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On the propagation of the local Rayleigh condition for the hydrostatic Euler equations

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Abstract

In this thesis we study the propagation in time of the local Rayleigh condition for the two-dimensional hydrostatic Euler equations. Under certain assumptions, we get lower bounds for certain functionals that measure the validity of the local Rayleigh condition, from which we deduce the formation of singularities or the collapse of the local Rayleigh condition in finite time. Additionally, we find necessary conditions for global solvability in the framework of the local well-posedness result by Masmoudi and Wong [20]. Finally, we establish the finite time blow up of certain smooth solutions to the Semi-Lagrangian equations introduced by Brenier in [1].

1. RESUMEN

En esta tesis se estudia la propagación de la condición local de Rayleigh para la ecuación hidrostática de Euler dos-dimensional. Bajo ciertas hipótesis se obtienen cotas inferiores para algunos funcionales que cuantifican la validez de la condición local de Rayleigh, de lo que se deduce la formación de singularidades o el colapso a tiempo finito de la condición de Rayleigh. Adicionalmente, se deducen condiciones necesarias que satisfacen las soluciones globales (en el marco del resultado de existencia local de Masmoudi y Wong [20]) de la ecuación hidrostática de Euler. Finalmente, se establece la formación de singularidades a tiempo finito de ciertas soluciones suaves de las ecuaciones semi-lagrangianas introducidas por Brenier en [1].

2. CONCLUSIONES

Se concluye que las condiciones iniciales que producen soluciones globales (en el marco del resultado de existencia local de Masmoudi y Wong [20]) de la ecuación hidrostática de Euler son una clase fuertemente restricta de funciones, razón por la cual el autor conjetura que dicha clase está compuesta por funciones independientes de la variable horizontal, las que corresponden a soluciones estacionarias. Lo anterior es respaldado por la restrictiva condición obtenida por Brenier en [1] que satisfacen las soluciones globales de la ecuación hidrostática de Euler. Cabe mencionar que el resultado principal de esta tesis no permite distinguir entre la formación de singularidades y el colapso de la condición local de Rayleigh, lo que queda para futuro estudio.

3. INTRODUCTION

In this thesis we will study the propagation in time of the local Rayleigh condition for the two-dimensional hydrostatic Euler equations. More precisely, we will begin by introducing Rayleigh's stability condition for the incompressible Euler equations and its relation with the local Rayleigh condition for the hydrostatic Euler equations. Secondly, we will give an overview of the previous known results concerning the smooth solutions to the hydrostatic Euler equations that satisfy the local Rayleigh condition; in Section 4.3, we will present the local well-posedness result obtained by Masmoudi and Wong [20], in Section 4.4 we will show the mathematical justification of the formal limit (that is, the convergence of the solutions to the rescaled Euler equation towards the solutions to the hydrostatic Euler equations) obtained by Masmoudi and Wong [20], in Section 4.5, we will show the stability result in Sobolev spaces obtained by Masmoudi and Wong [20] and in Section 4.7, we will deduce the necessary condition for global solvability obtained by Brenier in [1]. On the other hand, we will present in Section 4.2 the linear and nonlinear instability of the hydrostatic Euler equations around certain shear flows for which the local Rayleigh condition do not hold (originally obtained by Renardy [22] and Han-Kwan and Nguyen [13] respectively). Additionally, in Section 4.6 we will show the previous known results concerning the singularity formation for certain smooth solutions that do not satisfy the Rayleigh condition (see [7], [3] and [18]). In Chapter 5 we will show the original results obtained in this thesis; first, we will show under certain assumptions that the solution to the hydrostatic Euler equations will develop singularities or collapse the local Rayleigh condition. Secondly, we will deduce necessary conditions for global solvability in the framework of the local well-posedness result by Masmoudi and Wong [20]. Finally, we will establish the finite time blow-up of certain solutions to the semilagrangian equations introduced by Brenier in [1].

3.1. Rayleigh's stability condition

In this section we will introduce the Rayleigh stability condition for the $3D$ Euler equations, which establishes a necessary condition for the existence of certain unstable solutions to the linearization of the $3D$ Euler equations around a shear flow. The local Rayleigh condition is obtained as an analogue of the above condition for the case of the hydorstatic Euler equations.

In order to introduce the Rayleigh stability condition we will proceed as in [6] and [5], where the original work by Rayleigh [21] is presented. Let us consider a $3D$ inviscid incompressible fluid with homogeneous density governed by the $3D$ Euler equations. That is, the velocity $\mathbf{u}$ satisfies the following equations in $\mathbb{R}^2 \times (0,1)$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \cdot \mathbf{n}|_{z=0,1} = 0, \end{cases} \tag{3.1.1}$$

if we assume that $\mathbf{u}$ is given by the perturbation of a shear flow $\begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix}$, we get the following equations for the perturbation $\tilde{\mathbf{u}}$ in $\mathbb{R}^2 \times (0,1)$:

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + U \partial_x \tilde{\mathbf{u}} + \dfrac{dU}{dz} \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \nabla p = 0, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \tilde{\mathbf{u}} \cdot \mathbf{n}|_{z=0,1} = 0, \end{cases} \tag{3.1.2}$$

where

$$\mathbf{u} = \begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix} + \tilde{\mathbf{u}} \tag{3.1.3}$$

and w is the third component of $\tilde{\mathbf{u}}$. Now, by neglecting the convection term of the perturbation, we obtain:

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + U \partial_x \tilde{\mathbf{u}} + \dfrac{dU}{dz} \begin{pmatrix} \tilde{w} \\ 0 \\ 0 \end{pmatrix} + \nabla \tilde{p} = 0, \quad \text{in } \mathbb{R}^2 \times (0,1), \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \tilde{\mathbf{u}} \cdot \mathbf{n}|_{z=0,1} = 0, \end{cases} \tag{3.1.4}$$

where $\tilde{w}$ is the third component of $\tilde{\mathbf{u}}$. Applying divergence to the above equation yields: $-\Delta \tilde{p} = 2\frac{dU}{dz} \partial_x \tilde{w}$. Moreover, by taking the laplacian to the third component of 3.1.4 and using the above identity, we obtain:

$$\partial_t \Delta \tilde{w} + U \partial_x \Delta \tilde{w} = \frac{d^2 U}{dz^2} \partial_x \tilde{w}. \tag{3.1.5}$$

Since the coefficients of the linear differential equations (3.1.4) and (3.1.5) are independent of x, y and t, we may look for solutions of the form:

$$\begin{cases} \tilde{\mathbf{u}} = \mathbf{f}(z) e^{i(\alpha x + \beta y - \alpha c t)}, \\ \tilde{p} = g(z) e^{i(\alpha x + \beta y - \alpha c t)}, \end{cases} \tag{3.1.6}$$

where $\mathbf{f}$ and g are complex functions (the physical quantities should be considered equal to the real part of the above complex solutions). We will say that the above particular solutions are **linearly stable** if the real part of $i\alpha c$ is nonnegative (and so, the solutions decay) or **linearly unstable** if the real part of $i\alpha c$ is negative i.e. when the solutions grow exponentially in time.

By plugging the particular form of $\tilde{\mathbf{u}}$ and $\tilde{p}$ (3.1.6) into (3.1.4), we get the following eigenvalue problem:

$$\begin{cases} i\alpha(U-c)f_1 + \dfrac{dU}{dz}f_3 + i\alpha g = 0, \\ i\alpha(U-c)f_2 + i\beta g = 0, \\ i\alpha(U-c)f_3 + \partial_z g = 0, \\ i\alpha f_1 + i\beta f_2 + \partial_z f_3 = 0, \\ f_3 = 0, \quad \text{in } z = 0, 1, \end{cases} \tag{3.1.7}$$

where $\mathbf{f}$ is an eigenfunction with eigenvalue c (observe that we added the boundary condition from (3.1.4)). For a fixed shear flow $U(z)$, we will say that the flow is **linearly stable** if the real part of $i\alpha c$ is nonnegative for all real numbers α and β or **linearly unstable** if the real part of $i\alpha c$ is negative for at least one pair of real numbers α and β. Following the method shown in [23] (originally for viscous flows), let us introduce the following variables:

$$\begin{aligned} \tilde{\alpha} &= \sqrt{\alpha^2 + \beta^2} \\ \tilde{f} &= \frac{\alpha f_1 + \beta f_2}{\tilde{\alpha}} \\ \tilde{g} &= \frac{\tilde{\alpha}}{\alpha} g. \end{aligned}$$

By taking the sum of the first equation in (3.1.7) and the second equation in (3.1.7) multiplied by $\frac{\beta}{\alpha}$, we obtain: $i\tilde{\alpha}(U-c)\tilde{f} + \frac{dU}{dz}f_3 + i\tilde{\alpha}\tilde{g} = 0$. Multiplying the third equation in (3.1.7) by $\frac{\tilde{\alpha}}{\alpha}$ yields $i\tilde{\alpha}(U-c)f_3 + \partial_z\tilde{g} = 0$. Thus, we obtain the following eigenvalue problem:

$$\begin{cases} i\tilde{\alpha}(U-c)\tilde{f} + \dfrac{dU}{dz}f_3 + i\tilde{\alpha}\tilde{g} = 0, \\ i\tilde{\alpha}(U-c)f_3 + \partial_z\tilde{g} = 0, \\ i\tilde{\alpha}\tilde{f} + \partial_z f_3 = 0, \\ f_3 = 0 \quad \text{in } z = 0, 1, \end{cases} \tag{3.1.8}$$

which corresponds to the particular case when $\beta = f_2 = 0$ in (3.1.7). Thus, without loss of generality, we may consider $2D$ perturbations in order to study the linear instability of the $3D$ Euler equations.

Let us consider the stream function ψ of the $2D$ perturbation $\tilde{\mathbf{u}}$, that is, set $\partial_z \psi = f_1$ and $-\partial_x \psi = f_3$. Because of (3.1.6), we may assume that the stream function has the following form (note that ϕ may be a complex function):

$$\psi(x, z, t) = \phi(z) e^{i\alpha(x-ct)}. \tag{3.1.9}$$

Plugging the above into (3.1.7) (assuming that $\beta = f_2 = 0$), we get:

$$\begin{cases} (U-c)\dfrac{d\phi}{dz} - \dfrac{dU}{dz}\phi + g = 0, \\ \quad -\alpha^2(U-c)\phi + \partial_z g = 0, \\ \qquad\qquad\qquad \alpha\phi = 0, \quad \text{in } z = 0, 1, \end{cases} \tag{3.1.10}$$

from which we get **Rayleigh's stability equation**:

$$(U-c)\left(\frac{d^2\phi}{dz^2} - \alpha^2\phi\right) - \frac{d^2U}{dz^2}\phi = 0. \tag{3.1.11}$$

Let us point out that the above equation is independent of the sign of α. Hence, we may assume that $\alpha > 0$ without loss of generality, which implies that $c_i > 0$ if the flow is linearly unstable. Moreover, if $U(z)$ produces a linearly unstable flow associated to an eigenfunction $\phi(z)$ and eigenvalue $c_r + ic_i$, then $\bar{\phi}$ is an eigenfunction with eigenvalue $c_r - ic_i$ for the same value of α. Thus, every linearly unstable solution produces (by conjugation) a linearly stable solution. From now on, we will only consider unstable solutions for which $c_i > 0$.

Now we are ready to state Rayleigh's inflexion point theorem:

Theorem 1. *Assume that the shear flow U is linearly unstable, then $\frac{d^2U}{dz^2}$ cannot remain positive in the whole domain.*

Proof. Since U is linearly unstable, equation (3.1.11) is satisfied, which implies:

$$\left(\frac{d^2\phi}{dz^2}\bar{\phi} - \alpha^2|\phi|^2\right) - \frac{d^2U}{dz^2}\frac{|\phi|^2}{U-c} = 0.$$

Integrating in $z \in [0, 1]$, we obtain:

$$\int_0^1 \left(-\left|\frac{d\phi}{dz}\right|^2 - \alpha^2|\phi|^2\right) - \frac{d^2U}{dz^2}\frac{|\phi|^2}{U-c}dz = 0. \tag{3.1.12}$$

Finally, the imaginary part of the above equation is given by:

$$c_i \int_0^1 \frac{d^2U}{dz^2}\frac{|\phi|^2}{|U-c|^2}dz = 0, \tag{3.1.13}$$

where c_i is the imaginary part of c, which implies that U'' has at least one inflexion point. This concludes the proof of Theorem 1.

□

In addition to the previous theorem, we will present two further criteria for instability. The following theorem is known as **Fjørtoft's criterion for instability**, which was proved in [8].

Theorem 2. *Assume that the shear flow U is linearly unstable. If z_i satisfies $\frac{d^2U}{dz^2}(z_i) = 0$ then $\frac{d^2U}{dz^2}(U - U(z_i))$ cannot remain nonnegative in the whole domain.*

Proof. Let z_i be a point where $\frac{d^2U}{dz^2}$ vanishes and denote by c_r the real part of c. By taking the real part of equation (3.1.12), we get:

$$\int_0^1 \left(-\left|\frac{d\phi}{dz}\right|^2 - \alpha^2|\phi|^2 \right) dz = \int_0^1 \frac{d^2U}{dz^2} \frac{(U - c_r)}{|U - c|^2} |\phi|^2 dz.$$

Since U is linearly unstable Equation (3.1.13) is satisfied, which yields:

$$\int_0^1 \left(-\left|\frac{d\phi}{dz}\right|^2 - \alpha^2|\phi|^2 \right) dz = \int_0^1 \frac{d^2U}{dz^2} \frac{(U - U(z_i))}{|U - c|^2} |\phi|^2 dz,$$

which completes the proof of Theorem 2. □

Finally, we will present Howard's semicirlce theorem, which was proved originally in [14].

Theorem 3. *Assume that the shear flow U is linearly unstable. Then, every unstable solution satisfies:*

$$\left(c_r - \frac{1}{2}(U_M + U_m) \right)^2 + c_i^2 \leq \left(\frac{1}{2}(U_M - U_m) \right)^2, \; c_i > 0.$$

Where $U_M = \max U$ and $U_m = \min U$. That is, the complex number $c = c_r + ic_i$ (with $c_i > 0$) lies in the semicircle centered at $(\frac{1}{2}(U_M + U_m), 0)$ with radius $\frac{1}{2}(U_M - U_m)$.

Proof. Since U is linearly unstable, we have:

$$(U(z) - c)\frac{d^2\phi}{dz^2} - \phi\frac{d^2U}{dz^2} = \alpha^2(U(z) - c)\phi(z),$$

thanks to (3.1.11), which is equivalent to:

$$\frac{d}{dz}\left((U(z) - c)^2 \frac{d}{dz}\left(\frac{\phi(z)}{U(z) - c} \right) \right) = \alpha^2(U(z) - c)\phi(z),$$

because $(U(z)-c)^2\frac{d}{dz}\left(\frac{\phi}{U-c}\right) = (U(z)-c)\frac{d\phi}{dz} - \phi(z)\frac{dU}{dz}$. Multiplying by $\overline{\left(\frac{\phi(z)}{U(z)-c}\right)}$ yields:

$$\overline{\left(\frac{\phi(z)}{U(z)-c}\right)}\frac{d}{dz}\left((U(z)-c)^2\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right) = (U(z)-c)^2\alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2.$$

Integrating the above in $z \in [0,1]$, we get (using the boundary condition for ϕ):

$$\int_0^1 (U(z)-c)^2\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz = 0.$$

The real and imaginary parts of the above expression are given by:

$$\int_0^1 \left((U(z)-c_r)^2 - c_i^2\right)\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz = 0,$$

$$-2c_i\int_0^1 (U(z)-c_r)\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz = 0,$$

which implies:

$$\int_0^1 \left(U(z)^2 - (c_r^2+c_i^2)\right)\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz = 0,$$

which yields

$$\begin{aligned}
0 &\geq \int_0^1 (U(z)-U_m)(U(z)-U_M)\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz \\
&= \int_0^1 \left(U^2 - (U_M+U_m)U + U_MU_m\right)\left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz \\
&= H\int_0^1 \left(\left|\frac{d}{dz}\left(\frac{\phi(z)}{U(z)-c}\right)\right|^2 + \alpha^2\left|\frac{\phi(z)}{U(z)-c}\right|^2\right)dz,
\end{aligned}$$

where

$$H = c_r^2 + c_i^2 - (U_M+U_m)c_r + U_MU_m,$$

from which we get:

$$H = c_i^2 + \left(c_r - \frac{1}{2}(U_M+U_m)\right)^2 - \left(\frac{1}{2}(U_M-U_m)\right)^2 \leq 0.$$

The proof of Theorem 3 is now complete. □

4. THE HYDROSTATIC EULER EQUATIONS

4.1. Introduction to the hydrostatic Euler equations

The hydrostatic Euler equations are derived from the incompressible Euler equations by means of the hydrostatic approximation. More precisely, let us consider the three-dimensional incompressible Euler equations in $\mathbb{R}^2 \times (0,1)$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = 0, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \cdot \mathbf{n}|_{z=0,1} = 0, \end{cases} \tag{4.1.1}$$

where $\mathbf{u} \in \mathbb{R}^3$ is the velocity field and P is the scalar pressure. The hydrostatic approximation of equations (4.1.1) consists in neglecting the material derivative of the third component of the velocity field $\mathbf{u}$ in (4.1.1), namely:

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + w \partial_z \mathbf{v} + \nabla P = 0, \\ \partial_z P = 0, \\ \nabla \cdot \mathbf{v} + \partial_z w = 0, \\ w|_{z=0,1} = 0, \end{cases} \tag{4.1.2}$$

where ∇ denotes the two-dimensional gradient. Equations (4.1.2) are also known as the **formal hydrostatic limit** of the Euler equations (4.1.1).

In this thesis we will consider the two-dimensional hydrostatic Euler equations in a periodic channel $\Omega = \mathbb{T} \times (0,1)$:

$$\begin{cases} u_t + u u_x + v u_y + P_x = 0, \\ u_x + v_y = 0, \\ P_y = 0, \\ v|_{y=0,1} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{4.1.3}$$

where (u, v) is the velocity field, P is the scalar pressure and u satisfies the local Rayleigh condition $\partial_y^2 u > 0$. The mathematical justification of the formal hydrostatic limit has been established in several articles under the local Rayleigh condition (see for instance [2], [11] and [20]). On the other hand, if the initial condition does not satisfies the local Rayleigh condition, the convergence may not hold, as shown in [12], [11]. Furthermore, the ill-posedness of the linearization of (4.1.3) around certain shear flows was proved in [22], as well as the ill-posedness of (4.1.3) around certain shear flows, which was proved in [13].

The local existence of solutions to (4.1.3) has been proven in [16] and [17] in the analytic setting. Nevertheless, the finite time blow-up of solutions to (4.1.3) was established in [18] (under the assumption $u_0(x_0, y, t) \equiv C$, for $y \in [0, 1]$, which is incompatible with the local Rayleigh condition) and [3] (under the assumption of parity, which is again incompatible with the local Rayleigh condition. See also [7], where the finite time blow-up of certain solutions to the unsteady Prandtl equations is proved by similar methods).

Equivalently, we may consider the vorticity formulation of the hydrostatic Euler equations (4.1.3)

$$\begin{cases} \omega_t + u\omega_x + v\omega_y = 0, \\ (u, v) = \nabla^{\perp}\mathcal{A}\omega, \\ \omega|_{t=0} = \omega_0, \end{cases} \tag{4.1.4}$$

where the vorticity ω is defined by

$$\omega = u_y. \tag{4.1.5}$$

Let us point out that any solution to (4.1.3) satisfies $\int_0^1 u(x, y, t)dy \equiv k$ (see Proposition 1). Moreover, $\tilde{u}(x, y, t) = u(x + kt, y, t) - k$ solves (4.1.3) with a slightly different initial data and $\int_0^1 \tilde{u}(x, y, t)dy \equiv 0$. Hence, without loss of generality, we may assume that the solution to (4.1.3) satisfies

$$\int_0^1 u(x, y, t)dy \equiv 0. \tag{4.1.6}$$

The following proposition summarizes some properties that will be used in the rest of this thesis.

Proposition 1. *Assume that the initial vorticity $\omega_0 \in H^4(\mathbb{T} \times (0, 1))$ satisfies the local Rayleigh condition (i.e. $\partial_y\omega_0 > 0$). Let $\omega \in C([0, T]; H^4(\mathbb{T} \times (0, 1)))$ be the solution to the problem (4.1.4) that satisfies the local Rayleigh condition in $[0, T]$. Then we have*

$$\mathcal{A}(\omega) = (1-y)\int_0^y z\omega(x,z,t)dz + y\int_y^1 (1-z)\omega(x,z,t)dz, \tag{4.1.7}$$

$$u(x,y,t) = \int_0^1 z\omega(x,z,t)dz - \int_y^1 \omega(x,z,t)dz, \tag{4.1.8}$$

$$v(x,y,t) = (1-y)\int_0^y z\omega_x(x,z,t)dz + y\int_y^1 (1-z)\omega_x(x,z,t)dz, \tag{4.1.9}$$

$$\partial_x \int_0^1 u dy = 0, \tag{4.1.10}$$

$$\partial_t \int_0^1 u dy = 0, \tag{4.1.11}$$

$$P_x(x,t) = -\partial_x \left(\int_0^1 u^2(x,y,t)dy \right), \tag{4.1.12}$$

$$\partial_t \|u\|_2^2 = 0, \tag{4.1.13}$$

$$\|u\|_\infty \leq \frac{3}{2}\|\omega_0\|_\infty. \tag{4.1.14}$$

Proof. By (4.1.6), we may assume that the stream function $\mathcal{A}(\omega)$ solves

$$\begin{cases} -\partial_y^2 \mathcal{A}(\omega) = \omega, \\ \mathcal{A}(\omega)|_{y=0,1} = 0, \end{cases} \tag{4.1.15}$$

which implies

$$\begin{aligned} \mathcal{A}(\omega) &= -\int_0^y \int_0^z \omega(x,s,t)dsdz + y\int_0^1 \int_0^z \omega(x,s,t)dsdz \\ &= (1-y)\int_0^y z\omega(x,z,t)dz + y\int_y^1 (1-z)\omega(x,z,t)dz, \end{aligned}$$

from which we get (4.1.7). Next, (4.1.8) and (4.1.9) follow directly from (4.1.7). The incompressibility condition and the boundary value in (4.1.3) implies (4.1.10). Next, by (4.1.10), (4.1.3), the x-periodicity and integration by parts,

$$\begin{aligned} \partial_t \int_0^1 u dy &= \partial_t \int_{\mathbb{T}\times(0,1)} u dx dy \\ &= -\int_{\mathbb{T}\times(0,1)} \partial_x \left(\frac{u^2}{2} + P \right) + v\omega dx dy \\ &= -\int_{\mathbb{T}\times(0,1)} u u_x dx dy \\ &= -\int_{\mathbb{T}\times(0,1)} \partial_x \left(\frac{u^2}{2} \right) dx dy \\ &= 0, \end{aligned} \tag{4.1.16}$$

which proves (4.1.11). Let us prove (4.1.12). By (4.1.3), (4.1.11) and integration by parts, we have

$$\begin{aligned} 0 &= \int_0^1 (u_t + uu_x + vu_y + P_x)\, dy \\ &= \int_0^1 (uu_x + vu_y)\, dy + P_x \\ &= 2\int_0^1 uu_x dy + P_x, \end{aligned}$$

which implies (4.1.12). By (4.1.3), the x-periodicity and integration by parts

$$\begin{aligned} \partial_t \int_{\mathbb{T}\times(0,1)} u^2 dxdy &= 2\int_{\mathbb{T}\times(0,1)} uu_t dxdy \\ &= -2\int_{\mathbb{T}\times(0,1)} (u^2 u_x + vuu_y + uP_x) dxdy \\ &= -3\int_{\mathbb{T}\times(0,1)} u^2 u_x dxdy + 2\int_{\mathbb{T}} \left(\int_0^1 u_x dy\right) P(x,t) dx \\ &= 0, \end{aligned}$$

from which we get (4.1.13). Applying (4.1.8), we get

$$\begin{aligned} \|u\|_\infty &\le \|\omega\|_\infty \int_0^1 z dz + \|\omega\|_\infty \\ &= \frac{3}{2}\|\omega\|_\infty \\ &= \frac{3}{2}\|\omega_0\|_\infty. \end{aligned}$$

This completes the proof of Proposition 1. □

4.2. Linear and nonlinear instability

In this section we will present the linear and nonlinear instability of the hydrostatic Euler equation around certain shear flows. We will first establish the linear instability as shown in the work by Renardy [22]. Let us proceed as in Section 3.1, we will consider the hydrostatic Euler equations in $\mathbb{T}\times(0,1)$:

$$\begin{cases} u_t + uu_x + vu_y + P_x = 0, \\ P_y = 0, \\ u_x + v_y = 0, \\ v|_{y=0,1} = 0. \end{cases} \tag{4.2.1}$$

Assume that the solution to the above problem u is given by a perturbed shear flow U (that is, $u = U + \tilde{u}$), then the perturbation $\tilde{u}$ satisfies the following equations:

$$\begin{cases} \tilde{u}_t + (U + \tilde{u})\tilde{u}_x + \tilde{v}(U' + \tilde{u}_y) + P_x = 0, \\ P_y = 0, \\ \tilde{u}_x + \tilde{v}_y = 0, \\ \tilde{v}|_{y=0,1} = 0, \end{cases} \tag{4.2.2}$$

By neglecting the convection terms, we get the following linear differential equation:

$$\begin{cases} \tilde{u}_t + U\tilde{u}_x + \tilde{v}U'(y) + \tilde{P}_x = 0, \\ \tilde{P}_y = 0, \\ \tilde{u}_x + \tilde{v}_y = 0, \\ \tilde{v}|_{y=0,1} = 0, \end{cases} \tag{4.2.3}$$

If we derive with respect to y the first equation in (4.2.3), we get

$$\tilde{u}_{ty} + U\tilde{u}_{xy} + \tilde{v}U''(y) = 0. \tag{4.2.4}$$

Thus, the stream function ψ satisfies:

$$\psi_{tyy} + U(y)\psi_{xyy} - \psi_x U''(y) = 0. \tag{4.2.5}$$

Since the coefficients of the above linear differential equation are independent of x and t, we may look for solutions of the form:

$$\psi(x, y, t) = \phi(y)e^{i\alpha(x-ct)}, \tag{4.2.6}$$

By plugging (4.2.6) into (4.2.5), we obtain:

$$\begin{cases} (U - c)\dfrac{d^2\phi}{dy^2} = \phi(y)\dfrac{d^2U}{dy^2}, \\ \phi|_{y=0,1} = 0, \end{cases} \tag{4.2.7}$$

which is equivalent to (assuming that the imaginary part of c is not equal to 0):

$$\frac{d}{dy}\left((U - c)^2 \frac{d}{dy}\left(\frac{\phi}{U - c}\right)\right) = 0,$$

which implies:

$$\frac{d}{dy}\left(\frac{\phi}{U - c}\right) = \frac{K_1}{(U(y) - c)^2},$$

for certain constant K_1, which yields the following compatibility condition:

$$\int_0^1 \frac{dy}{(U(y)-c)^2} = 0.$$

The real and imaginary parts of the above equation are given by:

$$\int_0^1 \frac{U(y)^2 + c_r^2 - c_i^2 - 2U(y)c_r}{|U(y)-c|^4} dy = 0,$$
$$\int_0^1 \frac{2c_i(U(y)-c_r)}{|U(y)-c|^4} dy = 0.$$

The above computations yields the following theorem, which is the main result presented in [22].

Theorem 4. *Assume that the shear flow U satisfies*

$$\int_0^1 \frac{dy}{(U(y)-c)^2} = 0,$$

for certain complex number $c = c_r + ic_i$, where $c_i \neq 0$. Then the linearized hydrostatic Euler equations (4.2.3) admit solutions of the form:

$$\psi(x,y,t) = \phi(y)e^{i\alpha(x-ct)},$$

where $\nabla^\perp \psi = (\tilde{u}, \tilde{v})$, which yields the linear instability, since the stream function has an exponential growth in time given by αc_i.

Remark 1. *Note that in the above theorem α can be chosen arbritrarily (independently of c).*

Now we will present the nonlinear instability result for the $2D$ Euler equation obtained by Grenier in [11].

Theorem 5. *Let $\mathbf{U} = \begin{pmatrix} U(z) \\ 0 \end{pmatrix} \in C^\infty(\mathbb{T} \times (-1,1))$ be a shear flow. If $\mathbf{U}$ is linearly unstable (in the sense that there exists an eigenvalue of a linearized Euler operator with a nonnegative real part), then it is also nonlinearly unstable in the following sense: There exists positive constants δ_0 and C_s (for $s \in \mathbb{N}$) such that for every s arbitrarily large and every $\delta > 0$ arbitrarily small there exists a smooth solution $\mathbf{u}(t,x,y)$ of 3.1.1 such that*

$$\|\mathbf{u}(0,x,y)\|_{H^s(\mathbb{T}\times(-1,1))} \leq \delta$$

and a time

$$T_\delta \leq C_s \log\left(\delta^{-1}\right) + C_s$$

such that

$$\|\mathbf{u}(T_\delta, x, y) - \mathbf{U}\|_{L^\infty(\mathbb{T}\times(-1,1))} \geq \delta_0$$

and

$$\|\mathbf{u}(T_\delta, x, y) - \mathbf{U}\|_{L^2(\mathbb{T}\times(-1,1))} \geq \delta_0.$$

Finally, we will present the nonlinear instability result for the hydrostatic Euler equation around a shear flow that may be chosen to be analytic. The above result was obtained by Han-Kwan and Nguyen in [13].

Theorem 6. *There exists a stationary shear flow $U(z)$ such that the following holds. For all $s \in \mathbb{N}$, $\alpha \in (0,1]$, and $k \in \mathbb{N}$, there are families of solutions $(\omega_\varepsilon)_{\varepsilon>0}$ of 4.2.1, times $t_\varepsilon = O(\varepsilon|\log(\varepsilon)|)$, and $(x_0, z_0) \in \mathbb{T}\times(-1,1)$ such that*

$$\lim_{\varepsilon\to 0} \frac{\|\omega_\varepsilon - \frac{dU}{dz}\|_{L^2([0,t_\varepsilon]\times\Omega_\varepsilon)}}{\|\omega_\varepsilon|_{t=0} - \frac{dU}{dz}\|^\alpha_{H^s([0,t_\varepsilon]\times\Omega_\varepsilon)}} = +\infty,$$

with $\Omega_\varepsilon = B(x_0, \varepsilon^k)\times B(z_0, \varepsilon^k)$.

4.3. Local existence and uniqueness in Sobolev spaces under the local Rayleigh condition

In this section we will state the local existence and uniqueness of the solution to (4.1.3) in Sobolev spaces obtained by Masmoudi and Wong in [20]:

Theorem 7. *Let $s \geq 4$ and $m > 0$. Suppose that $\omega_0 \in H^s(\mathbb{T}\times(0,1))$ satisfies $2m < \partial_y\omega_0 < \frac{1}{2m}$. Then there exist a time $T > 0$ such that problem (4.1.4) has a unique solution ω satisfying:*

- $m < \omega_y < \dfrac{1}{m}, \quad$ *in* $[0,T]$.
- $\sup\limits_{t\in[0,T]} \|\omega(t)\|_{H^s(\mathbb{T}\times(0,1))} < \infty$.

4.4. Mathematical justification of the formal hydrostatic limit under the local Rayleigh condition

In order to justify the formal hydrostatic limit equations (4.1.3) we may proceed as follows:

Let us consider the two-dimensional Euler equations in $\mathbb{T}\times(0,\varepsilon)$

$$\begin{cases} \tilde{u}^\varepsilon_t + \tilde{u}^\varepsilon \tilde{u}^\varepsilon_x + \tilde{v}^\varepsilon \tilde{u}^\varepsilon_y + P^\varepsilon_x = 0, \\ \tilde{v}^\varepsilon_t + \tilde{u}^\varepsilon \tilde{v}^\varepsilon_x + \tilde{v}^\varepsilon \tilde{v}^\varepsilon_y + \tilde{P}^\varepsilon_y = 0, \\ \tilde{u}^\varepsilon_x + \tilde{v}^\varepsilon_y = 0, \\ \tilde{v}^\varepsilon|_{y=0,1} = 0, \\ \tilde{u}^\varepsilon|_{t=0} = \tilde{u}^\varepsilon_0. \end{cases} \tag{4.4.1}$$

By plugging the following rescale:

$$\begin{aligned} \varepsilon\tilde{y} &= y, \\ \tilde{u}^\varepsilon(x,\tilde{y},t) &= u(x,\varepsilon\tilde{y},t), \\ \varepsilon\tilde{v}^\varepsilon(x,\tilde{y},t) &= v(x,\varepsilon\tilde{y},t), \\ \tilde{p}^\varepsilon(x,\tilde{y},t) &= p(x,\varepsilon\tilde{y},t), \end{aligned}$$

into (4.4.1), we get the **rescaled 2D Euler equations** in $\mathbb{T}\times(0,1)$:

$$\begin{cases} u^\varepsilon_t + u^\varepsilon u^\varepsilon_x + v^\varepsilon u^\varepsilon_y + P^\varepsilon_x = 0, \\ \varepsilon^2(v^\varepsilon_t + u^\varepsilon v^\varepsilon_x + v^\varepsilon v^\varepsilon_y) + P^\varepsilon_y = 0, \\ u^\varepsilon_x + v^\varepsilon_y = 0, \\ v^\varepsilon|_{y=0,1} = 0, \\ u^\varepsilon|_{t=0} = u^\varepsilon_0. \end{cases} \tag{4.4.2}$$

Equivalently, we may consider the vorticity formulation of the rescaled Euler equations (4.4.2):

$$\begin{cases} \omega^\varepsilon_t + u^\varepsilon \omega^\varepsilon_x + v^\varepsilon \omega^\varepsilon_y = 0, \\ (u^\varepsilon, v^\varepsilon) = \nabla^\perp \mathcal{A}^\varepsilon(\omega^\varepsilon), \\ \omega^\varepsilon|_{t=0} = \omega^\varepsilon_0, \end{cases} \tag{4.4.3}$$

where the rescaled vorticity ω^ε is defined by:

$$\omega^\varepsilon = \partial_y u^\varepsilon - \varepsilon^2 \partial_x v^\varepsilon, \tag{4.4.4}$$

and the rescaled stream function $\mathcal{A}^\varepsilon(\omega^\varepsilon)$ satisfies:

$$\begin{cases} -\partial^2_y \mathcal{A}^\varepsilon(\omega^\varepsilon) - \varepsilon^2 \partial^2_x \mathcal{A}^\varepsilon(\omega^\varepsilon) = \omega^\varepsilon, \quad \text{in } \mathbb{T}\times(0,1), \\ \mathcal{A}^\varepsilon(\omega^\varepsilon)|_{y=0,1} = 0. \end{cases} \tag{4.4.5}$$

In this section we will justify the convergence of the solutions to the rescaled $2D$ Euler equations (4.4.2) towards the solution to the hydrostatic Euler equation (4.1.3) in Sobolev spaces, which was obtained by Masmoudi and Wong [20]. In order to prove the convergence result, we will need the following two results:

Lemma 1. *Let $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ be as in Theorem 7 and let ω be the solution to (4.1.4) for which $\sup_{t\in[0,T]}\|\omega\|_{H^4(\mathbb{T}\times(0,1))} \leq B$. Let $\omega_0^\varepsilon \in H^4(\mathbb{T}\times(0,1))$ and ω^ε be the solution to 4.4.3, suppose that:$\|u^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}\times(0,1))}, \|v^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}\times(0,1))} \leq M$ for a certain constant M. Then:*

$$\sup_{t\in[0,T]}\|\omega-\omega^\varepsilon\|^2_{L^2(\mathbb{T}\times(0,1))} \leq C\left(m,M,T,B\right)\left(\|\omega_0-\omega_0^\varepsilon\|^2_{L^2(\mathbb{T}\times(0,1))}+\varepsilon^4\right),$$

where $m<\omega_y<\frac{1}{m}$ in $[0,T]$.

Proof. Denote by D_t^ε the material derivative with respect to the incompressible flow $(u^\varepsilon,v^\varepsilon)$, that is: $D_t^\varepsilon f=\partial_t f+u^\varepsilon\partial_x f+v^\varepsilon\partial_y f$. The material derivative of $|\omega-\omega^\varepsilon|^2$ and $\frac{1}{\omega_y}$ is given by:

$$\begin{aligned}
&D_t^\varepsilon(|\omega-\omega^\varepsilon|^2)=2(\omega-\omega^\varepsilon)((u^\varepsilon-u)\omega_x+(v^\varepsilon-v)\omega_y,\\
&D_t^\varepsilon\left(\frac{1}{\omega_y}\right)=(u^\varepsilon-u)\partial_x\left(\frac{1}{\omega_y}\right)+(v^\varepsilon-v)\partial_y\left(\frac{1}{\omega_y}\right)+\frac{1}{\omega_y}\left(\frac{\omega\omega_x}{\omega_y}-u_x\right).
\end{aligned}$$

Thus:

$$\begin{aligned}
\partial_t\int_{\mathbb{T}\times(0,1)}\frac{|\omega-\omega^\varepsilon|^2}{\omega_y}dxdy&=2\int_{\mathbb{T}\times(0,1)}(\omega-\omega^\varepsilon)\left((u^\varepsilon-u)\frac{\omega_x}{\omega_y}+(v^\varepsilon-v)\right)dxdy\\
&\quad+\int_{\mathbb{T}\times(0,1)}|\omega-\omega^\varepsilon|^2\left(D_t^\varepsilon\left(\frac{1}{\omega_y}\right)\right)dxdy.
\end{aligned}$$

The second term is equal to

$$\begin{aligned}
2\int_{\mathbb{T}\times(0,1)}(\omega-\omega^\varepsilon)(v^\varepsilon-v)dxdy&=2\varepsilon^2\int_{\mathbb{T}\times(0,1)}v_x^\varepsilon(v^\varepsilon-v)dxdy\\
&=2\varepsilon^2\int_{\mathbb{T}\times(0,1)}v^\varepsilon v_x dxdy\\
&=-2\varepsilon^2\int_{\mathbb{T}\times(0,1)}u_x^\varepsilon\partial_y\mathcal{A}(v_x)dxdy\\
&=2\varepsilon^2\int_{\mathbb{T}\times(0,1)}(u^\varepsilon-u)\partial_y\mathcal{A}(v_{xx})dxdy, \qquad (4.4.6)
\end{aligned}$$

thanks to the identity $-\partial_y^2\mathcal{A}(v_x)=v_x$. On the other hand, we have

$$\left\|D_t^\varepsilon\left(\frac{1}{\omega_y}\right)\right\|_{L^\infty(\mathbb{T}\times(0,1)\times[0,T])},\left\|\frac{\omega_x}{\omega_y}\right\|_{L^\infty(\mathbb{T}\times(0,1)\times[0,T])}\leq C\left(m,M,T,B\right). \qquad (4.4.7)$$

Estimate (4.4.7) and equality (4.4.6) lead to:

$$\begin{aligned}
\partial_t\int_{\mathbb{T}\times(0,1)}\frac{|\omega-\omega^\varepsilon|^2}{\omega_y}dxdy&\leq C\left(m,M,T,B\right)\\
&\quad\times\left(\int_{\mathbb{T}\times(0,1)}\left(|\omega-\omega^\varepsilon|^2+|u-u^\varepsilon|^2\right)dxdy+\varepsilon^4\right).
\end{aligned}$$

By Poincare's inequality

$$\begin{aligned}\int_{\mathbb{T}\times(0,1)} |u-u^\varepsilon|^2 dxdy &\leq C\int_{\mathbb{T}\times(0,1)} |\partial_y(u-u^\varepsilon)|^2 dxdy\\ &= C\int_{\mathbb{T}\times(0,1)} |\omega-\omega^\varepsilon-\varepsilon^2 v_x^\varepsilon|^2 dxdy\\ &= C\int_{\mathbb{T}\times(0,1)} \left(|\omega-\omega^\varepsilon|^2+\varepsilon^4|v_x^\varepsilon|^2-2\varepsilon^2 v_x^\varepsilon(\omega-\omega^\varepsilon)\right) dxdy\\ &\leq \frac{1}{2}\int_{\mathbb{T}\times(0,1)} |u-u^\varepsilon|^2 dxdy + C\int_{\mathbb{T}\times(0,1)} (|\omega-\omega^\varepsilon|^2+\varepsilon^4 u_{xx}^2) dxdy,\end{aligned}$$

where we have used that

$$\begin{aligned}-2\varepsilon^2\int_{\mathbb{T}\times(0,1)} (\omega-\omega^\varepsilon)v_x^\varepsilon dxdy &= -2\varepsilon^2\int_{\mathbb{T}\times(0,1)} (\partial_y(u-u^\varepsilon)+\varepsilon^2 v_x^\varepsilon)v_x^\varepsilon dxdy\\ &= -2\varepsilon^4\int_{\mathbb{T}\times(0,1)} |v_x^\varepsilon|^2 dxdy - 2\varepsilon^2\int_{\mathbb{T}\times(0,1)} u_{xx}^\varepsilon(u-u^\varepsilon)dxdy\\ &= -2\int_{\mathbb{T}\times(0,1)} \left(\varepsilon^4|v_x^\varepsilon|^2 dxdy+\varepsilon^2|\partial_x(u-u^\varepsilon)|^2\right) dxdy\\ &\quad + 2\varepsilon^2\int_{\mathbb{T}\times(0,1)} u_{xx}(u-u^\varepsilon)dxdy.\end{aligned}$$

Finally, we have obtained:

$$\partial_t\int_{\mathbb{T}\times(0,1)} \frac{|\omega-\omega^\varepsilon|^2}{\omega_y} dxdy \leq C(m,M,T,B)\left(\int_{\mathbb{T}\times(0,1)} \frac{|\omega-\omega^\varepsilon|^2}{\omega_y}+\varepsilon^4\right),$$

which concludes the proof of Lemma 1 by applying Grönwall's inequality.

□

Proposition 2. *Let $s\geq 4$, $m>0$, $\omega_0^\varepsilon\in H^s(\mathbb{T}\times(0,1))$ such that $m<\partial_y\omega_0^\varepsilon<\frac{1}{m}$. Let ω^ε be the solution to 4.4.3, then:*

$$\partial_t\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))} \leq C(1+\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))})\|\omega^\varepsilon\|^2_{H^s(\mathbb{T}\times(0,1))}.$$

Proof. Let us point out first that the global existence in $H^s(\mathbb{T}\times(0,1))$ of the solution to (4.4.3) is well known due to the standard theory of the $2D$ Euler equations, the main point is that we would like to establish a uniform bound with respect to ε for the growth of $\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}$. Let us denote by $\partial^\alpha=\partial^{(\alpha_1,\alpha_2)}=\partial_x^{\alpha_1}\partial_y^{\alpha_2}$ and $D_t^\varepsilon=\partial_t+u^\varepsilon\partial_x+v^\varepsilon\partial_y$. Now for $\alpha_1<s$ and $\alpha_2\leq s$ such that $\alpha_1+\alpha_2\leq s$ we have:

$$-D_t^\varepsilon\left(\partial^\alpha\omega^\varepsilon\right)=\sum_{\beta<\alpha_1+\alpha_2}\binom{\alpha}{\beta}\left(\partial^{\alpha-\beta}u^\varepsilon\partial^\beta\omega_x^\varepsilon+\partial^{\alpha-\beta}v^\varepsilon\partial^\beta\omega_y^\varepsilon\right).$$

Thus

$$\begin{aligned}&\partial_t \int_{\mathbb{T}\times(0,1)} |\partial^\alpha \omega^\varepsilon|^2 \, dxdy \\ &= 2 \sum_{\beta<\alpha_1+\alpha_2} \int_{\mathbb{T}\times(0,1)} \partial^\alpha \omega^\varepsilon \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} u^\varepsilon \partial^\beta \omega_x^\varepsilon + \partial^{\alpha-\beta} v^\varepsilon \partial^\beta \omega_y^\varepsilon\right) dxdy.\end{aligned}$$

It is easy to see that we will have either

$$\|\partial^{\alpha-\beta} u^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \|u^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))},$$

or

$$\|\partial^\beta \omega_x^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}.$$

Analogously we will have either

$$\|\partial^{\alpha-\beta} v^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \|v^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))},$$

or

$$\|\partial^\beta \omega_y^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}.$$

Moreover, it is easy to see that the following estimates hold:

$$\begin{aligned}\|u^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))} &\leq C(s)\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}, \\ \|\partial^\alpha v^\varepsilon\|_{L^2(\mathbb{T}\times(0,1))} &\leq C(s)\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}, \text{ for } \alpha \neq (s,0).\end{aligned}$$

By applying the above estimates, we get:

$$\partial_t \int_{\mathbb{T}\times(0,1)} |\partial^\alpha \omega^\varepsilon|^2 \, dxdy \leq C(s)\|\omega^\varepsilon\|^3_{H^s(\mathbb{T}\times(0,1))}.$$

The material derivatives of $\frac{1}{\omega_y^\varepsilon}$ and $\partial_x^s \omega^\varepsilon$ are given by:

$$\begin{aligned}D_t^\varepsilon \left(\frac{1}{\omega_y^\varepsilon}\right) &= \frac{u_y^\varepsilon \omega_x^\varepsilon - u_x^\varepsilon \omega_y^\varepsilon}{\omega_y^\varepsilon} \frac{1}{\omega_y^\varepsilon}, \\ D_t^\varepsilon \left(\partial_x^s \omega^\varepsilon\right) &= -\left(\partial_x^s u^\varepsilon \omega_x^\varepsilon + \partial_x^s v^\varepsilon \omega_y^\varepsilon + \sum_{0<k<s} \binom{s}{k} \left(\partial_x^{s-k} u^\varepsilon \partial_x^k \omega_x^\varepsilon + \partial_x^{s-k} v^\varepsilon \partial_x^k \omega_y^\varepsilon\right)\right),\end{aligned}$$

from which we get:

$$\begin{aligned}&\partial_t \int_{\mathbb{T}\times(0,1)} \frac{|\partial_x^s \omega^\varepsilon|^2}{\omega_y^\varepsilon} dxdy \\ &= \int_{\mathbb{T}\times(0,1)} -2\partial_x^s \omega^\varepsilon \left(\partial_x^s u^\varepsilon \frac{\omega_x^\varepsilon}{\omega_y^\varepsilon} - \frac{1}{2}\partial_x^s \omega^\varepsilon D_t^\varepsilon \left(\frac{1}{\omega_y^\varepsilon}\right)\right) dxdy \\ &- 2\int_{\mathbb{T}\times(0,1)} \partial_x^s \omega^\varepsilon \left(\partial_x^s v^\varepsilon + \sum_{0<k<s} \binom{s}{k} \left(\partial_x^{s-k} u^\varepsilon \partial_x^k \omega_x^\varepsilon + \partial_x^{s-k} v^\varepsilon \partial_x^k \omega_y^\varepsilon\right)\right) dxdy.\end{aligned}$$

The last term is bounded by:

$$\int_{\mathbb{T}\times(0,1)} \partial_x^s \omega^\varepsilon \left(\partial_x^{s-k} u^\varepsilon \partial_x^k \omega_x^\varepsilon + \partial_x^{s-k} v^\varepsilon \partial_x^k \omega_y^\varepsilon \right) dxdy \leq C(s) \|\omega^\varepsilon\|^3_{H^s(\mathbb{T}\times(0,1))}.$$

On the other hand, the highest order term is equal to:

$$\begin{aligned} \int_{\mathbb{T}\times(0,1)} \partial_x^s \omega^\varepsilon \partial_x^s v^\varepsilon dxdy &= \int_{\mathbb{T}\times(0,1)} \partial_x^s u_y^\varepsilon \partial_x^s v^\varepsilon - \varepsilon^2 \partial_x^s v^\varepsilon \partial_x^s v_x^\varepsilon dxdy \\ &= 0. \end{aligned}$$

Finally, the other terms are bounded by:

$$\begin{aligned} &\int_{\mathbb{T}\times(0,1)} -2\partial_x^s \omega^\varepsilon \left(\partial_x^s u^\varepsilon \frac{\omega_x^\varepsilon}{\omega_y^\varepsilon} - \frac{1}{2} \partial_x^s \omega^\varepsilon D_t^\varepsilon \left(\frac{1}{\omega_y^\varepsilon} \right) \right) dxdy \\ &\leq C(s) \left(\frac{\|\omega^\varepsilon\|^3_{H^s(\mathbb{T}\times(0,1))}}{m} + \frac{\|\omega^\varepsilon\|^4_{H^s(\mathbb{T}\times(0,1))}}{m^2} \right), \end{aligned}$$

from which we obtain:

$$\partial_t \int_{\mathbb{T}\times(0,1)} |\partial^\alpha \omega^\varepsilon|^2 \, dxdy \leq C(s,m) \left(1 + \|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))} \right) \|\omega^\varepsilon\|^3_{H^s(\mathbb{T}\times(0,1))},$$

which yields uniform bounds for $\|\omega^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))}$. This proves Proposition 2.

□

Now we are ready to prove the convergence result.

Theorem 8. *Set $s \geq 4$ and $s' < s$. Let $\omega_0^\varepsilon \in H^s(\mathbb{T}\times(0,1))$ and ω^ε be the solution to 4.4.2. Suppose that $\sup_{\varepsilon>0} \|\omega_0^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))} \leq M$, $\omega_0 \in H^s(\mathbb{T}\times(0,1))$ and that $m < \partial_y \omega_0 < \frac{1}{m}$. Let ω be the solution to (4.1.3) that satisfies $\sup_{t\in[0,T']} \|\omega\|_{H^s(\mathbb{T}\times(0,1))} \leq N$, then there exists a positive time T such that:*

$$\sup_{t\in[0,T]} \|\omega - \omega^\varepsilon\|_{H^{s'}(\mathbb{T}\times(0,1))} \leq C(m,s,s',T,M,N) \left(\|\omega_0 - \omega_0^\varepsilon\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))} + \varepsilon^{2\left(1-\frac{s'}{s}\right)} \right).$$

Proof. By Proposition 2, there exists a positive time T such that:

$$\sup_{t\in[0,T]} \sup_{\varepsilon>0} \|\omega_0^\varepsilon\|_{H^s(\mathbb{T}\times(0,1))} \leq C(M,s,T).$$

By the interpolation inequality in Sobolev spaces, we have:

$$\begin{aligned} \|\omega - \omega^\varepsilon\|_{H^{s'}(\mathbb{T}\times(0,1))} &\leq C(s) \|\omega - \omega^\varepsilon\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))} \|\omega - \omega^\varepsilon\|^{\frac{s'}{s}}_{H^s(\mathbb{T}\times(0,1))} \\ &\leq C(m,s,s',M,N,T) \left(\|\omega_0 - \omega_0^\varepsilon\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))} \varepsilon^{2\left(1-\frac{s'}{s}\right)} \right), \end{aligned}$$

where we have applied Lemma 1, which completes the proof of Theorem 8.

□

4.5. Stability under the local Rayleigh condition

In this section we will present the stability result in Sobolev spaces obtained by Masmoudi and Wong [20].

Theorem 9. *Let $s \geq 4$ and $\omega_0, \omega_0' \in H^s(\mathbb{T} \times (0,1))$ such that $2m < \partial_y \omega_0, \partial_y \omega_0' < \frac{1}{2m}$. Let ω and ω' be the unique solutions to 4.1.4 with initial condition ω_0 and ω_0' respectively. Let $T > 0$ be a positive time such that $\|\omega\|_{H^s(\mathbb{T}\times(0,1))} + \|\omega'\|_{H^s(\mathbb{T}\times(0,1))} \leq M$ and $m < \partial_y \omega, \partial_y \omega' < \frac{1}{m}$, then if $s' < s$:*

$$\sup_{t\in[0,T]} \|\omega - \omega'\|_{H^{s'}(\mathbb{T}\times(0,1))} \leq C(m, M, s, s', T)\|\omega_0 - \omega_0'\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))}$$

Proof. Analogously to the proof of Lemma 1, we can first get a bound of the form:

$$\sup_{t\in[0,T]} \|\omega - \omega'\|_{L^2(\mathbb{T}\times(0,1))} \leq C(m, M, s, s', T)\|\omega_0 - \omega_0'\|_{L^2(\mathbb{T}\times(0,1))}.$$

Secondly, by the interpolation inequality in Sobolev spaces we have:

$$\begin{aligned}\|\omega - \omega'\|_{H^{s'}(\mathbb{T}\times(0,1))} &\leq C(s)\|\omega - \omega'\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))}\|\omega - \omega'\|^{\frac{s'}{s}}_{H^s(\mathbb{T}\times(0,1))} \\ &\leq C(m, s, s', M, T)\|\omega_0 - \omega_0'\|^{1-\frac{s'}{s}}_{L^2(\mathbb{T}\times(0,1))},\end{aligned}$$

which ends the proof of Theorem 9.

□

4.6. Singularity formation without the Rayleigh condition

In this section we will present the finite time blow-up for the hydrostatic Euler equations (4.1.3) and for the Prandtl equations (4.6.3). More precisely, in Theorem 10 and 12 we will show the finite time blow-up results for the hydrostatic Euler equations obtained by Wong [18] and by Cao, Ibrahim, Nakanishi and Titi [3] (originally presented for the inviscid primitive equations without rotation) respectively. In Theorem 11, we will prove the singularity formation result for the Prandtl equations obtained by E and Engquist [7].

It is worth mentioning that certain local conditions such as the local Rayleigh condition $\partial_y^2 u > 0$ for the hydrostatic Euler equations (4.1.3) and the monoticity condition $\partial_y u > 0$ for the Prandtl equations (4.6.3) prevents some scenarios that lead to the formation of singularities, for instance: the assumption $\partial_y u_0(x_0, y) \equiv 0$ in Theorem 10 and the parity assumption in Theorems 11 and 12.

Let us first prove the following blow-up result by Wong [18]:

Theorem 10. *Let $u_0(x,y)$ be a smooth function in $\mathbb{T}\times(0,1)$ satisfying*

- $u_0(x_0,y)=0, \qquad \forall y\in(0,1),$
- $\partial_y\partial_x u_0(x_0,0)=0,$
- $\partial_x\partial_y^2 u_0(x_0,y)<0, \quad \forall y\in(0,1),$

for a certain point $x_0\in\mathbb{T}$. Let u be a smooth solution to (4.1.3), then there exists a time $T^*>0$ such that either $P_{xx}(x,T^*)$ becomes discontinuos or $|u_x(\tilde{X}(t),1,t)|\to\infty$ as $t\to T^*$, where $\tilde{X}(t)$ is the x component of the characteristic curve starting at $(x_0,1)$. Moreover, $T^*\leq\frac{-3}{\partial_x u_0(x_0,1)}$.

Remark 2. *Let us point out that the local existence and uniqueness of the problem (4.1.3) was established by Kukavica, Temam, Vicol and Ziane [17] in the analytic setting. On the other hand, Theorem 10 establishes the finite time blow-up of certain smooth solutions to (4.1.3) for initial data that may be chosen to be analytic. Similar results for the 3D primitive equations with rotation were obtained by Ghoul, Ibrahim, Lin and Titi [9] and by Ibrahim, Lin and Titi [15].*

In order to prove the above Theorem, we will need the following Lemma (see [19]).

Lemma 2. *Let $f\in C^2([0,1])$ be a scalar function satisfying the following conditions:*

- $f'(0)=0.$
- $f''>0.$
- $\int_0^1 f(y)dy=0.$

Then, $f(1)>0$ and

$$\int_0^1 f^2(y)dy\leq\frac{1}{3}f^2(1).$$

Now let us prove Theorem 10:

Proof of Theorem 10. Let $(X(x_0,y,t),Y(x_0,y,t))$ be the characteristic curve starting at (x_0,y) (4.1.3), that is, $(X(x_0,y,t),Y(x_0,y,t))$ solve the following equations:

$$\begin{cases}\dfrac{d}{dt}X(x_0,y,t)=u(X(x_0,y,t),Y(x_0,y,t),t), & t\in(0,T),\\ X(x_0,y,0)=x_0,\\ \dfrac{d}{dt}Y(x_0,y,t)=v(X(x_0,y,t),Y(x_0,y,t),t), & t\in(0,T),\\ Y(x_0,y,0)=y.\end{cases}\tag{4.6.1}$$

Let $\tilde{X}_y(t) = X(x_0, y, t)$ be the solution to:

$$\begin{cases} \dfrac{d^2}{dt^2}\tilde{X}_y = -P_x(\tilde{X}_y, t), & t \in (0,T), \\ \tilde{X}_y(0) = x_0, & \\ \dfrac{d}{dt}\tilde{X}_y(0) = u(X(x_0,y,0), Y(x_0,y,0,t)) = u_0(x_0,y) = 0. & \end{cases} \tag{4.6.2}$$

Assume that $P_{xx}(x,t)$ remains continuous, then by the standard existence and uniqueness theorem for second order nonlinear differential equations we deduce that $\tilde{X}_y(t)$ is y-independent (and so $u(\tilde{X}_y(t), Y(x_0,y,t), t) = \frac{d}{dt}\tilde{X}_y(t)$ is y-independent) because they solve the same equation with the same initial data. Let us denote by $\tilde{X}(t)$ the solution to (4.6.2). Set $f(y,t) = -u_x(\tilde{X}(t), y, t)$, let us show that f satisfies the hypothesis of Lemma 2; The first condition follows directly from $\partial_y f(0,t) = -\omega_x(\tilde{X}(t), 0, t) = 0$. The third condition is a direct consequence of the boundary value of v: $\int_0^1 f(y)dy = v|_{y=0}^{y=1} = 0$. Finally, let us verify the second condition: First observe that $D_t(\omega_x) = -u_x\omega_x - v_x\omega_y$, which implies $\partial_t\partial_y f(0,t) = u_x(\tilde{X}(t),0,t)\partial_y f(0,t)$, from which we get $\partial_y f(0,t) = 0$, thanks to $\partial_y f(0,0) = 0$. On the other hand, we have $\partial_y^2 f(y,t) = -\omega_{xy}(\tilde{X}(t), y, t)$. Furthermore, the material derivative of ω_{xy} is equal to:

$$D_t(\omega_{xy}) = u_{xx}\omega_y - \omega_x^2 - \omega\omega_{xx} - v_x\omega_{yy},$$

moreover, since $u(\tilde{X}(t), y, t)$ is y-independent, its vertical derivatives will be identically 0 i.e. $u_y(\tilde{X}(t), y, t) = \omega(\tilde{X}(t), y, t) = \omega_y(\tilde{X}(t), y, t) = \omega_{yy}(\tilde{X}(t), y, t) = 0$, which implies:

$$\begin{aligned} \frac{d}{dt}\omega_{xy}(\tilde{X}(t), Y(x_0,y,t), t) &= -\omega_x^2(\tilde{X}(t), Y(x_0,y,t), t) \\ &\leq 0, \end{aligned}$$

for all $y \in (0,1)$. Thus, $\partial_y^2 f(y,t) > 0$, since $\partial_y^2 f(y,0) > 0$. Applying Lemma 2, we get:

$$\begin{aligned} \int_0^1 f^2(y,t)dy &\leq \frac{1}{3}f^2(1,t) \\ &= \frac{1}{3}u_x^2(\tilde{X}(t), 1, t). \end{aligned}$$

Set $g(t) = -u_x(\tilde{X}(t), 1, t)$, then the time derivative of g is given by:

$$
\begin{aligned}
\frac{dg}{dt} &= g^2(t) + P_{xx}(\tilde{X}(t), t) \\
&= g^2(t) - 2\int_0^1 \left(u u_{xx}(\tilde{X}(t), y, t) + u_x^2(\tilde{X}(t), y, t) \right) dy \\
&= g^2(t) - 2\frac{d\tilde{X}}{dt}\int_0^1 u_{xx}(\tilde{X}(t), y, t) dy - 2\int_0^1 f^2(y, t) dy \\
&\geq g^2(t) - \frac{2}{3} g^2(t) \\
&= \frac{1}{3} g^2(t),
\end{aligned}
$$

dividing by g^2 and integrating in $t \in [0, T]$, we get

$$
\begin{aligned}
\int_0^T \frac{g'(t)}{g^2(t)} dt &= \frac{1}{g(0)} - \frac{1}{g(T)} \\
&\geq \frac{T}{3},
\end{aligned}
$$

from which we obtain:

$$
\begin{aligned}
g(T) &\geq \frac{1}{\frac{1}{g(0)} - \frac{T}{3}} \\
&= \frac{3g(0)}{3 - g(0)T},
\end{aligned}
$$

which concludes the proof of Theorem 10.

□

Let us introduce the framework of the blow-up result by E and Engquist [7] for the Prandtl equations:

We will consider the Prandtl equations in $\mathbb{R} \times (0, \infty)$:

$$
\begin{cases}
u_t + u u_x + v u_y + P_x(x, t) = \partial_y^2 u, \\
u_x + v_y = 0, \\
\lim\limits_{y \to \infty} u(x, y, t) = U(x, t), \\
u, v|_{y=0} = 0, \\
u|_{t=0}(x, y) = u_0(x, y),
\end{cases}
\tag{4.6.3}
$$

where U and P are prescribed in $(0, T)$ and related by Bernoulli's law:

$$
\partial_t U + \partial_x \left(\frac{U^2}{2}(x, t) + P(x, t) \right) = 0, \qquad \text{in } t \in (0, T).
$$

If we assume that the initial condition u_0 is odd in x and u solves (4.6.3) with $U = P = 0$, then u admits the following representation: $u(x, y, t) = x\tilde{b}(x, y, t)$, since the oddness of u is propagated in time. Moreover, if we define b by

$$b(y, t) = -u_x(0, y, t), \tag{4.6.4}$$

then b satisfies the following evolution equation in $\mathbb{R} \times (0, T)$:

$$\begin{cases} b_t - b^2 + b_y \displaystyle\int_0^y b(z, t)dz = \partial_y^2 b, \\ \qquad\qquad b(0, t) = 0, \\ \qquad\qquad b|_{t=0} = b_0(y). \end{cases} \tag{4.6.5}$$

The following proposition will be useful to show the finite time blow up result.

Proposition 3. *Let u be an odd smooth solution to (4.6.3). Assume that $-\partial_x u_0(0, y)$ is nonnegative and compactly supported. Define b by (4.6.4) and E by*

$$E = \int_0^\infty \left(\frac{1}{4} b^3 - \frac{1}{2} b_y^2 \right) dy. \tag{4.6.6}$$

Then, as long as b remains bounded the following identities hold:

$$\partial_t \int_0^\infty b^2 dy = -2 \int_0^\infty b_y^2 dy + 3 \int_0^\infty b^3 dy, \tag{4.6.7}$$

$$\partial_t \int_0^\infty \frac{1}{2} b_y^2 dy = -\int_0^\infty b_{yy}^2 dy + \frac{3}{2} \int_0^\infty b_y^2 b dy, \tag{4.6.8}$$

$$\partial_t \frac{1}{3} \int_0^\infty b^3 dy = -2 \int_0^\infty b b_y^2 b dy + \frac{4}{3} \int_0^\infty b^4 dy, \tag{4.6.9}$$

$$\partial_t E = \int_0^\infty \left(\left(b_{yy} + \frac{3}{4} b^2 \right)^2 + \frac{7}{16} b^4 \right) dy. \tag{4.6.10}$$

Proof. By integration by parts,

$$\begin{aligned} \partial_t \int_0^\infty b^2 dy &= 2 \int_0^\infty b b_{yy} + \int_0^\infty \left(2b^3 - 2 b b_y \int_0^y b(z) dz \right) dy \\ &= -2 \int_0^\infty b_y^2 dy + 3 \int_0^\infty b^3 dy, \end{aligned}$$

which proves (4.6.7). The proof of (4.6.9) is completely analogous. The y derivative of the first equation in (4.6.5) is given by:

$$b_{yt} - b b_y - b_{yy} \int_0^y b(z, t) dz = \partial_y^3 b,$$

from which we get (4.6.8) after applying integration by parts. Finally, by (4.6.9) and (4.6.8) :

$$\begin{aligned}\partial_t E &= \int_0^\infty b_{yy}^2 dy - \frac{3}{2}\int_0^\infty b_y^2 b dy - \frac{3}{2}\int_0^\infty b b_y^2 b dy + \int_0^\infty b^4 dy \\ &= \int_0^\infty b_{yy}^2 - 3b_y^2 b + b^4 dy = \int_0^\infty b_{yy}^2 + \frac{3}{2} b_{yy} b^2 + b^4 dy \\ &= \int_0^\infty \left(\left(b_{yy} + \frac{3}{4} b^2 \right)^2 + \frac{7}{16} b^4 \right) dy,\end{aligned}$$

thanks to the identity:

$$\int_0^\infty b b_y^2 dy = -\int_0^\infty (b^2 b_{yy} + b b_y^2) dy.$$

This completes the proof of Proposition 3. □

Remark 3. *Note that* $\partial_t \int_0^\infty b^2 dy = -2\int_0^\infty b_y^2 dy + 3\int_0^\infty b^3 dy \le 3\|b\|_\infty \int_0^\infty b^2 dy$, *which implies that* $\|b\|_{L^2}$ *stays bounded as long as* b *does. Moreover,* $\partial_t \int_0^\infty b dy = -b_y(0,t) + 2\int_0^\infty b^2 dy$, *from which we deduce that* $\int_0^y b(z) dz$ *will remain bounded as well.*

With the above notation at our disposal, we will present the finite time blow-up of certain smooth solutions to the Prandtl equations:

Theorem 11. *Let* u *be an odd smooth solution to (4.6.3) with* $U = P = 0$. *Assume that* $-\partial_x u_0(0,y)$ *is nonnegative and compactly supported. Let* b *and* E *be defined by (4.6.4) and (4.6.6) respectively. Assume that* $E(0) > 0$, *then there exists a time* T^* *such that either* $\|b\|_\infty \to \infty$ *as* $t \to T^*$ *or* $E(t) \to \infty$ *as* $t \to T^*$.

Proof. Set $\alpha \in (1, \frac{5}{4})$, then:

$$\left(\int_0^\infty b^2 dy \right)^{1+\alpha} \partial_t \left(\frac{E}{\left(\int_0^\infty b^2 dy \right)^\alpha} \right) = \left(\int_0^\infty b^2 dy \partial_t E - \alpha E \partial_t \int_0^\infty b^2 dy \right).$$

The first term is equal to

$$\begin{aligned}\int_0^\infty b^2 dy \partial_t E &= \int_0^\infty b^2 dy \int_0^\infty \left(\left(b_{yy} + \frac{3}{4} b^2 \right)^2 + \frac{7}{16} b^4 \right) dy \\ &\ge \frac{7}{16} \left(\int_0^\infty b^3 dy \right)^2 + \left(\int_0^\infty \left(b_{yy} b + \frac{3}{4} b^3 \right) dy \right)^2 \\ &= \left(\int_0^\infty b^3 dy \right)^2 - \frac{3}{2} \int_0^\infty b^3 dy \int_0^\infty b_y^2 dy + \left(\int_0^\infty b_y^2 dy \right)^2.\end{aligned}$$

The second term is given by:

$$\alpha E\partial_t \int_0^\infty b^2 dy = \frac{3}{4}\alpha \left(\int_0^\infty b^3 dy\right)^2 - 2\alpha \int_0^\infty b^3 dy \int_0^\infty b_y^2 dy + \alpha \left(\int_0^\infty b_y^2 dy\right)^2.$$

Thus

$$\begin{aligned}\partial_t \left(\frac{E}{\left(\int_0^\infty b^2 dy\right)^\alpha}\right) &\geq \left(\frac{5}{4}-\alpha\right)\left(\int_0^\infty b^3 dy\right)^2 + 2\left(\frac{5}{4}-\alpha\right)\int_0^\infty b^3 dy \int_0^\infty b_y^2 dy \\ &\geq 0, \end{aligned} \tag{4.6.11}$$

where we have used that

$$\int_0^\infty b_y^2 dy \leq \frac{1}{2}\int_0^\infty b^3 dy, \tag{4.6.12}$$

thanks to $\partial_t E \geq 0$ and $E(0) > 0$. By (4.6.7), (4.6.6), (4.6.12) and (4.6.11)

$$\begin{aligned}\partial_t \int_0^\infty b^2(t) dy &\geq 12E(t) \\ &\geq 12\left(\frac{E(0)}{\left(\int_0^\infty b_0^2 dy\right)^\alpha}\right)\left(\int_0^\infty b^2(t) dy\right)^\alpha, \end{aligned} \tag{4.6.13}$$

which implies

$$\begin{aligned}\int_0^t \frac{\partial_s \|b(s)\|_2^2}{\|b(s)\|_2^{2\alpha}} ds &= \frac{1}{\alpha-1}\left(\frac{1}{\|b_0\|_2^{2(\alpha-1)}} - \frac{1}{\|b(t)\|_2^{2(\alpha-1)}}\right) \\ &\geq 12\left(\frac{E(0)}{\left(\int_0^\infty b_0^2 dy\right)^\alpha}\right) t, \end{aligned}$$

from which we get:

$$\left(\frac{1}{\|b_0\|_2^{2(\alpha-1)}} - 12(\alpha-1)\left(\frac{E(0)}{\left(\int_0^\infty b_0^2 dy\right)^\alpha}\right) t\right)^{-1} \leq \|b(t)\|_2^{2(\alpha-1)}, \tag{4.6.14}$$

which yields the finite time blow-up of $E(t)$ if $\|b(t)\|_\infty$ remains bounded(thanks to (4.6.13)) . This concludes the proof of Theorem 11.

□

Finally, we will introduce the necessary background to prove the finite time blow-up result by Cao, Ibrahim, Nakanishi and Titi [3] for the inviscid primitive equations. Let us consider the inviscid primitive equations without rotation in $\mathbb{R}^2 \times (0, H)$:

$$\begin{cases} u_t + uu_x + vu_y + wu_z + P_x = 0, \\ v_t + uv_x + vv_y + wv_z + P_y = 0, \\ u_x + v_y + w_z = 0, \\ P_z + T = 0, \\ T_t + uT_x + vT_y + wT_z = k_1\left(\partial_x^2 T + \partial_y^2 T\right) + k_2\partial_z^2 T, \\ w|_{z=0,H} = 0, \\ u|_{t=0}(x,y) = u_0(x,y), \end{cases} \tag{4.6.15}$$

where (u, v, w) is the velocity vector field, P is the scalar pressure and T is the temperature. If the initial temperature is given by a constant T_0 and the initial velocity field is y-independent, then equations (4.6.15) are reduced to:

$$\begin{cases} u_t + uu_x + wu_z + P_x = 0, \\ u_x + w_z = 0, \\ P_z + T_0 = 0, \\ w|_{z=0,H} = 0, \\ u|_{t=0}(x,y) = u_0(x,y), \end{cases} \tag{4.6.16}$$

which are equivalent to equations (4.1.3) when $T_0 \equiv 0$. If we assume that u_0 is odd in x, then the oddness will be propagated by any solution u to (4.6.16). Moreover, the x derivative of the first equation in (4.6.16) is given by:

$$u_{xt} + uu_{xx} + u_x^2 + vu_{xy} + v_x u_y + P_{xx} = 0, \tag{4.6.17}$$

where $P_{xx} = -2\int_0^1 (uu_{xx}(x,z,t) + u_x^2(x,z,t))dz$ (see (4.1.12)), which is reduced to the following equation in $x = 0$:

$$\begin{aligned} 0 &= u_{xt} + u_x^2 + vu_{xy} + P_{xx} \\ &= u_{xt} + u_x^2 - \int_0^y u_x(0,z,t)dz u_{xy} - 2\int_0^1 u_x^2(x,z,t)dz, \end{aligned}$$

since the oddness of u is propagated in time. Define $b(y,t)$ by

$$b(y,t) = u_x(0,y,t), \tag{4.6.18}$$

then b satisfies:

$$\begin{cases} b_t + b^2 - b_y \displaystyle\int_0^y b(z,t)dz - 2\int_0^1 b^2(z,t)dz = 0, & \text{in } (0,1)\times(0,T), \\ \displaystyle\int_0^1 b(z,t)dz = 0, \\ b|_{t=0} = b_0(y). \end{cases} \tag{4.6.19}$$

Moreover, if we assume that $b(y,t) = \frac{a(y)}{1-t}$, then a satisfies:

$$\begin{cases} a(y) + a^2(y) - \dfrac{da}{dy}\displaystyle\int_0^y a(z)dz - 2\int_0^1 a^2(z)dz = 0, \quad y \in (0,1), \\ \displaystyle\int_0^1 a(z)dz = 0. \end{cases} \tag{4.6.20}$$

Now, let us state the singularity formation result showed in [3]:

Theorem 12. *Let $u_0 \in C^\infty(\mathbb{T}\times(0,1))$ be an odd function in x. Assume that $\partial_x u_0(0,y)$ solves (4.6.20). Let u be a smooth solution to (4.1.3), then u_x blows up in finite time.*

In order to prove Theorem 12, we will first show the following two propositions:

Proposition 4. *Let $b, \tilde{b} \in L^2((0,T);H^2(0,1))$ be solutions to (4.6.19) with the same initial data $b_0 \in H^2(0,1)$, then $v = w$ in $(0,T)$.*

Proof. First note that:

$$\begin{aligned} &\partial_t \int_0^1 |b-\tilde{b}|^2 dy \\ &= 2\int_0^1 (b-\tilde{b})\left(\tilde{b}^2 - b^2 + b_y\int_0^y (b-\tilde{b})dz + (b_y - \tilde{b}_y)\int_0^y \tilde{b}dz\right)dy \\ &+ 4\int_0^1 (b-\tilde{b})\int_0^1 (\tilde{b}^2 - b^2)dzdy \\ &= \int_0^1 |b-\tilde{b}|^2(2b+\tilde{b}) + 4(b-\tilde{b})\int_0^1 (\tilde{b}^2 - b^2)dz + 2(b-\tilde{b})b_y\int_0^y (b-\tilde{b})dzdy \\ &\le \int_0^1 |b-\tilde{b}|^2 dy(6\|b\|_{L^\infty} + 5\|\tilde{b}\|_{L^\infty} + 2\|b_y\|_{L^\infty}) \\ &\le 8\int_0^1 |b-\tilde{b}|^2 dy(\|\tilde{b}\|_{H^2} + \|b\|_{H^2}). \end{aligned}$$

By Grönwall's inequality :

$$\int_0^1 |b-\tilde{b}|^2 dy(t=t') \leq \int_0^1 |b-\tilde{b}|^2 dy|_{t=0} \exp\left(8\int_0^{t'} \|\tilde{b}\|_{H^2} + \|b\|_{H^2} dt\right)$$
$$= 0.$$

This concludes the proof of Proposition 4. □

Proposition 5. *Problem 4.6.19 admits a solution of the form*

$$b = \frac{a(y)}{1-t}. \tag{4.6.21}$$

Proof. By plugging (4.6.21) into (4.6.19), we get:

$$\begin{cases} a(y) + a^2(y) - \dfrac{da}{dy}\displaystyle\int_0^y a(z)dz - 2\int_0^1 a^2(z)dz = 0, \quad y \in (0,1), \\ \displaystyle\int_0^1 a(z)dz = 0. \end{cases} \tag{4.6.22}$$

Define I by:

$$I = \sqrt{\frac{1}{4} + 2\int_0^1 a^2(z)dz}, \tag{4.6.23}$$

then:

$$\left(a + \frac{1}{2} + I\right)\left(a + \frac{1}{2} - I\right) = \frac{da}{dy}\int_0^y a(z)dz. \tag{4.6.24}$$

In order to solve (4.6.24) by separation of variables, we will assume that $|a+\frac{1}{2}| \leq I$. Moreover, if we consider $\int_0^y a(z)dz > 0$ for $y \in (0,1)$, then (4.6.24) implies $\frac{da}{dy} < 0$. Thus, a provides a decreasing bijection from $(0,1)$ to $(-\frac{1}{2}-I, -\frac{1}{2}+I)$. Furthermore, the bijection a satisfies:

$$\frac{da}{dy}\frac{1}{\left(a+\frac{1}{2}+I\right)\left(a+\frac{1}{2}-I\right)} = \frac{1}{\int_0^y a(z)dz},$$

which implies:

$$\frac{da}{dy}\frac{1}{2I}\left(\frac{-\frac{1}{2}+I}{a+\frac{1}{2}-I} + \frac{\frac{1}{2}+I}{a+\frac{1}{2}+I}\right) = \frac{a(y)}{\int_0^y a(z)dz},$$

which is equivalent to:

$$\frac{d}{dy}\left(\log\left(\left|a+\frac{1}{2}-I\right|^{\frac{1}{2}-\frac{1}{4I}}\left|a+\frac{1}{2}+I\right|^{\frac{1}{2}+\frac{1}{4I}}\right) - \log\left(\int_0^y a(z)dz\right)\right) = 0,$$

from which we get

$$\int_0^y a(z)dz = C\left|a+\frac{1}{2}-I\right|^{\frac{1}{2}-\frac{1}{4I}}\left|a+\frac{1}{2}+I\right|^{\frac{1}{2}+\frac{1}{4I}}, \tag{4.6.25}$$

for a certain constant C to be determined. By plugging (4.6.25) into (4.6.22), we obtain:

$$\left(a+\frac{1}{2}-I\right)\left(a+\frac{1}{2}+I\right)=\frac{da}{dy}C\left|a+\frac{1}{2}-I\right|^{\frac{1}{2}-\frac{1}{4I}}\left|a+\frac{1}{2}+I\right|^{\frac{1}{2}+\frac{1}{4I}},$$

which yields:

$$\frac{dy}{da}=-C\left|a+\frac{1}{2}-I\right|^{-\frac{1}{2}-\frac{1}{4I}}\left|a+\frac{1}{2}+I\right|^{-\frac{1}{2}+\frac{1}{4I}},$$

from which we obtain:

$$\begin{aligned}y(a)&=\int_{-\frac{1}{2}+I}^{a}\frac{dy}{da'}da'\\&=-C\int_{-\frac{1}{2}+I}^{a}\left|a'+\frac{1}{2}-I\right|^{-\frac{1}{2}-\frac{1}{4I}}\left|a'+\frac{1}{2}+I\right|^{-\frac{1}{2}+\frac{1}{4I}}da'.\end{aligned}$$

In order to compute C, we can use the monoticity of a (and its inverse) and the change of variable $2Is=a+\frac{1}{2}+I$:

$$\begin{aligned}y\left(-\frac{1}{2}-I\right)&=1\\&=-C\int_{-\frac{1}{2}+I}^{-\frac{1}{2}-I}\left|a+\frac{1}{2}-I\right|^{-\frac{1}{2}-\frac{1}{4I}}\left|a+\frac{1}{2}+I\right|^{-\frac{1}{2}+\frac{1}{4I}}da\\&=C\int_0^1 s^{\frac{1}{2}+\frac{1}{4I}-1}(1-s)^{\frac{1}{2}-\frac{1}{4I}-1}ds=CB\left(\frac{1}{2}+\frac{1}{4I},\frac{1}{2}-\frac{1}{4I}\right),\end{aligned}$$

where B denotes the beta function. Thus y is given by:

$$y(a)=\frac{1}{B\left(\frac{1}{2}+\frac{1}{4I},\frac{1}{2}-\frac{1}{4I}\right)}\int_{\frac{a}{2I}+\frac{1}{2}+\frac{1}{4I}}^{1}s^{\frac{1}{2}+\frac{1}{4I}-1}(1-s)^{\frac{1}{2}-\frac{1}{4I}-1}ds,$$

which defines a implicitely. Finally, by construction $b(y,t)=\frac{a(y)}{1-t}$ solves problem (4.6.19).

□

Finally, let us prove Theorem 12:

Proof of Theorem 12. Let u be a smooth solution to (4.1.3). Since $\partial_x u_0(0,y)$ solves (4.6.20), $\frac{\partial_x u_0(0,y)}{1-t}$ solves (4.6.19). Moreover, by Proposition 4 $u(0,y,t)\equiv\frac{\partial_x u_0(0,y)}{1-t}$, because $u_x(0,y,t)$ and $\frac{\partial_x u_0(0,y)}{1-t}$ solve problem (4.6.19) with the same initial data. This completes the proof of Theorem 12.

□

4.7. Semilagrangian equations and Brenier's condition for global solvability

In this section we will introduce the semilagrangian formulation of the hydrostatic Euler equations, which were originally introduced by Brenier in [1]. Moreover, we will deduce the necessary condition for global solvability obtained in the article previously mentioned (see Theorem 13).

Let us consider an initial vorticity ω_0 that satisfies the local Rayleigh condition $\partial_y \omega_0 > 0$ and

$$\omega_0(x,0) \equiv k, \tag{4.7.1}$$
$$\omega_0(x,1) \equiv k+1, \tag{4.7.2}$$

for certain constant k (note that (4.7.1) and (4.7.2) are propagated in time, because ω satisfies a transport equation), then problem (4.1.4) is equivalent to the following problem in $\Omega = \mathbb{T} \times (0,1)$ (see Proposition 6):

$$\begin{cases} v_t + \partial_x \left(\dfrac{v^2}{2} + P \right) = 0, \\ \partial_t h_a + \partial_x (v h_a) = 0, \\ \displaystyle\int_0^1 h_a da = 1, \\ \partial_a P = 0, \\ (\nu, h_a)|_{t=0} = (\nu_0, \partial_a h_0), \end{cases} \tag{4.7.3}$$

where v and h_a are defined by

$$\omega(x, h(x,t,a), t) \equiv k + a, \quad \text{for } a \in [0,1], \tag{4.7.4}$$
$$v(x,t,a) = u(x, h(x,t,a), t), \quad \text{for } a \in [0,1], \tag{4.7.5}$$

in this case we assume that $\partial_a h_0$ is positive, which is equivalent to the validity of the local Rayleigh condition. Equations (4.7.3) are known as the semilagrangian equations. In [1], the local existence of solutions to (4.1.4) was proved in the class C (which is defined as the set of C^1-functions in $\mathbb{T} \times (0,1)$ that satisfy the local Rayleigh condition, (4.7.1) and (4.7.2)).

The semilagrangian equations (4.7.3) have a natural extension to higher dimensions, namely

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla P = 0, \\ \partial_t h_a + \nabla \cdot (\mathbf{v} h_a) = 0, \\ \displaystyle\int_0^1 h_a da = 1, \\ \partial_a P = 0, \\ \partial_i v_j - \partial_j v_i = 0, \qquad \text{for } i, j \leq d, \\ (\mathbf{v}, h_a)|_{t=0} = (\mathbf{v}_0, \partial_a h_0), \end{cases} \tag{4.7.6}$$

where $\mathbf{v} : \mathbb{T}^d \times [0,1] \to \mathbb{R}^d$ is the semilagrangian velocity, P is the scalar pressure and h_a is a positive function in $\mathbb{T}^d \times [0,1]$. Let us point out that the semilagrangian equations (4.7.3) establish a vertical change of coordinates by means of the level curves of the vorticity ω, where the injectivity follows by the local Rayleigh condition. Hence, the extension of the semilagrangian equations to higher dimensions have no clear physical meaning.

Remark 4. *The first equation in (4.7.6), is equivalent to:*

$$\partial_t \mathbf{v} + \nabla \left(\frac{|\mathbf{v}|^2}{2} + P \right) = 0, \tag{4.7.7}$$

thanks to the curl free condition $\partial_i v_j = \partial_j v_i$.

Remark 5. *Since $\partial_i v_j = \partial_j v_i$, there exists a potential ϕ such that*

$$\nabla \phi = \mathbf{v}(x,t,a) - \mathbf{V}(a), \tag{4.7.8}$$

where ϕ satisfies

$$\phi_t + \frac{|\mathbf{v}|^2}{2} + P(x,t) = 0. \tag{4.7.9}$$

Remark 6. *If $(\mathbf{v}, h_a)$ is a smooth solution to (4.7.6), then it is easy to see that the following quantities are time independent:*

$$\mathbf{V}(a) = \int_{\mathbb{T}^d} \mathbf{v}(x,t,a) dx, \tag{4.7.10}$$

$$H_a(a) = \int_{\mathbb{T}^d} h_a(x,t,a) dx. \tag{4.7.11}$$

The following proposition summarizes how the change of variable works between the hydrostatic Euler equations (4.1.4) and the semilagrangian equations (4.7.3).

Proposition 6. *Let ω be a smooth solution to (4.1.4) that satisfies the local Rayleigh condition, (4.7.1) and (4.7.2) in $[0,T]$. Let (v, h_a) be a smooth solution to (4.7.3)*

in $[0,T]$. *Assume that* ω, h, u *and* v *satisfy (4.7.4) and (4.7.5) in* $[0,T]$, *then*

$$h_a(x,t,a) = \frac{1}{\omega_y(x,h(x,t,a),t)}, \tag{4.7.12}$$

$$h_x(x,t,a) = -\frac{\omega_x}{\omega_y}(x,h(x,t,a),t), \tag{4.7.13}$$

$$v_a(x,t,a) = (k+a)h_a, \tag{4.7.14}$$

$$v_x(x,t,a) = \left(u_x - \frac{\omega\omega_x}{\omega_y}\right)(x,h(x,t,a),t), \tag{4.7.15}$$

$$v(x,t,a) = -\frac{k+1}{2} + (k+a)h - \frac{1}{2}\int_0^1 h^2 db + \int_a^1 h db, \tag{4.7.16}$$

$$\mathcal{A}(\omega)(x,h(x,t,a),t) = -v(x,t,a)h(x,t,a) + \frac{1}{2}(k+a)h^2 - \frac{1}{2}\int_0^a h^2(x,t,b)db, \tag{4.7.17}$$

$$h_t(x,t,a) - \partial_x\left(\mathcal{A}(\omega)(x,h(x,t,a),t)\right) = 0, \tag{4.7.18}$$

$$v_t(x,t,a) = \left(u\left(\frac{\omega\omega_x}{\omega_y} - u_x\right)(x,h(x,t,a),t) - P_x(x,t)\right), \tag{4.7.19}$$

where $0 \le t \le T$.

Proof. First observe that (4.7.12), (4.7.13), (4.7.14) and (4.7.15) follow directly by (4.7.4) and (4.7.5). Applying (4.1.8) and the change of variable $z = h(x,t,b)$

$$\begin{aligned} u(x,h,t) &= \int_0^1 z\omega(x,z,t)dz - \int_{h(x,t,a)}^1 \omega(x,z,t)dz \\ &= \int_0^1 (k+b)hh_b db - \int_a^1 (k+b)h_b db \\ &= -\frac{k+1}{2} - \frac{1}{2}\int_0^1 h^2 db + (k+a)h + \int_a^1 h db, \end{aligned}$$

which proves (4.7.16). Next, by (4.1.4), we have $-\mathcal{A}(\omega)(x,y,t) = \int_0^y u(x,z,t)dz$, which implies

$$\begin{aligned} -\mathcal{A}(\omega)(x,h,t) &= \int_0^{h(x,t,a)} u(x,z,t)dz \\ &= \int_0^a v(x,t,b)h_b(x,t,b)db \\ &= v(x,t,a)h(x,t,a) - \int_0^a (k+b)hh_b db \\ &= v(x,t,a)h(x,t,a) - \frac{1}{2}(k+a)h^2 + \frac{1}{2}\int_0^a h^2(x,t,b)db, \end{aligned}$$

which gives (4.7.17). Let us prove (4.7.18). First observe that $h(x,t,a) = \int_0^a h_b db$, because $h(x,0,t) \equiv 0$. Thus, (4.7.18) follows by integrating the second equation of (4.7.3) in $b \in [0,a]$. Finally, by (4.7.18)

$$
\begin{aligned}
v_t &= u_t(x,h,t) + (k+a)h_t \\
&= -uu_x(x,h,t) - v(x,h,t)(k+a) - P_x(x,t) + (k+a)(v(x,h,t) - u(x,h,t)h_x) \\
&= -uu_x(x,h,t) - P_x(x,t) - (k+a)u(x,h,t)h_x,
\end{aligned}
$$

which leads to (4.7.19) thanks to (4.7.13). This concludes the proof of Proposition 6.

□

The next proposition establishes the time independence of $\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}|^2 h_a dxda$ and an explicit expression for the pressure P in (4.7.6).

Proposition 7. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6), then*

$$
\partial_t \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}|^2 h_a dxda = 0, \tag{4.7.20}
$$

$$
P(x,t) = (-\Delta)^{-1}\left(\nabla\cdot\nabla\cdot\int_0^1 (\mathbf{v}\otimes\mathbf{v})h_b db\right). \tag{4.7.21}
$$

Proof. First observe that

$$
\begin{aligned}
\partial_t \int_0^1 h_a da &= -\nabla\cdot\int_0^1 \mathbf{v}h_a da \\
&= 0.
\end{aligned} \tag{4.7.22}
$$

By (4.7.7), (4.7.22) and integration by parts

$$
\begin{aligned}
\partial_t \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}|^2 h_a dxda &= \int_{\mathbb{T}^d\times(0,1)} 2\mathbf{v}\cdot\mathbf{v_t}h_a + |\mathbf{v}|^2\partial_t h_a dxda \\
&= \int_{\mathbb{T}^d\times(0,1)} -\mathbf{v}\cdot\nabla\left(|\mathbf{v}|^2 + 2P\right)h_a - \nabla\cdot(\mathbf{v}h_a)|\mathbf{v}|^2 dxda \\
&= \int_{\mathbb{T}^d\times(0,1)} -\nabla\cdot(\mathbf{v}|\mathbf{v}|^2 h_a) + 2P\nabla\cdot(\mathbf{v}h_a) dxda \\
&= 2\int_{\mathbb{T}^d} P\nabla\cdot\left(\int_0^1 \mathbf{v}h_a da\right) dxda \\
&= 0,
\end{aligned}
$$

which proves (4.7.20). Now, let us prove (4.7.21). First observe that

$$
\begin{aligned}
\partial_t \int_0^1 \mathbf{v}h_a da &= -\int_0^1 \left(\nabla\left(\frac{|\mathbf{v}|^2}{2} + P\right)h_a + \mathbf{v}\nabla\cdot(\mathbf{v}h_a)\right) da \\
&= -\nabla P - \nabla\cdot\int_0^1 (\mathbf{v}\otimes\mathbf{v})h_a da,
\end{aligned}
$$

thanks to $\partial_i v_j = \partial_j v_i$. Applying divergence, we get:

$$0 = -\Delta P - \nabla \cdot \nabla \cdot \int_0^1 (\mathbf{v} \otimes \mathbf{v}) h_a da,$$

which implies (4.7.21). This concludes the proof of Proposition 7.

□

The rest of this section follows the procedure presented in [1] for proving the following necessary condition for global solvability of (4.7.6).

Theorem 13. *Let $(\mathbf{v}, h_a)$ be a global smooth solution to (4.7.6). Let V and H_a be defined by (4.7.10) and (4.7.11) respectively, then:*

$$\int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}|^2 h_a dx da = \int_0^1 |\mathbf{V}(a)|^2 H_a(a) da.$$

The next definition will be crucial in the proof of the next three propositions, which will be used for proving Theorem 13.

Definition 1. *Let $(\mathbf{v}', h_a') : \mathbb{T}^d \times (0,1) \to \mathbb{R}^d \times \mathbb{R}$ be a smooth pair of functions. We will say that $(\mathbf{v}', h_a')$ is an **admissible solution** for the semilagrangian equations (4.7.6) if satisfies the following properties:*

$$\partial_t h_a' + \nabla \cdot (\mathbf{v}' h_a') = 0, \tag{4.7.23}$$

$$\int_0^1 h_a' da = 1, \tag{4.7.24}$$

$$h_a(x,0,a) - h_a'(x,0,a) = h_a(x,T,a) - h_a'(x,T,a) = 0. \tag{4.7.25}$$

Proposition 8. *Let $(\mathbf{v}, h_a)$ be a solution to (4.7.6), then every admissible solution $(\mathbf{v}', h_a')$ to (4.7.6) satisfies:*

$$\int_0^T \int_{\mathbb{T}^d \times (0,1)} \left(|\mathbf{v} - \mathbf{V}|^2 h_a + |\mathbf{v} - \mathbf{v}'|^2 h_a' - |\mathbf{v}' - \mathbf{V}|^2 h_a' \right) dx da dt = 0.$$

In particular we have:

$$\int_0^T \int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}(x,t,a) - \mathbf{V}(a)|^2 h_a dx da dt \leq \int_0^T \int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}'(x,t,a) - \mathbf{V}(a)|^2 h_a' dx da dt.$$

Proof. Let ϕ be defined by (4.7.8). By (4.7.9) and integration by parts

$$
\begin{aligned}
0 &= 2(h_a - h'_a)\phi|_{t=0}^{t=T} \\
&= 2\int_0^T \int_{\mathbb{T}^d\times(0,1)} \partial_t\left((h_a - h'_a)\phi\right) dxdadt \\
&= -2\int_0^T \int_{\mathbb{T}^d\times(0,1)} \left(\phi\nabla\cdot((\mathbf{v}h_a - \mathbf{v}'h'_a)) + (h_a - h'_a)\left(\frac{|\mathbf{v}|^2}{2} + P(x,t)\right)\right) dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} 2\nabla\phi\cdot((\mathbf{v}h_a - \mathbf{v}'h'_a)) + (h'_a - h_a)\,|\mathbf{v}|^2 dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} 2\,(\mathbf{v}-\mathbf{V})\cdot(\mathbf{v}h_a - \mathbf{v}'h'_a) + (h'_a - h_a)\,|\mathbf{v}|^2 dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}|^2 h_a - 2\mathbf{v}\cdot\mathbf{V}h_a - 2\mathbf{v}\cdot\mathbf{v}'h'_a + 2\mathbf{v}'\cdot\mathbf{V}h'_a + |\mathbf{v}|^2 h'_a dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}-\mathbf{V}|^2 h_a + |\mathbf{v}-\mathbf{v}'|^2 h'_a - |\mathbf{v}'-\mathbf{V}|^2 h'_a + |\mathbf{V}|^2(h'_a - h_a) dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} \left(|\mathbf{v}-\mathbf{V}|^2 h_a + |\mathbf{v}-\mathbf{v}'|^2 h'_a - |\mathbf{v}'-\mathbf{V}|^2 h'_a\right) dxdadt,
\end{aligned}
$$

where we have used that $\int_0^1 h_a da = \int_0^1 h'_a da = 1$. This proves Proposition 8.

□

Proposition 9. *Let ζ be a smooth compactly supported function in $(0,T)$ and $\mathbf{q}\in\mathbb{R}^d$ be a fixed vector. Let $(\mathbf{v}, h_a)$ be a solution to (4.7.6), then:*

$$
\int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}(x+\mathbf{q}\zeta(t),t,a) - \mathbf{v}(x,t,a)|^2 h_a(x,t,a) dxdadt \le 4|\mathbf{q}|^2 \int_0^T |\zeta'(t)|^2 dt.
$$

Proof. Let $\mathbf{q}\in\mathbb{R}^d$ be a fixed vector and ζ be a smooth compactly supported function in $(0,T)$. Set $\mathbf{v}'(x,t,a) = \mathbf{v}(x-\mathbf{q}\zeta(t),t,a) + \mathbf{q}\zeta'$ and $h'_a(x,t,a) = h_a(x-\mathbf{q}\zeta(t))$, then $(\mathbf{v}', h'_a)$ is evidently an admissible solution. Now, Proposition 8 yields:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}(x-\mathbf{q}\zeta(t),t,a) + \mathbf{q}\zeta'(t) - \mathbf{v}(x,t,a)|^2 h_a(x-\mathbf{q}\zeta(t),t,a) dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}(x,t,a) + \mathbf{q}\zeta'(t) - \mathbf{v}(x+\mathbf{q}\zeta(t),t,a)|^2 h_a(x,t,a) dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} \left(|\mathbf{v}'-\mathbf{V}|^2 h'_a - |\mathbf{v}-\mathbf{V}|^2 h_a\right) dxdadt \\
&= \int_0^T \int_{\mathbb{T}^d\times(0,1)} \left(-2\zeta'(t)\mathbf{q}\cdot\mathbf{V}h'_a + 2\zeta'\mathbf{q}\cdot\mathbf{v}h_a + |\zeta'|^2|\mathbf{q}|^2\right) dxdadt \\
&= |\mathbf{q}|^2 \int_0^T |\zeta'|^2 dt,
\end{aligned}
$$

since $\int_{\mathbb{T}^d\times(0,1)} \mathbf{v}h_a dxda$ and $\int_{\mathbb{T}^d\times(0,1)} \mathbf{v}'h'_a dxda$ are time independent. Thus:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}(x,t,a) - \mathbf{v}(x+\mathbf{q}\zeta(t),t,a)|^2 h_a(x,t,a)dxdadt \\
&\leq 2\int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}(x,t,a) + \mathbf{q}\zeta'(t) - \mathbf{v}(x+\mathbf{q}\zeta(t),t,a)|^2 h_a(x,t,a)dxdadt \\
&+ 2|\mathbf{q}|^2 \int_0^T |\zeta'|^2 dt \\
&\leq 4|\mathbf{q}|^2 \int_0^T |\zeta'|^2 dt,
\end{aligned}$$

which proves Proposition 9.

□

Proposition 10. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6). Then there exists an admissible solution $(\mathbf{v}', h'_a)$ to (4.7.6) such that:*

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{v}'(x,t,a)h'_a(x,t,a)dxdt = \int_0^T \int_{\mathbb{T}^d} \mathbf{v}(x,t,a)h_a(x,t,a)dxdt, \quad \text{for all } a \in [0,1]. \tag{4.7.26}$$

$$\int_0^T \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}'(x,t,a)|^2 h'_a(x,t,a)dxdadt \leq \frac{C}{T} + T\|V_T\|^2, \tag{4.7.27}$$

where $V_T(a)$ and $\|V_T\|^2$ are defined by:

$$V_T(a) = \frac{1}{TH_a}\int_0^T \int_{\mathbb{T}^d} \mathbf{v}(x,t,a)h_a(x,t,a)dxdt, \tag{4.7.28}$$

$$\|V_T\|^2 = \int_0^1 |V_T(a)|^2 H_a(a)da. \tag{4.7.29}$$

Proof. Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6). Let us define the lagrangian trjectories by:

$$X(\alpha,t,a) = \alpha + \int_0^t \mathbf{v}(X(\alpha,s,a),s,a)ds.$$

Then for every smooth periodic function f, we have:

$$\int_{\mathbb{T}^d} f(x)h_a(x,t,a)dx = \int_{\mathbb{T}^d} f(X(x,t,a))h_a(x,0,a)dx,$$

where we have used that:

$$\partial_t \det\left(\nabla_\alpha X(\alpha,t,a)\right) = \nabla_x \cdot \mathbf{v}(X(\alpha,t,a),t,a)\det\left(\nabla_\alpha X(\alpha,t,a)\right),$$

which implies:

$$\partial_t \left(h_a(X(\alpha,t,a),t,a) \det\left(\nabla_\alpha X(\alpha,t,a)\right)\right) = 0.$$

Let $g(x,a) = X(x,T,a)$ be the final configuration, then:

$$V_T(a) = \frac{1}{TH_a(a)} \int_{\mathbb{T}^d} (g(x,a)-x) h_a(x,0,a) dx,$$

because:

$$\begin{aligned}\int_{\mathbb{T}^d} (g(x,a)-x) h_a(x,0,a) dx &= \int_{\mathbb{T}^d} \int_0^T \mathbf{v}(x,t,a) dt h_a(x,0,a) dx \\ &= \int_0^T \int_{\mathbb{T}^d} \mathbf{v}(x,t,a) h_a(x,t,a) dx dt.\end{aligned}$$

For $0 < r < \frac{1}{2}$ and $y \in \mathbb{T}^d$ let ξ be the following trajectory linking x with $g(x,a)$:

$$\xi(x,t,a,y) = \begin{cases} x + \frac{ty}{rT}, & if\ 0 \le t \le rT, \\ x + y + (g(x,a)-x)\frac{t-rT}{(1-2r)T}, & if\ rT \le t \le (1-r)T, \\ g(x,a) - \frac{(t-T)y}{rT}, & if\ (1-r)T \le t \le T. \end{cases} \tag{4.7.30}$$

Let h'_a and $\mathbf{v}' \in \mathbb{R}^d$ be defined implicitely by:

$$\begin{aligned}\int_{\mathbb{T}^d} f(x) h'_a(x,t,a) dx &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} f(\xi(x,t,a,y)) h_a(x,0,a) dx dy, \\ \int_{\mathbb{T}^d} f(x) \mathbf{v}' h'_a(x,t,a) dx &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \partial_t \xi(x,t,a,y) f(\xi(x,t,a,y)) h_a(x,0,a) dx dy,\end{aligned}$$

for every smooth function f in $\mathbb{T}^d$. From the above definition we see that $\mathbf{v}'$ and $h'_a > 0$ will be smooth functions that satisfy (4.7.23) and:

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \mathbf{v}'(x,t,a) h'_a(x,t,a) dx dt &= \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \partial_t \xi(x,t,a,y) h_a(x,0,a) dx dy dt \\ &= \int_{\mathbb{T}^d} (g(x,a)-x) h_a(x,0,a) dx = TH_a(a) V_T(a),\end{aligned}$$

from which we get (4.7.26). Now, let us write explicitly $\mathbf{v}'$ and h'_a:

$$h'_a(x,t,a) = \begin{cases} \int_{\mathbb{T}^d} h_a\left(x - \frac{yt}{rT}, 0, a\right) dy, & if\ 0 \le t \le rT, \\ H_a(a), & if\ rT \le t \le (1-r)T, \\ \int_{\mathbb{T}^d} h_a\left(x + \frac{t-T}{rT} y, T, a\right) dy, & if\ (1-r)T \le t \le T. \end{cases} \tag{4.7.31}$$

(Note that evidently $\int_0^1 h'_a da = 1$ and $h_a = h'_a$ for $t = 0, T$).

$$\mathbf{v}' h'_a(x,t,a) = \begin{cases} \int_{\mathbb{T}^d} \frac{y}{rT} h_a\left(x - \frac{yt}{rT}, 0, a\right) dy, & if\ 0 \le t \le rT, \\ H_a(a) \frac{V_T(a)}{1-2r}, & if\ rT \le t \le (1-r)T, \\ \int_{\mathbb{T}^d} -\frac{y}{rT} h_a\left(x + \frac{t-T}{rT} y, T, a\right) dy, & if\ (1-r)T \le t \le T. \end{cases} \tag{4.7.32}$$

For $0 \le t \le rT$ and $(1-r)T \le t \le T$, we have:

$$\int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}'(x,t,a)|^2 h'_a(x,t,a) dx da \le \frac{C}{r^2 T^2}.$$

On the other hand, for $rT \le t \le (1-r)T$, we have:

$$\int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}'(x,t,a)|^2 h'_a(x,t,a) dx da = \frac{1}{(1-2r)^2} \int_0^1 V_T^2(a) H_a da,$$

which yields:

$$\int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}'|^2 h'_a dx da \le \frac{C}{r^2 T^2} + \frac{1}{(1-2r)^2} \int_0^1 |V_T(a)|^2 H_a(a) da,$$

from which we get (4.7.27). This ends the proof of Proposition 10.

□

Theorem 14. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6) in $(0,T)$. Let $\mathbf{V}$ and H_a be defined by (4.7.10) and (4.7.11) respectively, then:*

$$|\|\mathbf{v}\| - \|\mathbf{V}\|| \le C \left(\frac{1}{T} + \frac{\|\mathbf{v}\|^{\frac{1}{2}}}{T^{\frac{1}{2}}} \right), \tag{4.7.33}$$

where

$$\|\mathbf{v}\|^2 = \int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}|^2 h_a dx da \quad and \quad \|\mathbf{V}\|^2 = \int_0^1 |\mathbf{V}|^2 H_a da.$$

Proof. Let $(\mathbf{v}', h'_a)$ be an admissible solution satisfying the properties mentioned in Proposition 10, then:

$$
\begin{aligned}
&T\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}|^2h_a dxda\\
&=\int_0^T\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}|^2h_a dxdadt\\
&=\int_0^T\int_{\mathbb{T}^d\times(0,1)}\left(|\mathbf{v}-\mathbf{V}|^2h_a+2\mathbf{v}\cdot\mathbf{V}h_a-|\mathbf{V}|^2h_a\right)dxdadt\\
&\le\int_0^T\int_{\mathbb{T}^d\times(0,1)}\left(|\mathbf{v}'-\mathbf{V}|^2h_a'+2\mathbf{v}\cdot\mathbf{V}h_a-|\mathbf{V}|^2h_a\right)dxdadt\\
&=\int_0^T\int_{\mathbb{T}^d\times(0,1)}\left(|\mathbf{v}'-\mathbf{V}|^2h_a'+2\mathbf{v}'\cdot\mathbf{V}h_a'-|\mathbf{V}|^2h_a'\right)dxdadt\\
&=\int_0^T\int_{\mathbb{T}^d\times(0,1)}\left(|\mathbf{v}'|^2h_a'-2\mathbf{v}'\cdot\mathbf{V}h_a'+|\mathbf{V}|^2h_a'+2\mathbf{v}'\cdot\mathbf{V}h_a'-|\mathbf{V}|^2h_a'\right)dxdadt\\
&=\int_0^T\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}'|^2h_a'dxdadt\\
&\le\frac{C}{T}+T\|V_T\|^2,
\end{aligned}
$$

from which we obtain:

$$
\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}|^2h_a dxda\le\frac{C}{T^2}+\|V_T\|^2.
$$

By (4.7.29)

$$
\begin{aligned}
\|V_T\|^2&=\int_0^1\frac{1}{T^2H_a^2(a)}\left|\int_0^T\int_{\mathbb{T}^d}\mathbf{v}h_a dxdt\right|^2H_a da\\
&\le\int_0^1\frac{1}{T^2H_a(a)}|\int_0^T\int_{\mathbb{T}^d}h_a dxdt|\int_0^T\int_{\mathbb{T}^d}|\mathbf{v}|^2h_a dxdtda\\
&=\int_{\mathbb{T}^d\times(0,1)}|\mathbf{v}|^2h_a dxda.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
|\|\mathbf{v}\|-\|V_T\||&=\|\mathbf{v}\|-\|V_T\|\\
&\le\frac{C}{T}.
\end{aligned}
\tag{4.7.34}
$$

On the other hand, we have:

$$
TH_a(a)(V_T(a)-\mathbf{V}(a))=\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}(v(x,t,a)-v(x',t,a))h_a(x,t,a)dxdx'dt,
$$

which implies:

$$\begin{aligned}
&\int_0^1 |V_T - \mathbf{V}(a)|^2 H_a da \\
&= \int_0^1 \frac{1}{T^2 H_a(a)} \left| \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (\mathbf{v}(x,t,a) - \mathbf{v}(x',t,a)) h_a(x,t,a) dx dx' dt \right|^2 da \\
&\leq \int_0^1 \frac{1}{T^2 H_a(a)} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |\mathbf{v}(x,t,a) - \mathbf{v}(x',t,a)|^2 h_a(x,t,a) dx dx' dt \cdot T H_a(a) da \\
&\leq \frac{8\tau}{T} \int_{\mathbb{T}^d \times (0,1)} |\mathbf{v}|^2 h_a dx da \\
&+ \frac{1}{T} \int_\tau^{T-\tau} \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |\mathbf{v}(x,t,a) - \mathbf{v}(x',t,a)|^2 h_a(x,t,a) dx dx' da dt,
\end{aligned}$$

for all $\tau \in (0, \frac{T}{2})$. Taking a suitable smooth function ζ compactly supported in $[0,T]$ such that $\zeta(t) = 1$ if $t \in (\tau, T-\tau)$, we can bound the second integral by applying Proposition 9:

$$\frac{1}{T} \int_{\mathbb{T}^d} \int_0^T \int_0^1 \int_{\mathbb{T}^d} |\mathbf{v}(x,t,a) - \mathbf{v}(x + \zeta(t)(x'-x), t, a)|^2 h_a(x,t,a) dx da dt dx' \leq \frac{1}{T} \frac{C}{\tau},$$

from which we get:

$$\int_0^1 |V_T - \mathbf{V}(a)|^2 H_a da \leq \frac{1}{T} \left(8\tau \|\mathbf{v}\|^2 + \frac{C}{\tau} \right),$$

where C is τ-independent. If we minimize the above in τ, we get that the minimum is attained when $8\|\mathbf{v}\|^2 \tau^2 = C$ (assuming that $2T^2 \|\mathbf{v}\|^2 > C$). Thus, we have obtained:

$$\int_0^1 |V_T - \mathbf{V}(a)|^2 H_a da \leq \frac{\tilde{C}}{T} \|\mathbf{v}\|. \tag{4.7.35}$$

Finally, by (4.7.35) and (4.7.34):

$$\begin{aligned}
|\|\mathbf{v}\| - \|\mathbf{V}\||^2 &= |\|\mathbf{v}\| - \|V_T\| + \|V_T\| - \|\mathbf{V}\||^2 \\
&\leq C \left(\frac{1}{T^2} + \frac{\|\mathbf{v}\|}{T} \right),
\end{aligned}$$

where we have used the reverse triangle inequality. This concludes the proof of Theorem 14.

□

Proof of Theorem 13. It is a direct consequence of Theorem 14, because global smooth solutions to (4.7.6) satisfy (4.7.33) for arbitrarily large T.

□

5. RESULTS

5.1. Introduction

In this chapter we will present the original results obtained in this thesis, which are available as a preprint in [4]. In the following subsections, we will state the three main results of this thesis, namely: the collapse of the local Rayleigh condition or the singularity formation for the hydrostatic Euler equations (4.1.3), the deduction of certain necessary conditions for the global solvability of the hydrostatic Euler equations (4.1.3) under the local Rayleigh condition and the finite time blow-up of the semilagrangian equations (4.7.6) for certain class of initial data.

5.1.1. Collapse of the local Rayleigh condition or singularity formation

Theorem 15. *Assume that the initial vorticity $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition (i.e. $\partial_y\omega_0 > 0$) and $\partial_x\omega_0$ is not identically 0. Let $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ be the solution to the problem (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Set*

$$E_1(t) = \int_{\mathbb{T}\times(0,1)} \frac{\omega\omega_x}{\omega_y} dxdy = \int_{\mathbb{T}\times(0,1)} \left(\frac{\omega\omega_x}{\omega_y} - u_x\right) dxdy \tag{5.1.1}$$

and

$$E_2(t) = \int_{\mathbb{T}\times(0,1)} u^2\frac{\omega\omega_x}{\omega_y} dxdy = \int_{\mathbb{T}\times(0,1)} u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x dxdy. \tag{5.1.2}$$

Assume that $E_1(0) \geq 0$ or $E_2(0) \geq 0$. Then there exists a positive time T^* such that at least one of the following properties holds:

- $\limsup_{t\to T^*} \|\omega(t)\|_{H^4(\mathbb{T}\times(0,1))} = \infty$,
- $\limsup_{t\to T^*} \left\|\frac{1}{\omega_y(t)}\right\|_{L^\infty(\mathbb{T}\times(0,1))} = \infty$.

Furthermore, as long as $\|\omega(t)\|_{H^4(\mathbb{T}\times(0,1))}$ and $\left\|\frac{1}{\omega_y(t)}\right\|_{L^\infty(\mathbb{T}\times(0,1))}$ remain bounded, we have the following estimates:

$$\log\left(\frac{1}{E_1(0)}\frac{1}{\frac{1}{E_1(0)} - t}\right) \leq \int_{\mathbb{T}\times(0,1)} \log\left(\frac{\partial_y\omega_0}{\omega_y(t)}\right) dxdy, \qquad \text{if } E_1(0) > 0, \tag{5.1.3}$$

$$\log\left(\frac{\|u\|_2^2}{E_2(0)}\frac{1}{\frac{\|u\|_2^2}{E_2(0)}-t}\right) \leq tC_1(\omega_0) + C_2(\omega_0)\int_{\mathbb{T}\times(0,1)} \log\left(\frac{\partial_y\omega_0}{\omega_y(t)}\right) dxdy, \text{ if } E_2(0) > 0, \tag{5.1.4}$$

$$\frac{1}{\frac{1}{E_1(0)}-t} \leq E_1(t) \leq E_1(0) + C_3(\omega_0)\int_0^t\int_{\mathbb{T}\times(0,1)} \omega_x^2\left(1+\frac{1}{\omega_y^2}\right) dxdyd\tau, \text{ if } E_1(0) > 0, \tag{5.1.5}$$

$$\frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(0)}-t} \leq E_2(t) \leq E_2(0) + C_4(\omega_0)\int_0^t\int_{\mathbb{T}\times(0,1)} \omega_x^2\left(1+\frac{1}{\omega_y^2}\right) dxdyd\tau, \text{ if } E_2(0) > 0, \tag{5.1.6}$$

where $C_1(\omega_0) = \left(\frac{3}{2}\frac{\|\omega_0\|_\infty}{\|u\|_2}\right)^4 |E_1(0)| + \frac{E_2(0)}{\|u\|_2^2}$, $C_2(\omega_0) = \left(\frac{3}{2}\frac{\|\omega_0\|_\infty}{\|u\|_2}\right)^4$, $C_3(\omega_0) = 2\|\omega_0\|_\infty^2 + \frac{2}{\pi^2}$ *and* $C_4(\omega_0) = 2\left(\frac{3}{\pi}\|\omega_0\|_\infty\right)^2 + 2\left(\frac{3}{2}\|\omega_0\|_\infty\right)^2 C_3(\omega_0)$.

Remark 7. *If* $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ *satisfies the local Rayleigh condition and* $\partial_x\omega_0 \equiv 0$*, then the solution to (4.1.4) is stationary. Conversely, the stationary solutions in* $H^4(\mathbb{T}\times(0,1))$ *that satisfy the local Rayleigh condition are* x*-independent (see Corollary 2).*

Remark 8. *Suppose that* $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ *is even in* x*, satisfies the local Rayleigh condition and* $\partial_x\omega_0$ *is not identically* 0*, then* $E_1(0) = 0$*, which implies the collapse in finite time (more precisely, the local Rayleigh condition will collapse or* ω *will develop singularities) of the solution to (4.1.4). Moreover, since* $\omega(x,y,t)$ *and* $\omega(-x,y,-t)$ *solve (4.1.4) with the same initial condition,* $\omega(x,y,t) \equiv \omega(-x,y,-t)$ *(thanks to the uniqueness). In that case, the solution will collapse going forward and backward in time.*

Remark 9. *It is worth mentioning that the set of initial conditions that satisfy the assumptions of Theorem 15 is nonempty. Indeed, if we choose* $\omega_0(x,y) = 2y - sin(2\pi x - y)$*, then* $\partial_x\omega_0 = -2\pi cos(2\pi x - y)$ *and* $\partial_y\omega_0 = 2 + cos(2\pi x - y)$*, which yields*

$$
\begin{aligned}
&E_1(0)\\
&= \int_{\mathbb{T}\times(0,1)} \frac{(2y - sin(2\pi x - y)) \cdot -2\pi cos(2\pi x - y)}{2 + cos(2\pi x - y)} dxdy\\
&= -2\pi \int_{\mathbb{T}\times(0,1)} (2y - sin(2\pi x - y))\, dxdy + 4\pi \int_{\mathbb{T}\times(0,1)} \frac{(2y - sin(2\pi x - y))}{2 + cos(2\pi x - y)} dxdy\\
&= -2\pi + \int_{\mathbb{T}\times(0,1)} \frac{4\pi \cdot 2y}{2 + cos(2\pi x - y)} dxdy\\
&= -2\pi + \frac{4\pi}{\sqrt{3}} \int_0^1 2y dy\\
&= 2\pi \left(\frac{2}{\sqrt{3}} - 1 \right)\\
&> 0.
\end{aligned}
$$

Remark 10. *Suppose that $\omega_0(x,y) \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition, $\partial_x\omega_0$ is not identically 0 but $E_1(0) < 0$, then $\tilde{\omega}_0(x,y) = \omega_0(-x,y)$ satisfies $E_1(0) > 0$ (the same applies for E_2). Hence, the assumptions of Theorem 15 are not so restrictive.*

5.1.2. Necessary conditions for global solvability

Theorem 16. *Assume that the initial vorticity $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition (i.e. $\partial_y\omega_0 > 0$). Let $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ be the solution to the problem (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Let E_1 and E_2 be defined by (5.1.1) and (5.1.2) respectively. Assume further that T can be chosen arbitrarily large, then we have:*

$$\int_0^\infty \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdydt = -E_2(0), \tag{5.1.7}$$

$$\int_0^\infty \int_{\mathbb{T}\times(0,1)} \left| u_x - \frac{\omega\omega_x}{\omega_y}\right|^2 dxdydt = -E_1(0), \tag{5.1.8}$$

$$\int_0^\infty \int_{\mathbb{T}^1} P_x^2 dxdt \le -2E_2(0) - \frac{9}{2}\|\omega_0\|_\infty^2 E_1(0), \tag{5.1.9}$$

$$\int_0^\infty |E_1(t)|dt \le \int_{\mathbb{T}\times(0,1)} \log\left(\frac{2\|\omega_0\|_\infty}{\partial_y\omega_0}\right) dxdy, \tag{5.1.10}$$

where $E_1(0), E_2(0) < 0$.

Remark 11. *Let us observe that in the semilagrangian formulation (4.7.3), the above energies are equal to (see Proposition 6):*

$$E_1(T) = -\int_{\mathbb{T}\times(0,1)} v_x h_a dxda = E_1(0) + \int_0^T \int_{\mathbb{T}\times(0,1)} v_x^2 h_a dxdadt,$$
$$E_2(T) = -\int_{\mathbb{T}\times(0,1)} v^2 v_x h_a dxda = E_2(0) + \int_0^T \int_{\mathbb{T}\times(0,1)} v_t^2 h_a dxdadt,$$

thanks to the change of coordinates $y = h(x,t,a)$. The above identities are a particular case of (5.6.2) and (5.6.3), when $d = 1$.

5.1.3. Finite time blow-up of the Semi-lagrangian equations

Theorem 17. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6). Set*

$$E_1(t) = -\int_{\mathbb{T}^d\times(0,1)} (\nabla\cdot\mathbf{v})\, h_a dxda \tag{5.1.11}$$

and

$$E_2(t) = \int_{\mathbb{T}^d\times(0,1)} (\mathbf{v}\cdot\mathbf{v}_t)\, h_a dxda. \tag{5.1.12}$$

Assume that $E_1(0) > 0$ or $E_2 > 0$, then the solution blows up in finite time. Moreover, as long as the solution exists, we have the following estimates:

$$d\log\left(\frac{d}{E_1(0)}\frac{1}{\frac{d}{E_1(0)} - t}\right) \leq \int_{\mathbb{T}^d\times(0,1)} \log(h_a(\tau))h_a(\tau)dxda|_{\tau=0}^{\tau=t}, \text{ if } E_1(0) > 0, \tag{5.1.13}$$

$$\frac{d}{\frac{d}{E_1(0)} - t} \leq E_1(t) = E_1(0) + \int_0^t \int_{\mathbb{T}^d\times(0,1)} |\nabla\mathbf{v}|^2 h_a dxdadt, \text{ if } E_1(0) > 0, \tag{5.1.14}$$

$$\frac{\|\mathbf{v}\|^2}{\frac{\|\mathbf{v}\|^2}{E_2(0)} - t} \leq E_2(t) = E_2(0) + \int_0^t \int_{\mathbb{T}^d\times(0,1)} |\partial_t\mathbf{v}|^2 h_a dxdadt, \text{ if } E_2(0) > 0, \tag{5.1.15}$$

where $\|\mathbf{v}\|^2 = \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}|^2 h_a dxda$.

The following corollary is a direct consequence of Theorem 17, Jensen's inequality and Lemma 9 (applied to the probability measure $\mu = h_a dxda$).

Corollary 1. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6) in $[0,T]$. Let E_1 be defined by (5.1.11). Suppose that $E_1(0) > 0$, then*

$$\exp\left(\int_{\mathbb{T}^d\times(0,1)} \partial_a h_0 \log\left(\partial_a h_0\right) dxda\right)\left(\frac{d}{E_1(0)}\frac{1}{\frac{d}{E_1(0)}-t}\right)^d$$
$$\leq \exp\left(\int_{\mathbb{T}^d\times(0,1)} h_a(T)\log\left(h_a(T)\right) dxda\right)$$
$$\leq \left(\int_{\mathbb{T}^d\times(0,1)} h_a(T)^{1+p} dxda\right)^{1/p},$$

for every $p \in (0,\infty)$. *Moreover*

$$\exp\left(\int_{\mathbb{T}^d\times(0,1)} h_a \log(h_a) dxda\right) = \lim_{p\to 0}\left(\int_{\mathbb{T}^d\times(0,1)} h_a^{1+p} dxda\right)^{\frac{1}{p}}. \tag{5.1.16}$$

Remark 12. *Corollary 1 gives a lower bound for the* $L^p(\mathbb{T}^d\times(0,1))$ *norm of* h_a *for* $p>1$, *which cannot be extended to* $p=1$, *because* $\int_{\mathbb{T}^d\times(0,1)} h_a dxda = 1$.

Remark 13. *In particular, smooth solutions to (4.7.6) satisfy the following equations in* $\mathbb{T}^d$*:*

$$\begin{cases} \partial_t \int_0^1 \mathbf{v} h_a da + \nabla\cdot\int_0^1 \mathbf{v}\otimes\mathbf{v} h_a da + \nabla P = 0, \\ \int_0^1 h_a da = 0, \\ \nabla\cdot\int_0^1 \mathbf{v} h_a da = 0, \\ \partial_i v_j - \partial_j v_i = 0, \quad \text{for } i,j\leq d, \end{cases} \tag{5.1.17}$$

which can be seen as an averaged version of the incompressible Euler equations in $\mathbb{T}^d$

$$\begin{cases} \partial_t \mathbf{v} + \nabla\cdot(\mathbf{v}\otimes\mathbf{v}) + \nabla P = 0, \\ \nabla\cdot\mathbf{v} = 0, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0. \end{cases} \tag{5.1.18}$$

5.2. Monoticity and lower bounds for E_1 and E_2

Throughout this section we will work in the framework of the local well-posedness result by Masmoudi and Wong [20], namely, we will assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition. We will show the monoticity of E_1 and E_2, as

well as the validity of certain lower bounds that will be useful for proving Theorem 15.

The following two lemmas are elementary, so the proof will be omitted.

Lemma 3. *Assume that $\omega_0 \in H^4(\mathbb{T} \times (0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Let $f, g \in C^1([0,T]; H^3(\mathbb{T} \times (0,1)))$. Denote by $D_t = \partial_t + u\partial_x + v\partial_y$ the material derivative, then:*

$$\partial_t \int_{\mathbb{T}\times(0,1)} f(x,y,t)dxdy = \int_{\mathbb{T}\times(0,1)} D_t f(x,y,t)dxdy, \tag{5.2.1}$$

$$D_t(fg) = fD_t g + gD_t f, \tag{5.2.2}$$

$$D_t\left(\frac{f}{g}\right) = \frac{gD_t f - fD_t g}{g^2}, \quad \text{if } g \text{ is strictly positive,} \tag{5.2.3}$$

$$D_t\left(\log(g)\right) = \frac{D_t g}{g}, \quad \text{if } g \text{ is strictly positive,} \tag{5.2.4}$$

where $0 \le t \le T$.

Lemma 4. *Assume that $\omega_0 \in H^4(\mathbb{T} \times (0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Denote by $D_t = \partial_t + u\partial_x + v\partial_y$ the material derivative, then:*

$$D_t(u) = -P_x, \tag{5.2.5}$$

$$D_t(\omega) = 0, \tag{5.2.6}$$

$$D_t(u_x) = -u_x^2 - v_x\omega - P_{xx}, \tag{5.2.7}$$

$$D_t(P_x) = P_{xt} + uP_{xx}, \tag{5.2.8}$$

$$D_t(\omega_x) = -u_x\omega_x - v_x\omega_y, \tag{5.2.9}$$

$$D_t(\omega_y) = u_x\omega_y - \omega\omega_x, \tag{5.2.10}$$

where $0 \le t \le T$.

Lemma 5. *Assume that $\omega_0 \in H^4(\mathbb{T} \times (0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Denote by $D_t = \partial_t + u\partial_x + v\partial_y$ the material derivative, then:*

$$D_t\left(\log\left(\frac{1}{\omega_y}\right)\right) = \frac{\omega\omega_x}{\omega_y} - u_x, \tag{5.2.11}$$

$$D_t\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) = \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 + P_{xx}, \tag{5.2.12}$$

$$D_t\left(u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right) = \left|u\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - P_x\right|^2 - uP_{xt}, \tag{5.2.13}$$

for $0 \le t < T$.

Proof. By Lemmas 3 and 4,

$$\begin{aligned} D_t\left(\log\left(\frac{1}{\omega_y}\right)\right) &= -\frac{u_x\omega_y - \omega\omega_x}{\omega_y} \\ &= \frac{\omega\omega_x}{\omega_y} - u_x, \end{aligned}$$

which gives (5.2.11). Next, by Lemmas 3 and 4,

$$\begin{aligned} D_t\left(\frac{\omega\omega_x}{\omega_y}\right) &= \frac{-u_x\omega\omega_x\omega_y - v_x\omega\omega_y^2 - u_x\omega\omega_x\omega_y + \omega^2\omega_x^2}{\omega_y^2} \\ &= \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 - \omega v_x - u_x^2. \end{aligned} \tag{5.2.14}$$

Thus, by (5.2.14) and (5.2.7)

$$D_t\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) = \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 + P_{xx},$$

from which we get (5.2.12). Finally, by Lemmas 3 and 4 and (5.2.12)

$$\begin{aligned} &D_t\left(u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right) \\ &= u^2\left(\left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 + P_{xx}\right) - 2uP_x\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) + P_x^2 - u^2P_{xx} - uP_{xt} \\ &= \left|u\left(\frac{\omega\omega_x}{\omega_y} - u_x\right)u - P_x\right|^2 - uP_{xt}, \end{aligned}$$

which implies (5.2.13). This completes the proof of Lemma 5. □

Proposition 11. *Assume that* $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ *satisfies the local Rayleigh condition. Let* $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ *be the solution to (4.1.4) that satisfies the local Rayleigh condition in* $[0,T]$. *Set*

$$E_1(t) = \int_{\mathbb{T}\times(0,1)}\left(\frac{\omega\omega_x}{\omega_y} - u_x\right)dxdy = \int_{\mathbb{T}\times(0,1)}\frac{\omega\omega_x}{\omega_y}dxdy \tag{5.2.15}$$

and

$$E_2(t) = \int_{\mathbb{T}\times(0,1)} \left(u^2 \left(\frac{\omega\omega_x}{\omega_y} - u_x \right) - uP_x \right) dxdy = \int_{\mathbb{T}\times(0,1)} u^2 \frac{\omega\omega_x}{\omega_y} dxdy. \quad (5.2.16)$$

Then

$$E_1 = \partial_t \int_{\mathbb{T}\times(0,1)} \log\left(\frac{1}{\omega_y}\right) dxdy, \quad (5.2.17)$$

$$\partial_t E_1 = \int_{\mathbb{T}\times(0,1)} \left| u_x - \frac{\omega\omega_x}{\omega_y} \right|^2 dxdy, \quad (5.2.18)$$

$$\partial_t E_2 = \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y} \right) \right|^2 dxdy, \quad (5.2.19)$$

where $0 \leq t < T$.

Proof. By Lemma 3, (5.2.11) and the x-periodicity

$$\partial_t \int_{\mathbb{T}\times(0,1)} \log\left(\frac{1}{\omega_y}\right) dxdy = \int_{\mathbb{T}\times(0,1)} \left(\frac{\omega\omega_x}{\omega_y} - u_x \right) dxdy, \quad (5.2.20)$$

which proves (5.2.17). Next, by Lemma 3, (5.2.12) and the x-periodicity

$$\begin{aligned} \partial_t \int_{\mathbb{T}\times(0,1)} \left(\frac{\omega\omega_x}{\omega_y} - u_x \right) dxdy &= \int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 + P_{xx} dxdy \\ &= \int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 dxdy, \end{aligned}$$

which gives (5.2.18). Finally, by Lemma 3 and (5.2.13)

$$\begin{aligned} \partial_t E_2 &= \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y} \right) \right|^2 - uP_{xt} dxdy \\ &= \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y} \right) \right|^2 + u_x P_t dxdy, \end{aligned}$$

from which we get (5.2.19), thanks to (4.1.10). This concludes the proof of Proposition 11.

□

Lemma 6. *Assume that* $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ *satisfies the local Rayleigh condition. Let* $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ *be the solution to (4.1.4) that satisfies the local Rayleigh condition in* $[0,T]$. *If* $\int_{\mathbb{T}\times(0,1)} \left| u_x - \frac{\omega\omega_x}{\omega_y} \right|^2 dxdy = 0$ *(at any time) then* ω *is stationary and independent of* x.

Proof. Assume that $\frac{\omega\omega_x}{\omega_y} - u_x \equiv 0$ at time $t = t_0$. Then,

$$\partial_y \left(\frac{u_x}{\omega}\right) = \frac{\omega\omega_x - u_x\omega_y}{\omega^2} = \frac{\omega_y}{\omega^2}\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) \equiv 0, \quad \text{if } \omega \neq 0. \tag{5.2.21}$$

Let us first assume that ω does not vanishes. By (4.1.10), there exists $y_0 \in [0,1]$ such that $u_x(x, y_0(x), t_0) = \frac{u_x(x,y_0(x),t_0)}{\omega(x,y_0(x),t_0)} = 0$. By (5.2.21), $u_x \equiv 0$ at time $t = t_0$. Thus, ω is x-independent at $t = t_0$. Moreover, $\omega(y, t_0)$ and $\omega(x, y, t_0 + t)$ satisfy (4.1.4) with the same initial condition, since the solution is unique, $\omega(x, y, t_0 + t) \equiv \omega(y, t_0)$ (see [20]).

Conversely, assume that ω vanishes in some points. Applying (5.2.21), we have

$$u_x(x, y, t_0) = \begin{cases} f(x, t_0)\omega(x, y, t_0) + c(x), & if \quad \omega > 0, \\ \tilde{f}(x, t_0)\omega(x, y, t_0) + \tilde{c}(x), & if \quad \omega < 0, \\ 0, & if \quad \omega = 0. \end{cases} \tag{5.2.22}$$

Since u_x is continuous, $c \equiv \tilde{c} \equiv 0$. Moreover,

$$\omega_x = \begin{cases} f(x, t_0)\omega_y, & if \quad \omega > 0, \\ \tilde{f}(x, t_0)\omega_y, & if \quad \omega < 0, \\ 0, & if \quad \omega = 0. \end{cases} \tag{5.2.23}$$

Therefore, the continuity of $\frac{\omega_x}{\omega_y}$ yields $f(x, t_0) = \tilde{f}(x, t_0)$. Thus, $u_x = f(x, t_0)\omega$ and $\omega_x = f(x, t_0)\omega_y$. Integrating in $y \in [0, 1]$, we get

$$\int_0^1 u_x dy = 0 = f(x, t_0) \int_0^1 \omega dy = f(x, t_0)u|_{y=0}^{y=1}, \tag{5.2.24}$$

$$\int_0^1 \omega_x dy = f(x, t_0) \int_0^1 \omega_y dy = \partial_x u|_{y=0}^{y=1}. \tag{5.2.25}$$

Applying (5.2.24), (5.2.25) and the positivity of ω_y, we get

$$\partial_x \left(|u(1) - u(0)|^2\right) = 2u|_{y=0}^{y=1}\partial_x u|_{y=0}^{y=1} \equiv 0. \tag{5.2.26}$$

Finally, if $f(x_0, t_0) \neq 0$, (5.2.24) implies $u(x_0, y, t_0)|_{y=0}^{y=1} = 0$. Moreover, by (5.2.26), $u(x, y, t_0)|_{y=0}^{y=1} \equiv 0$, which contradicts (5.2.25). Thus, $f \equiv u_x \equiv 0$ at time $t = t_0$. Furthermore, $\omega(y, t_0)$ and $\omega(x, y, t_0 + t)$ satisfy (4.1.4) with the same initial data. Then, by uniqueness $\omega(x, y, t_0 + t) \equiv \omega(y, t_0)$ (see [20]). This completes the proof of Lemma 6.

□

Corollary 2. *Let $\omega \in H^4(\mathbb{T} \times (0, 1))$ be a stationary solution to (4.1.4) that satisfies the local Rayleigh condition. Then, ω is x-independent. Conversely, if ω is an x-independent solution to (4.1.4), then ω is stationary.*

Proof. Let ω be a stationary solution, then by (5.2.18):

$$\partial_t E_1 = \int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 dxdy = 0.$$

By Lemma 6, ω is x-independent. On the other hand, if ω is an x-independent solution to (4.1.4), then

$$\omega_t = -u\omega_x - \mathcal{A}(\omega_x)\omega_y \equiv 0,$$

where $\mathcal{A}(\omega)$ is defined by (4.1.7), which concludes the proof of Corollary 2.

□

Lemma 7. *Assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. If $\int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdy = 0$ (at any time) then ω is stationary and independent of x.*

Proof. Assume that $P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right) \equiv 0$ at time $t = t_0$. Then,

$$P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right) = P_x + uu_x + \frac{v\omega\omega_y}{\omega_y} - \frac{\omega v\omega_y}{\omega_y} - \frac{\omega u\omega_x}{\omega_y} = \frac{\omega\omega_t}{\omega_y} - u_t. \quad (5.2.27)$$

Furthermore,

$$\partial_y\left(\frac{u_t}{\omega}\right) = \frac{\omega\omega_t - u_t\omega_y}{\omega^2} = \frac{\omega_y}{\omega^2}\left(\frac{\omega\omega_t}{\omega_y} - u_t\right) \equiv 0, \quad \text{if } \omega \neq 0. \quad (5.2.28)$$

Let us first assume that ω does not vanishes. By (4.1.11), there exists $y_0 \in [0,1]$ such that $u_t(x, y_0(x), t_0) = \frac{u_t(x,y_0(x),t_0)}{\omega(x,y_0(x),t_0)} = 0$. By (5.2.28), $u_t \equiv 0$ at time $t = t_0$. Moreover, $\omega(x,y,t_0)$ and $\omega(x,y,t_0+t)$ satisfy (4.1.4) with the same initial condition. Since the solution is unique, $\omega(x,y,t_0+t) \equiv \omega(x,y,t_0)$. Furthermore, by Corollary 2, ω is x-independent.

Conversely, assume that ω vanishes in some points. Applying (5.2.28), we have

$$u_t(x,y,t_0) = \begin{cases} f(x,t_0)\omega(x,y,t_0) + c(x), & if \quad \omega > 0, \\ \tilde{f}(x,t_0)\omega(x,y,t_0) + \tilde{c}(x), & if \quad \omega < 0, \\ 0, & if \quad \omega = 0. \end{cases} \quad (5.2.29)$$

Since u_t is continuous, $c \equiv \tilde{c} \equiv 0$. Moreover,

$$\omega_t = \begin{cases} f(x,t_0)\omega_y, & if \quad \omega > 0, \\ \tilde{f}(x,t_0)\omega_y, & if \quad \omega < 0, \\ 0, & if \quad \omega = 0. \end{cases} \quad (5.2.30)$$

Therefore, the continuity of $\frac{\omega_t}{\omega_y}$ yields $f(x,t_0) \equiv \tilde{f}(x,t_0)$. Thus, $u_t = f(x,t_0)\omega$ and $\omega_t = f(x,t_0)\omega_y$. Integrating in $y \in [0,1]$, we get

$$\int_0^1 u_t dy = 0 = f(x,t_0)\int_0^1 \omega dy = f(x,t_0)u|_{y=0}^{y=1}, \tag{5.2.31}$$

$$\int_0^1 \omega_t dy = f(x,t_0)\int_0^1 \omega_y dy = \partial_t u|_{y=0}^{y=1} = \frac{1}{2}\partial_x(u^2(0) - u^2(1)). \tag{5.2.32}$$

Applying (5.2.31), (5.2.32) and the positivity of ω_y, we get

$$\partial_t\left(|u(1)-u(0)|^2\right) = 2u|_{y=0}^{y=1}\partial_t u|_{y=0}^{y=1} = 2u|_{y=0}^{y=1}\partial_x\left(u^2(0)-u^2(1)\right) \equiv 0. \tag{5.2.33}$$

Multiplying by $u(1)+u(0)$, we obtain

$$\partial_x\left(\left|u^2(0)-u^2(1)\right|^2\right) \equiv 0. \tag{5.2.34}$$

Finally, if $f(x_0,t_0) \neq 0$, (5.2.31) implies $u(x_0,y,t_0)|_{y=0}^{y=1} = 0$. Moreover, by (5.2.34) and (5.2.32), $u(x,y,t_0)|_{y=0}^{y=1} \equiv 0$, which contradicts (5.2.32). Thus, $f \equiv u_t \equiv 0$ at time $t = t_0$. Furthermore, $\omega(x,y,t_0)$ and $\omega(x,y,t_0+t)$ satisfy (4.1.4) with the same initial data. Then, by uniqueness $\omega(x,y,t_0+t) \equiv \omega(x,y,t_0)$ (see [20]). Moreover, since ω is stationary, $\partial_t E_1 = 0$. Applying Lemma 6, we deduce that ω is x-independent. This concludes the proof of Lemma 7.

□

Lemma 8. *Let $f : [0,L) \to \mathbb{R}$ be a C^1 function and $C > 0$. Assume that $\partial_t f \geq Cf^2$ and $f(0) > 0$. Then there exists a positive time T^* such that $f \to \infty$ as $t \to T^*$. Moreover, $T^* \leq \frac{1}{Cf(0)}$ and f satisfies:*

$$f(t) \geq \frac{1}{C}\frac{1}{\frac{1}{Cf(0)} - t}, \tag{5.2.35}$$

for $t < T^*$.

Proof. Since f is positive we have:

$$\frac{\partial_s f}{f^2} \geq C.$$

Integrating the above we get:

$$\int_0^t \frac{f_s}{f^2} ds = \frac{1}{f(0)} - \frac{1}{f(t)} \geq C\int_0^t ds = Ct.$$

From which we obtain:

$$f(t) \geq \frac{1}{\frac{1}{f(0)} - Ct},$$

which concludes the proof. □

Proposition 12. *Assume that $\omega_0 \in H^4(\mathbb{T} \times (0,1))$ satisfies the local Rayleigh condition and $\partial_x \omega_0$ is not identically 0. Let $\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Then, E_1 satisfies:*

$$E_1(t) \geq \frac{1}{\frac{1}{E_1(0)} - t}, \qquad \text{if } E_1(0) > 0, \tag{5.2.36}$$

where $0 \leq t \leq T$.

Furthermore, if $E_1(0) = 0$ there exists a positive time t' (sufficiently small) such that $E_1(t') > 0$ and

$$E_1(t + t') \geq \frac{1}{\frac{1}{E_1(t')} - t}, \tag{5.2.37}$$

where $0 \leq t \leq T$.

Proof. By (5.2.18) and Hölder's inequality,

$$\begin{aligned} \partial_t E_1 &= \int_{\mathbb{T}\times(0,1)} \left| u_x - \frac{\omega\omega_x}{\omega_y} \right|^2 dxdy \\ &\geq \left(\int_{\mathbb{T}\times(0,1)} \left(u_x - \frac{\omega\omega_x}{\omega_y} \right) dxdy \right)^2 \\ &= E_1^2(t). \end{aligned} \tag{5.2.38}$$

If $E_1(0) > 0$ the result follows by applying Lemma 8. On the other hand, if $E_1(0) = 0$, Lemma 6 implies

$$\int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 dxdy > 0,$$

at every time. Therefore, for every positive time t' we have

$$\int_0^{t'} \int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 dxdydt = \int_0^{t'} \partial_t E_1(t) dt = E_1(t') - E_1(0) = E_1(t') > 0.$$

Choose t' such that $\omega \in C([0, T + t']; H^4(\mathbb{T} \times (0,1)))$ satisfies the local Rayleigh condition in $[0, T + t']$. Applying (5.2.36) to $\omega(t + t')$ we get

$$E_1(t' + t) \geq \frac{1}{\frac{1}{E_1(t')} - t},$$

for $0 \leq t \leq T$, which completes the proof of Proposition 12. □

Proposition 13. *Assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition and $\partial_x\omega_0$ is not identically 0. Let $\omega \in C([0,T];H^4(\mathbb{T}\times(0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Then, E_2 satisfies:*

$$E_2(t) \geq \frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(0)} - t}, \qquad \text{if } E_2(0) > 0, \tag{5.2.39}$$

where $0 \leq t \leq T$. Furthermore, if $E_2(0) = 0$ there exists a positive time t' (sufficiently small) such that $E_2(t') > 0$ and

$$E_2(t'+t) \geq \frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(t')} - t}, \tag{5.2.40}$$

where $0 \leq t \leq T$.

Proof. By (5.2.19), Hölder's inequality, (4.1.10) and the x-periodicity:

$$\begin{aligned}
&\|u\|_2^2 \cdot \partial_t E_2(t) \\
&= \left(\int_{\mathbb{T}\times(0,1)} u^2 dxdy\right) \cdot \int_{\mathbb{T}\times(0,1)} \left|P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdy \\
&\geq \left(\int_{\mathbb{T}\times(0,1)} u\left(P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right) dxdy\right)^2 \\
&= \left(\int_{\mathbb{T}} \left(\int_0^1 u dy\right) P_x dx + \int_{\mathbb{T}\times(0,1)} \partial_x\left(\frac{u^3}{3}\right) dxdy - \int_{\mathbb{T}\times(0,1)} u^2 \frac{\omega\omega_x}{\omega_y} dxdy\right)^2 \\
&= \left(\int_{\mathbb{T}\times(0,1)} u^2 \frac{\omega\omega_x}{\omega_y} dxdy\right)^2 \\
&= E_2^2(t).
\end{aligned} \tag{5.2.41}$$

Thus

$$\partial_t E_2 \geq \|u\|_2^{-2} E_2^2,$$

where $\|u\|_2^2$ is time independent thanks to (4.1.13). If $E_2(0) > 0$, Lemma 8 yields

$$E_2(t) \geq \frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(0)} - t},$$

where $0 \leq t \leq T$. On the other hand, if $E_2(0) = 0$, Lemma 7 implies

$$\int_{\mathbb{T}\times(0,1)} \left|P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdy > 0,$$

at every time, from which we get

$$\int_0^{t'}\int_{\mathbb{T}\times(0,1)}\left|P_x+u\left(u_x-\frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdyd\tau=\int_0^{t'}\partial_\tau E_2 d\tau=E_2(t')>0,$$

for every $t'>0$. Choose t' such that $\omega\in C([0,T+t'];H^4(\mathbb{T}\times(0,1)))$ satisfies the local Rayleigh condition in $[0,T+t']$. Applying (5.2.39) to $\omega(t+t')$ we get

$$E_2(t'+t)\geq\frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(t')}-t},$$

for $0\leq t\leq T$, which concludes the proof of Proposition 12.

□

5.3. Proof of Theorem 15: Collapse of the local rayleigh condition or singularity formation

Proof. Let us proceed by contradiction. Suppose that $\|\omega(t)\|_{H^4(\mathbb{T}\times(0,1))}$ and $\left\|\frac{1}{\omega_y(t)}\right\|_{L^\infty(\mathbb{T}\times(0,1))}$ remain bounded for every time $t>0$. Assume first that $E_1(0)\geq 0$. By Proposition 12, E_1 blows up in finite time. Moreover, by (5.2.18) and Poincare's inequality

$$\begin{aligned}
&E_1(\tau)|_{\tau=0}^{\tau=t}\\
&=\int_0^t\int_{\mathbb{T}\times(0,1)}\left|u_x-\frac{\omega\omega_x}{\omega_y}\right|^2 dxdyd\tau\\
&\leq 2\int_0^t\int_{\mathbb{T}\times(0,1)}u_x^2+\left(\frac{\omega\omega_x}{\omega_y}\right)^2 dxdyd\tau\\
&\leq 2\int_0^t\int_{\mathbb{T}\times(0,1)}\left(\frac{\omega_x^2}{\pi^2}+\|\omega_0\|_\infty^2\left(\frac{\omega_x}{\omega_y}\right)^2\right)dxdyd\tau\\
&\leq\left(\frac{2}{\pi^2}+2\|\omega_0\|_\infty^2\right)\int_0^t\int_{\mathbb{T}\times(0,1)}\omega_x^2\left(1+\frac{1}{\omega_y^2}\right)dxdyd\tau \qquad (5.3.1)\\
&\leq\left(\frac{2}{\pi^2}+2\|\omega_0\|_\infty^2\right)\int_0^t\|\omega_x(\tau)\|_{L^2(\mathbb{T}\times(0,1))}^2\left(1+\left\|\frac{1}{\omega_y(\tau)}\right\|_{L^\infty(\mathbb{T}\times(0,1))}^2\right)d\tau,
\end{aligned}$$

which yields a contradiction because t is arbritrary and E_1 blows up in finite time. Next, if $E_2(0)\geq 0$, by Proposition 13, E_2 blows up in finite time. Furthermore, by (5.2.19)

$$\begin{aligned}
E_2(\tau)|_{\tau=0}^{\tau=t}&=\int_0^t\int_{\mathbb{T}\times(0,1)}\left|P_x+u\left(u_x-\frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdyd\tau\\
&\leq 2\int_0^t\int_{\mathbb{T}\times(0,1)}\left(P_x^2+u^2\left|u_x-\frac{\omega\omega_x}{\omega_y}\right|^2\right)dxdyd\tau.
\end{aligned}$$

By (4.1.12), Hölder's inequality, Poincare's inequality and (4.1.14)

$$\begin{aligned}\int_{\mathbb{T}} P_x^2 dx &= 4\int_{\mathbb{T}} \left(\int_0^1 u u_x dy\right)^2 dx \\ &\leq 4\int_{\mathbb{T}}\int_0^1 u^2 dy \int_0^1 u_x^2 dy dx \\ &\leq 4\left(\frac{3}{2}\|\omega_0\|_\infty\right)^2 \frac{1}{\pi^2}\int_{\mathbb{T}\times(0,1)} \omega_x^2 dxdy \\ &= \left(\frac{3}{\pi}\|\omega_0\|_\infty\right)^2 \int_{\mathbb{T}\times(0,1)} \omega_x^2 dxdy.\end{aligned}$$

Applying (5.3.1) and (4.1.14), we get

$$\begin{aligned}&E_2(\tau)|_{\tau=0}^{\tau=t} \\ &\leq \left(2\left(\frac{3}{\pi}\|\omega\|_\infty\right)^2 + 2\left(\frac{2}{\pi^2} + 2\|\omega_0\|_\infty^2\right)\right)\int_0^t \int_{\mathbb{T}\times(0,1)} \omega_x^2\left(1+\frac{1}{\omega_y^2}\right) dxdyd\tau \quad (5.3.2) \\ &\leq \left(6\|\omega_0\|_\infty + \frac{4}{\pi^2}\right)\int_0^t \|\omega_x(\tau)\|^2_{L^2(\mathbb{T}\times(0,1))}\left(1 + \left\|\frac{1}{\omega_y(\tau)}\right\|^2_{L^\infty(\mathbb{T}\times(0,1))}\right) d\tau,\end{aligned}$$

which again yields a contradiction because t is arbritrary and E_2 blows up in finite time. From now on, we will assume that $\|\omega(t)\|_{H^4(\mathbb{T}\times(0,1))}$ and $\|\frac{1}{\omega_y(t)}\|_{L^\infty(\mathbb{T}\times(0,1))}$ remain bounded, so we can apply the local well-posedness result by Masmoudi and Wong [20]. First let us prove (5.1.3). Assume that $E_1(0) > 0$. By Proposition 12,

$$\frac{1}{\frac{1}{E_1(0)} - \tau} \leq E_1(\tau).$$

Applying (5.2.17) and integrating in $\tau \in [0,t]$, we get:

$$\begin{aligned}\log\left(\frac{1}{E_1(0)}\frac{1}{\frac{1}{E_1(0)} - t}\right) &= \int_0^t \frac{1}{\frac{1}{E_1(0)} - \tau} d\tau \\ &\leq \int_0^t E_1(\tau) d\tau \\ &= \int_0^t \partial_\tau \int_{\mathbb{T}\times(0,1)} \log\left(\frac{1}{\omega_y(\tau)}\right) dxdyd\tau \\ &= \int_{\mathbb{T}\times(0,1)} \log\left(\frac{\partial_y\omega_0}{\omega_y(t)}\right) dxdy,\end{aligned}$$

which completes the proof of (5.1.3). Let us now deduce the estimate (5.1.4). Assume that $E_2(0) > 0$. By Proposition 13,

$$\left(\frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(0)} - t}\right)^2 \leq E_2^2(t). \quad (5.3.3)$$

Moreover, by Hölder's inequality and (5.2.18),

$$\begin{aligned} E_2^2(t) &= \left(\int_{\mathbb{T}\times(0,1)} u^2 \left(\frac{\omega\omega_x}{\omega_y} - u_x \right) dxdy \right)^2 \\ &\leq \|u\|_4^4 \int_{\mathbb{T}\times(0,1)} \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|^2 dxdy \\ &= \|u\|_4^4 \partial_t E_1. \end{aligned} \tag{5.3.4}$$

Therefore, by (5.3.3), (5.3.4) and (4.1.14),

$$\begin{aligned} \frac{\|u\|_2^4}{\frac{\|u\|_2^2}{E_2(0)} - t} - E_2(0)\|u\|_2^2 &= \int_0^t \left(\frac{\|u\|_2^2}{\frac{\|u\|_2^2}{E_2(0)} - \tau} \right)^2 d\tau \\ &\leq \int_0^t E_2^2(\tau) d\tau \\ &\leq \int_0^t \|u\|_4^4 \partial_\tau E_1(\tau) d\tau \\ &\leq \left(\frac{3}{2}\|\omega_0\|_\infty \right)^4 (E_1(t) - E_1(0)) \\ &\leq \left(\frac{3}{2}\|\omega_0\|_\infty \right)^4 (E_1(t) + |E_1(0)|). \end{aligned}$$

Thus

$$\frac{1}{\frac{\|u\|_2^2}{E_2(0)} - t} \leq \left(\frac{E_2(0)}{\|u\|_2^2} + \left(\frac{3}{2}\frac{\|\omega_0\|_\infty}{\|u\|_2} \right)^4 |E_1(0)| \right) + \left(\frac{3}{2}\frac{\|\omega_0\|_\infty}{\|u\|_2} \right)^4 E_1(t).$$

Integrating in time, we get

$$\log \left(\frac{\|u\|_2^2}{E_2(0)} \frac{1}{\frac{\|u\|_2^2}{E_2(0)} - t} \right) \leq tC_1(\omega_0) + C_2(\omega_0) \int_{\mathbb{T}\times(0,1)} \log \left(\frac{\partial_y \omega_0}{\omega_y(t)} \right) dxdy,$$

which concludes the proof of (5.1.4). Finally, Propositions 12 and 13, (5.3.1) and (5.3.2) lead to (5.1.5) and (5.1.6). This completes the proof of Theorem 15.

□

5.4. Proof of Theorem 16: Necessary conditions for global solvability

In this section we will prove Theorem 16, which establishes necessary conditions for the global solvability of (4.1.3). The following proposition will be useful for proving Theorem 16.

Proposition 14. *Assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T];H^4(\mathbb{T}\times(0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Let E_1 and E_2 be defined by (5.1.1) and (5.1.2) respectively. Then, we have:*

$$\int_0^T \int_{\mathbb{T}} P_x^2 dxdt \leq 2E_2|_{t=0}^{t=T} + \frac{9}{2}\|\omega_0\|_\infty^2 E_1|_{t=0}^{t=T}. \tag{5.4.1}$$

Proof. By (5.2.19), (5.2.18) and (4.1.14)

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}} P_x^2 dxdt &= \int_0^T \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right) + u\left(\frac{\omega\omega_x}{\omega_y} - u_x\right)\right|^2 dxdydt \\
&\leq 2\int_0^T \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 + \left|u\left(\frac{\omega\omega_x}{\omega_y} - u_x\right)\right|^2 dxdydt \\
&\leq 2\int_0^T \partial_t E_2(t)dt + 2\left(\frac{3}{2}\|\omega_0\|_\infty\right)^2 \int_0^T \int_{\mathbb{T}\times(0,1)} \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 dxdydt \\
&= 2\int_0^T \partial_t E_2(t)dt + 2\left(\frac{3}{2}\|\omega_0\|_\infty\right)^2 \int_0^T \partial_t E_1(t)dt \\
&= 2E_2|_{t=0}^{t=T} + \frac{9}{2}\|\omega_0\|_\infty E_1|_{t=0}^{t=T},
\end{aligned}$$

which concludes the proof of Proposition 14.

□

Now we are ready to prove Theorem 16.

Proof of Theorem 16. By Theorem 15, $E_1(t)$ and $E_2(t)$ remain negative, for all $t > 0$. Moreover, by (5.2.18)and (5.2.19), $E_1(t)$ and $E_2(t)$ are convergent (since they are monotone and bounded). Furthermore, by (5.2.38) and (5.2.41),

$$\int_0^t E_1^2 d\tau \leq E_1|_{\tau=0}^{\tau=t}. \tag{5.4.2}$$

$$\int_0^t E_2^2 d\tau \leq \|u\|_2^2 E_2|_{\tau=0}^{\tau=t}. \tag{5.4.3}$$

Which implies that $E_1(t), E_2(t) \to 0$, as $t \to \infty$ (otherwise (5.4.2) and (5.4.3) would growth arbitrarily, which is a contradiction). Thus,

$$E_1|_{\tau=0}^{\tau=t} = \int_0^t \int_{\mathbb{T}\times(0,1)} \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 dxdyd\tau \to -E_1(0), \quad \text{as } t \to \infty \tag{5.4.4}$$

and

$$E_2|_{\tau=0}^{\tau=t} = \int_0^t \int_{\mathbb{T}\times(0,1)} \left| P_x + u\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 dxdyd\tau \to -E_2(0), \text{ as } t \to \infty, \quad (5.4.5)$$

which proves (5.1.7) and (5.1.8). Next, Proposition 14, (5.4.4) and (5.4.5) lead to (5.1.9). Finally, by Jensen's inequality

$$\begin{aligned} \exp\left(\int_{\mathbb{T}\times(0,1)} \log\left(\omega_y(t)\right) dxdy\right) &\leq \int_{\mathbb{T}\times(0,1)} \omega_y dxdy \\ &\leq 2\|\omega_0\|_\infty. \end{aligned} \quad (5.4.6)$$

Thus, (5.2.17) and (5.4.6) implies

$$\begin{aligned} \int_0^T |E_1(t)|dt &= -\int_0^T E_1(t)dt \\ &= \int_{\mathbb{T}\times(0,1)} \log\left(\frac{\omega_y(T)}{\partial_y\omega_0}\right) dxdy \\ &\leq \int_{\mathbb{T}\times(0,1)} \log\left(\frac{2\|\omega_0\|_\infty}{\partial_y\omega_0}\right) dxdy, \end{aligned}$$

which gives (5.1.10) because T is arbitrary. This completes the proof of Theorem 16.

□

5.5. Propagation of the local Rayleigh condition and H^1 control

It is worth mentioning that Theorem 15 does not give any information about what happens first; the collapse of the local Rayleigh condition or the formation of singularities, even if the lower bounds (5.1.3) and (5.1.4) hold, Theorem 15 does not guarantees that $\|\omega\|_{H^4(\mathbb{T}\times(0,1))}$ remains bounded before the collapse of the local Rayleigh condition. Nevertheless, if we assume that the vorticity ω does not vanishes, then we can control the $L^\infty([0,T]; H^1(\mathbb{T}\times(0,1)))$ norm of ω and the $L^\infty([0,T]; L^\infty(\mathbb{T}\times(0,1)))$ norm of $\frac{1}{\omega_y}$ by means of

$$M(T) = \int_0^T \left| \frac{\omega\omega_x}{\omega_y} - u_x \right|_\infty d\tau, \quad (5.5.1)$$

which is the main result of this section. The next proposition will be useful for proving Theorem 18.

Proposition 15. *Assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ be the solution to (4.1.4) that satisfies the*

local Rayleigh condition in $[0,T]$. *Assume further that* $|\omega_0|>0$ *in* $\mathbb{T}\times(0,1)$. *Then, we have:*

$$|u_x|_\infty \leq \left(\frac{\|\omega_0\|_\infty}{\min|\omega_0|}+1\right)\left|\frac{\omega\omega_x}{\omega_y}-u_x\right|_\infty, \tag{5.5.2}$$

$$\int_{\mathbb{T}\times(0,1)}\frac{\omega_x^2}{\omega_y}dxdy|_{t=T} \leq \int_{\mathbb{T}\times(0,1)}\frac{|\partial_x\omega_{0|}^2}{\partial_y\omega_0}dxdy\exp\left(C(\omega_0)\int_0^T\left|\frac{\omega\omega_x}{\omega_y}-u_x\right|_\infty dt\right), \tag{5.5.3}$$

where $C(\omega_0)=3+\frac{2\|\omega_0\|_\infty}{\min|\omega_0|}$.

Proof. Let x be any point in $\mathbb{T}$. By (4.1.10), there exists $y_0(x)\in[0,1]$ such that $u_x(x,y_0(x),t)=0$. The positivity of $|\omega|$ yields

$$\begin{aligned}\frac{u_x}{\omega}(x,\tilde{y},t) &= \int_{y_0(x)}^{\tilde{y}}\frac{\omega\omega_x-u_x\omega_y}{\omega^2}dy\\ &= \int_{y_0(x)}^{\tilde{y}}\frac{\omega\omega_x-u_x\omega_y}{\omega_y}\frac{\omega_y}{\omega^2}dy,\end{aligned}$$

from which we get:

$$\begin{aligned}|u_x|(x,\tilde{y},t) &\leq |\omega|(x,\tilde{y},t)\left|\frac{\omega\omega_x-u_x\omega_y}{\omega_y}\right|_\infty\int_{y_0(x)}^{\tilde{y}}\frac{\omega_y}{\omega^2}dy\\ &= \left|\frac{\omega\omega_x-u_x\omega_y}{\omega_y}\right|_\infty\left|\frac{\omega(x,\tilde{y},t)}{\omega(x,y_0(x),t)}-1\right|\\ &\leq \left(\frac{\|\omega_0\|_\infty}{\min|\omega_0|}+1\right)\left|\frac{\omega\omega_x-u_x\omega_y}{\omega_y}\right|_\infty,\end{aligned}$$

for every $\tilde{y}\in[0,1]$, which leads to (5.5.2). On the other hand, by Lemmas 3 and 4:

$$\begin{aligned}D_t\left(\frac{\omega_x^2}{\omega_y}\right) &= \frac{-2u_x\omega_x^2\omega_y-2\omega_xv_x\omega_y^2-u_x\omega_x^2\omega_y+\omega\omega_x^3}{\omega_y^2}\\ &= \left(\frac{\omega\omega_x}{\omega_y}-3u_x\right)\frac{\omega_x^2}{\omega_y}-2\omega_xv_x,\end{aligned}$$

integrating in $\mathbb{T}\times(0,1)$, we get

$$\partial_t\int_{\mathbb{T}\times(0,1)}\frac{\omega_x^2}{\omega_y}dxdy=\int_{\mathbb{T}\times(0,1)}\frac{\omega_x^2}{\omega_y}\left(\frac{\omega\omega_x}{\omega_y}-3u_x\right)dxdy, \tag{5.5.4}$$

where we have used that

$$\int_{\mathbb{T}\times(0,1)}\omega_xv_xdxdy=\int_{\mathbb{T}\times(0,1)}u_xu_{xx}dxdy=0,$$

thanks to integration by parts and the x-periodicity. Thus, by (5.5.4) and (5.5.2)

$$\begin{aligned}\partial_t \int_{\mathbb{T}\times(0,1)} \frac{\omega_x^2}{\omega_y} dxdy &= \int_{\mathbb{T}\times(0,1)} \frac{\omega_x^2}{\omega_y}\left(\frac{\omega\omega_x}{\omega_y} - 3u_x\right) dxdy \\ &\leq \left(\left|\frac{\omega\omega_x - u_x\omega_y}{\omega_y}\right|_\infty + 2\left|u_x\right|_\infty\right) \int_{\mathbb{T}\times(0,1)} \frac{\omega_x^2}{\omega_y} dxdy \\ &\leq \left(3 + \frac{2\|\omega_0\|_\infty}{\min|\omega_0|}\right) \left|\frac{\omega\omega_x - u_x\omega_y}{\omega_y}\right|_\infty \int_{\mathbb{T}\times(0,1)} \frac{\omega_x^2}{\omega_y} dxdy,\end{aligned}$$

By applying Grönwall's inequality, we get (5.5.3), which completes the proof of Proposition 15.

□

Let us state the main result of this section.

Theorem 18. *Assume that $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition. Let $\omega \in C([0,T]; H^4(\mathbb{T}\times(0,1)))$ be the solution to (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Let E_1 and E_2 be defined by (5.1.1) and (5.1.2) respectively. Assume further that $|\omega_0| > 0$ in $\mathbb{T}\times(0,1)$. Then, we have:*

$$\left\|\frac{1}{\omega_y(t)}\right\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \left\|\frac{1}{\partial_y\omega_0}\right\|_{L^\infty(\mathbb{T}\times(0,1))} \exp(M(t)), \quad \text{for } 0 \leq t \leq T, \tag{5.5.5}$$

$$\|\omega_y(t)\|_{L^\infty(\mathbb{T}\times(0,1))} \leq \|\partial_y\omega_0\|_{L^\infty(\mathbb{T}\times(0,1))} \exp(M(t)), \quad \text{for } 0 \leq t \leq T, \tag{5.5.6}$$

$$\left\|\frac{\omega_x(t)}{\sqrt{\omega_y(t)}}\right\|_{L^2(\mathbb{T}\times(0,1))} \leq \left\|\frac{\partial_x\omega_0}{\sqrt{\partial_y\omega_0}}\right\|_{L^2(\mathbb{T}\times(0,1))} \exp\left(\frac{C(\omega_0)}{2}M(t)\right), \text{ for } 0 \leq t \leq T, \tag{5.5.7}$$

$$E_1(T) \leq \left\|\frac{\partial_x\omega_0}{\sqrt{\partial_y\omega_0}}\right\|_{L^2(\mathbb{T}\times(0,1))} \|\omega_0\|_\infty \left\|\frac{1}{\partial_y\omega_0}\right\|_{L^\infty(\mathbb{T}\times(0,1))} \exp\left(\tilde{C}(\omega_0)M(T)\right), \tag{5.5.8}$$

$$E_2(T) \leq \frac{9}{4}\left\|\frac{\partial_x\omega_0}{\sqrt{\partial_y\omega_0}}\right\|_{L^2(\mathbb{T}\times(0,1))} \|\omega_0\|_\infty^3 \left\|\frac{1}{\partial_y\omega_0}\right\|_{L^\infty(\mathbb{T}\times(0,1))} \exp\left(\tilde{C}(\omega_0)M(T)\right), \tag{5.5.9}$$

where

$$M(t) = \int_0^t \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|_{L^\infty(\mathbb{T}\times(0,1))} d\tau, \tag{5.5.10}$$

$C(\omega_0) = 3 + \frac{2\|\omega_0\|_\infty}{\min|\omega_0|}$ *and* $\tilde{C}(\omega_0) = \frac{1}{2} + \frac{C(\omega_0)}{2}$.

Proof. Let α be any point in $\mathbb{T}\times(0,1)$. By (5.2.11)

$$\begin{aligned}\left|\log(\omega_y(X(\alpha,\tau),Y(\alpha,\tau),\tau))|_{\tau=0}^{\tau=t}\right| &= \left|\int_0^t \partial_\tau \log(\omega_y(X(\alpha,\tau),Y(\alpha,\tau),\tau))d\tau\right| \\ &= \left|\int_0^t \left(\frac{\omega\omega_x}{\omega_y} - u_x\right)(X(\alpha,\tau),Y(\alpha,\tau),\tau)d\tau\right| \\ &\leq \int_0^t \left|\frac{\omega\omega_x}{\omega_y} - u_x\right|_{L^\infty(\mathbb{T}\times(0,1))} d\tau,\end{aligned}$$

where $(X(\alpha,\tau),Y(\alpha,\tau))$ is the characteristic curve starting at α. Thus, applying exponential we get (5.5.5) and (5.5.6). From (5.5.3) we get (5.5.7). By Hölder's inequality, (5.5.5) and (5.5.7)

$$\begin{aligned}\int_{\mathbb{T}\times(0,1))} \frac{|\omega\omega_x|}{\omega_y} dxdy &\leq \left\|\frac{\omega_x}{\sqrt{\omega_y}}\right\|_{L^2(\mathbb{T}\times(0,1))} \left\|\frac{\omega}{\sqrt{\omega_y}}\right\|_{L^2(\mathbb{T}\times(0,1))} \\ &\leq \|\omega_0\|_{L^\infty(\mathbb{T}\times(0,1))} \left\|\frac{1}{\partial_y\omega_0}\right\|_{L^\infty(\mathbb{T}\times(0,1))}^{\frac{1}{2}} \\ &\times \exp\left(\frac{M(t)}{2} + \frac{C(\omega_0)M(t)}{2}\right) \left\|\frac{\partial_x\omega_0}{\sqrt{\partial_y\omega_0}}\right\|_{L^2(\mathbb{T}\times(0,1))},\end{aligned} \tag{5.5.11}$$

which yields (5.5.8) and (5.5.9) thanks to (4.1.14). This concludes the proof of Theorem 18.

□

5.6. Proof of Theorem 17: Finite time blow-up of the Semi-lagrangian equations

The aim of this section is to prove Theorem 17, which establishes the finite time blow-up of solutions to the semilagrangian equations (4.7.6) for certain class of initial data. The following proposition will be useful for proving Theorem 17.

Proposition 16. *Let $(\mathbf{v}, h_a)$ be a smooth solution to (4.7.6). Let E_1 and E_2 be defined by (5.1.11) and (5.1.12) respectively. Then, we have:*

$$E_1 = \partial_t \int_{\mathbb{T}^d\times(0,1)} \log(h_a) h_a dxda, \tag{5.6.1}$$

$$\partial_t E_1 = \int_{\mathbb{T}^d\times(0,1)} |\nabla\mathbf{v}|^2 h_a dxda, \tag{5.6.2}$$

$$\partial_t E_2 = \int_{\mathbb{T}^d\times(0,1)} |\partial_t\mathbf{v}|^2 h_a dxda. \tag{5.6.3}$$

Proof. By (4.7.6) and the x-periodicity

$$\begin{aligned}\partial_t \int_{\mathbb{T}^d\times(0,1)} \log(h_a) h_a dxda &= \int_{\mathbb{T}^d\times(0,1)} \frac{\partial_t h_a}{h_a} h_a + \log(h_a)\partial_t h_a dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} -\nabla\cdot(\mathbf{v}h_a) - \log(h_a)\nabla\cdot(\mathbf{v}h_a) dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} \nabla(\log(h_a))\cdot(\mathbf{v}h_a) dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} \nabla(h_a)\cdot\mathbf{v} dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} (-\nabla\cdot\mathbf{v}) h_a dxda,\end{aligned}$$

which proves (5.6.1). Next, let us compute the time derivative of E_1:

$$\begin{aligned}&\partial_t E_1(t) \\ &= \int_{\mathbb{T}^d\times(0,1)} \nabla\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2} + P\right) h_a + \nabla\cdot\mathbf{v}\nabla\cdot(\mathbf{v}h_a) dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} (\mathbf{v}\cdot\Delta\mathbf{v}) h_a + (\nabla\mathbf{v}\cdot\nabla\mathbf{v}) h_a + \nabla\cdot\mathbf{v}\nabla\cdot(\mathbf{v}h_a) dxda + \int_{\mathbb{T}^d} \Delta P \int_0^1 h_a dadx \\ &= \int_{\mathbb{T}^d\times(0,1)} (\mathbf{v}\cdot\Delta\mathbf{v}) h_a + |\nabla\mathbf{v}|^2 h_a + \nabla\cdot((\nabla\cdot\mathbf{v})\mathbf{v}h_a) - \nabla(\nabla\cdot\mathbf{v})\cdot\mathbf{v}h_a dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} (\mathbf{v}\cdot\Delta\mathbf{v}) h_a + |\nabla\mathbf{v}|^2 h_a - (\Delta\mathbf{v}\cdot\mathbf{v}) h_a dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} |\nabla\mathbf{v}|^2 h_a dxda,\end{aligned}$$

where we have used the identity $\nabla(\nabla\cdot\mathbf{v}) = \Delta\mathbf{v}$, which follows by the curl free condition $\partial_i v_j = \partial_j v_i$. Thus, we proved (5.6.2). Finally, let us compute the time derivative of E_2:

$$\partial_t E_2(t) = \int_{\mathbb{T}^d\times(0,1)} \mathbf{v}_t\cdot\mathbf{v}_t h_a + \mathbf{v}\cdot\mathbf{v}_{tt} h_a - \mathbf{v}\cdot\mathbf{v}_t\nabla\cdot(\mathbf{v}h_a) dxda. \tag{5.6.4}$$

The second term is equal to:

$$\begin{aligned}\int_{\mathbb{T}^d\times(0,1)} \mathbf{v}\cdot\mathbf{v}_{tt} h_a dxda &= -\int_{\mathbb{T}^d\times(0,1)} \mathbf{v}\cdot\nabla\left(\mathbf{v}\cdot\mathbf{v}_t + P_t\right) h_a dxda \\ &= \int_{\mathbb{T}^d} P_t\nabla\cdot\int_0^1 \mathbf{v}h_a dadx + \int_{\mathbb{T}^d\times(0,1)} \mathbf{v}\cdot\mathbf{v}_t\nabla\cdot(\mathbf{v}h_a) dxda \\ &= \int_{\mathbb{T}^d\times(0,1)} \mathbf{v}\cdot\mathbf{v}_t\nabla\cdot(\mathbf{v}h_a) dxda,\end{aligned}$$

thanks to (4.7.22), which leads to (5.6.3). Thus the proof of Proposition 16 is now complete. □

Now we can prove Theorem 17.

Proof of Theorem 17. First assume that (5.1.14) holds. Integrating in time and applying (5.6.1), we get:

$$\begin{aligned} d\log\left(\frac{d}{E_1(0)}\frac{1}{\frac{d}{E_1(0)}-t}\right) &= \int_0^t \frac{d}{\frac{d}{E_1(0)}-\tau}d\tau \\ &\leq \int_0^t E_1(\tau)d\tau \\ &= \int_{\mathbb{T}^d\times(0,1)} \log(h_a(\tau))h_a(\tau)dxda|_{\tau=0}^{\tau=t}, \end{aligned}$$

which implies (5.1.13). Next, by (5.6.2) and Hölder's inequality

$$\begin{aligned} \partial_t E_1 &= \int_{\mathbb{T}^d\times(0,1)} |\nabla\mathbf{v}|^2 h_a dxda \\ &\geq \frac{1}{d}\int_{\mathbb{T}^d\times(0,1)} |\nabla\cdot\mathbf{v}|^2 h_a dxda \\ &\geq \frac{1}{d}\left(\int_{\mathbb{T}^d\times(0,1)} |\nabla\cdot\mathbf{v}| h_a dxda\right)^2 \\ &\geq \frac{1}{d}E_1^2. \end{aligned} \tag{5.6.5}$$

Thus, Lemma 8, (5.6.2) and (5.6.5) lead to (5.1.14). Finally, by (5.6.3) and Hölder's inequality

$$\begin{aligned} \int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}|^2 h_a dxda\cdot\partial_t E_2 &\geq \left(\int_{\mathbb{T}^d\times(0,1)} |\mathbf{v}||\mathbf{v}_t| h_a dxda\right)^2 \\ &\geq \left(\int_{\mathbb{T}^d\times(0,1)} \mathbf{v}\cdot\mathbf{v}_t h_a dxda\right)^2 \\ &= E_2^2. \end{aligned} \tag{5.6.6}$$

Thus, Lemma 8, (5.6.6), (5.6.3) and (4.7.20) lead to (5.1.15). This concludes the proof of Theorem 17. □

5.7. Appendix

5.7.1. Integral lemma

The aim of this appendix is to prove the integral identity conerning the integral of the logarithm of a function.

Lemma 9. *Let (Ω, μ) be a measure space and let $f \in L^1(\Omega)$ be a positive function in Ω. Assume that $\mu(\Omega) = 1$, then*

$$\exp\left(\int_\Omega \log(f) d\mu\right) = \lim_{p\to 0}\left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}}. \tag{5.7.1}$$

Proof. Let us proceed as in [10], let p_n be any positive sequence converging to 0. By Jensen's inequality,

$$\exp\left(\int_\Omega \log(f) d\mu\right) \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}}, \quad \forall p > 0,$$

which implies

$$\exp\left(\int_\Omega \log(f) d\mu\right) \leq \liminf\left(\int_\Omega f^{p_n} d\mu\right)^{\frac{1}{p_n}}. \tag{5.7.2}$$

Since $\log t \leq t - 1$, for $t > 0$, we conclude that

$$\log\left(\int_\Omega f^{p_n} d\mu\right) \leq \int_\Omega (f^{p_n} - 1) d\mu. \tag{5.7.3}$$

Multiplying (5.7.3) by $\frac{1}{p_n}$ and applying exponential, we get

$$\left(\int_\Omega f^{p_n} d\mu\right)^{\frac{1}{p_n}} \leq \exp\left(\frac{1}{p_n}\int_\Omega (f^{p_n} - 1)\, d\mu\right),$$

which implies

$$\limsup\left(\int_\Omega f^{p_n} d\mu\right)^{\frac{1}{p_n}} \leq \limsup \exp\left(\frac{1}{p_n}\int_\Omega (f^{p_n} - 1)\, d\mu\right). \tag{5.7.4}$$

Now, let us compute $\limsup \exp\left(\frac{1}{p_n}\int_\Omega (f^{p_n} - 1)\, d\mu\right)$. Assume that $q_n \searrow 0$, then

$$\frac{1}{q_n}(t^{q_n} - 1) \searrow \log(t), \quad \forall t > 0\ . \tag{5.7.5}$$

Let $h_n : \Omega \to \mathbb{R}$ be a sequence of positive functions defined by

$$h_n(x) = \frac{1}{q_0}(f^{q_0} - 1) - \frac{1}{q_n}(f^{q_n} - 1).$$

By (5.7.5),

$$h_n(x) \nearrow \frac{1}{q_0}(f^{q_0} - 1) - \log(f).$$

Applying Lebesgue's monotone convergence theorem, we get

$$\int_\Omega h_n d\mu \nearrow \int_\Omega \frac{1}{q_0}(f^{q_0} - 1) - \log(f) d\mu.$$

Thus,

$$\int_\Omega \frac{1}{q_n}\left(f^{q_n}-1\right)d\mu \searrow \int_\Omega \log(f)d\mu.$$

Therefore,

$$\limsup \exp\left(\frac{1}{p_n}\int_\Omega \left(f^{p_n}-1\right)d\mu\right) = \exp\left(\int_\Omega \log(f)d\mu\right). \tag{5.7.6}$$

Finally, by (5.7.2), (5.7.4) and (5.7.6)

$$\limsup \left(\int_\Omega f^{p_n}d\mu\right)^{\frac{1}{p_n}} \leq \exp\left(\int_\Omega \log(f)d\mu\right) \leq \liminf \left(\int_\Omega f^{p_n}d\mu\right)^{\frac{1}{p_n}}.$$

This concludes the proof of Lemma 9.

□

5.7.2. Convergence of the pressure for global solutions

The aim of this appendix is to prove the convergence in $L^2(\mathbb{T})$ of the pressure $P(x,t)$ as $t\to\infty$ for the global solutions to (4.1.3) that satisfy the local Rayleigh condition $\partial_y^2 u > 0$.

Proposition 17. *Assume that the initial vorticity $\omega_0 \in H^4(\mathbb{T}\times(0,1))$ satisfies the local Rayleigh condition (i.e. $\partial_y\omega_0 > 0$). Let $\omega \in C([0,T];H^4(\mathbb{T}\times(0,1)))$ be the solution to the problem (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Assume further that T can be chosen arbitrarily large, then we have:*

$$\int_{\mathbb{T}} \left|P(x,t)+\|u\|^2_{L^2(\mathbb{T}\times(0,1))}\right|^2 dx \to 0 \quad \text{as } t\to\infty. \tag{5.7.7}$$

Proof. Since u is global and convex in y we have that $E_1(t) < 0$ for all $t \geq 0$ (see Theorem 16). Let E_3 be defined by:

$$E_3 = \int_{\mathbb{T}\times(0,1)} u^4\left(\frac{\omega\omega_x}{\omega_y}-u_x\right) - u^3P_x dxdy. \tag{5.7.8}$$

Applying (5.2.13), Lemma 3 and (5.2.5), we get

$$\begin{aligned}
&D_t\left(u^2\left(u^2\left(\frac{\omega\omega_x}{\omega_y}-u_x\right)-uP_x\right)\right)\\
&= u^2\left|u\left(\frac{\omega\omega_x}{\omega_y}-u_x\right)-P_x\right|^2 - u^3P_{xt} - 2uP_x\left(u^2\left(\frac{\omega\omega_x}{\omega_y}-u_x\right)-uP_x\right)\\
&= u^4\left|\frac{\omega\omega_x}{\omega_y}-u_x\right|^2 - 4u^3P_x\left(\frac{\omega\omega_x}{\omega_y}-u_x\right) + 3u^2P_x^2 - u^3P_{xt}\\
&= \left|u^2\left(\frac{\omega\omega_x}{\omega_y}-u_x\right)-2uP_x\right|^2 - u^2P_x^2 - u^3P_{xt}.
\end{aligned} \tag{5.7.9}$$

Next, observe that

$$P_t(x,t) = \partial_x \int_0^1 u^3(x,z,t)dz. \tag{5.7.10}$$

Then, by Lemma 3, (5.7.9) and (5.7.10)

$$\begin{aligned}
\partial_t E_3 &= \int_{\mathbb{T}\times(0,1)} \left| 2uP_x + u^2\left(u_x - \frac{\omega\omega_x}{\omega_y}\right)\right|^2 - u^2P_x^2 + P_t^2 dxdy \qquad (5.7.11)\\
&= \int_{\mathbb{T}\times(0,1)} \left| u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right|^2 + P_t^2 + 2u^2P_x^2 - 2u^3P_x\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) dxdy\\
&\geq \int_{\mathbb{T}\times(0,1)} \left| u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right|^2 + P_t^2 - \frac{1}{2}u^4\left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 dxdy. \qquad (5.7.12)
\end{aligned}$$

Set $C > \left(\frac{3}{2}\|\omega_0\|_\infty\right)^4$, by (5.7.12), (5.2.18), (4.1.14) and Hölder's inequality

$$\begin{aligned}
\partial_t(E_3 + CE_1) &\geq \int_{\mathbb{T}\times(0,1)} \left| u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right|^2 + P_t^2 + \frac{C}{2}\left|\frac{\omega\omega_x}{\omega_y} - u_x\right|^2 dxdy\\
&\geq \frac{1}{\|u\|_4^4}\int_{\mathbb{T}\times(0,1)} u^4 dxdy \int_{\mathbb{T}\times(0,1)} \left| u^2\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - uP_x\right|^2 dxdy + \frac{C}{2}E_1^2\\
&= \frac{1}{\|u\|_4^4}\left(\int_{\mathbb{T}\times(0,1)} \left(u^4\left(\frac{\omega\omega_x}{\omega_y} - u_x\right) - u^3P_x\right) dxdy\right)^2 + \frac{C}{2}E_1^2\\
&\geq \frac{E_3^2}{\|u\|_4^4} + \frac{C}{2}E_1^2\\
&\geq \frac{1}{C}\left(E_3^2 + \frac{C^2}{2}E_1^2\right)\\
&\geq \frac{1}{4C}\left(E_3 + CE_1\right)^2,
\end{aligned}$$

which implies $E_3(t) \leq -CE_1(t)$ for all $t > 0$, otherwise $E_3 + CE_1$ would blow-up in finite time, as a consequence of Lemma 8. Evidently E_3 is convergent (since $E_3(t) + CE_1(t)$ is increasing and bounded by 0). Furthermore, by (5.7.11) and Hölder's inequality:

$$\int_0^T \frac{E_3^2}{\|u\|_4^4} + \int_0^T \int_{\mathbb{T}} P_t^2 dxdt \leq E_3|_{t=0}^{t=T} + \int_0^\infty \int_{\mathbb{T}\times(0,1)} u^2P_x^2 dxdt, \tag{5.7.13}$$

from which we deduce that E_3 must convege to 0 (if not, then E_3 converges to a negative number, which makes the left hand side arbitrarily large, which is a contradiction). Applying (5.7.13) and (5.1.9), we have:

$$\int_0^\infty \int_{\mathbb{T}} P_t^2 dxdt < \infty. \tag{5.7.14}$$

Observe that

$$\begin{aligned}\int_{\mathbb{T}} \left|P + \|u\|_2^2\right|^2 dx|_{t=0}^{t=T} &= \int_0^T \partial_t \int_{\mathbb{T}} \left|P + \|u\|_2^2\right|^2 dxdt \\ &= 2\int_0^T \int_{\mathbb{T}} P_t \left(P + \|u\|_2^2\right) dxdt.\end{aligned}$$

Moreover, by (5.7.14), Poincare's inequality and (5.1.9)

$$\begin{aligned}\int_0^T \left|\partial_t \int_{\mathbb{T}} \left|P + \|u\|_2^2\right|^2 dx\right| dt &\leq 2\int_0^T \int_{\mathbb{T}} \left|P_t \left(P + \|u\|_2^2\right)\right| dxdt \\ &\leq \int_0^T \int_{\mathbb{T}} P_t^2 + \left|P + \|u\|_2^2\right|^2 dxdt \\ &\leq \int_0^T \int_{\mathbb{T}} P_t^2 + \pi^{-2} P_x^2 dxdt \\ &< \infty,\end{aligned}$$

which implies the convergence of $\int_{\mathbb{T}} |P + \|u\|_2^2|^2 \, dx$ as $t \to \infty$. On the other hand

$$\int_{\mathbb{T}} \left|P + \|u\|_2^2\right|^2 dx \leq \pi^{-2} \int_{\mathbb{T}} P_x^2 dx \text{ and } \int_0^\infty \int_{\mathbb{T}} P_x^2 dxdt < \infty,$$

lead to

$$\int_{\mathbb{T}} \left|P + \|u\|_2^2\right|^2 dx \to 0 \text{ as } t \to \infty.$$

(Note that the above yields the convergence of $\|u\|_4^4$ as $t \to \infty$, because $\|u\|_4^4 - 2\|P\|_2^2$ is time independent). This concludes the proof of Proposition 17.

□

5.7.3. Growth of certain smooth global solutions

The aim of this appendix is to establish a lower bound for $\|\omega_y(t)\|_{L^\infty(\mathbb{T}\times(0,1))}$ for global solutions to (4.1.3) that satisfy the local Rayleigh condition $\partial_y^2 u > 0$.

Proposition 18. *Assume that the initial vorticity $\omega_0 \in H^4(\mathbb{T} \times (0,1))$ satisfies the local Rayleigh condition $\partial_y \omega_0 > 0$ and $\partial_x \omega_0$ is not identically 0 on the boundary. Let $\omega \in C([0,T]; H^4(\mathbb{T} \times (0,1)))$ be the solution to the problem (4.1.4) that satisfies the local Rayleigh condition in $[0,T]$. Assume further that T can be chosen arbitrarily large, then we have:*

$$C(\omega_0)\sqrt{T} \leq \sup_{t\in[0,T]} \|\omega_y(t)\|_{L^\infty(\mathbb{T}\times(0,1))}, \tag{5.7.15}$$

where $C(\omega_0)$ is given by

$$C(\omega_0) = \sqrt{\sup_{x\in\mathbb{T}, y=0,1} -\omega_0 \partial_x \omega_0 \partial_y \omega_0(x,y)}.$$

Proof. Without loss of generality, assume that

$$\sup_{x\in\mathbb{T},y=0,1} -\omega_0\partial_x\omega_0\partial_y\omega_0(x,y) = -\omega_0\partial_x\omega_0\partial_y\omega_0(x_0,0) > 0.$$

Set $f(t) = -\omega\omega_x\omega_y(X(x_0,0,t),0,t)$, where $X(x_0,0,t)$ is the x component of the characteristic curve starting at $(x_0,0)$. By Lemma 3, (5.2.9) and (5.2.9)

$$\begin{aligned}
\partial_t f(t) &= \left(v_x\omega\omega_y^2 + \omega^2\omega_x^2\right)(X(0,x_0,t),0,t) \\
&= \omega^2\omega_x^2(X(0,x_0,t),0,t) \\
&= \frac{f^2(t)}{\omega_y^2(X(x_0,0,t),0,t)} \\
&\geq \frac{f^2(t)}{\|\omega_y(t)\|^2_{L^\infty(\mathbb{T}\times(0,1))}},
\end{aligned}$$

which implies:

$$\begin{aligned}
\frac{1}{f(0)} - \frac{1}{f(T)} &= \int_0^T \frac{\partial_t f}{f^2} dt \\
&\geq \int_0^T \frac{dt}{\|\omega_y(t)\|^2_{L^\infty(\mathbb{T}\times(0,1))}}, \\
&\geq \frac{T}{\sup_{t\in[0,T]}\|\omega_y(t)\|^2_{L^\infty(\mathbb{T}\times(0,1))}},
\end{aligned}$$

from which we get:

$$\left(\frac{1}{f(0)} - \frac{T}{\sup_{t\in[0,T]}\|\omega_y(t)\|^2_{L^\infty(\mathbb{T}\times(0,1))}}\right)^{-1} \leq f(T).$$

Since ω is a global smooth solution, f cannot blow-up in finite time, which yields:

$$f(0)T < \sup_{t\in[0,T]} \|\omega_y(t)\|^2_{L^\infty(\mathbb{T}\times(0,1))},$$

which proves Propostion 18.

□

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