# Finite dimensional Approximations of Operators related to Groups and their Applications 

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## Abstract

In the first part of this Thesis we will talk about the effective Lück approximation theorem. Let $G$ be a residually finite group and let $\left(N_{i}\right)$ be a chain of normal subgroups of $G$ of finite index with trivial intersection. Set $G_{i}=G / N_{i}$ and let $A \in \operatorname{Mat}_{n}(\mathbb{Q}[G])$. Let us denote by $A_{i} \in \operatorname{Mat}_{n \cdot\left|G_{i}\right|}(\mathbb{Q})$ the matrices that after choosing a basis represent the action given by right multiplication of $A$ on $\mathbb{C}\left[G_{i}\right]^{n} \cong \mathbb{C}^{n \cdot\left|G_{i}\right|}$. It was Wolfgang Lück who proved that

$$
\lim _{i \rightarrow \infty} \frac{1}{\left|G_{i}\right|} \operatorname{rk} A_{i}
$$

exists and is independent of the chain $\left(N_{i}\right)$. He even proved the the limit equals the von Neumann $\operatorname{rank} \operatorname{rk}_{G}(A)$. He did this by showing that for a self adjoint matrix $B$ over $\mathbb{Q}[G], 0 \in \mathbb{C}$ is not an accumulation point of eigenvalues of the matrices $B_{i}$. So what does that mean? For a matrix $C \in \operatorname{Mat}_{n}(\mathbb{C})$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ we denote by

$$
\mu_{C}=\sum_{i=1}^{n} \delta_{\lambda_{i}}
$$

the eigenvalue measure of $C$. Here $\delta_{x}$ is the Dirac measure at $x \in \mathbb{C}$. Lück showed that for a self adjoint matrix $A \in \operatorname{Mat}_{n}(\mathbb{Q}[G])$ the inequality

$$
\frac{1}{\left|G_{i}\right|} \mu_{A_{i}}((0, \lambda)) \leq \frac{b}{|\log \lambda|}
$$

holds for $0<\lambda<1$, where $b$ is some constant that only depends on $A$. We call this the effective Lück theorem. In the first part of the thesis we proof a generalization of this theorem.

Theorem. Let $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ be a free $*$-Algebra and let $A \in \operatorname{Mat}_{n}(\mathcal{A})$ be a normal matrix. Let $c \in \mathbb{R}_{>0}$. Then there is a function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ with $\lim _{\lambda \rightarrow 0^{+}} f(\lambda)=0$ such that for all $y \in \mathbb{C}$ and for all $*$-homomorphisms $\varphi: \mathcal{A} \rightarrow \operatorname{Mat}_{m}(\mathbb{C})$ with $\varphi\left(x_{j}\right) \in \operatorname{Mat}_{m}(\mathbb{Z})$ and $\left\|\varphi\left(x_{j}\right)\right\|_{1},\left\|\varphi\left(x_{j}\right)\right\|_{\infty} \leq c$ we have

$$
\frac{1}{m} \mu_{\varphi(A)}(B(0, \lambda) \backslash\{y\}) \leq f(\lambda)
$$

In the second part of this thesis we talk about twisted $\ell^{2}$-Betti numbers. Let $G$ be a discrete group, $\mathbb{C}[G]$ the complex group algebra and $\ell^{2}(G)$ be the group Hilbert space, that is the completion of $\mathbb{C}[G]$. Note that $\mathbb{C}[G]$ acts on $\ell^{2}(G)$ by right multiplication, that means we have an embedding $\rho: \mathbb{C}[G] \rightarrow \mathcal{B}\left(\ell^{2}(G)\right)$. We denote by $\mathcal{N}(G)$ the group von Neumann algebra, that is the weak closure of $\rho(\mathbb{C}[G])$ in $\mathcal{B}\left(\ell^{2}(G)\right)$ and by $\mathrm{rk}_{G}$ the von Neumann rank on $\operatorname{Mat}_{n}(\mathbb{C}[G])$. Let now $\sigma: G \rightarrow \mathrm{GL}_{k}(\mathbb{C})$ be a finite dimensional representation of $G$. We can define a twisting map $\tilde{\sigma}: \mathbb{C}[G] \rightarrow \operatorname{Mat}_{k}(\mathbb{C}[G])$ by

$$
\sum_{g \in G} a_{g} g \mapsto \sum_{g \in G} a_{g} \sigma(g) g
$$

We can extend this map entry wise to matrices over $\mathbb{C}[G]$. We prove the following theorem, that partially answers a question of Wolfgang Lück.

Theorem. Let $G$ be a sofic group and $\sigma: G \rightarrow \mathrm{GL}_{k}(\mathbb{C})$ be a representation of $G$. Then, for all matrices $A \in \operatorname{Mat}_{n \times m}(\mathbb{C}[G])$ we have

$$
k \cdot \operatorname{rk}_{G}(A)=\operatorname{rk}_{G}(\tilde{\sigma}(A))
$$

In the last part of this thesis we talk about convergence of eigenvalue measures related to groups. Let $G$ be a discrete residually finite group and let $A \in$ $\operatorname{Mat}_{n}(\mathbb{C}[G])$. Let $G \unrhd N_{1} \unrhd N_{2} \ldots$ be a chain of normal subgroups of finite index with trivial intersection. Set $G_{i}=G / N_{i}$. Then $A$ acts by right multiplication via reduction modulo $N_{i}$ on $\mathbb{C}\left[G_{i}\right]^{n}$. Since $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{\left|G_{i}\right|}$ as $\mathbb{C}$-vector spaces, this action can be represented by a matrix $A_{i} \in \operatorname{Mat}_{n \cdot\left|G_{i}\right|}(\mathbb{C})$. For every $i$ let now $\lambda_{1}^{(i)}, \ldots, \lambda_{n \cdot\left|G_{i}\right|}^{(i)}$ be the eigenvalues of $A_{i}$ and define

$$
\mu_{i}=\frac{1}{\left|G_{i}\right|} \sum_{k=1}^{n \cdot\left|G_{i}\right|} \delta_{\lambda_{k}^{(i)}}
$$

Where $\delta_{c}$ denotes the Dirac measure at $c \in \mathbb{C}$.
We now can ask the following questions:
(1) Does the limit $\lim _{i \rightarrow \infty} \mu_{i}(\{0\})$ exist?
(2) If the answer to the first question is yes, does the limit depend on the chain $\left\{N_{i}\right\}$ ?
(3) Let $\mathcal{N}(G)$ denote the group von Neumann algebra. We can consider $A$ as an element of the tracial von Neumann algebra $\operatorname{Mat}_{n}(\mathcal{N}(G))$ acting on the Hilbert space $\left(\ell^{2}(G)\right)^{n}$. Therefore we can define the Brown measure $\mu_{A}$ of $A$ and ask: Does the sequence $\mu_{i}$ converge weakly to $\mu_{A}$ ?

We prove the following two theorems.

Theorem. Let $G$ be a finitely generated abelian group. Then the answer to all of the above questions is yes.

Let $G=H_{3}(\mathbb{Z})=\langle a, b\rangle$ with

$$
a=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), b=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)
$$

be the discrete Heisenberg group. For $m \in \mathbb{N}$ we denote by

$$
N_{m}=\left\{\operatorname{Id}_{3}+m \cdot A \mid A \in \operatorname{Mat}_{3}(\mathbb{Z})\right\}
$$

the congruence kernel of $G$ modulo $m$.
Theorem. Let $G=H_{3}(\mathbb{Z})$ be the discrete Heisenberg group and $A=a-b \in$ $\mathbb{Z}[G]$. Let $p$ be an odd prime and set $G_{i}=G / N_{p^{i}}$. With the notation from above we have

$$
\mu_{i}(\{0\})=\frac{p}{p+1}
$$

This theorem shows that in general the answer to question (2) and (3) is no.

## Introduction and Conclusions

This thesis discusses some topics from the intersection of group theory, the theory of operator algebras and functional analysis. If $G$ is a discrete group we will consider the Hilbert space $\ell^{2}(G)$. On this Hilbert space the elements $a \in$ $\mathbb{C}[G]$ act as bounded operators, by left and by right multiplication. We want to investigate some spectral properties of these operators, especially approximation of these operators by matrices.

Let us describe more explicitly what we mean with approximation of operators by matrices.

Let $F$ be a finitely generated free group and let $X$ be a finite $F$-set, where $F$ acts on the right. Let $\mathbb{C}[F]$ be the group algebra with complex coefficients and $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$ be a matrix over $\mathbb{C}[F]$. Let us denote by $A_{X}: \mathbb{C}[X]^{n} \rightarrow \mathbb{C}[X]^{n}$ the linear operator induced by right multiplication with $A$ and for each $\lambda \in \mathbb{C}$ let $m_{X}^{A}(\lambda)$ be the multiplicity with which $\lambda$ appears as a root of the characteristic polynomial of $A_{X}$. The Brown measure of $A_{X}$ is given by

$$
\mu_{X}^{A}=\sum_{\lambda \in \mathbb{C}} \frac{m_{X}^{A}(\lambda)}{|X|} \delta_{\lambda}
$$

where $\delta_{\lambda}$ is the Dirac measure concentrated at $\lambda$. If the matrix $A$ is normal the measure $\mu_{X}^{A}$ is known as the spectral measure of $A_{X}$. In this thesis we want to investigate the measures $\mu_{X}^{A}$. The idea is that a sequence of finite $F$-sets can approximate some infinite algebraic structure, for example a sofic group. Let $N \unlhd F$ be a normal subgroup, $G=F / N$ be a sofic group and $\left\{X_{i}\right\}$ be a sofic approximation for $G$. We introduce the notion of sofic groups in the beginning of Chapter 3. In particular, amenable and residually finite groups are sofic. The easiest example for a sofic approximation is the following. Let $G$ be residually finite and let $\left(H_{i}\right), H_{i} \unlhd G$ be a chain of normal subgroups of finite index in $G$ with trivial intersection. We can define $X_{i}=G / H_{i}$, where $F$ acts naturally on $X_{i}$. Then $\left(X_{i}\right)$ is a sofic approximation for $G$. Given now a matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$ we obtain a sequence of measures $\mu_{X_{i}}^{A}$. We can then ask the following questions.
(1) Does the limit $\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})$ exist?
(2) If the answer to question (1) is yes, is the limit independent from the sofic approximation $\left(X_{i}\right)_{i}$ ?
(3) Let $\mu_{G}^{A}$ be the Brown measure of the operator $A_{G}$ on $\left(\ell^{2}(G)\right)^{n}$ given by right multiplication by $A$. Do the measures $\mu_{X_{i}}^{A}$ converge weakly to $\mu_{G}^{A}$ ?

First, we want to discuss the case when the matrix $A$ is normal. In this case we only work with spectral measures and the answer to all of the above question is yes. In [Kaz75] David Kazhdan discovered that the measures $\mu_{X_{i}}^{A}$ converge weakly to $\mu_{G}^{A}$. Thus the answer to question (3) is yes and to answer question (1) and (2) one only needs to check that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})=\mu_{G}^{A}(\{0\}) \tag{1}
\end{equation*}
$$

The theorem of Portmanteau already yields

$$
\limsup _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\}) \leq \mu_{G}^{A}(\{0\})
$$

This is often called Kazhdan's inequality. Later, in [Lüc94] Wolfgang Lück proved the other inequality for positive self adjoint matrices $A \in \operatorname{Mat}_{n}(\mathbb{Q}[F])$. Let us briefly describe his approach. Lück showed that the point $0 \in \mathbb{C}$ is not an accumulation point of eigenvalues of the operators $A_{X_{i}}$, or, in different words that the measures $\mu_{X_{i}}^{A}$ have small concentration close to 0 . Precisely he showed that there exists a constant $C=C(A) \in \mathbb{R}_{>0}$ that only depends on $A$ such that for every $\epsilon \in(0,1)$ and any finite $F$-set $X$ we have

$$
\begin{equation*}
\mu_{X}^{A}((0, \epsilon)) \leq \frac{C}{-\log \epsilon} \tag{2}
\end{equation*}
$$

An easy application of the theorem of Portmanteau then gives

$$
\liminf _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\}) \geq \mu_{G}^{A}(\{0\})
$$

Together with the inequality of Kazhdan this gives the equality in 1. Only minor changes in his proof are necessary to include not only self adjoint but also normal matrices. Lück's proof heavily relies on the fact that the coefficients in the matrix $A$ lie in $\mathbb{Q}$. It should be mentioned that Lück only considered the case of residually finite groups as in the previous example for sofic approximations. Using the same methods, in [Sch01] Thomas Schick extended this result to some larger class of groups and in [Dod+01] this result was extended by J. Dodziuk, P. Linnell, V. Mathai, T. Schick and S. Yates to matrices over $\overline{\mathbb{Q}}[F]$. In [ES05] G. Elek and E. Szabó considered the general case of sofic groups. However, to answer question (1) and (2) for an arbitrary matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$ new methods were necessary. In [Jai19] Andrei Jaikin-Zapirain used an algebraic approach, based on the notion of Sylvester matrix rank functions, to prove equation 1 for normal matrices $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$. Examples for Sylvester matrix rank functions are the von Neumann rank $\mathrm{rk}_{G}$ defined by

$$
\operatorname{rk}_{G} A=n-\operatorname{dim}_{G} \operatorname{ker} A_{G}
$$

where $\operatorname{dim}_{G}$ is the von Neumann dimension and the function

$$
\mathrm{rk}_{X}(A)=\frac{1}{|X|}=\mathrm{rk}_{\mathbb{C}} A_{X}
$$

for a finite $F$-set $X$. However Jaikin's method does not provide a bound as in 2 .
In this thesis we will prove that a bound for $\mu_{X_{i}}^{A}((0, \epsilon))$ as in 2 also exists in the case of matrices over $\mathbb{C}[F]$ and even in a more general setting. However our proof will not give an explicit bound but only its existence. Similar to the definition of the Brown measure, for any matrix $C \in \operatorname{Mat}_{k}(\mathbb{C})$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{k}$ we denote the eigenvalue measure of $C$ by

$$
\mu_{C}=\sum_{j=0}^{k} \delta_{\lambda_{j}}
$$

Our first main result will be the following.
Theorem A. Let $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ be a free $*$-algebra, $A \in \operatorname{Mat}_{m}(A)$ be normal and $c \in \mathbb{R}_{>0}$. Then there is a function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ with $\lim _{\lambda \rightarrow 0^{+}} f(\lambda)=$ 0 such that for every $*$-homomorphism $\varphi: \mathcal{A} \rightarrow \operatorname{Mat}_{n}(\mathbb{C})$ with $\varphi\left(x_{i}\right) \in \operatorname{Mat}_{n}(\mathbb{Z})$ and $\left\|\varphi\left(x_{i}\right)\right\|_{1},\left\|\varphi\left(x_{i}\right)\right\|_{\infty} \leq c$ for all $i$ and every $y \in \mathbb{C}$ we have

$$
\frac{1}{n} \mu_{\varphi(A)}(B(y, \lambda) \backslash\{y\})<f(\lambda)
$$

Note that Lück's result covers the case where for each $i \in\{1, \ldots, d\}$ the image $\varphi\left(x_{i}\right)$ is a permutation matrix and $y=0$. There is a relation between the spectral measure $\mu_{G}^{A}$ and the von Neumann dimension $\operatorname{dim}_{G}$ given by

$$
\mu_{G}^{A}(\{0\})=\operatorname{dim}_{G} \operatorname{ker} A_{G}
$$

Thus as a corollary we obtain:
Corollary A. [The sofic Lück approximation] Let $\left\{X_{i}\right\}$ be a sofic approximation of $G$ and let $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$. Then

$$
\lim _{i \rightarrow \infty} \frac{\operatorname{dim} \operatorname{ker} A_{X_{i}}}{\left|X_{i}\right|}=\operatorname{dim}_{G} \operatorname{ker} A_{G}
$$

To prove Theorem A we use methods similar to those used by Jaikin in [Jai19]. Jaikin considered a sofic approximation $\left\{X_{i}\right\}$ of some sofic group $G$ and showed that

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{X_{i}}=\mathrm{rk}_{G}
$$

as Sylvester matrix rank functions on $\mathbb{C}[F]$. In particular he considered $A_{G}$ to be an operator on the Hilbert space $\ell^{2}(G)$. We will use a more general approach that was already considered by Steffen Kionke in [Kio18] for group algebras $\mathbb{Q}[G]$. For every trace $\tau$ on $\mathcal{A}$ the GNS construction gives us some Hilbert space $\mathcal{H}_{\tau}$ on which $\mathcal{A}$ acts as bounded operators. This Hilbert space allows us to
define a rank function $\mathrm{rk}_{\tau}$ on $\mathcal{A}$. Having a point wise converging sequence of traces $\tau_{i}$ their limit $\tau=\lim _{i \rightarrow \infty} \tau_{i}$ is also a trace. We will consider the question if

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{\tau_{i}}=\mathrm{rk}_{\tau}
$$

holds.
In Chapter 3 we want to discuss an application of Corollary A. Although the rank function $\mathrm{rk}_{G}$ is defined in an analytic way, in some cases eg. if $G$ is amenable or locally indicable, it can be characterized algebraically. Therefore it is plausible to think that $\mathrm{rk}_{G}$ is rigid under some algebraic manipulations. For example, in [Jai19] Andrei Jaikin raises the following conjecture.

Conjecture 1 (The independence conjecture). Let $G$ be a group. Let $K$ be a field and let $\phi_{1}, \phi_{2}: K \rightarrow \mathbb{C}$ be two embeddings of $K$ into $\mathbb{C}$. Then for every matrix $A \in \operatorname{Mat}_{n \times m}(K[G])$

$$
\operatorname{rk}_{G}\left(\phi_{1}(A)\right)=\operatorname{rk}_{G}\left(\phi_{2}(A)\right)
$$

This conjecture was proved for sofic groups in [Jai19] and for locally indicable groups in [JL20]. We want to consider a different algebraic manipulation. For that let $\sigma: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ be a finite dimensional representation of $G$. We can define an algebra homomorphism

$$
\tilde{\sigma}: \mathbb{C}[G] \rightarrow \operatorname{Mat}_{n}(\mathbb{C}[G]), \sum_{g \in G} a_{g} g \mapsto \sum_{g \in G} a_{g} \sigma(g) g
$$

Obviously we can extend $\tilde{\sigma}$ entry wise to matrices over $\mathbb{C}[G]$.
The following conjecture is a rephrasing of a question raised by Lück in [Lüc18, Question 0.1].

Conjecture 2 (The Lück twisted conjecture). Let $G$ be a group and $\sigma: G \rightarrow$ $\mathrm{GL}_{k}(\mathbb{C})$ a homomorphism. Then for every matrix $A \in \operatorname{Mat}_{n \times m}(\mathbb{C}[G])$

$$
\operatorname{rk}_{G}(\tilde{\sigma}(A))=k \cdot \operatorname{rk}_{G}(A)
$$

Lück noticed that twisted representations appear when calculating $\ell^{2}$-Betti numbers of some fibrations of connected finite $C W$-complexes $F \rightarrow E \rightarrow B$ where $\pi_{1}(B) \cong G$ and the map $\pi_{1}(E) \rightarrow \pi_{1}(B)$ induced by the fibration is an isomorphism. We will talk about this in section 3.2.

Conjecture 2 was proved for torsion-free elementary amenable groups in [Lüc18] and for locally indicable groups in [KS21]. In our second main result we prove the conjecture for sofic groups.

Theorem B. The Lück twisted conjecture holds for sofic groups.
The main tool of our proof of Theorem B is the sofic Lück approximation proved in [Jai19], see Corollary A. Notice that the sofic case of the independence conjecture follows immediately from the sofic Lück approximation. However, in the case of the Lück twisted conjecture, this implication is not so direct.

In the last chapter we want to come back to our three questions from the beginning, but this time we want to consider the case when the matrix $A$ is not normal. We will only work in the case of residually finite approximations. Thus we can assume that $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ for some residually finite group $G$ and not $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$.

The problem of convergence of eigenvalues of non normal matrices has already been studied in the theory of random matrices, see [Sni02].

Since for matrices over $\mathbb{Q}[G]$ the methods used by Lück in [Lüc94] still work in the case of non normal matrices, a positive answer to question (3) would imply a positive answer to question (1) and (2).

We present an example that gives hope to a positive answer to question (3) also in the case of non normal matrices, at least for some groups. Let $G=\langle g\rangle$ be an infinite cyclic group and let

$$
A=\left(\begin{array}{cc}
g^{2}+3 g & 4 \\
g^{3} & -g^{4}+g
\end{array}\right)
$$

Let $H_{i}=\left\langle g^{5^{i}}\right\rangle \subseteq G$ and therefore $X_{i}=\mathbb{Z} /\left(5^{i}\right) \mathbb{Z}$ a cyclic group of order $5^{i}$. Note that we get the matrix $A_{X_{i}}$ by replacing $g$ in $A$ by the matrix

$$
\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & 0 \\
0 & & & \ddots & 1 \\
1 & 0 & \cdots & & 0
\end{array}\right)
$$

of dimension $\left|X_{i}\right|=5^{i}$. The following plot shows the eigenvalues of $A_{X_{i}}$ for $i=1,2,3$ :


In the graphics, the limit curve is the support of the Brown measure of the operator $A_{G}$. This will be a consequence of the next theorem.

Theorem C. Let $G$ be a finitely generated abelian group and let $G \unrhd H_{1} \unrhd H_{2} \ldots$ be a chain of normal subgroups of finite index with trivial intersection and set $X_{i}=G / H_{i}$. Let $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ and let $A_{X_{i}} \in \operatorname{Mat}_{n\left|X_{i}\right|}(\mathbb{C})$ be the matrix that represents the action of $A$ on $\mathbb{C}\left[X_{i}\right]^{n}$. Then the measures $\mu_{X_{i}}^{A}$ converge weakly and pointwise towards $\mu_{G}^{A}$.

In general, we obtain a negative answer to question (2) and therefore to question (3). For that we consider the discrete Heisenberg group $G=H_{3}(\mathbb{Z})$, which can be seen as the matrix group generated by the two matrices

$$
a=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad b=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)
$$

Our main result will be the following.
Theorem D. Let $G=H_{3}(\mathbb{Z})$ be the Heisenberg group and let $a-b \in \mathbb{Z}[G]$. Let $H_{i}=\operatorname{Id}_{3}+p^{i} \cdot \operatorname{Mat}_{3}(\mathbb{Z}) \cap G \unlhd G$ and consider the residual chain $G \unrhd H_{1} \unrhd$ $H_{2} \unrhd \ldots$ Set $X_{i}=G / H_{i} \cong H_{3}\left(\mathbb{Z} / p^{i} \mathbb{Z}\right)$. Let $A_{X_{i}} \in \operatorname{Mat} p_{p^{3 i}}(\mathbb{Z})$ be the matrix that represents the action of $a-b$ on $\mathbb{C}\left[X_{i}\right] \cong \mathbb{C}^{p^{3 i}}$. Then

$$
\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})=\frac{p}{p+1}
$$

In particular this limit depends on $p$ and therefore on the approximation $\left\{X_{i}\right\}$.

The structure of the thesis is the following. In Chapter 1 we present the necessary prerequisites for the thesis. In Chapter 2 we will present a proof of Theorem A. Especially here we need the results and notation from Chapter 1. Next, in Chapter 3, we explain our results about twisted $\ell^{2}$-Betti numbers. We will prove Theorem B in the first part of the chapter. In the second part we will present the application to fibrations we mentioned before. This chapter mainly needs Corollary A as a prerequisite. Last, in Chapter 4 we discuss our three questions from the beginning for non-normal matrices $A$. This chapter is almost self contained.

## Introducción y Conclusiones

Esta tesis discute algunos temas de la intersección de la teoría de grupos, la teoría de álgebras de operadores y el análisis funcional. Si $G$ es un grupo discreto, consideraremos el espacio de Hilbert $\ell^{2}(G)$. En este espacio de Hilbert, los elementos $a \in \mathbb{C}[G]$ actúan como operadores acotados, por multiplicación por la izquierda y por la derecha. Queremos investigar algunas propiedades espectrales de estos operadores, especialmente la aproximación de estos operadores por matrices.

Describamos más explícitamente lo que queremos decir con la aproximación de operadores por matrices.

Sea $F$ un grupo libre finitamente generado y sea $X$ un conjunto finito $F$ invariante, donde $F$ actúa por la derecha. Sea $\mathbb{C}[F]$ el álgebra del grupo con coeficientes complejos y sea $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$ una matriz sobre $\mathbb{C}[F]$. Denotamos por $A_{X}: \mathbb{C}[X]^{n} \rightarrow \mathbb{C}[X]^{n}$ el operador lineal inducido por la multiplicación por la derecha de $A$ y, para cada $\lambda \in \mathbb{C}$, sea $m_{X}^{A}(\lambda)$ la multiplicidad con la que $\lambda$ aparece como raíz del polinomio característico de $A_{X}$. La medida de Brown de $A_{X}$ está dada por

$$
\mu_{X}^{A}=\sum_{\lambda \in \mathbb{C}} \frac{m_{X}^{A}(\lambda)}{|X|} \delta_{\lambda}
$$

donde $\delta_{\lambda}$ es la medida de Dirac concentrada en $\lambda$. Si la matriz $A$ es normal, la medida $\mu_{X}^{A}$ se conoce como la medida espectral de $A_{X}$. En esta tesis queremos investigar las medidas $\mu_{X}^{A}$. La idea es que una secuencia de conjuntos finitos $F$ invariantes puede aproximar alguna estructura algebraica infinita, por ejemplo, un grupo sofico. Sea $N \unlhd F$ un subgrupo normal, $G=F / N$ un grupo sofico y $X_{i}$ una aproximación sofica para $G$. Introducimos la noción de grupos soficos al comienzo del Capítulo 3. En particular, los grupos amenables y residual finitos son soficos. El ejemplo más sencillo de una aproximación sofica es el siguiente. Sea $G$ residual finito y sea $\left(H_{i}\right), H_{i} \unlhd G$ una cadena de subgrupos normales de índice finito en $G$ con intersección trivial. Podemos definir $X_{i}=G / H_{i}$, donde $F$ actúa naturalmente en $X_{i}$. Entonces, $\left(X_{i}\right)$ es una aproximación sofica para $G$. Dada ahora una matriz $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$, obtenemos una secuencia de medidas $\mu_{X_{i}}^{A}$. Podemos hacer las siguientes preguntas:
(1) ¿Existe el límite $\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})$ ?
(2) Si la respuesta a la pregunta (1) es sí, ¿es el límite independiente de la aproximación sofica $\left(X_{i}\right)_{i}$ ?
(3) Sea $\mu_{G}^{A}$ la medida de Brown del operador $A_{G}$ en $\left(\ell^{2}(G)\right)^{n}$ dado por la multiplicación por la derecha por $A$. ¿Las medidas $\mu_{X_{i}}^{A}$ convergen débilmente a $\mu_{G}^{A}$ ?

En primer lugar, queremos discutir el caso en que la matriz $A$ es normal. En este caso, solo trabajamos con medidas espectrales y la respuesta a todas las preguntas anteriores es sí. En [Kaz75], David Kazhdan descubrió que las medidas $\mu_{X_{i}}^{A}$ convergen débilmente a $\mu_{G}^{A}$. Por lo tanto, la respuesta a la pregunta (3) es sí y para responder a la pregunta (1) y (2) solo es necesario verificar que

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})=\mu_{G}^{A}(\{0\}) \tag{3}
\end{equation*}
$$

El teorema de Portmanteau ya nos da

$$
\limsup _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\}) \leq \mu_{G}^{A}(\{0\})
$$

Esto a menudo se llama la desigualdad de Kazhdan. Más tarde, en [Lüc94], Wolfgang Lück demostró la otra desigualdad para matrices autoadjuntas y positivas $A \in \operatorname{Mat}_{n}(\mathbb{Q}[F])$. Veamos brevemente su enfoque. Lück mostró que el punto $0 \in \mathbb{C}$ no es un punto de acumulación de los autovalores de los operadores $A_{X_{i}} \mathrm{o}$, en otras palabras, que las medidas $\mu_{X_{i}}^{A}$ tienen una concentración pequeña cerca de 0 . Precisamente, demostró que existe una constante $C=C(A) \in \mathbb{R}_{>0}$ que solo depende de $A$, tal que para cada $\epsilon \in(0,1)$ y cualquier conjunto finito $F$-invariante $X$ tenemos:

$$
\begin{equation*}
\mu_{X}^{A}((0, \epsilon)) \leq \frac{C}{-\log \epsilon} \tag{4}
\end{equation*}
$$

Una fácil aplicación del teorema de Portmanteau da como resultado:

$$
\liminf _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\}) \geq \mu_{G}^{A}(\{0\})
$$

Junto con la desigualdad de Kazhdan, esto nos da la igualdad en (1). Solo se necesitan cambios menores en su prueba para incluir no solo matrices autoadjuntas sino también normales. La prueba de Lück depende en gran medida del hecho de que los coeficientes en la matriz $A$ estén en $\mathbb{Q}$. Cabe mencionar que Lück solo consideró el caso de grupos residualmente finitos como en el ejemplo anterior para aproximaciones soficas. Utilizando los mismos métodos, en [Sch01] Thomas Schick extendió este resultado a una clase más grande de grupos y en [Dod+01] este resultado fue extendido por J. Dodziuk, P. Linnell, V. Mathai, T. Schick y S. Yates a matrices sobre $\overline{\mathbb{Q}}[F]$. En [ES05] G. Elek y E. Szabó consideraron el caso general de grupos soficos. Sin embargo, para responder a las preguntas (1) y (2) para una matriz arbitraria $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$, se necesitaron nuevos métodos. En [Jai19] Andrei Jaikin-Zapirain utilizó un enfoque
algebraico, basado en la noción de funciones de rango de matriz de Sylvester, para probar la ecuación (1) para matrices normales $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$. Ejemplos de funciones de rango de matriz de Sylvester son el rango de von Neumann $\mathrm{rk}_{G}$ definido por

$$
\operatorname{rk}_{G} A=n-\operatorname{dim}_{G} \operatorname{ker} A_{G}
$$

donde $\operatorname{dim}_{G}$ es la dimensión de von Neumann y la función

$$
\operatorname{rk}_{X}(A)=\frac{1}{|X|} \operatorname{rk}_{\mathbb{C}} A_{X}
$$

para un conjunto finito $F$-set $X$. Sin embargo, el método de Jaikin no proporciona una cota como en (2).

En esta tesis, demostraremos que existe una cota para $\mu_{X_{i}}^{A}((0, \epsilon))$ como en 2 también en el caso de matrices sobre $\mathbb{C}[F]$ e incluso en un contexto más general. Sin embargo, nuestra prueba no dará una cota explícita sino solo su existencia. Similar a la definición de la medida de Brown, para cualquier matriz $C \in \operatorname{Mat}_{k}(\mathbb{C})$ con valores propios $\lambda_{1}, \ldots, \lambda_{k}$, denotamos la medida de valores propios de $C$ por:

$$
\mu_{C}=\sum_{j=0}^{k} \delta_{\lambda_{j}} .
$$

Nuestro primer resultado principal será el siguiente.
Theorem A. Sea $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ una $*$-álgebra libre, $A \in \operatorname{Matm}(A)$ normal $y c \in \mathbb{R}>0$. Entonces hay una función $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ con $\lim _{\lambda \rightarrow 0^{+}} f(\lambda)=$ 0 tal que para cada $*$-homomorfismo $\varphi: \mathcal{A} \rightarrow \operatorname{Mat}_{n}(\mathbb{C}) \operatorname{con} \varphi\left(x_{i}\right) \in \operatorname{Mat}_{n}(\mathbb{Z}) y$ $\left|\varphi\left(x_{i}\right)\right|_{1},\left|\varphi\left(x_{i}\right)\right|_{\infty} \leq c$ para todo $i$ y cada $y \in \mathbb{C}$ se tiene que

$$
\frac{1}{n} \mu_{\varphi(A)}(B(y, \lambda) \backslash\{y\})<f(\lambda)
$$

Nótese que el resultado de Lück cubre el caso donde para cada $i \in 1, \ldots, d$ la imagen $\varphi\left(x_{i}\right)$ es una matriz de permutación y $y=0$. Hay una relación entre la medida espectral $\mu_{G}^{A}$ y la dimensión de von Neumann $\operatorname{dim}_{G}$ dada por

$$
\mu_{G}^{A}(\{0\})=\operatorname{dim}_{G} \operatorname{ker} A_{G} .
$$

Por lo tanto, como corolario obtenemos:
Corollary A (La aproximación sofica de Lück). Sea $X_{i}$ una aproximación sofica de $G$ y sea $A \in \operatorname{Matn}(\mathbb{C}[F])$. Entonces,

$$
\lim _{i \rightarrow \infty} \frac{\operatorname{dim} \operatorname{ker} A_{X_{i}}}{\left|X_{i}\right|}=\operatorname{dim}_{G} \operatorname{ker} A_{G}
$$

Para demostrar el Teorema A utilizamos métodos similares a los utilizados por Jaikin en [Jai19]. Jaikin consideró una aproximación sofica $X_{i}$ de algún grupo sofico $G$ y mostró que

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{X_{i}}=\mathrm{rk}_{G}
$$

como funciones de rango de matriz de Sylvester en $\mathbb{C}[F]$. En particular, consideró que $A_{G}$ era un operador en el espacio de Hilbert $\ell^{2}(G)$. Usaremos un enfoque más general que ya fue considerado por Steffen Kionke en [Kio18] para álgebras de grupo $\mathbb{Q}[G]$. Para cada traza $\tau$ en $\mathcal{A}$ la construcción GNS nos da algún espacio de Hilbert $\mathcal{H}_{\tau}$ en el que $\mathcal{A}$ actúa como operadores acotados. Este espacio de Hilbert nos permite definir una función de $\operatorname{rango} \mathrm{rk}_{\tau}$ en $\mathcal{A}$. Teniendo una secuencia convergente puntualmente de trazas $\tau_{i}$, su límite $\tau=\lim _{i \rightarrow \infty} \tau_{i}$ también es una traza. Consideraremos la pregunta de si se cumple que

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{\tau_{i}}=\mathrm{rk}_{\tau}
$$

En el Capítulo 3 queremos discutir una aplicación del Corolario A. Aunque la función de rango $\mathrm{rk}_{G}$ está definida de manera analítica, en algunos casos, por ejemplo, si $G$ es amenable o localmente indicable, se puede caracterizar algebraicamente. Por lo tanto, es plausible pensar que $\mathrm{rk}_{G}$ es rígido bajo algunas manipulaciones algebraicas. Por ejemplo, en [Jai19], Andrei Jaikin plantea la siguiente conjetura.

Conjecture 1 (La conjetura de independencia). Sea $G$ un grupo. Sea $K$ un cuerpo y sean $\phi_{1}, \phi_{2}: K \rightarrow \mathbb{C}$ dos encajes de $K$ en $\mathbb{C}$. Entonces, para toda matriz $A \in \operatorname{Mat}_{n \times m}(K[G])$ se cumple que

$$
\operatorname{rk}_{G}\left(\phi_{1}(A)\right)=\operatorname{rk}_{G}\left(\phi_{2}(A)\right)
$$

Esta conjetura se demostró para grupos soficos en [Jai19] y para grupos localmente indicables en [JL20]. Queremos considerar una manipulación algebraica diferente. Para ello, sea $\sigma: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ una representación de dimensión finita de $G$. Podemos definir un homomorfismo de álgebras de la siguiente manera:

$$
\tilde{\sigma}: \mathbb{C}[G] \rightarrow \operatorname{Mat}_{n}(\mathbb{C}[G]), \sum_{g \in G} a_{g} g \mapsto \sum_{g \in G} a_{g} \sigma(g) g
$$

Obviamente, podemos extender $\tilde{\sigma}$ entrada por entrada a matrices sobre $\mathbb{C}[G]$. La siguiente conjetura es una reformulación de una pregunta planteada por Lück en [Lüc18, Question 0.1].

Conjecture 2 (La conjetura torcida de Lück). Sea $G$ un grupo y $\sigma: G \rightarrow$ $\mathrm{GL}_{k}(\mathbb{C})$ un homomorfismo. Entonces, para toda matriz $A \in \operatorname{Mat}_{n \times m}(\mathbb{C}[G])$ se cumple que

$$
\operatorname{rk}_{G}(\tilde{\sigma}(A))=k \cdot \operatorname{rk}_{G}(A)
$$

Lück notó que las representaciones torcidas aparecen al calcular los números $\ell^{2}$ de Betti de algunas fibraciones de $C W$-complejos finitos conectados $F \rightarrow$ $E \rightarrow B$ donde $\pi_{1}(B) \cong G$ y aplicación $\pi_{1}(E) \rightarrow \pi_{1}(B)$ inducido por la fibración es un isomorfismo. Hablaremos de esto en la sección 3.2.

La Conjetura 2 fue demostrada para grupos elementalmente amenables sin torsión en [Lüc18] y para grupos localmente indicables en [KS21]. En nuestro segundo resultado principal, probamos la conjetura para grupos soficos.

Theorem B. La Conjetura de Lück torcida es verdadera para grupos sofic.
La principal herramienta de nuestra prueba del Teorema B es la aproximación sofica de Lück demostrada en [Jai19], consulte el Corolario A. Observe que en el caso sofico, la Conjetura de independencia sigue inmediatamente de la aproximación sofica de Lück. Sin embargo, en el caso de la Conjetura torcida de Lück, esta implicación no es tan directa.

En el último capítulo queremos volver a nuestras tres preguntas del principio, pero esta vez queremos considerar el caso en que la matriz $A$ no es normal. Solo trabajaremos en el caso de aproximaciones residualmente finitas. Por lo tanto, podemos suponer que $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ para algún grupo residualmente finito $G$ y no $A \in \operatorname{Mat}_{n}(\mathbb{C}[F])$.

El problema de la convergencia de los autovalores de las matrices no normales ya ha sido estudiado en la teoría de las matrices aleatorias, vea [Sni02].

Ya que para matrices sobre $\mathbb{Q}[G]$ los métodos utilizados por Lück en [Lüc94] aún funcionan en el caso de matrices no normales, una respuesta positiva a la pregunta (3) implicaría una respuesta positiva a las preguntas (1) y (2).

Presentamos un ejemplo que da esperanza a una respuesta positiva a la pregunta (3) también en el caso de matrices no normales, al menos para algunos grupos. Sea $G=\langle g\rangle$ un grupo cíclico infinito y sea

$$
A=\left(\begin{array}{cc}
g^{2}+3 g & 4 \\
g^{3} & -g^{4}+g
\end{array}\right)
$$

Sea $H_{i}=\left\langle g^{5^{i}}\right\rangle \subseteq G$ y, por lo tanto, $X_{i}=\mathbb{Z} /\left(5^{i}\right) \mathbb{Z}$ un grupo cíclico de orden $5^{i}$. Nótese que obtenemos la matriz $A_{X_{i}}$ reemplazando $g$ en $A$ por la matriz

$$
\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & 0 \\
0 & & & \ddots & 1 \\
1 & 0 & \cdots & & 0
\end{array}\right)
$$

de dimensión $\left|X_{i}\right|=5^{i}$. El siguiente gráfico muestra los autovalores de $A_{X_{i}}$ para $i=1,2,3$ :


En las gráficas, la curva límite es el soporte de la medida de Brown del operador $A_{G}$. Esto será una consecuencia del siguiente teorema.

Theorem E. Sea $G$ un grupo abeliano finitamente generado y sea $G \unrhd H_{1} \unrhd$ $H_{2} \cdots$ una cadena de subgrupos normales de índice finito con intersección trivial, y sea $X_{i}=G / H_{i}$. Sea $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ y sea $A_{X_{i}} \in \operatorname{Mat}_{n\left|X_{i}\right|}(\mathbb{C})$ la matriz que representa la acción de $A$ en $\mathbb{C}\left[X_{i}\right]^{n}$. Entonces, las medidas $\mu_{X_{i}}^{A}$ convergen débilmente y puntualmente hacia $\mu_{G}^{A}$.

En general, obtenemos una respuesta negativa a la pregunta (2) y, por lo tanto, a la pregunta (3). Para ello, consideramos el grupo de Heisenberg discreto $G=H_{3}(\mathbb{Z})$, que se puede ver como el grupo de matrices generado por las dos matrices

$$
a=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \quad y \quad b=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)
$$

Nuestro resultado principal será el siguiente:
Theorem F. Sea $G=H_{3}(\mathbb{Z})$ el grupo de Heisenberg y sea $a-b \in \mathbb{Z}[G]$. Sea $H_{i}=\operatorname{Id}_{3}+p^{i} \cdot \operatorname{Mat}_{3}(\mathbb{Z}) \cap G \unlhd G$ y consideremos la cadena residual $G \unrhd H_{1} \unrhd$ $H_{2} \unrhd \ldots$. Definimos $X_{i}=G / H_{i} \cong H_{3}\left(\mathbb{Z} / p^{i} \mathbb{Z}\right)$. Sea $A_{X_{i}} \in \operatorname{Mat}_{p^{3 i}}(\mathbb{Z})$ la matriz que representa la acción de $a-b$ en $\mathbb{C}\left[X_{i}\right] \cong \mathbb{C}^{p^{3 i}}$. Entonces

$$
\lim _{i \rightarrow \infty} \mu_{X_{i}}^{A}(\{0\})=\frac{p}{p+1}
$$

En particular, este límite depende de $p$ y, por lo tanto, de la aproximación $X_{i}$.

La estructura de la tesis es la siguiente. En el Capítulo 1 presentamos los conocimientos previos necesarios para la tesis. En el Capítulo 2 presentamos una demostración del Teorema A. En particular, aquí se necesitan los resultados y la notación del Capítulo 1. A continuación, en el Capítulo 3, explicamos nuestros resultados sobre números $\ell^{2}$ de Betti torcidos. Demostramos el Teorema B en la primera parte del capítulo. En la segunda parte presentamos la aplicación a las fibraciones que mencionamos antes. Este capítulo necesita principalmente el Corolario A. Por último, en el Capítulo 4 discutimos nuestras tres preguntas iniciales para matrices $A$ no normales. Este capítulo es casi autocontenido.

## Chapter 1

## Preliminaries

In this chapter we give the preliminaries for later results. We will talk about necessary results from operator algebras, Sylvester rank functions, model theory and measure theory. We will start with fixing some notation.

### 1.1 Notation

We use the following notation and conventions.

| Rings | $R, S, Q$ |
| :--- | :--- |
| von Neumann algebras | $\mathcal{N}$ |
| Algebras | $\mathcal{A}$ |
| Groups | $G, H$ |
| morphisms | $\alpha, \beta, f$ |
| Hilbert space | $\mathcal{H}$ |
| von Neumann regular ring, *-regular ring | $\mathcal{U}$ |
| von Neumann algebra, algebra of unbound operators, |  |
| Hilbert space coming from GNS construction with respect to $\tau$ | $\mathcal{N}_{\tau}, \mathcal{U}_{\tau}, \mathcal{H}_{\tau}$ |

For matrices $A, B$ over a ring $R$ we mean by $A \oplus B$ the matrix $\left(\begin{array}{cc}A & 0 \\ 0 & B\end{array}\right)$. Linear operators of Hilbert spaces will act on the right. For a ring $R$ and $a \in R$ we denote by $\operatorname{Ann}_{r}(a)=\{b \in R \mid b a=0\}$ the right Annihilator of $a$. All rings will be unitary.

### 1.2 Operator Algebras

In this section we want to present the basic concepts and definitions of algebras of operators. We will not present any new results here. The road map is the following. We will start with the notion of tracial $*$-algebras in Section 1.2.1. We will then describe the GNS construction to show how to represent tracial
*-algebras in a canonical way as operators on some Hilbert space. Having this we will use von Neumanns bicommutant theorem to get a tracial von Neumann algebra that contains our initial *-algebra. In this setting we will talk in Section 1.2.2 about the spectral theorem of normal operators and about spectral measures. The spectral measures with the trace allows us to develop a dimension theory in 1.2 .3 . Last we will introduce an even larger algebra, the algebra of unbounded operators affiliated to a von Neumann algebra in Section 1.2.4. The advantage of this algebra is that it is von Neumann regular.

### 1.2.1 *-Algebras

Definition 1.2.1. A $*$-ring is a ring $R$ together with an involution $*: R \rightarrow R$ that means

- $(x+y)^{*}=x^{*}+y^{*}$
- $(x y)^{*}=y^{*} x^{*}$
- $1^{*}=1$
- $\left(x^{*}\right)^{*}=x$
for all $x, y \in R$.
For us $K$ will always be a field. When we talk about algebras, we always mean $K$-algebras. If $K$ is a $*$-ring, for example $\mathbb{C}$ with complex conjugation, then a *-algebra is an algebra $\mathcal{A}$ with an involution that satisfies $(\lambda \cdot a)^{*}=\lambda^{*} \cdot a^{*}$, where $a \in \mathcal{A}, \lambda \in K$.

Definition 1.2.2. A tracial $*$-algebra is a tuple $(\mathcal{A}, \tau)$, where $A$ is a $*$-algebra over a subfield $K \subseteq \mathbb{C}$ closed under complex conjugation and $\tau: \mathcal{A} \rightarrow K$ is a positive linear functional with the trace property, that means

- $\tau(a b)=\tau(b a)$
- $\tau(1)=1$
- $\tau\left(a^{*} a\right) \geq 0$.

We will call $\tau$ just a trace. The trace $\tau$ is called faithful if

$$
\tau\left(a a^{*}\right)=0 \Rightarrow a=0
$$

for all $a \in \mathcal{A}$. For us there will be two main examples of tracial $*$-algebras. The first one is the matrix algebra $\mathcal{A}=\operatorname{Mat}_{n}(K)$. The involution $(\cdot)^{*}$ is given by taking the adjoint matrix (transpose and complex conjugate) and $\tau=\frac{1}{n} \operatorname{Tr}$ is just the normalized trace. The second main example is $\mathcal{A}=K[G]$, the group algebra of a group $G$ over $K$. Let us just recall what the group ring is. The group ring $R[G]$ over a ring $R$ consists of all finite formal sums $\sum_{g \in G} r_{g} g$ with $r_{g} \in R$. The addition of two elements $a=\sum_{g \in G} r_{g} g$ and $b=\sum_{g \in G} s_{g} g$ is given by
$a+b=\sum_{g \in G}\left(r_{g}+s_{g}\right) g$. The product of $a$ and $b$ is $a \cdot b=\sum_{g, h \in G} r_{g} s_{h} g h$. If $R=K$ is a field we call $K[G]$ a group algebra. If $K$ is a subfield of $\mathbb{C}$ the involution $(\cdot)^{*}$ is defined as $\left(r_{g} g\right)^{*}=\overline{r_{g}} g^{-1}$. The trace $\tau: K[G] \rightarrow \mathbb{C}$ is defined as

$$
\begin{equation*}
\tau\left(\sum_{g \in G} r_{g} g\right)=r_{e} \tag{1.1}
\end{equation*}
$$

where $e \in G$ is the identity element in $G$. Traces on group algebras are usually called characters.
Given a tracial $*$-algebra $(\mathcal{A}, \tau)$, it is easy to see that the matrix $\operatorname{algebra} \operatorname{Mat}_{n}(\mathcal{A})$ is also a tracial $*$-algebra. For a matrix $M=\left(m_{i, j}\right) \in \operatorname{Mat}_{n}(\mathcal{A})$ we define $\operatorname{Tr}_{\tau}(M)=\frac{1}{n} \sum_{i=1}^{n} \tau\left(m_{i, i}\right)$ to extend the notion of trace to matrices over $*$-algebras. For any ring $R$ we denote the free $*$-ring over $R$ in $n \in \mathbb{N}$ free variables $X=$ $\left\{x_{1}, \ldots, x_{n}, x_{1}^{*}, \ldots, x_{n}^{*}\right\}$ by $R\langle X\rangle$. It consists of all finite linear combinations of finite words in $X$ and coefficients in $R$. If $R=K$ is a field we will speak of the free $*$-algebra. Note that any $*$-algebra is a homomorphic image of a free $*$-algebra. When talking about $*$-homomorphisms we mean ring/algebra homomorphisms that commute with the involution.

Example 1.2.3. Let $\mathcal{A}=\mathbb{C}\left\langle x_{1}, \ldots, x_{n}, x_{1}^{*}, \ldots, x_{n}^{*}\right\rangle$ be a free $*$-algebra and $\alpha: \mathcal{A} \rightarrow \operatorname{Mat}_{m}(\mathbb{C})$ be a $*$-homomorphism. Then $\frac{1}{m} \operatorname{Tr} \circ \alpha: \mathcal{A} \rightarrow \mathbb{C}$ defines a trace on $\mathcal{A}$. This can be generalized to the case whenever we have a $*$-homomorphism from a $*$-algebra to a tracial $*$-algebra.

Using the involution $(\cdot)^{*}$ we can define the following "special" elements in a tracial $*$-algebra.

Definition 1.2.4. Let $\mathcal{A}$ be a tracial $*$-algebra. We call an element
(1) $u \in \mathcal{A}$ unitary, if $u u^{*}=u^{*} u=1$,
(2) $p \in \mathcal{A}$ projection, if $p=p^{*}$ and $p^{2}=p$,
(3) $v \in \mathcal{A}$ isometry, if $v v^{*}=1$,
(4) $w \in \mathcal{A}$ partial isometry, if both, $w w^{*}$ and $w^{*} w$ are projections in $A$,
(5) $v \in \mathcal{A}$ idempotent, if $v^{2}=v$,
(6) $a \in \mathcal{A}$ self-adjoint, if $a^{*}=a$ and
(7) $n \in \mathcal{A}$ normal, if $n n^{*}=n^{*} n$.

Note that every unitary is an isometry, every isometry is a partial isometry and every projection is a partial isometry.

Of course these notions are familiar in a context of operator theory. Since $*-$ algebras are kind of hard to treat, we are looking for a way to represent them as algebras of operators acting on some Hilbert space $\mathcal{H}$, such that the involution $(\cdot)^{*}$ just becomes taking the adjoint operator. Let us just define what a Hilbert space is.

Definition 1.2.5. Let $K=\mathbb{C}$ or $K=\mathbb{R}$. A Hilbert space $\mathcal{H}$ is a $K$-vector space with a scalar product $\langle\cdot, \cdot\rangle$ which is complete with respect to the norm induced by the scalar product.

Remember that the norm of an element $v \in \mathcal{H}$ is defined by

$$
\|v\|^{2}=\langle v, v\rangle
$$

Given a Hilbert space $\mathcal{H}$ we denote by $\mathcal{B}(\mathcal{H})$ its algebra of bounded linear operators. For us, linear operators of Hilbert spaces will act on the right.
Corresponding to our two main examples of tracial $*$-algebras we also have two main examples of Hilbert spaces. Corresponding to $\mathcal{A}=\operatorname{Mat}_{n}(K), K \subseteq \mathbb{C}$ there is an obvious Hilbert space that $\mathcal{A}$ acts on which is $\mathcal{H}=\mathbb{C}^{n}$. So let us consider our second main example $\mathcal{A}=K[G]$, where $G$ is a group. There is a natural $\mathbb{C}$-vector space that $\mathcal{A}$ acts on by multiplication which is $V=\mathbb{C} \otimes_{K} \mathcal{A}=\mathbb{C}[G]$. The basis elements are just the $g \in G$. Also we can define a scalar product on $V:$ For $a=\sum_{g \in G} a_{g} g$ and $b=\sum_{g \in G} b_{g} g$ where $a_{g}, b_{g} \in \mathbb{C}$ we define

$$
\begin{equation*}
\langle a, b\rangle=\sum_{g \in G} \bar{a}_{g} b_{g} . \tag{1.2}
\end{equation*}
$$

If $G$ is finite we have $V \cong \mathbb{C}^{|G|}$ and this is already a Hilbert space. However, if $G$ is infinite, the space $V$ is not complete. When we take the completion of $V$ with respect to the norm defined above, we end up with a Hilbert space denoted by $\ell^{2}(G)$ which consists of infinite square summable formal sums:

$$
\ell^{2}(G)=\left\{\left.\sum_{g \in G} c_{g} g\left|c_{g} \in \mathbb{C}, \sum_{g \in G}\right| c_{g}\right|^{2}<\infty\right\}
$$

The scalar product is defined in the same way as for $\mathbb{C}[G]$. Note that $\mathbb{C}[G]$ is a dense subspace of $\ell^{2}(G)$.

So how exactly does an element $a \in \mathcal{A}=\mathbb{C}[G]$ act as a linear operator on $\ell^{2}(G)$ ? There are two ways how $G$ can act on $\ell^{2}(G)$. Let $g, h \in G$. We will consider $g$ as an operator acting on $\ell^{2}(G)$ and $h=1 \cdot h \in \ell^{2}(G)$ as a vector. First we have the left regular representation of $G$ given by $\lambda_{G}: G \rightarrow \mathcal{B}\left(\ell^{2}(G)\right), g \mapsto$ $\lambda_{G}(g): h \mapsto g^{-1} \cdot h$. On the other side we have the right regular representation $\rho_{G}: G \rightarrow \mathcal{B}\left(\ell^{2}(G)\right), g \mapsto \rho_{G}(g): h \mapsto h \cdot g$. Note that the two actions commute. So given a group $G$ there is always a Hilbert space that $\mathcal{A}=\mathbb{C}[G]$ acts on. We just take $V=\mathbb{C}[G]$ as a vector space and complete it with respect to the norm given by the inner product defined in 1.2. We would like to generalize this construction to arbitrary $*$-algebras. Let $K \subseteq \mathbb{C}$ be a field and $(\mathcal{A}, \tau)$ a tracial
*-algebra over $K$. Then again we can consider the $\mathbb{C}$-vector space $V=\mathbb{C} \otimes \mathcal{A}$, but this time we do not have an inner product. However, we can use the trace $\tau$ to define one. For that let us go back to the group case $\mathcal{A}=K[G]$ for a group $G$. Here we have the trace defined by 1.1 and the inner product defined by 1.2. There is a relation between these, given by

$$
\langle a, b\rangle=\tau\left(b^{*} a\right)
$$

for $a, b \in \mathbb{C}[G]$. So the trace on $\mathbb{C}[G]$ already defines the inner product on $\ell^{2}(G)$. We want to generalize this in the next proposition to the so called GNSconstruction after Gelfmark, Naimark and Segal.
Proposition 1.2.6. Let $(\mathcal{A}, \tau)$ be a tracial $*$-algebra over $\mathbb{C}$ such that

$$
\begin{equation*}
\mathcal{A}=\left\langle\left\{x \in \mathcal{A} \left\lvert\, \sup _{k}\left(\tau\left(x x^{*}\right)^{k}\right)^{\frac{1}{k}}<\infty\right.\right\}\right\rangle \tag{1.3}
\end{equation*}
$$

Then there is a Hilbert space $\mathcal{H}=\mathcal{H}_{\tau}$ and a cyclic representation $\rho_{\tau}=\rho: \mathcal{A} \rightarrow$ $\mathcal{B}\left(\mathcal{H}_{\tau}\right)$ with generator $e \in \mathcal{H}_{\tau}$, such that

$$
\langle e . \rho(a), e\rangle=\tau(a), a \in \mathcal{A}
$$

Here cyclic means that e. $\rho(A)$ is dense in $\mathcal{H}$. Moreover, if there is another cyclic representation $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ with generator $e^{\prime} \in \mathcal{H}$ for some Hilbert space $\mathcal{H}$ such that $\left\langle e^{\prime} \cdot \pi(a), e^{\prime}\right\rangle=\tau(a)$, then $\pi$ and $\rho$ are unitarily equivalent that means there is a unitary operator $U \in \mathcal{U}(\mathcal{H})$ such that

$$
U \rho(a)=\pi(a) U
$$

for all $a \in \mathcal{A}$.
Proof. We only want to give a sketch of the construction, for details see [NS06, Chapter 7] Consider the set $N:=\left\{n \in \mathcal{A} \mid \tau\left(n^{*} n\right)=0\right\}$. We want to see that this is a left ideal of $\mathcal{A}$. So let $n \in N, a \in \mathcal{A}$. Then

$$
\tau\left((a n)^{*}(a n)\right)^{2}=\tau\left(n^{*}\left(a^{*} a n\right)\right)^{2} \leq \tau\left(\left(a^{*} a n\right)\left(a^{*} a n\right)^{*}\right) \tau\left(n n^{*}\right)=0
$$

by the Cauchy-Schwarz inequality. In fact, since $\tau$ is tracial, $N$ is also a right ideal. Consider now the quotient $\mathcal{A} / N$ as a vector space with an inner product defined by

$$
\langle a+N, b+N\rangle_{\tau}=\tau\left(b^{*} a\right)
$$

This is well defined since if $a, b \in \mathcal{A}, m, n \in N$ then

$$
\langle a+m, b+n\rangle_{\tau}=\tau\left(\left(b^{*}+n^{*}\right)(a+m)\right)=\tau\left(b^{*} a\right)+\tau\left(n^{*} a\right)+\tau\left(b^{*} m\right)+\tau\left(n^{*} m^{*}\right)
$$

and each of the last three terms is zero because of the Cauchy-Schwarz inequality. Now, define a norm on $\mathcal{A} / N$ by $\|a+N\|^{2}=\langle x+N, a+N\rangle$ and let $\mathcal{H}_{\tau}$ be the completion of $\mathcal{A} / N$ with respect to this norm. Set $e_{\tau}=1_{\mathcal{A}}+N$. The right regular representation $r: \mathcal{A} \rightarrow \operatorname{End}(\mathcal{A}), a \mapsto(r(a): x \mapsto x a)$ gives a
representation $\rho: \mathcal{A} \rightarrow \operatorname{End}(\mathcal{A} / N), a \mapsto \rho(a): x+N \mapsto x a+N$, since $N$ is a both sided ideal. It is left to show that the operators $\rho(a)$ are actually bounded. This ensures condition 1.3. One can show that in the limit we get

$$
\|(b+N) \cdot \rho(a)\|^{2} \leq \tau\left(b^{*} b\right) \limsup _{n \rightarrow \infty}\left(\tau\left(\left(a^{*} a\right)^{n}\right)^{\frac{1}{n}}\right)
$$

and therefore

$$
\|\rho(a)\| \leq \limsup _{n}\left(\tau\left(\left(a^{*} a\right)^{n}\right)^{\frac{1}{n}}\right)<\infty
$$

Since $e_{\tau} . \rho(\mathcal{A})$ is dense in $\mathcal{H}_{\tau}$ the representation is cyclic. Let now $\pi$ be another cyclic representation with $\left\langle e^{\prime} . \pi(a), e^{\prime}\right\rangle=\left\langle e_{\tau} . \rho(a), e_{\tau}\right\rangle$. Then the map defined on the dense subspace $e_{\tau} . \rho(\mathcal{A})$ by $e_{\tau} . a \mapsto e^{\prime} . a$ is a well defined isometry which extends to $\mathcal{H}_{\tau}$.

Remark 1.2.7. Since $N$ is a both sided $*$-ideal we also get a left action $\lambda: A^{\mathrm{op}} \rightarrow$ $\mathcal{B}\left(\mathcal{H}_{\tau}\right), a \mapsto \lambda(a): x+N \mapsto a \cdot x+N$. As in the group case these two actions commute.
Remark 1.2.8. Obviously the maps $\lambda, \rho: \mathcal{A} \rightarrow \mathcal{B}\left(\mathcal{H}_{\tau}\right)$ factor through the $*$ algebra $\mathcal{A} / N$. Therefore, in many situations, by replacing $\mathcal{A}$ by $\mathcal{A} / N$, we can assume that the trace $\tau$ on $\mathcal{A}$ is faithful.

A small difference to the group case does exist. Since all group elements $g \in G$ are by definition of the involution unitaries, we can define $\lambda_{G}$ in a way that it is a homomorphism $G \rightarrow U(\mathcal{H})$, where the latter denotes the group of unitary operators. In the general algebra case, since we let operators act from the right, left multiplication gives a map $\lambda: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ that is an antihomomorphism that means a homomorphism from $\mathcal{A}^{\text {op }}$. However this does not cause any troubles. To avoid this difference one can see $\mathcal{H}$ as a left- and right- $\mathcal{A}$-bimodule and omit the maps $\mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$.
Note that we put an additional condition, namely that $\mathcal{A}$ is generated by elements $x \in \mathcal{A}$ with $\sup \left(\tau\left(x x^{*}\right)^{k}\right)^{\frac{1}{k}}<\infty$. This condition ensures that $\mathcal{A}$ acts on $\mathcal{H}_{\tau}$ as bounded operators. In the case where $\mathcal{A}=K[G]$ is a group algebra this is always true since the elements of $G$ are unitaries. From now on we will always assume that this additional condition holds.

We have just seen that we can always represent a tracial $*$-algebra $(\mathcal{A}, \tau)$ in the algebra of bounded operators of some Hilbert space $\mathcal{H}$. Furthermore, if the trace $\tau$ on $\mathcal{A}$ is faithful, this representation is an embedding. Furthermore we have seen that this Hilbert space is an $\mathcal{A}$-bimodule, that means we have a left and a right action and these two actions commute. This allows us to consider more interesting structures.

Definition 1.2.9. Let $\mathcal{H}$ be a Hilbert space. Then we have the following three topologies on $\mathcal{B}(\mathcal{H})$ :
(1) Let $T \in \mathcal{B}(\mathcal{H})$. The topology induced by the operator norm $\|T\|=$ $\sup \|v T\|$ is called the norm topology.
$v \in H$,
$\|v\|=1$
(2) The coarsest topology for which all evaluation maps $E_{v}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{H}, T \mapsto$ $v T$ with $v \in \mathcal{H}$ are continuous is called the strong operator topology.
(3) The coarsest topology for which all maps $E_{u, v}: \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}, T \mapsto\langle u, v T\rangle$ with $v, u \in \mathcal{H}$ are continuous is called the weak operator topology.

For a subset $M \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ we denote by $M^{\prime}=\{T \in$ $\mathcal{B}(\mathcal{H}) \mid S T=T S$ for all $S \in M\}$ the commutant of $M$. The following theorem connects the commutant, the strong and the weak operator topology.

Theorem 1.2.10. [Kam19, von Neumann bicommutant theorem, 1.19] Let M be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The following are equivalent.
(1) $M$ is closed in the weak operator topology.
(2) $M$ is closed in the strong operator topology.
(3) $\left(M^{\prime}\right)^{\prime}=M$.

Definition 1.2.11. A unital $*$-subalgebra $\mathcal{N}$ of $\mathcal{B}(\mathcal{H})$ that satisfies one and therefore all of the above conditions is called a von Neumann algebra.

Although we will not use it, let us just mention that a unital $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ that is closed in the norm topology is called $C^{*}$-algebra.

Let us come back to our two running examples. The algebra $\mathcal{A}=\operatorname{Mat}_{n}(\mathbb{C})$ is already a von Neumann algebra. For the group case we have to work a little bit more. So let $G$ be a group, $\mathcal{A}=\mathbb{C}[G]$ and $\tau$ be the trace on $\mathcal{A}$ as in 1.1. By 1.2.6 and the discussion before we have $*$-representation $\rho, \lambda: \mathcal{A} \rightarrow \mathcal{B}\left(l^{2}(G)\right)$. The Group von Neumann algebra $\mathcal{N}(G)$ is defined as all bounded operators on $\ell^{2}(G)$, that commute with the left action of $G$. That means we have $\mathcal{N}(G)=(\lambda(G))^{\prime}$. Obviously we have $\rho(G) \subseteq \mathcal{N}(G)$. For a general tracial $*$-algebra $(\mathcal{A}, \tau)$ we can do exactly the same and define $\mathcal{N}_{\tau}=(\lambda(\mathcal{A}))^{\prime} \subseteq \mathcal{B}\left(\mathcal{H}_{\tau}\right)$, where $\mathcal{H}_{\tau}$ is the Hilbert space from 1.2.6. A von Neumann algebra is called tracial if it has a faithful trace. Note that for a tracial $*$-algebra the trace $\tau$ of $\mathcal{A}$ can be extended to $\mathcal{N}_{\tau}$ using the scalar product of the Hilbert space $\mathcal{H}_{\tau}$ as in 1.2.6, and the resulting trace is faithful. Note that we can see $\mathcal{N}_{\tau}$ as a subspace of $\mathcal{H}_{\tau}$ via the map $a \mapsto\left(1_{\mathcal{A}}\right) . a$. Here $1_{\mathcal{A}}$ is seen as the vector in $\mathcal{H}_{\tau}$ that represents the class of $1_{\mathcal{A}} \in \mathcal{A}$. From now on we will see $\mathcal{H}_{\tau}$ as a right $\mathcal{N}_{\tau}$ module and will write $v . n_{1} n_{2}$, omitting the representations for $n_{1}, n_{2} \in \mathcal{N}_{\tau}, v \in \mathcal{H}_{\tau}$.
Remark 1.2.12. By 1.2 .8 the map $\rho: \mathcal{A} \rightarrow \mathcal{N}_{\tau}$ is injective and we can see $\mathcal{A}$ as a subalgebra of $\mathcal{N}_{\tau}$.

### 1.2.2 Operators on Hilbert Spaces

In this section we want to recall basic facts about bounded and unbounded operators on Hilbert spaces. A good source for more details is [Rud91]. We already defined in 1.2.4 some notions of operators with special properties in *-algebras. We now want to translate some of these notions to operators on Hilbert spaces. In the following let $\mathcal{H}$ and $\mathcal{H}^{\prime}$ be Hilbert spaces.

Definition 1.2.13. An operator $v \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$ is called an isometry, if $\|x . v\|=$ $\|x\|$ for all $x \in \mathcal{H}$.

Isometries are operators that behave well with the inner products on $\mathcal{H}$ and $\mathcal{H}^{\prime}$. In fact we have the following lemma:
Lemma 1.2.14. Let $v \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$ be a isometry. then for all $x, y \in \mathcal{H}$ we have

$$
\langle x . v, y . v\rangle=\langle x, y\rangle .
$$

Proof. We have

$$
\begin{aligned}
\|x\|^{2}+2 \operatorname{Re}(\langle x, y\rangle)+\|y\|^{2} & =\|x\|^{2}+\langle x, y\rangle+\overline{\langle x, y\rangle}+\|y\|^{2} \\
& =\langle x+y, x+y\rangle \\
& =\|x+y\|^{2} \\
& =\|(x+y) \cdot v\|^{2} \\
& =\langle x \cdot v+y \cdot v, x \cdot v+y \cdot v\rangle \\
& =\|x \cdot v\|^{2}+2 \operatorname{Re}(\langle x \cdot v, y \cdot v\rangle)+\|y \cdot v\|^{2}
\end{aligned}
$$

From this we get $\operatorname{Re}(\langle x . v, y \cdot v\rangle)=\operatorname{Re}(\langle x, y\rangle)$. Using the linearity of $v$ and the same argument as above we get

$$
\begin{aligned}
\operatorname{Im}(\langle x, y\rangle & =\operatorname{Re}(-i\langle x, y\rangle) \\
& =\operatorname{Re}(\langle-i x, y\rangle) \\
& =\operatorname{Re}(\langle(-i x) \cdot v, y \cdot v\rangle) \\
& =\operatorname{Re}(\langle-i x \cdot v, y \cdot v\rangle) \\
& =\operatorname{Re}(-i\langle x \cdot v, y \cdot v\rangle) \\
& =\operatorname{Im}(\langle x \cdot v, y \cdot v\rangle)
\end{aligned}
$$

We can now show that our definition from 1.2 .4 behaves well with the definition we gave in this chapter.
Proposition 1.2.15. Let $v \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$. Then $v$ is an isometry if and only if $v v^{*}=I d_{\mathcal{H}}$.
Proof. Assume first that $v$ is an isometry. Then

$$
\langle x . v, y . v\rangle=\left\langle x . v v^{*}, y\right\rangle
$$

and therefore

$$
\left\langle x .\left(\operatorname{Id}_{\mathcal{H}}-v v^{*}\right), y\right\rangle=0
$$

for all $x, y \in \mathcal{H}$. Therefore $\operatorname{Id}_{\mathcal{H}}=v v^{*}$.
On the other hand let us assume that $\mathrm{Id}_{\mathcal{H}}=v v^{*}$ holds. Then

$$
\left.\|x\|^{2}=\langle x, x\rangle=\left\langle x . v v^{*}, x\right)\right\rangle=\langle x . v, x . v\rangle=\|x . v\|^{2} .
$$

So $v$ is an isometry.

We now want to explain what a partial isometry is.
Definition 1.2.16. Let $w \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$. $w$ is called a partial isometry, if for all $x \in \operatorname{ker}(u)^{\perp}$ we have $\|x . w\|=\|x\|$.

The space $\operatorname{ker}(w)^{\perp}$ is called initial space and the space $\operatorname{Im}(w)$ is called final space of $w$. So a partial isometry $w \in \mathcal{H}$ is an isometry between its initial space and its final space. In the following we will collect some characterizations of partial isometries.

Proposition 1.2.17. Let $v \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$. Then the following are equivalent:
(1) $v$ is a partial isometry.
(2) $v^{*}=v^{*} v v^{*}$.
(3) $v=v v^{*} v$.
(4) $v v^{*}$ is a projection.
(5) $v^{*} v$ is a projection.
(6) $v^{*} \in \mathcal{B}\left(\mathcal{H}^{\prime}, \mathcal{H}\right)$ is a partial isometry.

Further $\operatorname{im}(v)$ is closed in $\mathcal{H}^{\prime}, v^{*} v$ is the projection onto $\operatorname{im}(v)$ and $v v^{*}$ is the projection onto $\operatorname{ker}(v)^{\perp}$.
Proof. (1) $\Rightarrow(2)$ : Let $v$ be a partial isometry and fix $x \in \mathcal{H}^{\prime}$. Assume first that $y \in \operatorname{ker}(v)$. Then

$$
\left\langle x \cdot v^{*} v v^{*}, y\right\rangle=\left\langle x \cdot v v^{*}, y \cdot v\right\rangle=0=\langle x, y \cdot v\rangle=\left\langle x \cdot v^{*}, y\right\rangle .
$$

On the other hand, when $y \in \operatorname{ker}(v)^{\perp}$ we have

$$
\left\langle x \cdot v^{*} v v^{*}, y\right\rangle=\left\langle x \cdot v^{*} v, y \cdot v\right\rangle=\left\langle x \cdot v^{*}, y\right\rangle .
$$

In the last equality we used that for any operator $t \in \mathcal{B}\left(\mathcal{H}, \mathcal{H}^{\prime}\right)$ we have $\operatorname{ker}(t)^{\perp}=$ $\overline{\operatorname{im}\left(t^{*}\right)}$ and that $v$ is an isometry on $\operatorname{ker}(v)^{\perp}$.
$(2) \Leftrightarrow(3)$ : Follows by taking adjoints.
$(2) \Rightarrow(4)$ : Obviously $v v^{*}$ is self-adjoint. Further we have

$$
v v^{*} v v^{*}=v\left(v^{*} v v^{*}\right)=v v^{*}
$$

$(3) \Rightarrow(5)$ : Exactly as in $(2) \Rightarrow(4)$.
(5) $\Rightarrow$ (1) Let $x \in \operatorname{ker}(v)^{\perp}=\overline{\operatorname{im}\left(v^{*}\right)}$. Let $\left(x_{n}\right)_{n} \in \mathcal{H}^{\prime}$ such that $\lim _{n \rightarrow \infty}\left(x_{n}\right) \cdot v^{*}=x$.

Then we have

$$
\begin{aligned}
\|x \cdot v\|^{2} & =\lim _{n \rightarrow \infty}\left\|\left(x_{n}\right) \cdot v^{*} v\right\|^{2}=\lim _{n \rightarrow \infty}\left\langle\left(x_{n}\right) \cdot v^{*} v,\left(x_{n}\right) \cdot v^{*} v\right\rangle \\
& =\lim _{n \rightarrow \infty}\left\langle\left(x_{n}\right) \cdot v^{*} v v^{*} v, x_{n}\right\rangle=\lim _{n \rightarrow \infty}\left\langle\left(x_{n}\right) \cdot v^{*} v, x_{n}\right\rangle \\
& =\lim _{n \rightarrow \infty}\left\langle\left(x_{n}\right) \cdot v^{*},\left(x_{n}\right) \cdot v^{*}\right\rangle=\langle x, x\rangle \\
& =\|x\|^{2}
\end{aligned}
$$

$(4) \Rightarrow(6)$ : Exactly as in $(5) \Rightarrow(1)$.
$(6) \Rightarrow(3)$ : Exactly as in $(1) \Rightarrow(2)$. To see that $\operatorname{im}(v)$ is closed let $x \in \overline{\operatorname{im}(v)}$. Let $\left(x_{n}\right)_{n} \in \mathcal{H}$ such that $\lim _{n \rightarrow \infty}\left(x_{n}\right) \cdot v=x$ Then we have

$$
x \cdot v^{*} v=\lim _{n \rightarrow \infty}\left(x_{n}\right) \cdot v v^{*} v=\lim _{n \rightarrow \infty}\left(x_{n}\right) \cdot v=x
$$

and therefore $x \in \operatorname{im}(v)$. Next let us show that $v^{*} v$ is the projction onto $\operatorname{im}(v)$. Let $x=y . v \in \operatorname{im}(v)$. Then $x \cdot v^{*} v=y \cdot v v^{*} v=y . v=x$. On the other hand let $x \in \operatorname{im}(v)^{\perp}=\operatorname{ker}\left(v^{*}\right)$. Then we have $x \cdot v^{*} v=0$. The last thing that is left to show is that $v v^{*}$ is the projection onto $\operatorname{ker}(v)^{\perp}=\overline{\operatorname{im}\left(v^{*}\right)}$. Since $v^{*}$ is also a partial isometry we know that $\operatorname{im}\left(v^{*}\right)=\overline{\operatorname{im}\left(v^{*}\right)}$ and that $v v^{*}$ is the projection onto $\operatorname{im}\left(v^{*}\right)=\operatorname{ker}(v)^{\perp}$ by the previous statement.

Next we want to recall the spectral theorem for normal operators $A \in \mathcal{B}(\mathcal{H})$. The proofs and more details can be found in [Rud91]. We will motivate this theorem by looking at the finite dimensional case. So let $\mathcal{H}=\mathbb{C}^{n}$ and $A \in$ $\operatorname{Mat}_{n}(\mathbb{C})$ be a normal matrix. We will work with row vectors, so $A$ acts by right multiplication. Let $\lambda_{1}, \ldots, \lambda_{n}$ be the eigenvalues of $A$ and let $v_{1}, \ldots, v_{n} \in \mathbb{C}^{n}$ be normalized corresponding eigenvectors. Note that since $A$ is normal we have $\left\langle v_{i}, v_{j}\right\rangle=0$ for $i \neq j$. Let $U$ be the unitary matrix with columns $v_{i}$ and $D$ be the diagonal matrix with the $\lambda_{i}$ 's on the diagonal. Then we have $U A=D U$ or equivalently $A=U^{*} D U$. This last equality gives

$$
\begin{equation*}
A=\sum_{i=1}^{n} \lambda_{i} E_{i} \tag{1.4}
\end{equation*}
$$

where $E_{i}=\bar{v}_{i}^{\mathrm{tr}} v_{i}$ is the orthogonal projection onto the linear subspace generated by $v_{i}$. Note that using 1.4 we can make sense of $f(A)$ for all functions $f \in C(U)$ where $\sigma(A) \subseteq U$ is an open neighbourhood of the spectrum of $A$. We just set

$$
\begin{equation*}
f(A)=\sum_{i=1}^{n} f\left(\lambda_{i}\right) E_{i} \tag{1.5}
\end{equation*}
$$

We want to generalize this now to bounded operators on infinite dimensional Hilbert spaces.

Definition 1.2.18. Let $\mathcal{B}$ be a $\sigma$-algebra on a set $\Omega$ and let $\mathcal{H}$ be a Hilbert space. A projection valued measure on $\mathcal{B}$ is a map $E: \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$ such that
(1) $E(\emptyset)=0, E(\Omega)=1$.
(2) $E(\omega)$ is a projection for each $\omega \in \mathcal{B}$.
(3) $E\left(\omega \cap \omega^{\prime}\right)=E(\omega) E\left(\omega^{\prime}\right)$.
(4) If $\omega \cap \omega^{\prime}=\emptyset$ then $E\left(\omega \cup \omega^{\prime}\right)=E(\omega)+E\left(\omega^{\prime}\right)$.
(5) For all $x, y \in \mathcal{H}$ the function $E_{x, y}: \mathcal{B} \rightarrow \mathbb{C}, \omega \mapsto\langle E(\omega)(x), y\rangle$ is a complex measure on $\Omega$.

If $\Omega$ is a compact or a locally compact Hausdorff space one usually requires
( $5^{\prime}$ ) Each measure $E_{x, y}$ is a regular Borel measure.
We already have seen an example of a projection valued measure. Consider again a normal matrix $A \in \operatorname{Mat}_{n}(\mathbb{C})$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ and corresponding normalized eigenvectors $v_{i}$. For a Borel set $S$ let $\left\{\mu_{1}, \ldots, \mu_{s}\right\}$ be the eigenvalues that lie in $S$ and let $w_{1}, \ldots w_{s}$ be the corresponding normalized eigenvectors. Then we define $E_{A}(S)=\sum_{i=1}^{s} \bar{w}^{\mathrm{tr}} w_{i}$.

To make sense of 1.4 and 1.5 in the infinite dimensional setting we need to define operator valued integrals with respect to a projection valued measure.
Theorem 1.2.19. [Rud91, Theorem 12.21] Let E be a projection valued measure on a $\sigma$-algebra $\mathcal{B}$ on a set $\Omega$. Then there exists an isometric $*$-isomorphism $\Phi$ from $\mathrm{L}^{\infty}(\Omega)$ to a closed commutative $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ which is related to $E$ by the formula

$$
\begin{equation*}
\langle\Phi(f) x, y\rangle=\int_{\Omega} f \mathrm{~d} E_{x, y} \tag{1.6}
\end{equation*}
$$

This justifies the notation

$$
\begin{equation*}
\Phi(f)=\int_{\Omega} f \mathrm{~d} E \tag{1.7}
\end{equation*}
$$

Moreover an operator $Q \in \mathcal{B}(\mathcal{H})$ commutes with every $E(\omega)$ if and only if it commutes with every $\Phi(f)$.

We have now all the necessary notation to formulate the spectral theorem.
Theorem 1.2.20. [Rud91, Theorem 12.23, 12, 29] Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and let $\sigma(T)$ be its spectrum. Then there exists a unique projection valued measure $E$ on the Borel sets of $\sigma(T)$ which satisfies

$$
\begin{equation*}
T=\int_{\sigma(T)} \lambda \mathrm{d} E(\{\lambda\}) . \tag{1.8}
\end{equation*}
$$

Further the projection valued measure satisfies the following properties:

- For every Borel set $\omega \subseteq \sigma(T)$ the projection $E(\omega)$ commutes with every operator $Q \in \mathcal{B}(\mathcal{H})$ that commutes with $T$.
- For $\lambda \in \sigma(T)$ we have

$$
\begin{equation*}
\operatorname{im}(E(\{\lambda\}))=\operatorname{ker}(T-\lambda \cdot \operatorname{Id}) . \tag{1.9}
\end{equation*}
$$

- $\lambda \in \sigma(T)$ is an eigenvalue of $T$ if and only if $E(\{\lambda\}) \neq 0$.

The projection valued measure $E$ associated to a normal operator $T$ is called the spectral resolution of $T$. By 1.2.19 we get for every measurable function $f$ on $\sigma(T)$ an operator

$$
\begin{equation*}
\Phi(f)=\int_{\sigma(T)} f \mathrm{~d} E(\{\lambda\}) \tag{1.10}
\end{equation*}
$$

This operator is usually denoted by $f(T)$. One thing we want to highlight for later use is that $E(\{0\})$ is exactly the projection onto the kernel of $T$.
Remark 1.2.21. Let us consider the case when $(\mathcal{N}, \tau)$ is a tracial von Neumann algebra with representation $\mathcal{H}_{\tau}$ coming from the GNS-construction. Every normal matrix $A \in \operatorname{Mat}_{n}(\mathcal{N})$ with spectral measure $E$ defines a complex measure denoted by

$$
\mu_{A, \tau}=\tau \circ E=E_{e, e}=\langle e . E(\cdot), e\rangle
$$

where $e=1_{\mathcal{A}} \in \mathcal{H}_{\tau}$ denotes the trace vector. If $A \in \operatorname{Mat}_{n}(\mathbb{C})$ with $\tau(A)=$ $\operatorname{Tr}(A)$ we get $\mu_{A, \tau}=\sum_{i=1}^{n} \delta_{\lambda_{i}}$, where $\lambda_{1}, \ldots, \lambda_{n}$ are the eigenvalues of $A$ and $\delta_{x}$ denotes the Dirac measure at $x \in \mathbb{C}$.

The last thing we want to mention here is the so called polar decomposition. We know that we can represent a complex number $z=x+y i \in \mathbb{C}$ also by its length $r=|z|=\sqrt{x^{2}+y^{2}}$ and its argument $\varphi$. We then have $z=r \exp i \varphi$. Something similar is possible for bounded operators on Hilbert spaces. For that let $T \in \mathcal{B}(\mathcal{H})$. Since $T T^{*}$ is a positive operator that means its spectrum is contained in $\mathbb{R}_{\geq 0}$, by 1.10 we can form the operator $|T|=\left(T T^{*}\right)^{\frac{1}{2}}$, which is still positive self-adjoint. We now define a partial isometry $U$ by the following steps. For $v=w \cdot|T| \in \operatorname{im}(|T|)$ we set $v . U=w \cdot T$. We can extend this to vectors $v \in \overline{\operatorname{im}(|T|})$. Further, for $v \in \operatorname{im}(|T|)^{\perp}$ we set $v . U=0$. Note that $\overline{\operatorname{im}(|T|})^{\perp}=\operatorname{ker}(|T|)=\operatorname{ker}(U)=\operatorname{ker}(T)$. We have the following theorem.

Theorem 1.2.22. Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a positive operator $P$ and a partial isometry $U$ such that $T=P U$. More precisely $P=|T|$ and $\operatorname{ker}(P)=\operatorname{ker}(U)=\operatorname{ker}(T)$

### 1.2.3 The von Neumann Dimension

We now want to use the theory we introduced before to define a type of dimension of subspaces of some Hilbert spaces. In the following let $\mathcal{A}=(\mathcal{A}, \tau)$ be a tracial $*$-algebra and $\mathcal{H}=\mathcal{H}_{\tau}$ the Hilbert space from the GNS construction. Let $\mathcal{N}=\mathcal{N}_{\tau}=(\lambda(A))^{\prime}$
Remark 1.2 .23 . We can extend the action of $\lambda$ diagonally to $\mathcal{H}^{n}$ for $n \in \mathbb{N}$. It is easy to see that the algebra of all bounded operators of $\mathcal{H}^{n}$ that commute with the diagonal action of $\lambda$ is given by $\operatorname{Mat}_{n}(\mathcal{N})$. This algebra is called the amplified von Neumann algebra.

Definition 1.2.24. Let $(\mathcal{A}, \tau)$ be a tracial $*$-algebra and $\mathcal{H}_{\tau}$ be the Hilbert space from the GNS construction. A finitely generated Hilbert $\mathcal{A}$-module is a
closed subspace $V \leq \mathcal{H}_{\tau}^{n}$ of $\mathcal{H}_{\tau}^{n}$ for some $n \in \mathbb{N}$ that is invariant under the left $\mathcal{A}$-action.

The following provides an easy example of a Hilbert $\mathcal{A}$-module.
Example 1.2.25. Let $(\mathcal{A}, \tau), n \in \mathbb{N}$ and $A \in \operatorname{Mat}_{n}(\mathcal{A})$. Denote by $r_{A}: \mathcal{H}_{\tau}^{n} \rightarrow$ $\mathcal{H}_{\tau}^{n}$ the operator given by right multiplication by $A$ with respect to $\rho$. Then $\operatorname{ker} r_{A}$ is a finitely generated Hilbert $\mathcal{A}$-module.

It is well known that closed subspaces of Hilbert spaces have orthorgonal complements. That means for a finitely generated Hilbert $\mathcal{A}$-module $V \leq \mathcal{H}^{n}$ we have $\mathcal{H}^{n}=V \oplus V^{\perp}$ and so $V^{\perp}$ is also a finitely generated Hilbert $\mathcal{A}$-module. Since both, $V$ and $V^{\perp}$ are $\lambda(\mathcal{A})$ invariant, the projection

$$
\operatorname{pr}_{V}: \mathcal{H}^{n}=V \oplus V^{\perp} \rightarrow V \leq \mathcal{H}^{n}, v=v_{1}+v_{2} \mapsto v_{1}
$$

commutes with the $\lambda(\mathcal{A})$ action and is therefore given by right multiplication by a matrix $P_{V} \in \operatorname{Mat}_{n}(\mathcal{N})$. We are now ready to define the von Neumann dimension of a finitely generated Hilbert $\mathcal{A}$-module.

Definition 1.2.26. Let $V \leq \mathcal{H}_{\tau}^{n}$ be a finitely generated Hilbert $\mathcal{A}$-module. We define the von Neumann dimension of $V$ as

$$
\begin{equation*}
\operatorname{dim}_{\tau} V:=\operatorname{Tr}\left(P_{V}\right)=\sum_{i=1}^{n}\left\langle e_{i} P_{V}, e_{i}\right\rangle \tag{1.11}
\end{equation*}
$$

where $e_{i}$ is the vector having the generator $e \in \mathcal{H}_{\tau}$ in the $i$-th coordinate and zeros in all others.

We call a sequence $V_{1} \rightarrow V \rightarrow V_{2}$ of finitely generated Hilbert $\mathcal{A}$-modules weakly exact if the kernel of the second map coincides with the closure of the image of the first map. We have the following result about the von Neumann dimension and weakly exact sequences.
Proposition 1.2.27. Let $0 \rightarrow V_{1} \rightarrow V \rightarrow V_{2} \rightarrow 0$ be a weakly exact sequence of Hilbert- $(\mathcal{A}, \tau)$-modules. Then

$$
\operatorname{dim}_{\tau} V_{1}+\operatorname{dim}_{\tau} V_{2}=\operatorname{dim}_{\tau} V
$$

Proof. Let $V \leq \mathcal{H}^{n}, V_{1} \leq \mathcal{H}^{n_{1}}, V_{2} \leq \mathcal{H}^{n_{2}}$ be finitely generated Hilbert $\mathcal{A}$ modules. We can assume that $n=n_{1}=n_{2}$ since for $k \leq m$ we can embed $\mathcal{H}^{k}$ into $\mathcal{H}^{m} \cong \mathcal{H}^{k} \oplus \mathcal{H}^{m-k}$ into the first $k$ components and this embedding does not alter the von Neumann trace. The proof works in three steps. Let us first assume that $f: V_{1} \rightarrow V_{2}$ is an $\mathcal{A}$-equivariant isometry between finitely generated Hilbert $\mathcal{A}$-modules. We can extend $f$ to a partial isometry $\bar{f}: \mathcal{H}^{n}=V_{1} \oplus V_{1}{ }^{\perp}, v+w \mapsto$ $f(v)$. Since $f$ is $\mathcal{A}$-equivariant, it commutes with the $\lambda(\mathcal{A})$-action and therefore $\bar{f}$ is given by right multiplication by a matrix $M_{\bar{f}} \in \operatorname{Mat}_{n}(\mathcal{N})$. Since $\bar{f}$ is a partial isometry we know that $\bar{f}^{*} \bar{f}$ is the projection onto $\operatorname{im}(\bar{f})=V_{2}$ and $\bar{f} \bar{f}^{*}$ is the projection onto $\operatorname{ker}(\bar{f})^{\perp}=V_{1}$. Since $\operatorname{Tr}$ has the trace property we have $\operatorname{dim}_{\tau} V_{2}=\operatorname{Tr}\left(\bar{f}^{*} \bar{f}\right)=\operatorname{Tr}\left(\bar{f} \bar{f}^{*}\right)=\operatorname{dim}_{\tau} V_{1}$.

Let now $f: V_{1} \rightarrow V_{2}$ be a weak isomorphism that means $\operatorname{ker}(f)=0$ and $\overline{\operatorname{im}(f)}=V_{2}$. Let $f=P U$ be the polar decomposition of $f$. Then $U$ is a partial isometry with $\operatorname{im}(U)=V_{2}$ and $\operatorname{ker}(U)^{\perp}=V_{1}$. Thus we are in the same situation as in the previous case. Let now

$$
0 \rightarrow V_{1} \xrightarrow{i} V \xrightarrow{f} V_{2} \rightarrow 0
$$

be a weak exact sequence. The theorem follows now, since

$$
V \rightarrow \overline{\operatorname{im}(i)}+V_{2}, v \mapsto\left(v \cdot P_{\overline{\operatorname{im}(i)}}, v . f\right)
$$

is a weak isomorphism and the von Neumann dimension is additive with respect to direct products.

Until now we have seen that a trace $\tau$ on a $*$-algebra $\mathcal{A}$ provides us with a lot of further structure. We now want to introduce one more function, which will be one of the main objects of interests in this thesis.

Definition 1.2.28. Let $(\mathcal{A}, \tau)$ be a tracial $*$-algebra. Let $A \in \operatorname{Mat}_{m, n}(\mathcal{A})$ and let $r_{A}: \mathcal{H}_{\tau}^{m} \rightarrow \mathcal{H}_{\tau}^{n}$ be the map given by right multiplication by $A$. We define the von Neumann rank of $A$ as

$$
\operatorname{rk}_{\tau}(A)=\operatorname{dim}_{\tau} \overline{\operatorname{im} r_{A}}=n-\operatorname{dim}\left(\operatorname{ker} r_{A}\right)
$$

The function $\mathrm{rk}_{\tau}$ is a Sylvester matrix rank function. We will learn more about these functions in Chapter 1.3.3. If $\mathcal{A}=\mathbb{C}[G]$ with trace defined as in 1.1 we will write $\mathrm{rk}_{G}$ instead of $\mathrm{rk}_{\tau}$.

Question 1.2.29. Let us denote by $\operatorname{Ch}(\mathcal{A})$ the space of traces of a*-algebra $\mathcal{A}$ and by $\mathbb{P}(\mathcal{A})$ the space of Sylvester matrix rank functions. Both spaces carry the topology of pointwise convergence. The above constructions give us a map $C h(\mathcal{A}) \rightarrow \mathbb{P}(\mathcal{A})$. Is this map continuous?

It is easy to see that in this generality the answer is simply no.
Example 1.2.30. $G=\langle g\rangle \cong \mathbb{Z}$ be the infinite cyclic group and $\mathcal{A}=\mathbb{C}[G]$ be the group algebra. For every $z \in \mathbb{C}$ with $|z|=1$ we get a character $\tau_{z}$ of $G$ with $\tau_{z}\left(g^{k}\right)=z^{k}$. Let $a=(g-z) \in \mathcal{A}$. From $\tau_{z}\left(a a^{*}\right)=0$ we see that $\mathcal{H}_{\tau_{z}} \cong \mathbb{C}$ is one-dimensional and the $\mathcal{A}$ action is just given by multiplication by $z$. Let now $\left(z_{i}\right)_{i}$ be a sequence in $S^{1}$ that converges to 1 . Then the characters $\tau_{z_{i}}$ converge to the trivial character $\mathbb{1}_{G}$. However the element $1-g \in \mathbb{C}[G]$ acts as multiplication by some non zero number on $\mathcal{H}_{\tau_{z_{i}}}$, but as the zero operator on $\mathcal{H}_{\mathbb{1}_{G}}$.

However, in some cases the answer to Question 1.2.29 is positive. We will come back to this at a later point.

### 1.2.4 Unbounded operators affiliated to a finite von Neumann algebra

Given a tracial $*$-algebra $(\mathcal{A}, \tau)$ we already introduced a Hilbert space $\mathcal{H}_{\tau}$ and a von Neumann algebra $\mathcal{N}_{\tau}$. In this subsection we want to introduce the algebra of unbounded operators, affiliated to a finite von Neumann algebra.

Definition 1.2.31. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{N}=\left(\mathcal{N}^{\prime}\right)^{\prime} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. The von Neumann algebra is called finite, if every isometry $V \in \mathcal{N}$ is already unitary.

Note that a von Neumann algebra with a faithful trace is already finite. This applies to the von Neumann algebras we get from the GNS construction. Before we introduce unbounded operators affiliated to a finite von Neumann algebra, we want to briefly describe what unbounded operators are and what difficulties arise with them. Let us start with an example. Consider the Hilbert space $\mathcal{H}=\ell^{2}(\mathbb{N})=\left\{\left(a_{1}, a_{2}, \ldots\right) \mid \sum a_{i}^{2} \leq \infty\right\}$. Denote by $e_{n} \in \mathcal{H}$ the basis vector that has 1 in the $n$-th coordinate and zeros elsewhere. Let $M$ be the linear operator defined by $M\left(e_{n}\right)=n e_{n}$. It is easy to see that $\|M\|=\infty$ and that the spectrum $\sigma(M)$ of $M$ is unbounded. As a consequence the operator $M$ is not defined everywhere on $\mathcal{H}$, but only on a subspace $\mathcal{D}(M)$, called the domain of $M$. We will call an operator $T$ closed, if its graph $\Gamma(T)=\{(x, x T) \mid x \in \mathcal{D}(T)\} \subseteq \mathcal{H} \oplus \mathcal{H}$ is closed. The operator $T$ is called densely defined, if $\mathcal{D}(T)$ is dense in $\mathcal{H}$. Dealing with unbounded operators is not that easy. For example the set of unbounded operators does not form an algebra, since every unbounded operator is only defined on its own domain. Furthermore the definitions of the adjoint is not straight forward. All these problems are solved when we look at unbounded operators affilliated to a finite von Neumann algebra.

Definition 1.2.32. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra. We denote by $\mathcal{U}$ the set of all unbounded operators $(T, \mathcal{D}(T))$ of $\mathcal{H}$ that satisfy

- $T$ is densely defined,
- $T$ is closed,
- $T$ is affiliated to $\mathcal{N}$ that means we have

$$
T U=U T
$$

for all unitaries $U \in \mathcal{N}^{\prime}$.
Proposition 1.2.33. [Rei, Theorem 11.20] The set $\mathcal{U}$ is a complex *-algebra that contains $\mathcal{N}$ as a subalgebra.
$\mathcal{U}$ is called the ring of unbound operators affiliated to $\mathcal{N}$. If $\mathcal{N}=\mathcal{N}(G)$ is a group von Neumann algebra we will write $\mathcal{U}(G)$ for $\mathcal{U}$. When $(A, \tau)$ is a tracial *-algebra and $\mathcal{N}_{\tau}$ its von Neumann algebra representation we will write $\mathcal{U}_{\tau}$ for
$\mathcal{U}$. Before describing some properties of the $\operatorname{ring} \mathcal{U}$ we want to give two more algebraic ways to construct it starting from $\mathcal{N}$. The first method is to see that $\mathcal{U}$ is the Ore localization of $\mathcal{N}$.

Definition 1.2.34. Let $R$ be a ring and $T \subseteq R$ a multiplicative set of non-zerodivisors. A right ring of fractions of $R$ with respect to $T$ is a ring $S$ in which $R$ embeds and that satisfies
(1) Every element of $T$ becomes invertible in $S$.
(2) Every element of $S$ can be written as $r t^{-1}$ with $r \in R, t \in T$.

In the commutative setting, we know that such a ring of fractions always exists. However in the non-commutative setting we have a problem. Let $R$ be a non commutative domain and $S=R \backslash\{0\}$ and consider the set $R S^{-1}=\left\{r s^{-1} \mid\right.$ $r \in R, s \in S\}$. We want to define a multiplication on this set, however we run into a problem. Let $r_{1} s_{1}^{-1}, r_{2} s_{2}^{-1} \in R S^{-1}$. To define $r_{1} s_{1}^{-1} r_{2} s_{2}^{-1}$ we need to find a way to pass $s_{1}^{-1}$ past $r_{2}$. So we want to find $r_{3} \in R, s_{3} \in S$ such that $s_{1}^{-1} r_{2}=r_{3} s_{3}^{-1}$. Multiplying by $s_{1}$ from the left and by $s_{3}$ from the right gives exactly the Ore condition.

Definition 1.2.35. Let $R$ be a ring and let $S \subseteq R$ be a multiplicative subset of non-zero divisors of $R$. The right Ore condition for $S$ states that for every $s \in S, r \in R$ we have $r S \cap s R \neq \emptyset$. The left Ore condition is defined similarly.

By the problem defined above, a ring of fraction does not exist with respect to any multiplicative set $T$.

Theorem 1.2.36. [GW04, Theorem 6.2] Let $R$ be a ring and let $T \subseteq R$ be a set of non-zero-divisors. Then there exists a right ring of fractions $R T^{-1}$ if and only if $(R, T)$ satisfy the right Ore condition.

A ring of fractions satisfies the following universal property.
Proposition 1.2.37. [GW04, Proposition 6.3, 6.5] Let $R$ be a ring, $T \subseteq R$ a set of non-zero-divisors that satisfies the right Ore condition and $S=R T^{-1}$ be a ring of fractions. Let $\varphi: R \rightarrow R^{\prime}$ be a $T$ inverting ring homomorphism. Then there is a unique extension $\bar{\varphi}: S \rightarrow R^{\prime}$ of $\varphi$.

Let us now come back to our situation. We have the following characterization of $\mathcal{U}_{\tau}$.

Proposition 1.2.38. [Rei, Proposition 2.8] Let $\mathcal{N}$ be a finite von Neumann algebra and let $T \subseteq \mathcal{N}$ be the set of non-zero divisors. Then $T$ satisfies both Ore conditions and $\mathcal{N} S^{-1} \cong \mathcal{U}$.

This characterization allows us to extend the rank function $\mathrm{rk}_{\tau}$ from $\mathcal{N}_{\tau}$ to $\mathcal{U}_{\tau}$. It is easy to see that the set $T \subseteq \mathcal{N}_{\tau}$ of non-zero divisors is exactly given by those elements $a \in \mathcal{N}_{\tau}$ with $\operatorname{rk}_{\tau}(a)=1$. Otherwise we would have
$P_{\operatorname{ker}(a)} \cdot a=0 \in \mathcal{N}$. Thus for an element $u=a b^{-1} \in \mathcal{U}_{\tau}$ with $a, b \in \mathcal{N}_{\tau}$ and $\mathrm{rk}_{\tau}(b)=1$ we can set

$$
\mathrm{rk}_{\tau}(u)=\mathrm{rk}_{\tau}(a)
$$

The last characterization of $\mathcal{U}_{\tau}$ we want to give is to see it as the completion of $\mathcal{N}_{\tau}$ with respect to its rank metric. The rank metric on $\mathcal{N}_{\tau}$ is defined by

$$
\mathrm{d}(a, b)=\mathrm{rk}_{\tau}(a-b)
$$

Since $\tau$ is faithful on $\mathcal{N}_{\tau}$, it is easy to see that from $d(a, b)=0$ it follows that $a=b$. We have the following proposition.
Proposition 1.2.39. [Tho08, Lemma 2.2] The completion of a tracial von Neumann algebra $(\mathcal{N}, \tau)$ is naturally identified with the algebra $\mathcal{U}_{\tau}$.

We want briefly sketch how to see that the rank completion of $\mathcal{N}_{\tau}$ and its Ore Localization are isomorphic. For that let $T \in \mathcal{N}_{\tau}$ be a non-zero-divisor. We want to find a sequence $\left(T_{n}\right)_{n} \in \mathcal{N}_{\tau}$ that represents $\frac{1}{T}$ in $\left(\overline{\mathcal{N}_{\tau}}\right)_{\mathrm{rk}}$, where the latter one denotes the rank completion of $\mathcal{N}_{\tau}$. By the polar decomposition and the fact that $T$ is a non-zero-divisor, we can write $T$ as a product of an isometry and a positive operator. Since $\mathcal{N}_{\tau}$ is a finite von Neumann algebra, every isometry is already unitary and therefore invertible. Thus we can assume that $T$ is already a positive operator. Let $E$ be the projection valued measure coming from the spectral resolution of $T$. Let $P_{n}=1-E\left(\left(0, \frac{1}{n}\right)\right)$ be a projection and consider the operator $T_{n}=P_{n} T P_{n}$. If we consider $T_{n}$ as an operator on $\mathcal{H}_{n}=\mathcal{H}_{\tau} P_{n}$ it is bounded from below and therefore invertible. Then the operators $T_{n}^{-1} \oplus 0: \mathcal{H}_{n} \oplus \mathcal{H}_{n}{ }^{\perp}=\mathcal{H}_{\tau} \rightarrow \mathcal{H}_{\tau}$ converge in the rank metric to the inverse of $T$.

We have seen three ways to think about the algebra $\mathcal{U}_{\tau}$. We have an analytic approach as unbound operators, an algebraic approach as the Ore localization of $\mathcal{N}_{\tau}$ and we can see it as the rank completion of $\mathcal{N}_{\tau}$. It is helpful to have all three approaches in mind. The algebra $\mathcal{U}_{\tau}$ is very large in general. The reason why we introduced it is that it is von Neumann regular [Rei, Proposition 2.4] We will present this definition and some interesting properties of von Neumann regular rings in the next chapter.

## 1.3 von Neumann regular rings, epic rings and Sylvester rank functions

In this chapter we want to introduce some more advanced algebraic concepts. Especially we want to talk about Sylvester rank functions and von Neumann regular rings, things we already mentioned in Chapter 1.2.

### 1.3.1 Von Neumann regular rings

In this section we want to explain what von Neumann regular rings are. Good sources to learn more details about von Neumann regular rings are [Rot08] and [Goo79]. We will start with its definition.

Definition 1.3.1. Let $R$ be a ring. An element $x \in R$ is called von Neumann regular if there is $y \in R$ with $x y x=x$. A ring $\mathcal{U}$ is called von Neumann regular if all its elements are von Neumann regular.

As mentioned in the previous chapter, the algebra of unbound operators affiliated to a finite von Neumann algebra is von Neumann regular. This will be our main example. Obviously division rings are von Neumann regular.

We have the following characterization of von Neumann regular rings.
Proposition 1.3.2. [Goo'79, Theorem 1.1][Ste'75, Chapter 1, 12.1] Let $R$ be a ring. Then the following are equivalent:
(1) $R$ is von Neumann regular.
(2) Every finitely generated right (left) ideal is generated by an idempotent.
(3) every right (left) $R$-module is flat.

We further have the following properties.
Proposition 1.3.3. [Goo'79, Theorem 1.7, Theorem 1.11, Proposition 2.6] Let $\mathcal{U}$ be a von Neumann regular ring. Then the following holds.
(1) For every $n \in \mathbb{N}$, the ring $\operatorname{Mat}_{n}(\mathcal{U})$ is von Neumann regular.
(2) All finitely generated submodules of a projective left (right) module $P$ are direct summands of $P$.
(3) Let $P$ be a countably generated projective left (right) $\mathcal{U}$-module. Then $P$ is a direct sum of cyclic submodules, each of which isomorphic to a principal left ideal of $\mathcal{U}$.

The concept of a von Neumann regular ring already gives many interesting properties, however when we combine it with a specific involution, it becomes an even more powerful concept.

Definition 1.3.4. Let $(R, *)$ be a $*$-ring. The involution $*$ is called proper if for all $r \in R: r r^{*}=0$ implies $r=0$. Further the involution is called $n$-positive definite if $\sum_{i=1}^{n} r_{i} r_{i}^{*}=0$ implies $r_{1}=\ldots=r_{n}=0$. The involution is called positive definite if it is $n$-positive definite for all $n$.

Obviously $\mathbb{C}$ and therefore $\operatorname{Mat}_{n}(\mathbb{C})$ are positive definite. Since the trace on $\operatorname{Mat}_{n}\left(\mathcal{U}_{\tau}\right)$ is faithful, it is easy to see that the algebra $\mathcal{U}_{\tau}$ is positive definite. Combining the notions of von Neumann regular rings and proper involution we get the following definition.
Definition 1.3.5. Let $(R, *)$ be a $*$-ring. We call $R *$-regular if it is von Neumann regular and its involution is proper.

To be consistent with the notation of $\mathcal{U}_{\tau}$ from the previous chapter we will denote a $*$-regular ring by $\mathcal{U}$. We have the following proposition, which can be found in [Jai19].

Proposition 1.3.6. Let $\mathcal{U} a *$-regular ring and $x \in \mathcal{U}$. Then
(1) $\mathcal{U} x=\mathcal{U} x^{*} x$ and $x \mathcal{U}=x x^{*} \mathcal{U}$.
(2) There exist unique projections e, $f \in \mathcal{U}$ such that $\mathcal{U} x=\mathcal{U}$ e and $f \mathcal{U}=x \mathcal{U}$. We put $e=R P(x)$ and $f=L P(x)$.
(3) There exists a unique $y \in e \mathcal{U} f$ such that $y x=e$ and $x y=f$. We put $y=x^{[-1]}$ and call it the relative inverse of $x$.
(4) $R P(x)=R P\left(x^{*} x\right)=L P\left(x^{*}\right)$ and $\left(x^{*}\right)^{[-1]}=\left(x^{[-1]}\right)^{*}$.
(5) $\left(x^{*} x\right)^{[-1]}=x^{[-1]}\left(x^{*}\right)^{[-1]}$ and $x^{[-1]}=\left(x^{*} x\right)^{[-1]} x^{*}$.
(6) If $x$ is self-adjoint then $x$ commutes with $x^{[-1]}$.

We have the following alternative description of the relative inverses.
Proposition 1.3.7. Let $\mathcal{U}$ be $a *$-regular ring and $x \in \mathcal{U}$. Then the relative inverse of $x$ is the unique element $y$, such that both $x y$ and $y x$ are projections with $x y x=x$ and $y x y=y$.

Proof. It follows directly from the previous proposition that the relative inverse has the given properties. For example using (5) we get

$$
y=y y^{*} x^{*}=y f^{*}=y f=y x y
$$

On the other hand, from $x y x=x$ it follows $\mathcal{U} x \subset \mathcal{U} y x \subset \mathcal{U} x y x=\mathcal{U} x$ and therefore $\mathcal{U} x=\mathcal{U} y x$. Similarly we get $x \mathcal{U}=x y \mathcal{U}$ and therefore, by the uniqueness in 1.3.6(2), we have $y x=e$ and $x y=f$. Last we have

$$
y=y x y=y x y x y=e y f \in e \mathcal{U} f
$$

and therefore $y=x^{[-1]}$.
Let us consider again the example of a tracial $*$-algebra $(\mathcal{A}, \tau)$. We have seen in the previous chapter that we can map $\mathcal{A}$ to a $*$-regular ring by first representing it as bounded operators on the Hilbert space $\mathcal{H}_{\tau}$, then take the completion with respect to the weak operator topology to get a von Neumann algebra and then taking the Ore localization. We have the following diagram:


The issue with this construction is that only knowing $\mathcal{A}$ does not yield much information about $\mathcal{U}_{\tau}$. The situation is comparable with embedding $\mathbb{Z}$ into a field. One would naturally choose $\mathbb{Q}$ and not $\mathbb{R}, \mathbb{C}$ or some function field over $\mathbb{C}$. We have the following proposition.

Proposition 1.3.8. Let $R$ be $a *$-subring of $a *$-regular ring $\mathcal{U}$. Then there is a smallest $*$-regular subring of $\mathcal{U}$ containing $R$, which is denoted by $\mathcal{R}(R, \mathcal{U})$.
Proof. Let $\left\{\mathcal{U}_{i}\right\}_{i \in I}$ be the set of all $*$-regular subrings of $\mathcal{U}$ that contain $R$.
Claim. The ring $\mathcal{R}(R, \mathcal{U})=\bigcap_{i \in I} \mathcal{U}_{i}$ is a *-regular ring.
To see this we have to show that for each $x \in \mathcal{R}(R, \mathcal{U})$ its relative inverse is also in $\mathcal{R}(R, \mathcal{U})$. For each $i$ let $y_{i} \in \mathcal{U}_{i}$ be the relative inverse of $x$ in $\mathcal{U}_{i}$. By proposition 1.3 .7 we have $y_{i}=y_{j}=x^{[-1]} \in \mathcal{U}$ for all $i, j \in I$. Therefore the claim follows.

Define now $R_{0}=R$ and $R_{i+1}$ as the subring of $\mathcal{U}$ generated by $R_{i}$ and all its relative inverses. By construction the ring

$$
\bigcup_{i \in \mathbb{N}} R_{i}
$$

is *-regular and contains $R$. Therefore

$$
\mathcal{R}(R, \mathcal{U})=\bigcup_{i \in \mathbb{N}} R_{i}
$$

Definition 1.3.9. The ring $\mathcal{R}(R, \mathcal{U})$ is called the $*$-regular closure of $R$ in $\mathcal{U}$.
Using this we can complete our diagram from above. We have


Let us consider a more explicit example. Let $G=\mathbb{Z}=\langle z\rangle$ the cyclic group and $\mathcal{A}=\mathbb{C}[G]$ with the trace $\tau$ as in 1.1. We then get $\mathcal{H}_{\tau}=\ell^{2}(\mathbb{Z})$. Fourier transformation gives us $\ell^{2}(\mathbb{Z}) \cong \mathrm{L}^{2}[-\pi, \pi]$. Multiplication by $z \in G$ acts like a shift on a basis element $k \in \mathbb{Z} \subseteq \ell^{2}(\mathbb{Z}): k \mapsto k . z=k+1$. In $\mathrm{L}^{2}[-\pi, \pi]$ this corresponds to shifting a basis vector $\frac{e^{i k x}}{\sqrt{2 \pi}}$ to $\frac{e^{i(k+1) x}}{\sqrt{2 \pi}}$. Note that $\mathbb{C}[\mathbb{Z}] \cong$ $\mathbb{C}\left[z, z^{-1}\right]$. This gives the following diagram:


Here $\mathrm{L}[-\pi, \pi]$ is the algebra of measurable functions. For more details about this example see [Kam19, Example 2.26]. Generalizing the previous example, we get for any torsion-free elementary amenable group $G$ the following diagram.


In general it is very hard to find an explicit description of $\mathcal{N}(G)$ or $\mathcal{U}(G)$, however the algebra $\mathcal{R}(\mathbb{C}[G], \mathcal{U}(G))$ is easier to handle.

Let us collect some more properties about *-regular rings. First let us show that $*$-regular rings are closed under quotients.

Proposition 1.3.10. Let $\mathcal{U}$ be $a *$-regular ring and let $I$ be a both-sided ideal of $\mathcal{U}$. Then the ring $\mathcal{U} / I$ is $*$-regular.

Proof. Let us first show that $I$ is $*$-closed. For that let $x \in I$. Thus we have $x x^{*} \in x x^{*} \mathcal{U}=x \mathcal{U} \subseteq I$ and therefore $x^{*} \in x^{*} \mathcal{U}=x^{*} x \mathcal{U} \subseteq I$. Moreover, by the same argument, if $x^{*} x \in I$, then $x^{*} U=x^{*} x \mathcal{U} \subseteq I$ and therefore the involution * is proper on $\mathcal{U} / I$.

Remember from Proposition 1.3.2 that in a von Neumann regular ring every finitely generated ideal is generated by an idempotent. In a *-regular ring we have the following stronger property.

Proposition 1.3.11. [Neu16, Part 2, Chapter 4, Theorem 4.5] Let $\mathcal{U}$ be $a$ *regular ring. Then every finitely generated right (left) ideal is generated by a unique projection.

We already mentioned that it makes sense not only to map a $*$-algebra in any $*$-regular ring $\mathcal{U}$, but somehow in the smallest possible $*$-regular subring of $\mathcal{U}$. We want to formalize this in the next subsection.

### 1.3.2 Epic $R$-Rings

In this section we want to introduce the notion of epic homomorphisms. We follow [Ste75], where more details can be found. Let $\varphi: R \rightarrow S$ be a ring homomorphism. We say that $\varphi$ is epic if for any two ring homomorphisms $\alpha_{1}, \alpha_{2}: S \rightarrow Q$, from $\alpha_{1} \circ \varphi=\alpha_{2} \circ \varphi$ already follows that $\alpha_{1}=\alpha_{2}$. More generally then $\varphi$ being epic, we call an element $s \in S$ dominated by $\varphi$, if from $\alpha_{1} \circ \varphi=\alpha_{2} \circ \varphi$ it follows that $\alpha_{1}(s)=\alpha_{2}(s)$. The set of elements of $S$ that are dominated by $\varphi$ is called the dominion of $\varphi$ and is denoted $\operatorname{by} \operatorname{dom}(\varphi)$. So $\varphi$ is epic if and only if the dominion of $\varphi$ is $S$. Let us look at some examples.

## Example 1.3.12.

(1) Let $\varphi: R \rightarrow S$ be a surjective ring homomorphism. Then $\varphi$ is epic.
(2) The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic.
(3) The inclusion $\varphi: \mathbb{Z} \hookrightarrow \mathbb{Q}[i]$ is not epic. For that let $\alpha_{1}: \mathbb{Q}[i] \rightarrow \mathbb{C}$ be the standard embedding and $\alpha_{2}=\sigma \circ \alpha_{1}$ where $\sigma$ denotes the complex conjugation. The dominion of $\varphi$ is given by $\mathbb{Q}$.

An epic $R$-ring is a pair $(S, \varphi)$ where $\varphi: R \rightarrow S$ is an epic ring homomorphism. We call two epic $R$-rings $\left(S_{1}, \varphi_{1}\right)$ and $\left(S_{2}, \varphi_{2}\right)$ isomorphic, if there is an isomorphism $\alpha: S_{1} \rightarrow S_{2}$ with $\alpha \circ \varphi_{1}=\varphi_{2}$. We have the following characterization of epic homomorphism.

Lemma 1.3.13. Let $\varphi: R \rightarrow S$ be a ring homomorphism. Then the following are equivalent.
(1) $s \in \operatorname{dom}(\varphi)$.
(2) For all $S$-bimodules $M$ and for all $x \in M$ such that $r x=x r$ for all $r \in \varphi(R)$, we have $s x=x s$.
(3) $1 \otimes s=s \otimes 1 \in S \otimes_{R} S$.
(4) For all right $S$-modules $M$ and $N$ and $\alpha \in \operatorname{Hom}_{R}(M, N)$ we have $\alpha(x s)=$ $\alpha(x) s$ for all $x \in M$.

As a direct corollary we have the following useful characterization.
Corollary 1.3.14. Let $\varphi: R \rightarrow S$ be a ring homomorphism. Then $\operatorname{dom}(\varphi)=$ $\left\{s \in S \mid 1 \otimes s=s \otimes 1 \in S \otimes_{R} S\right\}$. Further is $\varphi$ epic if and only if the multiplication map

$$
m: S \otimes_{R} S \rightarrow S, s_{1} \otimes s_{2} \mapsto s_{1} s_{2}
$$

is an isomorphism of $S$-bimodules.
Let us now prove the lemma.
Proof. (1) $\Rightarrow(2)$ : Define the ring $Q=S \times M$ where the multiplication is given by $\left(s_{1}, m_{1}\right)\left(s_{2}, m_{2}\right)=\left(s_{1} s_{2}, s_{1} m_{2}+m_{1} s_{2}\right)$. For one fixed $x \in M$ let $\alpha, \beta: S \rightarrow \mathbb{Q}$ be given by

$$
\alpha(s)=(s, 0) \quad \text { and } \quad \beta(s)=(s, s x-x s) .
$$

Now we have

$$
\alpha \circ \varphi=\beta \circ \varphi \Leftrightarrow r x=x r \quad \forall r \in \varphi(R) .
$$

If $b \in \operatorname{dom}(\varphi)$, we have $\alpha(b)=\beta(b)$, which is equivalent to $b x=x b$.
$(2) \Rightarrow(3)$ : Follows immediately with $M=S \otimes_{R} S$ and $x=1 \otimes 1 \in M$. (3) $\Rightarrow$ (4) : Fix $x \in M$ and define $\gamma: S \times_{R} S \rightarrow N, s_{1} \otimes s_{2} \mapsto \alpha\left(x s_{1}\right) s_{2}$. From $1 \otimes s=s \otimes 1$ we get $\alpha(x s)=\alpha(x) s$.
$(4) \Rightarrow(1):$ Let $Q$ be a ring and let $\alpha, \beta: S \rightarrow Q$ be ring homomorphisms with $\alpha \circ \varphi=\beta \circ \varphi$. Let us consider $Q$ as a $S$-right module with multiplication given by $q \cdot s=q \cdot \beta(s)$. Because $\alpha \circ \varphi=\beta \circ \varphi$ we have $\alpha \in \operatorname{Hom}_{R}(S, Q)$, where we consider both, $S$ and $Q$ as $S$-right modules. Therefore we have

$$
\alpha(b)=\alpha(1 \cdot b)=\alpha(1) \cdot b=1 \cdot \beta(b)=\beta(b)
$$

We have the following characterization of von Neumann regular rings.
Proposition 1.3.15. Let $\varphi: \mathcal{U} \rightarrow S$ be a ring homomorphism and $\mathcal{U}$ be von Neumann regular. If $\varphi$ is epic it is already surjective.

Obviously the other implication is also true.

Proof. We know that $\mathcal{U}_{1}=\varphi(\mathcal{U}) \subset S$ is von Neuman regular. Tensoring the monomorphism $\mathcal{U}_{1} \hookrightarrow S$ with the left $\mathcal{U}_{1}$-modules $S / \mathcal{U}_{1}$ and $S$ we get the following commutative diagram.


Note that the horizontal maps are injective by proposition 1.3.2. By $1 \in \mathcal{U}_{1}$ we have

$$
\beta(\bar{s})=1 \otimes \bar{s}=\gamma(1 \otimes s)=\gamma(s \otimes 1)=s \otimes \overline{1}=0
$$

Since $\beta$ is injective we get $\bar{s}=0$ and therefore $s \in \mathcal{U}_{1}$ for all $s \in S$.
A direct consequence of the previous proposition is that for every von Neumann regular ring $\mathcal{U}$ and every ring homomorphism $\varphi: \mathcal{U} \rightarrow S$ we have $\operatorname{dom}(\varphi)=\varphi(\mathcal{U})$. Further we have the following corollary which can be found in [Jai19].

Corollary 1.3.16. Let $R$ be a ring and let $f_{1}: R \rightarrow \mathcal{U}_{1}, f_{2}: R \rightarrow \mathcal{U}_{2}$ be an epic homomorphism into von Neumann regular rings $\mathcal{U}_{1}$ and $\mathcal{U}_{2}$. Let $\gamma_{1}: \mathcal{U}_{1} \rightarrow S$ and $\gamma_{2}: \mathcal{U}_{2} \rightarrow S$ be homomorphisms with $\gamma_{1} \circ f_{1}=\gamma_{2} \circ f_{2}$. Then $\operatorname{im}\left(\gamma_{1}\right)=$ $\operatorname{im}\left(\gamma_{2}\right)$.

Proof. Let $\alpha, \beta: S \rightarrow Q$ be ring homomorphisms, such that $\alpha \circ \gamma_{1} \circ f_{1}=$ $\beta \circ \gamma_{1} \circ f_{1}$. We want to show that $\operatorname{dom}\left(\gamma_{1} \circ f_{1}\right)=\operatorname{im}\left(\gamma_{1}\right)$. Since $f_{1}$ is epic, we have $\alpha \circ \gamma_{1}=\beta \circ \gamma_{1}$, in other words for all $u \in \mathcal{U}_{1}: \alpha\left(\gamma_{1}(u)\right)=\beta\left(\gamma_{1}(u)\right)$. Therefore we get $\operatorname{im}\left(\gamma_{1}\right) \subseteq \operatorname{dom}\left(\gamma_{1} \circ f_{1}\right)$. On the other hand we can show that $\operatorname{dom}\left(\gamma_{1} \circ f_{1}\right) \subseteq \operatorname{dom}\left(\gamma_{1}\right)=\operatorname{im}\left(\gamma_{1}\right)$. For that let $s \in \operatorname{dom}\left(\gamma_{1} \circ f_{1}\right)$ and let $\alpha, \beta$ : $S \rightarrow Q$ such that $\alpha \circ \gamma_{1}=\beta \circ \gamma_{1}$. In particular we have $\alpha \circ \gamma_{1} \circ f_{1}=\alpha \circ \gamma_{1} \circ f_{1}$ and therefore $\alpha(s)=\beta(s)$. Thus $\operatorname{dom}\left(\gamma_{1} \circ f_{1}\right) \subseteq \operatorname{dom}\left(\gamma_{1}\right)$. In a similar way we get $\operatorname{dom}\left(\gamma_{2} \circ f_{2}\right)=\operatorname{im}\left(\gamma_{2}\right)$. Since $\gamma_{1} \circ f_{1}=\gamma_{2} \circ f_{2}$, we have $\operatorname{im}\left(\gamma_{1}\right)=\operatorname{im}\left(\gamma_{2}\right)$.

When we talk about *-rings, in the notion of epic we have to replace homomorphism by $*$-homomorphism. It is then easy to see that the dominion of a *-homomorphism is closed under the $*$-operation.

We want to finish this last subsection with showing why it was useful to introduce the $*$-regular closure.

Proposition 1.3.17. Let $R$ be $a *$-ring, $\mathcal{U}$ be $a *$-regular ring and let $\varphi: R \rightarrow \mathcal{U}$ be $a *$-homomorphism. Then $\varphi$ is epic if and only if $\mathcal{U}=\mathcal{R}(\varphi(R), \mathcal{U})$.

Proof. " $\Rightarrow$ :" Assume first that $\varphi$ is epic. Then the inclusion $\varphi(R) \hookrightarrow \mathcal{U}$ is epic and therefore the inclusion $i: \mathcal{R}(\varphi(R), \mathcal{U}) \hookrightarrow \mathcal{U}$ is epic. Since $\mathcal{R}(\varphi(R), \mathcal{U})$ is von Neumann regular we get $\operatorname{dom}(i)=\operatorname{im}(i)$ and therefore $\mathcal{R}(\varphi(R), \mathcal{U})=\mathcal{U}$.
$" \Leftarrow: "$ Let $S=\left\{u \in \mathcal{U} \mid 1 \otimes u=u \otimes 1 \in \mathcal{U} \otimes_{\varphi(R)} \mathcal{U}\right\}$ be the dominion of $\varphi$. It is easy to see that $S$ is a $*$-subring of $\mathcal{U}$ that contains $\varphi(R)$. We now want to show
that $S$ is closed under taking relative inverses. For that let $r \in S$ be self-adjoint and let $s \in \mathcal{U}$ be its relative inverse. By 1.3.6 we have $r s=s r$. That gives us

$$
\begin{aligned}
1 \otimes s & =1 \otimes s r s \\
& =1 \otimes r s s \\
& =r \otimes s s \\
& =r s r \otimes s s+s s \otimes r(1-s r) \\
& =r s \otimes r s s+s s r \otimes(1-s r) \\
& =s r \otimes s r s+s r s \otimes(1-s r) \\
& =s \otimes r s+s \otimes(1-r s) \\
& =s \otimes 1 .
\end{aligned}
$$

We have just seen that all the relative inverses of self-adjoint elements $r \in S$ belong to $S$. Let now $x \in S$ be any element. Then $x^{*} x$ is self-adjoint, so its relative inverse belongs to $S$. By 1.3 .6 we have $x^{[-1]}=\left(x^{*} x\right)^{[-1]} x^{*} \in S$ since $S$ is a $*$-subring of $\mathcal{U}$. Therefore $S$ is a $*$-regular subring of $\mathcal{U}$ which implies $S=\mathcal{U}=\mathcal{R}(\varphi(R), \mathcal{U})$.

### 1.3.3 Sylvester rank functions

## Sylvester matrix rank functions

In linear algebra we consider finite dimensional vector spaces over fields. We learn that after choosing a basis a linear map between two $K$-vector spaces $V$ and $W$ can be represented my a matrix $A$ over $K$. In this setting it is easy to define the dimension of $V$ and $W$ or the rank of the matrix $A$. But what happens when we replace $K$ by an arbitrary ring $R$ and instead of $K$ vector spaces we consider $R$-modules? Is there still some notion or rank and dimension? These questions lead us to the definition of Sylvester rank functions. These were originally introduced by P. Malcomson in [Mal80] under the name algebraic rank functions. Most of the results we present in this section can be found in [Lóp21]. We will begin with Sylvester matrix rank functions. In the following we denote by $\operatorname{Mat}(R)$ the set of all matrices over a ring $R$.

Definition 1.3.18. Let $R$ be a unitary ring. A Sylvester matrix rank function on $R$ is a map rk : $\operatorname{Mat}(R) \rightarrow \mathbb{R}_{\geq 0}$ such that
(1) $\operatorname{rk}(1)=1, \mathrm{rk}\left(\begin{array}{ccc}0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0\end{array}\right)=0$.
(2) $\operatorname{rk}(A \cdot B) \leq \operatorname{rk}(A), \operatorname{rk}(B)$ for all matrices $A, B$ over $R$ that can be multiplied.
(3) $\operatorname{rk}(A \oplus B)=\operatorname{rk}(A)+\operatorname{rk}(B)$ for all matrices $A, B$ of the appropriate dimension.
(4) $\operatorname{rk}\left(\left(\begin{array}{cc}A & C \\ 0 & B\end{array}\right)\right) \geq \operatorname{rk}(A)+\operatorname{rk}(B)$ for all matrices $A, B, C$ of the appropriate dimension.

For a ring $R$ we denote by $\mathbb{P}(R)_{\text {Mat }}$ the space of all Sylvester matrix rank functions on $R$. Let us consider some examples.

Example 1.3.19. (1) A division ring $\mathcal{D}$ has only one Sylvester matrix rank function $\mathrm{rk}_{\mathcal{D}}$, which can be defined as the inner rank. For a matrix $A \in$ $\operatorname{Mat}_{n, m}(R)$ over a ring $R$ the inner rank of $A$ is given by

$$
\rho(A)=\min \left\{k \in \mathbb{N} \mid \exists B \in \operatorname{Mat}_{n, k}(R), C \in \operatorname{Mat}_{k, m}(R): A=B C\right\}
$$

Note that the inner rank is not for all rings a Sylvester matrix rank function. The rings where the inner rank is a Sylvester matrix rank function are called Sylvester domains.
(2) Let $R$ be a ring. If there are $m, n \in \mathbb{N}, n \neq m$ with $R^{n} \cong R^{m}$, then $\mathbb{P}(R)=\emptyset$ since

$$
n=\operatorname{rk}\left(\operatorname{Id}_{n}\right)=\operatorname{rk}\left(\operatorname{Id}_{m}\right)=m
$$

which is obviously nonsense.
(3) Let $\mathcal{D}$ be a division ring and $R=\operatorname{Mat}_{n}(\mathcal{D})$. Then $\mathbb{P}(\mathcal{D})=\left\{\frac{1}{n} \mathrm{rk}_{\mathcal{D}}\right\}$.

The set $\mathbb{P}(R)$ of Sylvester matrix rank function on a ring $R$ is a compact convex subset of all real valued functions on the matrices over $R$. We denote by $\mathrm{E}(R) \subset \mathbb{P}(R)$ the set of extreme points, that means the set of all rank functions that can not be expressed as a non trivial convex combination of points in $\mathbb{P}(R)$. We have $\mathbb{P}(R)=\langle\overline{\mathrm{E}(R)}\rangle$, where the latter one is the convex hull of $\mathrm{E}(R)$. For an arbitrary ring $R$ it is hopeless to determine $\mathbb{P}(R)$ or $\mathrm{E}(R)$. However in some special cases we can do it.

Example 1.3.20. (1) Let $\mathcal{D}$ be a division ring and let $R=\operatorname{Mat}_{n_{1}}(\mathcal{D}) \oplus$ $\operatorname{Mat}_{n_{2}}(\mathcal{D})$ Let $\pi_{i}$ be the projection onto the $i$ th summand and let $\mathrm{rk}_{i}=$ $\frac{1}{n_{i}} \circ \pi_{i}$. Then $\mathrm{E}(R)=\left\{\mathrm{rk}_{1}, \mathrm{rk}_{2}\right\}$.
(2) Let $G$ be a finite group and $R=\mathbb{C}[G]$. Let $\varphi_{1}, \ldots, \varphi_{n}$ be the irreducible representations of $G$ with dimensions $d_{1}, \ldots, d_{n}$. Let $\mathrm{rk}_{i}=\frac{1}{d_{i}} \mathrm{rk}_{\mathbb{C}} \circ \varphi_{i}$. Then $\mathrm{E}(R)=\left\{\mathrm{rk}_{1}, \ldots, \mathrm{rk}_{n}\right\}$.
(3) Let $R=\mathbb{Z}$. Besides the $\mathbb{Q}$-rank $\mathrm{rk}_{\mathbb{Q}}$ for each $p \in \mathbb{P}$ we have a map $\mathbb{Z} \rightarrow$ $\mathbb{Z} / p \mathbb{Z} \cong \mathbb{F}_{p}$ and therefore a rank function $\mathrm{rk}_{p}$. We have $\left\{\mathrm{rk}_{\mathbb{Q}}\right\} \cup\left\{\mathrm{rk}_{\mathbb{F}_{p}} \mid\right.$ $p \in \mathbb{P}\} \subseteq E(\mathbb{Z})$. However there are more extreme rank functions. See [Lóp21, Section 2.3] for details.

The most fruitful source of Sylvester matrix rank functions comes from ring homomorphisms. For that let $\varphi: R \rightarrow S$ be a ring homomorphism. Then we get a map $\varphi^{\#}: \mathbb{P}(S) \rightarrow \mathbb{P}(R)$ with $\varphi^{\#}(\mathrm{rk})(A)=\operatorname{rk}(\varphi(A))$, where $A$ is a matrix over $R$. We want to collect some properties of Sylvester matrix rank functions.

Proposition 1.3.21. [Lóp21, Proposition 1.2.2][Jai19, Proposition 5.1]. Let $R$ be a ring and $\mathrm{rk} \in \mathbb{P}(R)$. Then for all matrices $A, B$ of appropriate size the following holds.
(1) If $A$ has dimension $n \times m$ then $\operatorname{rk}(A) \leq m, n$.
(2) If $A$ has dimension $n \times n$ and is invertible then $\operatorname{rk}(A)=n$.
(3) $\operatorname{rk}(A+B) \leq \operatorname{rk}(A)+\operatorname{rk}(B)$.
(4) $\operatorname{rk}\left(\left(\begin{array}{ll}A & 0\end{array}\right)\right)=\operatorname{rk}\left(\binom{A}{0}\right)=\operatorname{rk}(A)$ for any zero matrix.
(5) If $A \in \operatorname{Mat}_{n, m}(R), B \in \operatorname{Mat}_{m, k}(R)$ then

$$
\operatorname{rk}(A B) \geq \operatorname{rk}(A)+\operatorname{rk}(B)-m
$$

(6) Multiplying by a an invertible square matrix does not change the rank.

## Sylvester module rank functions

When we turn back to matrices over a field $K$ the rank of a matrix can also be defined as the dimension of the image of the linear map given by multiplication by $A$. Thus, the notion of the rank of a matrix is connected to the notion of the dimension of a related vector space. In the general setting we also have such a connection, which leads to the following definition.
Definition 1.3.22. Let $R$ be a ring. A Sylvester module rank function dim on $R$ is a function that assigns to every finitely presented $R$-module a non negative real number and that satisfies the following conditions.
(1) $\operatorname{dim}(0)=0, \operatorname{dim}(R)=1$.
(2) $\operatorname{dim}\left(M_{1} \oplus M_{2}\right)=\operatorname{dim}\left(M_{1}\right)+\operatorname{dim}\left(M_{2}\right)$.
(3) If $M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$ is exact then

$$
\operatorname{dim}\left(M_{1}\right)+\operatorname{dim}\left(M_{3}\right) \geq \operatorname{dim}\left(M_{2}\right) \geq \operatorname{dim}\left(M_{3}\right)
$$

For a ring $R$ we denote by $\mathbb{P}_{\text {Mod }}(R)$ the set of Sylvester module rank functions. There is a duality between the notion of Sylvester module rank functions and Sylvester matrix rank functions. In fact we have the following proposition.

Proposition 1.3.23. [Lóp21, Proposition 1.2.8] Let $R$ be a ring. There exists a bijective correspondence between $\mathbb{P}_{\text {Mod }}(R)$ and $\mathbb{P}_{\text {Mat }}(R)$.

- Let $\operatorname{dim} \in \mathbb{P}_{\text {Mod }}(R)$. For $A \in \operatorname{Mat}_{n, m}(R)$ we set

$$
\operatorname{rk}(A)=m-\operatorname{dim}\left(R^{m} / R^{n} A\right)
$$

Then $\mathrm{rk} \in \mathbb{P}_{\text {Mat }}(R)$.

- Let $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$. For any finitely presented $R$-module $M$ with presentations $M=R^{m} / R^{n} A$ for some matrix $A \in \operatorname{Mat}_{n, m}(R)$ we set

$$
\operatorname{dim}(M)=m-\operatorname{rk}(A)
$$

Then $\operatorname{dim}(M)$ does not depend on the presentation of $M$ and $\operatorname{dim} \in$ $\mathbb{P}_{\text {Mod }}(R)$.

Given a rank function rk we will write $\operatorname{dim}_{\mathrm{rk}}$ for the associated module rank function. Similarly we will use $\mathrm{rk}_{\mathrm{dim}}$. When $R=\mathcal{D}$ is a division ring, we know that $\mathbb{P}_{\mathrm{Mat}}(\mathcal{D})=\left\{\operatorname{rk}_{\mathcal{D}}\right\}$. Further we know that modules over division rings are free which means they have a basis which naturally gives a dimension function $\operatorname{dim}_{\mathcal{D}} \in \mathbb{P}_{\text {Mod }}(\mathcal{D})$. Obviously we have $\operatorname{dim}_{\mathcal{D}}=\operatorname{dim}_{\mathrm{rk}_{\mathcal{D}}}$.
As in the case for matrix rank functions, given a ring homomorphism $\varphi: R \rightarrow S$ we also get a map $\varphi^{\#}: \mathbb{P}_{\text {Mod }}(S) \rightarrow \mathbb{P}_{\text {Mod }}(R)$. Given dim $\in \mathbb{P}_{\text {Mod }}(S)$ and a finitely presented $R$-module $M$ we define $\varphi^{\#}(\operatorname{dim})(M)=\operatorname{dim}\left(M \otimes_{\varphi(R)} S\right)$. Note that we used $\varphi^{\#}$ for both the map between module and matrix rank functions. This makes sense since the map commutes with the duality from the previous proposition we have $\varphi^{\#}\left(\operatorname{dim}_{\mathrm{rk}}\right)=\operatorname{dim}_{\varphi \#(\mathrm{rk})}$ and similarly for $\varphi^{\#}\left(\mathrm{rk}_{\mathrm{dim}}\right)$.

## Regular rank functions

We know that if $\mathcal{D}$ is a division ring, then $\mathbb{P}_{\text {Mod }}(\mathcal{D})=\left\{\operatorname{dim}_{\mathcal{D}}\right\}$, where $\operatorname{dim}_{\mathcal{D}}$ denotes the usual dimension function. However this dimension function is "better", meaning it has some special properties. We have for an exact sequence

$$
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
$$

of $\mathcal{D}$-modules that

$$
\operatorname{dim}_{\mathcal{D}}(A)+\operatorname{dim}_{\mathcal{D}}(C)=\operatorname{dim}_{\mathcal{D}}(B)
$$

This leads to the following definition.
Definition 1.3.24. Let $R$ be a ring. $\operatorname{dim} \in \mathbb{P}_{\text {Mod }}(R)$ is called exact, if for every surjection $\varphi: M \rightarrow N$ of finitely presented $R$-modules we have

$$
\operatorname{dim}(M)-\operatorname{dim}(N)=\inf \{\operatorname{dim}(L) \mid L \text { is finitely presented and } L \rightarrow \operatorname{ker}(\varphi)\}
$$

In this section we want to focus on Sylvester rank functions on von Neumann regular rings and more general regular rank functions. For a ring $R$ a rank function $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$ is called regular, if there is a ring homomorphism $\varphi$ : $R \rightarrow \mathcal{U}$ into a von Neumann regular ring with $\operatorname{rk} \in \operatorname{im}\left(\varphi^{\#}\right)$. Let us start with the following lemma.

Lemma 1.3.25. Every Sylvester module rank function dim of a von Neumann regular ring $\mathcal{U}$ is exact.

Proof. Let $\varphi: M \rightarrow N$ be a surjection of finitely presented $\mathcal{U}$ modules. By 1.3.2 and [Rot08, Theorem 3.56] the modules $M, N$ are projective. As a consequence the sequence

$$
0 \rightarrow \operatorname{ker}(\varphi) \rightarrow M \rightarrow N \rightarrow 0
$$

splits. Thus we have $M \cong \operatorname{ker} \varphi \oplus N$ and the result follows.
For any ring $R$, we call $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(R)$ faithful if $\operatorname{ker}(\mathrm{rk})=\{r \in R \mid \operatorname{rk}(r)=$ $0\}=\{0\}$. The following lemma shows that faithful Sylvester matrix rank functions on regular rings behave similar to rank functions on division rings.

Lemma 1.3.26. [Lóp21, Lemma 1.3.12] Let $\mathcal{U}$ be a von Neumann regular ring and $\mathrm{rk} \in \mathbb{P}_{\mathrm{Mat}}(\mathcal{U})$ be faithful. A square matrix $A \in \operatorname{Mat}_{n}(\mathcal{U})$ is invertible if and only if $\operatorname{rk}(A)=n$.

For later use we note the following result.
Lemma 1.3.27. [Jai19, Proposition 5.10] Let $R$ be an algebra over $K$ and let rk be a regular Sylvester matrix rank function on $R$. Then for every $A, B \in$ $\operatorname{Mat}_{n \times m}(R)$ and every $\epsilon>0$ we have

$$
|\{\lambda \in K \mid \operatorname{rk}(A)-\operatorname{rk}(A-\lambda B) \geq \epsilon\}| \leq \frac{\operatorname{rk}(A)}{\epsilon}
$$

Let us understand what this result means in a very specific case, namely $R=\mathbb{C}, B=\operatorname{Id}_{n}$ the identity matrix, $A \in \operatorname{Mat}_{n}(\mathbb{C})$ and $\epsilon=d \in \mathbb{N}$. Then $\operatorname{rk}(A)-\operatorname{rk}\left(A-\lambda \operatorname{Id}_{n}\right)>0$ means that $\lambda$ is an eigenvalue of $A$. Moreover $\operatorname{rk}(A)-$ $\operatorname{rk}\left(A-\lambda \operatorname{Id}_{n}\right)$ is just the dimension of the kernel of $(A-\lambda \mathrm{Id})$. So in this special case the lemma is obviously true.

## Extension of Sylvester module rank functions

Until now Sylvester module rank functions are only defined on finitely presented modules. However we know that for a division $\operatorname{ring} \mathcal{D}$ we can define $\operatorname{dim}_{\mathcal{D}}$ for all $\mathcal{D}$ modules. It is more or less easy to extend exact rank functions to all $R$-modules. Let us first define what we mean by an extension of rank functions.

Definition 1.3.28. Let $R$ be a ring and $\operatorname{dim} \in \mathbb{P}_{\text {Mod }}(R)$. An extension $\widetilde{\operatorname{dim}}$ of $\operatorname{dim}$ is a function $\mathbb{R}-\operatorname{Mod} \rightarrow \mathbb{R}_{\geq 0} \cup\{+\infty\}$ such that
(1) $\operatorname{dim}=\widetilde{\operatorname{dim}}$ on finitely presented $R$-modules.
(2) $\widetilde{\operatorname{dim}}(A \oplus B)=\widetilde{\operatorname{dim}}(A)+\widetilde{\operatorname{dim}}(B)$ for $R$-modules $A$ and $B$.
(3) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of $R$ modules then

$$
\widetilde{\operatorname{dim}}(A)+\widetilde{\operatorname{dim}}(C) \geq \widetilde{\operatorname{dim}}(B)
$$

It was Hanfeng Li in [Li21] who proved the following result.

Proposition 1.3.29. [Li21, Theorem 3.3] Let $R$ be a ring and $\operatorname{dim} \in \mathbb{P}_{\text {Mod }}(R)$. Then an extension $\widetilde{\operatorname{dim}}$ of $\operatorname{dim}$ exists and is unique.

We want briefly sketch how to extend the function dim. Extending dim to finitely generated modules is not a big problem. For a finitely generated $R$-module $M$ we set

$$
\widetilde{\operatorname{dim}}(M)=\inf \{\operatorname{dim} \widetilde{M} \mid \widetilde{M} \text { is finitely presented and maps onto } M\}
$$

To extend dim to arbitrary $R$-modules is a little bit more difficult. When the rank function dim is exact we can set

$$
\operatorname{dim} M=\sup \left\{\operatorname{dim} M^{\prime} \mid M^{\prime} \leq M, M^{\prime} \text { is finitely generated }\right\}
$$

This was done in [Vir19]. In the general case the definition of $\widetilde{\operatorname{dim}} M$ for an arbitrary $R$-module is

$$
\widetilde{\operatorname{dim}} M=\sup _{M_{1}} \inf _{M_{2}}\left(\operatorname{dim} M_{2}-\operatorname{dim} M_{2} / M_{1}\right)
$$

where $M_{1} \leq M_{2} \leq M$ and $M_{2}$ is finitely generated.
Proposition 1.3.30. [Li21, Theorem 8.1] Let $\varphi: R \rightarrow S$ be an epic ring homomorphism. Let $\operatorname{dim}_{1}, \operatorname{dim}_{2} \in \mathbb{P}_{\text {Mod }}(S)$ such that for all finitely presented $R$-modules $M$ we have

$$
\operatorname{dim}_{1}\left(M \otimes_{\varphi(R)} S\right)=\operatorname{dim}_{2}\left(M \otimes_{\varphi(R)} S\right)
$$

Then $\widetilde{\operatorname{dim}_{1}}=\widetilde{\operatorname{dim}_{2}}$.
Until now we have seen that given any Sylvester module rank function on a ring $R$ it can be extended to arbitrary modules over $R$. We now want to introduce a different extension. For that let $K$ be a subfield of $\mathbb{C}$, let $\mathcal{A}$ be a $K$ algebra and let rk be a Sylvester matrix rank function on $\mathcal{A}$. If $E \leq \mathbb{C}$ is a field containing $K$, can we extend rk to $E \otimes_{K} \mathcal{A}$ ? This was first answered positively by Jaikin in [Jai19]. Jaikin constructs two types of extensions. On the one hand he considers the case where $E$ is an algebraic extension of $K$. This construction works for all rank functions. On the other side he constructs for a regular rank functions an extension for $E=K(t)$, where $t \in \mathbb{C} \backslash K$ is transcendental. In [JL21] Jiang and Li unify these two constructions for any field extension $E$ of $K$. We will briefly describe their construction here. For proofs and more details see [JL21].

Let us fix a field $K$ and a field $E$ that contains $K$. Let $\mathcal{A}$ be a $K$-algebra and let $\mathrm{rk} \in \mathbb{P}(\mathcal{A})$. Let $\operatorname{dim}$ be the module rank function associated to rk. Last let $\mathcal{F}:=\left\{V \otimes_{K} \mathcal{A} \mid V\right.$ is a finite dimensional $K$-linear subspace of $\left.E\right\}$. Denote by $\hat{\mathcal{F}}$ the non zero elements of $\mathcal{F}$. Then the set $\mathcal{F}$ is a finite approximation system for $E \otimes_{K} \mathcal{A}$ and fulfills the following conditions.
(1) Each $W \in \mathcal{F}$ is a finitely generated free left $\mathcal{A}$-module.
(2) For every $W, V \in \mathcal{F}$ one has $W \cap V, W+F \in \mathcal{F}$.
(3) For any $W, V \in \mathcal{F}$ with $V \subseteq W$ one has $W=V \oplus V^{\prime}$ for some $V^{\prime} \in \mathcal{F}$. In particular $\operatorname{dim}(V) \leq \operatorname{dim}(W)$.
(4) Every finitely generated left $\mathcal{A}$-submodule of $E \otimes_{K} \mathcal{A}$ is contained in some $W \in \mathcal{F}$.
(5) For any finite subset $F$ of $E \otimes_{K} \mathcal{A}$ and every $\epsilon>0$ there is a $W \in \hat{\mathcal{F}}$ that is $(S, \epsilon)$-invariant in the sense that one has $\operatorname{dim}(V) \leq(1+\epsilon) \operatorname{dim}(W)$ for some $V$ containing $W+W F$.

Let now $A=\left(a_{i, j}\right) \in \operatorname{Mat}_{n \times m}\left(E \otimes_{K} \mathcal{A}\right)$ and $W \in \hat{\mathcal{F}}$. By condition (4) there is a $V \in \hat{\mathcal{F}}$, such that $W a_{i, j} \subseteq V$ for every $i, j$. Then we have

$$
W^{n} A \subseteq V^{m}
$$

Since $W, V$ are free $\mathcal{A}$-modules, we can choose a basis $w_{1}, \ldots, w_{k}$ for $W$ and $v_{1}, \ldots, v_{r}$ for $V$. Then there is some matrix $B \in \operatorname{Mat}_{k \times r}(\mathcal{A})$ such that

$$
\left(\begin{array}{c}
w_{1} A \\
\vdots \\
w_{k} A
\end{array}\right)=B \cdot\left(\begin{array}{c}
v_{1} \\
\vdots \\
v_{r}
\end{array}\right)
$$

We set $\operatorname{rk}_{W}(A)=\operatorname{rk}(B)$. Jiang and Li show that $\operatorname{rk}_{W}(A)$ does not depend on the choice of bases. Further they show that

$$
\mathrm{rk}_{\mathcal{F}}:=\inf _{W \in \hat{\mathcal{F}}} \frac{\operatorname{rk}_{W}(A)}{\operatorname{dim} W}
$$

exists and extends rk. The construction of Jiang and Li behaves well with the composition of field extensions. Also, as mentioned before, it coincides with the construction of Jaikin in his two cases. For a rank function rk on a $K$ algebra $\mathcal{A}$ and a field extension $E$ of $K$ we denote the extension $\mathrm{rk}_{\mathcal{F}}$ of rk by rk. In general it might be hard to recognize the extension rk in the following sense. Given a rank function $\mathrm{rk} \in \mathbb{P}(\mathcal{A})$ there might be many rank functions $\mathrm{rk}_{1}, \mathrm{rk}_{2}, \ldots \in \mathbb{P}(E \otimes \mathcal{A})$ that extend the rank function rk. But we can not say if $\mathrm{rk}_{i}=$ rk for some $i$. However in the cases when $E=\bar{K}$ is the algebraic closure of $K$ or $E=K(t)$ we can say something. We will begin with the algebraic case.
Theorem 1.3.31. Let $\mathrm{rk}_{1}, \mathrm{rk}_{2}$ be two rank functions on $R=\bar{K} \otimes_{K} \mathcal{A}$. Let $i: \mathcal{A} \rightarrow R$ be the inclusion and assume that $i^{\#}\left(\mathrm{rk}_{1}\right)=i^{\#}\left(\mathrm{rk}_{2}\right)$ and $\mathrm{rk}_{1} \leq \mathrm{rk}_{2}$. Then

$$
\mathrm{rk}_{1}=\mathrm{rk}_{2} \in \mathbb{P}(R)
$$

Proof. Let $A \in \operatorname{Mat}_{n \times m}(R)$ be a matrix over $R$. Since $A$ has only finitely many entries, we can find a finite Galois extension $E / K$ such that $A$ is a matrix over $E \otimes_{K} \mathcal{A}$. Let $G=\left\{\sigma_{1}, \ldots, \sigma_{d}\right\}$ be the Galois group of the extension $E / K$. Further let $\alpha \in E$ be primitive element, that means we have $E=K(\alpha)$.

Since $\operatorname{Mat}\left(E \otimes_{K} \mathcal{A}\right) \cong E \otimes_{K} \operatorname{Mat}(\mathcal{A})$ and every element $e \in E$ is of the form $e=f_{0} \alpha^{0}+\ldots f_{d-1} \alpha^{d-1}$ for some $f_{i} \in K$ we can write the matrix $A$ as a sum $A=\sum_{i=0}^{d-1} \alpha^{i} \otimes A_{i}$ for some matrices $A_{i} \in \operatorname{Mat}_{n \times m}(\mathcal{A})$. Consider now the matrix $B:=\sigma_{1}(A) \oplus \ldots \oplus \sigma_{d}(A)$. Let

$$
C=\left(\begin{array}{ccc}
\sigma_{1}(\alpha) & & \\
& \ddots & \\
& & \sigma_{d}(\alpha)
\end{array}\right)
$$

Using the same decomposition as above we can write $B=\sum_{i=0}^{d-1}\left(C^{i} \otimes A_{i}\right)$. The matrix $C$ is similar over $E$ to a matrix over $K$. That is because the characteristic polynomial of $C$ is the minimal polynomial of $\alpha$. Thus the matrix $C$ is similar to the companion matrix of this polynomial. That means we have an invertible matrix $T \in \operatorname{Mat}_{d}(E)$ such that $T C T^{-1} \in \operatorname{Mat}_{d}(K)$. Note that the matrices $T \otimes \operatorname{Id}_{n}$ and $T^{-1} \otimes \operatorname{Id}_{m}$ are invertible over $R$. Thus multiplying with them does not change the rank. We have

$$
\begin{aligned}
\sum_{j=1}^{d} \operatorname{rk}_{1}\left(\sigma_{j}(A)\right) & =\operatorname{rk}_{1}(B) \\
& =\operatorname{rk}_{1}\left(T \otimes \operatorname{Id}_{n} \cdot B \cdot T \otimes \operatorname{Id}_{m}\right) \\
& =\operatorname{rk}_{1}\left(T \otimes \operatorname{Id}_{n} \cdot \sum_{i=0}^{d-1}\left(C^{i} \otimes A_{i}\right) \cdot T \otimes \operatorname{Id}_{m}\right) \\
& =\operatorname{rk}_{1}\left(\sum_{i=0}^{d-1} T C^{i} T^{-1} \otimes A_{i}\right) \\
& =\operatorname{rk}_{2}\left(\sum_{i=0}^{d-1} T C^{i} T^{-1} \otimes A_{i}\right) \\
& =\operatorname{rk}_{2}\left(T \otimes \operatorname{Id}_{n} \cdot \sum_{i=0}^{d-1}\left(C^{i} \otimes A_{i}\right) \cdot T \otimes \operatorname{Id}_{m}\right) \\
& =\operatorname{rk}_{2}\left(T \otimes \operatorname{Id}_{n} \cdot B \cdot T \otimes \operatorname{Id}_{m}\right) \\
& =\operatorname{rk}_{2}(B) \\
& =\sum_{j=1}^{d} \operatorname{rk}_{2}\left(\sigma_{j}(A)\right)
\end{aligned}
$$

Since we assumed that $\mathrm{rk}_{1} \leq \mathrm{rk}_{2}$, we have $\mathrm{rk}_{1}\left(\sigma_{j}(A)\right) \leq \mathrm{rk}_{2}\left(\sigma_{j}(A)\right)$. Since their sum is equal we have equality for each $j$.

Let us now look at the transcendental case $E=K(t)$. The following result is a reformulation of [Jai19, Proposition 7.7], based on [JL21, Section 9.2].

Proposition 1.3.32. Let $\mathcal{U}$ be a von Neumann regular $K$-algebra. Let rk $\in$ $\mathbb{P}(\mathcal{U})$ and $\mathrm{rk}^{\prime} \in \mathbb{P}\left(K(t) \otimes_{K} \mathcal{U}\right)$ be a rank function that extends rk . If $\mathrm{rk}^{\prime}\left(I d_{n}+t A\right)=$ $n$ for any matrix $A \in \operatorname{Mat}_{n}(\mathcal{U})$, then $\mathrm{rk}^{\prime}=\tilde{\mathrm{rk}}$.

The following is a direct consequence of the previous proposition.
Corollary 1.3.33. [Jai19, Corollary 7.8] Let $\left\{\mathrm{rk}_{i}\right\}_{i \in \mathbb{N}}$ be a family of regular Sylvester matrix rank functions on some algebra $\mathcal{A}$. For each $i \in \mathbb{N}$ let $\mathrm{rk}_{i} \in$ $\mathbb{P}\left(K(t) \otimes_{K} \mathcal{A}\right)$ be the natural transcendental extension of $\mathrm{rk}_{i}$. Let $\omega$ be a non principal ultrafilter on $\mathbb{N}$. Then $\lim _{\omega} \tilde{r k}_{i}$ is the natural transcendental extension of $\lim _{\omega} \mathrm{rk}_{i}$.

Further we have the following result which can be found in [Li21].
Proposition 1.3.34. [Li21, Proposition 9.6] Let $\mathcal{A}$ be a $K$-algebra, rk be a regular rank function on $\mathcal{A}$ and $\left\{x_{i}\right\}_{i \in \mathbb{N}}$ a sequence of distinct points in $K$. For each $i$ let $\pi_{i}: K[t] \otimes_{K} \mathcal{A} \rightarrow \mathcal{A}$ be the quotient map defined by sending $t$ to $x_{i}$. Then for each matrix $A \in \operatorname{Mat}_{n \times m}\left(K[t] \otimes_{K} \mathcal{A}\right)$ we have

$$
\tilde{\operatorname{rk}}(A)=\lim _{i \rightarrow \infty} \operatorname{rk}\left(\pi_{i}(A)\right)
$$

## Epic $*$-regular $R$-rings

We want to introduce one more object that unifies all concepts we have described in this chapter.

Definition 1.3.35. Let $R$ be a $*$-ring. An epic $*$-regular $R$-ring is a triple $(\mathcal{U}, \mathrm{rk}, \varphi)$ such that
(1) $\mathcal{U}$ is a $*$-regular ring.
(2) rk is a faithful Sylvester matrix rank function on $\mathcal{U}$.
(3) $\varphi: R \rightarrow \mathcal{U}$ is an epic $*$-homomorphism.
(4) $\mathcal{R}(\varphi(R), \mathcal{U})=\mathcal{U}$.

Note that by proposition 1.3 .17 (3) and (4) are equivalent. Let us consider some examples.

Example 1.3.36. (1) Let $R=\mathbb{Z}$ with $a^{*}=a$ for $a \in \mathbb{Z}$. Then the triple $\left(\mathbb{Q}, \mathrm{rk}_{\mathbb{Q}}, i\right)$, where $i: \mathbb{Z} \rightarrow \mathbb{Q}$ is the inclusion, is a $*$-regular $\mathbb{Z}$-ring.
(2) Let $(\mathcal{A}, \tau)$ be a tracial $*$-algebra and let $\mathcal{U}_{\tau}$ be as in the previous chapter the ring of unbound operators affiliated to $\mathcal{A}$. Remember that we have a $\operatorname{map} \varphi: \mathcal{A} \rightarrow \mathcal{U}_{\tau}$. Let $\mathcal{R}_{\tau}=\mathcal{R}\left(\varphi(\mathcal{A}), \mathcal{U}_{\tau}\right)$. Then $\left(\mathcal{R}_{\tau}, \mathrm{rk}_{\tau}, \varphi\right)$ is an epic *-regular $\mathcal{A}$-ring.

We will say that two epic $*$-regular $R$-rings $\left(\mathcal{U}_{1}, \mathrm{rk}_{1}, \varphi_{1}\right)$ and $\left(\mathcal{U}_{2}, \mathrm{rk}_{2}, \varphi_{2}\right)$ are isomorphic, if there exists a $*$-isomorpism $f: \mathcal{U}_{1} \rightarrow \mathcal{U}_{2}$ such that the diagram

commutes and $\mathrm{rk}_{1}=\mathrm{rk}_{2} \circ f$. The following result shows that the rank function on a $*$ regular $R$-ring is completely defined by its values on $R$.

Lemma 1.3.37. [Jai19, Corollary 6.2] Let $\mathcal{U}$ be $a *$-regular ring and $R$ be $a *$-subring of $\mathcal{U}$. Assume that $\mathcal{U}=\mathcal{R}(\varphi(R), \mathcal{U})$. Then for any $r_{1}, \ldots, r_{k} \in$ $\operatorname{Mat}_{n \times m}(\mathcal{U})$, there is a matrix $M \in \operatorname{Mat}_{a \times b}(R)$ and matrices $v_{1}, \ldots, v_{k} \in$ $\operatorname{Mat}_{n \times b}(R)$ such that for any $t_{1}, \ldots, t_{k} \in \operatorname{Mat}_{n}(R)$ and every Sylvester matrix rank function rk on $\mathcal{U}$

$$
\operatorname{rk}\left(t_{1} r_{1}+\ldots+t_{k} r_{k}\right)=\operatorname{rk}\binom{M}{t_{1} v_{1}+\ldots+t_{k} v_{k}}-\operatorname{rk}(M)
$$

Using this result we can even show that the values of rk on $R$ do not only determine rk, but determine the epic *-regular $R$ ring up to isomorphism.

Theorem 1.3.38. [Jai19, Theorem 6.3] Let $\left(\mathcal{U}_{1}, r k_{1} \varphi_{1}\right)$ and $\left(\mathcal{U}_{2}, r k_{2} \varphi_{2}\right)$ be two epic $*$-regular $R$-rings. Then, $\left(\mathcal{U}_{1}, r k_{1} \varphi_{1}\right)$ and $\left(\mathcal{U}_{2}, r k_{2} \varphi_{2}\right)$ are isomorphic if and only if $r k_{1} \circ \varphi_{1}=r k_{2} \circ \varphi_{2} \in \mathbb{P}(R)$.

Proof. One direction follows directly from the definition of isomorphisms for epic *-regular $R$-rings. So let us assume that $\mathrm{rk}_{1} \circ \varphi_{1}=\mathrm{rk}_{2} \circ \varphi_{2} \in \mathbb{P}(R)$. Let $\varphi_{0}$ : $R \rightarrow \mathcal{U}_{1} \times \mathcal{U}_{2}, r \mapsto\left(\varphi_{1}(r), \varphi_{2}(r)\right)$. Since $\mathcal{U}_{1} \times \mathcal{U}_{2}$ is $*$-regular, we can define $\mathcal{U}_{0}=$ $\mathcal{R}\left(\varphi_{0}(R), \mathcal{U}_{1} \times \mathcal{U}_{2}\right)$. For $i=1,2$ we have the following commutative diagram.


Since $\pi_{i} \varphi_{0}=\varphi_{i}$ is epic, by Proposition $1.3 .15, \pi_{i}$ is surjective. So let us show that $\pi_{i}$ is injective. Note that $\pi_{1}^{\#}\left(\mathrm{rk}_{1}\right), \pi_{2}^{\#}\left(\mathrm{rk}_{2}\right) \in \mathbb{P}\left(\mathcal{U}_{0}\right)$ are Sylvester matrix rank functions on $\mathcal{U}_{0}$. Let $B \in \operatorname{Mat}_{n}(R)$ and $A=\varphi_{0}(B) \in \operatorname{Mat}_{n}\left(\mathcal{U}_{0}\right)$. Since

$$
\begin{aligned}
\pi_{1}^{\#}\left(\mathrm{rk}_{1}(A)\right)=\mathrm{rk}_{1}\left(\pi_{1}(A)\right)= & \mathrm{rk}_{1}\left(\pi_{1}\left(\varphi_{0}(B)\right)\right)=\operatorname{rk}_{1}\left(\varphi_{1}(B)\right) \\
& =\operatorname{rk}_{2}\left(\varphi_{2}(B)\right)=\operatorname{rk}_{2}\left(\pi_{2}\left(\varphi_{0}(B)\right)\right)=\pi_{2}^{\#}\left(\mathrm{rk}_{2}\right)(A)
\end{aligned}
$$

the previous lemma implies

$$
\pi_{1}^{\#} \mathrm{rk}_{1}=\pi_{2}^{\#} \mathrm{rk}_{2} \in \mathbb{P}\left(\mathcal{U}_{0}\right)
$$

Let now $u=\left(u_{1}, u_{2}\right) \in \mathcal{U}_{0}$ such that $\pi_{1}(u)=0$. Obviously we have $u_{1}=0$ and therefore

$$
0=\operatorname{rk}_{1}\left(u_{1}\right)=\pi_{1}^{\#} \operatorname{rk}_{1}\left(u_{1}, u_{2}\right)=\pi_{2}^{\#} \operatorname{rk}_{2}\left(u_{1}, u_{2}\right)=\operatorname{rk}_{2}\left(u_{2}\right)
$$

Since $r k_{2}$ is faithful we obtain $u_{2}=0$. Hence, $\pi_{1}$ is injective and in the same way it follows that $\pi_{2}$ is injective. Therefore, $f=\pi_{2} \circ \pi_{1}^{-1}: \mathcal{U}_{1} \rightarrow \mathcal{U}_{2}$ is a $*$-isomorphism. It is left to show that $f \circ \varphi_{1}=\varphi_{2}$ and $\mathrm{rk}_{1}=\mathrm{rk}_{2} \circ f$. Indeed we have

$$
f \circ \varphi_{1}=\pi_{2} \circ \pi_{1}^{-1} \circ \varphi_{1}=\pi_{2} \circ \pi_{1}^{-1} \circ \pi_{1} \circ \varphi_{0}=\pi_{2} \varphi_{0}=\varphi_{2}
$$

and
$\mathrm{rk}_{2} \circ f=\mathrm{rk}_{2} \circ \pi_{2} \circ \pi_{1}^{-1}=\pi_{2}^{\#}\left(\mathrm{rk}_{2}\right) \circ \pi_{1}^{-1}=\pi_{1}^{\#}\left(\mathrm{rk}_{1}\right) \circ \pi_{1}^{-1}=\mathrm{rk}_{1} \circ \pi_{1} \circ \pi_{1}^{-1}=\mathrm{rk}_{1}$.

### 1.4 Ultralimits and Ultrafilters

In this chapter we want to introduce ultrafilters, ultralimits and ultraproducts. For a bounded sequence $\left(a_{i}\right)_{i \in \mathbb{N}} \subseteq \mathbb{C}$ we know that there is always a convergent subsequence, although the sequence $\left(a_{i}\right)$ itself may not converge. Choosing an ultrafilter on $\mathbb{N}$ avoids this problem. We will see that for any bounded sequence $\left(a_{i}\right)_{i \in \mathbb{N}}$ and any ultrafilter $\omega$ on $\mathbb{N}$ the ultralimit $\lim _{\omega} a_{i}$ does always exist and is unique. Using this property allows us also do define the ultralimit of functions, for example the ultralimit of traces, measures or rank functions. In this section we will first define filters and ultrafilters. We will then present the definition and some properties of ultralimits. Also we will see that for some sequence $\left(a_{i}\right)$ the limit $a=\lim _{i \rightarrow \infty} a_{i}$ exists, if and only if $a=\lim _{\omega} a_{i}$ for every non principal ultrafilter $\omega$ on $\mathbb{N}$. Last we will present the tracial ultraproduct of von Neumann algebras, which is an important object whenever it comes to approximating von Neumann algebras. For us the most important application of ultrafilters will be the ultralimit of measures and rank functions.

We will start with the definition of a filter. For simplicity we will only consider filters and ultrafilters on $\mathbb{N}$. We will denote by $\mathcal{P}(\mathbb{N})$ the powerset of $\mathbb{N}$. For a more detailed introduction see [LW15].

Definition 1.4.1. A filter on $\mathbb{N}$ is a subset $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ such that
(1) $\emptyset \notin \mathcal{F}$.
(2) $A \subseteq B, A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.
(3) $A_{1}, \ldots, A_{n} \in \mathcal{F} \Rightarrow \bigcap_{i} A_{i} \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an ultrafilter if for all $A \subseteq \mathbb{N}$ we have $A \in \mathcal{F}$ or $\mathbb{N} \backslash A \in \mathcal{F}$.

We will denote an ultrafilter usually by the letter $\omega$. An important example of a filter is the Frechet filter

$$
\mathcal{F}=\{A \in \mathbb{N}| | \mathbb{N} \backslash A \mid<\infty\}
$$

The easiest but least helpful examples for ultrafilters are so called principal utrafilters. These are ultrafilters of the form $\omega_{n}=\{A \subseteq \mathbb{N} \mid n \in A\}$. Although we have not defined ultralimits yet let us briefly say why these are not helpful. For a sequence $\left(a_{i}\right)_{i \in \mathbb{N}}$ and a principal ultrafilter $\omega_{n}$ we have

$$
\lim _{\omega_{n}} a_{i}=a_{n}
$$

Thus the limit does not depend at all on the actual values $a_{i}$. The existence a of non principal ultrafilter uses the axiom of choice. One basically starts with the Frechet filter $\mathcal{F}$ and then chooses for each subset $A \in \mathbb{N}$ with $A, A^{c} \notin \mathcal{F}$ if either $A \in \mathcal{F}$ or $A^{c} \in \mathcal{F}$. We have the following lemma.
Lemma 1.4.2. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and $A \subseteq \mathbb{N}, A \neq \emptyset$, such that $A, A^{c} \notin \mathcal{F}$. Then $\mathcal{F} \cup\{A\}$ can be extended to a filter.

Proof. Define $\mathcal{G}=\mathcal{F} \cup\{A \cap B \mid B \in \mathcal{F}\}$ and set $\mathcal{H}=\{B \subseteq \mathbb{N} \mid C \subseteq$ $B$ for some $C \in \mathcal{G}\}$. It is easy to see that $\mathcal{H}$ is a filter containing $A$ and $\mathcal{F}$.

Lemma 1.4.3. Every filter $\mathcal{F}$ is contained in an ultrafilter.
Proof. The proof uses Zorn's Lemma. Let $\Omega$ be the set of all filters on $\mathbb{N}$ that contain the filter $\mathcal{F}$. Then $\Omega$ is partially ordered by inclusion. Let $\mathcal{C}$ be a chain in $\Omega$. Then the union over $\mathcal{C}$ is a filter and an upper bound for $\mathcal{C}$. Therefore, we can apply Zorns lemma and we get a maximal element $\omega \in \Omega$. We want to show that $\omega$ is an ultrafilter. For that assume that there is a set $A \subseteq \mathbb{N}$ with $A, A^{c} \notin \omega$. By the previous lemma $\omega \cup\{A\}$ could be extended to a filter, which would contradict the maximality of $\omega$. Therefore, $\omega$ is an ultrafilter.

Non principal ultrafilters do not contain finite sets.
Lemma 1.4.4. Let $\omega$ be a non principal ultrafilter. Then, $\omega$ contains the Frechet filter.
Proof. Since $\omega$ is non principal, for each $n \in \mathbb{N}$ we have $\mathbb{N} \backslash\{n\} \in \omega$. Therefore, for each finite set $N=\left\{n_{1}, \ldots, n_{k}\right\}$ we have $\mathbb{N} \backslash N=\bigcap_{i} \mathbb{N} \backslash\left\{n_{i}\right\} \in \omega$.

Using an ultrafilter, one can construct an ultralimit.
Definition 1.4.5. Let $\omega$ be a non principal ultrafilter on $\mathbb{N}$. If $\left(x_{n}\right)_{n \in \mathbb{N}}$ is a sequence of points in a metric space $(X, d)$ and $x \in X$ then $x=\lim _{\omega} x_{n}$ is called the $\omega$-limit of $\left(x_{n}\right)_{n \in \mathbb{N}}$ if for all $\epsilon>0$ we have

$$
\left\{i \in \mathbb{N} \mid d\left(x, x_{i}\right) \leq \epsilon\right\} \in \omega
$$

If the ultrafilter $\omega$ is clear from the context we will call $x=\lim _{\omega} x_{n}$ the ultralimit of the $x_{n}$.

From this definition our previous statement about ultralimits with respect to principal ultrafilters becomes clear. It is a standard result of model theory that if $X$ is compact, an ultralimit always exists which is obviously not true for the usual limit. We will show the case for $X=[a, b] \subseteq \mathbb{R}$.
Lemma 1.4.6. Let $\left(x_{n}\right)_{n}$ be a sequence in $[a, b] \subseteq \mathbb{R}$ with $a, b \in \mathbb{R}$ and let $\omega$ be a non principal ultrafilter on $\mathbb{N}$ Then the ultralimit $x=\lim _{\omega} x_{n}$ exists and is unique.

Proof. Without loss of generality assume $X=[0,1]$. Let $A_{1}=\left\{n \in \mathbb{N} \mid x_{n} \geq\right.$ $\left.\frac{1}{2}\right\}$. Then either $A_{1} \in \omega$ or $A_{1}^{c} \in \omega$. Without loss of generality lets assume $A_{1} \in \omega$. Define then $B_{2}=\left\{n \in \mathbb{N} \left\lvert\, x_{n} \geq \frac{3}{4}\right.\right\}$. Then again we have $B_{2} \in \omega$ or $B_{2}^{c} \in \omega$. If $B_{2} \in \omega$ set $A_{2}=B_{2}$, otherwise set $A_{2}=B_{2}^{c} \cap A_{1}$. We can do this iteratively and get a set $A_{n}=\left\{n \in \mathbb{N} \left\lvert\, \frac{k}{2^{n}} \leq x_{n}<\frac{k+1}{2^{n}}\right.\right\} \in \omega$ for some $k$. Therefore the ultralimit always exists. Assume now that we have $c \neq d \in[0,1]$ that both satisfiy the condition of 1.4.5. Let $\epsilon=\frac{|c-d|}{3}$. Then both, $C=\left\{n \in \mathbb{N}| | x_{n}-c \mid \leq \epsilon\right\}$ and $D=\left\{n \in \mathbb{N}| | x_{n}-d \mid \leq \epsilon\right\}$ belong to $\omega$. But we have $D \subseteq C^{c}$ and therefore $C^{c} \in \omega$ which is a contradiction.

The ultralimit has the following properties, we know from usually limits. For simplicity we will again only consider the case $X=\mathbb{R}$.

Lemma 1.4.7. Let $\omega$ be a non prinicpal ultrafilter on $\mathbb{N}$ and let $\left(a_{n}\right)_{n \in \mathbb{N}}$ and $\left(b_{n}\right)_{n \in \mathbb{N}}$ be bounded sequences in $\mathbb{R}$. Then the following holds.
(1) $\lim _{\omega} a_{n}+b_{n}=\lim _{\omega} a_{n}+\lim _{\omega} b_{n}$
(2) if $a_{n} \leq b_{n}$ for all $n$, then $\lim _{\omega} a_{n} \leq \lim _{\omega} b_{n}$.

Proof. Let $a=\lim _{\omega} a_{n}$ and $b=\lim _{\omega} b_{n}$. For every $\epsilon>0$ set $A_{\epsilon}=\{n \in \mathbb{N} \mid$ $\left.\left|a-a_{n}\right| \leq \epsilon\right\}$ and similar for $B_{\epsilon}$. Further set $C_{\epsilon}=A_{\epsilon} \cap B_{\epsilon}$. Then $A_{\epsilon}, B_{\epsilon}, C_{\epsilon} \in \omega$. For (1) see that the set $\left\{n \in \mathbb{N}\left||a+b|-\left|a_{n}+b_{n}\right|\right\}\right.$ contains $C_{\epsilon / 2}$ and therefore belongs to $\omega$.
For (2) assume for a contradiction that $a-b \geq \epsilon>0$ and see that $C_{\epsilon / 2}=\emptyset$, which can not be since $C_{\epsilon / 2}$ belongs to $\omega$.

Considering the ultralimit of a bounded sequence is comparable to choose a convergent subsequence. The following lemma connects ultralimits to classical limits.

Lemma 1.4.8. Let $\left(a_{n}\right)_{n}$ be a a bounded sequence of real numbers. Then the limit $\lim _{n \rightarrow \infty} a_{n}$ exists and is equal to $a \in \mathbb{R}$ if and only if for every non principal ultrafilter $\omega$ on $\mathbb{N}$ we have $a=\lim _{\omega} a_{n}$.
Proof. Since every non principal ultrafilter contains the Frechet filter the "only if" part is clear. Assume now that $a$ is not the limit of the $a_{n}$. That means there exists a $\epsilon>0$ such that the set $A=\left\{n \in \mathbb{N}| | a-a_{n} \mid>\epsilon\right\}$ is infinite. As usual let $\mathcal{F}$ be the Frechet filter. By 1.4.2 we can extend $\mathcal{F} \cup\{A\}$ to a filter and by 1.4.3 we can find a ultrafilter $\omega$ that contains $\mathcal{F} \cup\{A\}$. But then $\lim _{\omega} a_{n} \neq a$.

Next we want to introduce the tracial ultraproduct of von Neumann algebras. For that let $\left(\mathcal{N}_{n}, \tau_{n}\right)$ be tracial von Neumann algebras and $\omega$ be a non principal ultrafilter on $\mathbb{N}$. The tracial ultraproduct of the $\mathcal{N}_{n}$ is the quotient space

$$
\begin{equation*}
\mathcal{N}_{\omega}=\prod_{n \in \mathbb{N}}^{\mathrm{b}} \mathcal{N}_{n} / I_{\omega} \tag{1.12}
\end{equation*}
$$

where

$$
\mathcal{N}_{\omega}=\prod_{n \in \mathbb{N}}{ }^{\mathrm{b}} \mathcal{N}_{n}=\left\{\left(a_{n}\right)_{n} \in \prod_{n \in \mathbb{N}} \mathcal{N}_{n} \mid \sup _{n}\left\|a_{n}\right\| \leq \infty\right\}
$$

and

$$
I_{\omega}=\left\{\left(a_{n}\right)_{n} \in \prod_{n \in \mathbb{N}}^{\mathrm{b}} \mathcal{N}_{n} \mid \lim _{\omega} \tau\left(a^{*} a\right)=0\right\}
$$

The most interesting case for us will be when $\left(d_{n}\right)_{n}$ is a sequence of natural numbers and $\mathcal{N}_{n}=\operatorname{Mat}_{d_{n}}(\mathbb{C})$ with $\tau_{n}=\frac{1}{d_{n}} \operatorname{Tr}$. This algebra is a tracial von Neumann algebra with trace given by $\tau_{\omega}=\lim _{\omega}^{n} \tau_{n}$. More information about this can be found in [AH14].

The last thing we want to consider in this section is the ultraproduct of rank functions. For that let $R$ be a ring, let $\left(\mathrm{rk}_{i}\right)_{i \in \mathbb{N}} \subseteq \mathbb{P}_{\mathrm{Mat}}(R)$ be a sequence of rank functions of $R$ and as usual let $\omega$ be a non principal ultrafilter on $\mathbb{N}$. For any matrix $A \in \operatorname{Mat}_{n \times m}(R)$ the numbers $\operatorname{rk}_{i}(A)$ are bounded by max $\{m, n\}$, thus the $\operatorname{limit}^{\operatorname{rk}_{\omega}}(A)=\lim _{\omega} r k_{i}(A)$ always exists.
Proposition 1.4.9. The function $r k_{\omega}$ is a Sylvester matrix rank function on $R$.
Proof. One checks each property of a Sylvester matrix rank function easily with 1.4.7.

### 1.5 Measures on compact subsets of $\mathbb{C}$

In this section we briefly want to explain some measure theory. We already had some contact with measures when we talked about the spectral theorem. In this section we want to discuss the notion of convergence of measures and we will give a sufficient condition to pass from weak convergence to point wise convergence of a series of measures. We will start with explaining the weak convergence of measures and will then state the Riesz Representation Theorem for measures. In this section let $X \subseteq \mathbb{C}$ be a compact set and let $C(X)$ be the algebra of continuous functions $f: X \rightarrow \mathbb{C}$. We denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $X$.
Definition 1.5.1. Let $\left(\mu_{i}\right)_{i \in \mathbb{N}}$ be a sequence of Borel measures on $X$ with $\mu_{i}(X)=c \in \mathbb{R}$ for all $i$. We say that the measures $\mu_{i}$ converge weakly to some limit measure $\mu$ on X , if for all continuous functions $f \in C(S)$ we have

$$
\lim _{i \rightarrow \infty} \int_{S} f d \mu_{i}=\int_{S} f d \mu
$$

Example 1.5.2. Let $X=[-1,1]$ be a closed interval and let $\mu_{n}=\delta_{\frac{1}{n}}$ be the Dirac measure at $\frac{1}{n} \in \mathbb{R}$. Then the measures $\mu_{n}$ converge weakly to ${ }^{n} \mu=\delta_{0}$. Note further that $\mu([-1,0])=1 \neq 0=\lim _{n \rightarrow \infty} \mu_{n}([-1,0])$. This example shows that weak convergence does not imply convergence for all Borel sets.

We have the following characterization of weak convergence, known as the theorem of Portmanteau.

Proposition 1.5.3. [Els13, Chapter 8, 4.10] Let $X$ be a compact subspace of $\mathbb{C}$ and $\mu, \mu_{n}(n \in \mathbb{N})$ be Borel measures on $X$ with $\mu(X), \mu_{n}(X) \leq \infty$. The following are equivalent.
(a) For every continuous function $X \rightarrow \mathbb{C}$ we have

$$
\lim _{n \rightarrow \infty} \int_{X} f d \mu_{n}=\int_{X} f d \mu
$$

(b) For all closed sets $A \subseteq X$ we have

$$
\limsup _{n \rightarrow \infty} \mu_{n}(A) \leq \mu(A)
$$

(c) For all open sets $U \subseteq X$ we have

$$
\liminf _{n \rightarrow \infty} \mu_{n}(U) \geq \mu(U)
$$

(d) For all Borel sets $S$ with $\mu(\delta(S))=0$ where $\delta(S)$ denotes the boundary of $S$ we have

$$
\lim _{n \rightarrow \infty} \mu_{n}(S)=\mu(S)
$$

We now want to explain the Riesz Representations Theorem for measures.
Proposition 1.5.4. [Els13, Chapter 8, 2.19] Let $X \subseteq \mathbb{C}$ be compact and let $I: C(X) \rightarrow \mathbb{C}$ be a positive linear functional. Then there exists exactly one finite measure $\mu$ on $X$ such that for all measureable sets $A \subseteq X$ we have $\mu(A)=$ $\sup _{K}\{\mu(K) \mid K \subseteq A, K$ compact $\}$ and

$$
I(f)=\int_{X} f d \mu
$$

for all $f \in C(X)$. In particular we have

$$
\begin{gathered}
\mu(K)=\inf \left\{I(f) \mid f \in C(X), f \geq \chi_{K}\right\} \quad \text { when } K \text { is compact. } \\
\mu(A)=\sup \{\mu(K) \mid K \subseteq A, K \text { compact }\} \quad \text { when } A \text { is measurable. }
\end{gathered}
$$

We just want to describe one use case of this proposition which will be helpful later when we talk about approximation of spectral measures in a finite von Neumann algebra. Assume we have a series of measures $\mu_{n}(X)$ such that for all $f \in C(X)$ the limit

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{X} f d \mu_{n} \tag{1.13}
\end{equation*}
$$

exists. Then, 1.13 defines a linear functional on $C(X)$. Thus, by the previous theorem there exists a limit measure $\mu$ on $X$ such that the $\mu_{n}$ converge weakly towards $\mu$. Moreover this limit measure is unique.

In contrast to weak convergence of measures, there is also the notion of strong convergence. That is $\left.\lim _{n \rightarrow \infty} \mu_{( } A\right)=\mu(A)$ for all Borel sets $A$. However, for later use, we are more interested in the convergence in points. In particular we are interested in conditions that ensure $\lim _{n \rightarrow \infty} \mu_{n}(\{\lambda\})=\mu(\{\lambda\})$ for measures $\mu_{n}$ that converge weakly to $\mu$. For simplicity let $y \in \mathbb{C}$ and $X=\overline{B(y, d)} \subseteq \mathbb{C}$ be the closed ball with radius $d \in \mathbb{R}$ around $y$. Let $\mu_{n}$ be a series of finite measures on $X$ that converge weakly to $\mu$. We have the following sufficient condition.

Proposition 1.5.5. Assume that there is a function $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$such that $\lim _{\lambda \rightarrow 0} f(\lambda)=0$ and $\mu_{n}(B(x, \lambda) \backslash\{x\}) \leq f(\lambda)$ for all $n \in \mathbb{N}$. Then for any $x \in X$ we have

$$
\lim _{n \rightarrow \infty} \mu_{n}(\{x\})=\mu(\{x\})
$$

Proof. Since the measures $\mu_{n}$ converge weakly towards $\mu$, part (c) of Proposition 1.5.3 already gives us $\lim \sup \mu_{n}(\{x\}) \leq \mu(\{x\})$. Thus we only have to show the other inequality. For that we have for every $\lambda>0$

$$
\begin{aligned}
\lim \inf \mu_{n}(\{x\}) & =\liminf \mu_{n}(B(x, \lambda) \backslash(B(x, \lambda) \backslash\{x\})) \\
& \left.=\liminf \mu_{n}(B(x, \lambda))-\lim \inf \mu_{n}(B(x, \lambda) \backslash\{x\})\right) \\
& \geq \liminf \mu_{n}(B(x, \lambda))-f(\lambda) \\
& \geq \mu(B(x, \lambda))-f(\lambda)
\end{aligned}
$$

Since this holds for all $\lambda$ we get

$$
\lim \inf \mu_{n}(\{x\}) \geq \lim _{\lambda \rightarrow 0} \mu(B(x, \lambda))-f(\lambda)=\mu(\{x\})
$$

We now want to consider the question if the converse is also true. That means given that $\lim _{n \rightarrow \infty} \mu_{n}(0)=\mu(\{0\})$, is there a function $f$ as in Proposition 1.5.5?

In fact we have the following theorem.
Theorem 1.5.6. Let $\left(\mu_{n}\right)_{n \in \mathbb{N}}$ be a sequence of weakly convergent measures on a compact set $X \subseteq \mathbb{C}$ with weak limit $\mu$ and let $x \in X$. Then the following are equivalent:
(1) $\lim _{n \rightarrow \infty} \mu_{n}(\{x\})=\mu(\{x\})$.
(2) There is a function $f: \mathbb{R}_{+} \rightarrow R_{+}$with $\lim _{\lambda \rightarrow 0} f(\lambda)=0$ such that for all $n \in \mathbb{N}$ we have

$$
\mu_{n}(B(x, \lambda) \backslash\{x\}) \leq f(\lambda)
$$

Proof. We have already seen $(2) \Rightarrow(1)$ in the previous proposition. So let us show $(1) \Rightarrow(2)$. The proof works by contradiction. So let us assume that for all functions $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$with $\lim _{\lambda \rightarrow 0} f(\lambda)=0$ there is $n \in \mathbb{N}$ such that

$$
\mu_{n}(B(x, \lambda) \backslash\{x\})>f(\lambda)
$$

The assumption $\lim _{n \rightarrow \infty} \mu_{n}(\{0\})=\mu(\{0\})$ implies $\lim _{\omega} \mu_{n}(\{0\})=\mu(\{0\})$ for all non principal ultrafilters $\omega$. Consider the function ${ }^{\omega} f(\lambda)=\sup \mu_{n}(B(x, \lambda) \backslash\{x\})$. Then by definition we have $\mu_{n}(B(x, \lambda) \backslash\{x\}) \leq f(\lambda)$ for all $n$. Further we have $f(\lambda) \geq 0$ for all $\lambda$. Since the function is monotonically increasing, by our assumption we have $\lim _{\lambda \rightarrow 0} f(\lambda)>0$. Thus there exists $c>0$ such that $f(\lambda)=\sup _{n} \mu_{n}(B(x, \lambda) \backslash\{x\})>c$. Set $\epsilon=c / 2$ and define

$$
I(\lambda):=\left\{n \in \mathbb{N} \mid \mu_{n}(B(x, \lambda) \backslash\{x\})>\epsilon\right\} .
$$

Claim. The set $\mathcal{I}=\{A \subseteq \mathbb{N} \mid I(\lambda) \subseteq A$ for some $\lambda>0\}$ is a filter on $\mathbb{N}$ that contains no finite sets.

Proof. Obviously we have $\mathbb{N} \in \mathcal{I}$ and since each $I(\lambda) \neq \emptyset$ we have $\emptyset \neq \mathcal{I}$. Let now $A \subseteq B \subseteq \mathbb{N}$ and assume $A \in \mathcal{I}$. Then there is some $\lambda>0$ such that $I(\lambda) \subseteq A$. But then $I(\lambda) \subseteq B$ and therefore $B \in \mathcal{I}$. Since for $\lambda_{1} \leq \lambda_{2}$ we have $I\left(\lambda_{1}\right) \subseteq I\left(\lambda_{2}\right)$ the set $\mathcal{I}$ is closed under finite intersections. For the last part it is enough to show that each $I(\lambda)$ is an infinite set. For that note that for each $n \in \mathbb{N}$ there is $\lambda(n)>0$ such that $\mu_{n}(B(x, \lambda) \backslash\{x\})<\epsilon$. Assume now that $I(\lambda)=\left\{n_{1}, \ldots, n_{k}\right\}$. Set $\lambda^{\prime}=\min \left\{\lambda\left(n_{1}\right), \ldots, \lambda\left(n_{k}\right)\right\}$. But then $I\left(\lambda^{\prime}\right)$ would be empty, which can not be.

Thus we can construct a non principal ultrafilter $\omega$ that contains $\mathcal{I}$.
Claim. For all $\lambda>0$ we have

$$
\lim _{\omega} \mu_{n}((B(x, \lambda) \backslash\{x\}) \geq \epsilon
$$

Proof. Assume that $\lim _{\omega} \mu_{n}(B(x, \lambda) \backslash\{x\})=\delta<\epsilon$. Let $\mu_{n}((B(x, \lambda) \backslash\{x\})=$ $m(n)$. That means that for any $\kappa>0$ the set $J(\lambda)=\{n \in \mathbb{N}| | \delta-m(n) \mid<\kappa\}$ belongs to the ultrafilter $\omega$. But for $\kappa \leq \frac{|\epsilon-\delta|}{2}$ we have $J(\lambda) \cap I(\lambda)=\emptyset$, which is a contradiction.

Consider now the linear functional $I_{\omega}$ on $C(X)$ given by

$$
I_{\omega}(g)=\lim _{\omega} \int_{X} g d \mu_{n} .
$$

By 1.5.4 there exists a measure $\mu_{\omega}$ on $X$ such that

$$
I_{\omega}(g)=\int_{X} g d \mu_{\omega} .
$$

Let us calculate $\mu_{\omega}(\{x\})$. Again by the properties of $\mu_{\omega}$ we have

$$
\begin{aligned}
\mu_{\omega}(\{x\}) & =\inf \left\{I_{\omega}(g) \mid g \in C(X), g \geq \chi_{\{x\}}\right\} \\
& =\inf \left\{\lim _{\omega} \int_{X} g d \mu_{n} \mid g \in C(X), g \geq \chi_{\{x\}}\right\} \\
& \geq \inf \left\{\lim _{\omega} g(x) \cdot \mu_{n}(\{0\})+\int_{X \backslash\{x\}} g d \mu_{n} \mid g \in C(X), g \geq \chi_{\{x\}}\right\} .
\end{aligned}
$$

Without loss of generality we can assume that $g(x)=1$. Let $C=\{h \in C(X) \mid$ $\left.h \geq \chi_{\{x\}}, h(x)=1\right\}$. Fix $\delta>0$. By continuity, for each $h \in C$ there is a $\lambda(h)>0$ such that $h(y) \geq 1-\delta$ for all $y \in B(x, \lambda(h))$. With this notation we get

$$
\begin{aligned}
\mu_{\omega}(\{x\}) & =\inf \left\{\lim _{\omega} g(x) \cdot \mu_{n}(\{x\})+\int_{X \backslash\{x\}} g d \mu_{n} \mid g \in C(X), g \geq \chi_{\{x\}}\right\} \\
& =\lim _{\omega} \mu_{n}(\{x\})+\inf \left\{\lim _{\omega} \int_{X \backslash\{x\}} g d \mu_{n} \mid g \in C\right\} \\
& \geq \lim _{\omega} \mu_{n}(\{x\})+\inf \left\{\lim _{\omega} \int_{B(x, \lambda(g)) \backslash\{x\}} g d \mu_{n} \mid g \in C\right\} \\
& \geq \lim _{\omega} \mu_{n}(\{x\})+\inf _{g}\left\{\lim _{\omega}(1-\delta) \cdot \mu_{n}(B(x, \lambda(g)) \backslash\{x\})\right\} \\
& \geq \lim _{\omega} \mu_{n}(\{x\})+(1-\delta) \cdot \epsilon \\
& >\lim _{\omega} \mu_{n}(\{x\}) .
\end{aligned}
$$

Since the measures $\mu_{n}$ converge weakly to $\mu$, by the uniqueness property in 1.5.4, we have $\mu=\mu_{\omega}$. By assumption we have $\mu(\{0\})=\lim _{n \rightarrow \infty} \mu_{n}(\{0\})$. Since this implies $\lim _{\omega^{\prime}} \mu_{n}(\{0\})=\mu(\{0\})$ for all non principal ultrafilters $\omega^{\prime}$ we have a contradiction.

## Chapter 2

## Effective Lück approximation Theorem

### 2.1 Motivation

In this chapter we want to discuss the effective Lück approximation theorem. To motivate this problem we will discuss the easiest case first. For that, let $G$ be a residually finite group and let $G \unrhd N_{1} \unrhd N_{2} \ldots$ be a chain of normal subgroups in $G$ of finite index with trivial intersection. Let $G_{i}=G / N_{i}$ and fix a matrix $A \in \operatorname{Mat}_{n}(\mathbb{Z}[G])$. The group $G$ acts by right multiplication on the groups $G_{i}$ and this action can be extended linearly to an action of $\operatorname{Mat}_{n}(\mathbb{Z}[G])$ on $\left(\mathbb{C}\left[G_{i}\right]\right)^{n}$ by right multiplication. Note that, as $\mathbb{C}$-vector spaces, we have $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{\left|G_{i}\right|}$. Therefore the matrix $A$ induces a matrix $A_{i} \in \operatorname{Mat}_{\left|G_{i}\right| \cdot n}(\mathbb{Z})$ that represents the action of $A$ on $\mathbb{C}^{\left|G_{i}\right| \cdot n}$. We can now ask the following questions:
(1) Does $\lim _{i \rightarrow \infty} \frac{1}{\left|G_{i}\right|} \mathrm{rk}_{\mathbb{C}}\left(A_{i}\right)$ exist?
(2) If the answer to question (1) is yes, is the limit independent of the chain $N_{1} \unrhd N_{2} \unrhd \ldots ?$
(3) If the answer to question (2) is yes, is $\lim _{i \rightarrow \infty} \frac{1}{\left|G_{i}\right|} \mathrm{rk}_{\mathbb{C}}\left(A_{i}\right)=\mathrm{rk}_{G}(A)$ ?

In this specific case, Wolfgang Lück showed in 1994 in [Lüc94] that the answer to all questions is yes. In this chapter we want to generalize these questions.

### 2.2 The approximation property of traces

The situation described in the previous section is very specific and we want to introduce a more general viewpoint. For that let us remember how $\mathrm{rk}_{G}$ was defined in 1.2.28 We considered $\operatorname{Mat}_{n}(\mathbb{C}[G])$ as a subalgebra of the amplified von Neumann algebra $\operatorname{Mat}_{n}(\mathcal{N}(G))$ which sits in $\mathcal{B}\left(\ell^{2}(G)^{n}\right)$, the algebra of
bounded operators on $\left(\ell^{2}(G)\right)^{n}$. On the amplified group von Neumann algebra $\operatorname{Mat}_{n}(\mathcal{N}(G))$ we defined a trace by

$$
\tau(a)=\sum_{i=1}^{n}\left\langle 1_{i} a, 1_{i}\right\rangle,
$$

where $1_{i} \in \ell^{2}(G)^{n}$ was the vector having $1_{G}$ in the $i$-th coordinate and zeros elsewhere. Note that the quotient maps $G \rightarrow G_{i}$ give us representations

$$
\varphi_{i}: \operatorname{Mat}_{n}(\mathbb{C}[G]) \rightarrow \operatorname{Mat}_{n}\left(\mathbb{C}\left[G_{i}\right]\right)=\operatorname{Mat}_{n}\left(\mathcal{N}\left(G_{i}\right)\right)=\mathcal{B}\left(\ell^{2}\left(G_{i}\right)^{n}\right)
$$

Using the notation of 1.2.28 we have the following lemma.
Lemma 2.2.1. Let $\mathcal{A}$ be $a *$-algebra over a subfield $K$ of $\mathbb{C}$ and let $\varphi: \mathcal{A} \rightarrow$ $\operatorname{Mat}_{n}(K)$ be $a *$-homomorphism. Consider the trace $\tau=\frac{1}{n} \operatorname{Tr}_{\mathbb{C}} \circ \varphi$ on $\mathcal{A}$. Then

$$
\mathrm{rk}_{\tau}=\frac{1}{n} \mathrm{rk}_{\mathbb{C}} \circ \varphi \in \mathbb{P}(\mathcal{A})
$$

Further, for a normal element $a \in \operatorname{Mat}_{n}(\mathcal{A})$ we have

$$
\mu_{a, \tau}=\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i}}
$$

where $\lambda_{i}$ are the eigenvalues of $\varphi(a)$ and $\mu_{a, \tau}=\tau \circ E_{a, \tau}$ as in 1.2.21. We call this measure the normalized eigenvalue measure.

Proof. To determine $\mathrm{rk}_{\tau}$ we have to determine $H_{\tau}$ and $\mathcal{N}_{\tau}$ first. For simplicity we will only determine $\operatorname{rk}_{\tau}(a)$ for $a \in \mathcal{A}$. We will use the notation of 1.2.6. Let us determine $N=\left\{a \in \mathcal{A} \mid \tau\left(a^{*} a\right)=0\right\}$. For $a \in N$ we have

$$
\begin{aligned}
0 & =\tau\left(a^{*} a\right) \\
& =\frac{1}{n} \operatorname{Tr}_{\mathbb{C}}\left(\varphi\left(a^{*} a\right)\right) \\
& =\frac{1}{n} \operatorname{Tr}_{\mathbb{C}}\left(\varphi\left(a^{*}\right) \varphi(a)\right) .
\end{aligned}
$$

Since the usual trace on $\operatorname{Mat}_{n}(\mathbb{C})$ is faithful we get $\varphi(a)=0$ and therefore $N=\operatorname{ker}(\varphi)$. As a result we have that $\mathcal{A} / N$ is isomorphic to some subalgebra of $\operatorname{Mat}_{n}(\mathbb{C})$. The norm on $\mathcal{A} / N$ induced by $\tau$ is just the Frobenius norm on $\operatorname{Mat}_{n}(\mathbb{C})$. From that it follows easily that $\mathcal{H}_{\tau}$ is a subspace of $\operatorname{Mat}_{n}(\mathbb{C}) \cong \mathbb{C}^{n^{2}}$ and $\mathcal{N}_{\tau}$ is a subalgebra of $\operatorname{Mat}_{k}(\mathbb{C})$ the action is given by matrix multiplication from the right. Let now $a \in \mathcal{A}$ and denote by $r_{a}=\varphi(a)$ the induced operator on $\mathcal{H}_{\tau}$. Since for any matrix $C \in \operatorname{Mat}_{d}(\mathbb{C})$ we have $\operatorname{Ann}_{r}(C) \cong \operatorname{ker}(C)^{d}$ as $\mathbb{C}$ vector spaces we obtain

$$
\operatorname{dim}_{\mathbb{C}} \operatorname{Ann}_{r}(C)=d \cdot \operatorname{dim}_{\mathbb{C}} \operatorname{ker} C
$$

Let $P \in \operatorname{Mat}_{n}(\mathbb{C})$ be the projection onto $\operatorname{ker} \varphi(A)$. Note that $P$ as an operator on $\mathcal{H}_{\tau} \cong \operatorname{Mat}_{n}(\mathbb{C})$ is the projection onto $\operatorname{Ann}_{r}(\varphi(a))$. By the definition of $\mathrm{rk}_{\tau}$ it is enough to see that

$$
\begin{aligned}
\operatorname{dim}_{\tau} \operatorname{ker} r_{a} & =\operatorname{dim}_{\tau} \operatorname{Ann}_{r}(\varphi(a)) \\
& =\tau(P) \\
& =\frac{1}{n} \operatorname{Tr}_{\mathbb{C}}(P) \\
& =\frac{1}{n} \operatorname{dim}_{\mathbb{C}} \operatorname{ker} \varphi(a)
\end{aligned}
$$

The statement about the spectral measures follows directly from the uniqueness in 1.5.4.

Let us come back to the situation from the beginning of this chapter. Let $G,\left\{N_{i}\right\}$ and $\left\{G_{i}\right\}$ be a residually finite group, a chain of normal subgroups with trivial intersection and the associated quotient groups respectively. Let $\varphi_{i}: \mathbb{Z}[G] \rightarrow \operatorname{Mat}_{\left|G_{i}\right|}(\mathbb{Z})$ be the homomorphism induced by the action of $G$ on $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{\left|G_{i}\right|}$. Let $\tau_{i}=\frac{1}{\left|G_{i}\right|} \operatorname{Tr}_{\mathbb{C}} \circ \varphi_{i}$ and $A \in \operatorname{Mat}_{n}(\mathbb{Z}[G])$.

Lemma 2.2.2. We have

$$
\lim _{i \rightarrow \infty} \tau_{i}=\tau_{G}
$$

where $\tau_{G}$ denotes the regular character as in 1.1.
Proof. Since $\tau_{i}, \tau_{G}$ are linear for all $i$, it is enough to consider elements $g \in G$. Note that

$$
\tau_{i}(g)= \begin{cases}1 & g \in N_{i} \\ 0 & g \notin N_{i}\end{cases}
$$

and

$$
\tau_{G}(g)= \begin{cases}1 & g=e_{G} \\ 0 & g \neq e_{G}\end{cases}
$$

Therefore the result follows since $\bigcap N_{i}=\left\{1_{G}\right\}$.
Knowing this, our three questions are just one special situation of question 1.2.29, which asked if the assignment $\tau \mapsto \mathrm{rk}_{\tau}$ is continuous with respect to point wise convergence. We now want to consider this question in more detail. The following definition first appeard similar in [Kio18].

Definition 2.2.3. Let $\mathcal{A}$ be an algebra and let $\mathcal{C}$ be a class of traces on $\mathcal{A}$. We say that $\mathcal{C}$ has the approximation property if for every point wise converging sequence of traces $\left(\tau_{i}\right)_{i \in \mathbb{N}}, \tau_{i} \in \mathcal{C}$ with $\lim _{i \rightarrow \infty} \tau_{i}=\tau$ we have

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{\tau_{i}}=\mathrm{rk}_{\tau}
$$

We will see examples for $\mathcal{C}$ in the next chapter. In 1.2 .30 we have seen a converging sequence of traces $\lim _{i \rightarrow \infty} \tau_{i} \rightarrow \tau$ with $\lim _{i \rightarrow \infty} \mathrm{rk}_{\tau_{i}} \neq \mathrm{rk}_{\tau}$. Thus the class of all traces does not have the approximation property. So why should the convergence of the traces imply the convergence of the ranks in some cases? A more analytic approach can answer this question. Let $(\mathcal{A}, \tau)$ be a tracial *-algebra and let $A \in \operatorname{Mat}_{n}(\mathcal{A})$ be a normal matrix over $\mathcal{A}$. We have seen in the previous chapters that $\operatorname{rk}_{\tau}(A)=n-\operatorname{dim}_{\tau} \operatorname{ker}(A)=n-\tau\left(\operatorname{Pr}_{\operatorname{ker}(A)}\right)=$ $n-\tau\left(E_{A, \tau}(\{0\})\right)=n-\mu_{A, \tau}(\{0\})$, where $E_{A, \tau}$ denotes the spectral measure of the operator on $\mathcal{H}_{\tau}$ given by right multiplication by $A$. We will write $\mu_{A, \tau}$ for the complex measure $\tau \circ E_{A, \tau}$ as in 1.2.21.

Therefore a measure theoretic reformulation of the point wise convergence of rank functions is given by

$$
\lim _{i \rightarrow \infty} \mu_{A, \tau_{i}}(\{0\})=\mu_{A, \tau}(\{0\})
$$

Remark 2.2.4. Note that for any matrix $A \in \operatorname{Mat}_{n \times m}(\mathcal{A})$ and any trace $\tau$ we have $\operatorname{ker} A=\operatorname{ker} A A^{*}$ as subspaces of $\mathcal{H}_{\tau}^{n}$ and therefore $\operatorname{rk}_{\tau}(A)=\operatorname{rk}_{\tau}\left(A A^{*}\right)$. Therefore we can always assume that the matrix $A$ is positive self-adjoint, in particular normal.

When talking about convergence of measures, pointwise convergence is usually not the first thing what one asks for. The starting point is usually the weak convergence. Remember that for weak convergence we have to check

$$
\lim _{i \rightarrow \infty} \int_{S} f d \mu_{i}=\int_{S} f d \mu
$$

for all $f \in C(S)$. To check the convergence of these integrals for all kind of continuous functions would be a lot of work, however it is not necessary. We have the following theorem.

Theorem 2.2.5 (Stone-Weierstrass-Approximation). Let $S$ be a compact metric space and let $C(S, \mathbb{C})$ be the algebra of continuous functions on $S$. Let $R \subseteq C(S, \mathbb{C})$ be a complex unital subalgebra that is closed under complex conjugation and separates points in $X$, which means for all $x \neq y$ in $S$ there is a $g \in R$ with $g(x) \neq g(y)$. Then $R$ is dense in $C(S, \mathbb{C})$.

We are now ready to prove the following proposition.
Proposition 2.2.6. Let $\mathcal{A}$ be $a *$-algebra and let $\tau,\left(\tau_{i}\right)_{i \in \mathbb{N}}$ be traces on $\mathcal{A}$ such that $\lim _{i \rightarrow \infty} \tau_{i}=\tau$. Let $A \in \operatorname{Mat}_{n}(\mathcal{A})$ be a normal matrix and denote by $\rho_{\tau_{i}}(A) \in$ $\mathcal{B}\left(\mathcal{H}_{\tau_{i}}^{n}\right)$ the operator given by right multiplication by $A$ and similar for $\tau$. Assume that there is a constant $c$ such that for all $i \in \mathbb{N}\left\|\rho_{\tau_{i}}(A)\right\|,\left\|\rho_{\tau}(A)\right\| \leq c$. Then the measures $\mu_{A, \tau_{i}}$ converge weakly to $\mu_{A, \tau}$.

Proof. Since the operator norm of all operators $\rho_{\tau}(A), \rho_{\tau_{i}}(A)$ is bounded by $c$, the measures $\mu_{A, \tau}, \mu_{\mathcal{A}, \tau_{i}}$ are supported on $S=\overline{B(0, c)} \subseteq \mathbb{C}$. Consider the
algebra $R$ generated by all functions $f_{n, m}(x)=x^{n} \bar{x}^{m}$. By the previous theorem the algebra $R$ is dense in $C(S, \mathbb{C})$, therefore it is enough to show that for all functions $f \in R$ we have

$$
\lim _{i \rightarrow \infty} \int_{S} f d \mu_{A, \tau_{i}}=\int_{S} f d \mu_{A, \tau}
$$

Since integrals are linear it suffices to show that for each $n, m$ we have

$$
\lim _{i \rightarrow \infty} \int_{S} x^{n} \bar{x}^{m} d \mu_{A, \tau_{i}}=\int_{S} x^{n} \bar{x}^{m} d \mu_{A, \tau}
$$

Note that by the definition of $\mu_{A, \tau_{i}}$ and $\mu_{A, \tau}$ we have

$$
\lim _{i \rightarrow \infty} \int_{S} x^{n} \bar{x}^{m} d \mu_{A, \tau_{i}}=\lim _{i \rightarrow \infty} \tau_{i}\left(A^{n}\left(A^{*}\right)^{m}\right)=\tau\left(A^{n}\left(A^{*}\right)^{m}\right)=\int_{S} x^{n} \bar{x}^{m} d \mu_{A, \tau}
$$

Remark 2.2.7. By the Theorem of Portmanteau 1.5.3 the above already gives

$$
\lim _{i \rightarrow \infty} \mu_{A, \tau_{i}}(\{0\}) \leq \mu_{A, \tau}(\{0\})
$$

for any normal matrix $A$ over $\mathcal{A}$. Thus, for any arbitrary matrix $A$ over $\mathcal{A}$, we have

$$
\begin{aligned}
& \operatorname{rk}_{\tau}(A)=\operatorname{rk}_{\tau}\left(A A^{*}\right)=\mu_{A A^{*}, \tau}(\{0\}) \geq \\
& \quad \limsup _{i \rightarrow \infty} \mu_{A A^{*}, \tau_{i}}(\{0\})=\limsup _{i \rightarrow \infty} \operatorname{rk}_{\tau_{i}}\left(A A^{*}\right)=\limsup _{i \rightarrow \infty} \operatorname{rk}_{\tau_{i}}(A) .
\end{aligned}
$$

This inequality is known as Kazhdan Inequality.
Having the weak convergence of the measures and keeping in mind Theorem 1.5.6 ,to show that $\lim _{i \rightarrow \infty} \mu_{A, \tau_{i}}(\{0\})=\mu_{A, \tau}(\{0\})$ we need a function that bounds the measure of $B(0, \lambda) \backslash\{0\}$ in terms of $\lambda$.

Definition 2.2.8. Let $\mathcal{A}$ be a $*$-algebra and let $\mathcal{C}$ be a class of traces on $\mathcal{A}$. We say that $\mathcal{C}$ has the effective approximation property if for any normal matrix $A \in \operatorname{Mat}_{n}(\mathcal{A})$ there is a function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\lim _{\lambda \rightarrow 0} f(\lambda)=0$ such that for all $\tau \in \mathcal{C}$ and for all $x \in \mathbb{C}$

$$
\mu_{A, \tau}(B(x, \lambda) \backslash\{x\})<f(\lambda)
$$

### 2.3 Effective approximation property for integer valued bounded traces

In this section we want to introduce some classes of traces and consider the question whether they have the effective approximation property. Let us first
consider the group case. Let $F=\left\langle x_{1}, \ldots, x_{m}\right\rangle$ be a finitely generated free group and let $\mathcal{A}=\mathbb{C}[F]$ be the group algebra. If $X$ is a finite $F$-set, from the permutation action of $F$ on $X$ we get an induced homomorphism $f_{X}$ : $\mathbb{C}[F] \rightarrow \operatorname{Mat}_{|X|}(\mathbb{C})$. Then the character $\tau=\frac{1}{|X|} \operatorname{Tr} \circ f_{X}$ is called a permutation character. A character $\tau$ on $\mathbb{C}[F]$ is called a sofic if there exists a sequence of permutation characters $\tau_{i}$ such that

$$
\tau=\lim _{i \rightarrow \infty} \tau_{i}
$$

One example of a class of traces $\mathcal{C}$ on $\mathcal{A}$ would be the class of all permutation traces/characters. Wolfgang Lück showed that the class of all permutation characters on $\mathbb{Q}[F]$ has the approximation property (see 2.4.1). A character $\tau$ on $\mathbb{C}[F]$ is called unitary if there is a representation $\varphi: F \rightarrow U_{n}(\mathbb{C})$ with finite image such that

$$
\tau=\frac{1}{\left|X_{i}\right|} \operatorname{Tr} \circ \varphi .
$$

Here $n \in \mathbb{N}$ and $U_{n}(\mathbb{C})$ denotes the group of unitary matrices of dimension $n$. We call $\tau$ unitary of degree $d \in \mathbb{N}$, if there is a numberfield $K$ with $|K: \mathbb{Q}|=d$ and $\varphi(F) \in \operatorname{Mat}_{n}(K)$. We call a character $\tau$ on $\mathbb{C}[F]$ hyperlinear (of degree d) if there exists a sequence of unitary characters $\left(\tau_{i}\right)$ (of degree d) with $\lim _{i \rightarrow \infty} \tau_{i}=\tau$. Since permutation matrices are unitary, every sofic character is hyperlinear of degree 1. In [Kio18] Steffen Kionke shows that for fixed d, the class of unitary characters of degree $d$ on $\mathcal{A}=\overline{\mathbb{Q}}[F]$ has the approximation property.

Let us now consider the case of general $*$-algebras. Obviously every matrix over some algebra has only finitely many entries. Therefore we can restrict ourselves to finitely generated algebras. Since every $*$-algebra is the image of a free $*$-algebra, we can assume that $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{m}, x_{m}^{*}\right\rangle$.

Let $b \in \mathbb{R}_{>0}$. A trace $\tau$ on $\mathcal{A}$ is called $b$-bounded integer valued, if there is a $*$-homomorphism $\varphi: \mathcal{A} \rightarrow \operatorname{Mat}_{n}(\mathbb{C})$ with $\varphi\left(x_{j}\right) \in \operatorname{Mat}_{n_{i}}(\mathbb{Z})$ for each $j$ and such that the row- and column-sum norms of $\varphi\left(x_{j}\right)$ are bounded by $b$.:

$$
\begin{equation*}
\left\|\varphi\left(x_{j}\right)\right\|_{1},\left\|\varphi\left(x_{j}\right)\right\|_{\infty} \leq b . \tag{2.1}
\end{equation*}
$$

Note that since for every matrix $A \in \operatorname{Mat}_{n}(\mathbb{C})$, we have

$$
\|A\|_{2} \leq \sqrt{\|A\|_{1} \cdot\|A\|_{\infty}},
$$

this condition already gives a uniform bound on the operator norm of the $\varphi\left(x_{j}\right)$. To extend this definition to traces $\tau$ on arbitrary algebras $\mathcal{A}$, we choose a surjective $*$ - homorphism $\varphi: \mathcal{F} \rightarrow \mathcal{A}$ from a free $*$-algebra $\mathcal{F}$ to $\mathcal{A}$ and define the trace $\tau$ on $\mathcal{A}$ to be $b$-bounded integer valued, if the induced trace $\tau \circ \varphi$ on $\mathcal{F}$ is so. The main result of this chapter is the following.

Theorem A. Let $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ be a free $*$-algebra, and let $\mathcal{C}$ be the class of b-bounded integer valued traces on $\mathcal{A}$. Then $\mathcal{C}$ has the effective approximation property.

Note that this is a generalization of Lück's Lemma 2.4.1. To prove this theorem we will make use of traces and limit measures.

Example 2.3.1. Let $G$ be a sofic group (see 3.1 .1 for a definition), $A \in$ $\operatorname{Mat}_{n}(\mathbb{C}[G])$ and let $\tau_{G}$ be the regular character on $G$. Let $F$ be a finitely generated free group with a surjective group homomorphism $\alpha: F \rightarrow G$. Let $B \in \operatorname{Mat}_{n}(\mathbb{C}[F])$ be a preimage of $A$ under $\alpha$ and let $\left(X_{i}\right)$ be a sofic approximation of $G$. Each finite $F$-set $X_{i}$ gives a trace $\tau_{i}=\frac{1}{\left|X_{i}\right|} \operatorname{Tr} \circ \varphi_{i}: \mathbb{C}[F] \rightarrow \mathbb{C}$, where $\varphi_{i}: \mathbb{C}[F] \rightarrow \operatorname{Mat}_{\left|X_{i}\right|}(\mathbb{C})$ is the map that represents the action of $F$ on $\mathbb{C}\left[X_{i}\right] \cong \mathbb{C}^{\left|X_{i}\right|}$ after choosing a basis. In [Jai19] Andrei Jaikin proved that

$$
\lim _{i \rightarrow \infty} \frac{1}{\left|X_{i}\right|} \operatorname{rk}\left(\alpha_{i}(B)\right)=\operatorname{rk}_{G}(A)
$$

In the rest of this chapter we want to prove Theorem A. We will now briefly describe the structure of the proof. In the following $K$ will denote different subfields of $\mathbb{C}$. We will always consider the free $*$-algebra $\mathcal{A}=K\left\langle x_{1}, x_{1}^{*}, \ldots, x_{m}, x_{m}^{*}\right\rangle=$ $K\left\langle X, X^{*}\right\rangle$ over $K$ and the class $\mathcal{C}$ of $b$-bounded integer valued traces on $\mathcal{A}$. Let $\tau \in \mathcal{C}$. We will start with the case $K=\mathbb{Q}$. For matrices $A$ over $\mathbb{Q}\left\langle X, X^{*}\right\rangle$ we will first give an explicit bound for $\mu_{A, \tau_{i}}(B(0, \lambda) \backslash\{0\})$ in Proposition 2.4.1. We have seen that this implies the approximation property. We will then show that the class $\mathcal{C}$ also has the approximation property for $K=\mathbb{C}$ using 1.3.31 and 1.3.32 in the following way. Set $K_{1}=\mathbb{Q}$ and for each $i \geq 1$ let $K_{2 i}=\overline{K_{i}}$ be the algebraic closure of $K_{i}$ in $\mathbb{C}$ and $K_{2 i+1}=K_{2 i}(\lambda)$ where $\lambda \in \mathbb{C} \backslash K_{2 i}$ is some complex number that is transcendental over $K_{2 i}$. Thus, to prove the approximation property for $K_{2 i}$ we can use the approximation property over $K_{2 i-1}$ and Theorem 1.3.31. To prove the approximation property from for $K_{2 i+1}$ we can use the approximation property over $K_{2 i}$ and 1.3.32. Note here that for $\lambda \in \mathbb{C} \backslash K_{2 i}$ the field $K_{2 i+1}$ is isomorphic to the field $K_{2 i}(t)$ of rational functions in one variable. Since any matrix $A \in \mathbb{C}\left\langle X, X^{*}\right\rangle$ has only finitely many entries, there is an $n \in \mathbb{N}$ such that $A$ is a matrix over $K_{2 n}\left\langle X, X^{*}\right\rangle$. Thus we can show the approximation property for all matrices over $\mathbb{C}\left\langle X, X^{*}\right\rangle$. This is the same strategy that Jaikin uses in his proof in [Jai19]. Having the approximation property and some converging sequence $\left(\tau_{i}\right) \subseteq \mathcal{C}$ we will first show that for every normal matrix $A$ over $\mathcal{A}$ there is a function $f$ as in 2.2.8, such that

$$
\mu_{\tau_{i}, A}(B(0, \lambda) \backslash\{0\}) \leq f(\lambda)
$$

Note that this function still depends on the sequence $\tau_{i}$. By considering the operator $A \otimes 1-1 \otimes A$, we will show that there is such a function $f$ such that

$$
\mu_{\tau_{i}, A}(B(y, \lambda) \backslash\{y\}) \leq f(\lambda)
$$

for all $y \in \mathbb{C}$. Note that this function still depends on the sequence $\left(\tau_{i}\right)$. However we will see that this is already enough to prove Theorem A.

### 2.4 The Base Case

In this section we want to present a proof for A over the field $\mathbb{Q}$. This proof goes back to Wolfgang Lück [Lüc94].

Proposition 2.4.1 (Lück's Lemma). Let $A \in \operatorname{Mat}_{n}(\mathbb{Z})$ with $\|A\| \leq d$ and eigenvalues $\lambda_{1}, \ldots, \lambda_{s}$ with multiplicities $m_{1}, \ldots, m_{s}$. Let $\mu_{A}=\sum_{i=1}^{r} m_{i} \delta_{\lambda_{i}}$ be the eigenvalue measure of $A$. Then for $\lambda \in(0,1)$

$$
\mu_{A}(B(0, \lambda) \backslash\{0\}) \leq \frac{\log (d) \cdot n}{|\log (\lambda)|}
$$

Proof. Assume that $\lambda_{1}, \ldots, \lambda_{r}$ are the non zero eigenvalues and that $\left\|\lambda_{1}\right\| \leq$ $\left\|\lambda_{2}\right\| \leq \cdots \leq\left\|\lambda_{r}\right\|$. Further let $\left\|\lambda_{i}\right\| \leq \lambda$ for $i \in\{1, \ldots, t\}$.

Then we have

$$
\mu_{A}(B(0, \lambda) \backslash\{0\})=m_{1}+\ldots+m_{t} .
$$

Let $R=\sum_{i=1}^{r} m_{i}$. The characteristic polynomial of $A$ is given by

$$
p(x)=x^{n-R}\left(x-\lambda_{1}\right)^{m_{1}} \cdots\left(x-\lambda_{r}\right)^{m_{r}} .
$$

Therefore we have

$$
\frac{p(x)}{x^{n-R}}=\left(x-\lambda_{1}\right)^{m_{1}} \cdots\left(x-\lambda_{r}\right)^{m_{r}} \in \mathbb{Z}[x]
$$

which gives us

$$
\lambda_{1}^{m_{1}} \cdots \lambda_{r}^{m_{r}} \in \mathbb{Z} \backslash\{0\} .
$$

This gives

$$
\begin{aligned}
1 & \leq\left\|\lambda_{1}^{m_{1}} \cdots \lambda_{r}^{m_{r}}\right\| \\
& \leq\left\|\lambda_{1}^{m_{1}}\right\| \cdots\left\|\lambda_{r}^{m_{r}}\right\| \\
& \leq\left\|\lambda_{1}^{m_{1}}\right\| \cdots\left\|\lambda_{t}^{m_{t}}\right\| d^{n} \\
& \leq \lambda^{m_{1}+\cdots+m_{t}} \cdot d^{n} .
\end{aligned}
$$

By taking the logarithm we obtain the following. Note that $\log (\lambda)<0$.

$$
\begin{aligned}
& 1 \leq \lambda^{m_{1}+\ldots+m_{t}} \cdot d^{n} \\
\Leftrightarrow & 0 \leq\left(m_{1}+\ldots m_{t}\right) \cdot \log (\lambda)+n \cdot \log (d) \\
\Leftrightarrow & |\log (\lambda)| \cdot\left(m_{1}+\ldots+m_{t}\right) \leq \log (d) \cdot n \\
\Leftrightarrow & m_{1}+\ldots+m_{t} \leq \frac{\log (d) \cdot n}{|\log (\lambda)|} .
\end{aligned}
$$

Corollary 2.4.2. Let $\mathcal{A}=\mathbb{Q}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ and $\mathcal{C}$ be the class of $b$-bounded integer valued traces on $\mathcal{A}$. Then $\mathcal{C}$ has the approximation property.

Proof. Set $\mathcal{A}_{\mathbb{Z}}=\mathbb{Z}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ and let $A \in \operatorname{Mat}_{m}(\mathcal{A})$ be a normal matrix over $\mathcal{A}$. Since every element from $\mathcal{A}$ is a finite linear combination of the $x_{i}, x_{i}^{*}$ and the matrix $A$ contains only finitely many entries, we can find a $q \in \mathbb{N}$ such that $q \cdot A \in \operatorname{Mat}_{m}\left(\mathcal{A}_{\mathbb{Z}}\right)$. Let now $\tau=\frac{1}{n} \operatorname{Tr} \circ \varphi \in \mathcal{C}$ with $\varphi: \mathcal{A} \rightarrow \operatorname{Mat}_{n}(\mathbb{Q})$. Let $\mu_{\varphi(q A)}$ be as in the previous proposition. Since $\mu_{\varphi(q A)}=m \cdot \mu_{\tau, A}$ The result follows from the previous proposition and 1.5.6.

So we have done our first step. We already have all the ingredients to apply Theorem 1.3.31, however to apply Theorem 1.3.32 we have to check a condition; namely that for any $n \times n$ matrix $A$ over $K\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$, where $K \subseteq \mathbb{C}$ is an algebraically closed field and for any $\lambda \in \mathbb{C} \backslash K$ we have $\operatorname{rk}_{\tau}(A-\lambda I d)=n$. This is often called the algebraic eigenvalue property.

### 2.5 The strong eigenvalue property

In this subsection we want to provide a key ingredient to the algebraic eigenvalue property. Let us fix some notation. Let $X=\left\{x_{1}, \ldots, x_{d}\right\}, K \subseteq \mathbb{C}$ be a subfield closed under complex conjugation and let $\mathcal{A}=K\left\langle X, X^{*}\right\rangle$ be the free $*$-algebra on $X$ over $K$. Fix $b>0$ and let

$$
\begin{equation*}
F_{b}=\left\{f: X \rightarrow \operatorname{Mat}_{n_{f}}(\mathbb{Z}) \mid n_{f} \in \mathbb{N},\left\|f\left(x_{i}\right)\right\|_{1},\left\|f\left(x_{i}\right)\right\|_{\infty}<b \text { for all } i\right\} . \tag{2.2}
\end{equation*}
$$

Note that each $f \in F_{b}$ extends uniquely to a $*$-algebra homomorphism $\mathcal{A} \rightarrow$ $\operatorname{Mat}_{n_{f}}(K)$ which we will also denote by $f$. Set

$$
\mathcal{V}=\mathcal{V}_{b}=\prod_{f \in F_{b}} \operatorname{Mat}_{n_{f}}(K)
$$

and

$$
\bar{f}=(f)_{f \in F_{b}}: \mathcal{A} \rightarrow \mathcal{V} .
$$

Obviously $\bar{f}$ is injective, thus we will identify $\mathcal{A}$ with its image in $\mathcal{V}$. We write an element $a \in \mathcal{V}$ in the form $a=\left(a_{f}\right)$, with $a_{f} \in \operatorname{Mat}_{n_{f}}(K)$. Note that $\mathcal{V}$ is a *-regular ring, thus we can consider $\mathcal{R}(\bar{f}(\mathcal{A}), \mathcal{V})$, the regular closure of $\mathcal{A}$ in $\mathcal{V}$. Note further that every projection $\pi_{f}: \mathcal{V} \rightarrow \operatorname{Mat}_{n_{f}}(K)$ gives a rank function $\mathrm{rk}_{f}=\frac{1}{n_{f}} \mathrm{rk}_{K} \circ \pi_{f}$ on $\mathcal{V}$ and therefore on $\mathcal{A}$. In this section we want to prove the following theorem. It first appeared in a similar form in Andrei Jaikin's paper [Jai19]. Jaikin called it the strong eigenvalue property.
Theorem 2.5.1. Let $A, B \in \operatorname{Mat}_{n \times m}(\mathcal{R}(\bar{f}(\mathcal{A}), \mathcal{V}))$ For every $\epsilon>0$ the set

$$
S_{\epsilon}(A, B)=\left\{\lambda \in \bar{K} \mid \text { there is } f \in F_{b} \text { such that } \mathrm{rk}_{f}(B)-\mathrm{rk}_{f}(B-\lambda A) \geq \epsilon\right\}
$$

is finite.
The case where for each map $f$ the image $f(x)_{i}$ is a permutation matrix was already proved by Jaikin in [Jai19, Chapter 8]. We will follow his proof to obtain this slightly more general result. Let us explain one special case of the
previous theorem. Assume $A, B$ are square matrices over some group algebra $\mathbb{C}[G]$ and assume that $B$ is the identity matrix. Then $\lambda \in S_{\epsilon}(A, B)$ implies that $\lambda^{-1}$ is an eigenvalue of $f(A)$ for some $f \in F_{b}$. Thus the theorem states that in all possible representations $f \in F_{b}$ the matrix $A$ has only finitely many eigenvalues where the dimension of the eigenspace is greater then $\epsilon$. The proof of this result is very technical. In the next subsections we will introduce the necessary tools.

### 2.5.1 Reduction of the problem

In this subsection we want to show that it is enough to prove the theorem for matrices $A, B \in \operatorname{Mat}_{n \times m}\left(\bar{f}\left(\mathcal{O}\left\langle X, X^{*}\right\rangle\right)\right)$ where $\mathcal{O}=\mathbb{Z}$ or $\mathcal{O}=Q[x]$ for some algebraically closed subfield $Q$ of $\mathbb{C}$. Let us assume the notation from the previous section. For now we can assume that $K$ is algebraically closed. So let $\mathcal{U}=\mathcal{R}\left(\bar{f}\left(K\left\langle X, X^{*}\right\rangle\right), \mathcal{V}\right)$ and $A, B \in \operatorname{Mat}_{n \times m}(\mathcal{U})$. We will use the rank functions $\mathrm{rk}_{f} \in \mathbb{P}(\mathcal{U})$ also for rank function on $\mathcal{V}$ and $K\left\langle X, X^{*}\right\rangle$. By Corollary 1.3.37 we can find matrices $M \in \operatorname{Mat}_{a \times b}\left(\bar{f}\left(K\left\langle X, X^{*}\right\rangle\right)\right)$ and matrices $v_{1}, v_{2} \in$ $\operatorname{Mat}_{n \times b}\left(\bar{f}\left(K\left\langle X, X^{*}\right\rangle\right)\right)$ such that for any $\lambda \in K \subseteq K\left\langle X, X^{*}\right\rangle$ and any rank function rk on $\mathcal{U}$ we have

$$
\operatorname{rk}(B-\lambda A)=\operatorname{rk}\binom{M}{v_{1}-v_{2}}-\operatorname{rk}(M)=\operatorname{rk}\left(B^{\prime}-\lambda A^{\prime}\right)-\operatorname{rk}(M),
$$

where

$$
B^{\prime}=\binom{M}{v_{1}} \quad \text { and } \quad A^{\prime}=\binom{0}{v_{2}} .
$$

Therefore we have

$$
\begin{aligned}
\operatorname{rk}(B)-\operatorname{rk}(B-\lambda A) & =\operatorname{rk}(B-0 \cdot A)-\operatorname{rk}(B-\lambda A) \\
& =\operatorname{rk}\left(B^{\prime}-A^{\prime}\right)-\operatorname{rk}(M)-\operatorname{rk}\left(B^{\prime}-\lambda A^{\prime}\right)+\operatorname{rk}(M) \\
& =\operatorname{rk}\left(B^{\prime}\right)-\operatorname{rk}\left(B^{\prime}-\lambda A^{\prime}\right) .
\end{aligned}
$$

Thus we have shown that it is enough to show the theorem for matrices over $K\left\langle X, X^{*}\right\rangle$.

So let $A, B \in \operatorname{Mat}_{a \times b}\left(\bar{f}\left(K\left\langle X, X^{*}\right\rangle\right)\right)$. Since $A, B$ have only finitely many entries, we can assume that $K$ has finite transcendental degree over $\mathbb{Q}$. Therefore it is enough to consider the cases
(1) $K=\overline{\mathbb{Q}}$, in this case put $T=\mathbb{Q}$ and
(2) $K=\overline{Q(x)}$ for some algebraically closed subfield $Q$ of $\mathbb{C}$ with $x \in \mathbb{C} \backslash \mathbb{Q}$ and we can assume that the theorem holds for $\mathcal{A}=Q\left\langle X, X^{*}\right\rangle$. In this case we put $T=Q(x)$.

In each case we can find a finite Galois extension $E$ of $T$, such that $A, B$ are matrices over $E\left\langle X, X^{*}\right\rangle$. Let $d$ be the degree of this extension and $G=$
$\left\{\sigma_{1}, \ldots, \sigma_{d}\right\}$ its Galois group. Let $\bar{A}=\bigoplus_{i=1}^{d} \sigma_{i}(A)$ and similar for $\bar{B}$. Note that if for some $\lambda \in \bar{E}$ and some $f \in F_{b}$ we have

$$
\operatorname{rk}_{f}(B)-\operatorname{rk}_{f}(B-\lambda A) \geq \epsilon
$$

then

$$
\operatorname{rk}_{f}(\bar{B})-\operatorname{rk}_{f}(\bar{B}-\lambda \bar{A}) \geq d \cdot \epsilon
$$

Just like in the proof of 1.3 .31 there is an invertible matrix $J$ over $E\left\langle X, X^{*}\right\rangle$, such that $A^{\prime}=J \bar{A} J^{-1}, B^{\prime}=J \bar{B} J^{-1}$ are matrices over $T\left\langle X, X^{*}\right\rangle$. Since

$$
\mathrm{rk}_{f}(\bar{B})-\mathrm{rk}_{f}(\bar{B}-\lambda \bar{A})=\mathrm{rk}_{f}\left(B^{\prime}\right)-\mathrm{rk}_{f}\left(B^{\prime}-\lambda A^{\prime}\right)
$$

it is enough to consider matrices over $T\left\langle X, X^{*}\right\rangle$. Note that $T$ is the fraction field of $\mathcal{O}$, where $\mathcal{O}=\mathbb{Z}$ or $\mathcal{O}=Q[x]$. Since multiplying with some constant does not change the rank, we can multiply $A, B$ with some element $r \in \mathcal{O}$, such that $r A, r B \in \mathcal{O}\left\langle X, X^{*}\right\rangle$. Thus we can assume that $A, B \in \mathcal{O}\left\langle X, X^{*}\right\rangle$.

### 2.5.2 Dedekind domains

We want to start with some general results about Dedekind domains.
Definition 2.5.2. An integral domain $\mathcal{O}$ which is not a field is called a Dedekind domain if
(1) $R$ is Noetherian.
(2) Every nonzero prime ideal of $R$ is maximal.
(3) $R$ is integrally closed in its field of fractions.

There will be two main examples of Dedekind domains we care about. For any algebraic number field $K$, its ring of integers $\mathcal{O}_{K}$ is a Dedekind domain. The second example is slightly more complicated. For that let $Q$ be an algebraically closed field and let $K$ be a finite extension of $Q(x)$, where $Q(x)$ is the function field in one variable over $Q$. Set $\mathcal{O}_{K}$ to be the integral closure of $Q[x]$ in $K$. Then $\mathcal{O}_{K}$ is a Dedekind domain. For more information see [SZ14]. For the rest of this section let us fix an arbitrary Dedekind domain $\mathcal{O}$ with field of fractions $K$. We first want to collect some facts about Dedekind domains.

Proposition 2.5.3. [NarO4, Theorem 1.12] Every proper non zero ideal I of $R$ can be represented uniquely up to order in the form

$$
I=P_{1} \cdots P_{r}
$$

where $P_{i}$ are prime ideals of $R$.
Further we have the following structure theorem for finitely generated modules over Dedekind domains.

Proposition 2.5.4. [NarO4, Theorem 1.32] Let $M$ be a finitely generated $\mathcal{O}$ module. Let $M^{\text {tors }} \leq M$ be the submodule of all torsion elements. Then $M$ can be decomposed as

$$
M=\mathcal{O}^{k} \oplus I \oplus M^{\mathrm{tors}}
$$

where $k \in \mathbb{N}$ and $I$ is an ideal of $\mathcal{O}$. Further we have $M^{\text {tors }} \cong \mathcal{O} / I_{1} \oplus \ldots \oplus \mathcal{O} / I_{r}$ for ideals $I_{1}, \ldots, I_{r}$ of $\mathcal{O}$.

We now want to introduce the notion of length functions on $\mathcal{O}$-modules.
Definition 2.5.5. A length function on $\mathcal{O}-\operatorname{Mod}$ is a function $l: \mathcal{O}-\operatorname{Mod} \rightarrow$ $\mathbb{R}_{\geq 0} \cup\{\infty\}$ that satisfies
(LF1) If $0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$ is an exact sequence of $\mathcal{O}$-modules then $l\left(M_{1}\right)+l\left(M_{3}\right)=l\left(M_{2}\right)$.
(LF2) $l(M)=\sup \{l(L): L \leq M$ and $L$ is finitely generated $\}$
If $\mathcal{O}=\mathbb{Z}$ we can set $l(M)=\log _{2}|M|$ for a $\mathbb{Z}$-module $M$. Let us fix a length function $l$ on $\mathcal{O}-\bmod$ such that $l(M)=\infty$ if $M$ is finitely generated non Artinian and $l(M)<\infty$ if $M$ is finitely generated Artinian. For more information about length functions on Dedekind domains see [NR65, Section 7].

For each maximal ideal $P$ of $\mathcal{O}$ we denote by

$$
\mathcal{O}_{P}=\left\{\left.\frac{a}{b} \in K \right\rvert\, a, b \in \mathcal{O}, b \notin P\right\}
$$

the localization of $\mathcal{O}$ at $P$.
Lemma 2.5.6. $\mathcal{O}_{P}$ is a discrete valuation ring.
Proof. By construction $\mathcal{O}_{P}$ is a local Dedekind domain with maximal ideal $P$. Thus it is enough to show that $P$ is principal. For that let $a \in P \backslash P^{2}$ and consider the ideal $(a) \leq \mathcal{O}_{P}$ which is generated by $a$. Since $\mathcal{O}_{P}$ is local we have $(a)=P^{k}$ for some $k \in \mathbb{N}$. Since $a \notin P^{2}$ we have $k=1$ and therefore $P=(a)$.

For any $\mathcal{O}_{P}$ module $V$ we put $l_{P}(V)=l(V)$.
Lemma 2.5.7. $l_{P}$ is a length function on $\mathcal{O}_{P}$ and we have

$$
l(V)=\sum_{P \leq \mathcal{O}} l_{P}\left(\mathcal{O}_{P} \otimes_{\mathcal{O}} V\right)
$$

where $P$ runs over all prime ideals of $\mathcal{O}$.
Proof. The fact that $l_{P}$ is a length function follows from the definitions. So let us consider the summation formula. By (LF2) it is enough to consider finitely generated $\mathcal{O}$-modules. By 2.5.4 we can assume that $V \cong \mathcal{O}^{k} \oplus I \oplus \mathcal{O} / I_{1} \oplus \ldots \oplus$ $\mathcal{O} / I_{r}$ for ideals $I, I_{1}, \ldots, I_{r}$ of $\mathcal{O}$. If $l(V)=\infty$ the module $V$ is not Artinian and therefore $k>0$ or $I \neq 0$ and therefore the result is true. So let us assume that $l(V)<\infty$. Since $\mathcal{O}$ is not a field, we have that $l(\mathcal{O})=l(I)=\infty$ and therefore $V=\mathcal{O} / I_{1} \oplus \ldots \oplus \mathcal{O} / I_{r}$ where $I_{j}$ are ideals in $\mathcal{O}$. Thus the statement follows from 2.5.3.

Definition 2.5.8. A fractional ideal $I$ of $\mathcal{O}$ is a $\mathcal{O}$-submodule of $K$ such that there exists a $d \in \mathcal{O}$ with $d I \subseteq \mathcal{O}$. We denote by $I_{\mathcal{O}}$ the set of all fractional ideals, also called the ideal group of $\mathcal{O}$.

Each fractional ideal $I$ can be uniquely written as

$$
I=\prod_{i=1}^{r} P_{i}^{n_{i}}
$$

with different prime ideals $P_{i} \leq \mathcal{O}$ and nonzero $n_{i} \in \mathbb{Z}$. We put $I_{+}=\prod_{n_{i}>0} P_{i}^{n_{i}}$ and $I_{-}=\prod_{n_{i}<0} P_{i}^{-n_{i}}$. Note that $I_{+}, I_{-} \leq \mathcal{O}$. Thus it makes sense to define

$$
\operatorname{deg}_{+}(I)=l\left(\mathcal{O} / I_{+}\right) \quad \text { and } \quad \operatorname{deg}_{P,+}(I)=l_{P}\left(\mathcal{O} / I_{+}\right)
$$

For an element $0 \neq a \in K$ we define

$$
\operatorname{deg}_{+}(a)=\operatorname{deg}_{+}(a \mathcal{O}) \quad \text { and } \quad \operatorname{deg}_{P,+}(a)=l_{P}(a \mathcal{O})
$$

Next, we want to generalize the notion of $\operatorname{deg}_{+}$to matrices over $K$. So let $M \in \operatorname{Mat}_{n \times m}(K)$ and let

$$
V=\left(\mathcal{O}^{m}+\mathcal{O}^{n} M\right) / \mathcal{O}^{n} M
$$

By 2.5.4 we can write

$$
V=V / V^{\mathrm{tors}} \oplus V^{\mathrm{tors}}
$$

where $V^{\text {tors }}$ denotes the torsion part of $V$. Thus we can define

$$
\operatorname{deg}_{+}(M)=l\left(V^{\text {tors }}\right) \quad \text { and } \quad \operatorname{deg}_{P,+}(M)=l\left(\mathcal{O}_{P} \otimes_{\mathcal{O}} V^{\text {tors }}\right)
$$

Remark 2.5.9. Note that $K \otimes_{\mathcal{O}} V / V^{\text {tors }} \cong K^{m-\mathrm{rk}_{K}(M)}$.
For non zero ideals $I, J$ of $\mathcal{O}$ we have

$$
I / I J \cong \mathcal{O} / J
$$

Thus by 2.5 .4 we get

$$
\begin{equation*}
\left(V / V^{\text {tors }}\right) /\left(J \cdot\left(V / V^{\text {tors }}\right)\right) \cong(\mathcal{O} / J)^{m-m \mathrm{rk}(M)} \tag{2.3}
\end{equation*}
$$

We now want to compute $\operatorname{deg}_{+}(M)$ for a matrix $M \in \operatorname{Mat}_{n \times m}(\mathcal{O})$. When we write $V=\mathcal{O}^{m} / \mathcal{O}^{n} M$ we automatically define a homomorphism $\varphi: \mathcal{O}^{m} \rightarrow V$ with $\operatorname{ker}(\varphi)=\mathcal{O}^{n} M$. Thus, the matrix $M$ defines one generating system for $\operatorname{ker}(\varphi)$ given by the rows of $M$. But that means that when we multiply $M$ with an invertible matrix we just change the generating system for $\operatorname{ker}(\varphi)$. Thus for $S \in \mathrm{GL}_{n}(\mathcal{O}), T \in \mathrm{GL}_{m}(\mathcal{O})$ we have $V=\mathcal{O}^{m} / \mathcal{O}^{n} M=\mathcal{O}^{m} / \mathcal{O}^{n} S M T$. Further, if $M$ is a diagonal matrix it is easy to calculate $\operatorname{deg}_{+}(M)$. For principal ideal domains we have the Smith normal form. For a proof see [Jac95].

Proposition 2.5.10. Let $R$ be a principal ideal domain and $M \in \operatorname{Mat}_{n \times m}(R)$ Then there exist matrices $S \in \mathrm{GL}_{n}(R)$ and $T \in \mathrm{GL}_{m}(R)$, such that $A=S M T$ is a rectangular diagonal matrix with diagonal $\left(a_{1}, a_{2}, \ldots, a_{s}, 0, \ldots, 0\right)$ such that $a_{i}$ divides $a_{i+1}$ for all $i$. Further we have $a_{i}=\frac{d_{i}}{d_{i-1}}$, where $d_{i}$ is the greatest common divisor of all $i \times i$ minors of $M$ and $d_{0}=1$.

Thus if $R$ is a prinicipal ideal domain, $M \in \operatorname{Mat}_{n \times m}(R)$ with Smith normal form $A$ with non zero diagonal entries $a_{1}, \ldots, a_{s}$ we get $\operatorname{deg}_{+}(M)=\operatorname{deg}_{+}(A)=$ $\sum_{i=1}^{s} l\left(R / a_{i} R\right)$.

Lemma 2.5.11. Let $M \in \operatorname{Mat}_{n \times m}(\mathcal{O})$ be a non zero matrix of rank $s$ over $K$. Let $I$ be the ideal generated by all $s \times s$ minors of $M$. Then $\operatorname{deg}_{+}(M)=l(\mathcal{O} / I)$.

Proof. Let us first assume that $\mathcal{O}$ is a principal ideal domain and use the same notation for $a_{i}$ and $d_{i}$ as in 2.5.10, especially we have $d_{s} \mathcal{O}=I$. From 2.5.10 we know that $\operatorname{deg}_{+}(M)=l\left(\bigoplus_{i=1}^{s} \mathcal{O} / a_{i} \mathcal{O}\right)$. Note that we have $\prod_{i=1}^{s} a_{i}=d_{s}$. From 2.5.5 we get $l\left(\bigoplus_{i=1}^{s} \mathcal{O} / a_{i} \mathcal{O}\right)=l\left(\mathcal{O} / d_{s} \mathcal{O}\right)$. Let us now assume that $\mathcal{O}$ is a Dedekind domain. Thus for any prime ideal $P$ of $\mathcal{O}$ we have

$$
\operatorname{deg}_{P,+}(M)=l_{P}\left(\mathcal{O}_{P} / I\right)
$$

From 2.5.7 we obtain

$$
\operatorname{deg}_{+}(M)=\sum_{P} \operatorname{deg}_{P,+}(M)=\sum_{P} l_{P}(\mathcal{O} / I)=l(\mathcal{O} / I)
$$

The following lemma gives bounds for $\operatorname{deg}_{+}\binom{M}{\alpha \operatorname{Id}_{m}}$ where $M \in \operatorname{Mat}_{n \times m}(\mathcal{O}), \alpha \in$ $\mathcal{O}$.

Lemma 2.5.12. Let $M \in \operatorname{Mat}_{n \times m}(\mathcal{O})$ and $0 \neq \alpha \in \mathcal{O}$. Then
$\operatorname{deg}_{+}(\alpha)\left(m-\operatorname{rk}_{K}(M)\right) \leq \operatorname{deg}_{+}\binom{M}{\alpha \operatorname{Id}_{m}} \leq \operatorname{deg}_{+}(\alpha)\left(m-\operatorname{rk}_{K}(M)\right)+\operatorname{deg}_{+}(M)$.
Proof. Let $V=\mathcal{O}^{n} / \mathcal{O}^{m} M=V / V^{\text {tors }} \oplus V^{\text {tors }}$. Thus, by 2.3 we have

$$
\begin{aligned}
\mathcal{O}^{m} /\left(\mathcal{O}^{n} M+\mathcal{O}^{m} \alpha\right) & \cong\left(\mathcal{O}^{m} / \mathcal{O}^{n} M\right) /\left(\left(\mathcal{O}^{m} / \mathcal{O}^{n} M\right) \alpha\right) \\
& \cong\left(V / V^{\text {tors }}\right) /\left(V / V^{\text {tors }} \alpha\right) \oplus V^{\text {tors }} /\left(V^{\text {tors }} \alpha\right) \\
& \cong(\mathcal{O} / \alpha)^{m-\mathrm{rk}_{K}(M)} \oplus V^{\text {tors }} /\left(V^{\text {tors }} \alpha\right)
\end{aligned}
$$

Since $\operatorname{deg}_{+}\binom{M}{\alpha \operatorname{Id}_{m}}=l\left(\mathcal{O}^{m} /\left(\mathcal{O}^{n} M+\mathcal{O}^{m} \alpha\right)\right)$ The statement follows from $l\left((\mathcal{O} / \alpha)^{m-\mathrm{rk}_{K}(M)}\right)=\operatorname{deg}_{+}(\alpha)\left(m-\operatorname{rk}_{K}(M)\right)$ and $\operatorname{deg}_{+}(M)=l\left(V / V^{\text {tors }}\right)$.

Proposition 2.5.13. Let $M_{1}, M_{2} \in \operatorname{Mat}_{n \times m}(\mathcal{O})$ and let $0 \neq \alpha \in K$. Define $m_{1}=\operatorname{rk}_{K}\left(M_{2}-\alpha M_{1}\right)$ and $m_{2}=\operatorname{rk}_{K}\left(M_{2}\right)$. Assume $m_{2}>m_{1}$. Then

$$
\operatorname{deg}_{+}(\alpha) \leq \frac{\operatorname{deg}_{+}\left(M_{2}\right)}{m_{2}-m_{1}}
$$

Proof. Since $\operatorname{deg}_{+}(\alpha)=\sum_{P} \operatorname{deg}_{P,+}(\alpha)$ and $\operatorname{deg}_{+}\left(M_{2}\right)=\sum_{P} \operatorname{deg}_{P,+}\left(M_{2}\right)$ it is enough to show

$$
\operatorname{deg}_{P,+}(\alpha) \leq \frac{\operatorname{deg}_{P,+}\left(M_{2}\right)}{m_{2}-m_{1}}
$$

for each prime ideal $P$ of $\mathcal{O}$. Let $\alpha=\frac{a}{b}$ with $a, b \in \mathcal{O}$, thus we have $\operatorname{deg}_{+}(\alpha)=$ $\operatorname{deg}_{+}(a)$. Note that if $\alpha \notin \mathcal{O}_{P}$, then $a$ is invertible in $\mathcal{O}_{P}$, so $\operatorname{deg}_{P,+}(\alpha)=0$. Therefore we can assume that $\alpha \in \mathcal{O}_{P}$. By applying 2.5.12 to matrices over $\mathcal{O}_{P}$ we get

$$
\begin{equation*}
\operatorname{deg}_{P,+}(\alpha)\left(m-m_{1}\right) \leq \operatorname{deg}_{P,+}\binom{M_{2}-\alpha M_{1}}{\alpha \operatorname{Id}_{m}} \tag{2.4}
\end{equation*}
$$

Since $\left(\mathcal{O}^{n}\left(M_{2}-\alpha M_{1}\right)+\mathcal{O}^{m} \alpha=\mathcal{O}^{n} M_{2}+\mathcal{O}^{m} \alpha\right.$, by applying 2.5.12 again we get

$$
\begin{equation*}
\operatorname{deg}_{P,+}\binom{M_{2}-\alpha M_{1}}{\alpha \operatorname{Id}_{m}}=\operatorname{deg}_{P,+}\binom{M_{2}}{\alpha \operatorname{Id}_{m}} \leq \operatorname{deg}_{P,+}(\alpha)\left(m-m_{2}\right)+\operatorname{deg}_{P,+}\left(M_{2}\right) \tag{2.5}
\end{equation*}
$$

Taking 2.4 and 2.5 together we get our result.
So what have we have done so far? Let us consider the situation from the previous proposition. We found a bound for $\operatorname{deg}_{+}(\alpha)$ only in terms of $M_{2}$ and $m_{2}-m_{1}$. Note that with respect to Theorem 2.5 . 1 the value $m_{2}-m_{1}$ is connected to the threshold $\epsilon$. In the case $\mathcal{O}=\mathbb{Z}, M_{2}=\operatorname{Id}_{n}$ and $M_{1} \in \operatorname{Mat}_{n}(\mathbb{Z})$ the previous theorem gives us a bound for $\operatorname{deg}_{+}\left(\lambda^{-1}\right)$ where $\lambda$ is an eigenvalue of $M_{1}$.

In the next two sections we want to apply these results first in the case of rings of integers of number fields and function fields.

### 2.5.3 Rings of integers of number fields

In this section we want to apply the results from the previous section to the special case where $K$ is a finite extension of $\mathbb{Q}$ and $\mathcal{O}=\mathcal{O}_{K}$ is its ring of integers. The length function $l$ is defined as

$$
l(M)=\log _{2}(|M|)
$$

for any $\mathcal{O}$-module $M$. If we work with two extensions $\mathbb{Q} \leq K_{1} \leq K_{2}$, for $\alpha \in K_{1}$ we will write $\operatorname{deg}_{+}^{K_{1}}(\alpha)$ and $\operatorname{deg}_{+}^{K_{2}}(\alpha)$, depending on what base field we consider. Since $\mathcal{O}_{K_{2}} \cong \mathcal{O}_{K_{1}}^{\left|K_{2}: K_{1}\right|}$ as $\mathcal{O}_{K_{1}}$ modules we have

$$
\begin{equation*}
\operatorname{deg}_{+}^{K_{2}}(\alpha)=\left|K_{2}: K_{1}\right| \cdot \operatorname{deg}_{+}^{K_{1}}(\alpha) \tag{2.6}
\end{equation*}
$$

For $0 \neq \alpha \in \mathcal{O}_{K}$ let $\alpha_{1}, \ldots, \alpha_{n} \in \overline{\mathbb{Q}}$ be the roots of the minimal polynomial of $\alpha$ over $\mathbb{Q}$. The norm $N_{K / \mathbb{Q}}(\alpha)$ of $\alpha$ over $K$ is

$$
N_{K / \mathbb{Q}}(\alpha)=\left(\prod_{i=1}^{n} \alpha_{i}\right)^{|K: \mathbb{Q}(\alpha)|}=\left(\prod_{i=1}^{n} \alpha_{i}\right)^{\frac{|K: \mathbb{Q}|}{n}}
$$

Recall that

$$
\left|N_{K / \mathbb{Q}}(\alpha)\right|=\left|\mathcal{O}_{K} / \mathcal{O}_{K} \alpha\right|=2^{\operatorname{deg}_{+}(\alpha)}
$$

Let us put $\lceil\alpha\rceil=\max _{i}\left|\alpha_{i}\right|$. Then we get

$$
\begin{equation*}
\operatorname{deg}_{+}(\alpha)=\log _{2}\left(\left|N_{K / \mathbb{Q}}(\alpha)\right|\right) \leq|K: \mathbb{Q}| \cdot \log _{2}(\lceil\alpha\rceil) \tag{2.7}
\end{equation*}
$$

Note that $\lceil\alpha\rceil \geq 1$ and that for $\alpha, \beta \in \mathcal{O}_{K}$ we have

$$
\begin{equation*}
\lceil\alpha+\beta\rceil \leq\lceil\alpha\rceil+\lceil\beta\rceil \quad \text { and } \quad\lceil\alpha \cdot \beta\rceil \leq\lceil\alpha\rceil \cdot\lceil\beta\rceil \tag{2.8}
\end{equation*}
$$

For a non-zero matrix $M=\left(m_{i, j}\right) \in \operatorname{Mat}_{n \times m}(K)$ we define

$$
\lceil M\rceil=\max _{j} \sum_{i}\left\lceil m_{i, j}\right\rceil .
$$

If $M$ is a zero matrix we put $\lceil M\rceil=1$. We are now ready to estimate $\operatorname{deg}_{+}(M)$ for a matrix $M \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}\right)$.

Proposition 2.5.14. Let $M \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}\right)$. Then

$$
\operatorname{deg}_{+}(M) \leq m \cdot|K: \mathbb{Q}| \cdot \log _{2}(\lceil M\rceil)
$$

Proof. Let $\operatorname{rk}_{K}(M)=s$. Lemma 2.5.11 gives us $\operatorname{deg}_{+}(M)=l\left(\mathcal{O}_{K} / I\right)$ where $I$ is the ideal generated by all $s \times s$-minors of $M$. So let $y$ be such a minor. Then we have

$$
\operatorname{deg}_{+}(M)=l\left(\mathcal{O}_{K} / I\right) \leq l\left(\mathcal{O}_{K} / \mathcal{O}_{K} y\right)=\operatorname{deg}_{+}(y)
$$

By the properties 2.8 we get

$$
\lceil y\rceil \leq\lceil M\rceil^{s} \leq\lceil M\rceil^{m}
$$

Thus, using 2.7, we obtain

$$
\operatorname{deg}_{+}(M) \leq \operatorname{deg}_{+}(y) \leq|K: \mathbb{Q}| \cdot \log _{2}(\lceil y\rceil) \leq|K: \mathbb{Q}| \cdot m \cdot \log _{2}(\lceil M\rceil)
$$

Let us consider the ring $R=\mathcal{O}_{K}\left\langle X, X^{*}\right\rangle$ again. Let $W$ be the set of all finite words in $X$ and $X^{*}$. So $W$ is a basis for $K\left\langle X, X^{*}\right\rangle$. A word $w \in W$ is of length $n$, if $w=y_{i_{1}} \cdots y_{i_{n}}$ for $y_{j} \in X \cup X^{*}$. For an element $a=\sum_{w \in W} a_{w} w \in R$ with $a_{w} \in \mathcal{O}_{K}$ we define

$$
\lceil a\rceil=\sum_{w \in W}\left\lceil a_{w}\right\rceil
$$

Just as before, for a matrix $M=\left(m_{i, j}\right) \in \operatorname{Mat}_{n \times m}(R)$ we define

$$
\lceil M\rceil=\max _{j} \sum_{i}\left\lceil m_{i, j}\right\rceil
$$

The following lemma follows directly from the definitions. Recall that $F_{b}$ was defined in equation 2.2 as

$$
F_{b}=\left\{f: X \rightarrow \operatorname{Mat}_{n_{f}}(\mathbb{Z}) \mid n_{f} \in \mathbb{N},\left\|f\left(x_{i}\right)\right\|_{1},\left\|f\left(x_{i}\right)\right\|_{\infty}<b \text { for all } i\right\}
$$

Lemma 2.5.15. Let $f \in F_{b}$ and $M \in \operatorname{Mat}_{n \times m}(R)$. Let $t$ be the length of the longest word $w$ that appears in the coefficients of $M$. Then

$$
\lceil f(M)\rceil \leq b^{t}\lceil M\rceil
$$

If we apply the last lemma to the previous proposition, we obtain

$$
\operatorname{deg}_{+}(f(M)) \leq m \cdot n_{f} \cdot|K: Q| \cdot \log _{2}\left(b^{t}\lceil M\rceil\right)
$$

for $M \in \operatorname{Mat}_{n \times m}(R)$ and $f \in F_{b}, f: R \rightarrow \operatorname{Mat}_{n_{f}}(R)$. Thus we obtain that the torsion part of $\left(\mathcal{O}^{n_{f} \cdot m} /\left(\mathcal{O}^{n_{f} \cdot n} f(M)\right)\right.$ ) grows linearly in $n_{f}$.

We now want to see what this means for possible elements of $S_{\epsilon}\left(M_{1}, M_{2}\right)$. For $\alpha \in \overline{\mathbb{Q}}$ we define

$$
N_{+}(\alpha)=2^{\operatorname{deg}_{+}^{\mathbb{Q}(\alpha)}(\alpha)}=\left|\mathcal{O}_{\mathbb{Q}(\alpha)} /\left(\mathcal{O}_{\mathbb{Q}(\alpha)} \alpha\right)_{+}\right|
$$

Corollary 2.5.16. Let $M_{1}, M_{2} \in \operatorname{Mat}_{n \times m}(R), f \in F_{b}, \alpha \in K$ and $t$ be the length of the longest word that appears in $M_{2}$. We put

$$
m_{1}=\operatorname{rk}_{K}\left(f\left(M_{2}-\alpha M_{1}\right)\right) \quad \text { and } \quad m_{2}=\operatorname{rk}_{K}\left(f\left(M_{2}\right)\right) .
$$

Assume that $m_{2}>m_{1}$. Then

$$
N_{+}(\alpha) \leq b^{t} \cdot\left\lceil M_{2}\right\rceil^{\frac{m \cdot n_{f} \cdot|Q(\alpha): Q|}{m_{2}-m_{1}}}
$$

Proof. Using all the previous results we obtain

$$
\begin{aligned}
& \operatorname{deg}_{+}^{\mathbb{Q}(\alpha)}(\alpha) \stackrel{(2.6)}{=} \operatorname{deg}_{+}^{K}(\alpha) \\
&|K: \mathbb{Q}(\alpha)|(2.5 .13) \\
& \leq \operatorname{deg}_{+}^{K}\left(f\left(M_{2}\right)\right) \\
&|K: \mathbb{Q}(\alpha)| \cdot\left(m_{2}-m_{1}\right)(2.5 .14) \\
& \leq \\
& \frac{m \cdot n_{f} \cdot|K: \mathbb{Q}| \cdot \log _{2}\left(\left\lceil f\left(M_{2}\right)\right\rceil\right)}{|K: \mathbb{Q}(\alpha)| \cdot\left(m_{2}-m_{1}\right)} \stackrel{(2.5 .15)}{\leq} \frac{m \cdot n_{f} \cdot|\mathbb{Q}(\alpha): \mathbb{Q}| \cdot \log _{2}\left(b^{t}\left\lceil M_{2}\right\rceil\right)}{m_{2}-m_{1}}
\end{aligned}
$$

### 2.5.4 Function fields

Let $Q$ be an algebraically closed subfield of $\mathbb{C}$ and let $K$ be a finite extension of $Q(x)$. In this section we want to apply our results about Dedekind domains
to the integral closure of $Q[x]$ in $K$, which we denote by $\mathcal{O}=\mathcal{O}_{K}$. The length function $l$ on $\mathcal{O}_{K}$ modules is given by $l(M)=\operatorname{dim}_{Q} M$ for a $\mathcal{O}_{K}$-module $M$. Since we will also consider the ring $\mathcal{O}^{\prime}$ which will be the integral closure of $Q\left[x^{-1}\right]$ in $K$, we will write $\operatorname{deg}_{+} \mathcal{O}_{K}$ if the base ring is not clear.

In this section we need the notion of valuations. A good source to learn about those is [SZ14].
Definition 2.5.17. A $Q$-valuation on $K$ is a homomorphism $v: K \backslash\{0\} \rightarrow$ $(\mathbb{R},+)$ such that
(1) $v(q)=0$ for all $q \in Q \backslash\{0\}$ and
(2) $v(a+b) \geq \min \{v(a), v(b)\}$ for all $a, b \in K$.

Further we extend $v$ to $K$ by defining $v(0)=\infty$.
A valuation is called trivial if $v(a)=0$ for all $a \in K$. From now on lets assume that all valuations are non trivial. Two valuations $v_{1}, v_{2}$ are called equivalent if there is an $r \in R_{>0}$ such that $r v_{1}=v_{2}$. Let $C_{K}$ be the set of all equivalent classes of $Q$-valuations of $K$. In each class $v \in C_{K}$ there is exactly one valuation which takes values in $\mathbb{Z}$. By abuse of notation we will also write $v$ for this valuation. Further put $\mathcal{O}_{v}=\{k \in K \mid v(k) \geq 0\}$ and $P_{v}=\{k \in K \mid v(k)>0\}$. Then $\mathcal{O}_{v}$ is a discrete valuation domain with maximal ideal $P_{v}$. Further, the field $\mathcal{O}_{v} / P_{v}$ is isomorphic to $Q$, where the isomorphism comes from the inclusion $Q \hookrightarrow \mathcal{O}_{v}$. Thus for any $k \in \mathcal{O}_{v}$ there is exactly one $q \in Q$ such that $k-q \in P_{v}$. We define $k(v)=q$ for $k \in \mathcal{O}_{v}$ and $k(v)=\infty$ for $k \in K \backslash \mathcal{O}_{v}$. This way we can consider the elements of $K$ as functions over $C_{K}$.

The following lemma can be found in [SZ14, Theorem 2.5.2].
Lemma 2.5.18. Let $Q$ be an algebraically closed subfield of $\mathbb{C}, K$ a finite extension of $Q(x)$ and $0 \neq \alpha \in K$. Then

$$
|K: Q(a)|=\sum_{v \in C_{K}, v(a)>0} v(a)=-\sum_{v \in C_{K}, v(a)<0} v(a) .
$$

Corollary 2.5.19. Let $Q$ be an algebraically closed field, $K / Q(a)$ a finite extension and $a_{1}, a_{2} \in K$. Assume $a_{1}-a_{2} \notin Q$. Then

$$
\left|K: Q\left(a_{1}-a_{2}\right)\right| \leq\left|K: Q\left(a_{1}\right)\right|+\left|K: Q\left(a_{2}\right)\right|
$$

Proof. Let $v \in L_{K}$. Since $v\left(\alpha_{1}-\alpha_{2}\right)=v\left(\alpha_{1}+\alpha_{2}\right) \geq \min \left\{v\left(\alpha_{1}\right), \alpha_{2}\right\}$, if $v\left(\alpha_{1}-\right.$ $\left.\alpha_{2}\right)<0$ we get $v\left(\alpha_{1}\right)<0$ or $v\left(\alpha_{2}\right)<0$. Therefore,

$$
\begin{aligned}
& \left|K: Q\left(\alpha_{1}-\alpha_{2}\right)\right| \stackrel{(2.5 .18)}{=}-\sum_{v \in C_{K}, v\left(\alpha_{1}-\alpha_{2}\right)<0} v\left(\alpha_{1}-\alpha_{2}\right) \leq \\
& -\sum_{v \in C_{K}, v\left(\alpha_{1}-\alpha_{2}\right)<0} \min \left\{v\left(\alpha_{1}\right), v\left(\alpha_{2}\right)\right\} \leq \\
& -\sum_{v \in C_{K}, v\left(\alpha_{1}\right)<0} v\left(\alpha_{1}\right)-\sum_{v \in C_{K}, v\left(\alpha_{2}\right)<0} v\left(\alpha_{2}\right) \stackrel{(2.5 .18)}{=}\left|K: Q\left(\alpha_{1}\right)\right|+\left|K: Q\left(\alpha_{2}\right)\right|
\end{aligned}
$$

As mentioned before let us denote by $\mathcal{O}_{K}^{\prime}$ the elements of $K$ that are integtral over $Q\left[x^{-1}\right]$. The following lemma connects our notion of $\operatorname{deg}_{+}$with the notion of valuations.

Lemma 2.5.20. Let $\alpha \in K$. Then

$$
\operatorname{deg}_{+}^{\mathcal{O}_{K}}(a)=\sum_{v \in C_{K}, v(a) \geq 0, v(x) \geq 0} v(a) \text { and } \operatorname{deg}_{+}^{\mathcal{O}_{+}^{\prime}}(a)=\sum_{v \in C_{K}, v(a) \geq 0, v(x) \leq 0} v(a)
$$

In particular we have

$$
\max \left\{\operatorname{deg}_{+}^{\mathcal{O}_{K}}(a), \operatorname{deg}_{+}{ }^{\mathcal{O}_{K}^{\prime}(a)}\right\} \leq|K: \mathbb{Q}(a)| \leq \operatorname{deg}_{+}^{\mathcal{O}_{K}}(a)+\operatorname{deg}_{+}^{\mathcal{O}_{K}^{\prime}}(a)
$$

Proof. We will only proove the formula for $\operatorname{deg}_{+} \mathcal{O}_{K}$. The other formula is obtained in the same way. Note that by [SZ14, Theorem 2.21, Application E] we have $\mathcal{O}_{K}=\left\{a \in K \mid v(a) \geq 0\right.$ for all $v \in C_{K}$ with $\left.v(x) \geq 0\right\}$ and every maximal ideal of $\mathcal{O}_{K}$ is of the form $P_{v}=\{a \in K \mid v(a)>0\}$ for some $v \in C_{K}$ that satisfies $v(x) \geq 0$. Thus we obtain

$$
(\alpha)_{+}=\prod_{v(x) \geq 0, v(\alpha) \geq 0} P_{v}^{v(\alpha)}
$$

Since $\mathcal{O}_{v} / P_{v}^{n} \cong \bigoplus_{i=1}^{n} Q$ we obtain

$$
\operatorname{deg}_{+}^{\mathcal{O}_{K}}(\alpha)=\sum_{v \in C_{K}, v(a) \geq 0, v(x) \geq 0} v(a)
$$

As in the previous section we now want to find an invariant of matrices $M$ over $\mathcal{O}_{K}$ and $\mathcal{O}_{K}\left\langle X, X^{*}\right\rangle$ that serves as an upper bound for $\operatorname{deg}_{+} \mathcal{O}_{K}$ and does not increase when we pass from matrices over $\mathcal{O}_{K}\left\langle X, X^{*}\right\rangle$ to matrices over $\mathcal{O}_{K}$.

So let $M=\left(M_{i, j}\right) \in \operatorname{Mat}_{n \times m}(K)$. We define

$$
D_{K}(M)=\sum_{v \in C_{K}} \min \left\{\left\{v\left(m_{i, j}\right)\right\}, 0\right\}
$$

Let us first show that for matrices over $\mathcal{O}_{K}$ we do get a bound on $\operatorname{deg}_{+}{ }^{\mathcal{O}_{K}}$.
Lemma 2.5.21. Let $M \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}\right)$. Then

$$
\operatorname{deg}_{+}^{\mathcal{O}_{K}}(M) \leq m D_{K}(M)
$$

Proof. First we want to use Lemma 2.5.11. Let $k=\operatorname{rk}_{K}(M)$ and let $I$ be the ideal of $\mathcal{O}_{K}$ generated by all $k \times k$ minors of $M$. In particular let $y$ be one $k \times k$ minor. Then we get

$$
\operatorname{deg}_{+}^{\mathcal{O}_{K}}(M)=l\left(\mathcal{O}_{K} / I\right) \leq l\left(\mathcal{O}_{K} / \mathcal{O}_{K} y\right)=\operatorname{deg}_{+}^{\mathcal{O}_{K}}(y)
$$

Now since $y$ is a $k \times k$ minor, it is a sum with $k$ summands, where each summand is a product with $k$ factors and each factor is an entry of $M$. Since $v(a b)=$ $v(a)+v(b)$ and $v(a+b) \geq \min \{v(a), v(b)\}$ we get

$$
v(y) \geq k \cdot \min _{i, j}\left\{v\left(m_{i, j}\right)\right\} .
$$

Putting everything together we get

$$
\begin{aligned}
\operatorname{deg}_{+}^{\mathcal{O}_{K}}(M) & \leq \operatorname{deg}_{+}^{\mathcal{O}_{K}}(y) \\
& -\sum_{v \in C_{K}, v(y)<0} v(y) \leq-k \sum_{v \in C_{K}, v(y)<0} \sum_{i, j}^{(2.5 .20)}|K: Q(y)| \stackrel{(2.5 .18)}{\leq} \min _{i, j}\left\{v\left(m_{i, j}\right)\right\} \leq m D_{K}(M) .
\end{aligned}
$$

We now want to extend the notion of $D_{K}$ to matrices over $R=\mathcal{O}_{K}\left\langle X, X^{*}\right\rangle$. As in the previous section let $W$ be the set of all finite words in $X \cup X^{*}$. For any element $f=\sum_{w \in W} f_{w} w$ we put $v(f)=\min _{w}\left\{v\left(f_{w}\right)\right\}$. If $M=\left(m_{i, j}\right) \in \operatorname{Mat}_{n \times m}(R)$ we define

$$
D_{K}(M)=\min _{i, j}\left\{\left\{v\left(m_{i, j}\right)\right\}, 0\right\} .
$$

Since the maps $f \in F_{b}$ map the elements $x \in X$ to matrices over $\mathbb{Z}$ and we have $v(z)=0$ for all $z \in \mathbb{Z}$ the next lemma follows directly from the definitions.

Lemma 2.5.22. Let $M \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}\left\langle X, X^{*}\right\rangle\right)$ and let $f \in F_{b}$. Then

$$
D_{K}(f(M)) \leq D_{K}(M) .
$$

Let now $Q(x) \leq K \leq L$ be two finite extensions of $Q(x)$. By restricting we can see any valuation $v$ on $L$ as a valuation on $K$. By [SZ14, Theorem 2.2.1] the restriction map

$$
\operatorname{res}_{L / K}: C_{L} \rightarrow C_{K}
$$

is onto. Further, by [SZ14, Theorem 2.5.2], for any $v \in C_{K}$ and any $\tilde{v} \in C_{L}$ with $\operatorname{res}_{L / K}(\tilde{v})=v$ there exists a number $e_{\tilde{v}}$ such that

$$
|L: K|=\sum_{\tilde{v} \in C_{L}, \operatorname{res}(\tilde{v})=v} e_{\tilde{v}}
$$

and for every $\alpha \in K$ we have

$$
\begin{equation*}
\tilde{v}(\alpha)=e_{\tilde{v}} v(\alpha) . \tag{2.9}
\end{equation*}
$$

From these equations we obtain

$$
\begin{equation*}
D_{L}(M)=|L: K| D_{K}(M) . \tag{2.10}
\end{equation*}
$$

We are now ready to prove the equivalent of 2.5.16 for the case of function fields.

Corollary 2.5.23. Let $Q$ be an algebraically closed field and let $E$ be an algebraic closure of $Q(x)$. Let $K / Q(x)$ be a finite subextension of $E / Q(x)$ and $a \in E$. Let $\mathcal{O}_{K}$ be the integral closure of $Q[x]$ in $K$ and $\mathcal{O}_{K}^{\prime}$ the integral closure of $Q\left[x^{-1}\right]$ in $K$. Let $M_{1}, M_{2} \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}\left\langle X, X^{*}\right\rangle\right)$ and let $0 \neq d \in K$, such that $d M_{1}, d M_{2} \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}^{\prime}\left\langle X, X^{*}\right\rangle\right)$. Let $f \in F_{b}$. We put

$$
m_{1}=\operatorname{rk}_{E}\left(f\left(M_{2}-a M_{1}\right)\right) \text { and } m_{2}=\operatorname{rk}_{E}\left(f\left(M_{2}\right)\right)
$$

Suppose that $m_{2} \geq m_{1}$. Then

$$
|K(a): Q(a)| \leq \frac{m \cdot n_{f} \cdot|K(a): K|\left(D_{K}\left(M_{2}\right)+D_{K}\left(d M_{1}\right)\right)}{m_{2}-m_{1}}
$$

Proof. Let $L=K(a), \mathcal{O}_{L}$ the integral closure of $Q[x]$ in $L$ and let $\mathcal{O}_{L}^{\prime}$ be the integral closure of $Q\left[x^{-1}\right]$ in $L$. Using (2.5.13),(2.5.21) and (2.5.22) we get

$$
\operatorname{deg}_{+}^{\mathcal{O}_{L}}(a) \leq \frac{\operatorname{deg}_{+}^{\mathcal{O}_{L}}\left(f\left(M_{2}\right)\right)}{m_{2}-m_{1}} \leq \frac{m \cdot n_{f} \cdot D_{L}\left(f\left(M_{2}\right)\right)}{m_{2}-m_{1}} \leq \frac{m \cdot n_{f} \cdot D_{L}\left(M_{2}\right)}{m_{2}-m_{1}}
$$

Similarly we get

$$
\operatorname{deg}_{+} \mathcal{O}_{L}^{\prime}(a) \leq \frac{m \cdot n_{f} \cdot D_{L}\left(d M_{2}\right)}{m_{2}-m_{1}}
$$

Therefore (2.5.20) gives us

$$
\begin{aligned}
|L: Q(a)| \leq & \operatorname{deg}_{+}^{\mathcal{O}_{L}}(a)+\operatorname{deg}_{+}^{\mathcal{O}_{L}^{\prime}}(a) \leq \frac{m \cdot n_{f} \cdot\left(D_{L}\left(M_{2}\right)+D_{L}\left(d M_{2}\right)\right)}{m_{2}-m_{1}} \\
& =\frac{m \cdot n_{f} \cdot|L: K| \cdot\left(D_{K}\left(M_{2}\right)+D_{K}\left(d M_{2}\right)\right)}{m_{2}-m_{1}}
\end{aligned}
$$

### 2.5.5 Two finiteness results

In this section we want to show two finiteness results that allow us to apply the theory we developed so far. Both statements can be found in [Jai19, Chapter 8]. We will begin with an algebraic result.

Proposition 2.5.24. For given $k \in \mathbb{N}$ and $C \in \mathbb{R}$ there are only finitely many algebraic numbers $\alpha$ that satisfy
(1) $|\mathbb{Q}(\alpha): \mathbb{Q}|=k$ and
(2) $N_{+}(\alpha-i) \leq C$ for $i \in\{0, \ldots, k\}$.

Proof. Let $K=\mathbb{Q}(\alpha)$ and $\mathcal{O}_{K}$ its ring of integers. Write $\alpha=\frac{a}{b}$ with $a, b \in \mathcal{O}_{K}$. Let $\sigma_{j}: K \rightarrow \overline{\mathbb{Q}}, j \in\{0, \ldots, k\}$ be $k$ different embeddings of $K$ into $\overline{\mathbb{Q}}$. Let $a_{j}=\sigma_{j}(a)$ and $b_{j}=\sigma_{j}(b)$ and consider the polynomial

$$
f(x)=\prod_{j=1}^{k}\left(a_{j}-b_{j} x\right)=\sum_{j=0}^{k} c_{j} x^{j}
$$

Since we included all the possible embeddings of $K$ into $\overline{\mathbb{Q}}$ we have $f(x) \in \mathbb{Z}[x]$. Note that for $z \in \mathbb{Z}$ we have $f(z)=N_{K / \mathbb{Q}}(a-z b)$ and therefore $|f(z)|=$ $\left|\mathcal{O}_{K} / \mathcal{O}_{K}(a-z b)\right|$. Thus, with $A=\left|\mathcal{O}_{K} /\left(\mathcal{O}_{K} a+\mathcal{O}_{K} b\right)\right|$, we obtain

$$
\begin{gathered}
|f(i)|=\left|\mathcal{O}_{K} / \mathcal{O}_{K}(a-i b)\right|=A \cdot\left|\left(\mathcal{O}_{K} a+\mathcal{O}_{K} b\right) / \mathcal{O}_{K}(a-i b)\right|= \\
A \cdot\left|\left(\mathcal{O}_{K}(a-i b)+\mathcal{O}_{K} b\right) / \mathcal{O}_{K}(a-i b)\right|= \\
A \cdot\left|\mathcal{O}_{K}+\mathcal{O}_{K}(\alpha-i) / \mathcal{O}_{K}(\alpha-i)\right|=A \cdot N_{+}(\alpha-i)
\end{gathered}
$$

Put now $m_{i, j}=(j-1)^{i-1}$ for $i, j \in\{0, \ldots, k\}$ and consider the matrix $M=$ $\left(m_{i, j}\right)$. Note that $M$ is invertible and that

$$
\left(c_{0}, c_{1}, \ldots, c_{k}\right) M=(f(0), f(1), \ldots, f(k))
$$

Therefore, we have
$\left(c_{0}, c_{1}, \ldots, c_{k}\right)=(f(0), f(1), \ldots, f(k)) M^{-1}=A \cdot\left( \pm N_{+}(\alpha), \ldots, \pm\left(N_{+}(\alpha-k)\right) M^{-1}\right.$.
Consider now the polynomial $\tilde{f}(x)=\frac{1}{A} f(x)=\sum_{i=0}^{k} c_{i}^{\prime} x^{i}$. Obviously $\tilde{f}(\alpha)=0$ and $c_{i}^{\prime}=\frac{c_{i}}{A}$. Note that

$$
\left(c_{0}^{\prime}, \ldots, c_{k}^{\prime}\right)=\left( \pm N_{+}(\alpha), \ldots, \pm\left(N_{+}(\alpha-k)\right) M^{-1}\right.
$$

Since we assumed that each $N_{+}(\alpha-i)$ is bounded by $C$, there are only finitely many such polynomials $\tilde{f}$. Thus, the number of all their possible roots $\alpha$ is also finite.

We now want to prove a finiteness result for valuations on function fields. For that let $Q$ be an algebraically closed field and let $E$ be an algebraic closure of $Q(x)$. As in the previous section for any extension $K / \mathbb{Q}(x)$ we denote by $C_{K}$ the set of $Q$ valuations of $K$ over $Q$. Remember that for finite extensions $Q(x) \leq K \leq L$ we have surjective maps $\operatorname{res}_{L / K}: C_{L} \rightarrow C_{K}$. Since $E$ is the union of its finite dimensional subextensions we get
with surjective maps $\operatorname{res}_{E} / K: C_{E} \rightarrow C_{K}$. Remember that we interpreted the elements $\alpha \in L$ as functions on $C_{L}$. Thus for $\tilde{v} \in C_{E}$ and and $\alpha \in L \leq E$ we have $\alpha(\tilde{v})=\alpha\left(\operatorname{res}_{E / L}(\tilde{v})\right.$.

We have the following result.
Proposition 2.5.25. Let $C \in \mathbb{R}$ and let $K / Q(x)$ be a finite subextension of $E / Q(x)$. For every $i \in \mathbb{N}$ let $a_{i} \in E$ such that
(1) $\left|K\left(a_{i}\right): Q\left(a_{i}\right)\right| \leq C$
(2) $\left|K\left(a_{i}\right): K\right| \leq C$.

Assume that all $a_{i}$ are different. Let $\mathcal{S}=\left\{\widetilde{v} \in C_{E} \mid\left\{a_{i}(\widetilde{v})\right\}\right.$ is finite $\}$. Then $\operatorname{res}_{E / K}(S)$ contains at most $C(2 C+1)$ valuations.

Proof. Let $\tilde{v}_{1}, \ldots, \tilde{v}_{N} \in \mathcal{S}$ such that $\operatorname{res}_{E / K}\left(\tilde{v}_{i}\right) \neq \operatorname{res}_{E / K}\left(\tilde{v}_{i}\right)$ for $i \neq j$. Since $\tilde{v_{1}} \in S$, the set $\left\{\alpha_{i}\left(\tilde{v}_{1}\right) \mid i \in \mathbb{N}\right\}$ is finite. Therefore, there is a $\beta_{1} \in Q \cup\{\infty\}$ such that the set $J_{1}=\left\{i \in \mathbb{N} \mid \alpha_{i}\left(\tilde{v}_{1}\right)=\beta_{1}\right\}$ is infinite. Now since $J_{1}$ is infinite and $\tilde{v}_{2} \in S$, there is $\beta_{2} \in Q \cup\{\infty\}$ such that the set $J_{2}=\left\{i \in J_{1} \mid \alpha_{i}\left(\tilde{v}_{2}\right)=\beta_{2}\right\}$ is infinite. Repeating this procedure we get for each $j \in\{0, \ldots, N\}$ a $\beta_{j} \in Q \cup\{\infty\}$ and an infinite subset $J_{i} \in \mathbb{N}$ such that

$$
\alpha_{j}\left(\tilde{v}_{s}\right)=\beta_{s} \text { for each } j \in J_{i}, s \in\{1, \ldots, i\}
$$

By reordering the $\alpha_{i}$ and the $\tilde{v}_{j}$ we can assume that $1,2 \in J_{N}, \beta_{1}, \ldots, \beta_{N_{1}} \in$ $Q, \beta_{N_{1}+1}, \ldots, \beta_{N}=\{\infty\}$. Therefore, for $j \in\{1, \ldots, N\}$ we have $\alpha_{1}\left(\tilde{v}_{j}\right)=\alpha_{2}\left(\tilde{v}_{j}\right)$ and therefore $\left(\alpha_{1}-\alpha_{2}\right)\left(\tilde{v}_{j}\right)=0$. Let $L=K\left(\alpha_{1}, \alpha_{2}\right)$. We have

$$
\begin{aligned}
& \left|L: Q\left(\alpha_{1}-\alpha_{2}\right)\right| \stackrel{(2.5 .19)}{\leq}\left|L: Q\left(\alpha_{1}\right)\right|+\left|L: Q\left(\alpha_{2}\right)\right| \leq \\
& \quad\left|L: K\left(\alpha_{1}\right)\right| \cdot\left|K\left(\alpha_{1}\right): Q\left(\alpha_{1}\right)\right|+\left|L: K\left(\alpha_{2}\right)\right| \cdot\left|K\left(\alpha_{2}\right): Q\left(\alpha_{2}\right)\right| \leq \\
& \quad\left|K\left(\alpha_{2}\right): K\right| \cdot\left|K\left(\alpha_{1}\right): Q\left(\alpha_{1}\right)\right|+\left|K\left(\alpha_{1}\right): K\right| \cdot\left|K\left(\alpha_{2}\right): Q\left(\alpha_{2}\right)\right| \leq 2 C^{2}
\end{aligned}
$$

By Lemma 2.5.18 the function $\alpha_{1}-\alpha_{2}$ has at most $2 C^{2}$ zeros in $C_{L}$. Since we assumed that $\operatorname{res}\left(\tilde{v}_{i}\right) \neq \operatorname{res}\left(\tilde{v}_{j}\right)$ for $i \neq j$ we get obtain $N_{1} \leq 2 C^{2}$. Since $\left|K\left(\alpha_{1}\right): Q\left(\alpha_{1}\right)\right| \leq C$, again by Lemma 2.5.18 there are less then $C$ valuations $v \in C_{K\left(\alpha_{1}\right)}$ with $v\left(\alpha_{1}\right)<0$. Since $v\left(\alpha_{1}\right)<0$ implies $\alpha_{1}(v)=\infty$ we obtain $N-N_{1} \leq C$ and therefore $N \leq 2 C^{2}+C$.

### 2.5.6 Proof of Theorem 2.5.1

We are now ready to proof the main theorem of this section. We have already seen that it is enough to prove the theorem for matrices $A, B \in \operatorname{Mat}_{n \times m}\left(\bar{f}\left(\mathcal{O}\left\langle X, X^{*}\right\rangle\right)\right)$ where
(1) $\mathcal{O}=\mathbb{Z}$.
(2) $\mathcal{O}=Q([x])$ where $Q$ is an algebraically closed subfield of $\mathbb{C}$ and the theorem holds for $K=Q$.

In both cases let $K$ be the field of fractions of $\mathcal{O}$ and $E$ be an algebraic closure of $K$. Assume that the set $S_{\epsilon}(A, B)$ is infinite. For each $j \in \mathbb{N}$ let us choose a $\lambda_{j} \in S_{\epsilon}(A, B)$ such that $\lambda_{i} \neq \lambda_{j}$ for $i \neq j$. For each $j$ let now $\varphi_{j} \in F_{b}$ such that

$$
\begin{equation*}
\operatorname{rk}_{\varphi_{j}}(B)-\operatorname{rk}_{\varphi_{j}}\left(B-\lambda_{j} A\right) \geq \epsilon \tag{2.11}
\end{equation*}
$$

For simplicity we set $\mathrm{rk}_{j}=\mathrm{rk}_{\varphi_{j}}, A_{j}=\varphi_{j}(A), B_{j}=\varphi_{j}(B)$ and $n_{j}=n_{\varphi_{j}}$. Let $L_{j}$ be the extension of $K$ generated by $\lambda_{j}$. For each $\sigma \in \operatorname{Gal}(E / K)$ we have
$\sigma\left(A_{j}\right)=A_{j}$ and $\sigma\left(B_{j}\right)=B_{j}$. Therefore we have

$$
\begin{aligned}
& \mathrm{rk}_{j}(B-\lambda A)=\frac{1}{n_{j}} \mathrm{rk}_{L_{j}}\left(B_{j}-\lambda A_{j}\right)= \\
& \frac{1}{n_{j}} \mathrm{rk}_{\sigma\left(L_{j}\right)}\left(\sigma\left(B_{j}-\lambda A_{j}\right)\right)=\frac{1}{n_{j}} \mathrm{rk}_{\sigma\left(L_{j}\right)}\left(B_{j}-\sigma(\lambda) A_{j}\right)= \\
& \operatorname{rk}_{j}(B-\sigma(\lambda) A) .
\end{aligned}
$$

Therefore, by 1.3.27, we obtain $\left|L_{j}: K\right| \leq \frac{n}{\epsilon}$. Put $C_{1}=\frac{n}{\epsilon}$. Let us fix a non principal utralfilter $\omega$ on $\mathbb{N}$ and set $\mathrm{rk}_{\omega}=\lim _{\omega} \mathrm{rk}_{j}$.

Case $K=\mathbb{Q}, \mathcal{O}=\mathbb{Z}$.
We will first consider the algebraic case. We want to apply 2.5.24. By proposition 1.3.27 there is $C_{2} \in \mathbb{Z}_{\geq 0}$ such that for all $i \in\left\{0, \ldots, C_{1}\right\}$ we have

$$
\mathrm{rk}_{\omega}(B)-\mathrm{rk}_{\omega}\left(B+\left(C_{2}-i\right) A\right) \leq \frac{\epsilon}{4} .
$$

Therefore, by definition of the ultralimit, the set

$$
\begin{equation*}
J=\left\{j \in \mathbb{N} \left\lvert\, \mathrm{rk}_{j}(B)-\mathrm{rk}_{j}\left(B+\left(C_{2}-i\right) A\right) \leq \frac{\epsilon}{2}\right. \text { for every } i=0, \ldots, C_{1}\right\} \tag{2.12}
\end{equation*}
$$

belongs to the ultrafilter $\omega$ and is therefore infinite. Thus we obtain for every $j \in J$ and $i \in\left\{0, \ldots, C_{1}\right\}$,

$$
\begin{gathered}
\mathrm{rk}_{L_{j}}\left(B_{j}+\left(C_{2}-i\right) A_{j}\right)-\mathrm{rk}_{L_{j}}\left(B_{j}+\left(C_{2}-i\right) A_{j}-\left(\lambda_{j}+C_{2}-i\right) A_{j}\right)= \\
\mathrm{rk}_{L_{j}}\left(B_{j}+\left(C_{2}-i\right) A_{j}\right)-\mathrm{rk}_{L_{j}}\left(B_{j}-\lambda_{j} A_{j}\right)= \\
n_{j} \cdot\left(\mathrm{rk}_{j}\left(B+\left(C_{2}-i\right) A\right)-\mathrm{rk}_{j}\left(B-\lambda_{j} A\right)\right) \geq \\
n_{j} \cdot\left(\mathrm{rk}_{j}\left(B+\left(C_{2}-i\right) A\right)-\mathrm{rk}_{j}(B)+\epsilon\right) \geq \\
n_{j} \cdot\left(-\frac{\epsilon}{2}+\epsilon\right)=n_{j} \frac{\epsilon}{2}
\end{gathered}
$$

We can now apply 2.5.16 with $M_{1}=A, M_{2}=B+\left(C_{2}-i\right) A$ and $\alpha=\lambda_{j}+C_{2}-i$. Remember that $t$ is the length of the largest word that appears in $M_{2}$. This way we get for each $j \in J, 0 \leq i \leq C_{1}$,

$$
\begin{aligned}
N_{+}\left(\lambda_{j}+C_{2}-i\right) \leq & \left(b^{t}\left\lceil B+\left(c_{2}-i\right) A\right\rceil\right)^{\frac{m \cdot n_{j} j\left|Q\left(\lambda_{j}\right): Q\right|}{k_{L_{j}}\left(w B_{j}+\left(\left(C_{2}-i\right) A_{j}\right)-k_{L_{j}} L_{j}\left(B_{j}\right)-\lambda_{j} A_{j}\right)}} \leq \\
& \left(c^{l}\left\lceil B+\left(C_{2}-i\right) A\right\rceil\right)^{\frac{2 m \cdot| |\left(\lambda_{j}\right): Q \mid}{\epsilon}} \leq \\
& \left(c^{l}\lceil B\rceil+\left(C_{2}+C_{1}\right)\lceil A\rceil\right)^{\frac{2 m C_{1}}{\epsilon}} .
\end{aligned}
$$

Note that the last expression does not depend on $j$ anymore. Since $C_{2} \in \mathbb{Z}$ we have $\mathbb{Q}\left(\lambda_{j}+C_{2}\right)=\mathbb{Q}\left(\lambda_{j}\right)$ and therefore $\left|\mathbb{Q}\left(\lambda_{j}+C_{2}\right): \mathbb{Q}\right| \leq C_{1}$. Thus we can apply Proposition 2.5.24 and get that there are only finitely many $\lambda_{j}$, which is a contradiction.

Case $K=Q(x), \mathcal{O}=Q[x]$.
We now want to prove our second case. For that let $L / Q(x)$ be a subextension of $E / Q(x)$. For $v \in C_{L}$ we put $\mathcal{O}_{v}=\{a \in L: v(a) \geq 0\}$ and $P_{v}=\{a \in$ $L: v(a)>0\}$. Denote by $\varphi_{v}: \mathcal{O}_{v} \rightarrow Q$ the $Q$-algebra homomorphism coming from the reduction modulo $P_{v}$. For every Sylvester matrix rank function rk on $\mathcal{O}_{v}\left\langle X, X^{*}\right\rangle$ we define $\mathrm{rk}_{v}=\operatorname{rko} \varphi_{v} \in \mathbb{P}\left(Q\left\langle X, X^{*}\right\rangle\right)$. Put $\operatorname{Spec}(\mathcal{O})=\left\{v \in C_{K}:\right.$ $\left.\mathcal{O} \subseteq \mathcal{O}_{v}\right\}=\left\{v \in C_{K}: v(x) \geq 0\right\}$.
Lemma 2.5.26. Let $M \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}\left\langle X, X^{*}\right\rangle\right)$. For almost all $v \in \operatorname{Spec}(\mathcal{O})$ we have $\operatorname{rk}_{\omega}(M)-\operatorname{rk}_{\omega, v}(M) \leq \frac{\epsilon}{4}$.

Proof. By 1.3.33 $\mathrm{rk}_{\omega}$ is the natural transcendental extension of the restriction of $\mathrm{rk}_{\omega}$ to $Q\left\langle X, X^{*}\right\rangle$. Thus by 1.3.34 we get

$$
\lim _{i \rightarrow \infty} \operatorname{rk}_{\omega, v}=\mathrm{rk}_{\omega}
$$

Let now

$$
C_{2}=\max \left\{C_{1}, \frac{m C_{1}\left(D_{K}(B)+D_{K}(d b)\right)}{\epsilon}\right\} \quad \text { and } \quad C_{3}=C_{2}\left(2 C_{2}+1\right)+1
$$

By the previous lemma we can choose $C_{3}$ different $v_{i} \in \operatorname{Spec}(\mathcal{O}), i \in\left\{1, \ldots, C_{3}\right\}$ such that

$$
\operatorname{rk}_{\omega}(B)-\operatorname{rk}_{\omega, v}(B) \leq \frac{\epsilon}{4}
$$

Remember that $\mathrm{rk}_{\omega}=\lim _{\omega} \mathrm{rk}_{j}$ and therefore $\mathrm{rk}_{\omega, v}=\lim _{\omega} \mathrm{rk}_{j, v}$. Therefore the set

$$
J=\left\{J \in \mathbb{N} \left\lvert\, \operatorname{rk}_{j}(B)-\operatorname{rk}_{j, v_{i}}(B) \leq \frac{\epsilon}{2}\right. \text { for } i \in\left\{1, \ldots, C_{3}\right\}\right\}
$$

belongs to $\omega$ and is infinite. For each $i \in\left\{1, \ldots, C_{3}\right\}$ let $\tilde{v}_{i} \in C_{E}$ such that $\operatorname{res}_{E / K}\left(\tilde{v}_{i}\right)=v_{i}$.
Lemma 2.5.27. There exists an $i \in\left\{1, \ldots, C_{3}\right\}$ such that

$$
\left\{\lambda_{j}\left(\tilde{v}_{i}\right) \mid j \in J\right\}
$$

is infinite.
Proof. We have to show that the set $\left\{\lambda_{j}, j \in J\right\}$ satisfies the conditions of 2.5.25 Remember that $K=Q(x)$ and we have already seen that $\left|K\left(\lambda_{j}\right): K\right| \leq C_{1}$. Further we have for each $j \in J$

$$
\begin{aligned}
\left|K\left(\lambda_{j}\right): Q\left(\lambda_{j}\right)\right| & \stackrel{(2.5 .23)}{\leq} \frac{m \cdot n_{f} \cdot|K(a): K|\left(D_{K}\left(M_{2}\right)+D_{K}\left(d M_{1}\right)\right)}{\operatorname{rk}_{E}\left(\varphi_{j}(B)\right)-\operatorname{rk}_{E}\left(\varphi_{j}(B)-\lambda_{j} \varphi_{j}(A)\right)} \\
& \stackrel{(2.11)}{\leq} \frac{m \cdot C_{1}\left(D_{K}\left(M_{2}\right)+D_{K}\left(d M_{1}\right)\right)}{\epsilon}
\end{aligned}
$$

Thus, by 2.5.25, the set $\operatorname{res}_{E / K}(S)$ has only $C_{3}-1$ elements where $S=\{\tilde{v} \in$ $C_{E} \mid\left\{\lambda_{j}(\tilde{v})\right\}_{j \in J}$ is finite $\}$. Therefore there exsits an $i \in\left\{1, \ldots, C_{3}\right\}$ such that the set $\left\{\lambda_{j}\left(\tilde{v}_{i}\right) \mid j \in J\right\}$ is infinite.

Let now $\tilde{v}=\tilde{v}_{i}$, where $\left\{\lambda_{j}\left(\tilde{v}_{i}\right) \mid j \in J\right\}$ is infinite. Let

$$
I=\left\{j \in J \mid \lambda_{j}(\tilde{v}) \neq \infty\right\}
$$

that means $\tilde{v}\left(\lambda_{j}\right) \geq 0$. Obviously $I$ is infinite as well. The rank of a finite matrix over $\mathcal{O}_{\tilde{v}}$ cannot increase after the reduction modulo $\operatorname{ker}\left(\phi_{\tilde{v}}\right)=P_{\tilde{v}}=\{a \in E \mid$ $\tilde{v}(a)>0\}$. We get for each $j \in I$,

$$
\begin{aligned}
& \operatorname{rk}_{Q}\left(f _ { i _ { j } } \left(\varphi_{\tilde{v}}(B)-\right.\right.\left.\left.\lambda_{j}(\tilde{v}) \varphi_{\tilde{v}}(A)\right)\right)=\operatorname{rk}_{Q}\left(f_{i_{j}}\left(\varphi_{\tilde{v}}\left(B-\lambda_{j} A\right)\right)\right) \\
& \leq \operatorname{rk}_{E}\left(f_{i_{j}}\left(B-\lambda_{j} A\right)\right)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{gathered}
\operatorname{rk}_{Q}\left(f_{i_{j}}\left(\varphi_{\tilde{v}}(B)\right)\right)-\operatorname{rk}_{Q}\left(f_{i_{j}}\left(\varphi_{\tilde{v}}(B)-\lambda_{j}(\tilde{v}) \varphi_{\tilde{v}}(A)\right)\right) \geq \\
\operatorname{rk}_{E}\left(f_{i_{j}}(B)\right)-\frac{\epsilon}{2} \cdot i_{j}-\operatorname{rk}_{E}\left(f_{i_{j}}\left(B-\lambda_{j} A\right)\right) \geq \frac{\epsilon}{2} \cdot i_{j}
\end{gathered}
$$

This means that $S_{\frac{\epsilon}{2}}\left(\varphi_{\tilde{v}}(A), \varphi_{\tilde{v}}(B)\right)$ is infinite. Since we assumed that our theorem holds over $Q$ we have a contradiction.

### 2.6 The strong algebraic eigenvalue property

In this section we want to verify the conditions of 1.3.32. For that let $K$ be an algebraically closed field, $\mathcal{A}=K\left\langle x_{1}, x_{1}^{*} \ldots, x_{r}, x_{r}^{*}\right\rangle$ and $\mathcal{C}$ the class of $b$ bounded integer valued traces on $\mathcal{A}$. Let $\left(\tau_{i}\right)$ be a converging sequence in $\mathcal{C}$ with representations $\varphi_{i}: \mathcal{A} \rightarrow \operatorname{Mat}_{n_{i}}(K)$. Set $\tau=\lim _{i \rightarrow \infty} \tau_{i}$. Assume that the class $\mathcal{C}$ has the approximation property that means we have

$$
\lim _{i \rightarrow \infty} \frac{1}{n_{i}} \mathrm{rk}_{\mathbb{C}} \circ \varphi_{i}=\mathrm{rk}_{\tau}
$$

as rank functions on $\mathcal{A}$. Let $\mathcal{R}_{\tau}=\mathcal{R}\left(\mathcal{A}, \mathcal{U}_{\tau}\right)$ be the $*$-regular closure of $\mathcal{A}$ in $\mathcal{U}_{\tau}$. We want to prove the following theorem.

Theorem 2.6.1. Let $A \in \operatorname{Mat}_{n}\left(\mathcal{R}_{\tau}\right)$ and $\lambda \in \mathbb{C} \backslash K$. Then

$$
\mathrm{rk}_{\tau}\left(A-\lambda \mathrm{Id}_{n}\right)=n
$$

This means that the point spectrum of the matrix $A$, seen as an operator on $\mathcal{H}_{\tau}^{n}$ does not contain numbers that are transcendental over $K$, or equivalently, all eigenvalues of $A$ are algebraic over $K$. Let us for a moment assume that $A \in \operatorname{Mat}_{n}(\mathcal{A})$ is normal. For simplicity write $A_{i}=\varphi_{i}(A)$. We will see in Theorem 2.7.4 that from $\mathrm{rk}_{\tau}=\lim _{i \rightarrow \infty} \frac{1}{n_{i}} \mathrm{rk}_{\mathbb{C}} \circ \varphi_{i}$ we obtain that the spectral measures $\mu_{A, \tau_{i}}$ not only converge weakly towards $\mu_{A, \tau}$, but also that for any $z \in \mathbb{C}$ we have

$$
\lim _{i \rightarrow \infty} \mu_{A, \tau_{i}}(\{z\})=\mu_{A, \tau}(\{z\})
$$

Note that the $A_{i}$ are matrices over the algebraically closed field $K$. In particular, the support of the eigenvalue measures associated to the $A_{i}$ lies in $K$. Thus, we have

$$
\begin{aligned}
\operatorname{rk}_{\tau}\left(A-\lambda \mathrm{Id}_{n}\right) & =n-\operatorname{dim}_{\tau} \operatorname{ker}\left(A-\lambda \operatorname{Id}_{n}\right) \\
& =n-\mu_{A-\lambda \mathrm{Id}, \tau}(\{0\}) \\
& =n-\mu_{A, \tau}(\{\lambda\}) \\
& =n-\lim _{i \rightarrow \infty} \mu_{A, \tau_{i}}(\{\lambda\} \\
& =n-\lim _{i \rightarrow \infty} 0 \\
& =n
\end{aligned}
$$

However, when $A$ is not normal, we can not argue in this way. We do not know if the eigenvalues of the matrices $A_{i}$ describe the eigenvalues of the operator associated to the matrix $A$. By abuse of notation we will denote this operator also by $A$. Therefore we will scan the operator associated to $A$ indirectly for eigenvalues by checking the dimension of its centralizer inside the space of Hilbert-Schmidt-operators. We will start with the definition of this space, the definition of the centralizer and we will see how to define a von Neumann dimension on this space. In this section we will follow [Jai19, Chapter 9].

### 2.6.1 The space of Hilbert-Schmidt-Operators

Definition 2.6.2. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $X=\left\{x_{1}, x_{2}, \ldots\right\}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called Hilbert-Schmidt operator, if

$$
\sum_{x \in X}\|x A\|^{2}<\infty
$$

We denote by $\mathrm{HS}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ the set of Hilbert-Schmidt operators of $\mathcal{H}$.
It is important to note that $\operatorname{HS}(\mathcal{H})$ forms an ideal in $\mathcal{B}(\mathcal{H})$. If $Y$ is another orthonormal basis of $H$ and $A \in \operatorname{HS}(\mathcal{H})$ we have

$$
\sum_{x \in X}\|x A\|^{2}=\sum_{y \in Y}\left\|y A^{*}\right\|^{2}=\sum_{y \in Y}\|y A\|^{2}
$$

We now want to define a module structure on the space $\operatorname{HS}(\mathcal{H})$. We have the following result.

Proposition 2.6.3. Let $\mathcal{H}$ be a Hilbert space. The space $\mathcal{H}^{*} \otimes \mathcal{H}$ is isometrically isomorphic to $\mathrm{HS}(\mathcal{H})$, the space of Hilbert Schmidt operators of $\mathcal{H}$.

Proof. Let $v, u \in \mathcal{H}$. Let $\delta_{v} \in \mathcal{H}^{*}$ be the element defined by $w . \delta_{v}=\langle w, v\rangle$ for $w \in \mathcal{H}$. It is enough to consider the elements $\delta_{v} \otimes u \in \mathcal{H}^{*} \otimes \mathcal{H}$. Define $\Psi: \mathcal{H}^{*} \otimes \mathcal{H} \rightarrow \operatorname{HS}(\mathcal{H})$ by $(w)\left[\Psi\left(\delta_{v} \otimes u\right)\right]=\langle w, v\rangle u$. This is obviously injective
and since the image contains a dense subset it is also surjective. Let $\left\{e_{1}, e_{2}, \ldots\right\}$ be an orthonormal basis of $\mathcal{H}$. Then we have

$$
\begin{aligned}
\left\|\Psi\left(\delta_{v} \otimes u\right)\right\|^{2} & =\sum_{i}\left\langle\left\langle e_{i}, v\right\rangle u,\left\langle e_{i}, v\right\rangle, u\right\rangle \\
& =\sum_{i}\left\langle e_{i}, v\right\rangle \overline{\left\langle e_{i}, v\right\rangle}\langle u, u\rangle \\
& =\|v\|^{2} \cdot\|u\|^{2} \\
& =\left\|\delta_{v} \otimes u\right\|^{2}
\end{aligned}
$$

The bounded inverse theorem gives the result.
Now consider the algebra $\mathcal{A} \otimes_{\sigma} \mathcal{A}$, where $\sigma: \mathbb{C} \rightarrow \mathbb{C}$ is the complex conjugation. In the following we will write $\mathcal{H}=\mathcal{H}_{\tau}$, where $\mathcal{H}_{\tau}$ is the Hilbert space coming from the GNS construction with respect to $\tau$. We turn $\mathcal{H}^{*} \otimes \mathcal{H}$ into a right $\mathcal{A} \otimes_{\sigma} \mathcal{A}$ module by defining

$$
\left(\delta_{v} \otimes u\right)(a \otimes b)=\delta_{v a} \otimes u b
$$

for $v, u \in \mathcal{H}$. Remember that $\mathcal{H}_{\tau}$ was basically the completion of some quotient of $\mathcal{A}$ with respect to the norm induced by $\tau$. In particular, the orbit of the vector coming from $1 \in \mathcal{A}$ is dense in $\mathcal{H}_{\tau}$. By abuse of notation we will denote this vector by $e \in \mathcal{H}_{\tau}$. From that it follows that the orbit of the vector $\delta_{e} \otimes e$ under the $\mathcal{A} \otimes_{\sigma} A$ action is dense in $\mathcal{H}^{*} \otimes \mathcal{H}$. Thus we can define a trace on $\mathcal{A} \otimes_{\sigma} A$ by

$$
\begin{equation*}
\tilde{\tau}(a \otimes b)=\left\langle\left(\delta_{e} \otimes e\right)(a \otimes b), \delta_{e} \otimes b\right\rangle \tag{2.13}
\end{equation*}
$$

Lemma 2.6.4. We have

$$
\tilde{\tau}(a \otimes b)=\overline{\tau(a)} \tau(b)
$$

Proof.

$$
\begin{aligned}
\tilde{\tau}(a \otimes b) & =\left\langle\left(\delta_{e} \otimes e\right)(a \otimes b), \delta_{e} \otimes b\right\rangle \\
& =\left\langle\delta_{e . a}, \delta_{e}\right\rangle \cdot\langle e . b, e\rangle \\
& =\langle e, e . a\rangle \cdot \tau(b) \\
& =\left\langle e . a^{*}, e\right\rangle \cdot \tau(b) \\
& =\overline{\tau(a)} \tau(b)
\end{aligned}
$$

Thus, by 1.2 .6 , we get that $\mathcal{H}^{*} \otimes \mathcal{H}$ is isometrically isomorphic to $\mathcal{H}_{\tilde{\tau}}$. We now also want to define a right $\mathcal{A} \otimes_{\sigma} \mathcal{A}$ action on $\operatorname{HS}(\mathcal{H})$. For $v \in \mathcal{H}, A \in \operatorname{HS}(\mathcal{H})$ and $a, b \in \mathcal{A}$ we set

$$
(w)[A(a \otimes b)]=\left[\left(w a^{*}\right) A\right] b
$$

Lemma 2.6.5. Let $\Psi: \mathcal{H}^{*} \otimes \mathcal{H} \rightarrow \operatorname{HS}(\mathcal{H})$ be as above. Then $\Psi$ is a $A \otimes_{\sigma} A$ module isomorphism.

Proof. It is left to show that $\Psi$ interchanges with the $\mathcal{A} \otimes_{\sigma} \mathcal{A}$ action. So let $v, u, w \in H, a, b \in \mathcal{A}$.

$$
\begin{aligned}
(w)\left[\Psi\left(\left(\delta_{v} \otimes u\right)(a \otimes b)\right)\right] & =(w)\left[\Psi\left(\delta_{v a} \otimes u b\right)\right] \\
& =\langle w, v a\rangle(u b) \\
& =\left(\left\langle w a^{*}, v\right\rangle u\right) b \\
& =(w)\left[\Psi\left(\delta_{v} \otimes u\right)(a \otimes b)\right]
\end{aligned}
$$

Thus, having identified $\operatorname{HS}(\mathcal{H})$ with $\mathcal{H}_{\tilde{\tau}}$ we have a von Neumann dimension for closed invariant subspaces of $\operatorname{HS}(\mathcal{H})$. As stated before, for $A \in \mathcal{U}_{\tau}$ we are interested in the centralizer of $A$ in $\operatorname{HS}\left(\mathcal{H}_{\tau}\right)$. But what exactly is the centralizer of $A$ ? If $A \in \mathcal{N}_{\tau}$, since $\operatorname{HS}(\mathcal{H})$ is an ideal, we can just define

$$
\mathrm{C}_{\mathrm{HS}(\mathcal{H})}(A)=\{B \in \operatorname{HS}(\mathcal{H}) \mid A B=B A\}
$$

However, if $A \in \mathcal{U}_{\tau}$, for $D \in \operatorname{HS}(\mathcal{H})$, the product $A D$ or $D A$ might not be defined. Remember that the definition of a Hilbert Schmidt operator is totally independent of $\mathcal{A}$. Thus we have to define the centralizer differently. We know that $\mathcal{U}_{\tau}$ can be seen as the Ore localization of $\mathcal{N}_{\tau}$. Thus, for any $A \in \mathcal{U}_{\tau}$ there are $B_{1}, C_{1}, B_{2}, C_{2} \in \mathcal{N}_{\tau}$, where $C_{1}, C_{2}$ are non-zero divisors, such that $A=B_{1} C_{1}^{-1}=C_{2}^{-1} B_{2}$. Thus we can define

$$
\mathrm{C}_{\mathrm{HS}(\mathcal{H})}(A)=\left\{D \in \operatorname{HS}(\mathcal{H}) \mid C_{2} D B_{1}=C_{1} D B_{2}\right\}
$$

Let us just check that this definition is independent of the choice of $C_{i}, B_{i}$. For example assume we also have $A=B_{3} C_{3}^{-1}$. We want to show that from

$$
C_{2} D B_{1}=B_{2} D C_{1}
$$

we also get

$$
C_{2} D B_{3}=B_{2} D C_{3}
$$

In fact we have

$$
\begin{aligned}
B_{2} D C_{3} & =B_{2} D C_{1} C_{1}^{-1} C_{3} \\
& =C_{2} D B_{1} C_{1}^{-1} C_{3} \\
& =C_{2} D B_{3} C_{3}^{-1} C_{3} \\
& =C_{2} D B_{3}
\end{aligned}
$$

The next proposition shows that $\mathrm{C}_{\mathrm{HS}(\mathcal{H})}(A)$ is actually a $\mathcal{A} \otimes_{\sigma} A$ Hilbert sub module of $\mathrm{HS}(\mathcal{H})$.

Proposition 2.6.6. Let $A=B_{1} C_{1}^{-1}=C_{2}^{-1} B_{2} \in \mathcal{U}_{\tau}$. Then we have

$$
\Psi\left(\operatorname{ker}\left(A^{*} \otimes 1-1 \otimes A\right)\right)=C_{\mathrm{HS}(H)}(A)
$$

Proof. Since $\left(1 \otimes C_{1}\right)$ and $\left(C_{2}^{*} \otimes 1\right)$ are injective and commute we have

$$
\begin{aligned}
\operatorname{ker}\left(A^{*} \otimes 1-1 \otimes A\right) & =\operatorname{ker}\left(B_{2}^{*}\left(C_{2}^{-1}\right)^{*} \otimes 1-1 \otimes B_{1} C_{1}^{-1}\right) \\
& \left.=\operatorname{ker}\left(B_{2}^{*}\left(C_{2}^{-1}\right)^{*} \otimes 1-1 \otimes B_{1} C_{1}^{-1}\right)\left(C_{2}^{*} \otimes 1\right)\left(1 \otimes C_{1}\right)\right) \\
& =\operatorname{ker}\left(B_{2}^{*} \otimes C_{1}-C_{2}^{*} \otimes B_{1}\right)
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
& \delta_{v} \otimes u \in \operatorname{ker}\left(B_{2}^{*} \otimes C_{1}-C_{2}^{*} \otimes B_{1}\right) \\
\Longleftrightarrow & \delta_{v B_{2}^{*}} \otimes u C_{1}=\delta_{v C_{2}^{*}} \otimes u B_{1} \\
\Longleftrightarrow & \left\langle w, v B_{2}^{*}\right\rangle u C_{1}=\left\langle w, v C_{2}^{*}\right\rangle u B_{1} \quad \text { for all } \quad w \in \mathcal{H} \\
\Longleftrightarrow & w B_{2} \Psi\left(\delta_{v} \otimes u\right) C_{1}=w C_{2} \Psi\left(\delta_{v} \otimes u\right) B_{1} \quad \text { for all } \quad w \in \mathcal{H} \\
\Longleftrightarrow & \Psi\left(\delta_{v} \otimes u\right) \in \mathrm{C}_{\mathrm{HS}(H)}(A)
\end{aligned}
$$

### 2.6.2 An application of the approximation property

In the previous section we have seen that for $A \in \mathcal{U}_{\tau}$ we can see $\mathrm{C}_{\mathrm{HS}\left(\mathcal{H}_{\tau}\right)}(A)$ as Hilbert $\mathcal{A} \otimes_{\sigma} \mathcal{A}$ module. We now want to see that we can approximate the dimension of this module. Let us repeat some notation. Let $K$ be a subfield of $\mathbb{C}$ closed under complex conjugation. Let $\mathcal{A}$ be a finitely generated free *algebra over $K$ and let $\tau=\lim _{i \rightarrow \infty} \tau_{i}$ be the limit of a sequence of converging $b$-bounded integer valued trace ons $\mathcal{A}$. That means we have $*$-homomorphism $\varphi_{i}: \mathcal{A} \rightarrow \operatorname{Mat}_{n_{i}}(K)$ such that for every $a \in \mathcal{A}$ we have

$$
\tau(a)=\lim _{i \rightarrow \infty} \frac{1}{n_{i}} \operatorname{Tr}\left(\varphi_{i}(a)\right)
$$

Let $\mathcal{V}_{0}=\mathcal{U}_{\tau}$ and $\mathcal{V}_{i}=\operatorname{Mat}_{n_{i}}(\mathbb{C})$ for $i \geq 1$. For the definition of $\mathcal{U}_{\tau}$ see section 1.2.4. Let $\varphi_{0}: \mathcal{A} \rightarrow \mathcal{V}_{0}$ be the natural $*$-homomorphism of $\mathcal{A}$ to $\mathcal{V}_{0}$. For simplicity let $\mathcal{R}_{\tau}=\mathcal{R}\left(\varphi_{0}(\mathcal{A}), \mathcal{U}_{\tau}\right)$ be the $*$-regular closure of $\mathcal{A}$ under $\varphi_{0}$ in $\mathcal{V}_{0}$. Consider now the algebra

$$
\mathcal{V}=\prod_{j=0}^{\infty} \mathcal{V}_{j}
$$

This algebra is $*$-regular and further we have a map

$$
\varphi=\left(\varphi_{j}\right): \mathcal{A} \rightarrow \mathcal{V}
$$

This allows us to define and consider

$$
\mathcal{U}=\mathcal{R}(\varphi(\mathcal{A}), \mathcal{V})
$$

the $*$-regular closure of the image of $\mathcal{A}$ under $\varphi$ in $\mathcal{V}$. Let $\pi_{i}: V \rightarrow V_{i}$ be the projection map. For $i \geq 1$ we define $\mathrm{rk}_{i}=\frac{1}{n_{i}} \mathrm{rk}_{\mathbb{C}} \circ \pi_{i} \in \mathbb{P}(\mathcal{V})$. Note that with $\tau_{i}=\frac{1}{n_{i}} \operatorname{Tr} \circ \varphi_{i}$ we have $\mathrm{rk}_{i} \circ \varphi_{i}=\mathrm{rk}_{\tau_{i}} \in \mathbb{P}(\mathcal{A})$. Further we define $\mathrm{rk}_{0}=\mathrm{rk}_{\tau} \circ \pi_{0}$. Remember we extended $\mathrm{rk}_{\tau}$ from $\mathcal{A}$ first to $\mathcal{N}_{\tau}$ and then to $\mathcal{U}_{\tau}$. We have the following proposition.

Proposition 2.6.7. The following holds:
(a) The projection $\pi_{0}: \mathcal{V} \rightarrow \mathcal{V}_{0}$ restricts to a surjective $*$-homomorphism $\pi_{0} \mid \mathcal{U}: \mathcal{U} \rightarrow \mathcal{R}_{\tau}$.
(b) For $i \geq 1$, the projection $\pi_{i}$ satisfies $\pi_{i}(\mathcal{U}) \subseteq \operatorname{Mat}_{n_{i}}(K)$.
(c) If the class $\mathcal{C}$ has the approximation property, for any $z=\left(z_{i}\right) \in \operatorname{Mat}_{n}(\mathcal{U})$

$$
\lim _{i \rightarrow \infty} \operatorname{rk}_{i}(z)=\mathrm{rk}_{0}(z)
$$

Proof. Obviously we have $\varphi_{0}=\varphi \circ \pi_{0}: \mathcal{A} \rightarrow V_{i}$. Note now that $\varphi_{0}: \mathcal{A} \rightarrow \mathcal{R}_{\tau} \subseteq$ $V_{0}$ and $\varphi: \mathcal{A} \rightarrow \mathcal{U}$ are epic $*$-homomorphism. Therefore, the first part follows from 1.3.16. Part (b) follows by the same argument and the fact that $\operatorname{Mat}_{n_{i}}(K)$ is *-regular. The last statement follows directly from 1.3.37.

Let now as before $\mathcal{A}=K\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ and let $\tau=\lim _{j \rightarrow \infty} \tau_{j}$ be the limit of $b$-bounded integer valued traces on $\mathcal{A}$ with representation maps $\varphi_{j}: \mathcal{A} \rightarrow$ $\operatorname{Mat}_{n_{j}}(K)$. Let $\mathcal{B}=K\left\langle y_{1}, y_{1}^{*}, \ldots, y_{2 d}, y_{2 d^{*}}\right\rangle$ and let $\alpha: \mathcal{B} \rightarrow \mathcal{A} \otimes_{\sigma} A$ defined by $\alpha\left(y_{i}\right)=x_{i} \otimes 1$ for $i \leq d$ and $\alpha\left(y_{i}\right)=1 \otimes x_{i-d}$ for $i \geq d+1$. For $j \geq 1$ define now $\tilde{\varphi}: \mathcal{A} \otimes_{\sigma} A \rightarrow \operatorname{Mat}_{n_{j}}(K) \otimes_{\sigma} \operatorname{Mat}_{n_{j}}(K) \cong \operatorname{Mat}_{n_{j}^{2}}(K)$ by $\tilde{\varphi}_{j}(a \otimes b)=\overline{\varphi_{j}(a)} \otimes \varphi_{j}(b)$. Then the maps $\tilde{\varphi}_{j}$ are representation maps that yield $b^{\prime}$-bounded integer valued traces $\tilde{\tau}_{j}$ with $\lim _{j \rightarrow \infty} \tilde{\tau}_{j}=\tilde{\tau}$ defined in 2.13. Via the map $\alpha$ we can lift all our maps to a free $*$-algebra, however, for simplicity we will stick with the notion of $\mathcal{A} \otimes \mathcal{A}$. We are now ready to prove the main result of this section.

Proposition 2.6.8. Assume that for all $b$ the class of b-bounded integer valued traces has the approximation property on finitely generated free $*$-algebras over $K$. Let $z=\left(z_{i}\right) \in \operatorname{Mat}_{n}(\mathcal{U})$. Then

$$
\operatorname{dim}_{\tilde{\tau}} \mathrm{C}_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}\left(z_{0}\right)=\lim _{j \rightarrow \infty} \frac{\operatorname{dim}_{\mathbb{C}} \mathrm{C}_{\operatorname{Mat}_{n}\left(\mathcal{V}_{j}\right)}\left(z_{j}\right)}{n_{j}^{2}}
$$

Proof. For simplicity let us assume $n=1$. We have seen in 2.6.6 that

$$
\operatorname{dim}_{\tilde{\tau}} \mathrm{C}_{\mathrm{HS}\left(\mathcal{H}_{\tau}\right)}\left(z_{0}\right)=1-\operatorname{rk}_{\tilde{\tau}}\left(z_{0}^{*} \otimes 1-1 \otimes z_{0}\right)
$$

and similarly

$$
\frac{\operatorname{dim}_{\mathbb{C}} \mathrm{C}_{j}\left(z_{j}\right)}{n_{j}^{2}}=\frac{1}{n_{j}^{2}} \operatorname{rk}_{\mathbb{C}}\left(z_{j}^{*} \otimes 1-1 \otimes z_{j}\right)
$$

Thus the result follows by 2.6 .7 (c) applied to the algebra $\mathcal{B}$ and the trace $\tilde{\tau} \circ \alpha$ with approximation maps $\tilde{\varphi_{j}} \circ \alpha$.

### 2.6.3 A lower bound for the dimension of the centralizer

In the previous two sections we have seen how to define the centralizer of some matrix $A \in \mathcal{R}\left(\varphi(\mathcal{A}), \mathcal{U}_{\tau}\right)$ and also that we can approximate its dimension. But remember that our goal was to show that the matrix $A$ has no transcendental eigenvalues. So what is the connection between the dimension of the centralizer of $A$ and its eigenvalues. For that let us first consider the case of a complex matrix $A \in \operatorname{Mat}_{n}(\mathbb{C})$. Since conjugation does not change the dimension of the centralizer we can assume that $A$ is already in Jordan normal form. Consider the following matrices:

$$
A_{1}=\left(\begin{array}{ccc}
\lambda_{1} & 1 & 0 \\
0 & \lambda_{1} & 1 \\
0 & 0 & \lambda_{1}
\end{array}\right), \quad A_{2}=\left(\begin{array}{cc}
\lambda_{1} & 1 \\
0 & \lambda_{1}
\end{array}\right) \quad \text { and } \quad A_{3}=\left(\begin{array}{cc}
\lambda_{2} & 1 \\
0 & \lambda_{2}
\end{array}\right)
$$

for $\lambda_{1}, \lambda_{2} \in \mathbb{C}$. A straight forward caluclation shows

$$
\begin{aligned}
& \mathrm{C}_{\operatorname{Mat}_{n}(\mathbb{C})}\left(A_{1}\right)=\left\{\left.\left(\begin{array}{ccc}
a_{1} & a_{2} & a_{3} \\
0 & a_{1} & a_{2} \\
0 & 0 & a_{1}
\end{array}\right) \right\rvert\, a_{i} \in \mathbb{C}\right\} \\
& \mathrm{C}_{\mathrm{Mat}_{n}(\mathbb{C})}\left(A_{1} \oplus A_{2}\right)=\left\{\left.\left(\begin{array}{ccccc}
a_{1} & a_{2} & a_{3} & d_{1} & d_{2} \\
0 & a_{1} & a_{2} & 0 & d_{1} \\
0 & 0 & a_{1} & 0 & 0 \\
0 & c_{1} & c_{2} & b_{1} & b_{2} \\
0 & 0 & c_{1} & 0 & b_{1}
\end{array}\right) \right\rvert\, a_{i}, b_{i}, c_{i}, d_{i} \in \mathbb{C}\right\} \\
& \mathrm{C}_{\operatorname{Mat}_{n}(\mathbb{C})}\left(A_{1} \oplus A_{3}\right)=\left\{\left.\left(\begin{array}{ccccc}
a_{1} & a_{2} & a_{3} & 0 & 0 \\
0 & a_{1} & a_{2} & 0 & 0 \\
0 & 0 & a_{1} & 0 & 0 \\
0 & 0 & 0 & b_{1} & b_{2} \\
0 & 0 & 0 & 0 & b_{1}
\end{array}\right) \right\rvert\, a_{i}, b_{i} \in \mathbb{C}\right\}
\end{aligned}
$$

It seems that every eigenvalue contributes to the dimension of the centralizer. So how can we generalize this? For $\lambda \in \mathbb{C}$ we define

$$
n_{\lambda, i}(A)=\operatorname{dim}_{\mathbb{C}} \operatorname{ker}(A-\lambda)^{i}
$$

The following result describes completely the relation between the dimension of the centralizer of $A$ and the Jordan structure of $A$. Let us denote by $\sigma_{p}(A)$ the point spectrum of $A$ that is in the finite dimensional case just the eigenvalues of $A$.

Proposition 2.6.9. [CM93, Theorem 6.13] Let $A \in \operatorname{Mat}_{n}(\mathbb{C})$ with $\sigma_{p}=\lambda_{1} \ldots, \lambda_{k}$. Then we have

$$
\operatorname{dim}_{\mathbb{C}} \mathrm{C}_{\operatorname{Mat}_{n}(\mathbb{C})}(A)=\sum_{j=1}^{k} \sum_{i=0}^{n}\left(n_{\lambda_{j}, i+1}-n_{\lambda_{j}, i}\right)^{2}
$$

Let us first note the following corollary.

Corollary 2.6.10. Let $K$ be a subfield of $\mathbb{C}, A \in \operatorname{Mat}_{n}(K)$ and let $\epsilon<\frac{1}{2}$. Put $\delta=\frac{1+\sqrt{1-2 \epsilon}}{2}$. Assume that $\operatorname{dim}_{\mathbb{C}} \mathrm{C}_{\text {Mat }_{n}(\mathbb{C})}(A) \geq n^{2}(1-\epsilon)$. Then there exists exactly one $\lambda \in \sigma_{p}(A)$ with $n_{\lambda, 1}(A) \geq \delta n$. Moreover $\lambda \in K$.

Proof. We know that $\sum_{\lambda \in \sigma_{p}(A)} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}-n_{\lambda, i}\right)=n$. Let $a=\max \left\{n_{\lambda, 1}(A) \mid \lambda \in\right.$ $\left.\sigma_{p}(A)\right\}$. Then we have $\sum_{\lambda \in \sigma_{p}(A)} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}-n_{\lambda, i}\right)^{2} \leq a^{2}+(n-a)^{2}$. The existence follows then from $a^{2}+(n-a)^{2} \geq n^{2}(1-\epsilon)$. The uniqueness follows because $\delta>\frac{1}{2}$. Obviously $\lambda$ is algebraic over $K$ and every Galois conjugate $\lambda^{\prime}$ of $\lambda$ also satisfies $n_{\lambda^{\prime}, 1} \geq \delta n$. Therefore it follows $\lambda \in K$ by the uniqueness.

We would like to state the result of 2.6 .9 also for the case $A \in \operatorname{Mat}_{n}\left(\mathcal{U}_{\tau}\right)$. Just as before let us define

$$
n_{\lambda, i}(A)=\operatorname{dim}_{\tau} \operatorname{ker}(A-\lambda)^{i}
$$

For each eigenvalue $\lambda \in \sigma_{p}(A)$ of $A$ we now want to construct an $A$-invariant subspace of the centralizer of $A$. In the following we will write $A-\lambda$ instead of $A-\lambda \mathrm{Id}_{n}$. Define

$$
U_{A, \lambda, j}=\operatorname{ker}(A-\lambda)^{j+1} \cap\left(\operatorname{ker}(A-\lambda)^{j}\right)^{\perp}
$$

Obviously $U_{A, \lambda, j}$ is closed and invariant under the left $\mathcal{A}$ action, however the operator $A$ might not be bounded when restricted to $U_{A, \lambda, j}$. Therefore, for any $\epsilon>0$, let $U_{A, \lambda, j}^{\epsilon}$ be a closed, $\mathcal{A}$-left invariant subspace of $U_{A, \lambda, j}$ such that

- $\operatorname{dim}_{\tau} U_{A, \lambda, j}^{\epsilon} \geq \operatorname{dim}_{\tau} U_{A, \lambda, j}-\epsilon$ and
- $A$ is bounded when restricted to $U_{A, \lambda, j}^{\epsilon},\left(U_{A, \lambda, j}^{\epsilon}\right) A, \ldots,\left(U_{A, \lambda, j}^{\epsilon}\right) A^{j}$.

In the same way we define a subspace $U_{A^{*}, \bar{\lambda}, j}^{\epsilon}$. By definition we have

$$
\begin{equation*}
\operatorname{dim}_{\tau} U_{A, \lambda, j}^{\epsilon}, \operatorname{dim}_{\tau} U_{A^{*}, \bar{\lambda}, j}^{\epsilon} \geq n_{\lambda, j+1}(A)-n_{\lambda, j}(A)-\epsilon \tag{2.14}
\end{equation*}
$$

For $v \in U_{A^{*}, \bar{\lambda}, j}^{\epsilon}$ and $u \in U_{A, \lambda, j}^{\epsilon}$ we put

$$
w_{j}(v, u)=\sum_{i=0}^{j}\left(\delta_{v\left(A^{*}-\bar{\lambda}\right)^{j-i}} \otimes(u)(A-\lambda)^{i} \in \mathcal{H}_{\tau}^{*} \otimes \mathcal{H}_{\tau}\right)
$$

and define $W_{A, \lambda, j}^{\epsilon}$ as the closed subspace of $H_{\tau}^{*} \otimes H_{\tau}$ generated by the vectors $\left\{w_{j}(v, u) \mid v \in U_{A^{*}, \bar{\lambda}, j}^{\epsilon}, u \in U_{A, \lambda, j}^{\epsilon}\right\}$.

Lemma 2.6.11. Let $A \in \mathcal{U}_{\tau}$. Then the following holds.
(a) For every $\epsilon \geq 0$ we have $\Psi\left(W_{A, \lambda, j}\right)^{\epsilon} \subseteq \mathrm{C}_{\mathrm{HS}\left(\mathcal{H}_{\tau}\right)^{n}}(A)$.
(b) Let $k \in \mathbb{N}, \lambda_{1}, \ldots \lambda_{k} \in \sigma_{p}(A)$ and $n_{1}, \ldots, n_{k} \in \mathbb{N}$. Then

$$
\lim _{\epsilon \rightarrow 0} \operatorname{dim}_{\tilde{\tau}}\left(\sum_{i=1}^{k} \sum_{j=0}^{n_{i}} \Psi\left(W_{A, \lambda, j}^{\epsilon}\right)\right)=\sum_{i=1}^{k} \sum_{j=0}^{n_{i}}\left(n_{\lambda_{i}, j+1}(A)-n_{\lambda_{i}, j}(A)\right)^{2} .
$$

Proof. (1) For simplicity let us assume that $\lambda=0$. Let $v \in U_{A^{*}, 0, j}^{\epsilon}, u \in U_{A, 0, j}^{\epsilon}$. Represent $A=B_{1}^{-1} C_{1}=C_{2} B_{2}^{-1}$ with $B_{1}, B_{2}, C_{1}, C_{2} \in \mathcal{N}_{\tau}$. Then we have

$$
\begin{gathered}
C_{1} \Psi\left(\sum_{i=0}^{j}\left(\delta_{v\left(A^{*}\right)^{j-i}} \otimes u(A)^{i}\right) B_{2}-B_{1} \Psi\left(\sum _ { i = 0 } ^ { j } \left(\delta_{v\left(A^{*}\right)^{j-i}} \otimes u(A)^{i} C_{2}=\right.\right.\right. \\
\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i} C_{1}^{*}\right\rangle u(A)^{i} B_{2}-\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i} B_{1}^{*}\right\rangle u(A)^{i} C_{2}= \\
\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i} C_{1}^{*}\left(\left(B_{1}^{*}\right)^{-1} B_{1}^{*}\right)\right\rangle u(A)^{i} B_{2}-\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i} B_{1}^{*}\right\rangle u(A)^{i} C_{2} B_{2}^{-1} B_{2}= \\
\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i+1} B_{1}^{*}\right\rangle u(A)^{i} B_{2}-\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i} B_{1}^{*}\right\rangle u(A)^{i+1} B_{2}= \\
B_{1}\left(\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i+1}\right\rangle u(A)^{i}-\sum_{i=0}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i}\right\rangle u(A)^{i+1}\right) B_{2}= \\
B_{1}\left(\sum_{i=1}^{j}\left\langle\cdot, v\left(A^{*}\right)^{j-i+1}\right\rangle u(A)^{i}-\sum_{i=0}^{j-1}\left\langle\cdot, v\left(A^{*}\right)^{j-i}\right\rangle u(A)^{i+1}\right) B_{2}=0 .
\end{gathered}
$$

(2) Fix $\epsilon>0$. Let

$$
\rho: \oplus_{i=1}^{k} \oplus_{j=1}^{n_{i}} \delta_{U_{A^{*}, \bar{\lambda}_{i}, j}^{\epsilon}} \otimes U_{A, \lambda, j} \rightarrow \sum_{i=1}^{k} \sum_{j=1}^{n_{i}} \Psi\left(W_{A, \lambda_{i}, j}^{\epsilon}\right)
$$

be the continuous linear map defined on the generating set $\left\{\delta_{v_{i, j}} \otimes u_{i, j} \mid\right.$ $\left.v_{i, j} \in U_{A^{*}, \lambda_{i}, j}^{\epsilon}, u_{i, j} \in U_{A, \lambda_{i}, j}^{\epsilon}\right\}$ by

$$
\rho\left(\delta_{v_{i, j}} \otimes u_{i, j}\right)=\Psi\left(w_{j}\left(v_{i, j}, u_{i, j}\right)\right)
$$

Note that $\rho$ is bounded. Furthermore for different $\lambda$ and different $j$ the spaces $U_{A, \lambda, j}$ and $\delta_{U_{A, \bar{\lambda}, j}}$ are all disjoint. Therefore the sums

$$
U^{\epsilon}=\sum_{i=1}^{k} \sum_{j=0}^{n_{i}} U_{A, \lambda_{i}, j}^{\epsilon} \text { and } V^{\epsilon}=\sum_{i=1}^{k} \sum_{j=0}^{n_{i}} \delta_{U_{A^{*}, \bar{\lambda}_{i}, j}^{\epsilon}}
$$

are direct sums of closed subspaces of $H_{\tau}$ or $H_{\tau}^{*}$ respectively. Since the map

$$
\alpha: \oplus_{i=1}^{k} \oplus_{j=0}^{n_{i}}: \delta_{U_{A^{*}, \overline{\lambda_{i}}, j}} \otimes U_{A, \lambda, j} \rightarrow V^{\epsilon} \otimes U^{\epsilon}
$$

is a weak monomorphism, $\rho=\Psi \circ \alpha$ is also a weak monomorphism. The result follows then by 2.14

The following follows directly from the previous lemma.
Proposition 2.6.12. Let $A \in \operatorname{Mat}_{n}\left(U_{\tau}\right)$. Then

$$
\operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}(A) \geq \sum_{\lambda \in \sigma_{p}} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2}
$$

### 2.6.4 The centralizer dimension property

In the previous section we have seen a lower bound for $\operatorname{dim}_{\tilde{\tau}} \mathrm{C}_{H S\left(\mathcal{H}_{\tau}\right)}(A)$ for $A \in \operatorname{Mat}_{n}\left(\mathcal{U}_{\tau}\right)$. We now want to see that for matrices $A \in \mathcal{R}_{\tau}=\mathcal{R}\left(\mathcal{A}, \mathcal{U}_{\tau}\right)$ the inequality in 2.6 .12 is actually an equality. For that we will first use that by Proposition 2.6 .8 we can approximate the dimension of the centralizer by the dimensions of the centralizers of some matrices. We will then use Proposition 2.6.9 to calculate these dimension. Since the dimension of the approximation matrices goes to infinity we need the following result.

Proposition 2.6.13. Let $A \in \operatorname{Mat}_{n}(\mathbb{C})$ and $k \in \mathbb{N}$. Then

$$
\operatorname{dim}_{\mathbb{C}} \mathrm{C}_{\operatorname{Mat}_{n}(\mathbb{C})}(A)-\sum_{\lambda \in \sigma_{p}(A), n_{\lambda, 1}(A) \geq \frac{n}{k}} \sum_{i=0}^{k-1}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \leq \frac{n^{2}}{k}
$$

Proof. Let $\lambda \in \mathbb{C}$ and $s \geq 0$. Note that $n_{\lambda, k}(A)-n_{\lambda,(k-1)}(A)$ is the number of Jordan blocks of the Jordan normal form of $A$ related to the eigenvalue $\lambda$ of size at least $k$. Therefore we have

$$
n_{\lambda, i}(A)-n_{\lambda, i-1}(A) \geq n_{\lambda, i+1}(A)-n_{\lambda, i}(A)
$$

for all $i$. Note further that for $k>n$ we have

$$
n_{\lambda, k}(A)-n_{\lambda,(-1}(A)=0
$$

Having this in mind we obtain

$$
\begin{aligned}
& \sum_{i=s}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \\
& \leq\left(n_{\lambda, s+1}(A)-n_{\lambda, s}(A)\right) \sum_{i=s}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right) \\
& \quad \leq \min \left\{n_{\lambda, 1}, \frac{n}{s+1}\right\} \cdot n_{\lambda, n}
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
& \operatorname{dim}_{\mathbb{C}} C_{\operatorname{Mat}_{n}(\mathbb{C})}(A)-\sum_{\substack{\lambda \in \sigma_{p}(A) \\
n_{\lambda, 1}(A) \geq \frac{n}{k}}} \sum_{i=0}^{k-1}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \\
&=\sum_{\substack{\lambda \in \sigma_{p}(A) \\
n_{\lambda, 1}(A) \geq \frac{n}{k}}} \sum_{i=k}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2}+\sum_{\substack{\lambda \in \sigma_{p}(A) \\
n_{\lambda, 1}(A)<\frac{n}{k}}} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \\
& \leq \frac{n}{k} \sum_{\lambda \in \sigma_{p}(A)} n_{\lambda, n}(A) .
\end{aligned}
$$

Let now as in the previous section $\tau$ be the limit of $b$-bounded integer valued traces $\tau_{i}$ on $\mathcal{A}$ with approximation maps $\varphi_{i}$. Let $\tilde{\tau}$ be as in Section 2.6.2. We now have all the necessary ingredients to prove the following theorem.

Theorem 2.6.14. Let $\mathcal{R}_{\tau}=\mathcal{R}\left(\mathcal{A}, \mathcal{U}_{\tau}\right)$ and $A \in \operatorname{Mat}_{n}\left(\mathcal{R}_{\tau}\right)$. Assume the class of b-bounded integer valued traces has the approximation property for finitely generated free $*$-algebras over $K$. Then

$$
\operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}(A)=\sum_{\lambda \in K} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2}
$$

Proof. By Proposition 2.6 .12 we already know the " $\geq$ "part. So let us show the other direction. We will use the notation introduced in the beginning of Section 2.6.2. By Proposition 2.6.7 there is an element $z=\left(z_{i}\right) \in \operatorname{Mat}_{n}(\mathcal{U})$ such that $z_{0}=A$ and $z_{i} \in \operatorname{Mat}_{n \cdot n_{j}}(K)$ for all $i \geq 1$. For every $\epsilon>0$ let

$$
S_{\epsilon}=\left\{\lambda \in K \mid \exists j: \mathrm{rk}_{j}(z-\lambda) \leq(1-\epsilon) n\right\}
$$

By 2.5.1 the set $S_{\epsilon}$ is finite for every $\epsilon$. Further we have

$$
\begin{array}{r}
\left|\operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}\left(z_{0}\right)-\lim _{j \rightarrow \infty} \sum_{\lambda \in S_{\frac{1}{k}}} \frac{1}{n_{j}} \sum_{i=0}^{k-1}\left(n_{\lambda, i+1}\left(z_{j}\right)-n_{\lambda, i}\left(z_{j}\right)\right)^{2}\right| \stackrel{(2.6 .13)}{\leq} \\
\lim _{j \rightarrow \infty}\left|\operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}\left(z_{0}\right)-\frac{1}{n_{j}^{2}} \operatorname{dim}_{\mathbb{C}} \mathrm{C}_{\mathcal{V}_{j}}\left(z_{j}\right)\right|+\frac{n^{2}}{k} \stackrel{\text { (2.6.8) }}{=} \frac{n^{2}}{k} .
\end{array}
$$

By Proposition 2.6.7 we have for every $\lambda \in K, k \in \mathbb{N}$

$$
n_{\lambda, k}\left(z_{0}\right)=\lim \frac{n_{\lambda, k}\left(z_{j}\right)}{n_{j}}
$$

Therefore, we get

$$
\begin{aligned}
& \operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}\left(z_{0}\right)= \lim _{k \rightarrow \infty} \lim _{j \rightarrow \infty} \sum_{\lambda \in S_{\frac{1}{k}}} \sum_{i=0}^{k-1}\left(n_{\lambda, i+1}\left(z_{j}\right)-n_{\lambda, i}\left(z_{j}\right)\right)^{2} \\
& \lim _{k \rightarrow \infty} \sum_{\lambda \in S_{\frac{1}{k}}} \sum_{i=0}^{k-1} \lim _{j \rightarrow \infty} \frac{1}{n_{j}^{2}}\left(n_{\lambda, i+1}\left(z_{j}\right)-n_{\lambda, i}\left(z_{j}\right)\right)^{2}= \\
& \lim _{k \rightarrow \infty} \sum_{\lambda \in S_{\frac{1}{k}}} \sum_{i=0}^{k-1}\left(n_{\lambda, i+1}\left(z_{0}\right)-n_{\lambda, i}\left(z_{0}\right)\right)^{2} \leq \\
& \sum_{\lambda \in K} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}\left(z_{0}\right)-n_{\lambda, i}\left(z_{0}\right)\right)^{2}
\end{aligned}
$$

The proof of Theorem 2.6.1 is now easy. We have

$$
\begin{aligned}
\operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}(A) & \stackrel{(2.6 .12)}{\geq} \sum_{\lambda \in \sigma_{p}} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \\
& \geq \sum_{\lambda \in K} \sum_{i=0}^{\infty}\left(n_{\lambda, i+1}(A)-n_{\lambda, i}(A)\right)^{2} \\
& \stackrel{(2.6 .14)}{=} \operatorname{dim}_{\tilde{\tau}} C_{\operatorname{Mat}_{n}\left(\operatorname{HS}\left(\mathcal{H}_{\tau}\right)\right)}(A)
\end{aligned}
$$

and therefore $n_{(\lambda, 1)}(A)=\operatorname{dim}_{\tau} \operatorname{ker}(A-\lambda)=0$ for all $\lambda \in \mathbb{C} \backslash K$.

### 2.7 Proof of Theorem A

In this section we want to prove Theorem A. First we will prove that the class of $b$-bounded integer valued traces on free $*$-algebras has the approximation property.

Theorem 2.7.1. Let $\mathcal{A}$ be a finitely generate free $*$-algebra over $\mathbb{C}$. Let $b \in \mathbb{R}_{>0}$ and let $\mathcal{C}$ be the class of all b-bounded integer valued traces on $\mathcal{A}$. Then $\mathbb{C}$ has the approximation property.

Proof. Let as before $\mathcal{A}=\mathbb{C}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ and $A \in \operatorname{Mat}_{n}(\mathcal{A})$. Let $\left(\tau_{i}\right) \subseteq \mathcal{C}$ be a converging sequence and let Let $\tau=\lim _{i \rightarrow \infty} \tau_{i}$. Set $K_{1}=\mathbb{Q}$ and for $i \geq 1$ set $K_{2 i}=\overline{K_{2 i-1}}$ and $K_{2 i+1}=K_{2 i}\left(t_{i}\right)$ where $t_{i} \in \mathbb{C} \backslash K_{2 i}$ is one coefficient of an entry of $A$ that is transcendental over $K_{2 i}$. Since $A$ has only finitely many entries, there is a $k \in \mathbb{N}$ such that $A$ is a matrix over $K_{2 k}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$. Set $\mathcal{A}_{i}=K_{i}\left\langle x_{1}, x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle \leq \mathcal{A}$. Let us denote by $\tau^{i}$ and $\varphi_{j}^{i}$ the restriction of $\tau$ and $\varphi_{j}$ to $\mathcal{A}_{i}$ and by $\mathcal{C}_{i}$ the class of all $b$-bounded integer valued traces on
$\mathcal{A}_{i}$. By 2.4.2 we know that $\mathcal{C}_{0}$ has the approximation property defined in 2.2.3. That means we have for any matrix $A_{0} \in \operatorname{Mat}_{n}\left(\mathcal{A}_{0}\right)$

$$
\mathrm{rk}_{\tau_{0}}\left(A_{0}\right)=\lim _{j \rightarrow \infty} \frac{1}{n_{j}} \mathrm{rk}_{K_{0}} \varphi_{n}^{0}\left(A_{0}\right)
$$

We can reformulate this by saying that for every non principal ultrafilter $\omega$ with $\mathrm{rk}_{\omega, 0}=\lim _{\omega} \frac{1}{n_{j}} \mathrm{rk}_{K_{0}} \circ \varphi_{j}^{0}$ we have

$$
\mathrm{rk}_{\tau_{0}}=\mathrm{rk}_{\omega, 0} \in \mathbb{P}_{\mathrm{rk}}\left(\mathcal{A}_{0}\right)
$$

Let us generalize this notation and define $\mathrm{rk}_{\omega, i}=\lim _{\omega} \frac{1}{n_{j}} \mathrm{rk}_{K_{i}} \circ \varphi_{j}^{i} \in \mathbb{P}_{\mathrm{rk}}\left(\mathcal{A}_{i}\right)$.
Claim. Let $i \geq 1$ and assume that $r k_{\tau_{2 i-1}}=r k_{\omega, 2 i-1} \in \mathbb{P}_{r k}\left(\mathcal{A}_{2 i-1}\right)$. Then

$$
r k_{\tau_{2 i}}=r k_{\omega, 2 i} \in \mathbb{P}_{r k}\left(\mathcal{A}_{2 i}\right)
$$

Proof. We want to apply Theorem 1.3.31. Note that $\mathcal{A}_{2 i}=K_{2 i} \otimes_{K_{2 i-1}} \mathcal{A}_{2 i-1}$. Let $\alpha: \mathcal{A}_{2 i-1} \hookrightarrow \mathcal{A}_{2 i}$ be the inclusion. Note that $\mathrm{rk}_{\tau_{2 i-1}}=\alpha^{\#}\left(\mathrm{rk}_{\tau_{2 i}}\right)$ and $\operatorname{rk}_{\omega, 2 i-1}=\alpha^{\#}\left(\mathrm{rk}_{\omega, 2 i}\right)$. Last, by 2.2.7 we have

$$
\mathrm{rk}_{\tau_{2 i}} \geq \mathrm{rk}_{\omega, 2 i}
$$

Thus the statement follows by 1.3.31.
Claim. Let $i \geq 1$ and assume that $r k_{\tau_{2 i}}=r k_{\omega, 2 i} \in \mathbb{P}_{r k}\left(\mathcal{A}_{2 i}\right)$. Then

$$
r k_{\tau_{2 i+1}}=r k_{\omega, 2 i+1} \in \mathbb{P}_{r k}\left(\mathcal{A}_{2 i+1}\right)
$$

Proof. Note that $\mathcal{A}_{2 i+1}=K_{2 i}\left(t_{i}\right) \otimes_{K_{2 i}} \mathcal{A}_{2 i}$. We want to apply 1.3.32. That means, we want to show that both, $\mathrm{rk}_{\tau_{2 i+1}}$ and $\mathrm{rk}_{\omega, 2 i+1}$, are the natural transcendental extensions of $\mathrm{rk}_{\tau_{2 i}}$ and $\mathrm{rk}_{\omega, 2 i}$ respectively. Obviously we have $\mathrm{rk}_{K_{2 i+1}}=$ $\widetilde{\mathrm{rk}}_{2 i}$ and therefore, by 1.3.33 also $\mathrm{rk}_{\omega, 2 i+1}=\widetilde{\mathrm{rk}}_{\omega, 2 i}$. On the other hand, by 2.6.1 and 1.3.32 we get

$$
\mathrm{rk}_{\tau_{2 i+1}}=\widetilde{\mathrm{rk}}_{\tau_{2 i}}
$$

Thus the statement follows.
By applying these two claims iteratively, we obtain

$$
\mathrm{rk}_{\tau_{2 k}}=\mathrm{rk}_{\omega, 2 k} \in \mathbb{P}_{\mathrm{rk}}\left(\mathcal{A}_{2 k}\right)
$$

which implies that $\mathcal{C}_{k}$ has the approximation property for all $k$.
Lemma 2.7.2. Let $\mathcal{A}$ and $\mathcal{C}$ class of traces on $\mathcal{A}$ that satisfies the approximation property. Let $\left(\tau_{i}\right) \subseteq \mathcal{C}$ be a sequence of converging traces on $\mathcal{A}$ with $\lim _{i \rightarrow \infty} \tau_{i}=\tau$. Let $A$ be a normal matrix over $\mathcal{A}$. Then there is a function $f: \mathbb{R}_{>0} \xrightarrow{i \rightarrow \infty} \mathbb{R}_{>0}$ with $\lim _{\lambda \rightarrow 0^{+}} f(\lambda)=0$ such that

$$
\mu_{\tau_{i}, A}(B(0, \lambda) \backslash\{0\} \leq f(\lambda)
$$

Proof. Since $\mathcal{C}$ satisfies the approximation property

$$
\lim _{i \rightarrow \infty} \operatorname{rk}_{\tau_{i}}(A)=\operatorname{rk}_{\tau}(A)
$$

which is equivalent to

$$
\lim _{i \rightarrow \infty} \mu_{\tau_{i}, A}(\{0\})=\mu_{\tau, A}(\{0\})
$$

Since the measures $\mu_{\tau_{i}, A}$ converge weakly to $\mu_{\tau_{i}, A}$ the result follows from 1.5.6.

In the previous setting, we now want to show the existence of such a function $f$ that bounds $\mu_{\tau_{i}, A}(B(y, \lambda) \backslash\{y\})$ for all points $y \in \mathbb{C}$. We will consider the operator $A \otimes 1-1 \otimes A$ to move areas where the measures $\mu_{\tau_{i}, A}$ have large density to zero. The following lemma describes what happens in the case of matrices over $\mathbb{C}$.

Lemma 2.7.3. Let $A \in \operatorname{Mat}_{n}(\mathbb{C})$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$. The Kronecker product $B=A \otimes \operatorname{Id}_{n}-\operatorname{Id}_{n} \otimes A \in \operatorname{Mat}_{n^{2}}(\mathbb{C})$ has the eigenvalues $\lambda_{i, j}=\lambda_{i}-\lambda_{j}$ for $i, j \in\{1, \ldots, n\}$. In particular the multiplicity of the eigenvalue 0 of $B$ is at least $n$.

Proof. Follows directly by the definition of the Kronecker product and the Jordan normal form.

The idea to consider the differences of all eigenvalues goes back to Andreas Thom in [Tho08].

Theorem 2.7.4. Let $\mathcal{A}$ be a finitely generated free *-algebra and let $\mathcal{C}$ be the class of b-bounded integer valued traces on $\mathcal{A}$. Let $\left(\tau_{i}\right) \subseteq \mathcal{C}$ be a sequence of converging traces. Then there is a function $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ with $\lim _{\lambda \rightarrow 0^{+}} f(\lambda)=0$ such that for all $y \in \mathbb{C}$

$$
\mu_{\tau_{i}, A}(B(y, \lambda) \backslash\{y\} \leq f(\lambda)
$$

Proof. Let $\tau=\lim _{i \rightarrow \infty} \tau_{i}$. Consider the $*$-homomorphism $\varphi: \overline{\mathcal{A}}=\mathbb{C}\left\langle y_{1}, y_{1}^{*}, \ldots, y_{2 d}, y_{2 d}^{*}\right\rangle \rightarrow$ $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ defined by

$$
\varphi\left(y_{j}\right)= \begin{cases}x_{j} \otimes 1 & j \leq d \\ 1 \otimes x_{j-d} & j>d\end{cases}
$$

We then get a trace $\bar{\tau}$ on $\overline{\mathcal{A}}$ given by $\bar{\tau}=(\tau \otimes \tau) \circ \varphi$, where $(\tau \otimes \tau)\left(a_{1} \otimes a_{2}\right)=$ $\tau\left(a_{1}\right) \cdot \tau\left(a_{2}\right)$. Note that $\bar{\tau}=\lim _{i \rightarrow \infty} \overline{\bar{\tau}_{i}}$ where $\overline{\tau_{i}}=\left(\tau_{i} \otimes \tau_{i}\right) \circ \varphi$.

Let $A \in \operatorname{Mat}_{n}(\mathcal{A})$ be normal matrix. Consider a preimage $\bar{A}$ of the element $A \otimes 1-1 \otimes A \in \operatorname{Mat}_{n}(\mathcal{A}) \otimes_{\mathbb{C}} \operatorname{Mat}_{n}(\mathcal{A})$ under $\varphi$. Note that $\mu_{A, \tau_{i}}$ is the normalized eigenvalue measure of $A_{i}=\varphi_{i}(A)$ and $\mu_{\bar{A}, \overline{\tau_{i}}}$ is the normalized eigenvalue measure of $A_{i} \otimes \operatorname{Id}_{n_{i}}-\operatorname{Id}_{n_{i}} \otimes A_{i}$. Here we see the matrix tensor products as
the Kronecker product. Let $d_{i}(\lambda)$ be the number of eigenvalues of $A_{i}$ that lie in $B(x, \lambda) \backslash\{x\}$. That means we have

$$
\mu_{A, \tau_{i}}(B(x, \lambda) \backslash\{x\})=\frac{d_{i}(\lambda)}{n_{i}} .
$$

By the previous lemma we have

$$
\begin{equation*}
\mu_{\bar{A}, \overline{\bar{T}}_{i}}(B(0,2 \lambda) \backslash\{0\}) \geq \frac{d_{i}(\lambda)^{2}-d_{i}(\lambda)}{n_{i}^{2}}=\frac{d_{i}(\lambda)}{n_{i}} \cdot \frac{d_{i}(\lambda)-1}{n_{i}} . \tag{2.15}
\end{equation*}
$$

Assume now that for all $\lambda$ we have

$$
\sup _{i} \mu_{A, \tau_{i}}(B(x, \lambda) \backslash\{x\})=\frac{d_{i}(\lambda)}{n_{i}}>\epsilon .
$$

In particular, for each $\lambda$ there must be infinitely many $i \in \mathbb{N}$ such that $\mu_{A, \tau_{i}}(B(x, \lambda) \backslash$ $\{x\}) \geq \frac{\epsilon}{2}$. Thus, since $\left|n_{i}\right| \xrightarrow{i \rightarrow \infty} \infty$, for each $\lambda$ there exists an $i \in \mathbb{N}$ such that $\mu_{A, \tau_{i}}(B(x, \lambda) \backslash\{x\}) \geq \frac{\epsilon}{2}$ and $\frac{1}{n_{i}} \leq \frac{\epsilon}{8}$ Therefore, by 2.15 we get

$$
\begin{equation*}
\sup _{i} \mu_{\bar{A}, \overline{\tau_{i}}}(B(0,2 \lambda) \backslash\{0\}) \geq \frac{\epsilon^{2}}{4}-\frac{d_{i}(\lambda)}{n_{i}^{2}} \geq \frac{\epsilon^{2}}{4}-\frac{\epsilon}{8} \geq \frac{\epsilon}{8} . \tag{2.16}
\end{equation*}
$$

Since the traces $\bar{\tau}_{i}$ are $b^{\prime}$-bounded integer valued for some $b^{\prime} \in \mathbb{R}_{>0}$ by 2.7 .1 we have

$$
\lim _{i \rightarrow \infty} \mathrm{rk}_{\tilde{\tau}_{i}}=\mathrm{rk}_{\tilde{\tau}}
$$

and therefore

$$
\lim _{i \rightarrow \infty} \mu_{A, \bar{\tau}_{i}}(\{0\})=\mu_{A, \bar{\tau}}(\{0\}) .
$$

Thus, by 2.7.2, there is a function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\lim _{\lambda \rightarrow 0} f(\lambda)=0$ such that

$$
\mu_{\bar{A}, \overline{\tau_{i}}}((B(0, \lambda) \backslash\{0\}) \leq f(\lambda) .
$$

But this is a contradiction to Equation 2.16.
We finally have all the ingredients to prove our main result.
Proof of Theorem $A$. For a contradiction we can assume that there is a $c \in \mathbb{R}_{>0}$, a matrix $A \in \operatorname{Mat}_{m}(\mathcal{A})$, a point $y \in \mathbb{C}$ and a constant $\epsilon>0$ such that for each $i \in \mathbb{N}$ there is a $*$-homomorphism $\varphi_{i}: \mathcal{A} \rightarrow \operatorname{Mat}_{n_{i}}(\mathbb{C})$ with $\varphi\left(x_{j}\right) \in \operatorname{Mat}_{n_{i}}(\mathbb{Z})$ and $\left\|\varphi\left(x_{j}\right)\right\|_{1},\left\|\varphi\left(x_{j}\right)\right\|_{\infty}<c$ and

$$
\frac{1}{n_{i}} \mu_{\varphi_{i}(A)}\left(B\left(y, \frac{1}{i}\right) \backslash\{y\}\right) \geq \epsilon .
$$

From this situation we want to obtain a contradiction to Theorem 2.7.4. The following lemma will be helpful.

Lemma 2.7.5. Let $S$ be a countable set and let $\left(f_{n}\right), f_{n}: S \rightarrow \mathbb{C}$ be a sequence of functions, such that for all $n \in \mathbb{N}$ and $x \in S$ there is $c \in R_{>0}$ such that $\left\|f_{n}(x)\right\| \leq c$. Then there is a point wise convergent subsequence $\left(f_{n_{k}}\right)$.

Proof. Since $S$ is countable we can write $S=\left\{s_{k} \mid k \in \mathbb{N}\right\}$. The proof works by induction. By the theorem of Bolzano Weierstrass we can choose a subsequence $\left(f_{1,1}, f_{1,2}, \ldots\right)$ of $\left(f_{n}\right)$ such that $f_{1, i}\left(s_{1}\right)$ converges. So let us assume we have a subsequence $\left(f_{n, 1}, f_{n, 2}, \ldots\right)$ such that for all $i \in\{1, \ldots, n\} f(n, j)\left(s_{i}\right)$ converges. Then $f_{n, j}\left(s_{n+1}\right)$ is also a bounded sequence and we can choose a subsequence $f_{n+1, j}$ that converges for all $s \in\left\{s_{i} \mid 1 \leq i \leq n+1\right\}$. By induction we can find a subsequence that converges for all $s \in S$.

We want to apply this lemma to our situation. For that let $t_{1}, \ldots, t_{r}$ be all the numbers that appear as coefficients in the matrix $A$ that are transcendental over $\mathbb{Q}$. The field $K=\overline{\mathbb{Q}\left(t_{1}, \ldots, t_{r}\right)}$ is countable, and since the algebra $\mathcal{A}^{\prime}=$ $K\left\langle x_{1} x_{1}^{*}, \ldots, x_{d}, x_{d}^{*}\right\rangle$ is finitely generated it is also countable. Thus we can replace $\mathcal{A}$ with $\mathcal{A}^{\prime}$ and the previous lemma yields us a point wise convergent sequence of $b$-bounded integer valued traces $\tau_{i_{k}}=\frac{1}{n_{i_{k}}} \operatorname{Tr} \circ \varphi_{i_{k}}$. Thus we have a contradiction to 2.7.4 which finishes the proof of Theorem A.

## Chapter 3

## Twisted $\ell^{2}$-Betti numbers

In this chapter we want to talk about twisted $\ell^{2}$-Betti numbers. We will use some notation of the previous chapters, however, instead of talking about general *-algebras from now on we will focus on group algebras. Let us recall some notation first. For a group $G$ we denote by

- $\mathbb{C}[G]$ the complex group algebra,
- $\ell^{2}(G)$ the group Hilbert space that is the completion on $\mathbb{C}[G]$,
- $\mathcal{N}(G)$ the group von Neumann algebra and by
- $\mathcal{U}(G)$ the algebra of unbounded operators affiliated to $\mathcal{N}(G)$.

Remember that in the first chapter we defined a rank function $\mathrm{rk}_{G}$ on $\mathcal{U}(G)$. In this chapter we want to study what happens when we twist $\mathrm{rk}_{G}$ with some finite dimensional representation. So what do we mean by that? Let $\sigma: G \rightarrow \mathrm{GL}_{k}(\mathbb{C})$ be a representation of $G$. We can define a twisting map

$$
\begin{equation*}
\tilde{\sigma}: \mathbb{C}[G] \rightarrow \operatorname{Mat}_{k}(\mathbb{C}[G]), \sum_{g \in G} a_{g} g \mapsto \sum_{g \in G} a_{g} \sigma(g) g \tag{3.1}
\end{equation*}
$$

As always we can extend $\tilde{\sigma}$ entry wise to matrices over $\mathbb{C}[G]$. So given any matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ is there any relation between $\operatorname{rk}_{G}(A)$ and $\operatorname{rk}_{G}(\tilde{\sigma}(A))$ ?

It was Wolfgang Lück in [Lüc18] who conjectured the following.
Conjecture 2. Let $G$ be a group and let $\sigma: G \rightarrow \mathrm{GL}_{k}(\mathbb{C})$ be a homomorphism. Then for any matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$

$$
\frac{1}{k} \mathrm{rk}_{G}(\tilde{\sigma}(A))=\operatorname{rk}_{G}(A)
$$

Thus, Lück conjectured that for any representation $\sigma$ of $G$ the rank functions $\mathrm{rk}_{G}$ and $\mathrm{rk}_{G, \sigma}=\frac{1}{k} \mathrm{rk}_{G} \circ \tilde{\sigma}$ are equal. Conjecture 2 was proven in [Lüc18] for elementary torsion-free amenable groups and in [KS21] for locally indicable groups. In this chapter we want to prove the conjecture for sofic groups.

Theorem B. Let $G$ be a sofic group and $\sigma: G \rightarrow \mathrm{GL}_{k}(\mathbb{C})$ be a representation of $G$. Then, for all matrices $A \in \operatorname{Mat}_{n \times m}(\mathbb{C}[G])$ we have

$$
k \cdot \operatorname{rk}_{G}(A)=\operatorname{rk}_{G}(\tilde{\sigma}(A))
$$

First, lets explain a different view on twisting. For that let $G, \sigma$ and $A$ be as above. The matrix $A$ can be seen as a map

$$
\mathbb{C}[G]^{n} \rightarrow \mathbb{C}[G]^{n},\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(a_{1}, \ldots, a_{n}\right) A
$$

Now, consider the $\mathbb{C}$-vector space $C^{k} \otimes_{\mathbb{C}} \mathbb{C}[G]^{n}$. As a $\mathbb{C}$-vector space this is isomorphic to $\mathbb{C}[G]^{n \cdot k}$. For that let $\left(e_{1}, \ldots, e_{k}\right)$ be the standard basis for $\mathbb{C}^{k}$ and let $\alpha: \mathbb{C}^{k} \otimes_{\mathbb{C}} \mathbb{C}[G]^{n} \rightarrow \mathbb{C}[G]^{n \cdot k}$ be the isomorphism defined by

$$
e_{i} \otimes\left(0, \ldots, 0, a_{j}, 0, \ldots, 0\right) \rightarrow\left(0, \ldots, 0, a_{j}, 0, \ldots, 0\right)
$$

where in the latter the entry $a_{j}$ is in the $i \cdot n+j$ th coordinate. On the vector space $C^{k} \otimes_{\mathbb{C}} \mathbb{C}[G]$ we can define two different $G$-module structures. First define for $v \in \mathbb{C}^{k}, a \in \mathbb{C}[G]$

$$
(v \otimes a) \cdot{ }_{1} g=v \otimes a g
$$

Second, we define

$$
(v \otimes a) \cdot_{2} g=v \sigma(g) \otimes a g
$$

If we pass via $\alpha$ to $\mathbb{C}[G]^{k}$, the first action is given by right multiplication by the matrix $\operatorname{Id}_{k} \otimes g=g \cdot \operatorname{Id}_{k} \in \operatorname{Mat}_{k}(\mathbb{C}[G])$ while the second action is given by right multiplication by the matrix $\sigma(g) \otimes g=g \cdot \sigma(g)=\tilde{\sigma}(g) \in \operatorname{Mat}_{k}(\mathbb{C}[G])$. This construction gives the origin of the map $\tilde{\sigma}$. In the first part of this chapter we want to prove Conjecture 2 when the group $G$ is sofic. We will recall the notion of a sofic group later. In the second part of this chapter we want briefly explain the topological view point on $\ell^{2}$-Betti numbers and we will explain how twisting naturally occurs when dealing with fibrations. This chapter is based on the article [BJ22].

### 3.1 Twisted rank function for sofic groups

In this section we want to prove Conjecture 2 for sofic groups. We already mentioned the word sofic when we talked about sofic traces. Let us start with the definition of a sofic group.

Definition 3.1.1. Let $G$ be a finitely generated group. Write $G=F / N$ where $F$ is a finitely generated free group and $N$ a normal subgroup of $F$. We say that $G$ is sofic if there is a family $\left\{X_{k} \mid k \in \mathbb{N}\right\}$ of finte $F$-sets such that for any $w \in F$

$$
\lim _{k \rightarrow \infty} \frac{\left|\operatorname{Fix}_{X_{k}}(w)\right|}{\left|X_{k}\right|}=1 \text { if } w \in N \text { and } \frac{\left|\operatorname{Fix}_{X_{k}}(w)\right|}{\left|X_{k}\right|}=0 \text { if } w \notin N
$$

An arbitrary group $G$ is called sofic if each of its finitely generated subgroups is sofic.

The family $\left\{X_{k}\right\}$ is called a sofic approximation for $G=F / N$. Examples of sofic groups are residually finite groups and amenable groups. At the moment there is no non sofic group known, however people believe that non sofic groups do exist. It is not difficult to see that a group is sofic if and only if its regular character $\tau_{G}$ is sofic. Note that the action of $F$ on $X_{k}$ gives us maps $\varphi_{k}$ : $F \rightarrow \operatorname{Mat}_{\left|X_{k}\right|}(\mathbb{Z})$ which extends linearly to a $*$-homomorphism $\varphi_{k}: \mathbb{C}[F] \rightarrow$ $\operatorname{Mat}_{\left|X_{k}\right|}(\mathbb{C})$. For a matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ by abuse of notation we will also write $A$ for the lifted matrix over $\mathbb{C}[F]$. In the previous chapter we have seen that

$$
\begin{equation*}
\operatorname{rk}_{G}(A)=\lim _{k \rightarrow \infty} \frac{1}{\left|X_{k}\right|} \operatorname{rk}_{\mathbb{C}}\left(\varphi_{k}(A)\right) \tag{3.2}
\end{equation*}
$$

This equation is usually called the sofic Lück approximation. Note that from this equation it follows that we can change the sofic approximation of $G$ without changing the limit. This will be a key ingredient to our proof of Conjecture 2. The proof will work in two steps. In the first one we will assume that all coefficients that appear in $A$ and $\tilde{\sigma}(A)$ are algebraic over $\mathbb{Q}$. In the second step we will then extend our results to the general case.

### 3.1.1 The algebraic case

In this section we want to prove the following theorem.
Theorem 3.1.2. Let $G$ be a sofic group, $\sigma: G \rightarrow \mathrm{GL}_{k}(\overline{\mathbb{Q}})$ be a representation and $A \in \operatorname{Mat}_{n}(\overline{\mathbb{Q}}[G])$. Then

$$
\operatorname{rk}_{G}(A)=\frac{1}{k} \operatorname{rk}_{G}(\tilde{\sigma}(A))
$$

We will first proof some auxiliary results. As mentioned before the sofic Lück approximation will play a crucial role in the proof. So let $F$ be a finitely generated free group and $X$ a finite $F$-set. For $x \in X$ denote by $F_{x}$ the stabilizer of $x$ in $F$. Let $\mathbb{F}$ be any field and $V$ be right $\mathbb{F}[F]$ module of dimension $k$ over $\mathbb{F}$. As in the beginning of this chapter we want to define two different $\mathbb{F}[F]$ module structures on $V \otimes_{\mathbb{F}} \mathbb{F}[X]$. Let $v \in V, x \in X$ and $f \in F$. First define

$$
(v \otimes y) \cdot_{1} f=v \otimes y f
$$

and second

$$
(v \otimes y) \cdot 2 f=v f \otimes y f
$$

Denote the corresponding $\mathbb{F}[F]$-modules by $\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{1}$ and $\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{2}$ respectively. We have the following lemma.

Lemma 3.1.3. Assume that for any $x \in X, v \in V$ and $f \in F_{x}$ we have $v f=v$. Then

$$
\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{2} \cong\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{1} \cong \mathbb{F}[X]^{k} \quad \text { as } \mathbb{F}[F] \text {-modules. }
$$

Proof. Assume first that the action of $F$ on $X$ is transitive. So fix $y \in X$ such that $X=y \cdot F$. For each $x \in X$ choose a $f_{x}$ such that $x=y f_{x}$ and define the map

$$
\alpha:\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{1} \rightarrow\left(V \otimes_{\mathbb{F}} \mathbb{F}[X]\right)_{2} \quad \text { by } \quad \alpha\left(v \otimes y f_{x}\right)=v f_{x} \otimes y f_{x}
$$

for any $v \in V, x=y f_{x} \in X$. Let us first show that this map is independent of the choice of $f_{x}$. Assume that $x=y f=y g$. Then $f g^{-1} \in F_{y}$ and therefore $\left(v f g^{-1}\right) g \otimes y g=v f \otimes y f$. So let us show that $\alpha$ interchanges with the $F$-action. For $g \in F$ we have
$\alpha\left(\left(v \otimes y f_{x}\right) \cdot{ }_{1} g\right)=\alpha\left(v \otimes y f_{x} g\right)=v f_{x} g \otimes y f_{x} g=\left(v f_{x} \otimes y f_{x}\right) \cdot{ }_{2} g=\alpha\left(\left(v \otimes y f_{x}\right)\right) \cdot{ }_{2} g$.
Since the linear extension of $\alpha$ is clearly bijective, $\alpha$ is a $\mathbb{F}[F]$-module isomorphism.

If the action of $F$ on $X$ is not transitive, let $X_{1}, \ldots, X_{l}$ be the different orbits of the action. We then get

$$
\mathbb{F}[X] \cong \mathbb{F}\left[X_{1}\right] \times \ldots \times \mathbb{F}\left[X_{l}\right]
$$

and the construction above yields an isomorphism on each factor.

Later we want to apply the previous result when the finite $F$ set $X$ belongs to a sofic approximation of a group $G$.

Next, we want to introduce so called $S$-integers. For that let $K$ be a finite extension of $\mathbb{Q}$ and denote by $\mathcal{O}_{K}$ its ring of integers. Note that $\mathcal{O}_{K}$ is a dedekind domain, therefore every prime ideal $P$ of $\mathcal{O}_{K}$ is maximal. Let us fix a prime ideal $P$ in $\mathcal{O}_{K}$. We define a map

$$
v_{P}: \mathcal{O}_{K} \rightarrow \mathbb{N}_{0} \cup\{\infty\}, a \mapsto \max \left\{n \in \mathbb{N} \mid a \in P^{n}\right\}
$$

for $a \neq 0$ and $v_{P}(0)=\infty$. Since every element $x \in K$ can be written as $x=\frac{a}{b}, a, b \in \mathcal{O}_{K}$ we can extend $v_{P}$ to a map

$$
v_{P}: K \rightarrow Z \cup\{\infty\} \quad \text { by } \quad v_{P}\left(\frac{a}{b}\right)=v(a)-v(b)
$$

It is easy to see that $v_{P}$ restricted to $K^{*}$ is a group homomorphism. Further we have for all $a, b \in K$ :

$$
v(a+b) \geq \min \{v(a), v(b)\}
$$

The function $v_{P}$ is called the valuations associated to $P$. Let now $S \subseteq \operatorname{Spec}\left(\mathcal{O}_{K}\right)$ be a set of prime ideals of $\mathcal{O}_{K}$. The ring $\mathcal{O}_{K, S}$ of $S$-integers of $K$ is defined by

$$
\mathcal{O}_{K, S}=\left\{x \in K \mid v_{P}(x) \geq 0 \text { for all } P \in \operatorname{Spec}(K) \backslash S\right\}
$$

We are now ready to proof Theorem 3.1.2.

Proof of 3.1.2. Since $G$ is finitely generated we can find a finite extension $K$ of $\mathbb{Q}$, such that $\sigma(G) \subseteq \operatorname{Mat}_{m}(K)$. Let $\mathcal{O}_{K}$ be the ring of integers of $K$. Since each image of $\sigma(g)$ of a generator $g$ of $G$ is of the form $\frac{a}{b}$ with $a, b \in \mathcal{O}_{K}$ we can choose a finite set $S \subseteq \operatorname{Spec}\left(\mathcal{O}_{K}\right)$ such that $\sigma(G) \leq \operatorname{GL}_{k}\left(\mathcal{O}_{K, S}\right)$ and $A \in$ $\operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K, S}[G]\right)$. Since multiplying by a nonzero constant does not change the rank of a matrix, we can assume that $A$ and $\tilde{\sigma}(A)$ have entries in $\mathcal{O}_{K}[F]$. Let now $\left\{X_{i}\right\}$ be a sofic approximation for $G$. By the Lück approximation theorem we know that

$$
\begin{equation*}
\operatorname{rk}_{G}(A)=\lim _{i \rightarrow \infty} \frac{1}{\left|X_{i}\right|} \mathrm{rk}_{\mathbb{C}}\left(f_{X_{i}}(A)\right) \tag{3.3}
\end{equation*}
$$

Fix an infinite collection $\left\{P_{i} \mid i \in \mathbb{N}\right\}$ of maximal ideals in $\mathcal{O}_{K, S}$ and put $\mathbb{F}_{i}=\mathcal{O}_{K, S} / P_{i}$. Obviously we have

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left|\mathbb{F}_{i}\right| \rightarrow \infty \tag{3.4}
\end{equation*}
$$

Let $\sigma_{i}: F \rightarrow \mathrm{GL}_{k}\left(\mathbb{F}_{i}\right)$ be the composition of the quotient map $F \rightarrow F / N=G$, the map $\sigma: G \rightarrow \operatorname{GL}_{k}\left(\mathcal{O}_{K}\left[\frac{1}{c}\right]\right)$ and the reduction map $\mathcal{O}_{K, S} \rightarrow \mathbb{F}_{i}$. Now, put $N_{i}=\operatorname{ker} \sigma_{i}$ and consider the family of finite $F$-sets given by $Y_{i}=X_{i} \times F / N_{i}$.
Claim. The family $\left\{Y_{i} \mid i \in \mathbb{N}\right\}$ is a sofic approximation for $G$.
Proof. Observe that $N \subseteq N_{i}$ for every $N$. Therefore, if $w \in N$, we have $\operatorname{Fix}_{Y_{i}}(w)=\operatorname{Fix}_{X_{i}}(w) \times F / N_{i}$. On the other hand, when $w \notin N$, then $\operatorname{Fix}_{Y_{i}}(w) \subseteq$ $\operatorname{Fix}_{X_{i}}(w) \times F / N_{i}$. Since $\left\{X_{i}\right\}$ is a sofic approximation for $G$ so is $\left\{Y_{i}\right\}$.

Via the $\operatorname{map} \sigma_{i}$ we can, as in 3.1.3, define the two $\mathbb{F}_{i}[F]$-modules $\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]\right)_{1}$ and $\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]\right)_{2}$ for each $i \in \mathbb{N}$. Furthermore, by Lemma 3.1.3, these two modules are actually isomorphic. Now, consider the finite dimensional $\mathbb{F}_{i}$ vector space $\mathbb{F}_{i}\left[Y_{i}\right] \cong \mathbb{F}_{i}^{\left|Y_{i}\right|}$. Reduction modulo $P_{i}$ gives an action of $\mathcal{O}_{K, S}[F]$ on $\mathbb{F}_{i}\left[Y_{i}\right]$. Let $\rho_{Y_{i}, P_{i}}: \mathcal{O}_{K}\left[\frac{1}{c}\right][F] \rightarrow \operatorname{Mat}_{\left|Y_{i}\right|}\left(\mathbb{F}_{i}\right)$ be the representation associated to this action. We define a Sylvester matrix rank function $\mathrm{rk}_{Y_{i}, P_{i}}$ on $\mathcal{O}_{K}\left[\frac{1}{c}\right][F]$ by

$$
\operatorname{rk}_{Y_{i}, P_{i}}(B)=\frac{\operatorname{rk}_{\mathbb{F}_{i}}\left(\rho_{Y_{i}, P_{i}}(B)\right)}{\left|Y_{i}\right|}
$$

for any matrix $B$ over $\mathcal{O}_{K, S}[F]$.
Claim. For any matrix $B$ over $\mathcal{O}_{K, S}[F]$ we have

$$
k \cdot r k_{Y_{i}, P_{i}}(B)=r k_{Y_{i}, P_{i}}(\tilde{\sigma}(B))
$$

Proof. By identifying $\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]$ with $\mathbb{F}_{i}\left[Y_{i}\right]^{k}$ we obtain

$$
k \cdot \operatorname{rk}_{Y_{i}, P_{i}}(B)=\frac{1}{\left|Y_{i}\right|}\left(\operatorname{dim}_{\mathbb{F}_{i}}\left(\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]^{m}\right)_{1} /\left(\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]^{n}\right)_{1} \cdot 1 B\right)\right)\right)
$$

and

$$
\operatorname{rk}_{Y_{i}, P_{i}}(\tilde{\sigma}(B))=\frac{1}{\left|Y_{i}\right|}\left(\operatorname{dim}_{\mathbb{F}_{i}}\left(\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]^{m}\right)_{2} /\left(\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]^{n}\right)_{2} \cdot{ }_{2} B\right)\right)\right)
$$

Since $\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]\right)_{1}$ and $\left(\mathbb{F}_{i}^{k} \otimes_{\mathbb{F}_{i}} \mathbb{F}_{i}\left[Y_{i}\right]\right)_{2}$ are isomorphic as $\mathbb{F}_{i}[F]$ modules, the result follows.

Let us compare $\operatorname{rk}_{Y_{i}}(B)=\frac{1}{\left|Y_{i}\right|} \mathrm{rk}_{K} f_{Y_{i}}(B)$ and $\operatorname{rk}_{Y_{i}, P_{i}}(B)$.
Claim. Let $B \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K, S}[F]\right)$. Then there exists a constant $C$ depending only on B such that

$$
\left|r k_{Y_{i}}(B)-r k_{Y_{i}, P_{i}}(B)\right| \leq \frac{C}{\log _{2}\left|\mathbb{F}_{i}\right|}
$$

Proof. We want to use the same notation as introduced in Section 2.5.3. So for any element $\alpha \in K$ let $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{C}$ be the roots of the minimal polynomial of $\alpha$ over $\mathbb{Q}$. For any element $b=\sum_{h \in F} a_{h} h\left(a_{h} \in K\right)$ of the group algebra $K[F]$ we put

$$
\lceil b\rceil=\sum_{h \in F}\left\lceil a_{h}\right\rceil
$$

We also define

$$
\lceil B\rceil=\max _{j} \sum_{i}\left\lceil b_{i j}\right\rceil
$$

Since multiplication by a constant does not change the rank of $B$, we can assume that $B \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K}[F]\right)$ For each $i \in \mathbb{N}$ set $M_{i}=\mathcal{O}_{K}\left[Y_{i}\right]^{m} / \mathcal{O}_{K}\left[Y_{i}\right]^{n} f_{Y_{i}}(B)$. By the structure theorem of finitely generated modules over Dedekind domains we have

$$
M_{i} \cong M_{i} / M_{i}^{\text {tors }} \oplus M_{i}^{\text {tors }}
$$

Here $M_{i}^{\text {tors }}$ is the torsion part of the $\mathcal{O}_{K}$ module $M_{i}$. Recall that

$$
\operatorname{rk}_{Y_{i}}(B)=m-\frac{\operatorname{dim}_{K}\left(K \otimes_{\mathcal{O}_{K}} M_{i}\right)}{\left|Y_{i}\right|}
$$

and

$$
\mathrm{rk}_{Y_{i}, P_{i}}(B)=m-\frac{\operatorname{dim}_{\mathbb{F}_{i}}\left(\mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}\right)}{\left|Y_{i}\right|}
$$

Note further that

$$
\left(K \otimes_{\mathcal{O}_{K}} M_{i}\right)=\left(K \otimes \mathcal{O}_{K} M_{i} / M_{i}^{\text {tors }}\right) \oplus\left(K \otimes_{\mathcal{O}_{K}} M_{i}^{\text {tors }}\right)=K^{\left|Y_{i}\right|\left(m-\mathrm{rk}_{Y_{i}}(B)\right)}
$$

and

$$
\begin{aligned}
\left(\mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}\right)=\left(\mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i} / M_{i}^{\mathrm{tors}}\right) \oplus\left(\mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}^{\text {tors }}\right)= \\
\mathbb{F}_{i}^{\left|Y_{i}\right|\left(m-\mathrm{rk}_{Y_{i}}(B)\right)} \oplus \mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}^{\text {tors }}
\end{aligned}
$$

All together we get

$$
\begin{gathered}
\left|\mathrm{rk}_{Y_{i}}(B)-\mathrm{rk}_{Y_{i}, P_{i}}(B)\right|=\left|\left(m-\frac{\operatorname{dim}_{K}\left(K \otimes \mathcal{O}_{K} M_{i}\right)}{\left|Y_{i}\right|}\right)-\left(m-\frac{\operatorname{dim}_{\mathbb{F}_{i}}\left(\mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}\right)}{\left|Y_{i}\right|}\right)\right| \\
=\left|\frac{\operatorname{dim}_{\mathbb{F}_{i}}\left(\mathbb{F}_{i}^{\left|Y_{i}\right|\left(m-\mathrm{rk}_{Y_{i}}(B)\right)} \oplus \mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}^{\text {tors }}\right)}{\left|Y_{i}\right|}-\frac{\operatorname{dim}_{K} K^{\left|Y_{i}\right|\left(m-\mathrm{rk}_{Y_{i}}(B)\right)} \mid}{\left|Y_{i}\right|}\right| \\
=\frac{\operatorname{dim}_{\mathbb{F}_{i}} \mathbb{F}_{i} \otimes_{\mathcal{O}_{K}} M_{i}^{\text {tors }}}{\left|Y_{i}\right|} \leq \log _{\left|\mathbb{F}_{i}\right|}\left|M_{i}^{\text {tors }}\right|=\frac{\log _{2}\left|M_{i}^{\text {tors }}\right|}{\left|Y_{i}\right| \log _{2}\left|\mathbb{F}_{i}\right|}
\end{gathered}
$$

By 2.5.14 and 2.5.15 we have

$$
\log _{2}\left|M_{i}^{\text {tors }}\right| \leq m\left|Y_{i}\right||K: \mathbb{Q}| \log _{2}\lceil B\rceil
$$

Thus, with $C=m|K: \mathbb{Q}| \log _{2}\lceil B\rceil$ we obtain

$$
\left|\mathrm{rk}_{Y_{i}}(B)-\mathrm{rk}_{Y_{i}, P_{i}}(B)\right| \leq \frac{C}{\log _{2}\left|\mathbb{F}_{i}\right|}
$$

We can now finish the proof of 3.1.2. Let $\epsilon>0$ and let $B \in \operatorname{Mat}_{n \times m}\left(\mathcal{O}_{K, S}[F]\right.$ that maps onto $A$. Since $\left\{Y_{i}\right\}$ is a sofic approximation for $G$, we can choose a $j_{1} \in \mathbb{N}$ such that for all $i \geq j_{1}$ we have

$$
\begin{equation*}
\left|\mathrm{rk}_{G}(B)-\operatorname{rk}_{Y_{i}}(B)\right| \leq \frac{\epsilon}{4 k} \quad \text { and } \quad\left|\mathrm{rk}_{G}(\tilde{\sigma}(B))-\operatorname{rk}_{Y_{i}}(\tilde{\sigma}(B))\right| \leq \frac{\epsilon}{4} \tag{3.5}
\end{equation*}
$$

By 3.4 and 3.1.1 we can choose an $j_{2} \in \mathbb{N}$ such that for all $i \geq j_{2}$ we have

$$
\begin{equation*}
\left|\operatorname{rk}_{Y_{i}}(B)-\operatorname{rk}_{Y_{i}, P_{i}}(B)\right| \leq \frac{\epsilon}{4 k} \quad \text { and } \quad\left|\operatorname{rk}_{Y_{i}}(\tilde{\sigma}(B))-\operatorname{rk}_{Y_{i}, P_{i}}(\tilde{\sigma}(B))\right| \leq \frac{\epsilon}{4} \tag{3.6}
\end{equation*}
$$

Taking everything together we obtain for every $i \geq j_{1}, j_{2}$ :

$$
\begin{aligned}
& \left|\mathrm{rk}_{G}(\tilde{\sigma}(B))-k \cdot \mathrm{rk}_{G}(B)\right| \leq\left|\mathrm{rk}_{G}(\tilde{\sigma}(B))-\mathrm{rk}_{Y_{i}}(\tilde{\sigma}(B))\right|+ \\
& \quad\left|\operatorname{rk}_{Y_{i}}(\tilde{\sigma}(B))-\mathrm{rk}_{Y_{i}, P_{i}}(\tilde{\sigma}(B))\right|+\left|\mathrm{rk}_{Y_{i}, P_{i}}(\tilde{\sigma}(B))-k \cdot \mathrm{rk}_{Y_{i}, P_{i}}(B)\right|+ \\
& \quad k \cdot\left|\mathrm{rk}_{Y_{i}, P_{i}}(B)-\mathrm{rk}_{Y_{i}}(B)\right|+k \cdot\left|\mathrm{rk}_{Y_{i}}(B)-\mathrm{rk}_{G}(B)\right| \leq \epsilon
\end{aligned}
$$

Since $\epsilon$ was arbitrary this finishes the proof.

### 3.1.2 The general case

In this section we want to show that Conjecture 2 holds for sofic groups. The proof of the general case uses again the sofic Lück approximation as well as the already proven algebraic case. The idea is to approximate all transcendental coefficients that appear in a given matrix $A \in \operatorname{Mat}_{n \times m}(\mathbb{C}[G])$ by algebraic numbers. However we can not just choose any algebraic numbers for this approximation. The numbers need to fulfill some algebraic properties. We first need an auxiliary result.

## Density of algebraic points in a variety

In this section we want to prove the following theorem.
Theorem 3.1.4. Let $I$ be an ideal of $\overline{\mathbb{Q}}\left[x_{1}, \ldots, x_{n}\right]$ and let

$$
V(I)=\left\{x \in \mathbb{C}^{n} \mid f(x)=0 \text { for all } f \in I\right\}
$$

Then $V(I) \cap \overline{\mathbb{Q}}^{n}$ is dense in $V(I)$ with respect to the euclidean topology.
For the proof we need some terminology. We will call a field $\mathbb{K}$ real, if there is a total ordering $\leq$ on $\mathbb{K}$ such that for all $x, z, y \in \mathbb{K}$ :

- $x \leq y \Rightarrow x+y \leq y+z$,
- $0 \leq x, 0 \leq y \Rightarrow 0 \leq x y$.

We call a field $\mathbb{K}$ real closed, if $\mathbb{K}$ is real and $\mathbb{K}[\sqrt{-1}]=\overline{\mathbb{K}}$. Every subfield of $\mathbb{R}$ is real. Examples for real closed fields are $\mathbb{R}$ and $\mathbb{R}_{\text {alg }}=\mathbb{R} \cap \overline{\mathbb{Q}}$.

By a boolean combination of polynomial equations and inequalities over a field $\mathbb{K}$ we mean a finite combination of conjunction, disjunction and negation of polynomial equations and inequalities. Since we only consider a finite number of polynomials we can assume that all ocurring polynomials are in $\mathbb{K}\left[X_{1}, \ldots, X_{n}\right]$ for one fixed $n \in \mathbb{N}$.

Let now $\mathbb{K}$ be a real closed field and $a \in \mathbb{K}$. We define

$$
\begin{array}{lll}
\operatorname{sign}(a)=-1 & \text { if } & a<0 \\
\operatorname{sign}(a)=0 & \text { if } & a=0 \\
\operatorname{sign}(a)=1 & \text { if } & a>0
\end{array}
$$

We are now ready to formulate the Tarski-Seidenberg Principle which can be found in [BCR13, Theorem 1.4.2].
Theorem 3.1.5 (Tarski-Seidenberg Principle). Let $\mathbb{F}$ be a real field (eg. $\mathbb{Q}$ ) and $f_{1}(X, Y), \ldots, f_{m}(X, Y)$ be a sequence of polynomials in $n+1$ variables over $\mathbb{F}$, where $Y=\left(Y_{1}, \ldots, Y_{n}\right)$. Let $\epsilon$ be a function from $\{1, \ldots, m\}$ to $\{-1,0,1\}$. Then there exists a boolean combination $\mathcal{B}(Y)$ of polynomial equations and inequalities in the variables $Y$ and coefficients in $\mathbb{F}$ such that for every real closed field $\mathbb{K}$ containing $\mathbb{F}$ and every $y \in \mathbb{K}^{n}$ the system

$$
\operatorname{sign}\left(f_{i}(X, y)\right)=\epsilon(i), i \in\{1, \ldots, m\}
$$

has a solution $x \in \mathbb{K}$ if and only if $\mathcal{B}(y)$ holds true in $\mathbb{K}$.
Definition 3.1.6. Let $\mathbb{K}$ be a real closed field. A semi-algebraic subset of $\mathbb{K}^{n}$ is a set of the form

$$
\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_{i}}\left\{x \in \mathbb{K}^{n} \mid f_{i, j}(x) *_{i, j} 0\right\}
$$

where $*_{i, j} \in\{=,<\}$ and $f_{i, j} \in \mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$.

Obviously all algebraic sets are semi-algebraic. Further note that open and closed balls in $\mathbb{K}^{n}$ are semi-algebraic. It is easy to see that finite unions and intersections of semi-algebraic sets are again semi-algebraic. Note further that every semi-algebraic set can be described by a boolean combination of polynomial equations and inequalities. The following lemma is also obvious.

Lemma 3.1.7. Let $A \subseteq \mathbb{C}^{n}$ be an algebraic set. Then the image of $A$ under the identification $\mathbb{C} \cong \mathbb{R}^{\overline{2} n}$ is a semialgebraic set.

Proposition 3.1.8. [BCR13, Theorem 4.1.1] Let $\mathbb{K}, \mathbb{K}_{1}$ be real closed fields with $\mathbb{K} \subseteq \mathbb{K}_{1}$. Let $\mathcal{B}(X), X=\left(X_{1}, \ldots, X_{n}\right)$ be a boolean combination of polynomial equations and inequalities over $\mathbb{K}$. If $\mathcal{B}(x)$ holds true for some $x \in \mathbb{K}_{1}^{n}$, then $\mathcal{B}(y)$ holds true for some $y \in \mathbb{K}^{n}$.

Proof. We proceed by induction on $n$. For $n=0$ there is nothing to show. So let $n \geq 1$ and assume that the statement holds for $n-1$. By the Tarski-Seidenberg Principle there exists a boolean combination $\mathcal{C}\left(X_{1}, \ldots, X_{n-1}\right)$ with coefficients in $\mathbb{K}$ such that for every real closed field $\mathbb{K}_{2}$ containing $\mathbb{K}$ and every $x=\left(x_{1}, \ldots, x_{n-1}\right) \in \mathbb{K}_{2}$, the system $\mathcal{B}\left(x, X_{n}\right)$ has a solution $x_{n} \in \mathbb{K}_{2}$, if and only if $\mathcal{C}(x)$ holds true. Since we have a solution for $\mathcal{B}$ over $\mathbb{K}_{1}$, by our induction hypothesis there is a $y \in \mathbb{K}^{n-1}$ such that $\mathcal{C}(y)$ holds true. Then again by the Tarski-Seidenberg principle there is a $y_{n} \in \mathbb{K}$ such that $\left(y, y_{n}\right) \in \mathbb{K}^{n}$ is a solution for $\mathcal{B}(X)$.

With this proposition we are ready to prove the main theorem of this section.
Proof of Theorem 3.1.4. We identify $\overline{\mathbb{Q}}$ and $\mathbb{C}$ with $\mathbb{R}_{\text {alg }}^{2}$ and $\mathbb{R}^{2}$, respectively, in the usual way $\left(\overline{\mathbb{Q}}=\mathbb{R}_{\mathrm{alg}}+\mathbb{R}_{\mathrm{alg}} i, \mathbb{C}=\mathbb{R}+\mathbb{R} i\right)$. Then $V(I)$ becomes a set of zeros in $\mathbb{R}^{2 n}$ of some polynomials $g_{1}, \ldots, g_{m}$ over $\mathbb{R}_{\text {alg }}$ in $2 n$-variables $x_{1}, \ldots, x_{2 n}$.

Let $y=\left(y_{1}, \ldots, y_{2 n}\right) \in V(I)$ and $\epsilon>0$. Consider

$$
\mathcal{B}=\left(g_{1}=0\right) \wedge \ldots \wedge\left(g_{m}=0\right) \wedge\left(\sum_{i=1}^{2 n}\left(x_{i}-y_{i}\right)^{2} \leq \epsilon^{2}\right)
$$

Then $\mathcal{B}(y)$ holds. By Proposition 3.1.8, $\mathcal{B}(z)$ holds for some $z \in \mathbb{R}_{\text {alg }}^{2 n}$. Therefore,

$$
\left(V(I) \cap O_{\epsilon}(y)\right) \cap \overline{\mathbb{Q}}^{n} \neq \emptyset
$$

and so, $\overline{\mathbb{Q}}^{n} \cap V(I)$ is dense in $V(I)$.

## Proof of Theorem B

We are now ready to finish the proof.
Proof of Theorem B. Since the matrix $A$ has only finitely many entries we can assume that $G$ is finitely generated. Thus, we may choose $t=\left(t_{1}, \ldots, t_{l}\right) \in$ $\mathbb{C}^{l} \backslash \mathbb{Q}^{l}$, such that with $R=\overline{\mathbb{Q}}\left[t_{1}, \ldots, t_{l}\right]$ the matrices $A$ and $\tilde{\sigma}(A)$ are matrices
over $R[G]$. Let $I$ be the kernel of the homomorphism $\overline{\mathbb{Q}}\left[x_{1}, \ldots, x_{l}\right] \rightarrow R$ that maps $x_{i} \mapsto t_{i}$. Thus we have $R \cong \overline{\mathbb{Q}}\left[x_{1}, \ldots, x_{l}\right] / I$. For any matrix $C$ over $R[G]$ and any point $p=\left(p_{1}, \ldots, p_{l}\right) \in V(I)$ let $C(P)$ be the image of the matrix $C$ when sending $t_{i}$ to $p_{i}$. Thus we have $A(t)=A$ and $\tilde{\sigma}(A(t))=\tilde{\sigma}(A)$. Note further that we have $\tilde{\sigma}(A)(p)=\tilde{\sigma}(A(p))$
Claim. Let $\left(s_{i}\right)$ be a sequence of points in $V(I)$ such that $\lim _{i \rightarrow \infty} s_{i}=t$. Then for any matrix $C \in \operatorname{Mat}_{n \times m}(R[G])$ we have

$$
r k_{G}(C) \leq \liminf _{i \rightarrow \infty} r k_{G}\left(C\left(s_{i}\right)\right)
$$

Proof. We want to use the notion of traces and their approximation. Note that for each $i \in N$ we get a maps $f_{i}: R[G] \rightarrow \bar{C}[G]$ defined by sending $t \rightarrow s_{i}$. Thus we can define the traces $\tau_{i}=\tau_{G} \circ f_{i}$ on $R[G]$ Further, since $R[G] \subseteq \mathbb{C}[G]$ we also have the trace $\tau_{G}$ on $R[G]$. Remember that for any matrix $C \in \operatorname{Mat}_{n}(\mathbb{C})$ the trace $\tau_{G}(C)$ is defined by

$$
\tau_{G}(C)=\sum_{i=1}^{n}\left\langle 1_{i} C, 1_{i}\right\rangle
$$

where $1_{i} \in \ell^{2}(G)^{n}$ is the vector having $1_{G}$ in the $i$-th coordinate and zeros elsewhere. Since $\lim _{i \rightarrow \infty} s_{i}=t$ we get

$$
\lim \tau_{i}(C)=\tau_{G}(C)
$$

Consider the matrix $T=C C^{*}$ and let $\rho_{i}: \operatorname{Mat}_{n}(R[G]) \rightarrow \mathcal{B}\left(\mathcal{H}_{\tau_{i}}^{n}\right)=\operatorname{Mat}_{n}\left(\mathcal{B}\left(\mathcal{H}_{\tau_{i}}\right)\right)$ be the right representation coming from the GNS-representation with respect to $\tau_{i}$ and similarly $\rho: R[G] \rightarrow \mathcal{B}\left(\ell^{2}(G)\right)$ with respect to $\tau_{G}$. Let $\mu_{i}$ be the spectral measure associated to the operator $\rho_{i}(T)$ and $\mu$ be the spectral measure associated to the operator $\rho(T)$. By 2.2.6 we get that the measures $\mu_{i}$ converge weakly towards the measure $\mu$. Thus, by the theorem of Portmanteau, we have

$$
\limsup _{i \rightarrow \infty} \mu_{i}(\{0\}) \leq \mu(\{0\})
$$

However, we have

$$
\mu(\{0\})=\operatorname{dim}_{G} \operatorname{ker} \rho(T)=\operatorname{dim}_{G} \operatorname{ker} \rho(C)=n-\operatorname{rk}_{G}(C)
$$

and

$$
\mu_{i}(\{0\})=\operatorname{dim}_{G} \operatorname{ker} \rho_{i}(T)=\operatorname{dim}_{G} \operatorname{ker} \rho_{i}(C)=n-\operatorname{rk}_{G}\left(C\left(s_{i}\right)\right)
$$

From these equations the result follows.
We still have to show the other inequality.
Claim. Let $s \in V(I)$. Then, for any matrix $C$ over $R[G]$ we have

$$
r k_{G}(C(s)) \leq r k_{G}(C)
$$

Proof. Here we want to use the sofic Lück approximation. So let $F$ be a finitely generated free group and $N \unlhd F$ such that $G=F / N$. Let $\left\{X_{i}\right\}$ be a sofic approximation for $G$ and let $B$ be a matrix over $R[F]$ that maps onto $C$. Note that $B(s)$ maps onto $C(s)$. Let $f_{X_{i}}: \mathbb{F} \rightarrow \operatorname{Mat}_{\left|X_{i}\right|}(\mathbb{Z})$ be the map that represents the action of $F$ on $\mathbb{C}\left[X_{i}\right]=\mathbb{C}^{\left|X_{i}\right|}$. Since for each $i$ the matrix $f_{X_{i}}(B(s))$ is an image of the matrix $f_{X_{i}}(B)$, we obtain

$$
\operatorname{rk}_{\mathbb{C}}\left(f_{X_{i}}(B(s))\right) \leq \operatorname{rk}_{\mathbb{C}}\left(f_{X_{i}}(B)\right)
$$

for each $i$. Thus, with $\mathrm{rk}_{X_{i}}=\frac{\mathrm{rk}_{\mathrm{C}} \circ f_{X_{i}}}{\left|X_{i}\right|}$ we obtain

$$
\begin{aligned}
& \operatorname{rk}_{G}(C)=\operatorname{rk}_{G}(B)=\lim _{i \rightarrow \infty} \operatorname{rk}_{X_{i}}(B) \geq \\
& \qquad \lim _{i \rightarrow \infty} \operatorname{rk}_{X_{i}}(B(s))=\operatorname{rk}_{G}(B(s))=\operatorname{rk}_{G}(C(s))
\end{aligned}
$$

By 3.1.4 we can choose a sequence $\left(s_{i}\right)$ of points in $V(I) \cap \overline{\mathbb{Q}}^{l}$ such that $\lim _{i \rightarrow \infty} s_{i}=t$. From the previous two claims we deduce that

$$
\lim _{i \rightarrow \infty} \operatorname{rk}_{G}\left(A\left(s_{i}\right)\right)=\operatorname{rk}_{G}(A) \quad \text { and } \quad \lim _{i \rightarrow \infty} \operatorname{rk}_{G}\left(\tilde{\sigma}(A)\left(s_{i}\right)\right)=\operatorname{rk}_{G}(\tilde{\sigma}(A))
$$

By 3.1.2 we have $k \cdot \operatorname{rk}_{G}\left(A\left(s_{i}\right)\right)=\operatorname{rk}_{G}\left(\tilde{\sigma}(A)\left(s_{i}\right)\right)$ for each $i \in \mathbb{N}$. Thus we obtain

$$
\operatorname{rk}_{G}(\tilde{\sigma}(A))=\lim _{i \rightarrow \infty} \operatorname{rk}_{G}\left(\tilde{\sigma}(A)\left(s_{i}\right)\right)=\lim _{i \rightarrow \infty} k \cdot \operatorname{rk}_{G}\left(A\left(s_{i}\right)\right)=k \cdot \operatorname{rk}_{G}(A)
$$

### 3.2 Twisted rank functions and fibrations

The von Neumann rank function $\mathrm{rk}_{G}$ of a group $G$ is especially interesting when $G=\pi_{1}(X)$ is the fundamental group of some finite CW-complex $X$. For a CWcomplex $X$ the $n$-th Betti number is the dimension of the $n$-th homology group with coefficients in $\mathbb{C}$. The universal covering $\tilde{X}$ of $X$ has the structure of a $G$ -CW-complex, that means we have an action of $G$ on $\tilde{X}$ that permutes the open $n$-cells. The $n$-th $\ell^{2}$-Betti number $b_{n}^{(2)}(X)$ of a $G$-CW complex $X$ is defined as the von Neumann dimension $\operatorname{dim}_{G}$ of the $n$-th homology of $\ell^{2}(G) \otimes_{\mathbb{Z}[G]} C(X, \mathbb{C})$, the cellular chain complex of $X$ with local coefficients in $\ell^{2}(G)$.

Given a fibration

$$
F \rightarrow E \rightarrow B
$$

spectral sequences are a tool to calculate the homology of $E$, given the homology of $F$ and $B$. The situation is similar when we want to calculate the $\ell^{2}$ homology. We will see that in some cases during the calculation the twisted von Neumann rank function occurs. We will prove the following theorem.

Theorem 3.2.1. Let

$$
F \rightarrow E \rightarrow B
$$

be a fibration of connected $C W$-complexes of finite type and $d$ be a natural number. Suppose that the $n$-th $\ell^{2}$-Betti number of the universal covering of $B$ with respect to the action of the fundamental group $b_{n}^{(2)}(\tilde{B})$ vanishes for $n \leq d$. Assume further that $G=\pi_{1}(E)$ is sofic and the induced homomorphism $\pi_{1}(E) \rightarrow \pi_{1}(B)$ $i s$ an isomorphism. Then $b_{n}^{(2)}(\tilde{E})=0$ for $n \leq d$.

This theorem first appeared in [Lüc18]. Lück showed that Theorem B implies Theorem 3.2.1. In this section we will first define Betti numbers and $\ell^{2}$-Betti numbers. We will then give a short introduction to fibrations and spectral sequences. Lastly we will present a proof of Theorem 3.2.1

### 3.2.1 Betti numbers and $\ell^{2}$-Betti numbers

In this section we want to give the basic topological definitions and results that are necessary to talk about Betti numbers and $\ell^{2}$-Betti numbers. A great introduction to the topic is [Kam19]. In the following let $X$ be a finite CWcomplex. We denote by $X^{n}$ the $n$-skeleton of $X$ and by $e_{\alpha}^{n}$ the open $n$-cells, where $\alpha$ runs over some index set. Having a CW-complex $X$ we can consider the cellular chain complex $C_{*}(X, \mathbb{C})$ and its cellular homology

$$
H_{*}(X, \mathbb{C})=H\left(C_{*}(X, \mathbb{C})\right)
$$

Each $H_{n}(X, \mathbb{C})$ is a finite $\mathbb{C}$ vector space, thus it makes sense to define the $n$th Betti number of $X$ as

$$
b_{n}(X)=\operatorname{dim}_{\mathbb{C}} H_{n}(X, \mathbb{C})
$$

Let now $G$ be a countable discrete group that acts by homeomorphisms on $X$. We call the action cellular, if
(1) $G$ permutes the open $n$-cells that means $e_{\alpha}^{n}=g e_{\beta}^{n}$ for all $g \in G$ and
(2) if $g e_{\alpha}^{n} \cap e_{\alpha}^{n} \neq \emptyset$ then $g x=x$ for all $x \in e_{\alpha}^{n}$.

Definition 3.2.2. A $G$-CW-complex is a CW-complex $X$ with a cellular action of $G$ on $X . X$ is called

- finite type if it has finitely many equivariant $n$-cells for every $n$,
- finite if it has finitely many equivariant $n$-cells all together,
- proper if all stabilizer groups are finite,
- free if all stabilizer groups are trivial.

We now obtain the $\ell^{2}$-chain complex of $X$ by tensoring the original chain complex $C_{*}(X, \mathbb{C})$ with $\ell^{2}(G)$. Thus we have

$$
\begin{equation*}
C_{*}^{(2)}(X)=\ell^{2}(G) \otimes_{\mathbb{C}[G]} C_{*}(X, \mathbb{C}) \tag{3.7}
\end{equation*}
$$

We have the following theorem.

Theorem 3.2.3. [Kam19, Theorem 3.11] The $\ell^{2}$-chain complex defines a functor from proper finite type $G$ - $C W$-complexes to Hilbert $\mathcal{N}(G)$-modules.

Remark 3.2.4. Remember that with a Hilbert $\mathcal{N}(G)$ module $H \leq\left(\ell^{2}(G)\right)^{n}$ there comes a projection $p \in \operatorname{Mat}_{n}(\mathcal{N}(G))$ onto $H$. Thus we can also consider the projective $\mathcal{N}(G)$ module $M=\mathcal{N}(G)^{n} p$. This gives a bijection between Hilbert $\mathcal{N}(G)$ modules and projective $\mathcal{N}(G)$ modules. Thus, instead of working with $\ell^{2}(G) \otimes C_{*}(X, \mathbb{C})$ we could also set

$$
C_{*}^{(2)}(X)=\mathcal{N}(G) \otimes_{\mathbb{C}[G]} C_{*}(X, \mathbb{C})
$$

One can extend this argument also to finitely generated modules over $\mathcal{U}(G)$. For a detailed description see [Rei].

Now as we have the $\ell^{2}$-chain complex we can proceed as before to define $\ell^{2}$-Betti numbers. We denote by

$$
H_{*}^{(2)}(X)=H_{*}\left(C_{*}^{(2)}(X)\right)
$$

the $\ell^{2}$-homology of $X$ where

$$
H_{n}^{(2)}(X)=\operatorname{ker}\left(d_{n}\right) / \overline{\operatorname{imag}}\left(d_{n+1}\right)
$$

and $d_{n}: C_{n}^{(2)}(X) \rightarrow C_{n-1}^{(2)}(X)$ denote the chain maps. Note further that we have to take the closure on the image to ensure that the projection onto $\operatorname{imag}\left(d_{n+1}\right)$ exists in $\operatorname{Mat}_{n}(\mathcal{N}(G))$. We can now define the $n$-th $\ell^{2}$-Betti number as

$$
b_{n}^{(2)}=\operatorname{dim}_{G}\left(H_{n}^{(2)}(X)\right)
$$

### 3.2.2 Fibrations and Spectral Sequences

In this section we briefly want to recall the notions of fibrations and spectral sequences. For more details see [Kir]. In the following let $E$ and $B$ be topological spaces.

Definition 3.2.5. A continuous map $p: E \rightarrow B$ is called a fibration if it has the homotopy lifting property that means for every space $Y$ and commuting diagram

we can lift the homotopy $G$ to a homotopy $\tilde{G}: Y \times I \rightarrow E$ such that the diagram

commutes.

Obviously direct products of spaces give rise to fibration. Probably the most famous example of a fibration is the so called Hopf fibration $S^{1} \rightarrow S^{3} \xrightarrow{p} S^{2}$. Fibrations are interesting, because unlike arbitary continuous maps they give rise to a long exact sequence in homotopy. Let $p: E \rightarrow B$ be a fibration and let $B$ be path connected. Let $a, b \in B$ and let $\alpha: I \rightarrow B$ be a path from $a$ to $b$. Let $E_{a}=p^{-1}(a)$. Thus, the inclusion $E_{a} \subseteq E$ gives us a diagram

where $G(e, t)=\alpha(t)$ Note that $G(e, 1)$ is a map from $E_{a}$ to $E_{b}$. In fact one can show that the homotopy $\alpha_{*}=[G(e, 1)]$ depends only on the homotopy class of $\alpha$. Furthermore, one can show that $\alpha_{*}$ is a homotopy equivalence. We have the following theorem.

Theorem 3.2.6. [Kir, Theorem 6.12] Let $p: E \rightarrow B$ be a fibration and let $B$ be path connected. Then all fibers $E_{a}=p^{-1}(a)$ are homotopy equivalent. Moreover every path $\alpha: I \rightarrow B$ defines a homotopy class of of homotopy equivalences $E_{\alpha(0)} \rightarrow E_{\alpha(1)}$ which depends only on the homotopy class of $\alpha$ relative to its endpoints in such a way that multiplication of path corresponds to composition of homotopy equivalences. In particular there is a well defined group homomorphism

$$
\pi_{1}\left(B, b_{0}\right) \rightarrow\left\{\text { homotopy classes of self homotopy equivalences } E_{b_{0}} \rightarrow E_{b_{0}}\right\}
$$

As a direct consequence of this theorem, by applying the homology functor, we get an action of $\pi_{1}\left(B, b_{0}\right)$ on $H_{n}\left(E_{b_{0}}, M\right)$, for any coefficient group $M$.

We now want to introduce spectral sequences.
Definition 3.2.7. A homology spectral sequence is a sequence of bigraded chain complexes

$$
\left\{E_{p, q}^{r}, d^{r}\right\}_{r}
$$

where $(p, q) \in \mathbb{Z} \times \mathbb{Z}, r \in \mathbb{N}$. The differentials $d^{r}$ have bidegree $(-r, r-1)$ that means $d^{r}\left(E_{p, q}^{r}\right) \subseteq E_{p-r, q+r-1}^{r}$ and we have isomporhisms

$$
E_{p, q}^{r} \cong \frac{\operatorname{ker} d^{r}: E_{p, q}^{r} \rightarrow E_{p-r, q+r-1}^{r}}{\operatorname{imag} d^{r}: E_{p+r, q-r+1}^{r} \rightarrow E_{p, q}^{r}}
$$

Before we can use spectral sequences we have to talk about convergence of spectral sequences. Let us collect some definitions first.
Definition 3.2.8. Let $R$ be a ring.

- A graded $R$ module $A_{*}$ can be either thought of a collection of $R$-modules $\left\{A_{k}\right\}_{k \in \mathbb{Z}}$ or as a module $A=\bigoplus_{k \in \mathbb{Z}} A_{k}$. A homomorphism of a graded $R$ module is an element of $\prod_{k} \operatorname{Hom}\left(A_{k}, B_{k}\right)$.
- A filtration of an $R$-module $A$ is an increasing union

$$
0 \subseteq \ldots \subseteq F_{-1} \subseteq F_{0} \subseteq F_{1} \subseteq \cdots \subseteq F_{p} \subseteq \ldots \subseteq A
$$

of submodules. A filtration is convergent, if $\bigcup_{k} F_{k}=A$ and $\bigcap_{k} F_{k}=0$.

- Given a filtration $F=\left\{F_{k}\right\}$ of an $R$-module $A$, the associated graded module is given by $\operatorname{Gr}(A, F)_{*}$ with

$$
\operatorname{Gr}(A, F)_{k}=F_{k} / F_{k-1}
$$

We can now define what it means if a spectral sequence converges.
Definition 3.2.9. Let $\left\{E_{p, q}^{r}\right\}$ be a spectral sequence and let $A=A_{*}$ be a graded module. We say the spectral sequence converges to $A$ and write

$$
E_{p, q}^{2} \Rightarrow A_{p+q}
$$

if:

- For every $p, q$ there exists $r_{0}$ such that $d_{p, q}^{r}$ is the zero map for all $r>r_{0}$.
- There is a convergent filtration of $F$ of $A_{*}$ such that for each $n$ the limit $E_{p, n-p}^{\infty}=\underset{r \rightarrow \infty}{\operatorname{colim}} E_{p, n-p}^{r}$ is isomorphic to the associated graded module $\operatorname{Gr}\left(A_{*}, F\right)_{p}$.

We will only consider first quadrant spectral sequences that means $E_{p, q}^{r}=0$ if $p<0$ or $q<0$. In this case we actually get a stronger notion of convergence. Namely, for each pair $(p, q)$ there exists an $r_{0}$ such that for all $r>r_{0}$ we have $E_{p, q}^{r}=E_{p, q}^{\infty}$.

### 3.2.3 Twisted $\ell^{2}$-Betti numbers

In this section we want to prove Theorem 3.2.1. For that let us first introduce twisted $\ell^{2}$-Betti numbers. Remember that for a $G$-CW-complex $X$ we defined the $n$-th $\ell^{2}$-Betti number as

$$
b_{n}^{(2)}(X)=\operatorname{dim}_{G}\left(H_{n}^{(2)}(X)\right)
$$

where $H_{n}^{(2)}(X)$ is the $n$-th homology group of the $\ell^{2}$-chain complex

$$
\mathcal{N}(G) \otimes_{\mathbb{C}[G]} C_{*}(X, \mathbb{C})=C_{*}^{(2)}(X)
$$

For a finite dimensional representation $\sigma: G \rightarrow \mathrm{GL}_{m}(\mathbb{C})$ we can consider the twisted chain complex

$$
C_{*, \sigma}^{(2)}(X)=\left(\mathbb{C}^{m} \otimes_{\mathbb{C}} \mathcal{N}(G)\right) \otimes_{\mathbb{C}[G]} C_{*}(X, \mathbb{C})
$$

where the $G$-action on $\left(\mathbb{C}^{n} \otimes_{\mathbb{C}} \mathcal{N}(G)\right)$ is given by

$$
(v \otimes x)=v \sigma(g) \otimes x g
$$

Thus we define the $n$-th twisted $\ell^{2}$-Betti number of $X$ as

$$
b_{n, \sigma}^{(2)}(X)=\operatorname{dim}_{G} H_{n, \sigma}^{(2)}(X)=\operatorname{dim}_{G} H_{n}^{(2)}\left(C_{*, \sigma}^{(2)}(X)\right)
$$

The chain map $d_{n}: C_{n}^{(2)}(X) \rightarrow C_{n-1}^{(2)}(X)$ is given by multiplication by a matrix $A_{n}$ over $\mathbb{Z}[G]$. Thus the twisted chain map $d_{n, \sigma}: C_{n, \sigma}^{(2)}(X) \rightarrow C_{n-1, \sigma}^{(2)}(X)$ is given by multiplication by the matrix $\tilde{\sigma}(A)$ as in 3.1. Therefore, Theorem B implies that if $G$ is sofic we have

$$
\begin{equation*}
b_{n, \sigma}^{(2)}(X)=m \cdot b_{n}^{(2)}(X) \tag{3.8}
\end{equation*}
$$

Consider a fibration

$$
F \rightarrow E \xrightarrow{p} B
$$

of path connected CW-complexes of finite type. Let $G=\pi_{1}(B)$. Since a fibration gives rise to a long exact sequence in homotopy groups we have maps

$$
\pi_{1}(F) \rightarrow \pi_{1}(E) \rightarrow \pi_{1}(B)=G
$$

Thus on the $\operatorname{ring} \mathcal{N}(G)$ we can define a $\pi_{1}(F), \pi_{1}(E)$ and $\pi_{1}(B)=G$-module structure. For the spaces $F, E, B$ it makes sense to consider homology with local coefficients in $\mathcal{N}(G)$. Remember that by definition of local coefficients, one passes to the universal covers, since we need an action of the fundamental group. That means we consider the homology of the chain complex

$$
H(B ; \mathcal{N}(G))=\mathcal{N}(G) \otimes_{\pi_{1}(B)} H_{n}(C(\tilde{B}))
$$

and similarity for $F$ and $E$. By [Kir, Theorem 5.12] homology with local coefficients describes an ordinary homology theory. Therefore we have the following theorem.

Proposition 3.2.10. [Kir, Theorem 9.6] Let $F \rightarrow E \rightarrow B$ be a fibration of $C W$-complexes of finite type where $B$ is path connected. Let $\pi_{1}(B)=G$. Then there exists a spectral sequence

$$
H_{p}\left(B ; H_{q}(F ; \mathcal{N}(G))\right) \cong E_{p, q}^{2} \Rightarrow H_{p+q}(E ; \mathcal{N}(G))
$$

Let us explain what each term actually means. We have

$$
H_{q}(F ; \mathcal{N}(G))=H_{q}\left(\mathcal{N}(G) \otimes_{\pi_{1}(F)} C_{*}(\tilde{F})\right)
$$

With that we obtain

$$
H_{p}\left(B ; H_{q}(F ; \mathcal{N}(G))\right)=H_{p}\left(H_{q}(F ; \mathcal{N}(G)) \otimes_{\pi_{1}(B)} C_{*}(\tilde{B})\right)
$$

The action of $G$ on $F$ comes from the fiber transport. This action gives an action of $G$ on $C_{n}(F)$. The action of $G$ on $C_{*}(F) \otimes_{\pi_{1}(B)} \mathcal{N}(G)$ is then given by

$$
(\delta \otimes a) \cdot g=\delta g \otimes a g
$$

This comes from the definition of local coefficient systems. A special situation occurs when $\pi_{1}(B) \cong \pi_{1}(E)$. Since a fibration gives rise to a long exact sequence in homotopy groups, we obtain $\pi_{1}(F)=1$ that means $F$ is simply connected. Therefore we have $\tilde{F}=F$ and by the universal coefficient theorem

$$
\begin{align*}
& H_{q}(F, \mathcal{N}(G))=H_{q}\left(\mathcal{N}(G) \otimes_{\pi_{1}(F)} C_{*}(\tilde{F})\right)= \\
& \quad H_{q}\left(\mathcal{N}(G) \otimes_{\mathbb{C}} C_{*}(F, \mathbb{C})\right) \cong \mathcal{N}(G) \otimes_{\mathbb{C}} H_{q}(F, \mathbb{C}) \tag{3.9}
\end{align*}
$$

In this case the $G$ action on $H_{q}(F, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{N}(G)$ is again given diagonally. We are now ready to prove our main Theorem 3.2.1.
Proof of 3.2.1. We want to calculate $b_{n}^{(2)}(\tilde{E})=\operatorname{dim}_{G}\left(H_{n}(E ; \mathcal{N}(G))\right)$. By 3.2.10 we have a homological spectral sequence

$$
H_{p}\left(B ; H_{q}(F ; \mathcal{N}(G))\right) \cong E_{p, q}^{2} \Rightarrow H_{p+q}(E ; \mathcal{N}(G))=H_{p+q}^{(2)}(\tilde{E})
$$

By 3.9 we have

$$
H_{q}(F ; \mathcal{N}(G)) \cong \mathcal{N}(G) \otimes_{\mathbb{C}} H_{q}(F, \mathbb{C})
$$

Note that we have $H_{q}(F, \mathbb{C}) \cong \mathbb{C}^{m}$ for some $m$ and the action of $G$ on $H_{q}(F, \mathbb{C})$ by fiber transport gives a representation $\sigma: G \rightarrow \mathrm{GL}_{m}(\mathbb{C})$. By 3.8 we obtain

$$
\begin{equation*}
\operatorname{dim}_{G} H_{p}\left(B, H_{q}(F ; \mathcal{N}(G))\right)=b_{n, \sigma}^{(2)}(\tilde{B})=m \cdot b_{n}^{(2)}(\tilde{B})=m \cdot \operatorname{dim}_{G} H_{p}^{(2)}(\tilde{B}) \tag{3.10}
\end{equation*}
$$

Thus by our assumption, we have

$$
E_{p, q}^{2}=0
$$

for all $p \leq d$ and $q \in \mathbb{Z}$. Thus we obtain

$$
E_{p, q}^{\infty}=0
$$

for these $p, q$ and therefore

$$
H_{n}^{(2)}(\tilde{E})=0
$$

for $n \leq d$.

## Chapter 4

## Limit eigenvalue distributions in groups

### 4.1 The limit eigenvalue distribution associated to residual chains

In this chapter we want to consider a similar problem as in Chapter 2. Let $G$ be a residually finite group and let $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ be a matrix over the complex group ring. Let $G \unrhd N_{1} \unrhd N_{2} \ldots \unrhd$ a chain of normal subgroups of finite index with trivial intersection and set $G_{i}=G / N_{i}$. By right multiplication we get an action of $G$ on $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{\left|G_{i}\right|}$. This action extends linearly to matrices over $\mathbb{C}[G]$. Let $A_{i} \in \operatorname{Mat}_{n \cdot\left|G_{i}\right|}(\mathbb{C})$ be the matrix that represents the linear operator on $\mathbb{C}[G]^{n} \cong \mathbb{C}^{n \cdot\left|G_{i}\right|}$ given by right multiplication by $A$. Let now $\lambda_{1}, \ldots, \lambda_{n \cdot\left|G_{i}\right|}$ be the eigenvalues of $A_{i}$. We define the regularized eigenvalue measure of $A_{i}$ as

$$
\begin{equation*}
\mu_{A_{i}}=\frac{1}{\left|G_{i}\right|} \sum_{j=1}^{n \cdot\left|G_{i}\right|} \delta_{\lambda_{j}} \tag{4.1}
\end{equation*}
$$

$\delta_{z}$ is the Dirac measure at $z \in \mathbb{C}$. Note that in the measure the eigenvalues appear with multiplicities. We are now interested in the following questions:
(1) Does the limit $\lim _{i \rightarrow \infty} \mu_{A_{i}}(\{0\})$ exist?
(2) If the answer to question (1) is yes, is the limit independent of the chain $\left(N_{i}\right)_{i}$ ?
(3) Let $\mu_{A}$ be the Brown measure of the operator on $\ell^{2}(G)$ given by right multiplication by $A$. Do the measures $\mu_{A_{i}}$ converge weakly to $\mu_{A}$ ?

For a definition of the Brown measure see Section 4.5. So what is the connection to the effective Lück approximation? For that assume that the matrix $A$ is
normal. In this case we know that $\mu_{A_{i}}(\{0\})=n-\frac{1}{\left|G_{i}\right|} \mathrm{rk}_{\mathbb{C}}\left(A_{i}\right)$. Further the Brown measure is just the spectral measure and therefore the answer to all questions is yes. In this section we want to focus on the case when $A$ is not normal. Of course we could ask all these questions for general $*$-algebras with a converging series of characters as well, however we will see that already in the case described above things do not work out as nice as before. Let us first consider a small computable example. Let $G=\langle g\rangle$ be an infinite cyclic group and let

$$
A=\left(\begin{array}{cc}
g^{2}+3 g & 4 \\
g^{3} & -g^{4}+g
\end{array}\right)
$$

Let $N_{i}=\left\langle g^{5^{i}}\right\rangle \subseteq G$ and therefore $G_{i} \cong \mathbb{Z} /\left(5^{i}\right) \mathbb{Z}$ a cyclic group of order $5^{i}$. Note that we get the matrix $A_{i}$ by replacing $g$ in $A$ by the matrix

$$
\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{4.2}\\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & 0 \\
0 & & & \ddots & 1 \\
1 & 0 & \cdots & & 0
\end{array}\right)
$$

of dimension $\left|G_{i}\right|=5^{i}$. The following plot shows the eigenvalues of $A_{i}$ for $i=1,2,3$ :




These graphics show that at least there is some hope that the eigenvalue measures do converge to some limit measure. We will first focus on the questions (1) and (2). We will show that in the case when $G$ is a finitely generated abelian group the answer to both questions is yes. Later we will give an example to answer question (2) in general in the negative. Last we will focus on the Brown measure. We will show that for finitely generated abelian groups the answer to question (3) is yes, but the same counter example of question (2) will show that the answer is negative in the general case. This chapter is based on [Bos22].

### 4.2 Some Representation Theory

In this section we want briefly repeat some definitions and facts from representation theory of finite groups. We will only work with complex representations.

In the following let $G$ be a finite group.
Definition 4.2.1. A (complex) $n$ dimensional representation of $G$ is a group homomorphism $\rho: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ for some $n \in \mathbb{N}$. The representation $\rho$ is called irreducible if there is no invariant subspace $V \leq \mathbb{C}^{n}$, that means no subspace $V$ such that $v \cdot \rho(g) \in V$ for all $v \in V, g \in G$.

One important example is the regular representation. For that note that $\mathbb{C}[G]$ is isomorphic to $\mathbb{C}^{|G|}$ as a $\mathbb{C}$-vector space. The regular representation is the map $\rho_{\text {reg }}: G \rightarrow \operatorname{Perm}(|G|) \leq \mathrm{GL}_{|G|}(\mathbb{C})$ that represents the permutation action given by multiplication of $G$ on itself.

Given a $n$ dimensional representation $\rho$ and a matrix $A \in \mathrm{GL}_{n}(\mathbb{C})$ the map $\rho_{A}: G \rightarrow \mathrm{GL}_{n}(\mathbb{C}), g \mapsto A \rho(g) A^{-1}$ is also a representation of $G$. This leads to the following definition.

Definition 4.2.2. Two $n$-dimensional representations $\rho, \pi: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ of $G$ are called equivalent, if there is a matrix $A \in \mathrm{GL}_{n}(\mathbb{C})$ such that $\rho(g)=$ $A \pi(g) A^{-1}$ for all $g \in G$. For two equivalent representations $\rho$ and $\pi$ we write $\rho \cong \pi$

Since we are interested in spectral properties and conjugation does not change eigenvalues, it makes sense to consider equivalence classes of representations. Here for each equivalence class it does not matter which representative we choose to make calculations. Since $G$ is a finite group, the image $\rho(G) \leq \mathrm{GL}_{n}(G)$ of $G$ under a representation is a finite subgroup of $\mathrm{GL}_{n}(\mathbb{C})$. It is a well known theorem that every finite subgroup of $\mathrm{GL}_{n}(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{U}_{n}(\mathbb{C})$, the group of unitary matrices. Therefore we can restrict ourselves to unitary representations that means we can assume that $\rho(G) \leq \mathrm{U}_{n}(\mathbb{C})$. Let now $\rho$ be an $n$ dimensional and $\pi$ be an $m$ dimensional representation of $G$. We can then easily build an $n+m$ dimensional representation of $G$ defined by $\pi \oplus \rho: G \rightarrow \mathrm{GL}_{n+m}(\mathbb{C}),(\pi \oplus \rho)(g)=\pi(g) \oplus \rho(g)$. The representation $\pi \oplus \rho$ is called the direct sum of $\pi$ and $\rho$. Expanding the notation for vector spaces we will write $\rho^{n}=\rho \oplus \rho \oplus \ldots \oplus \rho$.

We have the following theorem.
Theorem 4.2.3. [Isa94, Maschke's Theorem 1.9, 1.10] Every representation $\rho$ of $G$ is equivalent to a direct sum of irreducible representations.

For the regular representation $\rho_{\text {textreg }}$ we can specify this result. The following theorem is also well known and follows directly from the above and [Isa94, Theorem 2.11, Theorem 1.17.]

Theorem 4.2.4. Let $G$ be a finite group and $M$ be a set of representatives of all irreducible representations of $G$. For $\rho \in M$ let $d_{\rho}$ be its dimension. Then

$$
\begin{equation*}
\rho_{\mathrm{reg}} \cong \bigoplus_{\rho \in M} \rho^{d_{\rho}} . \tag{4.3}
\end{equation*}
$$

Remark 4.2.5. Given a representation $\rho: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ we can define a right module structure on $\mathbb{C}^{m}$ given by

$$
v g=v \cdot \rho(g)
$$

Thus when the representation is clear we will use this notation.
To identify irreducible representations, one often uses character theory.
Definition 4.2.6. Let $G$ be a group and let $\rho: G \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ be a representation of $G$. The character $\tau$ induced by $\rho$ is the function $\tau=\tau_{\rho}=\operatorname{Tr}_{\mathbb{C}} \circ \rho: G \rightarrow \mathbb{C}$. The character $\tau$ is called irreducible if the underlying representation $\rho$ is irreducible.

Since the trace of matrices is invariant under conjugation we directly get that equivalent representations have the same character. The other direction is also true. In fact we have the following result.
Lemma 4.2.7. [Isa94, Corollary 2.9] Let $\rho, \pi$ are two representations of $G$. Then $\rho$ and $\pi$ are equivalent if and only if they afford the same character.

Also note that the direct sum of two representations gives the sum of the two characters. For two characters $\tau, \chi: G \rightarrow \mathbb{C}$ we define

$$
\begin{equation*}
[\tau, \chi]=\frac{1}{|G|} \sum_{g \in G} \tau(G) \overline{\chi(g)} \tag{4.4}
\end{equation*}
$$

We have the following result.
Lemma 4.2.8. [Isa94, Corollary 2.17] A character $\tau$ of $G$ is irreducible if and only if

$$
[\tau, \tau]=1
$$

### 4.3 The Abelian Case

In this section we want to answer question (1) and (2) in the case of finitely generated abelian groups. For simplicity we will only consider the case $G \cong \mathbb{Z}$ explicitly. At the end of the chapter we will explain briefly the necessary changes for the general case. Let $\left\{N_{m}\right\}$ be a chain of normal subgroups in $G=\langle t\rangle$ with trivial intersection and set $G_{m}=G / N_{m}$. Each $G_{m}$ is a cyclic group of finite order. Put $n_{m}=\left|G_{m}\right|$ and consider the $n_{m}$ dimensional $\mathbb{C}$ vector space $V_{m}=\mathbb{C}^{n_{m}} \cong \mathbb{C}\left[G_{m}\right]$. The action of $G$ on $V_{m}$ is completely determined by the action of $t$ on $V_{m}$. Since $t$ acts as a shift on $\mathbb{C}\left[G_{m}\right]$, the action of $t$ on $V_{m}$ is given by right multiplication with the matrix

$$
T_{m}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & 0 \\
0 & & & \ddots & 1 \\
1 & 0 & \cdots & & 0
\end{array}\right)
$$

Thus we obtain algebra homomorphisms $\rho_{m}: \mathbb{C}[G] \rightarrow \operatorname{Mat}_{n_{m}}(\mathbb{C})$ by sending $t$ to $T_{m}$ and extending linearly. Obviously we can extend $\rho_{m}$ to matrices, that means we have $\rho_{m}: \operatorname{Mat}_{n}(\mathbb{C}[G]) \rightarrow \operatorname{Mat}_{n \cdot n_{m}}(\mathbb{C})$. Let now $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ and $A_{m}=$ $\rho_{m}(A)$. Since $\mathbb{C}[G]$ is abelian, we can define the determinant on $\operatorname{Mat}_{n}([\mathbb{C}[G])$. Having a determinant we can also define the characteristic polynomial of $A$

$$
p(y)=\operatorname{det}\left(y \operatorname{Id}_{n}-A\right) \in \mathbb{C}[G][y]
$$

By identifying $\mathbb{C}[G]$ with $\mathbb{C}\left[t^{ \pm 1}\right]$ we obtain a polynomial $p(t, y) \in \mathbb{C}\left[t^{ \pm 1}, y\right]$.
Proposition 4.3.1. The characteristic polynomial of $A_{m}$ is given by $\chi_{m}(y)=$ $\prod_{\zeta^{n} m=1} p(\zeta, y)$

For the proof the following lemma will be helpful.
Lemma 4.3.2. [Sil, Theorem 1] Let $R$ be a commutative subring of $\operatorname{Mat}_{n}(\mathbb{C})$ and let $M \in \operatorname{Mat}_{m}(R)$. Then $\operatorname{det}_{\mathbb{C}}(M)=\operatorname{det}_{\mathbb{C}}\left(\operatorname{det}_{R}(M)\right)$.

Proof of the Proposition 4.3.1. Using the previous lemma, we just have to calculate the characteristic polynomial of $\rho_{m}(p(t, y))$, where $p(t, y) \in \mathbb{C}[G][y]$ is the characteristic polynomial of $A$. Since the matrix $T_{m}$ is conjugated to the diagonal matrix with the $n_{m}$-th roots of unity on the diagonal the result follows.

Note that for each $\zeta \in \mathbb{C}$ the polynomial $p(\zeta, y)$ is of fixed degree $n$. For a polynomial $f \in \mathbb{C}[y]$ with roots $\lambda_{1}, \ldots, \lambda_{n}$ counted with multiplicities let us denote by $\mu_{f}$ the uniform measure on $\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$. With this notation we can write the normalized eigenvalue measure of $A_{m}$ as

$$
\begin{equation*}
\mu_{A_{m}}=\frac{1}{n_{m}} \sum_{\zeta^{n_{m}=1}} \mu_{p(\zeta, y)} \tag{4.5}
\end{equation*}
$$

We can define a limit measure $\mu$ by

$$
\begin{equation*}
\mu=\frac{1}{2 \pi} \int_{S^{1}} \mu_{p(\zeta, y)} d \zeta \tag{4.6}
\end{equation*}
$$

That means that for a Borel set $K \subseteq \mathbb{C}$ we have

$$
\mu(K)=\frac{1}{2 \pi} \int_{S^{1}} \mu_{p(\zeta, y)}(K) d \zeta
$$

We now want to show that the measures $\mu_{A_{m}}$ converge weakly towards the measure $\mu$. We will need the following lemma.

Lemma 4.3.3. [HM87, Theorem B] Consider a polynomial

$$
a(x)=x^{n}+a_{1} x^{n-1}+\ldots+a_{n}=\prod_{i=1}^{s}\left(x-\lambda_{i}\right)^{m_{i}} \in \mathbb{C}[x]
$$

for distinct $\lambda_{1}, \ldots, \lambda_{s}$. Let $\epsilon>0$ be given such that for $i \neq j$ we have $B\left(\lambda_{i}, \epsilon\right) \cap$ $B\left(\lambda_{j}, \epsilon\right)=\emptyset$. Then there exists $\delta>0$ such that for $b_{i} \in B\left(a_{i}, \delta\right)$ for all $i$ the polynomial

$$
b(x)=x^{n}+b_{1} x^{n-1}+\ldots+b_{n}
$$

has exactly $m_{j}$ roots in $B\left(\lambda_{j}, \epsilon\right)$ for each $j$ where multiplicities are counted.
Theorem 4.3.4. The measures $\mu_{A_{m}}$ converge weakly to the measure $\mu$ from Equation 4.6.
Proof. For $\zeta \in S^{1}$ let us denote by $h(\zeta)=\left(h_{1}(\zeta), \ldots, h_{n}(\zeta)\right)$ the roots of $p(\zeta, y)$ counted with multiplicities. By Lemma 4.3.3, for each $\zeta$ we can order the roots of $p(\zeta, y)$ in a way such that $h(\zeta)$ as a function $h: S^{1} \rightarrow \mathbb{C}^{n}$ is continuous. Let now $f: \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function with compact support. We have

$$
\int_{\mathbb{C}} f d \mu=\frac{1}{2 \pi} \int_{S^{1}} \sum_{i=1}^{n} f\left(h_{i}(\zeta)\right) d \zeta=\lim _{m \rightarrow \infty} \sum_{\zeta^{n} m=1} \sum_{i=1}^{n} f\left(h_{i}(\zeta)\right)=\lim _{m \rightarrow \infty} \int_{\mathbb{C}} f d \mu_{A_{m}}
$$

since the functions $f\left(h_{i}(\zeta)\right)$ are continuous.
We will see later that the measure $\mu$ is exactly the Brown measure $\mu_{A}$ associated to $A$. The following Theorem answers our questions (1) and (2).
Theorem 4.3.5. Let $\lambda \in \mathbb{C}$. Then

$$
\lim _{m \rightarrow \infty} \mu_{A_{m}}(\{\lambda\})=\mu(\{\lambda\})
$$

Proof. Note that for polynomials $f_{1}, f_{2} \in \mathbb{C}[y]$ we have $\mu_{f_{1} \cdot f_{2}}=\mu_{f_{1}}+\mu_{f_{2}}$. Remember that we denoted by $p(t, y) \in \mathbb{C}\left[t^{ \pm 1}, y\right]$ the characteristic polynomial of the matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$. We can write $p(t, y)=(y-\lambda)^{r} \cdot p^{\prime}(t, y)$ where $p^{\prime}$ has $y$-degree equal to $n-r$ and $(y-\lambda)$ does not divide $p^{\prime}(t, y)$. For a polynomial $f \in \mathbb{C}[y]$ and $c \in \mathbb{C}$ let us denote by $m_{f}(c)$ the multiplicity with which $c$ appears as a root of $f$.

By definition we have

$$
\begin{gathered}
\mu(\{\lambda\})=\frac{1}{2 \pi} \int_{S^{1}} \mu_{p(\zeta, y)}(\{\lambda\}) d \zeta=\frac{1}{2 \pi} \int_{S^{1}} \mu_{p^{\prime}(\zeta, y)}+\mu_{(y-\lambda)^{r}}(\{\lambda\}) d \zeta= \\
\frac{1}{2 \pi} \int_{S^{1}} m_{p^{\prime}(\zeta, y)}(\lambda) d \zeta+r \leq \frac{1}{2 \pi}(n-r) \int_{\substack{\zeta \in S^{1} \\
p^{\prime}(\zeta, \lambda)=0}} 1 d \zeta+r=r
\end{gathered}
$$

Here the last equality holds since we are integrating over a finite set which has Lebesgue measure 0 . Obviously for each $\zeta \in S^{1}$ we have $\mu_{p(\zeta, y)}(\{\lambda\}) \geq r$. By the theorem of Portmanteau we have $\limsup _{m \rightarrow \infty} \mu_{A_{m}}(\{\lambda\}) \leq \mu(\{\lambda\})$. Putting everything together we obtain

$$
r \leq \limsup _{m \rightarrow \infty} \mu_{A_{m}}(\{\lambda\}) \leq \mu(\{\lambda\}) \leq r
$$

and therefore $\lim _{m \rightarrow \infty} \mu_{A_{m}}(\{\lambda\})=\mu(\{\lambda\})$

Let us briefly explain what needs to be done in the case if $G$ is an arbitrary finitely generated abelian group. If $G \cong \mathbb{Z}^{k}$ and $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$, we can identify $\mathbb{C}[G]$ with $\mathbb{C}\left[t_{1}^{ \pm 1}, \ldots, t_{k}^{ \pm 1}\right]$ and obtain a characteristic polynomial $p \in$ $\mathbb{C}\left[t_{1}^{ \pm 1}, \ldots, t_{k}^{ \pm 1}, y\right]$ of $A$. Then for every vector $\zeta=\left(\zeta_{1}, \ldots, \zeta_{k}\right)$ we obtain a polynomial $p\left(\zeta_{1}, \ldots, \zeta_{k}, y\right) \in \mathbb{C}[y]$. To obtain the limit measure as in e Equation 4.6 we need to integrate over the $k$-dimensional torus $T^{k} \cong S^{1} \times \ldots \times S^{1}$. If $G$ is not free abelian but also has a torsion part we can write $G \cong \mathbb{Z}^{k} \times H$ where $H$ is a finite abelian group. Considering our chain $\left(N_{m}\right)_{m \in \mathbb{N}}$ of normal subgroups with trivial intersection there exists $l \in \mathbb{N}$ such that $N_{l} \cap H=\{1\}$ and therefore $N_{l} \cong \mathbb{Z}^{k}$. Put $L=N_{l}$. We can consider $\mathbb{C}[G]$ as a right $\mathbb{C}[L]$-module and to the matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ we can associate a matrix $\tilde{A} \in \operatorname{Mat}_{n \cdot|G: L|}(\mathbb{C}[L])$ that mirrors the action of $A$ on $\mathbb{C}[G]^{n}$ seen as a $\mathbb{C}[L]$-module. We then have $\mu_{A}=\frac{1}{|G: L|} \mu_{\tilde{A}}$. Since for $i>l$ we have $N_{i} \leq N_{l}$ we also obtain $\mu_{A_{i}}=\frac{1}{|G: L|} \mu_{\tilde{A}_{i}}$. Thus the case when $G$ has torsion follows from the result for $L$.

### 4.4 A Counter Example in the Heisenberg Group

In this section we want to give an explicit example to show that the answer to question (2) is no in general. Our example comes from the Heisenberg group. For a unital commutative ring $R$ let $G=H_{3}(R)$ be the Heisenberg group over $R$. It can be seen as the subgroup of $\mathrm{GL}_{3}(R)$, generated by matrices $a, b$ given by

$$
a=\left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \text { and } b=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)
$$

Note that

$$
[a, b]=a^{-1} b^{-1} a b=c=\left(\begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

and

$$
[G, G]=\langle c\rangle=\mathrm{Z}(G)
$$

Further we have

$$
\left(\begin{array}{ccc}
1 & x & z  \tag{4.7}\\
0 & 1 & y \\
0 & 0 & 1
\end{array}\right)=b^{y} a^{x} c^{z}
$$

We will only consider the case $R=\mathbb{Z}$ and $R=\mathbb{Z} / n \mathbb{Z}$. The goal of this section is to prove the following theorem:
Theorem D. Let $G=H_{3}(\mathbb{Z})$ be the Heisenberg group and let $a-b \in \mathbb{Z}[G]$. Let $N_{i}=\mathrm{Id}_{3}+p^{i} \cdot \operatorname{Mat}_{3}(\mathbb{Z}) \cap G \unlhd G$ and consider the residual chain $G \unrhd N_{1} \unrhd N_{2} \unrhd \ldots$. Set $G_{i}=G / N_{i} \cong H_{3}\left(\mathbb{Z} / p^{i} \mathbb{Z}\right)$. Note that $\left|G_{i}\right|=p^{3 i}$. Let $A_{i} \in \operatorname{Mat}_{p^{3 i}}(\mathbb{Z})$ be the matrix that represents the action of $a-b$ on $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{3^{3 i}}$. Let $\mu_{A_{i}}$ be the regularized eigenvalue measure of $A_{i}$ as in 4.1. Then

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \mu_{A_{i}}(\{0\})=\frac{p}{p+1} \tag{4.8}
\end{equation*}
$$

In particular the limit depends on the prime $p$. Thus for each prime $p$ we get a different limit measure. The proof of the theorem works with the same approach as in the abelian case. Note that the matrix $A_{i}$ represents the action of $a-b$ in the regular representation of $G_{i}$. Further we know from 4.2.4 that the regular representation of a finite group is unitary equivalent to the direct sum of all its irreducible representations where each irreducible representation appears with multiplicity of its dimension. We will show that the matrix that represents the action of $a-b$ in any irreducible representation is either nilpotent, and therefore gives only the eigenvalue 0 , or is a root of a scalar matrix, and gives therefore no eigenvalue equal to 0 . We then just have to count in how many irreducible representations $a-b$ acts as a nilpotent matrix and add up their dimensions. First let us repeat how the equivalence classes of irreducible representations look like.

### 4.4.1 Irreducible Representations of $H_{3}(\mathbb{Z} / n \mathbb{Z})$

In this section we want to explain the irreducible representations of $G=H_{3}(\mathbb{Z} / n \mathbb{Z})$ up to equivalency. These results are not new and we will follow [GH01] to present them. We will use the same notation for $a, b, c \in G$ as above. Let $N=\langle b, c\rangle \leq G$ and $\rho: G \rightarrow \mathrm{GL}_{m}(\mathbb{C})$ any representation of $G$. Since $c$ is central in $G$ the subgroup $N$ is abelian and therefore $\rho(N) \subseteq \mathrm{GL}_{m}(\mathbb{C})$ is a set of commuting matrices. We know that commuting matrices are simultaneously diagonalizable that means we can choose a basis $v_{1}, \ldots, v_{m}$ of $\mathbb{C}^{m}$, such that

$$
\begin{equation*}
v_{i} b=\lambda_{i} v_{i} \quad \text { and } \quad v_{i} c=\mu_{i} v_{i} \tag{4.9}
\end{equation*}
$$

for all $i \in\{1, \ldots, m\}$ and some eigenvalues $\lambda_{i}, \mu_{i} \in \mathbb{C}$. Since $b^{n}=c^{n}=1_{G} \in G$ we know that $\lambda_{i}^{n}=\mu_{i}^{n}=1$. Fix now one $v=v_{i}$ and consider the vector space $V \leq \mathbb{C}^{m}$ generated by $\left\{v, v a, v a^{2}, \ldots, v a^{n-1}\right\}$.

Let $\omega=\exp \left(2 \pi \mathrm{i} \frac{1}{n}\right)$. Then we have

$$
\begin{equation*}
v b=\omega^{s} v \quad \text { and } \quad v c=\omega^{k} v \tag{4.10}
\end{equation*}
$$

for some $k, s \in\{0, \ldots, n-1\}$ Note that the subspace $V$ is $G$-invariant. We have

$$
\begin{equation*}
\left(v\left(a^{j}\right)\right) b=v\left(a^{j} b\right)=v\left(b a^{j} c^{j}\right)=v\left(b c^{j} a^{j}\right)=\left((v b) c^{j}\right) a^{j}=\omega^{j \cdot k+s} \cdot v a^{j} \tag{4.11}
\end{equation*}
$$

So the vectors $v a^{j}$ are eigenvectors for $\rho(b)$. Since $a$ and $c$ commute they are also eigenvectors for $\rho(c)$.

Theorem 4.4.1. Let $V=\left\langle v, v a, \ldots, v a^{n-1}\right\rangle$. Then the action of $G$ on $V$ via $\rho$ gives an irreducible representation of $G$.

Proof. Let $d_{1}=\operatorname{gcd}(n, k)$ and $d_{2}=\frac{n}{d_{1}}$.
Because of

$$
\omega^{k \cdot d_{2}}=\omega^{k \cdot \frac{n}{d_{1}}}=\omega^{k \cdot \frac{n}{\operatorname{gcd}(n, k)}}=\omega^{n \cdot \frac{k}{\operatorname{gcd}(n, k)}}=1
$$

we have

$$
\left(v a^{j}\right)\left(a^{d_{2}} b\right)=\left(v a^{j}\right)\left(b a^{d_{2}} c^{d_{2}}\right)=\omega^{k \cdot d_{2}}\left(v a^{j}\right)\left(b a^{d_{2}}\right)=\left(v a^{j}\right)\left(b a^{d_{2}}\right) .
$$

That means that $b$ and $a^{d_{2}}$ commute as operators on $V$. Therefore the operators induced by $b$ and $a^{d_{2}}$ on $V$ have the same eigenvectors what means that

$$
v a^{d_{2}}=\omega^{\bar{r}} v \quad \text { where } \omega^{\bar{r} d_{1}}=1 .
$$

Here the latter equality follows from

$$
v \omega^{\bar{r} \cdot d_{1}}=v\left(a^{d_{2}}\right)^{d_{1}}=v a^{n}=v .
$$

From that it follows that $n \mid \bar{r} \cdot d_{1}$ and therefore $d_{2} \mid \bar{r}$. That means that in the generating set $\left\{v, v a, v a^{2}, \ldots, v a^{n-1}\right\}$ are at most $d_{2}$ linear independent vectors. By 4.11 the vectors $\left\{v, v a, \ldots, v a^{d_{2}-1}\right\}=\mathcal{B}$ are eigenvectors for $b$ for different eigenvalues. Thus $\mathcal{B}$ forms a basis for $V$. It is now easy to describe the action of $G$ on $V$ by giving explicitly the matrix representations with respect to $\mathcal{B}$. We have $c \mapsto \omega^{k} \cdot \operatorname{Id}_{d_{2}}$ and

$$
a \mapsto\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{4.12}\\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
\omega^{r \cdot d_{2}} & 0 & 0 & \ldots & 0
\end{array}\right), \quad b \mapsto \omega^{s} \cdot\left(\begin{array}{cccc}
1 & 0 & \ldots & 0 \\
0 & \omega^{k} & \ldots & 0 \\
\vdots & & \ddots & \vdots \\
0 & \ldots & 0 & \omega^{\left(d_{2}-1\right) \cdot k}
\end{array}\right)
$$

with

$$
\begin{equation*}
r \in\left\{0, \ldots, d_{1}-1\right\}, r \cdot d_{2}=\bar{r} . \tag{4.13}
\end{equation*}
$$

We denote this representation by $\rho_{k, r, s}$. Having the matrix representation it is easy to calculate the associated character $\tau_{k, r, s}$. By 4.7 we just have to consider the expression $\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)$. Since $\rho_{k, r, s}(c)$ is a scalar matrix we have $\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)=\omega^{k \cdot z} \tau_{k, r, s}\left(b^{y} a^{x}\right)$. From the structure of $\rho_{k, r, s}(a)$ we directly obtain $\tau\left(b^{y} a^{x}\right)=0$ if $d_{2}$ does not divide $x$. Thus, if $d_{2}$ does divide $x, \rho_{k, r, s}(a)$ is also a scalar matrix. In this case we obtain

$$
\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)=\omega^{k z+r x} \tau\left(b^{y}\right) .
$$

Now $b$ always acts as a diagonal matrix, thus we obtain

$$
\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)=\omega^{k z+r x+s y} \sum_{i=0}^{d_{2}-1} \omega^{i k y} .
$$

Assume now that $\omega^{k y} \neq 1$. Then we have

$$
\sum_{i=0}^{d_{2}-1} \omega^{i k y}=\frac{\omega^{k y d_{2}}-1}{\omega^{k y}-1}=\frac{\left(\omega^{k d_{2}}\right)^{y}-1}{\omega^{k y}-1}=\frac{1-1}{\omega^{k y}-1}=0
$$

Lets sum up what we have until now. We have $\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)=0$ if

- $d_{2} \nmid x$ or
- $\omega^{k y} \neq 1$.

Thus, to get a non zero value, from the latter we obtain that $n$ divides $k y$ which implies $k=0$ or $d_{2} \mid y$. Thus we obtain the following.

$$
\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)= \begin{cases}0 & \text { if } d_{2} \nmid x \vee\left(d_{2} \nmid y \wedge k \neq 0\right)  \tag{4.14}\\ d_{2} \omega^{k z+r x+s y} & \text { else }\end{cases}
$$

Having this we want to apply 4.2 .8 to show that the characters $\tau_{k, r, s}$ and therefore the representations are irreducible. If $k=0$ we have $d_{1}=n$ and therefore the representation is one dimensional. In this case, for each $b^{y} a^{x} c^{z} \in G$ we have

$$
\tau_{0, r, s}\left(b^{y} a^{x} c^{z}\right)=\omega^{r x+s y}
$$

Thus we have

$$
\tau_{0, r, s}\left(b^{y} a^{x} c^{z}\right) \overline{\tau_{0, r, s}\left(b^{y} a^{x} c^{z}\right)}=1
$$

and

$$
\left[\tau_{0, r, s}, \tau_{0, r, s}\right]=1
$$

and therefore $\rho_{0, r, s}$ is irreducible by 4.2.8. If $k \neq 0$ we have

$$
\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right) \neq 0
$$

if and only if $d_{2} \mid x$ and $d_{2} \mid y$ where $x, y, z \in\{0, \ldots, n-1\}$. In this case we have

$$
\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right) \overline{\tau_{k, r, s}\left(b^{y} a^{x} c^{z}\right)}=\left(d_{2}\right)^{2}
$$

Thus we obtain

$$
\left[\tau_{k, r, s}, \tau_{k, r, s}\right]=\frac{1}{n^{3}} \sum_{z, d_{2}\left|x, d_{2}\right| y} d_{2}^{2}=\frac{1}{n^{3}} \cdot n \cdot d_{1} \cdot d_{1} \cdot d_{2}^{2}=1
$$

Again by 4.2 .8 the representation is $\rho_{k, r, s}$ is irreducible.

We have seen that the irreducible representations are completely described by the choices of $k, s \in\{0, \ldots, n\}$ and $r \in\left\{0, \ldots, d_{1}\right\}$ Now have a closer look at 4.14. Note that in the situation $d_{2} \mid y$, say for example $d_{2} \cdot u=y$, we have $\omega^{s y}=\exp \left(2 \pi i \frac{s d_{2} u}{d_{1} d_{2}}\right)=\exp \left(2 \pi i \frac{s u}{d_{1}}\right)$. Thus the value of $\tau_{k, r, s}$ and therefore the equivalence class of the representation $\rho_{k, r, s}$ depends only on $s \bmod d_{1}$. Therefore the irreducible characters of $G=H_{3}(\mathbb{Z} / n \mathbb{Z})$ are given by
$\left\{\rho_{k, r, s}: G \rightarrow \mathrm{GL}_{d_{2}}(\mathbb{C}) \mid k \in\{0, \ldots, n\}, r, s \in\left\{0, \ldots, d_{1}\right\}, d_{1}=\operatorname{gcd}(n, k), d_{2}=\frac{n}{d_{1}}\right\}$.

### 4.4.2 An almost nilpotent element in $\mathbb{C}\left[H_{3}(\mathbb{Z} / n \mathbb{Z})\right]$

In this section we want to use the irreducible representations of $G=H_{3}(\mathbb{Z} / n \mathbb{Z})$ to present an element in the integral group ring $\mathbb{Z}[G]$ that is almost nilpotent. Here by almost nilpotent we mean that the eigenvalue $\lambda=0$ has a large algebraic multiplicity.

Proposition 4.4.2. Let $n \in \mathbb{N}$ and $G=H_{3}(\mathbb{Z} / n \mathbb{Z})$. Let $\rho_{k, r, s}$ be an irreducible representation of $G$, using the above notation. Let $d_{1}=\operatorname{gcd}(n, k)$ and $d_{2}=\frac{n}{d_{1}}$. Then

$$
\rho_{k, r, s}\left((a-b)^{n}\right)=\left((-1)^{d_{2}-1} \omega^{(s-r) \cdot d_{2}}-1\right) \cdot \operatorname{Id}_{d_{2}}
$$

Proof. The proof is an explicit calculation. In the following we will use

- the equality

$$
\begin{aligned}
\left(1-a^{-1} b\right) a & =a-a^{-1} b a \\
& =a-b b^{-1} a^{-1} b a \\
& =a-b c^{-1} \\
& =a\left(1-a^{-1} b c^{-1}\right)
\end{aligned}
$$

- the fact that $c \in G$ is central and
- $G$ has exponent $n$.

We have

$$
\begin{aligned}
(a-b)^{n} & =\left(a\left(1-a^{-1} b\right)\right)^{n} \\
& =a\left(1-a^{-1} b\right) \cdot a\left(1-a^{-1} b\right) \cdot \ldots \cdot a\left(1-a^{-1} b\right) \\
& =a^{n}\left(1-a^{-1} b c^{-(n-1)}\right) \cdot\left(1-a^{-1} b c^{-(n-2)}\right) \cdot \ldots \cdot\left(1-a^{-1} b\right) \\
& =\prod_{i=1}^{n}\left(1-a^{-1} b c^{-(n-i)}\right) \\
& =(-1)^{n} \prod_{i=1}^{n}\left(a^{-1} b c^{-(n-i)}-1\right) \\
& =(-1)^{n} \prod_{i=1}^{n}\left(a^{-1} b c^{i}-1\right) \\
& =(-1)^{n} \prod_{i=1}^{n} c^{i} c^{-i}\left(a^{-1} b c^{i}-1\right) \\
& =(-1)^{n} \prod_{i=1}^{n} c^{i}\left(a^{-1} b-c^{-i}\right) \\
& =(-1)^{n} c^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}\left(a^{-1} b-c^{i}\right)
\end{aligned}
$$

The above calculation took place in the group ring $\mathbb{Z}[G]$. We now want to pass to the irreducible representation $\rho_{k, s, r}$ Therefore let $\bar{a}=\rho_{k, s, r}(a), \bar{b}=\rho_{k, s, r}(b)$ and $\bar{c}=\rho_{k, s, r}(c)$. Note that $\bar{c}$ is now a scalar matrix with $\bar{c}^{d_{2}}=\mathrm{Id}$. We have

$$
\bar{c}^{\frac{n(n+1)}{2}}=(-1)^{n+1} \cdot \mathrm{Id}
$$

and therefore we get

$$
\begin{equation*}
(\bar{a}-\bar{b})^{n}=(-1)^{n} \bar{c}^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}\left(\bar{a}^{-1} \bar{b}-\bar{c}^{i}\right)=-\prod_{i=1}^{n}\left(\bar{a}^{-1} \bar{b}-\bar{c}^{i}\right) \tag{4.15}
\end{equation*}
$$

Here $\bar{c}$ behaves like a primitive $d_{2}$-th root of unity, that means it commutes with $\bar{a}^{-1} \bar{b}$ and it satisfies $\bar{c}^{d_{2}}=\operatorname{Id}$ and $\bar{c}^{m} \neq \mathrm{Id}$ for $m \in\left\{1, \ldots, d_{2}-1\right\}$. Therefore the last expression in 4.15 is just a cyclotomic polynomial. That means we get

$$
\begin{align*}
(\bar{a}-\bar{b})^{n} & =-\prod_{i=1}^{n}\left(\bar{a}^{-1} \bar{b}-\bar{c}^{i}\right)  \tag{4.16}\\
& =-\left(\left(\bar{a}^{-1} \bar{b}\right)^{d_{2}}-1\right)^{d_{1}} \tag{4.17}
\end{align*}
$$

Note that we have

$$
\left(\bar{a}^{-1} \bar{b}\right)^{d_{2}}=\left(\bar{a}^{-1}\right)^{d_{2}} \bar{b}^{d_{2}} \bar{c}^{\frac{d_{2}\left(d_{2}-1\right)}{2}}
$$

Thus we get
$(\bar{a}-\bar{b})^{n}=-\left(\left(\bar{a}^{-1} \bar{b}\right)^{d_{2}}-1\right)^{d_{1}}=\left(\bar{a}^{d_{2}} \bar{b}^{d_{2}} \bar{c}^{\frac{d_{2}\left(d_{2}-1\right)}{2}}-1\right)^{d_{1}}=\left((-1)^{d_{2}+1} \cdot \bar{a}^{-d_{2}} \bar{b}^{d_{2}}-1\right)^{d_{1}}$
In this case an easy calculation shows that

$$
\bar{a}^{-d_{2}}=\omega^{-r d_{2}} \cdot \operatorname{Id}_{d_{2}} \quad \text { and } \quad \bar{b}^{d_{2}}=\omega^{s \cdot d_{2}} \cdot \operatorname{Id}_{d_{2}}
$$

and therefore

$$
\begin{equation*}
(\bar{a}-\bar{b})^{n}=\left((-1)^{d_{2}+1} \cdot \omega^{(s-r) \cdot d_{2}}-1\right) \cdot \operatorname{Id}_{d_{2}} \tag{4.18}
\end{equation*}
$$

Note that this is always a scalar matrix. That means that the matrix $\rho_{k, r, s}(a-b)$ has either only 0 as an eigenvalue or only non zero eigenvalues. We now want to check in which cases this is the zero matrix. Remember that $\omega$ is a primitive $n$-th root of unity, $n=d_{2} \cdot d_{1}$ and $r, s \in\left\{0, \ldots, d_{1}-1\right\}$. That means we have $0 \leq(s-r) \cdot d_{2}<n$. So if $d_{2}$ is odd then $(a-b)^{n}=0$ if and only if $s=r$.

### 4.4.3 $p$ odd prime

We are now ready to proof our main theorem. In this section let $p$ be an odd prime.

Theorem D. Let $G=H_{3}(\mathbb{Z})$ be the Heisenberg group and let $a-b \in \mathbb{Z}[G]$. Let $N_{i}=\mathrm{Id}_{3}+p^{i} \cdot \operatorname{Mat}_{3}(\mathbb{Z}) \cap G \unlhd G$ and consider the residual chain $G \unrhd N_{1} \unrhd N_{2} \unrhd \ldots$. Set $G_{i}=G / N_{i} \cong H_{3}\left(\mathbb{Z} / p^{i} \mathbb{Z}\right)$. Note that $\left|G_{i}\right|=p^{3 i}$. Let $A_{i} \in \operatorname{Mat}_{p^{3 i}}(\mathbb{Z})$ be the matrix that represents the action of $a-b$ on $\mathbb{C}\left[G_{i}\right] \cong \mathbb{C}^{3^{3 i}}$. Let $\mu_{A_{i}}$ be the regularized eigenvalue measure of $A_{i}$. Then

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \mu_{A_{i}}(\{0\})=\frac{p}{p+1} \tag{4.19}
\end{equation*}
$$

Proof. Let us fix $m \in \mathbb{N}$ and set $n=p^{m}$. We want to analyze the regular representation of the group $G=H_{3}(\mathbb{Z} / n \mathbb{Z})$. We know that the regular representations decomposes as a direct sum of equivalence classes of irreducible representations, where each irreducible representations appears with the multiplicity of its dimension. We use the notation from the previous section.

For each $i \in\{0, \ldots, m-1\}$ we have

$$
\begin{equation*}
\left(p^{m-i}-p^{m-i-1}\right) \cdot p^{i} \cdot p^{i} \tag{4.20}
\end{equation*}
$$

irreducible representations of dimension $d_{2}=p^{m-i}$ each one appearing with the multiplicity of $p^{m-i}$. Note that with the notation of Section 4.4.1 we have $d_{1}=\frac{n}{d_{2}}=p^{i}$. In Equation 4.20 the first factor represents the choices for $k$, to actually get a $d_{2}$ dimensional representation, and the last two factors represent the choices for $r$ and $s$. In addition we have $p^{2 m}$ one dimensional representations, when the center of $G$ acts trivially, that means $k=0$. We now want to count how often the eigenvalue 0 appears in the regular representations of the element $a-b \in \mathbb{Z}[G]$. For that have us let a look on Equation 4.18. This gives us that the element $a-b$ is nilpotent in the irreducible representations if and only if $s=r$. In this case the only eigenvalue of $a-b$ is 0 . If $s \neq r$ then $a-b$ is a root of some non zero scalar matrix and has therefore no eigenvalue equal to 0 . Therefore we just have to count all the cases when $s=r$ and add up their total dimensions. Let us sum up the multiplicity of the eigenvalue 0 in the regular representation. Let us decompose $\mathbb{C}[G]$ as $\mathbb{C}[G] \cong M_{1} \oplus M_{2}$, where $M_{2}$ is the direct sum of all one dimensional irreducible representations of $G$. The following sum describes the algebraic multiplicity of 0 in the representation of $a-b$ on $M_{1}$.

$$
S_{1}=\sum_{i=0}^{m-1} p^{m-i} \cdot p^{m-i} \cdot\left(p^{m-i}-p^{m-i-1}\right) \cdot p^{i}=p^{3 m}\left(1-\frac{1}{p}\right) \sum_{i=0}^{m-1}\left(\frac{1}{p^{2}}\right)^{i}
$$

Let us explain this equation. The first $p^{m-i}$ is the dimension of the irreducible representation. The second $p^{m-i}$ is its multiplicity with which it appears in the regular representation. The third factor is the number of choices we have for $k$ to get a $p^{m-i}$ dimensional irreducible representation. The last factor represents the choices of $r$ and $s$ such that $r=s$. Further we have $S_{2}=p^{m}$ times the eigenvalue 0 in the one dimensional representations in $M_{2}$, that means when $k=0, r=s$.

We are now interested in the normalized multiplicity of the eigenvalue 0 , that means in $\mu_{A_{m}}(\{0\})=\frac{S_{1}+S_{2}}{p^{3 m}}$. Using the above equations we get

$$
\mu_{A_{m}}(\{0\})=\left(1-\frac{1}{p}\right) \cdot\left(\sum_{i=0}^{m-1}\left(\frac{1}{p^{2}}\right)^{i}\right)+\frac{1}{p^{2 m}}
$$

We are interested in the limit behaviour, that means in $\lim _{m \rightarrow \infty} \mu_{A_{m}}(\{0\})$. Using the geometric series we get

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \mu_{A_{m}}(\{0\}) & =\lim _{m \rightarrow \infty}\left(1-\frac{1}{p}\right) \cdot \sum_{i=0}^{m-1}\left(\frac{1}{p^{2}}\right)^{i} \\
& =\left(1-\frac{1}{p}\right) \cdot \frac{p^{2}}{p^{2}-1} \\
& =\frac{p-1}{p} \cdot \frac{p^{2}}{(p+1)(p-1)} \\
& =\frac{p}{p+1}
\end{aligned}
$$

### 4.5 The Brown measure in group algebras

In this section we want briefly discuss a generalization of the spectral measure, called the Brown measure. We have seen that in a tracial von Neumann algebra $(\mathcal{N}, \tau)$, the projection valued spectral measure $E_{a}$ of a normal operator $a \in \mathcal{N}$ concatenated with the trace $\tau$ gives a complex measure

$$
\begin{equation*}
\mu_{A}=\tau \circ E_{A} \tag{4.21}
\end{equation*}
$$

In [Bro83] Lawrence Brown generalized this and constructed a complex measure for any operator $A \in \mathcal{N}$. We will here only give the definitions and some properties. More details can be found in [MS17].

### 4.5.1 Fuglede-Kadison determinant and Brown measure

Definition 4.5.1. Let $(\mathcal{N}, \tau)$ be a tracial von Neumann algebra. If $a \in \mathcal{N}$ is invertible we define the Fuglede-Kadison determinant of $a$ as

$$
\Delta(a)=\exp (\tau(\log |a|))
$$

where $|a|=\left(a^{*} a\right)^{\frac{1}{2}}$. More general, for arbitrary $a \in \mathcal{N}$ we define

$$
\Delta(a)=\lim _{\epsilon \rightarrow 0} \exp \left(\tau\left(\log \left(|a|+\epsilon^{2}\right)\right)\right)
$$

and set $\Delta(a)=0$ if the above limit tends to $-\infty$.

Remark 4.5.2. The Fuglede Kadison determinant has the follwoing properties.
(1) We have $\Delta(a)=\int_{\mathbb{R}_{>0}} \log |a| d \mu_{|a|}$, where $\mu_{|a|}$ is the complex spectral measure as in 4.21.
(2) $\Delta(a b)=\Delta(a) \Delta(b)$ for all $a, b \in \mathcal{N}$.
(3) $\Delta(a)=\Delta\left(a^{*}\right)=\Delta(|a|)$.
(4) $\Delta(u)=1$ if $u \in \mathcal{N}$ is unitary.
(5) $\Delta(\lambda a)=|\lambda| \Delta(a)$ for $\lambda \in \mathbb{C}$.
(6) The function $a \mapsto \delta(a)$ is upper semi continuous in the norm topology.

If $A \in \operatorname{Mat}_{n}(\mathbb{C})$ and $\tau$ is the normalized trace on $\operatorname{Mat}_{n}(\mathbb{C})$ we obtain

$$
\Delta(A)=\sqrt[n]{\operatorname{det}(A)}
$$

We are now ready to define the Brown measure. We will skip all the analytic details in the definition. Details can be found in [MS17].

Definition 4.5.3. Let $(\mathcal{N}, \tau)$ be a tracial von Neumann algebra and $a \in \mathcal{N}$. Then we have
(1) The function $z \mapsto \log \delta(a-z)$ is subharmonic $(z \in \mathbb{C})$.
(2) The corresponding Riesz measure

$$
\mu_{a}:=\frac{1}{2 \pi} \nabla^{2} \log \Delta(a-z)
$$

is a probability measure on $\mathbb{C}$ with support contained in the spectrum of a. Here $\nabla^{2}=\frac{\delta^{2}}{\delta z_{r}^{2}}+\frac{\delta^{2}}{\delta z_{i}^{2}}$ is the Laplace operator, where $z_{r}$ and $z_{i}$ are the real and imaginary part of $z=z_{r}+i z_{i} \in \mathbb{C}$.
(3) For all $\lambda \in \mathbb{C}$ we have

$$
\int_{\mathbb{C}} \log (\lambda-z) d \mu_{a}(z)=\log \Delta(a-z)
$$

and this characterizes $\mu_{a}$.
(4) For all functions $f \in C^{2}(K)$ with compact $K \subseteq \mathbb{C}$ we have

$$
\int_{K} f(z) d \mu_{a}(z)=\frac{1}{2 \pi} \int_{K} \log \Delta(a-z) \nabla^{2} f(z) d z
$$

The measure $\mu_{a}$ is called the Brown measure of $a \in \mathcal{N}$.

Remark 4.5.4. - If $a \in \mathcal{N}$ is normal, then $\mu_{a}$ coincides with the complex spectral measure from 4.21 .

- If $A \in \operatorname{Mat}_{n}(\mathbb{C})$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ and $\tau$ is the normalized trace on $\operatorname{Mat}_{n}(\mathbb{C})$ we obtain the normalized eigenvalue measure.

$$
\mu_{A}=\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i}}
$$

- For all $k \in \mathbb{N}$ we have

$$
\int_{\mathbb{C}} x^{k} d \mu_{a}=\tau\left(a^{k}\right) \quad \text { and } \quad \int_{\mathbb{C}} \bar{x}^{k} d \mu_{a}=\tau\left(\left(a^{*}\right)^{k}\right.
$$

### 4.5.2 Convergence of eigenvalue measures

In this section we want to address the third question we raised at the beginning of this chapter. Let us formulate it again. Let $G$ be a residually finite group, $\left(N_{i}\right)_{i}$ a chain of normal subgroups of finite index with trivial intersection and $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$. Set $G_{i}=G / N_{i}$ and denote by $A_{i} \in \operatorname{Mat}_{n\left|G_{i}\right|}(\mathbb{C})$ the matrix that represents the action of $A$ on $\mathbb{C}\left[G_{i}\right]^{n} \cong \mathbb{C}^{n\left|G_{i}\right|}$ by right multiplication. Denote by $\mu_{A}, \mu_{A_{i}}$ the Brown measures of $A$ and $A_{i}$. Do we then have

$$
\lim _{i \rightarrow \infty} \mu_{A_{i}}=\mu_{A}
$$

in the weak sense? In this generality we have already proven that the answer is simply no. For that let us consider again our counterexample $A=a-b$ from the Heisenberg group. We calculated that the limit $\lim _{i \rightarrow \infty} \mu_{A_{i}}(\{0\})$ depends on the chain $\left(N_{i}\right)_{i}$. However all the matrices $A_{i}$ are matrices over $\mathbb{Z}$. Thus Proposition 2.4.1 would imply that weak convergence of the measures $\mu_{A_{i}}$ would already imply convergence in 0 . But since for different chains $\left(N_{i}\right)$ we get different values $\lim _{i \rightarrow \infty} \mu_{A_{i}}(\{0\})$ there can not be a common limit measure. So what is going on, or better what is going wrong here? For that let $\omega$ be a non principal ultrafilter on $\mathbb{N}$, let $\left(d_{n}\right)_{n}$ be a sequence of natural numbers and let $\mathcal{N}_{\omega}$ be the tracial ultraproduct of the tracial von Neumann algebras $\left(\operatorname{Mat}_{d_{n}}(\mathbb{C}), \frac{1}{d_{n}} \operatorname{Tr}\right)$ as in 1.12. In the following let $\left(A_{n}\right) \in\left(\operatorname{Mat}_{d_{n}}(\mathbb{C})\right)_{n}$ be a sequence of matrices that converges in $*$-moments that means for every non commutative polynomial $f \in \mathbb{C}\langle X, Y\rangle$ in two variables we have that the limit

$$
\lim _{n \rightarrow \infty} \frac{1}{d_{n}} \operatorname{Tr}\left(f\left(A_{n}, A_{n}^{*}\right)\right)
$$

exists. Let us consider five sequences of matrices that converge in $*$-moments. The matrices of all sequences will be of the shape

$$
M_{n}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \\
& & & \ddots & 0 \\
& & & & 1 \\
a_{n} & & & & 0
\end{array}\right) \in \operatorname{Mat}_{d_{n}}(\mathbb{C})
$$

The sequences will only differ by the parameter $a_{n}$.

- For the first sequence $A_{1, n}$, let $a_{n}=1$ for all $n \in \mathbb{N}$.
- For the second sequence $A_{2, n}$ let $a_{n}=0$ for all $n \in \mathbb{N}$.
- For the third sequence $A_{3, N}$, let $a_{n}=\epsilon$ for some $\epsilon>0, n \in \mathbb{N}$.
- For the fourth sequence $A_{4, n}$ let $a_{n}=\frac{1}{n}$ for all $n \in \mathbb{N}$.
- For the fifth sequence $A_{5, n}$ let $a_{n}=\left(\frac{1}{n}\right)^{d_{n}}$.

Note that all these sequences represent the same element $A$ in $\mathcal{N}_{\omega}$, and this element has one unique Brown measure denoted by $\mu_{A}$. Let us calculate

$$
\mu_{i}=\lim _{n \rightarrow \infty} \mu_{A_{i, n}}
$$

for $i \in\{1, \ldots, 5\}$. The characteristic polynomial of the matrix $A_{i, n}$ is given by

$$
f_{i, n}=x^{d_{n}}+(-1)^{d_{n}+1} a_{n}
$$

Thus it is easy to see that $\mu_{1}=\mu_{3}=\mu_{4}$ is the uniform probability measure on the unit circle whereas $\mu_{2}=\mu_{5}=\delta_{0}$ is the Dirac measure at 0 . Note that the element $A \in \mathcal{N}_{\omega}$ is unitary, therefore its Brown measure, which is in this case the complex spectral measure, is given by the measures $\mu_{1}=\mu_{3}=\mu_{4}$. The problem is that the eigenvalues of non normal matrices are not stable under small perturbations. The sequences $\left(A_{2, n}\right)_{n}$ and $\left(A_{3, n}\right)_{n}$ differ only by an arbitrary small value $\epsilon$, however since the dimension grows the set of eigenvalues changes completely. However we do have some positive results.

Let us again consider the case $G \cong \mathbb{Z}$. We have seen that for any matrix $A \in$ $\operatorname{Mat}_{n}(\mathbb{C}[G])$ and any chain of normal subgroups of finite index the eigenvalue measures $\mu_{A_{i}}$ do converge to some limit measure, independent of the chain, compare Equation 4.6. A natural question is if this measure is actually the Brown measure of the matrix $A$ as an operator on $\left(\ell^{2}(G)\right)^{n}$. The answer to this is positive.

Theorem 4.5.5. Let $G=\langle t\rangle$ The Brown measure of a matrix $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ is given by the measure $\mu$ from 4.6.

Proof. Let $r_{t}$ be the operator on $\ell^{2}(G)$ given by right multiplication with $t \in G$. Then $r_{t}$ is unitary, especially normal. Thus there exists a projection valued measure $E$, such that

$$
r_{t}=\int_{S^{1}} \lambda d E(\lambda)
$$

We can use the same spectral measure to decompose the von Neumann algebra $\operatorname{Mat}_{n}(\mathcal{N}(G))$ as a direct integral. For details see [Dyk+16]. With that we get that

$$
A=\int_{S^{1}} A(\lambda) d E(\lambda)
$$

where we see $A=A(t) \in \operatorname{Mat}_{n}(\mathbb{C}[G]) \cong \operatorname{Mat}_{n}\left(C\left[t^{ \pm 1}\right]\right)$. By $[\operatorname{Dyk}+16$, Theorem 5.6] we get for the Brown measure $\mu_{A}$ of $A$ that

$$
\left.\mu_{A}=\frac{1}{2 \pi} \int_{S^{1}} \mu_{A(\lambda}\right) d \lambda
$$

where $\mu_{A(\lambda)}$ it the uniform distribution on all eigenvalues of $A(\lambda)$ counted with multiplicities. But this is exactly our measure $\mu$ from 4.6.

Again this can be generalized to the finitely generated free abelian groups. This, together with Theorem 4.3.5, Theorem 4.3.4 and the discussion at the end of section 4.3 yields the following.

Theorem C. Let $G$ be a finitely generated abelian group and let $G \unrhd N_{1} \unrhd N_{2} \ldots$ be a chain of normal subgroups of finite index with trivial intersection and set $G_{i}=G / N_{i}$. Let $A \in \operatorname{Mat}_{n}(\mathbb{C}[G])$ and let $A_{i} \in \operatorname{Mat}_{n\left|G_{i}\right|}(\mathbb{C})$ be the matrix that represents the action of $A$ on $\mathbb{C}\left[X_{i}\right]^{n}$. Then the measures $\mu_{A_{i}}$ converge weakly and pointwise towards $\mu_{A}$.

We further can show that the Brown measure does detect eigenvalues. For simplicity let us consider the von Neumann algebra $\mathcal{N}_{\omega}$ again. Note that we can embed the $\mathcal{N}(G)$ into $\mathcal{N}_{\omega}$ if the group $G$ is sofic. We have the following result.

Proposition 4.5.6. [HSO9, Proposition 6.5] Let $T \in \mathcal{N}_{\omega}$ and let $P \in \mathcal{N}_{\omega}$ be a projection such that $P T P=P T$ (remember that operators act from the right). Then

$$
\mu_{T}=\tau(P) \mu_{P T P}+(1-\tau(P)) \mu_{P^{\perp} T P^{\perp}}
$$

Corollary 4.5.7. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T \in \mathcal{N}_{\omega}$. Then

$$
\mu_{T}(\{\lambda\}) \geq \operatorname{dim}_{\tau} \operatorname{ker}(T-\lambda)
$$

Proof. Obviously $\operatorname{ker}(T-\lambda)$ is closed and by the double commutant theorem the projection $P$ onto $\operatorname{ker}(T-\lambda)$ belongs to $\mathcal{N}_{\omega}$. Further $P$ satisfies the conditions of the previous theorem which gives us the result.

The following question remains open.
Question 4.5.8. Let $T \in \mathcal{N}_{\omega}$ and let $\lambda$ be an eigenvalue of $T$. Let $\left(T_{n}\right)_{n}$ be a series of matrices, such that $T=\left[\left(T_{n}\right)\right]$. In which situations does $\lambda$ appear as an eigenvalue of one of the $T_{n}$ ?

Results in this direction could give a way easier proof of 2.6.1. We want to finish with one last positive approximation result of Brown measures.

Proposition 4.5.9. Let $\left(M_{n}\right), M_{n} \in \operatorname{Mat}_{d_{n}}(\mathbb{C})$ be a series of matrices that converges in $*$-moments. Let $M=\left[\left(M_{n}\right)\right] \in \mathcal{N}_{\omega}$. Let $\mu$ be the Brown measure of $M$ and let $\mu_{n}$ be the normalized eigenvalue measure of $M_{n}$. Then for all polynomials $f \in \mathbb{C}[X]$ we have

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{C}} f d \mu_{n}=\int_{\mathbb{C}} f d \mu
$$

Proof. This follows directly from the last point of 4.5.4.

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