Universidad Autónoma de Madrid
Escuela de Doctorado

Non-existence, strong ill-posedness and loss of regularity for active scalar equations

Luis Martínez Zoroa,

Programa de Doctorado en matemáticas
Centro realización Tesis/Institución ICMAT.
Departamento UAM/Facultad de matemáticas.

Madrid, 2023

Agradecimientos

En primer lugar, me gustaría mostrar mi agradecimiento a mi director, Diego Córdoba, tu curiosidad, optimismo y pasión son contagiosos, y puedo decir con sinceridad que reunirse contigo es siempre un placer. No puedo imaginar un mejor director.

Me gustaría también dar las gracias al Doctor D. Fernando Quirós por la tarea de ser mi tutor y por la gran paciencia que ha mostrado conmigo estos años.

A mi familia, que siempre me han querido y apoyado, especialmente a mi padre y a mi madre: no habría podido llegar hasta aquí sin vosotros, y sois un referente de la clase de persona que quiero ser. También quiero mencionar a otras dos personas que han ayudado a avivar mi interés por las matemáticas, mi tío Antonio y mi abuelo Procopio.

I want to thank Wojciech Ożański for his very valuable feedback, but also because working with him has been a very pleasant experience that I think has helped me become a better researcher.

I would also like to thank Professor Charles Fefferman for all his feedback, specially the very useful comments for chapter 2.

Finally, I would like to thank my girlfriend Dora: You bring me joy and make me want to be better, and everything feels like it is going to work out when I am with you. Moja čebula.

Contents

0.1 Summary

In this thesis we study the behaviour of several active scalar equations in spaces where well-posedness is not expected, namely we study 2D-Euler, the Surface Quasi-Geostrophic equation (SQG) and the generalized Surface Quasi-Geostrophic equation (gSQG) . Even though one expects some kind of bad behaviour to happen, such as non-uniqueness, wild norm growth or non-existence of solutions, it is hard to predict what the behaviour will be for a specific model. Furthermore, some counter-intuitive phenomena are possible, such as global existence when wild norm growth is possible.

In chapter 2, we study the SQG equation both in H^s and in C^k. For H^s, $s \in (\frac{3}{2}, 2]$ and C^k, $k \geq 2$ a natural number, we obtain wild norm growth as well as non existence of solutions. The same tools we apply can be used to obtain similar results in other critical spaces, such as $W^{1,\infty}$, as well as some other supercritical spaces (for example H^s with $s \leq \frac{3}{2}$).

In chapter 3, we study the generalized Surface Quasi-Geostrophic equation with more singular velocities than SQG in the spaces $C^{k,\beta}$. For this family of equations, the low regularity of the velocity suggests the possibility of ill-posedness, but a important cancelation in the evolution equation makes it so that there is local well-posedness in H^s. When considering the spaces $C^{k,\beta}$, it is unclear if this cancellation is enough to obtain well-posedness. The results in chapter 3 show that not only there is wild norm growth for this family of equations, but in fact there is non-existence of solutions.

In chapter 4 we study the 2D-Euler equation, and find initial conditions that produce global unique classical solutions with instant gap loss of regularity, i.e., they start in some space H^s ($s \in (0, 1)$) and for all $t > 0$ the solution is, at most, in some given space $H^{s'}$ with $s' < s$.

Finally, in chapter 5 we study the SQG equation again, but this time we add some fractional diffusion. Diffusion in general makes it harder for any kind of ill-posedness to occur, since it has a regularizing effect in the solutions. Despite this, we prove that for Sobolev spaces below the critical regularity strong ill-posedness can happen and in fact we prove non-existence of uniformly bounded solutions.

0.2 Resumen

En esta tesis estudiamos el comportamiento de diversas ecuaciones de escalar activo, en concreto 2D-Euler, la ecuación Cuasi-Geostrófica Superficial (SQG) y la ecuación Cuasi-Geostrófica Superficial generalizada (gSQG) en espacios donde se desconoce si el problema está bien propuesto. A pesar de que lo esperable es que se observe un comportamiento indeseado, como no unicidad, crecimiento salvaje de la norma o no existencia de soluciones, es difícil hacer predicciones sobre el comportamiento dada una ecuación concreta. Además, existen modelos para los cuales se producen fenómenos contraintuitivos, como la existencia global de soluciones a pesar de existir crecimiento salvaje de la norma.

En el capítulo 2, estudiamos la ecuación SQG en los espacios H^s y C^k. En H^s, $s \in (\frac{3}{2}, 2]$, y C^k, $k \geq 2$ un numero natural, demostramos que puede tener lugar un crecimiento salvaje de la norma e incluso que es posible la no existencia de soluciones. Las herramientas utilizadas servirían para demostrar resultados similares en otros espacios críticos como $W^{1,\infty}$ y en otros espacios supercríticos como H^s con $s \leq \frac{3}{2}$.

En el capítulo 3, estudiamos la ecuación Cuasi-Geostrófica Superficial generalizada en los espacios $C^{k,\beta}$ cuando la velocidad es más irregular que en SQG. Para esta familia de ecuaciones, la baja regularidad de la velocidad sugiere que el problema podría estar mal propuesto, pero una cierta cancelación en la ecuación de evolución permite demostrar que el problema está bien propuesto en H^s. Cuando se consideran los espacios $C^{k,\beta}$, no es evidente si dicha cancelación es suficiente como para demostrar que el problema está bien propuesto. Los resultados obtenidos en el capítulo 3 demuestran que el problema no está bien propuesto en $C^{k,\beta}$ y, de hecho, puede darse tanto el crecimiento salvaje de la norma como la no existencia de soluciones.

En el capítulo 4 estudiamos la ecuación 2D-Euler, y encontramos condiciones iniciales que producen una solución clásica única y global, pero dicha solución sufre instantáneamente un salto en su regularidad, más concretamente, la solución empieza perteneciendo al espacio H^s ($s \in (0, 1)$) pero, para cualquier $t > 0$, está como mucho en $H^{s'}$ para un cierto $s' < s$.

Finalmente, en el capítulo 5 estudiamos una vez más la ecuación SQG, pero esta vez le añadimos una difusión fraccionaria. La difusión en general favorece que el problema esté bien propuesto, dado que produce un efecto regularizador sobre la solución. A pesar de esto, demostramos que, para espacios de Sobolev con regularidad supercrítica, el problema está fuertemente mal propuesto e incluso se da la no existencia de soluciones uniformemente acotadas.

Chapter 1

Introduction

On this thesis we will study the behaviour of active scalar equations, i.e., PDEs of the form

$$\frac{\partial}{\partial t}f + v(f)\cdot\nabla f = 0, \tag{1.1}$$
$$f(x,0) = f_0(x),$$

where $v(f)) = (v_1(f), v_2(f))$ is a given operator.

Many important equations can be written in this form, such as 2D-Euler ($v(f) = \nabla^{\perp}\Delta^{-1}f$), the surface quasi-geostrophic equation ($v(f) = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}f$) or the Prandtl equation ($v_1(f) = f$, $\frac{\partial v_1(f)}{\partial x_1} = -\frac{\partial v_2(f)}{\partial x_2}$), and extra terms can be added to the equation to model specific phenomena, such as diffusion or external forces. One property that all these equations we mentioned have in common, is that

$$\frac{\partial v_1(f)}{\partial x_1} = -\frac{\partial v_2(f)}{\partial x_2}$$

i.e., they produce an incompressible flow. Even though this is not necessary, all the equations that we will consider in this thesis produce incompressible flows. A very important property of such equations is that, given a solution $f(x,t)$, if we define

$$\frac{\partial}{\partial t}\phi(x,t) = v(f)(x = \phi(x,t), t)$$
$$\phi(x,0) = x$$

then (assuming v is regular enough) $f(x,t) = f_0(\phi^{-1}(x,t))$. This in particular implies that the L^p norms ($1 \leq p \leq \infty$) are conserved.

In general, we say that an evolution equation is (locally) well-posed in some space X if, for any initial conditions $f_0(x) \in X$, the following conditions are fulfilled:

- Existence: There exists a solution $f(x,t) \in X$ for $t \in [0,\epsilon)$ for some $\epsilon > 0$.
- Uniqueness: $f(x,t)$ is the only solution fulfilling $f(x,t) \in X$ for the time interval $[0,\epsilon)$.
- Continuity: The solution (and, in particular, the time of existence) depends continuously on the initial conditions, i.e., given a solution $f(x,t)$ that exists for $t \in [0,\epsilon)$, for each $t_0 \in [0,\epsilon)$, we have that
$$\lim_{||f(x,0)-\tilde{f}(x,0)||_X \to 0} \sup_{t\in[0,t_0]} ||f(x,t) - \tilde{f}(x,t)||_X = 0$$
with $\tilde{f}(x,t)$ another solution to the evolution equation.

This definition of well-posedness, which is the well-posedness in the sense of Hadamard, is unfortunately in general a little too restrictive when talking about active scalar equations. The reason for this is that some spaces, and in particular $C^{k,\alpha}$, have the property that there exist functions $f(x)$ such that

$$\lim_{c\to 0}||f(x+c)-f(x)||_{C^{k,\alpha}} \neq 0,$$

so that, very often, active scalar equations are not well-posed in $C^{k,\alpha}$, even when the solutions exist and have nice properties. An example in that regard is the 2D-Euler equation, where initial conditions in $C^{k,\alpha}$ produce unique global solutions but we do not have well-posedness in the sense of Hadamard. Keeping this in mind, we will consider a different definition of well-posedness through this thesis that is better suited for active scalars equation:

Definition 1. We say that an active scalar equation as in (1.1) is well-posed in a Banach space X if, for $f(x,0) \in X$

- Existence: There exists a solution $f(x,t) \in X$ for $t \in [0,\epsilon)$ for some $\epsilon > 0$.
- Uniqueness: $f(x,t)$ is the only solution fulfilling $f(x,t) \in X$ for the time interval $[0,\epsilon)$.
- Norm control: There exists a continuous function

  $$H : A \to (0,\infty)$$

  with

  $$A := \{(a,b) \in \mathbb{R}^2 : b > a \geq 0\}$$

  such that, for any solution $f(x,t)$, we have that, if $t_1 \in [t_0, t_0 + H(\lambda_1,\lambda_2)]$, then

  $$||f(x,t_0)||_X \leq \lambda_1 \to ||f(x,t_1)||_X \leq \lambda_2.$$

The norm control condition allows us to assure that, if an active scalar equation is well-posed, then the norm of the solutions cannot grow in a very wild way. Note that in particular $H(a,b)$ gives us a lower bound for the time of existence for any solution with initial conditions with $||f_0(x)||_X = a$.

The main goal of this thesis is to study the behaviour of active scalar equations in spaces where we do not have local well-posedness in the sense of Definition 1. When this happens, we expect that we have either non-existence of solutions, non-uniqueness or a wild behaviour of the evolution of the norm, and in this thesis we will focus on showing either wild behaviour of the norm or non-existence of solutions.

We will distinguish between several different kinds of ill-posedness, depending on how bad the behaviour of the solutions is.

Definition 2. We say that an evolution equation is mildly ill-posed in the space X if there exists a constant c such that for any $\epsilon > 0$ we can find a solution $f(x,t)$ such that

$$||f(x,0)||_X \leq \epsilon,\ \sup_{t\in[0,\epsilon]}||f(x,t)||_X \geq c.$$

Definition 3. We say that an evolution equation is strongly ill-posed in the space X if for any $\epsilon > 0$ we can find a solution $f(x,t)$ such that

$$||f(x,0)||_X \leq \epsilon,\ \sup_{t\in[0,\epsilon]}||f(x,t)||_X \geq \frac{1}{\epsilon}.$$

There are a few relevant comments regarding these two definitions. First, mild ill-posedness implies that a continuous function $H(a,b)$ as in Definition 1 does not exist, but it could be possible to define $H(a,b)$ for b big enough or for $b-a$ big enough. This, however, does not always imply a very wild behaviour of the solution. For example, when one studies the evolution of perturbations around a stationary radial solution $g(r) \in C^{k,\alpha}$ for 2D-Euler, we obtain a system of the form

$$\frac{\partial}{\partial t}f(x,t) + v(f)\cdot\nabla(f(x,t)+g(r)) + v(g)\cdot\nabla f(x,t) = 0,$$

which can be mildly ill-posed in $C^{k,\alpha}$ despite the fact that $f(x,t)$ exists for all time and is in $C^{k,\alpha}$ if $f(x,0) \in C^{k,\alpha}$ and in fact $||f(x,t)+g(r)||_{C^{k,\alpha}}$ is continuous in time.

Another thing to keep in mind is that, usually, to show strong or mild ill-posedness we find initial conditions $f_0(x) \in Y \cap X$, with Y a space where we actually have well-posedness, to ensure that there exists some kind of solution. When this is not an option (for example, when it is not known if the evolution equation is well-posed in any space), one can consider similar definitions that work by contradiction: We find initial conditions such that, if a solution exists, then the norm of the solution will grow in the way specified by Definitions 2 or 3.

The wild growth of the norm that occurs when an evolution equation is strongly ill-posed in a space X suggests the possibility of finding initial conditions where no solution exists at all: If a solution grows infinitely fast, it could leave the space X instantly. There are several possible definitions of what non-existence even means, but the basic one is the following:

Definition 4. Given an evolution equation, we say that there is non-existence of uniformly bounded solutions in the space X if, for all $\epsilon > 0$ we can find initial conditions $f_0(x)$ with $||f_0(x)||_X \leq \epsilon$ such that, for any solution $f(x,t)$ with $f(x,0) = f_0(x)$ and any $\delta > 0$ we have

$$\text{ess-sup}_{t\in[0,\delta]}||f(x,t)||_X = \infty.$$

Throughout this thesis we will obtain several different results regarding loss of regularity/non-existence of solutions, some stronger than others, but all of them will, at least, imply a result like Definition 4.

In order to show strong ill-posedness, we will usually consider what we call pseudo-solutions, so it is important to clarify what we mean exactly by a pseudo-solution.

Definition 5. Given an evolution equation

$$\frac{\partial f}{\partial t} = H(f), \quad f(t=0) = f_0(x)$$

with $H(f)$ some operator, we say that $\bar{f}(x,t)$ is a pseudo-solution to the evolution equation with initial conditions $f_0(x)$ if it fulfills

$$\frac{\partial \bar{f}}{\partial t} = H(\bar{f}) + F(x,t), \quad \bar{f}(t=0) = f_0(x)$$

for some function $F(x,t)$.

Although this definition of pseudo-solution is very general (since $F(x,t)$ can be basically anything we want), we are only actually interested in pseudo-solutions with the source term $F(x,t)$ small in an appropriate norm. In order to obtain strong ill-posedness in a certain space, we will find (usually explicit) pseudo-solutions that exhibit the desired behaviour we want to show (i.e., arbitrarily fast norm growth), and with $F(x,t)$ smooth and small enough. In general one expects that, if $F(x,t)$ is sufficiently small, the pseudo-solution will have the same qualitative behaviour as an actual solution to the evolution equation, which would then imply strong ill-posedness.

One thing to keep in mind is that, even though one expects that

$$\lim_{F(x,t)\to 0}\bar{f}(x,t) = f(x,t)$$

this actually depends on the specific evolution equation we are considering, the specific space in which $F(x,t)$ tends to 0, and the properties of $\bar{f}(x,t)$, so in particular proving convergence can be difficult if we do not know $\bar{f}(x,t)$ explicitly.

As an example we can consider the evolution equation

$$\frac{\partial f}{\partial t} = f^2 + H(f)$$

for $f(x) : [0,\pi] \to \mathbb{R}$, and with H the Hilbert transform. This evolution is locally well-posed in C^α for $\alpha \in (0,1)$, but the appearance of the Hilbert transform suggest that it might be ill-posed in C^1.

This can actually be proved by considering the family of pseudo-solutions

$$\bar{f}_{N,K}(x,t) := \sum_{i=1}^{K} \frac{\cos(Nix - t)}{Ni^2}$$

which fulfil

$$\frac{\partial \bar{f}_{N,K}}{\partial t} = H(\bar{f}_{N,K})$$
$$||\bar{f}_{N,K}(x,t=0)||_{C^1} \leq C$$
$$||\bar{f}_{N,K}(x,t)||_{C^1} \geq c_0 t \ln(K)$$

for some $C, c_0 > 0$, for $t \in [0, \frac{\pi}{4}]$.

Furthermore, it is easy to check that

$$||\bar{f}^2_{N,K}(x,t)||_{H^1} \leq \frac{C}{N}$$

for some $C > 0$ depending on K.

One can then use this to prove that, if we define

$$\frac{\partial f_{N,K}}{\partial t} = f^2_{N,K} + H(f_{N,K})$$
$$f_{N,k}(x,0) = \bar{f}_{N,k}(x,0)$$

then, for any fixed K, for N big and $t \in [0, \frac{\pi}{4}]$ we get

$$||f_{N,k}(x,t) - \bar{f}_{N,k}(x,t)||_{H^1} \leq \frac{Ct}{N}$$

for some C depending on K. This plus the properties of $\bar{f}_{N,k}(x,t)$ (in particular, the fact that the function is $\frac{2\pi}{N}$-periodic) allows then us to show that, for N big

$$||f_{N,k}(x,t)||_{C^1} \geq \frac{c_0}{2} t \ln(K)$$

which proves strong ill-posedness.

1.1 Overview of the thesis

Chapter 2 will cover the Surface Quasi-geostrophic equation, obtaining strong ill-posedness and non existence of solutions in C^k ($k \geq 2$ a natural number) and H^s ($s \in (\frac{3}{2}, 2)$), as well strong ill-posedness and non-existence of uniformly bounded solutions in H^2 (see [39]).

Chapter 3 studies the generalized Quasi-geostrophic equation for singular kernels, and we manage to show strong ill-posedness and non-existence of solutions in $C^{k,\beta}$ with the specific values of k and β depending the specific kernel considered (see [40]).

In Chapter 4 we deal with the 2D-Euler equation, and we construct unique, global solutions $\omega(x,t)$ that lose some regularity instantly, and more precisely

$$\omega(x,0) \in H^s,\ \omega(x,t) \notin H^{s'}$$

for all $t > 0$, for some $s' < s$, $s \in (0,1)$ (see [41]).

Finally in Chapter 5 we consider the Surface Quasi-geostrophic equation with (supercritical) fractional diffusion, and construct global unique solutions $w(x,t)$ with

$$||w(x,0)||_{H^s} \leq \epsilon,\ \sup_{[0,\epsilon)} ||w(x,t)||_{H^s} = \infty$$

and with $w(x,t) \in C^\infty$ for all $t > 0$ (see [38]).

Chapter 2

Strong ill-posedness and Non-existence results for SQG

2.1 Introduction

In this chapter we will focus on the study of the Surface Quasi-Geostrophic equation, from now on the SQG equation.

We say a function $\theta(x,t) : \mathbb{R}^2 \times [0,T) \to \mathbb{R}$ is a solution to the SQG equation with initial conditions $\theta(x,0) = \theta_0(x)$ if the equation

$$\frac{\partial \theta}{\partial t} + v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2} = 0 \tag{2.1}$$

is fulfilled for every $x \in \mathbb{R}^2$ and $\theta(x,t)$ is (pointwise) differentiable for $(x,t) \in \mathbb{R}^2 \times [0,T)$. The velocity field $v = (v_1, v_2)$ is defined by

$$v_1 = -\frac{\partial}{\partial x_2} \Lambda^{-1} \theta = -\mathcal{R}_2 \theta$$

$$v_2 = \frac{\partial}{\partial x_1} \Lambda^{-1} \theta = \mathcal{R}_1 \theta$$

where $\mathcal{R}_i$ are the Riesz transforms in 2 dimensions, with the integral expression

$$\mathcal{R}_j \theta = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} \frac{(x_j - y_j)\theta(y)}{|x-y|^3} dy_1 dy_2$$

for $j = 1, 2$. We denote $\Lambda^\alpha f \equiv (-\Delta)^{\frac{\alpha}{2}} f$ by the Fourier transform $\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$.

This model arises in a geophysical fluid dynamics context (see [60] and [88]) and its mathematical analysis was initially treated by Constantin, Majda and Tabak in [27] motivated by the number of traits it shares with 3-D incompressible Euler system, where they already established local existence in H^s (see also [28] for bounded domains) and in the case of $C^{k,\alpha}$ ($k \geq 1$ and $1 > \alpha > 0$), see [98] by Wu. In the critical Sobolev space H^2 Chae and Wu [21] proved local existence for a logarithmic inviscid regularization of SQG (see also [66]). Finite time formation of singularities for smooth initial data with finite energy remains an open problem for both SQG and 3-D incompressible Euler equations.

Due to incompressibility and the transport structure of SQG the L^p ($1 \leq p \leq \infty$) norms of the scalar θ and the L^2 norm of the velocity field $v = (v_1, v_2)$ (kinetic energy) are conserved quantities of the system (2.1) for sufficiently regular solutions. Global existence of weak solutions in L^2 was proven by Resnick in [89] (see also [29] in the case of bounded domains) and extended by Marchand in [84] to the class of initial data in L^p with $p > \frac{4}{3}$. However non-uniqueness of weak solutions was obtained by Buckmaster, Shkoller and Vicol in [12] for solutions such that $\Lambda^{-1}\theta \in C_t^\sigma C_x^\beta$ with $\frac{1}{2} < \beta < \frac{4}{5}$ and $\sigma < \frac{\beta}{2-\beta}$.

One of the main objectives of this chapter is to construct solutions in $\mathbb{R}^2$ of SQG that initially are in $C^k \cap L^2$ ($k \geq 2$) but are not in C^k for $t > 0$. Note that if we consider a velocity field $v(\theta) = \nabla^{\perp}\Lambda^{-(1+\epsilon)}(\theta)$ with $\epsilon > 0$, then we have local existence in C^k for (2.1). We also prove strong ill-posedness in H^s for supercritical spaces in the range $s \in (\frac{3}{2}, 2)$ and for the critical space H^2. Moreover we construct solutions that are initially in H^s for $s \in (\frac{3}{2}, 2)$ but are not in H^s for $t > 0$, and that are unique in a certain sense that we will specify later. For the SQG equation, there were no strong ill-posedness results in H^s and C^k prior to the ones obtained in this chapter. There are ill-posedness results for active scalars with more singular velocities obtained by Kukavica, Vicol and Wang in [78] and, in the case of SQG, in [55] Elgindi and Masmoudi a mild ill-posedness result is obtained for perturbations of a stationary solution. This, however, does not imply mild or strong ill-posedness for SQG. A few days after the results of this chapter appeared on the arXiv, Jeong and Kim [64] posted an article on the arXiv with a similar result in the case of the critical Sobolev space H^2.

There are some remarkable results regarding norm growth in the periodic setting for SQG. Kiselev and Nazarov [73] showed that there exists initial conditions with arbitrarily small norm in H^s ($s \geq 11$) that become large after a long period of time. Recently, He and Kiselev proved in [59] an exponential in time growth for the C^2 norm

$$\sup_{t\leq T}|\nabla^2\theta|_{L^\infty} \geq \exp\gamma T \qquad \text{for } \gamma(\theta_0) > 0.$$

On the other hand, numerical simulations suggested the existence of solutions with very fast growth of $|\nabla\theta|$ starting with a smooth profile by a collapsing hyperbolic saddle scenario (see [27], [87] and [26]). Such a scenario cannot develop a singularity as shown analytically in [35] and [37], where a double exponential bound on $|\nabla\theta|$ is obtained. A different blow-up scenario was proposed in [90] where the fast growth of $|\nabla\theta|$ is associated to a cascade of filament instabilities.

2.1.1 The main theorems

In this chapter we prove the following results:

Theorem 2.1.1. (Strong ill-posedness in C^k) For any $c_0 > 0$, $M > 0$, $2 \leq k \in \mathbb{N}$ and $t_* > 0$, there exists $\theta_0(x) \in H^{k+\frac{1}{4}} \cap C^k$ with $||\theta_0(x)||_{C^k} \leq c_0$ such that the unique solution $\theta(x,t) \in H^{k+\frac{1}{4}}$ to the SQG equation (2.1) with initial conditions $\theta_0(x)$ satisfies $||\theta(x,t_*)||_{C^k} \geq Mc_0$.

Theorem 2.1.2. (Non existence in C^k) Given $c_0 > 0$, $t_* > 0$ and $2 \leq k \in \mathbb{N}$, there exists $\theta_0(x) \in H^{k+1/8} \cap C^k$ for the SQG equation (2.1) such that $||\theta_0||_{C^k} \leq c_0$ and the unique solution $\theta(x,t) \in H^{k+1/8}$ exists and satisfies that $||\theta(x,t)||_{C^k} = \infty$ for all $t \in (0, t_*]$.

In fact, for the initial conditions given by Theorem 2.1.2 there is no solution $\theta(x,t) \in L^\infty_t L^2_x$ to (2.1) with those initial conditions and $||\theta(x,t)||_{C^k} \leq M(t)$, for any $M(t) : \mathbb{R}_+ \to \mathbb{R}_+$, even if we allow for $||M(t)||_{L^\infty} = \infty$. For more details see Remark 2 after Theorem 2.2.2.

Theorem 2.1.3. (Strong ill-posedness in H^s) For any $c_0 > 0$, $M > 0$, $s \in (\frac{3}{2}, 2]$ and $t_* > 0$, there exists a H^β function $\theta_0(x)$ with $||\theta_0(x)||_{H^s} \leq c_0$ such that the only solution $\theta(x,t) \in H^\beta$, with $\beta(s) > 2$ to the SQG equation (2.1) with initial conditions $\theta_0(x)$ satisfies $||\theta(x,t_*)||_{H^s} \geq Mc_0$.

Remark 1. The purpose of this chapter is not to obtain the optimal range of Sobolev spaces in which strong ill-posedness is achieved. There are refinements to the methods used in Theorem 2.1.3 that would allow us to decrease the lower bound in the interval of ill-posedness.

Theorem 2.1.4. (Non existence in H^s in the supercritical case) For any t_*, $c_0 > 0$ and $s \in (\frac{3}{2}, 2)$ we can find initial conditions $\theta_0(x)$, with $||\theta_0(x)||_{H^s} \leq c_0$ such that there exists a solution $\theta(x,t)$ to (2.1) with $\theta(x,0) = \theta_0(x)$ satisfying $||\theta(x,t)||_{H^s} = \infty$ for all $t \in (0, t_*]$. Furthermore, it is the only solution with initial conditions $\theta_0(x)$ such that $\theta(x,t) \in L^\infty_t C^{\alpha_1}_x \cap L^\infty_t L^2_x$ ($0 < \alpha_1 < \frac{1}{2}$) with the property that $||\theta(x,t)||_{H^{\alpha_2}} \leq M(t)$ ($1 < \alpha_2 \leq \frac{3}{2}$) for some function $M(t)$.

Theorem 2.1.5. (Non uniform existence in H^2) For any $c_0 > 0$ there exist initial conditions $\theta(x,0)$ with $||\theta(x,0)||_{H^2} \leq c_0$ such that there is no solution $\theta(x,t)$ to (2.1) satisfying

$$\text{ess-sup}_{t\in[0,\epsilon]}||\theta(x,t)||_{H^2} = M$$

for any $\epsilon, M > 0$.

The proof of Theorems 2.1.4 and 2.1.5 can be adapted to work in the critical spaces $W^{1+\frac{2}{p},p}$, $p \in (1,\infty]$, but we will not go into detail since that is not the goal of the chapter. For more information regarding the necessary changes to adapt the proof for these cases, see Remark 5 after Theorem 2.4.2.

2.1.2 The strategy of the proof

Ill-posedness in critical spaces for the incompressible Euler equations was already considered in the papers by Bourgain and Li (see [9] and [8]) obtaining strong ill-posedness for the velocity in the 2D and 3D Euler equations in C^k, $k \geq 1$ and for vorticity in the space $H^{d/2}$ (d the dimension). In fact, they obtained stronger results: in [8] they obtain a velocity u satisfying, for $0 < t_0 \leq 1$

$$\text{ess-sup}_{0<t<t_0} ||u(x,t)||_{C^k} = \infty,$$

$$||u(x,0)||_{C^k} \leq c_0$$

and in [9] the vorticity ω satisfies

$$\text{ess-sup}_{0<t<t_0} ||\omega(x,t)||_{\dot{H}^{\frac{d}{2}}} = \infty,$$

$$||\omega(x,0)||_{\dot{H}^{\frac{d}{2}}} \leq c_0,$$

that is to say, they obtained non-existence of uniformly bounded solutions in H^1 for the vorticity and in C^k for the velocity. Later, analogous results were obtained by Elgindi and Masmoudi in [55] and Elgindi and Jeong in [54] with a different approach. Recently, Kwon proved in [77] that there is still strong ill-posedness in H^1 for a regularized version of the 2D incompressible Euler equations.

Our strategy in this chapter for proving strong ill-posedness for SQG differs from the previous works mentioned above since there is no global existence result for SQG in H^s. More precisely for Theorems 2.1.1, 2.1.2, 2.1.3 and 2.1.4, we construct solutions by perturbing radial stationary solutions $\theta = \theta(r)$ and, in order to obtain precise bounds of the errors, we consider an explicit in time family of pseudo-solutions of SQG for $t \in [0,T]$, namely

$$\begin{aligned}\bar{\theta}_{\lambda,J,N}(r,\alpha,t) :=& \lambda f_1(r)\\ &+ \lambda f_2(N^{1/2}(r-1)+1)\sum_{j=1}^{J}\frac{\sin(Nj\alpha - \lambda tNj\frac{v_\alpha(f_1)}{r} - \lambda C_0 t - \frac{\pi}{2}j)}{N^k j^{k+1}},\end{aligned}$$

where (r,α) are the polar coordinates, f_i are smooth compactly supported radial functions, $v_\alpha(f_1)$ is the angular velocity generated by the function f_1, the parameters fulfil $(\lambda, J, N) \in (\mathbb{R}_+, \mathbb{N}, \mathbb{N})$ and C_0 is a constant that arises from the velocity operator. This $\bar{\theta}_{\lambda,J,N}$ fulfills the evolution equation

$$\frac{\partial\bar{\theta}_{\lambda,J,N}}{\partial t} + \frac{\partial\bar{\theta}_{\lambda,J,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1)}{r} + \lambda C_0 H(\bar{\theta}_{\lambda,J,N}) = 0,$$

where H denotes the Hilbert transform with respect to the α variable, and for any fixed λ and J, as N becomes big, this pseudo-solution becomes a good approximation of SQG. The ill-posedness arises from the unboundedness of the operator H in the $C^k \cap L^2$ spaces. Note however that the appearance of an unbounded operator in our evolution equation does not imply directly ill-posedness, and for example in the Burger-Hilbert's equation

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} + H(f) = 0,$$

although the L^∞ norm has a fast growth (see [14]) as long as the solution is $C^{1,\delta}$, Bressan and Nguyen [11] proved the surprising result of global existence in $L^2 \cap L^\infty$.

We denote by $\theta_{\lambda,J,N}(r,\alpha,t)$ the unique $H^{k+\frac{1}{4}}$ solution of (2.1) satisfying initially

$$\theta_{\lambda,J,N}(r,\alpha,0) = \bar{\theta}_{\lambda,J,N}(r,\alpha,0).$$

We will prove that, for any fixed λ and J, for sufficiently large N we have

$$||\theta_{\lambda,J,N}(r,\alpha,t)-\bar{\theta}_{\lambda,J,N}(r,\alpha,t)||_{H^k}\leq CtN^{-(\frac{1}{4}+a(k))}$$

where $a(k)>0$ and the constant C depends only on the parameters λ, J, k and T. With this bound and the properties of the pseudo-solution we obtain

$$||\theta_{\lambda,J,N}(r,\alpha,t)||_{C^k}\geq \tilde{C}\lambda^2\ln(J)t$$

where $\tilde{C}$ is a universal constant.

Once we have solutions with arbitrarily large growth in norm we prove non-existence of solutions in C^k by considering the following initial conditions

$$\theta(x,0)=\sum_{n\in\mathbb{N}}T_{R_n}(\bar{\theta}_{\lambda_n,J_n,N_n}(x,0))$$

with $T_R(f(x_1,x_2)):=f(x_1+R,x_2)$. By choosing appropriately the parameters $(\lambda_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$, $(N_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ we can show that the unique solution $\theta(x,t)\in H^{k+\frac{1}{8}}$ with this initial data will leave C^k instantaneously. In particular the solution $\theta(x,t)$ is not in C^k for any time $t\in(0,T]$.

In the case of strong ill-posedness in Sobolev spaces, Theorem 2.1.3, we will use a similar strategy for $s\in(\frac{3}{2},2)$, although the proofs are more involved since we do not have any existence result for the supercritical Sobolev spaces. However, in the critical case (Theorem 2.1.5) it is not clear that a suitable pseudo-solution could be constructed by perturbing a radial solution. In order to overcome this obstacle we need a different strategy. In this case our initial data is similar to the one consider in [9] with the following expression

$$\theta_{c,J,b}(x,0)=\sum_{j=1}^{J}c\frac{f(b^{-j}r)b^j\sin(2\alpha)}{j},\quad \frac{1}{2}>b>0,$$

where the radial function $0\leq f\in C^\infty$ has $\mathrm{supp}(f)\in[\frac{1}{2},\frac{3}{2}]$, $c>0$ and $J\in\mathbb{N}$. The main difficulty when considering this type of initial conditions is that the usual energy estimates only give existence for a short time interval which does not provide enough growth in H^2. To obtain improved time intervals of existence we decompose our solution as a sum of pseudo-solutions with initial conditions

$$c\frac{f(b^{-j}r)b^j\sin(2\alpha)}{j}$$

for $j=1,...,J$. To finish the proof we perturb this solution with a small H^2 function localized around the origin that will experience very large norm growth.

The chapter is organized as follows. First in section 2.2 we prove strong ill-posedness and non existence for the space C^k. In section 2.3 we show strong ill-posedness and non existence for Sobolev spaces in the supercritical case. Finally in section 2.4 we prove strong ill-posedness and non-existence of uniformly bounded solutions for the critical H^2 space.

2.1.3 Notation

In this chapter we will consider functions $f(x):\mathbb{R}^2\to\mathbb{R}$ in C^k with k a positive integer and H^s with s a positive real number. These spaces allow many different equivalent norms, but we will specifically use

$$||f(x)||_{C^k}=\sum_{i=0}^{k}\sum_{j=0}^{i}||\frac{\partial^i f(x)}{\partial^j x_1\partial^{i-j}x_2}||_{L^\infty}$$

and for H^s, when s is a positive integer we will use

$$||f(x)||_{H^s}=\sum_{i=0}^{s}\sum_{j=0}^{i}||\frac{\partial^i f(x)}{\partial^j x_1\partial^{i-j}x_2}||_{L^2},$$

where the derivative is understood in the weak sense.

For non integer s , the standard way of defining the norm is by

$$||f(x)||_{H^s} = ||\mathcal{F}^{-1}\Big[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\Big]||_{L^2},$$

where $\mathcal{F}$ is the Fourier transform. We will not require to use this definition to compute the norm in these spaces through this chapter. For s a positive integer, we will sometimes write

$$||f(x)1_A||_{H^s},$$

where 1_A is the characteristic function in the set A. This is a slight abuse of notation since the function $f(x)1_A$ may not be in H^s, but we will use this as a more compact notation to write

$$\sum_{i=0}^{s}\sum_{j=0}^{i}(\int_A(\frac{\partial^i f(x)}{\partial^j x_1\partial^{i-j}x_2})^2 dx)^{\frac{1}{2}}.$$

Analogously, we will use

$$||f(x)1_A||_{C^k} := \sum_{i=0}^{k}\sum_{j=0}^{i}\text{ess-sup}_{x\in A}(\frac{\partial^i f(x)}{\partial^j x_1\partial^{i-j}x_2}).$$

We will work both in normal cartesian coordinates and in polar coordinates, using the change of variables $x_1 = r\cos(\alpha)$, $x_2 = r\sin(\alpha)$. We will sometimes define $f(x)$ as a function in the variable (x_1, x_2) and then refer to $f(r,\alpha)$ in polar coordinates (or vice versa), and this is an abuse of notation since we should actually write, if $F(r,\alpha)$ is the change of variables that takes us from (r,α) to (x_1, x_2), $f(F(r,\alpha))$. Furthermore, given a function $f(r,\alpha)$ in polar coordinates, we define

$$||f(r,\alpha)||_{H^s} := ||f(F^{-1}(x))||_{H^s},$$

$$||f(r,\alpha)||_{C^k} := ||f(F^{-1}(x))||_{C^k}.$$

For two sets A_1, A_2, we will use $d(A_1, A_2)$ to refer to the distance between the sets

$$d(A_1, A_2) := \inf_{x\in A_1, y\in A_2}|x-y|.$$

2.2 Strong ill-posedness and non existence in C^k

To prove ill-posedness in C^k we construct fast growth solutions by perturbing in a suitable way a stationary smooth radial solution. In contrast, there are previous results ([16] and [17]) where the perturbation of a radial function led to global C^4 rotating solutions and enhanced lifespan of solutions respectively.

In this section we will show that, for a specific kind of perturbation we can predict the behaviour of the solution with a very small error. The perturbation will be composed of functions of the form

$$f(N^{1/2}(r-1)+1)\sin(Nn\alpha)$$

where f is a given smooth function and N, n are integers. Below we will obtain the properties that will allow us to work with this kind of functions.

2.2.1 Estimates on the velocity field.

In this section we will use the following expression of the velocity field

$$v(\theta(.))(x) = \frac{\Gamma(3/2)}{\pi^{3/2}}P.V.\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}\theta(y)}{|x-y|^3}dy_1dy_2$$

with $v = (v_1, v_2)$ and for a vector (a, b) we define $(a, b)^{\perp} := (-b, a)$.

We will omit the constant on the outside of the integral from now on, since all the results we will obtain would remain the same if we were to change $\frac{\Gamma(3/2)}{\pi^{3/2}}$ for an arbitrary (non-zero) constant.

Lemma 2.2.1. *Given natural numbers n, N and a L^∞ function $g_N(r):[0,\infty)\to\mathbb{R}$ with support in $(1-\frac{N^{-1/2}}{2},1+\frac{N^{-1/2}}{2})$ we have that, for $\theta(r,\alpha):=g_N(r)\sin(Nn\alpha)$, there exists a constant C (depending on n) such that, for N big enough and $r\in[1-N^{-\frac{1}{2}},1+N^{-\frac{1}{2}}]$*

$$|v_r(\theta(.,.))(r,\alpha)-\cos(Nn\alpha)\int_{\mathbb{R}\times[-\pi,\pi]}\frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha' dh|$$
$$\leq C||g_N||_{L^\infty}N^{-1/2}.$$

Analogously, for $\theta(r,\alpha)=g_N(r)\cos(Nn\alpha)$ we have that

$$|v_r(\theta(.,.))(r,\alpha)+\sin(Nn\alpha)\int_{\mathbb{R}\times[-\pi,\pi]}\frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha' dh|$$
$$\leq C||g_N(r)||_{L^\infty}N^{-1/2}.$$

Before we get into the proof, a couple of comments need to be made. First, v_r refers to the radial component of the velocity at a given point, that is to say, if we call $\hat{x}$ to the unitary vector in the direction of x then

$$v_r(\theta(.))(x)=P.V.\int_{\mathbb{R}^2}\frac{\hat{x}\cdot(x-y)^{\perp}\theta(y)}{|x-y|^3}dy_1dy_2.$$

However, the expression obtained in Lemma 2.2.1 requires us to work in polar coordinates. Therefore, considering a generic function $f(r)\sin(k\alpha)$ and making the usual changes of variables $(x_1,x_2)=r(\cos(\alpha),\sin(\alpha))$, $(y_1,y_2)=r'(\cos(\alpha'),\sin(\alpha'))$ we obtain

$$v_r(\theta(\ .\ ,\ .\ ))(r,\alpha)$$
$$=P.V.\int_{\mathbb{R}\times[-\pi,\pi]}(r')^2\frac{(\cos(\alpha)\sin(\alpha')-\sin(\alpha)\cos(\alpha'))f(r')\sin(k\alpha')}{|(r\cos(\alpha)-r'\cos(\alpha'))^2+(r\sin(\alpha)-r'\sin(\alpha'))^2|^{3/2}}d\alpha' dr'$$

$$\begin{aligned}
&=P.V.\int_{\mathbb{R}\times[-\pi,\pi]}(r')^2\frac{\sin(\alpha'-\alpha)}{|(r-r')^2+2rr'(1-\cos(\alpha-\alpha'))|^{3/2}}f(r')\sin(k\alpha')d\alpha' dr'\\
&=\cos(k\alpha)P.V.\int_{\mathbb{R}\times[-\pi,\pi]}(r')^2\frac{\sin(\alpha'-\alpha)f(r')\sin(k\alpha'-k\alpha)}{|(r-r')^2+2rr'(1-\cos(\alpha-\alpha'))|^{3/2}}d\alpha' dr'\\
&=\cos(k\alpha)P.V.\int_{\mathbb{R}\times[-\pi,\pi]}(r+h)^2\frac{\sin(\alpha')f(r+h)\sin(k\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh, \qquad (2.2)
\end{aligned}$$

where we have used trigonometric identities and eliminated the terms that are odd with respect to $\alpha'-\alpha$. Note that in the last line we have relabeled $\alpha'-\alpha$ as α' for a more compact notation. Analogously if $\theta(r,\alpha)=f(r)\cos(k\alpha)$ we obtain

$$v_r(\theta(\ .\ ,\ .\ ))(r,\alpha)=-\sin(k\alpha)\int_{\mathbb{R}\times[-\pi,\pi]}(r+h)^2\frac{\sin(\alpha')f(r+h)\sin(k\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh.$$

With this, we are now ready to start the proof of Lemma 2.2.1.

Proof. We need to find bounds for

$$\int_{\mathbb{R}\times[-\pi,\pi]}\frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha' dh$$
$$-\int_{\mathbb{R}\times[-\pi,\pi]}(r+h)^2\frac{\sin(\alpha')g_N(r+h)\sin(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh$$

with $g_N(r)$ satisfying our hypothesis. We will first focus on

$$I_A := |\int_A \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha' dh \tag{2.3}$$
$$-\int_A (r+h)^2\frac{\sin(\alpha')g_N(r+h)\sin(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh|$$

with $A := [-2N^{-1/2}, 2N^{-1/2}] \times [-2N^{-1/2}, 2N^{-1/2}]$. This is accomplished in several steps. It should be noted that the constant C may depend on n and it may change through the proof, as it is the name we use for a generic constant that is independent of N and g.

Step 1:

$$|\int_A (r+h)^2\frac{(\sin(\alpha')-\alpha')g_N(r+h)\sin(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh|$$
$$\leq C\int_A (r+h)^2\frac{|\alpha'|^3|g_N(r+h)|}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh$$
$$\leq C\int_A |g_N(r+h)|d\alpha' dh$$
$$\leq CN^{-1}||g_N||_{L^\infty}$$

Step 2: Defining

$$F(r,h,\alpha') := \frac{1}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}} - \frac{1}{|h^2+(r+h)r(\alpha')^2)|^{3/2}}$$

we estimate the following integral by

$$|\int_A (r+h)^2\alpha' g_N(r+h)\sin(Nn\alpha')F(r,h,\alpha')d\alpha' dh|$$
$$\leq C\int_A |\alpha'||g_N(r+h)|\frac{(\alpha')^4}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{5/2}}d\alpha' dh$$
$$\leq C\int_A |g_N(r+h)|d\alpha' dh$$
$$\leq CN^{-1}||g_N||_{L^\infty}.$$

Step 3:

$$|\int_A ((r+h)^2-r^2)\frac{\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+(r+h)r(\alpha')^2)|^{3/2}}d\alpha' dh|$$
$$\leq C\int_A |h|\frac{|\alpha'||g_N(r+h)|}{|h^2+(r+h)r(\alpha')^2)|^{3/2}}d\alpha' dh$$
$$\leq C\int_A \frac{|g_N(r+h)|}{|h^2+(r+h)r(\alpha')^2|^{1/2}}d\alpha' dh$$
$$\leq CN^{-1/2}||g_N||_{L^\infty}$$

Combining all these three steps we conclude

$$|\int_A \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r(r+h)(\alpha')^2|^{3/2}}d\alpha' dh - \int_A \frac{(r+h)^2 sin(\alpha')g_N(r+h)\sin(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh|$$
$$\leq C||g_N||_{L^\infty}N^{-1/2},$$

and to bound the contribution of the integral in A we also need

$$
\begin{aligned}
&|\int_A \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+(r+h)r(\alpha')^2)|^{3/2}} - \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2)|^{3/2}} d\alpha' dh| \\
&\leq C\int_A |\alpha'||g_N(r+h)|\frac{(\alpha')^2|h|}{|h^2+\frac{r^2}{2}(\alpha')^2)|^{5/2}} d\alpha' dh \\
&\leq C\int_A \frac{|g_N(r+h)|}{|h^2+\frac{r^2}{2}(\alpha')^2|^{1/2}} d\alpha' dh \\
&\leq CN^{-1/2}||g_N||_{L^\infty}.
\end{aligned}
$$

Therefore adding and subtracting

$$
\int_A \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+(r+h)r(\alpha')^2)|^{3/2}}
$$

to (2.3) we obtain that

$$
I_A \leq C||g_N||_{L^\infty} N^{-1/2}.
$$

Finally, we need to deal with the integral outside of A. First we bound the following integral

$$
\begin{aligned}
&\int_{\mathbb{R}\times[-\pi,\pi]\setminus A} \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}} d\alpha' dh \\
&= 2\int_{[-2N^{-1/2},2N^{-1/2}]}\int_{[2N^{-1/2},\pi]} \frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}} d\alpha' dh.
\end{aligned}
$$

To do this we compute, for fixed arbitrary $h \in (-2N^{-\frac{1}{2}}, 2N^{-\frac{1}{2}})$ and $r+h \in \text{supp}(g_N)$, the integral over an interval of the form $\alpha \in [k\frac{2\pi}{Nn} - \frac{\pi}{2Nn}, (k+1)\frac{2\pi}{Nn} - \frac{\pi}{2Nn}]$ (which we will denote by $[\alpha_k, \alpha_{k+1}]$). Note that it has the length of the period of $\sin(Nn\alpha)$ and that $\sin(Nn\alpha)$ is an even function around the point $k\frac{2\pi}{Nn} + \frac{\pi}{2Nn}$.

If we define

$$
H(\alpha', h, r) := \frac{\alpha'}{|h^2+r^2(\alpha')^2|^{3/2}}
$$

we have that

$$
\begin{aligned}
&= \int_{[\alpha_k,\alpha_{k+1}]} \sin(Nn\alpha')\Big(H(\frac{\alpha_k+\alpha_{k+1}}{2}, h, r) \qquad (2.4)\\
&+ \frac{\partial H(\frac{\alpha_k+\alpha_{k+1}}{2}, h, r)}{\partial\alpha'}(\alpha' - \frac{\alpha_k+\alpha_{k+1}}{2}) + \frac{\partial^2 H(c(\alpha'), h, r)}{\partial\alpha'^2}\frac{1}{2}(\alpha' - \frac{\alpha_k+\alpha_{k+1}}{2})^2 d\alpha'\Big) \\
&= \int_{[\alpha_k,\alpha_{k+1}]} \sin(Nn\alpha')\frac{\partial^2 H(c(\alpha'), h, r)}{\partial\alpha'^2}\frac{1}{2}(\alpha' - \frac{\alpha_k+\alpha_{k+1}}{2})^2 d\alpha' \\
&\leq C\int_{[\alpha_k,\alpha_{k+1}]} |\sin(Nn\alpha')|(\alpha' - \frac{\alpha_k+\alpha_{k+1}}{2})^2 \frac{1}{|h^2+r^2\alpha_k^2|^2} d\alpha' \\
&\leq C\Big(\frac{2\pi}{Nn}\Big)^3 \frac{1}{|h^2+r^2\alpha_k^2|^2},
\end{aligned}
$$

where we have used a second degree Taylor expansion around $\frac{\alpha_k+\alpha_{k+1}}{2}$ for H, and $c(\alpha')$ is where we need to evaluate the second derivative to actually obtain an equality. Now, adding over all the intervals $[\alpha_k, \alpha_{k+1}]$ with $\pi - \frac{2\pi}{Nn} \geq \alpha_k \geq 2N^{-1/2}$, we get the upper bound

$$\begin{aligned}
&\sum_{\alpha_k\geq 2N^{-1/2}}^{\pi-\frac{2\pi}{Nn}} C\Big(\frac{2\pi}{Nn}\Big)^3\frac{1}{|h^2+r^2\alpha_k^2|^2}\leq \sum_{k\geq\frac{N^{1/2}n}{\pi}}^{\infty} C\Big(\frac{2\pi}{Nn}\Big)^3\frac{1}{|h^2+r^2\alpha_k^2|^2}\\
&\leq C\Big(\frac{2\pi}{Nn}\Big)^3\int_{\frac{N^{1/2}n}{\pi}-1}^{\infty}\frac{1}{|h^2+r^2(x\frac{2\pi}{Nn}-\frac{\pi}{2Nn})^2|^2}dx\\
&\leq C\Big(\frac{2\pi}{Nn}\Big)^3\int_{\frac{N^{1/2}n}{\pi}-2}^{\infty}\frac{1}{|h^2+(rx\frac{2\pi}{Nn})^2|^2}dx\\
&\leq C\Big(\frac{2\pi}{Nn}\Big)^3\Big(\frac{2\pi}{Nn}\Big)^{-4}(\frac{2\pi}{N^{1/2}n})^3\leq CN^{-1/2},
\end{aligned}$$

where we took N big to pass from the third to the fourth line. The only contribution missing now from the integral in the α' variable, if we call α_{k_0} the smallest α_k such that $\alpha_k\geq 2N^{-1/2}$ and α_∞ the biggest one with $\pi\geq\alpha_\infty$, is

$$\int_{[2N^{-1/2},\alpha_{k_0}]\cup[\alpha_\infty,\pi]}\sin(Nn\alpha')\frac{\alpha'}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha',$$

but

$$|\int_{2N^{-1/2}}^{\alpha_{k_0}}\sin(Nn\alpha')\frac{\alpha'}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha'|\leq C, \tag{2.5}$$

$$|\int_{\alpha_{k_\infty}}^{\pi}\sin(Nn\alpha')\frac{\alpha'}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha'|\leq \frac{C}{N}. \tag{2.6}$$

Combining (2.5),(2.6) and the bound we obtained for (2.4) and integrating with respect to h we get

$$\begin{aligned}
&|2\int_{[-2N^{1/2},2N^{-1/2}]}\int_{[2N^{-1/2},\pi]}\frac{r^2\alpha' g_N(r+h)\sin(Nn\alpha')}{|h^2+r^2(\alpha')^2|^{3/2}}d\alpha' dh|\\
&\leq\int_{[-2N^{1/2},2N^{-1/2}]}C|g_N(r+h)|dh\leq C||g_N||_{L^\infty}N^{-1/2}.
\end{aligned}$$

The term

$$|\int_{\mathbb{R}\times[-\pi,\pi]\setminus A}(r+h)^2\frac{\sin(\alpha')g_N(r+h)\sin(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' dh|$$

is bounded in a similar fashion, integrating first with respect to α' in intervals of the form $[\alpha_k,\alpha_{k+1}]$ and then bounding the parts that are not covered exactly by said intervals (as in (2.5) and (2.6)), and with that we would be done.

□

Now that we have a manageable expression for the radial velocity we are ready to compute it explicitly (with some error) for some special kind of functions.

Lemma 2.2.2. *Given natural numbers n, N and a C^2 function $g_N(.):\mathbb{R}\to\mathbb{R}$ with support in the interval $(1-\frac{N^{-1/2}}{2},1+\frac{N^{1/2}}{2})$ satisfying $||g_N||_{C^i}\leq MN^{i/2}$ for $i=0,1,2$ there exists a constant $C_0\neq 0$ (independent of N, n and g_N) such that for $\tilde{x}\in[1-N^{-\frac{1}{2}},1+N^{-\frac{1}{2}}]$,*

$$\begin{aligned}
&|C_0 g_N(\tilde{x})-\int_{\mathbb{R}\times[-\pi,\pi]}g_N(\tilde{x}+h_1)\frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2|\\
&\leq CMN^{-1/2},
\end{aligned}\tag{2.7}$$

with C depending on n.

Proof. The strategy of this proof is to first show that

$$|\int_{\mathbb{R}\times[-\pi,\pi]}(g_N(\tilde{x}+h_1)-g_N(\tilde{x}))\frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2|$$
$$\leq CMN^{-1/2}, \tag{2.8}$$

and then prove that

$$I_{N,n}:=\int_{\mathbb{R}\times[-\pi,\pi]}\sin(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2 \tag{2.9}$$

is a Cauchy series with respect to N, satisfying

$$|I_{N_1,n}-I_{N_2,n}|\leq Csup(N_1,N_2)^{-1/2} \tag{2.10}$$

with C depending on n.

Combining both of these results and taking

$$C_0:=\lim_{N\to\infty}I_{N,n}$$

we obtain (2.7), and we only need to check that C_0 is different from zero and independent of n.

We first obtain bound (2.8), by noting that, due to parity

$$|\int_{[-2N^{-1/2},2N^{-1/2}]\times[-\pi,\pi]}(g_N(\tilde{x}+h_1)-g_N(\tilde{x}))\sin(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2|$$
$$=|\int_{[0,2N^{-1/2}]\times[-\pi,\pi]}\frac{(g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2|. \tag{2.11}$$

Next we fix some $h_1\in(0,2N^{-\frac{1}{2}})$ and obtain bounds for the integral with respect to h_2. This is done as in Lemma 2.2.1, dividing in periods of length $\frac{2\pi}{Nn}$ starting at $\frac{\pi}{2Nn}$, and approximating $\frac{h_2}{h_1^2+h_2^2}$ by its second order Taylor expansion, since the first two orders will cancel. That way, for the interval with $h_2\in[k\frac{2\pi}{Nn}+\frac{\pi}{2Nn},(k+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}]$ we obtain the bound

$$|\int_{k\frac{2\pi}{Nn}+\frac{\pi}{2Nn}}^{(k+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}}\sin(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{\frac{3}{2}}}dh_2|\leq C\Big(\frac{2\pi}{Nn}\Big)^3\frac{1}{(h_1^2+(\frac{k2\pi}{Nn})^2)^2}. \tag{2.12}$$

We add periods contained in the interval $[0,2N^{-1/2}]$ and we denote by $k_\infty=k_\infty(N,n)$ the biggest integer k such that $(k+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}\leq 2N^{-1/2}$ to obtain that

$$|\int_{\frac{5\pi}{2Nn}}^{(k_\infty+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}}\sin(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{\frac{3}{2}}}dh_2|$$
$$\leq\sum_{k=1}^{k_\infty}C\Big(\frac{2\pi}{Nn}\Big)^3\frac{1}{(h_1^2+(\frac{k2\pi}{Nn})^2)^2}\leq C\Big(\frac{2\pi}{Nn}\Big)^3\int_0^{k_\infty}\frac{1}{(h_1^2+(\frac{x2\pi}{Nn})^2)^2}dx$$
$$\leq C\Big(\frac{2\pi}{Nn}\Big)^3\int_0^{k_\infty}\frac{1}{(h_1+\frac{x2\pi}{Nn})^4}dx=C\Big(\frac{2\pi}{Nn}\Big)^3\int_{\frac{h_1Nn}{2\pi}}^{k_\infty+\frac{h_1Nn}{2\pi}}\frac{1}{(\frac{x2\pi}{Nn})^4}dx$$
$$\leq C\Big(\frac{2\pi}{Nn}\Big)^2\frac{1}{h_1^3}.$$

This allows us to bound the contribution to (2.11) when $h_1\geq\frac{2\pi}{Nn}$ by dividing it into three parts:

1) If $h_2\leq\frac{5\pi}{2Nn}$:

$$|\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}}\int_0^{\frac{5\pi}{2Nn}}(g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\cos(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_2dh_1|$$

$$\leq |C\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} \frac{5\pi}{2Nn} M h_1^2 N \frac{1}{h_1^2} dh_1| \leq CMN^{-1/2}.$$

2) If $\frac{5\pi}{2Nn} \leq h_2 \leq (k_\infty+1)\frac{2\pi}{Nn} + \frac{\pi}{2Nn}$:

$$|\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} \int_{\frac{5\pi}{2Nn}}^{(k_\infty+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}} (g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1|$$
$$\leq \int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} |g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x})| \, |\int_{\frac{5\pi}{2Nn}}^{(k_\infty+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}} \frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}} dh_2| dh_1$$
$$\leq C\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} M h_1^2 N \Big(\frac{2\pi}{Nn}\Big)^2 \frac{1}{h_1^3} dh_1 \leq CMN^{-1} log(N).$$

3) If $(k_\infty+1)\frac{2\pi}{Nn} + \frac{\pi}{2Nn} \leq h_2 \leq 2N^{-1/2}$:

$$|\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} \int_{(k_\infty+1)\frac{2\pi}{Nn}+\frac{\pi}{2Nn}}^{2N^{-1/2}} (g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1|$$
$$\leq C\int_{\frac{2\pi}{Nn}}^{2N^{-1/2}} M dh_1 \leq CMN^{-\frac{1}{2}}.$$

Finally, we bound the error when $h_1 \leq \frac{2\pi}{Nn}$:

1) If $|h_2| \leq 2N^{-1/2}$

$$|\int_0^{\frac{2\pi}{Nn}} \int_0^{2N^{-1/2}} (g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\cos(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1|$$
$$\leq \int_0^{\frac{2\pi}{Nn}} \int_0^{2N^{-1/2}} M h_1^2 N \frac{1}{(h_1^2+h_2^2)} dh_2 dh_1 \leq CMN^{-1/2}.$$

2) If $|h_2| \geq 2N^{-1/2}$

$$|\int_0^{\frac{2\pi}{Nn}} \int_{2N^{-1/2}}^{\pi} (g_N(\tilde{x}+h_1)+g_N(\tilde{x}-h_1)-2g_N(\tilde{x}))\frac{\sin(Nnh_2)h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1|$$
$$\leq \int_0^{\frac{2\pi}{Nn}} \int_{2N^{-1/2}}^{\pi} \frac{M}{2} h_1^2 N \frac{1}{(h_1^2+h_2^2)} dh_2 dh_1 \leq CMN^{-1}.$$

Combining all these bounds we obtain (2.8). Therefore, we have that it is enough to prove that

$$|C_0 g_N(\tilde{x}) - \int_{\mathbb{R}\times[-\pi,\pi]} g_N(\tilde{x})\cos(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_1 dh_2|$$
$$\leq CMN^{-1/2},$$

which is equivalent to studying the behaviour of $I_{N,n}$, defined as in (2.9).

We start by transforming the integral with a change of variables $\bar{h}_1 := Nnh_1$, $\bar{h}_2 := Nnh_2$, although we will relabel $\bar{h}_1, \bar{h}_2$ as h_1, h_2 to simplify the notation.

$$\begin{aligned}
&g_N(\tilde{x}) \int_{[-2N^{-1/2},2N^{-1/2}]\times[-\pi,\pi]} \sin(Nnh_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2 \\
&= g_N(\tilde{x}) \int_{[-2nN^{1/2},2nN^{1/2}]\times[-Nn\pi,Nn\pi]} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2.
\end{aligned}$$

If we compare the integral for different values of N, $N_1 \geq N_2$ we get

$$I_{N_1,n} - I_{N_2,n} = \int_{A\cup B} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_1dh_2$$

with

$$A = [-2nN_2^{1/2}, 2nN_2^{1/2}] \times [N_2n\pi, N_1n\pi] \cup [-2nN_2^{1/2}, 2nN_2^{1/2}] \times [-N_1n\pi, -N_2n\pi],$$

$$B = [2nN_2^{1/2}, 2nN_1^{1/2}] \times [-nN_1\pi, nN_1\pi] \cup [-2nN_1^{1/2}, -2nN_2^{1/2}] \times [-nN_1\pi, nN_1\pi].$$

To get an estimate for the integral on A we use symmetry to focus on $h_2 > 0$ and we separate the integral into three parts, $h_2 \in [2\pi k_0 + \frac{\pi}{2}, 2\pi(k_\infty+1)+\frac{\pi}{2}]$ (with $k_0 = k_0(N_2,n)$ the smallest integer with $2\pi k_0 + \frac{\pi}{2} \geq N_2n\pi$ and $k_\infty = k_\infty(N_1,n)$ the biggest one such that $(k_\infty+1)2\pi + \frac{\pi}{2} \leq N_1n\pi$), $h_2 \in [N_2n\pi, 2\pi k_0 + \frac{\pi}{2}]$ and $h_2 \in [(k_\infty+1)2\pi + \frac{\pi}{2}, N_1n\pi]$, and we estimate each part separately:

1) If $h_2 \in [2\pi k_0 + \frac{\pi}{2}, 2\pi(k_\infty+1)+\frac{\pi}{2}]$

$$\begin{aligned}
&\left|\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \int_{2\pi k_0+\frac{\pi}{2}}^{2\pi(k_\infty+1)+\frac{\pi}{2}} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_2dh_1\right| \\
&\leq \int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} C\sum_{k=k_0}^{k_\infty} \frac{1}{(h_1^2+(k2\pi)^2)^2}dh_1 \leq C\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \sum_{k=k_0}^{k_\infty} \frac{1}{(h_1+k2\pi)^4}dh_1 \\
&\leq C\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \int_{\frac{N_2n}{2}-\frac{5}{4}}^{\frac{N_1n}{2}} \frac{1}{(h_1+x2\pi)^4}dxdh_1 \leq C\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \frac{1}{(h_1+N_2n)^3}dh_1 \\
&\leq \frac{C}{N_2^{5/2}n^2},
\end{aligned}$$

2) If $h_2 \in [N_2n\pi, 2\pi k_0 + \frac{\pi}{2}]$

$$\left|\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \int_{N_2n\pi}^{2\pi k_0+\frac{\pi}{2}} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_2dh_1\right| \leq \frac{C}{N_2^{3/2}n},$$

3) If $h_2 \in [(k_\infty+1)2\pi + \frac{\pi}{2}, N_1n\pi]$

$$\left|\int_{-2nN_2^{1/2}}^{2nN_2^{1/2}} \int_{2\pi(k_\infty+1)+\frac{\pi}{2}}^{N_1n\pi} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_2dh_1\right| \leq \frac{C}{N_2^{3/2}n}.$$

For the integration in B we use a similar trick, using parity to consider only $h_2 \geq 0$ and separating in the parts $h_2 \leq \frac{5\pi}{2}$, $\frac{5\pi}{2} \leq h_2 \leq 2\pi(k_\infty+1)+\frac{\pi}{2}$ and $2\pi(k_\infty+1)+\frac{\pi}{2} \leq h_2 \leq N_1n\pi$, with $k_\infty = k_\infty(N_1, n)$ the biggest integer such that $(k_\infty+1)2\pi + \frac{\pi}{2} \le N_1 n\pi$: 1) If $\frac{5\pi}{2} \le h_2 \le 2\pi(k_\infty+1)+\frac{\pi}{2}$

$$\begin{aligned}
&|\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{\frac{5\pi}{2}}^{2\pi(k_\infty+1)+\frac{\pi}{2}} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1| \\
&\le \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} C\sum_{k=1}^{k_\infty} \frac{1}{(h_1^2+(k2\pi)^2)^2} dh_1 \le C\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \sum_{k=1}^{k_\infty} \frac{1}{(h_1+k2\pi)^4} dh_1 \\
&\le C\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_0^{\frac{N_1 n}{2}} \frac{1}{(h_1+x2\pi)^4} dx dh_1 \le C\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \frac{1}{h_1^3} dh_1 \\
&\le \frac{C}{N_2 n^2}
\end{aligned}$$

2) If $h_2 \le \frac{5\pi}{2}$

$$|\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_0^{\frac{5\pi}{2}} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1| \le \frac{C}{N_2 n^2},$$

3) If $2\pi(k_\infty+1)+\frac{\pi}{2} \le h_2 \le N_1 n\pi$

$$|\int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2\pi(k_\infty+1)+\frac{\pi}{2}}^{N_1 n\pi} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 dh_1| \le \frac{C}{N_2^{3/2} n}.$$

Putting together the estimates in the regions A and B we have that $lim_{N\to\infty} I_{N,n} = C_0(n)$, and that $|C_0 - I_{N,n}| \le CN^{-1/2}$. The only thing left to do is to prove that C_0 is indeed different from 0 and independent of n.

To prove that $C_0(n)$ is actually independent of n, it is enough to prove that, for two arbitrary integers n_1, n_2,

$$\lim_{N\to\infty} I_{N,n_1} - I_{N,n_2} = 0.$$

The proof is equivalent to that of (2.10), so we will omit it.

To prove that $C_0 \neq 0$, we start by focusing on the integral with respect to h_2 for any fixed h_1 on an interval of the form $[-K\pi, K\pi,]$ with $K \in \mathbb{N}$

$$\begin{aligned}
&\int_{[-K\pi,K\pi]} \sin(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}} dh_2 \\
&= \int_{[-K\pi,K\pi]} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}} dh_2 - \Big[sin(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}}\Big]_{h_2=-K\pi}^{K\pi} \\
&= \int_{[-K\pi,K\pi]} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}} dh_2 = 2\int_{[0,K\pi]} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}} dh_2,
\end{aligned}$$

and we can use this property to compute the integral in $[-2nN^{1/2}, 2nN^{1/2}] \times [-K\pi, K\pi]$ as

$$\begin{aligned}
&\int_{-2nN^{1/2}}^{2nN^{1/2}} \int_0^{K\pi} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}} dh_2 dh_1 = \int_0^{K\pi} \int_{-2nN^{1/2}}^{2nN^{1/2}} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}} dh_1 dh_2 \\
&= \int_0^{K\pi} cos(h_2) \int_{-\frac{2nN^{1/2}}{h_2}}^{\frac{2nN^{1/2}}{h_2}} \frac{1}{(x^2+1)^{1/2}} dx dh_2 = 2\int_0^{K\pi} cos(h_2) log(\frac{2nN^{1/2}}{h_2} + (1+\frac{4n^2N}{h_2^2})^{1/2}) dh_2
\end{aligned}$$

$$= 2\int_0^{K\pi} cos(h_2)(log(\frac{2nN^{1/2}}{h_2} + (1+\frac{4n^2N}{h_2^2})^{1/2}) - log(\frac{4nN^{1/2}}{h_2}))dh_2$$
$$+ 2\int_0^{K\pi} cos(h_2)log(\frac{4nN^{1/2}}{h_2})dh_2,$$

and we can evaluate the last line by checking the two integrals separately

$$\int_0^{K\pi} cos(h_2)log(\frac{4nN^{1/2}}{h_2})dh_2 = -\int_0^{K\pi} cos(h_2)log(h_2)dh_2$$
$$= -\Big[log(x)\cos(x) - Si(x)\Big]_0^{K\pi} = Si(K\pi) > 0,$$

where $Si(x) \equiv \int_0^x \frac{\sin(t)}{t}dt$ denotes the Sine integral function, and

$$|\int_0^{K\pi} cos(h_2)(log(\frac{2nN^{1/2}}{h_2} + (1+\frac{4n^2N}{h_2^2})^{1/2}) - log(\frac{4nN^{1/2}}{h_2}))dh_2|$$
$$\leq \int_0^{K\pi} \frac{h_2}{4nN^{1/2}}((1+\frac{4n^2N}{h_2^2})^{1/2} - \frac{2nN^{1/2}}{h_2})dh_2 \leq \frac{CK^3}{N}.$$

Furthermore, we can bound the integral outside of the interval $h_2 \in [-K\pi, K, \pi]$. The particular way we divide the integral depends on the parity of K and Nn. Here we will obtain the bounds in the case K even and Nn odd, the other cases being analogous:

$$|\int_{-2nN^{1/2}}^{2nN^{1/2}}\int_{K\pi}^{Nn\pi} cos(h_2)\frac{h_2}{(h_1^2+h_2^2)^{3/2}}dh_2dh_1|$$
$$\leq \int_{-2nN^{1/2}}^{2nN^{1/2}} \sum_{k=\frac{K}{2}}^{\frac{Nn-1}{2}-1} \frac{1}{(h_1^2+(2\pi k)^2)^2}dh_1 + \int_{-2nN^{1/2}}^{2nN^{1/2}}\int_{(Nn-1)\pi}^{Nn\pi} \frac{1}{h_1^2+h_2^2}dh_2dh_1$$
$$\leq C\int_0^{2nN^{1/2}} \sum_{k=\frac{K}{2}}^{\frac{Nn-1}{2}-1} \frac{1}{(h_1+2\pi k)^4}dh_1 + \frac{C}{N^{\frac{3}{2}}}$$
$$\leq C\int_0^{2nN^{1/2}} \frac{1}{(h_1+2\pi(\frac{K}{2}-1))^3}dh_1 + \frac{C}{N^{\frac{3}{2}}} \leq (\frac{C}{K-2})^2 + \frac{C}{N^{\frac{3}{2}}}.$$

Combining all these together we get that, for any $K \leq nN$

$$\int_{-2nN^{1/2}}^{2nN^{1/2}}\int_0^{nN\pi} cos(h_2)\frac{1}{(h_1^2+h_2^2)^{1/2}}dh_2dh_1$$
$$\geq Si(K\pi) - (\frac{C}{K-2})^2 - \frac{CK^3}{N} - \frac{C}{N^{\frac{3}{2}}}$$

and by taking K big enough so that $\frac{Si(K\pi)}{2} - (\frac{C}{K-2})^2 > 0$ and then N big enough so that $\frac{Si(K\pi)}{2} - \frac{CK^3}{N} - \frac{C}{N^{\frac{1}{2}}} > 0$ we are done. □

We can now combine both lemmas to obtain

Lemma 2.2.3. *Given natural numbers n, N and a C^2 function $g_N(.) : \mathbb{R} \to \mathbb{R}$ with support in the interval $(1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{1/2}}{2})$ and $||g_N||_{C^i} \leq MN^{i/2}$ for $i = 0, 1, 2$, we have that there exists a constant $C_0 \neq 0$ such that, for $r \in (1 - N^{-1/2}, 1 + N^{-1/2})$,*

$$|v_r(g_N(r)\cos(Nn\alpha)) - C_0 cos(Nn\alpha)g_N(r)| \leq CMN^{-1/2} \tag{2.13}$$

with C depending on n but not on N or g.

Analogously, we have that

$$|v_r(g_N(r)\cos(Nn\alpha)) + C_0\sin(Nn\alpha)g_N(r)| \leq CMN^{-1/2} \tag{2.14}$$

with C depending only on n.

Proof. We already know by Lemma 2.2.1 that

$$\begin{aligned}&|v_r(g_N(r)\cos(Nn\alpha)) - \cos(Nn\alpha)\int_{\mathbb{R}\times[-\pi,\pi]} \frac{r^2\alpha' g_N(r+h)\cos(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} d\alpha' dh| \\ &\leq C||g_N(r)||_{L^\infty} N^{-1/2}\end{aligned} \tag{2.15}$$

and, by a change of variables, we have that

$$\int_{\mathbb{R}\times[-\pi,\pi]} \frac{r^2\alpha' g_N(r+h)\cos(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} dh d\alpha' = \int_{\mathbb{R}}\int_{[-\pi,\pi]} \frac{\alpha' g_N(r+hr)\cos(Nn\alpha')}{|h^2 + \alpha'^2|^{3/2}} d\alpha' dh.$$

However, for any fixed $r \in [1/2, 3/2]$, we have $||g_N(r+rh)||_{C^i} \leq 2^i||g_N(r+h)||_{C^i}$ and thus applying Lemma 2.2.2 we get

$$|C_0 g_N(r) - \int_{\mathbb{R}}\int_{[-\pi,\pi]} \frac{\alpha' g_N(r+hr)\cos(Nn\alpha')}{|h^2 + \alpha'^2|^{3/2}} d\alpha' dh| \leq 2CMN^{-1/2}, \tag{2.16}$$

and combining (2.15) and (2.16) finishes the proof of (2.13).

We omit the proof of (2.14) since it is completely analogous to that of (2.13). □

All these results will allow us to compute locally the radial velocity with a small error, but we would like to also have decay as we go far away from $r = 1$. For that we have the following lemma.

Lemma 2.2.4. *Given a L^∞ function $g_N(.) : \mathbb{R} \to \mathbb{R}$ with support in the interval $(1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2})$, and let θ be defined as*

$$\theta(r, \alpha) := sin(Nn\alpha)g_N(r)$$

with N, n natural numbers.

Then there is a constant C (independent of g_N) such that, if N is big enough and $1/2 > |r - 1| \geq N^{-1/2}$ or $r \geq 3/2$, we have

$$|v_r(\theta)(r, \alpha)| \leq \frac{C||g_N||_{L^\infty}}{N^{3/2}|r-1|^2}.$$

Proof. To estimate $|v_r(\theta)(r, \alpha)|$ we will use expression (2.2) and therefore we need to find upper bounds for

$$|\int_{\mathbb{R}\times[-\pi,\pi]} (r+h)^2 \frac{\sin(\alpha')g_N(r+h)\cos(Nn\alpha')}{|h^2 + 2(r+h)r(1-\cos(\alpha'))|^{3/2}} d\alpha' dh|.$$

Let us fix h such that $r+h \in (1-\frac{N^{-1/2}}{2}, 1+\frac{N^{-1/2}}{2})$ and with $r \geq 1/2$. Using that $\int_{i\frac{2\pi}{Nn}}^{(i+1)\frac{2\pi}{Nn}} \sin(Nn\alpha)d\alpha = 0$ and a degree one Taylor expansion around $\alpha' = k\frac{2\pi}{Nn} + \frac{\pi}{Nn}$ for $\frac{\sin(\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}$ we can bound the integral over a single period

$$
\begin{aligned}
&|\int_{[k\frac{2\pi}{Nn},(k+1)\frac{2\pi}{Nn}]} \frac{\sin(\alpha')\cos(Nn\alpha')}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha'| \\
&\leq \int_{[k\frac{2\pi}{Nn},(k+1)\frac{2\pi}{Nn}]} \frac{C}{Nn}\frac{1}{|h^2+2(r+h)r(1-\cos(\alpha'))|^{3/2}}d\alpha' \\
&\leq \frac{C}{(Nn)^2}\frac{1}{|h+ck\frac{2\pi}{Nn}|^3},
\end{aligned}
$$

with c small and C big, where we used that $r+h, r \geq 1/2$ and that there exists $c>0$ such that $\frac{1}{c}(1-\cos(\alpha')) \geq (\alpha')^2$ if $\alpha' \in [-\pi,\pi]$. Adding over all the relevant periods we obtain

$$
\begin{aligned}
&\sum_{k=0}^{nN} \frac{C}{(Nn)^2}\frac{1}{|h+ck\frac{2\pi}{Nn}|^3} \leq \int_{-1}^{Nn} \frac{C}{(Nn)^2}\frac{1}{|h+cx\frac{2\pi}{Nn}|^3}dx \\
&\int_{-1}^{Nn} CNn\frac{1}{|h\frac{Nn}{2\pi c}+x|^3}dx \leq CNn\frac{1}{|h\frac{Nn}{2\pi c}-1|^2} \\
&= \frac{C}{Nn}\frac{1}{|h-\frac{2\pi c}{Nn}|^2} \leq \frac{C}{Nn}\frac{1}{h^2}.
\end{aligned}
$$

Furthermore, since the support of $g_N(r)$ lies in $(1-\frac{N^{-1/2}}{2}, 1+\frac{N^{-1/2}}{2})$ and $|r-1| > N^{-1/2}$ we have that $|h| \geq \frac{|r-1|}{2}$, so, by integrating in h we get

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{C}{Nn}(r+h)^2\frac{|g_N(r+h)|}{h^2}dh \\
&\leq \int_{r+h-1\in(-\frac{N^{-1/2}}{2},\frac{N^{-1/2}}{2})} \frac{C}{Nn|r-1|^2}||g||_{L^\infty}dh \leq \frac{C}{N^{3/2}n|r-1|^2}||g||_{L^\infty}.
\end{aligned}
$$

□

2.2.2 The pseudo-solution method for ill-posedness in C^k

As mentioned in Definition 5, we will say $\bar{\theta}$ is a pseudo-solution to the SQG equation if it fulfils that

$$
\begin{aligned}
\frac{\partial\bar{\theta}}{\partial t} + v_1(\bar{\theta})\frac{\partial\bar{\theta}}{\partial x_1} + v_2(\bar{\theta})\frac{\partial\bar{\theta}}{\partial x_2} + F(x,t) = 0 \\
v_1(\bar{\theta}) = -\frac{\partial}{\partial x_2}(-\Delta)^{1/2}\bar{\theta} = -\mathcal{R}_2\bar{\theta} \\
v_2(\bar{\theta}) = \frac{\partial}{\partial x_1}(-\Delta)^{1/2}\bar{\theta} = \mathcal{R}_1\bar{\theta} \\
\bar{\theta}(x,0) = \theta_0(x),
\end{aligned}
$$

for some $F(x,t)$.

We will work with initial conditions of the form

$$\lambda(f_1(r) + f_2(N^{1/2}(r-1)+1)\sum_{k=1}^{K}\frac{\sin(Nk\alpha)}{N^2k^3})$$

with N and K natural numbers, $1 > \lambda > 0$ and where f_1 and f_2 satisfy the following conditions:

- Both $f_1(r)$ and $f_2(r)$ are C^∞ functions.
- $f_2(r)$ has its support contained in the interval $(1/2, 3/2)$ and f_1 has its support in $(1/2, 3/2)\cup (M_1, M_2)$ with some M_1, M_2 big.
- $\frac{\partial f_1(r)}{\partial r} = 1$ in $(3/4, 5/4)$.
- $f_2(r) = 1$ in $(3/4, 5/4)$.
- $\frac{\partial^k \frac{v_\alpha(f_1)(r)}{r}}{\partial r^k}$ is 0 when $r = 1$, $k = 1, 2$, where $v_\alpha(f_1)$ is the velocity produced by f_1 in the angular direction.

We will use these pseudo-solutions to prove ill-posedness in C^2, and at the end of this section we will explain how to extend the proof to C^k, $k > 2$.

It is not obvious that the properties we require for f_1 can be obtained, so we need the following lemma.

Lemma 2.2.5. *There exists a C^∞ compactly supported function $g(\cdot) : [0,\infty) \to \mathbb{R}$ with support in $(2,\infty)$ such that $\frac{\partial^i \frac{v_\alpha(g(.))(r)}{r}}{\partial r^i}(r = 1) = a_i$ with $i = 1, 2$ and a_i arbitrary.*

Proof. We start by considering a C^∞ function $h(x) : \mathbb{R} \to \mathbb{R}$ which is positive, with support in $(-1/2, 1/2)$ and $\int h dx = 1$. We define the family of functions

$$f_{n_1,n_2}(r) := n_1 h(n_1(r - n_2)),$$

with $n_2 \geq n_1 \geq 2$, $n_1, n_2 \in \mathbb{N}$. These functions are C^∞ for any n_1, n_2 , and are supported in the interval $(n_2 - \frac{1}{2n_1}, n_2 + \frac{1}{2n_1})$. Now let us consider the associated family of vectors

$$V = \cup_{n_1,n_2} V_{n_1,n_2},$$

with

$$V_{n_1,n_2} := (\frac{\partial v_\alpha(f_{n_1,n_2})}{\partial r}(r = 1), \frac{\partial^2 v_\alpha(f_{n_1,n_2})}{\partial r^2}(r = 1)).$$

Note that to prove our lemma it is sufficient to show that this family is in fact a basis of $\mathbb{R}^2$. Before we can prove that this is the case, we need to find expressions for V_{n_1,n_2}. For our purposes it is enough to compute $\lambda_{n_1,n_2} V_{n_1,n_1}$ since these vectors will span the same space as long as $\lambda_{n_1,n_2} \neq 0$.

To begin with, we deduce the expression for v_α. Proceeding in a similar way as for v_r, and for simplicity only considering the case when $\theta(r, \alpha) = f(r)$ we get

$$\begin{aligned}
&v_\alpha(\theta(\cdot,\cdot))(r, \alpha)\\
&= P.V.\int_{\mathbb{R}^2} \hat{x}^\perp \frac{(x-y)^\perp \theta(y)}{|x-y|^3} dy_1 dy_2 = P.V.\int_{\mathbb{R}^2} \hat{x}^\perp \frac{(x-y)^\perp(\theta(y)-\theta(x))}{|x-y|^3} dy_1 dy_2\\
&= P.V.\int_{\mathbb{R}_+\times[-\pi,\pi]} r' \frac{(f(r') - f(r))(r - r'(\cos(\alpha)\cos(\alpha') + sin(\alpha)\cos(\alpha')))}{|(rcos(\alpha) - r'cos(\alpha'))^2 + (r\sin(\alpha) - r'sin(\alpha'))^2|^{3/2}} d\alpha' dr'\\
&= P.V.\int_{\mathbb{R}_+\times[-\pi,\pi]} r' \frac{r - r'cos(\alpha' - \alpha)}{|r^2 + (r')^2 - 2rr'cos(\alpha - \alpha'))|^{3/2}} (f(r') - f(r)) d\alpha' dr'. \qquad (2.17)
\end{aligned}$$

Moreover, since we will be considering functions with support in $(2,\infty)$, after relabeling $\alpha-\alpha'$ as α' we end up with the expression

$$P.V.\int_{2}^{\infty}\int_{-\pi}^{\pi} r'\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}(f(r')-f(r))dr'd\alpha'.$$

Furthermore, if we write

$$F(r,r',\alpha'):=r'\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}$$

for $r=1$, we can use differentiation under the integral sign and obtain

$$\frac{\partial^j v_{\alpha}(f(.))}{\partial r^j}(r=1)=\int_{(2,\infty)\times[-\pi,\pi]}\frac{\partial^j F}{\partial r^j}(r,r',\alpha')(r=1)f(r')dr'd\alpha'.$$

But for $f=f_{n_1,n_2}$ we have that

$$|\int_{(2,\infty)\times[-\pi,\pi]}\frac{\partial^j F}{\partial r^j}(r,r',\alpha')f_{n_1,n_2}(r')d\alpha'dr'-\int_{[-\pi,\pi]}\frac{\partial^j F}{\partial r^j}(r,n_2,\alpha')d\alpha'|\leq\frac{C}{n_1},$$

with C depending on r and, in particular, since $span(V)$ is a closed set, by taking $lim_{n_1\to\infty}V_{n_1,n_2}$ we get that

$$(\int_{[-\pi,\pi]}\frac{\partial F}{\partial r}(1,n_2,\alpha')d\alpha',\int_{[-\pi,\pi]}\frac{\partial^2 F}{\partial r^2}(1,n_2,\alpha')d\alpha')\in span(V)$$

Furthermore, we have that

$$\frac{\partial F}{\partial r}(r,r',\alpha')(r=1)$$
$$=r'\Big(\frac{1}{|r^2+(r')^2-2rr'cos(\alpha')|^{3/2}}-\frac{3(r-r'cos(\alpha'))^2}{|r^2+(r')^2-2rr'cos(\alpha')|^{5/2}}\Big)(r=1)$$

and, integrating with respect to α' we get

$$\int_{[-\pi,\pi]}\frac{\partial F}{\partial r}(r,n_2,\alpha')(r=1)d\alpha'=-\frac{\pi}{(r')^2}(1+O(\frac{1}{r'})).$$

With the second derivative we obtain

$$\frac{\partial^2 F}{\partial r^2}(r,r',\alpha')(r=1)$$
$$=-r'\Big(\frac{9(r-r'cos(\alpha'))}{|r^2+(r')^2-2rr'cos(\alpha')|^{5/2}}-\frac{15(r-r'cos(\alpha'))^3}{|r^2+(r')^2-2rr'cos(\alpha')|^{7/2}}\Big)(r=1).$$

Now, before we get into more details regarding this value, we note that

$$r'\Big(|\frac{9(r-r'cos(\alpha'))}{|r^2+(r')^2-2rr'cos(\alpha')|^{5/2}}-\frac{15(r-r'cos(\alpha'))^3}{|r^2+(r')^2-2rr'cos(\alpha')|^{7/2}}|\Big)(r=1)\leq\frac{1}{(r')^3}.$$

Therefore, we have that

$$(\frac{1}{(n_2)^2}+O(\frac{1}{(n_2)^3}),O(\frac{1}{(n_2)^3}))\in span(V),$$

and again since $span(V)$ is a closed set, the vector $(1,0)$ belongs to $span(V)$. Now we only need to prove that there exists a point r' such that

$$\int_{[-\pi,\pi]}-r'\Big(\frac{9(1-r'cos(\alpha'))}{|1+(r')^2-2r'cos(\alpha')|^{5/2}}-\frac{15(1-r'cos(\alpha'))^3}{|1+(r')^2-2r'cos(\alpha')|^{7/2}|}\Big)d\alpha'\neq 0,$$

so that we can find a vector V_{n_1,n_2} of the form (a,b) with $b\neq 0$. But, for example, using that, for $\delta>0$ and r' big

$$\frac{1}{(1+(r')^2-2r'cos(\alpha'))^{\delta}}+\frac{1}{(1+(r')^2+2r'cos(\alpha'))^{\delta}}-\frac{2}{(1+(r')^2)^{\delta}}\leq\frac{C}{(r')^{2(\delta+1)}}$$

one can check that

$$\begin{aligned}&\int_{[-\pi,\pi]}-r'\Big(\frac{9(1-r'cos(\alpha'))}{|1+(r')^2-2r'cos(\alpha')|^{5/2}}-\frac{15(1-r'cos(\alpha'))^3}{|1+(r')^2-2r'cos(\alpha')|^{7/2}|}\Big)d\alpha'\\&=\frac{C}{(r')^4}+O(\frac{1}{(r')^5})\end{aligned}$$

with $C\neq 0$, and taking r' big enough we are done.

□

Therefore, to obtain f_1 with the desired properties, we first consider a radial C^{∞} function $\tilde{f}_1(r)$ with support in $(\frac{1}{2},\frac{3}{2})$ and derivative 1 in $(\frac{3}{4},\frac{5}{4})$ and then define

$$f_1(r):=\tilde{f}_1(r)+\bar{f}_1(r)$$

with $\bar{f}_1(r)$ a C^{∞} function with support in $[2,M]$ such that

$$\frac{\partial^k\frac{v_{\alpha}(\tilde{f}_1(r)+\bar{f}_1(r))(r)}{r}}{\partial r^k}=0$$

for $r=1$, $k=1,2$ and such a function exists thanks to Lemma 2.2.5.

Once we choose specific f_1 and f_2, this family of initial conditions has some useful properties that we will use later. First, for any fixed K and λ our initial conditions are bounded in $H^{2+1/4}$ independently of the choice of N. Furthermore, the C^2 norm is bounded for any fixed λ independently of both N and K, and can be taken as small as we want by taking λ small.

For any such initial conditions, we consider the associated pseudo-solution

$$\bar{\theta}_{\lambda,K,N}(r,\alpha,t):=\lambda(f_1(r)+f_2(N^{1/2}(r-1)+1)\sum_{k=1}^{K}\frac{\sin(Nk\alpha-\lambda tNk\frac{v_{\alpha}(f_1)}{r}-\lambda C_0t)}{N^2k^3}),\qquad(2.18)$$

where C_0 is the constant from Lemmas 2.2.2 and 2.2.3. We do not add subindexes for f_1 and f_2 since we consider them fixed from now on. Furthermore, the constants appearing in most of our results will also depend on f_1 and f_2, but, since we consider them fixed, we will not mention this.

This function for $N\geq 4$ satisfies

$$\frac{\partial\bar{\theta}_{\lambda,K,N}(r,\alpha,t)}{\partial t}+\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_{\alpha}(\lambda f_1)}{r}+\frac{\partial\lambda f_1}{\partial r}\bar{v}_r(\bar{\theta}_{\lambda,K,N})=0\qquad(2.19)$$

with

$$\bar{v}_r(f(r)\cos(k\alpha+g(r)))=C_0f(r)\cos(k\alpha+g(r)+\frac{\pi}{2})$$

if $k\neq 0$, and $\bar{v}_r(f(r))=0$. Note that, for arbitrary fixed T, these functions satisfy that $||\bar{\theta}_{\lambda,K,N}||_{H^{2+1/4}}\leq C\lambda K$, with C depending only on T.

Furthermore, we can rewrite (2.19) as

$$\begin{aligned}&\frac{\partial\bar{\theta}_{\lambda,K,N}(r,\alpha,t)}{\partial t}+\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\bar{\theta}_{\lambda,K,N})}{r}+\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})\\&+\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{r}+\frac{\partial(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})\\&+\frac{\partial\lambda f_1}{\partial r}(\bar{v}_r(\bar{\theta}_{\lambda,K,N})-v_r(\bar{\theta}_{\lambda,K,N}))=0\end{aligned}$$

Therefore $\bar{\theta}$ is a pseudo-solution with source term

$$\begin{aligned}&F_{\lambda,K,N}(x,t)=\\&\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{r}+\frac{\partial(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})+\frac{\partial\lambda f_1}{\partial r}(\bar{v}_r(\bar{\theta}_{\lambda,K,N})-v_r(\bar{\theta}_{\lambda,K,N})).\end{aligned}$$

Next we would like to prove that this source term is, indeed, small enough to obtain the desired results. We start by proving bounds on L^2 and in H^3 for $F_{\lambda,K,N}(x,t)$.

Lemma 2.2.6. *For $t\in[0,T]$ and a pseudo-solution $\bar{\theta}_{\lambda,K,N}$ as in (2.18) the source term $F_{\lambda,K,N}(x,t)$ satisfies*

$$||F_{\lambda,K,N}(x,t)||_{L^2}\leq CN^{-(2+3/4)}$$

with C depending on K, λ and T.

Proof. We start bounding the term $\frac{\partial\lambda f_1}{\partial r}(\bar{v}_r(\bar{\theta}_{\lambda,K,N})-v_r(\bar{\theta}_{\lambda,K,N}))$. First we decompose each function

$$\begin{aligned}&\frac{\sin(Nk\alpha-\lambda tN\frac{v_\alpha(f_1)}{r}-\lambda C_0t)}{N^2k^3}\\&=\frac{\sin(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0t)-\cos(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0t)}{N^2k^3}\end{aligned}$$

and using that $\frac{\partial^k\frac{v_\alpha(f_1)}{r}}{\partial r^k}(r=1)=0$ for $k=1,2$, then for $r\in(1-2N^{-1/2},1+2N^{-1/2})$ we have that

$$|\frac{\partial\frac{v_\alpha(f_1)}{r}}{\partial r}|\leq\frac{C}{N}$$

and thus

$$||\frac{\partial cos(\lambda tNk\frac{v_\alpha(f_1)}{r}+\lambda C_0t)}{\partial r}||_{L^\infty}\leq C$$
$$||\frac{\partial\sin(\lambda tNk\frac{v_\alpha(f_1)}{r}+\lambda C_0t)}{\partial r}||_{L^\infty}\leq C.$$

Therefore, we can directly apply Lemma 2.2.3 to obtain

$$\begin{aligned}&|v_r(f_2(N^{1/2}(r-1)+1)\frac{\sin(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0t)}{N^2k^3})\\&-\bar{v}_r(f_2(N^{1/2}(r-1)+1)\frac{\sin(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0t)}{N^2k^3})|\leq\frac{C}{N^{5/2}k^3},\end{aligned}$$

$$|v_r(f_2(N^{1/2}(r-1)+1)\frac{\cos(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0 t)}{N^2k^3})$$
$$-\bar{v}_r(f_2(N^{1/2}(r-1)+1)\frac{\cos(Nk\alpha)\cos(\lambda tN\frac{v_\alpha(f_1)}{r}+\lambda C_0 t)}{N^2k^3})|\leq \frac{C}{N^{5/2}k^3}.$$

With this we can estimate

$$\int_{1-N^{-1/2}}^{1+N^{-1/2}}\int_{-\pi}^{\pi}(\frac{\partial\lambda f_1}{\partial r}(v_r(\bar{\theta}_{\lambda,K,N})-\bar{v}_r(\bar{\theta}_{\lambda,K,N})))^2 d\alpha dr\leq (||\frac{\partial f_1}{\partial r}||_{L^\infty})^2\frac{C}{N^{5+1/2}}.$$

For $r\in(1/2,1-N^{-1/2})\cup(1+N^{-1/2},\infty)$, we use that $\bar{v}$ is zero in those points and Lemma 2.2.4 to obtain

$$\begin{aligned}&\int_{1/2}^{1-N^{-1/2}}\int_{-\pi}^{\pi}(\frac{\partial\lambda f_1}{\partial r}(v_r(\bar{\theta}_{\lambda,K,N})-\bar{v}_r(\bar{\theta}_{\lambda,K,N})))^2 d\alpha dr\\ &\leq \frac{C}{N^4}(||\frac{\partial f_1}{\partial r}||_{L^\infty})^2\int_{1/2}^{1-N^{-1/2}}\int_{-\pi}^{\pi}(\frac{||f_2||_{L^\infty}}{N^{3/2}|r-1|^2})^2 d\alpha dr\\ &\leq \frac{C}{N^{5+1/2}}(||\frac{\partial f_1}{\partial r}||_{L^\infty})^2(||f_2||_{L^\infty})^2\end{aligned}$$

and similarly

$$\int_{1+N^{-1/2}}^{\infty}\int_{-\pi}^{\pi}(\frac{\partial\lambda f_1}{\partial r}(v_r(\bar{\theta})-\bar{v}_r(\bar{\theta})))^2 d\alpha dr\leq \frac{C}{N^{5+1/2}}(||\frac{\partial f_1}{\partial r}||_{L^\infty})^2(||f_2||_{L^\infty})^2.$$

Combining all of these inequalities we get

$$||\frac{\partial\lambda f_1}{\partial r}(v_r(\bar{\theta})-\bar{v}_r(\bar{\theta}))||_{L^2}\leq \frac{C}{N^{2+3/4}}$$

with C depending on λ, K and T.

For the term $\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{r}$ we simply use $||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}||_{L^\infty}\leq\frac{C}{N}$ and

$$||\frac{v_\alpha(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{r}1_{\mathrm{supp}(\bar{\theta}_{\lambda,K,N})}||_{L^2}\leq \frac{C}{N^{2+1/4}}$$

so

$$||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{r}||_{L^2}\leq \frac{C}{N^{3+1/4}}.$$

Similarly for $\frac{\partial(\lambda f_1-\bar{\theta}_{\lambda,K,N})}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})$ we have that

$$||v_r(\bar{\theta}_{\lambda,K,N})||_{L^2}\leq\frac{C}{N^{2+1/4}}\quad\text{and}\quad ||\frac{\partial(\bar{\theta}_{\lambda,K,N}-\lambda f_1)}{\partial r}||_{L^\infty}\leq\frac{C}{N},$$

so

$$||\frac{\partial(\bar{\theta}_{\lambda,K,N}-\lambda f_1)}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})||_{L^2}\leq\frac{C}{N^{3+1/4}},$$

which finishes the proof. □

Lemma 2.2.7. *For $t \in [0,T]$, given a pseudo-solution $\bar{\theta}_{\lambda,K,N}$ as in (2.18) the source term $F_{\lambda,K,N}(x,t)$ satisfies*

$$||F_{\lambda,K,N}(x,t)||_{H^3} \leq CN^{3/4}$$

with C depending on K, λ, and T.

Proof. To prove this we will use that, given the product of two functions, we have

$$||fg||_{H^3} \leq C(||f||_{L^\infty}||g||_{H^3} + ||f||_{C^1}||g||_{H^2} + ||f||_{C^2}||g||_{H^1} + ||f||_{C^3}||g||_{L^2}).$$

Furthermore, for the pseudo-solutions considered, we have that $||\bar{\theta}_{\lambda,K,N} - \lambda f_1||_{C^k} \leq CN^{k-2}$, $||\bar{\theta}_{\lambda,K,N} - \lambda f_1||_{H^k} \leq CN^{k-2-1/4}$, $||\lambda f_1||_{C^k} \leq C$ with the constants C depending on k, λ and K.
Therefore we have that, using the bounds for the support of $\bar{\theta}_{\lambda,K,N}$

$$\begin{aligned}
&||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}\frac{v_\alpha(\lambda f_1 - \bar{\theta}_{\lambda,K,N})}{r}||_{H^3} \\
&\leq C(||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}||_{L^\infty}||v_\alpha(\lambda f_1 - \bar{\theta}_{\lambda,K,N})||_{H^3} + ||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}||_{C^1}||v_\alpha(\lambda f_1 - \bar{\theta}_{\lambda,K,N})||_{H^2} \\
&+ ||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}||_{C^2}||v_\alpha(\lambda f_1 - \bar{\theta}_{\lambda,K,N})||_{H^1} + ||\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial\alpha}||_{C^3}||v_\alpha(\lambda f_1 - \bar{\theta}_{\lambda,K,N})||_{L^2}) \\
&\leq CN^{-1/4},
\end{aligned}$$

and analogously

$$\begin{aligned}
&||\frac{\partial(\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}v_r(\bar{\theta}_{\lambda,K,N})||_{H^3} \\
&\leq C(||\frac{\partial(\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{L^\infty}||v_r(\bar{\theta}_{\lambda,K,N})||_{H^3} + ||\frac{\partial(\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{C^1}||v_r(\bar{\theta}_{\lambda,K,N})||_{H^2} \\
&+ ||\frac{\partial(\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{C^2}||v_r(\bar{\theta}_{\lambda,K,N})||_{H^1} + ||\frac{\partial(\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{C^3}||v_r(\bar{\theta}_{\lambda,K,N})||_{L^2}) \\
&\leq CN^{-1/4},
\end{aligned}$$

and finally

$$\begin{aligned}
&||\frac{\partial\lambda f_1}{\partial r}(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^3} \\
&\leq C(||\frac{\partial\lambda f_1}{\partial r}||_{L^\infty}||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^3} + ||\frac{\partial\lambda f_1}{\partial r}||_{C^1}||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^2} \\
&+ ||\frac{\partial\lambda f_1}{\partial r}||_{C^2}||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^1} + ||\frac{\partial\lambda f_1}{\partial r}||_{C^3}||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{L^2}) \\
&\leq CN^{3/4}.
\end{aligned}$$

□

We can combine these two lemmas and use the interpolation inequality for Sobolev spaces to obtain that

$$||F_{\lambda,K,N}||_{H^{2+1/4}} \leq C(N^{-(2+3/4)})^{1/4}(N^{3/4})^{3/4} \leq CN^{-1/8}.$$

With this, we are ready to study how the solution to SQG with the same initial conditions as $\bar{\theta}_{\lambda,K,N}$ behaves. If we define

$$\Theta_{\lambda,K,N} := \theta_{\lambda,K,N} - \bar{\theta}_{\lambda,K,N},$$

with $\theta_{\lambda,K,N}$ the only $H^{2+\frac{1}{4}}$ solution to the SQG equation with the same initial conditions as $\bar{\theta}_{\lambda,K,N}$, we have that

$$\begin{aligned}&\frac{\partial\Theta_{\lambda,K,N}}{\partial t}+v_1(\Theta_{\lambda,K,N})\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2}\\&+v_1(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_2}\\&+v_1(\bar{\theta}_{\lambda,K,N})\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1}+v_2(\bar{\theta}_{\lambda,K,N})\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2}-F_{\lambda,K,N}(x,t)=0,\end{aligned}\tag{2.20}$$

and we have the following results regarding the evolution of $\Theta_{\lambda,K,N}$.

Lemma 2.2.8. *Let $\Theta_{\lambda,K,N}$ defined as in (2.20), then if $\theta_{\lambda,K,N}$ exists for $t\in[0,T]$, we have that*

$$||\Theta_{\lambda,K,N}(x,t)||_{L^2}\leq\frac{Ct}{N^{(2+3/4)}}$$

with C depending on λ, K and T.

Proof. We start by noting that

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{||\Theta_{\lambda,K,N}||_{L^2}^2}{2}=-\int_{\mathbb{R}^2}\Theta_{\lambda,K,N}\\&\Big((v_1(\Theta_{\lambda,K,N})+v_1(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1}+(v_2(\Theta_{\lambda,K,N})+v_2(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2}\\&+v_1(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_2}-F_{\lambda,K,N}(x,t)\Big)dx,\end{aligned}$$

but, by incompressibility, we have that

$$\int_{\mathbb{R}^2}\Theta_{\lambda,K,N}\Big((v_1(\Theta_{\lambda,K,N})+v_1(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1}+(v_2(\Theta_{\lambda,K,N})+v_2(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2}\Big)dx=0,$$

and therefore we get that

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{||\Theta_{\lambda,K,N}||_{L^2}^2}{2}\\&\leq|\int_{\mathbb{R}^2}\Theta_{\lambda,K,N}\Big(v_1(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_2}+F_{\lambda,K,N}(x,t)\Big)dx|\\&\leq||\Theta_{\lambda,K,N}||_{L^2}\Big(||\Theta_{\lambda,K,N}||_{L^2}||\bar{\theta}_{\lambda,K,N}||_{C^1}+||F_{\lambda,K,N}(x,t)||_{L^2}\Big),\end{aligned}$$

and using that $||F_{\lambda,K,N}||_{L^2}\leq\frac{C}{N^{(2+3/4)}}$, $||\bar{\theta}_{\lambda,K,N}||_{C^1}\leq C$ and integrating we get that

$$||\Theta_{\lambda,K,N}||_{L^2}\leq\frac{C(e^{Ct}-1)}{N^{(2+3/4)}}.$$

□

Lemma 2.2.9. *Let $\Theta_{\lambda,K,N}$ be defined as in (2.20), then for N big enough, $\theta_{\lambda,K,N}$ exists for $t\in[0,T]$ and*

$$||\Theta_{\lambda,K,N}(x,t)||_{H^{2+1/4}}\leq\frac{Ct}{N^{1/8}}$$

with C depending on λ, K and T.

Proof. It is enough to prove that

$$||D^{2+1/4}\Theta_{\lambda,K,N}||_{L^2} \leq \frac{Ct}{N^{1/8}}$$

since

$$||f||_{H^s} \leq C(||D^s f||_{L^2} + ||f||_{L^2})$$

with $D^s = (-\Delta)^{s/2}$ and we already have the result

$$||\Theta_{\lambda,K,N}||_{L^2} \leq \frac{Ct}{N^{(2+3/4)}}.$$

We will use the following result found in [81].

Lemma 2.2.10. *Let $s > 0$. Then for any $s_1, s_2 \geq 0$ with $s_1 + s_2 = s$, and any f, $g \in \mathcal{S}(\mathbb{R}^2)$, the following holds:*

$$||D^s(fg) - \sum_{|\mathbf{k}|\leq s_1} \frac{1}{\mathbf{k}!}\partial^{\mathbf{k}} f D^{s,\mathbf{k}} g - \sum_{|\mathbf{j}|\leq s_2} \frac{1}{\mathbf{j}!}\partial^{\mathbf{j}} g D^{s,\mathbf{j}} f||_{L^2} \leq C||D^{s_1}f||_{L^2}||D^{s_2}g||_{BMO} \tag{2.21}$$

where $\mathbf{j}$ and $\mathbf{k}$ are multi-indexes, $\partial^{\mathbf{j}} = \frac{\partial}{\partial x_1^{j_1}\partial x_2^{j_2}}$, $\partial_\xi^{\mathbf{j}} = \frac{\partial}{\partial \xi_1^{j_1}\partial \xi_2^{j_2}}$ and $D^{s,\mathbf{j}}$ is defined using

$$\widehat{D^{s,\mathbf{j}}f}(\xi) = \widehat{D^{s,\mathbf{j}}}(\xi)\hat{f}(\xi)$$
$$\widehat{D^{s,\mathbf{j}}}(\xi) = i^{-|\mathbf{j}|}\partial_\xi^{\mathbf{j}}(|\xi|^s).$$

Although this result is for functions in the Schwartz space $\mathcal{S}$, since we only consider compactly supported functions we can apply it to functions in H^s. We will consider $s = 2 + 1/4$, although we will just write s for brevity.

Then

$$\begin{aligned}
&\frac{d}{dt}\frac{||D^s\Theta_{\lambda,K,N}||^2_{L^2}}{2} = -\int_{\mathbb{R}^2} D^s\Theta_{\lambda,K,N} \\
&D^s\Big((v_1(\Theta_{\lambda,K,N}) + v_1(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1} + (v_2(\Theta_{\lambda,K,N}) + v_2(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2} \\
&+ v_1(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N})\frac{\partial\bar{\theta}_{\lambda,K,N}}{\partial x_2} + F_{\lambda,K,N})(x,t)\Big)dx.
\end{aligned}$$

We will focus for now on

$$\int_{\mathbb{R}^2} D^s\Theta_{\lambda,K,N} D^s\Big(v_1(\bar{\theta}_{\lambda,K,N}))\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1} + v_2(\bar{\theta}_{\lambda,K,N})\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2}\Big)dx.$$

Applying (2.21) with $s_2 = 1$, $g = v_i(\bar{\theta}_{\lambda,K,N}))$, $f = \frac{\partial\Theta_{\lambda,K,N}}{\partial x_i}$, $i = 1, 2$ we get that

$$\begin{aligned}
&(D^s\Theta_{\lambda,K,N}, D^s(fg) - \sum_{|\mathbf{k}|\leq s_1}\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}} f D^{s,\mathbf{j}} g - \sum_{|\mathbf{k}|\leq s_2}\frac{1}{\mathbf{k}!}\partial^{\mathbf{k}} g D^{s,\mathbf{k}} f)_{L^2} \\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||D^{s_1}f||_{L^2}||D^{s_2}g||_{BMO} \\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\bar{\theta}_{\lambda,K,N}||_{H^s}||\Theta_{\lambda,K,N}||_{H^s}.
\end{aligned}$$

Furthermore we have that

$$\begin{aligned}
&(D^s\Theta_{\lambda,K,N}, D^s(\frac{\partial\Theta_{\lambda,K,N}}{\partial x_1})v_1(\bar{\theta}_{\lambda,K,N}) + D^s(\frac{\partial\Theta_{\lambda,K,N}}{\partial x_2})v_2(\bar{\theta}_{\lambda,K,N}))_{L^2} \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial}{\partial x_1}(D^s\Theta_{\lambda,K,N})^2 v_1(\bar{\theta}_{\lambda,K,N})) + \frac{\partial}{\partial x_2}(D^s\Theta_{\lambda,K,N})^2 v_2(\bar{\theta}_{\lambda,K,N})dx = 0
\end{aligned}$$

and, for $i=1,2$, using that the operators $D^{s,c}$ are continuous from H^a to H^{a-s+c}, we have the following three estimates

1)

$$\begin{aligned}
&|(D^s\Theta_{\lambda,K,N},\sum_{|\mathbf{k}|=1}\frac{1}{\mathbf{k}!}\partial^{\mathbf{k}}v_i(\bar{\theta}_{\lambda,K,N}))D^{s,\mathbf{k}}\frac{\partial\Theta_{\lambda,K,N}}{\partial x_i})_{L^2}|\\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||v_i(\bar{\theta}_{\lambda,K,N})||_{H^{2+\epsilon}}||\Theta_{\lambda,K,N}||_{H^s}\\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\bar{\theta}_{\lambda,K,N}||_{H^s}||\Theta_{\lambda,K,N}||_{H^s},
\end{aligned}$$

2)

$$\begin{aligned}
&|(D^s\Theta_{\lambda,K,N},\sum_{|\mathbf{j}|=1}\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}}\frac{\partial\Theta_{\lambda,K,N}}{\partial x_i}D^{s,\mathbf{j}}v_i(\bar{\theta}_{\lambda,K,N}))_{L^2}|\\
&\leq C\sum_{|\mathbf{j}|=1}||D^s\Theta_{\lambda,K,N}||_{L^2}||\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}}\frac{\partial\Theta_{\lambda,K,N}}{\partial x_i}||_{L^{2/(3-s)}}||D^{s,\mathbf{j}}v_i(\bar{\theta}_{\lambda,K,N}))||_{L^{2/(s-2)}}\\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\Theta_{\lambda,K,N}||_{H^s}||\bar{\theta}_{\lambda,K,N}||_{H^s},
\end{aligned}$$

3)

$$\begin{aligned}
&|(D^s\Theta_{\lambda,K,N},\frac{\partial\Theta_{\lambda,K,N}}{\partial x_i}D^s v_i(\bar{\theta}_{\lambda,K,N}))_{L^2}|\\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\Theta_{\lambda,K,N}||_{H^s}||\bar{\theta}_{\lambda,K,N}||_{H^s}.
\end{aligned}$$

Most of the other terms are bounded in a similar way without any complication, although a comment needs to be made about bounding the terms

$$\int_{\mathbb{R}^2}D^s(\Theta_{\lambda,K,N})\Big(v_1(\Theta_{\lambda,K,N})\frac{\partial D^s\bar{\theta}_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial D^s\bar{\theta}_{\lambda,K,N}}{\partial x_2}\Big)dx.$$

At first glance one could think that, since we are considering $\bar{\theta}_{\lambda,K,N}$ bounded in $H^{2+1/4}$ but not in higher order spaces, we could have a problem bounding this integral. However, we actually have that

$$||\frac{\partial D^s\bar{\theta}_{\lambda,K,N}}{\partial x_i}||_{L^\infty}\leq C||D^s\bar{\theta}_{\lambda,K,N}||_{H^{2+\epsilon}}\leq C||\bar{\theta}_{\lambda,K,N}||_{H^{4+1/4+\epsilon}}\leq CN^{2+\epsilon}$$

$$||v_i(\Theta_{\lambda,K,N})||_{L^2}\leq CTN^{-(2+3/4)}$$

and thus

$$\begin{aligned}
&|\int_{\mathbb{R}^2}D^s(\Theta_{\lambda,K,N})\Big(v_1(\Theta_{\lambda,K,N})\frac{\partial D^s\bar{\theta}_{\lambda,K,N}}{\partial x_1}+v_2(\Theta_{\lambda,K,N})\frac{\partial D^s\bar{\theta}_{\lambda,K,N}}{\partial x_2}\Big)dx|\\
&\leq CT||D^s(\Theta_{\lambda,K,N})||_{L^2}N^{-3/4+\epsilon}\leq CT||D^s(\Theta_{\lambda,K,N})||_{L^2}N^{-1/8},
\end{aligned}$$

and combining all of this together plus similar bounds for the other terms, and using

$$||\bar{\theta}_{\lambda,K,N}||_{H^s}\leq C,\quad ||F_{\lambda,K,N}||_{H^s}\leq CN^{-1/8}$$

with C depending on λ, K and T, we get

$$\frac{d}{dt}||D^s\Theta_{\lambda,K,N}||^2_{L^2} \leq ||D^s\Theta_{\lambda,K,N}||_{L^2}(CN^{-1/8} + C||\Theta_{\lambda,K,N}||_{H^s} + C||\Theta_{\lambda,K,N}||^2_{H^s})$$

which gives us, using

$$||\Theta_{\lambda,K,N}||_{H^s} \leq C(||\Theta_{\lambda,K,N}||_{L^2} + ||D^s\Theta_{\lambda,K,N}||_{L^2}) \leq C(||D^s\Theta_{\lambda,K,N}||_{L^2} + N^{-(2+3/4)})$$

that

$$\frac{\partial}{\partial t}||D^s\Theta_{\lambda,K,N}||_{L^2} \leq (CN^{-1/8} + C||D^s\Theta_{\lambda,K,N}||_{L^2} + C||D^s\Theta_{\lambda,K,N}||^2_{L^2}).$$

Now, we restrict ourselves to $[0, T_*]$, with T_* the biggest time such that $||D^s\Theta_{\lambda,K,N}||_{L^2} \leq 1$ (or T if T_* is bigger than T or it does not exist). Integrating for those times we get

$$||D^s\Theta_{\lambda,K,N}||_{L^2} \leq \frac{C(e^{Ct}-1)}{N^{1/8}},$$

and since for N big enough we have that $T \leq T_*$ we are done.

□

Now we are finally prepared to prove strong ill-posedness in C^2 for the SQG equation.

Theorem 2.2.1. For any $c_0 > 0$, $M > 0$ and $t_* > 0$, we can find a $C^2 \cap H^{2+1/4}$ function $\theta_0(x)$ with $||\theta_0(x)||_{C^2} \leq c_0$ such that the only solution $\theta(x,t) \in H^{2+\frac{1}{4}}$ to the SQG problem (2.1) with initial conditions $\theta_0(x)$ will satisfy $||\theta(x,t_*)||_{C^2} \geq Mc_0$.

Proof. We will prove this by constructing a solution with the desired properties. We fix arbitrary $c_0 > 0$, $M > 0$ and t_*, and consider the pseudo-solutions $\bar{\theta}_{\lambda,K,N}$. First, note that, for any N, K natural numbers, for $\lambda > 0$ small enough our family of pseudo-solutions has a small initial norm in C^2, so we consider $\lambda = \lambda_0$ small so that $||\theta_{\lambda_0,K,N}(x,0)||_{C^2} \leq c_0$ for all K, N natural and such that $|\lambda_0 C_0 t_*| \leq \frac{\pi}{2}$.

These pseudo-solutions fulfill that, at time t, for $\alpha = \lambda_0 t\frac{v_\alpha(f_1)}{r}$

$$\begin{aligned}
|\frac{\partial^2\bar{\theta}_{\lambda_0,K,N}(x,t)}{\partial\alpha^2}| &= |\lambda_0 f_2(N^{1/2}(r-1)+1)\sum_{k=1}^{K}\frac{\sin(Nk\alpha - \lambda_0 tNk\frac{v_\alpha(f_1)}{r} - \lambda_0 C_0 t)}{k})| \\
&= |\lambda_0 f_2(N^{1/2}(r-1)+1)\sum_{k=1}^{K}\frac{\sin(-\lambda_0 C_0 t)}{k})| \\
&\geq \lambda_0|f_2(N^{1/2}(r-1)+1)|ln(K)|sin(-\lambda_0 C_0 t)|.
\end{aligned}$$

Furthermore, we can find $c > 0$ small such that , for $\alpha \in [\lambda_0 t\frac{v_\alpha(f_1)}{r} - c\frac{2\pi}{NK}, \lambda_0 t\frac{v_\alpha(f_1)}{r} + c\frac{2\pi}{NK}]$ we have

$$|\frac{\partial^2\bar{\theta}_{\lambda_0,K,N}(x,t)}{\partial\alpha^2}| \geq \lambda_0\frac{|f_2(N^{1/2}(r-1)+1)|ln(K)|sin(-\lambda_0 C_0 t)|}{2}.$$

Therefore by using that $f(r) = 1$ if $r \in (3/4, 5/4)$ and defining

$$B := \cup_{j\in\mathbb{N}}\Big[j\frac{2\pi}{N} + \lambda_0 t\frac{v_\alpha(f_1)}{r} - c\frac{2\pi}{NK}, j\frac{2\pi}{N} + \lambda_0 t\frac{v_\alpha(f_1)}{r} + c\frac{2\pi}{NK}\Big]$$

and $A := [1 - \frac{N^{-1/2}}{4}, 1 + \frac{N^{-1/2}}{4}]$, we obtain

$$\int_A\int_B \frac{1}{r^3}\Big(\frac{\partial^2\bar{\theta}_{\lambda_0,K,N}}{\partial\alpha^2}\Big)^2 d\alpha dr \geq \lambda_0^2\frac{ln(K)^2}{4(1+N^{-\frac{1}{2}})^2}|A||B||sin(-\lambda_0 C_0 t)|^2, \tag{2.22}$$

with $|A|, |B|$ the length of A and B respectively. We now consider K big enough such that

$$\lambda_0 ln(K)|sin(-\lambda_0 C_0 t_*)| \geq 16Mc_0,$$

and thus, for N big

$$\int_A\int_B \frac{1}{r^3}\Big(\frac{\partial^2\bar{\theta}_{\lambda_0,K,N}}{\partial\alpha^2}\Big)^2 d\alpha dr \geq 16M^2c_0^2|A||B|. \tag{2.23}$$

Now, we can use Lemmas 2.2.9 and 2.2.8 plus the interpolation inequality for Sobolev spaces to obtain that, for N big enough,

$$||\Theta_{\lambda_0,K,N}||_{H^2} \leq CtN^{-a-1/4}$$

for some $a > 0$ which can be computed explicitly but whose particular value is not relevant for this proof. With this we have that the solution $\theta_{\lambda_0,K,N}$ satisfies that, at $t = t_*$

$$\begin{aligned}
&\Big(\int_A\int_B \frac{1}{r^3}\Big(\frac{\partial^2\theta_{\lambda_0,K,N}}{\partial\alpha^2}\Big)^2 d\alpha dr\Big)^{1/2} \\
&= ||\frac{1}{r^2}\frac{\partial^2\theta_{\lambda_0,K,N}}{\partial\alpha^2}1_{A\times B}||_{L^2} \\
&\geq ||\frac{1}{r^2}\frac{\partial^2\bar{\theta}_{\lambda_0,K,N}}{\partial\alpha^2}1_{A\times B}||_{L^2} - ||\frac{1}{r^2}\frac{\partial^2\Theta_{\lambda_0,K,N}}{\partial\alpha^2}1_{A\times B}||_{L^2} \\
&\geq 4Mc_0|A|^{1/2}|B|^{1/2} - Ct_*N^{-a-1/4}
\end{aligned}$$

where we used that there is a constant C such that

$$||\frac{1}{r^2}\frac{\partial^2 g}{\partial\alpha^2}1_{A\times B}||_{L^2} \leq C||g1_{A\times B}||_{H^2}. \tag{2.24}$$

But $|A||B| \geq CN^{-1/2}$, so, taking N big enough we get

$$\begin{aligned}
&\Big(\int_A\int_B \frac{1}{r^3}\Big(\frac{\partial^2\theta_{\lambda_0,K,N}}{\partial\alpha^2}\Big)^2 d\alpha dr\Big)^{1/2} \\
&\geq 3Mc_0|A|^{1/2}|B|^{1/2}.
\end{aligned}$$

But

$$\sup_{x\in A\times B}|\frac{1}{r^2}\frac{\partial^2 g}{\partial\alpha^2}| \leq 2||g||_{C^2}, \tag{2.25}$$

so

$$\begin{aligned}
&\Big(\int_A\int_B \frac{1}{r^3}\Big(\frac{\partial^2\theta_{\lambda_0,K,N}}{\partial\alpha^2}\Big)^2 d\alpha dr\Big)^{1/2} \\
&\leq 2|A|^{1/2}|B|^{1/2}||\theta_{\lambda_0,K,N}||_{C^2},
\end{aligned}$$

and thus

$$||\theta_{\lambda_0,K,N}||_{C^2} \geq \frac{3Mc_0}{2}.$$

□

2.2.3 Non existence in C^k

Now we can prove the last result of this section.

Theorem 2.2.2. Given $c_0 > 0$, there are initial conditions $\theta_0 \in H^{2+1/8}\cap C^2$ for the SQG equation (2.1) such that $||\theta_0||_{C^2} \leq c_0$ and the only solution $\theta(x,t) \in H^{2+\frac{1}{8}}$ with $\theta(x,0) = \theta_0(x)$ satisfies that there exists a $t_* > 0$ with $||\theta(x,t)||_{C^2} = \infty$ for all t in the interval $(0,t_*)$.

Remark 2. We can actually prove that, for the initial conditions $\theta_0(x)$ obtained in Theorem 2.2.2, there is no solution in $L^\infty_t L^2_x$ such that $\theta(x,t) \in C^2$ for t in some small time interval (even if we allow ess-sup$_{t\in[0,\epsilon]}||\theta(x,t)||_{C^2} = \infty$), since, if we call $\theta_1(x,t)$ the solution found in Theorem 2.2.2 and $\theta_2(x,t)$ the new solution belonging pointwise in time to C^2 for a small time interval, we can obtain the bound

$$\frac{d||\theta_2(x,t) - \theta_1(x,t)||_{L^2}}{dt} \leq C||\theta_2(x,t) - \theta_1(x,t)||_{L^2}$$

which implies that $||\theta_2(x,t) - \theta_1(x,t)||_{L^2} = 0$.

Remark 3. The value of t_* can be made arbitrarily big if wanted with very small adjustments on the proof, but for simplicity we provide the proof without worrying about the specific value of t_*.

Proof. (of Theorem 2.2.2)

We consider a family of pseudo-solutions to the SQG equation

$$\bar{\theta}_n(x,t) = \bar{\theta}_{\lambda_n,K_n,N_n}(x,t)$$

for $n \in \mathbb{N}$, with $\bar{\theta}_{\lambda_n,K_n,N_n}$ defined as in (2.18). Although $\bar{\theta}_n$ depends on the choice of λ_n, K_n and N_n, we do not write the dependence explicitly to get a more compact notation. We start by fixing λ_n satisfying

$$\lambda_n \leq 2^{-n},$$

and such that $||\bar{\theta}_n(x,0)||_{C^2} \leq c_0$ independently of the choice of K_n and N_n.

Note that this already tells us that for any fixed arbitrary T, if $0 \leq t \leq T$ then

$$||\bar{\theta}_n(x,t)||_{H^{2+1/8}} \leq C2^{-n}(\frac{K_n}{N_n^{1/8}} + 1)$$

with C depending on T. We will only consider $N_n^{1/8} \geq K_n$, so that $||\bar{\theta}_n(x,t)||_{H^{2+1/8}} \leq C2^{-n}$. We fix now K_n so that $\lambda_n^2 ln(K_n) \geq 16n$. Note that then, as seen in the proof of Theorem 2.2.1, we have that there is a set $S_n = S_{\lambda_n,K_n,N_n,t}$, (see (2.22), $A \times B$ would give the desired set) with measure $|S_n| \geq \frac{c}{K_n N_n^{1/2}} > 0$ such that the function $\bar{\theta}_n(x,t)$ fulfils that

$$||\frac{1}{r^2}\frac{\partial^2\bar{\theta}_n(x,t)}{\partial\alpha^2}1_{S_n}||_{L^2} \geq 4n\frac{|S_n|^{1/2}|sin(\lambda_n C_0 t)|}{\lambda_n}. \tag{2.26}$$

Let us consider now the initial conditions

$$\theta((\lambda_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}, (N_n)_{n\in\mathbb{N}}, (R_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} T_{R_n}(\bar{\theta}_n(x,0))$$

with $T_R(f(x_1,x_2)) = f(x_1+R, x_2)$, with R_n yet to be fixed. We will refer to these initial conditions simply as $\theta(x,0)$ and to the unique $H^{2+\frac{1}{8}}$ solution to the SQG equation (2.1) with initial conditions $\theta(x,0)$, as $\theta(x,t)$ for a more compact notation, keeping in mind that the function depends on multiple parameters. Since $||\bar{\theta}_n(x,0)||_{H^{2+1/8}} \leq C2^{-n}$ we have that $||\theta(x,0)||_{H^{2+1/8}} \leq C$, and thus we can use the a priori bounds to assure the existence of $\theta(x,t)$ for some time interval $[0,t_{ex}]$ and also $||\theta(x,t)||_{H^{2+1/8}} \leq C$ for some big C for $t \in [0, \frac{t_{ex}}{2}]$. This also tells us that, in particular, $||v_j(\theta)||_{L^\infty} \leq v_{max}$ for some big constant v_{max} for $t \in [0, \frac{t_{ex}}{2}]$ and $j = 1,2$.

We restrict ourselves now to study the interval $t \in [0, t_{crit}]$ with

$$t_{crit} = min(\frac{t_{ex}}{2}, \frac{\pi}{sup_n(\lambda_n)C_0 2}).$$

By construction, the support of $\bar{\theta}_n(x,0)$ is contained in a disk of a certain radius D. Then, if we consider $R_n = R_{n-1} + 2D + 4v_{max}t_{crit} + D_n + D_{n-1}$ with $D_n, D_{n-1} > 0$, we have that

$$d(\text{supp}(1_{B_{D+2v_{max}t_{crit}}(-R_n,0)}\theta(x,t)), \text{supp}(\theta(x,t) - 1_{B_{D+2v_{max}t_{crit}}(-R_n,0)}\theta(x,t))) > D_n$$

and

$$\tilde{\theta}_n(x,t) := \theta(x,t)1_{B_{D+2v_{max}t_{crit}}(-R_n,0)}$$

is a pseudo-solution fulfilling

$$\frac{\partial \tilde{\theta}_n}{\partial t} + v_1(\tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_2} + \tilde{F}_n = 0,$$
$$v_1(\tilde{\theta}_n) = -\frac{\partial}{\partial x_2}\Lambda^{-1}\tilde{\theta}_n = -\mathcal{R}_2\theta,$$
$$v_2(\tilde{\theta}_n) = \frac{\partial}{\partial x_1}\Lambda^{-1}\tilde{\theta}_n = \mathcal{R}_1\theta,$$
$$\tilde{F}_n := v_1(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_2},$$
$$\tilde{\theta}_n(x,0) := \theta(x,0)1_{B_{D+2v_{max}t_{crit}}(-R_n,0)}.$$

If we now define $\Theta_n := \tilde{\theta}_n - T_{R_n}(\bar{\theta}_n)$ we get

$$\begin{aligned}
&\frac{\partial \Theta_n}{\partial t} + v_1(\Theta_n)\frac{\partial \Theta_n}{\partial x_1} + v_2(\Theta_n)\frac{\partial \Theta_n}{\partial x_2} \\
&+ v_1(\Theta_n)\frac{\partial T_{R_n}(\bar{\theta}_n)}{\partial x_1} + v_2(\Theta_n)\frac{\partial T_{R_n}(\bar{\theta}_n)}{\partial x_2} \\
&+ v_1(T_{R_n}(\bar{\theta}_n))\frac{\partial \Theta_n}{\partial x_1} + v_2(T_{R_n}(\bar{\theta}_n))\frac{\partial \Theta_n}{\partial x_2} - T_{R_n}(F_{\lambda_n,K_n,N_n}(x,t)) + \tilde{F}_n = 0,
\end{aligned} \tag{2.27}$$

with F_{λ_n,K_n,N_n} the source term of our pseudo-solution $\bar{\theta}_n = \bar{\theta}_{\lambda_n,K_n,N_n}$ and therefore satisfying the bounds given by Lemmas 2.2.8 and 2.2.9,

$$||F_{\lambda_n,K_n,N_n}||_{L^2} \leq \frac{C}{N_n^{2+3/4}}$$

and

$$||F_{\lambda_n,K_n,N_n}||_{H^{2+1/4}} \leq \frac{C}{N_n^{1/8}}.$$

It is easy to prove that

$$||v_i(\theta - \tilde{\theta}_n)1_{\text{supp}(\tilde{\theta}_n)}||_{L^\infty} \leq \frac{C}{(D_n)^2}$$

and in fact

$$||v_i(\theta - \tilde{\theta}_n)1_{\text{supp}(\tilde{\theta}_n)}||_{C^k} \leq \frac{C_k}{(D_n)^2} \tag{2.28}$$

since

$$d(supp(\tilde{\theta}_n), supp(\tilde{\theta} - \tilde{\theta}_n)) \geq D_n.$$

Taking, for example, $D_n = N_n^{\frac{2+3/4}{2}}$ to obtain that $||\tilde{F}_n||_{L^2} \leq \frac{C}{N_n^{2+\frac{3}{4}}}$ we can argue as in Lemma 2.2.8 to get that

$$||\Theta_n||_{L^2} \leq \frac{Ct}{N_n^{2+3/4}}$$

for all $t \in [0, t_{crit}]$. We can also estimate $||\Theta_n||_{H^{2+1/8}}$ as in Lemma 2.2.9, being the only difference that now we have the extra term $\tilde{F}_n$. Therefore, it is enough to obtain bounds for

$$\int_{\mathbb{R}^2} D^s(\Theta_n)(D^s(v_1(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_2}))dx_1 dx_2$$

with $s=2+1/8$.

Using Lemma 2.2.10 in the same way as we did in Lemma 2.2.9, we can decompose this integral in several terms that are easy to bound using (2.28) plus the term

$$\int_{\mathbb{R}^2} D^s(\Theta_n)(v_1(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_1}+v_2(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_2})dx_1dx_2$$

which is, in principle, too irregular to be bounded. However, using incompressibility and $\Theta_n=\tilde{\theta}_n-T_{R_n}(\bar{\theta}_n)$ we get

$$\begin{aligned}
&|\int_{\mathbb{R}^2} D^s(\Theta_n)(v_1(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_1}+v_2(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_2})dx_1dx_2|\\
&=|\int_{\mathbb{R}^2} D^s(T_{R_n}(\bar{\theta}_n))(v_1(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_1}+v_2(\theta-\tilde{\theta}_n)D^s\frac{\partial\tilde{\theta}_n}{\partial x_2})dx_1dx_2|\\
&=|\int_{\mathbb{R}^2} D^s(\tilde{\theta}_n)(v_1(\theta-\tilde{\theta}_n)D^s\frac{\partial T_{R_n}(\bar{\theta}_n)}{\partial x_1}+v_2(\theta-\tilde{\theta}_n)D^s\frac{\partial T_{R_n}(\bar{\theta}_n)}{\partial x_2})dx_1dx_2|\\
&\leq ||D^s\tilde{\theta}_n||_{L^2}\frac{C}{N_n^{2+\frac{3}{4}}}N_n^2\leq\frac{C}{N_n^{\frac{3}{4}}}.
\end{aligned}$$

Therefore, as in Lemma 2.2.9, we get

$$||\Theta_n||_{H^{2+1/8}}\leq\frac{Ct}{N_n^{1/8}}.$$

This combined with the L^2 norm and using the interpolation inequality for Sobolev spaces gives us

$$||\Theta_n||_{H^2}\leq\frac{Ct}{N_n^{19/68}}=\frac{Ct}{N_n^{1/4+a}}.$$

with $a>0$, for all $t\in[0,t_{crit}]$.

However, this means that, if we consider the polar coordinates around the point $(-R_n,0)$, which we will call (r_{R_n},α_{R_n}), and using (2.24)

$$\begin{aligned}
&||\frac{1}{r_{R_n}^2}\frac{\partial^2\theta(x,t)}{\partial\alpha_{R_n}^2}T_{R_n}(1_{S_n})||_{L^2}\\
&\geq ||T_{R_n}(\frac{1}{r^2}\frac{\partial^2\bar{\theta}(x,t)}{\partial\alpha^2}1_{S_n})||_{L^2}-||\tilde{\theta}_n-T_{R_n}(\bar{\theta}_n(x,t))||_{H^2}\\
&\geq 4n|S_n|^{1/2}\frac{|sin(\lambda_nC_0t)|}{\lambda_n}-\frac{Ct}{N_n^{1/4+a}}
\end{aligned}$$

but, using $C_0t\lambda_n\leq\frac{\pi}{2}$, $|S_n|\geq CK_n^{-1}N_n^{-1/2}$ and taking N_n big enough we get

$$||T_{R_n}(1_{S_n})\frac{1}{r_{R_n}^2}\frac{\partial^2\theta(x,t)}{\partial\alpha_{R_n}^2}||_{L^2}\geq cnt|S_n|^{1/2}$$

for some small constant c.

But then

$$\begin{aligned}
&||T_{R_n}(1_{S_n})\frac{1}{r_{R_n}^2}\frac{\partial^2\tilde{\theta}_n(x,t)}{\partial\alpha_{R_n}^2}||_{L^2}\\
&\leq ||T_{R_n}\tilde{\theta}_n(x,t)||_{C^2}|S_n|^{1/2}
\end{aligned}$$

and thus $||T_{R_n}(1_{S_n})\theta(x,t)||_{C^2} \geq cnt$ and we are done since we can do this for every n.

$\square$

Both results in this section can be obtained in C^m for $m \geq 2$, using the same method. To do it we consider pseudo-solutions of the form

$$\lambda(f_1(r) + f_2(N^{1/2}(r-1)+1)\sum_{k=1}^{K} \frac{\sin(Nk\alpha)}{N^m k^{m+1}}).$$

The proof follows the same method, except that this time we have that the associated source terms $F_{\lambda,K,N}$ of these pseudo-solutions fulfil $||F_{\lambda,K,N}||_{L^2} \leq \frac{C}{N^{m+3/4}}$, $||F_{\lambda,K,N}||_{H^k} \leq CN^{k-m-1/4}$, which gives us, by taking k big and using the interpolation inequality that

$$||F_{\lambda,K,N}||_{H^{m+\frac{1}{4}}} \leq CN^{-\frac{1}{2}+\delta}$$

for any $\delta > 0$.

Note also that analogous expressions as (2.24) and (2.25) exists for higher order derivatives in α, albeit with different constants.

2.3 Strong ill-posedness and non existence in supercritical Sobolev spaces

2.3.1 Pseudo-solutions for H^s

The proof for ill-posedness in supercritical Sobolev spaces follows a very similar strategy as before. We find an appropriate pseudo-solution with the desired properties, we find bounds for the source term and then we obtain bounds for the difference between the real solution and the pseudo-solution. This time, we will consider pseudo-solutions of the form

$$\bar{\theta}(r,\alpha,t) = f_1(r) + f_2(r)\frac{\sin(N\alpha - Nt\frac{v_\alpha(f_1(r))}{r}))r_0^\beta}{N^\beta}$$

with f_1, f_2 compactly supported C^∞ functions, $r_0 > 0$ and $v_\alpha(f_1(r))$ is the angular velocity generated by the function $f_1(r)$.

The choice of f_1, f_2 and r_0 will depend on the specific behaviour we want our pseudo-solutions to have. Before we start to specify how we choose them and how we will label the pseudo-solutions, we need the following technical lemma.

Lemma 2.3.1. *For any $\beta \in (\frac{3}{2}, 2)$ and $K, c > 0$, there exists a C^∞ radial function $f_1(r)$: $\mathbb{R}_+ \times [0,2\pi] \to \mathbb{R}$, with support in some $[a_1,a_2] \times [0,2\pi]$, $0 < a_1 < a_2$ depending on K, c and β such that $||f_1(r)||_{H^\beta} \leq c$, and $|\frac{\partial \frac{v_\alpha(f_1(\cdot))(r)}{r}}{\partial r}(r = \frac{a_1}{2})| \geq \frac{2K}{a_1}$.*

Proof. By Lemma 2.2.5, we can find a C^∞ function $g(r) : \mathbb{R}_+ \times [0,2\pi] \to \mathbb{R}$ with support in $r \in [2,M]$ such that $\frac{\partial \frac{v_\alpha(g(.))(r)}{r}}{\partial r}(r=1) = 1$. If we consider now the functions

$$g_{\lambda_1,\lambda_2}(r) := \frac{g(\lambda_1 r)}{\lambda_2\lambda_1^{\beta-1}}, \quad \lambda_1, \lambda_2 > 1$$

we have (for example using the interpolation inequalities for Sobolev spaces) that

$$||g_{\lambda_1,\lambda_2}(r)||_{H^\beta} \leq \frac{C}{\lambda_2} \tag{2.29}$$

with C depending on $||g(r)||_{H^2}$.

Furthermore, $v_\alpha(f(\lambda\cdot))(\frac{r}{\lambda}) = v_\alpha(f(\cdot))(r)$, $\frac{\partial v_\alpha(f(\lambda\cdot))}{\partial r}(\frac{r}{\lambda}) = \lambda\frac{\partial v_\alpha(f(\cdot))}{\partial r}(r)$, so

$$\frac{\partial \frac{v_\alpha(g_{\lambda_1,\lambda_2}(.))(r)}{r}}{\partial r}(r=\frac{1}{\lambda_1})=\frac{\lambda_1^{3-\beta}}{\lambda_2}=\frac{\lambda_1^{2-\beta}}{\frac{1}{\lambda_1}\lambda_2}.$$

Therefore it is enough to take g_{λ_1,λ_2} with λ_2 big enough so that $\frac{C}{\lambda_2}\leq c$ (C the constant in (2.29)) and then λ_1 big enough so that $\frac{\lambda_1^{2-\beta}}{\lambda_2}\geq K$ and g_{λ_1,λ_2} with $a_1=\frac{2}{\lambda_1}$, $a_2=\frac{M}{\lambda_1}$ will have all the properties desired. □

From now on we consider β a fixed value in the interval $(\frac{3}{2},2)$. The family of pseudo-solutions we consider to obtain ill-posedness in H^β is, for $N\in\mathbb{N}$

$$\bar{\theta}_{N,c,K}(r,\alpha,t)=f_{1,c,K}(r)+f_{2,c,K}(r)r_{c,K}^{\beta}\frac{\sin(N\alpha-Nt\frac{v_\alpha(f_1(r))}{r})}{N^\beta} \tag{2.30}$$

with $f_{1,c,K}$ the function given by Lemma 2.3.1 for the specific values of c and K considered and $r_{c,K}=\frac{a_1}{2}$ given by the lemma. By continuity, we have that there exists an interval $[r_{c,K}-\epsilon,r_{c,K}+\epsilon]$ such that if $\bar{r}\in[r_{c,K}-\epsilon,r_{c,K}+\epsilon]$ then

$$\frac{\partial\frac{v_\alpha(f_{1,c,K}(.))(r)}{r}}{\partial r}(r=\bar{r})\geq\frac{K}{2\bar{r}}. \tag{2.31}$$

We take $f_{2,c,K}$ to be a C^∞ function with support in $[r_{c,K}-\epsilon,r_{c,K}+\epsilon]\cap[\frac{r_{c,K}}{2},\frac{3r_{c,K}}{2}]$ and fulfilling $||f_{2,c,K}||_{L^2}=c$.

These pseudo-solutions fulfil the evolution equation

$$\frac{\partial\bar{\theta}_{N,c,K}}{\partial t}+\frac{v_\alpha(f_{1,c,K}(\cdot))}{r}\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}=0$$

and therefore they are pseudo-solutions with source term

$$\begin{aligned}&F_{N,c,K}\\&:=-(\frac{v_\alpha(\bar{\theta}_{N,c,K}(\cdot)-f_{1,c,K}(\cdot))}{r}\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}+v_r(\bar{\theta}_{N,c,K}(\cdot))\frac{\partial\bar{\theta}_{N,c,K}}{\partial r})\\&=-(\frac{v_\alpha(\bar{\theta}_{N,c,K}(\cdot)-f_{1,c,K}(\cdot))}{r}\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}+v_r(\bar{\theta}_{N,c,K}(\cdot)-f_{1,c,K}(\cdot))\frac{\partial\bar{\theta}_{N,c,K}}{\partial r}).\end{aligned} \tag{2.32}$$

Next we need to obtain bounds for our source term. To do this, we start with a lemma analogous to Lemma 2.2.4:

Lemma 2.3.2. *Given a L^∞ function $\tilde{g}_N:\mathbb{R}\to\mathbb{R}$ with support in the interval (a,b) then if we define g_N as*

$$g_N(r,\alpha):=sin(N\alpha+\alpha_0)\tilde{g}_N(r)$$

with N a natural number, then there is a constant C depending on (a,b) such that if $r>b$, then

$$|v_r(g_N)|(r,\alpha)\leq\frac{C||g_N||_{L^\infty}}{N|r-b|^2}.$$

Furthermore, we have that if $||\tilde{g}_N||_{C^i}\leq MN^i$ for $i=0,1,...,m$, then

$$|\frac{\partial^m v_r(g_N)}{\partial x_1^{m-i}\partial x_2^i}|(r,\alpha)\leq\frac{CMN^{m-1}}{|r-b|^2}, \tag{2.33}$$

with C depending on (a,b) and m.

Proof. The proof for the decay of the velocity it is analogous to that of Lemma 2.2.4. As for the higher derivatives, using that

$$v_r(w) = \cos(\alpha(x))v_1(w) + sin(\alpha(x))v_2(w),$$

one can obtain that

$$\begin{aligned}
&|\frac{\partial^m v_r(g_N)}{\partial x_1^{m-i}\partial x_2^i}(r,\alpha)| \\
&\leq |v_r(\frac{\partial^m g_N}{\partial x_1^{m-i}\partial x_2^i}(r,\alpha))(r,\alpha)| \\
&+ C\sum_{i=0}^{m-1}\sum_{j=0}^{i}|(\frac{\partial^i v_1(g_N)}{\partial^j x_1 \partial^{i-j} x_2})(r,\alpha)| \\
&+ C\sum_{i=0}^{m-1}\sum_{j=0}^{i}|(\frac{\partial^i v_2(g_N)}{\partial^j x_1 \partial^{i-j} x_2})(r,\alpha)|
\end{aligned}$$

with C depending on m, a and b, and using the decay for v_r, and

$$|v_1(w)(x)| \leq C\frac{||w||_{L^1}}{|d(x, supp(w))|^2}$$

$$|v_2(w)(x)| \leq C\frac{||w||_{L^1}}{|d(x, supp(w))|^2}$$

we obtain (2.33).

□

With this, we are now ready to obtain the bounds for our source term.

Lemma 2.3.3. *For $t \in [0, T]$ and a pseudo-solution $\bar{\theta}_{N,c,K}$ as in (2.30) then the source term $F_{N,c,K}(x,t)$ as in (2.32) satisfies*

$$||F_{N,c,K}(x,t)||_{L^2} \leq CN^{-(2\beta-1)}$$

with C depending on c, K and T.

Proof. In order to obtain the desired estimate we divide the source term into several parts. First we have

$$\begin{aligned}
&||\frac{v_\alpha(\bar{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot))}{r}\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}||_{L^2} \\
&\leq C||v_\alpha(\bar{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot))||_{L^2}||\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}||_{L^\infty} \leq \frac{C}{N^{2\beta-1}}
\end{aligned}$$

and analogously

$$\begin{aligned}
&||v_r(\bar{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot))\frac{\partial(\bar{\theta}_{N,c,K} - f_{1,c,K}(r))}{\partial r}||_{L^2} \\
&\leq ||v_r(\bar{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot))||_{L^2}||\frac{\partial(\bar{\theta}_{N,c,K} - f_{1,c,K}(r))}{\partial r}||_{L^\infty} \leq \frac{C}{N^{2\beta-1}}.
\end{aligned}$$

Finally, by using that $supp(f_{1,c,K}) \in [2r_{c,K}, a_2)$ (see Lemma 2.3.1 and the definition of the pseudo-solution), $\text{supp}(f_{2,c,K}) \in [\frac{r_{c,k}}{2}, \frac{3r_{c,k}}{2}]$ and together with Lemma 2.3.2 we have

$$
\begin{aligned}
&||v_r(\bar{\theta}_{N,c,K}(\cdot)-f_{1,c,K}(\cdot))\frac{\partial f_{1,c,K}(r)}{\partial r}||_{L^2}\\
&\leq \Big(\int_{2r_{c,k}}^{a_2}\frac{C}{N^{2+2\beta}(r-\frac{3r_{c,K}}{2})^4}rdr\Big)^{1/2}\leq \frac{C}{N^{1+\beta}}.
\end{aligned}
\tag{2.34}
$$

Combining all three bounds we obtain the desired result. □

Lemma 2.3.4. *For $t\in[0,T]$ and a pseudo-solution $\bar{\theta}_{N,c,K}$ as in (2.30) then the source term $F_{N,c,K}(x,t)$ as in (2.32) satisfies, for $k\in\mathbb{N}$*

$$
||F_{N,c,K}(x,t)||_{H^k}\leq \frac{C}{N^{2\beta-1-k}}
$$

with C depending on k, c, K and T.

Proof. We separate the source term in three different parts:

1) Using the properties of the support of $\bar{\theta}_{N,c,K}$

$$
\begin{aligned}
&||\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}\frac{v_\alpha(f_{1,c,K}-\bar{\theta}_{N,c,K})}{r}||_{H^k}\\
&\leq C\sum_{i=0}^{k}||\frac{\partial\bar{\theta}_{N,c,K}}{\partial\alpha}||_{C^i}||v_\alpha(f_{1,c,K}-\bar{\theta}_{N,c,K})||_{H^{k-i}}\\
&\leq \frac{C}{N^{2\beta-1-k}},
\end{aligned}
$$

2)

$$
\begin{aligned}
&||\frac{\partial(\bar{\theta}_{N,c,K}-f_{1,c,K})}{\partial r}v_r(\bar{\theta}_{N,c,K})||_{H^k}\\
&\leq C\sum_{i=0}^{k}||\frac{\partial(\bar{\theta}_{N,c,K}-f_{1,c,K})}{\partial r}||_{C^i}||v_r(\bar{\theta}_{N,c,K})||_{H^{k-i}}\\
&\leq \frac{C}{N^{2\beta-1-k}},
\end{aligned}
$$

3) To bound $||\frac{\partial f_{1,c,K}}{\partial r}v_r(\bar{\theta}_{N,c,K})||_{H^k}$, we just apply Lemma 2.3.2 as in (2.34) to obtain

$$
\begin{aligned}
&||\frac{\partial f_{1,c,K}}{\partial r}v_r(\bar{\theta}_{N,c,K})||_{H^k}\\
&\leq \sum_{i=0}^{k}\sum_{j=0}^{i}||\frac{\partial^i(\partial\frac{f_{1,c,K}}{\partial r}v_r(\bar{\theta}_{N,c,K}))}{\partial^j x_1\partial^{i-j}x_2}||_{L^2}\leq\frac{C}{N^{\beta+1-k}}\leq\frac{C}{N^{2\beta-1-k}}.
\end{aligned}
$$

□

And applying the interpolation inequality for Sobolev spaces (with L^2 and for example H^3) we obtain the following corollary:

Corollary 2.3.5. *For $t\in[0,T]$ and a pseudo-solution $\bar{\theta}_{N,c,K}$ as in (2.30) then the source term $F_{N,c,K}(x,t)$ as in (2.32) satisfies*

$$
||F_{N,c,K}(x,t)||_{H^{\beta+\frac{1}{2}}}\leq CN^{-(\beta-\frac{3}{2})}
$$

with C depending on c, K, T.

Now, as in the previous section, we define $\theta_{N,c,K}(x,t)$ to be the unique $H^{\beta+\frac{1}{2}}$ solution to (2.1) with initial conditions $\theta_{N,c,K}(x,0) = \bar{\theta}_{N,c,K}(x,0)$, and we denote

$$\Theta_{N,c,K} := \theta_{N,c,K} - \bar{\theta}_{N,c,k}. \tag{2.35}$$

The next step now is to find bounds for $\Theta_{N,c,K}$.

Lemma 2.3.6. *Let $\Theta_{\lambda,K,N}$ defined as in (2.35), then if $\theta_{\lambda,K,N}$ exists for $t \in [0,T]$, we have that*

$$||\Theta_{N,c,K}(x,t)||_{L^2} \leq \frac{Ct}{N^{(2\beta-1)}}$$

with C depending on λ, K and T.

Proof. As in the proof of Lemma 2.2.8, we obtain the equation

$$\begin{aligned}
&\frac{d}{dt}\frac{||\Theta_{N,c,K}||_{L^2}^2}{2} \leq |\int_{\mathbb{R}^2} \Theta_{N,c,K} \\
&\Big(v_1(\Theta_{N,c,K})\frac{\partial\bar{\theta}_{N,c,K}}{\partial x_1} + v_2(\Theta_{N,c,K})\frac{\partial\bar{\theta}_{N,c,K}}{\partial x_2} + F_{N,c,K}(x,t)\Big)dx| \\
&\leq ||\Theta_{N,c,K}||_{L^2}\Big(||\Theta_{N,c,K}||_{L^2}||\bar{\theta}_{N,c,K}||_{C^1} + ||F_{N,c,K}(x,t)||_{L^2}\Big).
\end{aligned}$$

By using that $||F_{N,c,K}||_{L^2} \leq \frac{C}{N^{(2\beta-1)}}$, $||\bar{\theta}_{\lambda,K,N}||_{C^1} \leq C$ and integrating it follows

$$||\Theta_{N,c,K}||_{L^2} \leq \frac{C(e^{Ct}-1)}{N^{(2\beta-1)}}.$$

□

Before obtaining the bounds for the higher order norms of $\Theta_{N,c,K}$ we need a couple of technical lemmas:

Lemma 2.3.7. *Given a C^1 function $h(x) : \mathbb{R}^2 \to \mathbb{R}$ with $||h||_{L^\infty} \leq M$, $||h||_{C^1} \leq MN$ and $a \in (0,1)$, then there exists a constant C depending on a such that*

$$||(-\Delta)^{a/2}(h(x))||_{L^\infty} \leq CMN^a.$$

Proof. Using the integral expression from the fractional Laplacian

$$(-\Delta)^{a/2}(h(.))(x) = C\int_{\mathbb{R}^2}\frac{(h(x)-h(z))}{|x-z|^{2+a}}dz$$

and dividing the integral into two parts depending on the value of $|x-z|$ we get

$$\int_{|x-z|\geq\frac{1}{N}}\frac{(h(x)-h(z))}{|x-z|^{2+a}}dz \leq CN^a||h||_{L^\infty} = CMN^a$$
$$\int_{|x-z|\leq\frac{1}{N}}\frac{(h(x)-h(z))}{|x-z|^{2+a}}dz \leq CN^{a-1}||h||_{C^1} = CMN^a$$

and we are done.

□

Lemma 2.3.8. *Given a C^1 function $h(x) : \mathbb{R}^2 \to \mathbb{R}$ with $||h||_{L^\infty} \leq M$, $||h||_{C^1} \leq MN$ and with support in the set $[-R,R]^2$ for some R, we have that there exists a constant C depending on R such that for $i = 1,2$*

$$||v_i(h(x))||_{L^\infty} \leq CM\log(N).$$

Furthermore, if $||h||_{C^n} \leq M$,$||h||_{C^{n+1}} \leq MN$ *for some natural number* n *we also have that, for* $i = 1, 2,\ k = 0, 2, ..., n$

$$||\frac{\partial^n v_i(h(x))}{\partial^{n-k} x_1 \partial^k x_2}||_{L^\infty} \leq CMlog(N).$$

Proof. The proof of the first part is the same as in Lemma 2.3.7 but using the kernel for v_i instead of the one for $(-\Delta)^{a/2}$. For the second part we just need to use that, for sufficiently regular functions we have that

$$\frac{\partial v_i(h(x))}{\partial x_j} = v_i\Big(\frac{\partial h(x)}{\partial x_j}\Big).$$

□

Lemma 2.3.9. *Let* $\Theta_{N,c,K}$ *defined as in (2.35), then we have that, for* N *large,* $\theta_{N,c,K}$ *exists for* $t \in [0, T]$ *and*

$$||\Theta_{N,c,K}(x,t)||_{H^{\beta+\frac{1}{2}}} \leq \frac{Ct}{N^{\beta-\frac{3}{2}}}$$

with C *depending on* λ, K *and* T.

Proof. The proof is very similar to that of Lemma 2.2.9. We will prove the inequality for the time interval $[0, T^*]$ with T^* the smallest time fulfilling $||\Theta_{N,c,K}(x,t)||_{H^{\beta+1/2}} = log(N)N^{-(\beta-\frac{3}{2})}$, (we can just consider $t \in [0, T]$ directly if $T_* > T$ or if it does not exists). Note also that, since we have local existence, obtaining this bound also ensures that we have existence for the times considered.

First we have that, for $s = \beta + \frac{1}{2}$

$$\begin{aligned}
&\frac{d}{dt}\frac{||D^s\Theta_{N,c,K}||^2_{L^2}}{2} = -\int_{\mathbb{R}^2} D^s\Theta_{N,c,K} \\
&D^s\Big((v_1(\Theta_{N,c,K}) + v_1(\bar{\theta}_{N,c,K}))\frac{\partial\Theta_{N,c,K}}{\partial x_1} + (v_2(\Theta_{N,c,K}) + v_2(\bar{\theta}_{N,c,K}))\frac{\partial\Theta_{N,c,K}}{\partial x_2} \\
&+ v_1(\Theta_{N,c,K})\frac{\partial\bar{\theta}_{N,c,K}}{\partial x_1} + v_2(\Theta_{N,c,K})\frac{\partial\bar{\theta}_{N,c,K}}{\partial x_2} + F_{N,c,K}(x,t)\Big)dx.
\end{aligned}$$

We start bounding

$$\int_{\mathbb{R}^2} D^s\Theta_{N,c,K} D^s\Big(v_1(\bar{\theta}_{N,c,K}))\frac{\partial\Theta_{N,c,K}}{\partial x_1} + v_2(\bar{\theta}_{N,c,K})\frac{\partial\Theta_{N,c,K}}{\partial x_2}\Big)dx.$$

Applying Lemma 2.2.10 with $s_1 = s - 1$, $s_2 = 1$, $f = v_i(\bar{\theta}_{N,c,K}))$, $g = \frac{\partial\Theta_{N,c,K}}{\partial x_i}$, $i = 1, 2$ we would get that

$$\begin{aligned}
&(D^s\Theta_{N,c,K}, D^s(fg) - \sum_{|\mathbf{k}|\leq s_1}\frac{1}{\mathbf{k}!}\partial^{\mathbf{k}} f D^{s,\mathbf{k}} g - \sum_{|\mathbf{j}|\leq s_2}\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}} g D^{s,\mathbf{j}} f)_{L^2} \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}||D^{s_1} f||_{BMO}||D^{s_2} g||_{L^2} \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}||\Theta_{N,c,K}||_{H^s},
\end{aligned}$$

where we used $||D^{s_1} v_i(\bar{\theta}_{N,c,K})||_{L^\infty} \leq C$. Furthermore we have that

$$\begin{aligned}
&(D^s\Theta_{N,c,K}, D^s(\frac{\partial\Theta_{N,c,K}}{\partial x_1})v_1(\bar{\theta}_{N,c,K}) + D^s(\frac{\partial\Theta_{N,c,K}}{\partial x_2})v_2(\bar{\theta}_{N,c,K}))_{L^2} \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial}{\partial x_1}(D^s\Theta_{N,c,K})^2 v_1(\bar{\theta}_{N,c,K}) + \frac{\partial}{\partial x_2}(D^s\Theta_{N,c,K})^2 v_2(\bar{\theta}_{N,c,K})dx = 0
\end{aligned}$$

and, for $i = 1, 2$, using that the operators $D^{s,\mathbf{k}}$ are continuous from H^a to $H^{a-s+\mathbf{k}}$,

$$\begin{aligned}
&|(D^s\Theta_{N,c,K}, \sum_{|\mathbf{k}|=1} \frac{1}{\mathbf{k}!}\partial^{\mathbf{k}} v_i(\bar{\theta}_{N,c,K}) D^{s,\mathbf{k}} \frac{\partial \Theta_{\lambda,K,N}}{\partial x_i})_{L^2}| \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}||v_i(\bar{\theta}_{N,c,K})||_{C^1}||\Theta_{N,c,K}||_{H^s} \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}||\Theta_{N,c,K}||_{H^s}
\end{aligned}$$

where we used $||v_i(\bar{\theta}_{N,c,K} - f_{1,c,K})||_{C^1} \leq Clog(N)N^{-\beta+1}$ (consequence of Lemma 2.3.8) and $||v_i(f_{1,c,K})||_{C^1} \leq C$.

We also have

$$\begin{aligned}
&|(D^s\Theta_{\lambda,K,N}, \sum_{|\mathbf{j}|=1} \frac{1}{\mathbf{j}!}\partial^{\mathbf{j}} \frac{\partial \Theta_{\lambda,K,N}}{\partial x_i} D^{s,\mathbf{j}} v_i(\bar{\theta}_{\lambda,K,N}))_{L^2}| \\
&\leq C \sum_{|\mathbf{j}|=1} ||D^s\Theta_{\lambda,K,N}||_{L^2}||\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}} \frac{\partial \Theta_{\lambda,K,N}}{\partial x_i}||_{L^2}||D^{s,\mathbf{j}} v_i(\bar{\theta}_{\lambda,K,N})||_{L^\infty} \\
&\leq C \sum_{|\mathbf{j}|=1} ||D^s\Theta_{\lambda,K,N}||_{L^2}||\Theta_{\lambda,K,N}||_{H^s}||D^{s-2}\partial^{\mathbf{j}} v_i(\bar{\theta}_{\lambda,K,N})||_{L^\infty}, \\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\Theta_{\lambda,K,N}||_{H^s}(\frac{NN^{s-2}log(N)}{N^\beta} + C) \\
&\leq C||D^s\Theta_{\lambda,K,N}||_{L^2}||\Theta_{\lambda,K,N}||_{H^s},
\end{aligned}$$

where we used Lemmas 2.3.7 and 2.3.8, the expression for $D^{s,\mathbf{j}}$ and the bounds for the derivatives of $\bar{\theta}_{N,c,K}$.

The last part to bound from the term with $v_i(\bar{\theta}_{N,c,K})$ is, for $i = 1, 2$

$$\begin{aligned}
&|(D^s\Theta_{N,c,K}, \frac{\partial \Theta_{N,c,K}}{\partial x_i} D^s v_i(\bar{\theta}_{N,c,K}))_{L^2}| \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}||\Theta_{N,c,K}||_{H^1}||D^{s-2}v_i(\Delta\bar{\theta}_{N,c,K})||_{L^\infty}. \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}N^{-(\beta-\frac{1}{2})}\frac{CN^{s-2}log(N)N^2}{N^\beta} \\
&\leq C||D^s\Theta_{N,c,K}||_{L^2}N^{-\beta+1}log(N) \leq C||D^s\Theta_{N,c,K}||_{L^2}N^{-\frac{1}{2}},
\end{aligned}$$

where we used that, for the times considered, using Lemma 2.3.6 and the interpolation inequality we have $||\Theta_{N,c,K}||_{H^1} \leq CN^{-(\beta-\frac{1}{2})}$ (the bound is actually better, but this is enough).

The rest of the terms not depending on $F_{N,c,K}$ are bounded in a similar fashion, and using $||\bar{\theta}_{\lambda,K,N}||_{H^s} \leq C$, $||F_{N,c,K}||_{H^s} \leq CN^{-(\beta-\frac{3}{2})}$ with C depending on c, K and T, we get

$$\frac{d}{dt}||D^s\Theta_{N,c,K}||^2_{L^2} \leq ||D^s\Theta_{N,c,K}||_{L^2}(CN^{-(\beta-\frac{3}{2})} + C||\Theta_{N,c,K}||_{H^s} + C||\Theta_{N,c,K}||^2_{H^s})$$

which gives us, using

$$||\Theta_{N,c,K}||_{H^s} \leq C(||\Theta_{N,c,K}||_{L^2} + ||D^s\Theta_{N,c,K}||_{L^2}) \leq C(||D^s\Theta_{N,c,K}||_{L^2} + N^{-(2\beta-1)})$$

that

$$\frac{d}{dt}||D^s\Theta_{N,c,K}||_{L^2} \leq (CN^{-(\beta-\frac{3}{2})} + C||D^s\Theta_{N,c,K}||_{L^2} + C||D^s\Theta_{\lambda,K,N}||^2_{L^2}),$$

and using $||D^s\Theta_{\lambda,K,N}||_{L^2} \leq \log(N)N^{-(\beta-\frac{3}{2})}$ and integrating we get

$$||D^s\Theta_{N,c,K}||_{L^2} \leq \frac{C(e^{Ct}-1)}{N^{\beta-\frac{3}{2}}}.$$

Now, taking N big enough we obtain that $T* \geq T$ and we are done.

□

2.3.2 Strong ill-posedness in supercritical Sobolev spaces

Now we are ready to prove strong ill-posedness in supercritical Sobolev spaces:

Theorem 2.3.1. (Strong ill-posedness in H^{β}) For any $c_0 > 0$, $M > 1$, $\beta \in (\frac{3}{2}, 2)$ and $t_* > 0$, we can find a $H^{\beta+\frac{1}{2}}$ function $\theta_0(x)$ with $||\theta_0(x)||_{H^{\beta}} \leq c_0$ such that the unique solution $\theta(x,t)$ in $H^{\beta+\frac{1}{2}}$ to the SQG equation (2.1) with initial conditions $\theta_0(x)$ is such that $||\theta(x,t_*)||_{H^{\beta}} \geq Mc_0$.

Proof. First we prove a bound for the pseudo-solution $\bar{\theta}_{N,c,K}$ defined in (2.30). More precisely

$$||f_{2,c,K}(r)\frac{r_{c,K}^{\beta}\sin(N\alpha)}{N^{\beta}}||_{L^2} \leq \frac{cr_{c,k}^{\beta}}{N^{\beta}},$$

and

$$||f_{2,c,K}(r)\frac{r_{c,K}^{\beta}\sin(N\alpha)}{N^{\beta}}||_{H^2} \leq \frac{Ccr_{c,k}^{\beta-2}}{N^{\beta-2}},$$

which in combination with the interpolation inequality for Sobolev spaces and the bounds for $f_{1,c,K}$ gives us

$$||\bar{\theta}_{N,c,K}(x,0)||_{H^{\beta}} \leq C_1 c$$

with C_1 depending only on β.

Furthermore, at time t we have that our pseudo-solution fulfils

$$||\bar{\theta}_{N,c,K}(x,t) - f_{1,c,K}||_{L^2} \leq \frac{cr_{c,k}^{\beta}}{N^{\beta}}$$

and we can find the lower bound for the H^1 norm of $\bar{\theta}_{N,c,K} - f_{1,c,K}$ by using

$$\begin{aligned}&\frac{\partial(\bar{\theta}_{N,c,K} - f_{1,c,K})}{\partial x_1}\\ &= \cos(\alpha)\frac{\partial(\bar{\theta}_{N,c,K} - f_{1,c,K})}{\partial r} - \frac{\sin(\alpha)}{r}\frac{\partial(\bar{\theta}_{N,c,K} - f_{1,c,K})}{\partial \alpha}\end{aligned}$$

which gives us, after some trigonometric manipulations and using (2.31) that, for N large

$$||\bar{\theta}_{N,c,K}(x,t) - f_{1,c,K}||_{H^1} \geq C\frac{ctKr_{c,K}^{\beta-1}}{N^{\beta-1}}$$

with C a constant.

Furthermore, since $supp(\bar{\theta}_{N,c,K} - f_{1,c,K}) \cap supp(f_{1,c,K}) = \emptyset$ we have that

$$||\bar{\theta}_{N,c,K}(x,t)||_{H^1} \geq ||\bar{\theta}_{N,c,K}(x,t) - f_{1,c,K}||_{H^1} \geq C\frac{ctKr_{c,K}^{\beta-1}}{N^{\beta-1}},$$

for sufficiently large N. On the other hand the interpolation inequality gives us

$$||\bar{\theta}_{N,c,K}(x,t)||_{H^1} \leq ||\bar{\theta}_{N,c,K}(x,t)||_{H^{\beta}}^{\frac{1}{\beta}}||\bar{\theta}_{N,c,K}(x,t)||_{L^2}^{\frac{\beta-1}{\beta}}$$

and using our bounds for $||\bar{\theta}_{N,c,K}(x,t)||_{L^2}$ and $||\bar{\theta}_{N,c,K}(x,t)||_{H^1}$ we get

$$||\bar{\theta}_{N,c,K}(x,t)||_{H^{\beta}} \geq C_2 cK^{\beta}t^{\beta}$$

with C_2 depending only on β. Therefore, by choosing c, K appropriately we have that, for all N big enough,

$$||\bar{\theta}_{N,c,K}(x,0)||_{H^{\beta}} \leq c_0,$$

$$||\bar{\theta}_{N,c,K}(x,t^*)||_{H^{\beta}} \geq 2Mc_0.$$

Now, considering the solution $\theta_{N,c,K}$ of (2.1) with initial conditions $\bar{\theta}_{N,c,K}(x,0)$, we know that

$$||\theta_{N,c,K}(x,0)||_{H^\beta} \leq c_0,$$

and, using Lemma 2.3.9,

$$\begin{aligned} &||\bar{\theta}_{N,c,K}(x,t^*) - \theta_{N,c,K}(x,t^*)||_{H^\beta} \\ &\leq ||\bar{\theta}_{N,c,K}(x,t^*) - \theta_{N,c,K}(x,t^*)||_{H^{\beta+\frac{1}{2}}} \leq \frac{Ct^*}{N^{\beta-\frac{3}{2}}} \end{aligned}$$

for large N, and so, by taking N big enough we can conclude

$$||\theta_{N,c,K}(x,t^*)||_{H^\beta} \geq ||\bar{\theta}_{N,c,K}(x,t^*)||_{H^\beta} - ||\bar{\theta}_{N,c,K}(x,t^*) - \theta_{N,c,K}(x,t^*)||_{H^\beta} \geq Mc_0.$$

□

2.3.3 Non existence in supercritical Sobolev spaces

In this section we prove the following theorem:

Theorem 2.3.2. (Non existence in H^β in the supercritical case) For any t_0, $c_0 > 0$ and $\beta \in (\frac{3}{2}, 2)$ we can find initial conditions $\theta_0(x)$, with $||\theta_0(x)||_{H^\beta} \leq c_0$ such that there exists a solution $\theta(x,t)$ to (2.1) with $\theta(x,0) = \theta_0(x)$ satisfying $||\theta(x,t)||_{H^\beta} = \infty$ for all $t \in (0, t_0]$. Furthermore, it is the only solution with initial conditions $\theta_0(x)$ that satisfies $\theta(x,t) \in L^\infty_t C^{\gamma_1}_x \cap L^\infty_t L^2_x$ ($0 < \gamma_1 < \frac{1}{2}$) with the property that $\theta(x,t) \in H^{\gamma_2}$ ($1 < \gamma_2 \leq \frac{3}{2}$) for $t \in [0, t_0]$.

Remark 4. In particular, if there is another solution in $\theta(x,t) \in L^\infty_t C^{\gamma_1}_x \cap L^\infty_t L^2_x$ then it cannot fulfil $\theta(x,t) \in H^\beta$ for $t \in (0, t^*]$ with any $0 < t^* \leq t_0$, even if we allow

$$\text{ess-sup}_{t\in[0,t^*]}||\theta(x,t)||_{H^\beta} = \infty.$$

Proof. Let's first note some of the properties that the pseudo-solutions $\bar{\theta}_{N,c,K}$ (for some fixed β) have:

- $\bar{\theta}_{N,c,K}(x,t)$ is in C^∞ for all $t \in [0, t_0]$, with $||\bar{\theta}_{N,c,K}(x,t)||_{C^k} \leq CcN^{k-\beta}$, $||\bar{\theta}_{N,c,K}(x,t)||_{H^k} \leq CcN^{k-\beta}$ for any natural $k \geq 2$, with the constant C depending on k, K and t_0. Also, for $\beta > s \geq 0$ we have $||\bar{\theta}_{N,c,K}(x,t)||_{H^s} \leq C_1cN^{s-\beta} + C_2c$ with C_1 depending on K, s and t_0 and C_2 a constant.
- For N large we have the lower bound $||\bar{\theta}_{N,c,K}(x,t)||_{H^\beta} \geq Cct^\beta K^\beta$ with C a constant.
- $\bar{\theta}_{N,c,K}(x,t)$ is supported in the disk of radius M centered at zero $B_M(0)$ for some M independent of the values of the parameters.

Furthermore, we have the following result.

Lemma 2.3.10. *Consider the equation*

$$\frac{\partial\tilde{\theta}_{N,c,K}}{\partial t} + (v_1(\tilde{\theta}_{N,c,K}) + v^{N,c,K}_{1,ext})\frac{\partial\tilde{\theta}_{N,c,K}}{\partial x_1} + (v_2(\tilde{\theta}_{N,c,K}) + v^{N,c,K}_{2,ext})\frac{\partial\tilde{\theta}_{N,c,K}}{\partial x_2} = 0$$

with initial conditions $\tilde{\theta}_{N,c,K}(x,0) = \bar{\theta}_{N,c,K}(x,0)$ *and such that*

$$\frac{\partial v^{N,c,K}_{2,ext}}{\partial x_2} = -\frac{\partial v^{N,c,K}_{1,ext}}{\partial x_1}$$

and

$$||v^{N,c,K}_{i,ext}||_{C^3} \leq CN^{-3}$$

with C depending on c and K.

Then for any $T > 0$ we have that if N is big enough, then for $t \in [0, T]$ there exists a unique $\tilde{\theta}_{N,c,K}(x,t) \in H^{\beta+\frac{1}{2}}$ and

$$||\bar{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)||_{L^2} \leq CtN^{-(2\beta-1)}$$

$$||\bar{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)||_{H^{\beta+\frac{1}{2}}} \leq CtN^{-(\beta-\frac{3}{2})}$$

with C depending on c, K and T.

We first note that local well-posedness of this equation in $H^{\beta+1}$ is straightforward since $v^{N,c,K}_{i,ext} \in C^3$ for $i = 1, 2$. As for the error bounds, they are obtained in the same way as in Lemmas 2.3.6 and 2.3.9, i.e., studying the evolution equation for $\bar{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)$ now with new terms depending on $v^{N,c,K}_{i,ext}\frac{\partial\tilde{\theta}_{N,c,K}(x,t)}{\partial x_i}$. These terms, however, are easily bounded by writing

$$\tilde{\theta}_{N,c,K}(x,t) = (\tilde{\theta}_{N,c,K}(x,t) - \bar{\theta}_{N,c,K}(x,t)) + \bar{\theta}_{N,c,K}(x,t)$$

and using the properties for $v^{N,c,K}_{i,ext}$ and $\bar{\theta}_{N,c,K}(x,t)$.

This lemma tells us that our pseudo-solutions defined in (2.30) stay close to other pseudo-solutions that have the same initial conditions and an error term in the velocity field (if the error term is small enough). Now, to obtain the initial conditions that will produce instantaneous loss of regularity, we consider

$$\theta(x,0) := \sum_{j=1}^{\infty} T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,0)),$$

with $T_R(f(x_1,x_2)) = f(x_1 + R, x_2)$, and R_j yet to be fixed.

We will refer to the solution of (2.1) with this initial conditions and $H^{\frac{3}{2}}$ regularity (if it exists) as $\theta(x,t)$, keeping in mind that it depends on the values for R_j, N_j, c_j, K_j, with $j \in \mathbb{N}$.

We start by fixing c_j and K_j with the following properties:

1)

$$||\bar{\theta}_{N_j,c_j,K_j}(x,0)||_{H^\beta} \leq c_0 2^{-j}, ||\bar{\theta}_{N_j,c_j,K_j}(x,0)||_{L^1} \leq c_0 2^{-j}. \tag{2.36}$$

2) If N_j is large enough then

$$||\bar{\theta}_{N_j,c_j,K_j}(x,t)||_{H^\beta} \geq tc_0 2^j \tag{2.37}$$

and

$$||\bar{\theta}_{N_j,c_j,K_j}(x,t)||_{H^{\frac{3}{2}}} \leq c_0 2^{-j}$$

for $t \in [0, t_0]$.

This gives us a bound for the velocity generated by $\sum_{j=1}^{\infty} T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,t))$, which we will call v_{max}.

As for R_j, we will consider $R_j = R_{j-1} + D_j + D_{j-1}$, $R_0 = 0$, and we will take $D_j = j^4 N_j^4 + 2M + 8v_{max}t_0$.

Now, we say that a sequence $\vec{v} = (v^j_i(x,t))_{i=1,2,j\in\mathbb{N}}$ is in the set $V_{(N_j)_{j\in\mathbb{N}},C_0}$ if

- $v^j_i(x,t) \in C^3$ for $t \in [0, t_0]$, with $||v^j_i(x,t)||_{C^3} \leq \frac{C_0}{j^4 N_j^3}$.
- $\frac{\partial v^j_1}{\partial x_1} = -\frac{\partial v^j_2}{\partial x_2}$.

Given two elements $\vec{v}_1, \vec{v}_2$ of $V_{(N_j)_{j\in\mathbb{N}},C_0}$, we will consider the distance

$$d(\vec{v}_1, \vec{v}_2) := \sup_{j\in\mathbb{N},i=1,2}\text{ess-sup}_{t\in[0,t_0]} j^4 N_j^3 ||v^j_{1,i}(x,t) - v^j_{2,i}(x,t)||_{C^3}.$$

Note that with this distance $V_{(N_j)_{j\in\mathbb{N}},C_0}$ is a complete metric space.

Furthermore, given an element $\vec{v} \in V_{(N_j)_{j\in\mathbb{N}},C_0}$ we define the sequence of functions $W(\vec{v}) = (W_j(\vec{v})(x,t))_{j\in\mathbb{N}} = (w_j(x,t))_{j\in\mathbb{N}}$ as the only sequence of $H^{\beta+\frac{1}{2}}$ functions for $t \in [0,t_0]$, satisfying

$$\frac{\partial w_j(x,t)}{\partial t} = -(v_1(w_j) + v_1^j(x,t))\frac{\partial w_j}{\partial x_1} - (v_2(w_j) + v_2^j(x,t))\frac{\partial w_j(x,t)}{\partial x_2} \tag{2.38}$$

$$w_j(x,0) = T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,0))$$

Note that Lemma 2.3.10 tells us that if $(N_j)_{j\in N}$ are big enough, the condition

$$||v^j_{i,ext}||_{C^3} \leq \frac{C_0}{j^4 N_j^3}$$

implies that there is a (unique in $H^{\beta+\frac{1}{2}}$) solution to (2.38) for $t \in [0,t_0]$ with

$$||T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,t)) - w_j(x,t)||_{L^2} \leq CtN_j^{-(2\beta-1)}$$

$$||T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,t)) - w_j(x,t)||_{H^{\beta+\frac{1}{2}}} \leq CtN_j^{-(\beta-\frac{3}{2})}. \tag{2.39}$$

We will call the set including these sequences $W_{(N_j)_{j\in\mathbb{N}},C_0}$.

Now we define the map v_{ext} that takes an element of $w \in W_{(N_j)_{j\in\mathbb{N}},C_0}$ to an element of $v_{ext}(w) \in V_{(N_j)_{j\in\mathbb{N}},C_0}$ as

$$v^{j_0}_{1,ext}((w_j)_{j\in\mathbb{N}}) = \frac{\partial}{\partial x_2}\left(T_{R_{j_0}}\phi(x)\Lambda^{-1}[(\sum_{j=1}^{\infty} w_j) - w_{j_0}]\right),$$

$$v^{j_0}_{2,ext}((w_j)_{j\in\mathbb{N}}) = -\frac{\partial}{\partial x_1}\left(T_{R_{j_0}}\phi(x)\Lambda^{-1}[(\sum_{j=1}^{\infty} w_j) - w_{j_0}]\right),$$

where $\phi(x)$ is a smooth C^∞ function with $\phi(x) = 1$ if $x \in B_{4v_{max}+M}(0)$ and $\phi(x) = 0$ if $|x| \geq 8v_{max} + M$.

Note that $||v^{j_0}_{i,ext}((w_j)_{j\in\mathbb{N}})||_{C^3} \leq \frac{Cc_0}{j_0^4 N_{j_0}^4}$, and thus $||v^{j_0}_{i,ext}((w_j)_{j\in\mathbb{N}})||_{C^3} \leq \frac{C_0}{j_0^4 N_{j_0}^3}$ if N_{j_0} is large. Furthermore, if $x \in B_{4v_{max}+M}(-R_{j_0},0)$, then

$$v^{j_0}_{i,ext}((w_j)_{j\in\mathbb{N}}) = v_i((\sum_{j=1}^{\infty} w_j) - w_{j_0}) \tag{2.40}$$

and, since for $(N_j)_{j\in\mathbb{N}}$ big enough $supp(W_{j_0}(v_{ext}(w)) \subset B_{4v_{max}+M}(-R_{j_0},0)$, we have that $W_{j_0}(v_{ext}(w))$ actually fulfils (2.38) with $v^i_{j_0}$ given by (2.40).

This allows us to define the operator G over a sequence v in the space $V_{(N_j)_{j\in\mathbb{N}},C_0}$ as

$$G(\vec{v}) = (v^j_{i,ext}(W(\vec{v})))_{i=1,2,j\in\mathbb{N}},$$

The operator G maps (for $(N_j)_{j\in\mathbb{N}}$ large) $V_{(N_j)_{j\in\mathbb{N}},C_0}$ to $V_{(N_j)_{j\in\mathbb{N}},C_0}$ and actually, if we can find a point $\vec{v} \in V_{(N_j)_{j\in\mathbb{N}},C_0}$ such that $G(\vec{v}) = \vec{v}$, then, for $(w_j)_{j\in\mathbb{N}} = W(\vec{v})$,

$$\theta(x,t) = \sum_{j=1}^{\infty} w_j(x,t)$$

is a solution to (2.1) with initial conditions

$$\theta(x,0) = \sum_{j=1}^{\infty} T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,0)).$$

If we now consider two sequences $\vec{v}^1 = (v_i^{1,j})_{i=1,2,j\in\mathbb{N}}, \vec{v}^2 = (v_i^{2,j})_{i=1,2,j\in\mathbb{N}} \in V_{(N_j)_{n\in\mathbb{N}},C_0}$ and we define for two elements of $W_{(N_j)_{n\in\mathbb{N}},C_0}$ $d(w^1,w^2) = sup_{t\in[0,t_0]}\sum_{j=1}^{\infty} ||w^1_j - w^2_j||_{L^2}$ we can compute $d(W_j(\vec{v}^1), W_j(\vec{v}^2))$, by defining $\tilde{w}_j = W(\vec{v}^1) - W(\vec{v}^2)$, since it fulfills the evolution equation

$$\begin{aligned}\frac{\partial \tilde{w}_j}{\partial t} &= -\frac{\partial (W_j(\vec{v}^1))}{\partial x_1} v_1(\tilde{w}_j) - \frac{\partial \tilde{w}_j}{\partial x_1} v_1(W_j(\vec{v}^2)) \\ &- \frac{\partial (W_j(\vec{v}^1))}{\partial x_2} v_2(\tilde{w}_j) - \frac{\partial \tilde{w}_j}{\partial x_2} v_2(W_j(\vec{v}^2)) \\ &- \frac{\partial (W_j(\vec{v}^1))}{\partial x_1}(v_1^{1,j} - v_1^{2,j}) - \frac{\partial \tilde{w}_j}{\partial x_1} v_1^{2,j} \\ &- \frac{\partial (W_j(\vec{v}^1))}{\partial x_2}(v_2^{1,j} - v_2^{1,j}) - \frac{\partial \tilde{w}_j}{\partial x_2} v_2^{2,j}.\end{aligned}$$

This gives us a bound for the evolution of the L^2 norm of $\tilde{w}_j$

$$\frac{\partial ||\tilde{w}_j||_{L^2}}{\partial t} \leq C||W_j(\vec{v}^1)||_{C^1}||\tilde{w}_j||_{L^2} + ||W_j(\vec{v}^1)||_{C^1}(||v_1^{1,j} - v_1^{2,j}||_{L^\infty} + ||v_2^{1,j} - v_2^{1,j}||_{L^\infty})$$

But for N_j large we can bound $||W_j(\vec{v}^1)||_{C^1}$ by some constant $\bar{C}_j$ using (2.39) , and thus we obtain, for $t \in [0, t_0]$

$$\begin{aligned}&||\tilde{w}_j(x,t)||_{L^2} \leq C\tilde{C}_j(e^{Ct_0} - 1)(||v_1^{1,j} - v_1^{2,j}||_{L^\infty} + ||v_2^{1,j} - v_2^{1,j}||_{L^\infty}) \\ &\leq C\tilde{C}_j(e^{Ct_0} - 1)\frac{d(\vec{v}^1, \vec{v}^2)}{j^4 N_j^3},\end{aligned}$$

and for N_j large

$$||\tilde{w}_j(x,t)||_{L^2} \leq \epsilon \frac{d(\vec{v}^1, \vec{v}^2)}{j^4}$$

with ϵ as small as we want. Adding over all j we obtain, for $t \in [0, t_0]$

$$d(W(\vec{v}^1), W(\vec{v}^2)) \leq C\epsilon d(\vec{v}^1, \vec{v}^2)$$

with ϵ arbitrarily small.

But now, if the N_j's are big enough, we have that, by the definition of v_{ext},

$$\begin{aligned}&d(G(\vec{v}^1), G(\vec{v}^2)) = d(v_{ext}(W(\vec{v}^1)), v_{ext}(W(\vec{v}^2))) \leq \frac{C}{inf_{j\in\mathbb{N}}} Cd(W(\vec{v}^1), W(\vec{v}^2)) \\ &\leq \frac{C\epsilon}{\inf_{j\in\mathbb{N}}(N_j)} d(\vec{v}^1, \vec{v}^2) \leq \frac{1}{2} d(\vec{v}^1, \vec{v}^2)\end{aligned}$$

so the map G is a contraction, and since $V_{(N_j)_{n\in\mathbb{N}}, C_0}$ is a (non-empty) complete metric space, there is a fixed point and therefore $w(x,t)$ is a solution to (2.1) with initial conditions

$$\theta(x,0) = \sum_{j=1}^{\infty} T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,0)).$$

Properties (2.36),(2.37) and (2.39) finish the proof that a solution with the desired properties of Theorem 2.3.2 exists.

For uniqueness in the space mentioned we call $\theta_1(x,t)$ the solution we constructed above and assume the existence of another solution $\theta_2(x,t) \in L^\infty_t C^{\gamma_1}_x \cap L^\infty_t L^2_x$ $(0 < \gamma_1 < \frac{1}{2})$ with the property that $\theta_2(x,t) \in H^{\gamma_2}$ $(1 < \gamma_2 \leq \frac{3}{2})$ for $t \in [0,t_0]$. In particular (since it is in $L^\infty_t C^{\gamma_1}_x$), there exists a certain constant $v_{2,max}$ such that $||v_i(\theta_2)||_{L^\infty} \leq v_{2,max}$. We start by studying the uniqueness for $t \in [0, \min(t^*, t_0)]$ with $t^* v_{2,max} = t_0 v_{max}$. In particular, we have that $\text{supp}(\theta_2(x,t)) \subset \cup_{j\in\mathbb{N}} T_{R_j}(B_{t_0 v_{max}+M}(0))$. We define

$$\theta_1^j(x,t) = 1_{B_{4t_0 v_{max}+M}(-R_j,0)}\theta_1(x,t)$$

$$\theta_2^j(x,t) = 1_{B_{4t_0 v_{max}+M}(-R_j,0)}\theta_2(x,t).$$

If we define $\Theta^j := \theta_2^j - \theta_1^j$, $\Theta := \theta_2 - \theta_1$, we get

$$\begin{aligned}
\frac{\partial \Theta^j}{\partial t} &= -\frac{\partial \theta_1^j}{\partial x_1} v_1(\Theta^j) - \frac{\partial \Theta^j}{\partial x_1} v_1(\Theta^j) - \frac{\partial \theta_1^j}{\partial x_2} v_2(\Theta^j) - \frac{\partial \Theta^j}{\partial x_2} v_2(\Theta^j) \\
&- \frac{\partial \Theta^j}{\partial x_1} v_1(\theta_1^j) - \frac{\partial \Theta^j}{\partial x_2} v_2(\theta_1^j) - \frac{\partial \theta_1^j}{\partial x_1} v_1(\Theta - \Theta^j) - \frac{\partial \Theta^j}{\partial x_1} v_1(\Theta - \Theta^j) \\
&- \frac{\partial \theta_1^j}{\partial x_2} v_2(\Theta - \Theta^j) - \frac{\partial \Theta^j}{\partial x_2} v_2(\Theta - \Theta^j) \\
&- \frac{\partial \Theta^j}{\partial x_1} v_1(\theta_1 - \theta_1^j) - \frac{\partial \Theta^j}{\partial x_2} v_2(\theta_1 - \theta_1^j)
\end{aligned}$$

which gives us

$$\frac{\partial ||\Theta_j||_{L^2}}{\partial t} \leq C||\theta_1^j||_{C^1}||\Theta_j||_{L^2} + C||\theta_1^j||_{C^1}\frac{||\Theta||_{L^2}}{j^4 N_j^4}$$

and by taking N_j large and integrating the above inequality big we get

$$||\Theta_j||_{L^2} \leq \frac{\epsilon ||\Theta||_{L^2}}{j^4}$$

and adding over all j and taking ϵ small

$$||\Theta||_{L^2} \leq \frac{||\Theta||_{L^2}}{2}$$

and thus $||\Theta||_{L^2}$ for $t \in [0,t^*]$. Iterating the argument allows us to prove $||\Theta||_{L^2} = 0$ for $t \in [0,t_0]$. □

2.4 Strong ill-posedness in the critical Sobolev space H^2

For this section, we will consider solutions of (2.1) that are in layers around zero, each one closer to the origin, so that within each layer one gets (in the limit) an evolution system of the form

$$\frac{\partial \bar{\theta}}{\partial t} + (v_1(\bar{\theta}) + K(t)x_1)\frac{\partial \bar{\theta}}{\partial x_1} + (v_2(\bar{\theta}) - K(t)x_2)\frac{\partial \bar{\theta}}{\partial x_2} = 0,$$

$$v_1 = -\frac{\partial}{\partial x_2}\Lambda^{-1}\bar{\theta} = -\mathcal{R}_2\bar{\theta},$$

$$v_2 = \frac{\partial}{\partial x_1}\Lambda^{-1}\bar{\theta} = \mathcal{R}_1\bar{\theta},$$

$$\bar{\theta}(x,0) = \theta_0(x).$$

But first we need to obtain an expression for $\frac{\partial v_i(\theta)(0)}{\partial x_j}$ $(i,j=1,2)$ for θ with support far away from 0. We consider first $i=1$. We have

$$v_1(\theta) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} \frac{(-x_2+y_2)\theta(y)}{|x-y|^3} dy_1 dy_2.$$

For θ with support far away from $x=0$ we can just differentiate under the integral sign and when we evaluate at $x=0$ this yields

$$\frac{\partial v_1(\theta)}{\partial x_1}(x=0) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} -3y_1 \frac{y_2\theta(y)}{|y|^5} dy_1 dy_2,$$

$$\frac{\partial v_1(\theta)}{\partial x_2}(x=0) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} (\frac{-3y_2^2\theta(y)}{|y|^5} - \frac{\theta(y)}{|y|^3}) dy_1 dy_2.$$

We will consider $\theta(x_1,x_2)$ satisfying $\theta(-x_1,x_2) = -\theta(x_1,x_2)$, $\theta(x_1,-x_2) = -\theta(x_1,x_2)$, so

$$\frac{\partial v_1(\theta)}{\partial x_1}(x=0) = \frac{4\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2_+} -3y_1 \frac{y_2\theta(y)}{|y|^5} dy_1 dy_2,$$

$$\frac{\partial v_1(\theta)}{\partial x_2}(x=0) = 0.$$

If we take a look at the expression for $\frac{\partial v_1(\theta)}{\partial x_1}$ in polar coordinates and combine all the constant into a certain $C_0 > 0$ we obtain

$$\frac{\partial v_1(\theta)}{\partial x_1}(x=0) = -C_0 P.V. \int_{\mathbb{R}_+\times[0,\pi/2]} \frac{\sin(2\alpha')\theta(r',\alpha')}{(r')^2} dr' d\alpha'.$$

The expressions for v_2 are obtained the same way and in fact we have

$$\frac{\partial v_2(\theta)}{\partial x_1}(x=0) = 0,$$

$$\frac{\partial v_2(\theta)}{\partial x_2}(x=0) = C_0 P.V. \int_{\mathbb{R}_+\times[0,\pi/2]} \frac{\sin(2\alpha')\theta(r',\alpha')}{(r')^2} dr' d\alpha'.$$

Analogously, the second derivatives of v_i all vanish.

We will be interested in studying the evolution of initial conditions of the form

$$\sum_{j=1}^{J} \frac{f(b^{-j}r)b^j \sin(2\alpha)}{j}$$

for $f(r)$ a positive C^∞ function with compact support and $\frac{1}{2} > b > 0$. More precisely, we would like to study the behaviour of the unique H^4 solution with said initial conditions when b tends to zero. One could think that we can just check the evolution of each of the terms $\frac{f(b^{-j}r)b^j\sin(2\alpha)}{j}$ and then add them together, hoping that the interaction between them gets small as $b \to 0$. However this is not true, and we get an interaction depending on $\frac{\partial v_i}{\partial x_i}$. To get specific results, we fix some positive radial function f in C^∞ with $supp(f) \subset \{r \in [1/2, 3/2]\}$ and $||f(r)\cos(2\alpha)||_{H^4} = 1$. We define $\theta_{c,J,b}$ as the unique H^4 solution of

$$\frac{\partial\theta_{c,J,b}}{\partial t} + v_1(\theta_{c,J,b})\frac{\partial\theta_{c,J,b}}{\partial x_1} + v_2(\theta_{c,J,b})\frac{\partial\theta_{c,J,b}}{\partial x_2} = 0,$$

with

$$v_1(\theta_{c,J,b}) = -\frac{\partial}{\partial x_2}\Lambda\theta_{c,J,b} = -\mathcal{R}_2\theta_{c,J,b},$$

$$v_2(\theta_{c,J,b}) = \frac{\partial}{\partial x_1}\Lambda\theta_{c,J,b} = \mathcal{R}_1\theta_{c,J,b},$$

$$\theta_{c,J,b}(x,0) = c\sum_{j=1}^{J}\frac{f(b^{-j}r)b^j\sin(2\alpha)}{j}, \quad \frac{1}{2} > b > 0. \tag{2.41}$$

Note that the odd symmetry is preserved in time.

A few comments need to be made regarding the properties of the transformation $h(r,\alpha) \to \frac{h(\lambda r,\alpha)}{\lambda}$ (or equivalently $h(x) \to \frac{h(\lambda x)}{\lambda}$). We have that

- If $\lambda > 1$, then $||\frac{h(\lambda r,\alpha)}{\lambda}||_{H^2} \leq ||h(r,\alpha)||_{H^2}$.
- If $h(r,\alpha,t)$ is a solution to (2.1) with initial conditions $h(r,\alpha,0)$, then $\frac{h(\lambda r,\alpha,t)}{\lambda}$ is a solution to (2.1) with initial conditions $\frac{h(\lambda r,\alpha,0)}{\lambda}$.
- For $i = 1,2$, $j = 1,2$ we have $v_i(\frac{h(\lambda\cdot,\cdot)}{\lambda})(\frac{r}{\lambda},\alpha) = \frac{1}{\lambda}v_i(h(\cdot,\cdot))(r,\alpha)$, $\frac{\partial v_i(\frac{h(\lambda\cdot,\cdot)}{\lambda})}{\partial x_j}(\frac{r}{\lambda},\alpha) = \frac{\partial v_i(h(\cdot,\cdot))}{\partial x_j}(r,\alpha)$.

The initial conditions in (2.41) fulfil that, taking c small and J big, they have an arbitrarily small H^2 norm and an arbitrarily big value of $|\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,t=0)|$. If $|\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,t)|$ remained big for a long enough time and θ remained sufficiently regular during that time, we could then use a small perturbation around $x=0$ to obtain a big growth in some H^s norm.

The main problem here is that we cannot assure existence for sufficiently long times using just the a priori bounds, so we need some extra machinery to be able to work with these solutions. For that we consider $\tilde{C}$ the constant fulfilling that, for any H^4 solution of SQG (2.1) we have

$$\frac{\partial||\theta(x,t)||_{H^4}}{\partial t} \leq \tilde{C}||\theta(x,t)||_{H^4}^2 \tag{2.42}$$

For fixed constants $t_0, K > 0$, we define $t^{crit}_{t_0,K,c,J,b}$ as the biggest time fulfilling that, for all times t satisfying $t^{crit}_{t_0,K,c,J,b} \geq t \geq 0$ we have

- $t \leq t_0$.
- If $|x| \in [\frac{1}{2}b^n, \frac{3}{2}b^n]$ for $1 \leq n \leq J$, then $|\phi_{c,J,b}(x,t)| \in [b^{n+\frac{1}{8}}, b^{n-\frac{1}{8}}]$, with $\phi_{c,J,b}(x,t)$ the flow given by
$$\frac{d\phi_{c,J,b}(x,t)}{dt} = v(\theta_{c,J,b}(x,t)).$$
- $||b^{-j}\theta_{c,J,b}(b^jx,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4} \leq \frac{1}{t_0\tilde{C}}$ for $1 \leq j \leq J$.
- $\int_0^t |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds \leq K$.

Let us make a few remark on these conditions. First, due to the odd symmetry of the solution and the initial conditions, $\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}$ is always negative and thus

$$\int_0^t |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds$$

is a strictly monotone function with respect to t. Note also that we can check that the norm

$$||b^{-j}\theta_{c,J,b}(b^jx,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4}$$

is continuous in time by checking the evolution equation for it and using that $\theta_{c,J,b}$ exists locally in time. Also, depending on the choice of parameters it may happen that $t^{crit}_{t_0,K,c,J,b}$ does not exists (the second and third condition may not be satisfied for $t=0$), so we will only consider $c < \frac{1}{\tilde{C}t_0}$ and $b < 2^{-8}$ to avoid that. Finally, if we only consider the typical a priori bounds, the second and third conditions could make $t^{crit}_{t_0,K,c,J,b}$ tend to zero as we make b small, which would be a problem for our purposes. However, we have the following lemma.

Lemma 2.4.1. *Fixed t_0, K, c and J fulfilling $c < \frac{e^{-6K}}{\tilde{C}t_0}$ and $K > max(1, t_0)$, we have that, if b is small enough, then the unique H^4 solution $\theta_{c,J,b}$ with initial conditions as in (2.41) satisfies*

$$||b^j \theta_{c,J,b}(b^{-j}x, t) 1_{[b^{\frac{1}{8}}, b^{-\frac{1}{8}}]}(r)||_{H^4} < \frac{1}{t_0 \tilde{C}}$$

for $1 \leq j \leq J$, $t \in [0, t^{crit}_{t_0,K,c,J,b}]$ and if $x \in [b^n \frac{1}{2}, b^n \frac{3}{2}]$ then $\phi_{c,J,b}(x,t) \in (b^{n+\frac{1}{8}}, b^{n-\frac{1}{8}})$ if $0 \leq t \leq t^{crit}_{t_0,K,c,J,b}$.

Proof. Before we get into the proof, we need to define

$$k_n(t) := |\frac{\partial v_1(\theta_{c,J,b} 1_{(b^{n+\frac{1}{8}}, \infty)}(r))}{\partial x_1}(0,t)|,$$

$$K_n(t) := \int_0^t k_n(s) ds.$$

We will study the evolution of $\theta_j := \theta_{c,J,b} 1_{[b^{j+\frac{1}{8}}, b^{j-\frac{1}{8}}]}(r)$ (these functions obviously depend on c, J and b, but we will omit this dependence to obtain a more compact notation). These functions satisfy the evolution equation

$$\frac{\partial \theta_j}{\partial t} + v_1(\theta_j)\frac{\partial \theta_j}{\partial x_1} + v_2(\theta_j)\frac{\partial \theta_j}{\partial x_2} + v_2(\theta_{c,J,b} - \theta_j)\frac{\partial \theta_j}{\partial x_2} + v_1(\theta_{c,J,b} - \theta_j)\frac{\partial \theta_j}{\partial x_1} = 0.$$

Furthermore, we have that $\theta'_j(x,t) = b^{-j}\theta_j(b^j x, t)$ fulfils the evolution equation

$$\frac{\partial \theta'_j}{\partial t} + v_1(\theta'_j)\frac{\partial \theta'_j}{\partial x_1} + v_2(\theta'_j)\frac{\partial \theta'_j}{\partial x_2} + v_2(\theta'^{,j}_{c,J,b} - \theta'_j)\frac{\partial \theta'_j}{\partial x_2} + v_1(\theta'^{,j}_{c,J,b} - \theta'_j)\frac{\partial \theta'_j}{\partial x_1} = 0, \tag{2.43}$$

with $\theta'^{,j}_{c,J,b}(x,t) := b^{-j}\theta_{c,J,b}(b^j x, t)$.

We want to obtain suitable bounds for the terms depending on $\theta'^{,j}_{c,J,b} - \theta'_j$. To do this we decompose $\theta'^{,j}_{c,J,b} - \theta'_j$ as

$$\theta'^{,j}_{c,J,b} - \theta'_j = \theta'_{+,j} + \theta'_{-,j}$$

with $\theta'_{+,j} = (\theta'^{,j}_{c,J,b} - \theta'_j)1_{[1,\infty]}(r)$ and $\theta'_{-,j} = (\theta'^{,j}_{c,J,b} - \theta'_j)1_{[0,1]}(r)$.

But $\theta'_{-,j}$ satisfies that $||\theta'_{-,j}||_{L^1} \leq Cb^3$, $d(supp(\theta'_j), supp(\theta'_{-,j})) \geq \frac{b^{\frac{1}{8}}}{2}$, which gives us, if we define $v_i^{-,j}(x) := v_i(\theta'_{-,j})(x)$

$$||v_i^{-,j}(x) 1_{supp(\theta'_j)}||_{C^4} \leq Cb^{3-\frac{6}{8}}.$$

For the term depending on $\theta'_{+,j}$, we use that, for $k \geq 1$

$$||\theta'^{,j}_{c,J,b} 1_{[b^{-k+\frac{1}{8}}, b^{-k-\frac{1}{8}}]}||_{L^1} \leq Cb^{-3k}$$

$$d(supp(\theta'^{,j}_{c,J,b} 1_{[b^{-k+\frac{1}{8}}, b^{-k-\frac{1}{8}}]}), supp(\theta'_j)) \geq \frac{b^{-k+\frac{1}{8}}}{2}$$

which gives us, after adding the contributions for all the k, if $|x| \leq b^{-\frac{1}{8}}$

$$|\frac{\partial^2 v_i(\theta'_{+,j})}{\partial^{2-j} x_1 \partial^j x_2}(x)| \leq Cb^{\frac{1}{2}}.$$

Therefore, using a second order Taylor expansion for the velocity we obtain that, for $|x| \leq b^{-\frac{1}{8}}$

$$v_1(\theta'_{+,j}) = -k_{j-1}(t)x_1 + v_1^{+,j,error}(x),$$

with $||v_1^{+,j,error}(x)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{L^\infty}\leq Cb^{\frac{1}{4}}$. Furthermore by computing the derivatives of $v_1(\theta^{'}_{+,j})$ we actually obtain $||v_1^{+,j,error}(x)1_{[b^{\frac{1}{8}},b^{\frac{1}{8}}]}(r)||_{C^4}\leq Cb^{\frac{1}{4}}$.

Analogously, we have

$$v_2(\theta^{'}_{+,j})=k_{j-1}(t)x_2+v_2^{+,j,error}(x),$$

with $||v_2^{+,j,error}(x)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{C^4}\leq Cb^{\frac{1}{4}}$.

Writing $v_i^{error}:=v_i^{+,j,error}(x)+v_i^{-,j}(x)$, we get that (2.43) is equivalent to

$$\frac{\partial\theta^{'}_j}{\partial t}+(v_1(\theta^{'}_j)+v_1^{error}-k_{j-1}x_1)\frac{\partial\theta^{'}_j}{\partial x_1}+(v_2(\theta^{'}_j)+v_2^{error}+k_{j-1}x_2)\frac{\partial\theta^{'}_j}{\partial x_2}=0,$$

with $||v_i^{error}||_{C^4}\leq Cb^{\frac{1}{4}}$. To obtain the evolution of the H^4 norm, we note that, with our definition of the H^4 norm

$$\frac{\partial||\theta^{'}_j||_{H^4}}{\partial t}=\sum_{i=0}^{4}\sum_{j=0}^{i}\frac{\partial||\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2}||_{L^2}}{\partial t}$$

and

$$\frac{\partial||\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2}||^2_{L^2}}{\partial t}$$
$$=2(\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2},\frac{\partial^i}{\partial^j x_1\partial^{i-j}x_2}[(v_1(\theta^{'}_j)+v_1^{error}-k_{j-1}x_1)\frac{\partial\theta^{'}_j}{\partial x_1}+(v_2(\theta^{'}_j)+v_2^{error}+k_{j-1}x_2)\frac{\partial\theta^{'}_j}{\partial x_2}])_{L^2}.$$

However, using $||v_i^{error}||_{C^4}\leq Cb^{\frac{1}{4}}$ and incompressibility we get, for $i=0,1,...,4,\ j=0,...,i$

$$|(\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2},\frac{\partial^i(v_1^{error}\frac{\partial\theta^{'}_j}{\partial x_1})}{\partial^j x_1\partial^{i-j}x_2}+\frac{\partial^i(v_2^{error}\frac{\partial\theta^{'}_j}{\partial x_2})}{\partial^j x_1\partial^{i-j}x_2})_{L^2}|\leq Cb^{\frac{1}{4}}||\theta^{'}_j||^2_{H^4}$$

and

$$|(\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2},\frac{\partial^i(k_{j-1}x_1\frac{\partial\theta^{'}_j}{\partial x_1})}{\partial^j x_1\partial^{i-j}x_2}-\frac{\partial^i(k_{j-1}x_2\frac{\partial\theta^{'}_j}{\partial x_2})}{\partial^j x_1\partial^{i-j}x_2})_{L^2}|\leq ik_{j-1}||\frac{\partial^i\theta^{'}_j}{\partial^j x_1\partial^{i-j}x_2}||^2_{L^2}$$

which gives us, by adding all the terms and including the contribution from the terms depending on $v_1(\theta^{'}_j)\frac{\partial\theta^{'}_j}{\partial x_1}$ and $v_2(\theta^{'}_j)\frac{\partial\theta^{'}_j}{\partial x_2}$

$$\begin{aligned}\frac{\partial||\theta^{'}_j||_{H^4}}{\partial t}&=\frac{\partial||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4}}{\partial t}\\&\leq(4k_{j-1}+Cb^{\frac{1}{4}})||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4}\\&+\tilde{C}||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||^2_{H^4},\end{aligned}\tag{2.44}$$

with $\tilde{C}$ given by (2.42).

Using that, by hypothesis

$$||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4}\leq\frac{1}{t_0\tilde{C}},$$

$$||b^j\theta_{c,J,b}(b^{-j}x,0)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4}\leq c$$

and integrating (2.44) we get

$$||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4} \leq ce^{4K_{j-1}(t)+(\frac{1}{t_0}+Cb^{\frac{1}{4}})t}$$

and using $K_{j-1}(t) \leq K$, and taking b small enough

$$||b^j\theta_{c,J,b}(b^{-j}x,t)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}(r)||_{H^4} < ce^{6K} < \frac{1}{\tilde{C}t_0},$$

which gives us the first property we wanted.

As for the bounds for $\phi_{c,J,b}(x,t)$, we again work in the equivalent problem with $\theta'_{c,J,b}$ and note that we just proved that

$$|v_i(\theta^{',j}_{c,J,b})(x)1_{[b^{\frac{1}{8}},b^{-\frac{1}{8}}]}| \leq (k_J(t)+Cb^{\frac{1}{4}})|x| + |v_i(\theta^{'}_j)|(x),$$

and since $|v_i(\theta^{'}_j)| \leq min(C, C|x|)$ (by using our bounds in H^4 plus $v_i(\theta^{'}_j)(x=0)=0$), integrating in time we have that, for b small, the particles under that flow starting in $[\frac{1}{2},\frac{3}{2}]$ will stay in $(e^{-C},e^{C}) \subset (b^{\frac{1}{8}},b^{-\frac{1}{8}})$, with C depending on K and t_0 and we are done by undoing the scaling and returning from θ'_j to θ_j. $\square$

Note that last lemma tells us that for b small enough, at $t = t^{crit}_{t_0,K,c,J,b}$, either $t = t_0$ or $\int_0^{t^{crit}_{t_0,K,c,J,b}} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds = K$. Our next goal is to prove that, if the right conditions are met, we will actually have $\int_0^{t^{crit}_{t_0,K,c,J,b}} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds = K$.

Lemma 2.4.2. *For fixed t_0, K and c fulfilling $c < \frac{e^{-6K}}{\tilde{C}t_0}$ and $K > max(1,t_0)$, we can find J and b such that at time $t = t^{crit}_{t_0,K,c,J,b}$ we have that* $\int_0^{t^{crit}_{t_0,K,c,J,b}} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds = K.$

Proof. We start by studying the trajectories of particles with $|x| \in [b^{J+\frac{1}{8}}, b^{-\frac{1}{8}}]$.

In the proof of Lemma 2.4.1 we obtained that, for $|x| \in [b^{\frac{1}{8}}, b^{-\frac{1}{8}}]$,

$$v_1(\theta^{',j}_{c,J,b}) = v(\theta^{'}_j) + v_1^{error}(x) - k_{j-1}(t)x_1 \tag{2.45}$$

$$v_2(\theta^{',j}_{c,J,b}) = v(\theta^{'}_j) + v_2^{error}(x) + k_{j-1}(t)x_2$$

(let us remember that here $\theta^{'}_j$ actually depends on c, J and b but we omit it), with $||v_i^{error}(x)||_{C^4} \leq C_1 b^{\frac{1}{4}}$ for $i = 1,2$, with C_1 depending on c, j and J, and $||v(\theta^{'}_j)||_{C^1} \leq C_2$ with C_2 depending on t_0. By returning to the original problem, we get that, for $|x| \in [b^{j+\frac{1}{8}}, b^{j-\frac{1}{8}}]$

$$v_1(\theta_{c,J,b}) = v(\theta_j) + v_1^{error,j}(x) - k_{j-1}(t)x_1 \tag{2.46}$$

$$v_2(\theta_{c,J,b}) = v(\theta_j) + v_2^{error,j}(x) + k_{j-1}(t)x_2$$

with $||v^{error}||_{C^1} \leq Cb^{\frac{1}{4}}$ and $||v(\theta_j)||_{C^1} \leq C_2$ with C_2 depending on t_0.

We are interested in studying the ϕ associated to this problem in polar coordinates for particles starting in $(r,\alpha) \in ([\frac{1}{2},\frac{3}{2}],[0,2\pi])$. We study separately the evolution of the radial coordinate and of the angular coordinate for simplicity.

For the radial coordinate, if we call $\phi^j_r(r_0,\alpha_0,t)$ the flow associated to (2.46) that gives us the radial coordinate of the particle that was initially in (r_0,α_0), using that $v(0) = 0$ and integrating in time, we have that,

$$\frac{\phi^j_r(r_0,\alpha_0,t)}{r_0} \leq e^{\int_0^t k_{j-1}(s)ds + C_1 b^{\frac{1}{4}}t + C_2 t} \leq e^{K + C_1 b^{\frac{1}{4}}t + C_2 t}.$$

As for the change in the angular coordinate, we are interested in finding bounds for how fast a particle can approach the lines $\alpha = i\frac{\pi}{2}$, $i = 0,1,2,3$. All four cases are equivalent, so we will consider $i = 0$. We have that

$$v_\alpha(r,0,t)=0$$

and, since for $i=1,2$ $||\frac{\partial v_\alpha}{\partial x_i}||_{L^\infty}\leq C(|k_{j-1}|+C_1 b+C_2)$ (with C a universal positive constant) we get, defining ϕ^j_α similarly as we did with $\phi^j_r(r_0,\alpha_0,t)$,

$$\frac{\phi^j_\alpha(r_0,\alpha_0,t)}{\alpha_0}\geq e^{-C(\int_0^t k_{j-1}(s)ds+C_1 b^{\frac{1}{4}}t+C_2 t)}\geq e^{-C(K+C_1 b^{\frac{1}{4}}t+C_2 t)}.$$

Now we are ready to obtain bounds for

$$\int_0^{t^{crit}_{t_0,K,c,J,b}}|\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds.$$

Since the transformation

$$\theta_{c,J,b}(x)\to\frac{\theta_{c,J,b}(\lambda x)}{\lambda}$$

does not change the value of $\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)$ and by linearity, we have that, for $s=0$ we can compute

$$|\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(x=0,t=0)|=\sum_{j=1}^{J}\frac{c}{j}\frac{\partial v_1(f(r)\cos(2\alpha))}{\partial x_1}(x=0)=C(\sum_{j=1}^{J}\frac{c}{j})\geq Cc\ln(J),$$

for some $C>0$.

For times $t>0$, writing for the flow map $\phi_{c,J,b}(x,t)=(\phi_{1,c,J,b}(x,t),\phi_{2,c,J,b}(x,t))$

$$\begin{aligned}
&|\frac{\partial v_1(\theta_{c,J,b}(r,\alpha,t))}{\partial x_1}|=C\int_{\mathbb{R}^2_+}y_1\frac{y_2\theta_{c,J,b}(y,t)}{|y|^5}dy_1dy_2\\
&=C\int_{\mathbb{R}^2_+}y_1\frac{y_2\theta_{c,J,b}(\phi^{-1}_{c,J,b}(y,t),0)}{|y|^5}dy_1dy_2\\
&=C\int_{\mathbb{R}^2_+}\phi_{1,c,J,b}(\tilde{y},t)\frac{\phi_{2,c,J,b}(\tilde{y},t)\theta_{c,J,b}(\tilde{y},0)}{|\phi_{c,J,b}(\tilde{y},t)|^5}d\tilde{y}_1d\tilde{y}_2\\
&=C\int_{\mathbb{R}^2_+}\phi_{1,c,J,b}(\tilde{y},t)\frac{\phi_{2,c,J,b}(\tilde{y},t)}{|\phi_{c,J,b}(\tilde{y},t)|^5}\frac{|\tilde{y}|^5}{\tilde{y}_1\tilde{y}_2}\frac{\tilde{y}_1\tilde{y}_2\theta_{c,J,b}(\tilde{y},0)}{|\tilde{y}|^5}d\tilde{y}_1d\tilde{y}_2
\end{aligned}$$

with C a constant, but (passing to polar coordinates to obtain the bound more easily)

$$\begin{aligned}
&\phi_{1,c,J,b}(x,t)\frac{\phi_{2,c,J,b}(x,t)}{|\phi_{c,J,b}(x,t)|^5}\frac{|x|^5}{x_1x_2}\\
&=\frac{\sin(2\phi^\alpha_{c,J,b}(r,\alpha))}{\sin(2\alpha)}\frac{r^3}{\phi^r_{c,J,b}(r,\alpha)}\geq e^{-C(K+c_1 b^{\frac{1}{4}}t+C_2 t)}
\end{aligned}$$

for some C, and thus

$$\begin{aligned}
&|\frac{\partial v_1(\theta_{c,J,b}(r,\alpha,t))}{\partial x_1}(x=0)|\\
&\geq Ce^{-C(K+c_1 b^{\frac{1}{4}}t+C_2 t)}\int_{\mathbb{R}^2_+}\frac{\tilde{y}_1\tilde{y}_2\theta_{c,J,b}(\tilde{y},0)}{|\tilde{y}|^5}d\tilde{y}_1d\tilde{y}_2
\end{aligned}$$

and integrating in time

$$\int_0^{t^{crit}_{t_0,K,c,J,b}} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds \geq t^{crit}_{t_0,K,c,J,b} Ccln(J)e^{-C(K+C_1 b^{\frac{1}{4}} t_0+C_2 t_0)}$$

To finish our prove, we just fix some K, t_0 and c fulfilling our hypothesis, we take J big enough so that

$$t_0 Ccln(J)e^{-C(K+C_2 t_0)} > K+1$$

and then take b small enough so that using Lemma 2.4.1 either $t_0 = t^{crit}_{t_0,K,c,J,b}$ or

$$\int_0^{t^{crit}_{t_0,K,c,J,b}} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds = K$$

and such that

$$t_0 Ccln(J)e^{-C(K+C_1 b^{\frac{1}{4}} t_0+C_2 t_0)} > K.$$

The result then follows by contradiction, since if we assume $t_0 = t^{crit}_{t_0,K,c,J,b}$ we obtain

$$\int_0^{t_0} |\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)|ds \geq t_0 Ccln(J)e^{-C(K+C_1 b^{\frac{1}{4}} t_0+C_2 t_0)} > K.$$

□

Corollary 2.4.3. *There are initial conditions $\theta^{initial}_{K,t_0,\tilde{c}} \in H^4$ with $||\theta^{initial}_{K,t_0,\tilde{c}}||_{H^2} \leq \tilde{c}$ such that there exists $0 < t^{crit}_{K,t_0,\tilde{c}} \leq t_0$ and a solution $\theta_{K,t_0,\tilde{c}}(x,t)$ to (2.1) with $\theta^{initial}_{K,t_0,\tilde{c}}$ as initial conditions fulfilling*

$$\int_0^{t^{crit}_{K,t_0,\tilde{c}}} \frac{\partial v_1(\theta_{K,t_0,\tilde{c}})}{\partial x_1}(0,s)ds = -K,$$

$$||\theta_{K,t_0,\tilde{c}}(x,t)||_{H^4} \leq M_{K,t_0,\tilde{c}}.$$

Furthermore we have $supp(\theta^{initial}_{K,t_0,\tilde{c}}) \subset \{r \in (a_1, \frac{3}{2})\}$, $supp(\theta_{K,t_0,\tilde{c}}(x,t)) \subset \{r \in (a_1,a_2)\}$ with a_1, a_2 depending on K, t_0 and $\tilde{c}$.

Proof. The initial conditions and solution are the ones obtained in Lemma 2.4.2, we only need to note that $||\theta_{c,J,b}||_{H^2} = c(\sum_{j=1}^{J} \frac{1}{j^2})^{\frac{1}{2}} \leq Cc$, and thus we need to take $Cc \leq \tilde{c}$ and then apply Lemma 2.4.2. As for the condition regarding the support, we just need to use that since the solution remains in H^4 the velocity is C^1 and that the velocity at $(x_1,x_2) = (0,0)$ is zero and thus particles can only approach the origin exponentially fast. □

Theorem 2.4.1. For any $c_0 > 0$, $M > 2$ and $t_* > 0$, we can find a $H^{2+\frac{1}{4}}$ function $\theta_0(x)$ with $||\theta_0(x)||_{H^2} \leq c_0$ such that the only solution $\theta(x,t) \in H^{2+\frac{1}{4}}$ to the SQG equation (2.1) with initial conditions $\theta_0(x)$ is such that there exists $t \leq t^*$ with $||\theta(x,t)||_{H^2} \geq Mc_0$.

Proof. We consider the pseudo-solution

$$\begin{aligned}\bar{\theta}_{M,t^*,c_0,N} &= \theta_{K=4M,t_0=t^*,\tilde{c}=\frac{c_0}{2}}(x,t) \\ &+ \frac{c_0}{4} g_1(e^{G(t)}N^{\frac{1}{2}}x_1)g_2(e^{-G(t)}N^{\frac{1}{2}}x_2)\frac{\sin(e^{G(t)}Nx_1)}{N^{\frac{3}{2}}}\end{aligned} \tag{2.47}$$

with $\theta_{K,t_0,\tilde{c}}$ given by Corollary 2.4.3 with $\tilde{c} = \frac{c_0}{2}$, $t_0 = t^*$ and $K = 4M$,

$$G(t) = -\int_0^t \frac{\partial v_1(\theta_{K,t_0,\tilde{c}})}{\partial x_1}(0,s)ds$$

and $g_1(x_1), g_2(x_2)$ C^∞ functions with support in $[-1,1]$ and $||g_i||_{L^2}=1$. We will define

$$f^1_{M,t^*,c_0}(x,t) := \theta_{K=4M,t_0=t^*,\tilde{c}=\frac{c_0}{2}}(x,t)$$

$$f^2_{c_0,N}(x,t) := \frac{c_0}{4} g_1(e^{G(t)}N^{\frac{1}{2}}x_1)g_2(e^{-G(t)}N^{\frac{1}{2}}x_2)\frac{\sin(e^{G(t)}Nx_1)}{N^{\frac{3}{2}}}$$

for a more compact notation.

These pseudo-solutions have the following properties:

- For N large, $||\bar{\theta}_{M,t^*,c_0,N}(t=0)||_{H^2} \leq c_0$.
- There exists a $t_{crit} \leq t^*$ (given by Corollary 2.4.3) such that, for N large, we have

  $$||\bar{\theta}_{M,t^*,c_0,N}(t=t_{crit})||_{H^2} \geq \frac{c_0}{8}e^{8M} > c_0 e^M$$

  where we used that, since $g_1, g_2 \in C^1$ and have compact support, for $\lambda > 1$

  $$\lim_{N\to\infty}||N^{\frac{1}{2}}g_1(\lambda N^{\frac{1}{2}}x_1)g_2(\lambda^{-1}N^{\frac{1}{2}}x_2)\cos(\lambda N x_1)||_{L^2} = \frac{1}{\sqrt{2}}||g(x_1)||_{L^2}.$$

Furthermore they fulfil the evolution equation

$$\frac{\bar{\theta}_{M,t^*,c_0,N}}{\partial t} + v_1(f^1_{M,t^*,c_0})\frac{\partial f^1_{M,t^*,c_0}}{\partial x_1} + v_2(f^1_{M,t^*,c_0})\frac{\partial f^1_{M,t^*,c_0}}{\partial x_2}$$
$$+ x_1\frac{\partial v_1(f^1_{M,t^*,c_0})}{\partial x_1}\frac{\partial f^2_{c_0,N}}{\partial x_1} + x_2\frac{\partial v_2(f^1_{M,t^*,c_0})}{\partial x_2}\frac{\partial f^2_{c_0,N}}{\partial x_2} = 0$$

and thus it is a pseudo-solution with source term

$$F_{M,t^*,c_0,N}(x,t) = F^1_{M,t^*,c_0,N}(x,t) + F^2_{M,t^*,c_0,N}(x,t) + F^3_{M,t^*,c_0,N}(x,t),$$

$$F^1_{M,t^*,c_0,N}(x,t) := -(v_1(f^2_{c_0,N})\frac{\partial f^2_{c_0,N}}{\partial x_1} + v_2(f^2_{c_0,N})\frac{\partial f^2_{c_0,N}}{\partial x_2}),$$

$$F^2_{M,t^*,c_0,N}(x,t) := -(v_1(f^2_{c_0,N})\frac{\partial f^1_{M,t^*,c_0}}{\partial x_1} + v_2(f^2_{c_0,N})\frac{\partial f^1_{M,t^*,c_0}}{\partial x_2}),$$

$$F^3_{M,t^*,c_0,N}(x,t) := (x_1\frac{\partial v_1(f^1_{M,t^*,c_0})(x=0)}{\partial x_1} - v_1(f^1_{M,t^*,c_0}))\frac{\partial f^2_{c_0,N}}{\partial x_1},$$
$$+ (x_2\frac{\partial v_2(f^1_{M,t^*,c_0}))(x=0)}{\partial x_2} - v_2(f^1_{M,t^*,c_0}))\frac{\partial f^2_{c_0,N}}{\partial x_2}.$$

As usual we want to find bounds for the source term for $t\in[0,t_{crit}]$. For $F^1_{M,t^*,c_0,N}(x,t)$ it is easy to obtain that

$$||F^1_{M,t^*,c_0,N}(x,t)||_{L^2} \leq CN^{-\frac{5}{2}}, \quad ||F^1_{M,t^*,c_0,N}(x,t)||_{H^3} \leq CN^{\frac{1}{2}}$$

with C depending on M and c_0.

For $F^2_{M,t^*,c_0,N}(x,t)$, using that $||f^2_{c_0,N}||_{L^1} \leq CN^{-\frac{5}{2}}$ and that the support of f^1_{M,t^*,c_0} lies away from 0, we get

$$||F^2_{M,t^*,c_0,N}(x,t)||_{L^2} \leq CN^{-\frac{5}{2}}, \quad ||F^2_{M,t^*,c_0,N}(x,t)||_{H^3} \leq CN^{-\frac{5}{2}}$$

with C depending on M, t^* and c_0.

Finally, for $F^3_{M,t^*,c_0,N}(x,t)$, using that, for $i = 1, 2$

$$x_i \frac{\partial v_i(f^1_{M,t^*,c_0})}{\partial x_1} - v_i(f^1_{M,t^*,c_0})$$

vanishes to second order around 0, that the third derivatives of $v_i(f^1_{M,t^*,c_0})$ are bounded around 0, and that $\text{supp}(f^2_{c_0,N}) \subset [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}] \times [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}]$, we get

$$||F^3_{M,t^*,c_0,N}||_{L^2} \leq CN^{-\frac{5}{2}}, \quad ||F^3_{M,t^*,c_0,N}||_{H^3} \leq CN^{\frac{1}{2}},$$

with C depending on M, t^* and c_0.

With all this combined and using the interpolation inequality, we get

$$||F_{M,t^*,c_0,N}||_{L^2} \leq CN^{-\frac{5}{2}}, \quad ||F_{M,t^*,c_0,N}||_{H^{2+\frac{1}{4}}} \leq CN^{-\frac{1}{4}}.$$

This allows us to obtain, in a similar way as in Lemmas 2.2.8, 2.2.9, 2.3.6 and 2.3.9 that, if $\theta_{M,t^*,c_0,N}(x,t)$ is the solution to (2.1) with $\theta_{M,t^*,c_0,N}(x,0) = \bar{\theta}_{M,t^*,c_0,N}(x,0)$ then

$$||\theta_{M,t^*,c_0,N}(x,t) - \bar{\theta}_{M,t^*,c_0,N}(x,t)||_{H^{2+\frac{1}{4}}} \leq CtN^{-\frac{1}{4}}$$

and this combined with the properties of $\bar{\theta}_{M,t^*,c_0,N}(x,t)$ finishes the proof.

□

Theorem 2.4.2. For any $c_0 > 0$ there exist initial conditions $\theta(x,0)$ with $||\theta(x,0)||_{H^2} \leq c_0$ such that there is no solution $\theta(x,t)$ to (2.1) satisfying

$$\text{ess-sup}_{t\in[0,\epsilon]}||\theta(x,t)||_{H^2} \leq M$$

for some $\epsilon, M > 0$.

Proof. After fixing some arbitrary $c_0 > 0$ we define

$$\bar{\theta}_{n,R,N}(x,t) := T_R(\bar{\theta}_{M=4^n,t^*=2^{-n},c_0=2^{-n},N}),$$

with $\bar{\theta}_{M,t^*,c_0,N}$ as in (2.47) and $T_R(f(x_1,x_2)) = f(x_1+R,x_2)$. We will also refer to the first time when

$$||\bar{\theta}_{n,R,N}(x,t)||_{H^2} \geq 2^n$$

(which we already know exists and is smaller than 2^{-n}) as $t_{crit,n}$.

We will study the initial conditions

$$\theta(x,0) = \sum_{n=1}^{\infty} \bar{\theta}_{n,R_n,N_n}(x,0), \tag{2.48}$$

which fulfil $||\theta(x,0)||_{H^2} \leq c_0$ if each N_n is big enough, and we will prove by contradiction that if we choose appropriately $(R_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ there cannot exists a solution $\theta(x,t)$ with these initial conditions that satisfies

$$\text{ess-sup}_{t\in[0,\epsilon]}||\theta(x,t)||_{H^2} \leq P \tag{2.49}$$

for some ϵ, P. Note also that $\bar{\theta}_{n,R_n,N_n}(x,0)$ is supported in $B_{\frac{3}{2}}(-R_n,0)$. We can assume that our L^2 norm is conserved, since this will be true if equation (2.49) holds (for the time intervals that we will consider). We will assume without loss of generality that $\epsilon \leq 1$, and we define v_{max} as the maximum velocity that a function f with $||f||_{H^2} \leq 1$ can produce. With this in mind, we write

$$R_n = D_n + D_{n+1} + 4v_{max}2^{n-1} + R_{n-1} + 3$$

with $D_n = N_n^4$ and we will prove that, if N_n is big enough, then any solution to (2.1) with initial conditions (2.48) will satisfy

$$\text{ess-sup}_{t\in[0,2^{-n}]}||\theta(x,t)||_{H^2} \geq 2^{n-1} \tag{2.50}$$

for any $n \in \mathbb{N}$. Note that with this definition of R_n, we have, for any $i \neq n$ that

$$d(\text{supp}(T_{R_n}(\bar{\theta}_{n,R_n,N_n}(x,0))), \text{supp}(T_{R_i}(\bar{\theta}_{i,R_i,N_i}(x,0)))) \geq 4v_{max}2^{n-1} + D_n$$

Now, we focus on the evolution of

$$\theta_n(x,t) := 1_{B_{D_n+2v_{max}2^{n-1}+\frac{3}{2}}(-R_n,0)}\theta(x,t)$$

and we will assume that

$$\text{ess-sup}_{t\in[0,2^{-n}]}||\theta(x,t)||_{H^2} < 2^{n-1} \tag{2.51}$$

and try to get to a contradiction.

Then if $t \in [0, 2^{-n}]$, we have that $\theta_n(x,t)$ will fulfil the evolution equation

$$\frac{\partial \theta_n}{\partial t} + (v_1(\theta_n) + v_1(\theta-\theta_n))\frac{\partial \theta_n}{\partial x_1} + (v_2(\theta_n) + v_2(\theta-\theta_n))\frac{\partial \theta_n}{\partial x_2} = 0, \tag{2.52}$$

$$\theta_n(x,0) = 1_{B_{D_n+2v_{max}2^{n-1}+\frac{3}{2}}(-R_n,0)}\theta(x,0)$$

and that $||v_i(\theta-\theta_n)1_{B_{v_{max}2^n}(R_n)}||_{C^3} \leq CN_n^{-4}$ since $d(\text{supp}(\theta-\theta_n), \text{supp}(\theta_n)) \geq N_n^4$.

But then we can argue as in Lemmas 2.3.6, 2.3.9 and 2.3.10 to show that, for $t \in [0, t_{crit,n}]$, if N_n is large, we can find a solution $\tilde{\theta}_n(x,t)$ $H^{2+\frac{1}{4}}$ fulfilling (2.52) and

$$||\tilde{\theta}_n(x,t) - T_{R_n}(\bar{\theta}_{n,R_n,N_n}(x,t))||_{H^{2+\frac{1}{4}}} \leq CN_n^{-\frac{1}{4}}.$$

But then, the regularity of $\tilde{\theta_n}$ plus the (assumed) regularity of θ_n allows us to show that both solutions are actually the same by studying the evolution of $\theta_n - \tilde{\theta}_n$. Since for some $t_{crit,n} \in [0, 2^{-n}]$ we have that

$$||T_{R_n}(\bar{\theta}_{n,R_n,N_n}(x,t_{crit,n}))||_{H^2} \geq 2^n,$$

and the H^2 norm of $T_{R_n}(\bar{\theta}_{n,R_n,N_n}(x,t))$ is continuous in time, we arrive to a contradiction by taking N_n big enough and repeating this argument for each $n \in \mathbb{N}$.

□

Remark 5. The proof can be adapted to work in the critical spaces $W^{1+\frac{2}{p},p}$ for $p \in (1,\infty]$. For this, note that it is easy to obtain a version of Corollary 2.4.3 but with small $W^{1+\frac{2}{p},p}$, since the function

$$\sum_{j=1}^{J} c\frac{f(b^{-j}r)b^j \sin(2\alpha)}{j}$$

has a $W^{1+\frac{2}{p},p}$ norm as small as we want by taking c small. As for the perturbation, we need to consider

$$\lambda g_1(N^b x_1)g_2(N^b x_2)\frac{\sin(Nx_1)}{N^{1+a}},$$

with $a = a(p), b = b(p) \geq 0$ values that keep the norm $W^{1+\frac{2}{p},p}$ bounded (but not tending to zero) as $N \to \infty$ (for example, in $W^{1,\infty}$ we consider $a = 0$) and $\lambda > 0$. Taking $b = \frac{1}{2}$ and arguing as in Theorems 2.4.1 and 2.4.2 allows us to obtain ill-posedness for a wide range of p, but we need to include some refinements to obtain the result for all $p \in (1,\infty]$. Namely, approximations for the velocity similar to those obtained in Lemma 2.2.3 are needed and we have to include one extra time dependent term in the pseudo-solution.

Chapter 3

Strong ill-posedness and Non-existence results for gSQG

3.1 Introduction

In this chapter we consider a generalization to the SQG equation. More precisely, we say a function $w(x,t):\mathbb{R}^2\times\mathbb{R}_+\to\mathbb{R}$, $w(x,t)\in H^s$, $s>2+\gamma$ is a solution to the generalized Surface Quasi-geostrophic equation with parameter γ (or to the γ-SQG equation) with initial conditions $w(x,0)=w_0(x)$ if the equation

$$\frac{\partial w}{\partial t}+v_{1,\gamma}\frac{\partial w}{\partial x_1}+v_{2,\gamma}\frac{\partial w}{\partial x_2}=0 \tag{3.1}$$

is fulfilled for every $x\in\mathbb{R}^2$, with $v=(v_{1,\gamma},v_{2,\gamma})$ defined by

$$v_{1,\gamma}=-\frac{\partial}{\partial x_2}\Lambda^{-1+\gamma}w,\ v_{2,\gamma}=\frac{\partial}{\partial x_1}\Lambda^{-1+\gamma}w.$$

As in the previous chapter, we denote $\Lambda^\alpha f\equiv(-\Delta)^{\frac{\alpha}{2}}f$ by the Fourier transform $\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\hat{f}(\xi)$.

This family of equations becomes the 2D incompressible Euler equations and the SQG equation (see [27], [19] and [18]) when $\gamma=-1,0$ respectively. For the entire range $\gamma\in(-1,1)$, it has been shown in [18] that this system is locally well-possed in H^s for $s>2+\gamma$. In [21] the authors proved local existence in the critical Sobolev space H^2 for a logarithmic inviscid regularization of SQG (see also [66] for the γ-SQG case). Regarding H^s norm growth see [73] where the authors show that there exists initial conditions with arbitrarily small H^s norm ($s\geq 11$) that become large after a long period of time. Finite time formation of singularities for initial data in H^s for $s>2+\gamma$ remains an open problem for the range $\gamma\in(-1,1)$. On the other hand, there are a few rigorous constructions of non-trivial global solutions in H^s (for some s satisfying $s>2+\gamma$) in [16], [58], [15] and [95].

For both 2D Euler and SQG, the critical Sobolev space has been studied in [9], [39], [54] and [65], where it has been established non-existence of uniformly bounded solutions in $H^{2+\gamma}$ (see also [78] and [77] for other ill-posedness results for active scalars). Furthermore, for $\gamma=0$, in a range of supercritical Sobolev spaces ($s\in(\frac{3}{2},2)$) non-existence of solutions in H^s is proved in [39].

Global existence of solutions in L^2 have already been obtained for SQG in [89] (see [18], for an extension in the case $\gamma\in(0,1)$), but uniqueness is not known and in fact there is non uniqueness of solutions for $\Lambda^{-1}w\in C^\sigma_tC^\beta_x$ with $\frac{1}{2}<\beta<\frac{4}{5}$ and $\sigma<\frac{\beta}{2-\beta}$ (see [12]).

Local well-posedness in $C^{k,\beta}\cap L^q$ ($k\geq 1$, $\beta\in(0,1)$, $q>1$) was established for SQG in [98], and recently the result was improved in [3], where the requirement $w\in L^q$ has been dropped. The same result as in [98] applies for the range $\gamma\in[-1,0)$ for $\beta\in[0,1]$ (for the a priori estimates see [19]). Nevertheless, as shown in [39] for $\gamma=0$, there is no local existence result when $\beta=0,1$ (in the case of 2D Euler equations see [8] and [55] for a proof of strong ill-posedness and non-existence of uniformly bounded solutions for the velocity v in C^k).

Global in time exponential growth of solutions was obtained in [59] for the range $\gamma \in (-1,1)$ in $C^{1,\beta}$, with $\beta \in [f(\gamma),1]$.

3.1.1 Main results

The aim of this chapter is to prove strong ill-posedness in $C^{k,\beta}$ ($k \geq 1$, $\beta \in (0,1]$ and $k+\beta > 1+\gamma$) of the γ-SQG equation for the range $\gamma \in (0,1)$. We also construct solutions in $\mathbb{R}^2$ of γ-SQG that initially are in $C^{k,\beta} \cap L^2$ but are not in $C^{k,\beta}$ for $t > 0$.

Theorem 3.1.1. (Strong ill-posedness) Given k a natural number, $\beta \in (0,1]$, $\gamma \in (0,1)$ and $\delta \in (0, \frac{1}{2})$ with $k+\beta-2\delta > 1+\gamma$, then for any $T, t_{crit,}\epsilon_1, \epsilon_2 > 0$, there exist a $H^{k+\beta+1-\delta}$ function $w(x,0)$ such that $||w(x,0)||_{C^{k,\beta}} \leq \epsilon_1$ and the only solution to (3.1) in $H^{k+\beta+1-\delta}$ with initial conditions $w(x,0)$ exists for $t \in [0,T]$ and fulfills that

$$||w(x,t_{crit})||_{C^{k,\beta}} \geq \frac{1}{\epsilon_2}.$$

Theorem 3.1.2. (Non-existence) Given k a natural number, $\beta \in (0,1]$, $\gamma \in (0,1)$ and $\delta \in (0, \frac{1}{2})$ with $k+\beta-2\delta > 1+\gamma$, then for any T and $\epsilon > 0$, there exist a $H^{k+\beta+1-\frac{3}{2}\delta}$ function $w(x,0)$ such that $||w(x,0)||_{C^{k,\beta}} \leq \epsilon$ and that the only solution to (3.1) in $H^{k+\beta+1-\frac{3}{2}\delta}$ with initial conditions $w(x,0)$ exists for $t \in [0,T]$ and fulfills that, for $t \in (0,T]$, $||w(x,t)||_{C^{k,\beta}} = \infty$.

Remark 6. Although technically we do not prove the results for the case $\beta = 0$, the result in $C^{k,1}$ actually gives us strong ill-posedness and non-existence in the space C^{k+1}.

3.1.2 Strategy of the proof

To obtain the ill-posedness result, we first focus on finding a pseudo-solution $\bar{w}$ for $\gamma - SQG$ that exhibits the behaviour we would like to show, mainly that it has a small $C^{k,\beta}$ norm initially and this norm grows a lot in a very short period of time. As in Definition 5, we say that $\bar{w}$ is a pseudo-solution to $\gamma-$SQG if it fulfils an evolution equation of the form

$$\frac{\partial \bar{w}}{\partial t} + v_{1,\gamma}\frac{\partial \bar{w}}{\partial x_1} + v_{2,\gamma}\frac{\partial \bar{w}}{\partial x_2} + F(x,t) = 0 \tag{3.2}$$

with $v = (v_{1,\gamma}, v_{2,\gamma})$ defined by

$$v_{1,\gamma} = -\frac{\partial}{\partial x_2}\Lambda^{-1+\gamma}\bar{w}, \; v_{2,\gamma} = \frac{\partial}{\partial x_1}\Lambda^{-1+\gamma}\bar{w}.$$

Again, in general we will only use this definition for $\bar{w}$ when F is small in a relevant norm. Once we have a pseudo-solution $\bar{w}$ with the desired behaviour, if F is small and both F and $\bar{w}$ are regular enough, then $\bar{w} \approx w$, with w the solution to (3.1) with the same initial conditions as $\bar{w}$, and therefore w shows the same fast growth as $\bar{w}$.

The details about how to find a pseudo-solution with the desired behaviour are somewhat technical, but the rough idea is to consider initial conditions that in polar coordinates have the form

$$w_N(r,\alpha,0) = f(r) + \frac{g(r,N\alpha)}{N^{k+\beta}},$$

that is, a radial function (which is a stationary solution to γ-SQG) plus a perturbation of frequency N in α. The evolution of $w_{pert,N}(r,\alpha,t) := w_N(r,\alpha,t) - f(r)$ satisfies

$$\frac{\partial w_{pert,N}}{\partial t} + v_\gamma(w_{pert,N}) \cdot \nabla w_{pert,N} + v_{r,\gamma}(w_{pert,N})\frac{\partial f(r)}{\partial r} + \frac{\partial w_{pert,N}}{\partial \alpha}\frac{v_{\alpha,\gamma}(f(r))}{r} = 0,$$

where $v_{r,\gamma}, v_{\alpha,\gamma}$ are the radial and angular components of the velocity respectively.

For very big N, we have that

$$v_\gamma(w_{pert,N}) \cdot \nabla w_{pert,N} \approx 0, \;\; v_{r,\gamma}(w_{pert,N}) \approx C_\gamma(-\Delta_\alpha)^{\frac{\gamma}{2}} H_\alpha(w_{pert,N})$$

where $(-\Delta_\alpha)^{\frac{\gamma}{2}}, H_\alpha$ are the fractional laplacian and the Hilbert transform respectively with respect to only the variable α. This suggest studying

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\partial f(r)}{\partial r} C_\gamma(-\Delta_\alpha)^{\frac{\gamma}{2}} H_\alpha(\tilde{w}) + \frac{\partial \tilde{w}}{\partial \alpha} \frac{v_{\alpha,\gamma}(f(r))}{r} = 0. \tag{3.3}$$

and using $\bar{w} = f(r) + \tilde{w}$. The system (3.3) is relatively simple to study, since it is linear and one dimensional in nature, and one can obtain explicit solutions where the $C^{k,\beta}$ norm grows arbitrarily fast. Then, once the candidate pseudo-solutions are found, a careful study of the errors involved allows us to obtain ill-posedness.

Moreover, to obtain non-existence, we consider an infinite number of fast growing solutions, and spread them through the plane so that the interactions between them become very small.

3.1.3 Outline of the chapter

The chapter is organized as follows. In Section 2, we set the notation used through the chapter. In Section 3, we obtain estimates on the velocity in the radial and angular direction. In section 4, we introduce the pseudo-solutions with the desired properties and establish the necessary estimates on the source term $F(x,t)$. Finally in section 5, we prove strong ill-posedness and non-existence for the space $C^{k,\beta}$.

3.2 Preliminaries and notation

3.2.1 Polar coordinates

Many of our computations and functions become much simpler if we use polar coordinates, so we need to establish some notation in that regard. For the rest of this subsection, we will refer to

$$F : \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{R}^2$$

$$(r, \alpha) \to (r\cos(\alpha), r\sin(\alpha))$$

the map from polar to cartesian coordinates. Note that the choice of $[0, 2\pi)$ for the variable α is arbitrary and any interval of the form $[c, 2\pi + c)$ would also work, and in fact we will sometimes consider intervals different from $[0, 2\pi)$. These changes in the domain will not be specifically mentioned since they will be clear by context.

Given a function $f(x_1, x_2)$ from $\mathbb{R}^2$ to $\mathbb{R}$, we define

$$f^{pol} : \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{R}$$

as $f^{pol}(r, \alpha) := f(F(r, \alpha))$.

For $r > 0$, we also have the following equalities

$$\begin{aligned}\frac{\partial f(x_1,x_2)}{\partial x_1} &= \cos(\alpha(x_1,x_2)) \frac{\partial f^{pol}}{\partial r}(F^{-1}(x_1,x_2)) \\ &- \frac{1}{r}\sin(\alpha(x_1,x_2)) \frac{\partial f^{pol}}{\partial \alpha}(F^{-1}(x_1,x_2)),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{\partial f(x_1,x_2)}{\partial x_2} &= \sin(\alpha(x_1,x_2)) \frac{\partial f^{pol}}{\partial r}(F^{-1}(x_1,x_2)) \\ &+ \frac{1}{r}\cos(\alpha(x_1,x_2)) \frac{\partial f^{pol}}{\partial \alpha}(F^{-1}(x_1,x_2)).\end{aligned} \tag{3.5}$$

Furthermore, for functions such that $\text{supp}(f^{pol}(r,\alpha)) \subset \{(r,\alpha) : r \geq r_0\}$ with $r_0 > 0$, we have that for $m = 0, 1, ...$, using (3.4) and (3.5)

$$||f||_{C^m} \leq C_{r_0,m}||f^{pol}||_{C^m},$$

where

$$||f^{pol}||_{C^m} = \sum_{k=0}^{m}\sum_{i=0}^{k}||\frac{\partial^k f^{pol}}{\partial r^i \partial \alpha^{k-i}}||_{L^\infty},$$

and similarly

$$||f||_{C^{m,\beta}} \leq C_{r_0,m,\beta}||f^{pol}||_{C^{m,\beta}}. \tag{3.6}$$

with

$$\begin{aligned}||f^{pol}(r,\alpha)||_{C^{m,\beta}} &= ||f^{pol}||_{C^m} \\ &+ \sum_{i=0}^{k}\text{sup}_\Omega \frac{|\frac{\partial^m f^{pol}}{\partial^i r \partial^{m-i}\alpha}(R,A) - \frac{\partial^m f}{\partial^i r\partial^{m-i}\alpha}(R+h_1, A+h_2)}{|h_1^2+h_2^2|^{\frac{\beta}{2}}}.\end{aligned}$$

where $\Omega := \{R, \in [0,\infty], A \in [0,2\pi], h_1 \in [-R,\infty], h_2 \in [-\pi,\pi]\}$

Furthermore, if we restrict ourselves to functions such that

$$\text{supp}(f^{pol}(r,\alpha)) \subset \{(r,\alpha) : r_1 \geq r \geq r_0\}$$

with $r_1 > r_0 > 0$ then for $m = 0, 1, ...$

$$||f||_{H^m} \leq C_{r_1,r_0,m}||f^{pol}||_{H^m},$$

with

$$||f^{pol}||_{H^m} = \sum_{k=0}^{m}\sum_{i=0}^{k}||\frac{\partial^k f^{pol}}{\partial r^i \partial \alpha^{k-i}}||_{L^2}.$$

Since we will need to compute integrals in polar coordinates, for a general set S we will use the notation

$$S^{pol} := \{(r,\alpha) : F(r,\alpha) \in S\}$$

and more specifically, we will use

$$B^{pol}_\lambda(R,A) := \{(r,\alpha) : |F(r,\alpha) - F(R,A)| \leq \lambda\}$$

with $|(x_1,x_2)| = |x_1^2 + x_2^2|^{\frac{1}{2}}$ (this is simply the set $B_\lambda(R\cos(A), R\sin(A))$ in polar coordinates). Also, note that, for $R \geq 2\lambda$ (which we will assume from now on) we have

$$B^{pol}_\lambda(R,A) \subset [R-\lambda, R+\lambda] \times [A - \arccos(1-\frac{\lambda^2}{R^2}), A + \arccos(1-\frac{\lambda^2}{R^2})].$$

We also define, for $h \in [-\lambda,\lambda]$,

$$S_{\lambda,R,A}(h) := \text{sup}(\tilde{\alpha} : (R+h, A+\tilde{\alpha}) \in B^{pol}_\lambda(R,A))$$

and defining

$$S_{\lambda,R,A,\infty} := \text{sup}_{h\in[-\lambda,\lambda]}(S_{\lambda,R,A}(h))$$

then for $\tilde{\alpha} \in [-S_{\lambda,R,A,\infty}, S_{\lambda,R,A,\infty}]$ we can define

$$P_{\lambda,R,A,+}(\tilde{\alpha}) := \text{sup}(h : (R+h, A+\tilde{\alpha}) \in B^{pol}_\lambda(R,A))$$

$$P_{\lambda,R,A,-}(\tilde{\alpha}) := \inf(h : (R+h, A+\tilde{\alpha}) \in B^{pol}_{\lambda}(R,A)).$$

When the values of λ, R and A are clear by context, we will just write $S(h), S_{\infty}, P_{+}(\tilde{\alpha})$ and $P_{-}(\tilde{\alpha})$. A property for $P_{+}(\tilde{\alpha})$ and $P_{-}(\tilde{\alpha})$ that we will need to use later on is that, for $R \in [\frac{1}{2}, \frac{3}{2}]$ and $\tilde{\alpha} \in [-S_{\lambda,R,A,\infty}, S_{\lambda,R,A,\infty}]$ we have

$$|P_{\lambda,R,A,+}(\tilde{\alpha}) + P_{\lambda,R,A,-}(\tilde{\alpha})| \leq C\lambda^2.$$

Which can be easily obtained using that, since

$$|F(R,A) - F(r,\alpha)| = |(R-r)^2 + 2Rr(1-\cos(A-\alpha))|^{\frac{1}{2}}$$

then

$$\begin{aligned} P_{\lambda,R,A,+}(\tilde{\alpha}) &= -R(1-\cos(\tilde{\alpha})) \\ &+ \frac{\sqrt{(2R(1-\cos(\tilde{\alpha}))^2 - 4(2R^2(1-\cos(\tilde{\alpha})) - \lambda^2)}}{2} \end{aligned}$$

$$\begin{aligned} P_{\lambda,R,A,-}(\tilde{\alpha}) &= -R(1-\cos(\tilde{\alpha})) \\ &- \frac{\sqrt{(2R(1-\cos(\tilde{\alpha}))^2 - 4(2R^2(1-\cos(\tilde{\alpha})) - \lambda^2)}}{2}, \end{aligned}$$

so

$$|P_{\lambda,R,A,+}(\tilde{\alpha}) + P_{\lambda,R,A,-}(\tilde{\alpha})| = 4R(1-\cos(\tilde{\alpha})) \leq C\tilde{\alpha}^2 \leq C\lambda^2.$$

3.2.2 Other notation

Given two sets $X, Y \subset \mathbb{R}^2$, we will use $d(X,Y)$ to refer to the distance between the two, that is

$$d(X,Y) := \inf_{x\in X, y\in Y} |x-y| = \inf_{x\in X, y\in Y} |(x_1-y_1)^2 + (x_2-y_2)^2|^{\frac{1}{2}}.$$

Furthermore, given a function f and a set X we define $d(f,X)$ as

$$d(\text{supp}(f), X).$$

Also, given a set X and a point x we define the set

$$X - x := \{y \in \mathbb{R}^2 : y + x \in X\}.$$

Working in polar coordinates, we will use the notation

$$X^{pol} - (r,\alpha) := \{(\tilde{r},\tilde{\alpha}) \in \mathbb{R}^2 : (\tilde{r}+r, \tilde{\alpha}+\alpha) \in X^{pol}\},$$

where we need to be careful since $X^{pol} - (r,\alpha) \neq (X - F(r,\alpha))^{pol}$.

We will also define, for A a regular enough set, $k \in \mathbb{N}$

$$||f(x)1_A||_{C^k} := \sum_{i=0}^{k}\sum_{j=0}^{i} \text{ess-sup}_{x\in A}(\frac{\partial^i f(x)}{\partial^j x_1 \partial^{i-j} x_2}),$$

$$||f(x)1_A||_{H^k} := \sum_{i=0}^{s}\sum_{j=0}^{i} (\int_A (\frac{\partial^i f(x)}{\partial^j x_1 \partial^{i-j} x_2})^2 dx)^{\frac{1}{2}}.$$

Finally we will use the notation

$$|f|_{C^{k,\beta}} := \sum_{i=0}^{k} \sup_{h_1,h_2\in\mathbb{R}} \frac{|\frac{\partial^k f}{\partial^i x_1 \partial^{k-i} \partial x_2}(y_1,y_2) - \frac{\partial^k f}{\partial^i x_1 \partial^{k-i} \partial x_2}(y_1+h_1, y_2+h_2)|}{|h_1^2 + h_2^2|^{\frac{\beta}{2}}}.$$

3.2.3 The velocity

We will be considering γ-SQG, so our scalar w will be transported with a velocity given by

$$v_{\gamma}(w(.))(x)=C(\gamma)P.V.\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}w(y)}{|x-y|^{3+\gamma}}dy_1dy_2.$$

Since the results are independent of the specific value of $C(\gamma)$, we will just assume $C(\gamma)=1$. Furthermore we will use the notation

$$v_{1,\gamma}(w(.))(x)=v_{\gamma}\cdot(1,0)=P.V.\int_{\mathbb{R}^2}\frac{(y_2-x_2)w(y)}{|x-y|^{3+\gamma}}dy_1dy_2,$$
$$v_{2,\gamma}(w(.))(x)=v_{\gamma}\cdot(0,1)=P.V.\int_{\mathbb{R}^2}\frac{(x_1-y_1)w(y)}{|x-y|^{3+\gamma}}dy_1dy_2.$$

The operators $v_{\gamma}, v_{1,\gamma}$ and $v_{2,\gamma}$ have several useful properties that we will be using later, namely the fact that they commute with cartesian derivatives $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ (as long as w is regular enough) and also that, for $i=1,2$

$$||v_{i,\gamma}(w)||_{H^k}\leq C_{k,\gamma}||w||_{H^{k+\gamma}}.$$

It is unclear (and in fact, untrue) whether these properties translate to the operators $v_{r,\gamma}$ and $v_{\alpha,\gamma}$ that give us the velocity in the radial and polar direction respectively. We can obtain, however, similar properties for these operators.

We start by noting that

$$v_{r,\gamma}(w)=\cos(\alpha(x))v_{1,\gamma}(w)+\sin(\alpha(x))v_{2,\gamma}(w), \tag{3.7}$$

$$v_{\alpha,\gamma}(w)=\cos(\alpha(x))v_{2,\gamma}(w)-\sin(\alpha(x))v_{1,\gamma}(w),$$

and since $\cos(\alpha(x))$ and $\sin(\alpha(x))$ are C^{∞} if we are not close to $r=0$, we have that, for $m\in\mathbb{Z}$

$$||v_{r,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^m}\leq C_m(||v_{1,\gamma}(w)||_{H^m}+||v_{2,\gamma}(w)||_{H^m})\leq C_{m,\gamma}||w||_{H^{m+\gamma}},$$

$$||v_{\alpha,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^m}\leq C_m(||v_{1,\gamma}(w)||_{H^m}+||v_{2,\gamma}(w)||_{H^m})\leq C_{m,\gamma}||w||_{H^{m+\gamma}}.$$

Furhtermore, if we differentiate with respect to $\frac{\partial}{\partial x_i}$, $i=1,2$ we get

$$\frac{\partial v_{r,\gamma}(w)}{\partial x_i}=v_{r,\gamma}(\frac{\partial w}{\partial x_i})+\frac{\partial\cos(\alpha(x))}{\partial x_i}v_{1,\gamma}(w)+\frac{\partial\sin(\alpha(x))}{\partial x_i}v_{2,\gamma}(w),$$
$$\frac{\partial v_{\alpha,\gamma}(w)}{\partial x_i}=v_{\alpha,\gamma}(\frac{\partial w}{\partial x_i})+\frac{\partial\cos(\alpha(x))}{\partial x_i}v_{2,\gamma}(w)-\frac{\partial\sin(\alpha(x))}{\partial x_i}v_{1,\gamma}(w).$$

With this, using induction and if we only consider $|x|\geq\frac{1}{2}$ we get that, for $m_1,m_2\in\mathbb{Z}$

$$|\frac{\partial^{m_1+m_2}v_{r,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}(x)-v_{r,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}})(x)|$$
$$\leq C\sum_{k=0}^{m_1+m_2-1}\sum_{j=0}^{k}|\frac{\partial^k v_{1,\gamma}(w)}{\partial x_1^j\partial x_2^{k-j}}(x)|+|\frac{\partial^k v_{2,\gamma}(w)}{\partial x_1^j\partial x_2^{k-j}}(x)|,$$

$$|\frac{\partial^{m_1+m_2}v_{\alpha,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}(x)-v_{\alpha,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}})(x)|$$

$$\leq C\sum_{k=0}^{m_1+m_2-1}\sum_{j=0}^{k}|\frac{\partial^k v_{1,\gamma}(w)}{\partial x_1^j\partial x_2^{k-j}}(x)|+|\frac{\partial^k v_{2,\gamma}(w)}{\partial x_1^j\partial x_2^{k-j}}(x)|,$$

and thus

$$\begin{aligned}
&||(\frac{\partial^{m_1+m_2}v_{r,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}-v_{r,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}}))1_{|x|\geq\frac{1}{2}}||_{L^\infty} \\
&\leq C(||v_{1,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{C^{m_1+m_2-1}}+||v_{2,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{C^{m_1+m_2-1}}), \\
&||(\frac{\partial^{m_1+m_2}v_{r,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}-v_{r,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}}))1_{|x|\geq\frac{1}{2}}||_{L^2} \\
&\leq C(||v_{1,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^{m_1+m_2-1}}+||v_{2,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^{m_1+m_2-1}})
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&||(\frac{\partial^{m_1+m_2}v_{\alpha,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}-v_{\alpha,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}}))1_{|x|\geq\frac{1}{2}}||_{L^\infty} \\
&\leq C(||v_{1,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{C^{m_1+m_2-1}}+||v_{2,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{C^{m_1+m_2-1}}); \\
&||(\frac{\partial^{m_1+m_2}v_{\alpha,\gamma}(w)}{\partial x_1^{m_1}\partial x_2^{m_2}}-v_{\alpha,\gamma}(\frac{\partial^{m_1+m_2}w}{\partial x_1^{m_1}\partial x_2^{m_2}}))1_{|x|\geq\frac{1}{2}}||_{L^2} \\
&\leq C(||v_{1,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^{m_1+m_2-1}}+||v_{2,\gamma}(w)1_{|x|\geq\frac{1}{2}}||_{H^{m_1+m_2-1}})
\end{aligned} \tag{3.9}$$

with C depending on m_1 and m_2.

3.3 Bounds for the velocity

Since we will work in polar coordinates, it will be necessary to obtain expressions for the velocity in the radial and angular direction. These expressions are, assuming $w(x)$ is a C^1 function with compact support and $\gamma\in(0,1)$

$$v^{pol}_{r,\gamma}(w)(r,\alpha)=$$
$$\int_{[-r,\infty]\times[-\pi,\pi]}\frac{(r+h)^2\sin(\alpha')(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}}d\alpha' dh$$

$$v^{pol}_{\alpha,\gamma}(w)(r,\alpha)=$$
$$\int_{[-r,\infty]\times[-\pi,\pi]}\frac{(r+h)(r-(r+h)\cos(\alpha'))(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}}d\alpha' dh.$$

These expressions, however, hide some cancellation of the kernel when we are far from the support of w. Therefore, given a C^1 function w with support in $B_\lambda(R\cos(A),R\sin(A))$, $\frac{3}{2}>R>\frac{1}{2}$, $\lambda\leq\frac{1}{100}$ we will use the expressions

$$v^{pol}_{r,\gamma}(w)(r,\alpha)=$$
$$\int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)}\frac{(r+h)^2\sin(\alpha')(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}}d\alpha' dh$$

$$v^{pol}_{\alpha,\gamma}(w)(r,\alpha)=$$

$$\int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)} \frac{(r+h)(r-(r+h)\cos(\alpha'))(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}} d\alpha' dh$$

when $(r,\alpha) \in B_{2\lambda}(R,A)$ and

$$v^{pol}_{r,\gamma}(w)(r,\alpha) = \int_{supp(w^{pol})-(r,\alpha)} \frac{(r+h)^2 \sin(\alpha') w^{pol}(r+h,\alpha'+\alpha)}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}} d\alpha' dh$$

$$v^{pol}_{\alpha,\gamma}(w)(r,\alpha) =$$
$$\int_{supp(w^{pol})-(r,\alpha)} \frac{(r+h)(r-(r+h)\cos(\alpha'))w^{pol}(r+h,\alpha'+\alpha)}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}} d\alpha' dh$$

when $(r,\alpha) \notin B_{2\lambda}(R,A)$.

Although the expression for $B^{pol}_{4\lambda}(r,\alpha)$ is not simple, it will be enough for our computations to use the properties we obtained in subsection 3.2.1.

We are particularly interested in obtaining the velocity produced by w with support very concentrated around some point far from $r=0$ (say $r=1$ for simplicity), and for this we start with the following technical lemma.

Lemma 3.3.1. *Given $\lambda \leq \frac{1}{100}$, and a C^1 function $w(x)$ with supp$(w) \subset B_\lambda(\cos(c),\sin(c))$, $c \in \mathbb{R}$, we have that if $(r,\alpha) \in B_{2\lambda}(\cos(c),\sin(c))$ then*

$$|v^{pol}_{r,\gamma}(w)(r,\alpha) - \int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)} \frac{r^2\alpha'(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|$$
$$\leq C||w||_{L^\infty}\lambda^{1-\gamma},$$

$$|v^{pol}_{\alpha,\gamma}(w)(r,\alpha) + \int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)} \frac{rh(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|$$
$$\leq C||w||_{L^\infty}\lambda^{1-\gamma},$$

with C depending on γ.

Remark 7. The result can be extended to functions with support concentrated around a point (r,α) with $r \neq 0$, although then the constant will depend on the specific value of r.

Proof. This result is very similar to Lemma 2.2.1, and the proof is analogous. We just need to take successive approximations of the kernel and bound the error produced by each such approximation. For example, for $(r,\alpha) \in B_{2\lambda}(\cos(c),\sin(c))$ we have that

$$|\int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)} \frac{(r+h)^2(\sin(\alpha')-\alpha')(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}} d\alpha' dh|$$
$$\leq |\int_{B^{pol}_{4\lambda}(r,\alpha)-(r,\alpha)} (r+h)^2 \frac{|\alpha'|^3(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+2r(r+h)(1-\cos(\alpha'))|^{(3+\gamma)/2}} d\alpha' dh|$$
$$\leq C\lambda^{2-\gamma}||w||_{L^\infty}$$

and thus we can substitute the $\sin(\alpha'-\alpha)$ by $\alpha'-\alpha$ with an error small enough for our bounds. Repeating this process for other parts of the kernel yields the desired result. □

Lemma 3.3.2. *Given a natural number N, $\frac{1}{2} > \delta > 0$ fulfilling $N^{-\delta} \leq \frac{1}{100}$ and $N^{-1+\delta} < \frac{1}{100}$, a function $f_{N,\delta}(x)$ with supp$(f_{N,\delta}) \subset B_{N^{-1+\delta}}(\cos(c_1),\sin(c_1))$ ($c_1 \in \mathbb{R}$), $||f^{pol}_{N,\delta}||_{C^j} \leq MN^{j(1-\delta)}$ for $j=0,1,2$ and $1>\gamma>0$, then if $w^{pol}_{N,\delta}(r,\alpha) := f^{pol}_{N,\delta}(r,\alpha)\cos(N\alpha+c_2)$ ($c_2 \in \mathbb{R}$) we have that for $(r,\alpha) \in B^{pol}_{2N^{-1+\delta}}(1,c_1)$*

$$\Big|\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{r^2\alpha'(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh$$
$$-f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{r^2\alpha'(\cos(N(\alpha'+\alpha)+c_2)-\cos(N\alpha+c_2))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh\Big|$$
$$\leq CMN^{\gamma-\delta}$$

$$\Big|\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(w^{pol}(r+h,\alpha'+\alpha)-w^{pol}(r,\alpha))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh \tag{3.10}$$
$$-f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(\cos(N(\alpha'+\alpha)+c_2)-\cos(N\alpha+c_2))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh\Big|$$
$$\leq CMN^{\gamma-\delta}$$

with C depending on γ and δ.

Proof. We will just consider the case $c_1, c_2 = 0$ for simplicity, and we will focus on obtaining (3.10), the other inequality being analogous. We need to find bounds for

$$\Big|\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(w^{pol}_{N,\delta}(r+h,\alpha'+\alpha)-f^{pol}_{N,\delta}(r,\alpha)\cos(N(\alpha'+\alpha)))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh\Big|$$
$$=\Big|\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-S(h)}^{S(h)} \frac{rh(f^{pol}_{N,\delta}(r+h,\alpha'+\alpha)-f^{pol}_{N,\delta}(r,\alpha))\cos(N(\alpha'+\alpha))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh\Big|$$
$$=\Big|\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-rS(s_2)}^{rS(s_2)} \frac{s_2(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))\cos(N(\frac{s_1}{r}+\alpha))}{|s|^{3+\gamma}}ds_1 ds_2\Big|$$

where we used the change of variables $s_1 = r(\alpha'-\alpha)$, $h = s_2$ and we define $|s| := |s_1^2+s_2^2|^{\frac{1}{2}}$. Furthermore,

$$\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-rS(s_2)}^{rS(s_2)} \frac{s_2\cos(N(\frac{s_1}{r}+\alpha))(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1 ds_2$$

$$=\cos(N\alpha)\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-rS(s_2)}^{rS(s_2)} \frac{s_2\cos(\frac{N}{r}s_1)(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1 ds_2$$

$$-\sin(N\alpha)\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-rS(s_2)}^{rS(s_2)} \frac{s_2\sin(\frac{N}{r}s_1)(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1 ds_2.$$

We will only check the term that is multiplied by $\cos(N\alpha)$, the other term being analogous. We start with the contribution when $(s_1,s_2)\in\mathcal{A} := \{|s_j| \leq \frac{4\pi r}{N} \text{ with } j=1,2\}$, which gives us

$$|\int_{\mathcal{A}} \frac{s_2\cos(\frac{N}{r}s_1)(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1 ds_2|$$
$$\leq CMN^{\gamma-\delta}.$$

Next we consider the integral in

$$\mathcal{B} := \{(s_1,s_2) : (s_1,s_2)\in B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha), |s_1| \leq \lfloor\frac{S(s_2)N}{2\pi}\rfloor\frac{2\pi r}{N}\}\setminus\mathcal{A},$$

with $\lfloor\cdot\rfloor$ the integer part.

We will focus on the contribution when $(s_1,s_2)\in\mathcal{B}\cap(s_1\geq\frac{4\pi r}{N},s_2\geq 0)$, since the other parts of the integral are bounded analogously. We start by computing the integral with respect to s_1.

For this we first note that, for an integer i, given a C^2 function $g(x)$ and a real number $\frac{N}{r}>0$ we have

$$|\int_{i\frac{2\pi r}{N}}^{(i+1)\frac{2\pi r}{N}}\cos(\frac{N}{r}x)g(x)dx|\leq(\frac{\pi r}{N})^3(sup_{x\in(i\frac{2\pi r}{N},(i+1)\frac{2\pi r}{N})}|g''(x)|)$$

where $g''(x)$ is the second derivative of $g(x)$. This bound is obtained simply by considering a second order Taylor expansion around the middle point of the interval and noting that the constant and linear terms vanish. Therefore, if $i\geq 2$, $s_2>0$

$$\begin{aligned}&\left|\int_{i\frac{2\pi r}{N}}^{(i+1)\frac{2\pi r}{N}}\frac{\cos(\frac{N}{r}s_1)(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1\right|\\&\leq(\frac{2\pi r}{N})^3(sup_{s_1\in(i\frac{2\pi r}{N},(i+1)\frac{2\pi r}{N})}\left|\frac{d^2}{ds_1^2}\frac{f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha)}{|s|^{3+\gamma}}\right|)\\&\leq CM(\frac{2\pi r}{N})^3\frac{1}{((\frac{i2\pi r}{N})^2+s_2^2)^{\frac{3+\gamma}{2}}}\\&\times\Big(N^{2-2\delta}+\frac{N^{1-\delta}}{((\frac{i2\pi r}{N})^2+s_2^2)^{\frac{1}{2}}}+\frac{N^{1-\delta}[(i+1)\frac{2\pi}{N}+s_2]}{((\frac{i2\pi r}{N})^2+s_2^2)}\Big)\\&\leq CM(\frac{2\pi r}{N})^3\frac{1}{(\frac{i2\pi r}{N}+s_2)^{3+\gamma}}\Big(N^{2-2\delta}+\frac{N^{1-\delta}}{(\frac{i2\pi r}{N}+s_2)}\Big).\end{aligned}$$

Adding over all the relevant values of i we get

$$\begin{aligned}&\sum_{i=2}^{\lfloor\frac{S(s_2)N}{2\pi}\rfloor}CM(\frac{2\pi r}{N})^3\frac{1}{(\frac{i2\pi r}{N}+s_2)^{3+\gamma}}\Big(N^{2-2\delta}+\frac{N^{1-\delta}}{\frac{i2\pi r}{N}+s_2}\Big)\\&\leq\int_1^\infty CM(\frac{2\pi r}{N})^3\frac{1}{(\frac{x2\pi r}{N}+s_2)^{3+\gamma}}\Big(N^{2-2\delta}+\frac{N^{1-\delta}}{\frac{x2\pi r}{N}+s_2}\Big)dx\\&\leq\frac{CM}{N^{2\delta}(\frac{2\pi r}{N}+s_2)^{2+\gamma}}+\frac{CM}{N^{1+\delta}(\frac{2\pi r}{N}+s_2)^{3+\gamma}},\end{aligned}$$

and multiplying by s_2 and integrating with respect to s_2 we obtain

$$\int_0^{4N^{-1+\delta}}s_2(\frac{CM}{N^{2\delta}(\frac{2\pi r}{N}+s_2)^{2+\gamma}}+\frac{CM}{N^{1+\delta}(\frac{2\pi r}{N}+s_2)^{3+\gamma}})ds_2\leq CMN^{\gamma-\delta}.$$

Finally, we need to bound the integral when

$$(s_1,s_2)\in\mathcal{C}:=B_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)\setminus(\mathcal{A}\cup\mathcal{B}).$$

For this we only need to use that in this set $|s|\geq 3N^{-1+\delta}$ and that

$$|\int_{[-rS(r),rS(r)]\setminus[-\lfloor\frac{S(r)N}{2\pi}\rfloor\frac{2\pi r}{N},\lfloor\frac{S(r)N}{2\pi}\rfloor\frac{2\pi r}{N}]}ds_1|\leq 2\frac{2\pi r}{N},$$

which gives that

$$|\int_{\mathcal{C}}\frac{s_2\cos(\frac{N}{r}s_1)(f^{pol}_{N,\delta}(r+s_2,\frac{s_1}{r}+\alpha)-f^{pol}_{N,\delta}(r,\alpha))}{|s|^{3+\gamma}}ds_1ds_2|$$

$$\leq |\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}} \frac{C|s_2|M}{N|N^{-1+\delta}|^{3+\gamma}} ds_2| \leq CMN^{\gamma-\delta-\delta\gamma}.$$

□

Lemma 3.3.3. *Given $\frac{1}{2} > \delta > 0$ and $1 > \gamma > 0$, for any natural number N fulfilling $N^{-\delta} \leq \frac{1}{100}$ and $N^{-1+\delta} < \frac{1}{100}$, a function $f_{N,\delta}(x)$ with $supp(f_{N,\delta}) \subset B_{N^{-1+\delta}}(\cos(c_1), \sin(c_1))$ ($c_1 \in \mathbb{R}$), $||f^{pol}_{N,\delta}||_{C^j} \leq MN^{j(1-\delta)}$ for $j = 0, 1, 2$ then we have that if $w^{pol}_{N,\delta}(r,\alpha) := f^{pol}_{N,\delta}(r,\alpha)\cos(N\alpha + c_2)$ ($c_2 \in \mathbb{R}$) there exist constants C, C_γ such that for $(r,\alpha) \in B_{2N^{-1+\delta}}(1, c_1)$*

$$|v^{pol}_{r,\gamma}(w_{N,\delta})(r,\alpha) - N^\gamma f^{pol}_{N,\delta}(r,\alpha) C_\gamma \sin(N\alpha + c_2)| \leq CMN^{\gamma-\delta},$$

$$|v^{pol}_{\alpha,\gamma}(w_{N,\delta})(r,\alpha)| \leq CMN^{\gamma-\delta},$$

with $C_\gamma \neq 0$ depending on γ and C depending on γ and δ.

Proof. Using Lemmas 3.3.1 and 3.3.2 yields

$$\begin{aligned}&|v^{pol}_{r,\gamma}(w_{N,\delta})(r,\alpha)\\ &- f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{r^2\alpha'(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|\\ &\leq CMN^{\gamma-\delta},\end{aligned}$$

$$\begin{aligned}&|v^{pol}_{\alpha,\gamma}(w_{N,\delta})(r,\alpha)\\ &- f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|\\ &\leq CMN^{\gamma-\delta},\end{aligned}$$

and therefore it is enough to prove

$$\begin{aligned}&|f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{r^2\alpha'(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh\\ &- N^\gamma C_\gamma \sin(N\alpha + c_2)| \leq CMN^{\gamma-\delta},\end{aligned} \tag{3.11}$$

$$\begin{aligned}&|f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|\\ &\leq CMN^{\gamma-\delta}.\end{aligned} \tag{3.12}$$

We start with (3.12), where by using the odd symmetry of the integrand with respect to h

$$\begin{aligned}&|f^{pol}_{N,\delta}(r,\alpha)\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)} \frac{rh(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} d\alpha' dh|\\ &= |f^{pol}_{N,\delta}(r,\alpha)\int_{-S_\infty}^{S_\infty}\int_{P_-(\alpha')}^{P_+(\alpha')} \frac{rh(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} dh d\alpha'|\\ &= |f^{pol}_{N,\delta}(r,\alpha)\int_{-S_\infty}^{S_\infty}\int_{P_-(\alpha')}^{-P_+(\alpha')} \frac{rh(\cos(N(\alpha'+\alpha)+c_2) - \cos(N\alpha + c_2))}{|h^2 + r^2(\alpha')^2|^{(3+\gamma)/2}} dh d\alpha'|\end{aligned}$$

$$\leq |M\int_{-S_\infty}^{S_\infty}\frac{CN^{-2+2\delta}}{N^{(-1+\delta)(2+\gamma)}}d\alpha'| \leq CMN^{(-1+\delta)(1-\gamma)} \leq CMN^{\gamma-\delta}$$

where we used that $|P_+(\alpha')+P_-(\alpha')| \leq CN^{-2+2\delta}$, $|S_\infty| \leq \arccos(1-16\frac{N^{-2+2\delta}}{r^2}) \leq CN^{-1+\delta}$ and that, for $h \in [P_-(\alpha'), -P_+(\alpha')]$

$$\frac{1}{|h^2+r^2(\alpha')^2|^{(2+\gamma)/2}} \leq \frac{C}{N^{(-1+\delta)(2+\gamma)}}.$$

For (3.11) we use

$$\begin{aligned}
&\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)}\frac{r^2\alpha'(\cos(N(\alpha'+\alpha)+c_2)-\cos(N\alpha+c_2))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha' dh\\
&=-\sin(N\alpha+c_2)\int_{-4N^{-1+\delta}}^{4N^{-1+\delta}}\int_{-rS(h_2)}^{rS(h_2)}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2\\
&=-\sin(N\alpha+c_2)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2\\
&+4\sin(N\alpha+c_2)\int_0^\infty\int_{r\tilde{S}(h_2)}^\infty\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2
\end{aligned}$$

where we just take

$$\tilde{S}(h)=\begin{cases}S(h), & \text{if } h\in[-4N^{-1+\delta},4N^{-1+\delta}]\\ 0 & \text{otherwise.}\end{cases}$$

But, we have that, for i a natural number,

$$\begin{aligned}
&|\int_{i\frac{2\pi r}{N}+r\tilde{S}(h_2)}^{(i+1)\frac{2\pi r}{N}+r\tilde{S}(h_2)}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1|\\
&\leq\frac{C}{N^2}\frac{1}{|(i\frac{2\pi r}{N}+r\tilde{S}(h_2))^2+h_2^2|^{(3+\gamma)/2}}
\end{aligned}$$

and thus

$$\begin{aligned}
&|\int_{r\tilde{S}(h_2)}^\infty\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1|\\
&\leq\sum_{i=0}^\infty\frac{C}{N^2}\frac{1}{|(i\frac{2\pi r}{N}+r\tilde{S}(h_2))^2+h_2^2|^{(3+\gamma)/2}}\\
&\leq\int_{-1}^\infty\frac{C}{N^2}\frac{1}{|x\frac{2\pi r}{N}+r\tilde{S}(h_2)+h_2|^{(3+\gamma)}}dx\\
&\leq\frac{C}{N|-\frac{2\pi r}{N}+r\tilde{S}(h_2)+h_2|^{(2+\gamma)}}\leq\frac{C}{N|r\tilde{S}(h_2)+h_2|^{(2+\gamma)}}
\end{aligned}$$

where we used for $h_2>0$, $r\geq\frac{1}{2}$ we have $r\tilde{S}(h_2)+h_2\geq CN^{-1+\delta}$. But then

$$\begin{aligned}
&|4\sin(N\alpha+c_2)\int_0^{N^{-1+\delta}}\int_{r\tilde{S}(h_2)}^\infty\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2|\\
&\leq\int_0^{N^{-1+\delta}}\frac{C}{N^{1+(-1+\delta)(2+\gamma)}}dh_2=CN^{\gamma-\delta-\delta\gamma}
\end{aligned}$$

and

$$|4\sin(N\alpha+c_2)\int_{N^{-1+\delta}}^{\infty}\int_{r\tilde{S}(h_2)}^{\infty}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2|$$
$$\leq|\int_{N^{-1+\delta}}^{\infty}\frac{C}{N|h_2|^{(2+\gamma)}}dh_2|\leq CN^{\gamma-\delta-\gamma\delta},$$

and therefore

$$|\int_{B^{pol}_{4N^{-1+\delta}}(r,\alpha)-(r,\alpha)}\frac{r^2\alpha'(\cos(N(\alpha'+\alpha)+c_2)-\cos(N\alpha+c_2))}{|h^2+r^2(\alpha')^2|^{(3+\gamma)/2}}d\alpha'dh$$
$$+\sin(N\alpha+c_2)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2|\leq CN^{\gamma-\delta-\gamma\delta}$$

and combined with (3.11) we get

$$|v^{pol}_{\alpha,\gamma}(w_{N,\delta})(r,\alpha)+f^{pol}_{N,\delta}(r,\alpha)\sin(N\alpha+c_2)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{h_1\sin(N\frac{h_1}{r})}{|h_1^2+h_2^2|^{(3+\gamma)/2}}dh_1dh_2|$$
$$\leq CMN^{\gamma-\delta}.$$

Furthermore

$$-\sin(N\alpha+c_2)\int_{\mathbb{R}^2}\frac{h_1\sin(\frac{N}{r}h_1)}{|h_1^2+h_2^2|^{\frac{3+\gamma}{2}}}dh_1dh_2$$
$$=-\sin(N\alpha+c_2)\left(\frac{N}{r}\right)^{\gamma}\int_{\mathbb{R}^2}\frac{h_1\sin(h_1)}{|h_1^2+h_2^2|^{\frac{3+\gamma}{2}}}dh_1dh_2,$$

and

$$\int_{\mathbb{R}^2}\frac{h_1\sin(h_1)}{|h_1^2+h_2^2|^{\frac{3+\gamma}{2}}}dh_1dh_2=\int_{-\infty}^{\infty}h_1\sin(h_1)\int_{-\infty}^{\infty}\frac{1}{(h_1^2+h_2^2)^{\frac{(3+\gamma)}{2}}}dh_2dh_1$$
$$=\int_{-\infty}^{\infty}\frac{h_1\sin(h_1)}{|h_1|^{2+\gamma}}\int_{-\infty}^{\infty}\frac{1}{(1+\lambda^2)^{\frac{(3+\gamma)}{2}}}d\lambda dh_1=K_{\gamma}2\int_{0}^{\infty}\frac{h_1\sin(h_1)}{|h_1|^{2+\gamma}}dh_1.$$

By using that $\frac{h_1}{|h_1|^{2+\gamma}}$ is monotone decreasing for $h_1>0$, $\sin(x+\pi)=-\sin(x)$, $\sin(x)>0$ if $x\in(0,\pi)$ and $K_{\gamma}>0$ we obtain

$$C_{\gamma}:=-K_{\gamma}2\int_{0}^{\infty}\frac{h_1\sin(h_1)}{|h_1|^{2+\gamma}}<0.$$

Thus

$$|v^{pol}_{r,\gamma}(w_{N,\delta})(r,\alpha)-f^{pol}_{N,\delta}(r,\alpha)\left(\frac{N}{r}\right)^{\gamma}C_{\gamma}\sin(N\alpha+c_2)|\leq CMN^{\gamma-\delta},$$

and since, for the values of r considered we have

$$|\left(\frac{N}{r}\right)^{\gamma}-N^{\gamma}|\leq CN^{\gamma-1+\delta}\leq CN^{\gamma-\delta}$$

we are done.

$\square$

Lemma 3.3.4. *Given $0<\delta<\frac{1}{2}$, $0<\gamma<1$, a natural number N such that $N^{-1+\delta}\leq\frac{1}{100}$ and a C^2 function $f_{N,\delta}$, satisfying*

$$supp(f_{N,\delta})\subset B_{N^{-1+\delta}}(\cos(c_1),\sin(c_1))$$

($c_1\in\mathbb{R}$) with $||f_{N,\delta}||_{C^j}\leq MN^{j(1-\delta)}$, $j=0,1,2$, then for any $x=(x_1,x_2)=(R\cos(A),R\sin(A))\in\mathbb{R}^2\setminus B_{2N^{-1+\delta}}(\cos(c_1),\sin(c_1))$ we have that

$$|v^{pol}_{r,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta},$$

$$|v^{pol}_{\alpha,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta}$$

with C depending only on γ.

Furthermore, if $f_{N,\delta}\in C^{k+2}$ for k an integer $k\geq 1$ and $||f_{N,\delta}(r,\alpha)||_{C^j}\leq MN^{j(1-\delta)}$ for $j=0,1,...,k$ then we have

$$|\frac{\partial^j v^{pol}_{r,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)}{\partial x_1^l\partial x_2^{j-l}}|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j},$$

$$|\frac{\partial^j v^{pol}_{\alpha,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)}{\partial x_1^l\partial x_2^{j-l}}|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j}$$

for $j=0,1,...,k+2$, $l=0,1,...,j$, with C depending on γ and j.

Proof. We will consider $c_1=0$ for simplicity and we will obtain the expression only for $v_{r,\gamma}$, $v_{\alpha,\gamma}$ being equivalent. That is to say, we want to compute

$$\begin{aligned}
&\int_{\text{supp}(f^{pol}_{N,\delta})}\frac{(r')^2\sin(\alpha'-A)f_{N,\delta}(r',\alpha')\sin(N\alpha')}{|(R-r')^2+2Rr'(1-\cos(A-\alpha'))|^{(3+\gamma)/2}}d\alpha'dr'\\
&=\cos(NA)\int_{\text{supp}(f^{pol}_{N,\delta})}\frac{(r')^2\sin(\alpha'-A)f_{N,\delta}(r',\alpha')\sin(N\alpha'-NA)}{|(R-r')^2+2Rr'(1-\cos(A-\alpha'))|^{(3+\gamma)/2}}d\alpha'dr'\\
&+\sin(NA)\int_{\text{supp}(f^{pol}_{N,\delta})}\frac{(r')^2\sin(\alpha'-A)f_{N,\delta}(r',\alpha')\cos(N\alpha'-NA)}{|(R-r')^2+2Rr'(1-\cos(A-\alpha'))|^{(3+\gamma)/2}}d\alpha'dr'\\
&=\cos(NA)\int_{\text{supp}(f^{pol}_{N,\delta})-(0,A)}\frac{(r')^2\sin(\bar\alpha)f_{N,\delta}(r',\bar\alpha+A)\sin(N\bar\alpha)}{|(R-r')^2+2Rr'(1-\cos(\bar\alpha))|^{(3+\gamma)/2}}d\bar\alpha dr'\\
&+\sin(NA)\int_{\text{supp}(f^{pol}_{N,\delta})-(0,A)}\frac{(r')^2\sin(\bar\alpha)f_{N,\delta}(r',\bar\alpha+A)\cos(N\bar\alpha)}{|(R-r')^2+2Rr'(1-\cos(\bar\alpha))|^{(3+\gamma)/2}}d\bar\alpha dr'
\end{aligned}$$

with f, R and A as in the hypothesis of the lemma. We will focus on the part depending on $cos(NA)$, the other term being analogous. First, a second order Taylor expansion and some computations give us, since $r'\in(\frac{1}{2},\frac{3}{2})$

$$\begin{aligned}
&|\int_{i\frac{2\pi}{N}+\frac{\pi}{2N}}^{(i+1)\frac{2\pi}{N}+\frac{\pi}{2N}}\frac{(r')^2\sin(\bar\alpha)f_{N,\delta}(r',\bar\alpha+A)\sin(N\bar\alpha)}{|(R-r')^2+2Rr'(1-\cos(\bar\alpha))|^{(3+\gamma)/2}}d\bar\alpha|\\
&\leq\int_{i\frac{2\pi}{N}+\frac{\pi}{2N}}^{(i+1)\frac{2\pi}{N}+\frac{\pi}{2N}}\Big(\frac{2\pi}{N}\Big)^2|\sin(N\bar\alpha)|\\
&\times\sup_{\bar\alpha\in[i\frac{2\pi}{N}+\frac{\pi}{2N},(i+1)\frac{2\pi}{N}+\frac{\pi}{2N}]}\Big(\Big|\frac{\partial^2}{\partial\bar\alpha^2}\frac{(r')^2\sin(\bar\alpha)f_{N,\delta}(r',\bar\alpha+A)}{|(R-r')^2+2Rr'(1-\cos(\bar\alpha))|^{(3+\gamma)/2}}\Big|\Big)d\bar\alpha\\
&\leq C\Big(\frac{2\pi}{N}\Big)^3\Big(\frac{||f_{N,\delta}(r,\alpha)||_{C^2}}{|(R-r')^2+2Rr'(1-\cos(\bar\alpha))|^{(2+\gamma)/2}}
\end{aligned}$$

$$+\frac{||f_{N,\delta}(r,\alpha)||_{C^1}}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(3+\gamma)/2}}$$
$$+\frac{||f_{N,\delta}(r,\alpha)||_{L^\infty}}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(4+\gamma)/2}}\Big).$$

Using that, for $(r',\bar{\alpha}) \in \text{supp}(f^{pol}_{N,\delta}) - (0,A)$

$$|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{\frac{1}{2}} \geq d((R,A),f_{N,\delta})$$

$$d((R,A),f_{N,\delta}) \geq N^{-1+\delta}$$

and the properties of $f_{N,\delta}$ we get then that

$$\begin{aligned}&|\int_{i\frac{2\pi}{N}+\frac{\pi}{2N}}^{(i+1)\frac{2\pi}{N}+\frac{\pi}{2N}} \frac{(r')^2\sin(\bar{\alpha})f_{N,\delta}(r',\bar{\alpha}+A)\sin(N\bar{\alpha})}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(3+\gamma)/2}}d\bar{\alpha}|\\ &\leq \frac{CMN^{-1-2\delta}}{d((R,A),f_{N,\delta})^{2+\gamma}},\end{aligned}$$

so that

$$\begin{aligned}&\Big|\int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}}\int_{-\lfloor\frac{S(r')N}{2\pi}\rfloor\frac{2\pi}{N}+\frac{\pi}{2N}}^{\lfloor\frac{S(r')N}{2\pi}\rfloor\frac{2\pi}{N}-\frac{3\pi}{2N}} \frac{(r')^2\sin(\bar{\alpha})f_{N,\delta}(r',\bar{\alpha}+A)\sin(N\bar{\alpha})}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(3+\gamma)/2}}d\bar{\alpha}dr'\Big|\\ &\leq \int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}} \frac{CMN^{-1-\delta}}{d((R,A),f_{N,\delta})^{2+\gamma}}dr' \leq \frac{CM}{N^2 d((R,A),f_{N,\delta})^{2+\gamma}}.\end{aligned}$$

As for the rest of the integral we have

$$\begin{aligned}&\Big|\int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}}\int_{\lfloor\frac{S(r')N}{2\pi}\rfloor\frac{2\pi}{N}-\frac{3\pi}{2N}}^{S(r')} \frac{(r')^2\sin(\bar{\alpha})f_{N,\delta}(r',\bar{\alpha}+A)\sin(N\bar{\alpha})}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(3+\gamma)/2}}d\bar{\alpha}dr'\Big|\\ &\leq \int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}} \frac{CMN^{-1}}{d((R,A),f_{N,\delta})^{2+\gamma}}dr' \leq \frac{CMN^{\delta}}{N^2 d((R,A),f_{N,\delta})^{2+\gamma}},\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}}\int_{-S(r')}^{-\lfloor\frac{S(r')N}{2\pi}\rfloor\frac{2\pi}{N}+\frac{\pi}{2N}} \frac{(r')^2\sin(\bar{\alpha})f_{N,\delta}(r',\bar{\alpha}+A)\sin(N\bar{\alpha})}{|(R-r')^2+2Rr'(1-\cos(\bar{\alpha}))|^{(3+\gamma)/2}}d\bar{\alpha}dr'\Big|\\ &\leq \int_{1-N^{-1+\delta}}^{1+N^{-1+\delta}} \frac{CMN^{-1}}{d((R,A),f_{N,\delta})^{2+\gamma}}dr' \leq \frac{CMN^{\delta}}{N^2 d((R,A),f_{N,\delta})^{2+\gamma}}\end{aligned}$$

and we are done.

To obtain the result for the derivatives, we first note that since

$$\begin{aligned}v_{1,\gamma}(w) &= \cos(\alpha)v_{r,\gamma}(w)-\sin(\alpha)v_{\alpha,\gamma}(w)\\ v_{2,\gamma}(w) &= \sin(\alpha)v_{r,\gamma}(w)+\cos(\alpha)v_{\alpha,\gamma}(w)\end{aligned} \tag{3.13}$$

then for $x=(x_1,x_2)=(R\cos(A),R\sin(A))$

$$|v^{pol}_{1,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta},$$

$$|v^{pol}_{2,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta}.$$

Furthermore, derivation commutes with the operators $v_{1,\gamma}$ and $v_{2,\gamma}$, so we can prove that

$$|\frac{\partial^j v^{pol}_{1,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)}{\partial x_1^l\partial x_2^{j-l}}|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j},$$

$$|\frac{\partial^j v^{pol}_{2,\gamma}(f_{N,\delta}(r,\alpha)\sin(N\alpha))(R,A)}{\partial x_1^l\partial x_2^{j-l}}|\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j}$$

by differentiating $f_{N,\delta}(r,\alpha)\sin(N\alpha)$ and applying our lemma for each individual term.

Then, using (3.7) and computing $\frac{\partial^j}{\partial x_1^l\partial x_2^{j-l}}v_{r,\gamma}$, $\frac{\partial^j}{\partial x_1^l\partial x_2^{j-l}}v_{\alpha,\gamma}$ we obtain, for $r\geq\frac{1}{2}$ that

$$\begin{aligned}&\frac{\partial^j}{\partial x_1^l\partial x_2^{j-l}}v_{r,\gamma}(R,A)\leq C\Big(\sum_{k=0}^{j}\sum_{l=0}^{k}|\frac{\partial^k v_{1,\gamma}}{\partial x_1^l\partial x_2^{k-l}}(R,A)|+|\frac{\partial^k v_{2,\gamma}}{\partial x_1^l\partial x_2^{k-l}}(R,A)|\Big)\\&\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j},\end{aligned}$$

$$\begin{aligned}&\frac{\partial^j}{\partial x_1^l\partial x_2^{j-l}}v_{\alpha,\gamma}(R,A)\leq C\Big(\sum_{k=0}^{j}\sum_{l=0}^{k}|\frac{\partial^k v_{1,\gamma}}{\partial x_1^l\partial x_2^{k-l}}(R,A)|+|\frac{\partial^k v_{2,\gamma}}{\partial x_1^l\partial x_2^{k-l}}(R,A)|\Big)\\&\leq C\frac{M}{|d(x,f_{N,\delta})|^{2+\gamma}}N^{-2+\delta+j},\end{aligned}$$

and we are done.

□

3.4 Pseudo-solutions considered and their properties

To obtain ill-posedness for the space $C^{k,\beta}$ for γ-SQG, we will add perturbations to a radial solution $f(r)$ (with $f(r)$ chosen so that it has some specific properties). These perturbations will be of the form

$$\lambda\sum_{l=0}^{L-1}f(N^{1-\delta}(r-1),N^{1-\delta}\alpha)\frac{\cos(N(M+l)(\alpha-\alpha^1)+\alpha^2+\frac{k\pi}{2})}{L(NM)^{k+\beta}},\tag{3.14}$$

with

- $f(r-1,\alpha)=g(r-1)g(\alpha)$, g a positive C^∞ function with support in $[-\frac{1}{2},\frac{1}{2}]$ and such that $f(x)=1$ if $x\in[-\frac{1}{4},\frac{1}{4}]$ and $||f(r-1,\alpha)||_{C^j}\leq 100^j$,
- $M,N,\lambda>0$, $\delta\in(0,\frac{1}{2})$, $L\in\mathbb{N}$ and $\alpha^1,\alpha^2\in\mathbb{R}$,
- $N^\delta\geq 100$, $N^{1-\delta}\geq 100$,
- $k\in\mathbb{N}$, $\beta\in(0,1]$, $\gamma\in(0,1)$,

- $k+\beta > 1+2\delta+\gamma$,
- $L < \frac{M}{2}$.

For compactness of notation, whenever we have $f, \delta, N, L, M, \lambda$ satisfying these properties we will say that they satisfy the usual conditions. From now on we will consider k, β, γ and δ fixed satisfying these properties, just so that we can avoid extra sub-indexes for these parameters. Due to this, one needs to keep in mind that in general the constants in the lemmas obtained might depend on the specific values of k, β, γ and δ. Before we study how this kind of perturbations will evolve with time, we start by obtaining some basic properties regarding the norms of (3.14).

Lemma 3.4.1. *Given a perturbation as in (3.14), which we will refer as $w_{k,\beta}$, with $f, \delta, N, L, M, \lambda$ satisfying the usual conditions we have that*

$$||w_{k,\beta}||_{C^j} \leq CK_j\lambda(NM)^{j-k-\beta}$$

$$|\frac{\partial^k w_{k,\beta}(r,\alpha)}{\partial^{k-i}x_1\partial^i x_2}| \leq \frac{C\lambda}{L|\sin(N\frac{\alpha-\alpha^1}{2})|(NM)^\beta} + C\lambda(NM)^{-\delta-\beta} + \frac{C\lambda L}{M(NM)^\beta}$$
$$|\frac{\partial^{k+1} w_{k,\beta}(r,\alpha)}{\partial^{k+1-i}x_1\partial^i x_2}| \leq \frac{C\lambda(NM)^{1-\beta}}{L|\sin(N\frac{\alpha-\alpha^1}{2})|} + C\lambda(NM)^{-\delta-\beta+1} + \frac{C\lambda(NM)^{1-\beta}L}{M}$$

with C a constant depending on f and K_j constants depending on j.

Proof. The bounds for the C^j norms can be obtained directly by using that, for functions with support concentrated around $r=1$, we have that

$$||f(x_1,x_2)||_{C^j} \leq K_j||f^{pol}(r,\alpha)||_{C^j}$$

and the bounds for the derivatives of $w^{pol}_{k,\beta}$ can be obtained by direct computation. For the other two inequalities, we have that

$$|\frac{\partial^k w_{k,\beta}(r,\alpha)}{\partial^{k-i}x_1\partial^i x_2}| \leq K_k||w^{pol}_{k,\beta}(r,\alpha)||_{C^k} \leq C\lambda(NM)^{-\beta-\delta}$$
$$+ C\lambda|\sum_{l=0}^{L-1} f(N^{1-\delta}(r-1), N^{1-\delta}\alpha)\frac{\partial^k}{\partial\alpha^k}(\frac{\cos(N(M+l)(\alpha-\alpha^1)+\alpha^2+\frac{k\pi}{2})}{L(NM)^{k+\beta}})|$$
$$\leq C\lambda(NM)^{-\beta-\delta} + \frac{C\lambda L}{M(NM)^\beta}$$
$$+ C\lambda|\sum_{l=0}^{L-1} f(N^{1-\delta}(r-1), N^{1-\delta}\alpha)\frac{\cos(N(M+l)(\alpha-\alpha^1)+\alpha^2)}{L(NM)^\beta}|$$

and we can compute $\sum_{l=0}^{L-1}\cos(N(M+l)(\alpha-\alpha^1)+\alpha^2)$ as

$$\sum_{l=0}^{L-1}\cos(N(M+l)(\alpha-\alpha^1)+\alpha^2)$$
$$= \frac{\sin(\frac{NL(\alpha-\alpha^1)}{2})}{\sin(N\frac{\alpha-\alpha^1}{2})}\cos(NM(\alpha-\alpha^1)+\alpha^2+\frac{N(L-1)(\alpha-\alpha^1)}{2})$$

which gives us

$$|\frac{\partial^k w_{k,\beta}(r,\alpha)}{\partial^{k-i}x_1\partial^i x_2}| \leq C\lambda(NM)^{-\beta-\delta} + \frac{C\lambda L}{M(NM)^\beta} + \frac{C\lambda}{|\sin(N\frac{\alpha-\alpha^1}{2})|L(NM)^\beta}.$$

The proof with $k+1$ derivatives is done analogously. □

This lemma tells us that these perturbations behave similarly to wave packets, with their amplitude and derivatives decreasing as one gets further from $\alpha^1 + j\frac{2\pi}{N}$. We will use this property to obtain upper bounds for the norms of these perturbations when several of them are placed appropriately far way from each other. For this, we first we need a short technical lemma.

Lemma 3.4.2. *Given a C^1 function $f(x):\mathbb{R}\to\mathbb{R}$ with $||f(x)||_{L^\infty}\leq M_1$ and $||f'(x)||_{L^\infty}\leq M_2$, we have that, for any $x,h\in\mathbb{R}$, $\beta\in(0,1)$*

$$\frac{|f(x)-f(x+h)|}{|h|^\beta}\leq 2^{1-\beta}M_1^{1-\beta}M_2^\beta.$$

Proof. We have the two trivial bounds

$$\frac{|f(x)-f(x+h)|}{|h|^\beta}\leq \frac{2M_1}{|h|^\beta},$$

$$\frac{|f(x)-f(x+h)|}{|h|^\beta}\leq \frac{|h|M_2}{|h|^\beta},$$

and thus it is enough to find a bound for

$$\sup_{h\in\mathbb{R}}(\min(\frac{2M_1}{|h|^\beta},\frac{|h|M_2}{|h|^\beta})).$$

But it is easy to see that the supremum is attained when $\frac{2M_1}{|h|^\beta}=\frac{|h|M_2}{|h|^\beta}$. Since this happens when $|h|=\frac{2M_1}{M_2}$, substituting $|h|$ in any of the upper bounds gives us

$$\frac{|f(x)-f(x+h)|}{|h|^\beta}\leq \frac{2M_1}{\left(\frac{2M_1}{M_2}\right)^\beta}=(2M_1)^{1-\beta}M_2^\beta.$$

□

Now we are ready to prove decay in space of the functions that we use as perturbations.

Lemma 3.4.3. *Given a function $g(x)$ of the form*

$$g^{pol}(r,\alpha)=\sum_{j=1}^{J}\lambda_j\sum_{l=0}^{L-1}f(N^{1-\delta}(r-1),N^{1-\delta}\alpha)\frac{\cos(N(M_j+l)(\alpha-\alpha_j^1)+\alpha_j^2+\frac{k\pi}{2})}{JL(NM_j)^{k+\beta}}$$

where $f,\delta,N,L,M_j,\lambda_j$ satisfy the usual conditions and with $\alpha_j^1\in[c\frac{\pi}{N},\frac{\pi}{N}]$ and $|\alpha_{j_1}^1-\alpha_{j_2}^1|\geq c\frac{\pi}{N}$ for some $c>0$ and $\frac{M_{j_1}}{M_{j_2}}\leq 2$ for $j_1,j_2\in\{1,2,...,J\}$ then we have that

$$|g|_{C^{k,\beta}}\leq C\bar{\lambda}(\frac{1}{J}+\frac{1}{(N\bar{M})^\delta}+\frac{L}{\bar{M}}+\frac{1}{cL})$$

with C depending on k, β and δ and where $\bar{M}:=sup_{j=1,...,J}(M_j)$, $\bar{\lambda}:=sup_{j=1,...,J}(\lambda_j)$.

Proof. We will compute bounds for the seminorm $|\cdot|_{C^\alpha}$ of an arbitrary k-th derivative of g, and we will refer to it simply as $g^{(k)}(x)$ since the specific derivative we consider is irrelevant for the proof and we will use d^k as notation for the specific k-th derivative for the same reason. We start by obtaining bounds for $||g^{(k)}||_{L^\infty}$. Since $|\alpha_{j_1}^1-\alpha_{j_2}^1|\geq c\frac{\pi}{N}$ and $\alpha_j^1\in[c\frac{\pi}{N},\frac{\pi}{N}]$, we have that for any α there is at most one j with

$$\min_{n\in\mathbb{Z}}|\alpha-\alpha_j^1-\frac{\pi n}{N}|<\frac{c\pi}{2N}. \tag{3.15}$$

For simplicity, assume that $j=1$ fulfils (3.15) (the proof when other values of j or no value of j fulfil (3.15) is equivalent).

Then, using Lemma 3.4.1 we obtain

$$
\begin{aligned}
&|g^{(k)}(r,\alpha)| \leq \\
&\leq C|\lambda_1 \sum_{l=0}^{L-1} d^k\big(f(N^{1-\delta}(r-1), N^{1-\delta}\alpha)\frac{\cos(N(M_1+l)(\alpha-\alpha_j^1)+\alpha_1^2+\frac{k\pi}{2})}{JL(NM_1)^{k+\beta}}\big)| \\
&+ C|\sum_{j=2}^{J}\lambda_j \sum_{l=0}^{L-1} d^k\big(f(N^{1-\delta}(r-1), N^{1-\delta}\alpha)\frac{\cos(N(M_j+l)(\alpha-\alpha_j^1)+\alpha_j^2+\frac{k\pi}{2})}{JL(NM_j)^{k+\beta}}\big)| \\
&\leq \frac{C\bar{\lambda}}{J(N\bar{M})^{\beta}} + \frac{C\bar{\lambda}}{(N\bar{M})^{\beta+\delta}} + \frac{C\bar{\lambda}L}{\bar{M}(N\bar{M})^{\beta}} + \frac{C\bar{\lambda}}{|\sin(\frac{c\pi}{2})|L(N\bar{M})^{\beta}} \\
&\leq \frac{C\bar{\lambda}}{J(N\bar{M})^{\beta}} + \frac{C\bar{\lambda}}{(N\bar{M})^{\beta+\delta}} + \frac{C\bar{\lambda}L}{\bar{M}(N\bar{M})^{\beta}} + \frac{C\bar{\lambda}}{cL(N\bar{M})^{\beta}} \\
&= \frac{C\bar{\lambda}}{(\bar{M}N)^{\beta}}(\frac{1}{J} + \frac{1}{(N\bar{M})^{\delta}} + \frac{L}{\bar{M}} + \frac{1}{cL}).
\end{aligned}
$$

Arguing the same way for any arbitrary $k+1$ derivative we obtain

$$
\begin{aligned}
&|g^{(k+1)}(r,\alpha)| \\
&\leq N\bar{M}\frac{C\bar{\lambda}}{(\bar{M}N)^{\beta}}(\frac{1}{J} + \frac{1}{(N\bar{M})^{\delta}} + \frac{L}{\bar{M}} + \frac{1}{cL}).
\end{aligned}
$$

and then a direct application of Lemma 3.4.2 gives us

$$
\frac{g^{(k)}(x)-g^{(k)}(x+h)}{|h|^{\beta}} \leq C\bar{\lambda}(\frac{1}{J} + \frac{1}{(N\bar{M})^{\delta}} + \frac{L}{\bar{M}} + \frac{1}{cL}).
$$

□

With this out of the way, we are ready to define the pseudo-solutions that we will use to prove ill-posedness. Namely, we define

$$
\begin{aligned}
&\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t) := \lambda_0 f_1(r) \\
&+\sum_{j=1}^{J}\sum_{l=0}^{L-1}\left(\lambda_j f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1))) \times \frac{\cos(\Xi_{j,l})}{JL(NM_j)^{k+\beta}}\right)
\end{aligned}
\tag{3.16}
$$

with

$$
\Xi_{j,l} := N(M_j+l)(\alpha-\alpha_j^1 - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)) + \alpha_j^2 + \frac{k\pi}{2} + t\lambda_0 C_\gamma N^{\gamma}(M_j+l)^{\gamma}
$$

$$
M_j = M(1+\frac{j}{J}),\ \lambda_0 = \frac{\pi M^{1-\gamma}}{2\tilde{t}N^{\gamma}C_\gamma\gamma},\ \lambda_j = \lambda(1+\frac{j}{J})^{\beta} \text{ for } j=1,...,J,
$$

$$
\alpha_j^1 = \frac{\pi}{2N}(1+\frac{j}{J})^{-1+\gamma} - \tilde{t}\lambda_0 v_{\alpha,\gamma}(f_1)(r=1),\ \alpha_j^2 = -(\frac{1}{\gamma}-1)\frac{\pi}{2}M(1+\frac{j}{J})^{\gamma}.
$$

The functions $f_{1,\gamma}(r)$ and $f_2(r-1,\alpha)$ and the values $k,\beta,\gamma,\delta,\lambda,N,M,J,L$ and $\tilde{t}$ will fulfil the following properties:

- $\lambda,N,M,J,L,\tilde{t}>0$, $\delta\in(0,\frac{1}{2}),\gamma\in(0,1)$ and $L,J,M\in\mathbb{N}$, $\frac{M}{J}\in\mathbb{N}$,

- $f_2(r-1,\alpha)=g(r-1)g(\alpha)$, g a positive C^∞ function with support in $[-\frac{1}{2},\frac{1}{2}]$ and such that $f(x)=1$ if $x\in[-\frac{1}{4},\frac{1}{4}]$ and $||f_2(r-1,\alpha)||_{C^j}\leq 100^j$,
- $N^\delta\geq 100$, $N^{1-\delta}\geq 100$, $\lambda_0\leq 1$ (i.e. $N^\gamma\geq\frac{\pi M^{1-\gamma}}{2\tilde{t}C_\gamma\gamma}$),
- $k\in\mathbb{N}$, $\beta\in(0,1]$, $\gamma\in(0,1)$,
- $k+\beta>1+2\delta+\gamma$,
- $L<\frac{M}{2}$,
- $\frac{\partial^i\frac{v^{pol}_{r,\gamma}(f_1)}{r}}{\partial r^i}(r=1)=0$ for $i=1,2$,
- $\frac{\partial f_1}{\partial r}=1$ if $r\in[\frac{3}{4},\frac{5}{4}]$,
- $\text{supp}(f_1)\subset\{r:r\in(\frac{1}{2},K_\gamma)\}$ for some K_γ depending only on γ.

As before, to avoid extra sub-indexes we consider k,β,δ and γ to be fixed, but all the results will apply as long as they fulfil the restrictions mentioned. The constants appearing in the lemmas might depend on our specific choice but the final results will not.

However it is not immediately obvious whether the conditions we impose over $f_{1,\gamma}$ are too restrictive, so we need the following lemma to assure us that a $f_{1,\gamma}$ with the desired properties exists.

Lemma 3.4.4. *There exists a C^∞ compactly supported function $g(.):[0,\infty)\to\mathbb{R}$ with support in $(2,\infty)$ such that $\frac{\partial^i\frac{v_{\alpha,\gamma}(g(.))(r)}{r}}{\partial r^i}(r=1)=a_i$ with $i=1,2$ and a_i arbitrary.*

We will omit the proof of this lemma since it is completely equivalent to that of Lemma 2.5 in [39]. With this, the existence of the desired f_1 is easy to prove, since we can just choose some C^∞ $\tilde{f}$ with support in $(\frac{1}{2},2)$ with the desired derivative in $r\in[\frac{3}{4},\frac{5}{4}]$ and then add some other C^∞ function given by Lemma 3.4.4 to cancel out the derivatives of $V_{\alpha,\gamma}$ around $r=1$.

Our next goal will be to prove that this family of pseudo-solutions is a good approximation for our solutions. For this we define $\bar{v}_{r,\gamma}$ as

$$\begin{aligned}&\bar{v}^{pol}_{r,\gamma}(f_2(N^{1-\delta}(r-1),N^{1-\delta}\alpha+c_1)\cos(NK\alpha+c_2))(r,\alpha)\\ &:=(NK)^\gamma C_\gamma f_2(N^{1-\delta}(r-1),N^{1-\delta}\alpha+c_1)\sin(NK\alpha+c_2),\end{aligned}$$

$$\bar{v}_{r,\gamma}(f(r))=0.$$

We will only use this definition for ease of notation and we will only apply this operator to our pseudo-solution so we do not have to worry about defining this for a more general function.

With this, the evolution equation for $\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}$is

$$\begin{aligned}&\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial t}\\ &=-v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1)\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}-\lambda_0\bar{v}^{pol}_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})\\ &=-v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1)\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}-\frac{\partial\lambda_0 f_1(r)}{\partial r}\bar{v}^{pol}_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})\end{aligned}$$

while on the other hand, if $w_{\lambda,N,M,J,L,\tilde{t}}$ is the solution to γ-SQG with the same initial conditions as $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$ then

$$\frac{\partial w^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial t}$$

$$= -\frac{v^{pol}_{\alpha,\gamma}(w^{pol}_{\lambda,N,M,J,L,\tilde{t}})}{r}\frac{\partial w^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha} - \frac{\partial w^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial r}v^{pol}_{r,\gamma}(w^{pol}_{\lambda,N,M,J,L,\tilde{t}})$$

and we can rewrite the evolution equation of $w^{pol}_{\lambda,N,M,J,L,\tilde{t}}$ in pseudo-solution form as

$$\frac{\partial \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial t}$$
$$= -\frac{v^{pol}_{\alpha,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})}{r}\frac{\partial \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha} - \frac{\partial \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial r}v_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})$$
$$- F^{pol}_{\lambda,N,M,J,L,\tilde{t}}$$

with

$$F^{pol}_{\lambda,N,M,J,L,\tilde{t}} = F^{pol}_1 + F^{pol}_2 + F^{pol}_3 + F^{pol}_4,$$

$$F^{pol}_1 := \frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1 - \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})}{r}\frac{\partial \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha},$$

$$F^{pol}_2 := (v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1) - \frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r})\frac{\partial \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha},$$

$$F^{pol}_3 := \frac{\partial(\lambda_0 f_1(r) - \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})}{\partial r}v_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}),$$

$$F^{pol}_4 := \frac{\partial \lambda_0 f_1(r)}{\partial r}\big(\bar{v}_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}) - v_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})\big).$$

The next step in our proof will be to show that $F_{\lambda,N,M,J,L,\tilde{t}}$ can be made as small as we need by choosing appropriately the parameters, namely we will show that it becomes small as we make N big.

Before we get to prove that, there are some basic properties of $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$ that we will need later on

- $$||\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)||_{C^{m,\beta'}} \leq C_1 + C_2\lambda(NM)^{m+\beta'-k-\beta}$$
  $$||\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t) - \lambda_0 f_1(r)||_{C^{m,\beta'}} \leq C_2\lambda(NM)^{m+\beta'-k-\beta}$$
  for any $m \in \mathbb{N}$, $\beta' \in [0,1]$, $t \in \mathbb{R}$, with C_1 and C_2 depending on m and β'.

- $$||\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)||_{H^m} \leq C_1 + C_2\lambda N^{-1+\delta}(NM)^{m-k-\beta}$$
  $$||\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t) - \lambda_0 f_1(r)||_{H^m} \leq C_2\lambda N^{-1+\delta}(NM)^{m-k-\beta}$$
  for any $m \in \mathbb{N}$, $\beta' \in [0,1]$, $t \in \mathbb{R}$, with C_1 and C_2 depending on m.

- $$||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t)||_{C^{m,\beta'}} \leq C_1 + C_2\lambda(NM)^{m+\beta'-k-\beta}$$
  $$||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t) - \lambda_0 f_1(\sqrt{x_1^2+x_2^2})||_{C^{m,\beta'}} \leq C_2\lambda(NM)^{m+\beta'-k-\beta}$$
  for any $m \in \mathbb{N}$, $\beta' \in [0,1]$, $t \in \mathbb{R}$, with C_1 and C_2 depending on m and β'.

- $$||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t)||_{H^m} \leq C_1 + C_2\lambda N^{-1+\delta}(NM)^{m-k-\beta}$$
$$||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t) - \lambda_0 f_1(\sqrt{x_1^2+x_2^2})||_{H^m} \leq C_2\lambda N^{-1+\delta}(NM)^{m-k-\beta}$$
for any $m \in \mathbb{N}$, $\beta' \in [0,1]$, $t \in \mathbb{R}$, with C_1 and C_2 depending on m.
- By using the interpolation inequality for sobolev spaces we also have
$$||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t) - \lambda_0 f_1(x_1^2+x_2^2)||_{H^m} \leq C_1\lambda N^{-1+\delta}(NM)^{m-k-\beta}$$
for any $m > 0$, $t \in \mathbb{R}$, with C_1 depending on m.

The bounds in polar coordinates are obtained by direct calculation and then we obtain from those the ones in cartesian coordinates using that the functions are compactly supported and with support far from the origin. Now, for our pseudo-solutions to be a useful approximation of the solution to γ-SQG, we need the source term to be small. For that we have the following lemmas.

Lemma 3.4.5. *For any fixed T, if $0 \leq t \leq T$ we have that*
$$||F_{\lambda,N,M,J,L,\tilde{t}}||_{L^2} \leq (1+\frac{1}{\tilde{t}})\frac{C}{N^{k+\beta+1}}$$
with C depending on T, λ, M, J and L.

Furthermore, for $m \in \mathbb{N}$, we have that
$$||F_{\lambda,N,M,J,L,\tilde{t}}||_{H^m} \leq C(1+\frac{1}{\tilde{t}})\frac{N^m}{N^{k+\beta+1}}$$
with C_m depending on T, λ, M, J, L and m. In fact, by interpolation, the inequality also holds for any $m > 0$.

Proof. We start by obtaining bounds for $||F_1||_{L^2}$. We have that
$$\begin{aligned}
&||F_1||_{L^2} \leq ||v_{\alpha,\gamma}(\lambda_0 f_1 - \bar{w}_{\lambda,N,M,J,L,\tilde{t}})1_{|x|\geq\frac{1}{2}}||_{L^2}||\frac{1}{r}\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}||_{L^\infty}\\
&\leq C||\lambda_0 f_1 - \bar{w}_{\lambda,N,M,J,L,\tilde{t}}||_{H^\gamma}||\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}||_{L^\infty}\\
&\leq C\frac{N^\gamma}{N^{k+\beta+1-\delta}}\frac{1}{N^{k+\beta-1}} \leq C\frac{1}{N^{k+\beta+1}}.
\end{aligned}$$

For F_2, using that the first two derivatives with respect to r of $\frac{v_{\alpha,\gamma}(\lambda_0 f_1)}{r}$ vanish at $r=1$ plus the fact that it is a radial function, we have that if $x \in \mathrm{supp}(\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha})$ then
$$|v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1) - \frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r}| \leq CN^{-3+3\delta},$$
and so
$$||F_2^{pol}||_{L^2} \leq C\lambda_0 N^{-3+3\delta}||\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}||_{L^2} \leq C\frac{N^{-3+4\delta}}{N^{k+\beta}} \leq \frac{C}{N^{k+\beta+1}}.$$

Similarly, for F_3 we have
$$||F_3||_{L^2} \leq ||\frac{\partial(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}} - \lambda_0 f_1(r))}{\partial r}||_{L^\infty}||v_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})1_{|x|\geq\frac{1}{2}}||_{L^2}$$

$$\leq ||\frac{\partial(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}-\lambda_0 f_1(r))}{\partial r}||_{L^\infty}||v_{r,\gamma}(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}-\lambda_0 f_1(r))||_{H^\gamma}$$
$$\leq C\frac{N^{1-\delta}}{N^{k+\beta}}\frac{N^\gamma}{N^{k+\beta+1-\delta}}\leq C\frac{1}{N^{k+\beta+1}}.$$

Finally, for F_4, we go back to cartesian coordinates and divide the integral in two different parts,

$$A_1 := B_{2N^{-1+\delta}}(\cos(t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)),\sin(t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)))$$
$$A_2 := \text{supp}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})\setminus A_1$$

we have

$$||F_4||_{L^2}$$
$$\leq ||\frac{\partial\lambda_0 f_1}{\partial r}(\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{A_1}||_{L^2}$$
$$+||\frac{\partial\lambda_0 f_1}{\partial r}(\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{A_2}||_{L^2}.$$

For the bound on A_1, using Lemma 3.3.3 and $|\frac{\partial f_1}{\partial r}|\leq C$ we get

$$||\frac{\partial\lambda_0 f_1(r)}{\partial r}(\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{A_1}||_{L^2}$$
$$\leq ||\lambda_0 f_1(r)||_{C^1}||\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{A_1}||_{L^\infty}|A_1|^{\frac{1}{2}}$$
$$\leq C\lambda_0\frac{N^{\gamma-\delta}}{N^{k+\beta+1-\delta}}=C\frac{1}{\tilde{t}N^{k+\beta+1}}$$

where we used that $\lambda_0=\frac{CN^{-\gamma}}{\tilde{t}}$ (the constant C depending on M).

For the integral in A_2 using Lemma 3.3.4 and the bounds on f_1 we have

$$\Big(\int_{A_2}(\frac{\partial\lambda_0 f_1}{\partial r}(\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})))^2dx_1dx_2\Big)^{\frac{1}{2}}$$
$$\leq \tilde{t}^{-1}CN^{-\gamma}\Big(\int_{A_2}(v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))^2dx_1dx_2\Big)^{\frac{1}{2}}$$
$$\leq \tilde{t}^{-1}\frac{C}{N^{k+\beta+\gamma}}\Big(\int_{2N^{-1+\delta}}^{\infty}\Big(N^{-2+\delta}\frac{C}{h^{2+\gamma}}\Big)^2hdh\Big)^{\frac{1}{2}}$$
$$\leq\frac{C}{\tilde{t}N^{k+\beta+1}}$$

For the proof for the bound in H^m, we use the that, since

$$\text{supp}(w^{pol}_{\lambda,N,M,J,L,\tilde{t}})\subset\{(r,\alpha):r\in[\frac{1}{2},K]\}$$

for some K, then

$$||w_{\lambda,N,M,J,L,\tilde{t}}||_{H^m}\leq||w^{pol}_{\lambda,N,M,J,L,\tilde{t}}||_{H^m}$$

and therefore we just need to find bound for

$$\sum_{k=0}^{m}\sum_{j=0}^{k}||\frac{\partial^k F_i}{\partial^j r\partial^{k-j}\alpha}||_{L^2}$$

with $i=1,2,3,4$.

For the bounds in H^m we will use that, given two functions f,g and $m\in\mathbb{Z}$ we have

$$||fg||_{H^m}\leq C\sum_{i=0}^{m}||f||_{C^i}||g||_{H^{m-i}}$$

with C depending on m. Combining this with (3.9) we have

$$\begin{aligned}||F_1||_{H^m}&\leq C\sum_{i=0}^{m}||v_{\alpha,\gamma}(\lambda_0 f_1-\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}})1_{|x|\geq\frac{1}{2}}||_{H^i}||\frac{1}{r}\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}||_{C^{m-i}}\\ &\leq C\sum_{i=0}^{m}\frac{N^iN^{-1+\delta+\gamma}}{N^{k+\beta}}\frac{N^{m-i+1}}{N^{k+\beta}}\leq C\frac{N^m}{N^{k+\beta+1}}.\end{aligned}$$

For F_2, using that, for $r\in B^{pol}:=\text{supp}(\frac{\partial\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha})$ we have that

$$\frac{\partial^i(v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1)-\frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r})}{\partial r^i}\leq CN^{(3-i)(-1+\delta)}$$

for $i=0,1,2$, and since $v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)(r=1)-\frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r}$ only depends on r, then, for $i=0,1,2$

$$||v_{\alpha,\gamma}(\lambda_0 f_1)(r=1)-\frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r}1_{x\in B}||_{C^i}\leq CN^{(3-i)(-1+\delta)}$$

and for higher derivatives we just use

$$||v_{\alpha,\gamma}(\lambda_0 f_1)(r=1)-\frac{v^{pol}_{\alpha,\gamma}(\lambda_0 f_1)}{r}1_{x\in B}||_{C^i}\leq C,$$

where the constant depends on i. With this we get

$$\begin{aligned}||F_2||_{H^m}&\leq C\sum_{i=0}^{m}||v_{\alpha,\gamma}(\lambda_0 f_1)(r=1)-\frac{v_{\alpha,\gamma}(\lambda_0 f_1)}{r}1_{x\in B}||_{C^i}||\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial\alpha}||_{H^{m-i}}\\ &\leq C\frac{N^{-3+4\delta+m}}{\tilde{t}N^{k+\beta}}\leq\frac{CN^m}{\tilde{t}N^{k+\beta+1}}.\end{aligned}$$

For F_3 we have

$$\begin{aligned}||F_3||_{H^m}&\leq C\sum_{i=0}^{m}||\frac{\partial(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}-\lambda_0 f_1(r))}{\partial r}||_{C^i}||v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})||_{H^{m-i}}\\ &\leq C\sum_{i=0}^{m}\frac{N^{i+1}}{N^{k+\beta}}\frac{N^{m-i-1+\delta+\gamma}}{N^{k+\beta}}\leq C\frac{N^m}{N^{k+\beta+1}}.\end{aligned}$$

As for F_4, the contribution obtained when integrating in A_2 is obtained again applying lemma 3.3.4

$$\begin{aligned}&||F_4 1_{A_2}||_{H^m}\\ &\leq C\sum_{i}||\lambda_0 f_1(r)||_{C^{i+1}}||\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})-v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{A_1}||_{C^{m-i}}|A_1|^{\frac{1}{2}}\end{aligned}$$

$$\leq C\lambda_0 \frac{N^{m+\gamma-\delta}}{N^{k+\beta+1-\delta}} = C\frac{N^m}{\tilde{t}N^{k+\beta+1}}.$$

For the contribution when we integrate F_4 over A_1 using (3.8) we have

$$\begin{aligned}
&||F_4 1_{x\in A_1}||_{H^m} \leq C\lambda_0 ||(\bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}) - v_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))1_{x\in A_1}||_{H^m} \\
&\leq C\lambda_0 \sum_{q=0}^{m}\sum_{j=0}^{q} ||\Big(\frac{\partial^q \bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})}{\partial x_1^j \partial x_2^{q-j}} - v_{r,\gamma}(\frac{\partial^q \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_1^j \partial x_2^{q-j}})\Big)1_{x\in A_1}||_{L^2} \\
&+ C\lambda_0 ||v_{1,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})1_{x\in A_1}||_{H^{m-1}} + C\lambda_0 ||v_{2,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})1_{x\in A_1}||_{H^{m-1}} \\
&\leq C\lambda_0 \sum_{q=0}^{m}\sum_{j=0}^{q} ||\Big(\frac{\partial^q \bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})}{\partial x_1^j \partial x_2^{q-j}} - v_{r,\gamma}(\frac{\partial^q \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_1^j \partial x_2^{q-j}})\Big)1_{x\in A_1}||_{L^2} \\
&+ C\frac{N^{-1+\delta}}{\tilde{t}N^{k+\beta}}N^{m-1}.
\end{aligned}$$

But then since

$$\frac{\partial^q f(r,\alpha)}{\partial x_1^j \partial x_2^{q-j}} = \sum_{p=0}^{q}\sum_{l=0}^{p} g_{q,j,p,l}(r,\alpha)\frac{\partial^p f(r,\alpha)}{\partial r^l \partial \alpha^{p-l}}$$

with $g_{m,j,q,l}$ in C^∞ and bounded if $r \geq \frac{1}{2}$, we have that

$$\begin{aligned}
&||\Big(\frac{\partial^q \bar{v}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})}{\partial x_1^j \partial x_2^{q-j}} - v_{r,\gamma}(\frac{\partial^q \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_1^j \partial x_2^{q-j}})\Big)1_{x\in A_1}||_{L^2} \\
&\leq \sum_{p=0}^{q}\sum_{l=0}^{p} ||\Big(g_{q,j,p,l}(r,\alpha)\frac{\partial^p \bar{v}^{pol}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})}{\partial r^l \partial \alpha^{p-l}} \\
&\quad - v^{pol}_{r,\gamma}\big(g_{q,j,p,l}(r,\alpha)\frac{\partial^p \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial r^l \partial \alpha^{p-l}}\big)\Big)1_{(r,\alpha)\in A_1^{pol}}||_{L^2}.
\end{aligned}$$

But applying Lemma 3.3.3 to each of the terms we obtain after differentiating, we get

$$\begin{aligned}
&\leq \sum_{p=0}^{q}\sum_{l=0}^{p} ||\Big(g_{q,j,p,l}(r,\alpha)\frac{\partial^p \bar{v}^{pol}_{r,\gamma}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})}{\partial r^l \partial \alpha^{p-l}} \\
&\quad - v^{pol}_{r,\gamma}\big(g_{q,j,p,l}(r,\alpha)\frac{\partial^p \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial r^l \partial \alpha^{p-l}}\big)\Big)1_{(r,\alpha)\in A_1^{pol}}||_{L^2} \\
&\quad \leq C\frac{N^q N^{\gamma-\delta}N^{-1+\delta}}{N^{k+\beta}},
\end{aligned}$$

and so

$$||F_4 1_{x\in A_1}||_{H^m} \leq C\frac{N^{-1+\delta}}{\tilde{t}N^{k+\beta}}N^{m-1} + C\lambda_0 \sum_{q=0}^{m}\sum_{j=0}^{q}\frac{N^q N^{\gamma-\delta}N^{-1+\delta}}{N^{k+\beta}} \leq \frac{CN^m}{\tilde{t}N^{k+\beta+1}},$$

and we are done.

□

Since we are interested in showing (arbitrarily) fast norm growth for γ-SQG, our solution should start with a very small norm that gets very big after a short period of time. Lemma 2.2.4 already gives us tools to show that the initial norm is small, and the next lemma will gives us a lower bound for the $C^{k,\beta}$ norm of our pseudo-solutions at time $\tilde{t}$.

Lemma 3.4.6. *There exists a set A (depending on λ, N, M, J and L) such that, if $x \in A$ then there exists unitary u depending on x and a constant C with*

$$
\begin{aligned}
&|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}| \\
&\geq \lambda(\frac{1}{2(MN)^{\beta}} - \frac{CL^2}{(NM)^{\beta}M} - C(NM)^{-(\delta+\beta)} - C(NM)^{-\beta}N^{-1+\delta})
\end{aligned}
$$

and a set B (depending on λ, N, M, J and L) such that if $x \in B$ then for all unitary v we have that

$$
|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}| \leq \lambda(\frac{1}{4(MN)^{\beta}} + C(NM)^{-(\delta+\beta)} + \frac{CL^2}{(NM)^{\beta}M})
$$

furthermore, there is a set $S_{M,N,\delta}$ with $|S_{M,N,\delta}| \geq C_1 MN^{2\delta}$,

$$
A = \cup_{s\in S_{M,N,\delta}} A_s,
$$

$$
B = \cup_{s\in S_{M,N,\delta}} B_s,
$$

$d(x,y) \leq \frac{4\pi}{NM}$ if $x \in A_s, y \in B_s$, and $|A_s|, |B_s| \geq \frac{C_2}{(NM)^2}$, with C_1 and C_2 constants.

Note that, in particular

$$
||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t}) - \lambda_0 f_1||_{C^{k,\beta}} \geq \lambda(\frac{1}{4(4\pi)^{\beta}} - \frac{CL^2}{M} - C(NM)^{-\delta} - CN^{-1+\delta})
$$

Proof. We start by finding the set A as well as the unitary vector v that gives us a big $k-th$ derivative.

For this, we first want to obtain accurate estimates for

$$
\frac{\partial^k(\bar{w}^{pol}_{\lambda,N,M,J,L,\delta,t}(r,\alpha,t)-\lambda_0 f_1)}{\partial \alpha^k}
$$

The definition (3.16) yields

$$
\begin{aligned}
&\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)}{\partial \alpha^k} = \frac{\partial^k(\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)-\lambda_0 f_1)}{\partial \alpha^k} \\
&= \sum_{i=0}^{k}\binom{k}{i}\sum_{j=1}^{J}\sum_{l=0}^{L-1}\Big(\frac{1}{JL(NM_j)^{k+\beta}} \\
&\times \frac{\partial^i \lambda_j f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)))}{\partial \alpha^i}\frac{\partial^{k-i}\cos(\Xi_{j,l})}{\partial \alpha^{k-i}}\Big),
\end{aligned}
$$

and so

$$
\begin{aligned}
&|\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)}{\partial \alpha^k} - \sum_{j=1}^{J}\sum_{l=0}^{L-1}\Big(\frac{1}{JL(NM_j)^{k+\beta}} \\
&\times \lambda_j f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)))\frac{\partial^k \cos(\Xi_{j,l})}{\partial \alpha^k}\Big)| \\
&\leq C\lambda(NM)^{-(\delta+\beta)}.
\end{aligned}
$$

Furthermore

$$\frac{\partial^k \cos(\Xi_{j,l})}{\partial\alpha^k}$$
$$= (N(M_j+l))^k \cos(N(M_j+l)(\alpha-\alpha_j^1 - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1))$$
$$+ \alpha_j^2 + t\lambda_0 C_\gamma N^\gamma (M_j+l)^\gamma)$$
$$= (N(M_j+l))^k \cos(N(M_j+l)(\alpha-\alpha_j^1(t)) + \alpha_j^2(t) + \alpha_{j,l}^3(t))$$

with

$$\alpha_j^1(t) := \alpha_j^1 - t\lambda_0 C_\gamma \gamma (NM_j)^{\gamma-1} + t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)$$
$$\alpha_j^2(t) := \alpha_j^2 + (1-\gamma)t\lambda_0 C_\gamma (NM_j)^\gamma$$
$$\alpha_{j,l}^3(t) = t\lambda_0 C_\gamma((N(M_j+l))^\gamma - (NM_j)^\gamma - \gamma l N^\gamma M_j^{\gamma-1}),$$

and we have

$$|\cos(N(M_j+l)(\alpha-\alpha_j^1(t)) + \alpha_j^2(t) + \alpha_{j,l}^3(t))$$
$$-\cos(N(M_j+l)(\alpha-\alpha_j^1(t)) + \alpha_j^2(t))|$$
$$\leq C|\alpha_{j,l}^3(t)| \leq Ct\gamma(1-\gamma)\lambda_0 C_\gamma (NM_j)^\gamma \frac{L^2}{(M_j)^2} = \frac{CtJL^2}{\tilde{t}M_j},$$

so

$$|\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,t)}{\partial\alpha^k}$$
$$-\sum_{j=1}^{J}\sum_{l=0}^{L-1}\Big(\frac{1}{JL(NM)^\beta}\lambda f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha - t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)))$$
$$\cos(N(M_j+l)(\alpha-\alpha_j^1(t)) + \alpha_j^2(t))\Big)| \leq C\lambda(NM)^{-(\delta+\beta)} + \lambda\frac{CtL^2}{\tilde{t}(NM)^\beta M}.$$

But we have that $\alpha_j^1(\tilde{t})=0$, $\alpha_j^2(\tilde{t})=0$, so that if $\alpha = i\frac{2\pi}{NM}$, $i\in\mathbb{Z}$, then

$$|\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}$$
$$-\frac{1}{(NM)^\beta}\lambda f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha-\lambda_0\tilde{t}v_{\alpha,\gamma}(f_1)(r=1)))|$$
$$\leq C\lambda(NM)^{-(\delta+\beta)} + \lambda\frac{CL^2}{(NM)^\beta M},$$

and in fact, if $\alpha \in [i\frac{2\pi}{N} - \frac{\pi}{16NM}, i\frac{2\pi}{N} + \frac{\pi}{16NM}]$ with $i\in\mathbb{Z}$, then

$$\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}$$

$$\geq \frac{1}{2(NM)^{\beta}}\lambda f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha-\lambda_0\tilde{t}v_{\alpha,\gamma}(f_1)(r=1)))$$
$$-C\lambda(NM)^{-(\delta+\beta)}-\lambda\frac{CL^2}{(NM)^{\beta}M}.$$

But since $f(N^{1-\delta}(r-1), N^{1-\delta}(\alpha-t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)))=1$ if $(r,\alpha)\in[1-\frac{N^{-1+\delta}}{4}, 1+\frac{N^{-1+\delta}}{4}]\times[t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)-\frac{N^{-1+\delta}}{4}, t\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)+\frac{N^{-1+\delta}}{4}]$ then defining

$$A^{pol}=\bigcup_{j=-\lfloor\frac{N^{\delta}M}{4}\rfloor}^{j=\lfloor\frac{N^{\delta}M}{4}\rfloor-1}\bigcup_{i=\lfloor\frac{-N^{\delta}}{64}+\frac{Nt\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)}{2\pi}\rfloor}^{i=\lfloor\frac{N^{\delta}}{64}+\frac{Nt\lambda_0 v_{\alpha,\gamma}(f_1)(r=1)}{2\pi}\rfloor}A^{pol}_{i,j}$$

with

$$A_{i,j}:=(1+\frac{j}{NM}, 1+\frac{j+1}{NM}]\times[i\frac{2\pi}{N}-\frac{\pi}{16NM}, i\frac{2\pi}{N}+\frac{\pi}{16NM}]$$

we have that, for $(r,\alpha)\in A^{pol}$,

$$\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}\geq\frac{\lambda}{2(NM)^{\beta}}-C\lambda(NM)^{-(\delta+\beta)}-\lambda\frac{CL^2}{(NM)^{\beta}M}.$$

Furthermore, the sets $A_{i,j}$ fulfil $|A_{i,j}|\geq C(NM)^{-2}$ for some $C>0$. Therefore, if we prove that there exists a unitary vector $u=(u_1,u_2)$ such that, if $x=(r\cos(\alpha), r\sin(\alpha))\in A$

$$\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}\approx\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}$$

in a suitable way, then we are done proving the existence of the desired set A. But

$$\frac{\partial f(x)}{\partial u}=u_1[\cos(\alpha(x))\frac{\partial f^{pol}(r(x),\alpha(x))}{\partial r}-\frac{\sin(\alpha(x))}{r}\frac{\partial f^{pol}(r(x),\alpha(x))}{\partial\alpha}]$$
$$+u_2[\sin(\alpha(x))\frac{\partial f^{pol}(r(x),\alpha(x))}{\partial r}+\frac{\cos(\alpha(x))}{r}\frac{\partial f^{pol}(r(x),\alpha(x))}{\partial\alpha}]$$

so that

$$\frac{\partial^k f(x)}{\partial u^k}=\sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}g_{i_1,i_2}(\alpha,r,u_1,u_2)\frac{\partial f^{pol}(r,\alpha)}{\partial r^{i_2}\partial^{i_1-i_2}\alpha}$$

with g_{i_1,i_2} C^{∞} and bounded as long as we only consider $r\geq\frac{1}{2}$.

Applying this formula to $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$ we get

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}-g_{k,0}(\alpha,r,u_1,u_2)\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}|$$

$$\leq C\lambda(NM)^{-(\delta+\beta)}$$

and it is easy to prove that $g_{k,0} = \frac{g_{k-1,0}(\cos(\alpha)u_2-\sin(\alpha)u_1)}{r}$, $g_{0,0} = 1$ and therefore taking $v = (-\sin(\alpha), \cos(\alpha))$ we get

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t}) - \lambda_0 f_1)}{\partial u^k} - \frac{1}{r^k}\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial \alpha^k}| \leq C\lambda(NM)^{-(\delta+\beta)}$$

and using $r \in A^{pol} \Rightarrow r \in [1 - \frac{N^{-1+\delta}}{4}, 1 + \frac{N^{-1+\delta}}{4}]$ plus the bounds for $w^{pol}_{\lambda,N,M,J,L,\tilde{t}}$ gives

$$\begin{aligned}&|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t}) - \lambda_0 f_1)}{\partial u^k} - \frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial \alpha^k}|\\ &\leq C\lambda(NM)^{-(\delta+\beta)} + C\lambda(NM)^{-\beta}N^{-1+\delta},\end{aligned}$$

and so, for $x \in A$

$$\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})}{\partial u^k} \geq \frac{\lambda}{2(NM)^\beta} - \lambda\frac{CL^2}{(NM)^\beta M} - C\lambda(NM)^{-(\delta+\beta)} - C\lambda(NM)^{-\beta}N^{-1+\delta},$$

which finishes the proof for the existence of the set A. For the set B, we remember that for $r \geq \frac{1}{2}$ we have

$$\begin{aligned}&|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t}) - \lambda_0 f_1)}{\partial u^k} - g_{k,0}(\alpha,r,u_1,u_2)\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial \alpha^k}|\\ &\leq C\lambda(NM)^{-(\delta+\beta)}\end{aligned}$$

and since $|g_{k,0}| \leq \frac{1}{r^k}$ we only need to find a sets $B_{i,j}$ with the desired size and distance to $A_{i,j}$ such that $|\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial \alpha^k}|$ is small. But

$$\begin{aligned}&|\frac{\partial^k \bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial \alpha^k}|\\ &\leq |\sum_{j=1}^{J}\sum_{l=0}^{L-1}(\frac{1}{JL(NM)^\beta}\lambda f_2(N^{1-\delta}(r-1), N^{1-\delta}(\alpha - \lambda_0 \tilde{t} v_{\alpha,\gamma}(f_1)(r=1)))\\ &\cos(N(M_j+l)\alpha))| + C\lambda(NM)^{-(\delta+\beta)} + \lambda\frac{CJL^2}{(NM)^\beta M}.\end{aligned}$$

and using

$$\sum_{l=0}^{L-1}\cos(N(M_j+l)\alpha) = \frac{\sin(L\frac{N\alpha}{2})}{\sin(\frac{N\alpha}{2})}\cos(NM_j\alpha + \frac{(L-1)}{2}N\alpha),$$

we obtain

$$\sum_{j=1}^{J}\frac{\sin(L\frac{N\alpha}{2})}{\sin(\frac{N\alpha}{2})}\cos(NM\frac{j}{J}\alpha+NM\alpha+\frac{(L-1)}{2}N\alpha)$$
$$=\frac{\sin(L\frac{N\alpha}{2})}{\sin(\frac{N\alpha}{2})}\frac{\sin(\frac{NM\alpha}{2})}{\sin(\frac{NM\alpha}{2J})}\cos(NM(1+\frac{1}{J})\alpha+\frac{(L-1)}{2}N\alpha+\frac{(J-1)NM\alpha}{2J}).$$

If now we define

$$f^{pol}_{L,N,M,J}(r,\alpha)$$
$$=\frac{\sin(L\frac{N\alpha}{2})}{\sin(\frac{N\alpha}{2})}\frac{\sin(\frac{NM\alpha}{2})}{\sin(\frac{NM\alpha}{2J})}\cos(NM(1+\frac{1}{J})\alpha+\frac{(L-1)}{2}N\alpha+\frac{(J-1)NM\alpha}{2J})$$

then we have that

- $f^{pol}_{L,N,M,J}$ is $\frac{2\pi}{N}-$periodic in the α variable.
- There exists $|\tilde{\alpha}|\leq\frac{2\pi}{NM}$ such that $f^{pol}_{L,N,M,J}(r,\tilde{\alpha})=0$.
- $|\frac{\partial f^{pol}_{L,N,M,J}(r,\alpha)}{\partial\alpha}|\leq\bar{C}LMNJ$ with $\bar{C}$ a constant,

which means that if $\alpha\in\cup_{i\in\mathbb{Z}}[\tilde{\alpha}+i\frac{2\pi}{N}-\frac{1}{4\bar{C}MN},\tilde{\alpha}+i\frac{2\pi}{N}+\frac{1}{4\bar{C}MN}]$ then $|f^{pol}_{L,N,M,J}(r,\alpha)|\leq\frac{JL}{4}$. Using this we have that, if $\alpha\in\cup_{i\in\mathbb{Z}}[\tilde{\alpha}+i\frac{2\pi}{N}-\frac{1}{4\bar{C}MN},\tilde{\alpha}+i\frac{2\pi}{N}+\frac{1}{4\bar{C}MN}]$ then

$$|\frac{\partial^k\bar{w}^{pol}_{\lambda,N,M,J,L,\tilde{t}}(r,\alpha,\tilde{t})}{\partial\alpha^k}|\leq\frac{\lambda}{4(MN)^{\beta}}+\lambda C(NM)^{-(\delta+\beta)}+\lambda\frac{CJL^2}{(NM)^{\beta}M},$$

so, for any unitary vector u

$$|\frac{\partial^k\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})}{\partial u^k}|\leq\frac{\lambda}{4(MN)^{\beta}}+\lambda C(NM)^{-(\delta+\beta)}+\lambda\frac{CJL^2}{(NM)^{\beta}M},$$

and defining now

$$B_{i,j}:=(1+\frac{j}{NM},1+\frac{j+1}{NM}]\times[\tilde{\alpha}+i\frac{2\pi}{N}-\frac{\pi}{4\bar{C}NM},\tilde{\alpha}+i\frac{2\pi}{N}+\frac{\pi}{4\bar{C}NM}]$$

and it is easy to check that $A_{i,j}$, $B_{i,j}$ have the desired properties. □

The previous lemma shows that our pseudo-solutions do have a big norm at time $\tilde{t}$, and although this will be enough to show ill-posedness, for our non-existence result we will build solutions such that the $C^{k,\beta}$ norm will be infinite for a period of time, and this requires us to obtain specific bounds about how fast our solution can change their $C^{k,\beta}$ norm.

Lemma 3.4.7. *We have that*

$$\frac{d||\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t)-\lambda_0 f_1(\sqrt{x_1^2+x_2^2})||_{C^{k,\beta}}}{dt}\leq\frac{C\lambda M}{\tilde{t}}$$

with C a constant.

Proof. First, since rotations do not change the $C^{k,\beta}$ norm, it is enough to study the evolution of the norm of

$$\sum_{j=1}^{J}\sum_{l=0}^{L-1}\Bigg(\lambda_j f_2(N^{1-\delta}(r(x)-1),N^{1-\delta}\alpha(x))$$
$$\frac{\cos(N(M_j+l)(\alpha(x)-\alpha_j^1)+\alpha_j^2+\frac{k\pi}{2}+t\lambda_0 C_\gamma N^\gamma(M_j+l)^\gamma}{JL(NM_j)^{k+\beta}}\Bigg)$$

which has time derivative

$$-\lambda_0 C_\gamma N^\gamma (M_j+l)^\gamma\sum_{j=1}^{J}\sum_{l=0}^{L-1}\Bigg(\lambda_j f_2(N^{1-\delta}(r(x)-1),N^{1-\delta}\alpha(x))$$
$$\frac{\sin(N(M_j+l)(\alpha(x)-\alpha_j^1)+\alpha_j^2+\frac{k\pi}{2}+t\lambda_0 C_\gamma N^\gamma(M_j+l)^\gamma}{JL(NM_j)^{k+\beta}}\Bigg),$$

but since this function has support in $r\geq\frac{1}{2}$, we can use (3.6) and it is enough to obtain bounds for the $C^{k,\beta}$ norm in polar coordinates. However, using the expression for λ_0 we easily obtain

$$||\lambda_0 C_\gamma N^\gamma (M_j+l)^\gamma\sum_{j=1}^{J}\sum_{l=0}^{L-1}\Bigg(\lambda_j f_2(N^{1-\delta}(r-1),N^{1-\delta}\alpha)$$
$$\frac{\sin(N(M_j+l)(\alpha-\alpha_j^1)+\alpha_j^2+\frac{k\pi}{2}+t\lambda_0 C_\gamma N^\gamma(M_j+l)^\gamma}{JL(NM_j)^{k+\beta}}\Bigg)||_{C^{k,\beta}}$$
$$\leq\frac{C\lambda M}{\tilde{t}}.$$

□

We only need one last technical result before we can go to prove our ill-posedness result. Namely, we need to obtain bounds for the error between our pseudo-solution and the real solution to γ-SQG with our initial conditions. We will, however, prove a slightly stronger result, where we show that the error remains small even if we compare to a solution to γ-SQG with a small error in the velocity. This will later on be necessary when we prove the non-existence of solutions in $C^{k,\beta}$.

Lemma 3.4.8. *Given a pseudo-solution $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t)$ and a function $v_{error}=(v_{1,error},v_{2,error})$ fulfilling*

$$||v_{error}||_{C^m}\leq\frac{N^m}{N^{k+\beta+2}}$$

for $m=0,1,...,k+2$ and

$$\frac{\partial v_{1,error}}{\partial x_1}+\frac{\partial v_{2,error}}{\partial x_2}=0,$$

we have that, for any fixed T,λ,M,J,L and $\tilde{t}$, if N is big enough, then the unique $H^{k+\beta+1-\delta}$ solution $\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x_1,x_2,t)$ to

$$\frac{\partial\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial t}+(v_\gamma(\tilde{w}_{\lambda,N,M,J,L,\tilde{t}})+v_{error})\cdot(\nabla\tilde{w}_{\lambda,N,M,J,L,\tilde{t}})=0, \tag{3.17}$$

$$\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0) = \bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)$$

exists for $t \in [0,T]$ and, if we define

$$W := \tilde{w}_{\lambda,N,M,J,L,\tilde{t}} - \bar{w}_{\lambda,N,M,J,L,\tilde{t}}$$

then

$$||W(x,t)||_{L^2} \leq C(1+\frac{1}{\tilde{t}})tN^{-k-\beta-1},$$

$$||W(x,t)||_{H^{k+\beta+1-\delta}} \leq C(1+\frac{1}{\tilde{t}})tN^{-\delta}.$$

with C depending on T, λ, M, J and L.

Furthermore, by interpolation, for any $s \in [0, k+\beta+1-\delta]$ we have that

$$||W(x,t)||_{H^s} \leq C(1+\frac{1}{\tilde{t}})tN^{-(k+\beta+1)+s}.$$

Proof. First we note that the evolution equation for W is

$$\frac{\partial W}{\partial t} + (v_\gamma(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}) + v_\gamma(W) + v_{error}) \cdot \nabla W$$
$$+ (v_\gamma(W) + v_{error}) \cdot \nabla \bar{w}_{\lambda,N,M,J,L,\tilde{t}} - F_{\lambda,N,M,J,L,\tilde{t}} = 0.$$

and (using the properties of $F_{\lambda,N,M,J,L,\tilde{t}}$ for N big) this evolution equation has local existence and uniqueness in $H^{k+\beta+1-\delta}$ under our assumptions for v_{error}. Furthermore, it is enough to prove our inequalities under the assumption $||W(x,t)||_{H^{k+\beta+1-\delta}} \leq CN^{-\delta}log(N)$, since then using the continuity in time of $||W||_{H^{k+\beta+1-\delta}}$ and taking N big would give us the result for the desired time interval.

For the L^2 norm, we can use incompressibility to obtain

$$\frac{d||W||^2_{L^2}}{dt} \leq 2\int |W\big(v_\gamma(W)\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}} - F_{\lambda,N,M,J,L,\tilde{t}} + v_{error}\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}}\big)|dx$$
$$\leq \int 2|Wv_\gamma(W)\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx| + \frac{C}{N^{k+\beta+1}}(1+\frac{1}{\tilde{t}})||W||_{L^2}.$$

To bound the integral term with $v_\gamma(W)$ we need to use two important properties that will also be key when working with the $H^{k+\beta+1-\delta}$ bounds. First, as in [18], using that, for an odd operator A (which in our case will be $v_{1,\gamma}$ and $v_{2,\gamma}$) we have

$$\int fA(f)g = -\frac{1}{2}\int f(A(gf) - gA(f))$$

and so

$$|\int Wv_\gamma(W)\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx|$$
$$= \frac{1}{2}|\int W\big(v_{i,\gamma}(W\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i}) - v_{i,\gamma}(W)\frac{\partial\bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i}\big)dx|$$

and using Corollary 1.4 in [81]

$$|\int Wv_\gamma(W)\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx| \leq ||W||^2_{L^2}||\nabla v_\gamma(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})||_{L^\infty} \leq C||W||^2_{L^2}$$

where we used that

$$||v_\gamma(\bar{w}_{\lambda,N,M,J,L,\tilde{t}})||_{C^{k',\beta'}} \leq CN^{k'+\beta'+\gamma-k-\beta}log(N)$$

which is obtained by applying Lemmas 2.3.7 and 2.3.8, the definition of v_γ and the properties of $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$. Then, after applying Gronwall we get

$$||W||_{L^2} \leq \frac{Ct}{N^{k+\beta+1}}(1+\frac{1}{\tilde{t}})$$

with C depending on λ, M, J, L and T.

The proof of the inequality for $H^{k+\beta+1-\delta}$ is very similar to that of Lemmas 2.2.9 and 2.3.9, so we will skip most of the details and focus on the few differences for the sake of briefness. The idea is to use that

$$\frac{\partial ||\Lambda^s W||_{L^2}^2}{\partial t} \leq 2|\int (\Lambda^s W)\Lambda^s(\frac{\partial W}{\partial t})dx|,$$

and then bound each of the integrals obtained from the equation for $\frac{\partial W}{\partial t}$. For example, for the term

$$|\int (\Lambda^s W)\Lambda^s(v_\gamma(W)\nabla \bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx|$$

we use Lemma 2.2.10 (which is proved in [81]) to get for $s = k+\beta+1-\delta$ the inequality

$$\begin{aligned}
&|\int (\Lambda^s W)\Lambda^s(v_\gamma(W)\cdot\nabla \bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx| \\
&\leq \sum_{|\mathbf{a}|\leq s-\gamma} |\int \frac{1}{\mathbf{a}!}(\Lambda^s W)\Lambda^{s,\mathbf{a}}(v_\gamma(W))\cdot\nabla\partial^{\mathbf{a}}\bar{w}_{\lambda,N,M,J,L,\tilde{t}})dx| \\
&+ \sum_{|\mathbf{b}|<\gamma} |\int \frac{1}{\mathbf{b}!}(\Lambda^s W)\partial^{\mathbf{b}}(v_\gamma(W))\cdot\nabla\Lambda^{s,\mathbf{b}}(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}))dx| \\
&+ C||(\Lambda^s W)||_{L^2}||v_\gamma(W)||_{H^{s-\gamma}}||\Lambda^\gamma\nabla\bar{w}_{\lambda,N,M,J,L,\tilde{t}}||_{L^\infty},
\end{aligned}$$

where we used the multi-index notation, $\mathbf{c} = (c_1, c_2)$, $|\mathbf{c}| = (c_1^2+c_2^2)^{\frac{1}{2}}$, $\mathbf{c}! = c_1!c_2!$, $\partial^{\mathbf{c}} = \partial_x^{\mathbf{c}} = \partial_{x_1}^{c_1}\partial_{x_2}^{c_2}$ and the operator $\Lambda^{s,\mathbf{c}}$ is defined via the Fourier transform as

$$\begin{aligned}
\widehat{\Lambda^{s,\mathbf{j}} f}(\xi) &= \widehat{\Lambda^{s,\mathbf{j}}}(\xi)\hat{f}(\xi) \\
\widehat{\Lambda^{s,\mathbf{j}}}(\xi) &= i^{-|\mathbf{j}|}\partial_\xi^{\mathbf{j}}(|\xi|^s).
\end{aligned}$$

Most of these terms can be bounded directly by $C||W||_{H^s}^2$ using the properties of v_γ, $\Lambda^{s,\mathbf{c}}$, and $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$ plus the assumptions for W (including the L^2 growth) and the interpolation inequality for Sobolev spaces.

A few terms, however, requires more careful consideration, namely,

$$|\int (\Lambda^s W)\Lambda^s(v_{i,\gamma}(W))\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i})dx|, \tag{3.18}$$

$$|\int (\Lambda^s W)\Lambda^s(v_{i,\gamma}(W))\frac{\partial W}{\partial x_i})dx|$$

for $i = 1, 2$, since $||\Lambda^s(v_\gamma(W)||$ cannot by bounded by $||W||_{H^s}$. We will just focus on (3.18) since the other term is done in exactly the same way. Here, we need to again act as in the L^2 case, rewriting (3.18) as

$$\frac{1}{2}|\int (\Lambda^s W)\big(v_{i,\gamma}[\Lambda^s(W)\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i}] - v_{i,\gamma}[\Lambda^s(W)]\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i})dx|.$$

We can then use again Lemma 2.2.10 to get

$$\frac{1}{2}|\int (\Lambda^s W)\big(v_{i,\gamma}[\Lambda^s(W)\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i}] - v_{i,\gamma}[\Lambda^s(W)]\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i})dx|$$

$$\leq C||W||_{H^s}||W||_{H^s}||v_{i,\gamma}(\frac{\partial \bar{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial x_i})||_{L^\infty} \leq C||W||^2_{H^s}.$$

Combining the bounds for all the terms we obtain

$$\frac{\partial ||\Lambda^s W||^2_{L^2}}{\partial t} \leq C||W||_{H^s}(||W||_{H^s} + (1+\frac{1}{\tilde{t}})\frac{C}{N^\delta})$$

and therefore, for $t \in [0,T]$

$$||W(x,t)||_{H^s} \leq Ce^{Ct}t||F||_{H^s} \leq C(1+\frac{1}{\tilde{t}})tN^{-\delta}$$

with C depending on T, λ, M, J and L. □

Combining all the technical results together we obtain the following.

Theorem 3.4.1. Given $T, t_{crit}, \epsilon_1, \epsilon_2, \epsilon_3 > 0$ and $t_{crit} \in (0,T]$, we can find λ, M, J, L and $\tilde{t}$ such that, if N is big enough, then for any v_{error} satisfying

$$||v_{error}||_{C^m} \leq \frac{N^m}{N^{k+\beta+2}}$$

for $m = 0, 1, ..., k+2$ and

$$\frac{\partial v_{1,error}}{\partial x_1} + \frac{\partial v_{2,error}}{\partial x_2} = 0$$

then the unique $H^{k+\beta+1-\delta}$ function $\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,t)$ satisfying

$$\frac{\partial \tilde{w}_{\lambda,N,M,J,L,\tilde{t}}}{\partial t} + (v_\gamma(\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}) + v_{error}) \cdot (\nabla \tilde{w}_{\lambda,N,M,J,L,\tilde{t}}) = 0 \tag{3.19}$$
$$\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0) = \bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)$$

exists for $t \in [0,T]$ and has the following properties.

- $||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{C^{k,\beta}} \leq \epsilon_1$,
- $||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,t)||_{C^{k,\beta}} \geq \frac{1}{\epsilon_2}$ if $t \in (t_{crit} - Ct_{crit}, t_{crit})$ with C depending on ϵ_1 and ϵ_2,
- $||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{H^{k+\beta+1-\frac{3}{2}\delta}}, ||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{L^1} \leq \epsilon_3$.

Proof. We first fix some parameters so the pseudo-solutions $\bar{w}_{\lambda,N,M,J,L,\tilde{t}}$ have some desirable properties. We fix $\tilde{t} = t_{crit}$ so that, by Lemma 3.4.6 we have

$$|\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,t_{crit})|_{C^{k,\beta}} \geq \lambda(\frac{1}{4(4\pi)^\beta} - -\frac{CL^2}{M} - C(NM)^{-\delta} - CN^{-1+\delta}).$$

Since we want $\tilde{w}$ to also have a very big $C^{k,\beta}$ norm, this suggest taking $\lambda \approx \frac{1}{\epsilon_2}$, and we will specifically consider $\lambda = \frac{1}{\epsilon_2}32(4\pi)^\beta$.

With λ fixed, we can now focus on assuring that our initial conditions have a norm as small as required. Using Lemmas 3.4.1, 3.4.3 and 3.4.6 plus our choice for α_1^j we know that

$$||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{C^{k,\beta}} \leq C\lambda_0 + C\lambda(\frac{1}{J} + \frac{1}{(NM)^\delta} + \frac{J}{L} + (NM)^{-\beta} + \frac{L}{M})$$
$$= C\frac{M^{1-\gamma}}{\tilde{t}N^\gamma} + C\lambda(\frac{1}{J} + \frac{1}{(NM)^\delta} + \frac{J}{L} + (NM)^{-\beta} + \frac{L}{M}).$$

and that there are sets A and B (depending on λ, N, M, J and L) such that if $x \in A$ then there exists unitary u depending on x with

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}|$$

$$\geq \lambda(\frac{1}{2(MN)^{\beta}}-\frac{CL^2}{(NM)^{\beta}M}-C(NM)^{-(\delta+\beta)}-C(NM)^{-\beta}N^{-1+\delta})$$

and a set B such that if $x\in B$ then for all unitary u we have that

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}|\leq \lambda(\frac{1}{4(MN)^{\beta}}+C(NM)^{-(\delta+\beta)}+\frac{CL^2}{(NM)^{\beta}M})$$

furthermore, there is a set $S_{M,N,\delta}$ such that its cardinal fulfils $|S_{M,N,\delta}|\geq C_1 MN^{2\delta}$ and

$$A=\cup_{s\in S_{M,N,\delta}}A_s$$

$$B=\cup_{s\in S_{M,N,\delta}}B_s$$

$d(x,y)\leq\frac{4\pi}{NM}$ if $x\in A_s, y\in B_s$, and $|A_s|,|B_s|\geq\frac{C_2}{(NM)^2}$, with C_1 and C_2 constants.

By taking $t=t_{crit}$ and $J^2=L$, $M=L^3=J^6$ and fixing J big we can then obtain that

$$||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{C^{k,\beta}}\leq C\frac{J^{6(1-\gamma)}}{t_{crit}N^{\gamma}}+\frac{\epsilon_1}{2}$$

and for $x\in A$ there exists u unitary such that

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}|\geq 14\frac{(4\pi)^{\beta}}{\epsilon_2(MN)^{\beta}} \tag{3.20}$$

and for $x\in B$ and any unitary vector u

$$|\frac{\partial^k(\bar{w}_{\lambda,N,M,J,L,\tilde{t}}(x,\tilde{t})-\lambda_0 f_1)}{\partial u^k}|\leq 10\frac{(4\pi)^{\beta}}{\epsilon_2(MN)^{\beta}}.$$

Note that, the choice of the parameters J,L and M depend only on ϵ_1 and ϵ_2.

We would like to obtain similar bounds for $\tilde{w}$, so we need to show that $\tilde{w}$ and $\bar{w}$ are close to each other in a useful way. First, using Lemma 3.4.8 we have

$$\sum_{i=0}^{k}\int(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2\leq C(1+\frac{1}{\tilde{t}})N^{-2(\beta+1)}$$

and in particular (including from now on $(1+\frac{1}{\tilde{t}})$ inside of the constant C since it is constant with respect to N), there exists A_s,B_s such that

$$\sum_{i=0}^{k}\int_{A_s}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2+\int_{B_s}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2\leq CN^{-2(\beta+1+\delta)}$$

so

$$inf_{x\in A_s}|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2||A_s|\leq\sum_{i=0}^{k}\int_{A_s}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2\leq CN^{-2(\beta+1+\delta)}$$

$$inf_{x\in B_s}|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2||B_s|\leq\sum_{i=0}^{k}\int_{B_s}(\frac{\partial^k(\tilde{w}-\bar{w})}{\partial x_1^i\partial x_2^{k-i}})^2\leq CN^{-2(\beta+\delta+1)}$$

and therefore

$$inf_{x\in A_s}|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})(x,t)}{\partial x_1^i\partial x_2^{k-i}})^2|\leq CN^{-2(\beta+\delta)}$$

$$inf_{x\in B_s}|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})(x,t)}{\partial x_1^i \partial x_2^{k-i}})^2| \leq CN^{-2(\beta+\delta)}.$$

Given a time $t\in[0,t_{crit}]$, we dconsider $x_A(t)\in A_s$, $x_B(t)\in B_s$ points fulfilling

$$|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})(x_A(t),t)}{\partial x_1^i \partial x_2^{k-i}})^2| \leq CN^{-2(\beta+\delta)}$$

$$|\sum_{i=0}^{k}(\frac{\partial^k(\tilde{w}-\bar{w})(x_B(t),t)}{\partial x_1^i \partial x_2^{k-i}})^2| \leq CN^{-2(\beta+\delta)}.$$

Now, if u is the unitary vector given by (3.20) for $x_B(t)$, we have that

$$\begin{aligned}
&|\frac{\partial^k\tilde{w}(x_A,t)-\tilde{w}(x_B,t)}{\partial^k u}|\frac{1}{|x_A-x_B|^\beta}\\
&\geq|\frac{\partial^k\bar{w}(x_A,t)-\bar{w}(x_B,t)}{\partial^k u}|\frac{1}{|x_A-x_B|^\beta}-CN^{-\delta}\\
&\geq|\frac{\partial^k\bar{w}(x_A,t_{crit})-\bar{w}(x_B,t_{crit})}{\partial^k u}|\frac{1}{|x_A-x_B|^\beta}-||\bar{w}(x,t)-\bar{w}(x,t_{crit})||_{C^{k,\beta}}-CN^{-\delta}\\
&\geq\frac{4}{\epsilon_2}-C\frac{\lambda J^6|t-\tilde{t}|}{\tilde{t}}-CN^{-\delta}
\end{aligned}$$

where we used Lemma 3.4.7 in the last inequality. Then if $|C\frac{\lambda J^6|t-\tilde{t}|}{\tilde{t}}|\leq\frac{2}{\epsilon_2}$, $|CN^{-\delta}|\leq\frac{1}{\epsilon_2}$ we get

$$||\tilde{w}(x,t)||_{C^{k,\beta}}\geq|\frac{\partial^k\tilde{w}(x_A,t)-\tilde{w}(x_B,t)}{\partial^k v}|\frac{1}{|x_A-x_B|^\beta}\geq\frac{1}{\epsilon_2}$$

and this will be true if we take N big enough and $|t-\tilde{t}|\leq\frac{\tilde{t}}{\lambda J^6|t-\tilde{t}|}=C(\epsilon_1,\epsilon_2)$.

The only thing we need to prove is that we can also obtain

$$||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{C^{k,\beta}}\leq\epsilon_1$$

$$||\tilde{w}(x,0)||_{H^{k+\beta+1-\frac{3}{2}\delta}}\leq\epsilon_3,$$

but

$$||\tilde{w}(x,0)||_{H^{k+\beta+1-\frac{3}{2}\delta}}\leq\frac{C}{N^\gamma}+\frac{C}{N^{\frac{\delta}{2}}}$$

with C depending on J,L and M, so taking N big enough

$$||\tilde{w}(x,0)||_{H^{k+\beta+1-\frac{3}{2}\delta}}\leq\epsilon_3$$

and analogously,

$$||\tilde{w}_{\lambda,N,M,J,L,\tilde{t}}(x,0)||_{C^{k,\beta}}\leq C\frac{J^{6(1-\gamma)}}{t_{crit}N^\gamma}+\frac{\epsilon_1}{2}$$

so again, taking N big enough finishes the proof.

□

3.5 Strong ill-posedness and non-existence of solutions

We are now ready to prove ill-posedness and non-existence of solutions. As mentioned earlier, these results hold for $k \in \mathbb{N}$, $\beta \in (0,1]$, $\gamma \in (0,1)$ with $k+\beta > 1+\gamma$ and δ is some constant $\delta \in (0,\frac{1}{2})$ such that $k+\beta+2\delta > 1+\gamma$.

Theorem 3.5.1. Given $T, t_{crit}, \epsilon_1, \epsilon_2 > 0$, there exists a function $w(x,0)$ such that $||w(x,0)||_{C^{k,\beta}} \leq \epsilon_1$ and the only solution to (3.1) in $H^{k+\beta+1-\delta}$ with initial conditions $w(x,0)$ exists for $t \in [0,T]$ and fulfills that

$$||w(x,t_{crit})||_{C^{k,\beta}} \geq \frac{1}{\epsilon_2}.$$

Proof. This is just a direct application of Theorem 3.4.1 with

$$v_{1,error} = v_{2,error} = 0$$

. □

Theorem 3.5.2. Given $t_0, \epsilon > 0$, there exist a function $w(x,0)$ such that $||w(x,0)||_{C^{k,\beta}} \leq \epsilon$ and that the only solution to (3.1) in $H^{k+\beta+1-\frac{3}{2}\delta}$ with initial conditions $w(x,0)$ exists for $t \in [0,t_0]$ and fulfills that, for $t \in (0,t_0]$, $||w(x,t)||_{C^{k,\beta}} = \infty$.

Proof. To obtain initial conditions with the desired properties, we will consider initial conditions of the form

$$\sum_{j=1}^{\infty}\sum_{i=1}^{G(j,\epsilon)} T_{R_{i,j}}(w_{i,j}(x))$$

where $T_R(f(x_1,x_2)) = f(x_1+R,x_2)$. We will first choose $w_{i,j}(x)$ and afterwards we will pick the values of $R_{i,j}$.

First, fixed j, we will restrict to choices for $w_{i,j}$ such that they are initial conditions given by Theorem 3.4.1 with $\frac{1}{\epsilon_2} = j$, $\epsilon_1 = \epsilon$ and $T = t_0$. Then if we choose some $t_{crit} = t_{crit,i,j}$ and we call $\tilde{w}_{i,j}$ a solution to (3.17) with the initial conditions given by $w_{i,j}(x)$ and an appropriate v_{ext} fulfilling $||v_{ext}||_{C^{k+2}} \leq C_{i,j}$, we would then have that for $t \in [t_{crit,i,j} - Ct_{crit,i,j}, t_{crit,i,j}]$

$$||\tilde{w}_{i,j}(x,t)||_{C^{k,\beta}} \geq j$$

for some C depending on ϵ and j. Therefore, we can, by choosing $t_{crit,i,j}$ appropriately, obtain, for any $t \in [\frac{1}{j}, t_0]$

$$\sup_{i=1,2,\dots,G(j,\epsilon)} ||\tilde{w}_{i,j}(x,t)||_{C^{k\beta}} \geq j$$

with $G(j,\epsilon)$ a finite number depending on j.

Furthermore, we can now choose ϵ_3 in Theorem 3.4.1 so that

$$||w_{i,j}(x)||_{H^{k+\beta+1-\frac{3}{2}\delta}} \leq \frac{c_0 2^{-j}}{G(j,\epsilon)}$$

$$||w_{i,j}(x)||_{L^1} \leq \frac{2^{-j}}{G(j,\epsilon)}$$

with c_0 a constant small enough so that any solution to γ-SQG with

$$||w_0(x)||_{H^{k+\beta+1-\frac{3}{2}\delta}} \leq c_0$$

exists for $t \in [0,t_0]$ and $||w(x,t)||_{H^{k+\beta+1-\frac{3}{2}\delta}} \leq 1$ for $t \in [0,t_0]$. Therefore we know that, independently of the choice of $R_{i,j}$, for $t \in [0,t_0]$ there exists a unique $H^{k+\beta+1-\frac{3}{2}\delta}$ solution to (3.1) with initial conditions

$$\sum_{j=1}^{\infty}\sum_{i=1}^{G(j)} T_{R_{i,j}}(w_{i,j}(x))$$

and, furthermore, if we call this solution $w_\infty(x,t)$ (which still depends on the choice of $R_{i,j}$, but we omit it for simplicity of notation), then we have that there is a constant v_{max} such that, for $t \in [0, t_0]$

$$||v_1(w_\infty)||_{L^\infty}, ||v_2(w_\infty)||_{L^\infty} \leq v_{max}.$$

With this, and using that there exists $D \in \mathbb{R}$ such that $\text{supp}(w_{i,j}(x)) \subset B_D(0)$, we have that, if we choose the $R_{i,j}$ so that $|R_{i_1,j_1} - R_{i_2,j_2}| \geq 4t_0 v_{max} + 2D + sup(P_{i_1,j_1}, P_{i_2,j_2})$ with $P_{i,j} > 0$ then we have that

$$w_{i,j,\infty}(x,t) := 1_{B_{D+2t_0 v_{max}}(-R_{i,j},0)} w_\infty(x,t)$$

fulfils for $t \in [0, t_0]$ the evolution equation

$$\frac{\partial w_{i,j,\infty}}{\partial t} + (v_\gamma(w_{i,j,\infty}) + v(w_\infty - w_{i,j,\infty})) \cdot (\nabla w_{i,j,\infty}) = 0$$

and

$$||v(w_\infty - w_{i,j,\infty}))||_{C^{k+2}} \leq \frac{C}{P_{i,j}^{2+\gamma}}.$$

But by the choice of $w_{i,j}(x)$ and using that the supports of the $w_{i,j\infty}$ are disjoint, we have that if

$$||v(w_\infty - w_{i,j,\infty}))||_{C^{k+2}} \leq C_{i,j} \tag{3.21}$$

then for $t \in (0, t_0]$

$$||w_\infty(x,t)||_{C^{k,\beta}} = \sup_{j \in \mathbb{N}, i=1,2,\ldots,G(j,\epsilon)} ||w_{i,j,\infty}(x,t)||_{C^{k,\beta}} = \infty$$

and taking $P_{i,j}$ big enough so that (3.21) is fulfilled finishes the proof.

□

Chapter 4

Loss of regularity for 2D Euler

4.1 Introduction

We consider the incompressible Euler equations

$$\begin{aligned} \partial_t v + (v \cdot \nabla) v + \nabla P = 0, \\ \operatorname{div} v = 0 \end{aligned} \tag{4.1}$$

in $\mathbb{R}^d \times \mathbb{R}_+$, with $d = 2, 3$, where $v(x,t) = (v_1(x,t), .., v_d(x,t))$ is the velocity field and $P = P(x,t)$ is the pressure function. In this chapter we study ill-posedness of the initial value problem for (4.1) with a given initial data $v_0(x) = v(x,0)$.

In order to illustrate the ill-posedness phenomena, we first note that the classical theory of the Euler equations goes back to the work of Lichtenstein [82] and Gunther [57], who showed local well-posedness in $C^{k,\alpha}$ ($k \geq 1$, $\alpha \in (0,1)$). This was extended to global-in-time well-posedness in the 2D case by Wolibner [97] and Hölder [61]. In the case of Sobolev spaces, Ebin and Marsden [52] proved, in a compact domain, local well-posedness in H^s for $s > \frac{d}{2} + 1$, and Bourguignon and Brezis [10] have generalized it to the space $W^{s,p}$ for $s > \frac{d}{p} + 1$. Moreover, Kato [68] extended the local well-posedness to $\mathbb{R}^d$ for initial data u_0 in H^s for $s > \frac{d}{2} + 1$, see the extension to the $W^{s,p}$ spaces, due to Kato and Ponce [69].

Remarkably, in the 2D case these local-in-time results can be easily extended for all times using the Beale-Kato-Majda criterion [6], since the vorticity is transported by the flow. The optimal bound for growth was obtained by Kiselev and Šverák [76] in a disk, see also the work by Zlatoš [104] and the lecture notes [71] by Kiselev for further results.

Moreover, it can be shown that the equations are not well-posed in some spaces, such as integer C^k spaces ($k \geq 1$). This was recently demonstrated by Bourgain and Li [8], and independently by Elgindi and Masmoudi [55], who showed strong ill-posedness and non-existence of uniformly bounded solutions for the initial velocity v_0 in C^k. Furthermore, nonexistence of uniformly bounded solutions in the critical Sobolev space $H^{\frac{d}{2}+1}$ was established in another work of Bourgain and Li [9]. Subsequently, Elgindi and Jeong [54] obtained analogous results with a different approach, and Jeong [65] gave a simpler proof and similar results for the critical space $W^{s,p}$. Recently, Kwon proved in [77] that there is still strong ill-posedness in H^2 for a regularized version of the 2D incompressible Euler equations. We also refer the reader to Misiołek and Yoneda [85] for a proof of a nonexistence result in critical Besov spaces in $d = 3$.

These results gave the first methods of studying ill-posedness and nonexistence of solutions to the Euler equations. Moreover, subsequently Elgindi [53] proved a remarkable result on singularity formation of the 3D axisymmetric Euler equations without swirl for $C^{1,\alpha}$ velocity, where $\alpha > 0$ is sufficiently small, and Elgindi, Ghoul, and Masmoudi [56] extended it to the finite energy case. We also refer the reader to the work of Chen and Hou [22], who provided evidence of a possibility of nearly self-similar blow near a boundary, as well as their subsequent impressive work [23].

In the case of supercritical Sobolev spaces DiPerna and Lions [49] show that for $d = 3$ and for every $p \geq 1$, there exists a shear flow solution to (4.1) with $v_0 \in W^{1,p}$ and $v(x,t) \notin W^{1,p}$ for $t > 0$. Using the structure of shear flows Bardos and Titi [5] showed the instantaneous loss of smoothness of weak solutions for the 3D Euler equations with initial data in the Holder space C^α with $\alpha \in (0,1)$. Note that these constructions rely strongly in the $2+\frac{1}{2}$ dimensional structure of the shear flows. At this point is worth mentioning the ground-breaking work of De Lellis and Székelyhidi Jr. [46, 47], where they show non-uniqueness of solutions in L^2 by the method of convex integration (see also the work of Wiedemann [96]). Very recently, using similar tools, Khor and Miao [70] use the method of convex integration to construct infinitely many distributional 3D solutions in H^β for $0<\beta<<1$ which has an instantaneous gap loss of Sobolev regularity.

From now on in the present work we will focus in solutions with sufficient regularity in the two dimensional case and use the vorticity formulation, which is obtained by taking the curl of the first equation of (4.1) and denoting the scalar function (vorticity) by $\omega := \operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1$, where ∂_1, ∂_2 denote partial derivatives with respect to x_1, x_2, respectively. The equation for the vorticity reads

$$\partial_t \omega + v\cdot\nabla\omega = 0. \tag{4.2}$$

According to the Biot-Savart law, there is a stream function ψ such that $v = (-\partial_2\psi, \partial_1\psi)$ and $-\Delta\psi = \omega$ which gives that $v[\omega] = -\Delta^{-1}\nabla^\perp\omega$, where $\nabla^\perp := (-\partial_2, \partial_1)$. Thus the velocity field v can be expressed as

$$v[\omega](x,t) = \frac{2}{\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp \omega(y,t)}{|x-y|^2} dy \tag{4.3}$$

where $(x_1,x_2)^\perp := (-x_2, x_1)$, although we will ignore the factor $\frac{2}{\pi}$ in our computations since both velocities produce the exact same qualitative behaviour.

In [103] Yudovich proved the existence and uniqueness of weak solutions for bounded vorticity in a bounded domain. This statement can be extended to $\mathbb{R}^2$ for solutions such that $\omega \in L^1\cap L^\infty$ (see discussions in [83] and [2]). Very recently Vishik [93, 94] showed that although there is existence of solutions with a force source the uniqueness fails if L^∞ is substituted by L^p with $p<\infty$ (see also [2]).

The main result in this chapter is to construct unique solutions of the 2D incompressible Euler equations (in vorticity formulation) in $\mathbb{R}^2\times[0,\infty)$ with initial vorticity in the super-critical Sobolev space H^β, $0<\beta<1$, which, at each time $t>0$, does not belong to any $H^{\beta'}$ such that

$$\beta' > \frac{(2-\beta)\beta}{2-\beta^2}. \tag{4.4}$$

Moreover these solutions are not in the Yudovich class but are the unique classical solution in the sense given by Definition 4.1.3.

We note that the only result to-date in the direction of proving instantaneous loss of regularity for 2D Euler in the supercritical regime with velocity $v(t)\in H^1$ for all $t\geq 0$ is the result of Jeong [63], who constructed solutions to the 2D Euler equations which belong to the Yudovich class but the derivative of the vorticity loses integrability continuously in time, i.e. $\omega \notin W^{1,p(t)}$, with $p(t)$ decreasing continuously in t, $1\leq p(0)<2$. In fact, it is shown in [54] that for this regularity the solution cannot have a jump in the regularity class. Furthermore, Alberti, Crippa and Mazzucato [1] show a gap loss of Sobolev regularity for a passive scalar that is driven by a non-Lipschitz incompressible velocity field, see also [43].

4.1.1 Main results

We are interested in showing loss of regularity for solutions with vorticity $\omega\in H^\beta$, but as the first step we will prove that there are initial conditions $\omega_0 \in C_c^\infty$ that are not big in H^β but become arbitrarily big in $H^{\beta'}$ for β' as in (4.4).

Theorem 4.1.1 (Norm inflation for smooth data.)**.** *Given $T,K>0$, $\beta\in(0,1)$ and $\beta' > \frac{(2-\beta)\beta}{2-\beta^2}$, there exist finite energy initial conditions $\omega_0\in C_c^\infty$ with $\|\omega_0\|_{H^\beta}\leq 1$ such that the only classical solution to 2D Euler with initial condition ω_0 fulfils $\|\omega\|_{H^{\beta'}}\geq K$ for $t\in[\frac{1}{T},T]$.*

We then consider an infinite number of rapidly growing solutions and use a gluing argument to find initial conditions that lose regularity instantly.

Theorem 4.1.2 (Loss of regularity in the supercritical regime)**.** *For any $\epsilon > 0$, $\beta \in (0,1)$ there exist finite energy initial conditions ω_0 such that there exists a unique global classical solution ω to the 2D Euler equations (see Definition 4.1.3) with those initial conditions such that*

$$\|\omega_0\|_{H^\beta} \leq \epsilon,$$
$$\|\omega(x,t)\|_{H^{\beta'}} = \infty \quad \text{for} \quad t \in (0,\infty), \beta' > \frac{(2-\beta)\beta}{2-\beta^2}.$$

Since the initial conditions from Theorem 4.1.1 are chosen so that $\omega_0 \in C_c^\infty$, for Theorem 4.1.1 we can use the usual definition of classical solutions for the $2D$ Euler equations without any trouble. However, Theorem 4.1.2 requires us to consider initial conditions with very low regularity, and so we need to be a little more precise regarding what we consider a classical solution to 2D Euler in such a situation.

Definition 4.1.3. We say that $\omega \in L^\infty([0,T); L^1 \cap L^p)$, where $p > 2$, is a classical solution to 2D Euler with initial conditions $\omega_0(x)$ if

$$\omega \in C^1_{x,t}(K) \qquad \text{for every } K = \overline{B_d(0)} \times [0,a] \subset \mathbb{R}^2 \times [0,T)$$

and

$$\partial_t \omega + v[\omega] \cdot \nabla \omega = 0,$$
$$\omega(x,0) = \omega_0(x).$$

Since ω is $C^1_{x,t}$ on each compact set this assures that the transport equation makes sense, that the L^p norms are conserved (whenever they are well defined) and that the support of ω is transported with the velocity $v[\omega]$.

Note that the initial conditions considered will in general not be in the Yudovich class (but in $L^1 \cap L^p$ for some $\infty > p > 2$), so it is unclear whether we have have locally in time a classical solution, much less if it is also global and unique, and we will resolve these problems by hand.

4.1.2 Ideas of the proof

In order to prove the norm inflation result, Theorem 4.1.1, we start by considering ω_0 consisting of a stationary radial function and a perturbation involving highly oscillatory angular behaviour

$$\omega_0(x) = f(r) + g(r)\frac{\cos(N\alpha)}{N^\beta}. \tag{4.5}$$

As N grows, the effects of the velocity produced by $g(r)N^{-\beta}\cos(N\alpha)$ become less and less relevant, and thus we can approximate the solution by

$$\partial_t \omega(x,t) + v[f(r)] \cdot \nabla \omega(x,t) = 0,$$

see Figure 4.1 below.

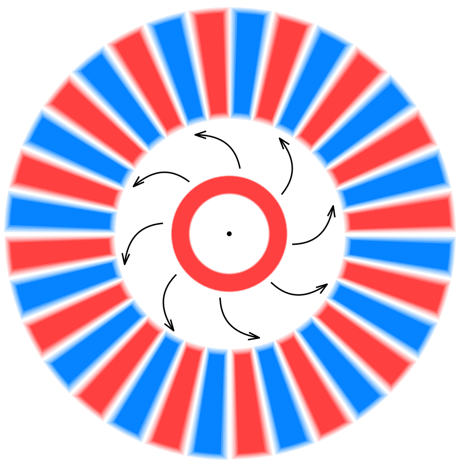

Figure 4.1: A sketch of the initial vorticity ω_0. Here the inner vorticity depends only on r, and so the resulting velocity field is only angular, which causes rotation of the outer part (here denoted by the arrows). The high frequency N in α of the outer part improves the control over the solution. We note that except for the inner part, the radial part of the vorticity must also include an outer part (supported far from the origin), which would guarantee zero average of ω_0.

This already allows us to obtain, in a fairly straightforward way, strong ill-posedness in H^β, $\beta \in (0,1)$, by choosing f to be small in H^β, but such that $\|v[f(r)]\|_{C^1}$ is large. However, in order to obtain $H^{\beta'}$ norm growth for some $\beta' < \beta$ (as in (4.4)), rather than merely for $\beta' = \beta$, we need to consider a more general family of initial conditions

$$\omega_0(x) = \omega_{rad}(0) + \omega_{osc}(0) := \lambda^{1-\beta} f(\lambda r) + \lambda^{1-\beta} N^{-\beta} g(\lambda r) \cos(N\alpha). \tag{4.6}$$

Note that such scaling with respect to $\lambda > 0$ preserves the $\dot{H}^\beta$ norm. As in (4.5), the periodicity parameter N allows us to improve our control over the behaviour of the solution and now the scaling parameter λ is compressing the timescale so that the growth happens faster. The appearance of the new parameter λ makes the control of the errors more challenging than in the case of SQG (chapter 2) . We will approximate the solution by a function of the form

$$\begin{aligned} \overline{\omega}(t) &= \overline{\omega_{rad}}(t) + \overline{\omega_{osc}}(t) \\ &:= \lambda^{1-\beta} f(\lambda r) + \lambda^{1-\beta} g(\lambda r) N^{-\beta} \cos\left(N \left(\alpha - \frac{1}{r} \int_0^t v_\alpha \left[f(\lambda r)\lambda^{1-\beta} + (error) \right] \mathrm{d}s \right) \right). \end{aligned} \tag{4.7}$$

We note that λ is related to N by a power law, which we describe in (4.10) below. We note that we will have that $N \gg \lambda$ for β close to 0 and $\lambda \gg N$ for β close to 1.

In order to keep track of the regularity of the solution $\omega(t)$ of the Euler equations (4.2) with initial data (4.6), we first show (in Section 4.4.1) that for any $T > 0$ we can choose λ large enough so that

$$\omega(t) = \omega_{osc}(t) + \omega_{rad}(t) \qquad \text{for } t \in [0,T],$$

where ω_{osc} and ω_{rad} remain localized in space. We also show that the influence of ω_{osc} on ω_{rad} is exponentially small in N, so that $\overline{\omega_{rad}}$ approximates ω_{rad},

$$\|\omega_{rad} - \overline{\omega_{rad}}\|_{L^2} \le \mathrm{e}^{-\frac{N}{2}} \qquad \text{on } [0,T].$$

This can be proved by an energy estimate on $W := \omega_{rad} - \overline{\omega_{rad}}$, which shows that $\|W\|_{L^2}$ grows exponentially in time of order $\mathrm{e}^{\lambda^{1-\beta} t}$, as well as by the localization of ω_{osc} and ω_{rad}, and a Paley-Wiener-type estimate, which shows that the growth of $\|W\|_{L^2}$ is dominated, on time interval $[0,T]$, by an $O(\mathrm{e}^{-N})$ smallness of the influence of ω_{osc} onto ω_{rad}, see Lemma 4.4.1 for details.

Next, in order to make sure that the evolution of ω_{osc} is governed, to a leading order, by $v[\overline{\omega_{rad}}]$ (i.e. that ω_{osc} can be approximated by $\overline{\omega_{osc}}$), we need to show that $v[\overline{\omega_{rad}}]$ can be approximated by $v[\omega_{rad}]$, and that its effect is not overpowered by $v[\omega_{osc}]$. We address the latter issue by proving that

$$\|\omega_{osc}(t)\|_{C^1} \le \lambda^{2-\beta} N^{1-\beta} \exp(C\lambda^{1-\beta}) \tag{4.8}$$

(see Lemma 4.4.2). We then show that ω_{osc} can be approximated by $\overline{\omega_{osc}}$ by noting that the oscillatory part $\overline{\omega_{osc}}$ of the pseudosolution (4.7) satisfies the same PDE as ω_{osc}, except that the velocity field is averaged over α, which allows us to use Lagrangian trajectories to show that

$$\|\omega_{osc} - \overline{\omega_{osc}}\|_{L^2} \le C\lambda^{2-3\beta} N^{-2\beta} \log N. \tag{4.9}$$

Indeed, the above estimate can be obtained by noting that the radius of the Lagrangian trajectory of $\overline{\omega_{osc}}$ remains constant throughout the flow, as well as using a version of the classical

Log-Lipschitz velocity estimate in polar coordinates (4.22)–(4.23), a resulting L^∞ radial velocity estimate (4.24) and the C^1 estimate (4.8).

At this point we pick any sufficiently small $\delta > 0$ such that

$$\beta_\delta := \frac{(2+\delta-\beta)\beta}{2+\delta-\beta^2} > \frac{(2-\beta)\beta}{2-\beta^2}$$

and we relate λ and N by

$$\lambda^{2-2\beta+\delta} = N^\beta. \tag{4.10}$$

Such choice suffices for the above arguments, as well as lets us observe the norm inflation claimed by Theorem 4.1.1.
Indeed, it shows that the right-hand side of (4.9) can be estimated by $C\|\overline{\omega_{osc}}\|_{L^2}\lambda^{-\delta/2}$, and consequently we can use a Sobolev interpolation argument to show that, for any $\beta' > \beta_\delta$ and sufficiently large λ,

$$\|\overline{\omega_{osc}}(t)\|_{\dot{H}^{\beta'}} \geq C\lambda^{\beta'(2-\beta)-\beta}N^{\beta'-\beta} \geq C\lambda^{\widetilde{\epsilon}} \qquad \text{for } t\in[1/T,T],$$

where $\widetilde{\epsilon} > 0$ is a positive (and small) number, and so the claim of Theorem 4.1.1 follows by taking λ sufficiently large.
We note that, in order to obtain the last inequality, one needs to be able to estimate from below the size of the H^s norms of the pseudosolution $\overline{\omega}(t)$ for $s\in(0,1)$. While we can use the explicit formula (4.7) for the pseudosolution, we note that it is merely "almost explicit", which makes the issue nontrivial. We show that the error term can be estimated in C^1 by a fractional power of the C^1 norm of the leading order term $f(\lambda r)\lambda^{1-\beta}$, but this by itself still does not suggest a way of computing a lower bound on $\|\overline{\omega}\|_{H^s}$ using an explicit formula, i.e. the Sobolev-Slobodeckij representation. Instead, we use the Sobolev interpolation $\|\cdot\|_{\dot{H}^r} \leq c\|\cdot\|_{\dot{H}^s}^{\frac{r-q}{s-q}}\|\cdot\|_{\dot{H}^q}^{\frac{s-r}{s-q}}$, and we choose $r=0$ and $q<0$. This way we can make use of the L^2 conservation of $\overline{\omega}$ to obtain a lower bound, and we need to estimate a negative Sobolev norm of $\overline{\omega}$ from above. We provide a subtle argument that provides robust estimate of such form, which can also take into account the error term, see Lemma 4.3.5 for details.

As for Theorem 4.1.2 we note that taking λ larger in the above argument increases the norm inflation, and ensures that it occurs on a larger time interval. Moreover, it also makes the solution more localized. Thus, for each j we can construct a solution ω_j to the 2D Euler equations (4.2) such that

$$\|\omega_j(\cdot,t)\|_{H^s} \geq 4^j \qquad \text{for } s > \frac{(2-\beta)\beta}{2-\beta^2}+\frac{1}{j}, t\in[4^{-j},1], \tag{4.11}$$

$$|\operatorname{supp}\omega_j| \leq 2^{-j} \quad \text{for } t\geq 0, \text{ with } \quad \operatorname{supp}\omega_j \subset B_1(0) \quad \text{for } t\in[0,2^j] \tag{4.12}$$

and

$$\|\omega_j(\cdot,t)\|_{L^p} = C \quad \text{for all } t\in[0,1],\ p\in[1,2/(1-\beta)]\supset[1,2]. \tag{4.13}$$

Thus considering the rescalings

$$\frac{1}{2^j}\omega_j\left(x,\frac{t}{2^j}\right), \tag{4.14}$$

we obtain the norm inflation of order 2^j on time interval $[2^{-j},2^j]$, which expands to $(0,\infty)$ as $j\to\infty$. We can therefore consider a series of the rescalings (4.14), translated in the x_1 direction by a rapidly increasing sequence distances R_j, defined by $R_0 := 0$, $R_{j+1} := R_j + D_j + D_{j+1}$ for some large D_j's, see Figure 4.2 below and (4.64). Let us denote the corresponding translations of (4.14) by $\tilde{\omega}_j(x,t)$.
In order to obtain the claimed gap loss of Sobolev regularity, we first perform a subtle limiting argument to show existence of a solution to the 2D Euler equations (4.2) with the corresponding initial data. In fact, we show strong convergence of the classical solution for a truncated initial condition (i.e. consisting of the first J pieces, $J\geq 0$) in $C_t^0H_x^4(K)$ for any compact set $K\subset\mathbb{R}^2\times[0,\infty)$, which gives us a limit ω_∞ that is a classical solution in the sense of Definition 4.1.3 above, see (4.66) for details.

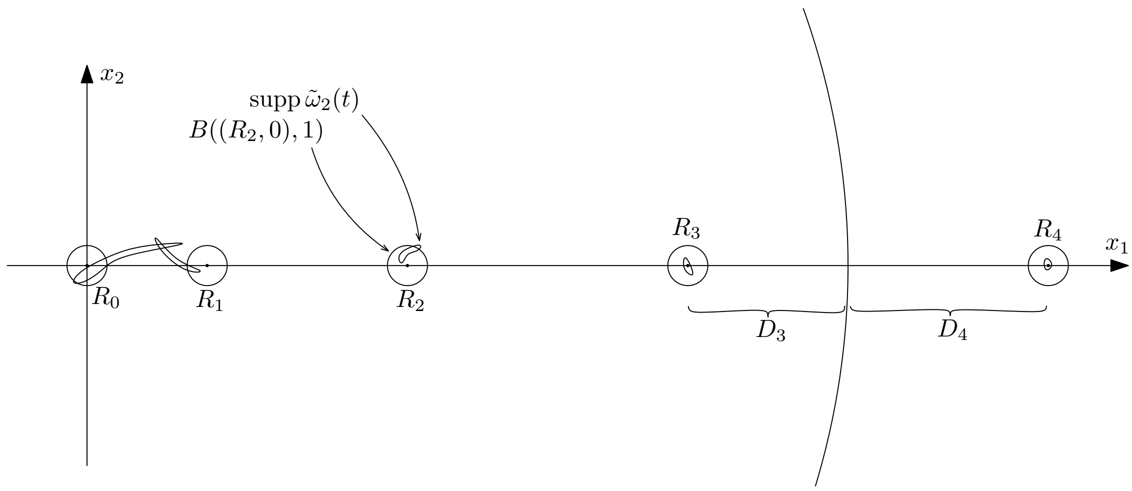

Figure 4.2: A sketch of the gluing argument. This shows the support of the first few individual pieces $\tilde{\omega}_j$ at some time $t \in (2^{-j}, 2^j)$, where $j = 4$. Note that, given j and $t \in [0, 2^j]$, supp $\tilde{\omega}_k(t) \subset B_1(R_k, 0)$ for $k \geq j$.

We can then observe that, given $t > 0$ and $\beta' > (2-\beta)\beta/(2-\beta^2)$, we can pick a sufficiently large j so that the norm inflation (4.11) implies arbitrarily large $H^{\beta'}$ norm of a j-th piece of ω_∞, and we need to make sure that the pieces do not interact with each other too much to affect this norm inflation.
To this end we note that the pieces are localized, in the sense that, given $t \in [0, 2^j)$, the support of $\tilde{\omega}_j$ is contained within $B_1(R_j, 0)$. This, together with (4.12) gives us an increasingly better control as $j \to \infty$. On the other hand, for the small values of j, we lose the control of the individual pieces (which can, for example, leave $B_1(R_j, 0)$ and interact with each other), but the support of all pieces has measure bounded by 1 and is included in $B_{R_j+D_j}(0)$, which implies that it is separated from further pieces, see Fig. 4.2 for a sketch. This can be obtained thanks to the L^p norm control (4.13), which implies a finite maximal speed v_{max}, and a choice of the D_j's (see (4.64)), as well as the fact that our $C_t^0 H_x^4$-loc argument lets us obtain property of our constructed limit ω_∞.
This control of the distances between pieces of ω_∞ lets us show that, given $t \in (2^{-j}, 2^j)$, the norm inflation of the j-th piece of ω_∞ is not affected by either the following pieces or by the sum of the previous pieces, see (4.70) for details. We emphasize that this argument implies not only that $H^{\beta'}$ regularity is lost instantly at $t = 0$, but also remains lost for all $t > 0$.
A similar argument can be used to show uniqueness of ω_∞, except that we need to make use of the both properties of the localization: the control of the distances between pieces and the measure of their supports. Moreover, we need to use Lagrange trajectories to keep track of the trajectories of the particles originating from each piece (see (4.71)). These facts, together with the C^1 bounds of each of the pieces at $t = 0$ (see (4.75)) and estimates of the Biot-Savart law (4.3), let us estimate the C^1 norm of the vorticity evolving from each piece (see (4.76) for details), given any solution in the sense of Definition 4.1.3, and establish a minimal growth of the R_j's (which involves 4 exponential functions in j, see (4.78)), that allows an L^2-based uniqueness proof (see (4.81)–(4.84) for the main setup). In fact, supposing there are two distinct solutions that coincide until some time $T \geq 0$, we pick a $j_0 \in \mathbb{N}$ (dependent on T) that identifies the piece after which the uniqueness is unlikely to occur. Namely we pick j_0 such that $2^{j_0} \sim T$ (e.g. $j_0 = 4$ in Fig. 4.2), which, for each $j \geq j_0$, allows us to efficiently control the C^1 norm of the vorticity originating from the j-th piece. As a result we can make the final choice of the initial distances between pieces (see (4.78)), such that, for each such j, the L^2 norms of the differences between the j-th pieces of the two distinct solutions can be estimated by a constant that is arbitrarily small with respect to j (see (4.82)). In order to make the resulting sum convergent, we simply pick j^{-2} (see (4.83)). On the other hand, we apply a rougher estimate for $j < j_0$ (see (4.84)) to obtain an L^2

estimate covering all such pieces at the same time. This gives uniqueness by a simple argument by contradiction (see (4.85) for details).

4.1.3 Outline of the chapter

In Section 4.2 we give some basic notation that we will use throughout the chapter, as well as some preliminary facts. In Section 4.3 we obtain some technical bounds related to the Biot-Savart law (4.3) as well as an upper bound on a negative Sobolev norm of functions used our construction. In section 4.4 we give the family of initial conditions that allows us to show Sobolev norm inflation and we prove such growth. Finally, in section 4.5, we show that a gluing argument allows us to build a global in time solution that losses regularity, and we show that it is the unique classical solution with the given initial conditions.

4.2 Notation and preliminaries

Throughout the chapter we will use functional norms, such as H^{β} for example, which refers to the spatial variables, that is $\|f(x,t)\|_{H^{\beta}}$ will refer to the spatial H^{β} norm for the specific time (or times) considered. We denote by ∂_t the partial derivative with respect to t, and by ∂_i the partial derivative with respect to x_i, $i=1,2$.
The only exception to this rule appears in Section 4.5 below, where we prove loss of regularity in a way that requires different treatment of of the space and time regularity. In order to avoid confusion, we will use sub-indexes to indicate the relevant variable for a norm; for example $\|f(x,t)\|_{C^1_x}$ would denote the spatial C^1 norm (for a fixed t) and $\|f(x,t)\|_{C^1_{x,t}}$ would denote the C^1 in both space and time.

We will use the following ODE fact:

$$\text{If } f'(t)\le cf(t)+b \text{ and } f(0)=0 \text{ then } f(t)\le \frac{b}{c}\left(\mathrm{e}^{ct}-1\right)\le bt\mathrm{e}^{ct}. \tag{4.15}$$

We will make use of polar coordinates, namely, given $(x_1,x_2)\in\mathbb{R}^2$, we define $(r,\alpha)\in[0,\infty)\times(-\pi,\pi]$ by $x_1=r\cos(\alpha)$, $x_2=r\sin(\alpha)$.
Moreover, given $f(r,\alpha)\colon\mathbb{R}^2\to\mathbb{R}$, we denote by

$$Af(r):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(r,\alpha)\,\mathrm{d}\alpha$$

the average of f with respect to α.
Since most of the specific computations will be performed in polar coordinates, we will often say that a function is $\frac{2\pi}{N}-$periodic if, in polar coordinates, $f(r,\alpha)=f(r,\alpha+\frac{2\pi}{N})$.
Moreover, we will use v_r and v_α to denote the radial and angular components of the velocity respectively.

Furthermore, we recall that,

$$\|v[\omega]\|_{W^{1,\infty}}\lesssim\|\omega\|_\infty\log\|\omega\|_{W^{1,\infty}} \tag{4.16}$$

for compactly supported ω, and so, if $\omega_0\in C_0^\infty(\mathbb{R}^2)$, then the unique solution ω of the Euler equations (4.2) satisfies

$$\|\omega(t)\|_{C^1}\le\|\omega_0\|_{C^1}+C\int_0^t\|v[\omega]\|_{C^1}\|\omega\|_{C^1}\le\|\omega_0\|_{C^1}+C\int_0^t\|\omega\|_{C^1}\|\omega\|_{L^\infty}\log\|\omega\|_{C^1}$$

for every $t>0$ (which can be proved by considering $\|\nabla\omega\|_{L^p}$ and taking $p\to\infty$). Thus, since $\|\omega(t)\|_{L^\infty}\le\|\omega_0\|_{L^\infty}\le\|\omega_0\|_{C^1}$, we obtain in particular that

$$\|\omega(t)\|_{C^1}\lesssim\mathrm{e}^{M\mathrm{e}^{CMt}}, \tag{4.17}$$

where $M := \|\omega_0\|_{C^1}$.
Finally, we recall the Sobolev-Slobodeckij characterization

$$\|f\|_{\dot{H}^s}^2 = C_s \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^{2+2s}} \mathrm{d}x\,\mathrm{d}y \qquad \text{for } s\in(0,1),$$

see [48, Proposition 3.4] for a proof. In particular, if $\{f_j\}_j$ is a family of disjointly supported functions in $\mathbb{R}^2$, then

$$\left\|\sum_j f_j\right\|_{\dot{H}^s}^2 \geq \sum_j \|f_j\|_{\dot{H}^s}^2 . \tag{4.18}$$

4.3 Velocity and vorticity estimates

In this section we study some properties of the vorticity function and the velocity fields given by the Biot-Savart law (4.3). We also estimate H^β norms of vorticity functions given in terms of an oscillatory ansatz.
First we note that if ω is a smooth solution of the Euler equations (4.2) with initial data ω_0, then

$$\omega(t) \text{ is } 2\pi/N\text{-periodic for all } t>0 \text{ if } \omega_0 \text{ is.} \tag{4.19}$$

Indeed, if $\omega(t)$ is not $2\pi/N$-periodic at any time $t>0$ then $\omega(R_{2\pi/N}x,t)$ is another solution to the Euler equations with the same ($2\pi/N$-periodic) initial data, which contradicts uniqueness, where R_α denotes the rotation operation by $\alpha\in[0,2\pi)$ in $\mathbb{R}^2$.

4.3.1 The Log-Lipschitz estimate

Lemma 4.3.1 (Log-Lipschitz continuity of v_r and v_α)**.** *Suppose that* supp $f \subset \Omega := B_R(0)\setminus B_{R/2}(0)$*. Then*

$$|v_r[f](x)-v_r[f](y)| \leq C\|f\|_\infty |x-y|\,(1+\log(R/|x-y|)) \tag{4.20}$$

and

$$|v_\alpha[f](x)-v_\alpha[f](y)| \leq C\|f\|_\infty |x-y|\,(1+\log(R/|x-y|)) \tag{4.21}$$

for any $x,y\in\Omega$.

Proof of Lemma 4.3.1. The proof is a modification of the classical proof (due to Yudovich [103]) of the log-Lipschitz bound on $v[f]$.
We first recall that by (4.3)

$$v_r[f](x) = \int_{\mathbb{R}^2} \widehat{x}\frac{(x-y)^\perp f(y)}{|x-y|^2}\mathrm{d}y$$

Let $x_1,x_2\in\Omega$ and $\delta := |x_1-x_2|$. Then

$$\begin{aligned}
|v_r[f](x_1)-v_r[f](x_2)| &\leq \int_{B_{2\delta}(x_1)} \frac{|f(y)|}{|x_1-y|}\mathrm{d}y + \int_{B_{2\delta}(x_1)} \frac{|f(y)|}{|x_2-y|}\mathrm{d}y \\
&\quad + \int_{\Omega\setminus B_{2\delta}(x_1)} |f(y)| \left| \widehat{x_1}\frac{(x_1-y)^\perp}{|x_1-y|^2} - \widehat{x_2}\frac{(x_2-y)^\perp}{|x_2-y|^2}\right| \mathrm{d}y \\
&\lesssim \|f\|_\infty \Bigg(\int_{B_{2\delta}(0)} |y|^{-1}\mathrm{d}y + \int_{B_{3\delta}(0)} |y|^{-1}\mathrm{d}y \\
&\quad + \int_{\Omega\setminus B_{2\delta}(x_1)} \left(\frac{|\widehat{x_1}-\widehat{x_2}|}{|x_1-y|} + \left|\frac{(x_1-y)^\perp}{|x_1-y|^2} - \frac{(x_2-y)^\perp}{|x_2-y|^2}\right| \right)\mathrm{d}y \Bigg) \\
&\lesssim \|f\|_\infty \left(\delta + \int_{\Omega\setminus B_{2\delta}(x_1)} \left(\frac{\delta}{R|x_1-y|} + \frac{\delta}{|x_*-y|}\right)\mathrm{d}y \right),
\end{aligned}$$

where x_* is a point between x_1, x_2. We now note that $R \gtrsim |x_1 - y|$ and that $|x_* - y| \sim |x_1 - y|$ to obtain

$$|v_r[f](x_1) - v_r[f](x_2)| \lesssim \|f\|_\infty \delta \left(1 + \int_{\Omega \setminus B_{2\delta}(x_1)} |x_1 - y|^{-2} \mathrm{d}y\right) \lesssim \|f\|_\infty \delta \left(1 + \log(R/\delta)\right),$$

as required.
A similar argument gives the same result for v_α. □

Corollary 4.3.2. *Suppose that* supp $f \subset \Omega := B_R(0) \setminus B_{R/K}(0)$ *for some* $K > 1$. *Then*

$$|v_r[f](x) - v_r[f](y)| \leq C\|f\|_\infty |x - y| \left(1 + \log(R/|x - y|)\right) \tag{4.22}$$

and

$$|v_\alpha[f](x) - v_\alpha[f](y)| \leq C\|f\|_\infty |x - y| \left(1 + \log(R/|x - y|)\right) \tag{4.23}$$

for any $x, y \in \Omega$.

This allows us to prove some improved control over the L^∞ bounds of velocities produced by $\frac{2\pi}{N}$−periodic functions.

Lemma 4.3.3. *If* supp $\omega \subset \Omega := B_R(0) \setminus B_{R/K}(0)$ *for some* $K > 1$, ω *is* $2\pi/N$*-periodic then*

$$\|v_r[\omega]\|_{L^\infty(\Omega)} \leq CR\|\omega\|_{L^\infty} \log(N)/N. \tag{4.24}$$

Given Corollary 4.3.2, we can prove (4.24) by noting that

$$A(v_r[\omega]) = 0$$

for any ω (by incompressibility). Moreover, ω is $2\pi/N$-periodic, which implies the same for $v_r[\omega]$. This means that, given $x \in \Omega$ there exists $y \in \Omega$ such that $v_r[\omega](y) = 0$ and $|x-y| \sim C \operatorname{diam}(\Omega)/N$. Thus an application of Corollary 4.3.2 gives

$$\begin{aligned} |v_r[\omega](x)| = |v_r[\omega](x) - v_r[\omega](y)| &\leq C\|\omega\|_\infty |x - y| \left(1 + \log(R/|x - y|)\right) \\ &\leq CR\|\omega\|_\infty \log(N)/N, \end{aligned} \tag{4.25}$$

as required.

4.3.2 An $\exp(-N)$ decay of the radial velocity of $2\pi/N$-periodic vorticities

Here we show that a compactly supported vorticity function that is $2\pi/N$-periodic generates a velocity field whose radial part decays exponentially fast as $N \to \infty$.

Lemma 4.3.4. *Let* $\omega \in L^\infty(\mathbb{R}^2)$ *be* $2\pi/N$*-periodic and such that* supp $\omega \subset B_{a_2}(0) \setminus B_{a_1}(0)$. *Then*

$$|v_r[\omega](r, \alpha)| \lesssim (a_2 - a_1)\|\omega\|_{L^\infty} \mathrm{e}^{-N} \tag{4.26}$$

for $r \in [0, a_1^2/12a_2] \cup [40a_2, \infty)$.

Proof. First note that if $\omega(r, \alpha) = g(r)\sin(N\alpha)$

$$v_r[\omega](r, \alpha) = \cos(N\alpha)\text{p.v.} \int_{\mathbb{R}} \int_{-\pi}^{\pi} (r + h)^2 \frac{\sin\alpha' g(r + h)\sin(N\alpha')}{h^2 + 2(r + h)r(1 - \cos\alpha')} \mathrm{d}\alpha' \mathrm{d}h, \tag{4.27}$$

and a similar formula holds if $\sin(N\alpha)$ is replaced by $\cos(N\alpha)$.
In order to analyze (4.27), we first consider

$$f(z) := \frac{\sin z}{C + (1 - \cos z)}$$

where $C>0$, and we note that f is holomorphic in $\mathbb{C}\setminus\{x+iy\colon x=2k\pi, y=-\log(1+C\pm\sqrt{C^2+2C})\}$ and 2π-periodic in the real direction. Thus, by the Cauchy Theorem,

$$\left|\int_{-\pi}^{\pi} f(z)\mathrm{e}^{iNz}\mathrm{d}z\right| = \left|\int_{-\pi}^{\pi} f(i\gamma+z)\mathrm{e}^{iN(i\gamma+z)}\mathrm{d}z\right| \leq 2\pi\mathrm{e}^{-\gamma N}\sup_{\mathbb{R}\times\{|\mathrm{Im}|\leq\gamma\}}|f|,$$

where

$$\gamma := \frac{1}{2}\log(1+C+\sqrt{C^2+2C}), \tag{4.28}$$

and we used the fact that $-\log(1+C-\sqrt{C^2+2C})=\log(1+C+\sqrt{C^2+2C})$. Since for $y\in[-\gamma,\gamma]$ we have $|\cos z|\leq\cosh y\leq\mathrm{e}^{\gamma}\leq\sqrt{1+C+\sqrt{C^2+2C}}$, and so

$$|C+1-\cos z|\geq C+1-\mathrm{e}^{\gamma}\geq C+1-\sqrt{1+C+\sqrt{C^2+2C}}\geq 1$$

for $C\geq 5$, we obtain that $|f|\lesssim 1$ for such C. In particular, since also $\gamma\geq 1$ for $C>5$, we obtain that

$$\left|\int_{-\pi}^{\pi} f(x)\sin(Nx)\mathrm{d}x\right|\leq 2\pi\mathrm{e}^{-N} \tag{4.29}$$

for such C.

Given $r>0$ we expand $\omega(r,\alpha)$ into Fourier series in α. Due to to $2\pi/N$-periodicity we have

$$\omega(r,\alpha)=\sum_{k\geq N}\left(g(r,k)\cos(k\alpha)+h(r,k)\sin(k\alpha)\right),$$

where

$$g(r,k)+ih(r,k) := \int_{-\pi}^{\pi}\omega(r,\alpha)\mathrm{e}^{ik\alpha}\mathrm{d}\alpha.$$

Clearly $|g(r,k)|,|h(r,k)|\leq 2\pi\|\omega\|_\infty$ for each r,k. Moreover, since $r\in[0,a_1^2/12a_2]\cup[40a_2,\infty)$, a direct computation shows that

$$C:=\frac{h^2}{2(r+h)r}\geq 5$$

for each $h\in[a_1-r,a_2-r]$. Thus, given $k\geq N$, we can apply (4.29) (and an analogous estimate for cos) to obtain

$$\begin{aligned}
|v_r[\omega](r,\alpha)| &\leq \int_{a_1-r}^{a_2-r}\frac{r+h}{2r}\sum_{k\geq N}\left(\left|\int_{-\pi}^{\pi}\frac{\sin\alpha' g(r+h,k)\cos(k\alpha')}{C+1-\cos\alpha'}\mathrm{d}\alpha'\right|\right.\\
&\qquad\left.+\left|\int_{-\pi}^{\pi}\frac{\sin\alpha' h(r+h,k)\sin(k\alpha')}{C+1-\cos\alpha'}\mathrm{d}\alpha'\right|\right)\mathrm{d}h\\
&\lesssim \|\omega\|_\infty\int_{a_1-r}^{a_2-r}\frac{r+h}{2r}\left(\sum_{k\geq N}\mathrm{e}^{-k}\right)\mathrm{d}h\\
&\lesssim (a_2-a_1)\|\omega\|_\infty\mathrm{e}^{-N},
\end{aligned}$$

as required. □

4.3.3 Sobolev norms for high frequency ansatz

In this section we prove a technical lemma that allows us to bound from above a negative-order homogeneous Sobolev norm of certain functions supported in an annulus in $\mathbb{R}^2$.

Lemma 4.3.5. *Given $\epsilon\in(0,1)$, $\delta\in(0,\epsilon)$, and $f\in C^2([1/2,4])$ with $f'>0$ in $[1/2,4]$, there exist $C,K_0\geq 1$ such that*

$$\omega_K(r,\alpha):=g(r)\cos(N\alpha-Kf(r)+f_{err}(r))$$

satisfies

$$\|\omega_K\|_{\dot{H}^{-\delta}}\leq CK^{-\delta}\|g\|_{C^1}$$

for every $g\in C_c^2((1/2,4))$, $K\geq K_0$, $N\in\mathbb{N}$ and $f_{err}\in C^1([1/2,4])$ such that $\|f_{err}\|_{C^1}\leq K^{1-\epsilon}$.

Proof. We first show that for $r \in (\frac{1}{4}, 6)$

$$|\Lambda^{-\delta}\omega_K| \leq CK^{-\delta}\|g\|_{L^\infty}. \tag{4.30}$$

We note that

$$\Lambda^{-\delta}\omega_K(r,\alpha) = C_\delta \int_{-\pi}^{\pi}\int_0^\infty \frac{\omega_K(r',\alpha')}{|(r-r')^2 + 2rr'(1-\cos(\alpha-\alpha'))|^{\frac{2-\delta}{2}}} r'\mathrm{d}r'\,\mathrm{d}\alpha'.$$

Using the change of variables $h = r' - r$, $\tilde{\alpha} = \alpha' - \alpha$, we can estimate the integral over the region $\{|\alpha'-\alpha| \leq 1/K\}$, by noting that $r, r+h = O(1)$, which implies that

$$\begin{aligned}
&\left|\int_{\alpha-\frac{1}{K}}^{\alpha+\frac{1}{K}}\int_0^\infty \frac{\omega_K(r',\alpha')}{|(r-r')^2 + 2rr'(1-\cos(\alpha-\alpha'))|^{\frac{2-\delta}{2}}} r'\,\mathrm{d}r'\mathrm{d}\alpha'\right| \\
&\qquad\leq C\|g\|_{L^\infty}\int_{-\frac{1}{K}}^{\frac{1}{K}}\int_{-\infty}^{\infty} \frac{1}{|h^2 + C\tilde{\alpha}^2|^{\frac{2-\delta}{2}}}\mathrm{d}h\,\mathrm{d}\tilde{\alpha} \\
&\qquad\leq C\|g\|_{L^\infty}\int_{-\frac{1}{K}}^{\frac{1}{K}}\int_{|\alpha|}^{\infty} \frac{1}{|h|^{2-\delta}}\mathrm{d}h\,\mathrm{d}\tilde{\alpha} \\
&\qquad\leq C\|g\|_{L^\infty}\int_{-\frac{1}{K}}^{\frac{1}{K}} |\tilde{\alpha}|^{-(1-\delta)}\mathrm{d}\tilde{\alpha} \leq C\|g\|_{L^\infty}K^{-\delta},
\end{aligned}$$

as claimed.
As for $|\tilde{\alpha}| > 1/K$, we first consider $h \in [0,4]$ and we divide this interval into $O(K)$ pieces of the form $[a, a + 2\pi/(Kf'(a+r))]$ and integrate by parts on each of them. Namely, given $a \in [0,4]$ we set

$$\begin{aligned}
u(\tilde{h}) &:= \int_a^{\tilde{h}} (r+h)g(r+h)\cos(N\alpha' - Kf(r+h) + f_{err}(r+h))\mathrm{d}h, \\
v(h) &:= |h^2 + 2r(r+h)(1-\cos\tilde{\alpha})|^{-\frac{2-\delta}{2}},
\end{aligned}$$

so that

$$\begin{aligned}
|v'(h)| &\leq C\frac{|2h + 2r(1-\cos\tilde{\alpha})|}{|h^2 + 2r(r+h)(1-\cos\tilde{\alpha})|^{\frac{4-\delta}{2}}} \leq C\left|h^2 + 2r(r+h)(1-\cos\tilde{\alpha})\right|^{-\frac{3-\delta}{2}}, \\
u'(h) &= (r+h)g(r+h)\cos(N\alpha' - Kf(r+h) + f_{err}(r+h)),
\end{aligned}$$

$u(a) = 0$, and we can estimate $u(h)$ for each $h \in (a, a + 2\pi/(Kf'(a+r))]$ by the brutal bound

$$|u(h)| \leq CK^{-1}\|g\|_{L^\infty}.$$

This gives that

$$\begin{aligned}
&\left|\int_a^{a+\frac{2\pi}{Kf'(a+r)}} \frac{g(r+h)\cos(N\tilde{\alpha} - Kf(r+h) + f_{err}(r+h))}{|h^2 + 2r(r+h)(1-\cos(\tilde{\alpha}))|^{\frac{2-\delta}{2}}}(r+h)\mathrm{d}h\right| \\
&= \left|\int_a^{a+\frac{2\pi}{Kf'(a+r)}} u'(h)v(h)\mathrm{d}h\right| \\
&\leq \frac{C\|g\|_{L^\infty}}{K}\int_a^{a+\frac{2\pi}{Kf'(a+r)}} v(h)\mathrm{d}h + v\left(a + \frac{2\pi}{Kf'(a+r)}\right)\left|u\left(a + \frac{2\pi}{Kf'(a+r)}\right)\right| \\
&\leq \frac{C\|g\|_{L^\infty}}{K}\int_a^{a+\frac{2\pi}{Kf'(a+r)}} |h^2 + C\tilde{\alpha}^2|^{-\frac{3-\delta}{2}}\mathrm{d}h + \frac{C\|g\|_{C^1}K^{-1-\epsilon}}{|(a + \frac{2\pi}{Kf'(a+r)})^2 + C\tilde{\alpha}^2|^{\frac{2-\delta}{2}}},
\end{aligned}$$

where we used the fact that $1-\cos\tilde\alpha \geq C(\tilde\alpha)^2$ as well as the fact that

$$\begin{aligned}\left|u\left(a+\frac{2\pi}{Kf'(a+r)}\right)\right| &= \Big|\int_a^{a+\frac{2\pi}{Kf'(a+r)}}\Big(g(r+h)\cos(N\alpha'-Kf(r+h)+f_{err}(r+h))\\ &\qquad -g(r)\cos(N\alpha'-Kf(r+a)-hKf'(r+a)+f_{err}(r+a))\Big)\mathrm{d}h\Big|\\ &\leq C\|g\|_{C^1}K^{-1-\epsilon},\end{aligned}$$

by adding and subtracting the mixed terms, and noting that the difference of the g's gives $C\|g\|_{C^1}K^{-2}$, the second order Taylor expansion of f gives the bound $C\|g\|_{L^\infty}K^{-2}$, and the assumption on f_{err} gives $C\|g\|_{L^\infty}K^{-1-\epsilon}$.
Thus, letting $a:=h_i$, where $h_0:=0$, $h_{i+1}:=h_i+\frac{2\pi}{Kf'(h_i+r)}$ for $i=0,\dots,i_0$, where i_0 is the largest integer such that $h_{i_0}\leq 4$, we obtain that $h\in(h_i,h_{i+1})$ for some $i\in\{0,\dots,i_0\}$ whenever $r+h\in \text{supp } g\cap[r,\infty)$, and

$$\begin{aligned}&\left|\int_0^4 \frac{g(r+h)\cos(N\alpha'-Kf(r+h)+f_{err}(r+h))}{|h^2+2r(r+h)(1-\cos\tilde\alpha)|^{\frac{2-\delta}{2}}}(r+h)\mathrm{d}h\right|\\ &\leq C\sum_{i=0}^{i_0}\left(\frac{\|g\|_{L^\infty}}{K}\int_{h_i}^{h_{i+1}}\frac{1}{|h^2+C\tilde\alpha^2|^{\frac{3-\delta}{2}}}\mathrm{d}h+\frac{\|g\|_{L^\infty}K^{-1-\epsilon}}{|h_{i+1}^2+C\tilde\alpha^2|^{\frac{2-\delta}{2}}}\right)\\ &\leq \frac{C\|g\|_{L^\infty}}{K}\int_0^{4+\frac{1}{K}}\frac{1}{|h^2+C\tilde\alpha^2|^{\frac{3-\delta}{2}}}\mathrm{d}h+\frac{C\|g\|_{C^1}K^{-\epsilon}}{|\tilde\alpha|^{1-\delta}}\end{aligned}$$

and a similar computation can be done for $h\in(-(r-1/8),0]$, which allows us to cover $r+h\in$ supp $g\cap(0,r)$. With this, in particular

$$\begin{aligned}&\left|\int_{\pi\geq|\tilde\alpha|\geq\frac{1}{K}}\int_{-r+\frac18}^4\frac{g(r+h)\cos(N\alpha'-Kf(r+h)+f_{err}(r+h))}{|h^2+2r(r+h)(1-\cos(\tilde\alpha))|^{\frac{2-\delta}{2}}}(r+h)\mathrm{d}h\,\mathrm{d}\tilde\alpha\right|\\ &\leq C\int_{\pi\geq|\tilde\alpha|\geq\frac1K}\left(\frac{\|g\|_{L^\infty}}{K}\int_0^{4+\frac1K}\frac{1}{|h^2+C\tilde\alpha^2|^{\frac{3-\delta}{2}}}\mathrm{d}h+\frac{\|g\|_{C^1}K^{-\epsilon}}{|\tilde\alpha|^{1-\delta}}\right)\mathrm{d}\tilde\alpha\\ &\leq C\|g\|_{C^1}\int_{\pi\geq|\tilde\alpha|\geq\frac1K}\left(\frac{1}{K|\tilde\alpha|^{2-\delta}}+\frac{1}{K^\epsilon|\tilde\alpha|^{1-\delta}}\right)\mathrm{d}\tilde\alpha\leq C\|g\|_{C^1}(K^{-1}K^{1-\delta}+K^{-\epsilon})\\ &\leq C\|g\|_{C^1}K^{-\delta}.\end{aligned}$$

Next we need to show some bounds for $r\leq\frac14$ and $r\geq 6$. For $r\in(0,1/4)$ we need $h\in(1/4,4)$ (so that $r+h\in$ supp g). Thus letting $h_0:=\frac14$, $h_{i+1}:=h_i+\frac{2\pi}{Kf'(r+h_i)}$ and letting $i_0\in\mathbb{N}$ be the largest integer such that $r+h_{i_0}\leq 4$, and applying integration by parts as before, we have

$$\begin{aligned}&\left|\int_{h_i}^{h_{i+1}}\frac{g(r+h)\cos(N\alpha'-Kf(r+h)+f_{err}(r+h))}{|h^2+2r(r+h)(1-\cos\tilde\alpha)|^{\frac{2-\delta}{2}}}(r+h)\mathrm{d}h\right|\\ &\leq \frac{C\|g\|_{L^\infty}}{K}\int_{h_i}^{h_{i+1}}\frac{1}{|h^2+Cr\tilde\alpha^2|^{\frac{3-\delta}{2}}}\mathrm{d}h'\\ &\qquad+\left(\frac{C}{|h_{i+1}^2+Cr\tilde\alpha^2|^{\frac{2-\delta}{2}}}\int_{h_i}^{h_{i+1}}g(r')\cos(N\alpha'-Kf(r+h)+f_{err}(r+h))\mathrm{d}h\right)\\ &\leq \frac{C\|g\|_{L^\infty}}{K}\int_{h_i}^{h_{i+1}}\frac{1}{|h^2+Cr\tilde\alpha^2|^{\frac{3-\delta}{2}}}\mathrm{d}h+\frac{C\|g\|_{C^1}K^{-1-\epsilon}}{|h_{i+1}^2+Cr\tilde\alpha^2|^{\frac{2-\delta}{2}}}.\end{aligned}$$

Thus, summing in i, and integrating in $\tilde\alpha\in\{|\tilde\alpha|\in(1/K,\pi)\}$ (recall that $\tilde\alpha=\alpha'-\alpha$), we obtain

$$|\Lambda^{-\delta}(\omega_K)(r,\alpha)|\leq C\|g\|_{C^1}(K^{-1}+K^{-\epsilon})\leq C\|g\|_{C^1}K^{-\delta}\quad\text{ for } r\in(0,1/4).$$

Similarly, for $r \in (6,\infty)$ we need $h \in (r-1/2, r-6)$, which gives the final bound of the form

$$|\Lambda^{-\delta}(\omega_K)(r,\alpha)| \le C\|g\|_{C^1}\left(\frac{K^{-1}}{(r-4)^{3-\delta}} + \frac{K^{-\epsilon}}{(r-4)^{2-\delta}}\right) \le C\|g\|_{C^1}\frac{K^{-\delta}}{(r-4)^{2-\delta}} \qquad \text{for } r>6.$$

Integrating the squares of the above pointwise estimates on $\Lambda^{-\delta}\omega_K$ gives the claimed L^2 bound. $\square$

4.4 Initial conditions and growth for smooth functions

Here we prove Theorem 4.1.1, that is we fix $\beta \in (0,1)$, $\beta' > (2-\beta)\beta/(2-\beta^2)$, and $K, T > 0$ and we construct $\omega_0 \in C_c^\infty(\mathbb{R}^2)$ such that $\|\omega_0\|_{H^\beta} \le 1$ and that the unique classical solution ω to the Euler equations admits growth $\|\omega\|_{H^{\beta'}} \ge K$ for $t \in [1/T, T]$.
To this end, we fix $\delta > 0$ sufficiently small so that

$$\beta_\delta := \frac{(2+\delta-\beta)\beta}{2+\delta-\beta^2} > \frac{(2-\beta)\beta}{2-\beta^2} \tag{4.31}$$

satisfies $\beta_\delta < \beta'$.
We will consider radial functions $f(r), g(r)$ such that $g \in C_c^\infty(\frac{1}{2}, 4)$ and $f \in C_c^\infty((a,b)\cup(c,d))$ and fulfilling

- $a \ge 10^3, d \le 10^{-4}$
- $\partial_r \frac{v_\alpha[f](r)}{r} \in (\frac{1}{M}, M)$ for some $M > 1$ when $r \in (\frac{1}{2}, 4)$,
- $\|f\|_{H^1}, \|g\|_{H^1} \le 1/20$
- $\int_0^\infty f(r) r \,\mathrm{d}r = 0.$

A function g fulfilling the requirement is trivial to obtain, but we need to justify that f with the required properties exists. For this, we first consider some arbitrary, positive $\tilde{f}(r) \in C_c^\infty(10^{-5}, 10^{-4})$. We will study, for $r \in (\frac{1}{2}, 4)$, $\partial_r \frac{v_\alpha[\lambda^2 \tilde{f}(\lambda\cdot)](r)}{r}$. First, we note that

$$v_\alpha[\lambda^2\tilde{f}(\lambda\cdot)](r) = \int_{-\pi}^{\pi}\int_0^\infty r' \frac{\lambda^2\tilde{f}(\lambda r')(r - r'\cos(\alpha))}{r^2 + (r')^2 + 2rr'(1-\cos(\alpha))}\mathrm{d}r'\,\mathrm{d}\alpha$$

so using the location of the support of $\tilde{f}(\lambda r)$ we have $v_\alpha(\tilde{f}(\lambda r)) \ge 0$. Furthermore, for $r \in (\frac{1}{2}, 4)$,

$$\begin{aligned}&\partial_r v_\alpha[\lambda^2\tilde{f}(\lambda\cdot)](r)\\ &= \int_{-\pi}^{\pi}\int_0^\infty r'\tilde{f}(\lambda r')\left(\frac{\lambda^2}{r^2+(r')^2+2rr'(1-\cos(\alpha))} - \frac{2\lambda^2(r-r'\cos(\alpha))^2}{(r^2+(r')^2+2rr'(1-\cos(\alpha)))^2}\right)\mathrm{d}r'\,\mathrm{d}\alpha\end{aligned}$$

and thus

$$\lim_{\lambda\to\infty} \partial_r v_\alpha[\lambda^2\tilde{f}(\lambda\cdot)](r) = -\frac{1}{r}\int_0^\infty 2\pi\tilde{f}(s)s\,\mathrm{d}s$$

so, if we take λ big enough then $\partial_r v_\alpha[\lambda^2 f(\lambda\cdot)](r) < 0$, and thus

$$\partial_r \frac{v_\alpha[\lambda^2\tilde{f}(\lambda\cdot)](r)}{r} < 0$$

for $r \in (\frac{1}{2}, 4)$, which implies that $-\lambda^2\tilde{f}(\lambda r)$ gives us the desired effect on the velocity for λ big, but this f would clearly not have zero average. To compensate for that, we now consider $\lambda^{-2}\tilde{f}(\frac{r}{\lambda})$ for $\lambda \ge 10^8$. It is easy to check that, as $\lambda \to \infty$, we have, for any $r \in (\frac{1}{2}, 4)$
$v_\alpha[\lambda^{-2}\tilde{f}(\frac{\cdot}{\lambda})](r) \to 0$ $\partial_r v_\alpha[\lambda^{-2}\tilde{f}(\frac{\cdot}{\lambda})](r) \to 0$ so that, for λ big enough

$$-\lambda^2 \tilde{f}(\lambda\cdot) + \lambda^{-2}\tilde{f}(\cdot/\lambda)$$

has the desired properties for the velocity and average value. Then, multiplication by some small constant $c > 0$ allows us to make the H^1 norm as small as we want.
We will thus consider some f and g with the desired properties and f and g will be fixed from now on, so in particular anything that depends only on the specific choice of f and g will just be a constant.
Given $\lambda > 0$ we now set

$$\omega_0 := \lambda^{1-\beta} f(\lambda r) + \lambda^{1-\beta} g(\lambda r) N^{-\beta} \cos(N\alpha), \tag{4.32}$$

where $f, g \in C_c^\infty(1/4, 2)$ are as above and N is related to λ via

$$\lambda^{2-2\beta+\delta} = N^\beta. \tag{4.33}$$

Note that in particular $\|\omega_0\|_{H^\beta} \leq 1$ for any $\lambda \geq 1$.
We denote by $\omega\colon \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ the unique solution of (4.2) with initial data ω_0. Before we can show the rapid growth of $\|\omega\|_{H^{\beta'}}$, we need to prove some basic properties of ω, which we discuss in Steps 1–3 below. We will keep in mind that $\lambda > 0$ is a large parameter that will be fixed in Step 4 (Section 4.4.4), where we will prove the growth of the $H^{\beta'}$ norm.

4.4.1 Step 1 - localization and control of ω_{rad}

We decompose ω into two parts, one that is mostly composed of highly oscillatory terms ω_{osc} and one that remains mostly radial ω_{rad}, namely, if $\phi(x,t)$ is the flow map given by $v[\omega]$ we define

$$\begin{aligned}\omega_{rad}(x,t) &:= \omega_{rad}(\phi^{-1}(x,t),0), \omega_{rad}(x,0) = \lambda^{1-\beta} f(\lambda r)),\\ \omega_{osc}(x,t) &:= \omega_{osc}(\phi^{-1}(x,t),0), \omega_{osc}(x,0) = \lambda^{1-\beta} g(\lambda r) N^{-\beta}\cos(N\alpha).\end{aligned}$$

Note that, with those definitions, we indeed have that

$$\omega(t) = \omega_{rad}(t) + \omega_{osc}(t). \tag{4.34}$$

We now show that these two parts barely change their support and, furthermore, ω_{rad} stays almost stationary.

Lemma 4.4.1. *For sufficiently large λ and $t \in [0,T]$, we have*

$$\begin{aligned}&\text{supp }\omega_{osc} \subset B_{6\lambda^{-1}}(0) \setminus B_{(4\lambda)^{-1}}(0),\\ &\text{supp }\omega_{rad} \subset B_{2b\lambda^{-1}}(0) \setminus B_{a\lambda^{-1}/2}(0) \cup B_{2c\lambda^{-1}}(0) \setminus B_{d\lambda^{-1}/2}(0),\end{aligned} \tag{4.35}$$

and furthermore

$$\|\omega_{rad} - \overline{\omega_{rad}}\|_{L^2} \leq \mathrm{e}^{-\frac{N}{2}} \tag{4.36}$$

with $\overline{\omega_{rad}} := \lambda^{1-\beta} f(\lambda r)$.

Proof. We first note that the claim of the lemma is valid at least for small times. We start by proving the localization, i.e., the bounds for the support of ω_{rad} and ω_{osc}. Noting that ω remains $2\pi/N$-periodic for all times, we can using Lemma 4.3.3 to deduce that for t such that (4.35) is satisfied, we have, for $r \in B_{2b\lambda^{-1}}(0)$

$$|v_r[w](r)| \leq C \log(N)(N\lambda)^{-\beta}$$

and, using the relationship between N and λ,

$$|v_r[w](r)| \leq C \log(N) \lambda^{-2+\beta-\delta}.$$

Thus (4.35) remains valid at least for $t \in [0, C\lambda^{1-\beta+\delta}(\log\lambda)^{-1}]$, and so taking λ large ensures that (4.35) holds until T.

As for (4.36), we note that, since $a \geq 10^3$, $d \leq 10^{-4}$, the assumption of Lemma 4.3.4 holds, and thus

$$\|v_r[\omega_{osc}]\|_{L^\infty(\text{supp }\omega_{rad})} \leq \|\omega_{osc}\|_\infty \mathrm{e}^{-N} \leq \|\omega_{osc,0}\|_\infty \mathrm{e}^{-N}.$$

Moreover, since

$$\partial_t \omega_{rad} + v(\omega_{rad}) \cdot \nabla \omega_{rad} + v(\omega_{osc}) \cdot \nabla(\omega_{rad}) = 0$$

letting $W := \omega_{rad} - \overline{\omega_{rad}}$ we see that

$$\partial_t W + v[W] \cdot \nabla W + v[W] \cdot \nabla \overline{\omega_{rad}} + v[\overline{\omega_{rad}}] \cdot \nabla W + v[\omega_{osc}] \nabla(W + \overline{\omega_{rad}}) = 0,$$

which gives us an evolution for the L^2 norm

$$\frac{\mathrm{d}\|W\|_{L^2}}{\mathrm{d}t} \leq C \left(\|v[W]\|_{L^2} \lambda^{2-\beta} + \mathrm{e}^{-N} \frac{\lambda^{3-2\beta}}{N^\beta} \right) \leq C \left(\|W\|_{L^2} \lambda^{1-\beta} + \mathrm{e}^{-N} \frac{\lambda^{3-2\beta}}{N^\beta} \right),$$

where we used that $\|v[W]\|_{L^2} \leq \frac{C}{\lambda}\|W\|_{L^2}$, since W is supported in a disc of radius $\frac{2b}{\lambda}$. In light of the ODE fact (4.15), this gives us a bound for the L^2 norm of

$$\|W(t)\|_{L^2} \leq C \mathrm{e}^{\lambda^{1-\beta} t - N} \frac{\lambda^{2-\beta}}{N^\beta} \tag{4.37}$$

which proves the second claim by taking λ large. $\square$

4.4.2 Step 2 - L^∞ control of $\nabla \omega_{osc}$

The H^s growth for our solutions will come from the effect of the velocity generated by ω_{rad} acting on ω_{osc}. However, we need to prove that this effect is not overpowered by the velocity generated by ω_{osc}. For that, we have the following lemma.

Lemma 4.4.2. *For sufficiently large* λ

$$\|\omega_{osc}(t)\|_{C^1} \leq \lambda^{2-\beta} N^{1-\beta} \exp(C\lambda^{1-\beta}) \qquad \textit{for } t \in [0, T]. \tag{4.38}$$

Proof. We first note that for any $C > 1$ (4.38) holds for some short time interval, say for $t \in [0, t_0]$. Moreover, observe that, for $t \in [0, t_0]$,

$$\partial_t \omega_{osc} + v[\omega_{osc}] \cdot \nabla \omega_{osc} + v[f(\lambda r)\lambda^{1-\beta} + \omega_{rad,err}] \cdot \nabla \omega_{osc} = 0,$$

with $\omega_{rad,err} := \omega_{rad} - f(\lambda r)\lambda^{1-\beta}$, and by Lemma 4.4.1 we have

$$\begin{aligned}
\frac{\mathrm{d}\|\omega_{osc}\|_{C^1}}{\mathrm{d}t} &\leq C(\|v[\omega_{osc}]\|_{C^1} + \lambda^{1-\beta})\|\omega_{osc}\|_{C^1} \\
&\leq C(\log(\|\omega_{osc}\|_{C^1})\|\omega_{osc}\|_{L^\infty} + \lambda^{1-\beta})\|\omega_{osc}\|_{C^1} \\
&\leq C(\log(\|\omega_{osc}\|_{C^1})\frac{\lambda^{1-\beta}}{N^\beta} + \lambda^{1-\beta})\|\omega_{osc}\|_{C^1}, \\
&\leq C\lambda^{1-\beta}\|\omega_{osc}\|_{C^1},
\end{aligned}$$

where we used the C^1 velocity estimate (4.16) in the second line, the L^∞ conservation of the vorticity in the third line, and the assumed bound of the C^1 norm in the last line. Thus

$$\|\omega_{osc}(t)\|_{C^1} \leq \|\omega_{osc}(t=0)\|_{C^1} \mathrm{e}^{C\lambda^{1-\beta} t} \leq \lambda^{2-\beta} N^{1-\beta} \mathrm{e}^{C\lambda^{1-\beta}}$$

for all $t \in [0, t_0]$, and a continuity argument completes the proof of (4.38). $\square$

4.4.3 Step 3 - L^2 control of the difference between ω_{osc} and the pseudosolution

In this section we show that the function

$$\overline{\omega_{osc}}(r,\alpha,t) := \lambda^{1-\beta} g(\lambda r) N^{-\beta} \cos\left(N\left(\alpha - \frac{1}{r}\int_0^t v_\alpha\left[f(\lambda r)\lambda^{1-\beta} + A(\omega_{osc}+\omega_{rad,err})\right] \mathrm{d}s\right)\right), \tag{4.39}$$

where

$$\omega_{rad,err} := \omega_{rad} - \overline{\omega_{rad}},$$

which is our guess for the behaviour of ω_{osc}, is actually a good approximation (in L^2) of ω_{osc}. We note that $\overline{\omega_{osc}} + \overline{\omega_{rad}}$ is the pseudosolution (4.7), which was discussed heuristically in the introduction (Section 4.1.2).

Note that this function corresponds to ω_{osc} advected with an averaged velocity, i.e.

$$\partial_t \overline{\omega_{osc}} + v[A\omega_{osc}] \cdot \nabla \overline{\omega_{osc}} + v[A\omega_{rad}] \cdot \nabla \overline{\omega_{osc}} = 0.$$

More precisely, we have the following lemma.

Lemma 4.4.3. *For λ big enough we have*

$$\|\omega_{osc,err}(t)\|_{L^2} \leq C\frac{1}{(\lambda N)^\beta}\frac{\lambda^{2-2\beta}\log N}{N^\beta} \leq C\|\overline{\omega_{osc}}(t)\|_{L^2}\lambda^{-\frac{\delta}{2}} \qquad \textit{for all } t \in [0,T], \tag{4.40}$$

where $\omega_{osc,err} := \omega_{osc} - \overline{\omega_{osc}}$.

Proof. To this end we define the flow maps between time s and time t (in polar coordinates)

$$\begin{aligned} \partial_t \phi(r,\alpha,s,t) &= (v[\omega]\circ\phi)(r,\alpha,s,t) \\ \phi(r,\alpha,s,s) &= (r\cos(\alpha), r\sin(\alpha)) \\ \partial_t \overline{\phi}(r,\alpha,s,t) &= (v[A\omega]\circ\overline{\phi})(r,\alpha,s,t) \\ \overline{\phi}(r,\alpha,s,s) &= (r\cos(\alpha), r\sin(\alpha)). \end{aligned}$$

Note that this definition allows for any $s,t \in \mathbb{R}$, but we will only be concerned with $t \in [0,s]$. We will denote the polar coordinates of ϕ by ϕ_r, ϕ_α (and analogously for $\overline{\phi}$), so that in particular, when we consider ω_{osc} in polar coordinates

$$\omega_{osc}(r,\alpha,s) = \omega_{osc}(\phi_r(r,\alpha,s,0), \phi_\alpha(r,\alpha,s,0), 0),$$

$$\overline{\omega_{osc}}(r,\alpha,s) = \omega_{osc}(\overline{\phi}_r(r,\alpha,s,0), \overline{\phi}_\alpha(r,\alpha,s,0), 0).$$

We first note that, since $v[A\omega]$ has no radial part,

$$\overline{\phi}_r(r,\alpha,s,t) = r$$

for all t. On the other hand, for ϕ_r we can use the L^∞ estimate (4.24) on $v_r[\omega]$ to obtain, for $r \in B_{6\lambda^{-1}}(0)$,

$$|\phi_r(r,\alpha,s,t) - r| \leq \left|\int_s^t \|v_r[\omega]\|_{L^\infty(B_{6\lambda^{-1}}(0))}\right| \leq C\lambda^{-\beta}N^{-1-\beta}\log(N) \tag{4.41}$$

for all $s,t \in [0,T]$.
As for the angular component we have

$$\partial_t(\overline{\phi}_\alpha - \phi_\alpha)(r,\alpha,s,t) = \frac{v_\alpha[A\omega-\omega]\circ\phi}{\phi_r(r,\alpha,s,t)} + \frac{v_\alpha[A\omega]\circ\overline{\phi}}{\overline{\phi}_r(r,\alpha,s,t)} - \frac{v_\alpha[A\omega]\circ\phi}{\phi_r(r,\alpha,s,t)}. \tag{4.42}$$

Noting that $A\omega - \omega$ has α-average zero, we can use Lemma 4.3.1 in the same way as in (4.25) to obtain that

$$\|v_\alpha[A\omega-\omega]\|_\infty \leq C\|\omega\|_\infty \frac{\log(N)}{\lambda N} \leq C\lambda^{-\beta}N^{-1-\beta}\log(N).$$

Moreover, since $v_\alpha[A\omega]$ does not depend on α we have

$$
\begin{aligned}
&|v_\alpha[A\omega]\circ\overline{\phi} - v_\alpha[A\omega]\circ\phi| \le |v_\alpha[A\omega](\overline{\phi}_r,0) - v_\alpha[A\omega](\phi_r,0)|\\
&\le \|v[A\omega]\|_{W^{1,\infty}(B_{6\lambda^{-1}}(0))}|\phi_r - \overline{\phi}_r|\\
&\le C\left(\|\omega_{osc}\|_{L^\infty}\log(\|\omega_{osc}\|_{W^{1,\infty}}) + \|v[\omega_{rad}]\|_{W^{1,\infty}(B_{6\lambda^{-1}}(0))}\right)\lambda^{-\beta}N^{-1-\beta}\log(N)\\
&\le C\left(\lambda^{1-\beta}N^{-\beta}\lambda^{1-\beta-\delta/2} + \lambda^{1-\beta}\right)\lambda^{-\beta}N^{-1-\beta}\log(N)\\
&\le C\lambda^{1-2\beta}N^{-1-\beta}\log(N),
\end{aligned}
$$

where we used (4.16) and (4.41) in the 3rd line, (4.36),(4.38) and the support separation (4.35) in the fourth line.
Finally, we have that, for $r \in (\frac{1}{4\lambda}, \frac{6}{\lambda})$

$$
\begin{aligned}
\frac{v_\alpha[A\omega]\circ\phi}{\phi_r(r,\alpha)} - \frac{v_\alpha[A\omega]\circ\phi}{\overline{\phi}_r(r,\alpha)} &\le C\lambda^2\lambda^{-\beta}N^{-1-\beta}\log(N)\lambda^{-\beta}\\
&\le C\lambda^{2-2\beta}N^{-1-\beta}\log(N)
\end{aligned}
$$

Thus, combining these bounds with (4.42) gives that

$$
\left|\partial_t(\overline{\phi}_\alpha - \phi_\alpha)\right| \le C\lambda^{2-2\beta}N^{-1-\beta}\log(N) \tag{4.43}
$$

for all $s,t\in[0,T]$, and in particular

$$
|\overline{\phi}_\alpha(r,\alpha,s,t) - \phi_\alpha(r,\alpha,s,t))| \le C\lambda^{2-2\beta}N^{-1-\beta}\log(N)
$$

Thus, since

$$
\begin{aligned}
\omega_{osc}(r,\alpha,s) &= g(\lambda\phi_r(r,\alpha,s,0))\lambda^{1-\beta}N^{-\beta}\cos(N\phi_\alpha(r,\alpha,s,0))\\
\overline{\omega_{osc}}(r,\alpha,s) &= g(\lambda r)\lambda^{1-\beta}N^{-\beta}\cos(N\overline{\phi}_\alpha(r,\alpha,s,0))
\end{aligned}
$$

we can apply both (4.41) and (4.43) to obtain

$$
\begin{aligned}
&|\omega_{osc} - \overline{\omega_{osc}}|\\
&\le C\frac{\lambda^{1-\beta}}{N^\beta}N\cdot\lambda^{2-2\beta}N^{-1-\beta}\log(N) + C\frac{\lambda^{1-\beta}}{N^\beta}\lambda\cdot\lambda^{-\beta}N^{-1-\beta}\log(N)\\
&\le C\frac{\lambda^{1-\beta}}{N^\beta}\frac{\lambda^{2-2\beta}\log(N)}{N^\beta},
\end{aligned}
$$

so integrating over the support of $\omega_{osc} - \overline{\omega_{osc}}$ we get

$$
\|\omega_{osc} - \overline{\omega_{osc}}\|_{L^2} \le C\|\omega_{osc}\|_{L^2}\frac{\lambda^{2-2\beta}\log(N)}{N^\beta}
$$

which proves (4.40), as required.

□

4.4.4 Step 4 - H^s norm inflation

Here we finish the proof of Theorem 4.1.1. Namely, we show that for sufficiently large λ the only solution to the 2D Euler equations (4.2) with initial conditions ω_0 given by (4.32) satisfies

$$
\|\omega\|_{H^{\beta'}} \ge K \qquad \text{for } t\in[1/T,T]. \tag{4.44}
$$

Remark 1. Note that since $\|\omega\|_{L^2} \le 1$, if $K > 1$ then $\|\omega\|_{H^s} \ge K$ for $s \ge \beta'$. Furthermore, note since $\|\omega_0\|_{H^\beta} \le 1$, independently of $\lambda > 0$, we are showing strong ill-posedness in H^β by considering $\epsilon\omega$ for small $\epsilon > 0$.

In order to see (4.44), we first recall that, by energy conservation and the form (4.32) of initial data

$$\|\omega_{osc}\|_{L^2} = \|\overline{\omega_{osc}}\|_{L^2} = \frac{1}{2}\|g\|_{L^2}(\lambda N)^{-\beta} \tag{4.45}$$

We set $\gamma := (\beta_\delta + \beta')/2$.
By interpolation and using Lemma 4.4.3

$$\begin{aligned}\|\omega_{osc,err}\|_{H^\gamma}^{\beta'} &\lesssim \|\omega_{osc,err}\|_{L^2}^{\beta'-\gamma}\|\omega_{osc,err}\|_{H^{\beta'}}^{\gamma}\\ &\lesssim \|\overline{\omega_{osc}}\|_{L^2}^{\beta'-\gamma}\lambda^{-\delta(\beta'-\gamma)/2}\left(\|\omega_{osc}\|_{H^{\beta'}} + \|\overline{\omega_{osc}}\|_{H^{\beta'}}\right)^{\gamma}\\ &\lesssim \left(\lambda^{-\beta-\frac{\delta}{2}}N^{-\beta}\right)^{\beta'-\gamma}\left(\|\omega_{osc}\|_{H^{\beta'}}^{\gamma} + \|\overline{\omega_{osc}}\|_{H^{\beta'}}^{\gamma}\right).\end{aligned} \tag{4.46}$$

We now show that, for some $\eta > 0$,

$$\|\overline{\omega_{osc}}\|_{\dot{H}^{-\eta}} \leq C\|\overline{\omega_{osc}}\|_{L^2}(N\lambda^{(2-\beta)})^{-\eta} \tag{4.47}$$

for $t \in [1/T, T]$. To this end, we first recall the definition (4.39) of $\overline{\omega_{osc}}$,

$$\overline{\omega_{osc}} = \lambda^{1-\beta}g(\lambda r)N^{-\beta}\cos\left(N\left(\alpha - \frac{1}{r}\int_0^t \left(v_\alpha[f](\lambda r)\lambda^{-\beta} + v_\alpha\left[A(\omega_{osc} + \omega_{rad,err})\right]\right)\mathrm{d}s\right)\right).$$

In order to apply Lemma 4.3.5, we note that

$$\|v[\omega_{osc}]\|_{L^\infty(B(0,4/\lambda))} \leq C\|\omega_{osc}\|_{L^\infty}\int_{B_{8/\lambda}(0)}|y|^{-1}\mathrm{d}y \leq C(N\lambda)^{-\beta}, \tag{4.48}$$

and we use the C^1 velocity estimate (4.16) together with the C^1 estimate (4.38) of ω_{osc} to obtain that

$$\|v[\omega_{osc}]\|_{C^1} \leq C\|\omega_{osc}\|_{L^\infty}\log\|\omega_{osc}\|_{C^1} \leq C\lambda^{1-\beta}N^{-\beta}\lambda^{1-\beta}\log(\lambda^{2-\beta}N^{1-\beta}) \leq \lambda^{1-\beta-\delta}, \tag{4.49}$$

recall (4.33) for the relation between λ and N. Moreover, since the supports of $\omega_{rad,err}$ and $g(\lambda\cdot)$ are at least C/λ apart (recall (4.35)), we can use (4.36) to obtain

$$|v[\omega_{rad,err}]|, |\nabla v[\omega_{rad,err}]| \leq C\|\omega_{rad,err}\|_{L^2} \leq \lambda^{-\beta-\delta} \qquad \text{in supp}\ \overline{\omega_{osc}} \tag{4.50}$$

Thus, given $t \in [1/T, T]$, letting $W(r), G(r), G_{err}(r)$ be defined by

$$\begin{aligned}W(\lambda r) &:= \overline{\omega_{osc}}(r,t),\\ G(\lambda r) &:= (\lambda r)^{-1}v_\alpha[f](\lambda r),\\ G_{err}(\lambda r) &:= -Nr^{-1}\int_0^t v_\alpha[A(\omega_{osc} + \omega_{rad,err})](r,s)\mathrm{d}s,\end{aligned}$$

we observe that

$$W = \lambda^{1-\beta}N^{-\beta}g\cos\left(N\alpha - Nt\lambda^{1-\beta}G + G_{err}\right).$$

Hence, since $W \in C_c^2([1/2, 4])$ and since (4.48)–(4.50) we get that

$$\|G_{err}\|_{C^1([1/2,4])} \leq CNt\lambda^{1-\beta-\delta} \leq (Nt\lambda^{1-\beta})^{1-\sigma}$$

for some small $\sigma > 0$, where we used the fact that $t \in [1/T, T]$ and λ is sufficiently large (and depends on T). This lets us use Lemma 4.3.5 to obtain that

$$\|W\|_{\dot{H}^{-\eta}} \leq C(N\lambda^{1-\beta})^{-\eta}\lambda^{1-\beta}N^{-\beta},$$

where the Sobolev norm is considered on $\mathbb{R}^2$, treating W as a radial function. This, together with the fact that $\|W\|_{\dot{H}^{-\eta}} = \lambda^{1+\eta}\|\overline{\omega_{osc}}\|_{\dot{H}^{-\eta}}$, gives (4.47), as required.
The upper bound (4.47), together with the L^2 conservation of $\overline{\omega_{osc}}$, let us use Sobolev interpolation (of L^2 in terms of $\dot{H}^{-\eta}$ and $\dot{H}^s$) to obtain a lower bound for $\|\overline{\omega_{osc}}\|_{\dot{H}^s}$ for $s \in (0,1]$. On the other

hand, a direct calculation shows that $\|\overline{\omega_{osc}}\|_{\dot H^1} \le CN\lambda^{2-\beta}\|\overline{\omega_{osc}}\|_{L^2}$ for $t\in[1/T,T]$, and so we can interpolate $\dot H^s$ between L^2 and $\dot H^1$ to obtain an upper bound on $\|\overline{\omega_{osc}}\|_{\dot H^s}$. Altogether we obtain

$$\|\overline{\omega_{osc}}\|_{H^s} \approx \|\omega_{osc}\|_{L^2}(N\lambda^{2-\beta})^s \qquad \text{for each } s\in[0,1]. \tag{4.51}$$

Applying interpolation again for ω_{osc} we obtain

$$\begin{aligned}
\|\omega_{osc}\|^{\gamma}_{H^{\beta'}} &\ge \frac{\|\omega_{osc}\|^{\beta'}_{H^\gamma}}{\|\omega_{osc}\|^{\beta'-\gamma}_{L^2}}\\
&\ge \frac{1}{\|\omega_{osc}\|^{\beta'-\gamma}_{L^2}}\left(C\|\overline{\omega_{osc}}\|^{\beta'}_{H^\gamma} - C\|\omega_{osc,err}\|^{\beta'}_{H^\gamma}\right)\\
&\ge C\|\overline{\omega_{osc}}\|^{\gamma}_{H^{\beta'}} - C\lambda^{-\delta(\beta'-\gamma)/2}\left(\|\omega_{osc}\|^{\gamma}_{H^{\beta'}} + \|\overline{\omega_{osc}}\|^{\gamma}_{H^{\beta'}}\right),
\end{aligned}$$

where, in the last inequality, we used (4.51) twice (with $s=\gamma$ and with $s=\beta'$) to estimate $\|\overline{\omega_{osc}}\|_{H^\gamma}$, as well as (4.46) to bound $\|\omega_{osc,err}\|_{H^\gamma}$ from above. Since the $\|\omega_{osc}\|_{H^{\beta'}}$ norm on the right-hand side can be absorbed by the left-hand side, and the last $\|\overline{\omega_{osc}}\|_{H^{\beta'}}$ norm is negligible in comparison with the first term on the right-hand side, we thus obtain that

$$\|\omega_{osc}\|_{H^{\beta'}} \ge C\|\overline{\omega_{osc}}\|_{H^{\beta'}}$$

for each $t\in[1/T,T]$. Hence, applying (4.51) again with $s=\beta'$ we obtain

$$\|\omega_{osc}\|_{H^{\beta'}} \ge C\frac{(N\lambda^{2-\beta})^{\beta'}}{(\lambda N)^{\beta}} \ge C\lambda^{\tilde\epsilon},$$

where $\tilde\epsilon>0$ is a small constant. Thus, choosing sufficiently large λ shows growth of $\|\omega_{osc}\|_{H^{\beta'}}$, and hence also of $\|\omega\|_{H^{\beta'}}$, due to the localization (4.35) and (4.18). In particular we obtain (4.44), as required.

4.5 Gluing: Loss of regularity

We are now ready to prove Theorem 4.1.2, that is, to show existence of a solution that loses regularity instantly, and furthermore it is the unique classical solution (as in Definition 4.1.3) and it is global in time.
By rescaling the initial data we can assume that $\epsilon=1$; thus, given $\beta\in(0,1)$, we need to find $\omega(x,0)$ such that there exists a unique global classical solution to 2D Euler (as in Definition 4.1.3) with this initial condition which satisfies

$$\|\omega(x,0)\|_{H^\beta}\le 1,$$

$$\|\omega(x,t)\|_{H^{\beta'}}=\infty \quad \text{for} \quad t\in(0,\infty),\ \beta'>\frac{(2-\beta')\beta}{2-\beta^2}. \tag{4.52}$$

First, given $j\ge1$, we will denote by $\omega_j(x,t)$ a smooth solution to 2D Euler equation given by Theorem 4.1.1 such that

$$\|\omega_j(x,t)\|_{H^s}\ge 4^j \text{ for } \ s>\frac{(2-\beta)\beta}{2-\beta^2}+\frac{1}{j},\ t\in\left[\frac{1}{4^j},1\right]. \tag{4.53}$$

Note also that by construction, we can choose the ω_j so that, for all $t\in[0,1]$,

$$|\operatorname{supp}\omega_j|\le 2^{-j},\ \operatorname{supp}\omega_j\subset B_1(0). \tag{4.54}$$

Moreover, setting $p:=2/(1-\beta)$ we can assume that $\|\omega_j(\cdot,t)\|_{L^p}=C$ for all $t\ge0$, where $C>0$ is a constant, by the L^p conservation and the form (4.32) of the initial data.

We will consider initial conditions $\omega(x,0)$ of the form

$$\omega(x,0) := \sum_{j=1}^{\infty} T_{R_j}\left(\frac{\omega_j(x,0)}{2^j}\right), \tag{4.55}$$

where $T_R(f(x_1,x_2)) = f(x_1 - R, x_2)$. For brevity, we will use the notation

$$\widetilde{\omega}_j(x,t) := T_{R_j}\left(\frac{\omega_j(x,t/2^j)}{2^j}\right),$$

where the R_j's remain to be fixed. Some properties to keep in mind are:

- $\widetilde{\omega}_j$ is a smooth global solution to 2D Euler with

$$\|\widetilde{\omega}_j(\cdot,t)\|_{H^s} \geq 2^j \qquad \text{for } s > \frac{(2-\beta)\beta}{2-\beta^2} + \frac{1}{j},\ t \in [2^{-j}, 2^j], \tag{4.56}$$

  due to (4.53). Furthermore, we have

$$\|\widetilde{\omega}_j(x,0)\|_{H^\beta} \leq 2^{-j}, \qquad \|\widetilde{\omega}_j(x,0)\|_{L^1 \cap L^p} \leq C2^{-j}, \tag{4.57}$$

  where we set $\|\cdot\|_{L^1\cap L^p} := \|\cdot\|_{L^1} + \|\cdot\|_{L^p}$ (recall $p = 2/(1-\beta) > 2$), as well as

$$|\operatorname{supp}\tilde{\omega}_j| \leq 2^{-j}, \operatorname{supp}\tilde{\omega}_j \subset B_1(R_j,0) \qquad \text{for } t \in [0,2^j]. \tag{4.58}$$

- Given the truncated initial conditions

$$\sum_{j=1}^{J} \widetilde{\omega}_j(x,0), \tag{4.59}$$

  we will refer to the unique global-in-time solution to the 2D Euler equations (4.2)–(4.3) with initial conditions given by (4.59) as $\omega_{tr,J}$, and, for any $t \in [0,T]$, $m, J \in \mathbb{N}$, there exists a constant $C_{m,J,T}$, independent of the choice of $(R_j)_{j\in\mathbb{N}}$ such that

$$\|\omega_{tr,J}(\cdot,t)\|_{H^m} \leq C_{m,J,T}, \qquad \|\widetilde{\omega}_J(\cdot,t)\|_{H^m} \leq C_{m,J,T}. \tag{4.60}$$

- Furthermore, noting that $\|v[f]\|_{L^\infty} \leq C_q(\|f\|_{L^1} + \|f\|_{L^q})$ for any f and $q > 2$ we deduce from (4.57) that

$$\|v[\omega_{tr,J}]\|_{L^\infty}, \|v[\widetilde{\omega}_J]\|_{L^\infty} \leq v_{max}$$

  for all $t \geq 0$, where v_{max} is some constant independent of J and of the choice of $(R_j)_{j\in\mathbb{N}}$. We also deduce from (4.60) that

$$\begin{aligned} |\nabla^k v[\omega_{tr,J}](x,t)| &\leq \frac{C_{k,J,T}}{(1+\operatorname{dist}(x, \operatorname{supp}\,\omega_{tj,J}))^{k+1}}, \\ |\nabla^k v[\widetilde{\omega}_J(x,t)]| &\leq \frac{C_{k,J,T}}{(1+\operatorname{dist}(x, \operatorname{supp}\,\widetilde{\omega}_J))^{k+1}} \end{aligned} \tag{4.61}$$

  for all $x \in \mathbb{R}^2$, $t \in [0,T]$.

- Moreover,

$$|\operatorname{supp}\,\omega_{tr,j}| = 1 - 2^{j+1} \leq 1 \qquad \text{for all } j \geq 1, \tag{4.62}$$

  as a property of the 2D Euler equations.

We will define $R_1 = 0$, $R_{j+1} = D_{j+1} + D_j + R_j$ and show that if $D_j > 0$ are big enough, then there exists a global solution ω_∞ with loss of regularity.

We first construct ω_∞ as a limit of $\omega_{tr,j}$ as $j \to \infty$. To this end, for any fixed $(D_j)_{j=1,\dots,J}$, we define inductively the "$J+1$-th approximation" by

$$\overline{\omega}_{tr,J+1} := \omega_{tr,J} + \widetilde{\omega}_{J+1}.$$

It fulfils an evolution equation of the form

$$\partial_t \overline{\omega}_{tr,J+1} + v[\overline{\omega}_{tr,J+1}] \cdot \nabla \overline{\omega}_{tr,J+1} + F = 0,$$

where

$$F := -v[\omega_{tr,J}] \cdot \nabla \widetilde{\omega}_{J+1} - v\left[\widetilde{\omega}_{J+1}\right] \cdot \nabla \omega_{tr,J}$$

and since, for $t \in [0, 2^{J+1}]$,

$$\operatorname{dist}\left(\operatorname{supp} \omega_{tr,J}, \operatorname{supp} \widetilde{\omega}_{J+1}\right) \geq D_{J+1} - 2v_{max} 2^{J+1} - 2$$

the H^m boundedness of the vorticity functions (4.60) and the decay (4.61) of the corresponding velocity fields v implies that

$$\|F(\cdot,t)\|_{H^4} \leq C_J (D_{J+1} - 2v_{max} 2^{J+1} - 2)^{-1} \to 0 \qquad \text{as } D_{J+1} \to \infty, \tag{4.63}$$

uniformly in $t \in [0, 2^{J+1}]$. We set

$$W_{J+1} := \omega_{tr,J+1} - \overline{\omega}_{tr,J+1},$$

and we use the evolution equation for W_{J+1},

$$\partial_t W_{J+1} = v[W_{J+1}] \cdot \nabla W_{J+1} - v[W_{J+1}] \cdot \nabla \overline{\omega}_{tr,J+1} + v\left[\overline{\omega}_{tr,J+1}\right] \cdot \nabla W_{J+1} - F$$

to obtain that

$$\frac{\mathrm{d}\|W_{J+1}\|_{H^4}}{\mathrm{d}t} \leq C(\|W_{J+1}\|_{H^4}^2 + \|W_{J+1}\|_{H^4} \|\overline{\omega}_{tr,J+1}\|_{H^5} + \|F\|_{H^4}),$$

where we used the velocity estimates

$$\begin{aligned}
&\|v[W_{J+1}]\|_{H^4(\operatorname{supp} \overline{\omega}_{tr,J+1})} \lesssim \|v[W_{J+1}]\|_{L^\infty(\operatorname{supp} \overline{\omega}_{tr,J+1})} + \|D^4 v[W_{J+1}]\|_{L^2} \leq \|W_{J+1}\|_{H^3} \\
&\|v[\overline{\omega}_{tr,J+1}]\|_{C^4(\operatorname{supp} W_{J+1})} \lesssim \|\overline{\omega}_{tr,J+1}\|_{H^7}
\end{aligned}$$

and that $|\operatorname{supp} \overline{\omega}_{tr,J+1}| \leq 1$ and $|\operatorname{supp} W_{J+1}| \leq 2$, due to (4.62). Thus, since $W_{J+1}(\cdot, 0) = 0$, and $\|\overline{\omega}_{tr,J+1}\|_{H^7} \leq \|\omega_{tr,J}\|_{H^7} + \|\widetilde{\omega}_{J+1}\|_{H^7} \leq C_J$ for $t \in [0, 2^{J+1}]$ (due to (4.60)) and since F vanishes in the limit D_{J+1} (recall (4.63)), we can find

$$\widetilde{D_{J+1}} \geq 4^{J+1}(v_{max} + 1) + 2 \tag{4.64}$$

such that

$$\|\omega_{tr,J+1}(\cdot,t) - \omega_{tr,J}(\cdot,t) - \widetilde{\omega}_{J+1}\|_{H^4} \leq 2^{-J-1} \tag{4.65}$$

for $D_{J+1} \geq \widetilde{D_{J+1}}$ and all $t \in [0, 2^{J+1}]$.
Given $a, d > 0$ we denote by

$$K = \overline{B_d(0)} \times [0, a]$$

an arbitrary compact set in space-time. Note that, for each such K the support of $\widetilde{\omega}_{J+1}$ is disjoint with K for sufficiently large J. Thus (4.65) implies that $\{\omega_{tr,J}\}_{J \geq 1}$ is Cauchy in $C_t^0 H_x^4(K)$, and so there exists $\omega_\infty \in C([0,\infty); H^4_{loc}(\mathbb{R}^2))$ such that we have that

$$\|\omega_{tr,J} - \omega_\infty\|_{C_t^0 H_x^4(K)} \to 0 \qquad \text{as } J \to \infty \tag{4.66}$$

for every K. Note that in particular $\omega_\infty \in C_t^0 C_x^2(K)$, and so, since $\omega_\infty \in C^0([0,a];L^1(\mathbb{R}^2))$ (a consequence of (4.65) and (4.57)) we see that, for each K, $D^\alpha v[\omega_\infty]$ exists at each point of K and each multiindex α with $|\alpha|\leq 2$, and

$$\begin{aligned}
&\|v[\omega_\infty]-v[\omega_{tr,J}]\|_{C_t^0C_x^2(K)}\\
&\leq C_K\left(\|\chi_{B_{2d}(0)^c}(\omega_\infty-\omega_{tr,J})\|_{C^0([0,a];L^1)}+\|\omega_\infty-\omega_{tr,J}\|_{C^0([0,a];C^2(B_{2d}(0)))}\right)\\
&\leq C_K\sum_{j\geq J}\left\|\chi_{B_{2d}(0)^c}(\omega_{tr,j+1}-\omega_{tr,j})\right\|_{C^0([0,a];L^1)}+o(1)\\
&\leq C_K\sum_{j\geq J}\left(2^{-j}+\|\widetilde{\omega}_{j+1}\|_{C^0([0,a];L^1)}\right)+o(1)\\
&\leq o(1)
\end{aligned}\tag{4.67}$$

as $J\to\infty$, where we used the Biot-Savart law (4.3) in the first inequality, (4.66) in the second inequality, (4.65) in the third and (4.57) in the fourth.

Having found the limit ω_∞ with convergence properties (4.66), (4.67), we can now take the limit $J\to\infty$ in the weak formulation of $\partial_t\omega_{tr,J}+v[\omega_{tr,J}]\cdot\nabla\omega_{tr,J}=0$ (which is obtained by multiplying by a smooth function that is compactly supported in K, and integrating) to obtain that $\partial_t\omega_\infty=-v[\omega_\infty]\cdot\nabla\omega_\infty\in C_t^0C_x^1(K)$. In particular $\omega_\infty\in C^1_{x,t}(K)$, which gives that ω_∞ is a classical solution of the Euler equations in the sense of Definition 4.1.3.

We now show that ω_∞ instantly loses regularity. Namely we show (4.52), for which it is sufficient to consider only $s\in\left(\frac{(2-\beta)\beta}{2-\beta^2},1\right)$. Given such s, and $\tau>0$ we fix $J\geq 1$ such that

$$s>\frac{(2-\beta)\beta}{2-\beta^2}+\frac{1}{J}\qquad\text{and}\qquad 2^{J+1}\geq\tau.\tag{4.68}$$

Using the short-hand notation

$$B_j:=B_{D_j}(R_j,0)\tag{4.69}$$

we obtain

$$\begin{aligned}
\|\omega_\infty(\cdot,\tau)\|_{H^s}&\geq\left\|\sum_{j\geq J+1}\widetilde{\omega}_j(\cdot,\tau)\right\|_{H^s}-\|\omega_{tr,J}\|_{H^s}-\sum_{j\geq J}\|\omega_{tr,j+1}(\cdot,\tau)-\omega_{tr,j}(\cdot,\tau)-\widetilde{\omega}_{j+1}(\cdot,\tau)\|_{H^s}\\
&\geq\left\|\sum_{j\geq J+1}\widetilde{\omega}_j(\cdot,\tau)\right\|_{\dot{H}^s}-C_{s,\tau}\\
&\geq\|\widetilde{\omega}_k(\cdot,\tau)\|_{\dot{H}^s}-C_{s,\tau}\geq 2^k-C_{s,\tau}
\end{aligned}\tag{4.70}$$

for any $k\geq J+1$, where we used (4.65) in the second inequality, as well as (4.18) and the fact that $\operatorname{supp}\widetilde{\omega}_j(\cdot,\tau)\subset B_1(R_j,0)$ for all $j\geq J+1$ (recall (4.54)) in the third inequality. Since $k\geq J+1$ is arbitrary, we obtain (4.52), as required.

In order to show that ω_∞ is the unique solution in the sense of Definition 4.1.3, we first denote by $\phi(x,t)$ the flow map of ω_∞, and we set

$$\omega_{\infty,j}(x,t):=\widetilde{\omega}_j(\phi^{-1}(x,t)),\qquad\omega_{\infty,\leq J}:=\sum_{j=1}^J\omega_{\infty,j}.\tag{4.71}$$

This allows us to decompose ω_∞ into pieces,

$$\omega_\infty=\sum_{j=1}^\infty\omega_{\infty,j},$$

where each piece satisfies

$$\partial_t \omega_{\infty,j} + v[\omega_\infty]\cdot\nabla\omega_{\infty,j} = 0. \tag{4.72}$$

In particular (recall (4.54))

$$|\operatorname{supp}\omega_{\infty,j}| \le 2^{-j} \qquad \text{for all times } t\ge 0. \tag{4.73}$$

We now show that, for each fixed $a>0$

$$\|\omega_{\infty,j}\|_{C^1} \le \mathrm{e}^{M_j \mathrm{e}^{\tilde C M_j a}} \qquad \text{and} \qquad \|\omega_{\infty,\le j}\|_{C^1} \le \mathrm{e}^{S_j \mathrm{e}^{\tilde C S_j a}} \tag{4.74}$$

for all $t\in[0,a]$ and j such that $2^{j-1}\ge a$, where $\tilde C>1$ is a universal constant and

$$M_j := \max\left(1, \|\widetilde\omega_j(\cdot,0)\|_{C^1}\right), \qquad S_j := \sum_{i=1}^{j} M_i. \tag{4.75}$$

To this end, we first apply the C^1 estimate (4.17) to $\widetilde\omega_j$ and $\omega_{tr,j}$, $j\ge 1$, to obtain

$$\|\widetilde\omega_j\|_{C^1} \le \mathrm{e}^{M_j \mathrm{e}^{\tilde C M_j a}} \qquad \text{and} \qquad \|\omega_{tr,j}\|_{C^1} \le \mathrm{e}^{S_j \mathrm{e}^{\tilde C S_j a}} \tag{4.76}$$

for all j and all $t\in[0,a]$, where $\tilde C>1$ is a constant. Thus, since for $2^j\ge a$

$$\begin{gathered} \omega_{\infty,\le j}(t) \ \text{ and } \ \omega_{tr,j}(t) \text{ remain supported in } B_{R_j+D_j}(0),\\ \omega_{\infty,j}(t) \ \text{ and } \ \widetilde\omega_j(t) \text{ remain supported in } B_j \end{gathered} \tag{4.77}$$

for $t\in[0,a]$ (recall (4.69) and (4.64)), we obtain that

$$\begin{aligned} \|\omega_{\infty,j}-\widetilde\omega_j\|_{H^4} &= \left\| \omega_\infty - \omega_{tr,j-1} - \sum_{k\ge j}\widetilde\omega_k \right\|_{H^4(B_j)}\\ &= \left\| \sum_{k\ge j}\left(\omega_{tr,k}-\omega_{tr,k-1}-\widetilde\omega_k\right)\right\|_{H^4(B_j)}\\ &\le \sum_{k\ge j}\|\omega_{tr,k}-\omega_{tr,k-1}-\widetilde\omega_k\|_{H^4} \le \sum_{k\ge j} 2^{-k} = 2^{-(j-1)} \end{aligned}$$

for all $t\in[0,a]$ and j such that $2^{j-1}\ge a$, where we used (4.65) in the last line. This and the first claim of (4.76) proves the first claim of (4.74), upon possibly taking $\tilde C$ larger. A similar calculation,

$$\begin{aligned} \|\omega_{\infty,\le j}-\omega_{tr,j}\|_{H^4} &= \left\| \omega_\infty - \omega_{tr,j} - \sum_{k\ge j+1}\widetilde\omega_k \right\|_{H^4(B_{R_j+D_j}(0))}\\ &\le \sum_{k\ge j+1}\|\omega_{tr,k}-\omega_{tr,k-1}-\widetilde\omega_k\|_{H^4(B_{R_j+D_j}(0))} \le \sum_{k\ge j+1} 2^{-k} = 2^{-j} \end{aligned}$$

for $t\in[0,a]$ and j such that $2^j\ge a$, together with (4.76) shows the second claim of (4.74), as required.

We emphasize that all of the above claims hold for each choice of the sequence $\{D_j\}_{j\ge1}$ satisfying $D_j\ge\widetilde{D_j}$, where $\widetilde{D_J}$ was defined by (4.64)–(4.65). We now prove uniqueness of ω_∞, provided that each D_j is chosen larger, namely that

$$D_j \ge \widetilde{D_j} + \exp\left(\exp\left(2M_j\exp\left(\tilde C M_j 2^j\right)\right)\right). \tag{4.78}$$

Indeed, suppose that there exists another classical solution $\tilde\omega_\infty$ of the Euler equations with initial data (4.55), and let $\tilde\omega_{\infty,j}$ be defined in the same way as $\omega_{\infty,j}$, but with the flow map given by $\tilde\omega_\infty$ so that

$$\tilde\omega_\infty(x,t) = \sum_{j=1}^{\infty}\tilde\omega_{\infty,j}(x,t), \quad \partial_t\tilde\omega_{\infty,j} + v[\tilde\omega_\infty]\cdot\nabla\tilde\omega_{\infty,j} = 0. \tag{4.79}$$

Note that $\tilde{\omega}_\infty$ conserves its L^p norms with time and in particular, it moves at most with speed v_{max}.
We let

$$T := \sup\{T' \geq 0\colon \omega_\infty(t) = \tilde{\omega}_\infty(t) \quad \text{for all } t \in [0,T']\}$$

and we set

$$W := \omega_\infty - \tilde{\omega}_\infty, \quad W_j := \omega_{\infty,j} - \tilde{\omega}_{\infty,j} \quad \text{and} \quad W_{\leq j} := \sum_{k=1}^{j} W_k.$$

Clearly

$$\partial_t W_j + v[\tilde{\omega}_\infty] \cdot \nabla W_j + v[W] \cdot \nabla \omega_{\infty,j} = 0. \tag{4.80}$$

In order to estimate $v[W]$ in L^2 we fix j_0 such that

$$2^{j_0-1} \geq T+1.$$

Note that, since supp $W_{\leq j-1} \subset B_{R_{j-1}+D_{j-1}}(0)$ and supp $W_k \subset B_k$ for $k \geq j \geq j_0$ (a consequence of (4.77)), and since $|x-y| \geq D_j - 2Tv_{max} - 2$ for $x \in B_j$ and $y \in B_{R_{j-1}+D_{j-1}}(0) \cup \bigcup_{k\geq j+1} B_k$ we have

$$\begin{aligned}
\|v[W]\|_{L^2(\text{supp } \omega_{\infty,j})} &\lesssim \left(\int_{\text{supp } \omega_{\infty,j}} \left(\int_{\text{supp } W_j} \frac{|W_j(y)|}{|x-y|} \mathrm{d}y \right)^2 \mathrm{d}x \right)^{1/2} + \frac{\|W\|_{L^2}}{D_j - 2(T+1)v_{max} - 2} \\
&\lesssim \|W_j\|_{L^2} + \frac{\|W\|_{L^2}}{D_j - 2(T+1)v_{max} - 2}
\end{aligned}$$

for each $t \in [T, T+1]$, where we used (4.73), as well as the fact that $1 = \chi_{B_2(y)}(x)$ under the first integral and Young's inequality $\|f * g\|_2 \leq \|f\|_2 \|g\|_1$ in the last line. Thus multiplying (4.80) by W_j and integrating we obtain the energy estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|W_j\|_{L^2} &\leq C \|W_j\|_{L^2} \|\omega_{\infty,j}\|_{C^1} + \frac{\|W\|_{L^2}}{D_j - 2(T+1)v_{max} - 2} \|\omega_{\infty,j}\|_{C^1} \\
&\leq C \|W_j\|_{L^2} \|\omega_{\infty,j}\|_{C^1} + \mathrm{e}^{-\mathrm{e}^{M_j \mathrm{e}^{\tilde{C} M_j 2^j}}} F
\end{aligned} \tag{4.81}$$

for $t \in [T, T+\epsilon]$, $j \geq j_0$, where $\epsilon \in (0,1)$,

$$U(t) := \sup_{s \in [T,t]} \|W(\cdot, s)\|_{L^2} \quad \text{for } t \in [T, T+\epsilon],$$

and we used the lower bound on D_j (4.78) (recall also (4.64)); note that the factor of 2 in (4.78) is used to absorb the upper bound (4.74) on $\|\omega_{\infty,j}\|_{C^1}$ norm. Thus, using the upper bound (4.74) again, the ODE fact (4.15) shows that

$$\|W_j\|_{L^2} \leq \epsilon\, \mathrm{e}^{C\epsilon \mathrm{e}^{M_j \mathrm{e}^{\tilde{C} M_j 2^j}}} \mathrm{e}^{-\mathrm{e}^{M_j \mathrm{e}^{\tilde{C} M_j 2^j}}} U \tag{4.82}$$

for all $t \in [T, T+\epsilon)$, $j \geq j_0$. Thus, taking $\epsilon \in (0,1)$ small enough so that, for each $j \geq j_0$, the product of the two exponential functions above is bounded by j^{-2}, we obtain

$$\|W_j\|_{L^2} \leq \frac{\epsilon}{j^2} U \tag{4.83}$$

for $t \in [T, T+\epsilon]$, $j \geq j_0$.
As for $j < j_0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|W_{<j_0}\|_{L^2} \leq C \|W_{<j_0}\|_{L^2} \|\omega_{<j_0}\|_{C^1} + \frac{\|W\|_{L^2}}{D_{j_0} - 2(T+1)v_{max} - 2} \|\omega_{<j_0}\|_{C^1}, \tag{4.84}$$

since supp $W_{<j_0} \subset B_{R_{j_0}+D_{j_0}}(0)$. Thus, applying the ODE fact (4.15) again, we obtain $\|W_{<j_0}\|_{L^2} \leq C_T \epsilon U$, where $C_T > 0$ depends on T. Adding this inequality to (4.83), for $j \geq j_0$, gives that $\|W\|_{L^2} \leq (C_T + C)\epsilon U$ for all $t \in [T, T+\epsilon]$, and so, taking sup gives

$$U(T+\epsilon) \leq (C_T + C)\epsilon U(T+\epsilon). \tag{4.85}$$

Taking ϵ sufficiently small, we thus obtain $U(T+\epsilon) = 0$, which proves uniqueness, as required.

Chapter 5

Loss of regularity for SQG with fractional diffusion

5.1 Introduction

As mentioned in chapter 2, the (inviscid) Surface Quasi-Geostrophic (SQG) equation is a significant active scalar model with various applications in atmospheric modeling [88], owing to its similarities with the 3D incompressible Euler equations (see [27]). In this chapter we consider the initial value problem for the dissipative 2D Surface Quasi-geostrophic equations (α-SQG) in the space-time domain $\mathbb{R}^2 \times \mathbb{R}_+$ which has the following form

$$\begin{aligned} \frac{\partial w}{\partial t} + v \cdot \nabla w + \Lambda^\alpha w = 0 \qquad \alpha \in (0,2] \\ v = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}), \quad \psi = \Lambda^{-1} w \end{aligned} \tag{5.1}$$

and as usual we denote $\Lambda^\alpha f \equiv (-\Delta)^{\frac{\alpha}{2}} f$ by the Fourier transform $\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$. Here the function $w = w(x,t)$ represents the potential temperature in a rapidly rotating and stratified flow driven by an incompressible velocity v (in chapter 2 we used θ for the scalar, which is the standard notation, but unfortunately here we need to use θ for the angle when working in polar coordinates). The velocity field can be written as $v = (-\mathcal{R}_2 w, \mathcal{R}_1 w)$, where $\mathcal{R}_i$ are the Riesz transforms in 2 dimensions, with the integral expression

$$\mathcal{R}_j w(x,t) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} \frac{(x_j - y_j) w(y,t)}{|x-y|^3} dy_1 dy_2$$

for $j = 1, 2$.
The equation (5.1) has been extensively studied since its introduction in [89]. In that work, the global existence of weak solutions in L^2 (finite energy) was demonstrated for $0 \leq \alpha \leq 2$. Further research on global existence of weak solutions in other spaces can be found in [84] and [4]. However, it should be noted that weak solutions are not unique below a certain regularity threshold [12]. The equation's scaling leads to three regimes to consider: sub-critical ($1 < \alpha \leq 2$), critical ($\alpha = 1$), and super-critical ($0 < \alpha < 1$). The global existence of unique smooth solutions in the sub-critical case has been established in [32], while the global well-posedness for the critical case with $\alpha = 1$ has been shown in [75], [13] and [31] using different techniques (see also [74], [45] and [30]).

5.1.1 Regularity in the Super-critical regime $\alpha \in (0,1)$

The problem of global regularity in the supercritical regime remains unresolved, despite the existence of eventual regularity results in [44], [42], [72], [80], [91] and [92]. The local well-posedness has been established for large data in H^s for $s \geq 2 - \alpha$ (see [86]) and for a number of functional spaces global well-posedness is present for small data (see [36], [20], [24], [34] [50], [51], [62], [67], [86], [91], [99], [100], [101] and [102]). In the case of large initial data global existence as $\alpha \to 1^-$

is shown in [42] (see also [25]). Additionally, there is a corresponding instant parabolic smoothing effect for sufficiently regular initial data [33], [34], [7], [50] and [51].
Recently in [25], a bound is obtained on the dimension of the spacetime singular set of the suitable weak solutions of (5.1) for a range of α's in the super-critical regime.

5.1.2 Main result

Our main result is to construct global unique solutions of (5.1) that lose regularity instantly in the super-critical regime.

Theorem 5.1.1. Given $\epsilon > 0$, $\alpha \in (0,1)$, $\beta \in (1, 2-\alpha)$, there exist initial conditions $w_0(x)$ with $||w_0||_{H^\beta} \leq \epsilon$ such that there exists a unique solution $w(x,t)$ to (5.1) with $w(x,t) \in L^\infty_t H^1_x$. This solution is global and smooth for any $t > 0$ and it fulfils

$$\lim_{n\to\infty} ||w(x,t_n)||_{H^\beta} = \infty$$

for some sequence of times $(t_n)_{n\in\mathbb{N}}$ that tends to zero.

Remark 8. One expects that, as α becomes bigger, instant loss of regularity should become harder and, in fact, if we consider L^∞ initial conditions the result obtained in [80] (see also [79]) shows that, in the critical case $\alpha = 1$, for $s \in (0,1)$, there exists at least one local weak solution that does not lose regularity, which suggests that there might not be instant loss of regularity for L^∞ functions in the case $\alpha \geq 1$. This is also supported by the global existence results for $\alpha = 1$ [13], [75] and [31].

Remark 9. The growth around the origin is at least logarithmic, i.e., there is an exponent $a > 0$ such that

$$\lim_{n\to\infty} \frac{||w(x,t_n)||_{H^\beta}}{|\ln(t_n)|^a} > 0.$$

We will however omit the proof of this fact in order to obtain a more readable chapter.

Remark 10. The solution $w(x,t)$ converges to the initial conditions $w(x,0)$ in the space $C^3_x(B_J(0))$ for any J as t tends to 0. This is a trivial consequence of (5.41).

5.1.3 Strategy of the proof

The motivation for studying this problem comes from the results obtained in chapter 2 where instant loss of regularity is shown in the inviscid case (see also [65] for a different proof): Since loss of regularity is possible in the inviscid case, maybe this phenomenon is still possible with some (possibly small) viscosity added, although this is not necessarily the case since the diffusion will fight the norm growth.
In order to obtain norm inflation in H^s with diffusion we need to consider similar (but more general) initial conditions in polar coordinates (r,θ) as in chapter 2, namely

$$w(r,\theta,0) \equiv w_{rad}(r,0) + w_{pert}(r,\theta,0) = \frac{f(\lambda r)}{\lambda^{\beta-1}} + g(\lambda r)\frac{\cos(N\theta)}{N^\beta \lambda^{s-1}}$$

where $N \in \mathbb{N}, \lambda \in \mathbb{R}$ are parameters to be fixed later and f, g are smooth radial functions and r, θ are the radius and angle in polar coordinates.
To obtain a reliable pseudo-solution, we aim to find an approximation of α-SQG that is simple enough to be solved explicitly, and the pseudo-solution will be obtained by solving the simplified evolution equation. This simplified evolution equation needs to be precise enough that the pseudo-solutions stay close to the actual solutions to α-SQG. When fractional dissipation is absent and λ is sufficiently large, we can rely on the following equations

$$\frac{\partial}{\partial t} w_{rad} = 0$$

$$\frac{\partial}{\partial t} w_{pert} + v(w_{rad}) \cdot \nabla w_{pert} = 0$$

to obtain a pseudo-solution that grows rapidly in time, and taking N big makes this pseudo-solution a good approximation of SQG. Unfortunately, this ansatz for the evolution would completely ignore the diffusion in the case $\alpha > 0$, which would make the pseudo-solution a very poor approximation of α-SQG. On the other hand, including the fractional diffusion in our simplified evolution equation already produces an equation that is too complicated for our purposes, and in particular it is hard to deal with the non-locality of Λ^α. Before we explain in some detail how to circumvent this, we will study what we call the naive pseudo-solution:

$$\bar{w}_{naive}(r,\theta,t) = e^{-C\lambda^\alpha t}\frac{g(\lambda r)}{\lambda^{s-1}} + e^{-C(N\lambda)^\alpha t} f(\lambda r)\frac{\cos(N(\theta - \frac{v_\theta g(\lambda r)}{r}\int_0^t e^{-C\lambda^\alpha s} ds))}{N^s \lambda^{s-1}}$$

with $v_\theta(g(\lambda r)) = \hat{\theta}\cdot v(g(\lambda r))$, which is obtained by using the (fully local) approximation

$$\Lambda^\alpha w_{rad} \approx C\lambda^\alpha w_{rad},$$
$$\Lambda^\alpha w_{pert} \approx C(\lambda N)^\alpha w_{pert}.$$

This ansatz (which is NOT a good approximation of α-SQG) actually gives some basic ideas of what the behaviour for the real solution is going to be. Namely, we see that the characteristic time for the decay of w_{pert} is $(\lambda N)^{-\alpha}$, while the "deformation time" (i.e., the time it would take for $||w_{pert}||_{H^\beta}$ to grow in the absence of diffusion) is of order $\lambda^{-2+\beta}$, which suggests considering $(\lambda N)^{-\alpha} \approx \lambda^{-2+\beta}$, so that the smoothing effects and the deformation effects have roughly the same strength. Note, in particular, that this already suggests that we can only have instantaneous loss of regularity if $\beta < 2-\alpha$, which is consistent with the fact that there is local well posedness in H^β for $\beta \geq 2-\alpha$.
However w_{naive} is not the right approximation, so to actually include the diffusion in our simplified evolution equation we will compute $\bar{\Lambda}^\alpha$, a local approximation of the diffusion, as well as $\bar{v}$, a local approximation of the velocity operator, to obtain the final version of our simplified equations

$$\frac{\partial}{\partial t}\bar{w}_{rad}(r,t) + \Lambda^\alpha \bar{w}_{rad}(r,t) = 0$$

$$\frac{\partial \bar{w}_{pert}}{\partial t} + v(\bar{w}_{rad}(r,t))\cdot\nabla(\bar{w}_{pert}) + \bar{v}(\bar{w}_{pert})\cdot\nabla\bar{w}_{rad}(r,t) + \bar{\Lambda}^\alpha(\bar{w}_{pert}) = 0,$$

which can be solved explicitly. This pseudo-solution, which depends on λ, N, β and α, grows very rapidly in H^β as long as N and λ are chosen correctly and $\beta < 2-\alpha$. Furthermore, if $\beta > 1$ (and again, N and λ chosen correctly), the pseudo-solution is a good approximation of α-SQG for all time $t > 0$. A gluing argument then allows us to combine an infinite number of these rapidly growing solutions to obtain the desired instant loss of regularity.

5.1.4 Outline of the chapter

This chapter is structured as follows. In Section 2, we introduce the basic notation that will be utilized throughout the chapter and derive necessary technical bounds to approximate the diffusion operator and to find and control the pseudo-solution. In Section 3, we present the pseudo-solution and analyze its essential properties. In Section 4, we demonstrate how a gluing argument can be employed to construct a unique global in time solution, despite a loss of regularity.

5.2 Technical lemmas

5.2.1 Notation and preliminaries

When a constant depends on several parameters (such as α, β, and γ), we will use the notation $C_{\alpha,\beta,\gamma}$ to indicate this dependence in this chapter.
We will, however, omit the sub-index if the parameter has been fixed at the time.
For many lemmas it will be necessary to work in polar coordinates, i.e., we will consider the change of coordinates

$$x_1 = r\cos(\theta), x_2 = r\sin(\theta).$$

Furthermore, if we call F_{polar} the function that gives us the change of coordinates from polar to cartesian coordinates, for some function $f(x)$ we will use the abuse of notation

$$f(r,\theta) = f(F_{polar}(r,\theta)).$$

Since θ will be our angle in polar coordinates, we will use w to refer to the scalar instead of θ. For $s \geq 0$, we will consider the H^s norms, which we will define as

$$||f||_{H^s} = ||f||_{L^2} + ||\Lambda^s f||_{L^2},$$

and sometimes we will use the fact that, for s an integer

$$||f||_{H^s} \approx \sum_{j=0}^{s}\sum_{i=0}^{j} ||\frac{\partial^j}{\partial x_1^i \partial x_2^{j-i}} f||_{L^2}.$$

Finally, we will sometimes consider the homogeneous Sobolev norms, defined as

$$||f||_{\dot{H}^s} = ||\Lambda^s f||_{L^2}.$$

5.2.2 Approximations for the fractional diffusion

As we mentioned in the introduction, in order to obtain an appropriate pseudo-solution, we need an approximation for the diffusion operator that is easier to work with. In particular we would like a local approximation for the operator. Doing this directly for Λ^α posses some difficulties due to the lack of integrability of the kernel of Λ^α, so we will first approximate $\Lambda^{-\alpha}$ and then use that to obtain information about Λ^α.

Lemma 5.2.1. *For any fixed parameters $\alpha \in (0,1]$, $P, \epsilon > 0$ there exists N_0 such that if $N > N_0$, then for any functions $f(r)$, $g(r)$ and $p(r)$ fulfilling supp$f(r) \subset (\frac{1}{2}, \frac{3}{2})$ and*

$$||g||_{C^5} \leq \ln(N)^P,\ ||f(r)||_{C^5} \leq \ln(N)^P ||f||_{L^\infty}, ||p(r)||_{C^5} \leq \ln(N)^P$$

then if we define

$$w(r,\theta) := f(r)\cos(N(\theta + g(r)) + p(r))$$

we have that for $\beta \in [0,3]$ there exist constants $K_\alpha > 0$ and $C_{\epsilon,\alpha,P}$ such that

$$||\Lambda^{-\alpha} w(r,\theta) - K_\alpha \frac{w(r,\theta)}{|(\frac{N}{r})^2 + (N)^2 g'(r)^2|^{\alpha/2}}||_{H^\beta} \leq C_{\epsilon,\alpha,P} N^{-1-\alpha+\epsilon+\beta} ||f||_{L^\infty}.$$

Furthermore if we have $f(r), g(r), p(r)$ with supp$f(r) \subset (\frac{1}{2\lambda}, \frac{3}{2\lambda})$ for some $\lambda \geq 1$ and such that we have

$$||g(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P,\ ||f(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P ||f(\frac{r}{\lambda})||_{L^\infty}, ||p(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P$$

then for $\beta \in [0,3]$ there exist constants $K_\alpha > 0$ and $C_{\epsilon,\alpha,P}$ such that

$$||\Lambda^{-\alpha} w(r,\theta) - K_\alpha \frac{w(r,\theta)}{|(\frac{N}{r})^2 + (N)^2 g'(r)^2|^{\alpha/2}}||_{H^\beta} \leq C_{\epsilon,\alpha,P,M} \lambda^{\beta-\alpha-1} N^{-1-\alpha+\epsilon+\beta} ||f||_{L^\infty}.$$

Proof. We will just consider $P = 1$ for simplicity since the proof is the same for other values of P. We start by proving the result for $\beta = 0$ and $\lambda = 1$. We will consider from now on that α is fixed with $\alpha \in (0,1)$, so we will omit the dependence of the contants with respect to α. First, we have that, in polar coordinates

$$\Lambda^{-\alpha} w(r,\theta) = \int_{-\pi}^{\pi}\int_0^\infty \frac{w(r',\theta')}{|(r-r')^2 + 2rr'(1-\cos(\theta-\theta'))|^{\frac{2-\alpha}{2}}} r'dr'd\theta'$$

$$= \int_{-\pi}^{\pi}\int_{-r}^{\infty} \frac{f(r+h)\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}}}(r+h)dhd\tilde{\theta}.$$

Since

$$\begin{aligned}
&||\Lambda^{-\alpha}w(r,\theta) - K_\alpha \frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}||_{L^2}\\
&\leq ||(\Lambda^{-\alpha}w(r,\theta) - K_\alpha \frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}})1_{r\in(\frac14,2)}||_{L^2}\\
&+||(\Lambda^{-\alpha}w(r,\theta) - K_\alpha \frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}})1_{r\notin(\frac14,2)}||_{L^2} = I_1 + I_2
\end{aligned}$$

we will study the operator when $r \in (\frac14, 2)$, so that we can bound I_1. First, using integration by parts with respect to $\tilde{\theta}$ and induction we get, for $k \geq 1$

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{\cos(N\tilde{\theta})}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}}}d\tilde{\theta} \\
&= \int_{-\pi}^{\pi} \frac{\cos(N\tilde{\theta}+k\frac{\pi}{2})}{N^k}\sum_{i=1}^{k}\frac{((r+h)r)^i P_{k,i}(\tilde{\theta})}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2+2i-\alpha}{2}}}d\tilde{\theta}
\end{aligned} \tag{5.2}$$

with

$$P_{k,i}(\tilde{\theta}) = \sum_{(j,l)\in S_{k,i}} c_{k,i,j,l}(\cos(\tilde{\theta}))^j(\sin(\tilde{\theta}))^l,$$

$$S_{k,i} := \{(j,l) \in (0\cup\mathbb{N})^2 : l \geq i-(k-i), j+l = i\}$$

and this, combined with the fact that, for $r \in (\frac12, 4)$, $(r+h) \in \text{supp}(f)$, we have

$$\frac{|\sin(\tilde{\theta})|}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac12}} \leq \frac{C|\sin(\tilde{\theta})|}{|1-\cos(\tilde{\theta})|^{\frac12}} \leq C,$$

implies that, for $(r+h) \in \text{supp}(f)$

$$\sum_{i=0}^{k}\frac{((r+h)r)^i P_{k,i}(\tilde{\theta})}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2+2i-\alpha}{2}}} \leq \frac{C_k}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2+k-\alpha}{2}}}.$$

Therefore we get, for any $\epsilon' > 0$

$$\begin{aligned}
&|\int_{-\pi}^{\pi}\int_{[-r,\infty]\setminus[-N^{-1+\epsilon'},N^{-1+\epsilon'}]} \frac{f(r+h)\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}}}(r+h)dhd\tilde{\theta}|\\
&\leq C_k\frac{||f||_{L^\infty}}{N^k}N^{(1-\epsilon')(2+k-\alpha)}
\end{aligned}$$

and this can be bounded by $C_{\epsilon'}N^{-1-\alpha}$ by taking k big enough.
We can therefore focus only on the integral when $(h,\tilde{\theta}) \in [-N^{-1+\epsilon'}, N^{-1+\epsilon'}]\times[-\pi,\pi]$, and in fact by symmetry it is enough to study $(h,\tilde{\theta}) \in [-N^{-1+\epsilon'}, N^{-1+\epsilon'}]\times[0,\pi]$. For this set, we will make a couple of approximations for our kernel that will only produce a small error, namely, we note that, using integration by parts with respect to $\tilde{\theta}$

$$|\int_{0}^{\pi}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} f(r+h)\cos(N\tilde{\theta})\Big(\frac{\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}}}$$

$$
\begin{aligned}
&- \frac{\cos(N\theta + Ng(r+h) + p(r+h))}{|h^2 + r(r+h)\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}\Big)(r+h)dhd\tilde{\theta}| \\
&\le C\frac{||f||_{L^\infty}}{N}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\int_0^\pi \Big|\frac{2r(r+h)\sin(\tilde{\theta})}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{4-\alpha}{2}}} - \frac{2r(r+h)\tilde{\theta}}{|h^2+r(r+h)\tilde{\theta}^2|^{\frac{4-\alpha}{2}}}\Big|d\tilde{\theta}dh \\
&\le C\frac{||f||_{L^\infty}}{N}\Big(\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\int_0^\pi \frac{1}{|h^2+\tilde{\theta}^2|^{\frac{1-\alpha}{2}}}d\tilde{\theta}dh\Big) \\
&\le C||f||_{L^\infty}N^{-2+\epsilon'}
\end{aligned}
$$

We also have, for large N,

$$
\begin{aligned}
&|\int_0^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} f(r+h)\Big(\frac{(r+h)\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+r(r+h)\tilde{\theta}^2|^{\frac{2-\alpha}{2}}} \\
&- \frac{rcos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}\Big)dhd\tilde{\theta}| \\
&\le ||f||_{L^\infty}C\int_0^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\Big(\frac{h\tilde{\theta}^2}{|h^2+r(r+h)\tilde{\theta}^2|^{\frac{4-\alpha}{2}}} + \frac{h}{|h^2+r(r+h)\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}\Big)dhd\tilde{\theta} \\
&\le ||f||_{L^\infty}CN^{-1+\epsilon'}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\frac{1}{h^{1-\alpha}}dh \le ||f||_{L^\infty}CN^{(-1+\epsilon')(1+\alpha)}.
\end{aligned}
$$

All these inequalities combined already give us

$$
\begin{aligned}
&|\Lambda^{-\alpha}w(r,\theta) - \int_{-\pi}^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\frac{f(r+h)\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r+h))}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}rdhd\tilde{\theta}| \\
&\le C_{\alpha,\epsilon'}||f||_{L^\infty}N^{-(1+\alpha)(1-\epsilon')},
\end{aligned}
$$

and furthermore, we have that

$$
\begin{aligned}
&|\int_{-\pi}^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\frac{f(r+h)\cos(N\tilde{\theta})(\cos(N\theta+Ng(r+h)+p(r+h))-\cos(N\theta+Ng(r+h)+p(r)))}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}rdhd\tilde{\theta}| \\
&\le C||f||_{L^\infty}\ln(N)N^{-1+\epsilon'}\int_0^\pi\int_0^{N^{-1+\epsilon'}}\frac{1}{(h+\tilde{\theta})^{2-\alpha}}dhd\tilde{\theta} \le C||f||_{L^\infty}\ln(N)N^{(1+\alpha)(1-\epsilon')},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\int_{-\pi}^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\frac{(f(r+h)-f(r))\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r))}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}rdhd\tilde{\theta}| \\
&\le C||f||_{L^\infty}\ln(N)N^{-1+\epsilon'}\int_0^\pi\int_0^{N^{-1+\epsilon'}}\frac{1}{(h+\tilde{\theta})^{2-\alpha}}dhd\tilde{\theta} \le C||f||_{L^\infty}\ln(N)N^{(1+\alpha)(1-\epsilon')},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\int_{-\pi}^\pi\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} f(r))\cos(N\tilde{\theta})\frac{1}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}}r \\
&(\cos(N\theta+Ng(r+h)+p(r))-\cos(N\theta+Ng(r)+Nhg'(r)+p(r))dhd\tilde{\theta}| \\
&\le C\ln(N)N^{-1+2\epsilon'}\int_0^\pi\int_0^{N^{-1+\epsilon'}}\frac{1}{(h+\tilde{\theta})^{2-\alpha}}dhd\tilde{\theta} \le C\ln(N)N^{(1+\alpha)(1-\epsilon')+\epsilon'}.
\end{aligned}
$$

Therefore, we just need to study

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \frac{f(r)\cos(N\tilde{\theta})\cos(N\theta+Ng(r)+Nhg'(r)+p(r))}{|h^2+r^2\tilde{\theta}^2|^{\frac{2-\alpha}{2}}} r dh d\tilde{\theta} \\
&= \int_{-r\pi}^{r\pi}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \frac{f(r)\cos(\frac{N}{r}\bar{\theta})\cos(N\theta+Ng(r)+Nhg'(r)+p(r))}{|h^2+\bar{\theta}^2|^{\frac{2-\alpha}{2}}} dh d\bar{\theta} \\
&= N^{-\alpha}\int_{-r\pi N}^{r\pi N}\int_{-N^{\epsilon'}}^{N^{\epsilon'}} \frac{f(r)\cos(\frac{1}{r}s_2)\cos(N\theta+Ng(r)+s_1 g'(r)+p(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2 \\
&= N^{-\alpha}\cos(N\theta+Ng(r)+p(r))\int_{-r\pi N}^{r\pi N}\int_{-N^{\epsilon'}}^{N^{\epsilon'}} \frac{f(r)\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2,
\end{aligned}$$

With this in mind, we want to show that

$$\begin{aligned}
H^* := \lim_{N\to} H_N := \lim_{N\to\infty} \int_{-r\pi N}^{r\pi N}\int_{-N^{\epsilon'}}^{N^{\epsilon'}} \frac{\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2 \\
= \frac{K_\alpha}{\left(\left(\frac{1}{r}\right)^2+g'(r)^2\right)^{\frac{\alpha}{2}}}
\end{aligned} \tag{5.3}$$

with $K_\alpha > 0$ and also that, for $N_2 \geq N_1$,

$$|H_{N_1} - H_{N_2}| \leq C_{\epsilon',\alpha} N_1^{-1+\epsilon'}, \tag{5.4}$$

so in particular $|H^* - H_N| \leq C_{\epsilon'} N^{-1+\epsilon'}$. We start by obtaining (5.4). Note that

$$H_{N_2} - H_{N_1} = \int_{A_{N_1,N_2}\cup B_{N_1,N_2}} \frac{\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2$$

with

$$A_{N_1,N_2} := [N_1^{\epsilon'}, N_2^{\epsilon'}] \times [-r\pi N_2, r\pi N_2] \cup [-N_2^{\epsilon'}, -N_1^{\epsilon'}] \times [-r\pi N_2, r\pi N_2]$$

$$B_{N_1,N_2} := [-N_1^{\epsilon'}, N_1^{\epsilon'}] \times [-r\pi N_2, -r\pi N_1] \cup [-N_1^{\epsilon'}, N_1^{\epsilon'}] \times [r\pi N_1, r\pi N_2].$$

We now bound the integral over A_{N_1,N_2}, we will focus on the part with $s_1 > 0$, the other half being analogous.
Applying integration by parts k times with respect to the variable s_2 we get

$$\begin{aligned}
&\left|\int_{N_1^{\epsilon'}}^{N_2^{\epsilon'}}\int_{-r\pi N_2}^{r\pi N_2} \frac{\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_2 ds_1\right| \\
&\leq \Big(\frac{C_k N_2^{\epsilon'}}{N_2^{2-\alpha}} + C_k \int_{N_1^{\epsilon'}}^{N_2^{\epsilon'}}\int_{-r\pi N_2}^{r\pi N_2} \frac{1}{|s_1^2+s_2^2|^{\frac{2-\alpha+k}{2}}} ds_2 ds_1\Big) \\
&\leq \Big(\frac{C_k N_2^{\epsilon'}}{N_2^{2-\alpha}} + C_k \int_{N_1^{\epsilon'}}^{N_2^{\epsilon'}}\int_{-r\pi N_2}^{r\pi N_2} \frac{1}{|s_1+s_2|^{2-\alpha+k}} ds_2 ds_1\Big) \\
&\leq C_k N_2^{-2+\alpha+\epsilon'} + C_k N_1^{-\epsilon'(k-\alpha)}
\end{aligned}$$

and by taking k big enough we can bounds this quantity by $C_{\epsilon'} N_1^{-2+\alpha+\epsilon'}$ and in particular by $C_{\epsilon'} N_1^{-1+\epsilon'}$.
For the integral over B_{N_1,N_2}, this time focusing on the part with $s_2 > 0$ and again applying integration by parts once with respect to s_2 we have

$$\left|\int_{-N_1^{\epsilon'}}^{N_1^{\epsilon'}}\int_{r\pi N_1}^{r\pi N_2} \frac{\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_2 ds_1\right|$$

$$\leq \frac{CN_1^{\epsilon'}}{N_1^{2-\alpha}} + C\int_{-N_1^{\epsilon'}}^{N_1^{\epsilon'}}\int_{r\pi N_1}^{r\pi N_2} \frac{1}{|s_1+s_2|^{3-\alpha}} ds_2 ds_1 \leq CN^{\epsilon'-2+\alpha}.$$

Next we need to show the last equality in (5.3).
But

$$\int_{-r\pi N}^{r\pi N}\int_{-N^{\epsilon'}}^{N^{\epsilon'}} \frac{\cos(\frac{1}{r}s_2)\cos(s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2$$
$$= 4\int_0^{r\pi N}\int_0^{N^{\epsilon'}} \frac{\cos(\frac{1}{r}s_2 + s_1 g'(r))}{|s_1^2+s_2^2|^{\frac{2-\alpha}{2}}} ds_1 ds_2$$
$$= 4\int_0^{L_{N,r}(A)}\int_0^{\frac{\pi}{2}} \frac{\cos(\frac{1}{r}R\sin(A) + Rcos(A)g'(r))}{R^{2-\alpha}} RdAdR$$

where we made the change of variables $s_1^1+s_2^2 = R^2$, $A = \arctan\left(\frac{s_2}{s_1}\right)$ and $L_{N,r}(A)$ is (given values A, N and r) the maximum value of R that is still in our domain of integration. The expression for $L_{N,r}(A)$ is complicated but we will just need to use that $L_{N,r}(A) \geq \min(N^{\epsilon'}, r\pi N)$. Note also that we can rewrite $\frac{1}{r}R\sin(A) + R\cos(A)g'(r) = R\lambda cos(A+\theta_0)$ with $\lambda = (\frac{1}{r^2} + g'(r)^2)^{\frac{1}{2}}$, $\theta_0 = arctan(-\frac{1}{rg'(r)})$. But

$$\int_0^{L_{N,r}(A)}\int_0^{\frac{\pi}{2}} \frac{\cos(\lambda R\cos(A+\theta_0))}{R^{1-\alpha}} dAdR$$
$$= \int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})}\int_\delta^{\frac{\pi}{2}} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} d\tilde{A}dR$$
$$+ \int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})}\int_0^{\delta} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dAdR$$

so then, using that, for any $S_2 \geq S_1 \geq 0$, $\Gamma \neq 0$, since $\alpha \in (0,1)$, we have

$$|\int_0^{S_1} \frac{\cos(\Gamma R)}{R^{1-\alpha}} dR| \leq \Gamma^{-\alpha} C_{max},\ |\int_{S_1}^{S_2} \frac{\cos(R)}{R^{1-\alpha}} dR| \leq \frac{C_{max}}{S_1^{1-\alpha}},$$

we get

$$|\int_0^{\delta}\int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A}| \leq \int_0^{\delta} C_{max}(\lambda\sin(\tilde{A}))^{-\alpha} d\tilde{A} \leq \frac{CC_{max}\delta}{(\lambda\delta)^{\alpha}}$$

and also

$$lim_{N\to\infty}\int_\delta^{\frac{\pi}{2}}\int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A} = \int_\delta^{\frac{\pi}{2}} P.V.\int_0^{\infty} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A}.$$

Thus

$$\frac{H^*}{4} = lim_{N\to\infty}\int_0^{\frac{\pi}{2}}\int_0^{L_{N,r}(A)} \frac{\cos(\lambda Rcos(A+\theta_0))}{R^{1-\alpha}} dRdA$$
$$= \lim_{\delta\to 0}(\lim_{N\to\infty}\int_\delta^{\frac{\pi}{2}}\int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A})$$
$$+ \lim_{\delta\to 0}(\lim_{N\to\infty}\int_0^{\delta}\int_0^{L_{N,r}(\tilde{A}-\theta_0-\frac{\pi}{2})} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A})$$
$$= \lim_{\delta\to 0}\int_\delta^{\frac{\pi}{2}} P.V.\int_0^{\infty} \frac{\cos(\lambda R\sin(\tilde{A}))}{R^{1-\alpha}} dRd\tilde{A}$$

$$= \lim_{\delta\to 0}\int_{\delta}^{\frac{\pi}{2}}(\lambda\sin(\tilde{A}))^{-\alpha}\int_0^{\infty}\frac{\cos(R)}{R^{1-\alpha}}dRd\tilde{A}$$
$$= \frac{1}{(\frac{1}{r^2}+g'(r)^2)^{\frac{\alpha}{2}}}\int_0^{\frac{\pi}{2}}\sin(\tilde{A})^{-\alpha}d\tilde{A}\int_0^{\infty}\frac{\cos(R)}{R^{1-\alpha}}dR,$$

so, if we prove that $\int_0^{\infty}\frac{\cos(R)}{R^{1-\alpha}}dR$ is positive we are done.
For this, we note that, for $n\in\mathbb{N}$

$$\int_{2\pi n}^{2\pi(n+1)}\frac{\cos(R)}{R^{1-\alpha}}dR = \int_{2\pi n}^{2\pi(n+1)}\frac{\sin(R)(1-\alpha)}{R^{2-\alpha}}dR > 0,$$

where we used integration by parts for the equality and the fact that the denominator is increasing and $\sin(x+\pi) = -\sin(x)$ for the inequality. In fact, the integral is also positive for $n = 0$ by taking the integral in $[\delta, 2\pi]$, applying integration by parts there and then taking δ small.
But then since, for $d\leq 2\pi$

$$\int_{2\pi n}^{2\pi n+d}\frac{\cos(R)}{R^{1-\alpha}}dR \leq \frac{C}{(2\pi n)^{1-\alpha}}$$

the limit trivially exists and is positive.
Combining all these bounds we have, for any $\epsilon' > 0$,

$$I_1 \leq C_{\epsilon'}||f||_{L^\infty}N^{(-1-\alpha)(1-\epsilon')},$$

so we just need to bound I_2. In order to bound the L^2 norm for $r\notin(\frac{1}{4},2)$, we can use integration by parts twice and $h\gtrsim r$, $r+h\approx 1$ to get

$$|\int_{-\pi}^{\pi}\frac{f(r+h)\cos(N\tilde{\theta})\cos(N\theta+Ng(r+h)+p(r))}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}}}(r+h)d\tilde{\theta}|$$
$$\leq \frac{C||f||_{L^\infty}}{N^2}\int_{-\pi}^{\pi}\frac{r}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|^{\frac{2-\alpha}{2}+1}}(1+\frac{r}{|h^2+2r(r+h)(1-\cos(\tilde{\theta}))|})d\tilde{\theta}$$
$$\leq \frac{C||f||_{L^\infty}}{N^2|h|^{3-\alpha}}(1+\frac{1}{|h|})$$

which gives us

$$||(\Lambda^{-\alpha}w(r,\theta))1_{r\in(0,\frac{1}{4})}||_{L^2} \leq C||(\Lambda^{-\alpha}w(r,\theta))1_{r\in(0,\frac{1}{4})}||_{L^\infty} \leq \frac{C||f||_{L^\infty}}{N^2}$$

and, for $r\geq 2$

$$|\Lambda^{-\alpha}w(r,\theta)| \leq \frac{C||f||_{L^\infty}}{N^2 r^{3-\alpha}}$$

so that

$$||(\Lambda^{-\alpha}w(r,\theta))1_{r\in(2,\infty)}||_{L^2} \leq \frac{C||f||_{L^\infty}}{N^2}$$

which finally gives us

$$||\Lambda^{-\alpha}w(r,\theta) - K_\alpha\frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}||_{L^2} \leq I_1+I_2 \leq C_{\epsilon'}N^{(-1-\alpha)(1-2\epsilon')}.$$

Now, taking for example $\epsilon' = \frac{\epsilon}{4}$ finishes the proof for L^2. Note also that we only needed the C^2 bounds of f and g to obtain this result. Next to obtain the bound for H^3 it is enough to check that we have a small L^2 norm for some arbitrary second derivative, that is to say, we want to show that

$$||\frac{\partial^3\Lambda^{-\alpha}w(r,\theta)}{\partial x_i\partial x_j\partial x_k} - \frac{\partial^3 K_\alpha\frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}}{\partial x_i\partial x_j\partial x_k}||_{L^2} \leq C_{\epsilon,\beta}N^{-1-\alpha+\epsilon+3}||f||_{L^\infty}.$$

We will consider $i=j=k=1$ for simplicity of notation. To bound this norm, we divide it in two different contributions

$$
\begin{aligned}
&||\frac{\partial^3\Lambda^{-\alpha}w(r,\theta)}{\partial x_1^3}-\frac{\partial^3 K_\alpha\frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}}{\partial x_1^3}||_{L^2}\\
&\leq ||\Lambda^{-\alpha}\frac{\partial^3 w(r,\theta)}{\partial x_1^3}-\frac{K_\alpha}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}\frac{\partial^3 w(r,\theta)}{\partial x_1^3}||_{L^2}\\
&+||\frac{\partial^2\frac{K_\alpha}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}}{\partial x_1^2}\frac{\partial w(r,\theta)}{\partial x_1}||_{L^2}+||\frac{\partial\frac{K_\alpha}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}}{\partial x_1}\frac{\partial^2 w(r,\theta)}{\partial x_1^2}||_{L^2}\\
&+||\frac{\partial^3\frac{K_\alpha}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}}{\partial^3 x_1}w(r,\theta)||_{L^2}\\
&\leq ||\Lambda^{-\alpha}\frac{\partial^3 w(r,\theta)}{\partial x_1^3}-\frac{K_\alpha}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}\frac{\partial^3 w(r,\theta)}{\partial x_1^3}||_{L^2}\\
&+\sum_{i=1}^{3}N^{-\alpha}||\frac{K_\alpha}{|(\frac{1}{r})^2+g'(r)^2|^{\alpha/2}}||_{C^i(r\in(\frac12,\frac32))}||w||_{H^{3-i}}.
\end{aligned}
$$

The first term of the contribution can be bounded by writing the derivatives in polar coordinates, dividing it in its different frequencies in θ (which now includes frequencies $N\pm1$, $N\pm2$ and $N\pm3$, which does not change the bounds for N big) and using the exact same bounds we obtained in L^2. The other term can be bound easily by direct computation by again writing the derivatives in polar coordinates, obtaining the desired bound for $\beta=3$. The interpolation inequality for Sobolev spaces then gives the result for $\beta\in[0,3]$.
Finally, the result for $\lambda>1$ follows directly from a scaling argument plus applying the lemma for $w(\frac{r}{\lambda},\theta)$ since

$$
\begin{aligned}
&||\Lambda^{-\alpha}w(r,\theta)-K_\alpha\frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}}||_{H^\beta}\\
&\leq \lambda^{-1-\alpha+\beta}||\Lambda^{-\alpha}w(\frac{r}{\lambda},\theta)-K_\alpha\frac{w(\frac{r}{\lambda},\theta)}{|(\frac{N}{r})^2+N^2g'(\frac{r}{\lambda})^2|^{\alpha/2}}||_{H^\beta}\\
&\leq C_{P,\epsilon,\alpha}M\lambda^{-1-\alpha+\beta}N^{-1-\alpha+\epsilon+\beta}||f||_{L^\infty}
\end{aligned}
$$

□

Corollary 5.2.2. *For any fixed parameters $\alpha\in(0,1]$, $P,\epsilon>0$ there exists N_0 such that if $N>N_0$, then for any $\lambda\geq1$ and functions $f(r)$, $g(r)$ and $p(r)$ fulfilling supp$f\subset(\frac{1}{2\lambda},\frac{3}{2\lambda})$ and*

$$||g(\frac{r}{\lambda})||_{C^6}\leq \ln{(N)}^P,\ ||f(\frac{r}{\lambda})||_{C^5}\leq \ln{(N)}^P||f||_{L^\infty}, ||p(\frac{r}{\lambda})||_{C^5}\leq \ln{(N)}^P$$

then for

$$w(r,\theta):=f(r)\cos(N(\theta+g(r))+p(r))$$

we have that for $\beta\in[0,3-\alpha]$ there exist constants $K_\alpha>0$ and $C_{P,\epsilon,\alpha,\beta}$ (depending on α and P,ϵ and α respectively) such that

$$||\Lambda^\alpha w(r,\theta)-K_\alpha^{-1}w(r,\theta)|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}||_{H^\beta}\leq C_{P,\epsilon,\alpha}\lambda^{-1+\alpha+\beta}N^{-1+\alpha+\epsilon+\beta}||f||_{L^\infty}.$$

Proof. This follows from the previous lemma. If we define, for w as in our statement, the operators

$$\bar{\Lambda}^\alpha(w(r,\theta)):=K_\alpha^{-1}w(r,\theta)|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2} \tag{5.5}$$

$$\bar{\Lambda}^{-\alpha}(w(r,\theta)):=K_\alpha\frac{w(r,\theta)}{|(\frac{N}{r})^2+N^2g'(r)^2|^{\alpha/2}} \tag{5.6}$$

we have that $\bar{\Lambda}^{-\alpha}(\bar{\Lambda}^{\alpha}w) = w$, $\Lambda^{-\alpha}(\Lambda^{\alpha}w) = w$, so

$$(\Lambda^{\alpha} - \bar{\Lambda}^{\alpha})(w) = -\Lambda^{\alpha}(\Lambda^{-\alpha} - \bar{\Lambda}^{-\alpha})\bar{\Lambda}^{\alpha}w$$

and since, for our choice of w, $\bar{\Lambda}^{\alpha}w$ fulfils the hypothesis of Lemma 5.2.1, we can apply it and get, for $\beta \in [0, 3]$

$$||(\Lambda^{-\alpha} - \bar{\Lambda}^{-\alpha})\bar{\Lambda}^{\alpha}w||_{H^{\beta}} \leq C_{\epsilon,\alpha,\beta,P}\lambda^{-1+\beta}N^{-1+\epsilon+\beta}||f||_{L^{\infty}}$$

and therefore

$$||(\Lambda^{\alpha} - \bar{\Lambda}^{\alpha})w||_{H^{\beta-\alpha}} = ||\Lambda^{\alpha}(\Lambda^{-\alpha} - \bar{\Lambda}^{-\alpha})\bar{\Lambda}^{\alpha}w||_{H^{\beta-\alpha}} \leq C_{\epsilon,\alpha,\beta,P}\lambda^{-1+\beta}N^{-1+\epsilon+\beta}||f||_{L^{\infty}},$$

which finishes the proof.

□

5.2.3 Other relevant bounds

Even though the most crucial technical bounds in this chapter are the ones we obtained for the fractional dissipation, we need some other technical lemmas in order to obtain a suitable pseudo-solution and control the errors between the pseudo-solution and the real solution to (5.1). Corollary 5.2.3 and Lemma 5.2.4 give useful local approximations for v. Lemma 5.2.5 and Corollary 5.2.6 give commutator estimates for the velocity of a highly oscillatory function, which will be useful when propagating the L^2 error between our pseudo-solution and the actual solution to (5.1). Lemma 5.2.7 proves some general decay bounds for radial functions, and finally Lemma 5.2.8 shows that we can find a radial function with several useful properties, that we will use to construct the initial conditions for our pseudo-solution.

Corollary 5.2.3. *For any fixed parameters $P, \epsilon > 0$ there exists N_0 such that if $N > N_0$, then for any $\lambda \geq 1$ and functions $f(r)$, $g(r)$ and $p(r)$ fulfilling supp$f(r) \subset (\frac{1}{2\lambda}, \frac{3}{2\lambda})$ and*

$$||g(\frac{r}{\lambda})||_{C^5} \leq \ln{(N)}^P \ \ ||f(\frac{r}{\lambda})||_{C^5} \leq \ln{(N)}^P||f||_{L^{\infty}}, ||p(\frac{r}{\lambda})||_{C^5} \leq \ln{(N)}^P$$

if we define

$$w(r,\theta) := f(r)\cos(N(\theta + g(r)) + p(r))$$

we have that for $\beta \in [0, 2]$ there exist constants $C_0 > 0$ and $C_{\epsilon,\beta,P}$ (depending on β, P and ϵ) such that

$$||v_1(w(r,\theta)) + \frac{\partial}{\partial x_2}\bar{\Lambda}^{-1}(w)(r,\theta)||_{H^{\beta}} \leq C_{\epsilon,\beta,P}N^{-1+\epsilon+\beta}||f||_{L^{\infty}},$$

$$||v_2(w(r,\theta)) - \frac{\partial}{\partial x_1}\bar{\Lambda}^{-1}(w)(r,\theta)||_{H^{\beta}} \leq C_{\epsilon,\beta,P}N^{-1+\epsilon+\beta}||f||_{L^{\infty}},$$

with $\bar{\Lambda}^{-1}$ defined as in (5.6).

Proof. This is a direct consequence of Lemma 5.2.1, since

$$v_1(w(r,\theta)) = -\frac{\partial}{\partial x_2}\Lambda^{-1}(w(r,\theta)), \quad v_2(w(r,\theta)) = \frac{\partial}{\partial x_1}\Lambda^{-1}(w(r,\theta))$$

and applying Lemma 5.2.1, we have

$$||\frac{\partial}{\partial x_i}\Lambda^{-1}(w(r,\theta)) - \frac{\partial}{\partial x_i}\bar{\Lambda}^{-1}(w(r,\theta))||_{H^{\beta}} \leq ||\Lambda^{-1}(w(r,\theta)) - \bar{\Lambda}^{-1}(w(r,\theta))||_{H^{\beta+1}}$$
$$\leq C_{\epsilon,P}\lambda^{-1+\beta}N^{-1+\epsilon+\beta}||f||_{L^{\infty}}.$$

□

Lemma 5.2.4. *For any fixed parameters $P, \epsilon > 0$ there exists N_0 such that if $N > N_0$, then for any $\lambda \geq 1$ and functions $f(r)$,$g(r)$ and $p(r)$ fulfilling supp$f(r) \subset (\frac{1}{2\lambda}, \frac{3}{2\lambda})$ and*

$$||g(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P \quad ||f(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P ||f(\frac{r}{\lambda})||_{L^\infty}, ||p(\frac{r}{\lambda})||_{C^5} \leq \ln(N)^P$$

if we define

$$w(r,\theta) := f(r)\cos(N(\theta + g(r)) + p(r))$$

we have that for $\beta \in [0,2]$ there exist constants $C_0 > 0$ and $C_{P,\epsilon}$ (depending on α and P and ϵ respectively) such that for N big enough,

$$||v_r(w(r,\theta)) + C_0 \frac{1}{(1 + r^2 g'(r)^2)^{\frac{1}{2}}} f(r)\sin(N(\theta + g(r)) + p(r))||_{H^\beta} \leq C_{\epsilon,\beta,P}\lambda^{-1+\beta}N^{-1+\epsilon+\beta}||f||_{L^\infty}$$

Proof. This proof is very similar to that of Lemma 5.2.1, but now we study the operator

$$\begin{aligned}
v_r(w)(r,\theta) &= v \cdot \hat{r} \\
&= \int_{[-r,\infty]\times[-\pi,\pi]} \frac{(r+h)^2 \sin(\theta')(w(r+h,\theta'+\theta) - w(r,\theta))}{|h^2 + 2r(r+h)(1-\cos(\theta'))|^{3/2}} d\theta' dh
\end{aligned}$$

so we will not delve too deeply into the details and mostly mention the key differences. Again, as before, we consider $P = 1$ for simplicity, and we start dealing with the case $\beta = 0, \lambda = 1$. First, using integration by parts k times with respect to θ' as in Lemma 5.2.1, we note that, for $r \in (\frac{1}{4}, 2)$ we have that,

$$\int_{(-r,\infty)\setminus[-N^{-1+\epsilon'},N^{-1+\epsilon'}]} \int_{-\pi}^{\pi} \frac{(r+h)^2\sin(\theta')(w(r+h,\theta'+\theta) - w(r,\theta))}{|h^2 + 2r(r+h)(1-\cos(\theta'))|^{3/2}} d\theta' dh| \leq C_{\epsilon'} N^{-1}||f||_{L^\infty},$$

and furthermore, integrating by parts 2 times with respect to θ' we obtain

$$\begin{aligned}
&|\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \int_{[N^{-\frac{1}{2}},\pi]\cup[-\pi,-N^{-\frac{1}{2}}]} \frac{(r+h)^2\sin(\theta')(w(r+h,\theta'+\theta) - w(r,\theta))}{|h^2 + 2r(r+h)(1-\cos(\theta'))|^{3/2}} d\theta' dh| \\
&= |\int_{[N^{-\frac{1}{2}},\pi]\cup[-\pi,-N^{-\frac{1}{2}}]} \int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \frac{f(r+h)\sin(N\theta')\sin(\theta')\sin(N\theta + Ng(r+h) + p(r+h))}{|h^2 + 2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}} (r+h)^2 dh d\theta'| \\
&\leq C\Big(\frac{1}{N}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \frac{||f||_{L^\infty}}{|(1-\cos(N^{-\frac{1}{2}}))|} dh + \int_{N^{-\frac{1}{2}}}^{\pi}\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}} \frac{1}{N^2}\frac{||f||_{L^\infty}}{|(1-\cos(\theta'))|^2} dh d\theta'\Big) \\
&\leq C||f||_{L^\infty} N^{-1+\epsilon'}.
\end{aligned}$$

So we can focus on the integral over $A := [-N^{-1+\epsilon}, N^{-1+\epsilon}] \times [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}]$. Then, we check that

$$\begin{aligned}
&|\int_A \Big(\frac{(r+h)^2\sin(\theta')(w(r+h,\theta'+\theta) - w(r,\theta))}{|h^2 + 2r(r+h)(1-\cos(\theta'))|^{3/2}} - \frac{r^2\theta'(w(r+h,\theta'+\theta) - w(r,\theta))}{|h^2 + r^2(\theta')^2)|^{3/2}}\Big) d\theta' dh| \\
&\leq C||f||_{L^\infty} N^{-1+\epsilon'}
\end{aligned}$$

so that we can work with the simplified version of the kernel, and we also have

$$\begin{aligned}
&|\int_A \frac{r^2\theta'(w(r+h,\theta'+\theta) - f(r)\sin(N\theta')\sin(N\theta + Ng(r) + Nhg'(r) + p(r)))}{|h^2 + r^2(\theta')^2)|^{3/2}} d\theta' dh| \\
&\leq C||f||_{L^\infty} N^{-1+3\epsilon'}.
\end{aligned}$$

Altogether, we obtain, for $r \in (\frac{1}{4}, 2)$,

$$|v_r(w)+f(r)\sin(N\theta+Ng(r)+p(r))\int_A \frac{r^2\theta'\sin(N\theta'+Nhg'(r)))}{|h^2+r^2(\theta')^2)|^{3/2}}d\theta' dh|\leq C_{\epsilon'}N^{-1+3\epsilon'}||f||_{L^\infty}.$$

Next, defining H_N as

$$H_N:=\int_A \frac{r^2\theta'\sin(N\theta'+Nhg'(r)))}{|h^2+r^2(\theta')^2)|^{3/2}}d\theta' dh=\int_{-\frac{N^{\epsilon'}}{r}}^{\frac{N^{\epsilon'}}{r}}\int_{-N^{\frac{1}{2}}}^{N^{\frac{1}{2}}}\frac{s_1\sin(s_1)\cos(rs_2g'(r))}{|s_1^2+s_2^2|^{3/2}}ds_1 ds_2.$$

We want to show that

$$H^*:=lim_{N\to\infty}H_N=\frac{C_0}{|1+r^2g'(r)^2|^{1/2}} \tag{5.7}$$

and that

$$|H^*-lim_{N\to\infty}H_N|\leq C_{\epsilon'}N^{-1+\epsilon'}||f||_{L^\infty}. \tag{5.8}$$

(5.8) is obtained exactly as in Lemma 5.2.1, by getting bounds for $|H_{N_2}-H_{N_1}|$ using integration by parts in the domain that remains after canceling out the integrals from H_{N_1} and H_{N_2}. As for (5.7), using that, for any $K\in\mathbb{R}$

$$|\int_0^K \frac{\sin(x)}{x}dx|\leq C,$$

and that, for $\lambda\neq 0$

$$\int_0^\infty \frac{\sin(\lambda x)}{x}dx=sign(\lambda)\int_0^\infty\frac{\sin(x)}{x}dx=sign(\lambda)\frac{C_0}{4}$$

we get that, using the change of variables $s_1=R\sin(A)$, $s_2=R\cos(A)$, basic trigonometric identities, checking carefully the convergence of the integrals, and using

$$\sin(\tan^{-1}(x))=\frac{x}{(1+x^2)^{\frac{1}{2}}}$$

we get

$$\begin{aligned}
H^* &= lim_{N\to\infty}\int_{-\frac{N^{\epsilon'}}{r}}^{\frac{N^{\epsilon}}{r}}\int_{-N^{\frac{1}{2}}}^{N^{\frac{1}{2}}}\frac{s_1\sin(s_1)\cos(rs_2g'(r))}{|s_1^2+s_2^2|^{3/2}}ds_1ds_2\\
&=\int_{-\pi}^{\pi}\sin(A)\int_0^\infty\frac{\sin(R(\sin(A)+\cos(A)rg'(r)))}{R}dRdA\\
&=\frac{C_0}{4}\int_{-\pi}^{\pi}\sin(A)\text{sign}(\sin(A)+\cos(A)rg'(r))dA\\
&=\frac{C_0}{4}\int_{-\pi}^{\pi}\sin(A)\text{sign}(\cos(A+\tan^{-1}(\frac{-1}{rg'(r)})))dA\\
&=-\frac{C_0}{4}\sin(\tan^{-1}(\frac{-1}{rg'(r)}))\int_{-\pi}^{\pi}cos(A+\tan^{-1}(\frac{-1}{rg'(r)}))\text{sign}(\cos(A+\tan^{-1}(\frac{-1}{rg'(r)})))dA\\
&=C_0\frac{1}{(1+r^2g'(r)^2)^{\frac{1}{2}}}
\end{aligned}$$

as we wanted to prove. The rest of the proof does not have any meaningful differences with Lemma 5.2.1, a bound for the decay of $v_r(w)$ is obtained for $r\notin(\frac{1}{4},2)$ to obtain the L^2 bound and then taking three derivatives, applying Leibniz rule and bounding each term gives us the bound for H^3. Then, the interpolation inequality gives then the result for $\beta\in(0,3)$.
For the case $\lambda>1$,

$$
\begin{aligned}
&||v_r(w(r,\theta))+C_0\frac{1}{(1+r^2g'(r)^2)^{\frac{1}{2}}}f(r)\sin(N(\theta+g(r))+p(r))||_{H^\beta}\\
&\leq \lambda^{-1+\beta}||v_r(w(\frac{r}{\lambda},\theta))+C_0\frac{1}{(1+r^2g'(\frac{r}{\lambda})^2)^{\frac{1}{2}}}f(\frac{r}{\lambda})\sin(N(\theta+g(\frac{r}{\lambda}))+p(\frac{r}{\lambda}))||_{H^\beta}\\
&\leq C_{\epsilon,\beta,P}M\lambda^{-1+\beta}N^{-1+\epsilon+\beta}||f||_{L^\infty}.
\end{aligned}
$$

□

Lemma 5.2.5. *Given $\epsilon>0$, $N\in\mathbb{N}>3$, $A>0$ then for any functions $g(r),f(r),p(r)\in C^1$, $f(r)\in L^2$ and if we define*

$$w(r,\theta):=f(r)\cos(N\theta+p(r))$$

then we have that for $i=1,2$

$$||1_{r\in[N^{-A},2]}[v_i(g(r)\sin(\theta)w(r,\theta))-g(r)\sin(\theta)v_i(w(r,\theta))]||_{L^2}\leq C_{\epsilon,A}N^{-1+\epsilon}||g||_{C^1}||w||_{L^2},$$

$$||1_{r\in[N^{-A},2]}[v_i(g(r)\cos(\theta)w(r,\theta))-g(r)\cos(\theta)v_i(w(r,\theta))]||_{L^2}\leq C_{\epsilon,A}N^{-1+\epsilon}||g||_{C^1}||w||_{L^2}.$$

Proof. We will only consider the first inequality and with $i=1$, since the other cases are analogous. We note now that

$$v_1(W(r,\theta))=P.V.\int_{-r}^{\infty}\int_{-\pi}^{\pi}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)W(r+h,\theta+\theta')dhd\theta'.$$

Now, since the principal value integral is defined using cartesian coordinates, we would like to show that for C^1 functions, there is a more suitable expression using polar coordinates, namely we would like to show that

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\Big(\int_{\mathbb{R}\setminus B_\epsilon(x)}\frac{y_2-x_2}{|x-y|^3}w(y)dy-\int_{||x|-|y||\geq\epsilon}\frac{y_2-x_2}{|x-y|^3}w(y)dy\Big)\\
&=\lim_{\epsilon\to 0}\int_{\{|x|+\epsilon\geq|y|\geq|x|-\epsilon\}\setminus B_\epsilon(x)}\frac{y_2-x_2}{|x-y|^3}w(y)dy=0
\end{aligned}
$$

but, writing the integrals in polar coordinates and cancelling all the terms with the wrong parity with respect to h or θ'

$$
\begin{aligned}
&|\int_{\{|y|+\epsilon\geq|x|\geq|y|-\epsilon\}\setminus B_\epsilon(x)}\frac{y_2-x_2}{|x-y|^3}w(x)dy|\\
&\leq C||w||_{L^\infty}\int_{\{|h|\geq\epsilon\}\setminus B_\epsilon(x)}\frac{h^2+\theta^2}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}dhd\theta'\leq C||w||_{L^\infty}\epsilon\ln(\epsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\int_{|y|+\epsilon\geq|x|\geq|y|-\epsilon\setminus B_\epsilon(x)}\frac{y_2-x_2}{|x-y|^3}(w(y)-w(x))dy|\\
&\leq||w||_{C^1}|\int_{|y|+\epsilon\geq|x|\geq|y|-\epsilon\setminus B_\epsilon(x)}\frac{1}{|x-y|}dy|\leq C||w||_{C^1}\epsilon\ln(\epsilon),
\end{aligned}
$$

so in particular we can write

$$
\begin{aligned}
v_1(W(r,\theta))&=P.V.\int_{-r}^{\infty}\int_{-\pi}^{\pi}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)W(r+h,\theta+\theta')dhd\theta'\\
&=\lim_{\epsilon\to 0}\int_{[-r,\infty]\setminus[-\epsilon,\epsilon]}\int_{-\pi}^{\pi}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)W(r+h,\theta+\theta')dhd\theta'.
\end{aligned}
$$

Now, using integration by parts k times with respect to θ', we have that, for $|\frac{h}{r}|\geq\frac{1}{2}$

$$
\begin{aligned}
&|\int_{-\pi}^{\pi}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{3/2}}(r+h)\cos(N\theta'+C)d\theta'|\\
&=|\frac{1}{r}\int_{-\pi}^{\pi}\frac{(1+\frac{h}{r})\sin(\theta+\theta')-\sin(\theta)}{|(\frac{h}{r})^2+2(1+\frac{h}{r})(1-\cos(\theta'))|^{3/2}}(1+\frac{h}{r})\cos(N\theta'+C)d\theta'|\\
&\leq\frac{1}{r}\int_{-\pi}^{\pi}\frac{C_k}{N^k}\frac{1}{|(\frac{h}{r})^2+2(1+\frac{h}{r})(1-\cos(\theta'))|^{\frac{1}{2}}}d\theta'\\
&\leq\frac{rC_k}{h^2N^k}.
\end{aligned}
$$

Analogously if $\frac{1}{2}\geq|\frac{h}{r}|\geq N^{-1+\epsilon}$, we can also perform integration by parts k times

$$
\begin{aligned}
&|\frac{1}{r}\int_{-\pi}^{\pi}\frac{(1+\frac{h}{r})\sin(\theta+\theta')-\sin(\theta)}{|(\frac{h}{r})^2+2(1+\frac{h}{r})(1-\cos(\theta'))|^{3/2}}cos(N\theta'+C)d\theta'|\\
&\leq\frac{C_k}{rN^k}\int_{-\pi}^{\pi}\sum_{i=0}^{k}\frac{1}{|(\frac{h}{r})^2+2(1+\frac{h}{r})(1-\cos(\theta'))|^{\frac{2+i}{2}}}d\theta'\\
&\leq\frac{C_k rN^{(1-\epsilon)k}}{h^2N^k}\leq\frac{C_k rN^{-k\epsilon}}{h^2}
\end{aligned}
$$

for any $k\in\mathbb{N}$ for some C_k.
Using this, for any $H(r)\in L^2$ and $p(r)\in C^1$ we get, for $r\in[N^{-A},2]$, and defining $v_{1,1}$

$$v_{1,1}(H(r)\cos(N\theta+p(r))):=\int_{-rN^{-1+\epsilon'}}^{rN^{-1+\epsilon'}}\int_{-\pi}^{\pi}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{3/2}}(r+h)\cos(N\theta+p(r))H(r+h)d\theta'dh$$

$$\leq\int_{|h|\geq rN^{-1+\epsilon'}}(\frac{C_kr}{h^2N^k}+\frac{C_kr}{h^2N^{k\epsilon}})|H(r+h)|dh\leq\frac{C_{\epsilon',A}}{N}||H||_{L^2}$$

which in particular implies $||1_{r\in[N^{-A},2]}g(r)\sin(\theta)v_{1,1}(w(r,\theta))||_{L^2}\leq\frac{C_{\epsilon',A}}{N}||g||_{L^\infty}||w||_{L^2}$ and by decomposing $sin(\theta)w(r,\theta)$ in its different Fourier modes and applying the previous inequality, $||1_{r\in[N^{-A},2]}v_{1,1}(g(r)\sin(\theta)w(r,\theta))||_{L^2}\leq\frac{C_{\epsilon',A}}{N}||g||_{L^\infty}||w||_{L^2}$.
If we now further divide the operator $v_1(W)$ as $v_1(W)=v_{1,1}(W)+v_{1,2}(W)+v_{1,3}(W)$, with

$$v_{1,2}(W):=\int_{-N^{-1+\epsilon'}}^{N^{-1+\epsilon'}}\int_{-rN^{-1+\epsilon'}}^{rN^{-1+\epsilon'}}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)w(r+h,\theta+\theta')dhd\theta'$$

$$v_{1,3}(W):=\int_{N^{-1+\epsilon'}}^{2\pi-N^{-1+\epsilon'}}\int_{rN^{-1+\epsilon'}}^{rN^{-1+\epsilon'}}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)w(r+h,\theta+\theta')dhd\theta'$$

using the previous bound it is enough to show that, for $i=2,3$

$$||1_{r\in[N^{-A},2]}[v_{1,i}(g(r)\sin(\theta)w(r,\theta))-g(r)\sin(\theta)v_{1,i}(w(r,\theta))]||_{L^2}\leq C_\epsilon N^{-1+\epsilon}||g||_{C^1}||f||_{L^2}.$$

But, for $i=2$

$$
\begin{aligned}
&||1_{r\in[N^{-A},2]}g(r)\sin(\theta)v_{1,2}(w(r,\theta))-v_{1,2}(g(r)\sin(\theta)w(r,\theta))||_{L^2}^2\\
&\leq||1_{r\in[0,2]}\int_{-N^{-1+\epsilon}}^{N^{-1'+\epsilon}}\int_{-N^{-1+\epsilon}}^{N^{-1+\epsilon}}\frac{(r+h)\sin(\theta+\theta')-r\sin(\theta)}{|h^2+2r(r+h)(1-\cos(\theta'))|^{\frac{3}{2}}}(r+h)
\end{aligned}
$$

$$[g(r+h)\sin(\theta+\theta')-g(r)\sin(\theta)]w(r+h,\theta+\theta')dhd\theta'||_{L^2}^2$$
$$\leq C||g||_{C^1}^2\int_0^2\int_0^{2\pi}\Big(\int_{|h|\leq rN^{-1+\epsilon'}}\int_{|\theta'|\leq N^{-1+\epsilon'}}\frac{(r+h)^2(|\theta'|+|h|)^2|f(r+h)|}{|h^2+2r(r+h)(1-\cos(\theta'))|^{3/2}}d\theta' dh\Big)^2 rdrd\theta$$

$$\leq C||g||_{C^1}^2\int_0^2\Big(\int_{|h|\leq rN^{-1+\epsilon'}}\int_{|\theta'|\leq N^{-1+\epsilon'}}\frac{|f(r+h)|}{|h^2+r^2(\theta')^2|^{1/2}}d\theta' dh\Big)^2 rdr$$
$$= C||g||_{C^1}^2\int_0^2\Big(\int_{|\tilde{h}|\leq N^{-1+\epsilon'}}\int_{|\theta'|\leq N^{-1+\epsilon'}}\frac{|f(r+r\tilde{h})|}{|\tilde{h}^2+(\theta')^2|^{1/2}}d\theta' d\tilde{h}\Big)^2 rdr$$
$$\leq C||g||_{C^1}^2\int_{|\tilde{h}_1|\leq N^{-1+\epsilon'}}\int_{|\theta_1'|\leq N^{-1+\epsilon'}}\frac{1}{|\tilde{h}_1^2+\theta_1^2|^{1/2}}\int_{|\tilde{h}_2|\leq N^{-1+\epsilon'}}\int_{|\theta_2'|\leq N^{-1+\epsilon'}}\frac{1}{|\tilde{h}_2^2+\theta_2^2|^{1/2}}$$
$$\int_0^2|f(r+r\tilde{h}_1)||f(r+r\tilde{h}_2)|rdrd\theta_2'd\tilde{h}_2d\theta_1'd\tilde{h}_1$$
$$\leq CN^{-2+2\epsilon'}||g||_{C^1}^2||f||_{L^2}^2$$

and similarly

$$||1_{r\in[N^{-A},2]}f(r)v_{1,3}(g(r)\cos(N\theta))-v_{1,3}(f(r)g(r)\cos(N\theta))||_{L^2}^2$$
$$\leq C||g||_{C^1}^2\int_0^2\Big(\int_{|\tilde{h}|\leq N^{-1+\epsilon'}}\int_{\pi\geq|\theta'|\geq N^{-1+\epsilon'}}\frac{|f(r+r\tilde{h})|}{|\tilde{h}^2+(\theta')^2|^{1/2}}d\theta' d\tilde{h}\Big)^2 rdr$$
$$\leq C||g||_{C^1}^2\int_0^2\Big(\int_{|\tilde{h}|\leq N^{-1+\epsilon'}}|f(r+r\tilde{h})|\ln(|h|+N^{-1+\epsilon})d\tilde{h}\Big)^2 rdr$$
$$\leq C||g||_{C^1}^2\int_{|\tilde{h}_1|\leq N^{-1+\epsilon'}}\ln(|h_1|+N^{-1+\epsilon})\int_{|\tilde{h}_2|\leq N^{-1+\epsilon'}}\ln(|h_2|+N^{-1+\epsilon})\int_0^2|f(r+r\tilde{h}_1)||f(r+r\tilde{h}_2)|rdrd\tilde{h}_2d\tilde{h}_1$$
$$\leq C\ln(N)^2N^{-2+2\epsilon'}||g||_{C^1}^2||f||_{L^2}^2.$$

We obtain now our result from combining all our inequalities since

$$||1_{r\in[N^{-A},2]}v_1(f(r)g(r)\cos(N\theta))-f(r)v_1(g(r)\cos(N\theta))||_{L^2}$$
$$\leq\sum_{i=1}^3||1_{r\in[N^{-A},2]}v_{1,i}(f(r)g(r)\cos(N\theta))-f(r)v_{1,i}(g(r)\cos(N\theta))||_{L^2}$$
$$\leq C_{\epsilon'}||f||_{L^2}||g||_{C^1}(N^{-1}+N^{-1+\epsilon'}+\ln(N)N^{-1+\epsilon'}).$$

□

Corollary 5.2.6. *Given $\epsilon>0$, $N\in\mathbb{N}>3$, $\lambda>1$, $A>0$ then for any functions $g(r),f(r),h(r)\in C^1$, $f(r)\in L^2$ and we have that for $i=1,2$, if we define*

$$w(r,\theta):=f(r)\cos(N\theta+h(r))$$

$$||1_{r\in[N^{-A}\lambda,2\lambda]}[v_i(g(r)\sin(\theta)w(r,\theta))-g(r)\sin(\theta)v_i(w(r,\theta))]||_{L^2}\leq C_{\epsilon,A}\lambda N^{-1+\epsilon}||g||_{C^1}||w||_{L^2},$$
$$||1_{r\in[N^{-A}\lambda,2\lambda]}[v_i(g(r)\cos(\theta)w(r,\theta))-g(r)\cos(\theta)v_i(w(r,\theta))]||_{L^2}\leq C_{\epsilon,A}\lambda N^{-1+\epsilon}||g||_{C^1}||w||_{L^2}.$$

Proof. To prove it is enough to note that, for $f(r)$, $g(r)$ as in our hypothesis, $f(\lambda r),g(\lambda r)$ would fulfil the hypothesis of Lemma 5.2.5, so again focusing on $i=1$ and $g(r)\sin(\theta)$, using the scaling properties of v and of the L^2 and C^1 norms we have

$$||1_{r\in[N^{-A}\lambda,2\lambda]}[v_1(g(r)\sin(\theta)w(r,\theta))-g(r)\sin(\theta)v_1(w(r,\theta))]||_{L^2}$$

$$= \lambda ||1_{r\in[N^{-A},2]}[v_1(g(\lambda r)\sin(\theta)w(\lambda r,\theta)) - g(\lambda r)\sin(\theta)v_1(w(\lambda r,\theta))]||_{L^2}$$
$$\leq C_\epsilon \lambda N^{-1+\epsilon}||g(\lambda r)||_{C^1}||w(\lambda r,\theta)||_{L^2} \leq C_\epsilon \lambda N^{-1+\epsilon}\lambda ||g(r)||_{C^1}\lambda^{-1}||w(r,\theta)||_{L^2}$$
$$= C_\epsilon \lambda N^{-1+\epsilon}||g(r)||_{C^1}||w(r,\theta)||_{L^2}.$$

□

Lemma 5.2.7. *Given a radial function $f(r) \in H^6$ we have that*

$$|\frac{\partial f}{\partial r}(r = r_0)| \leq C||f(r)||_{H^5} r_0^{-\frac{2}{5}}.$$

Furthermore, if $f(r) \in H^{6+2n}$, with n a natural number, we have that

$$|\frac{\partial f}{\partial r}(r = r_0)| \leq C_n||f(r)||_{H^{5+2n}} r_0^{-a_n}. \tag{5.9}$$

where $a_n = a_{n-1}\frac{2}{5} + \frac{2}{5}$, $a_0 = \frac{2}{5}$.

Proof. Fixed r_0, we define

$$r_{0,+} := \inf\{r : r \geq r_0, |f'(r)| \leq \frac{|f'(r_0)|}{2}\}$$

$$r_{0,-} := \sup\{r : r \leq r_0, |f'(r)| \leq \frac{|f'(r_0)|}{2}\}$$

so in particular

$$||f(r)||^2_{H^1} \geq \frac{r_0}{8}(r_0 - r_{0,-})f'(r_0)^2, \tag{5.10}$$

$$||f(r)||^2_{H^1} \geq \frac{r_0}{8}(r_{0,+} - r_0)f'(r_0)^2.$$

We assume that $f'(r_0), f''(r_0) \geq 0$, the other cases being analogous. Using that $|\frac{\partial^3 f(r)}{\partial r^3}| \leq ||f(r)||_{H^6}$, we have that, for $h > 0$,

$$f'(r_0 + h) \geq f'(r_0) + hf''(r_0) - ||\frac{\partial^3 f(r)}{\partial r^3}||_{L^\infty}\frac{h^2}{2} \geq f'(r_0) - ||f(r)||_{H^5}\frac{h^2}{2},$$

and thus

$$r_{0,+} - r_0 \geq \Big(\frac{f'(r_0)}{||f(r)||_{H^5}}\Big)^{\frac{1}{2}},$$

so

$$||f(r)||^2_{H^1} \geq \frac{r_0}{8}\Big(\frac{f(r_0)}{||f(r)||_{H^5}}\Big)^{\frac{1}{2}} f'(r_0)^2,$$

and in particular

$$\frac{8}{r_0}||f(r)||^{\frac{5}{2}}_{H^5} \geq f'(r_0)^{\frac{5}{2}}.$$

To obtain the bound for other spaces H^{5+2n}, we first note that we can assume that $r_0 - r_{0,-}, r_{0,+} - r_0 \leq \frac{r_0}{2}$, since for the points where $r_0 - r_{0,-} \geq \frac{r_0}{2}$ using (5.10) we have

$$|f'(r_0)| \leq \frac{C}{r_0}||f||_{H^1}.$$

With this in mind, we can finish the proof by induction using that if (5.9) is fulfilled for some n, since $|\frac{\partial^3 f(r)}{\partial r^3}| \leq C_n r^{-a_n}||f(r)||_{H^{7+2n}}$, then (again dealing with the case $f'(r_0), f''(r_0) \geq 0$, the other cases being analogous),

$$f'(r_0+h)\geq f'(r_0)+hf''(r_0)-||\frac{\partial^3 f(r)}{\partial r^3}||_{L^\infty(r\in(r_0,\frac{3r_0}{2}))}\frac{h^2}{2}\geq f'(r_0)-C_n\big(r_0\big)^{-a_n}||f(r)||_{H^{7+2n}}\frac{h^2}{2},$$

and thus

$$r_0-r_{0,+}\geq\Big(\frac{f'(r_0)}{C_n||f(r)||_{H^{5+2(n+1)}}r_0^{-a_n}}\Big)^{\frac{1}{2}},$$

so

$$||f(r)||^2_{H^1}\geq\frac{r_0}{8}\Big(\frac{f'(r_0)}{C_n||f(r)||_{H^{5+2(n+1)}}r_0^{-a_n}}\Big)^{\frac{1}{2}}f'(r_0)^2,$$

which implies that

$$C_{n+1}||f(r)||_{H^{5+2(n+1)}}r_0^{-\frac{2}{5}}r_0^{\frac{2}{5}(-a_n)}\geq f'(r_0).$$

□

Before we can define our pseudo-solution, we need one last technical lemma.

Lemma 5.2.8. *Given any a_i for $i=1,2$, there exists a C^∞ function $g(r)$ such that $\hat{g}(\hat{r})$ has support in $\hat{r}\in(c,\infty)$ for some $c>0$ and*

$$\frac{\partial}{\partial r}\frac{v_\alpha(g(r))}{r}(r=1)=a_1$$
$$\frac{\partial^2}{\partial r^2}\frac{v_\alpha(g(r))}{r}(r=1)=a_2,$$
$$\frac{\partial^3}{\partial r^3}\frac{v_\alpha(g(r))}{r}(r=1)=a_3,$$

where $\hat{g}$ is the Fourier transform of g and $\hat{r}$ is the radial variable in the frequency domain.

Proof. We start by choosing $h_1(r)$ smooth function with support in $r\in(\frac{1}{4},\frac{1}{2})$ fulfilling

$$\int_0^{\frac{1}{2}}sh_1(s)ds=1,$$

and then we define

$$h_2(r):=\frac{1}{r}\frac{\partial^2}{\partial r^2}rh_1(r),$$
$$h_3(r):=\frac{1}{r}\frac{\partial^4}{\partial r^4}rh_1(r).$$

Now, since for a generic radial function $h(r)$ we have

$$v_\alpha(h(\cdot))(r,\alpha)=P.V.\int_{\mathbb{R}_+\times[-\pi,\pi]}r'\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}(h(r')-h(r))d\alpha'dr',$$

then for $r\in(\frac{1}{2},\frac{3}{2})$ we have

$$\text{lim}_{\lambda\rightarrow\infty}v_\alpha(\lambda^2h_1(\lambda\cdot))=\text{lim}_{\lambda\rightarrow\infty}2\pi\int_{\mathbb{R}_+}\frac{r}{|r^2|^{3/2}}\lambda^2r'h_1(\lambda r')dr'=\frac{2\pi}{r^2},\tag{5.11}$$

and furthermore there is strong convergence in C^k for $r\in(\frac{1}{2},\frac{3}{2})$ for any fixed k. On the other hand, using integration by parts twice with respect to r', we have that, for $r\in(\frac{1}{2},\frac{3}{2})$, $\lambda>3$,

$$v_\alpha(\lambda^4h_2(\lambda\cdot))(r,\alpha)=P.V.\int_{\mathbb{R}_+\times[-\pi,\pi]}\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}\lambda^4r'h_2(\lambda r')d\alpha'dr'$$

$$= P.V. \int_{\mathbb{R}_+\times[-\pi,\pi]} \left(\frac{\partial^2}{\partial(r')^2}\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}\right)\lambda^2 r' h_1(\lambda r')d\alpha' dr'$$

and similarly, using integration by parts four times with respect to r',

$$v_\alpha(\lambda^6 h_3(\lambda\cdot))(r,\alpha) = P.V. \int_{\mathbb{R}_+\times[-\pi,\pi]} \frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}\lambda^6 r' h_3(\lambda r')d\alpha' dr'$$
$$= P.V. \int_{\mathbb{R}_+\times[-\pi,\pi]} \left(\frac{\partial^4}{\partial(r')^4}\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}\right)\lambda^2 r' h_1(\lambda r')d\alpha' dr'.$$

Direct computation then gives us that

$$\lim_{r'\to 0}\int_0^{2\pi} \frac{\partial^2}{\partial(r')^2}\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}d\alpha' = \frac{3\pi}{r^4},$$
$$\lim_{r'\to 0}\int_0^{2\pi} \frac{\partial^4}{\partial(r')^4}\frac{r-r'cos(\alpha')}{|r^2+(r')^2-2rr'cos(\alpha'))|^{3/2}}d\alpha' = \frac{135\pi}{4r^6},$$

and there is strong convergence in C^k for $r\in(\frac{1}{2},\frac{3}{2})$ for any fixed k, so

$$\lim_{\lambda\to\infty} v_\alpha(\lambda^4 h_2(\lambda\cdot))(r,\alpha)' = \int_{\mathbb{R}_+}\frac{3\pi}{r^4}\lambda^2 r' h_1(r')dr' = \frac{3\pi}{r^4}, \tag{5.12}$$

$$\lim_{\lambda\to\infty} v_\alpha(\lambda^6 h_3(\lambda\cdot))(r,\alpha) = \int_{\mathbb{R}_+}\frac{135\pi}{4r^6}\lambda^2 r' h_1(\lambda r')dr' = \frac{135\pi}{4r^6}, \tag{5.13}$$

with, again, strong convergence in C^k for $r\in(\frac{1}{2},\frac{3}{2})$ for any fixed k. Furthermore, if we now consider some C^∞ radial function $p(r)$ such that $p(r)=1$ if $r\geq 2$, $p(r)=0$ if $r\leq 1$, $1\geq p(r)\geq 0$, and define H_c as

$$\widehat{H_c(f(r))} = p(\frac{\hat{r}}{c})\hat{f}(\hat{r}),$$

we have that for any C^∞ function $H_c(f(r))$ tends strongly in C^k to $f(r)$ as c tends to 0, and furthermore $H_c(f(r))$ is radial and with Fourier transform supported in $\hat{r}\in(c,\infty)$. Using this plus (5.11),(5.12) and (5.13), we have that we can find smooth functions $g_1(r), g_2(r), g_3(r)$ with $\hat{g}_i$ supported in $\hat{r}\in(c,\infty)$ such that, for $r\in(\frac{1}{2},\frac{3}{2})$

$$\frac{\partial}{\partial r}\frac{v_\alpha(g_1)}{r}(r=1) = \frac{1}{r^4}+\epsilon, \frac{\partial^2}{\partial r^2}\frac{v_\alpha(g_1)}{r}(r=1) = -\frac{4}{r^5}+\epsilon, \frac{\partial^3}{\partial r^3}\frac{v_\alpha(g_1)}{r}(r=1) = \frac{20}{r^6}+\epsilon,$$
$$\frac{\partial}{\partial r}\frac{v_\alpha(g_2)}{r}(r=1) = \frac{1}{r^6}+\epsilon, \frac{\partial^2}{\partial r^2}\frac{v_\alpha(g_2)}{r}(r=1) = -\frac{6}{r^7}+\epsilon, \frac{\partial^3}{\partial r^3}\frac{v_\alpha(g_2)}{r}(r=1) = \frac{42}{r^8}+\epsilon,$$
$$\frac{\partial}{\partial r}\frac{v_\alpha(g_3)}{r}(r=1) = \frac{1}{r^8}+\epsilon, \frac{\partial^2}{\partial r^2}\frac{v_\alpha(g_3)}{r}(r=1) = -\frac{8}{r^9}+\epsilon, \frac{\partial^3}{\partial r^3}\frac{v_\alpha(g_3)}{r}(r=1) = \frac{72}{r^{10}}+\epsilon,$$

and since the vectors $(1,4,20)$, $(1,6,42)$ and $(1,8,72)$ are independent, evaluating at $r=1$ and taking ϵ small finishes the proof.

□

5.3 The pseudo-solution

We are now ready to define our pseudo-solution and obtain the necessary properties about it. First, taking

$$N^\alpha \ln(N) = \lambda^{2-\beta-\alpha} \tag{5.14}$$

for $t\in[0,\lambda^{-2+\beta}\ln(N)^3] = [0,(N\lambda)^{-\alpha}\ln(N)^2]$, we define

$$\bar{w}_{N,\beta}(r,\theta,t) := \bar{g}(r,t) + \bar{w}_{pert}(r,\theta,t), \tag{5.15}$$

where

$$\bar{g}(r,t) := \lambda \frac{g(r\lambda, t\lambda^{\alpha})}{\lambda^{\beta}},$$
$$\frac{\partial g(r,t)}{\partial t} = -\Lambda^{\alpha} g(r,t),$$
$$g(r,0) = g(r),$$
$$\bar{w}_{pert}(r,\theta,t) := f(\lambda r)\lambda \frac{\cos(N(\theta+\Theta(r,t)) - \int_0^t \frac{\partial \bar{g}(r,s)}{\partial r}\frac{C_0}{|1+r^2(\partial_r\Theta(r,s))^2|^{1/2}}ds)}{N^{\beta}\lambda^{\beta}} e^{-G(r,t)}$$

with
$$\Theta(r,t) := K\frac{v_{\alpha}(g(\lambda r,0))}{\lambda r} + \int_0^t \frac{v_{\alpha}(\bar{g}(r,s))}{r} ds,$$
$$G(r,t) := K_{\alpha}^{-1} N^{\alpha} \int_0^t \Big(\frac{1}{r^2} + (\frac{\partial}{\partial r}\Theta(r,s))^2\Big)^{\frac{\alpha}{2}} ds,$$

where we need to choose K, $g(r)$ and $f(r)$. First, we choose a smooth radial function $g(r)$ with $\text{supp}(\hat{g}) \subset \{\hat{r} \in (c,\infty)\}$ for some small $c > 0$ such that

$$\frac{\partial}{\partial r}\frac{v_{\alpha}(g(r))}{r}(r=1) = 1,$$
$$\frac{\partial^2}{\partial r^2}\frac{v_{\alpha}(g(r))}{r}(r=1) = 0,$$
$$\frac{\partial^3}{\partial r^3}\frac{v_{\alpha}(g(r))}{r}(r=1) = 1,$$

which exists thanks to Lemma 5.2.8. Note that this means that there is a small $\frac{1}{2} > \tilde{\epsilon} > 0$ such that $r_0 \in [1-\tilde{\epsilon}, 1+\tilde{\epsilon}]$, implies that

$$(\frac{\partial}{\partial r}\frac{v_{\alpha}(g(r))}{r})(r=r_0) - (\frac{\partial}{\partial r}\frac{v_{\alpha}(g(r))}{r})(r=1) \geq \frac{1}{10}(1-r_0)^2. \tag{5.16}$$

We would now like to choose K so that, if N is big enough, $G(r,t)$ is such that, for $t \in [0,(\lambda N)^{-\alpha}\ln(N)^2], r \in [\frac{1-\tilde{\epsilon}}{\lambda}, \frac{2-\tilde{\epsilon}}{2\lambda}] \cup [\frac{2+\tilde{\epsilon}}{2\lambda}, \frac{1+\tilde{\epsilon}}{\lambda}]$

$$G(\frac{1}{\lambda},t) \leq G(r,t). \tag{5.17}$$

For this, we note that it is enough to show that, for $t \in [0,(\lambda N)^{-\alpha}\ln(N)^2]$, $r \in [\frac{1-\tilde{\epsilon}}{\lambda}, \frac{2-\tilde{\epsilon}}{2\lambda}] \cup [\frac{2+\tilde{\epsilon}}{2\lambda}, \frac{1+\tilde{\epsilon}}{\lambda}]$,

$$\lambda^2 + (\frac{\partial}{\partial r}\Theta(r,t)(r=\frac{1}{\lambda}))^2 \leq \frac{1}{r^2} + (\frac{\partial}{\partial r}\Theta(r,t))^2.$$

But, using the definition of $\bar{g}$

$$\int_0^t \frac{\partial}{\partial r}\frac{v_{\alpha}(\bar{g}(\cdot,\tau))(r)}{r}d\tau = t\lambda^{3-\beta}\frac{\partial}{\partial(r\lambda)}\frac{v_{\alpha}(g(r\lambda,0))}{\lambda r} + \int_0^t \frac{\partial}{\partial r}\frac{v_{\alpha}(\bar{g}(\cdot,\tau)-\bar{g}(\cdot,0))(r)}{r}d\tau \tag{5.18}$$

and using $t \in [0,(\lambda N)^{-\alpha}\ln(N)^2], r \in [\frac{1-\tilde{\epsilon}}{\lambda}, \frac{1+\tilde{\epsilon}}{\lambda}]$ we have,

$$|\int_0^t \frac{\partial}{\partial r}\frac{v_{\alpha}(\bar{g}(\cdot,\tau)-\bar{g}(\cdot,0))(r)}{r}d\tau| = \lambda^{3-\beta}|\int_0^t \frac{\partial}{\partial(\lambda r)}\frac{v_{\alpha}(g(\lambda\cdot,\lambda^{\alpha}\tau)-g(\lambda\cdot,0))(r)}{\lambda r}d\tau|$$
$$\leq C\lambda^{3-\beta}\lambda^{\alpha}t^2 \leq Ct\lambda^{3-\beta}N^{-\alpha}\ln(N)^2,$$

so, for big N, $r \in [\frac{1-\tilde{\epsilon}}{\lambda}, \frac{2-\tilde{\epsilon}}{2\lambda}] \cup [\frac{2+\tilde{\epsilon}}{2\lambda}, \frac{1+\tilde{\epsilon}}{\lambda}]$, using the definition of $\Theta(r,t)$, (5.18) and (5.16) we have

$$\frac{1}{r^2} + (\frac{\partial}{\partial r}\Theta(r,t))^2 \geq \lambda^2(1-\tilde{\epsilon})^{-2} + ((K\lambda + t\lambda^{3-\beta})(1+\frac{\tilde{\epsilon}^2}{40}) - Ct\lambda^{3-\beta}N^{-\alpha}\ln(N)^2)^2$$

and also

$$\lambda^2+(\frac{\partial}{\partial r}\Theta(r,t))^2(r=\frac{1}{\lambda})\leq\lambda^2+(K\lambda+t\lambda^{3-\beta}+Ct\lambda^{(3-\beta)}N^{-\alpha}\ln(N)^2)^2$$

so by taking K,N big (and therefore λ big), gives us that (5.17) is fulfilled.
We now take $f(r)$ a smooth function with support in $r\in[1-\tilde{\epsilon},1+\tilde{\epsilon}]$ the small interval fulfilling (5.17), and such that $f(r)=1$ if $r\in[1-\frac{\tilde{\epsilon}}{2},1+\frac{\tilde{\epsilon}}{2}]$, $f(r)\leq 1$. Note that, since the maximum of $G(r,t)1_{r\in[\frac{1}{\lambda}(1-\tilde{\epsilon}),\frac{1}{\lambda}(1+\tilde{\epsilon})]}$ is in the interval $r\in[\frac{1}{\lambda}(1-\frac{\tilde{\epsilon}}{2}),\frac{1}{\lambda}(1+\frac{\tilde{\epsilon}}{2})]$, this ensures that

$$||\bar{w}_{pert}||_{L^\infty}=\frac{||f(\lambda r)||_{L^\infty}}{N^\beta\lambda^{\beta-1}}||1_{\mathrm{supp} f(\lambda r)}e^{-G(r,t)}||_{L^\infty}. \tag{5.19}$$

This pseudo-solution fulfills the evolution equation

$$\frac{\partial\bar{w}_{N,\beta}}{\partial t}+v(\bar{g}(r,t))\cdot\nabla(\bar{w}_{N,\beta})+\bar{v}_r(\bar{w}_{pert})\frac{\partial}{\partial r}\bar{g}(r,t)+\Lambda^\alpha(\bar{g}(r,t))+\bar{\Lambda}^\alpha(\bar{w}_{pert})=0$$

with

$$\bar{\Lambda}^\alpha h_1(r)\cos(N\theta+Nh_2(r)):=K_\alpha^{-1}h_1(r)\cos(N\theta+h_2(r))|(\frac{N}{r})^2+N^2h_2'(r)^2|^{\alpha/2}$$

and

$$\bar{v}_r(h_1(r)\cos(N\theta+h_2(r)):=-C_0\frac{h_1(r)\sin(N\theta+Nh_2(r))}{|1+r^2h_2'(r)^2|^{1/2}}$$

so that

$$\frac{\partial\bar{w}_{N,\beta}}{\partial t}+v(\bar{w}_{N,\beta})\cdot\nabla(\bar{w}_{N,\beta})+\Lambda^\alpha(\bar{w}_{N,\beta})+F_{N,\beta}(x,t)=0 \tag{5.20}$$

with

$$F_{N,\beta}(x,t):=F_1(x,t)+F_2(x,t)+F_3(x,t) \tag{5.21}$$

$$F_1(x,t)=(\bar{\Lambda}^\alpha-\Lambda^\alpha)(\bar{w}_{pert}),$$
$$F_2(x,t)=-v(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})$$
$$F_3(x,t)=(\bar{v}(\bar{w}_{pert})-v(\bar{w}_{pert}))\cdot\nabla\bar{g}(r,t).$$

Next we need to show that $F_{N,\beta}$ is small in suitable Sobolev spaces.

5.3.1 Bounds on the error term $F_{N,\beta}$

Lemma 5.3.1. *For any given $\epsilon>0$, there is N_0 such that if $N\geq N_0$, then for $F_{N,\beta}$ given by* (5.20) *and $s\in[0,2]$, we have that*

$$||F_{N,\beta}||_{H^s}\leq C_\epsilon\frac{N^\epsilon(\lambda N)^{s+\alpha}}{N^{\beta+1}\lambda^\beta}$$

with N,λ as in (5.14).

Proof. To prove this, we will just show that

$$||F_i||_{H^s}\leq C_\epsilon\frac{N^\epsilon(\lambda N)^{s+\alpha}}{N^{\beta+1}\lambda^\beta}.$$

for $i=1,2,3$, with F_i defined as in (5.21). We first focus on $F_1(x,t)$. To bound the H^s norms of this function, we will use Corollary 5.2.2, so for that we need to check if $\bar{w}_{pert}$ fulfils the hypothesis of the lemma. For this we note that

$$\bar{w}_{pert}(\frac{r}{\lambda},\theta,t)=f(r)\lambda\frac{\cos(N(\theta+\Theta(\frac{r}{\lambda},t))-\int_0^t\lambda^{2-\beta}\frac{\partial g(r,\lambda^\alpha s)}{\partial r}\frac{C_0}{|1+(r\partial_r\Theta(\frac{r}{\lambda},s))^2|^{1/2}}ds)}{N^\beta\lambda^\beta}e^{-G(\frac{r}{\lambda},t)}$$

$$G(\frac{r}{\lambda},t)=K_\alpha^{-1}(N\lambda)^\alpha\int_0^t\Big(\frac{1}{r^2}+(K\frac{\partial}{\partial r}\frac{v_\alpha(g(r,0))}{r}+\int_0^s\lambda^{2-\beta}\frac{\partial}{\partial r}\frac{v_\alpha(g(\cdot,\lambda^\alpha\tau))(r)}{r}d\tau)^2\Big)^{\frac{\alpha}{2}}ds,$$

$$\Theta(\frac{r}{\lambda},t)=K\frac{v_\alpha(g(r,0))}{r}+\lambda^{2-\beta}\int_0^t\frac{v_\alpha(g(r,\lambda^\alpha s))}{r}ds.$$

Since $f(r)e^{-G(\frac{r}{\lambda},t)}$ has support in $r\in(\frac{1}{2},\frac{3}{2})$, if we show that

$$||f(r)e^{-G(\frac{r}{\lambda},t)}||_{C^5}\leq||f(r)e^{-G(\frac{r}{\lambda},t)}||_{L^\infty}\ln{(N)}^P, \tag{5.22}$$

$$||K\frac{v_\alpha(g(r,0))}{r}+\int_0^t\frac{\lambda^{2-\beta}v_\alpha(g(r,\lambda^\alpha s))}{r}ds||\leq\ln{(N)}^P$$

$$||\int_0^t\lambda^{2-\beta}\frac{\partial g(r,\lambda^\alpha s)}{\partial r}\frac{C_0}{|1+(r\partial_r\Theta(\frac{r}{\lambda},s))^2|^{1/2}}ds)||_{C^5}\leq\ln(N)^P$$

we can apply Corollary 5.2.2.
For (5.22) we note that, for a function of the form $\tilde{h}(x)=h_1(x)e^{h_2(x)}$, we have the bound

$$||\tilde{h}(x)||_{C^i}\leq C_i||1_{\text{supp}(h_1(x))}e^{h_2(x)}||_{L^\infty}||h_1||_{C^i}(1+||h_2(x)||_{C^i})^i$$

but, using also that $f(r)$ is a fixed C^∞ function and (5.19)

$$\begin{aligned}&\frac{||f(r)e^{-G(\frac{r}{\lambda},t)}||_{C^5}}{||f(r)e^{-G(\frac{r}{\lambda},t)}||_{L^\infty}}\\&\leq C\frac{||1_{\text{supp}f(r)}e^{-G(\frac{r}{\lambda})(x)}||_{L^\infty}||f(r)||_{C^5}(1+||G(\frac{r}{\lambda},t)||_{C^5(1_{\text{supp}f(r)})})^5}{||f(r)||_{L^\infty}||1_{\text{supp}f(r)}e^{-G(\frac{r}{\lambda},t)}||_{L^\infty}}\\&\leq C(1+||G(\frac{r}{\lambda},t)||_{C^5(1_{\text{supp}f(r)})})^5\end{aligned}$$

so it is enough to obtain bounds for $||G(\frac{r}{\lambda},t)||_{C^5(1_{\text{supp}f(r)})}$. But, for $t\in[0,(\lambda N)^{-\alpha}\ln(N)^2]$

$$\begin{aligned}&||G(\frac{r}{\lambda},t)||_{C^5(1_{\text{supp}f(r)})}\\&=||K_\alpha^{-1}(N\lambda)^\alpha\int_0^t\Big(\frac{1}{r^2}+(K\frac{v_\alpha(g(r,0))}{r}+\int_0^s\lambda^{2-\beta}\frac{\partial}{\partial r}v_\alpha(g(\cdot,\lambda^\alpha\tau))(r)d\tau)^2\Big)^{\frac{\alpha}{2}}||_{C^5(1_{\text{supp}f(r)})}\\&\leq C\ln(N)^2\times\\&\text{sup}_{s\in[0,(N\lambda)^{-\alpha}\ln(N)^2]}||\Big(\frac{1}{r^2}+(K\frac{v_\alpha(g(r,0))}{r}+\int_0^s\lambda^{2-\beta}\frac{\partial}{\partial r}v_\alpha(g(\cdot,\lambda^\alpha\tau))(r)d\tau)^2\Big)^{\frac{\alpha}{2}}||_{C^5(1_{\text{supp}f(r)}))}\\&\leq C\ln(N)^2\times\\&\text{sup}_{s\in[0,(N\lambda)^{-\alpha}\ln(N)^2]}(1+||\frac{1}{r^2}+(K\frac{v_\alpha(g(r,0))}{r}+\int_0^s\lambda^{2-\beta}\frac{\partial}{\partial r}v_\alpha(g(\cdot,\lambda^\alpha\tau))(r)d\tau)^2||_{C^5(1_{\text{supp}f(r)}))})^5\\&\leq C\ln{(N)}^{32},\end{aligned}$$

where we used that $\frac{1}{r^2}+(K\frac{v_\alpha(g(r,0))}{r}+\int_0^s\lambda^{2-\beta}\frac{\partial}{\partial r}v_\alpha(g(\cdot,\lambda^\alpha\tau))(r)d\tau)^2>\frac{1}{4}$ for $r\in\text{supp}(f(r))$ in the fifth line.
On the other hand, for $t\in[0,(\lambda N)^{-\alpha}\ln(N)^2]$ we have

$$||K\frac{v_\alpha(g(r,0))}{r}+\int_0^t\frac{\lambda^{2-\beta}v_\alpha(g(r,\lambda^\alpha s))}{r}ds||_{C^5}\leq C\ln{(N)}^3,$$

$$\begin{aligned}&||\int_0^t\lambda^{2-\beta}\frac{\partial g(r,\lambda^\alpha s)}{\partial r}\frac{C_0}{|1+r^2(\partial_r\Theta(\frac{r}{\lambda},s))^2|^{1/2}}ds||_{C^5}\\&\leq C\ln(N)^3\text{sup}_{s\in[0,(\lambda N)^{-\alpha}\ln(N)^2]}||\frac{C_0}{|1+r^2(\partial_r\Theta(\frac{r}{\lambda},s))^2|^{1/2}}||_{C^5}\leq C\ln(N)^{33},\end{aligned}$$

so we can apply Corollary 5.2.2 and obtain that

$$||(\Lambda^{\alpha}-\bar{\Lambda}^{\alpha})\bar{w}_{pert}(r,\theta)||_{H^s}\leq C\lambda^{-1+s+\alpha}N^{-1+\alpha+\epsilon+s}||\frac{f(\lambda r)}{N^{\beta}\lambda^{\beta-1}}e^{-G(r,t)}||_{L^{\infty}}\leq C\frac{\lambda^{s+\alpha}N^{\epsilon+\alpha+s}}{\lambda^{\beta}N^{\beta+1}}.$$

Note also that, all the bounds we have obtained also give us, for any $\epsilon>0$, $s\in[0,5]$ and N big

$$||\bar{w}_{pert}(r,\theta)||_{H^s}\leq C\frac{\lambda^s N^{\epsilon+s}}{\lambda^{\beta}N^{\beta}},||\bar{w}_{pert}(r,\theta)||_{C^s}\leq C\frac{\lambda^s N^{\epsilon+s}}{\lambda^{\beta-1}N^{\beta}}. \tag{5.23}$$

Next, for $F_2(x,t)$, we note that

$$||F_2(x,t)||_{H^s}\leq||(\bar{v}-v)(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})||_{H^s}+||\bar{v}(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})||_{H^s}.$$

But, since we already checked the hypothesis for Lemma 5.2.3, we have, for $s=0,2$ and any $\epsilon>0$

$$||(\bar{v}-v)(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})||_{H^s}\leq C\sum_{i=0}^{s}||(\bar{v}-v)(\bar{w}_{pert})||_{H^i}||\nabla(\bar{w}_{pert})||_{C^{s-i}}\leq C\frac{N^{2\epsilon}(\lambda N)^s}{N^{\beta+1}\lambda^{\beta}}\lambda^{2-\beta}$$

$$=C\frac{N^{2\epsilon}(\lambda N)^{s+\alpha}}{N^{\beta+1}\lambda^{\beta}}\ln(N)\leq C\frac{N^{3\epsilon}(\lambda N)^{s+\alpha}}{N^{\beta+1}\lambda^{\beta}}$$

and interpolation gives the bound for $s\in(0,2)$. On the other hand we have

$$\bar{v}(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})=K_1\left(\frac{\partial}{\partial x_1}\frac{1}{|(\frac{N}{r})^2+(N\partial_r\Theta(r,t))^2|^{1/2}}\right)\bar{w}_{pert}(r,\theta,t)\frac{\partial}{\partial x_2}\bar{w}_{pert}(r,\theta,t)$$

$$-K_1\left(\frac{\partial}{\partial x_2}\frac{1}{|(\frac{N}{r})^2+(N\partial_r\Theta(r,t))^2|^{1/2}}\right)\bar{w}_{pert}(r,\theta,t)\frac{\partial}{\partial x_1}\bar{w}_{pert}(r,\theta,t)$$

and therefore, for $s\in[0,2]$,

$$\begin{aligned}
&||\bar{v}(\bar{w}_{pert})\cdot\nabla(\bar{w}_{pert})||_{H^s}\\
&\leq C\sum_{i=0}^{s}||\nabla\left(\frac{1}{|(\frac{N}{r})^2+(N\partial_r\Theta(r,t))^2|^{1/2}}\right)\bar{w}_{pert}(r,\theta)||_{C^i}||\bar{w}_{pert}(r,\theta)||_{H^{s+1-i}}\\
&\leq C\sum_{i=0}^{s}\lambda^{i+1-\beta}N^{i-\beta-1+2\epsilon}(\lambda N)^{s+1-i-\beta+\epsilon}=C\lambda^{1-\beta}N^{-\beta-1+2\epsilon}(\lambda N)^{s+1-\beta+\epsilon}\\
&=C\frac{(\lambda N)^s N^{3\epsilon}}{\lambda^{\beta}N^{\beta+1}}\lambda^{2-\beta}N^{1-\beta}\leq C\frac{(\lambda N)^{s+\alpha}N^{3\epsilon}}{\lambda^{\beta}N^{\beta+1}}.
\end{aligned}$$

Finally, for $F_3(x,t)$, we just have, for any $s\in[0,2]$, $\epsilon>0$, for N big enough

$$||(\bar{v}-v)(\bar{w}_{pert})\cdot\nabla\bar{g}(r,t)||_{H^s}\leq C\sum_{i=0}^{s}||(\bar{v}-v)(\bar{w}_{pert})||_{H^i}||\bar{g}||_{C^{s+1-i}}$$

$$\leq C\frac{N^{\epsilon}(N\lambda)^i}{\lambda^{\beta}N^{\beta+1}}\lambda^{s+2-i-\beta}\leq C\frac{N^{\epsilon}(N\lambda)^s}{\lambda^{\beta}N^{\beta+1}}\lambda^{2-\beta}\leq C\frac{N^{2\epsilon}(N\lambda)^{s+\alpha}}{\lambda^{\beta}N^{\beta+1}}$$

and this finishes the proof.

□

5.3.2 Using the pseudo-solution to control the solution

For the pseudo-solution $\bar{w}_{N,\beta}$ to be a useful tool, we need to show that, if we define $w_{N,\beta}$, the solution to (5.1) with the same initial conditions as $\bar{w}_{N,\beta}$, then $w_{N,\beta}\approx\bar{w}_{N,\beta}$. This sub-section will be devoted to show this.

Lemma 5.3.2. *There is $N_0, \epsilon_0, \delta > 0$ such that if $N \geq N_0$, $0 < \epsilon \leq \epsilon_0$ then for any $t \in [0, (N\lambda)^{-\alpha}\ln(N)^2]$ we have that*

$$||w_{N,\beta}(x,t) - \bar{w}_{N,\beta}(x,t)||_{L^2} \leq C_\epsilon \frac{N^\epsilon}{N^{\beta+1}\lambda^\beta}, \tag{5.24}$$

$$||w_{N,\beta}(x,t) - \bar{w}_{N,\beta}(x,t)||_{H^{2-\alpha+\delta}} \leq 1, \tag{5.25}$$

with $\bar{w}_{N,\beta}$ as in (5.15) *and $w_{N,\beta}$ a solution to* (5.1) *with the same initial conditions as $\bar{w}_{N,\beta}$.*

Proof. We first note that, since (5.1) is locally well-posed in $H^{2-\alpha+\delta}$, using continuity of $||w_{N,\beta}(x,t) - \bar{w}_{N,\beta}(x,t)||_{H^{2-\alpha+\delta}}$ we know that (5.25) will hold for at least some short time period $[0, T_{crit}]$. We start by showing that, for $t \in [0, T_{crit}] \cap [0, (N\lambda)^{-\alpha}\ln(N)^2]$, (5.24) holds.
For this, we define $W := w_{N,\beta}(x,t) - \bar{w}_{N,\beta}(x,t)$ and note that the evolution equation for W is

$$\frac{\partial W}{\partial t} + v(W)\cdot\nabla(W + \bar{w}_{N,\beta}(x,t)) + v(\bar{w}_{N,\beta}(x,t))\cdot\nabla W + \Lambda^\alpha W - F(x,t) = 0$$

so that, after using incompressibility,

$$\begin{aligned}\frac{\partial}{\partial t}||W||^2_{L^2} &= -\Big(2\int_{\mathbb{R}^2} W(v(W)\cdot\nabla\bar{w}_{N,\beta} - F(x,t))dx + ||W||^2_{\dot{H}^{\frac{\alpha}{2}}}\Big) \\ &\leq -\Big(2\int_{\mathbb{R}^2} W(v(W)\cdot\nabla(\bar{g}(r,t) - \bar{g}(0,t))) - F(x,t))dx\Big) + C\lambda^{2-\beta}N^{\beta-1}\ln(N)||W||^2_{L^2}.\end{aligned} \tag{5.26}$$

where we used that $||\bar{w}_{pert}(r,\theta,t)||_{C^1} \leq C\lambda^{2-\beta}N^{\beta-1}\ln(N)$. Now, we consider $\tilde{f}(r)$ a smooth function fulfilling $0 \leq \tilde{f}(r) \leq 1$, $\tilde{f}(r) = 1$ if $r \geq 2$, $\tilde{f}(r) = 0$ if $r \leq 1$ and we define

$$f_1(r) = (1 - \tilde{f}(\lambda^{3-\beta}r)), f_2(r) = 1 - \tilde{f}(r) - f_1(r),$$
$$f_3(r) = \tilde{f}(r) - f_4(r), f_4(r) = \tilde{f}(\lambda^{-\frac{1}{2}}r)$$

so in particular $f_1 + f_2 + f_3 + f_4 = 1$.
Now, by using the smoothness of $\bar{g}(r,t)$ and $\tilde{f}(r)$ (which in particular implies $\partial_r\tilde{f}(0) = \partial_r\bar{g}(0,t) = 0$) and the scaling properties of the C^k norms, we get that

$$||f_1(r)(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} \leq C\lambda^{-3+\beta}||\bar{g}(r,t)||_{C^2} + ||f_1(r)||_{C^1}C(\lambda^{-3+\beta})^2||\bar{g}(r,t)||_{C^2} \leq C,$$

$$||f_2(r)(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} \leq ||f_1(r)(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} + ||(1-\tilde{f}(r))(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} \leq C\lambda^{2-\beta}$$

and applying Lemma 5.2.7 to $g(r,t)$ we get

$$||f_3(r)\bar{g}(r,t)||_{C^1} \leq C_\epsilon\lambda^{-\frac{2}{3}+\epsilon}\lambda^{2-\beta}, ||f_4(r)\bar{g}(r,t)||_{C^1} \leq C_\epsilon\lambda^{\frac{3}{2}(-\frac{2}{3}+\epsilon)}\lambda^{2-\beta} = C_\epsilon\lambda^{\frac{3}{2}\epsilon}\lambda^{1-\beta} \leq C,$$

and thus

$$||f_3(r)(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} \leq C_\epsilon\lambda^{-\frac{2}{3}+\epsilon}\lambda^{2-\beta}, ||f_4(r)(\bar{g}(r,t) - \bar{g}(0,t))||_{C^1} \leq C.$$

Therefore

$$|\int_{\mathbb{R}^2} W(v(W)\cdot\nabla(f_1(r) + f_4(r)(\bar{g}(r,t) - \bar{g}(0,t)))dx| \leq C||W||^2_{L^2}$$

and by using Corollary 5.2.6, and the parity of the operator v_i for $i = 1, 2$,

$$\int_{\mathbb{R}^2} Wv_i(W)\frac{\partial}{\partial x_i}(f_2(r)(\bar{g}(r,t) - \bar{g}(0,t)))dx =$$
$$\frac{1}{2}\int_{\mathbb{R}^2} W[v_i(W)\frac{\partial}{\partial x_i}(f_2(r)(\bar{g}(r,t) - \bar{g}(0,t))) - v_i(W\frac{\partial}{\partial x_i}(f_2(r)(\bar{g}(r,t) - \bar{g}(0,t))))]dx$$

$$\leq CN^{-1}\lambda^{2-\beta}||W||_{L^2}^2 \leq C\lambda^{\alpha}N^{-1+\alpha}||W||_{L^2}^2,$$

$$\begin{aligned}&\int_{\mathbb{R}^2} Wv_i(W)\frac{\partial}{\partial x_i}(f_3(r)(\bar{g}(r,t)-\bar{g}(0,t)))dx\\ &= \frac{1}{2}\int_{\mathbb{R}^2} W[v_i(W)\frac{\partial}{\partial x_i}(f_3(r)(\bar{g}(r,t)-\bar{g}(0,t))) - v_i(W\frac{\partial}{\partial x_i}(f_3(r)(\bar{g}(r,t)-\bar{g}(0,t))))]dx\\ &\leq C\lambda^{\frac{1}{2}}N^{-1}\lambda^{-\frac{2}{3}+\epsilon}\lambda^{2-\beta}||W||_{L^2}^2 \leq C\lambda^{\alpha}N^{-1+\alpha}||W||_{L^2}^2.\end{aligned}$$

Combining all this we get that

$$\begin{aligned}\frac{\partial}{\partial t}||W||_{L^2}^2 &= -2\Big(\int_{\mathbb{R}^2} W(v(W)\cdot\nabla\bar{w}_{N,\beta} - F(x,t))dx + ||W||_{\dot{H}^{\frac{\alpha}{2}}}^2\Big)\\ &\leq C\lambda^{\alpha}N^{-1+\alpha}||W||_{L^2}^2 + C||W||_{L^2}||F(x,t)||_{L^2}.\end{aligned}$$

and using the bounds for $||F(x,t)||_{L^2}$ and integrating with time the evolution equation for $||W||_{L^2}$, we get, for any $\epsilon > 0$,

$$||W||_{L^2} \leq C_\epsilon \frac{N^\epsilon}{N^{\beta+1}\lambda^\beta}.$$

On the other hand, by integrating in time (5.26) we get, for any $t_0 \in [0,(N\lambda)^{-\alpha}\ln(N)]\cap[0,T_{crit}]$, any $\epsilon > 0$

$$\int_0^{t_0} ||W||_{H^{\frac{\alpha}{2}}}^2 \leq \sup_{t\in[0,t_0]}(N\lambda)^{-\alpha}\ln(N)(||F||_{L^2}||W||_{L^2} + ||W||_{L^2}^2) \leq C\frac{N^\epsilon}{(N^{\beta+1}\lambda^\beta)^2}. \tag{5.27}$$

To bound the growth of the higher order norm, we note that

$$\frac{\partial}{\partial t}||\Lambda^s W||_{L^2}^2 = -2\int_{\mathbb{R}^2}\Lambda^s(W)\Lambda^s[v(W)\cdot\nabla(W+\bar{w}_{N,\beta}(x,t)) + v(\bar{w}_{N,\beta}(x,t))\cdot\nabla W + \Lambda^\alpha W - F(x,t)]dx. \tag{5.28}$$

In order to bound the growth of the H^s norm with $s = 2-\alpha+\delta$ (we will now just write s instead of $2-\alpha+\delta$ for compactness of notation), we need to bound each of these terms under the assumption that (5.27) and (5.24) hold. First, we note that, as seen in [67],

$$||\Lambda^s(W)\Lambda^s[v(W)\cdot\nabla(W)]||_{L^1} \leq C||\Lambda^{2-\frac{\alpha}{2}}W||_{L^2}||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}||\Lambda^s W||_{L^2}$$

so, using our hypothesis for $||W||_{H^s}$, (5.24), the interpolation inequality for Sobolev spaces and some basic computations we get

$$||\Lambda^s(W)[v(W)\cdot\nabla(W)]||_{L^1} - \frac{1}{100}||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}^2 \leq \frac{1}{100},$$

and using our hypothesis for F and taking ϵ and δ small enough we get

$$||\Lambda^s(W)\Lambda^s(F)||_{L^1} \leq ||\Lambda^s(W)||_{L^2}\frac{N^\epsilon(\lambda N)^{2+\delta}}{N^{\beta+1}\lambda^\beta} \leq C||\Lambda^s(W)||_{L^2}(\lambda N)^\alpha N^{-\delta}.$$

For the rest of the terms, we need to use again Lemma 2.2.10, that is to say:

Lemma 5.3.3. *Let $s > 0$. Then for any $s_1, s_2 \geq 0$ with $s_1 + s_2 = s$, and any f, $g \in \mathcal{S}(\mathbb{R}^2)$, the following holds:*

$$||\Lambda^s(fg) - \sum_{|\mathbf{k}|<s_1}\frac{1}{\mathbf{k}!}\partial^{\mathbf{k}}f\Lambda^{s,\mathbf{k}}g - \sum_{|\mathbf{j}|\leq s_2}\frac{1}{\mathbf{j}!}\partial^{\mathbf{j}}g\Lambda^{s,\mathbf{j}}f||_{L^2} \leq C||\Lambda^{s_1}f||_{L^2}||\Lambda^{s_2}g||_{BMO} \tag{5.29}$$

where $\mathbf{j}$ *and* $\mathbf{k}$ *are multi-indexes,* $\partial^{\mathbf{j}} = \frac{\partial}{\partial x_1^{j_1} \partial x_2^{j_2}}$, $\partial_\xi^{\mathbf{j}} = \frac{\partial}{\partial \xi_1^{j_1} \partial \xi_2^{j_2}}$ *and* $\Lambda^{s,\mathbf{j}}$ *is defined using*

$$\widehat{\Lambda^{s,\mathbf{j}} f}(\xi) = \widehat{\Lambda^{s,\mathbf{j}}}(\xi)\hat{f}(\xi)$$
$$\widehat{\Lambda^{s,\mathbf{j}}}(\xi) = i^{-|\mathbf{j}|}\partial_\xi^{\mathbf{j}}(|\xi|^s).$$

Note that, even though this lemma is only valid for functions in $\mathcal{S}$ (the Schwartz space), an approximation argument allows to us use this lemma for the functions we will be considering. We will also use that, for any $\epsilon, s_1, s_2 \geq 0$, $s_1 + s_2 \leq 2$

$$||\bar{w}_{N,\beta}||_{C_{s_1}} \leq C_\epsilon \frac{(N\lambda)^{s_1+1-\beta}}{N} N^\epsilon + C\lambda^{s_1+1-\beta}$$
$$||\Lambda^{s_1}\bar{w}_{N,\beta}||_{C^{s_2}} \leq C_\epsilon \frac{(N\lambda)^{s_1+s_2+1-\beta}}{N} N^\epsilon + C\lambda^{s_1+s_2+1-\beta}$$

and if also $2 > s_1 > 1$, $\mathbf{k}$ with $|\mathbf{k}| = 1$

$$||\Lambda^{s_1,\mathbf{k}}\bar{w}_{N,\beta}||_{L^\infty} \leq N^\epsilon \frac{(\lambda N)^{s_1-\beta}}{N} + C\lambda^{s_1-\beta}$$

where we get these bounds from (5.23), the properties of $g(r,t)$, the scaling properties of C^s and

$$||\Lambda^{s_1} f||_{C^k} \leq C_{k,\epsilon}||f||_{C^{k+s_1+\epsilon}}, ||\Lambda^{s_2,\mathbf{k}} f||_{C^k} \leq C_{k,\epsilon}||f||_{C^{k+s_2-1+\epsilon}}$$

for $|\mathbf{k}| = 1$, $k > 0$, $s_2 > 1$, $s_1 \geq 0$. Now, applying Lemma 2.2.10 with $s_1 = 1+\delta$, $W = f$ we get

$$\begin{aligned}
&||\Lambda^s(W)\Lambda^s[v(W)\cdot\nabla\bar{w}_{N,\beta}(x,t)]||_{L^1}\\
&\leq ||\Lambda^s(W)\Lambda^s[v(W)]\cdot\nabla\bar{w}_{N,\beta}(x,t)]||_{L^1} + ||\Lambda^s(W)v(W)\cdot\Lambda^s[\nabla\bar{w}_{N,\beta}(x,t)]||_{L^1}\\
&+ ||\Lambda^s W||_{L^2}(\sum_{|\mathbf{k}|=1}||\partial^{\mathbf{k}}W||_{L^2}||\Lambda^{s,\mathbf{k}}\nabla\bar{w}_{n,\beta}||_{L^\infty} + ||\Lambda^{1+\delta}W||_{L^2}||\Lambda^{1-\alpha}\nabla\bar{w}_{n,\beta}||_{L^\infty}).
\end{aligned}$$

But, for $t \in [0, (\lambda N)^{-\alpha}\ln(N)^2]$, using the bounds for $\bar{w}_{N,\beta}$ and the interpolation inequality and taking δ small and N big

$$\begin{aligned}
&\int_0^t ||\Lambda^s(W)\Lambda^s[v(W)]\cdot\nabla\bar{w}_{N,\beta}(x,\tau)]||^2_{L^2}d\tau \leq C\lambda^{2-\beta}\int_0^t ||\Lambda^{\frac{\alpha}{2}}W||^{\frac{\alpha}{s}}_{L^2}||\Lambda^{s+\frac{\alpha}{2}}W||^{\frac{2s-\alpha}{s}}_{L^2}dt\\
&\leq C\lambda^{2-\beta}(\int_0^t ||\Lambda^{\frac{\alpha}{2}}W||^2_{L^2}dt)^{\frac{\alpha}{2s}}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}dt)^{\frac{2s-\alpha}{2s}} \leq C_\epsilon N^\epsilon\lambda^{2-\beta}(N^{\beta+1}\lambda^\beta)^{\frac{-\alpha}{s}}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}dt)^{\frac{2s-\alpha}{2s}}\\
&\leq C_\epsilon N^{\frac{\alpha}{2-\alpha}(1-\beta)+o(\delta)+o(\epsilon)}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}dt)^{\frac{2s-\alpha}{2s}} \leq \frac{1}{100}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}dt)^{\frac{2s-\alpha}{2s}}.
\end{aligned}$$

Similarly, for δ small and N big

$$\begin{aligned}
&\int_0^t ||\Lambda^s(W)v(W)\cdot\Lambda^s[\nabla\bar{w}_{N,\beta}(x,t)]||_{L^2}d\tau \leq C_\epsilon\lambda^{s+2-\beta}N^{s+1-\beta+\epsilon}\int_0^t ||\Lambda^s W||_{L^2}||W||_{L^2}d\tau\\
&\leq C_\epsilon\lambda^{s+2-\beta}N^{s+1-\beta+\epsilon}(\lambda^\beta N^{\beta+1})^{-1}N^\epsilon(N\lambda)^{-\frac{\alpha}{2}}N^\epsilon(N^{\beta+1}\lambda^\beta)^{-\frac{\alpha}{2s}}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}d\tau)^{\frac{2s-\alpha}{4s}}\\
&\leq C_\epsilon N^{-2(\beta-1)+o(\epsilon)+o(\delta)}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}d\tau)^{\frac{2s-\alpha}{4s}} \leq \frac{1}{100}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}d\tau)^{\frac{2s-\alpha}{4s}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t ||\Lambda^s W||_{L^2}\sum_{|\mathbf{k}|\leq 1}||\partial^{\mathbf{k}}W||_{L^2}||\Lambda^{s,\mathbf{k}}\nabla\bar{w}_{n,\beta}||_{L^\infty}d\tau \leq C_\epsilon\lambda^{s+1-\beta}N^{s-\beta+\epsilon}\int_0^t ||\Lambda^1 W||_{L^2}||\Lambda^s W||_{L^2}d\tau\\
&\leq C_\epsilon\lambda^{s+1-\beta}N^{s-\beta+\epsilon}N^\epsilon(N^{\beta+1}\lambda^\beta)^{-[\frac{1}{2}+\frac{\alpha}{2s}]}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}d\tau)^{\frac{1}{2}+\frac{2s-\alpha}{2s}}
\end{aligned}$$

$$\leq C_\epsilon N^{1-\beta+o(\epsilon)+o(\delta)} \leq \frac{1}{100}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}d\tau)^{\frac{1}{2}+\frac{2s-\alpha}{2s}}.$$

Moreover,

$$\int_0^t ||\Lambda^{1-\alpha}\nabla\bar{w}_{N,\beta}||_{L^\infty}||\Lambda^{1+\delta}W||_{L^2}||\Lambda^s W||_{L^2}d\tau \leq C_\epsilon\lambda^{3-\alpha-\beta}(1+N^{2-\alpha-\beta+\epsilon})\int_0^t ||\Lambda^{1+\delta}W||_{L^2}||\Lambda^s W||_{L^2}d\tau$$
$$\leq C_\epsilon\lambda^{3-\alpha-\beta}N^{2-\alpha-\beta+\epsilon}N^\epsilon(N^{\beta+1}\lambda^\beta)^{\frac{1-\frac{\alpha}{2}-\delta}{2-\alpha}+\frac{\alpha}{2s}}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}d\tau)^{\frac{1-\frac{\alpha}{2}+\delta}{2-\alpha}+\frac{2s-\alpha}{2s}}$$
$$\leq C_\epsilon N^{1-\beta+o(\epsilon)+o(\delta)} \leq \frac{1}{100}(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}d\tau)^{\frac{1-\frac{\alpha}{2}+\delta}{2-\alpha}+\frac{2s-\alpha}{2s}}.$$

Similarly,

$$\int_0^t ||\Lambda^s(W)\Lambda^s[v(\bar{w}_{N,\beta}(x,t))\cdot\nabla W]||_{L^1}d\tau$$
$$\leq \frac{1}{100}((\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||_{L^2}d\tau)^{\frac{1}{2}+\frac{2s-\alpha}{2s}}+2(\int_0^t ||\Lambda^{s+\frac{\alpha}{2}}W||^2_{L^2}d\tau)^{\frac{2s-\alpha}{2s}}).$$

Therefore, integrating (5.28) we get that, for $t\in[0,T_{crit}]\cap[0,(N\lambda)^{-\alpha}\ln(N)^2]$

$$||\Lambda^s W||^2_{L^2}\leq\frac{1}{10}, ||W||^2_{H^s}\leq\frac{1}{5}$$

and, since by continuity of the H^s norm we must have that $||W(x,T_{crit})||^2_{H^s}=1$, in particular it must be that $T_{crit}>(N\lambda)^{-\alpha}\ln(N)^2$, as we wanted to prove.

□

Lemma 5.3.2 allows us to show that our pseudo-solution is a very good approximation of the actual solution for $t\in[0,(\lambda N)^{-\alpha}\ln(N)^2]$. Furthermore, at $t^*=(\lambda N)^{-\alpha}\ln(N)^2$, we have that

$$||\bar{w}_{pert}(x,t^*)||_{L^2}\leq\frac{C}{(N\lambda)^\beta}\text{sup}_{r\in\text{supp}[\bar{w}_{pert}]}e^{-G(r,t^*)}\leq\frac{C}{(N\lambda)^\beta}e^{C(\lambda N)^\alpha(\lambda N)^{-\alpha}\ln(N)^2}\leq\frac{C}{N^{\beta+2}\lambda^\beta}$$

and for any $\epsilon>0$

$$||\bar{w}_{pert}(x,t^*)||_{H^2}\leq C_\epsilon(N\lambda)^{2-\beta}N^\epsilon\text{sup}_{r\in\text{supp}[\bar{w}_{pert}]}e^{-G(r,t^*)}\leq C(N\lambda)^{2-\beta}N^\epsilon e^{C(\lambda N)^\alpha(\lambda N)^{-\alpha}\ln(N)^2}$$
$$\leq C_\epsilon N^{-\beta+\epsilon}\lambda^{2-\beta}$$

so in particular, by interpolation, for small $\delta>0$

$$||w_{pert}(x,t)||_{H^{2-\alpha+\delta}}\leq N^{-\delta}. \tag{5.30}$$

Combining this with (5.25) and (5.24), we have that, for $t=(\lambda N)^{-\alpha}\ln(N)^2$,

$$||w_{N,\beta}(r,\theta,t)-\bar{g}(r,t)||_{L^2}\leq\frac{C}{N^{\beta+1}\lambda^\beta},$$

$$||w_{N,\beta}(r,\theta,t)-\bar{g}(r,t)||_{H^{2-\alpha+\delta}}\leq N^{-\delta}.$$

With this information, we can obtain the following lemma.

Lemma 5.3.4. *There is N_0,ϵ_0,δ such that if $N\geq N_0$, $\epsilon\leq\epsilon_0$ then for any $t\in[(N\lambda)^{-\alpha}\ln(N)^2,\lambda^{-\alpha}\ln(N)^3]$ we have that*

$$||w_{N,\beta}(x,t)-\bar{g}(r,t)||_{L^2}\leq C_\epsilon\frac{N^\epsilon}{N^{\beta+1}\lambda^\beta}, \tag{5.31}$$

$$||w_{N,\beta}(x,t)-\bar{g}(r,t)||_{H^{2-\alpha+\delta}}\leq 2. \tag{5.32}$$

where $w_{N,\beta}$ is the solution (5.1) *with the same initial conditions as $\bar{w}_{N,\beta}$ and $\bar{g}(r,t)$ is as in* (5.15).

Proof. We will omit the proof since it is completely analogous to that of Lemma 5.3.2: We note that (5.31) and (5.32) hold for $t = (N\lambda)^{-\alpha}\ln(N)^2$, we prove (5.31) for times when (5.32) holds, then we obtain an inequality like (5.27) and with that, the evolution equation for $w_{N,\beta}(x,t) - \bar{g}(r,t)$ and Lemma 2.2.10 we prove that (5.32) holds for $t \in [(N\lambda)^{-\alpha}\ln(N)^2, \lambda^{-\alpha}\ln(N)^3]$. □

Corollary 5.3.5. *There is $N_0, \delta > 0$ such that if $N \geq N_0$, then for any $t \in [\lambda^{-\alpha}\ln(N)^3, N^\delta]$ we have that*

$$||w_{N,\beta}(x,t)||_{H^{2-\alpha+\delta}} \leq CN^{-\delta}.$$

and for $t \in [N^{-\frac{\delta}{2}}, N^\delta]$

$$||w_{N,\beta}(x,t)||_{H^{2+\delta}} \leq CN^{-\frac{\delta}{2}}, \tag{5.33}$$

where $w_{N,\beta}$ is the solution to (5.1) *with the same initial conditions as $\bar{w}_{N,\beta}$.*

Proof. First, Lemma 5.3.4 allows us to show that, for $t = \lambda^{-\alpha}\ln(N)^3$ and for small $\delta > 0$, any $\epsilon > 0$

$$||w_{N,\beta}(x,t) - \bar{g}(r,t)||_{H^{2-\alpha+\delta}} \leq 2,$$
$$||w_{N,\beta}(x,t) - \bar{g}(r,t)||_{L^2} \leq \frac{C_\epsilon N^\epsilon}{\lambda^\beta N^{\beta+1}},$$

and, using that supp $\hat{\bar{g}}(\hat{r},0) \subset \{\hat{r} : \hat{r} \in (\lambda c, \infty)\}$ for some $c > 0$ (see Lemma 5.2.8), again for the same time

$$||\bar{g}(r,t)||_{H^s} \leq e^{C\lambda^{-\alpha}t}||\bar{g}(r,0)||_{H^s} = C\frac{\lambda^{s-\beta}}{N^3} \leq \frac{C\lambda^{s-2+\alpha}}{N^2},$$

and combining the three inequalities, taking ϵ small and using the interpolation inequality, we get that there is a small $\bar{\delta} > 0$ such that, for $t = \lambda^{-\alpha}\ln(N)^3$

$$||w_{N,\beta}(x,t)||_{H^{2+\alpha-\bar{\delta}}} \leq CN^{-\bar{\delta}}.$$

Finally, we note that

$$\begin{aligned}
&\frac{d}{dt}||w_{N,\beta}||^2_{\dot{H}^{2-\alpha+\bar{\delta}}} \leq 2||\Lambda^{2-\alpha+\bar{\delta}}(w_{N,\beta})\Lambda^{2-\alpha+\bar{\delta}}[v(w_{N,\beta})\cdot\nabla w_{N,\beta}]||_{L^1} - 2||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||^2_{L^2}\\
&\leq C||\Lambda^{2-\frac{\alpha}{2}}w_{N,\beta}||_{L^2}||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||_{L^2}||\Lambda^{2-\alpha+\bar{\delta}}w_{N,\beta}||_{L^2} - 2||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||^2_{L^2}\\
&\leq C(||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||_{L^2} + ||w_{N,\beta}||_{L^2})||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||_{L^2}||\Lambda^{2-\alpha+\bar{\delta}}w_{N,\beta}||_{L^2} - 2||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||^2_{L^2}\\
&\leq (C||\Lambda^{2-\alpha+\bar{\delta}}w_{N,\beta}||_{L^2} - 1)||\Lambda^{2-\frac{\alpha}{2}+\bar{\delta}}w_{N,\beta}||^2_{L^2} + C||\Lambda^{2-\alpha+\bar{\delta}}w_{N,\beta}||^2_{L^2}||w_{N,\beta}||^2_{L^2},
\end{aligned}$$

and integrating in time gives the desired bound.
Next, using the interpolation inequality and our bounds for $w_{N,\beta}$, we have that, for $t \in [\lambda^{-\alpha}\ln(N)^3, N^\delta]$

$$\begin{aligned}
&\frac{d}{dt}||w_{N,\beta}(x,t)||^2_{H^{2+\delta}} \leq C||w_{N,\beta}(x,t)||^3_{H^{2+\delta}} - 2||w_{N,\beta}(x,t)||^2_{\dot{H}^{2+\delta+\frac{\alpha}{2}}}\\
&\leq C||w_{N,\beta}(x,t)||^3_{H^{2+\delta}} + \frac{C}{\lambda^{2\beta}N^4} - 2||w_{N,\beta}(x,t)||^2_{H^{2+\delta+\frac{\alpha}{2}}}\\
&\leq C||w_{N,\beta}(x,t)||^3_{H^{2+\delta}} + \frac{C}{\lambda^{2\beta}N^4} - 2\frac{||w_{N,\beta}(x,t)||^3_{H^{2+\delta}}}{||w_{N,\beta}(x,t)||_{H^{2-\alpha+\delta}}} \leq -CN^\delta||w_{N,\beta}(x,t)||^3_{H^{2+\delta}} + \frac{C}{\lambda^{2\beta}N^4}.
\end{aligned}$$

Now, we note that if for some $t_0 \in [\lambda^{-\alpha}\ln(N)^3, N^\delta]$ we have $||w_{N,\beta}(x,t)||_{H^{2+\delta}} \leq N^{-1}$, then (5.33) holds trivially for $t \in [t_0, N^\delta]$ by integrating the equation. Therefore, it is enough to study the behaviour for t such that $||w_{N,\beta}(x,t)||_{H^{2+\delta}} \geq N^{-1}$, which in particular gives, for N big

$$N^\delta||w_{N,\beta}(x,t)||^3_{H^{2+\delta}} >> \frac{1}{\lambda^{2\beta}N^4}.$$

and with this, we have

$$\frac{d}{dt}||w_{N,\beta}(x,t)||_{H^{2+\delta}} \leq -CN^{\delta}||w_{N,\beta}(x,t)||^2_{H^{2+\delta}}$$

and integrating this equation gives the desired result.

□

The last computation we need to do regarding $w_{N,\beta}$ is to show that it exhibits a very fast growth in the H^{β} norm.

Lemma 5.3.6. *There is N_0 such that if $N \geq N_0$ then*

$$||w_{N,\beta}(x,t=\lambda^{-2+\beta}(\ln(N))^{\frac{1}{2}})||_{\dot{H}^{\beta}} \geq C\ln(N)^{\frac{\beta}{2}}||w_{N,\beta}(x,0)||_{H^{\beta}}$$

with $w_{N,\beta}$ the solution to (5.1) *with the same initial conditions as $\bar{w}_{N,\beta}$.*

Proof. We start by noting that, by using the scaling properties of the norms $\dot{H}^s$ plus the definition of $\bar{g}(r,t)$

$$||\bar{g}(r,t)||_{H^s} \leq C_s\lambda^{s-\beta}.$$

On the other hand, we have that, for $i=1,2$, by direct computation

$$||w_{N,\beta}(x,0)-\bar{g}(r,0)||_{H^i} \leq C\frac{(\lambda N)^i}{\lambda^{\beta}N^{\beta}},$$

and combining these inequalities plus the interpolation inequality gives us

$$||w_{N,\beta}(x,0)||_{H^{\beta}} \leq C.$$

On the other hand for $t=\lambda^{-2+\beta}(\ln(N))^{\frac{1}{2}})$, using Lemma 5.3.2 we have

$$||w_{N,\beta}(x,t)-\bar{g}(r,t)||_{L^2} \leq \frac{C}{\lambda^{\beta}N^{\beta}}$$

and

$$||w_{N,\beta}(x,t)-\bar{g}(r,t)||_{\dot{H}^1} \geq ||\bar{w}_{pert}||_{\dot{H}^1} - C_{\epsilon}\frac{N^{\epsilon}}{N^{\beta+1}\lambda^{\beta}}.$$

Furthermore using the expression for $\frac{\partial}{\partial x_1}$ in polar coordinates for $t\in[0,\lambda^{-2+\beta}\ln(N)^{\frac{1}{2}}]$,

$$||1_{\text{supp}(f(\lambda r))}\partial_r\Theta(r,t)||_{L^{\infty}} \leq C\ln(N)^{\frac{1}{2}},$$

so for $r\in\text{supp}(f(\lambda r))$ and $t=\lambda^{-2+\beta}\ln(N)^{\frac{1}{2}}$ $G(r,t)\leq C$, which, after some basic computations, gives, for $t=\lambda^{-2+\beta}\ln(N)^{\frac{1}{2}}$,

$$|f(\lambda r)\partial_r\Theta(r,t)e^{-G(r,t)}| \geq C|f(\lambda r)|\ln(N)^{\frac{1}{2}}.$$

Using all this we get, for N big,

$$\begin{aligned}
&||\bar{w}_{pert}||_{\dot{H}^1} \geq ||\frac{\partial}{\partial x_1}\bar{w}_{pert}||_{L^2} \geq ||cos(\theta)\frac{\partial}{\partial r}\bar{w}_{pert}||_{L^2} - \frac{C}{(N\lambda)^{\beta-1}}\\
&\geq ||cos(\theta)f(\lambda r)\lambda N\big(\frac{\partial}{\partial r}\Theta(r,t)\big)\frac{\sin(N(\theta+\Theta(r,t))-C_0\int_0^t\frac{\partial\bar{g}(r,s)}{\partial r}ds)}{N^{\beta}\lambda^{\beta}}e^{-G(r,t)}||_{L^2} - \frac{C}{(N\lambda)^{\beta-1}}\\
&\geq \frac{C}{(N\lambda)^{\beta-1}}||f(\lambda r)\big(\frac{\partial}{\partial r}\Theta(r,t)\big)e^{-G(r,t)}||_{L^2} - \frac{C}{(N\lambda)^{\beta-1}} \geq \frac{C\ln(N)^{\frac{1}{2}}}{(N\lambda)^{\beta-1}},
\end{aligned}$$

and using the interpolation inequality we get

$$||\bar{w}_{pert}||_{\dot{H}^{\beta}} \geq \frac{||\bar{w}_{pert}||^{\beta}_{\dot{H}^1}}{||\bar{w}_{pert}||^{\beta-1}_{L^2}} \geq C\ln(N)^{\frac{\beta}{2}},$$

so

$$\frac{||w_{N,\beta}(x,t)||_{\dot{H}^{\beta}}}{||w_{N,\beta}(x,0)||_{H^{\beta}}} \geq C\ln(N)^{\frac{\beta}{2}}$$

as we wanted to prove. $\square$

Remark 11. Lemma 5.3.6 would give us the growth around zero of the H^{β} norm, namely the final solution obtained in Theorem 5.4.2 will have a sequence of times t_n such that

$$\lim_{t_n\to 0}\frac{||w(x,t_n)||_{H^{\beta}}}{|\ln(t_n)|^{\frac{\beta}{2}}} > 0.$$

It should be noted that it is not the goal of this chapter to try and obtain the optimal explosion rate around $t=0$, and this rate can probably be improved substantially.

5.4 Loss of regularity

We will now use the previous results to obtain a more compact and usable theorem before we go on to prove the main theorem.

Theorem 5.4.1. For any $n \in \mathbb{N}$, $\alpha \in (0,1)$, $\beta \in (1,2-\alpha)$, there exists initial conditions $w_n(x,0)$, a solution $w_n(x,t)$ to (5.1) and $t_n \in [0,2^{-n}]$ such that

$$||w_n(x,0)||_{H^{\beta}} \leq 2^{-n}, ||w_n(x,t_n)||_{H^{\beta}} \geq 2^{n}.$$

Furthermore, there is small $\delta > 0$ such that, for $t \in [\frac{1}{n},1]$

$$||w_n(x,t)||_{H^{2+\delta}} \leq 2^{-n}$$

and for $t \in [0,1]$

$$||w_n(x,t)||_{H^{1+\delta}} \leq 2^{-n}, ||w_n(x,t)||_{L^1} \leq 2^{-n}$$
$$||w_n(x,t)||_{H^6} \leq C_n. \tag{5.34}$$

Proof. We start by fixing $\alpha \in (0,1)$ and $\beta \in (1,2-\alpha)$, and we consider the solutions $w_{N,\beta}(x,t)$ that we considered earlier. These solutions fulfil that

- $||w_{N,\beta}(x,0)||_{H^{\beta}} \leq C.$
- For $t = \lambda^{-2+\beta}\ln(N)$, N big

$$||w_{N,\beta}(x,t)||_{H^{\beta}} \geq C\ln(N)^{\frac{\beta}{2}}.$$

- There is some small $\delta > 0$ such that, for N big and $t \in [\lambda^{-\alpha}\ln(N)^3, N^{\delta}]$

$$||w_{N,\beta}(x,t)||_{H^{2+\delta}} \leq CN^{-\frac{\delta}{2}}.$$

- There is a small $\delta > 0$ such that for $t \in [0,N^{\delta}]$,

$$||w_{N,\beta}||_{H^{1+\delta}} \leq CN^{-\delta}, ||w_{N,\beta}||_{L^1} \leq CN^{-\delta}.$$

If we now consider

$$w_{N,K}(x,t) := \frac{w_{N,\beta}(Kx,K^{\alpha}t)}{K^{1-\alpha}}$$

we have that these functions are also solutions to (5.1) and,

- $||w_{N,K}(x,0)||_{H^{\beta}} \leq CK^{-2+\alpha+\beta}.$
- For $t = K^{-\alpha}\lambda^{-2+\beta}\ln(N)$, N big

$$||w_{N,K}(x,t)||_{H^{\beta}} \geq CK^{2-\alpha-\beta}\ln(N)^{\frac{\beta}{2}}.$$

- There is some small $\delta > 0$ such that, for N big and $t \in [K^{-\alpha}\lambda^{-\alpha}\ln(N)^3, K^{-\alpha}N^{\delta}]$

$$||w_{N,K}(x,t)||_{H^{2+\delta}} \leq CK^{\frac{\delta}{2}}N^{-\frac{\delta}{2}}.$$

- There is a small $\delta > 0$ such that, for $t \in [0, K^{-\alpha}N^{\delta}]$,

$$||w_{N,K}||_{H^{1+\delta}} \leq CK^{1+\delta-2+\alpha}N^{-\delta}, ||w_{N,K}||_{L^1} \leq CK^{-3+\alpha}N^{-\delta}.$$

so, for any fixed n, taking K big and then N big gives us all the inequalities but the bound in H^6. But then using Theorem 3.1 from [33] tells us that $w_{N,K}$ is in C^{∞} for any $t > 0$, and since our initial conditions are in C^{∞}, using the continuity of the H^6 norm for (5.1) finishes the proof. □

We are now ready to prove the main theorem of this chapter.

Theorem 5.4.2. Given $\epsilon > 0$, $\alpha \in (0,1)$, $\beta \in (1, 2-\alpha)$, there exists initial conditions $w(x,0)$ with $||w(x,0)||_{H^{\beta}} \leq \epsilon$ and a solution $w(x,t)$ to (5.1) such that $w(x,t) \in C^{\infty}$ for $t \in (0,\infty)$ and there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ converging to zero, with $\lim_{n\to\infty}||w(x,t_n)||_{H^{\beta}} = \infty$. Furthermore, this is the only solution with the given initial conditions that is in $L^{\infty}_t H^1_x$.

Proof. For this proof, we will be considering initial conditions of the form

$$w(x,0) = \sum_{j=1}^{\infty} T_{R_j}(w_{c_j}(x,0)) \tag{5.35}$$

with $T_R(f(x_1,x_2)) = f(x_1+R, x_2)$, $w_{c_J}(x,0)$ the initial conditions given by Theorem 5.4.1. In order to show properties of the solution given by (5.35), we will also consider a truncated initial conditions

$$\tilde{w}_J(x,0) = \sum_{j=1}^{J} T_{R_j}(w_{c_j}(x,0)) \tag{5.36}$$

and we will refer the solution with initial conditions given by (5.36) as $\tilde{w}_J(x,t)$. Fixed ϵ, which we will assume $\frac{1}{2} > \epsilon$ without loss of generality, we will choose $(c_j)_{j\in\mathbb{N}}$ so that they fulfil:

- $c_i > c_j$ if $i > j$.
- $2^{-c_1+1} \leq \epsilon$, so $||w(x,0)||_{H^{\beta}} \leq \epsilon$.
- If we define

$$S_j := \sum_{i=1}^{j} C_{c_i} \tag{5.37}$$

with C_{c_i} the constants given by (5.34), then we take c_j so that

$$c_j \geq je^{S_{j-1}+1}, 2^{c_j} \geq 2S_{j-1}.$$

- If t_{c_j} is the time given by Theorem 5.4.1 such that

$$||w_{c_j}(x,t_{c_j})||_{H^{\beta}} \geq 2^{c_j}$$

then $\frac{1}{c_{j+1}} \leq t_{c_j}$.

We will now divide the proof in four different steps.
Step 1)The goal of this step is to show the following claim:
For any choice of $(c_j)_{j\in\mathbb{N}}$ and $\epsilon' > 0$, we can choose $(R_j)_{j\in\mathbb{N}}$ such that, for $t \in [0,1]$, for any $J \in \mathbb{N}$

$$||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t) - T_{R_J}(w_{c_J}(x,t))||_{H^5} \leq \epsilon' 2^{-J-1}, \tag{5.38}$$

and $\tilde{w}_J(x,t) \in H^6$ for $t \in [0,1]$, and such that for $t \in [0,1]$

$$||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t)||_{C^3_x(B_J(0))} \leq \epsilon' 2^{-J} \tag{5.39}$$

where $B_J(0)$ is the ball of radius J centered at the origin.
We will show (5.38) and (5.39) by induction, by showing that, for any $J \in \mathbb{N}$, fixed $(c_j)_{j=1,...,J}$ and $\epsilon' > 0$, we can choose $(R_i)_{i=1,...,J}$ so that for any $j = 1, ..., J$

$$||\tilde{w}_j(x,t) - \tilde{w}_{j-1}(x,t) - T_{R_j}(w_{c_j}(x,t))||_{H^5} \leq \epsilon' 2^{-j-1}, \tag{5.40}$$

$$||\tilde{w}_j(x,t) - \tilde{w}_{j-1}(x,t)||_{C^3_x(B_J(0))} \leq \epsilon' 2^{-j}. \tag{5.41}$$

and $\tilde{w}_j(x,t) \in H^6$. For $J = 1$ this is trivial since $\tilde{w}_{J=0} = 0$, $\tilde{w}_{J=1}(x,t) = T_{R_1}(w_{c_1}(x,t))$ and $w_{J=1} \in C^\infty$ for all time. Now, for some arbitrary J we have that, if we define

$$\bar{w}_J(x,t) = \tilde{w}_{J-1}(x,t) + T_{R_J}(w_{c_J}(x,t))$$

we have that $\bar{w}_J(x,t)$ is a pseudo-solution for (5.1) with

$$F(x,t) = -v(\tilde{w}_{J-1}(x,t)) \cdot \nabla(T_{R_J}(w_{c_J}(x,t))) - v(T_{R_J}(w_{c_J}(x,t))) \cdot \nabla(\tilde{w}_{J-1}(x,t)).$$

Furthermore, both $\tilde{w}_{J-1}(x,t)$ and $T_{R_J}(w_{c_J}(x,t))$ are C^∞ functions for $t \in [0,1]$ since they are both solutions to (5.1) that are uniformly bounded in H^6, $\tilde{w}_{J-1}(x,t)$ by hypothesis and $T_{R_J}(w_{c_J}(x,t))$ by Theorem 5.4.1. But then we know that $\lim_{R_j \to \infty} ||F(x,t)||_{H^5} = 0$ by using that for any two functions $f_1(x), f_2(x) \in H^7$

$$\lim_{R \to \infty} ||f_1(x_1,x_2) f_2(x_1 + R, x_2)||_{H^5} = 0,$$

plus the fact that C^∞ solutions solutions to (5.1) are continuous in time with respect to the H^7 norm.
Now, to get (5.40), we just use that, if $\bar{w}_{error,c}(x,t)$ is a family of pseudo-solutions (which depends on the parameter c) with source term $F_{error,c}$ and fulfilling $||\bar{w}_{error,c}(x,t)||_{H^6} \leq C$ for $t \in [0,T]$ (C independent of c), $||F_{error,c}(x,t)||_{H^5} \leq c$ and we call $w_c(x,t)$ the solution of (5.1) with the same initial conditions as $\bar{w}_{error,c}$, then

$$\lim_{c \to 0} ||w_c(x,t) - \bar{w}_{error,c}(x,t)||_{H^5} = 0,$$

and therefore,

$$\lim_{R_J \to \infty} ||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t) - T_{R_J}(w_{c_J}(x,t))||_{H^5} = 0, \tag{5.42}$$

and so taking R_J big enough gives (5.40), and then since for $t \in [0,1]$ $w_J(x,t)$ is a H^5 solution to (5.1) with initial conditions in C^∞, it must also be in H^6.
Next, for (5.41), since we only need to prove the case $j = J$, we use

$$\begin{aligned}
&||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t)||_{C^3_x(B_J(0))} \\
&\leq ||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t) - T_{R_J}(w_{c_J}(x,t)||_{H^5} + ||T_{R_J}(w_{c_J}(x,t)||_{C^3_x(B_J(0))} \\
&\leq \epsilon' 2^{-J-1} + ||T_{R_J}(w_{c_J}(x,t)||_{C^3_x(B_J(0))}
\end{aligned}$$

and, as before, using the continuity in time with respect to the H^5 norm of smooth solutions to (5.1) gives us

$$\sup_{t \in [0,1]} \lim_{R_J \to \infty} ||T_{R_J}(w_{c_J}(x,t)||_{C^3_x(B_J(0))} = 0,$$

so taking R_j big enough finishes step 1.
Step 2)The goal of this step is to obtain the properties of $\lim_{J \to \infty} \tilde{w}_J(x,t)$:
First we note that,

$$||w_{c_j}(x,t)||_{H^{1+\delta}} \leq 2^{-c_j}, ||w_{c_j}(x,t)||_{L^1} \leq 2^{-c_j}$$

so there exists

$$w_\infty(x,t) := \lim_{J \to \infty} \tilde{w}_J(x,t)$$

and $\tilde{w}_J(x,t)$ tends to $w_\infty(x,t)$ in $H^{1+\delta} \cap L^1$. We would like to show that, for any $t \in [0,1]$

$$\frac{\partial}{\partial t} w_\infty(x,t) + v(w_\infty(x,t)) \cdot \nabla w_\infty(x,t) + \Lambda^\alpha(w_\infty(x,t)) = 0.$$

For this, we note that, for $j > J$, using the properties of Λ^{α} and v, for $t \in [0,1]$

$$||\Lambda^{\alpha} w_{\infty}(x,t) - \Lambda^{\alpha}\tilde{w}_j(x,t)||_{C^1_x(B_J(0))} \leq C(||w_{\infty}(x,t) - \tilde{w}_j(x,t)||_{C^2_x(B_J(0))} + ||w_{\infty}(x,t) - \tilde{w}_j(x,t)||_{L^1})$$
$$\leq C(\sum_{i=j}^{\infty} ||\tilde{w}_{i+1}(x,t) - \tilde{w}_i(x,t)||_{C^2_x(B_J(0))} + 2^{-j+1}) \leq C2^{-j},$$

and similarly

$$\begin{aligned}
&||v(w_{\infty}) \cdot \nabla w_{\infty} - v(w_j) \cdot \nabla w_j||_{C^1_x(B_J(0))} \\
&\leq C\mathrm{sup}_{i\geq j}||\tilde{w}_i||_{C^2_x(B_J(0))}(||w_{\infty}(x,t) - \tilde{w}_j(x,t)||_{C^2_x(B_J(0))} + ||w_{\infty}(x,t) - w_j(x,t)||_{L^1}) \\
&+ C\mathrm{sup}_{i\geq j}(||\tilde{w}_i||_{C^2_x(B_J(0))} + ||\tilde{w}_i||_{L^1})(||w_{\infty}(x,t) - \tilde{w}_j(x,t)||_{C^2_x(B_J(0))} \\
&\leq C2^{-j},
\end{aligned}$$

so that

$$\begin{aligned}
w_{\infty}(x,t_2) - w_{\infty}(x,t_1) &= \mathrm{lim}_{J\to\infty}(\tilde{w}_J(x,t_2) - \tilde{w}_J(x,t_1)) \\
&= -\mathrm{lim}_{J\to\infty}\int_{t_1}^{t_2}(v(\tilde{w}_J(x,s)) \cdot \nabla\tilde{w}_J(x,s) + \Lambda^{\alpha}(\tilde{w}_J(x,s)))ds \\
&= -\int_{t_1}^{t_2}(v(w_{\infty}(x,s)) \cdot \nabla w_{\infty}(x,s) + \Lambda^{\alpha}(w_{\infty}(x,s)))ds.
\end{aligned}$$

and using that

$$\begin{aligned}
&\mathrm{lim}_{J\to\infty}||(v(w_{\infty}(x,t)) \cdot \nabla w_{\infty}(x,t) + \Lambda^{\alpha}(w_{\infty}(x,t)) - (v(\tilde{w}_J(x,t)) \cdot \nabla\tilde{w}_J(x,t) \\
&+ \Lambda^{\alpha}(\tilde{w}_J(x,t))||_{C^0_t C^1_x([0,1]\times B_J(0))} = 0
\end{aligned}$$

and that $(v(\tilde{w}_J(x,t))\cdot\nabla\tilde{w}_J(x,t)+\Lambda^{\alpha}(\tilde{w}_J(x,t))$ is continuous in time with respect to the $C^1_x(B_J(0))$ norm, we get that the function

$$v(w_{\infty}(x,t)) \cdot \nabla w_{\infty}(x,t) + \Lambda^{\alpha}(w_{\infty}(x,t))$$

is also continuous in time (for $t \in [0,1]$) with respect to the $C^1_x(B_J(0))$ norm, so

$$\frac{\partial}{\partial t} w_{\infty}(x,t) = -(v(w_{\infty}(x,t)) \cdot \nabla w_{\infty}(x,t) + \Lambda^{\alpha}(w_{\infty}(x,t))).$$

holds.
Furthermore, for $t > 0$

$$||w_{\infty}(x,t) - \tilde{w}_J(x,t)||_{H^{2+\delta}} \leq \sum_{j=J}^{\infty}||w_{\infty}(x,t) - \tilde{w}_J(x,t) - T_{R_j}(w_{c_j}(x,t))||_{H^5} + \sum_{j=J}^{\infty}||w_{c_j}(x,t)||_{H^{2+\delta}}$$
$$\leq \epsilon' 2^{-J-1} + \sum_{j=J}^{\infty}||w_{c_j}(x,t)||_{H^{2+\delta}}.$$

Now, we note that, if $t \geq \frac{1}{c_j}$, then

$$||w_{c_j}(x,t)||_{H^{2+\delta}} \leq 2^{-j}$$

so, if

$$j_0(t) := \max(\{j \in \mathbb{N} : \frac{1}{c_j} < t\}, \{0\}) \tag{5.43}$$

then

$$\sum_{j=1}^{\infty}||w_{c_j}(x,t)||_{H^{2+\delta}} \leq \sum_{j=j_0(t)+1}^{\infty} 2^{-c_j} + S_{j_0(t)} \leq S_{j_0(t)} + 1$$

with S_j as in (5.37) and if $J > j_0(t)$

$$\sum_{j=J}^{\infty} ||w_{c_j}(x,t)||_{H^{2+\delta}} \leq \sum_{j=J}^{\infty} 2^{-c_j} \leq 2^{-J+1}.$$

This in particular means that, for any $t > 0$,

$$\lim_{J\to\infty} ||w_\infty(x,t) - \tilde{w}_J(x,t)||_{H^{2+\delta}} = 0 \quad (5.44)$$

and

$$||w_\infty(x,t)||_{H^{2+\delta}} \leq \epsilon' + S_{j_0(t)} + 1. \quad (5.45)$$

This implies that, for any $t_0 > 0$

$$\sup_{t\in[t_0,1]} ||w_\infty(x,t)||_{H^{2+\delta}} \leq \epsilon' + S_{j_0(t_0)} + 1 \quad (5.46)$$

and since $w_\infty(x,t)$ is a solution to (5.1), using Theorem 3.1 from [33] tells us that w_∞ is in C^∞ for any $t > 0$.
Finally, we have that, for $t = 1$

$$||w_\infty(x,t)||_{H^{2+\delta}} \leq \sum_{j=1}^{\infty} ||w_{c_j}(x,t)||_{H^{2+\delta}} + ||\tilde{w}_J(x,t) - \tilde{w}_{J-1}(x,t) - T_{R_J}(w_{c_J}(x,t)||_{H^{2+\delta}} \leq 2^{-c_1+1} + \epsilon',$$

so we can make the $H^{2+\delta}$ as small as we want by taking c_1 big and ϵ' small, and in particular $w_\infty(x,t)$ will be a global smooth solution to (5.1).
Step 3: We will now show that we have uniqueness, i.e. that, for any $M, \epsilon > 0$, if we assume there exists a solution $\tilde{w}(x,t)$ to (5.1) fulfilling

$$\sup_{t\in[0,\epsilon]} ||\tilde{w}(x,t)||_{H^1} \leq M$$

then, for $t \in [0,\epsilon]$ $\tilde{w}(x,t) = w_\infty(x,t)$.
For this, we note that, if we define $W(x,t) := \tilde{w}(x,t) - w_\infty(x,t)$ then

$$\frac{d}{dt}W(x,t) + v(W)\cdot\nabla(W + w_\infty) + v(w_\infty)\cdot\nabla W + \Lambda^\alpha W = 0$$

so in particular

$$\frac{\partial}{\partial t}||W||^2_{L^2} \leq -2\int_{\mathbb{R}^2} W(v(W)\cdot\nabla w_\infty(x,t))dx$$

but then we have

$$|\int_{\mathbb{R}^2} W(v(W)\cdot\nabla w_\infty(x,t))dx| \leq ||w_\infty(x,t)||_{C^1}||W||^2_{L^2} \quad (5.47)$$

$$|\int_{\mathbb{R}^2} W(v(W)\cdot\nabla w_\infty(x,t))dx| \leq ||w_\infty(x,t)||_{L^\infty}||W||_{L^2}||W||_{H^1},$$

so, for any $t_0 \in [0,t]$ we have

$$||W(x,t)||_{L^2} \leq Ct_0||W||_{H^1} e^{\int_{t_0}^t ||w_\infty(x,s)||_{C^1} ds}$$

but, for $t \leq 1$

$$||w_\infty(x,t)||_{C^1} \leq ||w_\infty(x,t)||_{H^{2+\delta}} \leq \epsilon' + S_{j_0(t_0)} + 1,$$

where we used (5.46). Now, we note that for $t = \frac{1}{c_j}$ we already know that

$$||W(x,t)||_{L^2} \leq \frac{C}{c_j}||W||_{H^1} e^{\epsilon' + S_{j_0(\frac{C}{c_j})} + 1} \leq \frac{1}{c_j}(M+1)e^{\epsilon' + S_{j_0(\frac{1}{c_j})} + 1}$$

and, by definition of j_0 (see (5.43)), we have that $j_0(\frac{1}{c_j}) \leq j-1$, so

$$||W(x,t)||_{L^2} \leq \frac{1}{c_j}(M+1)e^{\epsilon' + S_{j-1}+1} \leq \frac{C(M+1)}{j}$$

which tends to zero as j tends to infinity so, for $t \in [0,1]$ $||W(x,t)||_{L^2} = 0$ and therefore $\tilde{w}(x,t) = w_\infty(x,t)$. For $t > 1$, we just use that $\sup_{t\geq 1}||w_\infty(x,t)||_{C^1} \leq C$, and therefore (5.47) gives uniqueness.
Step 4: To end the proof, we need to show loss of regularity, and more precisely that there is a sequence of times t_n such that

$$\lim_{n\to\infty}||w_\infty(x,t_n)||_{H^\beta} = \infty.$$

But we chose our c_j so that

$$2^{c_j} \geq 2S_{j-1},$$

and if t_{c_j} is the time given by Theorem 5.4.1 such that

$$||w_{c_j}(x,t_{c_j})||_{H^\beta} \geq 2^{c_j}$$

then $\frac{1}{c_{j+1}} \leq t_{c_j}$.
Therefore, we have that

$$||w_\infty(x,t_{c_j})||_{H^\beta} \geq ||T_{R_j}(w_{c_j}(x,t_{c_j}))||_{H^\beta} - \sum_{i\in\mathbb{N}, i\neq j} ||T_{R_i}(w_{c_i}(x,t_{c_j}))||$$
$$-\sum_{i=0}^{\infty} ||\tilde{w}_j(x,t) - \tilde{w}_{j-1}(x,t) - T_{R_j}(w_{c_j}(x,t))||_{H^\beta} \geq 2^{c_j} - \sum_{i\in\mathbb{N}, i\neq j} ||T_{R_i}(w_{c_i}(x,t_{c_j}))|| - \epsilon'.$$

However,

$$\sum_{i=1}^{j-1} ||T_{R_i}(w_{c_i}(x,t_{c_j}))||_{H^\beta} \leq 2\sum_{i=1}^{j-1} ||T_{R_i}(w_{c_i}(x,t_{c_j}))||_{H^6} \leq S_{j-1} \leq 2^{c_j-2}$$

and

$$\sum_{i=j+1}^{\infty} ||T_{R_i}(w_{c_i}(x,t_{c_j}))||_{H^\beta} \leq \sum_{i=j+1}^{\infty} 2^{-c_i} \leq 1$$

so

$$||w_\infty(x,t_{c_j})||_{H^\beta} \geq 2^{c_j-1} - 1 - \epsilon'$$

and we are done. ☐

Chapter 6

Conclusions

6.1 Conclusions

The tools we have developed for this thesis allow us to show a variety of results regarding ill-posedness, non existence of solutions and loss of regularity, including non-existence of solutions for SQG in H^s and C^k (in chapter 2, non-existence of solution in $C^{k,\beta}$ for gSQG (in chapter 3), gap loss of regularity for 2D-Euler (in chapter 4) and non-existence of solution for dissipative SQG in H^s (in chapter 5).
Furthermore, the techniques applied here are versatile, which suggest that they could be applied to show similar results in other models, such as IPM, Prandtl or De Gregorio. Not only this, but since we manage to obtain a great deal of information about the qualitative and quantitative behaviour of the solutions, this could allow us to obtain more general results not related to ill-posedness, such as norm growth for long times, mixing or instability of solutions.

6.2 Conclusiones

Las herramientas que hemos usado en esta tesis nos permiten demostrar una variedad de resultados relacionados con el mal comportamientos de soluciones, la pérdida de regularidad y la no existencia de soluciones, incluyendo la no existencia de soluciones para SQG en H^s y C^k (en el capítulo 2), la no existencia de soluciones en $C^{k,\beta}$ para gSQG (en el capítulo 3, la existencia de salto de regularidad para 2D-Euler (en el capítulo 4) y la no existencia de soluciones para SQG con difusión fraccionaria en H^s (en el capítulo 5).
Además, las técnicas aplicadas parecen ser versátiles, lo cual sugiere que se pueden aplicar a otros modelos de importancia, como IPM, Prandtl o De Gregorio, y obtener resultados similares. No solo eso, pero dado que conseguimos obtener una gran cantidad de información, tanto cualitativa como cuantitativa, sobre el comportamiento de nuestras soluciones, en principio podríamos usar ideas similares para demostrar otro tipo de resultados, como el crecimiento de norma a tiempos largos, la inestabilidad de soluciones o el mezclado de un fluido.

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