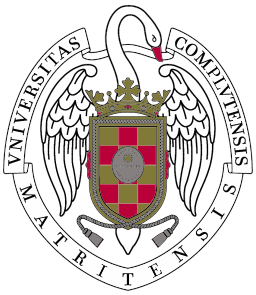
Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Acotación de soluciones de ecuaciones parabólicas logísticas no autónomas.

Boundedness of solutions of nonautonomous logistic parabolic equations.

Memoria para optar al grado de doctor presentada por

Marcos Molina Rodríguez

Director:

José María Arrieta Algarra

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UNIVERSIDAD COMPLUTENSE MADRID

DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

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estudiante en el Programa de Doctorado D9BE - Doctorado en investigación matemática,
de la Facultad de Ciencias Matemáticas de la Universidad Complutense de Madrid, como autor/a de la tesis presentada para la obtención del título de Doctor y titulada:

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Abstract

This dissertation focuses on studying under which conditions solutions of a particular nonautonomous logistic parabolic boundary value problem can be bounded or unbounded. Boundedness of solutions is the first step towards studying the asymptotic behaviour of solutions which is the ultimate goal of this research. Moreover, unboundedness results in this research also contribute to studying where are solutions exploding (in infinite time). Furthermore, boundedness of solutions is a necessary prerequisite in order to study periodic solutions and orbits. Therefore this research is the first and crucial step towards understanding asymptotic dynamics.

This research is divided into an introduction plus 4 chapters. The first chapter are technical results used in the rest of the dissertation, the second chapter presents a result on an autonomous problem. The third and fourth chapters are dedicated to the nonautonomous problem and to the conditions under which boundedness of solutions can be obtained.

Resumen

Esta tesis se centra en estudiar bajo qué condiciones las soluciones de un problema parabólico logístico no autónomo pueden ser acotadas o no acotadas. La acotación de las soluciones es el primer paso hacia el estudio del comportamiento asintótico de las soluciones, una cuestión muy importante y relevante para estos sistemas. Por otra parte, los resultados de no acotación de soluciones en esta investigación contribuyen a estudiar dónde explotan las soluciones (en tiempo infinito). Además, la acotación de las soluciones es un prerrequisito necesario para estudiar otros comportamientos dinámicos asintóticos de las soluciones, como la convergencia hacia el equilibrio, la existencia de órbitas periódicas o incluso soluciones cuasi o casi periódicas. Por lo tanto, esta investigación es el primer y crucial paso hacia la comprensión de la dinámica asintótica.

Esta investigación se divide en una introducción y 4 capítulos. El primer capítulo son resultados técnicos utilizados en el resto de la disertación, el segundo capítulo presenta un resultado sobre un problema autónomo. Los capítulos tercero y cuarto están dedicados al problema no autónomo y a las condiciones bajo las cuales puede obtenerse la acotación de las soluciones.

Introduction

This dissertation focuses on studying under which conditions solutions of nonautonomous logistic parabolic boundary value problem can be bounded or unbounded. Boundedness of solutions is the first step towards studying the asymptotic behaviour of solutions a very important and relevant question to address for this systems. On the other hand, unboundedness results in this research also contribute to studying where are solutions exploding (in infinite time). Furthermore, boundedness of solutions is a necessary prerequisite in order to study other asymptotic dynamical behaviour of solutions like convergence towards equilibria, existence of periodic orbits or even quasi or almost periodic solutions. Therefore this research is the first and crucial step towards understanding asymptotic dynamics.

Let us first introduce the specific nonautonomous logistic parabolic boundary value problem studied in this thesis by building upon the logistic differential equation. We will present both the mathematical point of view as well as the perspective from population dynamics with which the problem studied is intimately related to (see [10]).

Introduction

The logistic differential equation is closely related to population dynamics and it was first introduced by Pierre Verhulst in the mid XIX century in order to study human population. In its simplest form it can adopt the following expression:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P\left(1 - \frac{P}{B}\right) = \lambda P - \frac{\lambda}{B}P^2, \tag{0.0.1}$$

where P is the number of individuals in the population, which is assumed to be always positive, the variable t is the time, λ is the intrinsic growth rate of the population which is assumed to be positive and B is the carrying capacity of the media which is also assumed to be positive. The left hand side of the equation represents the variation of the population with respect to the time variable, whilst the right hand side of the equation has two opposing terms: a positive linear term (λP) and a negative nonlinear term ($-\lambda/BP^2$). Whenever P is less than the carrying capacity of the media, that is $P < B$, then the right hand side of the equation is positive, which implies that the derivative of P must also be positive and that the population grows. On the other hand, whenever $P > B$ derivative will be negative and the population decreases. Regarding equation (0.0.1), notice that

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P - \frac{\lambda}{B}P^2 = 0 \Leftrightarrow \begin{cases} P = 0, \\ P = B. \end{cases}$$

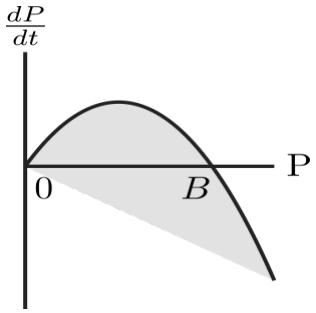


Figure 1: Graph of the derivative $\mathrm{d}P/\mathrm{d}t$.

That is, we have two equilibrium points: $P = 0$ which is unstable and $P = B$ which is a stable equilibrium. In particular, this implies that whenever the solution P is greater than the equilibrium point B, it forces itself to decrease. This is of special relevance since, unlike other popular models used in population dynamics, the logistic equation has a self–limitating component. This self–limitation is determined by the nonlinear term and is modulated by the exponent in P: the higher the exponent, the more aggressively the solutions limits itself.

The next natural step when modelling population dynamics is to consider that the population and the media have some spatial dependence. In particular the species is assumed to live in a certain habitat given by a domain Ω so that the population may depend on $x \in \Omega$ and the parameters appearing in the equation may also depend on the spatial variable. Moreover, it is natural to incorporate to the mathematical model the fact that the species may diffuse in the domain Ω. By incorporating these features, we arrive at the following semilinear parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = \lambda u(t,x) - n(x)|u(t,x)|^{\rho-1}u(t,x), & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{0.0.2}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded domain (the spatial location the species inhabits), $\rho > 1$ for a more general exponent, $u(t,x)$ is now the population density at time t and in point x, and u_0 is the initial population density which is considered non–negative since we are interested in positive solutions. We assume that the intrinsic growth rate is given by $\lambda > 0$ but the limitations of the environment are now x–dependent and expressed in the function $n(x) \geq 0$. Homogeneous Dirichlet boundary conditions imply that outside Ω the environment is hostile towards the species and no individual of that species is able to survive outside Ω. In the case $n(x) \equiv \beta > 0$, equation (0.0.2) shares some similarities with its ODE counterpart (0.0.1). In particular, it is self–limiting and, if $\lambda > 0$, the system has a nonnegative stable equilibrium which is given either by the trivial solution or, if λ is sufficiently large, by the unique positive solution (in case it exists) of

$$\begin{cases} -\Delta u_{eq} = \lambda u_{eq} - \beta u_{eq}^{\rho}, & x \in \Omega, \\ u_{eq} = 0, & x \in \partial\Omega. \end{cases}$$

More generally, if λ is large enough, the equilibrium will be given by the unique positive solution (in case it exists) of:

$$\begin{cases} -\Delta u_{eq} = \lambda u_{eq} - n(x) u_{eq}^{\rho}, & x \in \Omega, \\ u_{eq} = 0, & x \in \partial\Omega. \end{cases}$$

Observe that, the inclusion of the Laplace operator grants us now diffusion within Ω, that is, the solution now tends to redistribute throughout space. This is a relevant property of (0.0.2) since if

the solution is at any time concentrated in a particular spatial location, the parabolic operator will swiftly redistribute the population over the available space.

A different, but very interesting situation is when we allow the function $n(x)$ to vanish in certain region in Ω. Hence, let us define

$$K_0 := \{x \in \Omega : n(x) = 0\},$$

which is assumed to be compact. Therefore, three key effects take place in this reaction–diffusion model (0.0.2): the linear term λu pulls the solution towards $+\infty$, the self–limiting effect of the nonlinear term pulls down the solution (towards 0 or towards u_{eq}) wherever $n(x) > 0$ and the diffusion which tends to redistribute the solution. Hence, "inside" the set K_0 the nonlinear effect pulling the solution downwards disappears and solutions will generally grow. This implies that within Ω we will have two different zones: in K_0 solutions will tend to grow and outside K_0 solutions will tend to decrease due to the self–limiting property. That is, there is a competition between those two zones within Ω. Then, it is natural to ask whether the asymptotic behaviour of solutions of (0.0.2) will be roughly the same as in the case $n(x) \geq \beta > 0$ or, on the other hand, the presence of K_0 changes how solutions behave.

From a population dynamics perspective the set K_0 represents a gifted location where the species is able to grow whereas $\Omega \setminus K_0$ represents a location where the species encounters hardships such as predators, harsh terrain, low resources or intraspecies competition. We refer to K_0 as a "sanctuary" for the species. Then, by means of the diffusion, part of the growth attained in K_0 is migrated outside K_0 where the hard conditions threaten the growth of the species. It is natural to see that the size of K_0 and the strength of the intrinsic growth rate λ will play a key role determining whether the species keeps growing forever or remains bounded.

Whenever n vanishes inside Ω, equation (0.0.2) is often referred to as a degenerate logistic parabolic equation and has been extensively studied. The term degenerate is refered to the fact that $n(x)$ vanishes in some region. Taking into account the biological interpretation, it seems natural to consider non negative initial data, so that solutions are positive due to the maximum principle. Also, in many places in the literature it is considered the case where the function $n(x)$ is regular (at least continuous) and K_0 is the closure of a regular open set Ω_0, that is $K_0 = \bar{\Omega}_0$. With these conditions, the asymptotic behaviour of solutions of (0.0.2) has been studied. Denoting by λ_1^Ω as the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions on Ω, it is known that if $\lambda < \lambda_1^\Omega$ then 0 is the unique equilibrium which is globally asymptotically stable and therefore all solutions approach 0 when $t \to \infty$. If $\lambda > \lambda_1^\Omega$ then 0 becomes unstable, and solutions tend to grow away from it. As a matter of fact, if $\lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}$ then solutions tend to the unique globally asymptotically stable positive equilibrium u_{eq}. On the other hand, if $\lambda > \lambda_1^{\Omega_0}$ then solutions have been shown to grow and become unbounded as $t \to \infty$ only on K_0 but they remain bounded outside K_0, which will be referred to as "grow up". Hence, these results can be summarised as follows:

$$\begin{cases} \text{If } \lambda < \lambda_1^\Omega & \Rightarrow u \to 0, \\ \text{If } \lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0} & \Rightarrow u \to u_{eq}, \\ \text{If } \lambda_1^{\Omega_0} < \lambda & \Rightarrow u(x) \to \infty, \quad x \in \Omega_0. \end{cases} \tag{0.0.3}$$

Notice that the asymptotic behaviour of solutions is determined by λ, λ_1^Ω and $\lambda_1^{\Omega_0}$. We refer to the monograph [29] for a thorough and very complete study of problem (0.0.2) (even with a slightly more general formulation) for the "smooth" case, that is $n(x)$ and K_0 very regular. In this book, the author introduces the concept of metasolutions, which are the limiting profiles of the solution u of (0.0.2) when $t \to \infty$ when $\lambda > \lambda_1^{\Omega_0}$. These limiting profiles are, essentially speaking, singular

solutions to the elliptic equation $-\Delta u = \lambda u - n(x)|u|^{\rho-1}u$ with $u = \infty$ in Ω_0. Simplifying the more elaborate structures in [29], metasolutions are defined as zero on the boundary $\partial\Omega$, infinite on Ω_0 and outside Ω_0 as the solution of the singular problem

$$\begin{cases} -\Delta u = \lambda u - n(x)|u|^{\rho-1}u & \text{in } \Omega \setminus \bar{\Omega}_0, \\ u = \infty & \text{on } \partial\Omega_0. \end{cases} \tag{0.0.4}$$

Solutions of (0.0.4) are called large solutions. In the first part of the book, the author obtains the existence of minimal and maximal large solutions of (0.0.4) which are the essential parts of the definition of metasolutions as mentioned above. The minimal large solution is shown to be the limit $u_{\lambda,D,\infty} = \lim_{M\to\infty} u_{\lambda,D,M}$ where $u_{\lambda,D,M}$ is the solution of

$$\begin{cases} -\Delta u = \lambda u - n(x)|u|^{\rho-1}u & \text{in } D = \Omega \setminus \bar{\Omega}_0, \\ u = M & \text{on } \partial\Omega_0, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The maximal large solution is, in turn, obtained as the point–wise limit of minimal large solutions on a sequence of smaller domains $(D_n) \subset D$. Finally the author uses metasolutions for the autonomous parabolic logistic equation and arrives, roughly speaking, to (0.0.3) and a complete description of the dynamics of (0.0.2), amongst many other interesting results.

An extension of (0.0.3) to the case where K_0 is only assumed to be compact without any further regularity assumptions was studied in [7]. In [7], the role of $\lambda_1^{\Omega_0}$ is substituted by $\lambda_0(K_0)$, the characteristic value of the set K_0, which is defined as follows: consider $\{\Omega_\delta\}_{0<\delta\leq\delta_0}$ a nested decreasing family of smooth open sets satisfyng $K_0 = \cap_{0<\delta\leq\delta_0}\Omega_\delta$ and define

$$\lambda_0(K_0) := \lim_{\delta\to 0^+} \lambda_1^{\Omega_\delta}.$$

Notice that $\lambda_0(K_0)$ can be $+\infty$ if K_0 is "small", for example, when K_0 is a point in Ω. If K_0 is regular, that is, $K_0 = \bar{\Omega}_0$ for some regular open set Ω_0, then it is possible to show that $\lambda_0(K_0) = \lambda_1^{\Omega_0}$. In [7] authors show that if $\lambda < \lambda_0(K_0)$ then all solutions are bounded and if $\lambda \geq \lambda_0(K_0)$ solutions grow up without bound as time goes to $+\infty$. Moreover, in general terms observe that the smaller K_0 is, the larger $\lambda_0(K_0)$ is. Therefore, smaller K_0 imply that it is more likely to have $\lambda < \lambda_0(K_0)$ and thus, boundedness. And viceversa, the larger the set K_0, the smaller $\lambda_0(K_0)$ is and the more likely that solutions will grow. With this characteristic value, it has been shown that when $\lambda_1^\Omega < \lambda < \lambda_0(K_0)$ all solutions of (0.0.2) are bounded and tend to the unique globally asymptotically stable equilibrium. However, if $\lambda \geq \lambda_0(K_0)$ all solutions grow up, that is, all solutions of (0.0.2) become unbounded as $t \to \infty$.

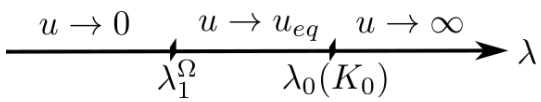


Figure 2: If $\lambda < \lambda_1^\Omega$ then solutions approach the stable equilibrium at 0. If $\lambda_1^\Omega < \lambda < \lambda_0(K_0)$ then solutions approach the unique positive globally asymptotically stable equilibrium u_{eq}. And, if $\lambda \geq \lambda_0(K_0)$ then solutions approach $+\infty$ as $t \to \infty$.

Therefore, whether regularity assumptions on K_0 are taken or not, the asymptotic behaviour of solutions of (0.0.2) is well researched and described. As a matter of fact, the asymptotic behaviour of solutions is intimately related to the relationship between λ and $\lambda_0(K_0)$.

Additionally it should be mentioned that there is plenty of research on the nonautonomous version of (0.0.2) and very specifically, on the remarkably relevant periodic problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = \lambda(t,x)u(t,x) - n(t,x)|u(t,x)|^{\rho-1}u(t,x), & t \in (0,T),\ x \in \Omega, \\ u(t,x) = 0, & t \in (0,T),\ x \in \partial\Omega, \end{cases} \tag{0.0.5}$$

where λ, n are T–periodic in time for some $T > 0$ and all $x \in \Omega$. Periodic boundary value problems (PBVP) have been studied extensively throughout the previous decades, see for instance [3, 12, 22] which are classical references for the existence of periodic solutions for semilinear equations.

Following a similar approach as in [29], authors in [1, 2] characterise the existence of positive periodic solutions for the PBVP associated to $u_t - \mathcal{L}u = \lambda(t,x)u - n(t,x)f(u)u$ where $\mathcal{L}$ is a uniformly elliptic operator, $f \in C^1$ and K_0 is assumed to be regular. In [1] authors impose strict restrictions on $n(t,x)$ assuming it is either strictly positive on Ω or it is static, that is, the set where n nullifies is not time–dependent, that is $\{x \in \Omega : n(t,x) = 0\} = K_0$ independent of t. However, very recently in [2] authors manage to generalise previous results droping the requirement that the set $\{x \in \Omega : n(t,x) = 0\}$ is independent of time.

There are many other type of PBVP derived from (0.0.2), which exceed the goals of this research and will be left for the reader to dive in further. See for instance [16] where the authors study problem (0.0.5) for λ constant and $n(t,x) = p(x)q(t)$ for some functions p, q, or for instance [13] where authors study the following problem where the time–dependent diffusion coefficient tends to zero:

$$\begin{cases} u_t - k(t)\Delta u = \lambda(t,x)u - n(t,x)u^2, & t \in (0,T), x \in \Omega, \\ u = 0, & t \in (0,T), x \in \partial\Omega. \end{cases}$$

We would like to stress that our focus is on general nonautonomous problems, non necessarily periodic problems.

Objective

This research started with the aim of addressing the general problem of asymptotic behaviour of the solutions of the following nonautonomous logistic parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = \lambda u(t,x) - n(t,x)|u(t,x)|^{\rho-1}u(t,x), & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{0.0.6}$$

where the logistic function n is allowed now to depend also on time, that is, $n = n(t,x)$. As mentioned above, a necessary first issue that must be addressed is the question of the boundedness of the solutions of (0.0.6), that is, to characterise when solutions are bounded as time goes to infinity or, on the contrary, solutions grow without bound. This is the question that we address in this thesis. We remark that the dependence of $n(t,x)$ on t is not prescribed and we are not restricting to the (very important) periodic case.

Moreover, we are considering rather general function $n(t,x)$ with not very restricted requirements on its smoothness.

If we define $K(t) := \{x \in \Omega : n(t,x) = 0\}$ then it is clear that the family of sets $K(t)$ might not be static. Observe that since $K(t)$ is now time dependent, $K(t)$ can expand, contract, rotate and can move smoothly inside Ω at varying speeds or even move in a pathological manner jumping from one place to another. Moreover, since $K(t)$ can change, $\lambda_0(K(t))$ can vary as well. Therefore, the inequalities $\lambda < \lambda_0(K(t))$ or $\lambda > \lambda_0(K(t))$ might not hold for all $t > 0$. This is specially relevant since the determining factor in the autonomous problem for deciding whether solutions are bounded or not is the relationship between λ and $\lambda_0(K_0)$. That is, the key factor in the asymptotic behaviour of solutions is the relation between these two time–independent scalars. In the nonautonomous case, we can no longer assume these relationships hold for every $t \geq 0$. Therefore, some immediate questions arise regarding the new possibilities the nonautonomous case offers for $K(t)$ and $\lambda_0(K(t))$.

First question arising is the following. If $\lambda < \lambda_0(K(t))$ for all $t > 0$, do the solutions remain bounded? or if $\lambda > \lambda_0(K(t))$ for all $t > 0$, do the solutions grow out of bounds as $t \to \infty$? This is the behaviour that we would expect if we extrapolate the results of the autonomous case to the nonautonomous one. And related to this question, if $\lambda < \lambda_0(K(t))$ for some intervals $I = (t_{2k}, t_{2k+1}]$ defined by a sequence of times $\{t_k\}_{k=1}^{\infty}$ with $t_1 < t_2 < t_3 \ldots$ while $\lambda > \lambda_0(K(t))$ for $t \in (t_{2k+1}, t_{2(k+1)}]$, how is the behaviour of the solutions? That is, can we decide the boundedness or unboundedness just with the relation between λ and $\lambda_0(K(t))$?

This first question motivates the following one: are there any new factors, apart from the relation between λ and $\lambda_0(K(t))$, that will influence the boundedness or unboundeness of the solutions? For instance, does the speed at which $K(t)$ moves impact the boundedness or unboundedness of solutions and hence, its asymptotic behaviour? Notice that $K(t)$ could be a rigid motion, that is, $K(t) = \gamma(t) + R(t)K_0$ where γ is a smooth curve and $R(t)$ is a rotation which varies continuously with time and not necessarily in a periodic or even almost periodic way. Observe that for this example $\lambda_0(K(t)) = \lambda_0(K_0)$. Hence, is it possible that $\lambda > \lambda_0(K(t)) \equiv \lambda_0(K_0)$ but solutions are bounded due to the fact that K_0 is moving?

Another interesting case is the case where $K(t)$ jumps from a compact set K_0 to another compact set K_1 at a sequence of times $\{t_{2k}\}_{k=1}^{\infty}$ and jumps back at $\{t_{2k+1}\}_{k=1}^{\infty}$ with $t_1 < t_2 < t_3 \ldots$, that is

$$K(t) = \begin{cases} K_0, & t \in [t_{2k}, t_{2k+1}), \quad k = 1, 2, \ldots \\ K_1, & t \in [t_{2k+1}, t_{2k+2}), \quad k = 1, 2, , \ldots \end{cases}$$

Moreover, in case the solution becomes unbounded as $t \to \infty$, where or how does the solution become unbounded? This latter question is not naive at all. It gave birth to the concept of metasolutions for the "smooth" case (see [29]). Moreover, for the autonomous case with general non–smooth set K_0 it was not completely answered (see [7]). Observe that having configurations of moving sets $K(t)$ where the solution blows up to infinity in subsets of $K(t)$, see Theorem 2.1.10 below, may be interpreted as having the nonautonomous counterpart of the metasolutions from [29].

Contents of this dissertation

Chapter 1 of the thesis is centred on stating clearly the conditions we are going to assume and developing some important tools used throughout this thesis. We start analysing the well-posedness of (0.0.2) under mild assumptions of the function $n(t,x)$. For this we follow the approach from [5]. Afterwards, we include positivity and comparison results, taken mainly from [6] that guarantee among other things that solutions of (0.0.2) are defined for all times $t \geq t_0$. We also include a parabolic Hopf lemma applied to our equation which shows that the solution with nonnegative and nontrivial initial data satisfies that the exterior normal derivative is strictly negative at all points

of the boundary, for all times strictly greater than the initial time. This result will be needed to obtain later in Chapter 3 that the character of boundedness or unboundedness of the solutions at a given $x \in \Omega$ is independent of the initial data and initial time. This implies in particular that if we find a particular nontrivial solution $u_p(x,t)$ of (0.0.2) which satisfies $t \to u_p(x,t)$ is a bounded function, then all solutions starting at any initial data will be bounded at x. Finally, we include in the last section of this chapter some results which measure the behaviour of the solutions when we perturb the nonlinearity and in particular when we perturb the logistic function $n(t,x)$. Important to mention that the perturbation of $n(t,x)$ is not necessarily small in L^∞ or L^1.

Chapter 2 is centred around the autonomous problem which, as mentioned above, has been extensively studied in the literature. One of the main goals of Chapter 2 is, in the case when solutions grow up, to narrow down the set where solutions become unbounded. For regular K_0 this has been already studied and is well documented (see [29]). However, for a general K_0 a satisfactory answer has not been given so far. Section 2.1 includes a summary of the main results presented in [7] in order for the reader to familiarise with concepts such as the characteristic value λ_0 and since it is one of the more relevant articles this thesis is based on. One of the goals of Chapter 2 is to give meaning to the solution of the equation $u_t - \Delta u = 0$ with Dirichlet boundary conditions in a compact set K_0 (actually in a subset of K_0), not necessarily smooth and probably with empty interior. This is done in Section 2.2. Afterwards, in Section 2.3, we obtain a comparison result between the problems (0.0.6) and the linear problem in K_0, proving that the solution to the latter is a subsolution of the former. This comparison result does not follow from standard comparison arguments since the set K_0 may be very pathological. We will rely on approximations of K_0 by the exterior by smooth open sets and using techniques like capacity, "fine topology" and its relation with the spectral and resolvent convergence of operators from Section 2.2. With this comparison result we give a characterization in terms of the spectra of the set where solutions grow up for a fixed λ. Finally in Section 2.4, a problem originally from [7] where the set K_0 has certain complicated or pathological structure is presented. This pathological problem is a specific case of problem (0.0.2) where K_0 compact is such that $\lambda_0(K_0) < \infty$ and has empty interior. The construction of such K_0 is included in this chapter. Since $\lambda_0(K_0) < \infty$, if $\lambda > \lambda_0(K_0)$ then solutions will grow up.

Chapter 3 is the main chapter of this dissertation. Several results are presented that study under which conditions the solutions of (0.0.6) are bounded or unbounded. Throughout this chapter we assume the following non degeneracy condition of the function n:

Assumption (N). *There exists a strictly increasing continuous function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\nu(0) = 0$, $\nu(s) > 0$ for $s > 0$ and*

$$n(t,x) \geq \nu(d(x,K(t))), \quad \text{for all } x \in \overline{\Omega}, t > t_0.$$

Without being exhaustive, the second section in Chapter 3 shows that boundedness or unboundedness of solutions is independent of initial data or initial time. This is relevant since it will allow us to choose initial data in suitable manners in order to analyse the boundedness properties. In Section 3.3, general results on boundedness and unboundedness are presented. In particular, the following sets are defined for $K(t)$:

$$K^{\sup} := \bigcap_{\tau_0 \geq t_0} \overline{\bigcup_{t \geq \tau_0} K(t)}, \quad \text{and} \quad K^{\inf} := \overline{\bigcup_{\tau_0 \geq t_0} \bigcap_{t \geq \tau_0} K(t)}.$$

Roughly speaking, the reader can think of $K^{\sup}$ and $K^{\inf}$ as the $\limsup_{t\to\infty} K(t)$ and $\liminf_{t\to\infty} K(t)$ respectively, that is, the upper and the lower limits of the family of sets $\{K(t)\}_{t\geq t_0}$. For example,

if we were to consider $K(t)$ an oscillating radius ball centred at 0 such that $K(t) = B(0, r(t))$ with $r : \mathbb{R} \to [1, 2]$ continuous and surjective:

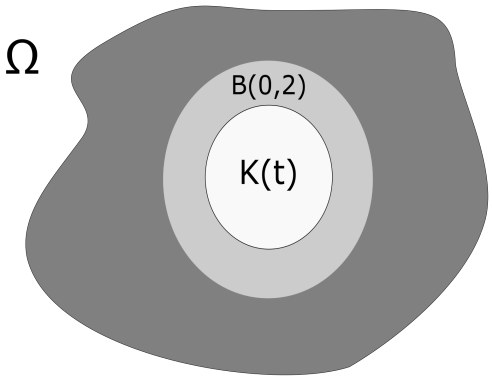


Figure 3: Domain Ω and a oscillating radius ball $K(t)$. The compact set $K(t)$ increases and decreases its radius.

Then we would have $K^{\sup} = \overline{B(0,2)}$ and $K^{\inf} = \overline{B(0,1)}$.

With these compact sets defined it is shown that, under fairly general assumptions on function $n(t, x)$, if $\lambda < \lambda_0(K^{\sup})$ then all solutions of (0.0.6) are bounded whilst if $\lambda > \lambda_0(K^{\inf})$ then all solution of (0.0.6) become unbounded as $t \to \infty$. This partially answers some of the questions that arose when presenting the nonautonomous problem. In particular, it shows that there is some parallelism with the autonomous case since we are able to ensure boundedness or unboundedness with a relationship involving λ and λ_0. However, it also shows that new elements appear in the nonautonomous scenario: $K^{\sup}$ and $K^{\inf}$.

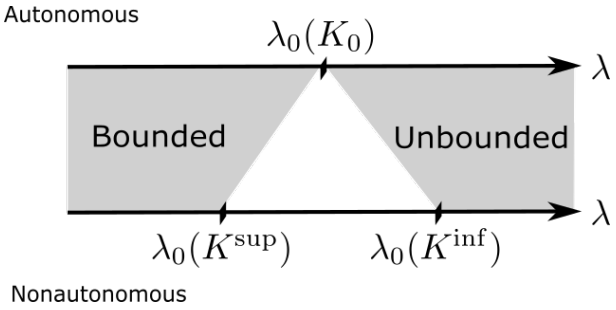


Figure 4: The nonautonomous case does present some parallelism with its autonomous counterpart. Moreover, λ and $\lambda_0(K_0)$ are no longer the only elements involved in determining boundedness or unboundedness of solutions and new scenarios arise in the nonautonomous case.

Section 3.4 is devoted to obtaining boundedness results for $\lambda \in (\lambda_0(K^{\sup}), \lambda_0(K^{\inf}))$ since this is the interval where the general results from the previous section don't apply. There are three distinct parts within Section 3.4. Let us go through each one of them.

In the first part, we analyse the case where the function $n(t, x)$ vanishes in all $\overline{\Omega}$ for some intervals of time and is strictly positive in all $\overline{\Omega}$ for the remaining times. That is, we study the case where $n \equiv 0$ for some time intervals $I = \bigcup_{k\in\mathbb{N}}(t_{2k}, t_{2k+1}] \subset \mathbb{R}^+$ for an unbounded sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_1 < t_2 < \ldots$ and $n \equiv 1$ for the rest of the times:

$$n(t,x) = \begin{cases} 0, & t \in I,\ x \in \overline{\Omega} \\ 1, & t \in \mathbb{R}^+ \setminus \bar{I},\ x \in \overline{\Omega}, \end{cases}$$

or, equivalently,

$$K(t) = \begin{cases} \overline{\Omega}, & t \in I, \\ \emptyset, & t \in \mathbb{R}^+ \setminus I. \end{cases}$$

It is shown that, under very general assumptions on the intervals I, for any $\lambda \in \mathbb{R}$ solutions of (0.0.6) with n as above are bounded. This result shows how strong the self–limiting effect is if acting on the whole domain Ω, that is, when $K(t) = \emptyset$. The techniques used in this subsection are mainly comparison. As a matter of fact, the solution of (0.0.6) is compared, in the time intervals $\mathbb{R}^+ \setminus \bar{I}$, to the solution of the ODE

$$\begin{cases} W(t)_t = \lambda W(t) - \nu_0 W(t)^\rho, & t > 0, \\ W(0) = W_0 > 0, & t = 0, \end{cases}$$

which is bounded by the function

$$W_\infty(t) = \left[\frac{\nu_0}{\lambda}\left(1 - e^{-\lambda(\rho-1)t}\right)\right]^{-\frac{1}{\rho-1}}.$$

This result is dedicated to obtaining bounds with the assumption that $n \equiv \beta > 0$, or equivalently, $K(t) = \emptyset$ for some time intervals on all of Ω. This assumption is quite limiting and, therefore, we would like to know how the solution of (0.0.6) behaves when we relax this assumption. In particular, instead of considering that $n(t,x) \geq \beta > 0$ on the whole domain Ω, we would like to study if any bounds on the solution can be obtained when considering $n(t,x) \geq \beta > 0$ for x in subsets of Ω.

Hence, the second part of Section 3.4 which is Subsection 3.4.1 is centred around obtaining local bounds in space and in time for solutions of (0.0.6). These bounds will be helpful when proving results later on in Section 3.4. In order to obtain these bounds, the following singular problem is introduced for some $a > 0$:

$$\begin{cases} -\Delta z = \lambda z - \beta z^\rho, & \text{in } B(0,a), \\ z = \infty, & \text{on } \partial B(0,a). \end{cases}$$

As mentioned before, singular problems associated to (0.0.2) and (0.0.6) have been extensively studied in the literature (see for example [14, 15, 17]). Following [17], with W_∞ above and z we are able to find solutions of singular problems which are supersolutions locally in space and time of (0.0.6). The techniques used in this subsection are mainly comparison of solutions.

Finally, the third part of Section 3.4 consists of Subsections 3.4.2 to 3.4.5 where we present four boundedness results. The first result is Theorem 3.4.9 which is an extension to the autonomous case where, roughly speaking, if $\lambda < \lambda_0(K(t))$ for all $t > 0$ then all solutions of (0.0.6) are bounded. This answers one of the questions we initially raised. Moreover, it is also an improvement with respect to the results in Section 3.3 where we showed that if $\lambda < \lambda_0(K^{\sup})$ then solutions are bounded. Since, in general, $\lambda_0(K^{\sup})$ is smaller than $\lambda_0(K(t))$ for all $t > 0$, it is a sharper result. The techniques used in this theorem are comparison results and the use of some results due to Mazja (see [30]) and Ladyzenskaja (see [25]).

For the second result (Section 3.3) we focus on the case where $K(t) = K_0$ for most of the time and $K(t)$ is larger for other times, where K_0 is a fixed set and $\lambda < \lambda_0(K_0)$. Notice that the autonomous problem with $n(x)$ vanishing on K_0 and $\lambda < \lambda_0(K_0)$ we have boundedness of solutions (see [7, 29]) and as a matter of fact, solutions converge to the unique positive equiibrium. Exploiting

this convergence to the autonomous problem and using appropriate perturbation techniques we will be able to show boundedness in the case the times where $K(t)$ is large (so that $\lambda_0(K(t)) < \lambda$) are small.

For the third result (Section 3.4), instead of imposing restrictions on the time where $K(t)$ is large, we impose restrictions on the size of $K(t)$, that is, if for given time intervals, $K(t)$ is sufficiently small, then all solutions of (0.0.6) are bounded. Notice that if $K(t)$ is very small, then this means that $K(t)$ is "near" the empty set. As a matter of fact, this result is proven using perturbation techniques with respect to the case where $n \equiv \beta > 0$ for some time intervals.

Our fourth and final boundedness result of Section 3.4 considers the case where $K(t)$ moves in a discontinuous way. In particular, whenever $K(t)$ jumps "regularly", then all solutions of (0.0.6) are bounded. Jumps must occurs infinitely many times, jumps must happen regularly in time and there must be a positive distance separating $K(t)$ before and after the jump for some positive time. A periodic jumping behaviour is an example of this case. This result is interesting as it since it shows that boundedness is obtained through jumping for all $\lambda \in \mathbb{R}$.

Section 3.5 is devoted to unboundedness results. Naturally, unboundedness cannot be achieved whenever we have conditions for boundedness. Therefore in order to seek for unbounded solutions we must assume no large jumps on $K(t)$, or generally speaking, some continuity on how $K(t)$ moves. Furthermore, we must also assume that $\lambda > \lambda_0(K(t))$ or otherwise we could obtain boundedness through results in the previous Section 3.4. The main result in this section is Theorem 3.5.1 which is the most general result on unboundedness we have obtained. Assumptions in Theorem 3.5.1 can be summarised as $K(t)$ being large enough to contain large enough subsets E (so that $\lambda > \lambda_0(E)$) and moving slow enough so that those subsets intersect in a ball with a minimum positive radius. The proof of Theorem 3.5.1 is a tailored construction of linear problems in order to obtain subsolutions which are shown to grow up. The most interesting consequence of this result, roughly speaking, is that we found the existence of a minimum "speed" at which $K(t)$ can move in order to obtain unboundedness of solutions. As a matter of fact, if we consider a problem with a continuously moving general $K(t)$ (not necessarily periodic) as in Figure 3, then solutions will be unbounded only if $K(t)$ moves slowly and if $\lambda > \lambda_0(K(t))$.

Therefore, the next natural question arising is: what happens if $K(t)$ moves quickly? Chapter 4 is dedicated to obtaining boundedness whenever $K(t)$ moves fast. The variation of (0.0.6) considered is the following:

$$\begin{cases} u_t - \Delta u = f(t,x,u,\epsilon) := \lambda u - n(t/\epsilon, x)|u|^{\rho-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega \ \dagger \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{0.0.7}$$

It is shown that if $K^{inf} = \emptyset$ and $\varepsilon > 0$ is small enough, then all solutions of (0.0.7) are bounded. Hence, proving that the minimum speed found in the unboundedness section is actually required for solutions to grow up.

Regarding the results above mentioned, there is some related research. In the following we will compare our findings with results found in [1, 2, 27, 36] and related papers.

In particular, Theorem 3.4.9 in this thesis is related in a non–trivial way to Lemma 7.0.7 in [36] (and later developed in [26, 34]). Let us fist introduce the main results and techniques in [36]. The author proves the existence of two ordered extremal equilibria (a minimum and a maximum) for a class of autonomous semilinear parabolic problems for both bounded and unbounded domains. These extremal equilibria provide uniform bounds on the attractor and therefore, determine the asymptotic behaviour of solutions. Let us briefly describe how the existence of this extremal

equilibria is obtained since it follows a similar approach when the author studies nonautonomous problems. The equation studied is the semilinear parabolic equation $u_t - \Delta u = f(x,u)$ and existence of these extremal equilibria is proven via a sub–supersolution method by bounding the nonlinearity f in the following manner:

$$uf(x,u) \leq C(x)u^2 + D(x)|u|,$$

which for positive solutions u is equivalent to

$$f(x,u) \leq C(x)u + D(x).$$

It is shown that if the semigroup generated by $\Delta + C(x)$ decays exponentially, the extremal equilibria exist.

For the nonautonomous parabolic semilinear equation $u_t - \Delta u = f(t,x,u)$ an analogous approach is used. For this equation equilibria need not exist since the nonlinearity depends on time. Hence, the role of equilibria is now taken by complete trajectories, which are solutions defined for all $t \in \mathbb{R}$. Moreover, the attraction concept used must also be modified for the nonautonomous case and is replaced by pullback attraction and, under certain assumptions, a forward attraction. In this nonautonomous context, the semigroup is substituted by a process U and the exponential decay property of the semigroup is substituted by exponential stability property of the process. Bounding the nonautonomous nonlinear term f in an analogous form as previously

$$f(t,x,u) \leq C(t,x)u + D(t,x) \tag{0.0.8}$$

and requiring that the process U_C (associated to the operator $\Delta + C(t,x)$) is exponentially stable, guarantee the existence of complete extremal trajectories. In particular, Lemma 7.0.7 from [36] gives a sufficient condition for the exponential stability of the process U_C which in turn provides the existence of complete bounded extremal trajectories for the nonautonomous linear homogeneous equation $v_t - \Delta v = C(t,x)v$. The sufficient condition for the exponential stability of the process U_C given by Lemma 7.0.7 in [36] is

$$\liminf_{t \to \pm\infty} \lambda_1(t) > 0, \tag{0.0.9}$$

where $\lambda_1(t)$ is, for every fixed $t \in \mathbb{R}$, the first eigenvalue of the problem

$$\begin{cases} -(\Delta + C(t,x))u = \lambda(t)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.10}$$

Let us see that, without taking into account regularity assumptions, Lemma 7.0.7 in [36] and its assumption (0.0.9) are related to Theorem 3.4.9 and its assumption $\lambda < \inf\limits_{t>t_0} \lambda_1^{\Omega_\delta(t)}$, where $\Omega_\delta(t)$ is an open bounded set such that, for every $t > t_0$, $\Omega_\delta(t) \supset K_0(t)$, $d(\Omega_\delta(t), K_0(t)) < \delta$ and such that $\inf\limits_{t>t_0} \lambda_1^{\Omega_\delta(t)}$ is close to $\lambda_0(K(t))$ so that $\lambda < \inf\limits_{t>t_0} \lambda_0(K_0(t))$ implies $\lambda < \inf\limits_{t>t_0} \lambda_1^{\Omega_\delta(t)}$. In particular, they are related in the sense that both assume an equivalent condition although they reach different results. Whilst from our Theorem 3.4.9 boundedness of solutions is guaranteed, from Lemma 7.0.7 and Theorems 7.1.4 and 8.3.2 the existence of complete extremal trajectories (not necessarily bounded) which pullback–attract solutions is guaranteed.

It can be seen that (0.0.8) is satisfied for f as in (0.0.6) with $A > 0$, $C_A(t,x)$ and $D_A(t,x)$ defined for each $t > t_0$ as follows:

$$C_A(t,x)=\begin{cases}\lambda, & x\in\Omega_\delta(t),\\ -A, & x\in\Omega\setminus\Omega_\delta(t),\end{cases}\qquad D_A(t,x)=\begin{cases}0, & x\in\Omega_\delta(t),\\ \beta\left(\frac{A+\lambda}{\nu(\delta)^{1/\rho}}\right)^{\rho'}, & x\in\Omega\setminus\Omega_\delta(t),\end{cases}\tag{0.0.11}$$

where $\beta>0$ and we have used Young's inequality and assumption **(N)** above. For a large enough A and proceeding as in Proposition 5.0.2 and Lemma 5.0.3 in [36] we have that the process U_{C_A} is exponentially stable. Therefore, the existence of complete extremal trajectories for (0.0.6) is guaranteed by Theorem 7.1.4 in [36]. Moreover, these trajectories pullback–attrack solutions by Theorems 8.1.1 and 8.3.2 in [36]. Notice that these complete trajectories may not be bounded. A notorious example of a not bounded extremal complete trajectory is found in [27] where authors prove the existence of unbounded solutions for

$$u_t-\Delta u=\lambda u-n(t,x)u^\rho,\tag{0.0.12}$$

where $\lambda>\lambda_1^\Omega$, $n(t,x)>0$ for all $t\in\mathbb{R}$ and $n(t,x)\to 0$ when $t\to 0$ for $x\in\Omega_0$. Moreover, authors in [27] show the existence of a time dependent set of positive states to which all solutions enter and remain for a suitable initial time (which is defined as pullback permanence). The particular case of equation (0.0.12) implies that $K(t)=\emptyset$ for all $t>t_0$ but $n_\infty=\lim_\infty t\to n(t)$ does vanish in a subset of Ω. Actually, (0.0.12) has not been considered in the present research due to assumption **(N)**.

Later results in [36] expand further on the exponential stability by providing exponential stability for the operators in perturbed equations $u_t-\Delta u=(C(t,x)+P(t,x))u$ and their respective processes U_{C+P}. While these perturbation results (in Proposition 7.0.8 and Corollary 7.0.10 in [36]) apply to a wide range of scenarios, they do not apply for example to the cases where the perturbation is large in $L^\infty(\mathbb{R},L^\infty(\Omega))$.

Also, the existence of solutions of periodic boundary value problems (PVBP) is related to this research in the following way: if there exists a periodic solution to the PVBP then it is in particular bounded. Therefore, with some additional assumptions in the very specific and very relevant case $f(t,x,u)$ is time periodic, our results could be derived from existence results for PVBP. We previously mentioned [1] where authors prove the existence of periodic solutions for a slightly modified version of (0.0.5) where $K(t)=K_0\times[0,T]$:

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)-\Delta u(t,x)=\lambda m(t,x)u(t,x)-\gamma n(t,x)|u(t,x)|^{\rho-1}u(t,x), & t\in(0,T),\ x\in\Omega,\\ u(t,x)=0, & t\in(0,T),\ x\in\partial\Omega,\end{cases}\tag{0.0.13}$$

where m,n are T–periodic functions, $T>0$ and $\lambda,\gamma\in\mathbb{R}$. Therefore, in the case $n(t,x)$ is periodic and $K(t)$ is static, existence results (and therefore, boundedness) were already proven in [1]. Very recently in [2], authors generalise results in [1] and characterise the existence of periodic solutions without imposing restrictions on the structure of K_0. In particular, authors prove a very interesting result where they give an if and only if characterization for the existence of periodic solutions in terms of the behavior of certain nontrivial periodic parabolic eigenvalue problems. This interesting technique is specific for periodic situations and do not apply directly to our most general non autonomous scenario.

Finally, let us mention that there are many interesting works in the literature addressing the general problema of the asymptotic behaviour of solutions for nonautonomous problems, analyzing the existence of different kind of asymptotic sets which respond to different concepts of attraction like pullback, forwards, uniform attractors and which describe the behavior of the systems for large

times. Without being exhaustive we would like to mention the monograph [11] and references therein for a nice and rather complete reference in this respect. We also mention the dissertations [36] to which we have made references above and to [33] which includes a nice introduction to attractors for nonautonomous systems.

Conclusions

One of the main conclusions of this thesis is that the non–autonomous case (0.0.6) presents a broader spectrum of scenarios than the autonomous counterpart (0.0.2). Moreover, new elements such as $K^{\inf}$ and $K^{\sup}$ appear and new elements have to be factored in (such as how $K(t)$ moves within $\overline{\Omega}$) when studying the boundedness or unboundedness of solutions. This variety of scenarios is the main reason we have focused on obtaining boundedness and unboundedness results instead of pursuing results on the asymptotic behaviour of solutions, which are the next natural step following this research.

The boundedness results obtained generally require less assumptions on n: both on its regularity and the geometry of $K(t)$. Because of this, some of the boundedness results such as Theorem 3.4.9 are similar to those in [7]. On the other hand, unboundedness results in this research do require a more specific setting for $K(t)$ and some regularity on $K(t)$. This shows that, generally speaking, in the nonautonomous context, boundedness is more likely than unboundedness. As a matter of fact, we might have a setting were solutions grow up (for example with a continuously moving large $K(t)$) and by increasing the speed at which $K(t)$ moves, solutions become bounded. Furthermore, another piece of evidence that shows that boundedness is more likely in the nonautonomous scenario is that we have successfully proven cases (see Subsection 3.4.5 and Chapter 4) where, even for $\lambda > \lambda_0(K(t))$ for all $t > 0$, all solutions are bounded.

Another main conclusion of this thesis is that, in order for solutions to grow up, there is a minimum time $K(t)$ must spend in a neighbourhood of $K(t)$ or a maximum speed at which $K(t)$ can move. That is, it cannot move very fast nor it can jump very far.

A main consequence of this thesis is that, since we have identified a wide range of scenarios where solutions are bounded with little assumptions on $K(t)$, these new scenarios can be studied in the context of asymptotic dynamics. Results in this thesis have been pursued with the goal of requiring as few assumptions as possible on the regularity of n, the geometry of $K(t)$ or the periodicity of any. It is clear that, under more restrictive assumptions such as regularity and periodicity on n, our results can be compared to other research.

Chapter 1

Preliminary material

In this chapter we define and develop the tools which will be used throughout this dissertation. These tools are mainly focused, although not limited to, comparing solutions of different problems.

Thus, in this chapter we want to establish

- Well–possedness for a wide range of non–autonomous parabolic equations.
- Comparison results for the solutions of those equations.
- A parabolic Hopf lemma.
- A perturbation result to estimate the L^∞ difference between solutions of the previously studied equations.

Our main goal is to study the behaviour of the solutions of an evolution problem, which may be non–autonomous, of the type

$$\begin{cases} u_t - \Delta u = \lambda u - n(t,x)|u|^{\rho-1}u, & x \in \Omega,\ t > t_0 \\ u = 0, & x \in \partial\Omega,\ t > t_0 \\ u(t_0, x) = u_0(x) \geq 0, & x \in \Omega \end{cases} \tag{1.0.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$, for some positive integer N, $\rho > 1, t_0, \lambda \in \mathbb{R}$, $n \in L^\infty(\mathbb{R} \times \Omega)$ and $u_0 \in L^q(\Omega)$ where $1 < q \leq \infty$ will be chosen appropriately large enough. We consider Dirichlet boundary conditions but the analysis could be developed for other boundary conditions.

Remark 1.0.1. *Although we are initially establishing $u_0 \in L^q(\Omega)$ for $1 < q \leq \infty$ we will see that for any $\tau > t_0$ the solution u of problem (1.0.1) will verify $u(\tau, \cdot) \in L^\infty(\Omega)$. Since we are concerned with the asymptotic evolution of the solutions of this problem, we will often request that $u_0 \in L^\infty(\Omega)$ as it would only imply taking $u_0 := u(\tau, \cdot)$ as the new initial data while not affecting the asymptotic behaviour of the solution.*

Equation (1.0.1) is a particular case of a more general class of parabolic nonautonomous equations:

$$\begin{cases} u_t - \Delta u = f(t,x,u), & x \in \Omega,\ t > t_0 \\ u = 0, & x \in \partial\Omega,\ t > t_0 \\ u(t_0, x) = u_0(x), & x \in \Omega \end{cases} \tag{1.0.2}$$

where $f : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ verifies

$$|f(t,x,u) - f(t,x,v)| \leq C\,|u-v|\left(|u|^{\rho-1} + |v|^{\rho-1} + 1\right), \quad \text{for } u, v \in \mathbb{R}, \tag{1.0.3}$$

uniformly for $t \geq t_0$ on bounded sets and $x \in \Omega$.

As a matter of fact, we can show

Lemma 1.0.2. *Function*

$$f(t,x,u) = \lambda u - n(t,x)|u|^{\rho-1}u \tag{1.0.4}$$

verifies (1.0.3).

Proof. Let $g(u) := u|u|^{\rho-1}$ then $g'(u) = \rho|u|^{\rho-1}$ and due to its monotonocity

$$\begin{aligned} \left||u|^{\rho-1}u - |v|^{\rho-1}v\right| &= \left|\int_v^u g'(s)ds\right| = \left|\int_v^u \left(\rho|s|^{\rho-1}\right) ds\right| \\ &\leq \rho \max\left(|u|^{\rho-1}, |v|^{\rho-1}\right)|u-v| \leq \rho\left(|u|^{\rho-1} + |v|^{\rho-1}\right)|u-v|. \end{aligned} \tag{1.0.5}$$

for any $u, v \in \mathbb{R}$. Therefore,

$$|f(t,x,u) - f(t,x,v)| \leq \lambda|u-v| + \|n\|_\infty \left||u|^{\rho-1}u - |v|^{\rho-1}v\right| \leq D\,|u-v|\left(1 + |u|^{\rho-1} + |v|^{\rho-1}\right),$$

where $D = \max(\lambda, \rho\|n\|_\infty)$. Thus, (1.0.3) follows.

□

1.1 Local existence

For problems of the type (1.0.2) with $u_0 \in L^q(\Omega)$ for $1 < q < \infty$ we follow [5] throughout this subsection in order to obtain local existence. We also refer to [11, 21, 36] for similar local existence results. All definitions and results in this subsection are obtained or immediately derived from the cited works.

First, let us state the general setting in which we will obtain local existence and uniqueness. Let

$$A : D(A) \subset X^0 \to X^0$$

be a linear operator on the Banach space X^0 with domain $D(A)$ such that $-A$ is sectorial (see [21]). Let $X^\alpha, \alpha \geq 0$ be the fractional power spaces asociated to A, $\|\cdot\|_{X^\alpha}$ their respective norms and e^{At} the analytic semigroup generated by A. Problem (1.0.2) can be written in abstract form as the following problem

$$\begin{cases} u_t - Au = F(t,u), \\ u(t_0) = u_0 \end{cases} \tag{1.1.1}$$

where

- A is the Laplace operator with homogeneous Dirichlet boundary conditions in Ω, a bounded and smooth domain.
- $u_0 \in L^q(\Omega)$ and
- The nonlinearity F is the Nemytskii operator associated to the function $f(t,x,u)$ acting between two spaces of the scale of fractional power spaces X^α, that is, $F(t,u) = f(t,x,u)$.

In this abstract setting, let us introduce mild solutions and local existence. Let us recall two definitions and a Theorem from [5].

Definition 1.1.1. *Let $t_0, \tau, \varepsilon \in \mathbb{R}$ such that $t_0 < \tau$ and $\varepsilon > 0$. We say that $u : [t_0, \tau] \to X^1$ is an ε–regular mild solution of (1.0.2) if $u \in C([t_0,\tau], X^1) \cap C((t_0,\tau], X^{1+\varepsilon})$, and $u(t)$ satisfies the variation of constants formula:*

$$u(t) = e^{A(t-t_0)}u_0 + \int_{t_0}^{t} e^{A(t-s)}F(s,u(s))ds, \quad t \in (t_0,\tau]. \tag{1.1.2}$$

We also introduce the following definition:

Definition 1.1.2. *For $\varepsilon \geq 0$, we will say that a map g is an ε–regular map relative to the pair (X^1, X^0) if there exist $\rho > 1$, $\gamma(\varepsilon)$ with $\rho\varepsilon \leq \gamma(\varepsilon) < 1$, and a constant c, such that $g : X^{1+\varepsilon} \to X^{\gamma(\varepsilon)}$ and*

$$\|g(u) - g(v)\|_{X^{\gamma(\varepsilon)}} \leq c\|u - v\|_{X^{1+\varepsilon}} \left(\|u\|^{\rho-1}_{X^{1+\varepsilon}} + \|v\|^{\rho-1}_{X^{1+\varepsilon}} + 1 \right), \ \forall u, v \in X^{1+\varepsilon}. \tag{1.1.3}$$

From this definiton, for $\varepsilon, \rho, \gamma(\varepsilon)$ and c positive real constants let $\mathcal{F} := \mathcal{F}(\varepsilon, \rho, \gamma(\varepsilon), c)$ be the family of functions F such that, for $t > 0$, $F(t,\cdot)$ is an ε–regular map relative to the pair (X^1, X^0),satisfying

$$\begin{cases} \|F(t,u) - F(t,v)\|_{X^{\gamma(\varepsilon)}} \leq c\|u - v\|_{X^{1+\varepsilon}} \left(1 + \|u\|^{\rho-1}_{X^{1+\varepsilon}} + \|v\|^{\rho-1}_{X^{1+\varepsilon}} \right), \\ \|F(t,u)\|_{X^{\gamma(\varepsilon)}} \leq c \left(1 + \|u\|^{\rho}_{X^{1+\varepsilon}} \right), \end{cases} \tag{1.1.4}$$

for all $u, v \in X^{1+\varepsilon}$. Finally, from [5] we have the following:

Theorem 1.1.3. *Let $F \in \mathcal{F}(\varepsilon, \rho, \gamma(\varepsilon), c)$. If $v_0 \in X^1$, there exists $r > 0$ and $\tau_0 > t_0$ with the property that for any $u_0 \in B_{X^1}(v_0, r)$ there exists a continuous function $u(\cdot, u_0) : [t_0, \tau_0] \to X^1$, with $u(t_0) = u_0$, which is a unique ε–regular mild solution starting at u_0 of the problem*

$$\begin{cases} u_t - Au = F(t,u), \quad t > t_0, \\ u(t_0) = u_0. \end{cases} \tag{1.1.5}$$

This solutions satisfies

$$\begin{aligned} &u \in C((t_0,\tau_0], X^{1+\theta}), \ 0 \leq \theta < \gamma(\varepsilon), \\ &t^\theta \|u(t,u_0)\|_{X^{1+\theta}} \xrightarrow{t \to t_0^+} 0, \ 0 < \theta < \gamma(\varepsilon). \end{aligned} \tag{1.1.6}$$

Moreover, if $u_0, w_0 \in B_{X^1}(v_0, r)$, then

$$t^\theta \|u(t,u_0) - u(t,w_0)\|_{X^{1+\theta}} \leq C\|u_0 - w_0\|_{X^1}, \ \forall t \in [t_0,\tau_0], \ 0 \leq \theta \leq \theta_0 < \gamma(\varepsilon). \tag{1.1.7}$$

Also, if $\gamma(\varepsilon) > \rho\varepsilon$, then r can be chosen arbitrarily large. That is, the time of existence is uniform on bounded sets of X^1.

If $t \to F(t,u)$, as a map from (t_0,∞) to $X^{\gamma(\varepsilon)}$, is locally Hölder continuous, uniformly on bounded sets of $u \in X^{1+\gamma(\varepsilon)}$, then

$$u \in C^1((t_0,\tau_0],X^{\gamma(\varepsilon)}) \cap C((t_0,\tau_0],X^{1+\gamma(\varepsilon)}), \tag{1.1.8}$$

and $u(\cdot,u_0)$ is a strict solution of (1.1.5).

The constants above depend on the following $\tau_0 = \tau_0(v_0,A,\varepsilon,\rho,\gamma(\varepsilon),c,M)$, $r = r(v_0,\varepsilon,\rho,\gamma(\varepsilon),c,M)$, $C = C(\theta_0,\varepsilon,\rho,\gamma(\varepsilon),M)$.

This is the result we will use on the following for local existence, uniqueness and regularity of solutions. In order to obtain local existence by applying Theorem 1.1.3, there are two main points that must be understood: the scale of fractional power spaces and under which circumstances the function F is ε–regular. Since problem (1.0.2) is stated for $u_0 \in L^q(\Omega)$ its scale of fractional power spaces associated to the laplacian with Dirichlet boundary conditions on Ω must be studied.

The scale of fractional power spaces

The setting in which $D(A) = L^q(\Omega)$ can be attained by shifting a more familiar scale of fractional power spaces in the following manner:

Let $E_q^0 = L^q(\Omega)$, $L = \Delta$ with Dirichlet boundary conditions and $\left\{E_q^\alpha\right\}_{\alpha\in\mathbb{R}}$ the scale of fractional power spaces associated to L. L is an unbounded operator in E_q^0 and $E_q^1 = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ is its domain. We have that,

$$L : D(L) = E_q^1 \subset E_q^0 \to E_q^0.$$

Moreover,

$$\begin{aligned} &E_q^\alpha \hookrightarrow H_q^{2\alpha}(\Omega), \quad \alpha \geq 0, \\ &E_q^{-\alpha} = \left(E_{q'}^\alpha\right)', \quad \alpha \geq 0, q' = \tfrac{q}{q-1}, \end{aligned}$$

where $H_p^l(\Omega)$ are the Bessel potentials which additionally verify the following relations:

$$\begin{aligned} &H_{p_1}^{l_1}(\Omega) \hookrightarrow H_{p_2}^{l_2}(\Omega), \quad \text{if } \tfrac{l_1}{N} - \tfrac{1}{p_1} \geq \tfrac{l_2}{N} - \tfrac{1}{p_2}, 1 < p_1 \leq p_2 < \infty, \\ &H_p^l(\Omega) \hookrightarrow C^\eta(\bar{\Omega}), \quad \text{if } l - \tfrac{N}{p} > \eta > 0, \\ &H_p^l(\Omega) = W^{l,p}(\Omega), \quad \text{if } p = 2, l \in \mathbb{R} \text{ or if } p > 1, l \in \mathbb{N}. \end{aligned}$$

Furthermore, the realization of L in E_q^α

$$L_\alpha : D(L_\alpha) = E_q^{\alpha+1} \subset E_q^\alpha \to E_q^\alpha,$$

is a sectorial operator, an isometry from $E_q^{\alpha+1}$ into E_q^α and $D(L_\alpha^\beta) = E_q^{\alpha+\beta}$ (see [21]).

Now, let us set $X_q^\alpha := E_q^{\alpha-1}$ for $\alpha \in \mathbb{R}$ then the sectorial operator

$$A := L_{-1} : X_q^1 = L^q(\Omega) \subset X_q^0 \to X_q^0$$

and the scale of fractional power spaces $\{X_q^\alpha\}_{\alpha\in\mathbb{R}}$ associated to A satisfy

$$\left.\begin{array}{ll} X_q^\alpha \hookrightarrow L^r(\Omega) & r \leq \frac{Nq}{N+2q-2q\alpha}, 1 \leq \alpha < 1+\frac{N}{2q}, \\ X_q^1 = L^q(\Omega), & \\ X_q^\alpha \hookleftarrow L^s(\Omega), & s \geq \frac{Nq}{N+2q-2q\alpha}, 1-\frac{N}{2q'} < \alpha \leq 1. \end{array}\right\} \tag{1.1.9}$$

Moreover, for the associated linear semigroup generated by A

$$\left\|e^{At}u_0\right\|_{X_q^\alpha} = \left\|A^\alpha e^{At}u_0\right\|_{X_q^0} \leq M\frac{e^{at}}{t^\alpha}\left\|u_0\right\|_{X_q^0}$$

where $\operatorname{Re}\sigma(A) < a < 0$ and for $\alpha > \beta$ there exists $M > 0$ such that

$$\left\|e^{At}u_0\right\|_{X_q^\alpha} \leq M\frac{e^{at}}{t^{\alpha-\beta}}\left\|u_0\right\|_{X_q^\beta}.$$

These type of estimates combined with Sobolev type inequalities yield, for $1 \leq s \leq r \leq \infty$, the following

$$\|e^{At}u_0\|_{L^r(\Omega)} \leq M\frac{e^{at}}{t^{\frac{N}{2}\left(\frac{1}{s}-\frac{1}{r}\right)}}\|u_0\|_{L^s(\Omega)}. \tag{1.1.10}$$

These are the necessary tools to understand the scale of fractional power spaces associated to our operator. Further analysis on F and its ε–regular properties remains in order to obtain local existence.

The nonlinearity F

Let us consider problem (1.0.2) with f satisfying (1.0.3). Let F be its Nemytskii operator. As illustrated in [5], local existence results are highly dependent on q and is more straightforward when q is large enough, that is, when it verifies the following hypothesis:

$$q > \max\left\{\frac{N}{2}\rho, \rho\right\}. \tag{1.1.11}$$

From [5] we have the following result.

Lemma 1.1.4. *Let f be as in (1.0.4) and q verifies (1.1.11). Then for $\varepsilon \in I := [0, \varepsilon_1]$ where $\varepsilon_1 > 0$, there exists $\gamma(\varepsilon)$ verifying $\rho\varepsilon < \gamma(\varepsilon) \leq 1$ such that for any $t > t_0$*

$$F(t,\cdot) : X_q^{1+\varepsilon} \to X_q^{\gamma(\varepsilon)},$$

and

$$\|F(t,u) - F(t,v)\|_{X_q^{\gamma(\varepsilon)}} \leq C\|u-v\|_{X_q^{1+\varepsilon}}\left(1 + \|u\|_{X_q^{1+\varepsilon}}^{\rho-1} + \|v\|_{X_q^{1+\varepsilon}}^{\rho-1}\right), \tag{1.1.12}$$

for $u, v \in X_q^{1+\varepsilon}$, uniformly for t on bounded sets. Therefore, F is ε–regular relative to the pair (X_q^0, X_q^1).

Corollary 1.1.5 (Local Existence and Uniqueness)**.** *Let q satisfy (1.1.11) and F be as above. Then F is an ε–regular map relative to the pair (X_q^1, X_q^0). Hence for $u_0 \in L^q(\Omega)$ problem (1.0.2) has a unique ε–regular solution for $t \in [t_0, \tau_0)$ and some $\tau_0 > t_0$. Moreover, for any $0 \leq \theta < 1$*

$$u \in C([t_0,\tau_0), L^q(\Omega)) \cap C((t_0,\tau_0), H_q^{2\theta}(\Omega)).$$

Moreover, if $f(t,x,u)$ is locally Hölder continuous in t, uniformly on bounded sets of u, then

$$u \in C^1((t_0,\tau_0), X_q^{\rho\theta}) \cap C((t_0,\tau_0), X_q^{1+\rho\theta}),$$

and $u(\cdot, u_0)$ *is a strict solution.*

Proof. It is derived immediately from Theorem 1.1.3 and Lemma 1.1.4.

□

Remark 1.1.6. *Function* $\gamma(\cdot)$ *in Lemma 1.1.4 for* q *satisfying (1.1.11) verifies* $\rho\varepsilon < \gamma(\varepsilon)$ *and* $\gamma(\varepsilon_1) = 1$*.Therefore, for* q *large enough, regularity from Theorem 1.1.3 and Corollary 1.1.5 holds for any* $0 \leq \theta < \gamma_{\max}(\varepsilon_1) = 1$*.*

As shown in [5], through bootstrap arguments the following result can be obtained:

Proposition 1.1.7 (Regularity)**.** *Let* $u_0 \in L^q(\Omega)$ *with* q *satisfying (1.1.11) and let* u *be the solution of (1.0.2) where* f *is as in (1.0.3). Then, for any* $0 < \eta < 1$

$$u \in C\left((t_0,\tau_0]; L^\infty(\Omega)\right) \cap C\left((t_0,\tau_0]; C^{1,\eta}(\bar{\Omega})\right).$$

Furthermore, if f *is Hölder continuous with respect to* t *and* x *uniformly for* u *in bounded sets and* $u_0 \in L^\infty(\Omega)$ *then* u *becomes classical for* $t > t_0$*.*

Remark 1.1.8. *As mentioned in Remark 1.0.1, our focus is long–term evolution of the equation and thus, we will often only require* $u_0 \in L^\infty(\Omega)$ *which will verify* $u_0 \in L^q(\Omega)$ *for any* $1 \leq q \leq \infty$*, granting us local existence, uniqueness and regularity.*

1.2 Positivity, Comparison and Global Existence of Solutions

Some key tools we will base our research on are comparison results between solutions of different equations similar to (1.0.2). Using these results we will obtain positivity, monotonicity properties and global existence of solutions. We will follow the Appendix in [6] throughout this section.

From Corollary 1.1.5 we have local existence of solutions of (1.0.2) for functions f verifying (1.0.3) and q verifying (1.1.11). Therefore, let us denote by $u(t, x, t_0, u_0, f)$ the solution of (1.0.2) in its maximal interval of existence $[t_0, \tau_0)$ and let $t_1 \in (t_0, \tau_0)$. Therefore, we have that

Theorem 1.2.1. *Let* $u(t, u_0, F)$ *be the solution of the following problem:*

$$\begin{cases} u_t - \Delta u = F(t,u), & t \in [t_0, t_1] \\ u(t_0) = u_0, \end{cases} \tag{1.2.1}$$

where $F(t,u)$ *is the Nemytskii operator of* $f : \mathbb{R} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ *which satisfies (1.0.3). Then, from [6] we have that:*

1. *Assume* $F(t,0) \geq 0$ *for every* t*. If* $u_0 \geq 0$ *then* $u(t, u_0, F) \geq 0$ *as long as the solution exists.*
2. *Let us assume that function* F *satisfies that for any* $r > 0$ *there exists a constant* $\beta = \beta(r) > 0$ *such that for every* $t \in [t_0, t_1]$*,* $F(t,\cdot) + \beta I$ *is increasing on the ball in* $X_q^{1+\varepsilon}$ *of radius* r*. Then,* $u_0, u_1 \in X_q^1$ *and* $u_0 \leq u_1$ *implies* $u(t, u_1, F) \geq u(t, u_0, F)$ *for as long as both solutions exist.*
3. *Assume that* F_0, F_1 *are such that for any* $r > 0$ *there exist a constant* $\beta = \beta(r) > 0$ *and an increasing function* $H(t,\cdot)$ *such that for every* $t \in [t_0, t_1]$*,*

$$F_1(t,\cdot) + \beta(r) I \geq H(t,\cdot) \geq F_0(t,\cdot) + \beta(r) I,$$

on the ball of radius r. Then, $u_0, u_1 \in X_q^1$, and $u_1 \geq u_0$ implies $u(t, u_1, F_1) \geq u(t, u_0, F_0)$ for as long as both solutions exist.

Remark 1.2.2. *Therefore, for non–negative initial data $u_0 \geq 0$ we have that $u(t, x; u_0) \geq 0$ and $f(t, x, u)$ in (1.0.1) can be rewritten as $f(t, x, u) = \lambda u - n(t, x)u^\rho$.*

Lemma 1.2.3. *The monotonicity requirements 2 and 3 in the previous Theorem are satisfied for real valued function $f(t, x, u)$ as in (1.0.4) and q satisfying (1.1.11).*

Proof. Let us first start with requirement 2. Let $u \in B(0, r) \subset X_q^{1+\varepsilon}$. Since q satisfies (1.1.11) then we have that for any $\eta \in (0, 2\varepsilon - N/q)$

$$X_q^{1+\varepsilon} \hookrightarrow H_q^{2\varepsilon} \hookrightarrow C^\eta(\overline{\Omega}) \hookrightarrow L^\infty(\overline{\Omega}).$$

Therefore, $u \in B(0, \tilde{r}) \subset L^\infty(\overline{\Omega})$ for some $\tilde{r} > 0$. Since $f(t, x, \cdot) \in C^1$:

$$\left|\frac{\partial f}{\partial u}\right| = \left|\lambda - n(t, x)\rho|u|^{\rho-1}\right| \leq \lambda + \|n\|_\infty \rho \tilde{r}^{\rho-1}, \quad |u| < \tilde{r},$$

then for $\beta > \lambda + \|n\|_\infty \rho \tilde{r}^{\rho-1}$ we have that

$$\frac{\partial(f + \beta u)}{\partial u} = \frac{\partial f}{\partial u} + \beta \geq -\lambda - \|n\|_\infty \rho \tilde{r}^{\rho-1} + \beta > 0, \quad |u| < \tilde{r}.$$

Hence, the function $u \mapsto f(t, x, u) + \beta u$ is increasing. Thus, if $u_0, u_1 \in X_q^{1+\varepsilon}$ and $u_0 \leq u_1$, $F(t, u_0) + \beta u_0 \leq F(t, u_1) + \beta u_1$.

Let us now see requirement 3. Let $u_0, u_1 \in B(0, r) \subset X_q^{1+\varepsilon}$ and let f_0, f_1 as in (1.0.4) and such that $f_0(t, x, u) \leq f_1(t, x, u)$ for all $t \in [t_0, t_1]$, $x \in \overline{\Omega}$ and $u \in \mathbb{R}$. Then, as before, $u_0, u_1 \subset B(0, \tilde{r}) \subset L^\infty(\overline{\Omega})$. Moreover, from the first part of this proof there exist $\beta_0 = \beta_0(\tilde{r})$ and $\beta_1 = \beta_1(\tilde{r})$ such that

$$u \longmapsto f_0(t, x, u) + \beta_0 u, \quad u \longmapsto f_1(t, x, u) + \beta_1 u, \quad |u| < \tilde{r}$$

are both increasing. Let us take $\beta := \max\{\beta_0, \beta_1\}$. Then, for $u \in B(0, \tilde{r}) \subset L^\infty(\overline{\Omega})$

$$u \longmapsto f_0(t, x, u) + \beta u, \quad u \longmapsto f_1(t, x, u) + \beta u,$$

are both increasing functions and

$$f_0(t, x, u) + \beta u \leq f_1(t, x, u) + \beta u.$$

Hence, taking for example $h(t, x, u) = f_0(t, x, u) + \beta u$ and letting $H(t, u)$ be its Nemytskii operator, we have that there exists $H(t, \cdot)$ such that requirement 3 above is satisfied.

□

Remark 1.2.4. *The previous Lemma 1.2.3 holds for f as in (1.0.2) verifying (1.0.3) with similar computations.*

Corollary 1.2.5. *Let $f_1(t, x, u) = \lambda u - n(t, x)|u|^{\rho-1}u$ with $n \in L^\infty(\mathbb{R} \times \Omega)$, $n(t, x) \geq 0$ for all t and a.e. $x \in \bar{\Omega}$, $f_2(t, x, u) = \lambda u$. Then, for as long as solutions of Theorem 1.2.1 exist, we have that:*

$$0 \leq |u(t, x, u_0, f_1)| \leq u(t, x, |u_0|, f_1) \leq u(t, x, |u_0|, f_2) = e^{\lambda(t-t_0)} e^{\Delta(t-t_0)} |u_0|, \tag{1.2.2}$$

a.e. $x \in \Omega$.

Proof. First, note that since $f_1(t,x,-u) = -f_1(t,x,u)$ we have, $-u(t,x,u_0,f_1) = u(t,x,-u_0,f_1)$. Thus, using Theorem 1.2.1

$$u(t,x,u_0,f_1) \leq u(t,x,|u_0|,f_1),$$
$$-u(t,x,u_0,f_1) = u(t,x,-u_0,f_1) \leq u(t,x,|u_0|,f_1).$$

Therefore,

$$0 \leq |u(t,x,u_0,f_1)| \leq u(t,x,|u_0|,f_1).$$

We wish to compare the solutions of $u(t,x,|u_0|,\lambda u - n(t,x)|u|^{\rho-1}u)$ and $u(t,x,|u_0|,\lambda u)$. The inequality required in Theorem 1.2.1 to obtain the comparison result between both solutions is for all $u \in \mathbb{R}$. Therefore, if we define

$$\tilde{f}_1(t,x,u) = \begin{cases} f_1(t,x,u), & u \geq 0, \\ 0, & u < 0 \end{cases} \quad \text{and} \quad \tilde{f}_2(t,x,u) = \begin{cases} f_2(t,x,u), & u \geq 0, \\ 0, & u < 0, \end{cases} \tag{1.2.3}$$

then, due to $\tilde{f}_1(t,x,u) \leq \tilde{f}_2(t,x,u)$ for all $(t,x,u) \in \mathbb{R} \times \Omega \times \mathbb{R}$ we have that $u(t,x,t_0,|u_0|,\tilde{f}_1) \leq u(t,x,t_0,|u_0|,\tilde{f}_2)$. Observe that $\tilde{f}_1$ and $\tilde{f}_2$ satisfy the third case in Theorem 1.2.1. Now, due to $|u_0|$ being non–negative we conclude that

$$u(t,x,t_0,|u_0|,f_1) = u(t,x,t_0,|u_0|,\tilde{f}_1) \leq u(t,x,t_0,|u_0|,\tilde{f}_2) \leq u(t,x,t_0,|u_0|,f_2),$$

that is,

$$0 \leq \left|u(t,x,t_0,u_0,\lambda u - n(t,x)|u|^{\rho-1}u)\right| \leq u(t,x,t_0,|u_0|,\lambda u - n(t,x)|u|^{\rho-1}u) \leq u(t,x,|u_0|,\lambda u).$$

Finally, observe that $u(t,x,|u_0|,\lambda u_0) = e^{\lambda(t-t_0)}e^{\Delta(t-t_0)}|u_0|$.

□

From the previous Corollary 1.2.5

$$0 \leq |u(t,x)| \leq U(t,x), \tag{1.2.4}$$

for $t \in [t_0,\infty)$ and a.e. $x \in \Omega$, and where we have denote by U the solution of the linear problem

$$\begin{cases} U_t - \Delta U = \lambda U, & x \in \Omega,\ t \in (t_0,\infty), \\ U = 0, & x \in \partial\Omega,\ t \in (t_0,\infty), \\ U(t_0) = |u_0|, & x \in \Omega \end{cases} \tag{1.2.5}$$

This solution can be expressed also as the linear semigroup $U(t,\cdot) = e^{\lambda(t-t_0)}e^{\Delta(t-t_0)}|u_0|$, which is globally defined and uniformly bounded in compact sets of $t > t_0$.

Corollary 1.2.6 (Global Existence)**.** *With the same assumptions as in Corollary 1.1.5 then we have that u is globally defined and bounded in compact sets of $t \in (t_0,+\infty)$. In particular, for $0 \leq u_0 \in L^q(\Omega)$ with q satisfying (1.1.11) and a.e. $x \in \Omega$, u verifies*

$$0 \leq u(t,x,u_0) \leq e^{\lambda t}e^{\Delta t}u_0(x),$$

which implies that

$$\begin{cases} \|u(t,\cdot,u_0)\|_{L^\infty(\Omega)} \leq Me^{(\lambda+a)(t-t_0)}\|u_0\|_{L^\infty(\Omega)}, & t > t_0 \\ \|u(t,\cdot,u_0)\|_{L^\infty(\Omega)} \leq M\frac{e^{(\lambda+a)(t-t_0)}}{(t-t_0)^{\frac{N}{2q}}}\|u_0\|_{L^q(\Omega)}, & t > t_0,\ 1 \leq q \leq \infty \end{cases} \tag{1.2.6}$$

where constant a is defined in (1.1.10).

Proof. Without any loss of generality, we can assume $u_0 \in L^\infty(\Omega)$. Otherwise, we can take $v_0 := u(t_0+\delta,\cdot)$ as the new initial data for $\delta > 0$ small which, by Proposition 1.1.7, we know that $v_0 \in L^\infty(\Omega)$.

We proceed by contradiction. Solution u exists for the time period $[t_0,\tau)$ where $\tau \leq \infty$. If we considered the case $\tau < \infty$ then

$$\lim_{t\to\tau}\|u(t,x)\|_\infty = \infty.$$

Since u verifies (1.2.4) for $t \in [t_0,\tau)$ and U exists and is bounded for all $t \geq t_0$ then we arrive at a contradiction:

$$0 \leq |u(t,x)| \leq U(t,x) \leq \|U(t,x)\|_\infty \leq e^{\lambda(t-t_0)}\|u_0\|_\infty,\ \forall t \in [t_0,\tau_0) \tag{1.2.7}$$

and thus, have global existence for solutions u.

□

1.3 Parabolic Hopf lemma

As mentioned earlier, comparison techniques are key tools in the development of the results contained in this thesis. Maximum principles are the basis for comparison between solutions. Maximum principles, and in particular, the strong maximum principle relies on the Hopf lemma. We will present a parabolic Hopf lemma for equation (1.0.1), whose proof relies on the standard parabolic Hopf lemma for linear problems, see [18, 31].

Proposition 1.3.1. *Let $\Omega \subset \mathbb{R}^N$ satisfy the interior sphere condition, $n \in L^\infty(\mathbb{R}\times\Omega)$ and let $u(t,x,t_0,u_0)$ be the solution of (1.0.1) with $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and $u_0 \not\equiv 0$. Then we have that*

$$\frac{\partial u}{\partial \vec{n}}(t,x,u_0) < 0, \quad \forall t > t_0, x \in \partial\Omega,$$

where $\vec{n}$ is the outward normal vector of $\partial\Omega$.

Proof. We will rely on the standard Hopf Lemma. Denoting by U the solution of the linear problem (1.2.5), by Corollary 1.2.5 we have that for $t \geq t_0$, $U(t) \in L^\infty(\Omega)$ and $\|U(t)\|_\infty \leq e^{\lambda(t-t_0)}\|u_0\|_\infty$. Moreover,

$$0 \leq u(t,x;u_0) \leq U(t,x;u_0), \quad \text{for } t > t_0, x \in \bar{\Omega}$$

Then,

$$\lambda u - n(t,x)|u|^{\rho-1}u \geq \lambda u - \|n\|_\infty\left(e^{\lambda(t-t_0)}\|u_0\|_\infty\right)^{\gamma-1}u \geq \lambda u - N_0 u, \quad t \in [t_0,t_1]$$

where

$$N_0 = \sup_{t\in[t_0,t_1]} \|n\|_\infty \left(\|u_0\|_\infty e^{\lambda(t-t_0)}\right)^{\gamma-1} = \|n\|_\infty \left(\|u_0\|_\infty e^{|\lambda|(t_1-t_0)}\right)^{\gamma-1}.$$

Now, the solution v of

$$\begin{cases} v_t - \Delta v - (\lambda - N_0)v = 0 & t \in (t_0,t_1),\ x \in \Omega, \\ v = 0, & t \in (t_0,t_1),\ x \in \partial\Omega, \\ v(t_0) = u_0 & x \in \Omega \end{cases} \tag{1.3.1}$$

verifies, using Theorem 1.2.1, $u(t,x) \geq v(t,x)$. Moreover, multiplying $e^{-(\lambda-N_0)(t-t_0)}$ to both sides of equation for v, we arrive at $v(t,x) = e^{(\lambda-N_0)(t-t_0)}w(t,x)$ where

$$\begin{cases} w_t - \Delta w = 0 & (t,x) \in (t_0,t_1)\times\Omega, \\ w = 0, & x \in \partial\Omega, \\ w(t_0) = u_0 & x \in \Omega. \end{cases} \tag{1.3.2}$$

Therefore,

$$u(t,x) \geq v(t,x) = e^{(\lambda-N_0)t}w(t,x), \quad \text{for all } (t,x) \in (t_0,t_1)\times\Omega. \tag{1.3.3}$$

Now, from Proposition 1.1.7 we have that, for any $0 < \eta < 1$ we have that $u(t,\cdot), v(t,\cdot), w(t,\cdot) \in C^{1,\eta}(\bar{\Omega})$ and $u(t,x) = v(t,x) = w(t,x)$ for $x \in \partial\Omega$. Moreover, v and w become classical for $t > t_0$ since the right hand sides of (1.3.1) and (1.3.2) are smooth in t, x and u. Therefore, taking (1.3.3) into account and the fact that $u(t,x) = v(t,x)$ for $x \in \partial\Omega$ we have the following

$$\frac{\partial u}{\partial \vec{n}} \leq \frac{\partial v}{\partial \vec{n}} = e^{(\lambda-N_0)t}\frac{\partial w}{\partial \vec{n}}, \quad \text{for } x \in \partial\Omega. \tag{1.3.4}$$

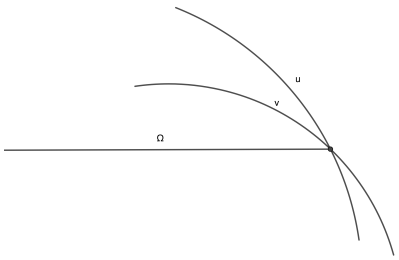


Figure 1.1: Solution u stays above v up to $\partial\Omega$.

Let us see (1.3.4). We have that

$$\frac{\partial u}{\partial(-\vec{n})} = \lim_{\delta\to 0}\frac{u(t,x-\delta\vec{n}) - u(t,x)}{\delta}, \quad \frac{\partial v}{\partial(-\vec{n})} = \lim_{\delta\to 0}\frac{v(t,x-\delta\vec{n}) - v(t,x)}{\delta}.$$

Hence, using that $u(t,x) = 0 = v(t,x)$ for any $x \in \partial\Omega$, we have

$$-\frac{\partial u}{\partial(\vec{n})}(t,x)+\frac{\partial v}{\partial(\vec{n})}(t,x)=\frac{\partial u}{\partial(-\vec{n})}(t,x)-\frac{\partial v}{\partial(-\vec{n})}(t,x)=\lim_{\delta\to 0}\frac{u(t,x-\delta\vec{n})-u(t,x)}{\delta}-\frac{v(t,x-\delta\vec{n})-v(t,x)}{\delta},$$

and since $u(t,x)\geq v(t,x)$ with $x\in\Omega$,

$$-\frac{\partial u}{\partial(\vec{n})}(t,x)+\frac{\partial v}{\partial(\vec{n})}(t,x)=\lim_{\delta\to 0}\frac{u(t,x-\delta\vec{n})}{\delta}-\frac{v(t,x-\delta\vec{n})}{\delta}\geq 0$$

which proves (1.3.4).

On the other hand, due to the parabolic maximum principle in [18] (see Theorem 11 Section 7.1) we have that the minimum is attained at the border. Therefore, we can apply the parabolic Hopf lemma (see and [31] Chapter 3 Theorem 7) to w, which is a classical solution, yielding

$$\frac{\partial w}{\partial\vec{n}}(t,x,u_0)<0,\quad \forall t>t_0, x\in\partial\Omega. \tag{1.3.5}$$

From where we get

$$\frac{\partial u}{\partial\vec{n}}(t,x,u_0)<0,\quad \forall t\in(t_0,t_1), x\in\partial\Omega. \tag{1.3.6}$$

Since we fixed $t_1>t_0$ arbitrarily,

$$\frac{\partial u}{\partial\vec{n}}(t,x,u_0)<0,\quad \forall t>t_0, x\in\partial\Omega. \tag{1.3.7}$$

This completes the proof of the proposition.

□

1.4 Perturbation Techniques and a Comparison Principle for Subdomains

In this subsection we will introduce two perturbation results for problems such as (1.0.1). The first result focuses on perturbations on the function $n(t,x)$ and the initial data u_0, whilst the second one focuses on the domain Ω. These results will help us in the analysis on the next chapters.

Since we are going to consider several modifications of problem (1.0.1) by changing its initial condition, function n or even the domain Ω we are going to denote by $P(u,t_0,u_0,n,\Omega)$ the problem

$$P(u,t_0,u_0,n,\Omega):=\begin{cases} u_t-\Delta u=\lambda u-n(t,x)|u|^{\rho-1}u, & t>t_0,\ x\in\Omega,\\ u(t,x)=0, & t>t_0,\ x\in\partial\Omega,\\ u(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{1.4.1}$$

Whenever the initial time t_0 is not relevant to the argument, we will just write $P(u,u_0,n,\Omega)$ instead.

Throughout this subsection we will consider $n\in L^\infty(\mathbb{R}\times\Omega)$, $u_0\in L^q(\Omega)$ for q verifying (1.1.11) and Ω bounded. Therefore, applying the results from Sections 1.1 and 1.2 all problems $P(u,t_0,u_0,n,\Omega)$ considered are well–posed, solutions are globally defined and estimates (1.2.6) hold true.

Proposition 1.4.1. *Let us consider problems $P(u,t_0,u_0,n,\Omega)$ and $P(\tilde{u},t_0,\tilde{u}_0,\tilde{n},\Omega)$. Let us assume there exist $R,N_0>0$ such that*

$$\begin{cases} \|u_0\|_{L^\infty(\Omega)}, \|\tilde{u}_0\|_{L^\infty(\Omega)} \leq R, \\ \|n\|_{L^\infty([t_0,T]\times\Omega)}, \|\tilde{n}\|_{L^\infty([t_0,T]\times\Omega)} \leq N_0. \end{cases}$$

Then, for any p such that $p > \frac{N}{2} + 1$

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^\infty(\Omega)} \leq C\left(\|u_0 - \tilde{u}_0\|_{L^\infty(\Omega)} + \|n - \tilde{n}\|^{1/p}_{L^1([t_0,T]\times\Omega)}\right),\ t \in [t_0, T], \tag{1.4.2}$$

where $C = C(T - t_0, \lambda, N_0, R, \rho, \Omega, M)$ and M is the constant in (1.1.10). Moreover,

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^\infty(\Omega)} \leq C\left((t-t_0)^{-r}\|u_0 - \tilde{u}_0\|_{L^q(\Omega)} + \|n - \tilde{n}\|^{1/p}_{L^1([t_0,T]\times\Omega)}\right),\ t \in (t_0, T], \tag{1.4.3}$$

where $r = r(q, N)$ and $C = C(T - t_0, \lambda, N_0, R, \rho, \Omega, M, r, q)$.

Proof. First we will prove (1.4.2). Computations for (1.4.3) are fairly similar. Solutions $u, \tilde{u}$ satisfy the Variation of Constants Formula, therefore for $T > t_0$, $t \in [t_0, T]$, $p > \frac{N}{2} + 1$ and a and M as in (1.1.10)

$$\begin{aligned}
\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_\infty &\leq \left\|e^{(\lambda I + \Delta)(t-t_0)}(u_0 - \tilde{u}_0)\right\|_\infty \\
&+ \int_{t_0}^t \left\|e^{(\lambda I+\Delta)(t-s)}(n(s,\cdot)|u|^{\rho-1}u - \tilde{n}(s,\cdot)|\tilde{u}|^{\rho-1}\tilde{u})\right\|_\infty ds, \\
&\leq M e^{(\lambda+a)(t-t_0)}\|u_0 - \tilde{u}_0\|_\infty \\
&+ \int_{t_0}^t M\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}\left\|(n(s,\cdot) - \tilde{n}(s,\cdot))|u|^{\rho-1}u\right\|_p ds \\
&+ \int_{t_0}^t M\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}\left\|\tilde{n}(s,\cdot)(|u|^{\rho-1}u - |\tilde{u}|^{\rho-1}\tilde{u})\right\|_p ds, \\
&\leq M e^{(\lambda+a)(t-t_0)}\|u_0 - \tilde{u}_0\|_\infty \\
&+ \int_{t_0}^t M\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}\|n(s,\cdot) - \tilde{n}(s,\cdot)\|_p \left(M e^{(\lambda+a)(s-t_0)}R\right)^\rho ds \\
&+ \int_{t_0}^t M\rho\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}N_0\left\||u - \tilde{u}|(1 + |u|^{\rho-1} + |\tilde{u}|^{\rho-1})\right\|_p ds,
\end{aligned} \tag{1.4.4}$$

where we have used (1.2.6) and (1.0.5) to obtain the last inequality.

Let us focus on estimates for some of the above elements in the equation. Since $p > 2$, for $s \in [t_0, T]$

$$\begin{aligned}\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_{L^p(\Omega)} &= \left(\int_\Omega |n(s,x)-\tilde{n}(s,x)||n(s,x)-\tilde{n}(s,x)|^{p-1}dx\right)^{1/p},\\ &\leq \left(\int_\Omega |n(s,x)-\tilde{n}(s,x)|\left(\|n(s,\cdot)\|_\infty+\|n(s,\cdot)\|_\infty\right)^{p-1}dx\right)^{1/p}, \qquad (1.4.5)\\ &\leq (2N_0)^{\frac{p-1}{p}}\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_{L^1(\Omega)}^{1/p}.\end{aligned}$$

Moreover,

$$\left\||u(s,\cdot)-\tilde{u}(s,\cdot)|(1+|u(s,\cdot)|^{\rho-1}+|\tilde{u}(s,\cdot)|^{\rho-1})\right\|_{L^p(\Omega)} \leq C_1\|u(s,\cdot)-\tilde{u}(s,\cdot)\|_\infty, \qquad (1.4.6)$$

where $C_1=\left(1+2\left(Me^{(\lambda+a)(T-t_0)}R\right)^{\rho-1}\right)|\Omega|^{1/p}$, from (1.2.6).

Therefore, using (1.4.5) and (1.4.6) in (1.4.4) we have

$$\begin{aligned}\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty &\leq Me^{(\lambda+a)(t-t_0)}\|u_0-\tilde{u}_0\|_\infty\\ &+\int_{t_0}^t M\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}\left(Me^{(\lambda+a)(s-t_0)}R\right)^\rho(2N_0)^{\frac{p-1}{p}}\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_{L^1(\Omega)}^{1/p}ds\\ &+\int_{t_0}^t M\rho\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}N_0\left(1+2\left(Me^{(\lambda+a)(s-t_0)}R\right)^{\rho-1}\right)|\Omega|^{1/p}\|u(s,\cdot)-\tilde{u}(s,\cdot)\|_\infty ds\\ &\leq Me^{(\lambda+a)(t-t_0)}\|u_0-\tilde{u}_0\|_\infty\\ &+M^{\rho+1}e^{(\rho+1)(\lambda+a)(T-t_0)}R^\rho(2N_0)^{\frac{p-1}{p}}\int_{t_0}^t\frac{\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_1^{1/p}}{(t-s)^{\frac{N}{2p}}}ds\\ &+\int_{t_0}^t M\rho\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}N_0\left(1+2\left(Me^{(\lambda+a)(T-t_0)}R\right)^{\rho-1}\right)|\Omega|^{1/p}\|u(s,\cdot)-\tilde{u}(s,\cdot)\|_\infty ds.\end{aligned} \qquad (1.4.7)$$

Let us focus on the first integral in (1.4.7) above. By Hölder's inequality we have that

$$\begin{aligned}\int_{t_0}^t\frac{\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_{L^1(\Omega)}^{\frac{1}{p}}}{(t-s)^{\frac{N}{2p}}}ds &\leq \left(\int_{t_0}^t(t-s)^{-\frac{N}{2p}\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_0}^t\|n(s,\cdot)-\tilde{n}(s,\cdot)\|_{L^1(\Omega)}ds\right)^{\frac{1}{p}},\\ &\leq \left(\frac{(T-t_0)^{1-\frac{N}{2(p-1)}}}{1-\frac{N}{2(p-1)}}\right)^{\frac{p-1}{p}}\|n-\tilde{n}\|_{L^1([t_0,T]\times\Omega)}^{\frac{1}{p}}.\end{aligned} \qquad (1.4.8)$$

Thus, using (1.4.8) in (1.4.7) we arrive at

$$\begin{aligned}\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty &\leq Me^{(\lambda+a)(t-t_0)}\|u_0-\tilde{u}_0\|_\infty \\ &+M^{\rho+1}e^{(\rho+1)(\lambda+a)(T-t_0)}R^\rho\left(2N_0\right)^{\frac{p-1}{p}}\left(\frac{(T-t_0)^{1-\frac{N}{2(p-1)}}}{1-\frac{N}{2(p-1)}}\right)^{\frac{p-1}{p}}\|n-\tilde{n}\|^{\frac{1}{p}}_{L^1([t_0,T]\times\Omega)} \\ &+\int_{t_0}^t M\rho\frac{e^{(\lambda+a)(t-s)}}{(t-s)^{\frac{N}{2p}}}N_0\left(1+2\left(Me^{(\lambda+a)(T-t_0)}R\right)^{\rho-1}\right)|\Omega|^{1/p}\|u(s,\cdot)-\tilde{u}(s,\cdot)\|_\infty ds.\end{aligned} \tag{1.4.9}$$

Notice that (1.4.9) above can be rewritten as

$$\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty \leq C_2\left(\|u_0-\tilde{u}_0\|_\infty+\|n-\tilde{n}\|^{\frac{1}{p}}_{L^1}\right)+C_3\int_{t_0}^t\frac{1}{(t-s)^{\frac{N}{2p}}}\|u(s,\cdot)-\tilde{u}(s,\cdot)\|_\infty ds, \tag{1.4.10}$$

where

$$C_2=\max\left\{Me^{|\lambda+a|(T-t_0)},M^{\rho+1}e^{(\rho+1)|\lambda+a|(T-t_0)}R^\rho\left(2N_0\right)^{\frac{p-1}{p}}\left(\frac{(T-t_0)^{1-\frac{N}{2(p-1)}}}{1-\frac{N}{2(p-1)}}\right)^{\frac{p-1}{p}}\right\},$$

and

$$C_3=M\rho e^{|\lambda+a|(T-t_0)}N_0\left(1+2\left(Me^{(\lambda+a)(T-t_0)}R\right)^{\rho-1}\right)|\Omega|^{1/p}.$$

That is, we have

$$\mu(t)\leq C_2\left(\|u_0-\tilde{u}_0\|_\infty+\|n-\tilde{n}\|^{\frac{1}{p}}_{L^1}\right)+C_3\int_{t_0}^t(t-s)^{-N/2p}\mu(s)ds,$$

where $\mu(t)=\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty$. From Chapter 7 of [21] we have the following Grönwall's inequality: if $a,b\geq 0$, $\alpha<1$, $\beta<1$, $0<T<\infty$ and

$$\mu(t)\leq a(t-t_0)^{-\alpha}+b\int_{t_0}^t(t-s)^{-\beta}\mu(s)ds,\quad t\in(t_0,T),$$

then

$$\mu(t)\leq a(t-t_0)^{-\alpha}\sum_{i=0}^\infty C_i'\left(b\,\Gamma(1-\beta)\right)^i(t-t_0)^{i(1-\beta)}, \tag{1.4.11}$$

where the series is convergent (see Lemma 7.1.2 in [21]) and

$$C_0'=1,\quad\text{and}\quad\frac{C_{i+1}'}{C_i'}=\frac{\Gamma(i(1-\beta)+1-\alpha)}{\Gamma(i(1-\beta)+2-\alpha-\beta)}.$$

Therefore, using Grönwall on (1.4.10), we arrive at (1.4.11) for $a=C_2\left(\|u_0-\tilde{u}_0\|_\infty+\|n-\tilde{n}\|^{\frac{1}{p}}_{L^1}\right)$, $b=C_3$, $\alpha=0$ and $\beta=N/2p$:

$$\begin{aligned}\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty &\leq C_2\left(\|u_0-\tilde{u}_0\|_\infty+\|n-\tilde{n}\|_{L^1}^{\frac{1}{p}}\right)\sum_{i=0}^{\infty}C_i'\,(C_3\,\Gamma(1-\beta))^i t^{i(1-\beta)},\\ &\leq C\left(\|u_0-\tilde{u}_0\|_\infty+\|n-\tilde{n}\|_{L^1}^{\frac{1}{p}}\right)\quad t\in[t_0,T],\end{aligned}$$

where $C=C_2\sum_{i=0}^{\infty}C_i'\,(C_3\,\Gamma(1-\beta))^i(T-t_0)^{i(1-\beta)}$, which concludes the proof of (1.4.2).

For (1.4.3), following similar steps as in (1.4.4) and using (1.2.6) we derive the following

$$\begin{aligned}\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_\infty &\leq \left\|e^{(\lambda I+\Delta)(t-t_0)}(u_0-\tilde{u}_0)\right\|_\infty+\int_{t_0}^{t}\left\|e^{(\lambda I+\Delta)(t-s)}(n(s,\cdot)|u|^{\rho-1}u-\tilde{n}(s,\cdot)|\tilde{u}|^{\rho-1}\tilde{u})\right\|_\infty ds,\\ &\leq M\frac{e^{|\lambda+a|(t-t_0)}}{(t-t_0)^{\frac{N}{2q}}}\|u_0-\tilde{u}_0\|_q+\int_{t_0}^{t}\left\|e^{(\lambda I+\Delta)(t-s)}(n(s,\cdot)|u|^{\rho-1}u-\tilde{n}(s,\cdot)|\tilde{u}|^{\rho-1}\tilde{u})\right\|_\infty ds.\end{aligned}$$

Steps (1.4.5) through (1.4.10) are the same with $\|u_0-\tilde{u}_0\|_q(t-t_0)^{-\frac{N}{2q}}$ instead of $\|u_0-\tilde{u}_0\|_\infty$. Applying Grönwall's inequality we obtain (1.4.3).

□

We obtain now comparison results when the domain is changed. We start with the following auxiliary result where we assume regularity on the function n.

Lemma 1.4.2. *Let us consider problems $P(v,t_0,v_0,n,\Omega)$ and $P(\tilde{v},t_0,\tilde{v}_0,n,\tilde{\Omega})$ such that $\tilde{\Omega}\subset\Omega$. Let us assume $n\in C^\infty(\mathbb{R}\times\Omega)$ and $v_0,\tilde{v}_0\in C_0^\infty(\tilde{\Omega})$ nonnegative such that $\tilde{v}_0\leq v_0$ in $\tilde{\Omega}$. Then, for $t>t_0$*

$$0\leq\tilde{v}(t,x,\tilde{v}_0)\leq v(t,x,v_0),\ \forall x\in\tilde{\Omega}.$$

Proof. Let $\gamma(t,x):=v(t,x)$ for $t>t_0$ and $x\in\partial\tilde{\Omega}$. Then we can restrict problem $P(v,v_0,n,\Omega)$ to $\tilde{\Omega}$ in the following manner

$$\begin{cases}v_t-\Delta v=\lambda v-n(t,x)|v|^{\rho-1}v, & t>t_0,\ x\in\tilde{\Omega},\\ v(t,x)=\gamma(t,x), & t>t_0,\ x\in\partial\tilde{\Omega},\\ v(0,x)=v_0(x), & x\in\tilde{\Omega},\end{cases}$$

where $\gamma(t,x)\geq 0$ for $t>t_0$ and $x\in\partial\tilde{\Omega}$ due to Theorem 1.2.1. Moreover, v and $\tilde{v}$ are classical solutions due to the regularity of n, v_0 and $\tilde{v}_0$. Therefore, applying a weak maximum principle for classical solutions (see Theorem 12 in [31]) we arrive at

$$0\leq\tilde{v}(t,x,\tilde{v}_0)\leq v(t,x,\tilde{v}_0),\ t>t_0,\ \forall x\in\tilde{\Omega}.$$

Finally, using Thereom 1.2.1 again we conclude the proof.

□

We obtain now the following general result.

Proposition 1.4.3. *Let us consider problems $P(u, u_0, n, \Omega)$ and $P(\tilde{u}, u_0, n, \tilde{\Omega})$ such that $\tilde{\Omega} \subset \Omega$. Let us assume $u_0 \in L^\infty(\Omega)$ and $n \in L^\infty(\mathbb{R} \times \Omega)$. Then*

$$0 \le \tilde{u}(t, x, u_0) \le u(t, x, u_0),\ t > t_0, \quad a.e.\ x \in \tilde{\Omega}.$$

Proof. Due to the density of C_0^∞ in L^p, for $1 \le p < \infty$, there exist sequences $(n_\varepsilon)_{\varepsilon>0}$ and $(u_0^\varepsilon)_{\varepsilon>0}$ such that

$$\begin{aligned} &n_\varepsilon(t, x) \in C^\infty(\mathbb{R} \times \Omega),\ \|n_\varepsilon - n\|_{L^1(\mathbb{R}\times\Omega)} \to 0,\ \varepsilon \to 0, \\ &u_0^\varepsilon(x) \in C_0^\infty(\Omega),\ \|u_0^\varepsilon - u_0\|_{L^q(\Omega)} \to 0,\ \varepsilon \to 0, \end{aligned}$$

where q satisfies (1.1.11). Let us consider problems $P(u_\varepsilon, u_0^\varepsilon, n_\varepsilon, \Omega)$ and $P(\tilde{u}_\varepsilon, u_0^\varepsilon, n_\varepsilon, \tilde{\Omega})$. By Lemma 1.4.2 we have that

$$\tilde{u}_\varepsilon(t, x, u_0^\varepsilon) \le u_\varepsilon(t, x, u_0^\varepsilon),\ \ t > t_0, \forall x \in \tilde{\Omega}.$$

Applying Proposition 1.4.1, we have $\tilde{u}_\varepsilon \to \tilde{u}$ and $u_\varepsilon \to u$ in $L^\infty(\Omega)$ uniformly for $t > t_0$ on bounded sets, which in particular implies point–wise convergence. Thus,

$$\tilde{u}(t, x, u_0) \le u(t, x, u_0),\ \ t > t_0,\ \text{a.e. } x \in \tilde{\Omega}.$$

□

Chapter 2

Autonomous degenerate logistic–type parabolic equations

As mentioned in the Introduction, in this chapter we study the asymptotic behaviour of the solutions of problem (0.0.2) where the function $n(x)$ vanishes in a compact subset K of $\bar{\Omega}$. That is, $K := \{x \in \overline{\Omega} : n(x) = 0\}$. Therefore, within K, equation (0.0.2) becomes linear.

Let us recall the autonomous problem (0.0.2)

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - n(x)|u|^{\rho-1}u, & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.0.1}$$

We present first a summary of the main concepts and results in [7] which apply to (2.0.1). One of the most relevant concepts from [7] and that we recall here is the characteristic value of a compact set K, denoted by $\lambda_0(K)$. In the case where $K = \overline{\Omega}_0$ with Ω_0 a smooth set, then $\lambda_0(K)$ coincides with the first eigenvalue of the Laplace operator in Ω_0 with Dirichlet boundary conditions. If K is an arbitrary compact set, it has a more involved definition, see [7] and Definition 2.1.1 below. This is a very relevant concept since its relationship with λ completely determines the asymptotic behaviour of solutions of (2.0.1). The characteristic value is similar to other concepts found in the literature such as the principal eigenvalue (see [28, 29]).

Following [7], given a compact set K, if its characteristic value $\lambda_0(K)$ is finite, then K will be referred to as a fat set whilst if it is not finite, it will be referred to as a slim set, see Definition 2.1.3 below. As mentioned in the Introduction of this thesis and seen in [7], whenever $\lambda < \lambda_0(K)$ then all solutions of (2.0.1) are bounded. And if $\lambda \geq \lambda_0(K)$ then all solutions grow up, that is, $\lim_{t\to\infty} \|u(t,\cdot)\|_\infty = \infty$. Therefore, since $\lambda_0(K) = \infty$ for any slim set K, all solutions associated to slim sets K will be bounded. Thus, grow up or unboundedness for $t \to \infty$ is only possible for fat sets K. Moreover, studying the asymptotic behaviour of grow up solutions implies studying where are solutions going to infinity, that is, studying the grow-up set: $G = \{x \in \Omega : u(x,t) \to +\infty \text{ as } t \to +\infty\}$, which it is known from [7] that it is independent of the initial condition.

Locating the set G, specially for the cases where K is not smooth, is an interesting topic related to the asymptotic behaviour of solutions of (2.0.1). In the case where K is such that there exists a large smooth open domain $B \subset K$ and $\lambda > \lambda_1(B)$, we are able to confirm that solutions grow up in B, that is $B \subset G$. This can be seen by comparing the solutions of $P(u, u_0, n, \Omega)$ to the solutions of $P(v, u_0, 0, B)$ (which is the linear problem defined in B)

$$P(u,u_0,n,\Omega)=\begin{cases}\frac{\partial u}{\partial t}-\Delta u=\lambda u-n(x)|u|^{\rho-1}u, & t>0,\ x\in\Omega,\\ u(t,x)=0, & t>0,\ x\in\partial\Omega,\\ u(0,x)=u_0(x)\geq 0, & x\in\Omega,\end{cases}$$

$$P(v,u_0,0,B)=\begin{cases}\frac{\partial v}{\partial t}-\Delta v=\lambda v, & t>0,\ x\in B,\\ v(t,x)=0, & t>0,\ x\in\partial B,\\ v(0,x)=u_0(x)\geq 0, & x\in B.\end{cases}$$

Let us recall the notation $P(u,t_0,u_0,n,\Omega)$ used in Section 1.4 and which refers to problem (0.0.6) and which can also be used for (0.0.2). By Theorem 1.2.1 we have $v(t,x)\leq u(t,x)$ for $x\in B$ and since it is a linear equation of a smooth domain B it is known that $v(t,\cdot)\to\infty$ in B. That is, if K is regular enough so that it may contain large bounded smooth domains inside, then roughly speaking it is possible to locate the grow up set G. However, there exist fat compact sets K with empty interior as it will be shown later on and which will be referred to as a pathological sets. For this pathological sets K, if $\lambda>\lambda_0(K)$ then all solutions of (2.0.1) grow up and as it is shown in [7], we have $\|u(t,\cdot)\|_{L^\infty(U)}\to\infty$ for any open set U with $K\subset U$. However, the set G is not known in detail.

One of the goals of this chapter is to give meaning to the solution of the equation $u_t-\Delta u=0$ with Dirichlet boundary conditions in a compact set K, not necessarily smooth and probably with empty interior. As a matter of fact, we will give a meaning to this problem in a subset of K which is the interior of K in the "fine topology" of $\mathbb{R}^n$ (in case it is nonempty), a set which we will denote by K^{fi}. This is accomplished in Section 2.2 and we will refer to this problem as $P(u,t_0,u_0,0,K)$. Afterwards our goal will be to obtain a comparison result between the problems $P(u,t_0,u_0,n,\Omega)$ and $P(v,t_0,u_0,0,K)$ which shows that $v(t,x)\leq u(t,x)$ for any $t>0$. This comparison result does not follow from standard comparison arguments since the set K may be very pathological. We will rely on approximations of K by the exterior by smooth open sets and using techniques like capacity, "fine topology" and its relation with the spectral and resolvent convergence of operators. With this comparison result we can give a characterization in terms of the spectra in K^{fi}, in case this set is non empty, of the grow up set of the solution for a fixed λ, see Proposition 2.3.3.

At the end of the chapter we include the construction and analysis of a pathological set constructed initially in [7] and which serves as an interesting example where standard comparison results between the solution in Ω and the solution in K do not apply, but the comparison we obtain in this chapter does.

2.1 Introduction to Slim and Fat sets

Let us recall from [7] the definition of the characteristic value of a compact set.

Definition 2.1.1. *Let $K\subset\mathbb{R}^n$ be a compact set and let $\{\Omega_\delta\}_{0<\delta\leq\delta_0}$ be a strictly nested family of smooth bounded open sets such that*

$$K=\bigcap_{0<\delta\leq\delta_0}\Omega_\delta. \tag{2.1.1}$$

Then, the **characteristic value** *of K is defined as the monotonic limit of the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions on Ω_δ:*

$$\lambda_0(K):=\lim_{\delta\to 0^+}\lambda_1(\Omega_\delta) \tag{2.1.2}$$

Remark 2.1.2. *i) It is not difficult to see that the definition is independent of the particular family* $\{\Omega_\delta\}_{0<\delta\leq\delta_0}$ *chosen. In particular we can take* $\Omega_\delta = \{x \in \mathbb{R}^n : d(x,K) < \delta\}$*. Also, we could define the characteristic value using nested sequences of domains instead of families of domains.*

ii) Observe that since $K \subset \Omega_{\delta_0}$ *and the limit above is increasing due to the monotonicity properties of the eigenvalues, we have* $0 < \lambda_1(\Omega_{\delta_0}) \leq \lambda_0(K)$*. We do not exclude the possibility that* $\lambda_0(K) = +\infty$*.*

Following [7], we classify compact sets according to their characteristic value as follows.

Definition 2.1.3. *Let* $K \subset \mathbb{R}^N$ *be a compact set.*

- *If* $0 < \lambda_0(K) < \infty$ *then* K *is a* **fat** *set,*
- *if* $\lambda_0(K) = \infty$ *then the set is* **slim**.

The characteristic value has some interesting properties which we now state:

Proposition 2.1.4. *Let* Ω *be a bounded open set,* $K \subset \Omega$ *be a compact set and* $\{\Omega_\delta\}_\delta$ *a strictly nested family of smooth open neighbourhoods of* K*. Then:*

1. *The definition of* $\lambda_0(K)$ *is independent of the family* $\{\Omega_\delta\}_\delta$*, which can be assumed to be* C^∞*–smooth.*
2. *If* $K = \overline{B}$ *where* B *is a bounded smooth open set, then*

$$\lambda_0(K) = \lambda_1(\Omega) \tag{2.1.3}$$

   where $\lambda_1(\Omega)$ *is the first eigenvalue of* $-\Delta$ *with Dirichlet boundary conditions on* Ω*.*

3. *If* $K_1 \subset K_2$ *then* $\lambda_0(K_2) \leq \lambda_0(K_1)$*. This monotonicity provides aditional properties:*
   (a) *Any compact superset of a fat set is fat. Therefore, if* K *contains a ball, it is fat. The reciprocal is not true (for instance, a single point is slim since for a family of balls centred at the point with decreasing radius, their first eigenvalues tend to infinite).*
   (b) *Any compact subset of a slim set is slim. Thus, if* $|K| = 0$ *then* K *is slim. However, the reciprocal is not true. There are slim sets with positive measure as will be shown in Example 2.1.6.*

We introduce now the space $H_0^1(K)$. For a given compact set $K \subset \mathbb{R}^N$ we define $H_0^1(K)$, a subspace of $H^1(\mathbb{R}^N)$ as

$$H_0^1(K) := \left\{\phi \in H^1(\mathbb{R}^N) : \phi(x) = 0 \text{ a.e. } x \in \mathbb{R}^N \setminus K\right\}. \tag{2.1.4}$$

Remark 2.1.5. *There are some immediate conclusions derived from (2.1.4) which help us have a better grasp on the concepts of slim and fat sets: it is always true that* $0 \in H_0^1(K)$ *whichever the* K*. However, for* K *such that* $|K| = 0$ *(*K *slim in particular) then* 0 *is the only function in* $H_0^1(K)$*, that is* $H_0^1(K) = \{0\}$*. Characterising when the space* $H_0^1(K)$ *is nontrivial is addressed in Proposition 2.1.7.*

Proposition 2.1.4 states that if $|K| = 0$ then K is slim. However, the reciprocal is not true, let us see this through a counterexample: that is let us see that there exist slim sets with positive measure.

Example 2.1.6. Consider the Smith-Volterra-Cantor set $S \subset [0,1]$ constructed by induction as follow

- Starting with $S_0 = [0,1]$, we construct S_1 removing the middle interval of length $1/4$, so that $S_1 = S_1^1 \cup S_1^2$, two intervals of length $3/8$ each.
- We continue by removing at each step the middle open interval of length $1/4^n$ from each of the 2^{n-1} intervals that constitutes S_{n-1}, obtaining

$$S_n = \bigcup_{k=1}^{2^n} S_n^k.$$

It is not difficult to see that this construction is possible and since all intervals S_n^k $k = 1, \ldots, 2^n$ have the same length, we have $1 \geq |S_n| = 2^n |S_n^k|$, $k = 1, 2, \ldots 2^n$, so that $|S_n^k| \leq 1/2^n$ for all $k = 1, \ldots 2^n$.

On the other hand, counting the intervals that have been taken out from S to construct S_n, we have

$$|S_n| = 1 - \frac{1}{4} - 2\frac{1}{4^2} - 2^2\frac{1}{4^3} - \ldots - 2^{n-1}\frac{1}{4^n} = \frac{1}{2} + \frac{1}{2^{n+1}}$$

- Finally, defining $S = \bigcap_{n\in\mathbb{N}} S_n$, we have that S is compact and $\mu(S) = \frac{1}{2}$. But $S \subset S_n$ for each $n \in \mathbb{N}$ and therefore $\lambda_0(S) \geq \lambda_0(S_n)$. But since S_n is a finite union of intervals, all of them of the same length, we have $\lambda_0(S_n) = \lambda_0(S_n^k)$ for each k. But

$$\lambda_0(S_n^k) = \frac{\pi^2}{|S_n^k|^2} \geq \pi^2 2^n \to +\infty \text{ as } n \to +\infty.$$

which shows that $\lambda_0(S) = +\infty$ and therefore S is slim.

□

The next result from [7] characterises when $H_0^1(K)$ is nontrivial:

Proposition 2.1.7. *A compact set $K \subset \mathbb{R}^N$ is fat if and only if the space $H_0^1(K)$ is different from $\{0\}$.*

Moreover, in this case,

$$\lambda_0(K) = \min\left\{ \frac{\int_{\mathbb{R}^N} |\nabla\phi|^2}{\int_{\mathbb{R}^N} |\phi|^2} : \phi \in H_0^1(K), \phi \neq 0 \right\}. \tag{2.1.5}$$

Also, the minimum is attained at least in a nonnegative function. Moreover, if ψ is a minimizer so is $|\psi|$.

Finally, if ψ is a minimizer, then ψ satisfies

$$\int_{\mathbb{R}^N} \nabla\psi\nabla\phi = \lambda_0(K) \int_{\mathbb{R}^N} \psi\phi, \ \ \forall \phi \in H_0^1(K). \tag{2.1.6}$$

It can be inmediately obtained that

Corollary 2.1.8. *A compact set $K \subset \mathbb{R}^N$ is slim if and only if $H_0^1(K) = \{0\}$.*

Remark 2.1.9. Equation (2.1.5) is the Raleigh quotient for $-\Delta$ with homogeneous Dirichlet boundary conditions in a compact set K. Formally this holds for any compact set K even if it has empty interior.

□

Let us introduce another result from [7] which provides an almost complete characterization of the boundedness or unboundedness of solutions of (2.0.1) in terms of the relation between λ and $\lambda_0(K)$.

Theorem 2.1.10. *We have the following:*

i) Assume K is slim. Then, for any $\lambda \in \mathbb{R}$ all solutions of (2.0.1) are bounded.

ii) Assume K is fat and $\lambda < \lambda_0(K)$ then all solutions of (2.0.1) are bounded.

iii) Assume K is fat and $\lambda \geq \lambda_0(K)$ then all solutions of (2.0.1) are unbounded in a neighbourhood of K and all solutions are bounded outside a neighbourhood of K.

2.2 Set capacity, quasi open sets, finely open sets and spectral properties

Let us introduce some auxiliary concepts and results that will be needed below. We do not pretend to be very complete in the exposition of these very deep concepts, but just state (without proofs) the ones that are needed for this work. Following [24] we have

Definition 2.2.1. *The (Newtonian) capacity of a set $E \subset \mathbb{R}^N$ is the number*

$$C_2(E) := \inf_{u \in \mathcal{W}(E)} \|u\|^2_{H^1(\mathbb{R}^N)}, \tag{2.2.1}$$

where $\mathcal{W}(E)$ is the set of admissible functions for E given by

$$\mathcal{W}(E) := \left\{ u \in H^1(\mathbb{R}^N) : u \geq 1 \ \text{ on a neighbourhood of } E \right\}. \tag{2.2.2}$$

Again following [24], the set of admissible functions in Definition 2.2.1 above can be restricted to those satisfying $0 \leq u \leq 1$.

Let us introduce the concept of quasi–open sets based on the 2–capacity (see Section 4.1 of [9]):

Definition 2.2.2. *A subset A of $\mathbb{R}^N$ is said to be* **quasi–open** *if for every $\varepsilon > 0$ there exists and open subset A_ε of $\mathbb{R}^N$ such that $C_2(A \triangle A_\varepsilon) < \varepsilon$ where $\triangle$ denotes the symmetric difference of sets, that is,*

$$A \triangle A_\varepsilon = (A \setminus A_\varepsilon) \cup (A_\varepsilon \setminus A).$$

Remark 2.2.3. *The capacity of the empty set is zero. Therefore, any open set $U \subset \mathbb{R}^N$ is quasi–open: for every $\varepsilon > 0$ there exists an open set (for instance U itself) such that $C_2(U \triangle U) = C_2(\emptyset) = 0 < \varepsilon$.*

Hence, every open set is quasi–open, yet not every quasi–open set is open as it will be shown below for the set U_∞ to be introduced.

We will also need some very basic concepts related to the "fine topology" in $\mathbb{R}^N$. This topology is defined as the weakest topology in $\mathbb{R}^N$ that makes all subharmonic functions in $\mathbb{R}^N$ continuous. This topology is finer than the usual one and the name "fine topology" comes from this fact. We refer to [8, 20] for an introduction to this topology. Following [19] we know that an open set is finely open and each finely open set is quasi open. As a matter of fact, Theorem 4.1.5 from [9] establishes that a set $A \subset \mathbb{R}^N$ is quasi open if and only if A is the union of a finely open set and a set of zero capacity.

The relevance of the fine topology in our problem can be explained as follows. After the standard open sets in $\mathbb{R}^N$, the next more general class of sets where we can give a meaning to the Laplace operator, including the spectrum and semigroup generation, is the family of "finely open" sets.

Moreover, as we will see below, in this class of sets the convergence of the spectrum of the Laplace operator has very nice properties.

We want to give a meaning to the problem $u_t = \Delta u$ with Dirichlet boundary condition in the compact set K which is the set where the function $n(x)$ vanishes. As a matter of fact, we will be able to do so in the fine interior of K, whenever this set is not empty.

Following [19], given a finely open nonempty bounded set D we consider $H_0^1(D) = \{u \in H^1(\mathbb{R}^n) : u = 0, \text{ a.e. } x \in \mathbb{R}^n \setminus D\}$ which is known to be dense in the space $L^2(D)$, which is identified with $L^2(D) = \{u \in L^2(\mathbb{R}^n) : u = 0, \text{ a.e. } x \in R^n \setminus D\}$, see [19, Proposition 2.1].

With this functional setting the standard bilinear form $a : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$, given by

$$a(u,v) = \int_{\mathbb{R}^n} \nabla u \nabla v$$

can be seen to be continuous and coercive and therefore generates an unbounded selfadjoint operator, denoted by $-\Delta_D$ (or $-\Delta$ when there is no ambiguity on the domain) given by

$$\begin{aligned} D(-\Delta_D) &:= \left\{u \in H_0^1(D) | \ \exists f \in L^2(D) \text{ such that } a(u,v) = \langle f, v\rangle_{L^2(D)} \ \ \forall v \in H_0^1(D)\right\}. \\ -\Delta u &:= f. \end{aligned}$$

Moreover, since $D \neq \emptyset$ we have, see [19], that the embedding $H_0^1(D) \hookrightarrow L^2(D)$ is compact, and therefore the standard theory of spectral decomposition of compact operators can be applied (to $(-\Delta_D)^{-1}$) and therefore the eigenvalues of $-\Delta_D$ are given by a sequence $0 < \lambda_1(D) \leq \lambda_2(D) \leq \ldots \to +\infty$.

This construction behaves well under perturbations of the domain. As a matter of fact, from [19, Theorem 2] we have that if D_i is a decreasing sequence of bounded smooth open sets so that

$$D = \left(\bigcap_{i=1}^{\infty} D_i\right)^{fi}$$

that is, D is the fine interior of the asymptotic limit of the (decreasing) sequence of D_i, then $\lambda_n(D_i) \to \lambda_n(D)$ for all $n \in \mathbb{N}$ as $i \to +\infty$. Moreover, also from [19, Theorem 2] the spectral projections and the resolvent operators converge in the uniform operator norm, that is

$$\sup_{f \in L^2(\mathbb{R}^n), \|f\| \leq 1} \|(-\Delta_{D_i})^{-1} f - (-\Delta_D)^{-1} f\|_{L^2(\mathbb{R}^n)} \to 0, \text{ as } i \to +\infty$$

The convergence of the resolvent operators, or equivalentely, the convergence of the eigenvalues and the spectral projections, implies the convergence of the linear semigroups generated by the operators Δ_D and Δ_{D_i} respectively. This means that for each $t > 0$ and each $u_0 \in L^2(\mathbb{R}^n)$, we have

$$\|e^{\Delta_{D_i} t} u_0 - e^{\Delta_D t} u_0\|_{L^2(\mathbb{R}^n)} \to 0, \text{ as } i \to +\infty \tag{2.2.3}$$

Remark 2.2.4. *If $\{D_i\}_{i \geq 1}$ is a decreasing family of open sets given, for instance $D_i = \{x \in \Omega : d(x,K) \leq \frac{1}{i}\}$, then $\cap_{i \geq 1} D_i = K$, $D = K^{fi}$ and $\lambda_1(D_i) \to \lambda_1(D)$ but from the definition of $\lambda_0(K)$ we also have that $\lambda_1(D_i) \to \lambda_0(K)$. Therefore, $\lambda_0(K) \equiv \lambda_1(D)$.*

2.3 Comparison Principle

In the previous section we have given a meaning to the equation $u_t = \Delta u$ with Dirichlet boundary conditions in a "fine open" set D. We want to apply this knowledge to obtain a comparison principle for our logistic degenerate equation.

Notice that if our set $K = \{x \in \Omega : n(x) = 0\}$ is the closure of a smooth open domain Ω_0, that is $K = \bar{\Omega}_0$, then standard comparison results show that if $v(t,x)$ is the solution of $P(v, \varphi_1^{\Omega_0}, 0, \Omega_0)$, that is

$$P(v, \varphi_1^{\Omega_0}, 0, \Omega_0) = \begin{cases} v_t - \Delta v = \lambda v, & x \in \Omega_0, t > 0, \\ v = 0, & x \in \partial\Omega_0 \\ v(x,0) = \varphi_1^{\Omega_0}(x), & x \in \Omega_0, \end{cases} \tag{2.3.1}$$

where $\varphi_1^{\Omega_0}$ is the first positive eigenfunction in Ω_0, we have

$$0 \leq v(t,x) = e^{(\lambda-\lambda_1)t}\varphi_1^{\Omega_0} \leq u(t, x, \varphi_1^{\Omega_0})$$

Therefore, if $\lambda > \lambda_1(\Omega_0) \equiv \lambda_0(K)$, we have $v(t,x) \to +\infty$ as $t \to +\infty$ for each $x \in \Omega_0$ and $u(t, x, \varphi_1^{\Omega_0})$ grow up to infinity in Ω_0 and therefore since the boundedness of solutions in independent of initial conditions (see [7]), $u(t, x, u_0) \to +\infty$ in Ω_0 for each $u_0 \geq 0$ but not identically 0.

In case K is not the closure of a smooth set (even may have empty interior), we cannot apply the same conclusion. But in this case, we can show:

Proposition 2.3.1. *If the fine interior of K is non empty, that is $K^{fi} \neq \emptyset$, and if $\lambda > \lambda_1(K^{fi}) \equiv \lambda_0(K)$, then for each initial data $u_0 \geq 0$, we have*

$$0 \leq v(t, x, u_0) = e^{\lambda t} e^{\Delta_{K^{fi}} t} u_0 \leq u(t, x, u_0), \ \ a.e. x \in \Omega$$

Proof. First, let Ω_j be a stricitly nested family of bounded open sets as in (2.1.1). Let us define the family of problems $P(u_j, u_0, n_j, \Omega)$ and $P(\underline{u}_j, u_0, 0, \Omega_j)$ for $j \geq 1$

$$P(u_j, u_0, n_j, \Omega) = \begin{cases} (u_j)_t - \Delta u_j = \lambda u_j - n_j(x) u^\rho, & x \in \Omega, t > 0, \\ u_j = 0, & x \in \partial\Omega, t > 0 \\ u_j = u_0, & x \in \Omega, t = 0 \end{cases} \tag{2.3.2}$$

and

$$P(\underline{u}_j, u_0, 0, \Omega_j) = \begin{cases} (\underline{u}_j)_t - \Delta \underline{u}_j = \lambda \underline{u}_j, & x \in \Omega_j, t > 0, \\ \underline{u}_j = 0, & x \in \partial\Omega_j, t > 0 \\ \underline{u}_j = u_0|_{\Omega_j}(x), & x \in \Omega_j, t = 0 \end{cases} \tag{2.3.3}$$

where

$$n_j(x) = \begin{cases} n(x), & x \in \Omega \setminus \Omega_j, \\ 0, & x \in \Omega_j. \end{cases} \tag{2.3.4}$$

Observe that $P(\underline{u}_j, u_0, n_j, \Omega_j)$ is the restriction of $P(u_j, u_0, n_j, \Omega)$ to Ω_j and therefore, from comparison result Theorem 1.2.1 we obtain $\underline{u}_j \leq u_j$.

Moreover, it is obvious that $\underline{u}_j = e^{\lambda t} e^{-\Delta_{\Omega_j} t} u_0$, which we know from the previous section that it converges (at least a.e.) to $v = e^{\lambda t} e^{-\Delta_{K^{fi}} t} u_0$.

We show below in Lemma 2.3.2 that $u_j \to u$. Hence, we prove the result accordingly to the following diagram:

$$\begin{array}{ccc} \underline{u}_j & \leq & u_j \\ j \to \infty \ \downarrow & & \downarrow \ j \to \infty \\ v & \leq & u \end{array}$$

This concludes the proof of the proposition.

□

Let us show now that $u_j \to u$.

Lemma 2.3.2. *Let u_j and u be the solutions of $P(u_j, u_0, n_j, \Omega)$ and $P(u_j, u_0, n, \Omega)$ respectively. Then, $u_j(t,x) \to u(t,x)$ pointwise on t uniformly in x.*

Proof. We will use Proposition 1.4.1. Recall that $n(x) = 0$ for all $x \in K$, then for any $j \geq 1$ we have

$$\begin{aligned}\|n(\cdot) - n_j(\cdot)\|_{L^1(\Omega)} &= \int_\Omega |n(x) - n_j(t,x)|ds = \int_\Omega |n(x) - n(x)\chi_{\Omega\setminus\Omega_j}|ds \\ &= \int_{\Omega\setminus K} |n(x) - n(x)\chi_{\Omega\setminus\Omega_j}|ds + \int_K |n(x) - n(x)\chi_{\Omega\setminus\Omega_j}|ds \\ &= \int_{\Omega\setminus K} |n(x) - n(x)\chi_{\Omega\setminus\Omega_j}|ds \leq \|n\|_{L^\infty}|\Omega_j \setminus K|.\end{aligned}$$

Therefore, by Proposition 1.4.1 there exists $C = C(T - t_0, \lambda, \|n\|_{L^\infty}, \|u_0\|_{L^\infty}, \rho, \Omega, M)$ such that

$$\|u(t,\cdot) - u_j(t,\cdot)\|_{L^\infty(\Omega)} \leq C\|n - \tilde{n}\|^{1/p}_{L^1([t_0,T]\times\Omega)} \leq C\|n\|_{L^\infty}|\Omega_j \setminus K|,\ t \in [t_0, T],$$

Thus, for every $T > t_0$ there exists a $j_0 \geq 1$ such that $|\Omega_j \setminus K| < \varepsilon$ and

$$\|u(T,\cdot) - u_j(T,\cdot)\|_{L^\infty(\Omega)} \leq C\|n\|_{L^\infty}\varepsilon, \quad \forall j \geq j_0.$$

This concludes the proof of the lemma.

□

With this comparison result, now it is not difficult to prove:

Proposition 2.3.3. *If $K^{fi} \neq \emptyset$ and we denote by (λ_k, φ_k), $k = 1, 2, \ldots$ the eigenvalues and eigenfunctions of $-\Delta_{K^{fi}}$ and we define the set*

$$K^{fi}_\lambda = \{x \in K^{fi} : \exists k \in N \text{ with } \lambda > \lambda_k \text{ and } \varphi_k(x) > 0\}$$

then u grows up to infinity a.e. in K^*_λ.

Proof. Just use the comparison result from Proposition 2.3.1 and use that $v(t,x,\varphi_k) = e^{(\lambda-\lambda_k)t}\varphi_k(x) \to \infty$ since $\lambda > \lambda_k$ and $\varphi_k(x) > 0$.

□

2.4 A Pathological set

In the spirit of an example given at [7] we provide a construction in $\mathbb{R}^N$, with $N \geq 3$ of two sets U_∞ and K_∞ with the property that $U_\infty \subset K_\infty$, K_∞ is a compact with empty interior, positive measure and fat. Moreover U_∞ is "quasi-open". First, let us introduce the following result from [32]

Lemma 2.4.1. *For any bounded open set U and $q \in U$, we have that*

$$\lambda_1(U \setminus \overline{B}(q,\varepsilon)) \to \lambda_1(U), \quad \text{as } \varepsilon \to 0. \tag{2.4.1}$$

We construct now the sets U_∞, K_∞.

Let $U_0 \subset \mathbb{R}^N$ be an open bounded smooth domain. Let us fix a small positive number $\eta > 0$ and let us consider $\mathcal{Q} = U_0 \cap \mathbb{Q}^N$ where $\mathbb{Q} \subset \mathbb{R}$ is the set of rational numbers. Since $\mathcal{Q}$ is numerable, we can see it as $\mathcal{Q} = \{q_i\}_{i=1}^{\infty}$.

Let $p_1 = q_1$, then due to (2.4.1) there exists $\varepsilon_1 > 0$ such that

- $B(p_1, 2\varepsilon_1) \subset \Omega_0$,
- $|B(p_1, \varepsilon_1)| \leq \eta/2$,
- $0 < \varepsilon_1 \leq 2^{-1}$,
- if $U_1 = U_0 \setminus \overline{B}(p_1, \varepsilon_1)$, then

$$\lambda_1(U_1) \leq \lambda_1(U_0) + \frac{1}{2}. \tag{2.4.2}$$

Thus, we end up with U_1 smooth such that $\lambda_1(U_0) < \lambda_1(U_1) \leq \lambda_1(U_0) + \frac{1}{2}$.

Now, if we choose $p_2 = q_{i_2}$ where q_{i_2} is the first point in $\mathcal{Q}$ after q_1 lying in U_1. That is, $q_1, q_2, \ldots, q_{i_2-1}$ are all in $B(p_1, \varepsilon_1)$ and q_{i_2} is the first element in $\mathcal{Q}$ starting from q_1 lying outside $B(p_1, \varepsilon_1)$. Again, due to (2.4.1) there exists $\varepsilon_2 > 0$ such that $B(p_2, \varepsilon_2) \subset U_1$, $|B(p_2, \varepsilon_2)| \leq \eta/2^2$, $0 < \varepsilon_2 \leq 2^{-2}$ and if $U_2 = U_1 \setminus \overline{B}(p_2, \varepsilon_2)$, then

$$\lambda_1(U_2) \leq \lambda_1(U_1) + \frac{1}{2^2}. \tag{2.4.3}$$

Thus, we end up with U_2 smooth such that $\lambda_1(U_1) < \lambda_1(U_2) \leq \lambda_1(U_1) + \frac{1}{2^2}$.

By induction we end up with a sequence of points $\{p_i\}_i \subset \mathcal{Q}$ and a sequence of numbers $\{\varepsilon_i\}_i$ such that $B(p_i, 2\varepsilon_i) \subset U_{i-1}$, $|B(p_i, \varepsilon_i)| \leq \eta/2^i$, $0 < \varepsilon_i \leq 2^{-i^2}$ and if $U_i = U_{i-1} \setminus \overline{B}(p_i, \varepsilon_i)$, then

$$\lambda_1(U_i) \leq \lambda_1(U_{i-1}) + \frac{1}{2^i} \leq \lambda_1(U_0) + \sum_{j=1}^{i} \frac{1}{2^j} \leq \lambda_1(U_0) + 1. \tag{2.4.4}$$

With U_i smooth such that $\lambda_1(U_{i-1}) \leq \lambda_1(U_i)$. Notice that all balls $B(p_i, \varepsilon_i)$ are pairwise disjoint and

$$\forall \varepsilon > 0, \forall m \in \mathbb{N}, \forall k \geq m \text{ we have that } B(p_m, \varepsilon) \cap \Omega_k^c \neq \emptyset. \tag{2.4.5}$$

We define

$$U_\infty := \bigcap_{i=0}^{\infty} U_i = U_0 \setminus \bigcup_{i=1}^{\infty} \overline{B}(p_i, \varepsilon_i), \qquad K_\infty := \bigcap_{i=0}^{\infty} \overline{U}_i = \overline{U}_0 \setminus \bigcup_{i=1}^{\infty} B(p_i, \varepsilon_i). \tag{2.4.6}$$

We can show,

Proposition 2.4.2. *We have the following properties of K_∞ and U_∞*

i) K_∞ is a nonempty compact set, with empty interior and fat.

ii) U_∞ is a nonempty quasi-open set with $U_\infty \subset K_\infty$.

iii) Although the interior of K_∞ is empty, its fine interior K_∞^{fi} (i.e. the interior in the fine topology) is non empty.

Proof. i) It is nonempty since the Lebesgue measure of

$$\left| \bigcup_{i=1}^{\infty} B(p_i, \varepsilon_i) \right| \leq \sum_{i=1}^{\infty} \eta/2^i \leq \eta \tag{2.4.7}$$

and thus $|K_\infty| \geq |U_0| - \eta > 0$ for η small enough. The interior of K_∞ is empty. In other case there would exist a point $y \in K_\infty$ and a number $R > 0$ such that $B(y,R) \subset K_\infty$ and, therefore, $B(y,R) \cap \bigcup_{i=1}^{\infty} B(p_i, \varepsilon_i) = \emptyset$. But since rationals are dense in $\mathbb{R}$ there must exist a $q_k \in B(y,R)$ and a $R' > 0$ such that $B(q_k, R') \subset B(y,R) \subset K$ and therefore $B(q_k, R') \cap \bigcup_{i=1}^{\infty} B(q_i, \varepsilon_i) = \emptyset$ which is a contradiction since $q_k \in \mathcal{Q}$.

To check that K_∞ is fat, we observe that if for $\delta > 0$ small enough we define $\Omega_\delta = \{x \in \mathbb{R}^n : d(x, K_\infty) < \delta\}$, we have that there exists $n_0 = n_0(\delta)$ such that $\Omega_\delta \supset U_n$ for each $n \geq n_0$. This is so because for fixed δ, if we choose n_0 such that $\frac{1}{2^{n_0}} < \delta$ by construction of the set K_∞, we have $\Omega_\delta \supset U_n$ for each $n \geq n_0$. This implies that $\lambda_1(\Omega_\delta) \leq \lambda_1(U_{n_0}) \leq \lambda_1(U_0) + 1$. This implies that $\lambda_0(K_\infty) \leq \lambda_1(U_0) + 1$

ii) That U_∞ is nonempty and $U_\infty \subset K_\infty$ is clear from construction and from the proof that K_∞ is nonempty.

To show that U_∞ is quasi-open we proceed as follows. Let $\varepsilon > 0$ arbitrary. Following the notation above, we have $U_\infty \subset \Omega_n$ for each $n \in \mathbb{N}$. Moreover

$$U_\infty \triangle \Omega_n = \Omega_n \setminus U_\infty = \bigcup_{i=n+1}^{\infty} \overline{B(p_i, \varepsilon_i)}. \tag{2.4.8}$$

Because the 2–capacity is an outer measure (see Theorem 3.2 from [24]]) and using Theorem 4.6 from [24] we have that for μ the Lebesgue measure and c depending only the doubling constant of the Lebesgue measure

$$\begin{aligned} C_2(U_\infty \triangle \Omega_n) &\leq \sum_{i=n+1}^{\infty} C_2(B(p_2, \varepsilon_i)) \leq c \sum_{i=n+1}^{\infty} \varepsilon_i^{-2} \mu(B(p_i, \varepsilon_i)) = cM \sum_{i=n+1}^{\infty} \varepsilon_i^{-2} \varepsilon_i^N, \\ &= cM \sum_{i=n+1}^{\infty} \varepsilon_i^{N-2}, \end{aligned} \tag{2.4.9}$$

which is convergent for $N > 2$ and $\varepsilon_i \leq 2^{-i}$ for all $i \in \mathbb{N}$ by construction. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$C_2(U_\infty \triangle \Omega_n) < \varepsilon, \quad \text{for all } n \geq n_0. \tag{2.4.10}$$

iii) From [23, Theorem 1.5], we have that the since U_∞ is quasi open, then U_∞ is the union of a fine open set and a set of 0 capacity. But since $C_2(U_\infty) > 0$, we have that its fine interior, $U_\infty^{fi} \neq \emptyset$. But $\emptyset \neq U_\infty^{fi} \subset K_\infty^{fi}$. This concludes the proof of the lemma.

□

Remark 2.4.3. *Although we do not have a proof, we conjecture that U_∞ is finely open and $K_\infty^{fi} = U_\infty$.*

Chapter 3

Non–autonomous degenerate logistic–type parabolic equations

3.1 Introduction

In this chapter, we study the asymptotic behaviour of the solutions of the problem (1.0.1) where the nonnegative function $n \in L^\infty(\mathbb{R} \times \overline{\Omega})$ nullifies in a time–dependent subset of $\overline{\Omega}$

$$K(t) := \left\{ x \in \overline{\Omega} : n(t,x) = 0 \right\}, \quad t > t_0. \tag{3.1.1}$$

Therefore, within $K(t)$, equation (1.0.1) becomes linear. We assume, as in the autonomous case, $K(t)$ is compact for every $t > t_0$. Moreover, throughout this chapter we will often assume the following:

Assumption (N). *There exists a strictly increasing continuous function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\nu(0) = 0$, $\nu(s) > 0$ for $s > 0$ and*

$$n(t,x) \geq \nu(d(x, K(t))), \quad \text{for all } x \in \overline{\Omega}, t > t_0.$$

Notice that $K(t)$ may be empty for some value of t. For the values of t when $K(t)$ is empty, we will assume that there exists $\nu_0 > 0$ such that $n(t,x) \geq \nu_0$ for all $x \in \overline{\Omega}$.

Assumption **(N)** ensures that $\lim_{t\to\infty} K(t)$ relates to $\lim_{t\to\infty} n(t,x)$ as illustrated later on in Remark 3.3.3. Hence, assumption **(N)** allows us to focus on $K(t)$ instead of $n(t,x)$.

In this chapter, we generalise the results in [7] for n non–autonomous and provide for the asymptotic behaviour of solutions of (1.0.1)

- Initial data independence,
- Comparison techniques with the autonomous case in [7] and results for the limit behaviour of $K(t)$,
- A local boundedness result,
- Several boundedness results applicable to different behaviours of $K(t)$,
- An unboundedness result,
- Examples.

Finally, we introduce an example we will use to illustrate the main results. Let us consider u the solution of problem (1.0.1) where n satisfies **(N)**. Let $K(t)$ be a regular set moving continuously as illustrated in the following Figure 3.1.

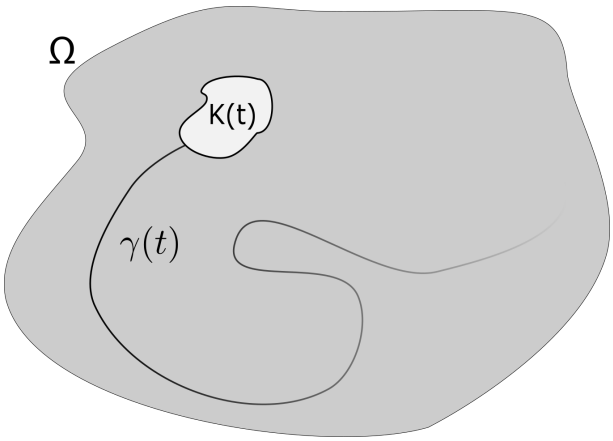


Figure 3.1: Domain Ω and a continuously moving $K(t)$. The compact sets $K(t)$ moves along a continuous curve $\gamma(t)$.

This moving $K(t)$ is able to display a wide range of asymptotic behaviours which can be summarised as:

- If $\lambda < \lambda_0(K(t))$ for all $t \geq t_0$, then u is bounded.
- If $\lambda \geq \lambda_0(K(t))$ for every $t \geq t_0$ and
  - $K(t)$ is static then, the solution u is unbounded.
  - $K(t)$ moves continuously and slowly in some sense then, u is unbounded.
  - $K(t)$ moves continuously when moving slowly in some sense, and jumps at certains times then, u is bounded.
  - $K(t)$ moves continuously and quickly in some sense then, u is bounded.

Although several additional examples will be presented throughout this chapter, the continuously moving $K(t)$ does illustrate most of the behaviours contemplated in this research.

3.2 Initial Data Independence

Throughout this chapter, we are going to consider several modifications of problem (1.0.1) by changing its initial condition, function n or even the domain Ω. Therefore, let us recall the notation introduced in Section 1.4 where we refer to (1.0.1) as $P(u, u_0, n, \Omega)$. Moreover, whenever it is relevant, we will also specify the initial time referring to problem (1.0.1) as $P(u, t_0, u_0, n, \Omega)$ and to its solution as $u(t, x; t_0, u_0)$. Furthermore, throughout this chapter we will choose suitable initial data for the proofs: initial data as large as needed as in Section 3.4 or specific functions such as in Section 3.5. As proven below (following [7]), boundedness and unboundedness are independent of initial data. Furthermore, the initial data u_0 can be considered in $L^\infty(\Omega)$.

Lemma 3.2.1. *Let u be the solution of (1.0.1) and let $\lambda \in \mathbb{R}$, $u_0 \geq 0$ non–zero and $\alpha > 0$. Then:*

- *If* $\alpha > 1$
$$\alpha u(t,x;u_0) \geq u(t,x;\alpha u_0).$$
- *If* $\alpha < 1$
$$\alpha u(t,x;u_0) \leq u(t,x;\alpha u_0).$$

Proof. We study $v(t,x) = \alpha u(t,x;u_0)$ which verifies the problem

$$\begin{cases} v_t - \Delta v = \lambda v - \frac{n(t,x)}{\alpha^{\rho-1}} v^\rho, & x \in \Omega, t > 0 \\ v(t,x) = 0, & x \in \partial\Omega, t > 0, \\ v(0,x) = \alpha u_0, & x \in \Omega, \end{cases} \tag{3.2.1}$$

so, by Theorem 1.2.1, since $\frac{n(t,x)}{\alpha^{\rho-1}} \leq n(t,x)$ if $\alpha > 1$ and $\frac{n(t,x)}{\alpha^{\rho-1}} \geq n(t,x)$ if $\alpha < 1$,

- if $\alpha > 1$, then v is a supersolution of $u(t,x;\alpha u_0)$, so $\alpha u(t,x;u_0) \geq u(t,x;\alpha u_0)$, and
- if $\alpha < 1$, then v is a subsolution of $u(t,x;\alpha u_0)$, so $\alpha u(t,x;u_0) \leq u(t,x;\alpha u_0)$.

Which proves the result.

□

Now, using the previous Lemma 3.2.1 it can be seen that:

Proposition 3.2.2. *The solution of (1.0.1) is bounded or unbounded independently of initial data and initial time.*

Proof. Let u_0, v_0 be two non–trivial, non–negative initial data, let $\tau_0 \geq t_0$ and let us denote by $u(t,\cdot;t_0,u_0)$ be the solution of (1.0.1) starting at t_0 with initial condition u_0.

First, we want to show for $t > \tau_0$ that $u(t,\cdot;\tau_0,u_0)$ is bounded iff $u(t,\cdot;\tau_0,v_0)$ is bounded. We let the process evolve a time $\delta > 0$ and by Proposition 1.1.7, $u(\tau_0+\delta,\cdot;\tau_0,u_0), u(\tau_0+\delta,\cdot;\tau_0,v_0) \in C^1(\bar{\Omega})$, moreover, these solutions nullify in $\partial\Omega$, they also satisfy $\frac{\partial u(\tau_0+\delta,\cdot;\tau_0,u_0)}{\partial \vec{n}}, \frac{\partial u(\tau_0+\delta,\cdot;\tau_0,v_0)}{\partial \vec{n}} < 0$ by Proposition 1.3.1 and both are strictly positive inside Ω. Therefore there exist $\alpha < 1 < \beta$ such that

$$\alpha u(\tau_0+\delta,\cdot;\tau_0,u_0) \leq u(\tau_0+\delta,\cdot;\tau_0,v_0) \leq \beta u(\tau_0+\delta,\cdot;\tau_0,u_0).$$

Hence, taking into account that $u(t,\cdot;\tau_0,v_0) = u(t,\cdot;\tau_0+\delta,u(\tau_0+\delta,\cdot;\tau_0,v_0))$, for $t > \tau_0+\delta$ by Theorem 1.2.1,

$$u(t,\cdot;\tau_0+\delta,\alpha u(\tau_0+\delta,\cdot;\tau_0,u_0)) \leq u(t,\cdot;\tau_0,v_0) \leq u(t,\cdot;\tau_0+\delta,\beta u(\tau_0+\delta,\cdot;\tau_0,u_0)).$$

But from Lemma 3.2.1 for $\alpha < 1 < \beta$ and $t > t_0+\delta$

$$\alpha u(t,\cdot;\tau_0,u_0) = \alpha u(t,\cdot;\tau_0+\delta,u(\tau_0+\delta,\cdot;\tau_0,u_0)) \leq u(t,\cdot;\tau_0+\delta,\alpha u(\tau_0+\delta,\cdot;\tau_0,u_0)),$$

$$u(t,\cdot;\tau_0+\delta,\beta u(\tau_0+\delta,\cdot;\tau_0,u_0)) \leq \beta u(t,\cdot;\tau_0+\delta,u(\tau_0+\delta,\cdot;\tau_0,u_0)) = \beta u(t,\cdot;\tau_0,u_0).$$

Hence, we arrive at

$$\alpha u(t,\cdot;\tau_0,u_0) \leq u(t,\cdot;\tau_0,v_0) \leq \beta u(t,\cdot;\tau_0,u_0), \quad t > \tau_0+\delta.$$

Thus, one solution is bounded if and only if the other is bounded.

Now let us take two different initial times $\tau_1, \tau_2 \geq t_0$ and two initial data u_0, v_0 as before. Without any loss of generality, we may assume $\tau_1 < \tau_2$. Let the process evolve from τ_1 to τ_2 with initial data u_0 and denote by $\tilde{u}_0 = u(\tau_2, \cdot; \tau_1, u_0)$. From the proof above, we know that

$$u(t, \cdot; \tau_2, \tilde{u}_0) = u(t, \cdot; \tau_1, u_0) \quad \text{is bounded for } t \geq \tau_2 \quad \Leftrightarrow \quad u(t, \cdot; \tau_2, v_0) \text{ is bounded.}$$

□

3.3 General results and setting

As seen for the autonomous case in [7], that is, $n(t,x) = n(x)$, $\forall t \geq t_0$ and therefore $K(t) = K$ is time independent, the key factor that decides whether the solutions are bounded or unbounded is the relation between λ and $\lambda_0(K)$: if $\lambda < \lambda_0(K)$ solutions remain bounded whereas if $\lambda \geq \lambda_0(K)$ solutions become unbounded as $t \to \infty$. In the latter case, this type of behaviour exhibited by a solution is often referred to as *grow up*.

In the nonautonomous case the situation is not so simple and, as we will see in this chapter, the relation between λ and $\lambda_0(K(t))$ will not be enough to decide the boundedness or unboundedness of the solutions. Note that $\lambda_0(K(t))$ is time dependent and we may not have $\lambda < \lambda_0(K(t))$ for all t or $\lambda > \lambda_0(K(t))$ for all t. Moreover, as we will see below, even in the case $\lambda > \lambda_0(K(t))$ for all t we may not have grow–up and if $\lambda < \lambda_0(K(t))$ may not have boundedness. Nevertheless, in several instances, we will be able to deduce the boundedness or unboundedness of the solutions of the non–autonomous case via comparison techniques with certain autonomous reference problems.

Let us see now an example of a comparison with autonomous reference problems.

Example 3.3.1. *Let $t_0 > 0$ and let $B(x_0, r) \subset \Omega$ be the ball of radius $r > 0$ centered at $x_0 \in \Omega$. Define*

$$n_1(t,x) := \begin{cases} 1, & t > t_0,\ x \in \overline{\Omega} \setminus \overline{B(x_0, rt_0/t)}, \\ \\ 0, & t > t_0,\ x \in \overline{B(x_0, rt_0/t)}. \end{cases}$$

In this case, $K(t) = \{x \in \Omega : n_1(t,x) = 0\}$ is the closed shrinking ball with center at x_0 and radius rt_0/t. From Proposition 2.1.4 we know that $\lambda_0(K(t)) = \lambda_1(B(x_0, rt_0/t))$, the first eigenvalue of the Laplace operator with Dirichlet boundary conditions in $B(x_0, rt_0/t)$. But we know that $\lambda_1(B(x_0, R)) = \lambda_1(B(x_0, 1))/R^2$. Therefore

$$\lim_{t\to\infty} \lambda_0(K(t)) = \lim_{t\to\infty} \lambda_1(B(x_0, rt_0/t)) = +\infty.$$

Thus, with λ fixed, we can find $\tau_0 > t_0$ such that $\lambda_0(K(t)) > \lambda$ for any $t \geq \tau_0$. Let us consider problems $P(u_1, t_0, u_0, n_1, \Omega)$ and $P(u_2, \tau_0, u_1(\tau_0), n_2, \Omega)$ where

$$n_2(t,x) = n_2(x) := \begin{cases} 1, & x \in \overline{\Omega} \setminus \overline{B(x_0, rt_0/\tau_0)}, \\ 0, & x \in \overline{B(x_0, rt_0/\tau_0)}. \end{cases}$$

Then, with the comparison techniques from Theorem 1.2.1, the function u_2 is a supersolution of u_1 for $t > \tau_0$ and u_2 is bounded since $\lambda < \lambda_0(B(rt_0/\tau_0))$. Therefore, u_1 is also bounded due to Proposition 3.2.2.

Hence, when trying to decide whether the solutions are bounded or not for the non–autonomous equation (1.0.1), it is natural to start by comparing λ and $\lambda_0(K(t))$. Notice that now the set K is time–dependent and therefore, the relation $\lambda < \lambda_0(K(t))$ or $\lambda \geq \lambda_0(K(t))$ may not hold true for every $t > t_0$.

In the previous example, function n_1 was increasing over time as the number of points where the function nullified decreased. That is, the greater the function n_1 grew, the smaller $K(t)$ became. The opposite relation between n and $K(t)$ holds true as well. Thus, greater compact sets $K(t)$ are associated to smaller functions n (specifically to functions nullified in a greater subset of $\overline{\Omega}$) and vice versa. Moreover, notice that in the previous example the time dependent function n_1 has been compared to a time independent function n_2 for $t \geq \tau_0$. Hence, our first approach will be to define time independent functions which can be compared to $n(t,x)$ in (1.0.1) and that can lead us to boundedness or unboundedness results as in the previous example. Also, the behaviour of the function $n(t,x)$ for large t and the "size" of the sets $K(t)$ as $t \to \infty$ will also be important to analize the behaviour of the solutions of our equations.

Therefore, let us define the following functions and sets which will be important in the analysis below:

$$\begin{aligned} \underline{n}_{\tau_0}(x) &:= \inf_{t\geq\tau_0} n(t,x), \quad \text{for } x \in \overline{\Omega}, \qquad K^{\sup}_{\tau_0} := \overline{\{x \in \overline{\Omega} : \underline{n}_{\tau_0}(x) = 0\}}, \quad \tau_0 \geq t_0 \\ \overline{n}_{\tau_0}(x) &:= \sup_{t\geq\tau_0} n(t,x), \quad \text{for } x \in \overline{\Omega}, \qquad K^{\inf}_{\tau_0} := \{x \in \overline{\Omega} : \overline{n}_{\tau_0}(x) = 0\}, \quad \tau_0 \geq t_0 \end{aligned} \tag{3.3.1}$$

Lemma 3.3.2. *With the notations above,*

$$\overline{\bigcup_{t\geq\tau_0} K(t)} \subset K^{\sup}_{\tau_0}, \qquad \bigcap_{t\geq\tau_0} K(t) = K^{\inf}_{\tau_0}.$$

Moreover, if **(N)** *holds, then* $\overline{\bigcup_{t\geq\tau_0} K(t)} = K^{\sup}_{\tau_0}$.

Proof. Indeed, let us see all inclusions.

Let $x \in \bigcup_{t\geq\tau_0} K(t)$, then $x \in K(t)$ for some $t \geq \tau_0$, which implies $n(t,x) = 0$ and therefore $\underline{n}_{\tau_0}(x) = 0$. Hence,

$$\bigcup_{t\geq\tau_0} K(t) \subset \{x \in \Omega : \underline{n}_{\tau_0}(x) = 0\} \Rightarrow \overline{\bigcup_{t\geq\tau_0} K(t)} \subset \overline{\{x \in \Omega : \underline{n}_{\tau_0}(x) = 0\}} = K^{\sup}_{\tau_0}.$$

Now, let us take $x \in K^{\inf}_{\tau_0}$, then $\sup_{t\geq\tau_0} n(t,x) = 0$. Therefore

$$0 \leq n(t,x) \leq \sup_{t\geq\tau_0} n(t,x) = 0, \ \ \forall t \geq \tau_0 \ \ \Rightarrow n(t,x) = 0, \ \ \forall t \geq \tau_0.$$

Hence, $x \in K(t)$ for every $t \geq \tau_0$ and thus, $x \in \bigcap_{t\geq\tau_0} K(t)$ and therefore, $K^{\inf}_{\tau_0} \subset \bigcap_{t\geq\tau_0} K(t)$.

Now, let $x \in \bigcap_{t\geq\tau_0} K(t)$, then $x \in K(t)$ for every $t \geq \tau_0$. Therefore, $\overline{n}_{\tau_0}(x) = 0$ which implies that $x \in K^{\inf}_{\tau_0}$ and, thus, $\bigcap_{t\geq\tau_0} K(t) \subset K^{\inf}_{\tau_0}$.

Finally, let us assume **(N)** holds and $x \in \{x \in \Omega : \underline{n}_{\tau_0}(x) = 0\}$ for some $\tau_0 \geq t_0$. Then, $\underline{n}_{\tau_0}(x) = \inf_{t \geq \tau_0} n(t,x) = 0$. Therefore, either

- $\exists t \geq \tau_0$ with $n(t,x) = 0$. This implies $x \in K(t)$ and therefore $x \in \bigcup_{t \geq \tau_0} K(t)$ or,
- $n(t,x) > 0$ for every $t \geq \tau_0$ and there exists a sequence $(t_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} n(t_i, x) = 0$, which implies that

$$0 \leq \lim_{i \to \infty} \nu(d(x, K(t_i))) \leq \lim_{i \to \infty} n(t_i, x) = 0.$$

Hence, $\lim_{i \to \infty} d(x, K(t_i)) = 0$ and, therefore, $x \in \overline{\bigcup_{t \geq \tau_0} K(t)}$.

Thus, $K^{\sup}_{\tau_0} \subset \overline{\bigcup_{t \geq \tau_0} K(t)}$. □

Remark 3.3.3. *Assumption* **(N)** *is relevant for Lemma 3.3.2 and also for this chapter. Otherwise we can construct the function*

$$n(t,x) = \begin{cases} 0, & t \geq t_0, x \in K_0, \\ \frac{1}{t - t_0 + 1}, & t \geq t_0,\ x \in \overline{\Omega} \setminus K_0, \end{cases}$$

where $K(t) \equiv K_0$ compact but $\underline{n}_{\tau_0}(x) = 0$ for all $\tau_0 \geq 0$ and all $x \in \overline{\Omega}$. Therefore $K^{\sup}_{\tau_0} = \bar{\Omega}$ but $\overline{\bigcup_{t \geq \tau_0} K(t)} = K_0 \subsetneq \bar{\Omega}$, and therefore the last inequality in Lemma 3.3.2 does not hold true.

From the definition it is clear that the family of sets $K^{\sup}_{\tau_0}$ is decreasing in τ_0, that is $K^{\sup}_{\tau_0} \subset K^{\sup}_{\tau_0'}$ if $\tau_0 \geq \tau_0'$ and the family of sets $K^{\inf}_{\tau_0}$ is increasing in τ_0, that is $K^{\inf}_{\tau_0} \supset K^{\inf}_{\tau_0'}$ if $\tau_0 \geq \tau_0'$. Moreover, it is also clear that both sets are compact and thus, $\lambda_0(K^{\sup}_{\tau_0}), \lambda_0(K^{\inf}_{\tau_0})$ are well defined.

We can show now

Lemma 3.3.4 (Comparison with the Autonomous Case)**.** *Let us consider problem (1.0.1). We have,*

- *If $\lambda < \lambda_0(K^{\sup}_{\tau_0})$ for some $\tau_0 \geq t_0$, then all solutions of (1.0.1) are bounded.*
- *If $\lambda \geq \lambda_0(K^{\inf}_{\tau_0})$ for some $\tau_0 \geq t_0$, then all solutions of (1.0.1) are unbounded.*

Proof. Let $\tau_0 \geq t_0$ and let $\underline{v}$ and $\overline{v}$ be the respective solutions of the autonomous problems $P(\underline{v}, u_0, \underline{n}^*_{\tau_0}, \Omega)$ and $P(\overline{v}, u_0, \overline{n}_{\tau_0}, \Omega)$ where

$$\underline{n}^*_{\tau_0} := \begin{cases} 0, & x \in K^{\sup}_{\tau_0}, \\ \underline{n}_{\tau_0}(x), & x \in \overline{\Omega} \setminus K^{\sup}_{\tau_0}. \end{cases}$$

Then, since

$$\overline{n}_{\tau_0}(x) \geq n(t,x) \geq \underline{n}_{\tau_0}(x) \geq \underline{n}^*_{\tau_0}(x),\ t \geq \tau_0,\ x \in \overline{\Omega},$$

by Theorem 1.2.1 we have that

$$\overline{v}(t,x) \leq u(t,x) \leq \underline{v}(t,x),\ t \geq \tau_0,\ \text{a.e. } x \in \Omega.$$

Now,

- If $\lambda < \lambda_0(K^{\sup}_{\tau_0})$ then $\underline{v}$ is bounded due to [7]. Thus u is a bounded solution.
- If $\lambda \geq \lambda_0(K^{\inf}_{\tau_0})$, then $\overline{v}$ is unbounded due to [7]. Thus u is an unbounded solution.

□

In order to capture the limit behaviour of $K(t)$, we introduce the following definition.

Definition 3.3.5. *Let $K(t)$ be a family of compact subsets of $\overline{\Omega}$ satisfying* **(N)** *and let $K^{\sup}_{\tau_0}, K^{\inf}_{\tau_0}$ be as in* (3.3.1)*. We define,*

$$K^{\sup} := \bigcap_{\tau_0 \geq t_0} K^{\sup}_{\tau_0} = \bigcap_{\tau_0 \geq t_0} \overline{\bigcup_{t \geq \tau_0} K(t)},$$

and

$$K^{\inf} := \overline{\bigcup_{\tau_0 \geq t_0} K^{\inf}_{\tau_0}} = \overline{\bigcup_{\tau_0 \geq t_0} \bigcap_{t \geq \tau_0} K(t)}.$$

It is clear that $K^{\sup}, K^{\inf}$ are compact sets and that $K^{\inf} \subset K^{\sup}$. Moreover, if the family of compact sets $K(t)$ is such that

$$K^{\sup} = \bigcap_{\tau_0 \geq t_0} \bigcup_{t \geq \tau_0} K(t), \qquad K^{\inf} = \bigcup_{\tau_0 \geq t_0} \bigcap_{t \geq \tau_0} K(t),$$

we can say that $K^{\sup}$ is the upper limit of the sets $K(t)$ and $K^{\inf}$ is the lower limit of the sets $K(t)$ when t goes to infinity.

As shown in the following example, it cannot be assumed that the characteristic value $K^{\sup}$ or $K^{\inf}$ coincides with the limit of the characteristic values of the family of sets $K(t)$.

Example 3.3.6. *Let us consider problem (1.0.1), $\Omega = (-2,2) \subset \mathbb{R}$ and $n(t,x) = \chi_{\Omega \setminus K(t)}$ where*

$$K(t) := \left[-1, -\frac{1}{t - t_0 + 1}\right] \bigcup \left[\frac{1}{t - t_0 + 1}, 1\right],$$

therefore

$$\lambda_0(K(t)) = \min\left\{\lambda_0\left(\left[-1, -\frac{1}{t - t_0 + 1}\right]\right), \lambda_0\left(\left[\frac{1}{t - t_0 + 1}, 1\right]\right)\right\}.$$

Hence,

$$\lim_{t \to \infty} \lambda_0(K(t)) = \lambda_0([-1,0]) = \lambda_0([0,1]),$$

but

$$\lambda_0\left(\lim_{t \to \infty} K(t)\right) = \lambda_0\left([-1,1]\right) < \lambda_0([-1,0]) = \lim_{t \to \infty} \lambda_0(K(t)).$$

Therefore, in order to obtain the most general result for the asymptotic behaviour of solutions of (1.0.1), we assume the following:

Assumption (K). *Let $K(t)$ be a family of compact sets. Then, we will assume that*

$$\lambda_0(K^{\sup}) = \lim_{\tau_0\to\infty} \lambda_0(K^{\sup}_{\tau_0}), \qquad \lambda_0(K^{\inf}) = \lim_{\tau_0\to\infty} \lambda_0(K^{\inf}_{\tau_0}).$$

Remark 3.3.7. *Whenever the family of compact sets $K(t)$ satisfies*

$$K^{\sup} = K^{\sup}_{\tau_0}, \quad K^{\inf} = K^{\inf}_{\tau_0}, \quad \text{for some } \tau_0 \geq t_0,$$

*then, assumption (***K***) is satisfied. In particular, if $K(t)$ is periodic in t we have that there is no dependency on τ_0 and thus,*

$$K^{\sup}_{\tau_0} = K^{\sup}_{t_0} \quad \text{and} \quad K^{\inf}_{\tau_0} = K^{\inf}_{t_0}, \quad \text{for any } \tau_0 \geq t_0.$$

Hence,

$$K^{\sup} = K^{\sup}_{t_0} \quad \text{and} \quad K^{\inf} = K^{\inf}_{t_0}.$$

Therefore, $K(t)$ periodic in t implies that $K(t)$ satisfies **(K)**.

We are in position to prove the main result of this section:

Proposition 3.3.8. *Let us consider problem (1.0.1) and let us assume* **(N)** *and* **(K)**. *If $\lambda < \lambda_0(K^{\sup})$ we have boundedness of solutions and if $\lambda > \lambda_0(K^{\inf})$ we have unboundedness of solutions. Moreover if $K^{\inf} = K^{\inf}_{\tau_0}$ for a certain $\tau_0 \geq t_0$, then we have unboundedness for $\lambda = \lambda_0(K^{\inf})$.*

Proof. If $\lambda < \lambda_0(K^{\sup}) = \lim_{\tau_0\to\infty} \lambda_0(K^{\sup}_{\tau_0})$ then we can choose τ_0 such that $\lambda < \lambda_0(K^{\sup}_{\tau_0})$. Applying Lemma 3.3.4 we have boundedness.

If $\lambda > \lambda_0(K^{\inf}) = \lim_{\tau_0\to\infty} \lambda_0(K^{\inf}_{\tau_0})$ then we can choose τ_0 such that $\lambda > \lambda_0(K^{\inf}_{\tau_0})$ and again from Lemma 3.3.4 we have unboundedness.

If $\lambda = \lambda_0(K^{\inf}) = \lambda_0(K^{\inf}_{\tau_0})$ then from Lemma 3.3.4 we also have unboundedness.

□

This proposition is expressing the importance of $K^{\sup}, K^{\inf}, \lambda_0(K^{\sup})$ and $\lambda_0(K^{\inf})$ in order to obtain boundedness or unboundedness of solutions in a rather general situation. That is, if $\lambda < \lambda_0(K^{\sup})$ we have boundedness and if $\lambda > \lambda_0(K^{\inf})$ we have unboundedness. Observe that the proposition does not say anything if $\lambda \in (\lambda_0(K^{\sup}), \lambda_0(K^{\inf}))$.

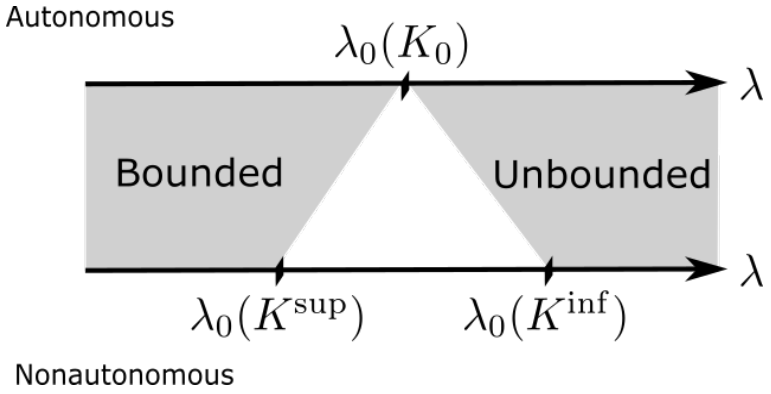


Figure 3.2: Impact of the time-dependency of $K(t)$ on the variety of scenarios.

Let us see some examples of sets $K^{\sup}$ and $K^{\inf}$.

Example 3.3.9 (A Ball of Changing Radius)**.** *Let $x_0 \in \Omega$ and $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$ and let*

$$n(t,x) = 1 - \chi_{K(t)}(x), \quad \text{for } t > 0,\ x \in \overline{\Omega},$$

where $K(t)$ is specified according to different cases.

Case 1. Consider:

$$K(t) = \overline{B\left(x_0, r\left(1 - \frac{1}{t+1}\right)\right)},\ t > 0.$$

Then, $K^{\sup} = K^{\inf} = \overline{B(x_0, r)}$. Hence, if $\lambda < \lambda_0(\overline{B(x_0, r)})$ we have boundedness of solutions and if $\lambda > \lambda_0(\overline{B(x_0, r)})$ we have unboundedness of solutions.

Case 2. Now, if $K(t)$ is a continuously shrinking ball, for example

$$K(t) = \overline{B\left(x_0, \frac{r}{t+1}\right)},\ t > 0,$$

then, $K^{\sup} = K^{\inf} = \{x_0\}$. Hence solutions are bounded for each $\lambda \in \mathbb{R}$ since $\lambda_0(K^{\sup}) = \infty$.

Case 3. Finally, if for some $\omega \in \mathbb{R}$ and $\omega \neq 0$

$$K(t) = \overline{B\left(x_0, 1 + |\sin(\omega t)|\right)},\ t > 0,$$

then, $K^{\inf} = \overline{B(x_0, 1)} \subsetneq K^{\sup} = \overline{B(x_0, 2)}$. Hence if $\lambda < \lambda_0(\overline{B(x_0, 2)})$ we have boundedness of solutions and if $\lambda > \lambda_0(\overline{B(x_0, 1)})$ we have unboundedness of solutions. But Proposition 3.3.8 does not provide any information about the behaviour of solutions when $\lambda \in (\lambda_0(\overline{B(x_0, 2)}), \lambda_0(\overline{B(x_0, 1)}))$.

We can conclude that the non–autonomous problem presents a broader spectrum of cases than its autonomous version. Also λ and $\lambda_0(K(t))$ are no longer the only elements involved in determining asymptotic behaviour of the solutions as, for example, $\lambda_0(K^{\sup})$ and $\lambda_0(K^{\inf})$ arise naturally in the non–autonomous setting.

Let us study other configurations of $K(t)$.

Example 3.3.10 (Rotating Cheese)**.** *Consider an open domain $\Omega \subset \mathbb{R}^2$ with $\overline{B(0, r_0)} \subset \Omega$ and let the following set expressed in polar coordinates*

$$\tilde{K} = \{(r, \theta) : 0 \leq r \leq r_0,\ \theta_0 \leq \theta \leq \theta_1\}.$$

Define $K(t)$ as the set $\tilde{K}$, rotating clockwise at certain angular speed $\omega > 0$, that is

$$K(t) = \{(r, \theta) : 0 \leq r \leq r_0,\ \theta_0 - \omega t \leq \theta \leq \theta_1 - \omega t\}.$$

In this particular case, $K^{\sup}$ is the entire circle $\overline{B(0, r_0)}$ as every point x in the circle verifies that $n(t, x)$ vanishes for some $t > t_0$.

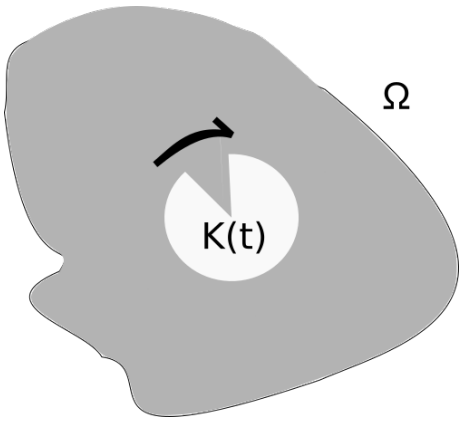


Figure 3.3: Domain Ω with rotating subdomain $K(t)$.

Furthermore, $K^{\inf} = \{0\}$*, the center of the circle, as it is the only point where* n *vanishes for every* $t > t_0$*. Thus, in this example we have that* $\lambda_0(K(t)) = \lambda_0(\tilde{K})$ *is constant and*

$$\lambda_0(\overline{B(0,r_0)}) = \lambda_0(K^{\sup}) < \lambda_0(K(t)) < \lambda_0(K^{\inf}) = \infty, \quad \text{for any } t > t_0.$$

Now, although from the previous proposition we can show that the solutions are bounded for $\lambda < \lambda_0(\overline{B(0,r_0)})$*, we cannot decide the behaviour for* $\lambda > \lambda_0(\overline{B(0,r_0)})$ *since we cannot apply Proposition 3.3.8 to arrive to a conclusion. In fact, as seen later on, whether the solution is bounded or not will also depend on the speed at which* $K(t)$ *rotates.*

Even though the main factors for long–term dynamics in the autonomous case are λ and $\lambda_0(K(t))$, the previous Example 3.3.10 illustrates how $\lambda_0(K(t))$ might be constant whilst $K(t)$ might not be so. Therefore, the geometry of $K(t)$ should be considered when studying asymptotic behaviour of solutions in the non–autonomous case.

In fact, the geometry of $K(t)$ is one of the main reasons for the variety of scenarios arising in the non–autonomous case. Even if $\lambda > \lambda_0(K(t))$ for all $t > t_0$ we might not obtain unboundedness of solutions as we would expect in the autonomous case. To see this, consider the following example.

Example 3.3.11 (Jumping Sets)**.** *Let us consider problem (1.0.1) and* $n(t,x)$ *such that* $K(t)$ *"jumps" from a compact set* K_0 *to another compact set* K_1 *periodically:*

$$K(t) = \begin{cases} K_0, & t \in (nt_0, nt_0 + T_1], n \in \mathbb{N}, \\ K_1, & t \in (nt_0 + T_1, (n+1)t_0], n \in \mathbb{N}, \end{cases}$$

where $0 < T_1 < t_0$ *and* K_0, K_1 *are compact disjoint sets as in Figure 3.25. This is an example of special significance as, following the autonomous logic, it is reasonable to expect the existence of unbounded solutions for* $\lambda > \lambda_0(K(t))$ *for all* $t > t_0$*. However, as shown later on, for every* $\lambda \in \mathbb{R}$ *the solutions remain bounded. The relationship between a jumping* $K(t)$ *and boundedness will be explored in the following Section 3.4.*

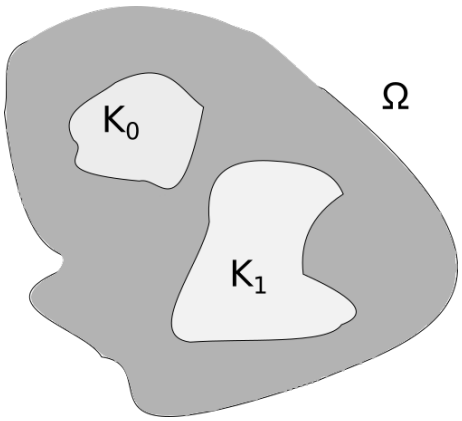


Figure 3.4: Domain Ω with "jumping" subdomains K_0 and K_1.

From the previous two examples, it is clear that the relation between λ and $\lambda_0(K(t))$ although important, does not completely assess the asymptotic behaviour of solutions of (1.0.1): the geometry of $K(t)$ and how $K(t)$ moves also have to be factored in. In fact, throughout this chapter we will present several examples of boundedness and unboundedness linked to the geometry of $K(t)$: not only its size, but also taking into consideration how $K(t)$ moves.

Finally, notice that, although we have presented several relatively simple examples, Proposition 3.3.8 can be applied to a wide range of problems.

3.4 Boundedness for $\lambda \in (\lambda_0(K^{\sup}), \lambda_0(K^{\inf}))$

The main objective of this section is to obtain boundedness results that go beyond Proposition 3.3.8 in the previous section. In this result, we have established boundedness for $\lambda < \lambda_0(K^{\sup})$ and unboundedness for $\lambda > \lambda_0(K^{\inf})$. This section is devoted to explore under which conditions we can obtain boundedness for $\lambda \in (\lambda_0(K^{\sup}), \lambda_0(K^{\inf}))$.

Let us start by presenting an example which illustrates the behaviour of the solutions of the Ball with Changing Radius.

Example 3.4.1. *Let u be the solution of (1.0.1) where $n : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ satisfies* **(N)** *and such that $K(t)$ is a continuous oscillating ball (at a fixed speed ω) as in Example 3.3.9 Case 3 defined as:*

$$K(t) = \overline{B(0, r(t))}, \qquad r(t) = 1 + |\sin(\omega t)|\,.$$

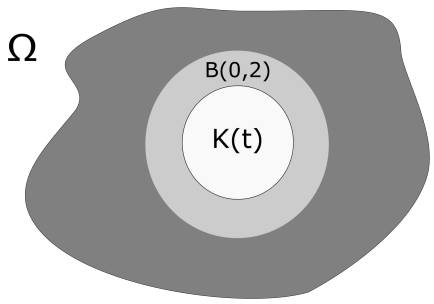


Figure 3.5: Domain Ω with $K(t)$ as an oscillating ball with maximum radius 2.

Then, since it is periodic in time, we know that

$$K^{\sup} = K^{\sup}_{t_0} = \overline{B(0,2)} \quad \textit{and} \quad K^{\inf} = K^{\inf}_{t_0} = \overline{B(0,1)}.$$

Moreover, from Lemma 3.3.4 and Corollary 3.3.8, the asymptotic behaviour of solutions of (1.0.1) for $K(t)$ has already been stablished for

$$\lambda < \lambda_0(\overline{B(0,2)}), \quad \textit{or} \quad \lambda_0(\overline{B(0,1)}) < \lambda.$$

Therefore, if we fix $\lambda \in (\lambda_0(\overline{B(0,2)}), \lambda_0(\overline{B(0,1)}))$, there will be times where $\lambda_0(K(t)) < \lambda$ and times where $\lambda_0(K(t)) > \lambda$, sparkling two opposing behaviours:

- *Whenever $\lambda < \lambda_0(K(t))$ we might expect solutions to stay bounded.*
- *However, whenever $\lambda > \lambda_0(K(t))$ we might expect solutions to grow.*

It seems reasonable to expect that if we spend most of the time in the regime $\lambda < \lambda_0(K(t))$, then solutions will be bounded, while if we spend most of the time in the regime $\lambda_0(K(t)) < \lambda$, then probably solutions may grow up and become unbounded.

For fixed $\lambda \in (\lambda_0(\overline{B(0,2)}), \lambda_0(\overline{B(0,1)}))$, we define $r_\lambda > 0$ such that $\lambda = \lambda_0(\overline{B(0,r_\lambda)})$ and consider sequences of time $(T_{0,i})_{i\in\mathbb{N}}, (T_{1,i})_{i\in\mathbb{N}}$ where

$$\begin{cases} \lambda_0(K(t)) < \lambda \ \left(\Leftrightarrow \overline{B(0,r_\lambda)} \subset K(t)\right), & \textit{for } t \in (T_{0,i}, T_{1,i}), \\ \lambda < \lambda_0(K(t)) \ \left(\Leftrightarrow K(t) \subset \overline{B(0,r_\lambda)}\right), & \textit{for } t \in (T_{1,i}, T_{0,i+1}). \end{cases}$$

Hence, the larger the intervals $(T_{1,i}, T_{0,i+1})$, and the smaller the intervals $(T_{0,i}, T_{1,i})$ are, the better chances to obtain that solutions are bounded. And viceversa: the smaller the intervals $(T_{1,i}, T_{0,i+1})$, and the larger the intervals $(T_{0,i}, T_{1,i})$ are, the better chances to obtain unboundedness.

To go ahead with the proof of such result we will use comparison results with appropriate piecewise constant $\tilde{K}(t)$.

This example motivates the following definition:

Definition 3.4.2. *We say that the family of compact sets $K(t)$ is* **bounded above by the pair of compact sets** *$K_1, K_0 \subset \overline{\Omega}$, denoted by $K(t) \subset (K_1, K_0)$, if*

(1) there exist time sequences $(T_{0,i})_{i\in\mathbb{N}}, (T_{1,i})_{i\in\mathbb{N}}$ with $T_{0,i} < T_{1,i} < T_{0,i+1} < T_{1,i+1} \to \infty$, $i = 1, 2, \cdots$ and with $T_{0,1} \geq t_0$ such that for every $i \in \mathbb{N}$

$$\begin{cases} K(t) \subset K_1, & T_{0,i} < t \leq T_{1,i}, \\ K(t) \subset K_0, & T_{1,i} < t \leq T_{0,i+1}, \end{cases}$$

(2) and there exist $\eta, \Xi > 0$ such that the sequences verify

$$\begin{cases} |T_{1,i} - T_{0,i}| \leq \Xi, & i \in \mathbb{N}, \\ |T_{0,i+1} - T_{1,i}| > \eta, & i \in \mathbb{N}. \end{cases} \tag{3.4.1}$$

The purpose of Definition 3.4.2, as illustrated in Example 3.4.1, is to construct a supersolution of our problem (1.0.1) with the use of sets K_0, K_1. If the supersolution is bounded, then we will obtain boundedness of solutions.

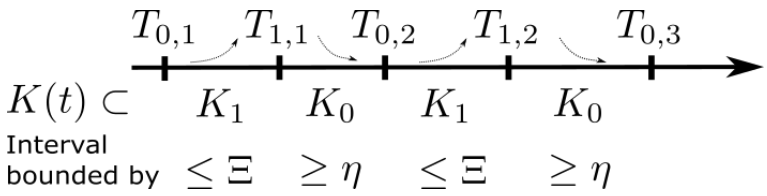


Figure 3.6: This figure illustrates how the sequences $(T_{0,i})_{i\geq 1}$ and $(T_{1,i})_{i\geq 1}$ in Definition 3.4.2 define the growth and shrink regimes.

Let us consider an important example where this supersolution can be easily constructed. The example is given by an "intermittent" nonlinearity, where the nonlinear term switches "on and off" according to the sequences of time $(T_{0,i})_{i\geq 1}, (T_{1,i})_{i\geq 1}$.

Let us assume **(N)** and let $K(t)$ be a family of compact sets bounded above by the pair $\overline{\Omega}, \emptyset$, that is, $K(t) \subset (\bar{\Omega}, \emptyset)$ in the sense of Definition 3.4.2.



Figure 3.7: Domain Ω with $K(t)$ "intermitent". The right figure represents the growth regime where $n(x) = 0$ and the left figure the fast–decaying regime where $n(x) > 0$.

In particular, we have that $K(t) = \emptyset$ for the sequence of intervals $(T_{1,i}, T_{0,i+1}]$ and $|T_{0,i+1} - T_{1,i}| > \eta$. Moreover, requiring hypothesis **(N)** we have $n(t,x) \geq \nu_0 > 0$ $\forall t \in (T_{1,i}, T_{0,i+1}]$ $\forall i \geq 0$ and $x \in \overline{\Omega}$. This includes in particular the case where

$$n(t,x) = \begin{cases} 0, & t \in (T_{0,i}, T_{1,i}],\ x \in \overline{\Omega} \\ 1, & t \in (T_{1,i}, T_{0,i+1}],\ x \in \overline{\Omega}. \end{cases}$$

We can show:

Proposition 3.4.3. *Let $K(t)$ be a family of compact sets satisfying $K(t) \subset (\overline{\Omega}, \emptyset)$. Assume also hypothesis* **(N)** *holds. Then, for all $\lambda \in \mathbb{R}$, all solutions of (1.0.1) are bounded. Moreover, there exist $\tau > 0$ and $M > 0$ independent of u_0 such that*

$$|u(t,x;t_0,u_0)| \leq M, \quad \forall t \geq \tau, \forall x \in \Omega.$$

Proof. Since the boundedness or unboundedness property of solutions is independent of initial conditions due to Propositions 1.1.7 and 3.2.2, we can take $u_0 \in L^\infty(\Omega)$.

First, we estimate the growth of u for the growth regime, that is, the interval $[T_{0,i}, T_{1,i})$. This is straightforward since from (1.2.6) we have

$$\|u(t,\cdot;T_{0,i},u(T_{0,i},\cdot;t_0,u_0))\|_\infty \leq Me^{(\lambda-\lambda_1(\Omega))(t-T_{0,i})}\|u(T_{0,i},\cdot;t_0,u_0)\|_\infty,\ t \in [T_{0,i},T_{1,i}), i \in \mathbb{N}.$$

Moreover, since $|T_{1,i} - T_{0,i}| \leq \Xi$, we have

$$\|u(t,\cdot;u(T_{0,i},\cdot;u_0))\|_\infty \leq Me^{|\lambda-\lambda_1(\Omega)|\Xi}\|u(T_{0,i},\cdot;u_0)\|_\infty,\ t\in[T_{0,i},T_{1,i}), i\in\mathbb{N}.$$

Due to the fact that in $[T_{1,i},T_{0,i+1})$ the nonlinear term is applied to all the domain, we can obtain a supersolution for the time intervals $[T_{1,i},T_{0,i+1})$ for $i\in\mathbb{N}$ by using the following ODE problem:

$$\begin{cases} W(t)_t = \lambda W(t) - \nu_0 W(t)^\rho, & t\in(0,T_{0,i+1}-T_{1,i}),\\ W(0) = \|u(T_{1,i},\cdot)\|_{L^\infty(\Omega)}, & t=0, \end{cases} \tag{3.4.2}$$

that is, $W(t-T_{1,i})\geq u(t,x)$ for $t\in(T_{1,i},T_{0,i+1}]$ for every $i\in\mathbb{N}$ and a.e. $x\in\Omega$. As a matter of fact the function $W(t)$ is a solution of

$$\begin{cases} W_t(t)-\Delta W(t) = \lambda W(t)-\nu_0 W(t)^\rho \geq \lambda W(t) - n(t,x)W(t)^\rho, & t\in(0,T_{0,i+1}-T_{1,i}),\ x\in\Omega,\\ W(t,x)\geq 0, & t\in(0,T_{0,i+1}-T_{1,i}),\ x\in\partial\Omega,\\ W(0)=\|u(T_{1,i},\cdot)\|_{L^\infty(\Omega)}, & t=0,\ x\in\Omega, \end{cases}$$

which shows that W is a supersolution of u. By direct inspection the solution W of (3.4.2) is:

$$W(t) = \left[\frac{\nu_0}{\lambda}\left(1-e^{-\lambda(\rho-1)t}\right)+e^{-\lambda(\rho-1)t}W(0)^{1-\rho}\right]^{-\frac{1}{\rho-1}}$$

which is bounded above by:

$$W_\infty(t) = \left[\frac{\nu_0}{\lambda}\left(1-e^{-\lambda(\rho-1)t}\right)\right]^{-\frac{1}{\rho-1}}. \tag{3.4.3}$$

The function $W_\infty(t-T_{1,i})$ is specially important not only because it bounds $u(t,x)$ for $t\in[T_{1,i},T_{0,i+1})$ for every $i\in\mathbb{N}$ and every $x\in\Omega$ but it also reaches, for a fixed amount of time $(T_{0,i+1}-T_{1,i})$, the same threshold $W_\infty(T_{0,i+1}-T_{1,i})$ wherever the solution u may start at time $T_{1,i}$. Now, since $|T_{0,i+1}-T_{1,i}|>\eta$ for every $i\in\mathbb{N}$, we have that $W_\infty(\eta)$ bounds $u(T_{0,i+1})$ for $i\in\mathbb{N}$ and a.e. $x\in\Omega$. In particular,

$$|u(T_{0,2},x)|\leq W_\infty(\eta),\quad \text{a.e. } x\in\Omega.$$

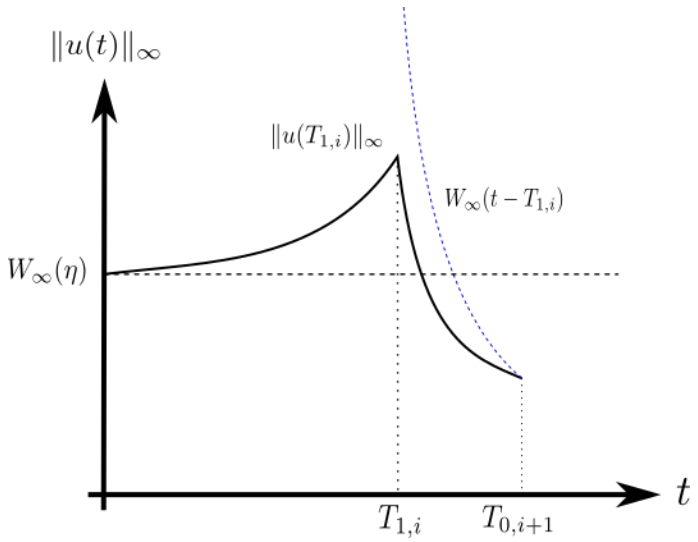


Figure 3.8: W_∞ bounds u for $t\in(T_{1,i},T_{0,i+1}]$.

Thus, if we define the function

$$\overline{u}(t) := \begin{cases} Me^{|\lambda-\lambda_1(\overline{\Omega})|(t-T_{0,i})}W_\infty(\eta), & t \in (T_{0,i}, T_{1,i}+\eta/2],\ i=2,3,\dots \\ W_\infty(\eta/2), & t \in (T_{1,i}+\eta/2, T_{0,i+1}],\ i=2,3,\dots. \end{cases}$$

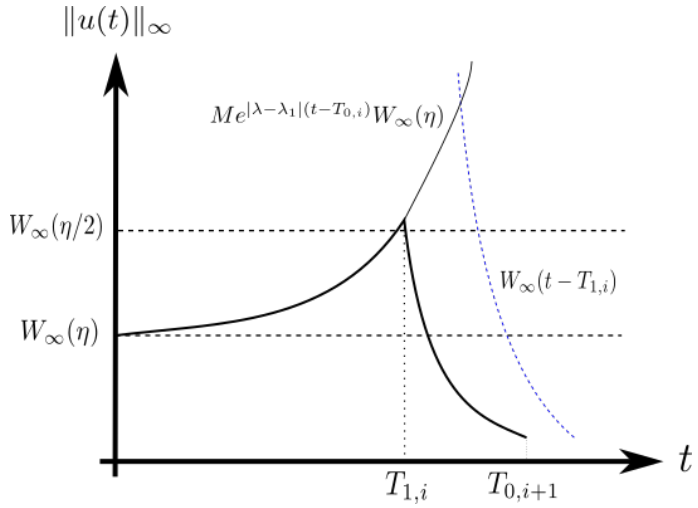


Figure 3.9: Construction of the upper bound for u.

Then, we have

$$|u(t,x;u_0)| \le \bar{u}(t), \quad t \ge T_{0,2},\ x \in \Omega$$

and in particular

$$|u(t,x;u_0)| \le M := \max_{s\in[T_{0,2},T_{0,3}]} \bar{u}(s), \quad \forall t \ge T_{0,2}.$$

This concludes the proof of the proposition.

□

Remark 3.4.4. *Observe that Proposition 3.4.3 applies to all $\lambda \in \mathbb{R}$. In this case, $\lambda_0(K^{\inf}) = \infty$ since $K^{\inf} = \emptyset$.*

3.4.1 Estimates of the solution locally in space and time

The effect of having the logistic term acting over the whole domain Ω during certain intervals of time, allows us to obtain global bounds in space with the use of the function $W_\infty(t)$ (see Proposition 3.4.3). In this subsection we will obtain bounds of the solutions which are local in space and time under certain conditions on the vanishing logistic term.

Let us start with a partial result that gives us estimates of the solution locally in space but globally in time. In order to accomplish this, we will need to consider the following singular elliptic Dirichlet problem

$$\begin{cases} -\Delta z = \lambda z - \beta z^\rho, & \text{in } B(0,a), \\ z = \infty, & \text{on } \partial B(0,a). \end{cases} \tag{3.4.4}$$

Problem (3.4.4) has been studied in numerous articles, see [14, 15, 17]. It is known that it has a unique positive radial solution denoted by $z_a(x)$ and this solution is used to construct a supersolution in any ball where the logistic term $n(t,x)$ is strictly positive. In particular, if $\overline{B(x_0,a)} \subset \Omega$ and $n(t,x) \geq \beta > 0$ for all $x \in B(x_0,a)$, $t \geq t_0$ then if $u_0(x) \in L^\infty(\Omega)$ is such that

$$0 \leq u_0(x) \leq z_a(0) = \inf_{y \in B(0,a)} z_a(y) \leq z_a(y), \quad \forall x \in B(x_0,a),\ \forall y \in B(0,a).$$

Hence, by Theorem 1.2.1, $z_a(x-x_0)$ is a supersolution of $u(t,x)$, that is,

$$0 \leq u(t,x) \leq z_a(x-x_0), \quad x \in B(x_0,a),\ t \geq t_0,$$

and therefore, if $M = \sup_{|y| \leq b} z_a(y)$ for $0 < b < a$ then

$$0 \leq u(t,x) \leq M, \quad x \in B(x_0,b),\ t \geq t_0.$$

Observe that $M = M(\lambda, \beta, \rho, a, b)$.

Now we can show the following result:

Lemma 3.4.5. *Let us consider problem (1.0.1) with $n(t,x)$ satisfying* **(N)** *and let $K \subset \overline{\Omega}$ be a compact set with $K(t) \subset K$ for every $t \geq t_0$ (for instance $K = K^{\sup}$).*

For each $\delta > 0$ small enough there exists $M = M(\Omega, \lambda, \rho, n, u_0, \delta) > 0$ satisfying

$$0 \leq u(t,x) \leq M, \quad \text{for } t > t_0, x \in \Omega \setminus \Omega_\delta, \tag{3.4.5}$$

where $\Omega_\delta = \{x \in \Omega : d(x,K) < \delta\}$ is an open neighbourhood of K.

Proof. Let us consider $\Omega_{\delta/2}$ and define

$$n_{\delta/2}(x) := \begin{cases} 0, & x \in \overline{\Omega_{\delta/2}}, \\ \nu(\delta/2), & x \in \Omega \setminus \overline{\Omega_{\delta/2}} \end{cases}$$

where $\nu(\cdot)$ is the function in **(N)**. It is clear that $n(t,x) \geq n_{\delta/2}(x)$, $\forall x \in \overline{\Omega_{\delta/2}}$, $t \geq t_0$. Moreover, from **(N)** we have $n(t,x) \geq \nu(d(x,K(t))$ for every $x \in \Omega \setminus \overline{\Omega_{\delta/2}}$ But since $d(x,K(t)) \geq \delta/2$ for $x \in \Omega \setminus \overline{\Omega_{\delta/2}}$ and ν is increasing, then we have $n(t,x) \geq \nu(\delta/2) > 0$. Hence, $n(t,x) \geq n_{\delta/2}(x)$ $\forall x \in \overline{\Omega}$ and $\forall t \geq t_0$ and by comparison $u(t,x;u_0) \leq \hat{u}(t,x,;u_0)$ where $\hat{u}$ is the solution of $P(\hat{u}, u_0, n_{\delta/2}, \Omega)$.

But estimates (3.4.5) were shown in [7] for $\hat{u}$ (since $\hat{u}$ solves an autonomous problem) as it has been shown above.

□

Note that the bounds obtained in the previous Lemma 3.4.5 apply only to points outside $K^{\sup}$. But in many situations we may have points $x_0 \in \Omega$ with $\overline{B(x_0,a)} \subset \Omega \setminus K(t)$ for $t \in (t_1,t_2)$ while $x_0 \in K(t)$ for other times. For instance, consider $K(t)$ is a continuously moving ball within a ring for which $K^{\sup}$ is the ring. If $x \in K^{\sup}$ we have times when $x \in K(t)$, but we have other times where $x \in \Omega \setminus K(t)$. It is very relevant to study the behaviour of the solution in these points.

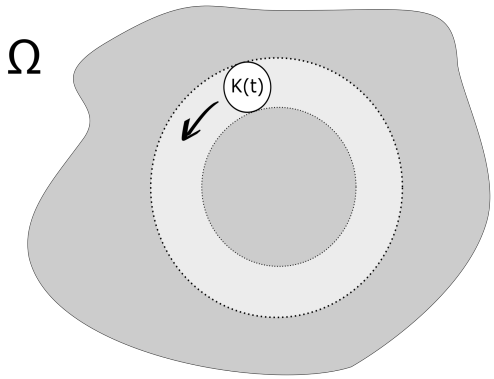


Figure 3.10: Domain Ω and a continuously moving $K(t)$. The compact sets $K(t)$ move along the ring $\bigcup_{t\geq t_0} K(s)$.

We have the following:

Proposition 3.4.6. *Let u be the solution of (1.0.1) for $0 \leq u_0 \in L^\infty(\Omega)$. Let $x_0 \in \Omega$ and $a > 0$ be such that $B(x_0, a) \subset \Omega$ and let n satisfy*

$$n(t,x) \geq \beta > 0,\ t \in (t_1, t_2),\ x \in B(x_0, a),$$

for some $\beta > 0$ and $t_1 < t_2$. Then, for any $\tau_0 \in (0, t_2 - t_1)$ and any $r \in (0, a)$ there exists $M = M(\lambda, \beta, \rho, \tau_0, r, a)$ such that

$$0 \leq u(t,x) \leq M, \quad \text{for } t \in [t_1 + \tau_0, t_2],\ x \in B(x_0, r).$$

Proof. Let us consider $U^*(t,x) = z_{a,x_0}(x) + W_\infty(t - t_1)$ for $t \in (t_1, t_2)$ and $x \in B(x_0, a)$ where $z_{a,x_0}(x) = z_a(x - x_0)$ and z_a is the unique positive solution of (3.4.4) and W_∞ is defined in (3.4.3). Then, since $z_a, W_\infty > 0$ and $\rho > 1$,

$$\begin{aligned} U_t^* - \Delta U^* &= W_{\infty t} - \Delta z_{a,x_0} = \lambda W_\infty - \beta W_\infty^\rho + \lambda z_{a,x_0} - \beta z_{a,x_0}^\rho \\ &= \lambda U^* - \beta\left(W_\infty^\rho + z_{a,x_0}^\rho\right) = \lambda U^* - \beta(W_\infty + z_{a,x_0})^\rho \left(\left(\tfrac{W_\infty}{W_\infty + z_{a,x_0}}\right)^\rho + \left(\tfrac{z_{a,x_0}}{W_\infty + z_{a,x_0}}\right)^\rho\right) \\ &\geq \lambda U^* - \beta(W_\infty + z_{a,x_0})^\rho \left(\left(\tfrac{W_\infty}{W_\infty + z_{a,x_0}}\right) + \left(\tfrac{z_{a,x_0}}{W_\infty + z_{a,x_0}}\right)\right) = \lambda U^* - \beta(U^*)^\rho, \end{aligned}$$

where we have used that $\rho > 1$ and therefore

$$\left(\frac{W_\infty}{W_\infty + z_{a,x_0}}\right)^\rho \leq \frac{W_\infty}{W_\infty + z_{a,x_0}}, \quad \left(\frac{z_{a,x_0}}{W_\infty + z_{a,x_0}}\right)^\rho \leq \frac{z_{a,x_0}}{W_\infty + z_{a,x_0}}.$$

Moreover $U^*(t,x) = \infty$ for $t \in (t_1, t_2)$, $x \in \partial B(x_0, a)$ and for $t = t_1$, $x \in B(x_0, a)$. Therefore, U^* is a supersolution for u in $(t_1, t_2) \times B(x_0, a)$ for any initial condition $u_0 \in L^\infty(\Omega)$. Notice that for any $\tau_0 \in (0, t_2 - t_1)$ and any $r \in (0, a)$ if $t \in [t_1 + \tau_0, t_2),\ x \in B(x_0, r)$ then

$$0 \leq u(t,x) \leq U^*(t,x) \leq \|W_\infty(\tau_0)\|_\infty + \|z_{a,x_0}(\cdot)\|_{L^\infty(B(0,r))} < \infty.$$

Which concludes the proof, choosing $M = \|W_\infty(\tau_0)\|_\infty + \|z_{a,x_0}(\cdot)\|_{L^\infty(B(0,r))}$.

□

Remark 3.4.7. *Other bounds for U^* when $x \to \partial B(x_0, a)$ or, $t \to t_2$ for $\beta = \beta(t)$ continuous, $\beta(t) > 0$ for every $t \in (t_1, t_2)$ and $\beta(t_2) = 0$ can be seen in [16].*

We can apply Proposition 3.4.6 to obtain bounds not only in balls but other subsets:

Corollary 3.4.8. *Let D be a subset of Ω, $\tau_0 > 0$, and $\delta > 0$. If $d\left(D, \bigcup_{t\in[t_1,t_2]} K(t)\right) > \delta > 0$ for any $t_1, t_2 \in \mathbb{R}$ with $t_2 - t_1 > \tau_0 > 0$, then, there exists $\Lambda = \Lambda(\lambda, \rho, \tau_0, \nu, \delta)$ such that*

$$0 \leq u(t, x) \leq \Lambda, \quad \forall t \in [t_1 + \tau_0, t_2],\ \forall x \in D.$$

Proof. Let us take the open set $\mathcal{U}$

$$\mathcal{U} := \bigcup_{z\in D} B\left(z, \frac{\delta}{2}\right) \supset D.$$

Since D is at least a distance δ away from $K(t)$ for $t \in [t_1, t_2]$ we know that $\mathcal{U}$ is at least a distance $\delta/2$ away from $K(t)$ for the same time interval. Therefore,

$$n(t, x) \geq \nu(d(x, K(t)) \geq \nu\left(\frac{\delta}{2}\right) > 0, \quad \forall t \in [t_1, t_2],\ x \in \mathcal{U}.$$

Moreover, applying Proposition 3.4.6 for every ball $B(z, \delta/2)$ with $z \in D$ and using the continuity of u in t we obtain that there exists an $\Lambda = \Lambda(\lambda, \rho, \tau, \nu(\delta/2))$ such that

$$0 \leq u(t, x) \leq \Lambda, \quad \forall t \in [t_1 + \tau, t_2],\ \forall x \in B\left(z, \frac{\delta}{4}\right),\ \forall z \in D,$$

which concludes the proof.

□

3.4.2 Boundedness for $\lambda < \lambda_0(K(t)), \forall t$

In this subsection we obtain one of the main results of this chapter which consists of a boundedness result for the case where, roughly speaking, $\lambda < \lambda_0(K(t))\ \forall t \geq t_0$. As a matter of fact, we will show the following important result:

Theorem 3.4.9. *Let us assume n satisfies* **(N)** *and that there exist $\tau_0 > 0$, $\delta > 0$ and a family of smooth bounded open sets $\{\Omega_\delta(t)\}_{t\geq t_0}$ with $\Omega_\delta(t) \subset \Omega$ such that for each $t \geq t_0 + \tau_0$*

$$\bigcup_{s\in[t-\tau_0,t+\tau_0]} K(s) \subset \Omega_\delta(t), \qquad \text{and} \qquad d\left(\bigcup_{s\in[t-\tau_0,t+\tau_0]} K(s), \partial\Omega_\delta(t)\right) \geq \delta. \tag{3.4.6}$$

Then, if $\lambda < \inf_{t\geq t_0} \lambda_1(\Omega_\delta(t))$, we have that all solutions of (1.0.1) are bounded.

Remark 3.4.10. *Condition (3.4.6) implies that $K(t)$ is at a distance at least $\delta > 0$ from $\partial\Omega$. See Corollary 3.4.13 below to extend this result to the case where $K(t) \cap \partial\Omega \neq \emptyset$.*

Proof (of Theorem 3.4.9). Let us define $t_1 = t_0 + \tau_0$, and in general, $t_{i+1} = t_i + \tau_0$ for $i \geq 1$, $\lambda_1 := \inf_{t \geq t_0} \lambda_1(\Omega_\delta(t))$ and let D_i be a $\delta/2$–neighbourhood of $\Omega \setminus \Omega_\delta(t_i)$, that is, $D_i = \{x \in \Omega : d(x, \Omega \setminus \Omega_\delta(t_i)) < \delta/2\}$ for $i \geq 1$. Now, from (3.4.6) we have that for any $x \in D_i$ and $i \geq 1$

$$d\left(x, \bigcup_{s \in [t_i - \tau_0, t_i + \tau_0]} K(s)\right) \geq \delta/2.$$

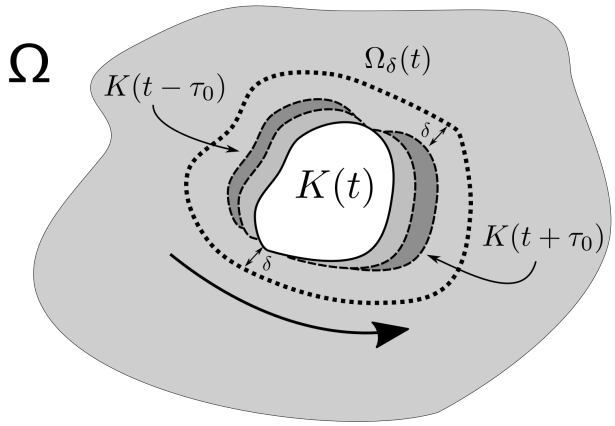


Figure 3.11: The neighbourhood $\Omega_\delta(t)$ contains $\cup_{s \in [t-\tau_0, t+\tau_0]} K(s)$ and its boundary is at a distance $\delta > 0$ from it.

Applying Corollary 3.4.8 to D_i for $i \geq 1$ we obtain that there exists a bound $\Lambda = \Lambda(\lambda, \rho, \tau_0, \nu, \delta)$ such that $u(t,x) \leq \Lambda$ for $t \in [t_i, t_i + \tau_0]$ and $x \in D_i$. In particular, since $\Omega \setminus \Omega_\delta(t_i) \subset D_i$ and $\partial\Omega_\delta(t_i) \subset D_i$, constant Λ is independent of i and verifies

$$u(t,x) \leq \Lambda, \qquad \text{for } t \in [t_i, t_{i+1}], \ \text{ a.e. } x \in \Omega \setminus \Omega_\delta(t_i), \ i \geq 1. \tag{3.4.7}$$

In particular we have $u(t,x) \leq \Lambda$, for $t \in [t_i, t_{i+1}]$, a.e. $x \in \partial\Omega_\delta(t_i)$ and any $i \geq 1$.

Let us prove by induction that

$$\exists \Lambda^* \quad \text{such that} \quad u(t_{i+1}, x) \leq \Lambda^*, \quad \forall x \in \Omega_\delta(t_i), \ i = 1, 2, \ldots. \tag{3.4.8}$$

Then, let us consider $i = 1$ and problem

$$\begin{cases} v_t - \Delta v = \lambda v, & x \in \Omega_\delta(t_1), \ t \in (t_1, t_2) \\ v = \Lambda, & x \in \partial\Omega_\delta(t_1), \ t \in (t_1, t_2) \\ v(t_1, x) = \Lambda^*, & x \in \Omega_\delta(t_1), \end{cases} \tag{3.4.9}$$

where Λ^* will be fixed later but satisfies $\Lambda^* \geq \max\{\Lambda, Me^{\lambda\tau_0}\|u_0\|_{L^\infty}\}$, where $Me^{\lambda\tau_0}\|u_0\|_{L^\infty} \geq \|u(t_1, \cdot)\|_{L^\infty(\Omega_\delta(t_1))}$ and M is as in (1.2.6). Then, by Theorem 1.2.1,

$$v(t,x) \geq u(t,x), \qquad t \in [t_1, t_2], \ \text{a.e. } x \in \Omega_\delta(t_1).$$

Now, v can be split as $v = v_1 + v_2$ where

$$\begin{cases} -\Delta v_1 = \lambda v_1, & x \in \Omega_\delta(t_1), \\ v_1 = \Lambda, & x \in \partial\Omega_\delta(t_1), \end{cases} \tag{3.4.10}$$

$$\begin{cases} v_{2_t} - \Delta v_2 = \lambda v_2, & x \in \Omega_\delta(t_1),\ t \in (t_1, t_2), \\ v_2 = 0, & x \in \partial\Omega_\delta(t_1),\ t \in (t_1, t_2), \\ v_2(t_1, x) = \Lambda^* - v_1(x), & x \in \Omega_\delta(t_1). \end{cases} \tag{3.4.11}$$

Observe that since $\lambda < \lambda_1(\Omega_\delta(t_1))$, v_1 is well defined and, by the maximum principle, $v_1(x) \geq 0$. Moreover, since $M_2 - v_1(x) \leq M_2$ then the solution of (3.4.11) is bounded above by the solution of the ODE

$$\begin{cases} \tilde{v}_{2_t} = \lambda \tilde{v}_2, t \in (t_1, t_2), \\ \tilde{v}_2(0) = M_2. \end{cases}$$

Therefore, we have $v_2(t, x) \leq \Lambda^* e^{(\lambda - \lambda_1)(t - t_1)}$ which is stricly decreasing in time. Hence, denoting by v_1^* the solution of the following problem

$$\begin{cases} -\Delta v_1^* = \lambda v_1^*, & \text{in } U, \\ v_1^* = 1, & \text{on } \partial U, \end{cases} \tag{3.4.12}$$

it is immediate that $v_1^* = \Lambda \cdot v_1$ where v_1 is the solution of (3.4.10). As it will be shown in the following auxiliary Lemma 3.4.12, we have the following important estimate:

$$\|v_1^*\|_{L^\infty(\Omega_\delta(t))} \leq C(|\Omega_\delta(t)|) \leq C(|\Omega|), \quad \forall t \geq t_0.$$

Therefore, choosing

$$\Lambda^* = \max\left\{\Lambda, M e^{\lambda \tau_0} \|u_0\|_{L^\infty}, \frac{\Lambda C(|\Omega|)}{1 - e^{(\lambda - \lambda_1)\tau_0}}\right\}, \tag{3.4.13}$$

we have that

$$\begin{aligned} u(t_2, x) &\leq v(t_2, x) \leq v_1(x) + v_2(t_2, x) \leq \Lambda \|v_1^*\|_{L^\infty(\Omega(t_1))} + \Lambda^* e^{(\lambda - \lambda_1)\tau_0} \\ &= \Lambda^* \left(\tfrac{\Lambda}{\Lambda^*} \|v_1^*\|_{L^\infty(\Omega(t_1))} + e^{(\lambda - \lambda_1)\tau_0}\right) \leq \Lambda^*, \quad x \in \Omega_\delta(t_1). \end{aligned} \tag{3.4.14}$$

Thus, we have that $u(t_2, \cdot) \leq \Lambda^*$ on $\Omega_\delta(t_1)$ and $u(t_2, \cdot) \leq \Lambda$ on $\Omega \setminus \Omega_\delta(t_1)$ and since $\Lambda \leq \Lambda^*$ from (3.4.13), we have $u(t_2, x) \leq \Lambda^*$ for $x \in \Omega$.

Let us assume now that $u(t_i, x) \leq \Lambda^*$ in $\Omega_\delta(t_i)$ for some $i \geq 2$ and let us prove that $u(t_{i+1}, x) \leq \Lambda^*$ in $\Omega_\delta(t_i)$. Notice that we have proven before that

$$u(t, x) \leq \Lambda, \qquad \text{for } t \in [t_i, t_{i+1}], \ \text{ a.e. } x \in \Omega \setminus \Omega_\delta(t_i).$$

In particular we have $u(t, x) \leq \Lambda$ for $t \in [t_i, t_{i+1}]$ and a.e. $x \in \partial\Omega_\delta(t_i)$.

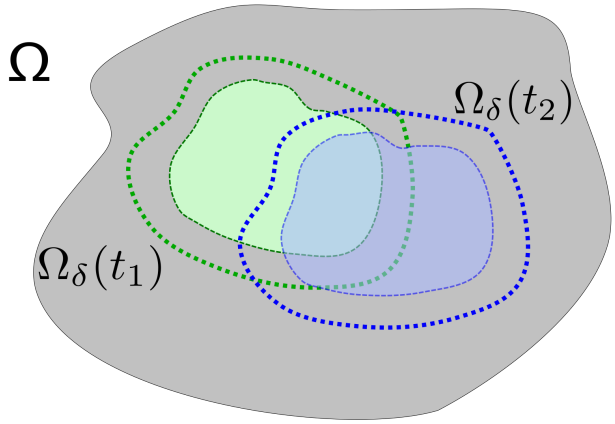


Figure 3.12: Neighbourhoods $\Omega_\delta(t_1)$ (dotted green) and $\Omega_\delta(t_2)$ (dotted blue). Solution u at time t_2 is bounded by M_1 outside $\Omega_\delta(t_1)$ and bounded by M_2 inside it. Hence, at time t_2 we can consider M_2 as initial data of a supersolution of u inside $\Omega_\delta(t_2)$.

Let us consider the following problem

$$\begin{cases} v_t - \Delta v = \lambda v, & x \in \Omega_\delta(t_i),\ t \in (t_i, t_{i+1}) \\ v = \Lambda, & x \in \partial\Omega_\delta(t_i),\ t \in (t_i, t_{i+1}) \\ v(t_i, x) = \Lambda^*, & x \in \Omega_\delta(t_i), \end{cases} \tag{3.4.15}$$

Observe that by Theorem 1.2.1, $v(t,x) \geq u(t,x)$ for $t \in [t_i, t_{i+1}]$ and a.e. $x \in \Omega_\delta(t_i)$. As we did in (3.4.14), the solution can be split $v = v_1 + v_2$ and since we already chose Λ^* in (3.4.13) we have:

$$\begin{aligned} u(t_{i+1}, x) &\leq v(t_{i+1}, x) \leq v_1(x) + v_2(t_{i+1}, x) \leq \Lambda \|v_1^*\|_{L^\infty(\Omega(t_i))} + \Lambda^* e^{(\lambda-\lambda_1)\tau_0} \\ &= \Lambda^* \left(\tfrac{\Lambda}{\Lambda^*} \|v_1^*\|_{L^\infty(\Omega(t_i))} + e^{(\lambda-\lambda_1)\tau_0} \right) \leq \Lambda^*, \quad x \in \Omega_\delta(t_i). \end{aligned}$$

This proves (3.4.8).

Thus, from (3.4.7) and (3.4.8) and since $\Lambda \leq \Lambda^*$, we have that $u(t_i, x) \leq \Lambda^*$ for all $x \in \Omega$ and all $i \geq 2$.

To bound the solution between t_i and t_{i+1} we use that if $t \in [t_i, t_{i+1}]$, $u(t,x) \leq U(t - t_i, x)$ where U solves

$$\begin{cases} U_t - \Delta U = \lambda U, & x \in \Omega,\ t \in (0, \infty) \\ U = 0, & x \in \partial\Omega,\ t \in (0, \infty) \\ U(0, x) = \Lambda^*, & x \in \Omega. \end{cases}$$

Hence $u(t,x) \leq \sup_{s \in [0, \tau_0]} \|U(s, \cdot)\|_{L^\infty} \leq e^{\lambda \tau_0} \Lambda^*$. This concludes the proof of the Theorem. □

Remark 3.4.11. *Conditions (3.4.6) can be easily met, for example, whenever $K(t) = \gamma(t) + R(t)K_0$, where $\gamma(t)$ is a $\mathcal{C}^1$ curve in Ω with $\sup_{t \geq t_0} |\gamma'(t)| < \infty$, $R(t)$ is a rigid motion for all $t \geq t_0$ and is continuous with respect to $t \geq t_0$ with the condition that $\gamma(t) + R(t)K_0 \subset \Omega\ \forall t \geq t_0$, and K_0 is a compact subset. If $\lambda < \lambda_0(K_0)$ and K_0 can be seen as the intersection of a family of nested open neighbourhoods of K_0, then there exists an open neighbourhood Ω_δ such that $\lambda < \lambda_1(\Omega_\delta)$ and $d(\partial\Omega_\delta, K_0) > 2\delta$ for some $\delta > 0$.*

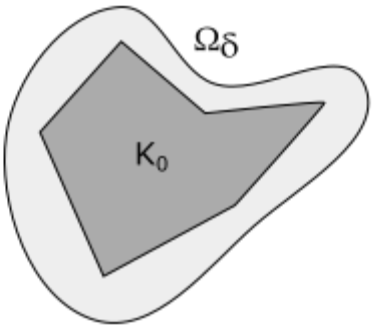


Figure 3.13: Static K_0 and Ω_δ.

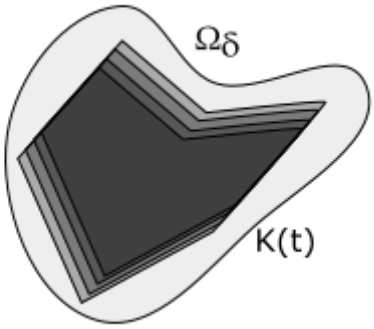


Figure 3.14: Moving $K(t)$ within Ω_δ.

Since $K(t)$ moves continuously, for any positive distance to the boundary of Ω_δ, let us consider δ, there exists a time τ such that

$$\bigcup_{[t-\tau,t+\tau]} K(s) \subset \Omega_\delta(t), \qquad \text{and} \qquad d\left(\bigcup_{[t-\tau,t+\tau]} K(s), \partial\Omega_\delta(t)\right) \geq \delta.$$

Finally, since there is a maximum speed at which it travels, there is a minimum time τ_0 which verifies all the above.

Let us prove now the auxiliary Lemma used in the proof of Theorem 3.4.9:

Lemma 3.4.12. *Let $\lambda \in \mathbb{R}$ and $U \subset \mathbb{R}^N$ a bounded smooth domain. Let v_1^* be the weak solution of problem (3.4.12). If the first eigenvalue of $-\Delta - \lambda I$ with Diritchlet boundary conditions is positive, that is,*

$$\lambda_1^* := \lambda_1(U) - \lambda > 0,$$

then

$$\|v_1^*\|_{L^\infty(U)} \leq C_M,$$

for a constant $C_M = C_M(|U|, N, \lambda, \lambda_1^*)$.

Proof. Let us define $\phi_k = (v_1^* - k)^+$ for $k \geq 1$ and $A_k := \{x \in U : v_1^*(x) > k\}$. Multiplying (3.4.12) by ϕ_k and integrating by parts leads us to

$$\int_U |\nabla \phi_k|^2 - \int_U \lambda |\phi_k|^2 = \int_U \lambda k \phi_k.$$

Since $\lambda_1^* > 0$, we have

$$\lambda_1^* \int_U |\phi_k|^2 \leq \int_U |\nabla \phi_k|^2 - \int_U \lambda |\phi_k|^2 = \int_U \lambda k \phi_k \quad \Rightarrow \quad \int_U |\phi_k|^2 \leq \frac{1}{\lambda_1^*} \int_U \lambda k \phi_k.$$

Hence,

$$\int_U |\nabla \phi_k|^2 \leq \left(1 + \frac{|\lambda|}{\lambda_1^*}\right) \left| \int_U \lambda k \phi_k \right|.$$

A sharpened Friedrich's inequality due to Mazja (see Corollary 4.11.2 in [30]) gives us the existence of a constant $C_M = C_M(|U|, N)$ satisfying

$$\|\phi_k\|^2_{L^{\frac{2N}{N-1}}(U)} \leq C_M \int_U |\nabla \phi_k|^2.$$

Therefore,

$$\|\phi_k\|^2_{L^{\frac{2N}{N-1}}(U)} \leq C_M \left(1 + \frac{|\lambda|}{\lambda_1^*}\right) \left| \int_U \lambda k \phi_k \right|.$$

However,

$$\left| \int_U \lambda k \phi_k \right| \leq |\lambda| k \int_{A_k} |\phi_k| \leq \lambda k |A_k|^{\frac{1}{2} + \frac{1}{2N}} \|\phi_k\|_{L^{\frac{2N}{N-1}}(A_k)},$$

where we have applied Hölder's inequality. Therefore,

$$\|\phi_k\|_{L^{\frac{2N}{N-1}}(U)} = \|\phi_k\|_{L^{\frac{2N}{N-1}}(A_k)} \leq C_M \left(1 + \frac{|\lambda|}{\lambda_1^*}\right) |\lambda| k |A_k|^{\frac{1}{2} + \frac{1}{2N}}.$$

Thus, for any $k \geq 1$ applying Hölder's inequality again

$$\|\phi_k\|_{L^1(A_k)} \leq \|\phi_k\|_{L^{\frac{2N}{N-1}}(A_k)} |A_k|^{\frac{1}{2} + \frac{1}{2N}} \leq C_M \left(1 + \frac{\lambda}{\lambda_1^*}\right) \lambda k |A_k|^{1 + \frac{1}{N}} \leq \gamma k |A_k|^{1 + \frac{1}{N}},$$

where $\gamma = C_M \left(1 + \frac{|\lambda|}{\lambda_1^*}\right) |\lambda|$. Applying Lemma 5.1 in [25] results in

$$\max_{x \in U} v_1^*(x) \leq C_M(|U|, N, \lambda, \lambda_1^*, \|v_1^*\|_{L^1(A_1)}).$$

Notice that, $v_1^* = \phi_1 + 1$, hence

$$\|v_1^*\|_{L^1(A_1)} \leq \|\phi_1\|_{L^1(A_1)} + |A_1| \leq |A_1| \left(\gamma |A_1|^{\frac{1}{N}} + 1\right).$$

Hence,

$$\max_{x \in U} v_1^*(x) \leq C_M(|U|, N, \lambda, \lambda_1^*).$$

□

Finally, as we mentioned above, we can consider the case where $K(t)$ touches $\partial\Omega$ with the aid of Proposition 1.4.3, that is

Corollary 3.4.13. *Let $R > 0$ such that $\overline{\Omega} \subset B(0, R)$. Let us assume n satisfies* **(N)** *and there exist $\tau_0 > 0, \delta > 0$ and a family of smooth bounded open sets $\{\Omega_\delta(t)\}_{t \geq t_0}$ with $\Omega_\delta(t) \subset B(0, R)$ such that for each $t \geq t_0 + \tau_0$, (3.4.6) in Theorem 3.4.9 is satisfied.*

Then, if $\lambda < \inf_{t \geq t_0} \lambda_1(\Omega_\delta(t))$, we have that all solutions of (1.0.1) are bounded.

Proof. If we define

$$\tilde{n}(t,x) := \begin{cases} n(t,x), & t \geq t_0,\ x \in \overline{\Omega}, \\ \nu(2R), & t \geq t_0,\ x \in B(0,R) \setminus \overline{\Omega}, \end{cases}$$

we have $\tilde{n}$ satiesfies **(N)** also with the same ν. Moreover, $P(\tilde{u}, u_0, \tilde{n}, B(0,R))$ is a supersolution of $P(u, u_0, n, \Omega)$ by Theorem 1.2.1.

But now we can apply Theorem 3.4.9 to $\tilde{u}$, obtaining boundedness for $\tilde{u}$ and therefore for u.

□

3.4.3 Boundedness for $K(t) \subset K_0$ most of the time and $\lambda < \lambda_0(K_0)$

In this subsection we consider the case where $K(t) \subset K_0$ for most of the time where K_0 is a fixed compact set with $\lambda < \lambda_0(K_0)$, but for other times we may have $K(t)$ large and therefore $\lambda > \lambda_0(K(t))$. That is, we will assume $K(t) \subset (\overline{\Omega}, K_0)$ (see Definition 3.4.2). We will regard this case as a perturbation of the case where $K(t) \subset K_0$ for all time. Therefore, let us recall some important ingredients of the autonomous case with $K(t) \subset K_0$ and $\lambda < \lambda_0(K_0)$:

Choosing $\delta > 0$ small enough and considering as usual the open set

$$\Omega_\delta := \{x \in \Omega : d(x, K_0) < \delta\}$$

we have that $K(t) \subset K_0 \subset \Omega_\delta \subset \overline{\Omega_\delta}$ and $\lambda < \lambda_0(\overline{\Omega_\delta})$.

If $n(t,x)$ satisfies **(N)** then with $\beta = \nu(\delta) > 0$ we have $n(t,x) \geq \beta \chi_{\Omega \setminus \overline{\Omega_\delta}}(x)$. This inequality is trivial if $x \in \overline{\Omega_\delta}$ since $n(t,x) \geq 0 = \beta \chi_{\Omega \setminus \overline{\Omega_\delta}}(x)\ \forall x \in \overline{\Omega_\delta}$. Moreover, if $x \in \Omega \setminus \overline{\Omega_\delta}$

$$n(t,x) \geq \nu(d(x, K(t))) \geq \nu(d(x, K_0)) \geq \nu(\delta) = \beta.$$

Standard comparison principles (Theorem 1.2.1) show that if $\tilde{u}$ solves $P(\tilde{u}, u_0, \tilde{n}, \Omega)$ where $\tilde{n}(x) = \beta \chi_{\Omega \setminus \overline{\Omega_\delta}}(x)$ then

$$0 \leq u(t,x;u_0) \leq \tilde{u}(t,x;u_0).$$

We can now prove:

Lemma 3.4.14. *With the hypothesis above, $\exists E > 0$ independent of initial data such that*

$$\limsup_{t \to \infty} \|u(t,\cdot;u_0)\|_{L^\infty} \leq E.$$

Proof. Since $0 \leq u(t,x;u_0) \leq \tilde{u}(t,x;u_0)$ and $\tilde{u}(t,x;u_0)$ solves $P(\tilde{u}, u_0, \tilde{n}, \Omega)$ which is an autonomous problem with $\lambda < \lambda_0(K_0)$ then from [7] we know that there exists a unique stationary solution $\tilde{\varphi}_\lambda$ which attracts all initial data in $L^\infty(\Omega)$. Taking $E = \|\tilde{\varphi}_\lambda\|_{L^\infty(\Omega)}$ we prove the result.

□

Remark 3.4.15. *With the notation above, constant E in the previous Lemma depends only on λ, ν, the chosen δ and K_0. Let us see this: let us assume we have two problems $P(u, u_0, n, \Omega)$ as before and problem $P(\hat{u}, u_0, \hat{n}, \Omega)$, where $\hat{n}(t,x)$ satisfies* **(N)** *for the same ν as $n(t,x)$, and where $\hat{K}(t) \subset K_0$ for all $t \geq t_0$ where the family of compact sets is defined as $\hat{K}(t) := \{x \in \Omega : \hat{n}(t,x) = 0\}$ for $t \geq t_0$.*

Then, as seen previously, for given λ and K_0 there exists a $\delta > 0$ and an open neighbourhood Ω_δ such that $K_0 \subset \Omega_\delta$ and $\lambda < \lambda_0(\overline{\Omega_\delta})$. Therefore, for $\beta = \nu(\delta)$ as before we have that

$$n(t,x) \geq \beta \chi_{\Omega \setminus \Omega_\delta}, \qquad \text{and} \qquad \hat{n}(t,x) \geq \beta \chi_{\Omega \setminus \Omega_\delta}.$$

By Theorem 1.2.1 we have that

$$u(t,x) \leq \tilde{u}(t,x), \qquad \text{and} \qquad \hat{u}(t,x) \leq \tilde{u}(t,x).$$

Hence, there exists $E = \|\varphi_\lambda\|_\infty > 0$ such that

$$\limsup_{t\to\infty} \|u(t,\cdot;u_0)\|_{L^\infty} \leq E, \qquad \text{and} \qquad \limsup_{t\to\infty} \|\hat{u}(t,\cdot;u_0)\|_{L^\infty} \leq E.$$

Let us consider now the configuration where $n(t,x)$ satisfies **(N)** and verifies that $K(t) \subset (\overline{\Omega}, K_0)$, that is, there exist $(T_{0,i})_{i\geq 1}, (T_{1,i})_{i\geq 1}$ with $T_{0,i} < T_{1,i} < T_{0,i+1} < T_{1,i+1} \to \infty$ where

$$K(t) \subset K_0, \qquad t \in (T_{1,i}, T_{0,i+1}]$$

with $|T_{0,i+1} - T_{1,i}| \geq \eta$ and $|T_{1,i} - T_{0,i}| \leq \Xi$. Let us recall Figure 3.6 where for this case we consider $K_1 = \overline{\Omega}$:

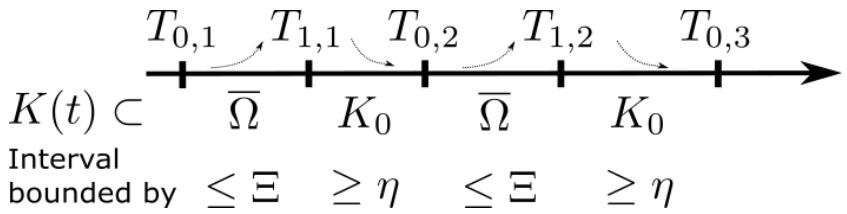


Figure 3.15: This figure illustrates how the sequences $(T_{0,i})_{i\geq 1}$ and $(T_{1,i})_{i\geq 1}$ in Definition 3.4.2 define the growth and shrink regimes.

Before we present the main result of this subsection, The following result shows that there exists a supersolution where the intervals $(T_{1,i}, T_{0,i+1}]$ are bounded without any additional assumptions on $(T_{0,i})_{i\geq 1}$ or $(T_{1,i})_{i\geq 1}$.

Lemma 3.4.16. *Let $A, B \subset \overline{\Omega}$ be two compact subsets and u be the solution of problem $P(u, u_0, n, \Omega)$ where $K(t) \subset (A,B)$ for sequences $(T_{0,i})_{i\in\mathbb{N}}, (T_{1,i})_{i\in\mathbb{N}}$. Then, there exists a supersolution $\bar{u}$ solution of $P(\bar{u}, \bar{u}_0, \bar{n}, \Omega)$ where $\bar{K}(t) := \{x : \bar{n}(t,x) = 0\}$ and $\bar{K}(t) \subset (A,B)$ for sequences $\left(\bar{T}_{0,i}\right)_{i\in\mathbb{N}}, \left(\bar{T}_{1,i}\right)_{i\in\mathbb{N}}$ and such that those sequences satisfy:*

$$\sup_{i\geq 1} \left|\bar{T}_{0,i+1} - \bar{T}_{1,i}\right| < \infty.$$

Proof. If $\sup_{i\geq 1} |T_{0,i+1} - T_{1,i}| < \infty$ we just take $\bar{n} \equiv n$ and we are finished. Let us assume $\sup_{i\geq 1} |T_{0,i+1} - T_{1,i}| = \infty$. Then, there exists a subsequence $i_j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \left|T_{0,i_j+1} - T_{1,i_j}\right| = \infty.$$

Let us define $J \subset \mathbb{N}$ as the indices such that

$$|T_{0,i+1} - T_{1,i}| > \Xi + 2\eta, \quad \forall i \in J.$$

Moreover, let us denote by m_i the number of times we can divide interval $(T_{1,i}, T_{0,i+1}]$ into subintervals of length $\Xi/2 + \eta$:

$$m_i := \left\lfloor \frac{|T_{0,i+1} - T_{1,i}|}{\frac{\Xi}{2} + \eta} \right\rfloor, \quad \forall i \in J.$$

Observe that $m_i \geq 2$ for $i \in J$. Then, we define, for $i \notin J$ $\bar{T}_{0,i,1} := T_{0,i}$ and $\bar{T}_{1,i,1} := T_{1,i}$ and for $i \in J$ and $1 \leq j \leq m_i - 1$, the sequences

$$\begin{cases} \bar{T}_{0,i,1} := T_{1,i} + \eta, \\ \bar{T}_{0,i,j+1} := \bar{T}_{1,i,j} + \eta, \end{cases} \qquad \begin{cases} \bar{T}_{1,i,j} := \bar{T}_{0,i,j} + \Xi/2, \\ \bar{T}_{1,i,m_i} := T_{0,i+1}. \end{cases}$$

Observe that the last subinterval has a slightly bigger upper bound on its length:

$$\left|\bar{T}_{1,i,m_i} - \bar{T}_{0,i,m_i}\right| \leq \Xi + \eta.$$

Let us define $\bar{n}$:

$$\bar{n}(t,x) := \begin{cases} 0, & \forall t \in (\bar{T}_{0,i,j}, \bar{T}_{1,i,j}],\ i \in J,\ 1 \leq j \leq m_i,\ x \in \overline{\Omega}, \\ n(t,x), & \text{elsewhere}, \end{cases}$$

which implies that $\bar{n}(t,x) \leq n(t,x)$ for all $t \geq t_0$ and all $x \in \overline{\Omega}$ and that $\bar{u}$ is a supersolution of u applying Theorem 1.2.1.

Now, observe that sequences $\left(\bar{T}_{0,i,j}\right)_{i\in\mathbb{N}}, \left(\bar{T}_{1,i,j}\right)_{i\in\mathbb{N}}$ satisfy the following:

- For every $i \notin J$,
$$|\bar{T}_{1,i,1} - \bar{T}_{0,i,1}| = |T_{1,i} - T_{0,i}| \leq \Xi,$$
and
$$\eta \leq |\bar{T}_{0,i+1,1} - \bar{T}_{1,i,1}| = |T_{0,i+1} - T_{1,i}| \leq \Xi + 2\eta.$$
- For every $i \in J$ and for $1 \leq j \leq m_i - 1$
$$|\bar{T}_{1,i,j} - \bar{T}_{0,i,j}| = \Xi/2, \quad \text{and} \quad |\bar{T}_{1,i,m_i} - \bar{T}_{0,i,m_i}| \leq \Xi + \eta,$$
and
$$|\bar{T}_{0,i+1,1} - \bar{T}_{1,i,1}| = \eta.$$

Therefore, $\bar{K}(t) \subset (A, B)$ and, renumbering the indices in the times sequences

$$\lim_{i\to\infty} \left|\bar{T}_{0,i} - \bar{T}_{1,i}\right| < \infty.$$

□

We can now prove the following result:

The following result shows that there exists a supersolution where the intervals $(T_{1,i}, T_{0,i+1}]$ are bounded without any additional assumptions.

Proposition 3.4.17. *Let $K(t) \subset (\overline{\Omega}, K_0)$. With the notation above and with $\lambda < \lambda_0(K_0)$, for fixed $\eta > 0$ there exists $\Xi_0 > 0$ such that if $0 \leq \Xi \leq \Xi_0$, then all solutions of (1.0.1) are bounded.*

Proof. As we have defined above before the statement of Lemma 3.4.14, let $\tilde{n}(t,x) = \beta\chi_{\Omega\setminus\overline{\Omega_\delta}}(x)$ for $t \geq t_0$, $x \in \overline{\Omega}$ and $\delta > 0$ small enough. For $i \geq 1$ let $\hat{u}_i(t,x;T_{0,i},u_0)$ be the solution starting at $T_{0,i}$ of problem $P(\hat{u}_i,u_0,\hat{n},\Omega)$ where we may assume without loss of generality that $T_{0,1} = t_0 \geq 0$ and for every $i \geq 1$ we define

$$\hat{n}(t,\cdot) := \begin{cases} n(t,\cdot), & t \in (T_{0,i},T_{1,i}], \\ \tilde{n}(t,\cdot), & t \in (T_{1,i},T_{0,i+1}]. \end{cases}$$

Since $\tilde{n}(t,x) \leq n(t,x)$ then $\hat{n}(t,x) \leq n(t,x)$ for every $t \geq t_0$ and $x \in \overline{\Omega}$. Therefore, by Theorem 1.2.1, $u(t,x;T_{0,i},u(T_{0,i},\cdot;t_0,u_0)) \leq \hat{u}_i(t,x;T_{0,i},u(T_{0,i},\cdot;t_0,u_0))$ for $t \in [T_{0,i},T_{0,i+1}]$ which implies that if $\hat{u}_i$ is bounded by a certain constant for $t \in [T_{0,i},T_{0,i+1}]$, so is u. Hence, we will focus on the boundedness of $\hat{u}_i$. In order to do so, we will compare $\hat{u}_i$ with $\tilde{u}$ and estimate the distance between them.

Observe that from Lemma 3.4.14 there exists $E = \|\tilde{\varphi}\|_\infty > 0$ such that

$$\limsup_{t\to\infty} \|\tilde{u}(t,\cdot;u_0)\|_{L^\infty} \leq E. \tag{3.4.16}$$

Let us fix our initial condition as $u_0(x) = 3E > 0$. Therefore, there exists a time $\hat{T} > T_{0,1}$ such that

$$\|\tilde{u}(t,\cdot;u_0)\|_{L^\infty(\Omega)} \leq 2E,\ t \geq \hat{T}.$$

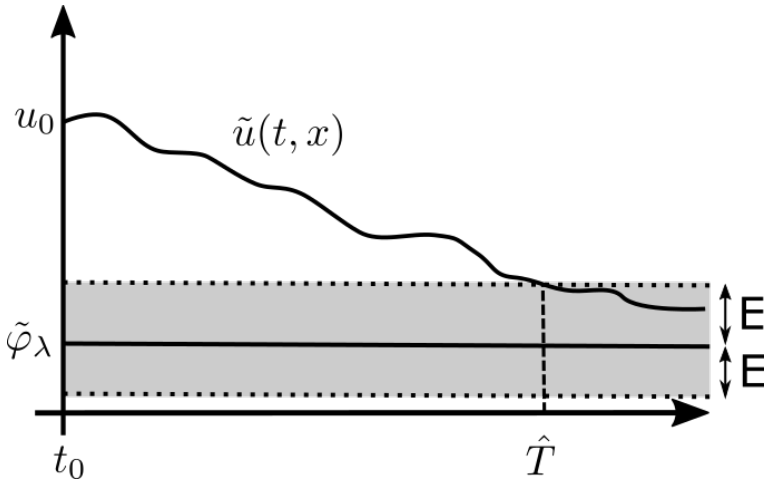


Figure 3.16: Solution $\tilde{u}$ enters an E–neighbourhood of $\tilde{\varphi}_\lambda$ in $L^\infty(\Omega)$ at time $\hat{T}$.

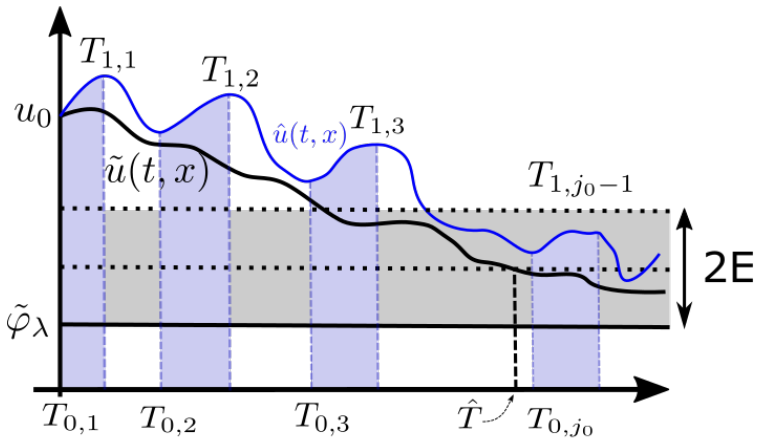


Figure 3.17: Solution $\hat{u}$ stays at a distance less than E of solution $\tilde{u}$ at time T_{0,j_0} in $L^\infty(\Omega)$. Shadowed in blue are the growth regimes, where solution $\hat{u}$ may grow for a time at most $\Xi > 0$.

Let us consider $\hat{u}_1$ first. Let $j_0 \in \mathbb{N}$ be the smallest $j \geq 2$ such that $T_{0,j} \geq \hat{T}$ and such that $j \geq \frac{\hat{T}}{\eta}$. Therefore, for $k \geq 1$ it is verified that $T_{0,(k+1)j_0} - T_{0,kj_0} \geq j_0\eta \geq \hat{T}$. Which implies that $T_{0,(k+1)j_0} \geq \hat{T} + T_{0,kj_0}$. Then,

$$\begin{aligned}
\|\tilde{n} - \hat{n}\|_{L^1((T_{0,1},T_{0,j_0})\times\Omega)} &= \int_{T_{0,1}}^{T_{0,j_0}} \int_\Omega |\tilde{n}(t,x) - \hat{n}(t,x)| dx dt \\
&= \sum_{k=1}^{j_0-1} \int_{T_{0,k}}^{T_{1,k}} \int_\Omega |\tilde{n}(t,x) - n(t,x)| dx dt \leq 2\|n\|_{L^\infty} |\Omega| \sum_{k=1}^{j_0-1} \int_{T_{0,k}}^{T_{1,k}} dt \\
&\leq 2j_0 \Xi |\Omega| \|n\|_\infty.
\end{aligned}$$

Therefore, by Proposition 1.4.1 we have for $t \in [T_{0,1}, T_{0,j_0}]$ and any $p > \frac{N}{2} + 1$

$$\|\tilde{u}(t,\cdot;u_0) - \hat{u}_1(t,\cdot;u_0)\|_{L^\infty(\Omega)} \leq C\left(2j_0\Xi\,|\Omega|\,\|n\|_\infty\right)^{1/p},$$

where $C = C(T_{0,j_0}, \lambda, R, d, \rho, \Omega, M)$, with $\|u_0\|_{L^\infty(\Omega)} \leq R$ and $\|n\|_{L^\infty(\Omega)} \leq d$. Thus, for $t \in [\hat{T} + T_{0,1}, T_{0,j_0}]$

$$\begin{aligned}
\|\hat{u}_1(t,\cdot;u_0)\|_{L^\infty} &\leq \|\tilde{u}(t,\cdot;u_0)\|_{L^\infty} + \|\tilde{u}(t,\cdot;u_0) - \hat{u}_1(t,\cdot;u_0)\|_{L^\infty} \\
&\leq 2E + C\left(2j_0\Xi\,|\Omega|\,\|n\|_\infty\right)^{1/p},
\end{aligned}$$

and, in particular, $\|\hat{u}_1(T_{0,j_0},\cdot;u_0)\|_{L^\infty} \leq 2E + C\left(2j_0\Xi\,|\Omega|\,\|n\|_\infty\right)^{1/p}$. Then, there exists $\Xi_0 = \Xi_0(E, j_0, |\Omega|, C, \|n\|_{L^\infty}, p)$ small such that

$$C\left(2j_0\Xi\,|\Omega|\,\|n\|_\infty\right)^{1/p} < E, \quad \forall 0 < \Xi \leq \Xi_0$$

and thus

$$\|u(T_{0,j_0},\cdot;u_0)\|_{L^\infty} \le \|\hat{u}_1(T_{0,j_0},\cdot;u_0)\|_{L^\infty} < 3E < u_0.$$

Therefore, if solution u starts at $u(T_{0,1},\cdot) \le u_0$ then at time $T_{0,2}$ it stays below u_0. We can iterate this process for every interval $[T_{0,j_0k}, T_{0,j_0(k+1)}]$: let us consider solutions $\hat{u}_{j_0}(t,\cdot;T_{0,j_0},u_0)$ and $\tilde{u}(t,\cdot;T_{0,j_0},u_0)$ which both start in u_0 at time T_{0,j_0}. Then,

$$\|\tilde{u}(t,\cdot;T_{0,j_0},u_0)\|_{L^\infty(\Omega)} \le 2E,\ t \ge \hat{T} + T_{0,j_0}$$

and, in particular, since j_0 was chosen so that $j_0 \ge \frac{\hat{T}}{\eta}$ we have that $T_{0,2j_0} \ge \hat{T} + T_{0,j_0}$. Thus,

$$\|\tilde{u}(t,\cdot;T_{0,j_0},u_0)\|_{L^\infty(\Omega)} \le 2E,\ t \ge T_{0,2j_0}.$$

Since we have proven already that $u(T_{0,j_0},\cdot) \le u_0$ by comparison (Theorem 1.2.1) we have that $u(t,x) \le \hat{u}_{j_0}(t,x)$ for $t \in [T_{0,j_0}, T_{0,2j_0}]$ and a.e. $x \in \Omega$. Replicating computations above, by Proposition 1.4.1 we have for $t \in [T_{0,j_0}, T_{0,2j_0}]$ and any $p > \frac{N}{2} + 1$

$$\|\tilde{u}(t,\cdot;u_0) - \hat{u}_{j_0}(t,\cdot;u_0)\|_{L^\infty(\Omega)} \le C\left(2j_0\Xi\,|\Omega|\,\|n\|_\infty\right)^{1/p} < E,$$

and, therefore,

$$\|\hat{u}_{j_0}(T_{0,2j_0},\cdot;u_0)\|_{L^\infty} \le \|\tilde{u}(T_{0,2j_0},\cdot;u_0)\|_{L^\infty} + \|\tilde{u}(T_{0,2j_0},\cdot;u_0) - \hat{u}_{j_0}(T_{0,2j_0},\cdot;u_0)\|_{L^\infty} < 3E.$$

Thus, $\|u(T_{0,2j_0},\cdot)\|_{L^\infty(\Omega)} \le \|\hat{u}_{j_0}(T_{0,2j_0},\cdot)\|_{L^\infty(\Omega)} \le 3E = u_0$. This computations can be iterate for every $k \ge 1$ for solutions $\tilde{u}(t,\cdot;T_{0,kj_0},u_0)$ and $\hat{u}_{kj_0}(t,\cdot;T_{0,kj_0},u_0)$. Therefore, by induction, we arrive at

$$\|u(T_{0,kj_0},\cdot)\|_{L^\infty(\Omega)} \le 3E, \quad \forall k \ge 1.$$

In order to prove that $\|u(t,\cdot)\|_{L^\infty(\Omega)}$ is bounded by some constant for every $t \ge t_0$ we can assume due to Lemma 3.4.16 without loss of generality that

$$\sup_{i\in\mathbb{N}} |T_{0,i+1} - T_{1,i}| \le A, \text{ or, equivalently since } |T_{1,i} - T_{0,i}| \le \Xi, \quad \sup_{i\in\mathbb{N}} |T_{0,i+1} - T_{0,i}| \le A$$

for some $A > 0$. Finally, since

$$\|u(T_{0,kj_0},\cdot)\|_{L^\infty(\Omega)} \le 3E, \quad \forall k \ge 1,$$

we have by (1.2.6)

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le 3EMe^{(\lambda-\lambda_1(\Omega))(\Xi+A)}, \quad \forall t \in [T_{0,kj_0}, T_{0,(k+1)j_0}], \forall k \ge 1.$$

Since this bound is independent of t or k, we have

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le 3EMe^{(\lambda-\lambda_1(\Omega))(\Xi+A)}, \quad \forall t \ge t_0,$$

which concludes the proof.

□

3.4.4 Boundedness for $K(t) \subset K_0$ for some time and $|K_0| << 1$

In this subsection, we consider that for some time, probably small, $K(t) \subset K_0$ and $|K_0| << 1$. That is, we consider $K(t) \subset (\overline{\Omega}, K_0)$ and obtain boundedness results when $|K_0| << 1$. Hence, in opposition to the previous subsection, we will not impose conditions on time (on Ξ specifically) but rather on K_0. Or in other words, since $K(t) \subset (\overline{\Omega}, K_0)$ then $K(t)$ has a growth regime for some times and a shrink regime for others. Instead of imposing a restriction on the time $K(t)$ stays in the growth regime, in this section we obtain boundedness results assuming conditions on how much does $K(t)$ shrink during its shrinking regime.

Therefore since $K(t) \subset (\overline{\Omega}, K_0)$, from Definition 3.4.2 we have times $T_{0,i} < T_{1,i} < T_{0,i+1} < \dots$ such that $K(t) \subset K_0$ for $t \in (T_{1,i}, T_{0,i+1}]$ and no restriction on $K(t)$ for times $t \in (T_{0,i}, T_{1,i}]$ for $i \geq 1$.

Observe that in Definition 3.4.2 there is no upper bound on η, which means that there are no assumptions on how long can $K(t)$ stay in a shrinking regime, which is natural given that we are seeking for boundedness results. For technical reasons we will need to consider the times in the shrinking regime to be bounded, that is $\sup_{i\geq 1} |T_{0,i+1} - T_{1,i}| = A < \infty$ for some $A > 0$. This can be assumed using Lemma 3.4.16. Moreover, we will require that $n(t,x) \geq \nu_0 \chi_{\Omega\setminus K_0}(x)$ for $t \in (T_{1,i}, T_{0,i+1}]$ and any $x \in \overline{\Omega}$ where $\nu_0 > 0$ is independent of time.

Proposition 3.4.18. *Let $K(t) \subset (\overline{\Omega}, K_0)$ and assume n satisfies* **(N)**. *With the notation above, there exists $\varepsilon_0 > 0$ such that if $|K_0| < \varepsilon_0$ then all solutions of (1.0.1) are bounded.*

Proof. Let us consider for every $i \geq 1$ $\hat{u}_i(t, x; T_{0,i}, w_0)$ the solution of problem $P(\hat{u}_i, w_0, \hat{n}, \Omega)$ starting at $\hat{u}_i(T_{0,i}) = w_0$ and where $\hat{n}$ is defined for every $i \geq 1$ as follows:

$$\hat{n}(t,\cdot) := \begin{cases} n(t,\cdot), & t \in (T_{0,i}, T_{1,i}], \\ n(t,\cdot) + \nu_0 \chi_{K_0}(\cdot), & t \in (T_{1,i}, T_{0,i+1}]. \end{cases}$$

Observe that since $n(t,x) + \nu_0 \chi_{K_0}(x) \geq \nu_0 > 0$ for any $x \in \overline{\Omega}$, $\hat{n}$ satisfies that $\hat{K}(t) \subset (\Omega, \emptyset)$ and therefore, from the previous Proposition 3.4.3, we know that there exists $W_\infty(\eta)$ such that for arbitrary initial condition w_0

$$\|\hat{u}_i(T_{0,i+1}, \cdot; T_{0,i}, w_0)\|_{L^\infty(\Omega)} \leq W_\infty(\eta), \ \forall i \geq 1.$$

Let us consider $\hat{u}_i$ in $t \in [T_{0,i}, T_{0,i+1}]$ for $i \geq 1$, then we have that

$$\begin{aligned} \|n - \hat{n}\|_{L^1((T_{0,i}, T_{0,i+1})\times\Omega)} &= \int_{T_{0,i}}^{T_{0,i+1}} \int_\Omega |n(t,x) - \hat{n}(t,x)| dx dt = \int_{T_{1,i}}^{T_{0,i+1}} \int_\Omega \nu_0 \chi_{K_0}(x) dx dt \\ &\leq \nu_0 |K_0| |T_{0,i+1} - T_{1,i}| \leq \nu_0 |K_0| \sup_{i\geq 1} |T_{0,i+1} - T_{1,i}| \leq \nu_0 |K_0| A, \end{aligned}$$

where $A > 0$ is given by Lemma 3.4.16. Therefore, by Proposition 1.4.1, for $t \in [T_{0,i}, T_{0,i+1}]$ and any $p > \frac{N}{2} + 1$

$$\|u(t,\cdot;T_{0,i},w_0)-\hat{u}_i(t,\cdot;T_{0,i},w_0)\|_{L^\infty(\Omega)}\leq C\left(\nu_0|K_0|A\right)^{1/p},$$

where $C=C(\Xi+A,\lambda,R,N_0,\rho,\Omega,M)$ with $R\geq\|w_0\|_{L^\infty(\Omega)}$ and $N_0\geq\|n\|_{L^\infty(\Omega)}$. Thus,

$$\begin{aligned}\|u(T_{0,i+1},\cdot;T_{0,i},w_0)\|_{L^\infty} &\leq \|\hat{u}_i(T_{0,i+1},\cdot;T_{0,i},w_0)\|_{L^\infty(\Omega)}\\ &\quad+\|u(T_{0,i+1},\cdot;T_{0,i},w_0)-\hat{u}_i(T_{0,i+1},\cdot;T_{0,i},w_0)\|_{L^\infty(\Omega)}\\ &\leq W_\infty(\eta)+C\left(\nu_0|K_0|A\right)^{1/p}.\end{aligned}$$

Hence, let us choose as our initial condition $w_0\equiv W_\infty(\eta)+1$ which fixes C above. Moreover, let $0<\varepsilon\leq\varepsilon_0$ where $\varepsilon_0>0$ verifies $C(\nu_0\varepsilon_0 A)^{1/p}<1/2$. Then, we have that for any $0<\varepsilon\leq\varepsilon_0$

$$\|u(T_{0,i+1},\cdot;T_{0,i},w_0)\|_{L^\infty}\leq W_\infty(\eta)+C\left(\nu_0|K_0|A\right)^{1/p}\leq W_\infty(\eta)+\tfrac{1}{2}<W_\infty(\eta)+1=w_0.$$

This implies that starting in w_0 at time $T_{0,i}$ the solution $u(t,x)$ stays below w_0 at time $T_{0,i+1}$ for $i\geq 1$. Now, observe that from (1.2.6) we have $\|u(t,\cdot)\|_{L^\infty}\leq Me^{(\lambda-\lambda_1(\Omega))(\Xi+A)}(w_0)$ for every $t\in[T_{0,i},T_{0,i+1}]$. Therefore,

$$\|u(t,\cdot)\|_{L^\infty(\Omega)}\leq Me^{(\lambda-\lambda_1(\Omega))(\Xi+A)}(W_\infty(\eta)+1),\quad\forall t\geq T_{0,1},\text{ a.e. }x\in\Omega,$$

and since from Proposition 3.2.2 we have that boundedness is independent of the initial time and initial data, we conclude the proof.

□

3.4.5 Boundedness for a jumping $K(t)$

This final boundedness result characterises boundedness with a discontinuity in $K(t)$ independently of the size of λ. Hence, it is a strong result in this research since it does not consider the linear growth provided by $K(t)$ and its relationship with λ: it only focuses on the motion of $K(t)$.

As a prototype of the situation we would like to address, we consider Example 3.3.11 in order to illustrate the discontinuity in $K(t)$ or "jumps" we are referring to. See Figure 3.4.5 below.

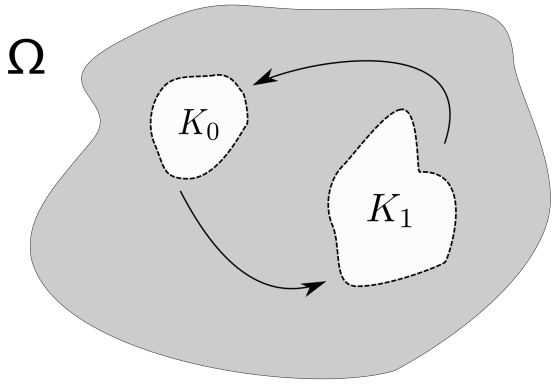


Figure 3.18: Domain Ω and a jumping $K(t)$ between compact subsets K_0 and K_1.

That is, by "jumps" or discontinuity in $K(t)$ we refer to a sequence of times $(t_j)_{j\in\mathbb{N}}$ with $t_1<t_2<t_3<\ldots\to\infty$ where the moving set $K(t)$ jumps at each t_j. As a matter of fact we have:

Theorem 3.4.19. *Let u be the solution of (1.0.1), $\lambda \in \mathbb{R}$ and let n satisfying* **(N)**. *Moreover, let us assume there exists an unbounded increasing sequence $(t_j)_{j\in\mathbb{N}}$ and $\tau_0, \delta, \Xi > 0$ such that*

$$\sup_{j\in\mathbb{N}} |t_{j+1} - t_j| = \Xi < \infty\,, \qquad d\left(\bigcup_{(t_j-\tau_0,t_j]} K(s), \bigcup_{(t_j,t_j+\tau_0)} K(s)\right) > \delta. \tag{3.4.17}$$

Then, all solutions of (1.0.1) are bounded.

Proof. In this proof we will use that $K(t)$ is bounded away from the boundary of Ω. Corollary 3.4.20 below, extends this result even when $K(t) \cap \partial\Omega \neq \emptyset$. Moreover, we can assume without loss of generality that $t_1 \geq t_0 + \tau_0$ from Proposition 3.2.2.

For each $j \in \mathbb{N}$ let $\Omega^+_\delta(t_j)$ be an open set with

$$\Omega^+_{\delta/3}(t_j) := \left\{ x \in \Omega : d\left(x, \bigcup_{(t_j,t_j+\tau_0)} K(s)\right) < \frac{\delta}{3} \right\} \supset \bigcup_{(t_j,t_j+\tau_0)} K(s),$$

which satisfies

$$d\left(\partial\Omega^+_\delta(t_j), \bigcup_{(t_j,t_j+\tau_0)} K(s)\right) = \delta/3, \tag{3.4.18}$$

and from (3.4.17) and (3.4.18) we have

$$d\left(\Omega^+_\delta(t_j), \bigcup_{(t_j-\tau_0,t_j]} K(s)\right) > \delta/3. \tag{3.4.19}$$

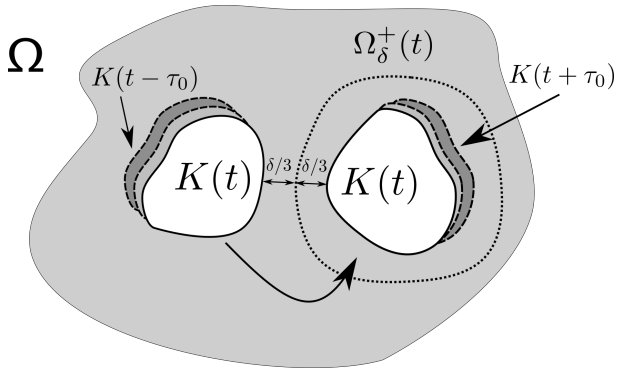


Figure 3.19: Neighbourhood $\Omega^+_\delta(t_j)$ stays at least at a distance $\delta/3$ from $\bigcup_{s\in(t_j-\tau_0,t_j]} K(s)$ and its boundary stays at least at the same distance from $\bigcup_{s\in(t_j,t_k+\tau_0,t_j)} K(s)$.

Let $j \in \mathbb{N}$, since (3.4.19) holds, operating as in Theorem 3.4.9 and applying Corollary 3.4.8 to the interval $(t_j - \tau_0, t_j)$ and the subset $\Omega^+_{\delta/3}(t_j)$ there exists a constant $\Lambda_1 = \Lambda_1(\lambda, \rho, \tau_0, \nu, \delta)$ such that

$$u(t_j, \cdot) \leq \Lambda_1, \qquad \text{for a.e. } x \in \Omega^+_\delta(t_j).$$

Applying again Corollary 3.4.8 to the interval $(t_j - \tau_0, t_j + \tau_0)$ and the subset $\overline{\Omega \setminus \Omega^+_{\delta/3}(t_j)}$ there exists a constant $\Lambda_1^* = \Lambda_1^*(\lambda, \rho, \tau_0, \nu, \delta)$ such that

$$u(t, \cdot) \leq \Lambda_1^*, \qquad \forall t \in [t_j, t_{j+1}],\ x \in \partial\Omega^+_\delta(t_j),$$

Let us take $\Lambda = \max\{\Lambda_1, \Lambda_1^*\}$. Observe that Λ does not depend on the inital data at $t = t_0$. Let us consider the following problem

$$\begin{cases} v_t - \Delta v = \lambda v, & x \in \Omega^+_\delta(t_j),\ t \in (t_j, t_j + \tau_0) \\ v = \Lambda, & x \in \partial\Omega^+_\delta(t_j),\ t \in (t_j, t_j + \tau_0) \\ v(t_j, x) = \Lambda, & x \in \Omega^+_\delta(t_j). \end{cases} \tag{3.4.20}$$

Hence, by Theorem 1.2.1, we have that $u(t, \cdot) \leq v(t, \cdot)$ for $t \in (t_j, t_j + \tau_0)$ and a.e. $x \in \Omega^+_\delta(t_j)$. As in Theorem 3.4.9, we can split $v = v_1 + v_2$ where

$$\begin{cases} -\Delta v_1 = \lambda v_1, & x \in \Omega^+_\delta(t_j), \\ v_1 = M_1, & x \in \partial\Omega^+_\delta(t_j), \end{cases}$$

$$\begin{cases} v_{2_t} - \Delta v_2 = \lambda v_2, & x \in \Omega^+_\delta(t_j),\ t \in (t_j, t_j + \tau_0), \\ v_2 = 0, & x \in \partial\Omega^+_\delta(t_j),\ t \in (t_j, t_j + \tau_0), \\ v_2(t_1, x) \leq M_1, & x \in \Omega^+_\delta(t_j), \end{cases}$$

and $v_1 = M_1 v_1^*$ where v_1^* is as in Lemma 3.4.12 and $v_2(t, \cdot) \leq M_1 M e^{(\lambda - \lambda_1)(t - t_j)} \leq M_1 M e^{\lambda \tau_0}$ for $t \in [t_j, t_j + \tau_0]$. Thus, there exists $M_2 = M_2(\lambda, N, |\Omega|, \nu, \rho, \tau_0, \delta)$, $M_2 \geq M_1$ such that for every $t \in [t_j, t_j + \tau_0]$ and a.e. $x \in \Omega^+_\delta(t_j)$:

$$u(t, x) \leq v(t, x) \leq v_1(t, x) + v_2(t, x) \leq M_1 \|v_1^*\|_{L^\infty(\Omega(t_1))} + M_1 e^{(\lambda - \lambda_1)\tau_0} =: M_2.$$

Therefore, we have obtained a bound M_2 for u in $\Omega^+_\delta(t_j)$ for any $j \in \mathbb{N}$. Let us find a bound for $u(t_j + \tau_0, \cdot)$ in $\Omega \setminus \Omega^+_\delta(t_j)$. Let $\mathcal{U}$ be an open $\delta/6$–neighbourhood of $\Omega \setminus \Omega^+_\delta(t_j)$, then, by Corollary 3.4.8 there exists $M_3 = M_3(\lambda, \rho, \tau_0, \nu, \delta)$ such that

$$u(t_j + \tau_0, x) \leq M_3, \quad x \in \mathcal{U}.$$

Therefore, $u(t_j + \tau_0, \cdot)$ is bounded in Ω by constant $M_0 := \max\{M_2, M_3\}$. For every $j \geq 1$ we have that

$$\|u(t_j + \tau_0, \cdot)\|_{L^\infty(\Omega)} \leq M_0.$$

Now, let us denote by U the solution of the linear problem $P(U, u(t_j + \tau_0, \cdot), 0, \Omega)$. Then, by Theorem 1.2.1

$$u(t, \cdot) \leq U(t, \cdot) \leq \|u(t_j + \tau_0, \cdot)\|_{L^\infty(\Omega)} e^{\lambda \Xi} \leq M_0 e^{\lambda \Xi}, \qquad t \in (t_j + \tau_0, t_{j+1} + \tau_0],\ \text{a.e. } x \in \Omega.$$

Which implies that u is always bounded by $M_0 e^{\lambda \Xi}$ since we chose an arbitrary $j \in \mathbb{N}$.

□

Corollary 3.4.20. *Let $R > 0$ such that $\overline{\Omega} \subset B(0, R)$. Then, assuming the same conditions as Theorem 3.4.19, we have that all solutions of (1.0.1) are bounded.*

Proof. If we define

$$\tilde{n}(t,x) := \begin{cases} n(t,x), & t \geq t_0,\ x \in \overline{\Omega}, \\ \nu(2R), & t \geq t_0,\ x \in B(0,R) \setminus \overline{\Omega}, \end{cases}$$

we have $\tilde{n}$ satiesfies **(N)** also with the same ν. Moreover, $P(\tilde{u}, u_0, \tilde{n}, B(0,R))$ is a supersolution of $P(u, u_0, n, \Omega)$ by Theorem 1.2.1.

But now we can apply Theorem 3.4.19 to $\tilde{u}$, obtaining boundedness for $\tilde{u}$ and therefore for u. □

3.5 Unboundedness

This section is devoted to obtaining conditions under which solutions grow up, that is, under which solutions become unbounded as $t \to \infty$.

Observe that we have proven before (Section 3.4.2) that if, roughly speaking, $\lambda < \lambda_0(K(t))$ $\forall t \geq t_0$ then we have boundedness of solutions. The equivalent situation for unboundedness would be to show that if $\lambda > \lambda_0(K(t))$ $\forall t \geq t_0$, the growth given by the linear part is strong enough to produce solutions which are unbounded as $t \to \infty$. But the previous section has also provided us with a clear counterexample to this hypothetical result.

Actually we may have $K_0, K_1 \neq \emptyset$ and $\lambda > \lambda_0(K_0)$, $\lambda > \lambda_0(K_1)$ but solutions remain bounded as $t \to \infty$ as detailed in Theorem 3.4.19 and seen in Example 3.3.11. This again proves that the size of $\lambda_0(K(t))$ is not enough to decide the boundedness vs unboundedness of solutions. Actually, the geometry and, as we will see here, the "velocity" at which the sets $K(t)$ move are also important parameters.

The assumptions in the following result can be interpreted as first of all having $K(t)$ large enough to allow the solution to grow inside these set. Moreover, the sets $K(t)$ move slowly enough so that the growth inside $K(t)$ is transported together with $K(t)$, allowing the function u to grow without bounds inside the slowly moving set $K(t)$.

We can state now:

Theorem 3.5.1. *Let us assume n satisfies* **(N)**. *Let us also assume that there exists a sequence of times $(t_i)_{i\geq 0}$ with $t_0 < t_1 < \ldots < t_i < \ldots$ and a family of nonempty, smooth open subsets*

$$E_i \subset \bigcap_{s \in [t_i, t_{i+1}]} K(s), \tag{3.5.1}$$

with rigid motions $T_i : \mathbb{R}^N \to \mathbb{R}^N$ such that $E_i := T_i(E_0)$, and such that there exists $(x_i)_{i\geq 0}$ and $r > 0$ such that $B(x_i, 2r) \subset E_i \cap E_{i+1}$ for $i = 1, 2, \ldots$. Assume also that $\lambda > \lambda_1(E_0)$.

Then, $\exists \tau > 0$ such that if $t_{i+1} - t_i \geq \tau$ $\forall i = 0, 1, 2, \ldots$ all solutions are unbounded.

Observe that having the sets E_i satisfying $\lambda > \lambda_0(E_i)$ is saying that the sets E_i are large enough and therefore $K(t)$ are large enough from (3.5.1). Also, intuitively, having $t_{i+1} - t_i \geq \tau$ with property (3.5.1) means that the velocity at which the sets $K(t)$ move is slow enough.

The key point to prove this result is to find subsolutions which are unbounded as $t \to \infty$. Since the nonlinear logistic term is applied outside $K(t)$, unbounded growth will only be possible

within $K(t)$. Therefore, we will consider the problem restricted to the sets E_i which will be linear problems. Under certain conditions the solutions of these linear problems will grow without bounds, providing us with the key factor to get the unboundedness for the nonlinear problem.

In this respect we have the following interesting auxiliary result on the growth of the linear problem and its relation with the first eigenfunctions.

Let us denote by ϕ_n^A the n–th eigenfunction of the laplacian with homogeneous Dirichlet boundary conditions on the domain A normalized in the L^2–norm and by λ_n^A its eigenvalue. Assume also that ϕ_1^A is chosen to be positive in A.

Lemma 3.5.2. *Let us consider the following linear problem*

$$\begin{cases} v_t - \Delta v = \lambda v, & t > 0,\ x \in E \\ v = 0, & t > 0,\ x \in \partial E, \\ 0 \le v(0,x) = v_0(x) \in L^\infty(E), & x \in E. \end{cases} \tag{3.5.2}$$

where E is a smooth bounded open connected domain.

Let $D \subset E$ be a smooth open subset with $\overline{D} \subset E$ and let $\lambda > \lambda_1^E$. Then, if $\gamma > 1$ is given, v_0 is not identically 0 and $\tau > 0$ is given by:

$$\tau = \max\left\{ \frac{N\lambda_2^E}{2e(\lambda_2^E - \lambda_1^E)} \left(\frac{2C_\infty \|v_0\|_{L^2(E)}}{\left\langle v_0, \phi_1^E \right\rangle_{L^2(E)} \inf\limits_D \phi_1^E} \right)^{\frac{2}{N}}, \frac{1}{\lambda - \lambda_1^E} \log\left(\frac{2\gamma \max\limits_D \phi_1^D}{\left\langle v_0, \phi_1^E \right\rangle_{L^2(E)} \inf\limits_D \phi_1^E} \right) \right\} \tag{3.5.3}$$

where $C_\infty = C_\infty(E,N)$ is the constant of the embedding $H^N(E) \hookrightarrow L^\infty(E)$, then

$$v(t,x;v_0) \ge \gamma \phi_1^D(x), \quad t \ge \tau,\ x \in D.$$

Proof. Let us define

$$I = \inf_D \phi_1^E > 0, \qquad \mu = \frac{I}{2\max\limits_D \phi_1^D}, \qquad \alpha_j = \left\langle v_0, \phi_j^E \right\rangle_{L^2(E)}, \quad j \in \mathbb{N}.$$

We have for $x \in D$

$$\begin{aligned} v(t,x) &= \sum_{j=1}^{\infty} e^{\left(\lambda - \lambda_j^E\right)t} \langle v_0, \phi_j^E \rangle_{L^2(E)} \phi_j^E(x) \\ &= e^{\left(\lambda - \lambda_1^E\right)t} \alpha_1 \phi_1^E(x) + \sum_{i=2}^{\infty} e^{\left(\lambda - \lambda_j^E\right)t} \alpha_j \phi_j^E(x) \\ &\ge e^{\left(\lambda - \lambda_1^E\right)t} \alpha_1 (\mu \phi_1^D + I/2) + \sum_{i=2}^{\infty} e^{\left(\lambda - \lambda_j^E\right)t} \alpha_j \phi_j^E(x) \end{aligned}$$

where we have used that

$$\phi_1^E(x) = \frac{\phi_1^E(x)}{2} + \frac{\phi_1^E(x)}{2} \ge \frac{I}{2} + \frac{I}{2} \ge \frac{I\phi_1^D}{2\max\limits_D \phi_1^D} + \frac{I}{2} = \mu\phi_1^D + \frac{I}{2}$$

Now, let us observe that

$$v(t,x) \ge e^{\left(\lambda - \lambda_1^E\right)t} \left(\alpha_1 \mu \phi_1^D(x) + \alpha_1 I/2 + J(x) \right),$$

where

$$J(x) = \sum_{j=2}^{\infty} e^{(\lambda_1^E - \lambda_j^E)t} \alpha_j \phi_j^E(x).$$

We want to arrive at an L^∞–estimate for J. If N is the dimension of the space and if $k > N/2$ we have the Sobolev inclusion $H^k(\Omega) \hookrightarrow L^\infty(\Omega)$. Thus, denoting by $C_\infty = C_\infty(N, E)$ the constant of the embedding $H^N(E) \hookrightarrow L^\infty(E)$, we have

$$\begin{aligned}
\|J\|^2_{L^\infty(E)} &\leq C_\infty^2 \left\| \sum_{j=2}^{\infty} e^{(\lambda_1^E - \lambda_j^E)t} \alpha_j \phi_j^E \right\|^2_{H^N(E)} = C_\infty^2 \sum_{j=2}^{\infty} e^{-2(\lambda_j^E - \lambda_1^E)t} \alpha_j^2 (\lambda_j^E)^N \\
&= C_\infty^2 \sum_{j=2}^{\infty} e^{-2(\lambda_j^E - \lambda_1^E)t} \left((\lambda_j^E - \lambda_1^E)t\right)^N \alpha_j^2 \frac{(\lambda_j^E)^N}{\left((\lambda_j^E - \lambda_1^E)t\right)^N} \\
&\leq C_\infty^2 \left(\frac{N}{2e}\right)^N \sum_{j=2}^{\infty} \alpha_j^2 \left(\frac{\lambda_j^E}{(\lambda_j^E - \lambda_1^E)t}\right)^N \\
&\leq C_\infty^2 \left(\frac{N}{2e}\right)^N \sup_{j \geq 2} \left(\frac{\lambda_j^E}{(\lambda_j^E - \lambda_1^E)t}\right)^N \|v_0\|^2_{L^2(E)}
\end{aligned}$$

where we are bounding

$$e^{-2(\lambda_j^E - \lambda_1^E)t} \left((\lambda_j^E - \lambda_1^E)t\right)^N \leq \sup_{a \geq 0} \{e^{-2a} a^N\} = \left(\frac{N}{2e}\right)^N, \quad \forall j \geq 2$$

and using

$$\sum_{j=2}^{\infty} \alpha_j^2 \leq \sum_{j=1}^{\infty} \alpha_j^2 = \|v_0\|^2_{L^2(E)}.$$

Moreover, since $\lambda_j^E \geq \lambda_2^E > \lambda_1^E$ for $j \geq 2$, we can easily check that $\frac{\lambda_j^E}{\lambda_j^E - \lambda_1^E}$ is decreasig in $j \geq 2$ and therefore $\frac{\lambda_j^E}{\lambda_j^E - \lambda_1^E} \leq \frac{\lambda_2^E}{\lambda_2^E - \lambda_1^E}$. Hence,

$$\|J\|^2_{L^\infty(E)} \leq C_\infty^2 \left(\frac{N}{2e}\right)^N \left(\frac{\lambda_2^E}{(\lambda_2^E - \lambda_1^E)t}\right)^N \|v_0\|^2_{L^2(E)}$$

Therefore, if we take

$$t \geq \frac{N \lambda_2^E}{2e(\lambda_2^E - \lambda_1^E)} \left(\frac{C_\infty 2 \|v_0\|_{L^2(E)}}{\alpha_1 I}\right)^{2/N}$$

we have $\|J\|^2_{L^\infty(E)} \leq \alpha_1 I/2$ and therefore,

$$v(t,x) \geq e^{(\lambda - \lambda_1^E)t} \left(\alpha_1 \mu \phi_1^D(x) + \alpha_1 I/2 - \|J\|_{L^\infty(E)}\right) \geq e^{(\lambda - \lambda_1^E)t} \alpha_1 \mu \phi_1^D(x).$$

Moreover, if $\gamma > 1$ is given and we choose $t \geq \frac{1}{\lambda - \lambda_1^E} \log\left(\frac{\gamma}{\alpha_1 \mu}\right)$, then $e^{(\lambda - \lambda_1^E)t} \alpha_1 \mu \geq \gamma$.

Hence, if we take

$$\tau = \max\left\{ \frac{N\lambda_2^E}{2e(\lambda_2^E - \lambda_1^E)} \left(\frac{C_\infty 2\|v_0\|_{L^2(E)}}{\alpha_1 I} \right)^{2/N}, \frac{1}{\lambda - \lambda_1^E} \log\left(\frac{\gamma}{\alpha_1 \mu} \right) \right\} > 0$$

we have $v(t,x) \geq \gamma \phi_1^D(x)$ for all $t \geq \tau$ and $x \in D$. This concludes the proof.

□

Let us consider now the following corollary, which will be important in the proof of the main result.

Corollary 3.5.3. *If in the setting of the previous proposition, we assume that $B(a,2r) \subset E$ and take $v_0 = \phi_1^{B(a,r)}$, then there exists a $\rho > 0$, independent of a, such that*

$$\|v(t,x,v_0)\|_{L^\infty(E)} \geq \rho, \quad t \geq 0$$

Proof. Let us fix $a_0 \in E$ such that $B(a_0,2r) \subset E$. Let us also fix $\gamma > 1$. Applying Proposition 3.5.2, with $D = B(a_0,r) \subset\subset E$, we have that the value of τ is independent of a. Hence, we have that

$$v(t,x,\phi_1^{B(a,r)}) \geq \gamma \phi_1^{B(a_0,r)}(x), \quad x \in B(a,r), \quad t \geq \tau$$

This implies that for all $a \in E$ with $B(a,2r) \subset E$, we have

$$\|v(t,\cdot,\phi_1^{B(a,r)})\|_{L^\infty(E)} \geq \gamma \|\phi_1^{B(a_0,r)}\|_{L^\infty(E)} = \gamma \|\phi_1^{B(0,r)}\|_{L^\infty(E)}, \quad \forall t \geq \tau.$$

Now, by comparison, with the problem posed in $B(a,r)$ and using that $\phi_1^{B(a,r)}$ is the first eigenfunction in $B(a,r)$, we have

$$v(t,x,\phi_1^{B(a,r)}) \geq e^{(\lambda - \lambda_1^{B(a,r)})t} \phi_1^{B(a,r)}(x), \quad t \geq 0, \quad x \in B(a,r)$$

But this implies that

$$\|v(t,\cdot,\phi_1^{B(a,r)})\|_{L^\infty(E)} \geq e^{-|\lambda - \lambda_1^{B(0,r)}|\tau} \|\phi_1^{B(0,r)}\|_{L^\infty(E)}, \qquad t \in [0,\tau]$$

Therefore, choosing

$$\rho = \min\{\gamma \|\phi_1^{B(0,r)}\|_{L^\infty(E)}, e^{-|\lambda - \lambda_1^{B(0,r)}|\tau} \|\phi_1^{B(0,r)}\|_{L^\infty(E)}\}$$

we prove the result.

□

We are in a position now to give a proof of the main result, Theorem 3.5.1.

Proof. (of Theorem 3.5.1). First let us state some immediate remarks derived from the assumptions. Since $B(x_i,2r) \subset E_i \cap E_{i+1}$ then $d(x_i,\partial E_i) \geq 2r$ and $d(x_i,\partial E_{i+1}) \geq 2r$. Moreover, since $T_i : \mathbb{R}^N \to \mathbb{R}^N$ are rigid motions and $\lambda > \lambda_0(E_0)$ then $\lambda > \lambda_0(E_i)$ for any $i \geq 0$, and if we define $\tilde{x}_i = T_i^{-1}(x_i)$ then we have $d(\tilde{x}_i,\partial E_0) \geq 2r$ for all $i = 0,1,2,\ldots$.

To simplify, let us denote by ψ the first eigenfuntion of $-\Delta$ in $B(0,r)$ with Dirichlet boundary conditions (which is explicitly known), that is $\psi = \phi_1^{B(0,r)}$. Also, denote by $\psi_i := \psi(x - x_i)$ the first

eigenfunction of $-\Delta$ in $B(x_i, r)$ with Dirichlet homogeneous boundary conditions and extended by zero outside $B(x_i, r)$ for every $i \geq 0$.

We are going to apply Proposition 3.5.2 for $i = 1, 2, \ldots$, with $E = E_i$, $D = B(x_i, r)$ and with initial condition $v_0 = \psi_{i-1}$, which has support in $B(x_{i-1}, r) \subset E_i$. The value of τ obtained in the proposition, see (3.5.3), is given for each i by:

$$\tau_i = \max \left\{ \frac{N\lambda_2^{E_i}}{2e(\lambda_2^{E_i} - \lambda_1^{E_i})} \left(\frac{2C_\infty \|\psi_{i-1}\|_{L^2(E_i)}}{\left\langle \psi_{i-1}, \phi_1^{E_i} \right\rangle_{L^2(E_i)} \inf\limits_{B(x_i,r)} \phi_1^{E_i}} \right)^{2/N}, \right.$$

$$\left. \frac{1}{\lambda - \lambda_1^{E_i}} \log \left(\frac{2\gamma \max\limits_{B(x_i,r)} \phi_1^{B(x_i,r)}}{\left\langle \psi_{i-1}, \phi_1^{E_i} \right\rangle_{L^2(E_i)} \inf\limits_{B(x_i,r)} \phi_1^{E_i}} \right) \right\} > 0$$

Observe that since all sets E_i are rigid transformations of the same set E_0, then $\lambda_k^{E_i} = \lambda_k^{E_0}$, for all $k = 1, 2, ..$, and the embedding constant C_∞ is independent of i. Moreover since $B(x_i, r) \subset E_i^r \equiv \{x \in E_i : d(x, \partial E_i) > r\}$ and the maps T_i are rigid motions, then $\inf\limits_{B(x_i,r)} \phi_1^{E_i} \geq \inf\limits_{E_i^r} \phi_1^{E_i} = \inf\limits_{E_0^r} \phi_1^{E_0}$, which is independent of i. Moreover, $\|\psi_{i-1}\|_{L^2(E_i)} = 1$ and $\max\limits_{B(x_i,r)} \phi_1^{B(x_i,r)} = \max\limits_{B(0,r)} \phi_1^{B(0,r)} > 0$ which are also independent of i. Also,

$$\left\langle \psi_{i-1}, \phi_1^{E_i} \right\rangle_{L^2(E_i)} = \int_{E_i} \psi_{i-1} \phi_1^{E_i} \geq \left(\int_{B(x_{i-1},r)} \psi_{i-1} \right) \inf_{B(x_{i-1},r)} \phi_1^{E_i}$$

$$\geq \left(\int_{B(0,r)} \psi \right) \inf_{E_i^r} \phi_1^{E_i} = \left(\int_{B(0,r)} \psi \right) \inf_{E_i^0} \phi_1^{E_0} > 0$$

which is also independent of i. Therefore, if we define

$$\tau = \max \left\{ \frac{N\lambda_2^{E_0}}{2e(\lambda_2^{E_0} - \lambda_1^{E_0})} \left(\frac{2C_\infty}{\int_{B(0,r)} \psi (\inf\limits_{E_0^r} \phi_1^{E_0})^2} \right)^{2/N}, \frac{1}{\lambda - \lambda_1^{E_0}} \log \left(\frac{2\gamma \max\limits_{B(0,r)} \psi}{\int_{B(0,r)} \psi (\inf\limits_{E_0^r} \phi_1^{E_0})^2} \right) \right\} > 0$$

we can apply the previous proposition for intervals of time larger or equal than τ.

Hence, if for $i = 1, 2 \ldots$, we have $t_i - t_{i-1} \geq \tau$ and we consider problems

$$\begin{cases} v_{i_t} - \Delta v_i = \lambda v_i, & x \in E_i,\ t \in (t_{i-1}, t_i] \\ v_i(t, x) = 0, & x \in \partial E_i,\ t \in (t_{i-1}, t_i] \\ v_i(t_{i-1}, x) = \gamma^{i-1} \psi_{i-1}, & x \in E_i, \end{cases} \tag{3.5.4}$$

then we have that $v_i(t_i, x) \geq \gamma^i \psi_i(x)$ in $B(x_{i+1}, r)$. If we extend the functions $v_i(t, \cdot)$ by 0 to all Ω and define the function $v(t, x)$ as $v(t, x) = v_i(t, x)$ $t \in [t_{i-1}, t_i)$, then by comparison it is not difficult to see that

$$v(t, x) \leq u(t, x, t_0, \psi_0(\cdot)), \quad t \geq t_0.$$

Indeed, if $t \in [t_0, t_1]$ we have that $v_1(t,x)$ is a subsolution of $u(t,x)$ and therefore $u(t_1,x) \geq v_1(t_1,x) \geq \gamma\psi_1(x)$. Observing that $\gamma\psi_1(x)$ is the initial condition of v_2, we get again that in $[t_1,t_2]$ the function $v_2(t,x)$ is a subsolution of $u(t,x)$. Therefore $u(t_2,x) \geq v_2(t_2,x) \geq \gamma^2\psi_2(x)$. By induction it is not difficult to see that in $[t_{i-1},t_i]$ we have that $v_i(t,x)$ is a subsolution of $u(t,x)$ and therefore $u(t_i,x) \geq \gamma^i\psi_i(x)$. This means that

$$u(t,x) \geq v(t,x), \qquad t \geq t_0, \quad x \in \Omega$$

and in particular

$$\|u(t_i,x)\|_{L^\infty(\Omega)} \geq \gamma^i \|\psi_i\|_{L^\infty(\Omega)} = \gamma^i \|\psi\|_{L^\infty(B(0,r))} \to \infty, \text{ as } i \to \infty$$

Moreover, with Corollary 3.5.3 we can actually see that $\|v_i(t,x)\|_{L^\infty(\Omega)} \geq \gamma^{i-1}\rho$ for each $t \in [t_{i-1},t_i]$. Hence, we get

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{L^\infty(\Omega)} = +\infty$$

□

The next result shows that the assumption in Theorem 3.5.1

$$\lambda > \lambda_1(E(t)), \quad t \geq t_0,$$

can also be weakened under some conditions, allowing that $\lambda < \lambda_1(E(t))$ for some short intervals of time.

Let us consider problem (1.0.1) where n satisfy **(N)** and $K(t)$ satisfies

$$\begin{cases} K(t) \supset K_1, & t \in (T_{0,i}, T_{1,i}], x \in \Omega, \\ K(t) \supset K_0, & t \in (T_{1,i}, T_{0,i+1}], x \in \Omega, \end{cases} \tag{3.5.5}$$

for some unbounded time sequences $t_0 \leq T_{0,i} \leq T_{1,i} \leq T_{0,i+1} \to \infty$ as $i \to \infty$ and for some sets K_0, K_1 which are the closure of smooth open sets $\Omega_0 \subset \Omega_1$.

Defining $\underline{n}$ as

$$\underline{n}(t,x) := \begin{cases} \|n\|_{L^\infty}\chi_{\Omega\setminus K_1}(x), & t \in (T_{0,i}, T_{1,i}],\ x \in \Omega,\ i = 1,2,\dots \\ \|n\|_{L^\infty}\chi_{\Omega\setminus K_0}(x), & t \in (T_{1,i}, T_{0,i+1}],\ x \in \Omega,\ i = 1,2,\dots, \end{cases} \tag{3.5.6}$$

we have $n(t,x) \leq \underline{n}$. Hence, by Theorem 1.2.1, the solution $\underline{u}$ of problem $P(\underline{u}, u_0, \underline{n}, \Omega)$ is a subsolution of u, the solution of (1.0.1).

We can show the following

Theorem 3.5.4. *With the notation above, if we assume*

$$\lambda_0(K_1) < \lambda < \lambda_0(K_0) < \infty. \tag{3.5.7}$$

and that there exist τ and η such that for the time sequences $t_0 \leq T_{0,i} < T_{1,i} < T_{0,i+1} \to \infty$ satisfy

$$\begin{cases} |T_{1,i} - T_{0,i}| > \tau, & i \in \mathbb{N}, \\ |T_{0,i+1} - T_{1,i}| \leq \eta, & i \in \mathbb{N}, \end{cases} \tag{3.5.8}$$

then, if τ is sufficiently large, the solution $u(t,x)$ of (1.0.1) is unbounded.

Proof. We want to exhibit a subsolution of $u(t,x)$ that grows up as time goes to infinity. Without any loss of generaliy, we can assume that

$$\lambda \in (\lambda_1^{\Omega_1}, \lambda_2^{\Omega_1}), \tag{3.5.9}$$

since, if greater or equal than $\lambda_2^{\Omega_1}$, we can take a smaller λ which verifies (3.5.9) and it would result in a subsolution of the original solution by Theorem 1.2.1. Let us consider the following family of problems for each $i \geq 1$:

$$\begin{cases} U_{i_t}(t,x) - \Delta U_i(t,x) = \lambda U_i(t,x), & t \in (T_{0,i}, T_{1,i}], \quad x \in \Omega_1 \\ U_i(t,x) = 0, & t \in (T_{0,1}, T_{1,i}], \quad x \in \partial\Omega_1 \\ U_i(T_{0,i}, x) = \phi_1^{\Omega_0}, & \qquad\qquad\quad x \in \Omega_1 \end{cases} \tag{3.5.10}$$

$$\begin{cases} W_{i_t}(t,x) - \Delta W_i(t,x) = \lambda W_i(t,x), & t \in (T_{1,i}, T_{0,i+1}], \quad x \in \Omega_0 \\ W_i(t,x) = 0, & t \in (T_{1,i}, T_{0,i+1}], \quad x \in \partial\Omega_0 \\ W_i(T_{1,i}, x) = \phi_1^{\Omega_0}, & \qquad\qquad\quad x \in \Omega_0. \end{cases} \tag{3.5.11}$$

Observe that U_i is posed in the large set Ω_1 and therefore the solution there tends to grow, while W_i is posed in the smaller set Ω_0 where the solution tends to decay. As a matter of fact,

$$W_i(t) = e^{(\lambda - \lambda_1^{\Omega_0})(t - T_{1,i})} \phi_1^{\Omega_0},$$

and if we define

$$\alpha = e^{(\lambda - \lambda_1^{\Omega_0})\eta} \in (0,1)$$

we have

$$W_i(t) \geq \alpha \phi_1^{\Omega_0}, \qquad t \in [T_{1,i}, T_{0,i+1}] \tag{3.5.12}$$

Applying now Proposition 3.5.2 to U_i, with $E = \Omega_1$, $D = \Omega_0$, and $U_{i,0} = \phi_1^{\Omega_0}$, we have that if $\gamma > 1$ and define

$$\tau = \max\{\frac{N\lambda_2^{\Omega_1}}{2e(\lambda_2^{\Omega_1} - \lambda_1^{\Omega_1})} \left(\frac{2C_\infty \|\phi_1^{\Omega_0}\|_{L^2(\Omega_1)}}{\left\langle \phi_1^{\Omega_0}, \phi_1^{\Omega_1} \right\rangle_{L^2(\Omega_1)} \inf\limits_{\Omega_0} \phi_1^{\Omega_1}} \right)^{2/N}, \frac{1}{\lambda - \lambda_1^{\Omega_1}} \log \left(\frac{2\frac{\gamma}{\alpha} \max\limits_{\Omega_0} \phi_1^{\Omega_0}}{\left\langle \phi_1^{\Omega_0}, \phi_1^{\Omega_1} \right\rangle_{L^2(\Omega_1)} \inf\limits_{\Omega_0} \phi_1^{\Omega_1}} \right)\}$$

which does not depend on i. Then, choosing $T_{1,i} - T_{0,i} \geq \tau$, we have

$$U_i(t) \geq \frac{\gamma}{\alpha} \phi_1^{\Omega_0}, \quad t \in [T_{0,i} + \tau, T_{1,i}]. \tag{3.5.13}$$

Also, notice that comparing problem (3.5.10)) with the same problem but posed in $\Omega_0 \subset \Omega_1$, we have that

$$U_i(t) \geq e^{(\lambda - \lambda_1^{\Omega_0})t} \phi_1^{\Omega_0} \geq e^{(\lambda - \lambda_1^{\Omega_0})\tau} \phi_1^{\Omega_0}, \quad t \in [T_{0,i}, T_{0,i} + \tau]. \tag{3.5.14}$$

Putting together (3.5.13) and (3.5.14), we get

$$U_i(t) \geq \rho \phi_1^{\Omega_0}, \qquad t \in [T_{0,i}, T_{1,i}] \tag{3.5.15}$$

where $\rho = \min\{\frac{\gamma}{\alpha}, e^{(\lambda - \lambda_1^{\Omega_0})\tau}\}$.

Let us construct the function $Z(t,x)$ where for every $i \in \mathbb{N}$:

$$Z(t,\cdot) = \begin{cases} \gamma^{i-1} U_i(t,\cdot), & t \in (T_{0,i}, T_{1,i}], \\ \frac{\gamma^i}{\alpha} W_i(t,\cdot), & t \in (T_{1,i}, T_{0,i+1}], \end{cases}$$

where we assume U_i, W_i are extended by 0 to the whole domain Ω. With the analysis above, we can conclude that $Z(t,x)$ is a subsolution of $u(t,x)$ and with (3.5.12) and (3.5.15) we get that $\lim_{t\to+\infty} \|Z(t,\cdot)\|_{L^\infty} = +\infty$, which proves the result.

□

3.6 Examples

Let us present some cases where the results of this chapter can be applied to study the asymptotic behaviour.

3.6.1 Continuously moving $K(t)$

Let us consider problem (1.0.1) where $n = \chi_{\Omega\setminus K(t)}$ and

$$K(t) = \gamma(t) + K_0$$

where for every $t \geq t_0$, $\gamma(t)$ is a C^1 smooth curve, and K_0 is the closure of a smooth bounded open set and K_0 and γ are such that $\bigcup_{t\geq t_0} K(t) \subset\subset \Omega$ and $\bigcap_{t\geq t_0} K(t) = \emptyset$.

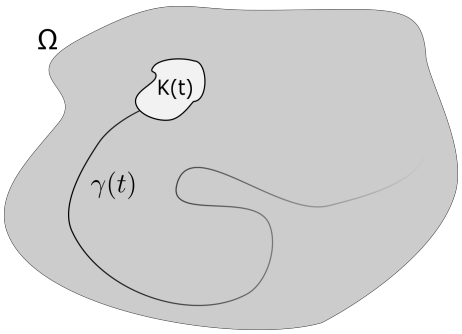


Figure 3.20: Domain Ω with continuously moving subdomain $K(t)$.

Global existence and well–posedness have already been stablished in Corollaries 1.1.5 and 1.2.6 for $u_0 \in L^q(\Omega)$ and q verifying (1.1.11). Moreover, n satisfies **(N)** trivially and let us assume $K(t)$ satisfies **(K)** and $K^{\sup} = K^{\sup}_{t_0}$. Moreover, it is clear that $K^{\inf} = \emptyset$. Therefore, we can apply Proposition 3.3.8:

- for $\lambda \in (-\infty, \lambda_0(K^{\sup}))$ all solutions are bounded, and
- since $\lambda_0(K^{\inf}) = \infty$ there are no garanteed unbounded solutions.

Therefore, it remains to study the case for $\lambda \in [\lambda_0(K^{\sup}), \infty)$.

(1) Observe that for $\lambda \in [\lambda_0(K^{\sup}), \lambda_0(K_0))$ we have that $\lambda < \lambda_0(K(t))$ for any $t \geq t_0$. Moreover, Remark 3.4.11 applies since $\gamma(\cdot)$ is continuous, there exists a $\delta > 0$ and an open neighbourhood $\Omega_\delta(t)$ of $K(t)$ such that for any $t \geq t_0$

$$\lambda < \lambda_0(\Omega_\delta(t)), \qquad \bigcup_{[t,t+\tau_0]} K(s) \subset \Omega_\delta(t), \qquad \text{and} \quad d\left(\bigcup_{s\in[t,t+\tau_0]} K(s), \partial\Omega_\delta(t) \right) > \delta, \quad (3.6.1)$$

for some $\tau_0 > 0$. Hence, conditions for Theorem 3.4.9 are met and for $\lambda \in [\lambda_0(K^{\sup}), \lambda_0(K_0))$ all solutions are also bounded.

(2) Finally, for $\lambda \in (\lambda_0(K_0), \infty)$:

- Let $(t_i)_{i\geq 0}$ be a sequence of times such that for $i \geq 0$

$$E_i := \bigcap_{s\in[t_i,t_{i+1}]} K(s), \quad \text{where } \lambda > \lambda_0(E_i).$$

Let us define $\tau_i = t_{i+1} - t_i$ for $i \geq 0$ and observe that the faster $K(t)$ moves, the closer the intersections E_i are to $K^{\inf}$ and thus, the closer $\lambda_0(E_i)$ is to $\infty = \lambda_0(K^{\inf})$. Or conversely, the slower $K(t)$ travles, the greater τ_i can be and still verify $\lambda_0(E_i) < \lambda < \infty$. If $K(t)$ moves slowly, that is, if $\sup_{t\geq t_0} |\gamma'(t)|$ is small, then $E_i \cap E_{i+1} \neq \emptyset$ and therefore, since K_0 is the closure of a bounded open set, there exist a sequence of points $(x_i)_{i\geq 0}$ and a $\delta > 0$ such that $B(x_i, 2r) \subset E_i \cap E_{i+1}$ for $i \geq 0$. Moreover, if it moves slowly enough, then for all $i \geq 0$ $\tau_i \leq \tilde{\tau}$ for $\tilde{\tau} > 0$ from Theorem 3.5.1 which implies that all solutions grow up.

- If $K(t)$ moves fast, as we will see in the next chapter, then all solutions are bounded for any $\lambda \in \mathbb{R}$.

3.6.2 Rotating cheese $K(t)$

A very closely related problem to the continuously moving $K(t)$ is the rotating cheese introduced in Example 3.3.10 we we can assume that $n(t,x) = \chi_{\Omega\setminus K(t)}(x)$, $\forall t \geq t_0$ and recall Figure 3.21:

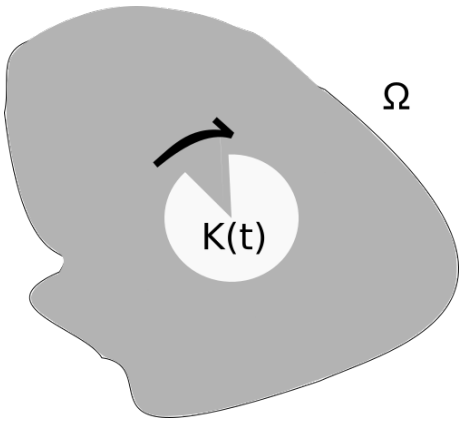


Figure 3.21: Domain Ω with rotating subdomain $K(t)$.

Now, as in the previous example, we have global existence and uniqueness due to Corollaries 1.1.5 and 1.2.6 for $u_0 \in L^q(\Omega)$ and q verifying (1.1.11). Moreover, n satisfies **(N)** since it is defined through the characteristic function and $K(t)$ satisfies **(K)** trivially since $K^{\sup} = K^{\sup}_{t_0}$. Furthermore, it is also immediate that $K^{\sup} = B(0, r_0)$ and $K^{\inf} = \{0\}$. Hence, Proposition 3.3.8 can be applied:

- for $\lambda \in (-\infty, \lambda_0(\overline{B(0,r_0)}))$ all solutions are bounded, and
- since $\lambda_0(\{0\}) = \infty$, there are no guaranteed grow up solutions. Which is the only difference between this example and the previous continuously moving $K(t)$ example.

A similar problem with identical conclusions arises even when $K(t)$ moves discontinuously between only two compact subsets of $\mathbb{R}^N$ one being contained within the other:

3.6.3 Oscillating radius $K(t)$

Let $K_0 \subset K_1 \subset \Omega \subset \mathbb{R}^N$ be compact sets as in Figure 3.22.

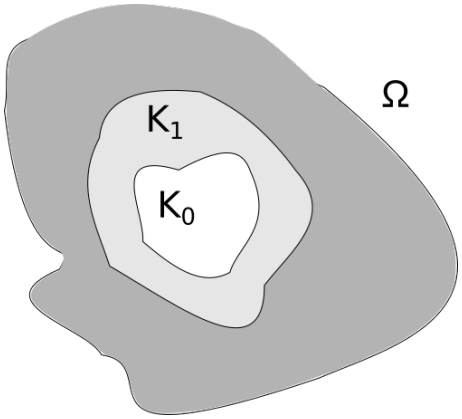


Figure 3.22: Domain Ω with contracting/expanding subdomains K_0 and K_1.

To simplify, let us consider $t_0 = 0$. Also, let $0 < T_1 < T$ and consider the function $n(t,x)$ obtained by extending the following function by periodicity to all $t \in [0,\infty)$

$$n(t,x) := \begin{cases} \chi_{\Omega\setminus K_1}(x), & t \in [0,T_1), x \in \Omega \\ \chi_{\Omega\setminus K_0}(x), & t \in [T_1,+T), x \in \Omega, \end{cases} \tag{3.6.2}$$

Now, as in the previous example, we have global existence and uniqueness due to Corollaries 1.1.5 and 1.2.6 for $u_0 \in L^q(\Omega)$ and q verifying (1.1.11). Moreover, n satisfies **(N)** since it is defined through the characteristic function and $K(t)$ satisfies **(K)** trivially since it can only be K_0 or K_1. Furthermore, it is also immediate that $K^{\sup} = K_1$ and $K^{\inf} = K_0$. Hence, Proposition 3.3.8 can be applied:

- for $\lambda \in (-\infty, \lambda_0(K_1))$ all solutions are bounded, and
- for $\lambda \in [\lambda_0(K_0), \infty)$ all solutions grow up.

Unlike with the previous example, we now have guaranteed unbounded solutions due to $K^{\inf}$ being fat (i.e. $\lambda_0(K^{\inf}) < \infty$). Therefore, it remains to study the case for $\lambda \in [\lambda_0(K_1), \lambda_0(K_0))$. Moreover, unlike the previous example, after applying Proposition 3.3.8 there is no interval in which it can be verified that $\lambda < \lambda_0(K(t))$ for $t \geq t_0$. This implies that we cannot invoke Theorem 3.4.9 to obtain bounded solutions. However, due to the nature of this more "static" $K(t)$ we are able to apply other results:

- Since $K(t)$ is bounded above by K_0, K_1, that is, $K(t) \subset (K_1, K_0)$ as according to Definition 3.4.2, by Proposition 3.4.17, there exists $\Xi_0 > 0$ independent of T_1 such that if $T_1 < \Xi_0$ then all solutions remain bounded.

- By Theorem 3.5.4, there exists $\tau > 0$ independent of T_1 such that if K_0 and K_1 are the closure of bounded open sets, T_1 and T are sufficiently large and $T_1 > \tau$ then all solutions are unbounded.

Therefore, depending on T_1 the corresponding problem will have bounded solutions or solutions that grow up. That is, it would seem that we would have the following setting:

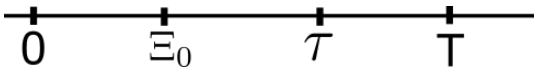


Figure 3.23: Setting with Ξ_0 and τ.

where if $T_1 < \Xi_0$ then all solutions are bounded and if $T_1 > \tau$ (and additional assumptions including: T is large enough and K_0, K_1 are the closure of smooth bounded open sets) then all solutions grow up.

Let $P(u_\varepsilon, u_0, n_\varepsilon, \Omega)$ be a family of problems where n_ε are defined as in (3.6.2) for $T_1 = \varepsilon > 0$. Since $K_0 \subset K_1$ it can be easily verified through Theorem 1.2.1 that if $\varepsilon_1 < \varepsilon_2$ then $u_{\varepsilon_1}(t,x) \leq u_{\varepsilon_2}(t,x)$ for all $t \geq 0$ and a.e. $x \in \Omega$. Therefore, for any $\varepsilon < \Xi_0$ we have that all solutions are bounded and for any $\varepsilon > \tau$ (plus the additional assumptions on T and K_0, K_1) all solutions grow up. Hence, we end up with the following scenario:

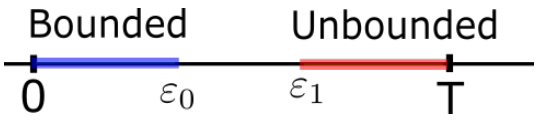


Figure 3.24: Monotonicity of solutions with respect to $\varepsilon \in [0, T]$.

where we have boundedness and unboundedness regions. It can be trivially verified (in any other case, a contradiction would rise) that there must exist a critical time T^* such that every solution with $T_1 < T^*$ must be bounded and every solution with $T_1 > T^*$ (plus the additional assumptions mentioned earlier) must grow up.

Remark 3.6.1. Due to the additional assumptions in Theorem 3.5.4, it cannot be assured that $T^* < T$ in every scenario. Nevertheless the existence of a critical time T^* is quite appealing since it offers the possibility of studying other aspects of problems (P_ε). In particular, our goal in the next chapter will be to study asymptotic behaviour of the time–homogenization of solutions, which is interesting because of the existence of T^*.

3.6.4 Jumping $K(t)$

Let $K_0, K_1 \subset \Omega \subset \mathbb{R}^N$ be compact sets such that $K_0 \cap K_1 = \emptyset$. Let us consider problem (1.0.1) and let n be defined as

$$n(t,x) := \begin{cases} \chi_{\Omega \setminus K_1}(x), & t \in [T_{0,i}, T_{1,i}), x \in \Omega, \\ \chi_{\Omega \setminus K_0}(x), & t \in [T_1, T_{0,i+1}), x \in \Omega \end{cases} \tag{3.6.3}$$

where the time sequences $(T_{0,i})_{i\in\mathbb{N}}, (T_{1,i})_{i\in\mathbb{N}}$ satisfy

$$0 < \inf_{i\in\mathbb{N}} |T_{1,i} - T_{0,i}| \leq \sup_{i\in\mathbb{N}} |T_{0,i+1} - T_{1,i}| < \infty,$$
$$0 < \inf_{i\in\mathbb{N}} |T_{0,i+1} - T_{1,i}| \leq \sup_{i\in\mathbb{N}} |T_{0,i+1} - T_{1,i}| < \infty.$$

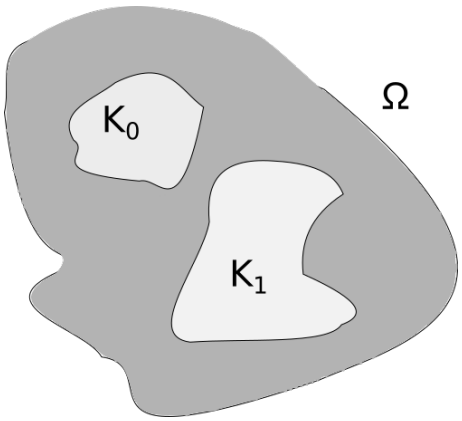


Figure 3.25: Domain Ω with "jumping" subdomains K_0 and K_1.

Again, global existence and well–posedness have already been stablish in Corollaries 1.1.5 and 1.2.6 for $u_0 \in L^q(\Omega)$ and q verifying (1.1.11). Moreover, $K(t)$ satisfies **(N)** since n is defined through the characteristic function and

$$K^{\sup} = K_{t_0}^{\sup} = K_0 \cup K_1, \quad K^{\inf} = K_{t_0}^{\inf} = \emptyset.$$

Hence, for $\lambda < \lambda_0(K^{\sup})$ all solutions are bounded by Proposition 3.3.8. Since $K^{\inf} = \emptyset \Rightarrow \lambda_0(K^{\inf}) = \infty$, and therefore it remains to study the case for $\lambda \in [\lambda_0(K^{\sup}), \infty)$.

In this example, it is straightforward that the time sequence $T_{0,i}$ verifies

$$\sup_{j\in\mathbb{N}} |T_{0,j+1} - T_{0,j}| \leq \sup_{j\in\mathbb{N}} |T_{0,j+1} - T_{1,j}| + \sup_{j\in\mathbb{N}} |T_{1,j} - T_{0,j}| < \infty.$$

Moreover, since K_0 and K_1 are compact, there is a positive distance $\delta > 0$ between them. Hence, if $\tau_0 = \min\left(\inf_{j\in\mathbb{N}} |T_{0,j+1} - T_{0,j}|, \inf_{j\in\mathbb{N}} |T_{1,j} - T_{0,j}|\right) > 0$ then we have that

$$d\left(\bigcup_{(T_{0,j}-\tau_0,T_{0,j})} K(s), \bigcup_{(T_{0,j},T_{0,j}+\tau_0)} K(s)\right) = d\left(\bigcup_{(T_{0,j}-\tau_0,T_{0,j})} K_1, \bigcup_{(T_{0,j},T_{0,j}+\tau_0)} K_0\right) = d\left(K_1, K_0\right) > \delta.$$

Now, Theorem 3.4.19 provides us the boundedness of u for every $\lambda \in \mathbb{R}$. That is, even if λ is much larger than $\lambda_0(K(t))$ we have boundedness of solutions in contrast to what an autonomous analogy would suggest. This reflects that in the non–autonomous case, solutions of equation (1.0.1) are more likely to be bounded.

3.7 Conclusions

The main conclusion of this chapter is that in the non–autonomous case of (1.0.1), more bounded solutions arise as shown in the first examples in the previous section where, for every $\lambda \in \mathbb{R}$, the solutions remain bounded. This is a crucial difference with the autonomous case as it illustrates that the geometry of $K(t)$ is key to study the asymptotic behaviour of the solutions. Those examples show that, whenever $K(t)$ moves under certain conditions (such as alternating between disconnected subsets or moving continuously at a fixed speed with $\lambda < \lambda_0(K(t))$ for all $t \geq t_0$) the solutions remain bounded. Therefore, a time varying $K(t)$ favours boundedness.

The other main conclusion of this chapter is that, in order to obtain grow up of solutions, there is a minimum time a large $K(t)$ must spend in a neighbourhood of $K(t)$. That is, it cannot move very fast nor it can jump.

From a population dynamics perspective, if the sanctuary is repeatedly being preyed on and/or moved, the projected population is more likely bounded. In order for the population to grow unbounded, the sanctuary must maintain a guaranteed static or nearly static core all year round. In particular, the population will be bounded everywhere where the species is being preyed on regularly, independently of the growth rate.

All the examples included in this chapter show the relevance of the geometry of $K(t)$ in the asymptotic behaviour of solutions. That is, we have found cases of bounded and unbounded solutions for $\lambda \in [\lambda_0(K^{\sup}), \lambda_0(K^{\inf}))$ (see Figure 3.2) even for the same $K(t)$ as in the case of the continuously moving $K(t)$.

Chapter 4

Averaging

4.1 Introduction

In Section 3.5 we have observed that unboundedness of solutions of (1.0.1) is intimately related to a slowly changing, non–jumping $K(t)$. In particular, Proposition 3.5.2, Theorems 3.5.1 and Theorem 3.5.4, all provide grow up based on the assumption that there is enough time for the linear part of the equation to pull the solution towards infinity. That is, if $K(t)$ is large enough and is allowed to stay a certain amount of time in large enough subsets (without jumping), then the solution is able to "invest in growing itself". From a population dynamics perspective, we could refer to this phenomenon as the sanctuary not changing drastically in order for the species to be able to reproduce indefinitely.

Therefore it seems natural to study how will solutions behave if $K(t)$ is large enough and moves quickly. First, we must specify what "moving fast" or "moving quickly" implies since, for example, $K(t)$ moves infinitely rapidly if it jumps and jumping $K(t)$ have already been associated to bounded solutions in Subsection 3.4.5.

Hence, our approach for this matter will have two main assumptions:

- $K(t)$ will move fast. As mentioned above, our working hypothesis is that a slow paced $K(t)$ is key for grow up. Hence, we will assume otherwise.
- $K(t)$ must not jump. As mentioned before and proven in Theorem 3.4.19, jumps automatically provide us with boundedness. Therefore, we must assume $K(t)$ does not jump in order to study a different scenario.

The problem associated to a fast paced $K(t)$ is obtained simply by $K(t/\varepsilon)$ and making ε small. This is the basis for homogenization techniques and, in the particular case of the time variable, is referred to as averaging when the limit $\varepsilon \to 0$ is taken. Hence, the problem we will study is:

$$\begin{cases} u_t - \Delta u = f(t,x,u,\epsilon) := \lambda u - n(t/\epsilon, x)|u|^{\rho-1}u, & x \in \Omega, t > 0, \\ u(t,x) = 0, & x \in \partial\Omega \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where specific assumptions will be detailed in later sections. When $\varepsilon \to 0$, $K(t/\varepsilon)$ is so rapidly moving that the averaging occurs. In fact, as seen later in this chapter, this solution u remains close to the solution of the following problem which is referred to as the averaged solution and the averaged problem respectively:

$$\begin{cases} v_t - \Delta v = f_0(x,v), & x \in \Omega, t > 0, \\ v(t,x) = 0, & x \in \partial\Omega \\ v(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where

$$f_0(x,v) := \lambda u - \bar{n}(x)u^\rho, \quad \bar{n}(x) := \lim_{T\to\infty} \frac{1}{T}\int_0^T n(\tau,x)d\tau, \quad \forall x \in \overline{\Omega}.$$

Since this is a new type of modification to our initial problem (1.0.1), we will introduce an ODE example first in order to understand and study the impact the high–frequency $K(t)$ has on the solutions. The example relies on the ability to compute the explicit solution to each problem.

Finally, in Section 4.2 we study the non–autonomous parabolic averaging problem and obtain boundedness of solutions u that stay close to v, the averaged solution, and therefore, implicitly proving that the time assumptions for unboundedness are necessary.

Initial example

As mentioned above, let us first introduce an ODE example in order to better understand how a highly oscillating $K(t)$ affects the behaviour of the solution.

Example 4.1.1. *For each $\epsilon > 0$ consider the non-autonomous scalar logistic ODE*

$$\begin{cases} \dot{u}_\epsilon = \lambda u_\epsilon - \varphi(t/\epsilon)u_\epsilon^2, & t > 0, \\ u_\epsilon(0) = u_0 \end{cases} \tag{4.1.1}$$

where $\varphi(t) = \varphi(t+T)$ for $t \geq 0$ is a nonnegative and nontrivial bounded T–periodic function. If $u(t,u_0)$ is the solution of the averaged equation

$$\begin{cases} \dot{u} = \lambda u - \overline{\varphi}u^2, & t > 0, \\ u(0) = u_0 \end{cases} \tag{4.1.2}$$

where $\overline{\varphi} = \frac{1}{T}\int_0^T \varphi(s)ds$, then

$$|u_\epsilon(t,u_0) - u(t,u_0)| \to 0 \quad as \quad \epsilon \to 0, \tag{4.1.3}$$

uniformly in $t \geq 0$.

Proof. Note that the explicit solutions of (4.1.1) and (4.1.2) are given by

$$u_\epsilon(t,u_0) = \frac{e^{\lambda t}}{u_0^{-1} + \int_0^t e^{\lambda s}\varphi(s/\epsilon)ds} \tag{4.1.4}$$

and

$$u(t,u_0) = \frac{e^{\lambda t}}{u_0^{-1} + \int_0^t e^{\lambda s}\overline{\varphi}ds}, \tag{4.1.5}$$

respectively. For each $\epsilon > 0$ we can explicit the initial data for which the solution in (4.1.4) is periodic of period ϵT. Indeed

$$\mu_\epsilon = u_\epsilon(\epsilon T, \mu_\epsilon) \iff \mu_\epsilon = \frac{e^{\lambda\epsilon T} - 1}{\int_0^{\epsilon T} e^{\lambda s}\varphi(s/\epsilon)ds}. \tag{4.1.6}$$

Using L'Hôpital's rule after a change of variable in the integral we get

$$\lim_{\epsilon\to 0}\mu_\epsilon = \lim_{\epsilon\to 0}\frac{\lambda T e^{\lambda\epsilon T}}{\int_0^T e^{\lambda\epsilon s}\varphi(s)ds + \epsilon\lambda\int_0^T e^{\lambda\epsilon s}s\varphi(s)ds} = \frac{\lambda}{\frac{1}{T}\int_0^T \varphi(s)ds}. \tag{4.1.7}$$

Which implies that $\lambda/\mu_\varepsilon \to \bar{\varphi}$ as $\varepsilon \to 0$. The distance between solutions u and u_ε can be estimated through the help of an additional solution. The previous limit just shown suggests to consider for that role the solution of the following autonomous problem:

$$\begin{cases} \dot{v}_\epsilon = \lambda v_\epsilon - \lambda {\mu_\epsilon}^{-1} v_\epsilon^2, \quad t > 0, \\ v_\epsilon(0) = u_0 \end{cases} \tag{4.1.8}$$

whose solution is given by

$$v_\epsilon(t, u_0) = \frac{e^{\lambda t}}{u_0^{-1} + \int_0^t e^{\lambda s}\lambda\mu_\epsilon^{-1}ds}. \tag{4.1.9}$$

Direct computations show that

$$|v_\epsilon(t, u_0) - u(t, u_0)| \to 0 \ \text{ as } \ \epsilon \to 0, \tag{4.1.10}$$

uniformly in $t \geq 0$.

Taking into account that $v_\epsilon(n\epsilon T, u_0) = u_\epsilon(n\epsilon T, u_0)$ for all $n \in \mathbb{N}$, it is not dificult to see that

$$|v_\epsilon(t, u_0) - u_\epsilon(t, u_0)| \to 0 \ \text{ as } \ \epsilon \to 0, \ \text{ uniformly in } t \geq 0. \tag{4.1.11}$$

Thus, from (4.1.10) and (4.1.11) we conclude

$$|u_\epsilon(t, u_0) - u(t, u_0)| \to 0 \ \text{ as } \ \epsilon \to 0.$$

uniformly in $t \geq 0$.

□

4.2 Averaging for the PDE

In this section we are interested in investigating the behaviour of problem (1.0.1) under a high-frequency perturbation. Let us recall problem (4.2.1) introduced at the start of this chapter:

$$\begin{cases} u_t - \Delta u = f(t, x, u, \epsilon) := \lambda u - n(t/\epsilon, x)|u|^{\rho-1}u, & x \in \Omega, t > 0, \\ u(t, x) = 0, & x \in \partial\Omega \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{4.2.1}$$

where $\epsilon > 0$ is a small parameter, $t_0 = 0$ for simplicity, $u_0 \in L^q(\Omega)$ with q satisfying (1.1.11) and n verifies the same assumptions as in previous chapters plus an additional assumption:

- $0 \leq n \in L^\infty((0, \infty) \times \overline{\Omega})$,
- n verifies **(N)** and **(K)**, and

- (additional assumption) n satisfies

$$\int_\Omega |n(t,x) - n(s,x)| \, \mathrm{d}x \leq C \, |t-s|^\zeta \,,$$

for any t, s such that $|t-s| < 1$, a.e. $x \in \overline{\Omega}$ and $\zeta > 0$.

Let us denote by u_ε the solution of problem (4.2.1) in order to reflect its dependency on the parameter ε. Moreover, let us define the averaged problem:

$$\begin{cases} v_t - \Delta v = f_0(x,v), & x \in \Omega, t > 0, \\ v(t,x) = 0, & x \in \partial\Omega \\ v(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{4.2.2}$$

where

$$f_0(x,v) := \lambda u - \bar{n}(x) u^\rho, \quad \bar{n}(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T n(\tau, x) d\tau, \quad \forall x \in \overline{\Omega}. \tag{4.2.3}$$

Let us study $\bar{n}(x)$ and problem (4.2.2) before progressing any further. First, let us observe that problem (4.2.2) is now an autonomous problem. In particular, results from [7] apply. Secondly, let K_a be the subset of $\overline{\Omega}$ where $\bar{n}$ vanishes. Then it is straightforward to see the following:

Lemma 4.2.1. *Let v be the solution of problem (4.2.2) above, let n verify assumptions above and let*

$$K_a := \{x \in \Omega : \bar{n}(x) = 0\}.$$

Then, $K^{\inf} \subset \overline{K_a}$. Moreover, if n is periodic in t, then $K^{\inf} = K_a$.

Remark 4.2.2. *Recall Definition that $K^{\inf}$ is defined in 3.3.5 as $K^{\inf} = \overline{\bigcup_{t>0} \bigcap_{s>t} K(s)}$.*

Proof. Let us see the first inclusion.

$$x_0 \in \bigcup_{\tau\geq 0} \bigcap_{s\geq\tau} K(s) = \bigcup_{\tau\geq 0} K_\tau^{\inf}.$$

Therefore, $x_0 \in K_{\tau_0}^{\inf}$ for some $\tau_0 > 0$ which implies that $n(x_0, t) = 0$ for every $t \geq \tau_0$. Hence, since $n \geq 0$

$$\begin{aligned} 0 \leq \bar{n}(x_0) &= \lim_{T\to\infty} \tfrac{1}{T} \int_0^T n(\tau, x_0) d\tau = \lim_{T\to\infty} \frac{1}{T} \left(\int_0^{\tau_0} n(\tau, x_0) d\tau + \int_{\tau_0}^T n(\tau, x_0) d\tau \right) \\ &\leq \lim_{T\to\infty} \tfrac{1}{T} \left(\tau_0 \|n\|_{L^\infty((0,\infty)\times\Omega)} + 0 \right) = 0. \end{aligned}$$

Thus,

$$\bigcup_{\tau\geq 0} \bigcap_{s\geq\tau} K(s) \subset K_a \Rightarrow K^{\inf} = \overline{\bigcup_{\tau\geq 0} \bigcap_{s\geq\tau} K(s)} \subset \overline{K_a}.$$

Let us see the other inclusion assuming $n(\cdot, x)$ is τ_0–periodic in time. Let $x_0 \in K_a$, therefore

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T n(\tau,x_0)d\tau=0.$$

Then, for any $\alpha_0>0$ there exists $j_0\in\mathbb{N}$ such that

$$0\le\frac{1}{j\tau_0}\int_0^{j\tau_0} n(\tau,x_0)d\tau<\alpha_0,\quad \forall j\ge j_0.$$

Since n is τ_0–periodic, we have for any $\alpha_0>0$ that there exists $j_0\in\mathbb{N}$ such that

$$0\le\frac{1}{j\tau_0}\int_0^{j\tau_0} n(s,x_0)ds=\frac{1}{\tau_0}\int_0^{\tau_0} n(s,x_0)ds<\alpha_0,\quad \forall j\ge j_0.$$

And since $n\ge 0$ and this is for all $\alpha_0>0$ then

$$\frac{1}{\tau_0}\int_0^{\tau_0} n(s,x_0)ds=0.$$

Therefore, $n(s,x)=0$ a.e. $s\ge 0$.

□

The following result is analogous to Lemma 3.4.14.

Corollary 4.2.3. *If $\lambda<\lambda_0(\overline{K_a})$ then all solutions of (4.2.2) are bounded and have a globally asymptotically stable equilibrium $\bar{\varphi}$, and there exist $E>0$ independent of initial data such that*

$$\limsup_{t\to\infty}\|v(t,\cdot;u_0)\|_{L^\infty(\Omega)}\le E.$$

Proof. Since problem (4.2.2) is an autonomous problem, by [7] we conclude the proof.

□

From Corollary 4.2.3, if $\lambda_1^\Omega<\lambda<\lambda_0(\overline{K_a})$ then problem (4.2.2) has an attracting equilibrium. We would like to view the nonautonomous problem (4.2.1) as a perturbation of (4.2.2), similarly as in Section 3.4.3. But we cannot apply Proposition 1.4.1 since if we denote by $n_\varepsilon(t,x)=n(t/\varepsilon,x)$ we do not have

$$\|n_\varepsilon-\bar{n}\|_{L^1([0,T]\times\Omega)}\to 0,\quad \varepsilon\to 0.$$

Instead, we will use the averaging theory developed by Henry, in particular Theorem 3.4.9 in [21].

Let us present the setting for the Theorem.

First, let q satisfy (1.1.11) and $q>\frac{N}{2}$ and recall Section 1.1 where we defined $X_q^1=L^q(\Omega)$, X_q^α as the scale of fractional power spaces associated to Δ with Dirichlet boundary conditions and let us also recall Lemma 1.1.4. Let $\theta\in[0,1)$ be such that $\theta\rho=1$, let $X_q^{1+\theta}=H_q^{2\theta}(\Omega)$, then the Nemytskii operators related to f and f_0 are defined as

$$F,F_0:(0,\infty)\times H_q^{2\theta}\to L^q(\Omega).$$

Problems (4.2.1) and (4.2.2) can be expressed as

$$\begin{cases} u_{\varepsilon_t} - \Delta u_\varepsilon = F(t/\varepsilon, u_\varepsilon) = \lambda u_\varepsilon - n(t/\varepsilon, x) u_\varepsilon^\rho, & t > 0 \\ u_\varepsilon(0) = u_0, & t = 0, \end{cases} \tag{4.2.4}$$

$$\begin{cases} v_t - \Delta v = F_0(v) = \lambda v - n_0(tx) v^\rho, & t > 0 \\ v(0) = u_0, & t = 0. \end{cases} \tag{4.2.5}$$

Finally, from [21] we have the following result:

Theorem 4.2.4. *Let us assume $F : (0, \infty) \times U \to L^q$ is continuous with U an open set in $H_q^{2\theta}$, and F is locally Hölder continuous in t, bounded and locally Lipschitz in u. Moreover, for any $u \in U$ let us assume that*

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_0^T F(s, u) ds - F_0(u) \right\|_{L^q} = 0.$$

Suppose there exists a solution $v(t)$ of problem (4.2.5) with $v(t) \in U$ for $0 \leq t \leq \tau < \infty$ for some $\tau > 0$. Then, given $\eta > 0$ there exists $\varepsilon_1 > 0$ such that if $0 < \varepsilon \leq \varepsilon_1$ then

$$\|u_\varepsilon(t) - v(t)\|_{H_q^{2\theta}} \leq \eta, \quad t \in [0, \tau].$$

Remark 4.2.5. *Let us observe that problems (4.2.1) and (4.2.2) both satisfy the previous Theorem assumptions. Indeed, $F(t, u) := F_1(t, u) = \lambda u - n(t, x) u^\rho$ is locally Lipchitz in u from Lemma 1.1.4. Moreover, it is locally Hölder in t since*

$$\begin{aligned} \|F(t, u) - F(s, u)\|_{L^q} &= \|(n(t, \cdot) - n(s, \cdot)) u^\rho\|_{L^q} \leq \|n(t, \cdot) - n(s.\cdot)\|_{L^p} \|u^\rho\|_{L^r} \\ &\leq \left(\int_\Omega |n(t, x) - n(s, x)|^p \, dx \right)^{1/p} \|u\|_{L^{r\rho}}^\rho \\ &\leq 2\|n\|_\infty \left(\int_\Omega |n(t, x) - n(s, x)| \, dx \right)^{1/p} \|u\|_{L^{r\rho}}^\rho \leq C|t - s|^{\zeta/p} \|u\|_{L^{r\rho}}^\rho \\ &\leq C|t - s|^\zeta \|u\|_{L^{r\rho}}^\rho, \end{aligned}$$

for some $C > 0$ and where we have used a Hölder inequality, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and assumed $|t - s| < 1$. If $|t - s| \geq 1$ then

$$\begin{aligned} \|F(t, u) - F(s, u)\|_{L^q} &\leq \left(\int_\Omega |n(t, x) - n(s, x)|^p \, dx \right)^{1/p} \|u\|_{L^{r\rho}}^\rho \leq 2\|n\|_\infty |\Omega|^{1/p} \|u\|_{L^{r\rho}}^\rho \\ &\leq 2\|n\|_\infty |\Omega|^{1/p} \|u\|_{L^{r\rho}}^\rho |t - s|^\zeta. \end{aligned}$$

Finally, the following condition is satisfied by definition in (4.2.3):

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_0^T F(s, u) ds - F_0(u) \right\|_{L^q} = 0.$$

We can now show the following result.

Proposition 4.2.6. *With the hypothesis above and $\lambda < \lambda_0(\overline{K_a})$, there exists $M > 0$ and $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_1$*

$$\limsup_{t\to\infty} \|u_\varepsilon(t,\cdot)\|_{L^\infty(\Omega)} \leq M.$$

Proof. From Proposition 3.2.2 we have that boundedness of u_ε is independent of initial data, therefore we may choose $u_0 = 3E \in L^\infty(\Omega)$ without any loss of generality. From Corollary 4.2.3 we have that there exists a time $\tau_0 > 0$ such that $\|v(t)\|_{L^\infty(\Omega)} \leq 2E$ for any $t \geq \tau_0$. That is, starting at $3E$ solution $v(t)$ becomes bounded by $2E$ for $t \geq \tau_0$.

We wish to estimate the distance between u_ε and v. Since $X_q^{1+\theta} = H_q^{2\theta} \hookrightarrow C^\eta(\bar{\Omega})$ for some $\eta > 0$ then

$$\|u_\varepsilon(t) - v(t)\|_{L^\infty(\Omega)} \leq \|u_\varepsilon(t) - v(t)\|_{X_q^{1+\theta}}.$$

Moreover, for the compact interval of time $[0, \tau_0]$ and for $E/2$ from Theorem 4.2.4 above we have that there exists $\varepsilon_1 > 0$ such that

$$\|u_\varepsilon(t) - v(t)\|_{X_q^{1+\theta}} \leq E/2, \quad \forall t \in [0,\tau_0],\ 0 < \varepsilon \leq \varepsilon_1.$$

Hence, we arrive at

$$\|u_\varepsilon(\tau_0)\|_{L^\infty} \leq \|u_\varepsilon(\tau_0) - v(\tau_0)\|_{L^\infty} + \|v(\tau_0)\|_{L^\infty} \leq \|u_\varepsilon(\tau_0) - v(\tau_0)\|_{X_q^{1+\theta}} + 2E \leq \frac{5}{2}E < 3E, \quad \forall 0 < \varepsilon \leq \varepsilon_1.$$

For every $k \geq 0$ we define $w_k(t,\cdot)$ as

$$\begin{cases} w_{k_t} - \Delta w_k = \lambda w_k - n(t/\varepsilon, x) w_k^\rho, & x \in \Omega,\ t \in (\tau_0 k, \tau_0(k+1)) \\ w_k = 0, & x \in \partial\Omega,\ t \in (\tau_0 k, \tau_0(k+1)) \\ w_k(\tau_0 k, x) = 3E \geq 0, & x \in \Omega. \end{cases}$$

It is evident that $w_0 = u_\varepsilon$ and hence, $\|w_0(\tau_0)\|_{L^\infty} \leq 5/2E < 3E = w_1(\tau_0)$. Moreover, by Theorem 1.2.1 we have that $u_\varepsilon(t) \leq w_1(t)$ for $t \in [\tau_0, 2\tau_0]$.

Since the arguments above can be replicated for any $w_k(\tau_0(k+q))$, by induction we have that

$$\|w_k(\tau_0(k+1))\|_{L^\infty} < w_{k+1}(\tau_0(k+1)), \quad \text{and} \quad u_\varepsilon(t) \leq w_{k+1}(t), \quad t \in [\tau_0 k, \tau_0(k+1)].$$

Thus, for every $k \geq 0$ we define $w(t,\cdot) := w_k(t,\cdot)$ for $t \in [\tau_0 k, \tau_0(k+1)]$ which is a supersolution of u_ε and is bounded

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq \|w(t)\|_{L^\infty(\Omega)} \leq \sup_{t\in[0,\tau_0]} \|w_0(t)\|_{L^\infty(\Omega)} = M < \infty, \quad \forall t > 0, 0 < \varepsilon \leq \varepsilon_1.$$

□

Corollary 4.2.7. *If $\overline{K_a}$ is slim (for instance $\overline{K_a} = \emptyset$) then all solutions of (4.2.1) are bounded for $\varepsilon > 0$ small enough, for all values of $\lambda \in \mathbb{R}$.*

Moreover, we may have the situation where $K(t)$ is a moving large ball so that solutions grow up if $K(t)$ moves slowly but if $K(t)$ moves fast and $\overline{K_a} = \emptyset$ then all solutions are bounded.

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