{\partial x^I}\left(b^\lambda_\gamma(y)y^\gamma_{1_\mu}\right)\Big|_x\frac{\partial^{|J'-I|}}{\partial x^{J'-I}}\left(b^\epsilon_\beta(y)y^\beta_{1_\nu}\right)\Big|_x \\
&= \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c^\alpha_{\lambda\epsilon}b^\lambda_\gamma(y)b^\epsilon_\beta(y)y^\gamma_{1_\mu}y^\beta_{1_\nu}\right)\Big|_x
\end{aligned} \tag{6.18}$$

for each $1\le\alpha\le m$, and $1\le\mu,\nu\le n$. By gathering (6.17) and (6.18), and by making use of Lemma 6.12, we conclude

$$\begin{aligned}
y^\alpha_J(\mu) - y^\alpha_J(\nu) &= A^\alpha_{\mu,J'+1_\nu} - A^\alpha_{\nu,J'+1_\mu} - \frac{\partial^{|J'|}}{\partial x^{J'}}\left(\left(\frac{\partial b^\alpha_\gamma}{\partial y^\beta}(y) - \frac{\partial b^\alpha_\beta}{\partial y^\gamma}(y)\right)y^\gamma_{1_\mu}y^\beta_{1_\nu}\right)\Big|_x \\
&= \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c^\alpha_{\lambda\epsilon}b^\lambda_\gamma(y)b^\epsilon_\beta(y)y^\gamma_{1_\mu}y^\beta_{1_\nu}\right)\Big|_x - \frac{\partial^{|J'|}}{\partial x^{J'}}\left(c^\alpha_{\lambda\epsilon}b^\lambda_\gamma(y)b^\epsilon_\beta(y)y^\gamma_{1_\mu}y^\beta_{1_\nu}\right)\Big|_x = 0.
\end{aligned}$$

Once we have defined $j^k_x s=(x^\mu,y^\alpha;y^\alpha_J)\in J^kP$, the last step is to check that $\Theta_k(j^k_x s)=j^{k-1}_x\sigma_A$, which is straightforward from (6.16) and Proposition 6.10.

Lastly, we show that Ξ_k is proper. Let $\left([j^k_{x_n} s_n]_G\right)_{n=1}^{\infty}$ be a sequence in $\left(J^kP\right)\big/G$ such that $\left(j^{k-1}_{x_n}\sigma_{A_n} = \Xi_k\left([j^k_{x_n} s_n]_G\right)\right)_{n=1}^{\infty}$ converges to some $j^{k-1}_x\sigma_A \in J^{k-1}C(P)$. Since $j^{k-2}_{x_n}\tilde{F}_{A_n} = j^{k-2}_{x_n}\hat{0}$ for each $n \in \mathbb{N}$, by continuity we conclude that $j^{k-2}_x\tilde{F}^A = j^{k-2}_x\hat{0}$, that is, $j^{k-1}_x\sigma_A \in \ker j^{k-2}\tilde{F} = \operatorname{im}\Xi_k$. Thus, there exists $j^k_x s \in J^kP$ such that $\Xi_k([j^k_x s]_G) = j^{k-1}_x\sigma_A$. In coordinates, we write $j^k_{x_n}s_n = (x^\mu_n, y^\alpha_n; (y_n)^\alpha_J)$, $j^{k-1}_{x_n}\sigma_{A_n} = (x^\mu_n, (A_n)^\alpha_\mu; (A_n)^\alpha_{\mu,J})$, $j^k_x s = (x^\mu, y^\alpha; y^\alpha_J)$ and $j^{k-1}_x\sigma_A = (x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J})$. Without loss of generality, we may assume that $y^\alpha_n = 0$ and $y^\alpha = 0$, $1 \le \alpha \le m$. This way, by recalling Proposition 6.10, we have

$$\lim_n \frac{\partial^{|J|}}{\partial x^J}\left(b^\alpha_\beta(y_n)(y_n)^\beta_{1_\mu}\right)\Big|_{x_n} = \lim_n (A_n)^\alpha_{\mu,J} = A^\alpha_{\mu,J} = \frac{\partial^{|J|}}{\partial x^J}\left(b^\alpha_\beta(y)y^\beta_{1_\mu}\right)\Big|_x$$

for each $0 \le |J| \le k-1$, $1 \le \alpha \le m$ and $1 \le \mu \le n$. Subsequently, by recurrence as above, it can be checked that $\lim_n (y_n)^\alpha_J = y^\alpha_J$ for each $0 \le |J| \le k$ and $1 \le \alpha \le m$ and, thus, $\left(j^k_{x_n}s_n\right)_{n=1}^{\infty}$ converges to $j^k_x s$.

By gathering the previous results, we obtain the geometry of the reduced configuration bundle.

Corollary 6.14. *For each $k \ge 2$, there is an isomorphism of fiber bundles covering the identity* id_X*,*

$$\left(J^kP\right)/G \simeq \ker j^{k-2}\tilde{F} = \left\{j^{k-1}_x\sigma_A \in J^{k-1}C(P) \mid j^{k-2}_x\tilde{F}^A = j^{k-2}_x\hat{0}\right\}.$$

Proof. Let us show that Ξ_k is an immersion, i.e., that $d\Xi_k$ is everywhere injective. Since the property is local we can work in a trivialization of $\pi_{P,X}$, that is, we assume that $P = X\times G$. Given two sections $s = (\mathrm{id}_X, f), s' = (\mathrm{id}_X, f') \in \Gamma(\pi_{P,X})$, we define $\tau : P \to P$ as $\tau(x,g) = \left(x, f'(x)f(x)^{-1}g\right)$ for each $x \in X$, which is a principal bundle automorphism such that $\tau\circ s = s'$. Besides, the jet extension $j^1\tau : J^1P \to J^1P$ is G-equivariant, i.e., $j^1\tau(j^1_x s)\cdot g = j^1\tau\left((j^1_x s)\cdot g\right)$ for each $g \in G$ and each $j^1_x s \in J^1P$. Hence, it induces an automorphism of the bundle of connections, $\hat{\tau} = [j^1\tau]_G : C(P) \to C(P)$. By taking the jet lift of these maps, we arrive at the following commutative diagram

$$\begin{array}{ccc} J^kP & \xrightarrow{\Theta_k} & J^{k-1}C(P) \\ {\scriptstyle j^k\tau}\downarrow & & \downarrow{\scriptstyle j^{k-1}\hat{\tau}} \\ J^kP & \xrightarrow{\Theta_k} & J^{k-1}C(P) \end{array}$$

Therefore, Θ_k has constant rank, since τ and $\hat{\tau}$ are isomorphisms and this construction can be done for every pair of sections $s, s' \in \Gamma(\pi_{P,X})$. By using this and the fact that π_k is a submersion, and by recalling (6.11), we conclude that Ξ_k has constant rank too. Since Ξ_k is injective (recall Proposition 6.9*(i)*), the Global Rank Theorem (see, for example, [60, Theorem 4.14]) ensures that Ξ_k is an immersion.

In short, Ξ_k is an injective immersion and proper (by *(ii)* of Theorem 6.13), whence it is an embedding (cf. [60, Proposition 4.22]). Hence, $\operatorname{im}\Xi_k = \ker j^{k-2}\tilde{F}$ is a submanifold of $J^{k-1}C(P)$ and, again by the Global Rank Theorem, $\Xi_k : \left(J^kP\right)\big/G \to \ker j^{k-2}\tilde{F}$ is a diffeomorphism.

Notation. Henceforth, we will denote

$$C^{k-1}_0(P) = \operatorname{im}\Xi_k = \ker j^{k-2}\tilde{F}.$$

6.4 Higher-order Euler–Poincaré reduction

At this point, we are ready to develop the Euler–Poincaré reduction for higher-order field theories. Given a principal G-bundle, $\pi_{P,X} : P \to X$, over a boundaryless manifold X and $k \geq 2$, we consider a k-th order Lagrangian density, $\mathfrak{L} = Lv : J^k P \to \bigwedge^n T^* X$, that is G-invariant, i.e.,

$$L \circ \Psi_g^{(k)} = L, \qquad \forall g \in G.$$

As a result, we have a *dropped* or *reduced Lagrangian*,

$$\mathfrak{l} = lv : \big(J^k P\big)\big/ G \to \bigwedge^n T^* X, \qquad \big[j_x^k s\big]_G \mapsto \mathfrak{l}\big(\big[j_x^k s\big]_G\big) = \mathfrak{L}\big(j_x^k s\big).$$

By means of Corollary 6.14, we may regard the reduced Lagrangian as defined on $C_0^{k-1}(P) = \ker j^{k-2}\tilde{F} \subset J^{k-1}C(P)$. For an open subset $\mathcal{U} \subset X$ with compact closure and $s \in \Gamma\left(\mathcal{U}, \pi_{P,X}\right)$, the corresponding *reduced section* is defined as

$$\sigma_A = \big[j^1 s\big]_G \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right).$$

A variation δs of the original section induces a variation $\delta\sigma_A$ of the reduced section. By construction, the original and the reduced variations satisfy $\mathfrak{L}\left(j^k s_t\right) = \mathfrak{l}\left(j^{k-1}\left(\overline{\sigma}_A\right)_t\right)$ for all $t \in (-\epsilon, \epsilon)$ and, thus,

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{U}} \mathfrak{L}\left(j^k s_t\right) = \left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{U}} \mathfrak{l}\left(j^{k-1}\left(\sigma_A\right)_t\right). \tag{6.19}$$

If we choose a fixed principal connection $A_0 \in \Omega^1(P, \mathfrak{g})$ on $\pi_{P,X}$, we can identify the bundle of connections with the modelling vector bundle,

$$C(P) \ni \sigma_A(x) \overset{1:1}{\longleftrightarrow} \overline{\sigma}_A(x) = \sigma_A(x) - \sigma_{A_0}(x) \in T^* X \otimes \tilde{\mathfrak{g}}. \tag{6.20}$$

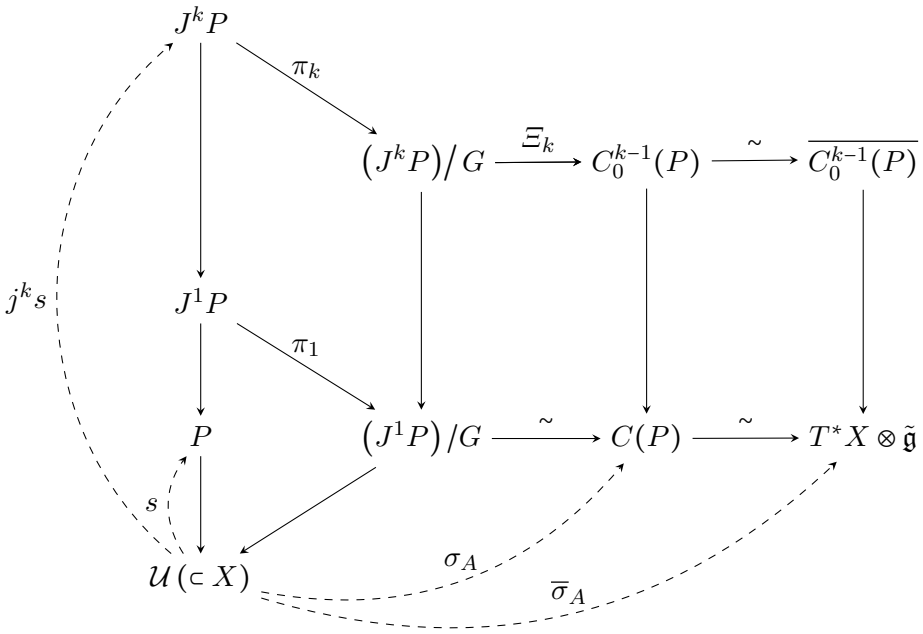
By taking the $(k-1)$-jet lift of this map, we can also identify the reduced configuration space with some subbundle of $J^{k-1}\left(T^* X \otimes \tilde{\mathfrak{g}}\right)$, that is,

$$J^{k-1}C(P) \supset C_0^{k-1}(P) \overset{\sim}{\longleftrightarrow} \overline{C_0^{k-1}(P)} \subset J^{k-1}\left(T^* X \otimes \tilde{\mathfrak{g}}\right),$$

so that the reduced Lagrangian could be regarded as

$$\mathfrak{l} : \overline{C_0^{k-1}(P)} \to \bigwedge^n T^* X.$$

The whole situation is summarized in the following commutative diagram:



Definition 6.4 *Let* $\nabla^{A_0} : \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right) \to \Gamma\left(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\right)$ *be the linear connection on the adjoint bundle,* $\pi_{\tilde{\mathfrak{g}},X}$*, induced by* A_0*. The* divergence of ∇^{A_0} *is minus the adjoint of* ∇^{A_0}*, i.e., the map* $\operatorname{div}^{A_0} : \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right) \to \Gamma\left(\pi_{\tilde{\mathfrak{g}}^*,X}\right)$ *implicitly defined by*

$$\int_X \left\langle \eta, \nabla^{A_0}\xi \right\rangle v = -\int_X \left\langle \operatorname{div}^{A_0}\eta, \xi \right\rangle v, \qquad \xi \in \Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right),\ \eta \in \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right).$$

Given $\overline{\sigma}_A \in \Gamma\left(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\right)$, we define the map $\operatorname{ad}^*_{\overline{\sigma}_A} : \Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right) \to \Gamma\left(\pi_{\tilde{\mathfrak{g}}^*,X}\right)$ as the coadjoint representation of $\tilde{\mathfrak{g}}^*$ and the dual pairing of TX and T^*X. In the following, since $\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}$ is a vector bundle, we identify its vertical bundle with itself, $V\left(T^*X\otimes\tilde{\mathfrak{g}}\right) \simeq T^*X\otimes\tilde{\mathfrak{g}}$, and analogous for its dual.

Theorem 6.15 (Higher-order Euler–Poincaré field equations). *Let* $k \geq 2$*,* $\pi_{P,X} : P \to X$ *be a principal* G*-bundle and* $\mathfrak{L} : J^kP \to \bigwedge^n T^*X$ *be a* k*-th order,* G*-invariant Lagrangian density. Let* $\mathfrak{l} : C_0^{k-1}(P) \to \bigwedge^n T^*X$ *be the reduced Lagrangian density, where* $C_0^{k-1}(P) \simeq \left(J^kP\right)/G \subset J^{k-1}(C(P))$ *is the reduced space, and consider an extension* $\hat{\mathfrak{l}} : J^{k-1}\left(C(P)\right) \to \bigwedge^n T^*X$ *of the reduced Lagrangian* $\mathfrak{l}$ *to the whole jet. Let* $A_0 \in \Omega^1(P,\mathfrak{g})$ *be a principal connection on* $\pi_{P,X}$*, which enables us to identify* $C(P) \simeq T^*X\otimes\tilde{\mathfrak{g}}$*, and let* ∇^{A_0} *be the linear connection on* $\pi_{\tilde{\mathfrak{g}},X}$ *induced by* A_0*.*

Given a (local) section $s \in \Gamma(\mathcal{U},\pi_{P,X})$*, where* $\mathcal{U} \subset X$ *is an open subset with compact closure, and the corresponding reduced section* $\sigma_A \in \Gamma\left(\mathcal{U},\pi_{C(P),X}\right)$*, the following statements are equivalent:*

(i) The variational principle $\delta\int_{\mathcal{U}} \mathfrak{L}\left(j^ks\right) = 0$ *holds for arbitrary variations of* s *such that* $j^{k-1}_x\,\delta s = 0$ *for each* $x \in \partial\mathcal{U}$*.*

(ii) The section s *satisfies the Euler–Lagrange field equations,* $\left(j^{2k}s\right)^*\mathcal{EL}(\mathfrak{L}) = 0$*.*

(iii) The variational principle $\delta\int_{\mathcal{U}} \mathfrak{l}\left(j^{k-1}\sigma_A\right) = 0$ *holds for variations of the form*

$$\delta\sigma_A = \nabla^{A_0}\xi - [\overline{\sigma}_A,\xi] \in \Gamma\left(\mathcal{U},\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\right),$$

where $\xi \in \Gamma(\mathcal{U},\pi_{\tilde{\mathfrak{g}},X})$ *is an arbitrary section such that* $j^{k-1}_x\xi = 0$ *for each* $x \in \partial\mathcal{U}$*.*

(iv) The reduced section $\overline{\sigma}_A$ *satisfies the* k-th order Euler–Poincaré field equations*:*

$$\left(\operatorname{div}^{A_0} - \operatorname{ad}^*_{\overline{\sigma}_A}\right)\left(\left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) = 0, \tag{6.21}$$

where $\mathcal{EL}\left(\hat{\mathfrak{l}}\right) \in \Omega^n\left(J^{2k-2}(C(P)),TX\otimes\tilde{\mathfrak{g}}^*\right)$ *is the Euler–Lagrange form of* $\hat{\mathfrak{l}}$*.*

Proof. The equivalence between (i) and (ii) is established in Theorem 6.6. In addition, the induced variations of the reduced section were computed in [12, Theorem 1]. Hence, the equivalence between (i) and (iii) is straightforward from this fact and equation (6.19). To conclude, let us show the equivalence between (iii) and (iv). Firstly, note that given a variation of the original section, together with the induced variation of the reduced section, we have $j^{k-1}_x(\sigma_A)_t \in C_0^{k-1}(P)$ for all $t \in (-\epsilon,\epsilon)$ and $x \in \mathcal{U}$. Hence, $\delta\left(j^{k-1}\sigma_A\right)(x) = \left(\delta\sigma_A\right)^{(k-1)}(x) \in T_{j^{k-1}_x\sigma_A}\left(C_0^{k-1}(P)\right)$. By applying the explicit expression of the reduced variations and Proposition 6.5, we obtain

$$\begin{aligned}
\delta\int_{\mathcal{U}} \hat{\mathfrak{l}}\left(j^{k-1}\sigma_A\right) &= \int_{\mathcal{U}} \left\langle \left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right), \delta\sigma_A \right\rangle \\
&= \int_{\mathcal{U}} \left\langle \left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right), \nabla^{A_0}\xi - [\overline{\sigma}_A,\xi] \right\rangle \\
&= -\int_{\mathcal{U}} \left\langle \operatorname{div}^{A_0}\left(\left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right) - \operatorname{ad}^*_{\overline{\sigma}_A}\left(\left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right), \xi \right\rangle = 0.
\end{aligned}$$

Since this holds for every $\xi \in \Gamma\left(\mathcal{U},\pi_{\tilde{\mathfrak{g}},X}\right)$ vanishing at the boundary, we conclude.

Remark 6.3 *Since (the $(k-1)$-th jet lift of) the reduced section lies in $C_0^{k-1}(P)$ and the infinitesimal reduced variations are tangent to this space, the solutions of the reduced equations do not depend on the choice of the extended Lagrangian $\hat{\mathfrak{l}}\colon J^{k-1}C(P)\to\bigwedge^n T^*X$. That is, if we have two such extensions $\hat{\mathfrak{l}}_1,\hat{\mathfrak{l}}_2$ with $\hat{\mathfrak{l}}_1|_{C_0^{k-1}(P)}=\hat{\mathfrak{l}}_2|_{C_0^{k-1}(P)}$, then a reduced section is critical with respect for $\hat{\mathfrak{l}}_1$ if and only if it is critical for $\hat{\mathfrak{l}}_2$. In other words, $C_0^{k-1}(P)\subset J^{k-1}C(P)$ may be regarded as a kind of holonomic constraint.*

Remark 6.4 *If we choose the connection $A_0=A$ itself, that is, the connection defined by the reduced section $\sigma_A=[j^1s]_G$, the reduced equation (6.21) has the simpler form*

$$\operatorname{div}^A\left(\left(j^{2k-2}\sigma_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right)\right)=0.$$

Let us find the local expression of the reduced equations. As in Section 6.3, let $\{B_\alpha\mid 1\le\alpha\le m\}$ be a basis of $\mathfrak{g}$, (x^μ,y^α) be normal bundle coordinates for $\pi_{Y,X}$ and (x^μ,A^α_μ) be bundle coordinates for $\pi_{C(P),X}$. Similarly, consider the adapted coordinates $(x^\mu,y^\alpha;y^\alpha_J)$ and $(x^\mu,A^\alpha_\mu;A^\alpha_{\mu,J})$ for J^kY and $J^{k-1}C(P)$, respectively. For the sake of simplicity, we fix the flat connection on $\pi_{Y,X}$ under these coordinates, i.e., $A_0=dy^\alpha\otimes B_\alpha$. This way, (x^μ,A^α_μ) may also be regarded as bundle coordinates on $\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}$, and we denote by (x^μ,v^μ_α) the bundle coordinates on the dual bundle, $\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}$.

Lemma 6.16. *With the above bundle coordinates, let $\eta=\eta^\mu_\alpha\,\partial_\mu\otimes B^\alpha\in\Gamma\left(\pi_{TX\otimes\tilde{\mathfrak{g}}^*,X}\right)$ and $\overline{\sigma}_A=\sigma^\alpha_\mu\,dx^\mu\otimes B_\alpha\in\Gamma\left(\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\right)$. Then we have*

$$\operatorname{div}^{A_0}(\eta)=\partial_\mu\eta^\mu_\alpha\,B^\alpha,\qquad \operatorname{ad}^*_{\overline{\sigma}_A}(\eta)=-c^\alpha_{\beta\gamma}\,\eta^\mu_\alpha\,\sigma^\beta_\mu\,B^\gamma.$$

Proof. Let $\xi=\xi^\alpha B_\alpha\in\Gamma\left(\pi_{\tilde{\mathfrak{g}},X}\right)$ and note that $\nabla^{A_0}\xi=\partial_\mu\xi^\alpha dx^\mu\otimes B_\alpha$. By definition of divergence and the integration by parts formula [9, Lemma 4.5], we finish

$$\begin{aligned}\int_X\left\langle\operatorname{div}^{A_0}(\eta),\xi\right\rangle v&=-\int_X\left\langle\eta,\nabla^{A_0}\xi\right\rangle v\\&=-\int_X\langle\eta^\mu_\alpha\partial_\mu\otimes B^\alpha,\partial_\mu\xi^\alpha dx^\mu\otimes B_\alpha\rangle\,v\\&=-\int_X(\eta^\mu_\alpha\partial_\mu\xi^\alpha)\,v=\int_X(\partial_\mu\eta^\mu_\alpha\xi^\alpha)\,v.\end{aligned}$$

For the second part, we have

$$\begin{aligned}\left\langle\operatorname{ad}^*_{\overline{\sigma}_A}(\eta),\xi\right\rangle&=-\langle\eta,\operatorname{ad}_{\overline{\sigma}_A}(\xi)\rangle=-\left\langle\eta^\mu_\alpha\,\partial_\mu\otimes B^\alpha,\left[\sigma^\alpha_\mu\,dx^\mu\otimes B_\alpha,\xi^\alpha B_\alpha\right]\right\rangle\\&=-\left\langle\eta^\mu_\alpha\,\partial_\mu\otimes B^\alpha,c^\alpha_{\beta\gamma}\,\sigma^\beta_\mu\,\xi^\gamma\,dx^\mu\otimes B_\alpha\right\rangle=-c^\alpha_{\beta\gamma}\,\eta^\mu_\alpha\,\sigma^\beta_\mu\,\xi^\gamma.\end{aligned}$$

Corollary 6.17 (Local equations). *With the above coordinates, we write*

$$\overline{\sigma}_A=\sigma^\alpha_\mu dx^\mu\otimes B_\alpha\in\Gamma\left(\mathcal{U},\pi_{T^*X\otimes\tilde{\mathfrak{g}},X}\right)$$

for the reduced section. Then the local expression of the k-th order Euler–Poincaré field equations is given by

$$\sum_{|J|=0}^{k-1}(-1)^{|J|}\left(\frac{\partial^{|J+1_\mu|}}{\partial x^{J+1_\mu}}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\overline{\sigma}_A\right)\right)-c^\gamma_{\beta\alpha}\sigma^\beta_\mu\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\gamma_{\mu,J}}\left(j^{k-1}\overline{\sigma}_A\right)\right)\right)=0,\qquad 1\le\alpha\le m.$$

Proof. From (6.4) and (6.2), it is clear that

$$\left(j^{2k-2}\overline{\sigma}_A\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}\right)=\sum_{|J|=0}^{k-1}(-1)^{|J|}\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\overline{\sigma}_A\right)\right)v\otimes dA^\alpha_\mu.$$

Recall that we are identifying $V^*\left(T^*X\otimes\tilde{\mathfrak{g}}\right)\simeq TX\otimes\tilde{\mathfrak{g}}^*$, i.e., $dA^\alpha_\mu\simeq\partial_\mu\otimes B^\alpha$. The result is now an straightforward consequence of (iv) of Theorem 6.15 and Lemma 6.16.

At last, note that a solution of the reduced equations lying in the reduced space corresponds to a flat principal connection and, thus, the reconstruction condition is automatically satisfied.

Theorem 6.18 (Reconstruction). *In the conditions of Theorem 6.15, let* $\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C(P),X}\right)$ *be a solution of the k-th order Euler–Poincaré field equations (6.21) such that* $A \in \Omega^1(P|_{\mathcal{U}}, \mathfrak{g})$ *has trivial holonomy on a domain and* $j^{k-1}\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C_0^{k-1}(P),X}\right)$. *Then there exists a section* $s \in \Gamma(\mathcal{U}, \pi_{P,X})$ *that is critical for the original variational problem defined by* $\mathfrak{L}$ *and such that* $\sigma_A = \left[j^1 s\right]_G$. *Furthermore, any critical section on* $\mathcal{U}$ *of* $\mathfrak{L}$ *is of the form* $s \cdot g$ *for certain* $g \in G$.

Proof. From the condition $j^{k-1}\sigma_A \in \Gamma\left(\mathcal{U}, \pi_{C_0^{k-1}(P),X}\right)$ we know that $A \in \Omega^1(P|_{\mathcal{U}}, \mathfrak{g})$ is a flat connection. Therefore, there exists a foliation of $P|_{\mathcal{U}}$ given by the integral leaves of A. The trivial holonomy ensures that each integral leaf intersects once and only once each fiber of $P|_{\mathcal{U}}$. Subsequently, given a integral leaf of A, it defines a section $s \in \Gamma(\mathcal{U}, \pi_{P,X})$ projecting onto σ_A. This section is critical thanks to theorem 6.15. Moreover, the remaining integral leaves of A are obtained from s as follows,

$$\mathcal{F}_g = s(\mathcal{U}) \cdot g \subset P|_{\mathcal{U}}, \qquad g \in G.$$

Hence, given another critical section, $\tilde{s} \in \Gamma(\mathcal{U}, \pi_{P,X})$, projecting onto σ_A, we conclude that $\tilde{s} = s \cdot g$ for some $g \in G$, since $\tilde{s}(\mathcal{U})$ must belong to the previous family.

If X is a simply connected manifold, then every flat connection has trivial holonomy and we have the following equivalence

$$\left(j^{2k}s\right)^* \mathcal{EL}(\mathfrak{L}) = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \left(\mathrm{div}^{A_0} - \mathrm{ad}^*_{\overline{\sigma}_A}\right)\left(\left(j^{2k-2}\overline{\sigma}_A\right)^* \mathcal{EL}\left(\hat{1}\right)\right) = 0, \\ \tilde{F}^A = 0. \end{cases}$$

In an arbitrary manifold the equivalence above is valid locally only.

Remark 6.5 *That fact that the reconstruction process requires the flatness of a connection is a characteristic trait in Field Theories that does not show up in Mechanics, i.e., when* $\dim X = 1$ *(see for example [12, 32] for comments on this situation). However, there is a significant difference from first order to higher-order reduction. Indeed, the flatness condition is a compatibility equation that must be incorporated by hand for* $k = 1$ *theories, whereas this condition comes directly from the geometry of the reduced phase space for* $k > 1$. *In other words, for higher-order Euler–Poincaré reduction the reconstruction is not inserted ad hoc, but it is intrinsic in the nature of the reduced sections.*

6.5 Noether theorem

The well-known Noether theorem establishes that infinitesimal symmetries of a Lagrangian density lead to conserved quantities for the dynamics of the system. As in the previous section, let $\mathfrak{L} = Lv : J^k P \to \bigwedge^n T^*X$ be a G-invariant k-th order Lagrangian on a principal G-bundle, $\pi_{P,X} : P \to X$. The aim of this section is to prove that the conservation laws of $\mathfrak{L}$ arising from its G-invariance are equivalent to the higher-order Euler–Poincaré field equations.

We start by recalling the definition of an infinitesimal symmetry and the Noether theorem for higher-order Lagrangian densities. (cf. [77, §10]).

Definition 6.5 *An* infinitesimal symmetry *of* $\mathfrak{L}$ *is a projectable vector field* $U \in \mathfrak{X}(P)$ *such that* $\pounds_{U^{(k)}}\mathfrak{L} = 0$, *where* $\pounds$ *denotes the Lie derivative and* $U^{(k)} \in \mathfrak{X}(J^k P)$ *is the k-th order prolongation of* U.

For a vertical vector field $V \in \mathfrak{X}(P)$, being an infinitesimal symmetry is equivalent to $\pounds_{V^{(2k-1)}}\Theta_{\mathfrak{L}} = 0$, for any covariant Cartan form, $\Theta_{\mathfrak{L}} \in \Omega^n(J^{2k-1}P)$, of $\mathfrak{L}$.

Theorem 6.19 (Noether theorem). *Let $s \in \Gamma(\pi_{P,X})$ and $U \in \mathfrak{X}(U)$ be a critical section and an infinitesimal symmetry of $\mathfrak{L}$, respectively. Then*

$$\mathrm{d}\left(\left(j^{2k-1}s\right)^* \iota_{U^{(2k-1)}}\Theta_{\mathfrak{L}}\right) = 0,$$

for any covariant Cartan form, $\Theta_{\mathfrak{L}}$, of $\mathfrak{L}$.

Henceforth, we focus on the covariant Cartan form given locally by (6.3). As our Lagrangian is G-invariant, each infinitesimal generator $\xi^* \in \mathfrak{X}(Y)$ of an element $\xi \in \mathfrak{g}$ is an infinitesimal symmetry of $\mathfrak{L}$. We define the *Noether current* as the $\mathfrak{g}^*$-valued form $\mathcal{J} \in \Omega^{n-1}\left(J^{2k-1}P, \mathfrak{g}^*\right)$ given by

$$\left\langle \mathcal{J}\left(j^{2k-1}_x s\right), \xi \right\rangle = \iota_{(\xi^*)^{(2k-1)}}\Theta_{\mathfrak{L}}\left(j^{2k-1}_x s\right), \qquad j^{2k-1}_x s \in J^{2k-1}P,\ \xi \in \mathfrak{g},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing, as usual. Observe that the Noether theorem ensures that

$$\mathrm{d}\left(\left(j^{2k-1}s\right)^* \mathcal{J}\right) = 0 \tag{6.22}$$

for every critical section $s \in \Gamma(\pi_{Y,X})$. We are ready to show the main theorem of this section.

Theorem 6.20. *Let $\pi_{P,X} : P \to X$ be a principal G-bundle, $\mathfrak{L} = Lv : J^kP \to \bigwedge^n T^*X$ be a G-invariant Lagrangian, $\mathfrak{l} = lv : C_0^{k-1}(P) \to \bigwedge^n T^*X$ be the reduced Lagrangian, and $\mathcal{J} \in \Omega^{n-1}\left(J^{2k-1}P, \mathfrak{g}^*\right)$ be the Noether current given by the Cartan form $\Theta_{\mathfrak{L}} \in \Omega^n(J^{2k-1}P)$ locally defined by (6.3). Consider a section, $s \in \Gamma(\pi_{Y,X})$ and the corresponding reduced section, $\sigma_A \in \Gamma\left(\pi_{C(P),X}\right)$. Then the Noether equation (6.22) holds if and only if the k-th order Euler–Poincaré field equations hold for the principal connection $A \in \Omega^1(P, \mathfrak{g})$, i.e.,*

$$\mathrm{div}^A\left(\left(j^{2k-2}\sigma_A\right)^* \mathcal{EL}\left(\hat{l}\right)\right) = 0,$$

where $\hat{l} : J^{k-1}(C(P)) \to \bigwedge^n T^*X$ is an extension of the reduced Lagrangian.

Proof. Let $(x^\mu, y^\alpha; y^\alpha_J)$ be adapted coordinates for $J^{2k}P$. Some conditions will be imposed on these coordinates along the proof. Firstly, suppose that (y^α) are normal coordinates on some neighborhood of the identity element. Hence, given a basis $\{B_\alpha \mid 1 \le \alpha \le m\}$ of $\mathfrak{g}$, we may use equation (6.12) and Proposition 6.2 (recall that $\xi^* \in \mathfrak{X}(P)$ is a vertical vector field) to obtain

$$(\xi^*)^{(2k-1)}_y = \xi^\beta a^\alpha_\beta(y)\partial_\alpha + \xi^\beta \frac{d^{|J|}a^\alpha_\beta(y)}{dx^J}\partial^J_\alpha, \qquad y = (x^\mu, y^\alpha) \in P.$$

If the coordinates are chosen so that the volume form is given by $v = d^nx = dx^1 \wedge \cdots \wedge dx^n$, the left interior product of the covariant Cartan form (6.3) by $(\xi^*)^{(2k-1)}$ reads

$$\iota_{(\xi^*)^{(2k-1)}}\Theta_{\mathfrak{L}} = \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}(-1)^{|J|}\xi^\beta \frac{d^{|J|}}{dx^J}\left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\right)\frac{d^{|I|}a^\alpha_\beta(y)}{dx^I}d^{n-1}x_\mu,$$

where $d^{n-1}x_\mu = \iota_{\partial_\mu}d^nx$. In coordinates, the critical section is written as $s(x^\mu) = (x^\mu, s^\alpha(x^\mu))$, for some local functions $s^\alpha \in C^\infty(X)$, $1 \le \alpha \le m$. From the previous expression and (6.2), we get

$$\left(j^{2k-1}s\right)^* \mathcal{J} = \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}(-1)^{|J|}\xi^\beta \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\left(j^k s\right)\right)\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu. \tag{6.23}$$

On the other hand, let $(x^\mu, A^\alpha_\mu; A^\alpha_{\mu,J})$ be adapted coordinates for $J^{2k-1}C(P)$. Observe that locally $C(P) \simeq T^*X \otimes \tilde{\mathfrak{g}}$, so we will use these coordinates for both spaces indifferently. For the reduced section, we write $\sigma_A(x^\mu) = \overline{\sigma}_A(x^\mu) = (x^\mu, \sigma^\alpha_\mu(x^\mu))$ for some local functions $\sigma^\alpha_\mu \in C^\infty(X)$, $1 \le \mu \le n$, $1 \le \alpha \le m$. Since $\sigma_A = \left[j^1 s\right]_G$, Proposition 6.10 yields

$$\sigma^\alpha_\mu = b^\alpha_\beta(s^\alpha)\partial_\mu s^\beta, \qquad 1 \le \mu \le n,\ 1 \le \alpha \le m.$$

From this and Lemma 6.4, we get

$$\begin{aligned}
&\frac{\partial}{\partial y^\alpha_{I+J+1_\mu}}\left(\frac{\partial^{|I+J+K|}}{\partial x^{I+J+K}}\left(b^\beta_\gamma(s^\alpha)\partial_\mu s^\gamma\right)\right) \qquad (6.24)\\
&\quad = \frac{\partial}{\partial y^\alpha_{I+J+1_\mu}}\left(\sum_{I^{(1)}+I^{(2)}=I+J+K}\frac{(I+J+K)!}{I^{(1)}!I^{(2)}!}\frac{\partial^{|I^{(1)}|}b^\beta_\gamma(s^\alpha)}{\partial x^{I^{(1)}}}\frac{\partial^{|I^{(2)}|}\partial_\mu s^\gamma}{\partial x^{I^{(2)}}}\right)\\
&\quad = \frac{(I+J+K)!}{K!(I+J)!}\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K},
\end{aligned}$$

for each $1 \le \mu \le n$, $1 \le \alpha, \beta \le m$, $0 \le |I| \le k-1$, $0 \le |J| \le k-|I|-1$ and $0 \le |K| \le k-|I|-|J|-1$. Moreover, by definition the extension of the reduced Lagrangian satisfies

$$\hat{l}(x^\mu, \sigma^\alpha_\mu(x^\mu); \partial_J\sigma^\alpha_\mu(x^\mu)) = \hat{l}\left(j^{k-1}_x\overline{\sigma}_A\right) = L\left(j^k_x s\right) = L(x^\mu, s^\alpha(x^\mu); \partial_J s^\alpha(x^\mu)),$$

where $\partial_J = \partial/\partial x^J$. By using this relation and equation (6.24), we obtain

$$\frac{\partial L}{\partial y^\alpha_{I+J+1_\mu}}\left(j^k s\right) = \sum_{|K|=0}^{k-|I|-|J|-1}\frac{(I+J+K)!}{K!(I+J)!}\frac{\partial\hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\sigma_A\right)\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K},$$

for each $1 \le \mu \le n$, $0 \le |I| \le k-1$ and $0 \le |J| \le k-|I|-1$. For brevity, we have denoted $A^\alpha_{\mu,J_0} = A^\alpha_\mu$ for $J_0 = (0, \dots, 0)$. By introducing this in (6.23) and by using Lemma 6.4, we obtain

$$\begin{aligned}
\left(j^{2k-1}s\right)^*\mathcal{J} &= \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}\sum_{|K|=0}^{k-|I|-|J|-1}(-1)^{|J|}\frac{(I+J+K)!}{K!(I+J)!}\xi^\beta \qquad (6.25)\\
&\qquad \frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\overline{\sigma}_A\right)\frac{\partial^{|K|}b^\beta_\alpha(s^\alpha)}{\partial x^K}\right)\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu\\
&= \sum_{|I|=0}^{k-1}\sum_{|J|=0}^{k-|I|-1}\sum_{|K|=0}^{k-|I|-|J|-1}\sum_{|L|=0}^{J}(-1)^{|J|}\frac{(I+J+K)!J!}{K!(I+J)!L!(J-L)!}\xi^\beta\\
&\qquad \frac{\partial^{|L|}}{\partial x^L}\left(\frac{\partial\hat{l}}{\partial A^\beta_{\mu,I+J+K}}\left(j^{k-1}\sigma_A\right)\right)\frac{\partial^{|J+K-L|}b^\beta_\alpha(s^\alpha)}{\partial x^{J+K-L}}\frac{\partial^{|I|}a^\alpha_\beta(s^\alpha)}{\partial x^I}d^{n-1}x_\mu.
\end{aligned}$$

In addition, we have $b^\beta_\alpha(s^\alpha)a^\alpha_\beta(s^\alpha) = m$. Hence, from Lemma (6.4) we get

$$\frac{\partial^{|M|}}{\partial x^M}\left(b^\beta_\alpha(s^\alpha)a^\alpha_\beta(s^\alpha)\right) = \sum_{M^{(1)}+M^{(2)}=M}\frac{M!}{M^{(1)}!M^{(2)}!}\frac{\partial^{|M^{(1)}|}b^\beta_\alpha(s^\alpha)}{\partial x^{M^{(1)}}}\frac{\partial^{|M^{(2)}|}a^\gamma_\beta(s^\alpha)}{\partial x^{M^{(2)}}} = 0, \qquad |M| > 0.$$

By introducing this in (6.25), we arrive at

$$\left(j^{2k-1}s\right)^*\mathcal{J} = m\sum_{|J|=0}^{k-1}(-1)^{|J|}\xi^\alpha\frac{\partial^{|J|}}{\partial x^J}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\overline{\sigma}_A\right)\right)d^{n-1}x_\mu.$$

Thence, the Noether conservation law, $\mathrm{d}\left(\left(j^{2k-1}s\right)^*\mathcal{J}\right) = 0$, locally reads

$$\sum_{|J|=0}^{k-1}(-1)^{|J|}\frac{\partial^{|J+1_\mu|}}{\partial x^{J+1_\mu}}\left(\frac{\partial\hat{l}}{\partial A^\alpha_{\mu,J}}\left(j^{k-1}\sigma_A\right)\right)=0,\qquad 1\le\alpha\le m, \tag{6.26}$$

where we have used that $\xi=\xi^\alpha B_\alpha\in\mathfrak{g}$ is arbitrary.

Lastly, fix $x_0=(x_0^\mu)\in X$ and assume that our adapted coordinates are chosen so that A is flat at $y=s(x_0)$, i.e., $A_{s(x_0)}=(dy^\alpha)_{s(x_0)}\otimes B_\alpha$. It is now clear that (6.26) are exactly the local equations computed in Corollary 6.17 for $\overline{\sigma}_A=\hat{0}$, i.e., $\sigma^\alpha_\mu=0$ for $1\le\mu\le n$ and $1\le\alpha\le m$.

6.6 Multivariate k-splines on Lie groups

Variational splines are piecewise polynomial functions that allow for interpolating while minimizing some cost functional. For this reason, they have many applications in different areas such as optimal control theory, medical imaging or robotics (cf., for instance, [59]). Here we focus on higher-order splines, i.e., those in which the cost functional depends on higher-order derivatives. Namely, the reduction theory for 1-dimensional k-splines on Lie groups developed in [40, §3.2] (see also [62]) is extended to multivariate functions on Lie groups (cf. [24]).

Let G be a Lie group endowed with a right-invariant, (pseudo-)Riemannian metric, $\mathbf{g}$. The inner product induced on the Lie algebra, $\mathfrak{g}$, is denoted by the same symbol, $\mathbf{g}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$, and the induced norms on TG and $\mathfrak{g}$ are denoted by $\|\cdot\|_{\mathbf{g}}:TG\to\mathbb{R}$ and $\|\cdot\|_{\mathfrak{g}}:\mathfrak{g}\to\mathbb{R}$, respectively. Similarly, the adjoint of ad_ξ induced by $\mathbf{g}$ is denoted by $\mathrm{ad}^\dagger_\xi:\mathfrak{g}\to\mathfrak{g}$, i.e.,

$$\mathbf{g}(\mathrm{ad}_\xi(\eta),\zeta)=\mathbf{g}(\eta,\mathrm{ad}^\dagger_\xi(\zeta)),\qquad \xi,\eta,\zeta\in\mathfrak{g}.$$

On the other hand, let $X=\mathbb{R}^n$ (recall Remark 6.1) with the standard (global) coordinates and volume form given by $(x^\mu)=(x^1,\dots,x^n)$ and $d^nx=dx^1\wedge\cdots\wedge dx^n$, respectively. The configuration bundle is given by

$$\pi_{P,X}:P=X\times G\to X,\quad (x,g)\mapsto x.$$

The bundle being trivial yields $J^kP=J^k(X,G)$, the family of k-jets of functions $\varsigma:X\to G$, whose elements are denoted by $j^k_x\varsigma$. Let $\nabla:\mathfrak{X}(G)\times\mathfrak{X}(G)\to\mathfrak{X}(G)$ be the Levi–Civita connection of $\mathbf{g}$. Its pull-back by a (local) function $\varsigma:X\to G$, which is a linear connection on $\pi_{\varsigma^*TG,X}:\varsigma^*TG\to X$, is denoted by $\varsigma^*\nabla$ and, given $1\le\mu\le n$, the corresponding covariant derivative is denoted by

$$\frac{\nabla}{\partial x^\mu}=\varsigma^*\nabla_{\partial_\mu}:\Gamma(\pi_{\varsigma^*TG,X})\to\Gamma(\pi_{\varsigma^*TG,X}).$$

Definition 6.6 *The Lagrangian for* multivariate k-splines *on G is given by*

$$L_s\left(j^k_x\varsigma\right)=\frac{1}{2}\kappa^\mu\left\|\left.\frac{\nabla^{k-1}}{\partial(x^\mu)^{k-1}}\right|_x(\partial_\mu\varsigma)\right\|^2_{\mathbf{g}},\qquad j^k_x\varsigma\in J^k(X,G), \tag{6.27}$$

for certain $\kappa^\mu\in\mathbb{R}$, $0\le\mu\le n$.

Given a (local) function $\varsigma:X\to G$, we set

$$\sigma=dR_{\varsigma^{-1}}\circ d\varsigma=(dR_{\varsigma^{-1}}\circ\partial_\mu\varsigma)\,dx^\mu\in\Gamma(\pi_{T^*X\otimes\mathfrak{g},X})=\Omega^1(X,\mathfrak{g}),$$

where $R_g:G\to G$ denotes the right multiplication by $g\in G$. This induces the following isomorphism,

$$J^1(X,G)\simeq G\ltimes(T^*X\otimes\mathfrak{g}),\quad j^1_x\varsigma\mapsto(\varsigma(x),\sigma(x)),$$

where $\ltimes$ denotes the fibered semidirect product given by the adjoint representation (cf. [21, §4.3.1]). Hence, the isomorphism Ξ_k given in (6.11) becomes

$$J^k(X,G)/G \ni j^k_x\varsigma \mapsto j^{k-1}_x\sigma \in C^{k-1}_0(P).$$

Observe that for the canonical principal connection on the trivial bundle $\pi_{P,X}$, the isomorphism (6.20) reduces to the identity and, thus, $\overline{C^{k-1}_0(P)} = C^{k-1}_0(P)$.

Proposition 6.21. *The Lagrangian (6.27) is right invariant. Furthermore, a (natural) extended reduced Lagrangian is*

$$\hat{l}_s\left(j^{k-1}_x\sigma\right) = \frac{1}{2}\kappa^\mu \left\|\xi^{k-1}_\mu(x)\right\|^2_{\mathfrak{g}}, \qquad j^{k-1}_x\sigma \in J^{k-1}C(P), \tag{6.28}$$

where $\sigma = \sigma_\mu dx^\mu$ and $\xi^j_\mu : X \to \mathfrak{g}$ are defined recursively as follows,

$$\xi^0_\mu = \sigma_\mu, \qquad \xi^j_\mu = \partial_\mu\xi^{j-1}_\mu + \frac{1}{2}\left(\mathrm{ad}^\dagger_{\sigma_\mu}\left(\xi^{j-1}_\mu\right) + \mathrm{ad}^\dagger_{\xi^{j-1}_\mu}(\sigma_\mu) - \mathrm{ad}_{\sigma_\mu}\left(\xi^{j-1}_\mu\right)\right), \ 1 \le j \le k-1.$$

Proof. Given a (local) function $\varsigma : X \to G$, suppose that

$$\frac{\nabla^{k-1}}{\partial(x^\mu)^{k-1}}(\partial_\mu\varsigma) = dR_\varsigma \circ \xi^{k-1}_\mu, \qquad 1 \le \mu \le n. \tag{6.29}$$

In such case, the Lagrangian (6.27) may be written as

$$L_s\left(j^k_x\varsigma\right) = \frac{1}{2}\kappa^\mu \left\|\left(dR_{\varsigma(x)}\right)_e\left(\xi^{k-1}_\mu(x)\right)\right\|^2_{\mathbf{g}} = \frac{1}{2}\kappa^\mu \left\|\xi^{k-1}_\mu(x)\right\|^2_{\mathfrak{g}},$$

where right invariance of $\mathbf{g}$ has been used. It is now clear that L is right invariant, since the maps ξ^{k-1}_μ do not depend on the value of ς, but only on its derivatives (regarded as elements of the Lie algebra). By recalling that $\sigma = dR_{\varsigma^{-1}} \circ d\varsigma$, i.e., $\sigma_\mu = dR_{\varsigma^{-1}} \circ \partial_\mu\varsigma$, the reduced Lagrangian reads

$$l_s\left(j^{k-1}_x\sigma\right) = L_s\left(j^k_x\varsigma\right) = \frac{1}{2}\kappa^\mu \left\|\xi^{k-1}_\mu(x)\right\|^2_{\mathfrak{g}}.$$

Although the reduced Lagrangian is computed for elements $j^{k-1}_x\sigma \in C^{k-1}_0(P)$, it can extended straightforwardly to the whole jet, $\hat{l}_s : J^{k-1}C(P) \to \mathbb{R}$, by taking arbitrary functions $\sigma_\mu : X \to \mathfrak{g}$, $1 \le \mu \le n$.

Lastly, let us check (6.29) by induction. For $k = 1$ it is straightforward. Now let $k > 1$. The formula for the Levi–Civita covariant derivative of the right-invariant metric $\mathbf{g}$ (cf. [40, Eq. (3.13)] or [58, §46.5]), together with the definition of pull-back connection, yield

$$\varsigma^*\nabla_{\partial_\mu}\varsigma^*U = \varsigma^*\left(\nabla_{d\varsigma(\partial_\mu)}U\right) = dR_\varsigma \circ \left(d\phi_U \circ \varsigma_\mu + \frac{1}{2}\mathrm{ad}^\dagger_{\sigma_\mu}(\phi_U) + \frac{1}{2}\mathrm{ad}^\dagger_{\phi_U}(\sigma_\mu) - \frac{1}{2}\mathrm{ad}_{\sigma_\mu}(\phi_U)\right) \circ \varsigma$$

for each $U \in \mathfrak{X}(G)$, where $\phi_U \in C^\infty(G,\mathfrak{g})$ is given by $\phi_U(g) = (dR_{g^{-1}})_g(U(g))$, $g \in G$. By using this and the induction hypothesis, we finish:

$$\begin{aligned}\frac{\nabla^{k-1}}{\partial(x^\mu)^{k-1}}(\partial_\mu\varsigma) &= \varsigma^*\nabla_{\partial_\mu}\left(\frac{\nabla^{k-2}}{\partial(x^\mu)^{k-2}}(\partial_\mu\varsigma)\right) = \varsigma^*\nabla_{\partial_\mu}\left(dR_\varsigma \circ \xi^{k-2}_\mu\right)\\ &= dR_\varsigma \circ \left(\partial_\mu\xi^{k-2}_\mu + \frac{1}{2}\mathrm{ad}^\dagger_{\sigma_\mu}\left(\xi^{k-2}_\mu\right) + \frac{1}{2}\mathrm{ad}^\dagger_{\xi^{k-2}_\mu}(\sigma_\mu) - \frac{1}{2}\mathrm{ad}_{\sigma_\mu}\left(\xi^{k-2}_\mu\right)\right) = dR_\varsigma \circ \xi^{k-1}_\mu.\end{aligned}$$

Let us compute explicitly the reduced equations for $k = 2$. This corresponds to the minimization of the accelerations, which in specific applications is directly related with the fuel expenditure of trajectories (cf. [51]). In such case, for each $\sigma = \sigma_\mu dx^\mu \in \Omega^1(X,\mathfrak{g})$, it is easy to check that $\xi^1_\mu = \partial_\mu\sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)$, $1 \le \mu \le n$.

Theorem 6.22. *The Euler–Poincaré field equations for* (6.28) *with* $k = 2$*, i.e.,*

$$\hat{l}_s\left(j_x^1\sigma\right) = \frac{1}{2}\kappa^\mu \left\|\partial_\mu\sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)\right\|^2_{\mathfrak{g}}, \qquad j_x^1\sigma \in J^1C(P),$$

are given by

$$\kappa^\mu\left(\partial_\mu - \mathrm{ad}^\dagger_{\sigma_\mu}\right)\left(\mathrm{ad}^\dagger_{\eta_\mu}(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu) + \partial_\mu\eta_\mu\right) = 0, \tag{6.30}$$

where $\eta_\mu = \partial_\mu\sigma_\mu + \mathrm{ad}^\dagger_{\sigma_\mu}(\sigma_\mu)$ *for each* $1 \le \mu \le n$*.*

Proof. Let $\sigma = \sigma_\mu dx^\mu \in \Omega^1(X,\mathfrak{g})$. The musical isomorphisms defined by $\mathbf{g}$ are denoted by $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ and $\flat : \mathfrak{g} \to \mathfrak{g}^*$. For each $1 \le \mu, \nu \le n$, it can be easily checked that

$$\frac{\partial\hat{l}_s}{\partial(\partial_\nu\sigma_\mu)}\left(j^1\sigma\right) = \delta^{\mu\nu}\kappa^\mu\eta_\mu^\flat, \qquad \partial_\nu\left(\frac{\partial\hat{l}_s}{\partial(\partial_\nu\sigma_\mu)}\left(j^1\sigma\right)\right) = \kappa^\mu\partial_\mu\eta_\mu^\flat. \tag{6.31}$$

As straightforward computation shows that $\mathrm{ad}^\dagger_\xi(\zeta) = \mathrm{ad}^*_\xi(\zeta^\flat)^\sharp$ for each $\xi, \zeta \in \mathfrak{g}$. Thus, for each $1 \le \mu \le n$ we have

$$\begin{aligned}\frac{\partial\hat{l}_s}{\partial\sigma_\mu}\left(j^1\sigma\right) &= \kappa^\mu\mathbf{g}\left(\eta_\mu, \mathrm{ad}^\dagger(\sigma_\mu) + \mathrm{ad}^\dagger_{\sigma_\mu}(\cdot)\right) \\ &= \kappa^\mu\left(\mathbf{g}\left(\mathrm{ad}(\eta_\mu),\sigma_\mu\right) + \mathbf{g}\left(\mathrm{ad}_{\sigma_\mu}(\eta_\mu),\cdot\right)\right) = \kappa^\mu\left(-\mathbf{g}\left(\mathrm{ad}_{\eta_\mu}(\cdot),\sigma_\mu\right) - \mathbf{g}\left(\mathrm{ad}_{\eta_\mu}(\sigma_\mu),\cdot\right)\right) \\ &= -\kappa^\mu\mathbf{g}\left(\mathrm{ad}^\dagger_{\eta_\mu}(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu),\cdot\right) = -\kappa^\mu\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat\right).\end{aligned} \tag{6.32}$$

By gathering (6.31) and (6.32), the Euler–Lagrange form of $\hat{\mathfrak{l}}_s = \hat{l}_s\, d^nx : J^1C(P) \to \bigwedge^n T^*X$ is readily seen to be

$$\begin{aligned}\left(j^2\sigma\right)^*\mathcal{EL}\left(\hat{\mathfrak{l}}_s\right) &= \left(\frac{\partial\hat{l}_s}{\partial\sigma_\mu}\left(j^1\sigma\right) - \partial_\nu\left(\frac{\partial\hat{l}_s}{\partial(\partial_\nu\sigma_\mu)}\left(j^1\sigma\right)\right)\right)d^nx\otimes\partial_\mu \\ &= -\kappa^\mu\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu\eta_\mu^\flat\right)d^nx\otimes\partial_\mu \in \Omega^n(X, TX\otimes\mathfrak{g}^*).\end{aligned} \tag{6.33}$$

On the other hand, since we have chosen A_0 as the canonical connection, we obtain the standard divergence, that is,

$$\mathrm{div}^{A_0} : \Gamma(\pi_{TX\otimes\mathfrak{g}^*,X}) \to C^\infty(X,\mathfrak{g}), \quad \eta^\mu_\alpha\,\partial_\mu\otimes B^\alpha \mapsto (\partial_\mu\eta^\mu_\alpha)B^\alpha,$$

where $\{B_\alpha \in \mathfrak{g} \mid 1 \le \alpha \le m\}$ is a basis of $\mathfrak{g}$ and $\{B^\alpha \in \mathfrak{g}^* \mid 1 \le \alpha \le m\}$ is its dual basis. In the same fashion, we have

$$\mathrm{ad}^*_\sigma : \Gamma\left(\pi_{TX\otimes\mathfrak{g}^*,X}\right) \to C^\infty(X,\mathfrak{g}^*), \quad \eta^\mu_\alpha\partial_\mu\otimes B^\alpha \mapsto \eta^\mu_\alpha\,\mathrm{ad}^*_{\sigma_\mu}(B^\alpha).$$

From this and (6.33), we obtain the Euler–Poincaré field equations for $\hat{l}_s$,

$$\begin{aligned}0 &= \kappa^\mu\left(\mathrm{div}^{A_0} - \mathrm{ad}^*_\sigma\right)\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu\eta_\mu^\flat\right)d^nx\otimes\partial_\mu \\ &= \kappa^\mu\left(\partial_\mu - \mathrm{ad}^*_{\sigma_\mu}\right)\left(\mathrm{ad}^*_{\eta_\mu}(\sigma_\mu^\flat) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu)^\flat + \partial_\mu\eta_\mu^\flat\right)d^nx.\end{aligned}$$

By applying $\sharp$, we conclude.

Corollary 6.23. *If the metric* $\mathbf{g}$ *is bi-invariant, then the Euler–Poincaré field equations for* 2*-splines* (6.30) *become*

$$\kappa^\mu\left(\partial_\mu^3\sigma_\mu + \left[\sigma_\mu, \partial_\mu^2\sigma_\mu\right]\right) = 0.$$

Proof. For a bi-invariant metric, we have $\mathrm{ad}^\dagger_\xi(\zeta) = -\mathrm{ad}_\xi(\zeta)$ for each $\xi, \eta \in \mathfrak{g}$. Therefore, $\eta_\mu = \partial_\mu\sigma_\mu$ for each $1 \le \mu \le n$ and we finish.

Remark 6.6 (Elastica) *The incorporation of elastic terms to the Lagrangian for multivariate k-splines is motivated by optimal control problems where the velocities (first order derivatives) have to be minimized together with their accelerations [8, 51, 79]. In such case, the Lagrangian is given by*

$$L_e\left(j_x^k\varsigma\right) = L_s\left(j_x^k\varsigma\right) + \frac{1}{2}\tau^\mu \left\|\partial_\mu\varsigma\right\|_{\mathbf{g}}^2, \qquad j_x^k\varsigma \in J^k(X,G), \tag{6.34}$$

for certain $\tau^\mu \in \mathbb{R}$, $1 \le \mu \le n$, with $L_s : J^k(X,G) \to \mathbb{R}$ as in (6.27). The reduced Lagrangian, as well as the reduced equations, are computed by adding the extra term in the above results. Namely, it is clear that

$$\hat{l}_e\left(j_x^{k-1}\sigma\right) = \hat{l}_s\left(j_x^{k-1}\sigma\right) + \frac{1}{2}\tau^\mu \left\|\sigma_\mu\right\|_{\mathfrak{g}}^2, \qquad j_x^{k-1}\sigma \in J^{k-1}C(P),$$

with $\hat{l}_s : J^{k-1}C(P) \to \mathbb{R}$ as in (6.28). The corresponding Euler–Poincaré field equations for $k = 2$ are given by

$$\kappa^\mu\left(\partial_\mu - \mathrm{ad}^\dagger_{\sigma_\mu}\right)\left(\mathrm{ad}^\dagger_{\eta_\mu}(\sigma_\mu) + \mathrm{ad}_{\eta_\mu}(\sigma_\mu) + \partial_\mu\eta_\mu\right) = \tau^\mu\left(\partial_\mu - \mathrm{ad}^\dagger_{\sigma_\mu}\right)\sigma_\mu.$$

Lastly, when the metric $\mathbf{g}$ is bi-invariant, the previous equations boil down to

$$\kappa^\mu\left(\partial_\mu^3\sigma_\mu + \left[\sigma_\mu, \partial_\mu^2\sigma_\mu\right]\right) = \tau^\mu\partial_\mu\sigma_\mu. \tag{6.35}$$

Conclusions and further directions

In this thesis, two new types of covariant reduction by symmetries for Lagrangian field theories have been developed. On the one hand, we have generalized the Lagrange–Poincaré reduction theory to account for local gauge symmetries, which have been modelled by fiberwise actions of Lie group bundles. As most physical systems are not invariant by the full jet extension of the fibered action, we have studied the reduction procedure when the Lagrangian is invariant only by an affine Lie group subbundle. In order to study the geometry of the reduced configuration bundle and to drop the variational principle, an Ehresmann connection equivariant by the fiberwise action has been utilized. This has led to the notion of generalized principal connection, a natural generalization of principal connections. Furthermore, the gauge reduction scheme has been applied to several examples, such as the classical case of global symmetries (Lagrange–Poincaré reduction) and Electromagnetism (including k-form electromagnetism and symmetry breaking by product groups). We have also considered non-Abelian gauge theories, leading to the well-known Utiyama theorem and the Yang–Mills equations. In this application, the fibered action is not free, so our theory cannot be applied a priori. This situation has been handled through a sort of unreduction-reduction procedure in which higher-order jets of functions with values on a Lie group have arisen. For this reason, the Lie groupoid structure of those jets has been studied using a trivialization. On the other hand, we have extended the covariant Euler–Poincaré reduction to higher-order theories. In this case, the geometry of the reduced configuration space has been identified canonically, i.e., without using any connections. Lastly, the reduced field equations have been applied to higher-order, multivariate splines on Lie groups.

The reduction schemes investigated in this work open new interesting questions for future work:

1. **Gauge reduction.** Even though the gauge examples examined have symmetries defined by an affine subbundle, the generalization of our theory to any subbundle $H \subset J^1\mathcal{G}$ is a valuable topic. However, in this case the reduction of the configuration bundle is more intricate, as Proposition 4.1 is not available. In addition, it would be interesting to connect the present reduction approach to some constructions of curved gauge theories [36, 57]. In short, curved gauge theories are constructed as standard gauge theories by substituting the principal bundle by a generalized principal bundle. For that reason, the fiberwise approach to local gauge symmetries here developed is suitable to analyze those theories. The singular actions in these cases could be tackled again with the unreduction-reduction idea used for Yang–Mills in Chapter 5. Lastly, the next natural step in the study of local symmetries would be to substitute the Lie group bundle of symmetries by a Lie groupoid [26, 64], which will allow to study symmetries that are not vertical, i.e., when the action moves the base point of the manifold, as well as local symmetries where the dimension of the Lie group is not constant, but it varies with the point of the base manifold.
2. **Hamiltonian reduction.** In the Hamiltonian formulation of mechanics, reduction is performed on each level set of the momentum map (cf. [67, 73]). An alternative approach to Hamiltonian

reduction that avoids the use of momentum maps consists of reducing the Poisson bracket [70]. This latter approach has been extended to covariant field theories yielding the Hamiltonian counterparts of the Euler–Poincaré and the Lagrange–Poincaré reduction theories, that is, the Lie–Poisson [14] and the Poisson–Poincaré [3] reduction theories, respectively. Of course, the Lagrangian and Hamiltonian viewpoints are related via the Legendre transform. It would be highly desirable to extend the covariant Poisson reduction to account for local gauge symmetries, yielding the Hamiltonian counterpart of the theory developed in Chapter 4.

3. **Higher-order reduction.** The higher-order Euler–Poincaré reduction developed in Chapter 6 constitutes the first step towards a comprehensive understanding of reduction in higher-order field theories, but much work to provide more general higher-order reduction schemes is yet to be done. For example, Euler–Poincaré reduction by a subgroup of the structure group, Lagrange–Poincaré reduction and gauge reduction could be extended to the higher-order realm. One paradigmatic example of application would be higher-order gauge theories [27].

4. **Multi-Dirac reduction.** The Lagrange–Dirac formulation of mechanics [94, 95] provides a unifying framework to deal with degenerate systems with nonholonomic constraints that allows for treating interconnected systems [52, 82], infinite dimensional systems [84] or systems with irreversible processes (thermodynamics) [45, 46], among others. Of course, reduction by symmetries has also been developed from the Dirac point of view [44, 83, 85, 96, 97]. Given the relevance of the Dirac approach in mechanics, it has been extended to covariant field theories by means of the so-called multi-Dirac structures [91]. Nevertheless, the different reduction schemes coming from Lagrangian field theories (Euler–Poincaré, Lagrange–Poincaré and gauge reduction) have not been explored in this context yet.

5. **Discrete reduction.** The construction of discrete theories mimicking the continuous ones has been a fruitful approach to obtain variational and geometric integrators for the dynamical equations of the systems (see, for example, [74] for Mechanics or [30, 61] for field theories). In the same vein, reduction by symmetries has also been analyzed in the discrete setting for both mechanics [5, 6, 68, 69, 85] and field theories [90]. The construction of variational integrators for gauge theories and higher-order theories is a big goal that will require a careful analysis of the discrete analogs of the objects introduced here.

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