Path Planning on Riemannian Manifolds with Applications to Quadrotor Load Transportation

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In the last century, the field of geometric mechanics has proven itself to be invaluable in applications related to physics, aerospace engineering, and robotics. At its core, geometric mechanics seeks to study mechanical systems from a geometric perspective. Often, this amounts to determining the properties of some dynamical system by taking advantage of the geometric properties of its configuration manifold.

One particular application that has become increasingly pervasive—and technologically feasible—in recent years is path planning in robotic systems, from robotic manipulators in assembly lines to teams of quadrotor UAVs carrying a payload. The first step in such a problem is to design a control scheme which causes the system to converge to some desired trajectory. Once a trajectory tracking control scheme has been developed, the natural next step is to design the trajectories themselves. In most cases, these trajectories must interpolate some set of knot points, and it is desirable that they minimize some quantity while doing so. A quantity of particular interest is the energy consumption, which in many applications is related to the magnitude of the controls. The magnitude of the controls, in turn, can often be related to the norm of the covariant acceleration of the state variables with respect to the Riemannian metric induced by the kinetic energy—a fundamentally geometric quantity. Thus, such path-planning problems have a natural correspondence with a certain variational problem on a Riemannian manifold, and can thus be studied using the tools of geometric mechanics.

This particular variational problem gives rise to curves called *Riemannian cubic polynomials*, which have studied extensively in the literature over the last 40 years. In this thesis, we are motivated by a slightly different problem. Specifically, one where, in addition to finding optimal interpolants, the resulting trajectories avoid certain prescribed obstacles in phase space, or avoid inter-agent collisions in the case of multi-agent systems. A strategy implemented in recent years to formulate this task as a variational problem (where the tools of geometric mechanics can be applied) is to augment the action functional which we

wish to minimize with an artificial potential function which grows large near obstacles. In that sense, it is expected that minimizers of the resulting variational problem will naturally seek to avoid collisions.

In this thesis, we study this variational problem—deriving necessary and sufficient conditions for optimality, reducing said equations by taking advantage of the symmetry of the configuration manifold and the associated Riemannian metric, and designing artificial potentials which guarantee obstacle or collision avoidance. We then apply these techniques to the application of teams of quadrotor UAVs transporting a payload via inflexible elastic cables. That is, we design a trajectory tracking control scheme, and then write energy-optimal trajectories in terms of the proposed variational problem.

Resumen

En el último siglo, el ámbito de la mecánica geométrica ha demostrado ser invaluable en aplicaciones relacionadas con la física, la ingeniería aeroespacial y la robótica. En esencia, la mecánica geométrica busca estudiar los sistemas mecánicos desde una perspectiva geométrica. A menudo, esto equivale a determinar las propiedades de algún sistema dinámico aprovechando las propiedades geométricas de su variedad de configuración.

Una aplicación particular que se ha vuelto cada vez más omnipresente y tecnológicamente factible en los últimos años es el planeamiento de trayectorias en en sistemas robóticos, desde manipuladores robóticos en líneas de ensamblaje hasta equipos de vehículos aéreos cuadrotoes no tripulados que transportan una carga. El primer paso en tal problema es diseñar un esquema de control que haga que el sistema converja a alguna trayectoria deseada. Una vez que se ha desarrollado tal esquema de control, el siguiente paso natural es diseñar las trayectorias mismas. En la mayoría de los casos, estas trayectorias deben interpolar algún conjunto de puntos, y es deseable que minimicen alguna cantidad al hacerlo. Una cantidad de particular interés es el consumo de energía, que en muchas aplicaciones está relacionado con la magnitud de los controles. La magnitud de los controles, a su vez, a menudo se puede relacionar con la norma de la aceleración covariante de las variables de estado con respecto a la métrica Riemanniana inducida por la energía cinética, una cantidad fundamentalmente geométrica. Por lo tanto, tales problemas de planeamiento de trayectorias tienen una correspondencia natural con un cierto problema variacional en una variedad Riemanniana y, como resultado, pueden estudiarse utilizando las herramientas de la mecánica geométrica.

Este problema variacional particular da lugar a curvas llamadas *polinomios cúbicos Riemannianos*, que se han estudiado extensamente en la literatura durante los últimos 40 años. En esta tesis, estamos

motivados por un problema ligeramente diferente. Específicamente, uno donde, además de encontrar interpolantes óptimos, las trayectorias resultantes evitan ciertos obstáculos prescritos en el espacio de fase, o evitan colisiones entre agentes en el caso de sistemas multiagente. Una estrategia implementada en los últimos años para formular esta tarea como un problema variacional (donde se pueden aplicar las herramientas de la mecánica geométrica) es aumentar el funcional de acción que queremos minimizar con una función de potencial artificial que crece cerca de los obstáculos. En ese sentido, se espera que los minimizadores del problema variacional resultante busquen naturalmente evitar colisiones.

En esta tesis, estudiamos este problema variacional, derivando las condiciones necesarias y suficientes para la optimización, reduciendo dichas ecuaciones aprovechando la simetría de la variedad de configuración y la métrica de Riemann asociada, y diseñando potenciales artificiales que garanticen evitar obstáculos o colisiones. Siempre buscamos hacerlo de una forma prácticamente útil, y luego aplicamos estas técnicas a la aplicación de equipos de drones cuatrorotores que transportan una carga a través de cables elásticos inflexibles. Es decir, diseñamos un esquema de control de seguimiento de trayectoria y luego escribimos trayectorias de energía óptima en términos del problema variacional propuesto.

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Chapter 1

Introduction

1.1 Motivation

In recent years, path planning strategies have become vital in the field of robotics, and more generally, in the field of control engineering. At its core, path planning seeks to construct trajectories that interpolate some set of knot points in an "optimal" way (for example, by minimizing some quantity such as the distance traveled, the time elapsed, or the energy consumed along the path), while completing some additional tasks such as avoiding obstacles in phase space or avoiding inter-agent collisions in the case of multi-agent systems. One of the most pervasive cases of path planning arises in rigid body systems. In this case, trajectories can be modelled as paths in $\mathrm{SO}(3) \times \mathbb{R}^3$ or $\mathrm{SE}(3)$, both of which are Lie groups with physically motivated Riemannian metrics. Finding optimal trajectories can then be interpreted as finding solutions to a variational problem on a Riemannian manifold. This was first introduced in [89], where necessary conditions for optimality were derived, giving rise to the notion of *Riemannian cubic polynomials*.

From that point on, Riemannian cubic polynomials were studied in their own right, both due to their importance in path planning strategies, and to the interesting geometric and analytic properties that they possessed. Sufficient conditions for optimality were first derived and studied in [29], and subsequently in [19], where the notion of bi-Jacobi fields and biconjugate points were first introduced. The existence and uniqueness of global minimizers was studied in [39], and for higher-order Riemannian polynomials in [40], utilizing tools of global analysis. Local uniqueness was discussed in [20], where the notion of biexponential maps were introduced. Reduction by symmetry was discussed on Lie groups in [1, 36], on Riemannian symmetric spaces in [37], and on Riemannian homogeneous spaces in [114].

Riemannian cubic polynomials, however, are unable to take the tasks of obstacle avoidance or inter-agent collision avoidance into consideration—which are vital components of path planning in many robotic systems. One strategy, introduced in [7], is to augment the action functional whose critical points are Riemannian cubic polynomials with an artificial potential that grows radially near the obstacles. The critical points of this new functional will be referred to as *modified Riemannian cubics* throughout the thesis. Necessary conditions for optimality were derived in [7, 8] for the obstacle avoidance task, and in [4, 23] for the collision avoidance task. Reduction by symmetry on Lie groups equipped with bi-invariant metrics and on symmetric spaces was considered in [9]. The study of modified Riemannian cubics is still very new, and many desirable results have not yet been obtained.

In this thesis, I seek to rigorously establish an analytic theory for modified Riemannian cubics. This comes first by proving theoretical results for modified Riemannian cubics (such as existence and uniqueness of global minimizers, finding sufficient conditions for optimality and studying the local minimizers, and applying reduction by symmetry on a wide variety of spaces) that have thus far been obtained only for Riemannian polynomials in the literature. Moreover, I aim to improve the real-world applicability of modified Riemannian cubics in the obstacle and collision avoidance tasks by studying the artificial potential design. In particular, sufficient conditions for avoidance in both the obstacle and collision avoidance task are obtained based on the magnitude of the artificial potential within certain regions surrounding the obstacles/other agents. Artificial potential models are provided which guarantee safety, and closed expressions are obtained for compact and connected Lie groups and homogeneous spaces, which encompass many of the most important applications to this work.

Finally, the application of teams of quadrotor UAVs transporting a payload is studied. Such systems have seen a great amount of attention in recent years, as technological advances in robotics are making them increasingly available in the public sector (see [42, 71, 73, 75, 112, 101, 31]). In particular, a model is provided which ensures the integrity of its payload by making use of inflexible elastic cables. A geometric control scheme is constructed, based on the strategies implemented in [75] and [70], which guarantees the exponential convergence of the state variables to some desired trajectory, and is robust to disturbances. It is then shown how modified Riemannian cubic appear naturally when considering the path planning of such

systems.

1.2 Outline of the Thesis and Main Contributions

In this section, an outline for the thesis is provided, and the main contributions of this work are discussed. In many cases, the results pertain to work that has been published or submitted for publication, in which case the references therein will be provided.

Chapter 2 is a background chapter which is dedicated to the study of Riemannian manifolds. Many of the important features of Riemannian geometry will be defined in Section 2.1-2.2, including the general notions of Riemannian metrics, connections on vector bundles, the Levi-Civita connection, geodesics, and the exponential map. Along this thesis, the predominant regularity that will be considered is Sobolev class H^2 regularity. The most important path spaces of this thesis will be defined in Section 2.3.1. In Section 2.3.2, it is shown that the length and energy functions give rise to geodesics as their critical points. While this is a classical result, I choose to define these functionals over certain closed subspaces of Sobolev class H^2 curves, which, to my knowledge, has not been studied in the literature and demands an alternative proof strategy. Moreover, in Section 2.3.3, it is shown that the energy functional satisfies the Palais-Smale condition, which provides an alternative treatment to the classical result that there exist global minimizers of the energy/length functionals on complete Riemannian manifolds. Finally, in Section 2.4, the notions of Jacobi fields and conjugate points are studied in their relation to the sufficient conditions for geodesics to be length-minimizing.

Chapter 3 is dedicated to the study of Lie groups. Sections 3.1 and 3.2 introduce the basic notions of Lie groups and Lie group actions. Important examples are given throughout the development of the concepts. In Section 3.3, Riemannian metrics on Lie groups are discussed, and the notion of a *Riemannian* $\mathfrak{g}$*-connection* is defined—which will be one of the principle tools in studying reduction by symmetry throughout the thesis. In Section 3.4, the process of reduction by symmetry and the associated Euler-Poincaré equations are discussed for geodesics. This is extended to the reduction with broken symmetry of mechanical Lagrangians in Section 3.4.1. While all of the results presented can be found in the literature, the perspective adopted of using H^2 regularity instead globally C^1, piecewise smooth curves is unique.

Furthermore, a Riemannian perspective is used in the case of reduction with broken symmetry, whereas the literature uses a Lagrangian formalism.

Chapter 4 studies Homogeneous spaces. **Many results obtained here pertain to the article [50], which has been conditionally accepted to the SIAM Journal on Applied Algebra and Geometry at the time of this thesis publication**. The *horizontal connection* is defined in Section 4.1, which is used in Section 4.1.1 for the purposed of reduction by symmetry on Riemannian homogeneous spaces. The perspective adopted is novel, and does not appear in any of the literature. Finally, in Section 4.2, Riemannian homogeneous spaces associated to Lie groups with bi-invariant metrics, and Riemannian symmetric spaces, are discussed. The unit sphere S^2 is given as an example, where the reduced Euler-Lagrange equations for mechanical Lagrangians are derived.

In Chapter 5, the variational path planning problem is introduced. In Section 5.1, necessary conditions for optimality are derived, giving rise to the so-called *modified Riemannian cubics*. While the necessary conditions align with those derived in the literature, the result is stronger and the proof strategy is distinct due to assuming less regularity of the curves in the admissible path space. The existence of global minimizers is studied in Section 5.2, which uses tools of global analysis discussed in Section 2.3.3, and follows a strategy similar to that implemented for Riemannian cubic polynomials in [39]. **These results were obtained in [47], which has been published in the IFAC-PapersOnLine journal**. Finally, in Section 5.3, sufficient conditions for optimality are studied. Local minimizers are divided into two classes: Ω-local minimizers and Q-local minimizers, which are studied separately. **This work has been published in [44] in the Journal of Geometric Mechanics**, although certain results are expanded upon within the thesis.

Chapter 6 is dedicated to the study of the obstacle avoidance and collision avoidance tasks. The development considered is quite distinct from the literature, and all of the results obtained are original. In particular, artificial potential design for the obstacle avoidance task is considered in 6.1, where sufficient conditions for avoidance are provided for point-obstacles and for relatively compact obstacles. In Section 6.3, interpolation on systems with impulse effects is considered as an application to the obstacle avoidance outlined in the previous sections. Finally, in Section 6.2, the collision avoidance task for multi-agent systems is discussed, where sufficient conditions for avoidance are similarly provided. **The results obtained in this**

chapter correspond to [47], which was published in the the IFAC-PapersOnLine journal, and to [48], which was published in the SIAM Journal on Control and Optimization.

In Chapter 7, reduction by symmetry is studied for modified Riemannian cubics. In Section 7.1, we study reduction with broken symmetry on Lie groups. In particular, reduction of the necessary conditions for optimality with partial symmetry on Lie groups equipped with left-invariant Riemannian metrics is considered in Section 7.1.1, and for bi-invariant metrics in 7.1.2. Applications to rigid body dynamics are considered. Reduction of necessary conditions on homogeneous spaces in considered in Section 7.2, first for general Riemannian homogeneous spaces in Section 7.2, and then on Riemannian homogeneous spaces whose underlying Lie groups admit bi-invariant metrics and Riemannian symmetric spaces in Section 7.2.2. As when applying reduction by symmetry of mechanical Lagrangians in Chapter 4, the framework employed is original and many of the results obtained are not found in the literature. Additionally, we return to the obstacle avoidance task, and show how the artificial potential can be designed in such cases. In particular, closed forms for the gradient vector field of the artificial potential which can be easily calculated in many important applications are provided. Reduced modified Riemannian cubics on the unit sphere S^2 are derived. **This work correspond to the article [50], which has been conditionally accepted to the SIAM Journal on Applied Algebra and Geometry at the time of this thesis publication**. Finally, the reduction of sufficient conditions for optimality on Lie groups is considered in Section 7.1.3, which has not been discussed in any of the literature.

In Chapter 8, we study the application of teams of quadrotor UAVs transporting a payload via inflexible elastic cables. In particular, in Section 8.2, the task of two quadrotor UAVs transporting a rigid rod is considered, which had not been explicitly studied in the literature. In Section 8.2.1, dynamics for the control system are derived using the Lagrange d'Alembert principle. In Section 8.2.2, we reduce the dynamical system to the corresponding system with inelastic cables, under the assumption that the spring constants and damping coefficients of the cables are large, using the techniques of singular perturbation theory. In Section 8.2.3, a control scheme for the reduced model is proposed. While this particular problem had not been studied in the literature, the control scheme closely resembles that given in [75]. In Section 8.2.4, stability analysis of the resulting closed-loop system is studied by constructing a Lyapunov function. The constructed

Lyapunov function again closely resembles that which was proposed in [75], however we rigorously prove the existence of stabilizing gains—a result which had not previously been obtained in the literature for quadrotor systems. In Section 8.2.5, we show that the control designed for the reduced model also yields favorable results when applied to the original elastic model. In Section 8.2.6, we additionally show that the control scheme is well-behaved in the presence of bounded unstructured disturbances. In Section 8.2.7, a connection is established between the trajectory tracking problem for the two quadrotor system and the variational path planning problem discussed in the previous chapters. In particular, modified Riemannian cubics can be chosen as the desired trajectories, and will in some sense minimize the energy consumption while also allowing for obstacle/collision avoidance tasks to be completed. **The results obtained here were largely based on [49], which has been published in the Journal of Nonlinear Science**. Section 8.3 repeats all of this analysis, but now for the problem of arbitrarily large teams of quadrotor UAVs transporting a rigid body load. **This, in turn, is based on the work [45], which has been conditionally accepted to the IEEE Transactions on Control Systems Technology Journal, and is presently undergoing the final stages of revision at the time of this thesis publication.**

For convenience, some commonly used notation along the thesis is provided in Table 1.1. Note that all differentiable manifolds are assumed to be smooth (in the sense of C^∞). When referring to curves, scalar fields, and vector fields on manifolds, the word "smooth" will occasionally be used to mean "of sufficient regularity" (for example, H^2-regularity) instead of C^∞, but the particular regularity will be stated if it not clear from context.

Notation	Description
$C^\infty(Q)$	Space of smooth scalar fields on the smooth manifold Q.
$C^\infty([a,b],Q)$	Space of smooth curves $q:[a,b]\to Q$.
$\pi : E \to Q$	Vector bundle with total space E, base space Q, and projection map π.
$\Gamma(E)$	Smooth sections on the vector bundle $\pi : E \to Q$.
$\Gamma(TQ)$	Smooth vector fields on smooth manifold Q.
$\Gamma(q)$	Smooth vector fields along the curve $q:[a,b]\to Q$
$\mathcal{L}_X$	Lie derivative along the vector field X on smooth manifold Q given by $\mathcal{L}_X(Y)$.
$[X,Y]$	Lie bracket of vector fields X, Y on smooth manifold Q.
$X(f)$	Application of vector field X on scalar field f given by $\mathcal{L}_X(f)$.
$\nabla_X u$	covariant derivative $u \in \Gamma(E)$ in the direction of $X \in \Gamma(TQ)$ with respect to the connection ∇.
$D_t X$	covariant derivative of $X \in \Gamma(q)$ along the curve $q \in C^\infty([a,b],Q)$.

Table 1.1: Commonly used notation

Chapter 2

Riemannian Geometry

Riemannian manifolds have proven to be the most natural setting for many applications in physics, engineering, and other fields where dynamical systems evolve on nonlinear configuration spaces. In the later part of this thesis, for example, we will concern ourselves with the application of teams of quadrotor UAVs transporting payloads via elastic cables—whose configuration space is a differentiable manifold with a natural choice of Riemannian structure. Variational principles such as Hamilton's principle of least action can be viewed through the lens of Riemannian geometry, which we use to derive the dynamical control system associated to the quadrotor system. Moreover, seeking integral curves which minimize the norm of the controls leads to an optimal control problem that can be restated as a variational principle on a Riemannian manifold.

In this chapter, a cursory overview of Riemannian geometry as it is used throughout the thesis is provided. The reader who is familiar with the basic notions of Riemannian geometry may skip the first two sections of this chapter entirely, or skim through them to familiarize themself with the notation employed here. In general, the material presented is highly non-exhaustive and, in many cases, proofs will not be provided. In Section 2.3, Hilbert manifolds are discussed, and the most important path spaces of this thesis will be constructed. Using this formalism, alternative proofs to classical results in Riemannian geometry related to the length and energy functionals are provided. References will be given throughout the chapter, which are recommended to readers who are new to the concepts. In general, we refer readers to [81, 103, 63, 53, 86, 92] for a more comprehensive treatment of Riemannian geometry. Basic knowledge of differentiable manifolds is assumed throughout the chapter (see [72, 15] for a detailed overview), and unless

stated otherwise, all manifolds are smooth. Einstein summation convention is used in most cases.

2.1 Riemannian Metrics

Riemannian metrics are the most basic and fundamental tool of Riemannian geometry, upon which all other major concepts will be constructed. They allow to extend the notions of length, angles, and distance from Euclidean geometry to non-linear spaces by taking advantage of the vector space structure of the tangent spaces of a smooth manifold (where Euclidean geometry may be used).

Definition 2.1.1. Let Q be a smooth manifold. A *Riemannian metric* g on Q is a 2-tensor field that is

(1) symmetric: $g_q(X, Y) = g_q(Y, X)$ for all $q \in Q$ and $X, Y \in T_qQ$.

(2) positive-definite: $g_q(X, X) > 0$ for all $q \in Q$ and $X \in T_qQ \setminus \{0\}$.

The pair (Q, g) is called a *Riemannian manifold*.

We may also interpret the Riemannian metric as a map $g : \Gamma(TQ) \times \Gamma(TQ) \to C^\infty(Q)$ given by $g(X, Y)(q) = g_q(X_q, Y_q)$ for all $q \in Q$ and $X, Y \in \Gamma(TQ)$. Observe that for any $q \in Q$, the restriction of the Riemannian metric to T_qQ, given by g_q, is an inner product on T_qQ. For this reason, we will often denote the metric by $\langle \cdot, \cdot \rangle$ instead of g, though it should be noted that the Riemannian metric is not an inner product itself, but rather a smooth family of inner products. We will often refer to the Riemannian metric as the "metric" when there is no chance for confusion, and suppress the base point q when it is clear which tangent space we are applying the Riemannian metric on. It should be noted that every smooth manifold can be equipped with a Riemannian metric [81], but this choice is not uniquely defined by the smooth structure, so we consider it to be an extra structure imposed on the smooth manifold.

Let $q \in Q$. We define the *norm* of $X \in T_qQ$ to be $\|X\| := \langle X, Y \rangle^{1/2}$. We say that $X, Y \in T_qQ$ are orthogonal if $\langle X, Y \rangle = 0$ and that a collection of vectors $X_1, \ldots, X_n$ is orthonormal if $\langle X_i, X_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Given any local frame $E_1, \ldots, E_{\dim(Q)}$ of TQ with dual frame $\xi^1, \ldots, \xi^{\dim(Q)}$, we may write the Riemannian metric in local coordinates as $g = g_{ij}\xi^i\xi^j$, where $g_{ij} = \langle E_i, E_j \rangle$. In particular, if $\partial_1, \ldots, \partial_{\dim Q}$ is a coordinate frame, we find that $g = g_{ij}dx^i dx^j$.

The structure preserving morphisms of a Riemannian manifold (Q, g) are called *isometries*, which are maps $\varphi : (Q, g) \to (\tilde{Q}, \tilde{g})$ which are diffeomorphisms (preserve the smooth manifold structure) and satisfy $\varphi^* \tilde{g} = g$ (preserve the metric). We say that (Q, g) and $(\tilde{Q}, \tilde{g})$ are *isometric* when there exists an isometry between them. We will often say that Q and $\tilde{Q}$ are isometric, or $Q \cong \tilde{Q}$ as Riemannian manifolds when the metrics are understood.

Riemannian manifolds frequently appear as the product of other Riemannian manifolds. If (Q, g) and $(\tilde{Q}, \tilde{g})$ are Riemannian manifolds, the product manifold $Q \times \tilde{Q}$ has a natural choice in Riemannian metric called the *product metric*, which is given by $\bar{g}(X + \tilde{X}, Y + \tilde{Y}) = g(X, Y) + \tilde{g}(\tilde{X}, \tilde{Y})$ for all $X, Y \in T_q Q$ and $\tilde{X},\ \tilde{Y} \in T_{\tilde{q}} \tilde{Q}$, where we have made the identification $T_{(q,\tilde{q})}(Q \times \tilde{Q}) \cong T_q Q \times T_{\tilde{q}} \tilde{Q}$. More compactly, we see that $\bar{g} = g \oplus \tilde{g}$ as tensor fields.

Riemannian metrics also provide a way to convert tangent vectors into cotangent vectors (and vice versa) through the so-called *musical morphisms*. We define the *flat map* $\flat : TQ \to T^*Q$ by $X^\flat(Y) := \langle X, Y \rangle$ for all $X, Y \in T_q Q$, and the *sharp map* $\sharp : T^*Q \to TQ$ as the inverse of the flat map. It is clear that we may consider the musical morphisms as maps between $\Gamma(TQ)$ and $\Gamma(T^*Q)$ as well by defining $X^\flat(Y)(q) = \langle X, Y \rangle$, where $X, Y \in \Gamma(TQ)$. A particular important consequence of this construction for us is the gradient vector field of a scalar field $V \in C^\infty(Q)$, defined as $\text{grad} V := dV^\sharp$. This is equivalent to the conditions that $dV(X) = \langle \text{grad} V, X \rangle$ for all $X \in TQ$. More generally, we may consider scalar fields $f \in C^\infty(Q^n)$ (where $Q^n := Q \times Q \times \cdots \times Q$ with n copies) and define the *gradient vector field of f with respect to the i^{th} argument* (or simply the i^{th} gradient vector field of f) by $\text{grad}_i f = \text{grad} f_i$ where $f_i(q) = f(q^1, \ldots, q^{i-1}, q, q^{i+1}, \ldots, q^n)$ for all $1 \leq i \leq n$. We similarly denote $d_i f = df_i$.

2.2 Connections

While Riemannian metric are widely considered to be the foundational tool of Riemannian geometry, an alternative (and independent) construction is that of connections. These are differential operators which generalize the notion of directional derivatives in Euclidean geometry, but unlike the Lie derivative, allow the introduction of parallelism and some meaningful notion of "straight lines," which will be discussed at length in the subsequent section. Like with Riemannian metrics, any smooth manifold can be equipped with

a connection, and this choice is not unique, so that we consider it to be an additional structure placed upon the manifold.

2.2.1 Connections on Vector Bundles

We begin with the notion of connections as defined on smooth vector bundles. It should be noted that other, more general definitions of connections exist (such as the Ereshmann connection), but for the purposes of this thesis, these constructions will not be necessary.

Definition 2.2.1. Consider a smooth vector bundle $\pi : E \to Q$. A *connection* on E is a map $\nabla : \Gamma(TQ) \times \Gamma(E) \to \Gamma(E)$ satisfying:

(1) $\nabla_{f_1 X_1 + f_2 X_2} u = f_1 \nabla_{X_1} u + f_2 \nabla_{X_2} u$ for all $f, g \in C^\infty(Q)$,

(2) $\nabla_X (c_1 u_1 + c_2 u_2) = c_1 \nabla_X u_1 + c_2 \nabla_X u_2$ for all $c_1, c_2 \in \mathbb{R}$,

(3) $\nabla_X (fu) = f \nabla_X u + X(f) u$ for all $f \in C^\infty(Q)$.

We say that $\nabla_X u$ is the *covariant derivative of* $u \in \Gamma(E)$ *in the direction of* $X \in \Gamma(TQ)$.

For some $q \in Q$, consider a *local frame* $E_1, \ldots, E_{\dim Q}$ of TQ on some open subset $U \subset Q$ containing $q \in Q$, and a local frame $\tilde{E}_1, \ldots, \tilde{E}_{\dim E}$ of E on some open subset $\tilde{U} \subset Q$ containing q. We call the $\dim(Q)^2 \dim(E)$ smooth functions $\Gamma_{ij}^k : U \cap \tilde{U} \to \mathbb{R}$ defined implicitly by $\nabla_{E_i} \tilde{E}_j = \Gamma_{ij}^k \tilde{E}_k$ the *Christoffel symbols* of ∇ with respect to the local frames E and $\tilde{E}$. Moreover, on $U \cap \tilde{U}$, we can locally express any $X \in \Gamma(TQ)$ and $u \in \Gamma(E)$ as $X = X^i E_i$ and $u = u^j \tilde{E}_j$, from which it follows by Definition 2.2.1 that

$$\nabla_X u = \left(X(u^k) + X^i u^j \Gamma_{ij}^k \right) \tilde{E}_k. \tag{2.1}$$

It is not difficult to see from (2.1) that $(\nabla_X Y)(q)$ depends only on X_q and the values of Y in an arbitrarily small neighborhood of $q \in Q$.

We define the *curvature endomorphism* $R : \Gamma(TQ) \times \Gamma(TQ) \times \Gamma(E) \to \Gamma(E)$ as

$$R(X, Y)u = \nabla_X \nabla_Y u - \nabla_{[X,Y]} u, \tag{2.2}$$

where $[\cdot,\cdot] : \Gamma(TQ) \times \Gamma(TQ) \to \Gamma(TQ)$ denotes the Lie bracket on Q. It is easily verified that R is $C^\infty(Q)$-multilinear, and that in the local frames $E_1, \ldots, E_{\dim Q}$ of TQ and $\tilde{E}_1, \ldots, \tilde{E}_{\dim E}$ of E, we have $R^k_{ijm} = X^j(\Gamma^k_{im}) - Y^m(\Gamma^k_{ij}) + \Gamma^k_{sj}\Gamma^s_{im} - \Gamma^k_{sm}\Gamma^s_{ij}$, where $R(E_j, E_k)\tilde{E}_i = R^k_{ijm}\tilde{E}_k$.

The curvature endomorphism has a number of important symmetries, one of which we provide now and the remaining we provide when discussing the Levi-Civita connection in Section 2.2.3. In particular, for all $X, Y, Z \in \Gamma(TQ)$ and $u \in \Gamma(E)$,

$$R(X,Y)u = -R(Y,X)u, \tag{2.3}$$

2.2.2 Affine Connections

In the case that $E = TQ$, we call a connection ∇ on E an *affine connection*. Consider an affine connection ∇ on Q and a local frame $E = \{E_1, \ldots, E_{\dim Q}\}$ of TQ. Equation (2.1) then takes the form

$$\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma^k_{ij})E_k. \tag{2.4}$$

where $X = X^i E_i$ and $Y = Y^j E_j$ in the local frame, and the Christoffel symbols are defined implicitly by $\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$. Equation (2.1) provides a way to construct affine connections on an arbitrary smooth manifold Q. The procedure is as follows: Cover Q in coordinate charts, and choose the Christoffel symbols Γ^k_{ij} in each one—providing a connection within each chart. We then patch these together using a partition of unity to define a connection on Q.

An affine connection ∇ on Q induces a unique connection on each tensor bundle $T^k_l(Q)$ satisfying the following properties:

(1) $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$

(2) $\nabla_X(\text{tr}(Y)) = \text{tr}(\nabla_X Y)$

(3) $\nabla_X \langle \omega, Y \rangle = \langle \nabla_X \omega, Y \rangle + \langle \omega, \nabla_X Y \rangle$, where $Y \in \Gamma(E)$, $\omega \in \Gamma^*(E)$, and

(4) $(\nabla_X F)(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k) = X(F(\omega^1, \ldots, \omega^l, Y_1, \ldots, Y_k))$
$- \sum_{j=1}^l F(\omega^1, \ldots, \nabla_X \omega^j, \ldots \omega^l, Y_1, \ldots, Y_k)$
$- \sum_{i=1}^k F(\omega^1, \ldots, \omega^l, Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k)$.

As with connections on vector bundles, $(\nabla_X Y)(q)$ depends only on X_q and the values of Y in an arbitrarily small neighborhood of q. However, this statement may be improved for affine connections. In particular, $(\nabla_X Y)(q)$ only depends on X_q and the values of Y along any curve tangent to X_q (for example, an integral curve of X). This suggests that we may be able to discuss covariant derivatives along vector fields along curves, rather than globally defined vector fields, which leads us to one of the most important constructions related to affine connections: covariant differentiation along curves. It is of such importance because, as we will see, it provides a way to meaningfully discuss acceleration and higher order vector fields along curves without needing to define higher order tangent bundles and jet bundles. The following was proven in [81], Lemma 4.9.

Lemma 2.2.2. *Let ∇ be an affine connection on Q. For each curve $\gamma : I \to Q$, ∇ determines a unique operator $D_t : \Gamma(\gamma) \to \Gamma(\gamma)$ satisfying:*

(1) $D_t(aX + bY) = aD_tX + bD_tY$ *for all* $a, b \in \mathbb{R}$

(2) $D_t(fX) = \dot{f}X + fD_tX$ *for all* $f \in C^\infty(I)$

(3) If X is extendible (that is, there exists a vector field $\tilde{X}$ on some neighborhood of $\gamma(I)$ such that $\tilde{X}_{\gamma(t)} = X(t)$ for all $t \in I$), then $D_tX = \nabla_{\dot{\gamma}}\tilde{X}$ for any extension of $\tilde{X}$ of X.

For any $X \in \Gamma(\gamma)$, D_tX is called the covariant derivative of X along γ.

We inductively denote the repeated application of covariant derivatives along a curve by $D_t^k X = D_t(D_t^{k-1}X)$. The velocity vector field $\dot{\gamma}$ is a valid vector field along γ, so that we may also consider the *covariant acceleration* of γ given by $D_t\dot{\gamma}$. The curves such that the covariant acceleration is identically zero are of particular importance, and in a sense can be thought to generalize the notion of "straight lines" in flat geometry.

Definition 2.2.3. Let Q be a smooth manifold equipped with an affine connection ∇. A *geodesic* of ∇ is a curve $\gamma : I \to Q$ such that $D_t\dot{\gamma} = 0$ for all $t \in I \subset \mathbb{R}$, where D_t denotes the covariant derivative along γ with respect to ∇.

Given a local coordinate system (q^i, ∂_i) on Q, it can be seen from (2.1) that the coordinates of any geodesic must satisfy $\ddot{q}^k + \sum_{i,j} \dot{q}^i \dot{q}^j \Gamma^k_{ij} = 0$, from which the existence and uniqueness of geodesics immediately follows by standard ODE theory. In particular, a geodesic is uniquely defined by its initial position and velocity, which gives rise to the *maximal geodesic* γ_X starting at $X \in TQ$, which has maximal domain. An important tool that is induced by geodesics is the *exponential map* $\exp : \mathcal{E} \to Q$ (with respect to an affine connection ∇) given by $\exp(X) = \gamma_X(1)$, where $\mathcal{E} = \{X \in TQ \mid \gamma_X \text{ is defined on } [0, 1]\}$. We denote the restriction of the exponential map to a particular tangent space T_pQ by $\exp_p$. We outline the important properties of the exponential map in the following lemma, which combines results discussed in [81]:

Lemma 2.2.4. *Suppose that Q is a smooth manifold equipped with an affine connection ∇. The following are properties of the exponential map with respect to ∇:*

(1) $\exp(tX) = \gamma_X(t)$ *for all* $t \in \mathbb{R}$.

(2) $\exp$ *is smooth.*

(3) For any $p \in Q$, there is a neighborhood $\mathcal{V}$ of the origin in T_pQ and a neighborhood U of p in Q such that $\exp_p : \mathcal{V} \to U$ *is a diffeomorphism.*

Q is called *geodesically complete* if every maximal geodesic is defined for all $t \in \mathbb{R}$. Equivalently, this means that $\exp_p$ is defined on the entire tangent space T_pQ for all $p \in Q$.

2.2.3 The Levi-Civita Connection

The most important and most frequently encountered connection in Riemannian geometry and its applications is the Levi-Civita connection, or Riemann connection. It is natural in that it preserves certain desirable properties of the traditional directional derivative from Euclidean geometry, and serves to link the two possible bases of Riemannian geometry discussed in previous sections: the Riemannian metric, and the affine connection. Before defining the connection, we discuss the properties that we wish for it to have.

Consider a Riemannian manifold $Q \subset \mathbb{R}^n$ be an embedded submanifold some $n \geq \dim(Q)$ (such an embedding is always possible). We may then consider the Euclidean connection ∇ given by the ordinary

directional derivative, and define the *tangential connection* ∇^T on Q via $\nabla^T_X Y = \pi \circ \nabla_{\tilde{X}} \tilde{Y}$ for all $X, Y \in \Gamma(TQ)$, where $\tilde{X}, \tilde{Y}$ are arbitrary extensions of X and Y (respectively) to $\mathbb{R}^n$, and $\pi : T\mathbb{R}^n \to TQ$ denotes orthogonal projection onto TQ. It can be seen that ∇^T is a connection on Q (and hence does not depend on the particular extensions $\tilde{X}$ and $\tilde{Y}$ of X and Y). The tangential connection is natural in that it extends the directional derivative from Euclidean geometry to Riemannian manifolds, which was our original intention with the definition of connections. However, defining the orthogonal projections requires an embedding of Q into some $\mathbb{R}^n$, and we would like to define connections intrinsically. Hence, we search for additional properties of ∇^T that can then be extrapolated to intrinsically defined manifolds. In particular, for all $X, Y, Z \in \Gamma(TQ)$, we see that ∇^T satisfies:

$$(1) \quad \nabla^T_X Y - \nabla^T_Y X = [X, Y] \qquad \text{(symmetry)} \tag{2.5}$$

$$(2) \quad X(g(Y, Z)) = g\left(\nabla^T_X Y, Z\right) + g\left(Y, \nabla^T_X Z\right) \qquad \text{(metric compatibility)} \tag{2.6}$$

Where g denotes the standard (Euclidean) inner product $g(X, Y) = X \cdot Y$ on $\mathbb{R}^n$. Given a Riemannian manifold (Q, g) (not necessary embedded in $\mathbb{R}^n$), we look for affine connections on Q which are symmetric and compatible with g, which leads to the following result, which is Theorem 5.4 in [81]:

Theorem 2.2.5. *Let (Q, g) be a Riemannian manifold. There exists a unique affine connection ∇ on Q that is torsion-free and compatible with g. We call this connection the Levi-Civita connection, or the Riemannian connection.*

In a local coordinate system (x^i, ∂_i) on Q, it can be shown that the Christoffel symbols corresponding to the Levi-Civita connection are uniquely defined in terms of the metric g and its partial derivatives. In particular,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{2.7}$$

where g^{kl} denotes the kl-coordinate of the inverse matrix to g_{ij}—that is $g_{mk} g^{kl} = \delta_{lm}$. We may express this in a coordinate invariant form using the *Koszul formula*:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g([Y, X], Z) - g([X, Z], Y) - g([Y, Z], X). \tag{2.8}$$

The curvature endomorphism R with respect to the Levi-Civita connection is called the *Riemannian curvature endomorphism*, and we further define the *Riemannian curvature tensor* by $Rm : \Gamma(TQ)^4 \to C^\infty(Q)$ given by

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle , \tag{2.9}$$

Which is a type $\binom{3}{1}$-tensor field. Let $X, Y, Z, W, V \in \Gamma(TQ)$. We have the following symmetry relations for the Riemannian curvature tensor and its covariant derivatives:

$$Rm(W, X, Y, Z) = -Rm(X, W, Y, Z), \tag{2.10}$$

$$Rm(W, X, Y, Z) = -Rm(W, X, Z, Y), \tag{2.11}$$

$$Rm(W, X, Y, Z) = Rm(Y, Z, W, X), \tag{2.12}$$

$$Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0, \tag{2.13}$$

$$(\nabla_W Rm)(X, Y, Z, V) + (\nabla_Z Rm)(X, Y, V, W) + (\nabla_V Rm)(X, Y, W, Z) = 0. \tag{2.14}$$

The symmetry relation (2.10) clearly follow from the (2.3), and the remaining identities are consequences of the symmetry and metric compatibility of the Levi-Civita connection. (2.14) is often referred to as the *Bianchi identity*, and the covariant derivative of the curvature tensor is as discussed in Section 2.2.2. The Bianchi identity takes a similar form for connections on a vector bundle $\pi : E \to Q$, where the curvature endomorphism is interpreted as a section on $T^*Q \otimes T^*Q \otimes \mathrm{End}(E)$, but such construction will not be necessary in this thesis.

We now show that the Levi-Civita connection is *natural*, by which we mean that it preserves isometries.

Theorem 2.2.6. *Suppose that $\varphi : (Q, g) \to (\tilde{Q}, \tilde{g})$ is an isometry. Let $\nabla, \tilde{\nabla}$ be the Levi-Civita connections on Q and $\tilde{Q}$, respectively, and Rm and $\tilde{Rm}$ be the corresponding Riemannian curvature tensors. Given a curve γ be a curve on Q, and define D_t to be the covariant derivative along γ with respect to ∇ and $\tilde{D}_t$ to be the covariant derivative along $\varphi \circ \gamma$ with respect to $\tilde{\nabla}$.*

(1) $\varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X} \varphi_* Y$ *for all* $X, Y \in \Gamma(TQ)$,

(2) $\varphi_* D_t X = \tilde{D}_t(\varphi_* X)$ *for all* $X \in \Gamma(\gamma)$.

(3) $\varphi^* \widetilde{Rm} = Rm$

The geodesics of the Levi-Civita connection are sometimes referred to as *Riemannian geodesics*, and the corresponding exponential map the *Riemannian exponential map*. It follows from Theorem 2.2.6 that isometries preserve Riemannian geodesics since $0 = \varphi_*(D_t\dot{\gamma}) = \tilde{D}_t(\varphi_*\dot{\gamma}) = \tilde{D}_t(\varphi \circ \gamma)'$. In terms of the Riemannian exponential map $\exp$ on Q and $\widetilde{\exp}$ on $\tilde{Q}$, this implies that $\varphi \circ \exp = \widetilde{\exp} \circ \varphi_*$.

2.3 Hilbert Manifolds and Global Analysis

Thus far, we have primarily discussed finite-dimensional Riemannian manifolds. Indeed, the configuration spaces used in our applications of interest and in the path-planning problem in general are all of finite dimension. However, infinite-dimensional manifolds associated to some finite-dimensional manifold of interest nonetheless appear in a very natural way. In particular, the space of curves of some regularity on a manifold forms an infinite-dimensional manifold, and functionals defined on these spaces are the foundation of the calculus of variations (and related notions like optimal control theory and Lagrangian mechanics). In Section 2.3.1, we discuss a particularly important example for the purposes of this thesis: Sobolev spaces of curves. We then define two very important functionals in the study of Riemannian geometry, the length and energy functionals, over these spaces in Section 2.3.2. Finally, in Section 2.3.3, we disucss an important application of these concepts to the field of Global analysis. In particular, how the existence of global minimizers of functionals defined on Hilbert manifolds can be proven under certain conditions.

2.3.1 Sobolev Spaces of Curves

One often views finite-dimensional smooth manifolds as spaces which are locally diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Infinite-dimensional manifolds are defined in much the same way, with $\mathbb{R}^n$ being replaced by some infinite-dimensional topological vector space equipped with some additional structure that allows for the notion of smoothness. Common choices include locally convex topological vector spaces, Fréchet spaces, Banach spaces, and Hilbert spaces (in decreasing order of generality), which are known as the *model spaces* for the manifold. Each type of model space comes with its own advantages and disadvantages, and is often determined by the problem of interest. In this thesis, the most natural choice in model space turns out to be Hilbert spaces (particularly Sobolev spaces).

Let $I \subset \mathbb{R}$ be a closed interval and $L^2(I, \mathbb{R}^n)$ denote the space of square integrable functions $f : I \to \mathbb{R}^n$. That is, $f \in L^2(I, \mathbb{R}^n)$ if and only if $\int_I \|f(x)\|^2 dx < +\infty$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Equivalently, we could say that $f = (f^1, \ldots, f^n)$ is of class $L^2(I, \mathbb{R}^n)$ if and only if f^i is of class $L^2(I, \mathbb{R})$ for all $1 \leq i \leq n$. L^2 becomes a Hilbert space when equipped with the inner product $\langle f, g \rangle = \int_I (f(t) \cdot g(t))\, dt$, where where $f \cdot g$ denotes the "dot product" on $\mathbb{R}^n$.

Let $k \geq 1$ and consider functions $f, \varphi : [a, b] \to \mathbb{R}^n$ such that $\frac{d^j}{dt^j}\varphi(a) = \frac{d^j}{dt^j}\varphi(b) = 0$ for all $0 \leq j \leq k$. It follows via integration by parts that

$$\int_a^b \left(f(t) \cdot \frac{d^k}{dt^k}\varphi(t) \right) dt = (-1)^k \int_a^b \left(\frac{d^k}{dt^k} f(t) \cdot \varphi(t) \right) dt,$$

The left-hand side of the above equation still makes sense if we assume f only to be integrable on $[a, b]$. If there exists an integrable function $g : [a, b] \to \mathbb{R}^n$ such that

$$\int_a^b \left(f(t) \cdot \frac{d^k}{dt^k}\varphi(t) \right) dt = (-1)^k \int_a^b (g(t) \cdot \varphi(t))\, dt$$

for all $\varphi : [a, b] \to \mathbb{R}^n$ which vanish at the endpoints along with its first k derivatives, we refer to g as the k^{th} weak derivative of f, and often denote $g = \frac{d^k}{dt^k} f$ when there is no confusion. We define the Sobolev space

$$H^k(I, \mathbb{R}^n) := \{f : I \to \mathbb{R}^n \mid f \text{ is } C^{k-1} \text{ and has } k^{\text{th}} \text{ weak derivative in } L^2(I, \mathbb{R}^n)\}.$$

It is well-known that $H^k(I, \mathbb{R}^n)$ becomes a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{H^k} = \sum_{j=0}^{k} \int_I \left(\frac{d^j}{dt^j} f(t) \cdot \frac{d^j}{dt^j} g(t) \right) dt = \sum_{j=0}^{k} \left\langle \frac{d^j}{dt^j} f, \frac{d^j}{dt^j} g \right\rangle_{L^2}.$$

The inner product $\langle \cdot, \cdot \rangle_{H^k}$ induces the norm

$$\|f\|_{H^k} = \left[\sum_{j=0}^{k} \int_I \left\| \frac{d^j}{dt^j} f(t) \right\|_{\mathbb{R}^n}^2 dt \right]^{1/2},$$

where $\|\cdot\|_{\mathbb{R}^n}$ is the Euclidean norm on $\mathbb{R}^n$. An alternative way to construct $H^k(I, \mathbb{R}^n)$ is to define it as the completion of the space of smooth functions $C^\infty(I, \mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^k}$. It can be shown that the two characterizations of $H^k(I, \mathbb{R}^n)$ are indeed equivalent.

Let Q be a finite-dimensional smooth manifold. We define $H^k(I, Q)$ as the set of curves $q : I \to H$ such that for all coordinate charts (U, φ) on Q such that $q(I') \subset U$ for some $I' \subset I$, the chart representation

$\varphi \circ q : I' \to \mathbb{R}^{\dim(Q)}$ is of Sobolev class $H^k(I', \mathbb{R}^{\dim(Q)})$. It can be seen that $H^k(I, Q)$ is an infinite-dimensional smooth manifold modelled on $H^k(I, \mathbb{R}^{\dim(Q)})$ ([63]). It should be noted that $H^k(I, Q)$ is not in general a Hilbert space, as it not generally a vector space and thus has no well-defined inner product structure. However, the tangent space $T_q H^k(I, Q)$ of Sobolev class H^k vector fields along a curve q on Q may be identified with $H^k(I, \mathbb{R}^{\dim(Q)})$ and hence is a Hilbert space.

Suppose that (Q, g) is a finite-dimensional complete Riemannian manifold and $q \in H^k(I, Q)$. We equip the vector space $\Gamma(q)$ of smooth vector fields along q with the norm

$$\|X\|_{H^k_g} := \left[\sum_{j=0}^{k} \int_I \|D_t^j X\|_g^2 dt \right]^{1/2} \tag{2.15}$$

where D_t^j denotes the j^{th} covariant derivative along q with respect to the Levi-Civita connection (by convention, we take $D_t^0 X = X$), and $\|\cdot\|_g$ denotes the norm induced by the Riemannian metric g. Denote the completion of $\Gamma(q)$ under $\|\cdot\|_{H^k_g}$ by $H^k_g(q)$. Consider an orthonormal basis of parallel vector fields $\{\xi_i\} \in \Gamma(q)$ with respect to g. It follows that if $X = X^i \xi_i$, then $D_t^j X = X^{i(j)} \xi_i$ for all $j \in \mathbb{N}$. If we let $A \cdot B := A^i B^i$ for all $A, B \in \Gamma(q)$, where $A = A^i \xi_i$ and $B = B^i \xi_i$, then we have

$$\|X\|_{H^k_g} := \left[\sum_{j=0}^{k} \int_I \left(X^{(j)} \cdot X^{(j)} \right) dt \right]^{1/2}, \tag{2.16}$$

from which it follows that $H^k_g(I, Q)$ can be identified with $H^k(I, \mathbb{R}^n)$. Hence any complete Riemannian metric on Q induces a Hilbert structure associated to $\Gamma(q)$ that coincides with that of $T_q H^k(I, Q)$. Moreover, it follows that the inner product

$$\langle X, Y \rangle_{H^k_g} := \sum_{j=0}^{k} \int_I g \left(D_t^j X, D_t^j Y \right) dt, \tag{2.17}$$

varies smoothly across the tangent spaces, and hence is a Riemannian metric on $H^k(I, Q)$[1] .

In applications, especially when ultimately interested in solutions to some order $2k$ ODE, it is often the case that you wish to consider curves of some specified regularity which satisfy a set of boundary

[1] For any Hilbert manifold M modelled on a Hilbert space H, any Riemannian structure placed on M is locally equivalent to the Hilbert structure on H. In particular, for any local coordinate chart (φ, U) on M and point $x \in U$, there exists a unique bounded, positive-definite, self-adjoint operator $g(x)$ on H such that $\langle X, Y \rangle_M = \langle g(x) \varphi_*(X), \varphi_*(Y) \rangle_H$ for all $X, Y \in T_x M$. Moreover, the map $x \mapsto g(x)$ is smooth on U.

values. For example, curves of Sobolev class H^{2k} regularity whose first k (covariant) derivatives (including positions as $k = 0$) satisfy some specified boundary conditions. For that reason, we define the path space:

$$\Omega^{(2k)} = \{q \in H^{2k}([a,b], Q) \mid q(a) = q_a,\ q(b) = q_b,\ D_t^j \dot{q}(a) = \xi_a^j \text{ and } D_t^j \dot{q}(b) = \xi_b^j \text{ for } j = 0, 1, \ldots, k-1\} \tag{2.18}$$

where $q_a, q_b \in Q$ and $\xi_a^j \in T_{q_a}Q$ and $\xi_b^j \in T_{q_b}Q$ for all $1 \leq j \leq k-1$.

It is easy to see that $\Omega^{(2k)}$ is the inverse image of $\Big((q_a, q_b), (\xi_a^0, \xi_b^0), \ldots (\xi_a^{k-1}, \xi_b^{k-1})\Big)$ under the map $F : H^{2k}([a,b], Q) \to TQ^k$ given by $F(q) = \big((q(a), q(b)), (\dot{q}(a), \dot{q}(b)), \ldots, (D^{k-1}\dot{q}(a), D^{k-1}\dot{q}(b))\big)$. Moreover, it can be shown that F is a smooth submersion, from which it follows by the implicit function theorem that $\Omega^{(2k)}$ is a closed submanifold of $H^{2k}([0,1], Q)$, and hence inherits its Hilbert structure. The tangent space $T_q\Omega^{(2k)}$ can be indentified with the space $X \in \mathring{H}_g^{2k}(q)$ of vector fields in $H_g^{2k}(q)$ which vanish at the endpoints along with their first k covariant derivatives. Hence we may equip $\Omega^{(2k)}$ with the Riemannian structure (2.17). We also consider the special case

$$\Omega^{(1)} = \{q \in H^1([a,b], Q) \mid q(a) = q_a,\ q(b) = q_b\} \tag{2.19}$$

which is itself a closed submanifold of $H^1([a,b], Q)$, and is of particular importance for geodesics. We will occasionally use the notation $\Omega^{(1),[a,b]}_{q_a,q_b}(Q)$ (and similar for higher-order path spaces) when it is necessary to refer to the boundary conditions, underlying manifold, and interval of integration.

2.3.2 Geodesics: The Length and Energy Functionals

In the later parts of this thesis, we will study the critical points of a functional $J : \Omega^{(2)} \to \mathbb{R}$, where $\Omega^{(2)}$ is as defined in (2.18) and Q is a finite-dimensional Riemannian manifold. We devote this section to studying the motivating example finding the critical points of the length and energy functionals, and their connection to geodesics.

Let (Q, g) be a Riemannian manifold and $\Omega^{(1)}$ be as defined in (2.19). We define the *length functional* $L : \Omega^{(1)} \to \mathbb{R}$:

$$L[q] = \int_a^b \|\dot{q}(t)\| dt, , \tag{2.20}$$

where $\|\cdot\|$ refers to the norm induced by g. The length functional can be used to define a metric on Q (in the sense of metric spaces). In particular, we consider the *Riemannian distance function* $d : Q \times Q \to \mathbb{R}$ given by

$$d(q_a, q_b) = \inf\{L(q) \mid q \in \Omega^{(1)}\}, \tag{2.21}$$

for all $q_a, q_b \in Q$ (here we have slightly abused notation by taking the endpoints in $\Omega^{(1)}$ to be q_a, q_b). If Q is connected, then the distance function is a well-defined metric on Q. That is, every connected Riemannian manifold is a metric space with respect to the Riemannian distance function induced by the Riemannian metric. One of the fundamental results of Riemannian geometry is the *Hopf-Rinow theorem*, which describes exactly when a connected Riemannian manifold is a complete metric space.

Theorem 2.3.1. *A connected Riemannian manifold is complete as a metric space if and only if it is geodesically complete with respect to the Levi-Civita connection.*

We now seek to find the critical points of L over $\Omega^{(1)}$. For this, we define the *Energy functional* $E : \Omega^{(1)} \to \mathbb{R}$ as

$$E[q] = \int_a^b \|\dot{q}(t)\|^2 dt. \tag{2.22}$$

It can be seen that both L and E are smooth on $\Omega^{(1)}$ and have the same critical point. However, E has nicer analytic properties, and so we switch our attention to calculating its critical points. To do this, we need only find the curves $q \in \Omega^{(1)}$ such that $dE[q]X = 0$ for all $X \in T_q\Omega^{(1)}$. A standard approach for calculating $dE[q]X$ is to consider an *admissible variation*[2] of q. That is, for some $\epsilon > 0$, we consider a family of curves $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to Q$ such that

(1) $\Gamma(s, \cdot) \in \Omega^{(1)}$ for all $s \in (-\epsilon, \epsilon)$,

(2) $\Gamma(0, t) = q(t)$ for all $t \in [a, b]$,

[2] Admissible variations (as they are defined in this thesis) are typically referred to as *proper* admissible variations in most texts on the Calculus of Variations. The adjective "proper" simply refers to the fact that the endpoints are fixed. For us, this is an immediate consequence of the fact that $\Gamma(s, \cdot) \in \Omega^{(1)}$, and so the term is redundant.

(3) $\partial_s \Gamma(s,t)\big|_{s=0} = X(t)$ for all $t \in [a,b]$.

X is called the *variational vector field* of the variation Γ. Along the thesis, we will often denote variations of q by q_s, and the corresponding variational vector field by δq (these are more standard notations in the Calculus of Variations). We then calculate $dE[q]X = \partial_s\big|_{s=0} E[q][\Gamma(s,\cdot)]$. It is also common to use the notation $\delta E[q] := dE[q]X$, which is often called the *first variation of* $E[q]$. Hence, in the language of the calculus of variations, the statement "$dE[q]X = 0$ for all vector fields $X \in T_q\Omega^{(1)}$" is stated as "$\delta E[q] = 0$ among all admissible variations of q." The later framework defines what is called a *variational principle*. We will switch between the two formalisms at various points throughout the thesis.

In the literature, the admissible path space is almost always taken to be the space of piece-wise smooth curves on Q rather than curves of H^1 regularity (which is more general). For that reason, we feel it instructive to prove the following result:

Theorem 2.3.2. *Let Q be a complete Riemannian manifold. A curve $q \in \Omega^{(1)}$ satisfies $\delta E[q] = 0$ among all admissible variations of q if and only if q is smooth and satisfies $D_t\dot{q}(t) = 0$ for all $t \in [a,b]$ (that is, q is a Riemannian geodesic).*

Before continuing with the proof, we need an important lemma regarding the commutativity of derivatives of an admissible variation, often referred to as the *Symmetry lemma*:

Lemma 2.3.3. *If Γ is an admissible variation of $q \in \Omega^{(1)}$, then*

$$D_s\partial_t\Gamma(s,t) = D_t\partial_s\Gamma(s,t) \tag{2.23}$$

for all $(s,t) \in (-\epsilon,\epsilon) \times [a,b]$.

Proof. Consider an admissible variation q_s of q with variational vector field $X \in T_q\Omega^{(1)}$. We calculate

$$\begin{aligned} dE[q]X &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \|\dot{q}_s\|\, dt = \int_a^b \frac{\partial}{\partial s}\Big|_{s=0} \|\dot{q}_s\| dt \\ &= \int_a^b \langle D_s\dot{q}_s, \dot{q}_s\rangle \Big|_{s=0} dt \\ &= \int_a^b \langle D_t X, \dot{q}\rangle\, dt \end{aligned}$$

Observe that $q([a,b])$ is bounded by the continuity of q. Since Q is complete, there exists an $r \in \mathbb{R}$ and $q_0 \in Q$ such that $q([a,b]) \subset \overline{B_r(q_0)}$, where $\overline{B_r(q_0)}$ denotes the closed geodesic ball of radius r centered at q_0. It is well-known that the space $C_0^\infty([a,b], \overline{B_r(q_0)})$ of smooth functions with compact support in $\overline{B_r(q_0)}$ is dense in $H^1([a,b], B_r(q_0))$ [53], hence there exists a sequence $(q_n) \subset C_0^\infty([a,b], \overline{B_r(q_0)})$ of smooth curves such that $q_n \to q$ with respect to the H^1 norm. We then find that

$$\begin{aligned} dE[q_n]X &= \int_a^b \langle D_t X, \dot{q}_n \rangle \, dt \\ &= \int_a^b \left[\frac{d}{dt} \langle X, \dot{q}_n \rangle - \langle X, D_t \dot{q}_n \rangle \right] dt \\ &= -\int_a^b \langle X, D_t \dot{q}_n \rangle \, dt \end{aligned}$$

Consider the vector field $X \in T_q\Omega^{(1)}$ given by $X(t) = f(t) D_t \dot{q}_n$ for some $f : [a,b] \to \mathbb{R}$ such that $f(t) \geq 0$ for all $t \in [a,b]$, and $f(a) = f(b) = 0$. Then $dE[q_n]X \to 0$ implies that $\int_a^b f(t) \|D_t \dot{q}_n\|^2 dt \to 0$, from which it follows that $D_t \dot{q}_n \to 0$. Therefore, q_n converges with respect to the H^2-norm, and so in fact $q_n \to q$ with respect to the H^2-norm by uniqueness. Hence, q is of H^2 regularity, and by the previous argument, we see that $dE[q]X = 0$ for all $X \in T_q\Omega^{(1)}$ is equivalent to $D_t \dot{q} = 0$. It follows immediately that q is smooth (by expanding in local coordinates and applying Gronwall's inequality, for instance). $\square$

We similarly see that the critical points of the length functional are smooth geodesics. It follows by the Hopf-Rinow theorem 2.3.1 that if Q is complete and connected, then any two points x and y in Q can be connected by a (not necessarily unique) minimal-length geodesic $\gamma_{x,y}$. In this case, the Riemannian distance between x and y can be defined by

$$d(x,y) = \int_0^1 \left\| \frac{d\gamma_{x,y}}{ds}(s) \right\| ds = L(\gamma_{x,y}). \tag{2.24}$$

Alternatively, it is easily seen that we may express the square of the Riemannian distance $d^2 : Q \times Q \to \mathbb{R}$ in terms of the energy functional as

$$d^2(x,y) = \int_0^1 \left\| \frac{d\gamma_{x,y}}{ds}(s) \right\|^2 ds = E(\gamma_{x,y}). \tag{2.25}$$

Moreover, if x and y belong to a geodesically convex neighborhood, then by Lemma 2.2.4, it follows that

we can write the inverse of the Riemannian distance by means of the Riemannian exponential:

$$d(x, y) = \|\exp_x^{-1} y\|. \tag{2.26}$$

This expression will be utilized frequently when designing artificial potentials in later chapter of the thesis.

2.3.3 Global Analysis: The Palais-Smale Condition

The notion of gradient descent as used in classical analysis can be extended to the context of Hilbert manifolds, and used to prove existence of minimizers of functionals satisfying certain properties. In the finite-dimensional Hilbert manifolds, the arguments for the existence of global minimizers can be treated in a way very similar to Euclidean spaces. For infinite-dimensional Hilbert manifolds, however, additional compactness assumptions are needed—the most interesting of which is referred to as the *Palais-Smale condition*:

Definition 2.3.4. Suppose that M is a Hilbert manifold with Riemannian metric, and $f : M \to \mathbb{R}$ is a smooth functional. We say that f satisfies the *Palais-Smale condition* if every sequence $\{q_n\} \subset M$ for which $|f(q_n)|$ is bounded, and $\|df_{q_n}\| \to 0$, has a convergent subsequence.

Note that in the above definition, $\|df_q\|$ refers to the *operator norm* of $df_q : T_qM \to \mathbb{R}$ with respect to the Riemannian structure on M. That is, $\|df_q\| = \{|df_q(X)| \;\big|\; X \in T_qM \text{ and } \|X\| = 1\}$. We have the following result of Palais and Terng ([92], Theorem $9.1.9$):

Theorem 2.3.5. *Suppose that M is a Hilbert manifold which is complete as a Riemannian manifold. Let $f : M \to \mathbb{R}$ be a smooth functional which is bounded below and satisfies the Palais-Smale condition. Then there exists a global minimizers of f in M.*

It is known that the completeness of Q with respect to its Riemannian metric implies the completeness of $\Omega^{(1)}$ with respect to the induced Riemannian metric (2.17) (see [5]). Moreover, the energy functional is clearly bounded below by 0. According to theorem 2.3.5, we can show that E has global minimizers in $\Omega^{(1)}$ provided that it satisfies the Palais-Smale condition. We show this now.

Proposition 1. *The energy functional* (2.22) *satisfies the Palais-Smale condition.*

Proof. Suppose that $\{q_n\} \subset \Omega^{(1)}$ is a Palais-Smale sequence for E. Let $\sup_{n\in\mathbb{N}} E[q_n] := c$. It follows that $\sup_{n\in\mathbb{N}} L[q_n] = \sqrt{c}$, so that $\{q_n\}$ is uniformly bounded over $[a,b]$. Hence we may consider a finite collection of charts (U_i, φ_i) on Q and an accompanying finite partition I_i of $[a,b]$ such that for each i, there exists a compact set $K_i \subset U_i$ such that $q_n(I_i) \subset K_i$. In particular, since $\varphi_i(K_i) \subset (\varphi \circ q_n)(I_i)$ is compact, the sequence $(\varphi \circ q_n)$ is bounded in $L^2([a,b], \mathbb{R}^n)$. Moreover, it is clear that $(\dot{q}_n)$ is bounded in $L^2_g(q_n)$ by the definition of the energy functional E. It follows that (q_n) is bounded in H^1, so that there exists some subsequence (which we again refer to by (q_n) for convenience) which converges weakly to a curve $q \in \Omega^{(1)}$.

Now observe that, since $\|dE[q_n]\|_{T^*_{q_n}\Omega^{(1)}} \to 0$, we additionally have $\|\text{grad}E[q_n]\| \to 0$ with respect to the norm induced by the Riemannian metric (2.17) on $\Omega^{(1)}$. In particular, if we let $A_n := \text{grad}E[q_n]$, we see that $D_t A_n \to 0$ in L^2, and

$$dE[q_n]X = \int_a^b \langle \dot{q}_n, D_t X\rangle \, dt,$$
$$\langle A_n, X\rangle = \int_a^b \langle D_t A_n, D_t X\rangle \, dt,$$

so that the vector field $Z_n := \dot{q}_n - D_t A_n$ satisfies

$$\int_a^b \langle Z_n, D_t X\rangle \, dt = 0$$

for all $X \in T_{q_n}\Omega^{(1)}$ and $n \in \mathbb{N}$. It is easily seen that Z_n is smooth and $D_t Z_n = 0$ for all $n \in \mathbb{N}$. In particular, since (Z_n) is bounded in L^2, it follows that (Z_n) is bounded in H^1. Hence, there exists a subsequence (Z_{n_k}) which converges strongly in L^2. Since $\dot{q}_{n_k} = Z_{n_k} + D_t A_{n_k}$, it follows that $q_{n_k} \to q \in \Omega^{(1)}$ with respect to the H^1 norm. □

2.4 Jacobi Fields and Conjugate Points

In section 2.3.2, we discussed how necessary conditions for optimality of a functional on a Hilbert manifold could be obtained via variational principles. In particular, for the length and energy functionals, these conditions returned the geodesic equations $D_t\dot{q} = 0$. We further saw that global minimizers indeed exist in section 2.3.3, which guarantees solutions to the geodesic equation on $\Omega^{(1)}$—that is, there exist

solutions to the boundary value problem $D_t\dot{q} = 0$, $q(a) = q_a$ and $q(b) = q_b$ and at least one of them realizes the infimum of the length/energy functionals over $\Omega^{(1)}$.

Presently, we have no way to verify that a solution to this BVP is indeed a minimizer. In general, this is a hard problem. Even in classical analysis, there are no reliable methods to calculating the global extrema of a function without very strong convexity conditions. Verifying that a solution is a local minima, however, is straight-forward (indeed, it is nothing more than the "second derivative test"). Once local minima are established, the global minima can be obtained by direct comparison (as long as there is a finite quantity of local minima). In this section, we develop techniques to test whether a critical point of the energy functional is a local minima—which amounts to calculating the *second variation* of the energy functional and defining the notions of Jacobi fields and conjugate points. The techniques used here will be expanded and adapted later when discussing higher order Riemannian polynomials later in the thesis.

Recall that given a function $f : M \to \mathbb{R}$ on a Hilbert manifold M, its differential at a point $p \in M$ is a linear map $df[p] : T_pM \to \mathbb{R}$. The *second differential* at a point $(p, X) \in TM$ is similarly given by a linear map $d^2f[p]X : T_X(T_pM) \to \mathbb{R}$. Through the identification $T_X(T_pM) \cong T_pM$, we may reinterpret the second differential at $p \in M$ as a bilinear map $d^2f[p] : T_pM \times T_pM \to \mathbb{R}$.

In particular, when $M = \Omega^{(1)}$ (as defined in (2.19)) and q is a geodesic on Q, we refer the map $I(X, Y) = d^2E[q](X, Y)$ as the *index form* of q. As we did with the first differential, we may calculate the index form by employing the calculus of variations. In particular, for some $\epsilon_r, \epsilon_s > 0$, we define a *two-parameter admissible variation* of q as a smoothly varying family of curves $\Gamma : (-\epsilon_r, \epsilon_r) \times (-\epsilon_s, \epsilon_s) \times [a, b] \to Q$ such that:

(1) $\Gamma(r, s, \cdot) \in \Omega^{(1)}$ for all $(r, s) \in (-\epsilon_r, \epsilon_r) \times (-\epsilon_s, \epsilon_s)$,

(2) $\Gamma(0, 0, t) = q(t)$ for all $t \in [a, b]$,

(3) $\partial_r\Gamma(r, s, \cdot)\big|_{(r,s)=(0,0)} = X \in T_q\Omega^{(1)}$,

(4) $\partial_s\Gamma(r, s, \cdot)\big|_{(r,s)=(0,0)} = Y \in T_q\Omega^{(1)}$.

We then have that $I(X, Y) = \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} E(\Gamma(r, s, \cdot))$. We now wish to calculate the index

form along a geodesic, but before doing so provide a lemma discussing the commutativity of second order (covariant) derivatives:

Lemma 2.4.1. *Suppose that Γ is an admissible variation of $q \in \Omega^{(1)}$, and $V \in T_q\Omega^{(1)}$. Then,*

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \dot{\Gamma})V. \tag{2.27}$$

Proof. See [81], Lemma 10.1. □

Theorem 2.4.2. *The index form along a geodesic $q \in \Omega^{(1)}$ is given by*

$$I(X,Y) = \int_a^b \left[\langle D_t X, D_t Y\rangle - \mathit{Rm}(X, \dot{q}, \dot{q}, Y)\right] dt, \tag{2.28}$$

for all $X, Y \in T_q\Omega^{(1)}$.

Proof. Let Γ be a two-parameter variation of q. Then,

$$\begin{aligned}
\frac{\partial}{\partial s}E(\Gamma) &= \int_a^b \left\langle D_s\dot{\Gamma}, \dot{\Gamma}\right\rangle dt, \\
&= \int_a^b \left\langle D_t\partial_s\Gamma, \dot{\Gamma}\right\rangle dt, \\
\frac{\partial^2}{\partial r \partial s}E(\Gamma) &= \int_a^b \left[\left\langle D_r D_t \partial_s\Gamma, \dot{\Gamma}\right\rangle + \left\langle D_r\dot{\Gamma}, D_s\dot{\Gamma}\right\rangle\right] dt, \\
&= \int_a^b \left[\left\langle D_t D_r \partial_s\Gamma + R(\partial_r\Gamma, \dot{\Gamma})\partial_s\Gamma, \dot{\Gamma}\right\rangle + \langle D_t\partial_r\Gamma, D_t\partial_s\Gamma\rangle\right] dt.
\end{aligned}$$

By equation (2.27). Observe that $\partial_t\left\langle D_r\partial_s\Gamma, \dot{\Gamma}\right\rangle = \left\langle D_t D_r\partial_s\Gamma, \dot{\Gamma}\right\rangle + \left\langle D_r\partial_s\Gamma, D_t\dot{\Gamma}\right\rangle$. Taking $(r,s) = (0,0)$ and recalling that q is a geodesic, we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial r\partial s}E(\Gamma) &= \int_a^b \left[\partial_t\left\langle D_r\partial_s\Gamma\Big|_{(r,s)=(0,0)}, \dot{q}\right\rangle + \langle R(X,\dot{q})Y,\dot{q}\rangle + \langle D_t X, D_t Y\rangle\right] dt, \\
&= \int_a^b \left[\langle D_t X, D_t Y\rangle - \mathrm{Rm}(X,\dot{q},\dot{q},Y)\right] dt,
\end{aligned}$$

using the fact that $D_r\partial_s\Gamma$ vanishes at the endpoints, and applying the symmetries of the curvature tensor. □

We now introduce some terminology to distinguish between different types of minimizers.

Definition 2.4.3. Consider a smooth functional $f : \Omega^{(1)} \to \mathbb{R}$. A curve $q \in \Omega^{(1)}$ is a:

(i) *Global minimizer* of f iff $f[q] \leq f[\tilde{q}]$ for all $\tilde{q} \in \Omega^{(1)}$.

(ii) Ω-local minimizer of f iff $f[q] \leq f[\tilde{q}]$ for all $\tilde{q}$ in some C^0 neighborhood of q (within $\Omega^{(1)}$).

(iii) Q-local minimizer of f iff for any $\tau \in [a, b]$, there exists an interval $[a^*, b^*] \subset [a, b]$ containing τ such that $q|_{[a^*,b^*]}$ is a minimizer of f on $\Omega^{(1),[a^*,b^*]}_{p_0,p_f}$, where $p_0 = q(a^*),\ p_f = q(b^*)$.

Sufficient conditions for optimality of a geodesic $q \in \Omega^{(1)}$ can now be understood in terms of the index form as follows:

Theorem 2.4.4. *Suppose that $q \in \Omega^{(1)}$ is a geodesic. If $I(X, X) > 0$ for all $X \in T_q\Omega^{(1)} \setminus \{0\}$, then q is an Ω-local minimizer of E.*

Proof. Consider an admissible variation q_s of q with variational vector field $\partial_s q_s|_{s=0} = X$ and define $g(s) = E[q_s(\cdot)]$. Observe that $g'(0) = 0$ since q is a geodesic, and $g''(0) = I(X, X) > 0$. By the second-derivative test, g has a local minimum at $s = 0$, so that $E[q] \leq E[q_s]$ for all $s \in (-\epsilon, \epsilon)$. Since this holds for all admissible variations, the result follows. □

In general, verifying $I(X, X) > 0$ for all $X \in T_q\Omega^{(1)} \setminus \{0\}$ via direct computation is not a feasible task. Moreover, while the condition is sufficient, it is not necessary—it may happen that there exist vector fields $X \in T_q\Omega^{(1)}$ such that $I(X, X) = 0$ and q is an Ω-local minimizer. What is certain is that if there exists $X \in T_q\Omega^{(1)}$ such that $I(X, X) < 0$, then q is certainly *not* an Ω-local minimizer. We now seek to characterize the edge case $I(X, X) = 0$ by discovering which kinds of vector fields lie in the kernel of the index form.

Theorem 2.4.5. *Suppose that $q \in \Omega^{(1)}$ is a geodesic. A vector field $J \in T_q\Omega^{(1)}$ belongs to the kernel of the index form along q if and only if it is smooth and*

$$D_t^2 J + R(J, \dot{q})\dot{q} = 0, \tag{2.29}$$

on $[a, b]$. We call a smooth solution J to (2.29) *a Jacobi field.*

In local coordinates (q^i, ∂_i), a Jacobi field may be expressed in the form

$$\ddot{J}^i + R^i_{jkl} J^j \dot{q}^k \dot{q}^l = 0, \tag{2.30}$$

where $J = J^i \partial_i$, $q = q^i$, and $R(\partial_i, \partial_j)\partial_k = R_{jkl}\partial_l$. It follows immediately that the set of Jacobi fields along a geodesic forms a $2\dim(Q)$-dimensional linear subspace of $\Gamma(q)$. Moreover, a Jacobi field is uniquely determined upon specifying the initial conditions $J(a)$ and $D_t J(a)$. The most important concept related to Jacobi fields is that of conjugate points, which we now define:

Definition 2.4.6. Suppose that $q \in \Omega^{(1)}$ is a geodesic. We say that $t_1, t_2 \in [a, b]$ are *conjugate points* if there exists a non-zero Jacobi field J with $J(t_1) = J(t_2) = 0$.

As we will see in the following theorem, conjugate points are precisely the obstructions to a geodesic being an Ω-local minimizers of the energy (or length) functional.

Theorem 2.4.7. *Let $q \in \Omega^{(1)}$ be a geodesic. Then q is an Ω-local minimizer of the energy (length) functional if and only if there does not exist a conjugate point to $t = a$ along q.*

Proof. See [63]. □

This fact turns out to be highly related to the invertibility of the exponential map. We have seen in Lemma 2.2.4 that the exponential map is a local diffeomorphism around the zero vector. More generally, the exponential map is a local diffeomorphism around an arbitrary tangent vector precisely until the geodesic corresponding to the exponential map at that vector reaches a conjugate point. This is summarized in the following lemma ([81], Proposition 10.11).

Lemma 2.4.8. *Suppose that $p \in Q, X \in T_pQ$. Then $\exp_p$ is a diffeomorphism on some neighborhood of X if and only if p is not conjugate to $\exp_p(X)$ along the geodesic $\exp(tX)$.*

Chapter 3

Riemannian Geometry on Lie Groups

In Chapter 2, we discussed the tools and results from Riemannian geometry that are important to this thesis. In this chapter, we consider a special case of Riemannian manifolds—Finite dimensional Lie groups equipped with invariant Riemannian metrics (and the related notions of Riemannian homogeneous spaces and symmetric spaces). Such spaces have proven themselves to be of utmost importance in many fields due to their presence in virtually all mechanical systems. In particular, Euclidean space $\mathbb{R}^3$, the special Orthogonal group $\mathrm{SO}(3)$ on $\mathbb{R}^3$, and the unit sphere S^2 in $\mathbb{R}^3$, are ubiquitous in the majority of robotic applications and, indeed, together compose the configuration space for the principal application of interest in this thesis (as we will see in Chapter 8).

Lie groups themselves are smooth manifolds, and so all the results discussed in the previous chapter apply equally well here. Still, their perversity warrants a study of their construction and properties—if only to later study the application with teams of quadrotor UAVs. However, it turns out that there is further motive to study them. Lie groups come with additional structure (namely a group structure) that, when compatible with the equipped Riemannian metric in a certain sense that we will soon develop, allows for a process called *reduction by symmetry* (or *Euler-Poincaré reduction*). Such a process turns out to be invaluable in the generation of curves that will be needed later in the thesis when discussing path planning strategies. In particular, it enables us to view ODEs evolving on the Lie group (a smooth manifold, which in general is not a vector space) as a lower-order ODE evolving on a vector space (together with a so-called reconstruction equation). Such a result allows for classical numerical integration schemes to be applied without resulting to local coordinate charts (which are known to cause erratic behavior [67]). Moreover, certain quantities such

as the Riemannian curvature endomorphism, the Levi-Civita connection, and the Riemannian exponential map can be calculated explicitly in some special cases—which greatly reduces computational complexity while increasing the precision of calculations.

This chapter is not intended to be a self-contained nor exhaustive overview of Riemannian geometry on Lie groups. For such expositions, we recommend the readers see [16, 34]. The reader who is already familiar with the basic notions of Riemannian geometry on Lie groups may skip much of this chapter, or glance through it to familiarize themself with the notation employed here. Familiarity with smooth manifolds and group theory are assumed, and Einstein summation convention is used throughout the chapter.

3.1 Lie Groups

Definition 3.1.1. A *Lie group* is a smooth manifold that is also a group, such that the group multiplication operation $(g, h) \mapsto gh$ and the group inversion operation $g \mapsto g^{-1}$ are smooth.

Unless stated otherwise, we will denote Lie groups by G, and the identity element of G will be denoted by e. Observe that the multiplication map can be used to induce two addition maps for any $g \in G$: Namely, the *left-translation map* $L_g : G \to G$ given by $L_g(h) = gh$ and the *right-translation map* $R_g : G \to G$ given by $R_g(h) = hg$ for all $h \in G$. It can be seen that both the left- and right-translations are diffeomorphisms of G onto itself. As such, given any local chart (U, φ) on G (say, a coordinate chart covering the identity element $e \in G$), it follows that $(L_g(U), \varphi \circ L_{g^{-1}}))$ is a chart covering $g \in G$. Hence, provided a single chart, the left-translation map (respectively, the right-translation map) allows to construct an atlas on G.

For any $h \in G$ and $X \in T_hG$, we may interpret the push-forward of the left-translation map as a linear map $L_{g^*} : T_hG \to T_{gh}G$. We will frequently denote $L_{g^*}(X) := gX \in T_{gh}G$ (similarly, we have $R_{g^*}(X) := Xg \in T_{hg}G$). Since left-translation is a diffeomorphism, we may apply its push-forward to entire vector fields. We say that a vector field $X \in \Gamma(TG)$ is *left-invariant* if $L_{g^*}X \equiv X \circ L_g$ for all $g \in G$. Equivalently, we have that $gX(h) = X(gh)$ for all $g, h \in G$. We further denote the vector space of left-invariant vector fields by $\mathfrak{X}_L(G)$. One of the most important features of left-invariant vector fields is

that $\mathfrak{X}_L(G) \cong T_eG$ as vector spaces under the isomorphism $\phi : T_eG \to \mathfrak{X}_L(G)$ defined by

$$\phi(\xi)(g) = L_{g*}\xi, \tag{3.1}$$

for all $\xi \in T_eG, g \in G$. Note that this identification may be made equally well with the similarly defined *right-invariant* vector fields. We often denote the left-invariant vector field associated to some $\xi \in T_eG$ by X_ξ—which satisfies $X_\xi(g) = g\xi$ for all $g \in G$.

One of the most important objects associated to a Lie group G is its *Lie algebra*, which will be denoted (unless stated otherwise) by $\mathfrak{g}$. There are some alternative (but equivalent) definitions of $\mathfrak{g}$, but in this thesis, we will choose the convention $\mathfrak{g} := T_eG$. Lie algebras are a separable concept from Lie groups, and in the most general setting, are defined as follows:

Definition 3.1.2. A *Lie algebra* is a real vector space V together with a bilinear operation $[\cdot,\cdot] : V \times V \to V$ called the *Lie bracket*, which satisfies:

(1) $[u, v] = -[v, u]$ (skew-symmetry),

(2) $[[u, v], w] + [[w, u], v] + [[v, w], u] = 0$ (Jacobi identity),

for all $u, v, w \in V$.

Given a Lie algebra $(V, [\cdot,\cdot])$, a vector subspace $U \subset V$ is said to be a *Lie subalgebra* of V provided that U is closed under the Lie bracket. That is, provided that $[u, w] \in U$ for all $u, w \in U$. Clearly, this just means that $(U, [\cdot,\cdot])$ is a Lie algebra in its own right. Let G be a Lie group with Lie algebra $\mathfrak{g}$. It can be seen that $\Gamma(TG)$ is a Lie algebra when equipped with the Lie bracket of vector fields $[\cdot,\cdot] : TG \times TG \to TG$ defined by $[X, Y] = \mathcal{L}_X Y$ for all $X, Y \in \Gamma(TG)$. Moreover, $\mathfrak{X}_L(G)$ is a Lie subalgebra of $\Gamma(TG)$. This follows from the fact that $L_{g^*}[X, Y] = [L_{g^*}X, L_{g^*}Y]$, since L_g is a diffeomorphism for all $g \in G$, together with the definition of left-invariance. In particular, under the isomorphism $\phi : \mathfrak{g} \to \mathfrak{X}_L(G)$, we may define a Lie bracket $[\cdot,\cdot]_\mathfrak{g}$ on $\mathfrak{g}$ by $[\xi, \eta]_\mathfrak{g} = [\phi(\xi), \phi(\eta)]$ for all $\xi, \eta \in \mathfrak{g}$. Hence, the Lie algebra $\mathfrak{g}$ of G is a Lie algebra in the sense of Definition 3.1.2 when equipped with the aforementioned Lie bracket.

Similarly, we say that a subgroup H of a Lie group G is a *Lie subgroup* if H is itself a Lie group, and the inclusion map $\iota : H \hookrightarrow G$ is a smooth immersion. We have immediately that if H is a Lie subgroup

of G, the Lie algebra $\mathfrak{h}$ of H is a Lie subalgebra of $\mathfrak{g}$. Observe that it is not necessarily the case that H is a submanifold of G. A Lie subgroup H of G that is also a submanifold of G will be called a *closed Lie subgroup*. This name comes from the fact that any closed Lie subgroup must be closed in the topological sense. Moreover, any closed subgroup of a Lie group is, in fact, a Lie subgroup (this result is often called Cartan's Theorem).

An important class of subgroups of Lie groups are the so-called *one-parameter subgroups*. The one parameter subgroup corresponding to some $\xi \in \mathfrak{g}$ is defined as the unique solution $\gamma_\xi : \mathbb{R} \to G$ to $\dot{g} = X_\xi(g)$ with $g(0) = e$, where X_ξ is the left-invariant vector field associated to ξ. Note that, in general, a one-parameter subgroup is not closed and hence need not be a Lie subgroup. It is, however, a *Lie group homomorphism* (that is, a smooth map which is also a group homomorphism). We define the *Lie exponential map* $\mathrm{Exp} : \mathfrak{g} \to G$ by

$$\mathrm{Exp}(\xi) = \gamma_\xi(1), \tag{3.2}$$

for all $\xi \in \mathfrak{g}$. The similarities between the definitions of the Riemannian exponential map $\exp$ as defined in Section 2.2.2 and the Lie exponential map Exp are apparent. We have simply changed the generating curves from geodesics to one-parameter subgroups. In Section 3.3, we will explore these similarities in more detail—in particular, to understand under what conditions the two coincide. For now, we state some important properties of the Lie exponential map, most of which are direct analogues of the properties of the Riemannian exponential map discussed in Lemma 2.2.4.

Lemma 3.1.3. *Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$. Then the following statements hold:*

(1) $\mathrm{Exp}(t\xi) = \gamma_\xi(t)$ *for all* $t \in \mathbb{R}$.

(2) $\mathrm{Exp}((t+s)\xi) = \mathrm{Exp}(t\xi)\mathrm{Exp}(s\xi)$ *for all* $s, t \in \mathbb{R}$.

(3) Exp *is a smooth.*

(4) Exp *is a diffeomorphism from some neighborhood of* $0 \in \mathfrak{g}$ *onto a neighborhood of* $e \in G$.

Since the Lie exponential map is a local diffeomorphism, we can define a local inverse. In particular, wherever defined, the inverse of the Lie exponential map $\mathrm{Log} : G \to \mathfrak{g}$ is called the *Logarithmic map*.

Matrix Lie groups (that is, Lie groups whose elements are $n \times n$ matrices for some fixed $n \in \mathbb{N}$) are particularly pervasive in applications. Along this thesis, we will only consider real matrices, but they can be defined over more general fields. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ of dimension $n \in \mathbb{N}$ is the set of all $n \times n$ real invertible matrices. It is easily seen that $\mathrm{GL}(n, \mathbb{R})$ is a Lie group, and that all other matrix Lie groups are necessarily subgroups of $\mathrm{GL}(n, \mathbb{R})$. Furthermore, the left-translation map (respectively, the right-translation map) is simply given by matrix multiplication from the left (right). This holds true even for the push-forward of left-translations, so that $L_{A^*}X = AX$, where $A \in \mathrm{GL}(n, \mathbb{R})$ and $X \in T_A\mathrm{GL}(n, \mathbb{R})$. It follows that the matrix commutator defined by $[A, B] = AB - BA$ for all $A, B \in \mathrm{GL}(n, \mathbb{R})$ is also the Lie bracket of vector fields on $\Gamma(T\mathrm{GL}(n, \mathbb{R}))$. In particular, the matrix commutator satisfies the conditions of a Lie bracket as defined in Definition 3.1.2. Hence, for any matrix Lie group, its Lie algebra will be a vector subspace of the space of all $n \times n$ real matrices, with the matrix commutator as its Lie bracket. It can be shown that the Lie exponential map on a matrix Lie group takes the form

$$\mathrm{Exp}(X) = \sum_{i=0}^{\infty} \frac{X^n}{n!} \tag{3.3}$$

for all X in the Lie algebra, which perfectly mirrors the Taylor series of the exponential function e^x expanded around $x = 0$.

Example 1. It is easily seen that any real vector space is a Lie group, where the group multiplication is just vector addition. Since the tangent spaces of any real vector space can be identified with the vector space itself, it follows that the Lie algebra is isomorphic (as a vector space) to the original vector space. Following the prior construction of the Lie algebra in terms of the Lie bracket of vector fields, it is clear that the Lie bracket on the Lie algebra of a vector space vanishes identically (i.e. the flows of vector fields always commute). It should, however, that other Lie brackets (and hence Lie algebras) may be considered over the same vector space. Of particular importance in mechanical applications is the Euclidean space $\mathbb{R}^3$, which becomes a Lie algebra when equipped with the Lie bracket $[x, y] = x \times y$ for all $x, y \in \mathbb{R}^3$, where $x \times y$ denotes the vector cross-product of x and y.

Example 2. A matrix Lie group that appears frequently throughout this thesis, and is ubiquitous in applications involving rigid bodies, is the *special orthogonal group* in dimension 3, denoted by $\mathrm{SO}(3)$. In particular,

$\mathrm{SO}(3)$ is the space of all 3×3 orthogonal matrices with determinant 1. That is, the matrices $R \in \mathrm{GL}(3, \mathbb{R})$ such that $R^T R = I$ and $\det(R) = 1$. These represent the *pure rotations* when viewing the elements of $\mathrm{SO}(3)$ as acting on vectors in $\mathbb{R}^3$. It can be seen that its Lie algebra, denoted by $\mathfrak{so}(3)$, is the space of real 3×3 skew-symmetric matrices equipped with the matrix commutator. The *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is a Lie algebra isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$ equipped with the cross-product, and is defined by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{3.4}$$

. It is easily verified that $\hat{x}y = x \times y$ and $[\hat{x}, \hat{y}] = \widehat{x \times y}$ for all $x, y \in \mathbb{R}^3$. The inverse of the hat map is called the *vee map*, and is denoted by $\vee\cdot : \mathfrak{so}(3) \to \mathbb{R}^3$. Similarly, the dual space $\mathfrak{so}(3)^*$ may be identified with $\mathbb{R}^3$ via the *breve map* $\breve{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)^*$ defined implicitly by the relation $\left\langle \breve{\Pi}, \hat{\Omega} \right\rangle_* = \Pi \cdot \Omega$ for all $\Pi, \Omega \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle_*$ denotes the natural pairing of $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$, and $\Pi \cdot \Omega$ denotes the standard dot product in $\mathbb{R}^3$. It is easy to see by identifying $\mathfrak{so}(3) \cong \mathbb{R}^3$ and $\mathfrak{so}(3)^* \cong \mathbb{R}^3$, and making use of the natural pairing on $\mathbb{R}^3$ (that is, the dot product), that $\left\langle \breve{\Pi}, \hat{\Omega} \right\rangle_* = \frac{1}{2}\mathrm{tr}\left(\hat{\Pi}\hat{\Omega}^T\right)$ for all $\Pi, \Omega \in \mathbb{R}^3$, where tr denotes the trace of a matrix. It follows that $\breve{\Pi}$ can be identified with $\frac{1}{2}\hat{\Pi}$ under the isomorphism $\mathfrak{so}(3)^* \to \mathfrak{so}(3)$ induced by the trace pairing.

3.2 Lie Group Actions

One of the most important aspects of groups is their use in understanding the symmetry properties of some set via *group actions*. That is, the types of groups that are able to act upon a set (together with the properties of the particular group actions) can illuminate important properties related to the structure and symmetry of the set itself. The same can be said for Lie groups, where now the sets that they act upon are smooth manifolds.

3.2.1 Lie groups acting on smooth manifolds

Definition 3.2.1. Let G be a Lie group and Q be a smooth manifold. A *left-action* of G on Q is a smooth map $\Phi : G \times Q \to Q$ such that

(1) $\Phi(e, q) = q$,

(2) $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$,

(3) The map $\Phi_g : Q \to Q$ defined by $\Phi_g(q) = \Phi(g, q)$ is a diffeomorphism,

for all $g, h \in G$, $q \in Q$.

Right-actions may be similarly defined, with $\Phi(g, \Phi(h, q)) = \Phi(hg, q)$ taking the place of condition (2) in Definition 3.2.1. We will frequently use the notation $\Phi(g, q) = gq$ (and $\Phi(g, q) = qg$ for right-actions), which mirrors the notation used for left-translation. Note that this is consistent, as the multiplication operator on a Lie group G can be interpreted as a left-action of G on itself, as is easily verified. Moreover, it is clear that any left-action $\Phi : G \times Q \to Q$ induces a right-action $\Psi : G \times Q \to Q$ given by $\Psi(g, q) = \Phi(g^{-1}, q)$.

Any left-action $\Phi : G \times Q \to Q$ induces an action $T\Phi : G \times TQ \to TQ$, called the *tangent lifted action*, given by $T\Phi(g, X) = \Phi_{g^*}X$ for all $g \in G, X \in TQ$. We will denote this action by $T\Phi(g, X) = gX$ whenever there is no chance for confusion. Similarly, Φ induces an action $T^*\Phi : G \times T^*Q \to T^*Q$, called the *cotangent lifted action*, which is given by $T^*\Phi(g, \omega) = \Phi^*_{g^{-1}}\omega$ for all $g \in G, \omega \in T^*Q$. Hence,

$$\left\langle T^*\Phi(g, \omega), X \right\rangle_* = \left\langle \Phi^*_{g^{-1}}\omega, X \right\rangle_* = \left\langle \omega, \Phi_{g^{-1}*}X \right\rangle_* = \left\langle \omega, T\Phi(g^{-1}, X) \right\rangle_*,$$

for all $q \in Q, g \in G, \omega \in T^*_qQ, X \in T_{gq}Q$, where $\langle \cdot, \cdot \rangle_*$ denotes the natural pairing of vectors and covectors.

Definition 3.2.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose that G acts on a smooth manifold Q on the left. For any $\xi \in \mathfrak{g}$, we define the *infinitesimal generator* associated to ξ, denoted by ξ_Q, as

$$\xi_Q(q) = \frac{d}{dt}\Big|_{t=0}(\text{Exp}(t\xi)q), \tag{3.5}$$

for all $q \in Q$, where $\text{Exp} : \mathfrak{g} \to G$ is the Lie exponential map on G.

Informally, we think of the infinitesimal generator as a map $\mathfrak{g} \times Q \to TQ$ which describes the *infinitesimal action* of $\mathfrak{g}$ on Q, despite not being a group action in the sense of Definition 3.2.1. Moreover,

the map $\mathfrak{g} \to \Gamma(TQ)$ which maps $\xi \mapsto \xi_Q$ is linear and satisfies $[\xi, \eta]_\mathfrak{g} = -[\xi_Q, \eta_Q]$ for all $\xi, \eta \in \mathfrak{g}$, where $[\cdot, \cdot]_\mathfrak{g}$ is the Lie bracket on $\mathfrak{g}$ and $[\cdot, \cdot]$ is the Lie bracket of vector fields. The infinitesimal generator induces another important map, called the *momentum map* $J : T^*Q \to \mathfrak{g}^*$, which is defined implicitly by the relation

$$\langle J(\omega), \xi \rangle_* = \langle \omega, \xi_Q(q) \rangle_* , \tag{3.6}$$

for all $q \in Q, \omega \in T_q^*Q, \xi \in \mathfrak{g}$. Momentum maps turn out to be a vital concept in the process of reduction by symmetry, as we will see in Section 3.4. Moreover, they serve as the connection between the symmetry of a manifold (invariance under group actions) and conserved quantities, a result which is often known as *Noether's Theorem* [55].

We now define some important classes of group actions:

Definition 3.2.3. Let G be a Lie group, Q be a smooth manifold, and $\Phi : G \times Q \to Q$ be a left-action. We say that Φ is:

(1) *transitive* if for every $p, q \in Q$, there exists some $g \in G$ such that $gp = q$.

(2) *faithful* if for all $g \neq e$, there exists some $q \in Q$ such that $gq \neq q$.

(3) *free* if $gq = q$ implies that $g = e$,

(4) *proper* if, whenever the sequences $(q_n), (g_n q_n) \subset Q$ converge in Q, the sequence $(g_n) \subset G$ has a convergent subsequence in G.

Of these, the most important for us along this thesis are the transitive group actions. They serve as the basis for *homogeneous spaces*, which we will study in Chapter 4 and revisit in Chapter 7 when discussing reduction by symmetry for path-planning.

3.2.2 Lie groups acting on themselves

One of the most important cases of a Lie group action is a Lie group acting on itself. As we have already mentioned, the left and right-translation maps can be viewed as left and right-actions of G on itself, respectively. In particular, we have the left-action $L : G \times G \to G$ given by $L(g, h) = L_g(h) = gh$ for all

$g, h \in G$ (and identically for right-actions with R_g replacing L_g). Of greater interest to us, however, is the so-called *conjugate left-action* $\Phi : G \times G \to G$ given by $\Phi(g,h) = ghg^{-1}$ for all $g, h \in G$. This action induces a number of important actions:

(1) The *Adjoint action* $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{Ad}(g,\xi) = \mathrm{Ad}_g\xi := T\Phi(g,\xi)$ for all $g \in G, \xi \in \mathfrak{g}$.

(2) The *Co-Adjoint action* $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ denoted by $\mathrm{Ad}^*(g,\omega) = \mathrm{Ad}^*_{g^{-1}}\omega$ and defined implicitly by the relation $\left\langle \mathrm{Ad}^*_{g^{-1}}\omega, \xi \right\rangle_* = \left\langle \omega, \mathrm{Ad}_{g^{-1}}\xi \right\rangle_*$ for all $g \in G, \omega \in \mathfrak{g}^*, \xi \in \mathfrak{g}$.

(3) The *adjoint action* $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $\mathrm{ad}(\xi,\eta) = \mathrm{ad}_\xi\eta := \xi_{\mathfrak{g}}(\eta)$ for all $\xi, \eta \in \mathfrak{g}$, where $\xi_{\mathfrak{g}}$ is the infinitesimal generator associated to ξ under the Adjoint action of G on $\mathfrak{g}$ (and where we've made the identification $T\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}$). In particular, we have $\mathrm{ad}_\xi(\eta) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(t\xi)}\eta$. It can be seen that, in fact, we have $\mathrm{ad}_\xi\eta = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}$.

(4) The *coadjoint action* $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined implicitly by $\left\langle \mathrm{ad}^*_\xi\omega, \eta \right\rangle_* = \langle \omega, \mathrm{ad}_\xi\eta \rangle_*$ for all $\xi, \eta \in \mathfrak{g}, \omega \in \mathfrak{g}^*$. It is not difficult to see that the coadjoint action satisfies $\mathrm{ad}^*_\xi\omega = -\xi_{\mathfrak{g}^*}(\omega)$ for all $\xi \in \mathfrak{g}, \omega \in \mathfrak{g}^*$, where $\xi_{\mathfrak{g}^*}$ denotes the infinitesimal generator associated to ξ under the Co-Adjoint action.

Example 3. Every matrix Lie group of dimension $G \subset \mathrm{GL}(n,\mathbb{R})$ has a natural left-action on $\mathbb{R}^n$ given by $\Phi(M,x) = Mx$ for all $M \in G$ and $x \in \mathbb{R}^n$. Due to the natural identification of $\mathbb{R}^n$ with its tangent and cotangent spaces, the tangent lifted and cotangent lifted actions of G on $\mathbb{R}^n$ are given by $T\Phi(M,x) = Mx$ and $T^*\Phi(M,x) = (M^{-1})^T x$. Moreover, the infinitesimal generator associated to $\Omega \in \mathfrak{g}$ at $M \in G$ is given by the matrix product $\Omega_Q(M) = \Omega M$.

The left-action $L : G \times G \to G$ induced by the left-translation map is given by $L(M,R) = L_M(R) = MR$ for all $M, R \in G$. The tangent lifted action is similarly given by $TL(M,\Omega) = L_{M^*}\Omega = M\Omega$ for all $M \in G, \Omega \in TG$. The cotangent lifted action satisfied $T^*L(M,\Pi) = L^*_{M^{-1}}\Pi = (M^{-1})^T\Pi$ for all $M \in G, \Pi \in T^*G$.

The Adjoint action is given by $Ad_M\Omega = M\Omega M^{-1}$ for all $M \in G, \Omega \in TG$. The Co-Adjoint action then satisfies $\left\langle Ad^*_{M^{-1}}\Pi, \Omega \right\rangle_* = \left\langle \Pi, M^{-1}\Omega M \right\rangle_*$ for all $M \in G, \Pi \in \mathfrak{g}^*, \Omega \in \mathfrak{g}$. Via the trace pairing, we

find that $\left\langle \Pi, M^{-1}\Omega M \right\rangle_* = \left\langle (M^{-1})^T \Pi M^T, \Omega \right\rangle_*$, from which it follows that $\mathrm{Ad}^*_{M^{-1}}\Pi = (M^{-1})^T \Omega M^T$. The adjoint action is simply the matrix commutator $\mathrm{ad}_\Omega X = [\Omega, X] = \Omega X - X\Omega$ for all $\Omega, X \in \mathfrak{g}$. It follows that the coadjoint action satisfies $\langle \mathrm{ad}^*_\Omega \Pi, X\rangle = \langle \Pi, \mathrm{ad}_\Omega X\rangle = \langle \Pi, \Omega X - X\Omega\rangle = \mathrm{tr}((\Omega^T\Pi - \Pi\Omega^T)X^T) = \left\langle -[\Pi, \Omega^T], X \right\rangle$ for all $\Omega, X \in \mathfrak{g}, \Pi \in \mathfrak{g}^*$. It follows that $\mathrm{ad}^*_\Omega \Pi = -[\Pi, \Omega^T]$. In particular, for $G = \mathrm{SO}(3)$, we see that:

(1) $\mathrm{Ad}_M \Omega = M\Omega M^T$,

(2) $\mathrm{Ad}^*_M \Pi = M^T \Pi M$,

(3) $\mathrm{ad}_\Omega X = \Omega X - X\Omega$,

(4) $\mathrm{ad}^*_\Omega \Pi = \Pi\Omega - \Omega\Pi$,

for all $M \in \mathrm{SO}(3), \Omega, X \in \mathfrak{so}(3), \Pi \in \mathfrak{so}(3)^*$. Note, however, that through the hat map (as defined in Example 2), we see that for all $M \in \mathrm{SO}(3)$ and $\Omega \in \mathbb{R}^3$,

$$\mathrm{Ad}_M \hat{\Omega} = \widehat{M\Omega}. \tag{3.7}$$

Hence, the Adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)$ may be identified with the left-translation action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. Similarly, using the breve map, we see that for all $M \in \mathrm{SO}(3), \Pi \in \mathbb{R}^3$, the Co-Adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$ satisfies

$$\mathrm{Ad}^*_{M^{-1}} \breve{\Pi} = \breve{M\Pi}, \tag{3.8}$$

so that the Co-Adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$ may also be identified with the left-translation action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. By definition, we have $\mathrm{ad}_{\hat{\Omega}}\hat{\eta} = \hat{\Omega}\hat{\eta} - \hat{\eta}\hat{\Omega} = \Omega$, which can be easily seen to satisfy:

$$\mathrm{ad}_{\hat{\Omega}}\hat{\eta} = \widehat{\Omega \times \eta} \tag{3.9}$$

for all $\Omega, \eta \in \mathbb{R}^3$. Finally, we see under these same identifications that the coadjoint action of $\mathfrak{so}(3)$ on $\mathfrak{so}(3)^*$ satisfies

$$\mathrm{ad}^*_{\hat{\Omega}} \breve{\Pi} = \widetilde{\Pi \times \Omega}, \tag{3.10}$$

for all $\Omega, \Pi \in \mathbb{R}^3$.

3.3 Invariant Metrics on Lie Groups

We now specialize our attention to Riemannian geometry on Lie groups. Of course, since every Lie group is by definition a smooth manifold, they can be equipped with a Riemannian metric (as discussed in Section 2.1)—at which point all the results discussed in Chapter 2 follow identically. What we are interested in, however, are metrics which in some sense preserve the structure of the Lie group.

3.3.1 Left-invariant metrics

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and consider some inner-product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. Left-translation then provides a Riemannian metric $\langle \cdot, \cdot \rangle$ on G via the relation: $\langle X, Y \rangle := \left\langle g^{-1}X, g^{-1}Y \right\rangle_{\mathfrak{g}}$ for all $g \in G, X, Y \in T_gG$. Such a Riemannian metric is called *left-invariant*, due to the fact that $\langle gX, gY \rangle = \langle X, Y \rangle$ for all $g, h \in G$ and $X, Y \in T_hG$. It follows immediately that there is a one-to-one correspondence between left-invariant Riemannian metrics on G and inner products on $\mathfrak{g}$, and that $L_g : G \to G$ is an isometry for all $g \in G$ by construction. We may similarly define right-invariant metrics via the relation $\langle X, Y \rangle := \left\langle Xg^{-1}, Yg^{-1} \right\rangle_{\mathfrak{g}}$. We have the following results from [34]:

Lemma 3.3.1. *Let G be a Lie group equipped with a left-invariant (or right-invariant) metric. Then G is complete as a Riemannian manifold*

Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$, and equip G with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Let ∇ be the Levi-Civita connection on G with respect to $\langle \cdot, \cdot \rangle$. Using the isomorphism $\phi : \mathfrak{g} \to \mathfrak{X}_L(G)$ (3.1), we construct an operator $\nabla^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by:

$$\nabla^{\mathfrak{g}}_{\xi}\eta := \nabla_{\phi(\xi)}\phi(\eta)(e), \tag{3.11}$$

for all $\xi, \eta \in \mathfrak{g}$. Although $\nabla^{\mathfrak{g}}$ is not a connection in the strict sense, we shall refer to it as the *Riemannian $\mathfrak{g}$-connection* corresponding to ∇, due to the fact that it has many similar properties:

Lemma 3.3.2. *$\nabla^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear, and for all $\xi, \eta, \sigma \in \mathfrak{g}$, the following relations hold:*

(1) $\nabla^{\mathfrak{g}}_{\xi}\eta - \nabla^{\mathfrak{g}}_{\eta}\xi = [\xi, \eta]_{\mathfrak{g}}$,

(2) $\left\langle \nabla^{\mathfrak{g}}_{\sigma}\xi, \eta \right\rangle + \left\langle \xi, \nabla^{\mathfrak{g}}_{\sigma}\eta \right\rangle = 0.$

Proof. The $\mathbb{R}$-bilinearity of $\nabla^{\mathfrak{g}}$ follows immediately from (3.11). For (1), observe that

$$\nabla^{\mathfrak{g}}_{\xi}\eta - \nabla^{\mathfrak{g}}_{\eta}\xi = \left(\nabla_{\phi(\xi)}\phi(\eta) - \nabla_{\phi(\eta)}\phi(\xi)\right)(e) = [\phi(\xi), \phi(\eta)](e) = [\xi, \eta]_{\mathfrak{g}}$$

since ∇ is the torsion-free. For (2), note that, since ∇ is compatible with $\langle \cdot, \cdot \rangle$, we have that

$$\begin{aligned}\mathcal{L}_{\phi(\sigma)} \langle \phi(\xi), \phi(\eta) \rangle (e) &= \left\langle \nabla_{\phi(\sigma)}\phi(\xi)(e), \phi(\eta)(e) \right\rangle + \left\langle \phi(\xi)(e), \nabla_{\phi(\sigma)}\phi(\eta)(e) \right\rangle \\ &= \langle \nabla^{\mathfrak{g}}_{\sigma}\xi, \eta \rangle + \langle \xi, \nabla^{\mathfrak{g}}_{\sigma}\eta \rangle ,\end{aligned}$$

where $\mathcal{L}$ stands for the Lie derivative of vector fields. But since $\phi(\xi), \phi(\eta)$ are left-invariant, $\langle \phi(\xi), \phi(\eta) \rangle$ is a constant function, so $\mathcal{L}_{\phi(\sigma)} \langle \phi(\xi), \phi(\eta) \rangle \equiv 0$. □

Remark 1. We may consider the Riemannian $\mathfrak{g}$-connection as an operator $\nabla^{\mathfrak{g}} : C^{\infty}([a,b], \mathfrak{g}) \times C^{\infty}([a,b], \mathfrak{g}) \to C^{\infty}([a,b], \mathfrak{g})$ in a natural way. Namely, if $\xi, \eta \in C^{\infty}([a,b], \mathfrak{g})$, we can write $(\nabla^{\mathfrak{g}}_{\xi}\eta)(t) := \nabla^{\mathfrak{g}}_{\xi(t)}\eta(t)$ for all $t \in [a,b]$. With this notation, Lemma 3.3.2 works identically if we replace $\xi, \eta, \sigma \in \mathfrak{g}$ with $\xi, \eta, \sigma \in C^{\infty}([a,b], \mathfrak{g})$. ◇

We define an operator $\mathrm{ad}^{\dagger} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}^{\dagger}_{\xi}\eta = (\mathrm{ad}^{*}_{\xi}\eta^{\flat})^{\sharp}$ for all $\xi, \eta \in \mathfrak{g}$. It follows that ad^{*} is bilinear, and satisfies $\left\langle \mathrm{ad}^{\dagger}_{\xi}\eta, \sigma \right\rangle = \left\langle \mathrm{ad}^{*}_{\xi}\eta^{\flat}, \sigma \right\rangle_{*} = \langle \eta^{\flat}, \mathrm{ad}_{\xi}\sigma \rangle_{*} = \langle \eta, \mathrm{ad}_{\xi}\sigma \rangle$ for all $\xi, \eta, \sigma \in \mathfrak{g}$. Hence, ad^{*} is nothing more than the adjoint of the adjoint action with respect to the Riemannian metric on G. Using this, and the Kozsul formula (2.8), we obtain the following decomposition of $\nabla^{\mathfrak{g}}$ (see Theorem 5.40 of [16], for instance)

Lemma 3.3.3. *The Riemannian $\mathfrak{g}$-connection satisfies:*

$$\nabla^{\mathfrak{g}}_{\xi}\eta = \frac{1}{2}\left([\xi, \eta]_{\mathfrak{g}} - \mathrm{ad}^{\dagger}_{\xi}\eta - \mathrm{ad}^{\dagger}_{\eta}\xi \right), \tag{3.12}$$

for all $\xi, \eta \in \mathfrak{g}$.

Given a basis $\{A_i\}$ of $\mathfrak{g}$, we may write any vector field $X \in \Gamma(TG)$ as $X = X^i \phi(A_i)$, where $X^i : G \to \mathbb{R}$ are the component functions. If $g \in H^k([a,b], G)$ and $X \in H^2_g(g)$, then we may equivalently

write $X = X^i g A_i$, where now $X^i : [a,b] \to \mathbb{R}$ is of Sobolev class H^2 and $gA_i =: L_g A_i$. We denote $\dot{X} := \dot{X}^i A_i$, which may be written in a coordinate-free fashion via $\dot{X}(t) = \frac{d}{dt}\left(g(t)^{-1}X(t)\right)$. We wish to understand how the Levi-Civita connection D_t along g is related to the Riemannian $\mathfrak{g}$-connection $\nabla^{\mathfrak{g}}$. This relation is summarized in the following result:

Lemma 3.3.4. *Consider a Lie group G with Lie algebra $\mathfrak{g}$ and left-invariant Levi-Civita connection ∇. Let $g \in \Omega$ and X be a C^1 vector field along g with $eta := g^{-1}X$. Then the following relation holds for all $t \in [a,b]$:*

$$D_t X(t) = g(t)\left(\dot{X}(t) + \nabla^{\mathfrak{g}}_{\xi}\eta(t)\right). \tag{3.13}$$

Proof. Let $\{A_i\}$ be a basis for $\mathfrak{g}$, and suppose that $\xi(t) := g(t)^{-1}\dot{g}(t) = \xi^i(t)A_i$ and $\eta(t) := g(t)^{-1}X(t) = \eta^i(t)A_i$. Expanding out the left-hand side of (3.13), and using that $L_{g^*}(\eta_j A_j) = \eta_j L_{g^*}A_j$, we obtain

$$D_t X(t) = D_t\left(\eta^j g A_j\right)(t) = g(t)\frac{\partial \eta^j}{\partial t}(t)A_i + \eta^j(t) D_t\left(gA_j\right).$$

Note that $g(t)A_i = \phi(A_i)(g(t))$ for all $t \in [a,b], 1 \le i \le n$. Hence,

$$D_t\left(gA_j(t)\right) = \xi^i(t)\left(\nabla_{\phi(A_i)}\phi(A_j)\right)(g(t)) = g(t)\xi^i(t)\left(\nabla_{\phi(A_i)}\phi(A_j)\right)(e) = g(t)\xi^i(t)\nabla^{\mathfrak{g}}_{A_i}A_j,$$

since $\phi(A_i)$ is left-invariant for all $1 \le i \le n$ and left-translation is an isometry. Equation (3.13) then follows by the bi-linearity of $\nabla^{\mathfrak{g}}$ together with the definition of $\dot{X}$ with respect to the basis $\{A_i\}$. □

Lemma 3.3.5. *Suppose that $\mathcal{Y}, \mathcal{Z} : [a,b] \to \mathfrak{g}$. Then:*

$$\frac{d}{dt}\langle \mathcal{Y}, \mathcal{Z}\rangle = \left\langle \dot{\mathcal{Y}}, \mathcal{Z}\right\rangle + \left\langle \mathcal{Y}, \dot{\mathcal{Z}}\right\rangle \tag{3.14}$$

Proof. Let $\mathcal{T} := \gamma^{-1}\dot{\gamma}$. From Lemma 3.3.4, the metric compatibility of ∇, and the left-invariance of the

metric, we have that:

$$\begin{aligned}
\frac{d}{dt}\langle \mathcal{Y}, \mathcal{Z}\rangle &= \frac{d}{dt}\langle \gamma\mathcal{Y}, \gamma\mathcal{Z}\rangle \\
&= \langle \nabla_T \gamma\mathcal{Y}, \gamma\mathcal{Z}\rangle + \langle \gamma\mathcal{Y}, \nabla_T \gamma\mathcal{Z}\rangle, \\
&= \left\langle \gamma\left(\dot{\mathcal{Y}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Y}\right), \gamma\mathcal{Z}\right\rangle + \left\langle \gamma\mathcal{Y}, \gamma\left(\dot{\mathcal{Z}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Z}\right)\right\rangle, \\
&= \left\langle \dot{\mathcal{Y}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Y}, \mathcal{Z}\right\rangle + \left\langle \mathcal{Y}, \dot{\mathcal{Z}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Z}\right\rangle, \\
&= \left\langle \dot{\mathcal{Y}}, \mathcal{Z}\right\rangle + \left\langle \mathcal{Y}, \dot{\mathcal{Z}}\right\rangle + \langle \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Y}, \mathcal{Z}\rangle + \langle \mathcal{Y}, \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{Z}\rangle, \\
&= \left\langle \dot{\mathcal{Y}}, \mathcal{Z}\right\rangle + \left\langle \mathcal{Y}, \dot{\mathcal{Z}}\right\rangle,
\end{aligned}$$

where in the last equality we've applied Lemma 3.3.2 in the context of Remark 1. □

Remark 2. Here we see a stark divergence from [1]. Namely, it is claimed in Proposition 5.3 that $\left\langle \dot{\mathcal{Y}}, \mathcal{Z}\right\rangle + \left\langle \mathcal{Y}, \dot{\mathcal{Z}}\right\rangle = 0$, implying that $\frac{d}{dt}\langle \mathcal{Y}, \mathcal{Z}\rangle = 0$. Considering the case where $\mathcal{Z} = \mathcal{Y}$, this would then imply that every curve in the Lie algebra has constant length, which cannot be true. Consider for instance a curve $\xi(t)$ of constant length c, and define $\eta(t) = f(t)\xi(t)$ for some smooth non-constant real-valued function f. Then $\langle \eta(t), \eta(t)\rangle = f(t)^2 \langle \xi(t), \xi(t)\rangle = (cf(t))^2$, which is non-constant by assumption.

In anticipation of working with variational principles, we momentarily switch our attention to variations. Consider a curve $g \in H^2([a,b], G)$ and an admissible proper variation $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to G$ of g with variational vector field $\frac{\partial}{\partial s}\Big|_{s=0}\Gamma(s,t) = \delta g(t)$. Denote the vector fields along the main and transverse curves of Γ as $T(s,t) := \frac{\partial}{\partial t}\Gamma(s,t)$ and $S(s,t) = \frac{\partial}{\partial s}\Gamma(s,t)$, respectively. Further denote their pullback (via left-translations) to the Lie algebra $\mathfrak{g}$ by $\mathcal{T}(s,t) := \Gamma(s,t)^{-1}T(s,t)$ and $\mathcal{S}(s,t) := \Gamma(s,t)^{-1}S(s,t)$ for all $(s,t) \in (-\epsilon, \epsilon) \times [a,b]$. Observe the following relations:

$$\begin{aligned}
T(0,t) &= \dot{g}(t), & \mathcal{T}(0,t) &= g(t)^{-1}\dot{g}(t), \\
S(0,t) &= \delta g(t), & \mathcal{S}(0,t) &= g(t)^{-1}\delta g(t).
\end{aligned}$$

Consider a basis $\{A_i\}_{i=1}^{\dim(G)}$ of $\mathfrak{g}$, and write $\mathcal{T} = a^i(s,t)A_i$ and $\mathcal{S} = b^i(s,t)A_i$ for some Sobolev class H^2 coordinate functions a^i, b^i on $(-\epsilon, \epsilon) \times [a,b]$. We denote $\dot{\mathcal{S}} = \frac{\partial b^i}{\partial t}(s,t)A_i$ and $\mathcal{T}' = \frac{\partial a^i}{\partial s}(s,t)A_i$. Then Lemma 3.3.4 gives the following corollary:

Corollary 1. *The relationships*

$$\nabla_T S(s,t) = \Gamma(s,t)\big(\dot{\mathcal{S}}(s,t) + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{S}(s,t)\big) \tag{3.15}$$

$$\nabla_S T(s,t) = \Gamma(s,t)\big(\mathcal{T}'(s,t) + \nabla^{\mathfrak{g}}_{\mathcal{S}}\mathcal{T}(s,t)\big) \tag{3.16}$$

hold for all $(s,t) \in (-\epsilon,\epsilon)\times[a,b]$ *among all proper variations* Γ *of g, with* $S,T,\mathcal{S},\mathcal{T}$ *as defined above. Moreover, we have that*

$$\dot{\mathcal{T}} = \mathcal{S}' + [\mathcal{T},\mathcal{S}]_{\mathfrak{g}}\,. \tag{3.17}$$

Proof. For equation (3.15), we need to only apply Lemma 3.3.4 with $g(t) = \Gamma(s,t)$ and $X(t) = S(s,t)$ for fixed $s \in (-\epsilon,\epsilon)$. Similarly, equation (3.16) follows with $g(t) = \Gamma(s,t)$ and $X(t) = T(s,t)$ for fixed $t \in [a,b]$. Finally, equation (3.17) follows from (3.15) and (3.16) together with the Symmetry Lemma 2.23. □

Remark 3. It is stated in lemma 5.2 of [1] that $\mathcal{T}' = \dot{\mathcal{S}}$ (so that, in particular, $[\mathcal{T},\mathcal{S}] \equiv 0$). We provide a counterexample to this assertion now. Consider $G = \mathrm{GL}(2,\mathbb{R})$ with Lie algebra $\mathfrak{gl}(2,\mathbb{R})$. Let $g : [0,1] \to \mathrm{GL}(2,\mathbb{R})$ be defined by $g(t) = \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix}$, and consider the admissible proper variation $\Gamma(s,t) = \begin{bmatrix} t+f(s,t) & -1 \\ 1+f(s,t) & 0 \end{bmatrix}$, where f is a smooth function such that $f(0,t) = f(s,0) = f(s,1) = 0$ and $1 + f(s,t) > c > 0$ for all $(s,t) \in (-\epsilon,\epsilon)\times[0,1]$ and some $c \in \mathbb{R}$. It follows that $\Gamma(s,t)^{-1} = \frac{1}{1+f(s,t)}\begin{bmatrix} 0 & 1 \\ -(1+f(s,t)) & t+f(s,t) \end{bmatrix}$ and so:

$$T = \frac{\partial}{\partial t}\Gamma = \begin{bmatrix} 1+f_t & 0 \\ f_t & 0 \end{bmatrix}, \qquad S = \frac{\partial}{\partial s}\Gamma = \begin{bmatrix} f_s & 0 \\ f_s & 0 \end{bmatrix},$$

$$\mathcal{T} = \frac{1}{1+f}\begin{bmatrix} f_t & 0 \\ tf_t - f - (t+1) & 0 \end{bmatrix}, \qquad \mathcal{S} = \frac{1}{1+f}\begin{bmatrix} f_s & 0 \\ f_s(t-1) & 0 \end{bmatrix},$$

where f_t, f_s denote the partial derivatives of f with respect to t and s, respectively. We show that $[\mathcal{T},\mathcal{S}] \neq 0$ for appropriate choice in f. As discussed in Section 3.1, the Lie bracket is simply the matrix commutator,

so in particular we need to show that $\mathcal{TS} \neq \mathcal{ST}$. This reduces to the condition

$$f_s(tf_t - f - (t+1)) = f_s f_t(t-1) \tag{3.18}$$

for all $(s,t) \in (\epsilon,\epsilon) \times [0,1]$. We choose $f(s,t) = st(t-1)$ for $s \in (-\frac{1}{2}, \frac{1}{2})$, which clearly satisfies all of the constraints imposed on f. It is easy to see that $f(s,\frac{1}{2}) = \frac{1}{4}s$, $f_t(s,\frac{1}{2}) = 0$, and $f_s(s,\frac{1}{2}) = -\frac{1}{4}$. By direct computation, we see that (3.18) is not satisfied at $t = \frac{1}{2}$ for any $s \in (-\frac{1}{2}, \frac{1}{2})$.

3.3.2 Bi-invariant Metrics

Now we wish to discuss another important class of Riemannian metrics on a Lie group, the so-called *bi-invariant* (or Ad-*invariant*) metrics. These are the Riemannian metrics $\langle \cdot, \cdot \rangle$ on G which are both left- and right-invariant. Unlike left- and right-invariant metrics, not every Lie group G admits a bi-invariant metric. The following Theorem from [34] provides necessary and sufficient conditions for the existence of a bi-invariant metric.

Lemma 3.3.6. *A connected Lie group admits a bi-invariant metric if and only if it is isomorphic to the Cartesian product of a compact Lie group and a finite-dimensional vector space. Moreover, such a metric is unique up to scalar multiplication.*

Despite this limitation, many important examples of Lie groups satisfy the conditions: In particular, two of our most important model spaces $\mathrm{SO}(3)$ and $\mathbb{R}^3$. This is because $\mathrm{SO}(3)$ is a compact Lie group, and $\mathbb{R}^3$ is a finite-dimensional vector space. Bi-invariant metrics have many nice properties that greatly simplify calculations in practice. First, it is clear that for all $g \in G$, $X, Y \in T_gG$, we have $\langle X, Y \rangle = \left\langle gXg^{-1}, gYg^{-1} \right\rangle = \langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y \rangle$ (this is why such metrics are also called Ad-invariant). Let $\xi, \eta, \sigma \in \mathfrak{g}$. Then, $\langle \eta, \sigma \rangle = \left\langle \mathrm{Ad}_{\mathrm{Exp}(t\xi)}\eta, \mathrm{Ad}_{\mathrm{Exp}(t\xi)}\sigma \right\rangle$. Differentiating at $t = 0$ and making use of Lemma 3.3.5, we see that $0 = \langle \mathrm{ad}_\xi \eta, \sigma \rangle + \langle \eta, \mathrm{ad}_\xi \sigma \rangle$, which implies that $\left\langle \mathrm{ad}_\xi^\dagger \eta, \sigma \right\rangle = \langle -\mathrm{ad}_\xi \eta, \sigma \rangle$. Hence, $\mathrm{ad}_\xi^\dagger \eta = -\mathrm{ad}_\xi \eta = [\eta, \xi]$ for all $\xi, \eta \in \mathfrak{g}$. Plugging this into (3.12), we obtain the following lemma:

Lemma 3.3.7. *Consider a Lie group G equipped with a bi-invariant metric. Let ∇ be the Levi-Civita connection and $\nabla^{\mathfrak{g}}$ be the corresponding Riemannian $\mathfrak{g}$-connection. Then:*

(1) $\nabla^{\mathfrak{g}}_{\xi}\eta = \frac{1}{2}[\xi, \eta]$,

(2) $R(\xi, \eta)\sigma = \frac{1}{4}[[\xi, \eta], \sigma]$,

for all $\xi, \eta, \sigma \in \mathfrak{g}$.

We now show that the Lie exponential map $\text{Exp} : \mathfrak{g} \to G$ agrees with the restriction of the Riemannian exponential map $\exp : TG \to G$ to the Lie algebra, where $\exp$ is taken with respect to the Levi-Civita connection of a bi-invariant metric. In particular, this implies that geodesics through the identity are just one-parameter subgroups (and vice verse).

Theorem 3.3.8. *Consider a Lie group* G *equipped with a bi-invariant metric. For all* $\xi \in \mathfrak{g}$*, we have* $\text{Exp}(\xi) = \exp_e(\xi)$*, where* Exp *is the Lie exponential map, and* $\exp_e : \mathfrak{g} \to G$ *is the Riemannian exponential map at the identity* $e \in G$ *with respect to the Levi-Civita connection.*

Theorem 3.3.8 turns out to be very useful in many applications on compact matrix Lie groups, due to the simple expression for the Lie exponential map (3.3). In cases where we are able to calculate the Logarithmic map $\text{Log} : G \to \mathfrak{g}$ directly (on some neighborhood of the identity element $e \in G$), we may also calculate the Riemannian distance explicitly using (2.26). In particular, we find that $d(e, g) = \|\text{Log}(g)\|$ wherever it is defined.

Example 4. It is well-known that the attitude of a rigid body can be modelled on $G = \text{SO}(3)$, equipped with the left-invariant Riemannian metric $\left\langle \dot{R}_1, \dot{R}_2 \right\rangle = \text{tr}(\dot{R}_1 \mathbb{M} \dot{R}_2^T)$ for all $R \in \text{SO}(3), \dot{R}_1, \dot{R}_2 \in T_R\text{SO}(3)$, where $\mathbb{M}$ is a symmetric positive-definite 3×3 matrix called the *coefficient of inertia matrix*. On the Lie algebra $\mathfrak{so}(3)$, the metric reduces to the inner-product

$$\left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle = \Omega_1^T \mathbb{J} \Omega_2, \tag{3.19}$$

for all $\Omega_1, \Omega_2 \in \mathbb{R}^3$, where we have made use of the hat map (3.4) to identify $\mathfrak{so}(3) \cong \mathbb{R}^3$, and $\mathbb{J}$ is a symmetric positive-definite 3×3 matrix called the *moment of inertia tensor*. In the case that $\mathbb{M} = \mathbb{J} = I$, the metric is in fact bi-invariant (and is the unique bi-invariant metric on $\text{SO}(3)$ up to scalar multiplication by Lemma 3.3.6). In this case, we have the following Lemma ([16], Proposition 5.7):

Lemma 3.3.9. $\mathrm{Exp} : \mathfrak{so}(3) \to \mathrm{SO}(3)$ *is diffeomorphism between* $\{\hat{\Omega} : \Omega \in \mathbb{R}^3,\ \Omega^T\Omega \leq \pi^2\}$ *and* $\{R \in \mathrm{SO}(3) :\ \mathrm{tr}(R) \neq -1\}$*, and the logarithmic map is given by*

$$\mathrm{Log}(R) = \begin{cases} 0, & R = I \\ \frac{\phi(R)}{\sin(\phi(R))}(R - R^T), & R \neq I \end{cases}$$

where $\phi(R) := \arccos(\frac{1}{2}(\mathrm{tr}(R) - 1))$*. Moreover, we have* $\|\mathrm{Log}(R)\| = \phi(R)$*.*

3.4 Reduction by Symmetry

Let G be a Lie group equipped with a left-invariant metric $\langle \cdot, \cdot \rangle$. We saw in Section 2.3.2 that the critical points of the energy functional $E : \Omega^1 \to \mathbb{R}$ are geodesics. That is, they are the smooth curves $g \in \Omega^1$ such that $D_t\dot{g} = 0$ on the full interval $[a, b]$, where D_t is the covariant derivative along g induced by the Levi-Civita connection of $\langle \cdot, \cdot \rangle$. Observe that from Lemma 3.3.4, this implies that $g\left(\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\dot{\xi}\right) \equiv 0$, where $\xi := g^{-1}\dot{g}$. Since left-translation is a diffeomorphism, we obtain the following reduction of geodesics:

Theorem 3.4.1. *Let* G *be a Lie group equipped with a left-invariant metric, and suppose that* $g : [a, b] \to G$ *is a geodesic with respect to the Levi-Civita connection induced by this metric. Then,* $\xi := g^{-1}\dot{g}$ *satisfies*

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = 0 \tag{3.20}$$

on $[a, b]$*.*

Proof. Although we have already seen that the result holds by a direct application of 3.3.4 to the geodesic equation, we find it instructive to derive (3.20) via the calculus of variations—that is, without assuming prior knowledge of the geodesic equation. To that end, suppose that $\Gamma : (-\epsilon, \epsilon) \times [a, b]$ is a one-parameter variation of g with variational vector field δg. We define:

$$T(s,t) := \frac{\partial}{\partial t}\Gamma(s,t), \qquad \mathcal{T}(s,t) := \Gamma^{-1}(s,t)T(s,t),$$
$$S(s,t) := \frac{\partial}{\partial s}\Gamma(s,t), \qquad \mathcal{S}(s,t) := \Gamma^{-1}(s,t)S(s,t).$$

Then, by repeated applications of Lemma 3.3.5 and Corollary 1,

$$\begin{aligned}
\delta E &= \int_a^b \langle \nabla_S T, T\rangle \big|_{s=0} dt, \\
&= \int_a^b \left\langle \mathcal{T}' + \nabla^{\mathfrak{g}}_{\mathcal{S}}\mathcal{T}, \mathcal{T}\right\rangle \big|_{s=0} dt, \\
&= \int_a^b \left\langle \dot{\mathcal{S}} + [\mathcal{T},\mathcal{S}], \mathcal{T}\right\rangle \big|_{s=0} dt + \int_a^b \left\langle \nabla^{\mathfrak{g}}_{\mathcal{S}}\mathcal{T}, \mathcal{T}\right\rangle \big|_{s=0} dt, \\
&= -\int_a^b \left\langle \dot{\mathcal{T}}, \mathcal{S}\right\rangle \big|_{s=0} dt + \int_a^b \frac{d}{dt}\left\langle \mathcal{S}, \mathcal{T}\right\rangle \big|_{s=0} dt + \int_a^b \left\langle \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{S}, \mathcal{T}\right\rangle \big|_{s=0} dt, \\
&= -\int_a^b \left\langle \dot{\mathcal{T}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{T}, \mathcal{S}\right\rangle \big|_{s=0} dt, \\
&= -\int_a^b \left\langle g(\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi), \delta g\right\rangle dt.
\end{aligned}$$

Since this must hold for all variational vector fields δg, and it is necessary that $\delta E = 0$ along a critical point, equation (3.20) follows immediately. □

Theorem 3.4.1 tells us that solving $D_t\dot{g} = 0$ on a Lie group with a left-invariant metric is equivalent to solving (3.20) (often called the *reduced geodesic equation*, or the *Euler-Poincaré equation* for a geodesic) together with the *reconstruction equation* $\dot{g} = g\xi$. Hence, we have replaced a second-order equation with two first-order equations. In mechanical applications, these equations represent the *kinematics* (the reconstruction equation) and the *dynamics* (the reduced geodesic equation) of some agent. Moreover, (3.20) evolves on the Lie algebra $\mathfrak{g}$, so that we have removed the non-linearity of the manifold from our dynamics. This allows for the use of classical numerical integration schemes in approximating solutions, which can only be done for equations evolving on non-linear spaces by using local coordinate systems (which often lead to poor behavior, especially when dealing with chart transitions). In all, the process of converting some ODE/variational principle evolving on a Lie group G into an equation evolving on $\mathfrak{g}$ (together with a reconstruction equation) is known as *reduction by symmetry*.

Remark 4. Reduction by symmetry may be framed more generally in the language of Lagrangian mechanics. That is, given a function $L : TG \to \mathbb{R}$ called the *Lagrangian*, we seek to minimize the action $\mathcal{A}[g] = \int_a^b L(g,\dot{g})dt$ on Ω. It is well-known that the critical points of $\mathcal{A}$ satisfy the *Euler-Lagrange equations* $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{g}}\right) + \dfrac{\partial L}{\partial g} = 0$. If the Lagrangian is *left-invariant*, meaning that $L(hg, h\dot{g}) = L(g,\dot{g})$ for all

$g, h \in G$, then we may consider the *reduced Lagrangian* $l : \mathfrak{g} \to \mathbb{R}$ given by $l(\xi) := L(e, \xi) = L(g, \dot{g})$, where $\xi := g^{-1}\dot{g}$. Then, the critical points of the *reduced action* $\mathcal{A}_{\text{red}}[\xi] = \int_a^b l(\xi)dt$ among variations of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$ (note that this is just (3.17)), where η is arbitrary admissible proper variation, satisfy the Euler-Poincaré equations $\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = \text{ad}^*_\xi \frac{\partial l}{\partial \xi}$. Moreover, it can be seen that g satisfies the Euler-Lagrange equations corresponding to L if and only if $\xi := g^{-1}\dot{g}$ satisfies the Euler-Poincaré equations corresponding to l [54]. The equation $\dot{g} = g\xi$ is known as the *reconstruction equation*. The reduced variational principle is an example of a *constrained* variational principle, since we are constraining the set of admissible variations.

The energy function E is precisely the action corresponding to the Lagrangian $L(g, \dot{g}) = \frac{1}{2}\|\dot{g}\|^2$. Since the Riemannian metric is left-invariant, it follows that the Lagrangian too is left-invariant, and so the reduced Lagrangian takes the form $l(\xi) = \frac{1}{2}\|\xi\|^2$. It is straight-forward to show that $\frac{\partial l}{\partial \xi} = \xi^\flat$, so that the Euler-Poincaré equations associated to l are given by $\dot{\xi}^\flat = \text{ad}^*_\xi \xi^\flat$, which is equivalent to (3.20) by Lemma 3.3.3. Notice that the Euler-Poincaré equations corresponding to l naturally live on the dual of the Lie algebra $\mathfrak{g}^*$. It is only through the metric that we are able to convert them into equations on $\mathfrak{g}$.

Example 5. Consider $G = \text{SO}(3)$ equipped with the left-invariant metric $\left\langle \dot{R}_1, \dot{R}_2 \right\rangle = \text{tr}(\dot{R}_1 \mathbb{M} \dot{R}_2^T)$ for all $R \in \text{SO}(3), \dot{R}_1, \dot{R}_2 \in T_R\text{SO}(3)$ and for some coefficient of inertia matrix $\mathbb{M}$. Suppose that $R : [a, b] \to \text{SO}(3)$ is a geodesic with respect to the Levi-Civita connection and define $\hat{\Omega} := R^{-1}\dot{R} \in \mathfrak{so}(3)$. Then, from Theorem 3.4.1, we have $\dot{\hat{\Omega}} = \text{ad}^\dagger_{\hat{\Omega}} \hat{\Omega}$. First observe that, from equation (3.19),

$$\left\langle \hat{\Omega}, \hat{\eta} \right\rangle = \Omega^T \mathbb{J} \eta = (\mathbb{J}\Omega) \cdot \eta$$

for all $\Omega, \eta \in \mathbb{R}^3$, so that we may identify $\hat{\Omega} \in \mathfrak{so}(3)$ with $\mathbb{J}\Omega \in \mathbb{R}^3$. Moreover, from Equation (3.9),

$$\begin{aligned} \left\langle \text{ad}^\dagger_{\hat{\Omega}} \hat{\sigma}, \hat{\eta} \right\rangle = \left\langle \hat{\sigma}, \text{ad}_{\hat{\Omega}} \hat{\eta} \right\rangle &= \left\langle \hat{\sigma}, \widehat{\Omega \times \eta} \right\rangle \\ &= \sigma^T \mathbb{J}(\Omega \times \eta) = (\mathbb{J}\sigma) \cdot (\Omega \times \eta) = (\mathbb{J}\sigma \times \Omega) \cdot \eta, \end{aligned}$$

so that we may identify $\text{ad}^\dagger_{\hat{\Omega}} \hat{\sigma} \in \mathfrak{so}(3)$ with $\mathbb{J}\sigma \times \Omega \in \mathbb{R}^3$. It follows that (3.20) is equivalent to $\mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega$, which is known as Euler's equation for a rigid body [55].

3.4.1 Reduction with Broken Symmetry

Consider the action functional $\mathcal{A} : \Omega^1 \to \mathbb{R}$ given by

$$\mathcal{A}[g] = \int_a^b \left(\frac{1}{2} \|\dot{g}(t)\| - V(g(t)) \right) dt, \tag{3.21}$$

where $V : G \to \mathbb{R}$ is a smooth scalar field called the *potential*. It is easy to see from Remark 4 that the critical points of (3.21) are the smooth solutions to the Euler-Lagrange equations

$$D_t \dot{g}(t) = -\text{grad}V(g(t)), \tag{3.22}$$

for all $t \in [a, b]$. In general, the process of reduction by symmetry used in Theorem 3.4.1 for the Energy functional (that is, $\mathcal{A}$ with $V \equiv 0$) will only apply to $\mathcal{A}$ when V is invariant under left-translations. However, this is seldom the case in applications, at which point the potential is said to "break the symmetry" of the energy functional. Rather, it is more common that V possesses a *partial symmetry*.

Definition 3.4.2. We say that the potential $V : G \to \mathbb{R}$ admits a partial symmetry over some smooth manifold Q if there exists a function $V_{\text{ext}} : G \times Q \to \mathbb{R}$ called the *extended potential* such that:

(1) for some $q_0 \in Q$ and for all $g \in G$, we have $V_{\text{ext}}(g, q_0) = V(g)$,

(2) there exists a right-action $\Phi : G \times Q \to Q$ such that V_{ext} is invariant under the left-action $\Psi : G \times (G \times Q) \to G \times Q$ given by $\Psi_h(g, q) = (hg, \Phi_{h^{-1}}(q))$ for all $g, h \in G, q \in Q$.

This leads to the following proposition.

Proposition 2. *Let G be a Lie group equipped with a left-invariant metric, and suppose that $g : [a, b] \to G$ is a critical point of the action function defined in* (3.21)*. Then g is smooth, and if the potential admits a partial symmetry over some smooth manifold Q, then $\xi := g^{-1}\dot{g}$ and $q := \Phi_g(q_0)$ satisfy*

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi} \xi = -J(d_2 V_{\text{ext}}(e, q))^{\sharp}, \tag{3.23}$$

$$\dot{q} = \xi_Q(q), \tag{3.24}$$

on $[a, b]$, where $J : T^*Q \to \mathfrak{g}^*$ is the momentum map corresponding to the right-action of G on Q.

Proof. g is a critical point of $\mathcal{A}$ if and only if $\delta\mathcal{A}[g] = 0$ for all admissible variations g_s of g. This is equivalent to the condition

$$\frac{\partial}{\partial s}\Big|_{s=0}\int_a^b \left(\frac{1}{2}\|\dot{g}_s\|^2 - V(g_s)\right) dt = \int_a^b \left(\langle D_s\dot{g}_s, \dot{g}_s\rangle - \frac{\partial}{\partial s}V(g_s)\right)\Big|_{s=0} dt$$

From the proof of Theorem 3.20, we know that

$$\int_a^b \langle D_s\dot{g}_s, \dot{g}_s\rangle \Big|_{s=0} dt = \int_a^b \langle D_t\delta g, \dot{g}\rangle \, dt$$

We now handle the term involving the potential.

$$\begin{aligned}\int_a^b \frac{\partial}{\partial s}V(g_s)\Big|_{s=0} dt &= \int_a^b \frac{\partial}{\partial s}V_{\text{ext}}(g_s, q_0)\Big|_{s=0} dt \\ &= \int_a^b \frac{\partial}{\partial s}V_{\text{ext}}(e, \Phi_{g_s}(q_0))\Big|_{s=0} dt \\ &= \int_a^b \langle d_2V_{\text{ext}}(e,q), \eta_Q(q)\rangle_* \, dt\end{aligned}$$

since $\frac{\partial}{\partial s}\Big|_{s=0}\Phi_{g_s}(q_0) = \frac{\partial}{\partial s}\Big|_{s=0}\Phi_{g^{-1}g_s}\circ\Phi_g(q_0) = \frac{\partial}{\partial s}\Big|_{s=0}\Phi_{\text{Exp}(s\eta)}(q) = \eta_Q(q)$. By definition, the momentum map $J : T^*Q \to \mathfrak{g}^*$ satisfies $\langle J(d_2V_{\text{ext}}(e,q), \eta\rangle_* = \langle d_2V_{\text{ext}}(e,q), \eta_Q(q)\rangle_*$, It follows that

$$\delta\mathcal{A} = \int_a^b \left(\langle D_t\delta g, \dot{g}\rangle - \left\langle gJ(d_2V_{\text{ext}}(e,q))^\sharp, \delta g\right\rangle\right) dt.$$

It is straightforward to show that g must be smooth via a similar bootstrapping method as applied in 3.4.1 (and noting that both V and J are smooth). Hence, integrating the first term by parts, we obtain

$$\int_a^b \left(\left\langle D_t\dot{g} + gJ(d_2V_{\text{ext}}(e,q))^\sharp, \delta g\right\rangle\right) dt = 0$$

for all $\delta g \in T_g\Omega^1$. Equation (3.23) follows from the left-invariance of the metric together with Lemma 3.3.4, whereas (3.24) is a direct consequence of the definition $q = \Phi_g(q_0)$. □

We now discuss two important special cases of Proposition 2.

Example 6. Consider 2 in the case where $Q = W$ is a finite-dimensional real vector space on which G acts on the right by linear transformations (that is, $\Phi_g : W \to W$ is linear for all $g \in G$). We implicitly

define a right-action $\Psi : G \times W^* \to W^*$ on the dual space via $\langle \Psi_g(w), x \rangle_* = \left\langle w, \Phi_{g^{-1}}(x) \right\rangle_*$ for all $w \in W^*, x \in W$. Consider the map $\rho_x : \mathfrak{g} \to W$ given by $\rho_x(\xi) = \xi_W(x)$, where ξ_W is the infinitesimal generator associated to Φ. Then the dual map $\rho_x^* : W^* \to \mathfrak{g}^*$ satisfies $\langle \rho_x^*(w), \xi \rangle_* = \langle w, \rho_x(\xi) \rangle_*$ for all $x \in W, w \in W^*, \xi \in \mathfrak{g}$. Hence, through the identification $T^*W = W \times W^*$, we see that the momentum map $J : W \times W^* \to \mathfrak{g}^*$ is given by $J(x, w) = \rho_x^*(w)$. Another common notation used here is the *diamond operator* $\diamond : W \times W^* \to \mathfrak{g}^*$ defined by $x \diamond w = \rho_x^*(w)$ for all $x \in W, w \in W^*$. In all, equations (3.23)-(3.24) take the form

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = -\rho_q^*(d_2 V_{\text{ext}}(e, q))^\sharp,$$
$$\dot{q} = \rho_q(\xi).$$

Example 7. We now consider the case where $Q = G$, which will be our central focus in Chapter 7 when applying reduction by symmetry in the path planning problem. We let G act on itself by right-translations. In this case, it can be seen that the infinitesimal generator takes the form $\xi_G(q) = R_{g^*}\xi := \xi g$, and the momentum map takes the form $J(\omega) = L_q^*\omega$ for all $q \in G, \omega \in T_q^*G$ [55]. Alternatively, we see that

$$\begin{aligned}
\langle J(\omega), \xi \rangle_* &= \left\langle L_q^*\omega, \xi \right\rangle_* = \langle \omega, L_{q^*}\xi \rangle_*, \\
&= \left\langle \omega^\sharp, L_{q^*}\xi \right\rangle = \left\langle L_{q^{-1}*}\omega^\sharp, \xi \right\rangle, \\
&= \left\langle (L_{q^{-1}*}\omega^\sharp)^\flat, \xi \right\rangle_*.
\end{aligned}$$

Hence, $J(\omega) = (L_{q^{-1}*}\omega^\sharp)^\flat$. It follows that equations (3.23)-(3.24) take the form

$$\dot{\xi} + \nabla^{\mathfrak{g}}_{\xi}\xi = -L_{q^{-1}*}\text{grad}_2 V_{\text{ext}}(e, q)),$$
$$\dot{q} = q\xi.$$

Chapter 4

Homogeneous Spaces

We now discuss spaces which are not necessarily Lie groups themselves, but nonetheless possess certain symmetries and invariances that allow for the similar results to be obtained: *Homogeneous spaces*. Let G be a connected Lie group. A homogeneous space H of G is a smooth manifold on which G acts transitively (as in Definition 3.2.3). Any Lie group is itself a homogeneous space, where the transitive action is given by left-translation (or right-translation).

Suppose that $\Psi : G \times H \to H$ is a transitive left-action, which we denote by $gx := \Phi_g(x)$. It can be shown that for any $x \in H$, we have $G/\text{Stab}(x) \cong H$ as differentiable manifolds, where $\text{Stab}(x) := \{g \in G \mid gx = x\}$ denotes the *stabilizer subgroup* (also called the *isotropy subgroup*) of x, and $G/\text{Stab}(x)$ denotes the space of equivalence classes determined by the equivalence relation $g \sim h$ if and only if $g^{-1}h \in \text{Stab}(x)$. Moreover, for any closed Lie subgroup $K \subset G$, the left-action $\Phi : G \times G/K \to G/K$ satisfying $\Phi_g([h]) = [gh]$ for all $g, h \in G$ is transitive, and so G/K is a homogeneous space. Hence, we may assume without loss of generality that $H := G/K$ is a homogeneous space of G for some closed Lie subgroup K.

Let $\pi : G \to H$ be the canonical projection map. We define the *vertical subspace* at $g \in G$ by $\text{Ver}_g := \ker(\pi_*|_g)$, from which we may construct the *vertical bundle* as $VG := \bigsqcup_{g \in G} \{g\} \times \text{Ver}_g$. Given a Riemannian metric $\langle \cdot, \cdot \rangle_G$ on G, we define the *horizontal subspace* at any point $g \in G$ (with respect to $\langle \cdot, \cdot \rangle_G$) as the orthogonal complement of Ver_g. That is, $\text{Hor}_g := \text{Ver}_g^{\perp}$. Similarly, we define the horizontal bundle as $HG := \bigsqcup_{g \in G} \{g\} \times \text{Hor}_g$. Both the vertical and horizontal bundles are vector bundles, and are in fact subbundles of the tangent bundle TG. It is clear that $T_gG = \text{Ver}_g \oplus \text{Hor}_g$ for all $g \in G$, so that the Lie algebra $\mathfrak{g}$ of G admits the decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K and

$\mathfrak{h} \cong T_{\pi(e)}H$. We denote the orthogonal projections onto the vertical and horizontal subspaces by $\mathcal{V}$ and $\mathcal{H}$, respectively.

A section $Z \in \Gamma(HG)$ is called a *horizontal vector field*. That is, $Z \in \Gamma(TG)$ and $Z(g) \in \mathrm{Hor}_g$ for all $g \in G$. A vector field $Y \in \Gamma(TG)$ is said to be π-related to some $X \in \Gamma(TH)$ if $\pi_* Y_g = X_{\pi(g)}$ for all $g \in G$. If in addition $Y \in \Gamma(HG)$, we say that Y is a *horizontal lift* of X. We further define a horizontal lift of a smooth curve $q : [a,b] \to H$ as a smooth curve $\tilde{q} : [a,b] \to G$ such that $\pi \circ \tilde{q} = q$ and $\dot{\tilde{q}}(t)$ is horizontal for all $t \in [a,b]$. We have the following result from [72]:

Lemma 4.0.1. *Let H be a homogeneous space of G. Then, and $X \in \Gamma(TH)$. Then:*

(1) For all $X \in \Gamma(TH)$, there exists a unique horizontal lift $\tilde{X}$ of X. That is, the map $\tilde{\cdot} : \Gamma(TH) \to \Gamma(HG)$ sending $X \mapsto \tilde{X}$ is $\mathbb{R}$-linear and injective.

(2) For all smooth curves $q : [a,b] \to H$ and $q_0 \in \pi^{-1}(\{q(a)\})$, there exists a unique horizontal lift $\tilde{q} : [a,b] \to H$ of q satisfying $\tilde{q}(a) = q_0$.

called the horizontal lift of X which is π-related to X.

In general, the horizontal lift map will not be surjective, as it need not be the case that $Z_g = Z_h$ whenever $\pi(g) = \pi(h)$ for $Z \in \Gamma(HG)$. The image of the horizontal lift map will be denoted by $\mathcal{B}(G) \subset \Gamma(HG)$, and its elements will be called *basic vector fields*. In particular, $\mathcal{B}(G) \cong \Gamma(TH)$ as an $\mathbb{R}$-vector space, and a vector field $Z \in \Gamma(TG)$ is basic if and only if it is the horizontal lift of some vector field in $\Gamma(TH)$. Basic vector fields are precisely those which can be pushed-forward to a smooth non-zero vector field on H under π. We may similarly call a curve $g : [a,b] \to G$ basic when it is a horizontal lift of some curve $q : [a,b] \to H$, and a vector field $\tilde{X} \in \Gamma(g)$ basic when it is horizontal and π-related to some $X \in \Gamma(q)$. That is, $(\pi_*)_{g(t)} \tilde{X}(t) = X(t)$ for all $t \in [a,b]$. However, it turns out that these distinctions are redundant, as any smooth curve $g : [a,b] \to G$ satisfying $\dot{g}(t) \in \mathrm{Hor}_{g(t)}$ for all $t \in [a,b]$ is necessarily the horizontal lift of some $q : [a,b] \to H$. Namely, it is the horizontal lift of $q := \pi \circ g$. Similarly, every horizontal vector field along a basic curve is basic. To see this, first observe that any curve $\tilde{\eta} : [a,b] \to \mathfrak{h}$ induces a curve $\eta : [a,b] \to T_{\pi(e)}H$ under $(\pi_*)_e$. Now notice that the left-action Φ_g on H

satisfies $\Phi_g \circ \pi = \pi \circ L_g$ for all $g \in G$. Taking the differential of both sides, we see that the following diagram commutes:

$$\begin{array}{ccc} \text{Hor}_g & \xrightarrow{d\pi_g} & T_{\pi(g)}(G/K) \\ {\scriptstyle L_{g^{-1}*}}\downarrow & & \downarrow{\scriptstyle \Phi_{g^{-1}*}} \\ \mathfrak{h} & \xrightarrow{d\pi_e} & T_{\pi(e)}(G/K). \end{array} \tag{4.1}$$

Since Φ is a transitive action, we may translate η to a vector field X along any curve $q : [a,b] \to H$ via $X(t) := \Phi_{\tilde{q}(t)*}(\eta(t))$ for all $t \in [a,b]$, where $\tilde{q}$ is a horizontal lift of q. X may then be horizontally lifted to a unique $\tilde{X}$ along $\tilde{q}$. Then $\tilde{X}$ is basic, and due to the identification (4.1), it follows that $L_{(\tilde{q}^{-1})*}\tilde{X} \equiv \tilde{\eta}$. This is summarized in the following lemma:

Lemma 4.0.2. *Suppose that $q : [a,b] \to H$ and $\tilde{q} : [a,b] \to G$ is a horizontal lift of q. Then, for any $\tilde{\eta} : [a,b] \to \mathfrak{h}$, there exists a unique $X \in \Gamma(q)$ such that its horizontal lift $\tilde{X}$ along $\tilde{q}$ satisfies $L_{\tilde{q}(t)^{-1}*}\tilde{X}(t) = \tilde{\eta}(t)$ for all $t \in [a,b]$.*

4.1 Riemannian Homogeneous Spaces

Consider a connected Lie group G and a homogeneous space $H = G/K$ of G. Since H is a smooth manifold, it can be equipped with a Riemannian metric. As when discussing Riemannian metrics on Lie groups in Section 3.3, we are interested in those metrics $\langle \cdot, \cdot \rangle_H$ which in some sense preserve the structure of the homogeneous space. In this case, we wish to choose $\langle \cdot, \cdot \rangle_H$ so that the canonical projection map $\pi : G \to H$ is a *Riemannian submersion*. That is, so that $\pi_*|_g$ is a linear isometry between Hor_g and $T_{\pi(g)}H$ for all $g \in G$. In such a case, we call H a *Riemannian homogeneous space*. It is clear that if H is a Riemannian homogeneous space, then $\langle \mathcal{H}(X), \mathcal{H}(Y) \rangle_G = \langle \pi_* X, \pi_* Y \rangle_H$ for all $X, Y \in T_g G, g \in G$. In particular, $\left\langle \tilde{X}, \tilde{Y} \right\rangle_G = \left\langle X, Y \right\rangle_H$ for all $X, Y \in T_g G, g \in G$. The metric $\langle \cdot, \cdot \rangle_H$ is said to be *G-invariant* if it is invariant under the left-action Φ_g for all $g \in G$. It can be shown that every homogeneous space $H = G/K$ that admits a G-invariant metric is *reductive*. That is, the Lie algebra admits a decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K, and $\mathfrak{h}$ satisfies $[\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}$. In particular, this implies that $\mathfrak{h} \cong T_{\pi(e)}(G/K)$ as vector spaces.

Lemma 4.1.1. *Suppose that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ is a reductive Lie algebra. Then $[\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}$.*

Proof. Since $\mathfrak{s}$ is Lie subalgebra of $\mathfrak{g}$, we have by definition that $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{s}$. Let $\xi, \eta \in \mathfrak{s}$ and $\sigma \in \mathfrak{h}$. Then,

$$\begin{aligned} 0 = \langle [\xi,\eta], \sigma \rangle &= \left\langle \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\mathrm{Exp}(t\xi)}(\eta), \sigma \right\rangle, \\ &= \frac{d}{dt}\Big|_{t=0} \left\langle \mathrm{Ad}_{\mathrm{Exp}(t\xi)}(\eta), \sigma \right\rangle. \end{aligned}$$

It follows that the map $t \mapsto \left\langle \mathrm{Ad}_{\mathrm{Exp}(t\xi)}(\eta), \sigma \right\rangle$ is constant. In particular, since $\left\langle \mathrm{Ad}_{\mathrm{Exp}(t\xi)}(\eta), \sigma \right\rangle \big|_{t=0} = \langle \eta, \sigma \rangle = 0$, we have $\left\langle \mathrm{Ad}_{\mathrm{Exp}(t\xi)}(\eta), \sigma \right\rangle = 0$ for all $\xi, \eta \in \mathfrak{s}, \sigma \in \mathfrak{m}, t \in \mathbb{R}$. Since the exponential map is a local diffeomorphism, and $\mathfrak{s} = \{X \in \mathfrak{g} \mid \mathrm{Exp}(tX) \in H \text{ for all } t \in \mathbb{R}\}$, it follows that there exists an open subset $V \subset G$ containing e such that $\mathrm{Ad}_g(\mathfrak{s}) \subset \mathfrak{s}$ for all $g \in V \cap H$. Since H is connected by assumption, there exists some finite collection of vectors $\xi_1, \ldots, \xi_n \in \mathfrak{h}$ such that $\mathrm{Exp}(\xi_1) \cdots \mathrm{Exp}(\xi_n) = h$. Hence, for all $\xi \in \mathfrak{h}$, $\mathrm{Ad}_h(\xi) = \mathrm{Ad}_{\mathrm{Exp}(\xi_1)} \circ \cdots \circ \mathrm{Ad}_{\mathrm{Exp}(\xi_n)}(\xi) \in \mathfrak{h}$. □

The following result establishes an equivalency between the existence of a G-invariant metrics on H, and the existence of a left-invariant metric on G for which H is a Riemannian homogeneous space.

Lemma 4.1.2. *If G is equipped with a left-invariant metric $\langle \cdot, \cdot \rangle_G$, and $\pi : G \to H$ is a Riemannian submersion, then $\langle \cdot, \cdot \rangle_H$ is G-invariant. Moreover, if $\langle \cdot, \cdot \rangle_H$ is G-invariant, then there exists a left-invariant metric $\langle \cdot, \cdot \rangle_G$ on G such that $\pi : G \to H$ is a Riemannian submersion.*

Proof. First suppose G is equipped with a left-invariant metric $\langle \cdot, \cdot \rangle_G$, and that $\pi : G \to H$ is a Riemannian submersion. Let $q \in H, X, Y \in T_qH, g \in G$, and let $\tilde{X}, \tilde{Y}$ be horizontal lifts of X, Y. Then,

$$\begin{aligned} \langle \Phi_{g*}X, \Phi_{g*}Y \rangle_H = \left\langle \pi_*^{-1} \circ \Phi_{g*}X, \pi_*^{-1} \circ \Phi_{g*}Y \right\rangle_G &= \left\langle \pi_*^{-1} \circ \Phi_{g*} \circ \pi_* \tilde{X}, \pi_*^{-1} \circ \Phi_{g*} \circ \pi_* \tilde{Y} \right\rangle_G \\ &= \left\langle L_{g*}\tilde{X}, L_{g*}\tilde{Y} \right\rangle_G \end{aligned}$$

where the last equality follows from (4.1). By the left-invariance of $\langle \cdot, \cdot \rangle_G$, we then have

$$\left\langle L_{g*}\tilde{X}, L_{g*}\tilde{Y} \right\rangle_G = \left\langle \tilde{X}, \tilde{Y} \right\rangle_G = \langle X, Y \rangle_H .$$

In particular, $\langle \cdot, \cdot \rangle_H$ is G-invariant.

Now suppose that H is equipped with a G-invariant metric $\langle \cdot, \cdot \rangle_H$. Since H is reductive, we may define the orthogonal projections $\mathcal{V} : \mathfrak{g} \to \mathfrak{s}$ and $\mathcal{H} : \mathfrak{g} \to \mathfrak{h}$, where $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$. Define an inner product on $\mathfrak{h}$ via $\langle \xi, \eta \rangle_{\mathfrak{h}} := \langle \pi_* \xi, \pi_* \eta \rangle_H$ for all $\xi, \eta \in \mathfrak{h}$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ be any inner product on $\mathfrak{s}$, and define an inner product on $\mathfrak{g}$ by $\langle \xi, \eta \rangle_{\mathfrak{g}} := \langle \mathcal{V}(\xi), \mathcal{V}(\eta) \rangle_{\mathfrak{s}} + \langle \mathcal{H}(\xi), \mathcal{H}(\eta) \rangle_{\mathfrak{h}}$. We extend $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ to a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle_G$ on G via left-translations. Now, for all $X, Y \in \mathrm{Hor}_g$,

$$\begin{aligned}
\left\langle \pi_*\big|_g X, \pi_*\big|_g Y \right\rangle_H &= \left\langle \Phi_{g^{-1*}} \circ \pi_*\big|_g X, \Phi_{g^{-1*}} \circ \pi_*\big|_g Y \right\rangle_H \\
&= \left\langle \pi_*^{-1}\big|_e \circ \Phi_{g^{-1*}} \circ \pi_*\big|_g X, \pi_*^{-1}\big|_e \circ \Phi_{g^{-1*}} \circ \pi_*\big|_g Y \right\rangle_{\mathfrak{h}} \\
&= \left\langle L_{g^*} \circ \pi_*^{-1}\big|_e \circ \Phi_{g^{-1*}} \circ \pi_*\big|_g X, L_{g^*} \circ \pi_*^{-1}\big|_e \circ \Phi_{g^{-1*}} \circ \pi_*\big|_g Y \right\rangle_G
\end{aligned}$$

By (4.1), we have that $L_{g^*} \circ \pi_*^{-1}\big|_e \circ \Phi_{g^{-1*}} \circ \pi_*\big|_g$ is the identity map on Hor_g, so that $\langle X, Y \rangle_G = \left\langle \pi_*\big|_g X, \pi_*\big|_g Y \right\rangle_H$ for all $X, Y \in \mathrm{Hor}_g$. Hence π is a Riemannian submersion, and so $(H, \langle \cdot, \cdot \rangle_H)$ is a Riemannian homogeneous space with respect to $(G, \langle \cdot, \cdot \rangle_G)$. □

In general, there need not exist G-invariant metrics on a homogeneous space. The following theorem from [72] describes a sufficient condition which is commonly fulfilled in application (indeed, this theorem will suffice for our application of interest).

Theorem 4.1.3. *Suppose that G and $H := G/K$ is a homogeneous space of G. If* $\mathrm{Stab}(x)$ *is compact for all $x \in H$, then there exist a G-invariant metric on H.*

Hence, if $\mathrm{Stab}(x)$ is compact for all $x \in H$ (which happens for example if H is compact, or equivalently, that G is compact), then by Lemma 4.1.2, there exists a left-invariant metric on G and a G-invariant metric on H for which H is a Riemannian homogeneous space. **For the remainder of the chapter, we suppose that G is a simply connected Lie group equipped with a left-invariant metric, and H is a Riemannian homogeneous space of G.** Denote the Levi-Civita connections on H and G with respect to these metrics by ∇ and $\tilde{\nabla}$, respectively. The following lemma outlines some useful properties of $\tilde{\nabla}$.

Lemma 4.1.4. *Let $X, Y \in \Gamma(TH)$, and $\tilde{X}, \tilde{Y} \in \mathcal{B}(G)$ be the horizontal lifts of X and Y, respectively.*

Further suppose that $V \in \Gamma(VG)$. Then, the following identities hold:

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}\mathcal{V}([\tilde{X}, \tilde{Y}]), \tag{4.2}$$

$$\tilde{\nabla}_{\tilde{X}}V = \tilde{\nabla}_V \tilde{X} \tag{4.3}$$

Proof. Equation (4.2) follows directly from [114]. To see (4.3), observe that $0 = [\tilde{X}, V] = \tilde{\nabla}_{\tilde{X}}V - \tilde{\nabla}_V \tilde{X}$, where the first equality follows from the fact that $\tilde{X}$ is π-related to X, and V is π-related to the 0 vector field on H, while the second follows from the fact that $\tilde{\nabla}$ is torsion-free. □

Apart from the Levi-Civita connection, there is another important connection that we will utilize throughout this thesis when working on (Riemannian) homogeneous spaces. We define the *horizontal connection* $\tilde{\nabla}^{\mathcal{H}} : \Gamma(TG) \times \Gamma(HG) \to \Gamma(HG)$ on the horizontal bundle by

$$\tilde{\nabla}^{\mathcal{H}}_W Z = \mathcal{H}(\tilde{\nabla}_W Z), \tag{4.4}$$

for all $W \in \Gamma(TG),\ Z \in \Gamma(HG)$. It is easy to see that $\tilde{\nabla}^{\mathcal{H}}$ satisfies all the conditions required to be a connection in the sense of Definition 2.2.1. Moreover, $\tilde{\nabla}^{\mathcal{H}}$ is simply the projection of the Levi-Civita connection on G onto the horizontal bundle. Observe that while $\tilde{\nabla}^{\mathcal{H}}$ is only a connection in the strict sense on the domain $\Gamma(TG) \times \Gamma(HG)$, it is still a well-defined operation on the larger domain $\Gamma(TG) \times \Gamma(TG)$. Here we outline some additional properties satisfied by $\tilde{\nabla}^{\mathcal{H}}$ and basic vector fields, which follow immediately from 4.1.4 and the fact that $\tilde{\nabla}$ is the metric compatible and torsion-free:

Lemma 4.1.5. *Let $X, Y \in \Gamma(TH)$, and $\tilde{X}, \tilde{Y} \in \mathcal{B}(G)$ be the horizontal lifts of X and Y, respectively. Further suppose that $W, Z \in \Gamma(HG)$, and $P \in \Gamma(TG)$. Then, the following identities hold:*

$$\tilde{\nabla}^{\mathcal{H}}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} \tag{4.5}$$

$$P\,\langle W, Z\rangle_G = \left\langle \tilde{\nabla}^{\mathcal{H}}_P W, Z \right\rangle_G + \left\langle W, \tilde{\nabla}^{\mathcal{H}}_P Z \right\rangle_G \tag{4.6}$$

$$\tilde{\nabla}^{\mathcal{H}}_W Z - \tilde{\nabla}^{\mathcal{H}}_Z W = \mathcal{H}([W, Z]) \tag{4.7}$$

Proof. Equation (4.5) follows immediately from the definition of $\tilde{\nabla}^{\mathcal{H}}$ and Lemma 4.1.4. To see equation (4.6), note that the metric compatibility (2.6) of $\tilde{\nabla}$ implies that $P\,\langle W, Z\rangle_G = \left\langle \tilde{\nabla}_P W, Z \right\rangle_G + \left\langle W, \tilde{\nabla}_P Z \right\rangle_G$.

Now since horizontal and vertical vectors are orthogonal with respect to $\langle \cdot, \cdot \rangle_G$, it follows that $\left\langle \tilde{\nabla}_P W, Z \right\rangle_G + \left\langle W, \tilde{\nabla}_P Z \right\rangle_G = \left\langle \mathcal{H}(\tilde{\nabla}_P W), Z \right\rangle_G + \left\langle W, \mathcal{H}(\tilde{\nabla}_P Z) \right\rangle_G$, from which the conclusion follows. Finally, equation (4.7) is a consequence of the fact that $\tilde{\nabla}$ is torsion-free (2.5). □

As in Section 3.3, we may define an operator $\tilde{\nabla}^{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ called the *Riemannian* $\mathfrak{h}$*-connection* via

$$\tilde{\nabla}^{\mathfrak{h}}_{\xi} \eta = \left(\tilde{\nabla}^{\mathcal{H}}_{\phi(\xi)} \phi(\eta) \right)(e). \tag{4.8}$$

It is clear that $\tilde{\nabla}^{\mathfrak{h}}_{\xi} \eta = \mathcal{H}(\tilde{\nabla}^{\mathfrak{g}}_{\xi} \eta)$, where $\tilde{\nabla}^{\mathfrak{g}}$ is the Riemannian $\mathfrak{g}$-connection corresponding to the Levi-Civita connection $\tilde{\nabla}$ on G. Therefore, we obtain the explicit expression

$$\tilde{\nabla}^{\mathfrak{h}}_{\xi} \eta = \frac{1}{2} \mathcal{H}([\xi, \eta]_{\mathfrak{g}} - \mathrm{ad}^{\dagger}_{\xi} \eta - \mathrm{ad}^{\dagger}_{\eta} \xi). \tag{4.9}$$

Given any C^1 curve g, we may further define the operator $\tilde{D}^{\mathcal{H}}_t X := \tilde{\nabla}^{\mathcal{H}}_{\dot{g}} X$ for all C^1 horizontal vector fields X along g. It is clear from Lemma 4.1.5 and 4.0.1 that if g is a basic curve and X is a C^1 horizontal vector field along $\pi \circ g$, then $\tilde{D}^{\mathcal{H}}_t \tilde{X} = \widetilde{D_t X}$. We now express Lemma 3.3.4 in terms of the Riemannian $\mathfrak{h}$-connection and horizontal connection:

Lemma 4.1.6. *Let $g : [a, b] \to G$ be a basic curve and X a smooth horizontal vector field along g. Suppose that $\xi(t) = g(t)^{-1} \dot{g}(t)$ and $\eta(t) = g(t)^{-1} X(t)$. Then the following relation holds for all $t \in [a, b]$:*

$$\tilde{D}^{\mathcal{H}}_t X(t) = g(t) \left(\dot{X}(t) + \tilde{\nabla}^{\mathfrak{h}}_{\xi(t)} \eta(t) \right). \tag{4.10}$$

Another important object in our studies is the type $(1, 2)$-tensor field $A : \Gamma(TG) \times \Gamma(TG) \to \Gamma(TG)$ defined by

$$A_X Y = \mathcal{H} \left(\tilde{\nabla}_{\mathcal{H}(X)} \mathcal{V}(Y) \right) + \mathcal{V} \left(\tilde{\nabla}_{\mathcal{H}(X)} \mathcal{H}(Y) \right). \tag{4.11}$$

As a direct consequence of Lemma 4.1.5 and the definition of A, we have the following lemma.

Lemma 4.1.7. *Let $X, Y \in B(G)$ and $V \in \Gamma(VG)$. Then,*

$$A_X Y = \frac{1}{2} \mathcal{V}([X, Y]) \tag{4.12}$$

$$A_X V = \tilde{\nabla}^{\mathcal{H}}_X V \tag{4.13}$$

A induces a 4-tensor field $\mathrm{Am} : \Gamma(TG)^4 \to \Gamma(TG)$ defined by

$$\mathrm{Am}(W, X, Y, Z) = \langle A_W A_X Y, Z \rangle_G . \tag{4.14}$$

Of particular interest to us are the symmetries of Am on the space of basic vector fields $B(G)$, which we show in the following lemma.

Lemma 4.1.8. *Let $W, X, Y, Z \in B(G)$. Then,*

$$\mathrm{Am}(W, X, Y, Z) = -\mathrm{Am}(W, Y, X, Z) \tag{4.15}$$

$$\mathrm{Am}(W, X, Y, Z) = -\mathrm{Am}(Z, X, Y, W) \tag{4.16}$$

$$\mathrm{Am}(W, X, Y, Z) = \mathrm{Am}(X, W, Z, Y) \tag{4.17}$$

Proof. Equation (4.15) is an immediate consequence of equation (4.12). To see equations (4.16) and (4.17), first observe that $\langle A_X Y, Z \rangle_G = 0$. By the metric compatibility of $\tilde{\nabla}$, it then follows that $0 = W \langle A_X Y, Z \rangle_G = \left\langle \tilde{\nabla}^{\mathcal{H}}_W A_X Y, Z \right\rangle_G + \left\langle A_X Y, \tilde{\nabla}_W Z \right\rangle_G$. Now, it is clear from equation (4.12) that $A_X Y$ is vertical, so that $\tilde{\nabla}^{\mathcal{H}}_W A_X Y = A_W A_X Y$ by equation (4.13). Moreover, equation (4.11) shows that $\left\langle A_X Y, \tilde{\nabla}_W Z \right\rangle_G = \langle A_X Y, A_W Z \rangle_G$. Hence, $\langle A_W A_X Y, Z \rangle_G = - \langle A_X Y, A_W Z \rangle_G$. Equation (4.16) now follows from

$$\mathrm{Am}(W, X, Y, Z) = - \langle A_X Y, A_W Z \rangle_G = \langle A_X Y, A_Z W \rangle_G = - \langle A_Z A_X Y, W \rangle_G = -\mathrm{Am}(Z, X, Y, W).$$

Similarly, equation (4.17) can be seen from:

$$\mathrm{Am}(W, X, Y, Z) = - \langle A_X Y, A_W Z \rangle_G = - \langle A_W Z, A_X Y \rangle_G = \langle A_X A_W Z, Y \rangle_G = \mathrm{Am}(X, W, Z, Y).$$

□

Since A is tensorial, we may also evaluate it along a vector field along some curve $g : [a, b] \to G$ by defining $A_t X = A_{\dot{g}} X$ for all $X \in \Gamma(g)$. Moreover, if $\xi = g^{-1}\dot{g}$ and $\eta = g^{-1} X$, then we have $A_t X = g A_\xi \eta$. Note, however, that the right-hand side of (4.11) is only well-defined in the case that $\dot{g}$ is horizontal, at which point it is clear that $A_t X = \tilde{D}^{\mathcal{H}}_t \mathcal{V}(X) + \mathcal{V}(\tilde{D}_t \mathcal{H}(X))$. To evaluate this quantity when

g is not horizontal, we need only consider for each $\tau \in [a,b]$ any vector fields $Z_\tau, Y_\tau \in \Gamma(TG)$ such that $Z_\tau(g(\tau)) = \dot{g}(\tau)$ and $Y_\tau = X(\tau)$, and then evaluate $(A_t X)(\tau) = (A_{Z_\tau} Y_\tau)(g(\tau))$.

In order to relate the second derivatives with respect to $\tilde{\nabla}^{\mathcal{H}}$, we now study the curvature endomorphism (in the sense of (2.2)) $Q : \Gamma(TG) \times \Gamma(TG) \times \Gamma(HG) \to \Gamma(HG)$ corresponding to $\tilde{\nabla}^{\mathcal{H}}$. Namely, we have

$$Q(W,X)Y = \tilde{\nabla}^{\mathcal{H}}_W \tilde{\nabla}^{\mathcal{H}}_X Y - \tilde{\nabla}^{\mathcal{H}}_X \tilde{\nabla}^{\mathcal{H}}_W Y - \tilde{\nabla}^{\mathcal{H}}_{[W,X]} Y \tag{4.18}$$

for all $W, X \in \Gamma(TG), Y \in \Gamma(HG)$. Moreover, we may define the horizontal curvature tensor Qm by $\mathrm{Qm}(W,X,Y,Z) := \langle Q(W,X)Y, Z\rangle_G$. Due to the fact that $[W,X]$ is not in general basic even when W, X are, the symmetries of Qm over $B(G)$ don't immediately follow from the symmetries of Rm or $\widetilde{\mathrm{Rm}}$. In fact, if $\tilde{W}, \tilde{X}, \tilde{Y} \in B(G)$ are the horizontal lifts of $W, X, Y \in \Gamma(TH)$, then we have from (4.13) and (4.3) that

$$Q(\tilde{W},\tilde{X})\tilde{Y} = \widetilde{R(W,X)Y} - 2A_{\tilde{Y}} A_{\tilde{W}} \tilde{X}. \tag{4.19}$$

Therefore,

$$\mathrm{Qm}(\tilde{W},\tilde{X},\tilde{Y},\tilde{Z}) = \mathrm{Rm}(W,X,Y,Z) - 2\mathrm{Am}(\tilde{Y},\tilde{W},\tilde{X},\tilde{Z}) \tag{4.20}$$

for all $W, X, Y, Z \in \Gamma(TH)$. Hence, we may understand the symmetries of Qm on $B(G)$ by studying those of Am (as in Lemma 4.1.8) and Rm (as in equations (2.10)-(2.14)). As we will see, most of the symmetries of Rm will be preserved. The only exception is that $\mathrm{Qm}(W,X,Y,Z) + \mathrm{Qm}(X,Y,W,Z) + \mathrm{Qm}(Y,W,X,Z)$ in general fails to vanish.

Lemma 4.1.9. *For all $W, X, Y, Z \in B(G)$, the following relations hold:*

$$\mathrm{Qm}(W,X,Y,Z) = -\mathrm{Qm}(X,W,Y,Z) \tag{4.21}$$

$$\mathrm{Qm}(W,X,Y,Z) = -\mathrm{Qm}(W,X,Z,Y) \tag{4.22}$$

$$\mathrm{Qm}(W,X,Y,Z) = \mathrm{Qm}(Y,Z,W,X) \tag{4.23}$$

Proof. Equation (4.21) follows immediately from the definition of Qm. Now let $\bar{W} := \pi_* W$, and similarly for $\bar{X}$ and $\bar{Y}$. Then, on one hand, using equation (4.15), we have:

$$\mathrm{Qm}(W,X,Y,Z) = \mathrm{Rm}(\bar{W},\bar{X},\bar{Y},\bar{Z}) + 2\mathrm{Am}(Y,X,W,Z)$$

On the other hand, using equations (4.16), (4.15), and (2.11),

$$\mathrm{Qm}(W,X,Z,Y) = \mathrm{Rm}(\bar{W},\bar{X},\bar{Z},\bar{Y}) + 2\mathrm{Am}(Z,X,W,Y) = -\mathrm{Rm}(\bar{W},\bar{X},\bar{Y},\bar{Z}) - 2\mathrm{Am}(Y,X,W,Z).$$

Adding the two equations yields equation (4.22). Similarly,

$$\mathrm{Qm}(W,X,Y,Z) = \mathrm{Rm}(\bar{W},\bar{X},\bar{Y},\bar{Z}) + 2\mathrm{Am}(Y,X,W,Z)$$

$$\mathrm{Qm}(Y,Z,W,X) = \mathrm{Rm}(\bar{Y},\bar{Z},\bar{W},\bar{X}) + 2\mathrm{Am}(Z,X,W,Y) = \mathrm{Rm}(\bar{W},\bar{X},\bar{Y},\bar{Z}) + 2\mathrm{Am}(Y,X,W,Z),$$

Subtracting these equations from each other yields equation (4.23). □

Finally, from equation (4.19), observe that the $(3,1)$-tensor field $\tilde{Q} : \Gamma(TG) \times \Gamma(TG) \times \Gamma(TG) \to \Gamma(HG)$ defined by

$$\tilde{Q}(\tilde{W},\tilde{X})\tilde{Y} = Q(\tilde{W},\tilde{X})\mathcal{H}(\tilde{Y}) + 2\mathcal{H}\big(A_{\tilde{Y}}A_{\tilde{W}}\tilde{X}\big) \tag{4.24}$$

satisfies $\tilde{Q}(\tilde{W},\tilde{X})\tilde{Y} = \widetilde{R(W,X)Y}$ for all $X,Y,Z \in B(G)$. Hence, $\tilde{Q}$ maps basic vector fields to basic vector fields, and we may interpret $\tilde{Q}$ as a sort of horizontal lift of the Riemannian curvature endomorphism R on H to G.

4.1.1 Reduction by Symmetry on Riemannian Homogeneous Spaces

Throughout the section, we assume that G is a Lie group with left-invariant metric, and $H := G/K$ is a Riemannian homogeneous space of G. For some smooth scalar field $V : H \to \mathbb{R}$, we consider the action function $\mathcal{A}_H : \Omega^{(1)}(H) \to \mathbb{R}$ on H as in (3.21). As usual, our aim is to find the critical points of $\mathcal{A}_H$ over $\Omega^{(1)}$. Since H is a Riemannian manifold, this may be done via variational analysis (as we did for the energy function in Theorem 2.3.2). However, in many cases, it will be more convenient to work on G—especially in cases where we can apply reduction by symmetry. To that end, we seek variational principles on G which are equivalent to the variational principle $\delta\mathcal{A}_H = 0$ on H (whose solutions are given by (3.22)). Ultimately, this equivalency will come from the relationship between the metrics on G and H. In particular, suppose that $g \in \Omega^{(1)}(G)$ and let $q := \pi \circ g$. It is clear that $q \in \Omega^{(1)}(H)$, and $\dot{q} = \pi_*\dot{g}$. Hence, $\|\dot{q}\|_H = \|\pi_*\dot{g}\|_H = \|\mathcal{H}(\dot{g})\|_G$. On the other hand, suppose that $q \in \Omega^{(1)}(H)$ and let g be a horizontal lift of

q. Then for some g_a, g_b, we have $g \in \Omega^{(1)}_{g_a,g_b}(G)$, and $\|\mathcal{H}(\dot{g})\|_G = \|\dot{g}\|_G = \|\dot{q}\|_H$. This motivates the action functional $\tilde{\mathcal{A}}_G : \Omega^{(1)}(G) \to \mathbb{R}$ given by

$$\tilde{\mathcal{A}}_G = \int_a^b \left(\frac{1}{2} \|\mathcal{H}(\dot{g}(t))\|_G^2 - \tilde{V}(g(t)) \right) dt, \tag{4.25}$$

where $\tilde{V} := V \circ \pi$. We have the following Proposition:

Proposition 3. *Let $q \in \Omega^{(1)}(H)$ and $g \in \Omega^{(1)}(G)$ satisfy $\pi \circ g = q$ on $[a, b]$. Then q is a critical point of $\mathcal{A}_H$ if and only if g is a critical point of $\tilde{\mathcal{A}}_G$.*

Proof. Suppose that q is a critical point of $\mathcal{A}_H$, and consider an admissible variation g_s of g. It follows that $q_s := \pi \circ g_s$ is an admissible variation of q, and $\dot{q}_s = \pi_* \dot{g}$. Hence,

$$\begin{aligned}
\delta \tilde{\mathcal{A}}_G[g] &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2} \|\mathcal{H}(\dot{g}_s)\|_G^2 - \tilde{V}(g_s) \right) dt, \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2} \|\dot{q}_s\|_H^2 - V(q_s) \right) dt, \\
&= \mathcal{A}_H[q] = 0.
\end{aligned}$$

Therefore, g is a critical point of $\tilde{\mathcal{A}}_G$. Now suppose that g is a critical point of $\tilde{\mathcal{A}}_G$, and consider an admissible variation q_s of q. Let $\tilde{q}_s$ be the horizontal lift of q_s at $g(a)$, and consider the family of curves defined by $g_s(t) = \tilde{q}_s(t) \tilde{q}_{\frac{s(t-a)}{b-a}}(b)^{-1} g(b)$ for all $(s, t) \in (-\epsilon, \epsilon) \times [a, b]$. Observe that

(1) $g_0(t) = \tilde{q}_0(t) \tilde{q}_0(b)^{-1} g(b) = g(t)$ for all $t \in [a, b]$,

(2) $g_s(a) = \tilde{q}_s(a) \tilde{q}_0(b)^{-1} g(b) = g(a)$ $s \in (-\epsilon, \epsilon)$,

(3) $g_s(b) = \tilde{q}_s(b) \tilde{q}_s^{-1}(b) g(b) = g(b)$, for all $s \in (-\epsilon, \epsilon)$.

(4) $\pi \circ g_s \equiv \pi \circ \tilde{q}_s \equiv q_s$, since $\tilde{q}_{\frac{s(t-a)}{b-a}}(b)^{-1}$ and $g(b)$ belong to the same equivalency class for all $(s, t) \in (\epsilon, \epsilon) \times [a, b]$.

It follows that g_s is indeed an admissible variation of g, so that

$$\begin{aligned}
\delta \mathcal{A}_H[q] &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2} \|\dot{q}_s\|_H^2 - V(q_s) \right) dt \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2} \|\mathcal{H}(\dot{g}_s)\|_G^2 - \tilde{V}(q_s) \right) dt \\
&= \tilde{\mathcal{A}}_G[g] = 0
\end{aligned}$$

□

From Proposition 3, we can generate critical points of $\mathcal{A}_H$ by finding critical points of $\tilde{\mathcal{A}}_G$ on then projecting them onto H. Assuming that $V \circ \pi$ admits a partial symmetry, we could then, in principle, apply symmetry reduction to $\tilde{\mathcal{A}}_G$. However, the inclusion of the horizontal projection $\mathcal{H}$ in $\tilde{\mathcal{A}}_G$ makes the analysis difficult. Our goal now is to find a *constrained* variational principle for $\mathcal{A}_G : \Omega^{(1)} \to \mathbb{R}$ defined by (3.21) on G with the potential $\tilde{V}$.

Definition 4.1.10. We call an admissible variation g_s of a basic curve $g \in \Omega^{(1)}(G)$ for which both $\dot{g}_s(t)$ and $\frac{\partial}{\partial s} g_s(t)$ are horizontal vector fields a *basic variation* of g.

Before proceeding to define the desired constrained variational principle for $\mathcal{A}_G$, we demonstrate some properties of the horizontal covariant derivative along basic variations that will be useful later.

Lemma 4.1.11. *Let $g : [a, b] \to G$ be a basic curve and g_s be a basic variation of g. Then,*

$$\tilde{D}_s^{\mathcal{H}} \dot{g}_s = \tilde{D}_t^{\mathcal{H}} \frac{\partial \dot{g}_s}{\partial s} \tag{4.26}$$

$$\tilde{D}_s^{\mathcal{H}} \tilde{D}_t^{\mathcal{H}} \dot{g}_s - \tilde{D}_t^{\mathcal{H}} \tilde{D}_s^{\mathcal{H}} \dot{g}_s = \tilde{Q}\left(\frac{\partial}{\partial s} \dot{g}_s, \dot{g}_s\right) \dot{g}_s. \tag{4.27}$$

Proof. Equation (4.26) follows directly from the definition of the horizontal connection and the Symmetry Lemma (2.23). To see (4.27), first note that, since g_s is a basic variation, there exists an admissible variation q_s of $\pi \circ g : [a, b] \to H$ such that $q : [a, b] \to H$ and a corresponding proper variation q_s such that g_s is the horizontal lift of q_s at $g(a)$. Hence, from Lemma 2.4.1,

$$\begin{aligned}
\tilde{D}_s^{\mathcal{H}} \tilde{D}_t^{\mathcal{H}} \dot{g}_s - \tilde{D}_t^{\mathcal{H}} \tilde{D}_s^{\mathcal{H}} \dot{g}_s &= \tilde{D}_s^{\mathcal{H}} \widetilde{D_t \dot{q}_s} - \tilde{D}_t^{\mathcal{H}} \widetilde{D_s \dot{q}_s} \\
&= \widetilde{D_s D_t \dot{q}_s} - \widetilde{D_t D_s \dot{q}_s} \\
&= R\left(\widetilde{\frac{\partial}{\partial s} \dot{q}_s}, \dot{q}_s\right) \dot{q}_s \\
&= \tilde{Q}\left(\frac{\partial}{\partial s} \dot{g}_s, \dot{g}_s\right) \dot{g}_s.
\end{aligned}$$

□

Proposition 4. *Suppose that $g \in \Omega^{(1)}(G)$ is a critical point of $\tilde{\mathcal{A}}_G$. If g is basic, then $\delta\mathcal{A}_G[g] = 0$ among all basic variations.*

Proof. Let g_s be a basic variation of g. Then,

$$\begin{aligned}\delta\mathcal{A}[g] &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2}\|\dot{g}_s(t)\|_G^2 - \tilde{V}(g_s(t))\right) dt \\ &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2}\|\mathcal{H}(\dot{g}_s(t))\|_G^2 - \tilde{V}(g_s(t))\right) dt \\ &= \delta\tilde{\mathcal{A}}[g] = 0.\end{aligned}$$

□

We now derive necessary conditions on g to solve the constrained variational principle on G.

Proposition 5. *A basic curve $g \in \Omega^{(1)}$ satisfies the variational principle $\delta\mathcal{A}_G[g] = 0$ among all basic variations of g if and only if g is smooth and satisfies*

$$\tilde{D}_t^{\mathcal{H}}\dot{g}(t) = -\text{grad}V(g(t)), \tag{4.28}$$

for all $t \in [a, b]$.

Proof. Let g_s be a basic variation of G with variational vector field δg. Then, from Lemmas 4.1.11 and 4.1.5,

$$\begin{aligned}\delta\mathcal{A}_G[g] &= \frac{\partial}{\partial s}\Big|_{s=0} \int_a^b \left(\frac{1}{2}\|\dot{g}_s(t)\|_G^2 - \tilde{V}(g_s(t))\right) dt \\ &= \int_a^b \left(\left\langle \tilde{D}_s^{\mathcal{H}}\dot{g}_s, \dot{g}_s\right\rangle_G - \left\langle \text{grad}\tilde{V}(g_s), \frac{\partial}{\partial s}\dot{g}_s\right\rangle_G\right)\Big|_{s=0} dt \\ &= \int_a^b \left(\left\langle \tilde{D}_t^{\mathcal{H}}\delta g, \dot{g}\right\rangle_G - \left\langle \text{grad}\tilde{V}(g), \delta g\right\rangle_G\right) dt.\end{aligned}$$

As in the proof of Theorem 3.4.1, the smoothness of g can be shown via a bootstrapping method. Integrating once by parts, we then obtain

$$\delta\mathcal{A}_G[g] = \int_a^b \left\langle \tilde{D}_t^{\mathcal{H}}\dot{g} + \text{grad}\tilde{V}(g), \delta g\right\rangle_G dt = 0$$

for all basic variational vector fields δg. Now let $h \in G,\ X \in \text{Ver}_h$, and observe that $\left\langle \text{grad}\tilde{V}(h), X \right\rangle_G = d\tilde{V}(X) = dV \circ \pi_*(X) = 0$. Hence $\text{grad}\tilde{V} \in \Gamma(HG)$. Moreover, if we consider any $\tilde{Y} \in \text{Hor}_h$, then:

$$\left\langle \text{grad}\tilde{V}(h), \tilde{Y} \right\rangle_G = d\tilde{V}(\tilde{Y}) = dV(Y) = \langle \text{grad}V(\pi(h)), Y \rangle_H = \left\langle \widetilde{\text{grad}V(\pi(h))}, \tilde{Y} \right\rangle_G,$$

where $Y = \pi_* \tilde{Y}$. It follows that $\text{grad}\tilde{V}$ is precisely the horizontal lift of $\text{grad}V$. Hence,

$$\int_a^b \left\langle \tilde{D}_t^{\mathcal{H}} \dot{g} + \text{grad}\tilde{V}(g), \delta g \right\rangle_G dt = \int_a^b \left\langle \tilde{D}_t^{\mathcal{H}} \dot{g} + \text{grad}\tilde{V}(g), \delta g \right\rangle_G dt.$$

It follows from 4.0.1 and the fact that G is simply connected that every horizontal variational vector field is the variational vector field of some basic variation, from which (4.28) follows. □

Corollary 2. *If $q \in \Omega^{(1)}(H)$ is a critical point of $\mathcal{A}_H$, then its horizontal lift $\tilde{q}$ satisfies equation* (4.28)*. Moreover, if $g \in \Omega^{(1)}(G)$ is smooth, basic, and satisfies* (4.28)*, then $\pi \circ g$ is a critical point of $\mathcal{A}_H$.*

Proof. The first direction is an immediate consequence of Propositions 4, 3, and 5. Now suppose that $g \in \Omega^{(1)}(G)$ is basic and satisfies (4.28). Since g is smooth and basic, the curve $q := \pi \circ g$ is smooth and satisfies $\tilde{D}_t^{\mathcal{H}} \dot{g} = \widetilde{D_t \dot{q}}$. Moreover, from the proof of Proposition 5, we have $\text{grad}\tilde{V}(g) = \widetilde{\text{grad}V(q)}$. Hence,

$$0 = \tilde{D}_t^{\mathcal{H}} \dot{g} + \text{grad}\tilde{V}(g) = \widetilde{D_t \dot{q} + \text{grad}V(q)},$$

so that $D_t \dot{q} + \text{grad}V(q) = 0$ on $[a, b]$, which are precisely the Euler-Lagrange equations for $\mathcal{A}_H$. □

Now that we have found a (constrained) variational principle on G which is equivalent to $\delta \mathcal{A}_H = 0$ on H, we seek to apply reduction by symmetry under the assumption that V admits a partial symmetry over some smooth manifold Q.

Proposition 6. *Let $g \in \Omega^{(1)}(G)$ be a smooth, basic solution to* (4.28)*, and suppose that $\tilde{V}$ admits a partial symmetry over some smooth manifold Q (as in Definition* (3.4.2)*. Then $\xi := g^{-1}\dot{g}$ and $q := \Phi_g(q_0)$ satisfy*

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{h}}_{\xi} \xi = -\mathcal{H}(J(d_2 \tilde{V}_{\text{ext}}(e, q))^{\sharp}), \tag{4.29}$$

$$\dot{q} = \xi_Q(q), \tag{4.30}$$

on $[a, b]$, where $J : T^*Q \to \mathfrak{g}^*$ is the momentum map corresponding to the right-action $\Phi : G \times Q \to Q$.

Proof. From Lemma 4.1.6, we have $\tilde{D}_t^{\mathcal{H}}\dot{g} = g(\dot{\xi} + \tilde{\nabla}_\xi^{\mathfrak{h}}\xi)$. Hence, if g_s is a basic variation of g, then from the proofs of Propositions 5 and 2,

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0}\int_a^b \left(\frac{1}{2}\|\dot{g}_s(t)\|_G^2 - \tilde{V}_{\text{ext}}(e, g(t0)\right) dt &= \int_a^b \left(\left\langle \tilde{D}_t^{\mathcal{H}}\dot{g}, \delta g\right\rangle_G - \frac{\partial}{\partial s}\Big|_{s=0}\tilde{V}(g_s(t))\right) dt \\ &= \int_a^b \left(\left\langle \dot{\xi} + \tilde{\nabla}_\xi^{\mathfrak{h}}\xi, \eta\right\rangle_G - \left\langle J(d_2\tilde{V}_{\text{ext}}(e,q))^\sharp, \eta\right\rangle_G\right) dt,\end{aligned}$$

where $\eta := g^{-1}\delta g \in \mathfrak{h}$. □

4.2 Bi-invariant metrics and Riemannian Symmetric Spaces

Suppose G is a Lie group equipped with a left-invariant metric, and $H := G/K$ is a Riemannian homogeneous space. It is easily seen that the metric $\langle\cdot,\cdot\rangle_G$ is right K-invariant. That is, $\langle Xk, Yk\rangle_G = \langle X, Y\rangle_G$ for all $k \in K$. In the case that the metric $\langle\cdot,\cdot\rangle_G$ is bi-invariant, we have the following lemma as an analogue to Lemma 3.3.7

Lemma 4.2.1. *Suppose that G is a Lie group equipped with a bi-invariant metric, and H is a Riemannian homogeneous space of G. Then, for all $\xi, \eta, \sigma \in \mathfrak{h}$,*

$$\tilde{\nabla}_\xi^{\mathfrak{h}}\eta = \frac{1}{2}\mathcal{H}\big([\xi,\eta]\big), \tag{4.31}$$

$$\tilde{Q}(\xi,\eta)\sigma = \frac{1}{4}\big(\mathcal{H}\big([\sigma,[\xi,\eta]]\big) - [\xi, \mathcal{V}([\eta,\sigma]) + [\eta, \mathcal{V}([\xi,\sigma])] + 2[\sigma, \mathcal{V}([\xi,\eta])]\big). \tag{4.32}$$

Proof. Equation (4.31) follows immediately from Lemma 3.3.7 together with the fact that $\tilde{\nabla}_\xi^{\mathfrak{h}}\eta = \mathcal{H}(\tilde{\nabla}_\xi^{\mathfrak{g}}\eta)$. To see equation (4.32), we have from Theorem 2.3 of [114] that $\tilde{Q}(\xi,\eta)\sigma = \mathcal{H}(\tilde{R}(\xi,\eta)\sigma) - A_\xi A_\eta\sigma + A_\eta A_\xi\sigma + 2A_\sigma A_\xi\eta$. Note that $\mathcal{H}(\tilde{R}(\xi,\eta)\sigma) = \frac{1}{4}\mathcal{H}([\sigma,[\xi,\eta]])$ from Lemma 3.3.7. From the definition of A, it is clear that if $u_1, u_2 \in \mathfrak{h}$ and $v \in \mathfrak{s}$, then $A_{u_1}u_2 = \mathcal{V}(\tilde{\nabla}_{u_1}^{\mathfrak{g}}u_2) = \frac{1}{2}\mathcal{V}([u_1,u_2])$ and $A_{u_1}v = \tilde{\nabla}_{u_1}^{\mathfrak{h}}v = \frac{1}{2}\mathcal{H}([u_1,v])$. Hence, $A_\xi A_\eta\sigma = \frac{1}{4}\mathcal{H}([\xi,\mathcal{V}([\eta,\sigma])])$, and similarly for $A_\eta A_\xi\sigma$ and $A_\sigma A_\xi\eta$. Since H is a Riemannian homogeneous space, and hence reductive, we have in particular that $[\mathfrak{h},\mathfrak{s}] \subset \mathfrak{h}$. Equation (4.32) follows immediately upon observing that $\mathcal{V}: \mathfrak{g} \to \mathfrak{s}$, and $\mathcal{H}$ acts as the identity map on $\mathfrak{h}$. □

Another important special case of spaces that appears frequently in applications are *Riemannian symmetric spaces*, which are Riemannian homogeneous spaces such that there exists an involutive automorphism

$\sigma : G \to G$ with $K = \{g \in G \ \ \sigma(g) = g\}$. It can be seen that every Riemannian symmetric space of the form G/K satisfies the *Cartan Decomposition*:

$$[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}, \quad [\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{s}. \tag{4.33}$$

The condition $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$ follows from the fact that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$, and $[\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}$ follows from the fact that every Riemannian homogeneous space is reductive. The novel condition, then, is $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{s}$, which is not in general true for Riemannian homogeneous spaces. Moreover, if G is simply connected (an assumption that we always make), then every Riemannian homogeneous space satisfying (4.33) is a Riemannian symmetric space. It is neither necessary nor sufficient that $\langle \cdot, \cdot \rangle_G$ be bi-invariant in order for G/K to be Riemannian symmetric, though it is the case in many important applications. In particular, we have the following decomposition in the case that H is a Riemannian symmetric space, we have the following result:

Lemma 4.2.2. *Suppose that G is a Lie group equipped with a left-invariant metric, and H is a Riemannian symmetric space of G. Then, for all $\xi, \eta, \sigma \in \mathfrak{h}$,*

$$\tilde{\nabla}^{\mathfrak{h}}_{\xi} \eta = 0 \tag{4.34}$$

$$\tilde{Q}(\xi, \eta)\sigma = -[[\xi, \eta], \sigma] \tag{4.35}$$

Proof. Observe that the condition $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{s}$ in the Cartan decomposition (4.33) also implies that $\mathrm{ad}^{\dagger}_{\mathfrak{h}}(\mathfrak{h}) \subset \mathfrak{s}$, since $\left\langle \mathrm{ad}^{\dagger}_{\xi} \eta, \sigma \right\rangle_G = \langle \eta, [\xi, \sigma] \rangle_G = 0$ for all $\xi, \eta, \sigma \in \mathfrak{h}$. Equation (4.34) then follows immediately from equation (4.9). Equation (4.35) follows directly from [90], together with the fact that $\widetilde{R(\xi, \eta)\sigma} = \tilde{Q}(\xi, \eta)\sigma$. □

Example 8. Consider the Lie group $G = \mathrm{SO}(3)$ equipped with the bi-invariant metric defined by $\left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle_{\mathrm{SO}(3)} = \Omega_1^T \Omega_2$ for all $\Omega_1, \Omega_2 \in \mathbb{R}^3$, where we've made use of the hat map (3.4) to identify $\mathfrak{so}(3) \cong \mathbb{R}^3$. We let $\mathrm{SO}(3)$ act on $\mathbb{R}^3$ from the matrix multiplication (i.e. $\Phi : \mathrm{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $(R, x) \mapsto Rx$). We let $K \subset \mathrm{SO}(3)$ be the Lie subgroup given by

$$K = \left\{ \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{array} \right] \ \middle| \ \theta \in \mathbb{R} \right\} \cong \mathrm{SO}(2).$$

It is clear that $K = \text{Stab}(e_3)$, where $e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and $\text{Stab}(e_3)$ denotes the stabilizer subgroup of the left-action Φ. Moreover, the restriction of Φ to the unit sphere S^2 (embedded in $\mathbb{R}^3$ via inclusion) is transitive. It follows that the quotient space $H = \text{SO}(3)/\text{SO}(2) \cong S^2$ is a homogeneous space of $\text{SO}(3)$, and the projection map $\pi : \text{SO}(3) \to S^2$ is given by $\pi(R) = Re_3$ for all $R \in \text{SO}(3)$.

We consider the orthonormal basis $\{\hat{\Omega}_1, \hat{\Omega}_2, \hat{\Omega}_3\}$ of $\mathfrak{g} = \mathfrak{so}(3)$ given by

$$\hat{\Omega}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \hat{\Omega}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \hat{\Omega}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.36}$$

It is clear from the definition of the hat map (3.4) that $\hat{\Omega}_1 = \hat{e}_1, \hat{\Omega}_2 = \hat{e}_2$, and $\hat{\Omega}_3 = \hat{e}_3$. Moreover, we have $\pi_*\hat{\Omega}_1 = -e_2$, $\pi_*\hat{\Omega}_2 = e_1$, and $\pi_*\hat{\Omega}_3 = 0$, so that, in particular, $\mathfrak{s} = \ker(\pi_*\big|_I) = \text{span}\{\hat{\Omega}_3\}$ and $\mathfrak{h} = \mathfrak{s}^\perp$. Define an inner product on $T_{e_3}S^2$ via the relation $\langle X, Y\rangle_{T_{e_3}S^2} := \left\langle \pi_*^{-1}X, \pi_*^{-1}Y \right\rangle_{\text{SO}(3)}$ for all $X, Y \in T_{e_3}S^2$. Since $\{e_1, e_2\}$ forms a basis for $T_{e_3}S^2$, it is clear that—letting $X = X^1e_1 + X^2e_2$ and $Y = Y^1e_1 + Y^2e_2$—that

$$\begin{aligned} \langle X, Y\rangle_{T_{e_3}S^2} &= X^1Y^1 \langle e_1, e_1\rangle_{T_{e_3}S^2} + X^1Y^2 \langle e_1, e_2\rangle_{T_{e_3}S^2} + X^2Y^1 \langle e_2, e_1\rangle_{T_{e_3}S^2} + X^2Y^2 \langle e_2, e_2\rangle_{T_{e_3}S^2}, \\ &= X^1Y^1 \left\langle \hat{\Omega}_1, \hat{\Omega}_1 \right\rangle_{\text{SO}(3)} + X^1Y^2 \left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle_{\text{SO}(3)} + X^2Y^1 \left\langle \hat{\Omega}_2, \hat{\Omega}_1 \right\rangle_{\text{SO}(3)} + X^2Y^2 \left\langle \hat{\Omega}_2, \hat{\Omega}_2 \right\rangle_{\text{SO}(3)}, \\ &= X^1Y^1 + X^2Y^2, \end{aligned}$$

so that $\langle \cdot, \cdot \rangle_{T_{e_3}S^2}$ is just the standard Euclidean metric with respect to the basis $\{e_1, e_2\}$. We extend this inner product to an $\text{SO}(3)$-invariant Riemannian metric on S^2 by left-action. That is,

$$\langle X, Y\rangle_{S^2} = \left\langle R^TX, R^TY \right\rangle_{T_{e_3}S^2} = R^TX \cdot R^TY = X \cdot Y \tag{4.37}$$

for all $X, Y \in T_qS^2$ and for any $R \in \text{SO}(3)$ such that $\pi(R) = q$. It is clear that with this metric, S^2 becomes a Riemannian homogeneous space, and from the matrix commutator relations,

$$[\hat{\Omega}_1, \hat{\Omega}_2] = \hat{\Omega}_3, \qquad [\hat{\Omega}_1, \hat{\Omega}_3] = -\hat{\Omega}_2, \qquad [\hat{\Omega}_2, \hat{\Omega}_3] = \hat{\Omega}_1,$$

it is immediately clear that the decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$ satisfies Cartan's Decomposition (4.33), so that S^2 is in fact a Riemannian homogeneous space of $\text{SO}(3)$.

Consider a curve $q : [a,b] \to S^2$, and let $R : [a,b] \to \mathrm{SO}(3)$ be a horizontal lift of q. Hence, $q(t) = R(t)e_3$ for all $t \in [a,b]$. Let $X \in \Gamma(q)$, and denote its horizontal lift by $\tilde{X}$. It follows that $\tilde{X} = R\hat{\Omega}$ for some curve $\hat{\Omega} : [a,b] \to \mathfrak{h}$. Then,

$$D_t X = \pi_*(\tilde{D}_t^{\mathcal{H}} \tilde{X}) = (\tilde{D}_t^{\mathcal{H}} \tilde{X}) e_3 = R(\dot{\hat{\Omega}} + \tilde{\nabla}^{\mathfrak{h}}_{R^T \dot{R}} \hat{\Omega}) e_3$$

Since S^2 is a Riemannian symmetric space, we have from (4.34) that $\tilde{\nabla}^{\mathfrak{h}}_{R^T \dot{R}} \hat{\Omega} = 0$. Hence,

$$R\dot{\hat{\Omega}} e_3 = R(\dot{\Omega} \times e_3) = R\dot{\Omega} \times Re_3 = R\dot{\Omega} \times q.$$

Let $\omega := R\Omega$. Then, on one hand, $\dot{\omega} = \dot{R}\Omega + R\dot{\Omega} = R(\Omega \times \Omega) + R\dot{\Omega} = R\dot{\Omega}$, so that $D_t X = \dot{\omega} \times q$. On the other hand, we have $X = \tilde{X} e_3 = R\hat{\Omega} e_3 = R\Omega \times q = \omega \times q$. Since X is everywhere orthogonal to q, this further implies that $\omega = q \times X$, so that $\dot{\omega} = \dot{q} \times X + q \times \dot{X}$. Therefore, $D_t X = (\dot{q} \times X) \times q - q \times (q \times \dot{X})$. It is easy to see from the vector triple product rule that $(\dot{q} \times X) \times q = 0$, so that

$$D_t X = -q \times (q \times \dot{X}), \tag{4.38}$$

which is precisely the orthogonal projection of $\dot{X}$ onto the tangent bundle. It follows that the Riemannian geodesics q on S^2 satisfy $q \times (q \times \ddot{q}) = 0$,. Alternatively, with $\omega = q \times \dot{q}$, we have $\dot{\omega} = q \times \ddot{q}$, from which it follows that $\dot{\omega}$ is in fact orthogonal to q (i.e., it lies in the tangent bundle). On the other hand, the geodesic equation says that $\dot{\omega} \times q = 0$, which means that $\dot{\omega}$ is parallel to q. This is only possible if, in fact, $\dot{\omega} = 0$. In summary, geodesics on S^2 can alternatively be characterized by the system of equations:

$$\dot{q}(t) = \omega(t) \times q(t),$$

$$\dot{\omega}(t) = 0.$$

Finally, under the identifications $\mathfrak{so}(3) \cong \mathbb{R}^3$ and $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ through the hat and breve maps, respectively, we have from [55] that the infinitesimal generator corresponding to the left-action of $\mathrm{SO}(3)$ on S^2 takes the form $\hat{\Omega}_Q(q) = \Omega \times q$. Hence, the momentum map is given by $J(q,p) = q \times p$. It follows that, in the context of Proposition 6, the Euler-Poincaré equations (4.29)-(4.30) take the form

$$\dot{\Omega} = -x \times d_2 \tilde{V}_{\text{ext}}(e,x),$$

$$\dot{x} = \Omega \times x,$$

together with the reconstruction equation $\dot{R} = R\hat{\Omega}$.

Chapter 5

Modified Riemannian Cubics

Path planning has become ubiquitous in fields such as robotics, industrial engineering, physics, biology, and related disciplines. Typically, we have an agent governed by some physical laws or control schemes, and we wish for it to connect some set of knot points (interpolating given positions and velocities, and potentially higher order derivatives [28, 29]) while minimizing some quantity such as time or energy (e.g. battery consumption). For such problems, the use of variationally defined curves has a rich history due to the regularity and optimal nature of the solutions. In particular, the so-called *Riemannian splines* [89] are a particularly ubiquitous choice in interpolant, which themselves are composed of Riemannian polynomials—satisfying boundary conditions in positions, velocities, and potentially higher-order derivatives—that are glued together. In Euclidean spaces, Riemannian splines are just cubic splines. That is, the minimizers of the total squared acceleration.

Riemannian polynomials are smooth and optimal in the sense that they minimize the average square magnitude of some higher-order derivative along the curve. This quantity is often related to the magnitude of the controller in control engineering applications (which itself is related to energy consumption). Moreover, Riemannian polynomials carry a rich geometry with them, which has been studied extensively in the literature (see [39, 18, 89] for a detailed account of Riemannian cubics and [40] for some results with higher-order Riemannian polynomials).

It is often the case that—in addition to interpolating points—there are obstacles or regions in space which need to be avoided. In this case, a typical strategy is to augment the action functional with an artificial potential term that grows large near the obstacles and small away from them (in that sense, the trajectories

which minimize the action are expected to avoid the obstacles) [68]. This was done for instance in [7] and [8], where necessary conditions for extrema in obstacle avoidance problems on Riemannian manifolds were derived. In addition to applications to interpolation problems on manifolds and to energy-minimum problems on Lie groups and symmetric spaces endowed with a bi-invariant metric. Along this chapter, we will adopt the same strategy. However, we will define the action functional on a more general admissible path space, with which the necessary conditions for optimality will be derived (and shown to be equivalent to those found in [7]) in Section 5.1. We then proceed to prove the existence of global minimizers in Section 5.2 using techniques from global analysis (as discussed in 2.3.3). Finally, in Section 5.3, we derive sufficient conditions for optimality and extend the framework of Bi-Jacobi fields and biconjugate points (as defined in [20]) to this context.

5.1 Necessary Conditions for Optimality

Consider a Riemannian manifold Q, and the space $\Omega := \Omega^{(2)}$ (2.18). The principle object of study along this chapter will be the functional $J : \Omega \to \mathbb{R}$ as:

$$J[q] = \int_a^b \Big(\frac{1}{2} \left\| D_t \dot{q}(t) \right\|^2 + V(q(t)) \Big) dt. \tag{5.1}$$

where $V : Q \to \mathbb{R}$ is a smooth non-negative function called the *artificial potential*, which we will later design in Chapter 6 for the obstacle avoidance and collision avoidance tasks. Of particular interest to us are the critical points of J—in particular the curves $q \in \Omega$ which minimize J. However, even in classical analysis, finding global minima often requires first finding all local minima, classifying them, and then comparing the local minima to determine which are global minima. The first step, which we handle in this section, is to derive necessary conditions for the critical points of J on Ω. We do this via variational analysis—in particular, we search for curves $q \in \Omega$ such that $\delta J[q] = 0$ (or equivalently, such that $dJ[q]X = 0$ for all $X \in T_q\Omega$). In the case where Ω is the space of piecewise smooth curves satisfying the relevant boundary conditions in positions and velocities, such necessary conditions were derived in [7], and the critical points were shown to be smooth. We extend this result to the case where the admissible path space consists of Sobolev class H^2 curves satisfying the same boundary conditions in the following

proposition.

Proposition 7. *A curve $q \in \Omega$ is a critical point of the functional J if and only if it is smooth on $[a,b]$ and satisfies:*

$$D_t^3 \dot{q} + R\left(D_t \dot{q}, \dot{q}\right) \dot{q} = -\text{grad}\, V(q(t)). \tag{5.2}$$

Proof. Let $q \in \Omega$ be a critical point of J and $X \in T_q\Omega$. Consider an admissible proper variation $\alpha : (-\epsilon, \epsilon) \times [a,b]$ of q such that $\frac{\partial}{\partial s}\big|_{s=0} \alpha(s,t) = X$. Since q is a critical point of J, we must have that $\delta J[q]X = \frac{\partial}{\partial s}\big|_{s=0} J[\alpha(s,\cdot)] = 0$. Hence,

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0} J[\alpha(s,\cdot)] &= \int_a^b \frac{\partial}{\partial s}\left[\frac{1}{2}\left\|D_t\dot{\alpha}\right\|^2 + V(\alpha)\right]\Big|_{s=0} dt \\
&= \int_a^b \left[\left\langle D_s D_t \dot{\alpha},\ D_t\dot{\alpha}\right\rangle + \frac{\partial}{\partial s} V(\alpha)\right]\Big|_{s=0} dt \\
&= \int_a^b \left[\left\langle R\left(\frac{\partial \alpha}{\partial s}, \dot{\alpha}\right)\dot{\alpha} + D_t D_s \dot{\alpha},\ D_t\dot{\alpha}\right\rangle + \left\langle \text{grad} V(\alpha),\ \frac{\partial \alpha}{\partial s}\right\rangle\right]\Big|_{s=0} dt.
\end{aligned}$$

Where we have suppressed the arguments of α, $\dot{\alpha} = \frac{\partial}{\partial t}\alpha$, and we have applied Lemma 2.4.1 in the last line. From (2.10)-(2.13), it follows that

$$\begin{aligned}
0 = dJ[q]X &= \int_a^b \left[\left\langle D_t\dot{\alpha}, D_t^2 \frac{\partial \alpha}{\partial s}\right\rangle + \left\langle R\left(D_t\dot{\alpha}, \dot{\alpha}\right)\dot{\alpha} + \text{grad} V(\alpha), \frac{\partial \alpha}{\partial s}\right\rangle\right]\Big|_{s=0} dt \\
&= \int_a^b \left[\left\langle D_t\dot{q}, D_t^2 X\right\rangle + \left\langle R(D_t\dot{q},\dot{q})\dot{q} + \text{grad} V(q), X\right\rangle\right] dt,
\end{aligned}$$

for all $X \in T_q\Omega$. By construction, the space $\Gamma(q)$ of smooth vector fields along q is dense in $H_g^1(q)$ (as defined in (2.17)), hence there exists a sequence $(Y_n) \subset \Gamma(q)$ of smooth vector fields along q such that $Y_n \to \dot{q}$ with respect to the H_g^1 norm. Observe that

$$\begin{aligned}
&\int_a^b \left[\left\langle D_t Y_n, D_t^2 X\right\rangle + \left\langle R(D_t Y_n, Y_n)Y_n + \text{grad} V(q), X\right\rangle\right] dt \\
&= \int_a^b \left[-\left\langle D_t^2 Y_n, D_t X\right\rangle + \left\langle R(D_t Y_n, Y_n)Y_n + \text{grad} V(q), X\right\rangle\right] dt \\
&= \int_a^b \left[\left\langle D_t^3 Y_n + R(D_t Y_n, Y_n)Y_n + \text{grad} V(q), X\right\rangle\right] dt \to 0,
\end{aligned}$$

as $n \to \infty$, where in the second line we have used the fact that $\langle D_t Y_n, D_t^2 X \rangle_{L^2} = -\langle D_t^2 Y_n, D_t X \rangle_{L^2}$ by definition of the weak derivative $D_t^2 X$ with $D_t X(a) = D_t X(b) = 0$, and in the last line we have integrated the resulting term by parts and used the fact that $X(a) = X(b) = 0$. Considering the vector field $X = f\left[D_t^3 Y_n + R(D_t Y_n, Y_n)Y_n + \text{grad}V(q)\right]$ for some $f : [a,b] \to \mathbb{R}$ satisfying $f(a) = f(b) = 0$ and $f(t) \geq 0$ for all $t \in [a,b]$, we find

$$\int_a^b f(t) \left\| D_t^3 Y_n + R(D_t Y_n, Y_n)Y_n + \text{grad}V(q(t)) \right\|^2 dt \to 0.$$

It follows that $D_t^3 Y_n \to -R(D_t Y_n, Y_n)Y_n - \text{grad}V(q)$ uniformly on $[a,b]$. On the other hand, $R(D_t Y_n, Y_n)Y_n \to R(D_t\dot{q}, \dot{q})\dot{q}$ in $L_g^2(q)$ by the smoothness of the curvature tensor, from which it follows that $D_t^3 Y_n$ converges in L_g^2. By the uniqueness of limits, it must be the case that $Y_n \to \dot{q}$ in H_g^3, so in fact q is of H^4 regularity. Repeating this argument inductively, it is clear that in fact q must, in fact, be smooth, and $D_t^3 Y_n + R(D_t Y_n, Y_n) + \text{grad}V(q) \to D_t^3\dot{q} + R(D_t\dot{q}, \dot{q}) + \text{grad}V(q)$ uniformly, from which we find (5.2). $\square$

We will call any smooth curve satisfying equation 5.2 a Riemannian cubic polynomial modified by V (or just a modified Riemannian cubic for short, when the artificial potential V is understood) following the nomenclature that the solutions to the 6.4 when $V \equiv 0$ are referred to as Riemannian cubic polynomials.

5.2 Existence of Global Minimizers

Due to standard ODE theory, the existence and uniqueness of solutions to (5.2) is guaranteed upon specifying initial conditions $q(a),\ \dot{q}(a),\ D_t\dot{q}(a),\ D_t^2\dot{q}(a)$. This is known as an *initial value problem* (IVP). Ultimately, however, we are concerned with the *boundary value problem* (BVP) given by solving (5.2) with the boundary data specified by the admissible path space Ω. Existence of solutions to BVPs is more challenging and case specific than for IVPs, and the uniqueness of solutions often fails even for the most regular ODEs. For example, for any choice of boundary conditions on the unit sphere S^2 (equipped with the round metric), there are at least two geodesics satisfying them [81]. Moreover, if the boundary conditions are antipodal points, then there are infinitely many geodesics satisfying them, and all of them are global minimizers of the length functional.

In this section, we seek to prove the existence of solutions to the BVP specified by (5.2) and the boundary data on Ω. In fact, we will prove the stronger result that there exists a global minimizer of J in Ω. We will provide two proofs, one known as the *direct method*, which amounts to establishing the weak lower semicontinuity of J on Ω, and one utilizing tools from global analysis 2.3.3. In particular, we will show that J satisfies the Palais-Smale condition, from which existence is guaranteed by Theorem 2.3.5. The strategies implemented here closely resemble those given in [39] to prove the existence of global minimizers in the case that $V \equiv 0$. Certain details are changed due to the inclusion of the artificial potential, and to fit with the framework outlined in Section 2.3. Before beginning the proof, we prove a lemma that will simplify the analysis considerably.

Lemma 5.2.1. *Let Q be an m-dimensional complete Riemannian manifold, and suppose that $\{q_n\} \subset \Omega$ is a sequence such that $\sup_{n\in\mathbb{N}} J[q_n] < +\infty$. Then, $\{q_n\}$ and $\{\dot{q}_n\}$ are uniformly bounded, and there exists a subsequence of $\{q_n\}$ which converges weakly to some $q \in \Omega$ with respect to the H^2 norm.*

Proof. Suppose that $\{q_n\}$ is such a sequence. Setting $G_a := g(v_a, v_a)$ and using the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} g\left(\dot{q}_n(t), \dot{q}_n(t)\right) &= g(\dot{q}_n(a), \dot{q}_n(a)) + \int_a^t \frac{d}{du} g\left(\dot{q}_n(u), \dot{q}_n(u)\right) du \\ &\leq G_a + 2\int_a^b \left|g\left(D_u\dot{q}_n(u), \dot{q}_n(u)\right)\right| du \\ &\leq G_a + 2\left[\int_a^b \left\|D_u\dot{q}_n(u)\right\|^2 dt\right]^{1/2}\left[\int_a^b \left\|\dot{q}_n(u)\right\|^2 du\right]^{1/2} \\ &\leq G_a + 2(b-a)\left[\sup_{k\in\mathbb{N}} J[q_k]\right]^{1/2}\left[\sup_{t\in[a,b]} \left\|\dot{q}_n(t)\right\|^2\right]^{1/2}, \end{aligned}$$

where we have used the fact that V is non-negative in the last inequality. Let $c := \sup_{k\in\mathbb{N}} J[q_k]$ and $G := \sup_{t\in[a,b]} \|\dot{q}_n(t)\|^2$. Taking the supremum of the inequality over $t \in [a, b]$, we have

$$G \leq G_a + 2(b-a)\sqrt{cG} \implies G \leq \left((b-a)\sqrt{c} + \sqrt{(b-a)^2 c + G_a}\right)^2 := r^2.$$

Now observe that the sequence of lengths of the curves similarly satisfies

$$L(q_n) = \int_a^b \|\dot{q}_n(t)\|\, dt \leq (b-a)\sqrt{G} \leq r(b-a).$$

Hence, the image of q_n is contained in the closed geodesic ball $\overline{B}_{r(b-a)}(q_n)$, which is compact and independent of $n \in \mathbb{N}$. Therefore, the sequence $\{q_n\}$ is uniformly bounded over $[a, b]$. Now observe that, for the Riemannian distance $d(\cdot, \cdot)$ and for all $a \leq t < \tau \leq b$,

$$d(q_n(t), q_n(\tau)) \leq L\left(q_n|_{(t,\tau)}\right) = \int_t^\tau \|\dot{q}_n(u)\| \, du \leq r(\tau - t),$$

where $q_n|_{(t,\tau)}$ denotes the restriction of q_n to the interval $(t, \tau) \subset [a, b]$. Hence, $\{q_n\}$ is equicontinuous on $[a, b]$, and by the Arzela-Ascoli Theorem, there exists a subsequence $\{q_{n_k}\} \subset \{q_n\}$ which converges uniformly to a continuous curve q_n satisfying the boundary conditions in position.

Let (U_i, φ_i) be a finite collection of charts on Q and I_i an accompanying finite partition of $[a, b]$ such that, for sufficiently large n, there exists a compact subset $K \subset U_i$ containing $q_n(I_i)$. In local coordinates, we may consider q_n to be a curve on $\mathbb{R}^m$ (however, we will abuse this notation by continuing to call it q_n both on the chart U_i and its image in $\mathbb{R}^m$). Hence, in K, we have

$$\begin{aligned} \|q_n\|_{H^2}^2 &= \|q_n\|_{L^2}^2 + \|\dot{q}_n\|_{L^2}^2 + \|\ddot{q}_n\|_{L^2}^2 \\ &= \int_{I_i} \|q_n(t)\|_{\mathbb{R}^m}^2 \, dt + \int_{I_i} \|\dot{q}_n(t)\|_{\mathbb{R}^m}^2 \, dt + \int_{I_i} \|\ddot{q}_n(t)\|_{\mathbb{R}^m}^2 \, dt. \end{aligned}$$

The first integral is bounded since φ_i is a homeomorphism and K is compact—hence $\varphi_i(K) \supset (\varphi_i \circ q_n)(I_i)$ is bounded in $\mathbb{R}^m$. For the second integral, note that for some scalars $\alpha, \ \beta$ we have $\alpha \|X\|_{\mathbb{R}^m}^2 \leq \|X\|^2 \leq \beta \|X\|_{\mathbb{R}^m}^2$ for all $X \in T_xQ$ with $x \in K$. Hence, the boundedness of the second integral is equivalent to the boundedness of

$$\int_{I_i} \|\dot{q}_n(t)\|^2 \, dt,$$

which follows immediately by the uniform boundedness of $\|\dot{q}_n\|^2$ on $[a, b]$ (and hence also on the subset I_i).

For the final integral, note that

$$D_t \dot{q}_n = \ddot{q}_n + \Gamma(q_n; \dot{q}_n, \dot{q}_n). \tag{5.3}$$

where $\Gamma : \mathbb{R}^{3m} \to \mathbb{R}^m$ is continuous in the first argument and bilinear in the last two—and is determined by the ordinary Christoffel Symbols induced by the connection and chart. It follows that the boundedness of the final integral is equvialent to the boundedness of

$$\int_{I_i} \|D_t\dot{q}_n(t)\|^2 \, dt.$$

Since Γ and V are uniformly bounded on I_i by continuity, and q_n and $\dot{q}_n$ are uniformly bounded, it follows by the assumption $\sup_{n\in\mathbb{N}} J[q_n] < +\infty$ that the above integral is bounded. Hence, (q_n) is bounded in H^2. Since H^2 is a Hilbert space, we then get the weak convergence of some subsequence of (q_n) to $q \in \Omega$ under the H^2 norm. □

Theorem 5.2.2. *The functional J attains its minimum in Ω.*

Proof. Suppose that $\{q_n\} \subset \Omega$ is a minimizing sequence, so that $\lim_{n\in\mathbb{N}} J[q_n] = \inf_{q\in\Omega} J[q] \geq 0$. Such a sequence must satisfy the assumptions of Lemma (5.2.1), so that there exists a subsequence of q_n (which we also denote by $\{q_n\}$ for convenience) which converges weakly to some $q \in \Omega$ with respect to the H^2 norm. It suffices to show that $J[q] \leq \liminf_{n\to\infty} J[q_n]$. As in the proof of Lemma 5.2.1, $\|\dot{q}_n\|^2$ is uniformly bounded, and so there exists a finite collection of charts (U_i, φ_i) on Q and I_i an accompanying finite partition of $[a, b]$ such that, for sufficiently large n, there exists a compact subset $K \subset U_i$ containing $q_n(I_i)$. In local coordinates, we may consider q_n to be a curve on $\mathbb{R}^m$ (as before, we will abuse this notation by continuing to call it q_n both on the chart U_i and its image in $\mathbb{R}^m$).

Observe first that $V(q_n)$ converges to $V(q)$ uniformly on $[a, b]$ since the interval is compact, V is continuous, and $q_n \to q$ uniformly. Hence, $J[q] \leq \liminf_{n\to\infty} J[q_n]$ is equivalent to showing that for all intervals I_i and $n \in \mathbb{N}$,

$$\int_{I_i} \|D_t\dot{q}(t)\|^2 \, dt \leq \liminf_{n\to\infty} \int_{I_i} \|D_t\dot{q}_n(t)\|^2 dt. \tag{5.4}$$

Moreover, the weak H^2 convergence of q_n to q implies that $\dot{q}_n \to \dot{q}$ uniformly, so that $\|\dot{q}_n\|^2 \to \|\dot{q}\|^2$. Hence, (5.4) is equivalent to $\|\dot{q}\|_{H^1_g} \leq \liminf_{n\to\infty} \|\dot{q}_n\|_{H^1_g}$, with $H^1_g(q)$ defined by (2.15). However, we showed that $H^1_g(q)$ and $H^1(I_i, \mathbb{R}^m)$ are equivalent as normed linear spaces, and as such induce the same weak topology. That is, the weak convergence of $\dot{q}_n$ to $\dot{q}$ with respect to the H^1 norm implies its weak convergence with respect to the H^1_g norm, from which $\|\dot{q}\|_{H^1_g} \leq \liminf_{n\to\infty} \|\dot{q}_n\|_{H^1_g}$ follows immediately.

□

We now provide an alternate (and stronger) proof of existence. Since Ω is a smooth and complete Hilbert manifold and $J : \Omega \to \mathbb{R}$ is smooth and bounded below, we have from Theorem 2.3.5 that there exists a global minimizer of J in Ω if J satisfies the Palais-Smale condition (2.3.4). We show this in the following theorem.

Theorem 5.2.3. *J satisfies the Palais-Smale condition on Ω.*

Proof. Suppose that $\{q_n\} \subset \Omega$ is a Palais-Smale sequence for J, and let $B_n(t)$ and $C_n(t)$ be vector fields along q_n satisfying

$$D_t^2 B_n(t) = R\left(D_t\dot{q}_n(t), \dot{q}_n(t)\right)\dot{q}_n(t) \text{ and } D_t^2 C_n(t) = \text{grad}V(q_n(t)).$$

Note that such vector fields can be shown to exist by, for example, moving to local coordinates and applying standard ODE estimates (Gronwall's inequality). From Lemma 5.2.1, some subsequence of q_n (again denoted by q_n) converges weakly to a curve $q \in \Omega$ with respect to the H^2 norm. Hence $\{q_n\}$ and $\{\dot{q}_n\}$ are uniformly bounded, and $D_t\dot{q}_n$ is bounded with respect to the L^2 norm. Therefore, $R\left(D_t\dot{q}_n(t), \dot{q}_n(t)\right)\dot{q}_n(t)$ and $\text{grad}V(q_n)$ are bounded in L^2. Hence B_n and C_n are bounded in H^2, so that there exists subsequences converging strongly in L^2 (which we denote again by A_n and B_n for convenience).

Observe that

$$\|dJ[q_n]\|_{T^*_{q_n}\Omega} \to 0 \implies \|\text{grad}J[q_n]\|_{T_{q_n}\Omega} \to 0,$$

with respect to the Riemannian structure on Ω induced by (2.17). Hence, for $A_n := \text{grad}J[q_n]$, we have $D_t^2 A_n \to 0$ in $L^2_g(q_n)$, and

$$\begin{aligned} dJ[q_n]X &= \int_a^b \left\langle D_t\dot{q}_n(t) + B_n(t) + C_n(t), D_t^2 X\right\rangle dt, \\ \text{grad}J[q_n]X &= \int_a^b \left\langle D_t^2 A_n, D_t^2 X\right\rangle dt, \end{aligned}$$

for all $X \in T_{q_n}\Omega$. It follows that, with $Z_n(t) := D_t\dot{q}_n(t) + B_n(t) + C_n(t) - D_t^2 A_n$,

$$\int_a^b \left\langle Z_n, D_t^2 X\right\rangle dt = 0,$$

for all $X \in T_{q_n}\Omega$ and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we may consider a sequence of smooth vector fields $(\bar{Z}_{n_k})_k \subset \Gamma(q_n)$ along q_n such that $\bar{Z}_{n_k} \to Z_n$ in $L^2_g(q_n)$. Therefore,

$$\int_a^b \left\langle \bar{Z}_{n_k}, D_t^2 X \right\rangle dt \to 0 \text{ as } k \to \infty, \text{ for all } X \in T_{q_n}\Omega.$$

Using the definition of weak derivative and integrating by parts once, we find that

$$\int_a^b \left\langle D_t^2 \bar{Z}_{n_k}, X \right\rangle dt \to 0 \text{ as } k \to \infty,$$

for all $X \in T_{q_n}\Omega$, from which it follows that $D_t^2 \bar{Z}_{n_k} \to 0$ uniformly. In particular, this implies that $D_t^2 \bar{Z}_{n_k}$ converges in $L^2_g(q_n)$, so that $\bar{Z}_{n_k} \to Z_n$ in $H^2_g(q_n)$. Repeating the above argument with Z_n in place of $\bar{Z}_{n_k}$, we find that $D_t^2 Z_n \equiv 0$ for all $n \in \mathbb{N}$. By standard ODE estimates, it can be seen that Z_n is smooth on $[a, b]$ for all $n \in N$.

To conclude the result, it suffices to show that the sequence $(D_t Z_n)$ is bounded in L^2, for this will immediately imply that some subsequence of (Z_n) converges strongly in L^2—and hence so will $D_t \dot{q}_n$ since $D_t^2 A_n, B_n$, and C_n all converge strongly in L^2. Defining the sequence of functions $\beta_n : \mathbb{R} \to \mathbb{R}$ by $\beta_n(t) := \|Z_n(t)\|^2$, we see by direct computation that $\frac{d^3}{dt^3}\beta_n(t) = 0$ for all $t \in [a, b]$, from which it follows immediately that $\beta_n(t)$ is a sequence of quadratic polynomials. Observe that:

(1) $\beta_n(t) \geq 0$ for all $t \in [a, b]$

(2) $\int_a^b \beta_n(t)dt = \int_a^b \|Z_n\|^2 dt$ is bounded since (Z_n) is bounded in L^2.

From [40], it follows that the sequences of coefficients of β_n are bounded. In particular, the quadratic terms given by $\frac{1}{2}\frac{d^2}{dt^2}\beta_n(t) = \|D_t Z_n(t)\|^2$ are bounded, so that $\|D_t Z_n(t)\|^2$ is uniformly bounded on $[a, b]$. In particular, this implies that $(D_t Z_n)$ is bounded in L^2, as desired. □

As a consequence of Theorem 5.2.3 and 2.3.5, we have the following:

Proposition 8. *Suppose that Q is connected and complete, and $V : Q \to \mathbb{R}$ is smooth and non-negative. Then there exists a curve $q^* \in \Omega$ satisfying* (5.2) *and such that $J[q^*] = \inf_{q \in \Omega} J[q]$.*

5.3 Sufficient Conditions for Optimality

We have seen that the critical points of J, the so-called modified Riemannian cubics, must satisfy equation (5.2). Ultimately, however, we are concerned only with the minimizers of J. Proposition 8 guarantees the existence of a global minimizer in Ω, but we do not yet know how to find such a curve. As discussed in Section 5.1, the problem of finding global minimizers is difficult. Indeed, even in the case of smooth bounded functions on the real line, global minima typically must be found by first finding all critical points and values, and then comparing them to determine the global minima.

In our case, finding all critical points amount to finding all smooth solutions to the BVP determined by (5.2) together with the boundary conditions specified by Ω. This, in general, is a difficult problem. Even if we are able to approximate critical points via numerical methods (e.g. a numerical integrator and a shooting method), there may be no systematic way to find all of them, and no way to know even when we have. Thus, we reduce our search to local minimizers of J, based on the hope that they will be well-behaved enough to be useful in path-planning strategies. For functions whose domain is a subset of some Euclidean space, this amounts to applying the second-derivative test at the critical points in question. For our problem, the principle is similar—we need only replace the second derivative with the second variation (or differential) along a modified Riemannian cubic, as was done for the energy functional along geodesics in Section 2.4. For consistency, we introduce the following types of minimizers:

Definition 5.3.1. A modified Riemannian cubic $q \in \Omega$ is a:

(i) *Global minimizer* of J iff $J[q] \leq J[\tilde{q}]$ for all $\tilde{q} \in \Omega$.

(ii) Ω-local minimizer of J iff $J[q] \leq J[\tilde{q}]$ for all $\tilde{q}$ in some C^1 neighborhood of q (within Ω).

(iii) Q-local minimizer of J iff for any $\tau \in [a, b]$, there exists an interval $[a^*, b^*] \subset [a, b]$ containing τ such that $q|_{[a^*,b^*]}$ is a global minimizer of J on $\Omega_{\xi,\eta}^{[a^*,b^*]}$, where $\xi = (q(a^*), \dot{q}(a^*))$, $\eta = (q(b^*), \dot{q}(b^*))$.

It should be noted that we have slightly abused our notation in the definition of a Q-local minimizer. Technically, we are concerned with minimizing the integral $\int_{a^*}^{b^*} \big(\frac{1}{2} \|D_t\|^2 + V(q(t))\big) dt$, which has differ-

ent limits of integration than J as defined in equation (5.1). We will continue to refer to integrals of this form by J throughout the chapter—and in every case, the limits of integration will match that of the boundary conditions defined by the admissible set $\Omega_{\xi,\eta}^{[a,b]}$ on which J is being discussed.

5.3.1 Ω-local minimizers

In order to understand when a critical point of J (that is, a modified Riemannian cubic) is an Ω-local minimizer of J, we utilize second variations. Suppose that $q \in \Omega$ is a modified Riemannian cubic, and let $I_r = (-\epsilon_r, \epsilon_r)$ and $I_s = (-\epsilon_s, \epsilon_s)$ for some $\epsilon_r, \epsilon_s > 0$. We consider a two-parameter variation $\alpha : I_r \times I_s \times [a,b] \to Q$ of q satisfying

(1) $\alpha(r,s,\cdot) \in \Omega$ for all $(r,s) \in I_r \times I_s$,

(2) $\alpha(\cdot,\cdot,t) \in C^\infty(I_r \times I_s, Q)$ for all $t \in [a,b]$,

(3) $\alpha(0,0,t) = q(t)$ for all $t \in [a,b]$,

(4) $\frac{d}{dr}\big|_{r=0}\alpha(r,0,\cdot) = X \in T_q\Omega$,

(5) $\frac{d}{ds}\big|_{s=0}\alpha(0,s,\cdot) = Y \in T_q\Omega$.

We calculate the second variation of J along α by $\delta^2 J[q] = \dfrac{\partial^2}{\partial r \partial s}\big|_{(r,s)=(0,0)} J(\alpha(r,s,\cdot))$. As for geodesics in Section 2.4, we define the *index form* $I : T_q\Omega \times T_q\Omega \to \mathbb{R}$ associated to the modified Riemannian cubic $q \in \Omega$ as $I(X,Y) = d^2J[q](X,Y)$ for all $X, Y \in T_q\Omega$, where we've considered the second differential of J as a bilinear map $d^2J : T_q\Omega \times T_q\Omega \to \mathbb{R}$ via the identification $T_X(T_q\Omega) \cong T_q\Omega$. It is clear that $I(X,Y) = \delta^2 J[q]$, where X and Y are the variational vector fields of the variation along which we consider $\delta^2 J[q]$ (note that every vector field in $T_q\Omega$ can be written as the variational vector field of some variation [81]). Mirroring 2.4.4, we show the utility of the index form (and hence the second variation) in the problem of classifying critical points of J with the following theorem.

Theorem 5.3.2. *Suppose that $q \in \Omega$ is a modified Riemannian cubic. If $I(X,X) > 0$ for all $X \in T_q\Omega \setminus \{0\}$, then q is an Ω-local minimizer of J.*

Proof. First, suppose that $I(X,X) > 0$ for all $X \in T_q\Omega$. For some $\epsilon > 0$, consider an admissible variation q_s of q with variational vector field $\partial_s q_s|_{s=0} = X$, where $s \in (-\epsilon, \epsilon)$. Let $f(s) := J[q_s]$, and observe that $f'(0) = 0$ since q is a modified Riemannian cubic. Moreover, $f''(0) = I(X,X) > 0$, so that by the second-derivative test, f has a local minimum at $s = 0$. It follows that $J[q] \leq J[q_s]$ for all $s \in (-\epsilon, \epsilon)$. Since this holds for all admissible variations, it follows that q is an Ω-local minimizer. □

We give an expression for the index form in the following proposition:

Proposition 9. *Let $q \in \Omega$ be a modified Riemannian cubic. Then the index form along q is given by*

$$I(X,Y) = \int_a^b \left[\left\langle D_t^2 X, D_t^2 Y \right\rangle + \left\langle F(X,\dot{q}) + \nabla_X \text{grad} V, Y \right\rangle \right] dt, \tag{5.5}$$

for all $X, Y \in T_q\Omega$, where

$$\begin{aligned} F(X,Y) &= (\nabla_Y^2 R)(X,Y)Y + (\nabla_X R)(\nabla_Y Y, Y)Y + R(R(X,Y)Y,Y)Y + R(X, \nabla_Y^2 Y)Y \\ &\quad + 2\left[(\nabla_Y R)(\nabla_Y X, Y)Y + (\nabla_Y R)(X, \nabla_Y Y)Y + R(\nabla_Y^2 X, Y)Y \right] \\ &\quad + 3\left[(\nabla_Y R)(X,Y)\nabla_Y Y + R(X,Y)\nabla_Y^2 Y + R(X, \nabla_Y Y)\nabla_Y Y \right] + 4R(\nabla_Y X, Y)\nabla_Y Y. \end{aligned} \tag{5.6}$$

Proof. Let α be a two parameter variation of q with $\frac{\partial \alpha}{\partial r}\Big|_{(r,s)=(0,0)} = X$ and $\frac{\partial \alpha}{\partial s}\Big|_{(r,s)=(0,0)} = Y$. We calculate $I(X,Y) = \delta^2 J[q] = \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} J(\alpha(r,s,\cdot)) = \frac{\partial}{\partial r}\Big|_{r=0} \frac{\partial}{\partial s}\Big|_{s=0} J(\alpha(r,s,\cdot))$. From Proposition 7, we see that

$$\frac{\partial}{\partial s}\Big|_{s=0} J[\alpha] = \int_a^b \left[\left\langle D_t \dot{\alpha}, D_t^2 \frac{\partial \alpha}{\partial s} \right\rangle + \left\langle R\left(D_t \dot{\alpha}, \dot{\alpha}\right)\dot{\alpha} + \text{grad} V(\alpha), \frac{\partial \alpha}{\partial s} \right\rangle \right] \Big|_{s=0} dt.$$

Therefore,

$$\begin{aligned} \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} J[\alpha] = \int_a^b \Big[&\left\langle D_r D_t \dot{\alpha}, D_t^2 \frac{\partial \alpha}{\partial s} \right\rangle + \left\langle D_t \dot{\alpha}, D_r D_t^2 \frac{\partial \alpha}{\partial s} \right\rangle \\ &+ \left\langle (D_r R)(D_t \dot{\alpha}, \dot{\alpha})\dot{\alpha}, \frac{\partial \alpha}{\partial s} \right\rangle + \left\langle R(D_r D_t \dot{\alpha}, \dot{\alpha})\dot{\alpha}, \frac{\partial \alpha}{\partial s} \right\rangle \\ &+ \left\langle R(D_t \dot{\alpha}, D_r \dot{\alpha})\dot{\alpha}, \frac{\partial \alpha}{\partial s} \right\rangle + \left\langle R(D_t \dot{\alpha}, \dot{\alpha}) D_r \dot{\alpha}, \frac{\partial \alpha}{\partial s} \right\rangle \\ &+ \left\langle D_r \text{grad} V(\alpha), \frac{\partial \alpha}{\partial s} \right\rangle + \left\langle R\left(D_t \dot{\alpha}, \dot{\alpha}\right)\dot{\alpha} + \text{grad} V(\alpha), D_r \frac{\partial \alpha}{\partial s} \right\rangle \Big] \Big|_{(r,s)=(0,0)} dt. \end{aligned}$$

Upon multiple applications of Lemmas 2.23 and 2.4.1, we find

$$\begin{aligned} D_r D_t \dot{\alpha}\Big|_{(r,s)=(0,0)} &= D_t^2 X + R(X,\dot{q})\dot{q}, \\ D_r D_t^2 \frac{\partial \alpha}{\partial s}\Big|_{(r,s)=(0,0)} &= D_t^2 \left(D_r \frac{\partial \alpha}{\partial s} \right) \Big|_{(r,s)=(0,0)} + (D_t R)(X,\dot{q})Y \\ &\quad + R(D_t X,\dot{q})Y + R(X, D_t\dot{q})Y + 2R(X,\dot{q})D_t Y. \end{aligned}$$

Moreover, if we define $\gamma_t(r) := \alpha(r,0,t)$, and denote $\dot{\gamma}_t(r) = \frac{d}{dr}\gamma_t(r)$, then $D_r\Big|_{r=0}\text{grad}V(\alpha(r,0,t))$ can be expressed as:

$$D_r\Big|_{r=0}\text{grad}V(\gamma_t(r)) = \nabla_{\dot{\gamma}_t(r)}\text{grad}V\Big|_{r=0} = \nabla_{X(t)}\text{grad}V.$$

It follows that

$$\begin{aligned} \frac{\partial^2}{\partial r \partial s}\Big|_{(r,s)=(0,0)} J[\alpha] = \int_a^b \Big[& \left\langle D_t^2 X, D_t^2 Y \right\rangle + \langle \nabla_X \text{grad}V(q), Y\rangle + \left\langle R(X,\dot{q})\dot{q}, D_t^2 Y \right\rangle \\ & + \langle (D_t R)(X,\dot{q})Y + R(D_t X,\dot{q})Y + R(X, D_t\dot{q})Y + 2R(X,\dot{q})D_t Y, D_t\dot{q}\rangle \\ & + \left\langle (\nabla_X R)(D_t\dot{q},\dot{q})\dot{q} + R(D_t^2 X,\dot{q})\dot{q} + R(R(X,\dot{q})\dot{q},\dot{q})\dot{q} + R(D_t\dot{q}, D_t X)\dot{q} + R(D_t\dot{q},\dot{q})D_t X, Y \right\rangle \\ & + \left\langle D_t^3\dot{q} + R(D_t\dot{q},\dot{q})\dot{q} + \text{grad}V(q), D_r \frac{\partial \alpha}{\partial s}\Big|_{(r,s)=(0,0)} \right\rangle \Big] dt. \end{aligned}$$

The final term vanishes identically since q is a modified Riemannian cubic. Upon applying the symmetries of the curvature tensor (2.10)-(2.14) as was done in [29], we obtain (5.5). □

When the artificial potential vanishes identically, we recover the index form for Riemannian cubics that was obtained in [29] and [19]. Observe, however, that (5.5) is defined for vector fields of Sobolev class H^2 along q which vanish at the endpoints along with their covariant derivatives—whereas the index form discussed in the literature assumes more regularity—in particular, that the vector fields are piecewise smooth (and globally C^1).

Unlike the case of Riemannian cubics and geodesics where the corresponding index form is symmetric, the artificial potential introduces an asymmetry. Namely, we can decompose I as $I(X,Y) = I_c(X,Y) + P_+(X,Y) + P_-(X,Y)$, where

- $I_c(X,Y) = \int_a^b \left[\left\langle \frac{D^2X}{dt^2}, \frac{D^2Y}{dt^2} \right\rangle + \langle Y, F(X,\dot{q}(t))\rangle \right] dt$ is the index form for Riemannian cubics.

- $P_+(X,Y) = \frac{1}{2}\int_a^b [\langle Y, \nabla_X \text{grad}V\rangle + \langle X, \nabla_Y \text{grad}V\rangle]\, dt$ is a symmetric bilinear form describing the symmetric contribution of the artificial potential.

- $P_-(X,Y) = \frac{1}{2}\int_a^b [\langle Y, \nabla_X \text{grad}V\rangle - \langle X, \nabla_Y \text{grad}V\rangle]\, dt$ is an antisymmetric bilinear form describing the antisymmetric contribution of the artificial potential.

From Theorem 5.3.2, we wish to determine when the index form corresponding to some modified Riemannian cubic is positive-definite. In general, such a task will not be computationally feasible to verify via the direct method (that is, calculating $I(X,X)$ for all $X \in T_q\Omega$), so we turn our attention to studying the vector fields which belong to the kernel of the index form. As we will see, this yields the analogue of Jacobi fields for modified Riemannian cubics.

Proposition 10. *A vector field $X \in T_q\Omega$ belongs to the kernel of I if and only if X is smooth and satisfies*

$$D_t^4 X + F(X,\dot{q}) + \nabla_X \text{grad}V(q) = 0 \tag{5.7}$$

for all $t \in [a,b]$.

Proof. Since $\Gamma(q)$ is dense in $H_g^2(q)$, we may consider a sequence $(X_n) \subset \Gamma(q)$ of smooth vector fields along q such that $X_n \to X$ with respect to the H_g^2 norm (2.15). It follows that $D_t^2 X_n \to D_t^2 X$ in L_g^2, and hence $F(X_n,\dot{q}) \to F(X,\dot{q})$ in L_g^2 since both the curvature endomorphism and q are smooth. Moreover, $\nabla_{X_n}\text{grad}V(q) \to \nabla_X \text{grad}V(q)$ uniformly since V is smooth. It follows that $\lim_{n\to\infty} I(X_n, Y) = I(X,Y) = 0$ for all $Y \in T_q\Omega$. From (5.5), we see upon applying the definition of weak derivative, integrating by parts, and using $Y(a) = Y(b) = D_tY(a) = D_tY(b) = 0$, that

$$\begin{aligned}
&\int_a^b \left[\left\langle D_t^2 X_n, D_t^2 Y\right\rangle + \langle F(X_n,\dot{q}) + \nabla_{X_n}\text{grad}V(q), Y\rangle\right] dt \\
&= \int_a^b \left[\left\langle D_t^4 X_n + F(X_n,\dot{q}) + \nabla_{X_n}\text{grad}V(q), Y\right\rangle\right] dt \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

for all $Y \in T_q\Omega$. It follows that $D_t^4 X_n \to -F(X_n,\dot{q}) - \nabla_{X_n}\text{grad}V(q)$ uniformly on $[a,b]$. Hence $D_t^4 X_n \to -F(X,\dot{q}) - \nabla_X \text{grad}V(q)$ with respect to the L_g^2 norm. By the uniqueness of limits, it must be the case that X is of H_g^4-regularity, and $D_t^4 X + F(X,\dot{q}) + \nabla_X \text{grad}V(q) \equiv 0$ on $[a,b]$. Moreover, this implies that, in fact, $F(X_n,\dot{q}) \to F(X,\dot{q})$ and $\nabla_{X_n}\text{grad}V(q) \to \nabla_X \text{grad}V(q)$ with respect to the H_g^2

norm—implying the convergence of X_n with respect to the H^6_g norm. Repeating this argument ad infinitum shows that X is smooth. $\square$

This motivates the following definition:

Definition 5.3.3. Let $q \in \Omega$ be a modified Riemannian cubic. We call any vector field X along q satisfying (5.7) for all $t \in [a, b]$ a *modified bi-Jacobi field*.

Observe that in the case where $V \equiv 0$, (5.7) reduced to the equation describing bi-Jacobi fields along a Riemannian cubic, as defined in [19]. Moreover, the ODE (5.7) is linear in X, so that (since V is smooth) all solutions are necessarily smooth and the existence and uniqueness of solutions on $[a, b]$ given initial values $X(0),\ D_tX(0),\ D_t^2X(0),\ D_t^3X(0)$ follows immediately (say, by moving to coordinate charts). In particular, the set of modified bi-Jacobi fields along a modified Riemannian cubic q forms a $4n$-dimensional vector space. Note that we do not make any assumptions about the initial or final values of bi-Jacobi fields—that is, it is not necessarily the case that a bi-Jacobi field X is an element of the tangent space $T_q\Omega$.

Definition 5.3.4. Two points $t = t_1 < t_2 \in [a, b]$ are said to be biconjugate along a modified Riemannian cubic q if there exists a non-zero modified bi-Jacobi field X such that

$$X(t_1) = X(t_2) = 0, \quad \text{and} \quad D_tX(t_1) = D_tX(t_2) = 0.$$

It is clear that the existence of biconjugate points is equivalent to the existence of a bi-Jacobi field $X \in T_{q|_{[t_1,t_2]}}\Omega^{[t_1,t_2]}_{\xi,\eta}$ for some $t_1, t_2 \in [a, b]$, where $q|_{[t_1,t_2]}$ denotes the restriction of q to the interval $[t_1, t_2] \subset [a, b]$ and $\xi = (q(t_1), \dot{q}(t_1)), \eta = (q(t_2), \dot{q}(t_2))$. Analogous to the case of geodesics and conjugate points ([63], Theorem 4.3.1), or Riemannian cubic polynomaials and biconjugate points ([19], Theorem 7.2), we now show that modified Riemannian cubics do not minimize past their biconjugate points.

Proposition 11. *Suppose that $q \in \Omega$ is a modified Riemannian cubic and that $a \leq t_1 < t_2 \leq b$ are biconjugate. Then q is not an Ω-local minimizer of J.*

Proof. Since t_1, t_2 are biconjugate, there exists a modified bi-Jacobi field U such that $U(t_1) = U(t_2) = D_tU(t_1) = D_tU(t_2) = 0$. Note that at least one of $D_t^2U(t_1),\ D_t^3U(t_1)$ are non-zero and at least one of

$D_t^2U(t_2),\ D_t^3U(t_2)$ are non-zero (otherwise, $U \equiv 0$). Consider the vector field $X(t) = \begin{cases} U(t), & t \in [t_1, t_2] \\ 0, & \text{otherwise} \end{cases}$.

It is clear that X is globally C^1, and smooth except possibly at the points $t = t_1, t_2$. We also see that $X(a) = X(b) = D_tX(a) = D_tX(b) = 0$, so that in particular $X \in T_q\Omega$. Moreover, since X satisfies (5.7) on $[a, t_1], [t_1, t_2]$, and $[t_2, b]$, it is clear that $I(X, X) = 0$. Now, let $Z, W \in \Gamma(q)$ be smooth vector fields such that

$$Z(t_1) = -D_t^3U(t_1), \qquad Z(t_2) = -D_t^3U(t_2), \qquad D_tZ(t_1) = D_tZ(t_2) = 0,$$
$$W(t_1) = W(t_2) = 0, \qquad D_tW(t_1) = D_t^2U(t_1), \qquad D_tW(t_2) = D_t^2U(t_2).$$

For $\delta > 0$, consider $Y(t) = \phi(t)Z(t) + \psi(t)W(t)$, where $\phi, \psi \in C^\infty(\mathbb{R})$ are such that

$$\phi(t_1) = \phi(t_2) = 1, \qquad \dot\phi(t_1) = \dot\phi(t_2) = 0, \qquad \text{supp}(\phi) \subseteq (t_1 - \delta, t_1 + \delta) \cup (t_2 - \delta, t_2 + \delta),$$
$$\psi(t_1) = \psi(t_2) = 0, \qquad \dot\psi(t_1) = \dot\psi(t_2) = 1, \qquad \text{supp}(\psi) \subseteq (t_1 - \delta, t_1 + \delta) \cup (t_2 - \delta, t_2 + \delta),$$

and $0 \leq \phi(t) \leq \psi(t) \leq 1$ for all $t \in \mathbb{R}$. Then, in particular, Y satisfies:

$$Y(t_1) = -D_t^3U(t_1), \qquad D_tY(t_1) = D_t^2U(t_1), \qquad Y(t_2) = -D_t^3U(t_2), \qquad D_tY(t_2) = D_t^2U(t_2),$$

with $\text{supp}(Y) \subset (t_1 - \delta, t_1 + \delta) \cup (t_2 - \delta, t_2 + \delta)$. Moreover, we find by direct computation that $I(X, Y) = -\|Y(t_1)\|^2 - \|D_tY(t_1)\|^2 - \|Y(t_2)\|^2 - \|D_tY(t_2)\|^2 < 0$, and that

$$\begin{aligned} I(Y, X) &= I_c(Y, X) + P_+(Y, X) + P_-(Y, X) \\ &= I_c(X, Y) + P_+(X, Y) - P_-(X, Y) \\ &= I(X, Y) - 2P_-(X, Y). \end{aligned}$$

Consider the 2-form $G(X, Y) = \langle Y, \nabla_X \text{grad}V\rangle - \langle X, \nabla_Y \text{grad}V\rangle$. Then we have:

$$|G(X, Y)| \leq \phi|G(X, Z)| + \psi|G(X, W)| \leq (\phi + \psi)(\|Z\|\|\nabla_W \text{grad}V\| + \|W\|\|\nabla_Z \text{grad}V\|),$$

hence

$$\begin{aligned} |P_-(X, Y)| &\leq \int_a^b |G(X, Y)|dt \\ &\leq 2\delta \max_{t\in[a,b]} \{\|Z(t)\|\|\nabla_W(t)\text{grad}V(q(t))\| + \|W(t)\|\|\nabla_Z(t)\text{grad}V(q(t))\|\} \\ &:= \delta C. \end{aligned}$$

It follows that $I(Y,X) \leq I(X,Y) + 2\delta C$. Now, for $\epsilon > 0$, we consider the vector field $U_\epsilon = X + \epsilon Y$. It is clear that $U_\epsilon \in T_q\Omega$, and by the bilinearity of I, we have

$$\begin{aligned} I(U_\epsilon, U_\epsilon) &= I(X,X) + \epsilon(I(X,Y) + I(Y,X)) + \epsilon^2 I(Y,Y) \\ &\leq 2\epsilon(I(X,Y) + \delta C) + \epsilon^2 I(Y,Y). \end{aligned}$$

Hence, if we choose δ so that $I(X,Y) + \delta C < 0$, then it is clear that $I(U_\epsilon, U_\epsilon) < 0$ for sufficiently small ϵ. □

We are now interested in the reverse problem. That is, we would like to show that if there are no biconjugate points along a modified Riemannian cubic q, then q is an Ω-local minimizer. Before proceeding to the result, we introduce a symmetry of modified bi-Jacobi fields that will help to simplify calculations. Let $\alpha_q(X,Y) := \langle D_t^3 X, Y\rangle + 2\langle R(D_tX, \dot{q})\dot{q}, Y\rangle + \langle D_tX, D_t^2 Y\rangle + 2\langle R(X,\dot{q})D_t\dot{q}, Y\rangle$.

Lemma 5.3.5. *If X and Y are bi-Jacobi fields along q such that $X(0) = Y(0) = 0$ and $D_tX(0) = D_tY(0) = 0$, then*

$$\frac{1}{2}\big(\alpha_q(X,Y) - \alpha_q(Y,X)\big) = P_-(X,Y) \tag{5.8}$$

Proof. Observe that

$$\langle D_tX + F(X,\dot{q}) + \nabla_X \text{grad}V, Y\rangle - \langle D_tY + F(Y,\dot{q}) + \nabla_Y \text{grad}V, X\rangle = 0$$

since X and Y are bi-Jacobi fields along q. We may separate this as

$$\langle D_tX + F(X,\dot{q}), Y\rangle - \langle D_tY + F(Y,\dot{q}), X\rangle = \langle \nabla_Y \text{grad}V, X\rangle - \langle \nabla_X \text{grad}V, Y\rangle. \tag{5.9}$$

In [19], it was shown that

$$\langle D_tX + F(X,\dot{q}), Y\rangle - \langle D_tY + F(Y,\dot{q}), X\rangle = D_t\left[\alpha_q(X,Y) - \alpha_q(Y,X)\right].$$

Using (5.9) and integrating over t from $t = a$ to $t = b$, we obtain the desired result. □

Proposition 12. *Suppose that $q \in \Omega$ is a modified Riemannian cubic, and $t = a$ and $t = t_0$ are not biconjugate along q for each $t_0 \in (a,b]$. Then q is an Ω-local minimizer.*

Proof. Suppose that $X \in H_g^2(q)$ and satisfies $X(a) = D_tX(a) = 0$. We will show that, for the bi-Jacobi field J along q satisfying $J(a) = D_tJ(a) = 0,\ J(b) = X(b),\ D_tJ(b) = D_tX(b)$—which exists and is uniquely defined since there are no biconjugate points along q (see [19]), we have $0 = I(J, J) \leq I(X, X)$ with equality if and only if $J = X$. The result then follows from the fact that, if there are no biconjugate points, there is no non-zero bi-Jacobi field in $T_q\Omega$—and hence the equality $J = X$ cannot hold for any non-zero $X \in T_q\Omega$.

Following the strategy implemented in [19], let $\{v_i\}_{i=1}^n$ be a basis for $T_{q_b}Q$ and consider the bi-Jacobi fields $\{J_i\}_{i=1}^{2n}$ defined by

$$\begin{aligned}
&J_i(a) = 0, && D_tJ_i(a) = 0, && \\
&J_i(b) = v_i, && D_tJ_i(b) = 0, && \text{for } i = 1, \ldots, n \\
&J_i(a) = 0, && D_tJ_i(a) = 0, && \\
&J_i(b) = 0, && D_tJ_i(b) = v_{i-n}, && \text{for } i = n+1, \ldots, 2n.
\end{aligned}$$

Since there are no biconjugate points along q, these $2n$ bi-Jacobi fields are uniquely defined and linearly independent, and hence form a basis for the vector space $J_q(a)$ of bi-Jacobi fields along q which vanish at $t = a$ along with their first covariant derivatives. Thus, $J = c^iJ_i$ for some real numbers $c^i,\ i = 1, \ldots 2n$. Moreover, it is clear that $(J_i(t_0), D_tJ_i(t_0))$ forms a basis for $T_{q(t_0)}Q \times T_{q(t_0)}Q$ for each $t_0 \in (a, b]$ since $t = a$ and $t = t_0$ are not bi-conjugate. Hence, we may write $(X(t), D_tX(t)) = \sum_{i=1}^{2n} f^i(t)(J_i(t), D_tJ_i(t))$ for all $t \in [a, b]$ where $f^i \in H^2([a, b], \mathbb{R})$ is such that $f^i(b) = c^i$ for all $i = 1, \ldots, 2n$, and $\sum_{i=1}^{2n} f_i'(t)J_i(t) = 0$ for all $t \in [a, b]$. Observe that

$$D_t^2X = \dot{f}^iD_tJ_i + f^iD_t^2J_i, \tag{5.10}$$

$$F(X, \dot{q}) = f^iF(J_i, V) + 2\dot{f}^iR(D_tJ_i, \dot{q})\dot{q}. \tag{5.11}$$

Hence,

$$\begin{aligned}
I(X, X) = \int_a^b \Big[&\|\dot{f}^iD_tJ_i\|^2 + \langle \dot{f}^iD_tJ_i, f^jD_t^2J_j\rangle + \|f^iD_t^2J_i\|^2 \\
&+ \langle f^iJ_i, f^jF(J_j, \dot{q}) + f^j\nabla_{J_j}\text{grad}V\rangle + 2\langle f^iJ_i, \dot{f}^jR(D_tJ_j, \dot{q})\dot{q}\rangle \Big] dt.
\end{aligned}$$

Observe that

$$\begin{aligned}\|f^i D_t^2 J_i\|^2 = D_t\Big[\langle f^i D_t J_i, f^j D_t^2 J_j\rangle - \langle f^i J_i, f^j D_t^3 J_j\rangle\Big] \\ + \langle f^i J_i, \dot{f}^j D_t^3 J_j\rangle + \langle f^i J_i, f^j D_t^4 J_j\rangle + \langle \dot{f}^i J_i, f^j D_t^3 J_j\rangle \\ - \langle \dot{f}^i D_t J_i, f^j D_t^2 J_j\rangle - \langle f^i D_t J_i, \dot{f}^j D_t^2 J_j\rangle\end{aligned}$$

Substituting this identity into $I(X,X)$ and making use of the fact that J_i is a bi-Jacobi field for each $i = 1,\dots,2n$, we obtain

$$\begin{aligned}I(X,X) = \big\langle c^i D_t J_i(T), c^j D_t^2 J_j(T)\big\rangle - \big\langle c^i J_i(T), c^j D_t^3 J_j(T)\big\rangle \\ + \int_a^b \Big(\|\dot{f}^i D_t J_i\|^2 + \dot{f}^i f^j\Big[\langle D_t^3 J_i, J_j\rangle - \langle D_t^3 J_j, J_i\rangle \\ + \langle D_t J_i, D_t^2 J_j\rangle - \langle D_t J_j, D_t^2 J_i\rangle + 2\langle R(D_t J_i, \dot{q})\dot{q}, J_j\rangle\Big]\Big) dt.\end{aligned}$$

The first line in the expansion is simply $I(J,J)$, which can be seen by integrating $I(J,J)$ twice by parts. Moreover, since we have $\dot{f}^i J_i = 0$, it follows that

$$\dot{f}^i f^j \langle R(D_t J_j, \dot{q})\dot{q}, \dot{f}^i f^j J_i\rangle = 0, \quad \dot{f}^i f^j \langle R(J_i, \dot{q}) D_t \dot{q}, J_j\rangle = 0, \quad \dot{f}^i f^j \langle R(J_j, \dot{q}) D_t \dot{q}, J_i\rangle = 0.$$

Hence we may add these terms into our expansion to utilize Lemma 5.3.5. That is,

$$I(X,X) = I(J,J) + \int_a^b \|\dot{f}^i D_t J_i\|^2 dt + \dot{f}^i f^j P_-(J_i, J_j).$$

However, $P_-(\cdot,\cdot)$ is a tensor field, and so $\dot{f}^i f^j P_-(J_i,J_j) = P_-(\dot{f}^i J_i, f^j J_j) = 0$ since $\dot{f}^i J_i = 0$. Therefore, $I(X,X) - I(J,J) = \int_a^b \|\dot{f}^i D_t J_i\|^2 dt \geq 0$. Moreover, if $\int_a^b \|\dot{f}^i D_t J_i\|^2 dt = 0$, it follows immediately that $\dot{f}^i D_t J_i = 0$. Since it also holds that $\dot{f}^i J_i = 0$, and $(J_i, D_t J_i)$ is a basis for $T_q Q \times T_q Q$, it must be the case that $\dot{f}^i(t) = 0$ for all $t \in [a,b]$, $i = 1,\dots,2n$. As $f^i(b) = c^i$ for all $i = 1,\dots,2n$, it follows that, in fact, $f^i(t) = c^i$ for all $t \in [a,b], i = 1,\dots,2n$. Therefore, $X \equiv J$ on $[a,b]$. □

Propositions 11 and 12 can be combined into the following corollary:

Corollary 3. *A modified Riemannian cubic $q \in \Omega$ is an Ω-local minimizer of J if and only if there are no biconjugate points along q.*

One might wish to understand the robustness of being an Ω-local minimizer. That is, if q is an Ω-local minimizer, is it also true that the restriction of q to a subset of $[a,b]$ is an Ω-local minimizer on its corresponding admissible set? This question is addressed in the following proposition.

Proposition 13. *Suppose that q is an Ω-local minimizer of J and let $[a^*,b^*] \subset [a,b]$. Then the curve $q|_{[a^*,b^*]}$ is an $\Omega_{\xi,\eta}^{a^*,b^*}$-local minimizer, where $\xi = (q(a^*), \dot{q}(a^*)),\ \eta = (q(b^*), \dot{q}(b^*)) \in TQ$.*

Proof. Since q is an Ω-local minimizer, there exists a C^1-neighborhood B of q contained in Ω such that $J[q] \leq J[\tilde{q}]$ for any $\tilde{q} \in B$. Let B^* be the set of curves $q^* \in \Omega_{\xi,\eta}^{a^*,b^*}$ such that the curve $\tilde{q}(t) = \begin{cases} q(t), & t \in [a,a^*) \cup (b^*,b] \\ q^*(t), & t \in [a^*,b^*] \end{cases}$ is contained in B. Clearly, B^* is a non-empty C^1-neighborhood of q^* contained in $\Omega_{\xi,\eta}^{a^*,b^*}$. Now, fix any $q^* \in B^*$, and the corresponding $\tilde{q} \in B$. Then by the additivity of integration and the fact that $J[q] \leq J[\tilde{q}]$, it follows that

$$\begin{aligned} J\big[q|_{[a,a^*]}\big] + J\big[q|_{[a^*,b^*]}\big] + J\big[q|_{[b^*,b]}\big] &\leq J\big[\tilde{q}|_{[a,a^*]}\big] + J\big[\tilde{q}|_{[a^*,b^*]}\big] + J\big[\tilde{q}|_{[b^*,b]}\big], \\ J\big[q|_{[a,a^*]}\big] + J\big[q|_{[a^*,b^*]}\big] + J\big[q|_{[b^*,b]}\big] &\leq J\big[q|_{[a,a^*]}\big] + J\big[q^*|_{[a^*,b^*]}\big] + J\big[q|_{[b^*,b]}\big], \\ J\big[q|_{[a^*,b^*]}\big] &\leq J\big[q^*|_{[a^*,b^*]}\big]. \end{aligned}$$

Hence, $q|_{[a^*,b^*]}$ is an $\Omega_{\xi,\eta}^{a^*,b^*}$-local minimizer of J. □

Denote by $Tq : [a,b] \to TQ$ the *tangent lift* of the curve $q \in \Omega$. That is, the curve lying in the tangent bundle TQ whose local coordinate expression is given by $(q^i, \dot{q}^i)$.

Proposition 14. *Suppose that q is an Ω-local minimizer of J, and let B be a C^1 neighborhood of q contained in Ω such that $J[q] \leq J[\tilde{q}]$ for any $\tilde{q} \in B$. If $q_0 \in B$ satisfies $J[q_0] = J[q]$ and $Tq_0(\tau) = Tq(\tau)$ for some $\tau \in (a,b)$, then $q_0 \equiv q$ on $[a,b]$.*

Proof. Suppose that $q_0 \in B$ is such that $Tq(\tau) = Tq_0(\tau)$ and $J[q] = J[q_0]$, and consider the curve $\tilde{q}(t) = \begin{cases} q(t), & t \in [a,\tau) \\ q_0(t), & t \in [\tau,b] \end{cases}$. Clearly, $\tilde{q}$ is contained in B and $J[q] = J[\tilde{q}]$, so that $\tilde{q}$ is an Ω-local minimizer

(on B, for instance) and hence is a critical point of J on Ω. But by Proposition 7, this implies that $\tilde{q}$ is smooth, and therefore

$$\begin{cases} q(\tau) = q_0(\tau), & \dot{q}(\tau) = \dot{q}_0(\tau) \\ D_t\dot{q}(\tau) = D_t\dot{q}_0(\tau), & D_t^2\dot{q}(\tau) = D_t^2\dot{q}_0(\tau). \end{cases}$$

By the uniqueness of solutions to equation (5.2) with the above initial conditions, we have that $q|_{[\tau,b]} \equiv q_0|_{[\tau,b]}$. Repeating this argument backwards in time, we reach the desired conclusion. □

Lemma 14 can be interpreted as saying that the tangent lifts of sufficiently nearby Ω-local minimizers cannot intersect away from the boundary points. Alternatively, this means that Ω-local minimizers are (locally) uniquely determined by a single internal position and velocity. If we were to consider Proposition 14 in the case that q is a *global* minimizer, we could of course extend B to be the entire space Ω. Together with Proposition 13, this leads to the following corollary.

Corollary 4. *Suppose that $q \in \Omega^{a,b}_{\xi,\eta}$ is a global minimizer of J, and let $[a^*, b^*] \subset [a, b]$ be a proper subinterval. Then, for $\mu = (q(a^*), \dot{q}(a^*))$, $\nu = (q(b^*), \dot{q}(b^*)) \in TQ$, the curve $q|_{[a^*,b^*]}$ is the unique minimizer of J on $\Omega^{a^*,b^*}_{\mu,\nu}$*

5.3.2 Q-local minimizers

We now wish to understand when critical points of J are Q-local minimizers. In Proposition 16, we will show that, in fact, *every* modified Riemannian cubic is a Q-local minimizer of J—just as every geodesic is a Q-local minimizer of the length functional (or equivalently, the energy functional). In particular, we will do this by first establishing a local uniqueness result for modified Riemannian cubics, based off of a similar result for Riemannian cubic polynomials provided in [20].

Consider the 3-tangent bundle $(TQ)^3 := \bigcup_{p\in Q} (T_pQ)^3$. It is easily seen (for example, by considering local coordinate charts and applying standard results of ODEs) that a modified Riemannian cubic is uniquely determined by initial conditions

$$q(a) = p, \quad \dot{q}(a) = v, \quad D_t\dot{q}(a) = y, \quad D_t^2\dot{q}(a) = z,$$

for $(p, v, y, z) \in (TQ)^3$. That is, there exists a neighborhood $B \subset (TQ)^3$ containing (p, v, y, z) and a $\delta > 0$ such that for each $(p_a, v_a, y_a, z_a) \in B$, there exists a unique modified Riemannian cubic $q : (a - \delta, a + \delta) \to Q$ with the conditions prescribed by (p_a, v_a, y_a, z_a) at $t = a$. If we consider a set $C \subset (T_pQ)^2$ defined such that $\{(p, v)\} \times C \subset B$, we may then define the *bi-exponential map* $\text{biexp}^t_{(p,v)} : C \to TQ$ by $\text{biexp}^t_{(p,v)}(y, z) = (q(t), \dot{q}(t))$, where q is the unique modified Riemannian cubic satisfying the initial conditions prescribed by $(p, v, y, z) \in (TQ)^3$, and $t \in (a, a + \delta)$.

The bi-exponential map serves as a connection between the initial value problem and the boundary value problem. As we will see below, the differential of the bi-exponential map determines when we have local uniqueness of solutions to equation (5.2) (that is, when modified Riemannian cubics are locally unique). Furthermore, there is an intimate connection between biconjugate points and the differential of the bi-exponential map, described by Lemma 5.3.7. Lemma 5.3.6 and Lemma 5.3.7 appear in [20] in the case that $V \equiv 0$, however, the proofs follow virtually identically in both cases, and so we will not provide them here.

Lemma 5.3.6. *Let $(p, v, y, z) \in (TQ)^3$ and $\text{biexp}^\tau_{(p,v)}$ be defined in a neighborhood C of $(y, z) \in (T_pQ)^2$ for some $\tau > 0$. If $\text{biexp}^\tau_{(p,v)}$ is not critical at (y, z), then there exists a neighborhood W_1 of (y, z), a neighborhood W_2 of $\text{biexp}^\tau_{(p,v)}(y, z)$, and a neighborhood V of (p, v, y, z) such that for each $(p_1, v_1) \in W_1$ and $(p_2, v_2) \in W_2$, there exists a unique modified Riemannian cubic q satisfying*

$$q(0) = p_1, \quad \dot{q}(0) = v_1, \quad q(\tau) = p_2, \quad \dot{q}(\tau) = v_2,$$

and $(q(0),\ \dot{q}(0),\ D_t\dot{q}(0),\ D_t^2\dot{q}(0)) \in V$.

Since we know that there exists a *unique* global minimizer of J independent of the boundary conditions chosen on Ω, it is clear that the unique modified Riemannian cubic given in Lemma 5.3.6 must be this minimizer. We will return to this point later, but now we wish to understand when exactly the biexponential map $\text{biexp}^\tau_{(p,v)}$ is not critical at some given $(y, z) \in (T_pQ)^2$. In the subsequent lemma, we establish a connection between this criticality of the biexponential map and the biconjugate points along the modified Riemannian cubics that it generates.

Lemma 5.3.7. *$biexp^{\tau}_{(p,v)}$ is not critical at $(y,z) \in (T_pQ)^2$ if and only if the points $t = a, \tau$ are not biconjugate along the modified Riemannian cubic $t \mapsto \pi \circ \text{biexp}^t_{(p,v)}(y,z)$, where $\pi : TQ \to Q$ is the canonical projection map.*

We now wish to establish that there are a finite number of points biconjugate to any $\tau \in [a,b)$—for this will immediately imply that biconjugate points are isolated (and hence, for sufficiently small $\delta > 0$, there will be no points biconjugate to τ in $[\tau, \tau+\delta]$). To that end, we return to our study of the index form $I : T_q\Omega \times T_q\Omega \to \mathbb{R}$ given by (5.5) along a modified Riemannian cubic $q \in \Omega$. Define the extended index of q, denoted by $\text{Ind}_0(q)$, as the dimension of the largest subspace of $T_q\Omega$ on which I is negative semi-definite.

Lemma 5.3.8. $\text{Ind}_0(q)$ *is finite.*

Proof. Assume towards contradiction that $\text{Ind}_0(q)$ is infinite. Under the identification $T_q\Omega \cong \mathring{H}^{2k}_g(q)$ (as discussed in Section 2.3), there then there exists a sequence $(X_n) \subset \mathring{H}^2_g(q)$ such that $I(X_n, X_n) \leq 0$, and (X_n) is orthonormal with respect to the $\mathring{H}^1_q$ product given by

$$\langle X, Y \rangle_{\mathring{H}^1_q} = \int_a^b \left[\langle X, Y \rangle + \langle D_t X, D_t Y \rangle \right] dt$$

for all $X, Y \in \mathring{H}^2_q$. Since $I(X_n, X_n) \leq 0$, we have that

$$\begin{aligned} \int_a^b \left\| D_t^2 X_n \right\|^2 dt &\leq \int_a^b \left| \langle X_n, F(X_n, \dot{q}) + \nabla_{X_n} \text{grad} V \rangle \right| dt \\ &\leq \int_a^b \|X_n\| \|F(X_n, \dot{q}) + \nabla_{X_n} \text{grad} V\| dt. \end{aligned}$$

Since grad$V$, q, and R are smooth, we may consider the family of linear maps $A_t : T_{q(t)}Q \to T_{q(t)}Q$ given by $A_t(Z) = F(Z, \dot{q}(t)) + \nabla_Z \text{grad} V(q(t))$ for all $Z \in T_{q(t)}Q$, together with the induced norm on $\text{End}(T_{q(t)}Q)$ given by $|A_t| = \sup_{\substack{Z \in T_{q(t)}Q \\ \|Z\|=1}} \|A_t(Z)\|$. Since $[a,b]$ is compact, it follows that $\sup_{t \in [a,b]} |A_t| < +\infty$. Hence, utilizing the Cauchy-Schwarz inequality,

$$\int_a^b \|X_n\| \|F(X_n, \dot{q}) + \nabla_{X_n} \text{grad} V\| dt \leq \sup_{t \in [a,b]} |A_t| \int_a^b \|X_n\|^2 dt \leq \sup_{t \in [a,b]} |A_t|,$$

since

$$\int_a^b \|X_n\|^2 dt = \|X_n\|^2_{L^2_g} \leq \|X_n\|^2_{H^1_g} = 1$$

by construction. Hence, (X_n) is bounded with respect to the $\mathring{H}^2_q$ norm. By the Rellich–Kondrachov theorem, there exists a subsequence (X_{n_k}) which converges strongly in $\mathring{H}^1_q$. However, this is impossible since (X_{n_k}) is orthonormal with respect to the product on $\mathring{H}^1_q$, and hence cannot be Cauchy. □

Corollary 5. *Suppose that $q \in \Omega$ is a modified Riemannian cubic. Then, for any $\tau \in [a,b]$, there are finitely many points biconjugate to τ along q.*

Proof. Suppose that there are infinitely many points biconjugate to τ. Then, at least one of the sets $\tau^- = \{t < \tau : \ t, \tau \text{ biconjugate}\}$ and $\tau^+ = \{t > \tau : \ t, \tau \text{ biconjugate}\}$ are infinite. Without loss of generality, suppose that τ^+ is infinite, and consider a sequence $(t_n) \subset \tau^+$. For each $n \in \mathbb{N}$, there exists a bi-Jacobi Field J_n along q such that $J_n(\tau) = J_n(t_n) = D_t J_n(\tau) = D_t J_n(t_n) = 0$. Consider the sequence of vector fields $X_n \in \mathring{H}^2_g(q)$ defined by $X_n(t) = \begin{cases} J_n(t), & t \in [\tau, t_n] \\ 0, & \text{otherwise} \end{cases}$. Then it is clear that $(X_n) \subset T_q\Omega$ since each X_n is C^1 on $[a,b]$, smooth except possibly at $t = \tau, t_n$, and satisfies $X_n(a) = X_n(b) = D_t X_n(a) = D_t X_n(b) = 0$. Moreover, (X_n) is a linearly independent sequence and $I(X_n, X_n) = 0$ for all $n \in \mathbb{N}$. However, this implies that $\text{Ind}_0(q)$ is infinite, which contradicts Lemma 5.3.8. □

In the following Proposition, we show that the bi-exponential map $\text{biexp}^t_{(p,v)}$ is not critical provided that t is sufficiently small. This follows from the fact that the set of points biconjugate to any fixed $\tau \in [a,b]$ is finite, and hence consists of isolated points. We use this idea to show that the restrictions of modified Riemannian cubics to sufficiently small intervals are unique.

Proposition 15. *Suppose that q is a modified Riemannian cubic and $\tau \in [a,b]$. Then there exists an interval $[a^*, b^*] \subset [a,b]$ containing τ such that $q|_{[a^*,b^*]}$ is the unique solution of* (5.2) *on $\Omega^{[a^*,b^*]}_{\xi,\eta}$, where $\xi = (q(a^*), \dot{q}(a^*)),\ \eta = (q(b^*), \dot{q}(b^*)) \in TQ$.*

Proof: From Corollary 5, there are a finite number of points biconjugate to $t = \tau$. Hence, for some $\delta > 0$, the interval $(\tau, \tau + \delta] \subset [a,b]$ contains no points biconjugate to τ. Consider $a^* = \tau$, $b^* = \tau + \delta$, $p = q(a^*),\ v = \dot{q}(a^*)$. By Lemma 5.3.7, $\text{biexp}^{|b^*-a^*|}_{(p,v)}$ is not critical at $(y,z) := (D_t\dot{q}(a^*), D^2_t\dot{q}(a^*)) \in (T_pQ)^2$. Therefore, by Lemma 5.3.6, the curve $q^* : [a^*, b^*] \to Q$ given by $q^*(t) = \pi \circ \text{biexp}^t_{(p,v)}(y,z)$ is the unique

modified Riemannian cubic satisfying $Tq^*(a^*) = \xi$ and $Tq^*(b^*) = \eta$. Since $q|_{[a^*,b^*]}$ is clearly another such modified Riemannian cubic, it follows immediately that $q|_{[a^*,b^*]} \equiv q^*$.

We are now in a position to prove that the modified Riemannian cubics are exactly the Q-local minimizers of J. In fact, we will show something stronger—namely that modified Riemannian cubics are locally the *unique* minimizers of J, as is the case for geodesics with the length functional.

Proposition 16. *$q \in \Omega$ is a modified Riemannian cubic if and only if it is a Q-local minimizer of J. Moreover, q is the locally unique minimizer.*

Proof. It is clear that if q is a Q-local minimizer of J, then it is a modified Riemannian cubic. For the other direction, fix $\tau \in [a, b]$. From Proposition 15, there exists a neighborhood $[a^*, b^*]$ containing τ such that $q|_{[a^*,b^*]}$ is the unique critical point of J on $\Omega_{\xi,\eta}^{[a^*,b^*]}$, where $\xi = Tq(a^*),\ \eta = Tq(b^*) \in TQ$. However, from Proposition 8, there exists a global minimizer of J on $\Omega_{\xi,\eta}^{[a^*,b^*]}$—which must itself be a critical point of J. Hence, $q|_{[a^*,b^*]}$ is this minimizer. □

Chapter 6

Artificial Potential Design

In Chapter 5, we considered a variational problem that gave rise to the so-called modified Riemannian cubics, and studied their analytic properties. This was done under the guise that such curves could be used in path planning strategies. Indeed, when the artificial potential vanishes identically, the resulting curves (Riemannian cubic polynomials) are smooth, energy-optimal, and satisfy some prescribed boundary conditions. Hence, they may be used to produce optimal trajectories in interpolation problems that necessitate the minimization of energy consumption.

However, this is insufficient in many real-world applications (especially in robotics and control engineering) where agents must additionally avoid obstacles in phase space, or inter-agent collisions in the case of multi-agent systems. This is precisely the role of the artificial potential. In principle, we would like to design V so that it grows large as an agent approaches an obstacle, and otherwise remains small. In that sense, modified Riemannian cubics will naturally avoid collision, and will approximate Riemannian cubic polynomials away from obstacles/other agents. In this chapter, we consider the obstacle and collision avoidance tasks in more detail. In particular, we seek to find sufficient conditions on the artificial potential under which the corresponding modified Riemannian cubics avoid collisions, and to design parameterized families of artificial potentials which can be used to provide safety guarantees in the obstacle/collision avoidance tasks.

6.1 The Obstacle Avoidance Task

In this section, we explore the task of obstacle avoidance. As in previous sections, we assume that Q is a complete and connected finite-dimensional Riemannian manifold. In the most general case, we consider an obstacle to be a region $U \subset Q$ that we wish for our trajectories to avoid. That is, we design the artificial potential function $V : Q \to \mathbb{R}$ such that solutions to the BVP (5.2) on Ω remain within the set $Q \setminus U$.

Recall that Q is a (complete) metric space when equipped with the Riemannian distance function $d : Q \times Q \to \mathbb{R}$ given by (2.21). Hence, for any subset $P \subset Q$ and $p \in Q$, we may define $d(p, P) = \inf\{d(p, q) \mid q \in P\}$. Moreover, for $r \in \mathbb{R}$, we define $B_r(P) = \{q \in Q \mid d(q, P) < r\}$. An *obstacle avoidance problem* $\mathcal{P}$ is a pair (P, r), where $P \subset Q$ is called the *obstacle* and $r > 0$ is called the *tolerance*. We also call the 4-tuple $\mathcal{P}_{\text{ext}} = (P, r, r^*, R)$ with $r < r^* \leq R$ an *extended obstacle avoidance problem*. With respect to an extended obstacle avoidance problem $\mathcal{P}_{\text{ext}}$, we define:

(1) the *Collision region* as $C_P := B_r(P)$, and say that $q \in \Omega$ avoids the obstacle P with tolerance r if $q(t) \in Q \setminus C_P$ for all $t \in [a, b]$. Moreover, we say that q collides with P if $q(t) \in C_P$ for some $t \in [a, b]$.

(2) the *Risk region* $C_P^* := B_{r^*}(P)$, and say that q is at risk of collision with p whenever $q(t) \in C_P^* \setminus C_P$.

(3) the *Safety region* $S_P := Q \setminus \overline{B_R}(P)$, where $\overline{B_R}(P)$ denotes the topological closure of $B_R(P)$.

We call an (extended) obstacle avoidance problem *feasible* with tolerance $s \in \mathbb{R}$ if there exists a curve $q \in \Omega$ avoiding the obstacle with tolerance s (note that this requires $s < \min\{d(P, q_a),\ d(P, q_b)\}$). For an extended obstacle avoidance problem, we define a *reference trajectory* to be a curve $q \in \Omega$ such that $q(t) \in S_P$ for all $t \in [a, b]$. Clearly, the existence of a reference trajectory is equivalent to saying that obstacle avoidance is feasible with the tolerance R of the Safety region.

6.1.1 Point-Obstacles

Consider the extended obstacle avoidance problem $\mathcal{P}_{\text{ext}} = (P, r, r^*, R)$. In the case that $P = \{p\}$ for some $p \in Q$, we call P a *point-obstacle*, and the corresponding extended obstacle avoidance problem

$\mathcal{P}_{\text{ext}} = (p, r, r^*, R)$ an extended point-obstacle avoidance problem. In Figure 6.1, we show the regions corresponding to an extended point-obstacle avoidance problem.

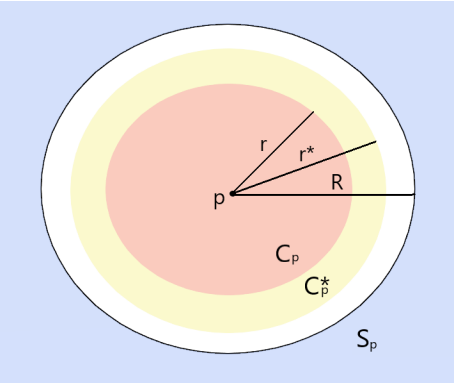


Figure 6.1: The regions C_p, C_p^*, and S_p taken with respect to the position of the point-obstacle $p \in Q$.

We now wish to derive conditions on the artificial potential V under which obstacle avoidance is guaranteed in a feasible point-obstacle avoidance problem. The strategy will be to first derive general conditions for obstacle avoidance in terms of upper and lower bounds on the potential and some reference trajectory.

Proposition 17. *Consider the extended point-obstacle avoidance problem $\mathcal{P}_{\text{ext}} = (p, r, r^*, R)$ and suppose that there exist constants $0 \leq V^- \leq V^*$ such that $V \geq V^*$ on C_p^* and $V \leq V^-$ on S_p. Further suppose that $\mathcal{P}_{\text{ext}}$ is feasible and $q \in \Omega$ is a reference trajectory. If $V^* > \frac{cv}{2(r^*-r)}$, where $A := \sup_{t\in[a,b]} ||D_t\dot{q}(t)||$, $c := (A^2 + V^-)(b-a)$, and $v := \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_a||^2}$, then any minimizer $q^* \in \Omega$ of J avoids p with tolerance r.*

Proof. Since our reference trajectory q remains in the S_p for all $t \in [a, b]$,

$$\begin{aligned} J(q) &= \int_a^b \left[||D_t\dot{q}(t)||^2 + V(q(t)) \right] dt \\ &\leq (b-a) \sup_{t\in[a,b]} \left\{ ||D_t\dot{q}(t)||^2 + V(q(t)) \right\} \\ &\leq \left(A^2 + V^-\right)(b-a) =: c \end{aligned}$$

Now suppose that $v^* = \sup_{t\in[a,b]} \{||\dot{q}^*(t)||\}$, and observe that, by the fundamental theorem of calculus and the

Cauchy-Schwarz inequality,

$$\begin{aligned}
||\dot{q}^*(t)||^2 &= ||v_a||^2 + \int_a^b \frac{d}{dt}||\dot{q}^*(t)||^2 dt \\
&= ||v_a||^2 + 2\int_a^b g\left(D_t\dot{q}^*(t), \dot{q}^*(t)\right) dt \\
&\leq ||v_a||^2 + 2\left[\int_a^b ||D_t\dot{q}^*(t)||^2 \, dt\right]^{1/2} \left[\int_a^b ||\dot{q}^*(t)||^2 \, dt\right]^{1/2} \\
&\leq ||v_a||^2 + 2v^*\sqrt{J(q^*)(b-a)} \\
&\leq ||v_a||^2 + 2v^*\sqrt{c(b-a)},
\end{aligned}$$

since $J(q^*) \leq J(q) \leq c$ by definition of minimizer. Taking the supremum over $t \in [a, b]$ on both sides of the inequality, we find $(v^*)^2 \leq ||v_a||^2 + 2v^*\sqrt{c(b-a)}$, so that $v^* \leq \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_a||^2} =: v$. Now suppose towards contradiction that q^* collides with p. That is, there exists $\tau \in [a, b]$ with $q^*(\tau) \in C_p$. It follows that for all $a \leq t \leq \tau$,

$$\begin{aligned}
d(q^*(t), p) &\leq d(q^*(t), q^*(\tau)) + d(q^*(\tau), p) \\
&\leq L\left(q^*|_{(\tau,t)}\right) + r \\
&= \int_\tau^t ||\dot{q}(t)||dt + r \\
&\leq v^*(t-\tau) + r \leq v(t-\tau) + r,
\end{aligned}$$

and so $q^*(t) \in C_p^*$ for all $t \in \left[\tau, \min\{\tau + \frac{r^*-r}{v}, b\}\right)$. By considering a similar argument with $t^* \leq t \leq b$, it can be seen that $q^*(t) \in C^*$ for all $t \in \left(\max\{\tau - \frac{r^*-r}{v}, a\}, \min\{\tau + \frac{r^*-r}{v}, b\}\right)$. Therefore, since V is positive-definite and $V^* < V$ on C_p^*,

$$\begin{aligned}
2V^*\left(\frac{r^*-r}{v}\right) &\leq \int_{\max\{\tau - \frac{r^*-r}{v},\, a\}}^{\min\{\tau + \frac{r^*-r}{v},\, b\}} V(q^*(t))dt \\
&\leq \int_a^b V(q^*(t))dt \\
&\leq J(q^*) \leq J(q) \leq c.
\end{aligned}$$

So that $V^* \leq \frac{cv}{2(r^*-r)}$. By contradiction, q^* avoids the obstacle with tolerance r. □

The conditions outlined in Proposition 17 depend intimately on the chosen reference trajectory. In particular, the smaller the value for $A = \sup_{t\in[a,b]} \|D_t\dot{q}(t)\|$, the smaller we must take V^* to guarantee collision avoidance. Generating such curves on a Riemannian manifold can be a complicated task, and so we wish to somehow remove the dependency of the conditions on the particular reference trajectory chosen. Moreover, it will often be the case that the collision region is determined by the application, whereas the tolerances for the risk and safety regions are somewhat arbitrary. That is, they are unlikely to correspond to any physical parameters of a system. Hence, we would ultimately like to find safety guarantees for the point-obstacle problem $\mathcal{P}$ rather than the extended point-obstacle problem $\mathcal{P}_{\text{ext}}$. We will accomplish this by constructing a family of artificial potentials satisfying certain desired properties.

Definition 6.1.1. We call a family of artificial potentials $V^k_{\tau,D} : Q \to \mathbb{R}$ of the form $V^k_{\tau,D}(q) = f^k_{\tau,D}(d(p,q))$ an *avoidance family* for the obstacle $p \in Q$, where $f^k_{\tau,D} : \mathbb{R} \to \mathbb{R}$ is a family of smooth non-negative functions parameterized by $k, D, \tau \in \mathbb{R}$ such that:

(1) $f^k_{\tau,D}(0) = \tau$ is the unique global maximum of f.

(2) $\lim_{k\to\infty} f^k_{\tau,D}(x) = 0$ for all $|x| > D$

(3) $\lim_{k\to\infty} f^k_{\tau,D}(x) = \tau$ for all $|x| < D$

(4) $0 \leq \lim_{k\to\infty} f^k_{\tau,D}(D) \leq \tau$.

Corollary 6. *Suppose that the point-obstacle avoidance problem $\mathcal{P} = (p, D)$ is feasible. Then for all $r < D$, there exists $\tau^* \in \mathbb{R}^+,\ k^* \in \mathbb{N}$ such that for all $\tau > \tau^*$ and $k > k^*$, every minimizer of J with $V = V^k_{\tau,D}$ avoids p with tolerance r, where $V^k_{\tau,D}$ is an avoidance family of artificial potentials.*

Proof. Fix $r < D$ and consider the extended obstacle avoidance problem $\mathcal{P}_{\text{ext}} = (p, r, D, D)$. Since $\mathcal{P}$ is feasible, there exists a reference trajectory $q \in \Omega$ of $\mathcal{P}_{\text{ext}}$ that remains in S_p for all $t \in [a, b]$. Fix $\epsilon, \tau^* > 0$ such that $\tau^* - \epsilon > \frac{cv}{2(D-r)}$ for $A := \sup_{t\in[a,b]} ||D_t\dot{q}(t)||$, $c := (A^2 + \epsilon)(b - a)$, and $v := \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_a||^2}$.

Now, since $\lim_{k\to\infty} V^k_{\tau,D}(x) = 0$ whenever $d(p, x) > D$, and $\lim_{k\to\infty} V^k_{\tau,D}(x) = \tau$ whenever $d(p, x) < D$, it follows that there exists k^* such that $k > k^*$ implies $V^k_{\tau^*,D}(x) > \tau^* - \epsilon$ for $d(p, x) < D$ and $V^k_{\tau^*,D}(x) < \epsilon$

for $d(p,x) > D$ (that is, for $x \in S_p$). Hence, for sufficiently large k, we may choose $V^* = \tau^* - \epsilon$ and $V^- = \epsilon$. It follows immediately by Proposition 17 that any minimizer of J avoids p with tolerance r. $\square$

Remark 5. In applications, distances can be calculated by, for instance, attaching a sensor to the agent. Due to technological limitations, it is often the case that distance measurements are only possible/reliable within some given ball of the sensor, the radius h of which is called the *sensing radius*. We may account for this by demanding that the potential vanish identically whenever the distance between the sensor and the obstacle exceeds h. This can be handled while still preserving the regularity and positive-definiteness of the obstacle avoidance potential by utilizing bump functions. In particular, we consider the family of artificial potentials given by

$$V^k_{\tau,D}(q) = \begin{cases} e\tau \exp\left(-\frac{1}{1-(d(p,q)/D)^{2k}}\right) & d(p,q) < D \\ 0 & \text{else.} \end{cases} \tag{6.1}$$

It is easily seen that $V^k_{\tau,D}$ is an avoidance family of artificial potentials, and when $D \leq h$, we satisfy the requirement that the potential vanishes identically outside of the sensing radius.

Remark 6. In many cases, it will be more convenient to use even powers of the Riemannian distance, as its gradient can be computed explicitly (see [9]). If $d^2 : Q \times Q \to \mathbb{R}$ and $V : Q \to \mathbb{R}$ is defined by $V(q) = f(d^2(q,p))$, then we have $\text{grad}V(q) = f'(d^2(q,p))\text{grad}_1 d^2(q,p)$. Provided that q is within some geodesically convex neighborhood of p on which the Riemannian exponential map is a diffeomorphism, we have from (2.26) that $d(q,p) = \| \exp_q^{-1}(p)\|$. This leads to the following lemma.

Lemma 6.1.2. *Let $q, p \in Q$ are contained within some geodesically convex neighborhood on which the Riemannian exponential map is a diffeomorphism. Then,*

$$\text{grad}_1 d^2(q,p) = -2\exp_q^{-1} p. \tag{6.2}$$

Proof. Let $X \in T_qQ$ and let $\alpha : (-\epsilon, \epsilon) \to Q$ be any curve such that $\alpha(0) = q$ and $\alpha'(0) = X$. Consider the family $\gamma : [0,1] \times (\epsilon, \epsilon) \to G$ of geodesics given by $\gamma(s,t) = \exp_{\alpha(s)}(t \exp^{-1}_{\alpha(s)}(p))$. From Lemma

2.2.4,

$$\begin{aligned}
&\gamma(s,0) = \alpha(s), && \frac{\partial\gamma}{\partial s}(s,0) = \alpha'(s)\\
&\gamma(s,1) = p, && \frac{\partial\gamma}{\partial t}(s,1) = 0\\
&\gamma(0,t) = \exp_q(t\exp_q^{-1}(p)), && \frac{\partial\gamma}{\partial t}(0,0) = \exp_q^{-1}(p)
\end{aligned}$$

for all $t \in (-\epsilon, \epsilon)$. Using Lemma 2.23, we find that

$$\begin{aligned}
\left\langle \text{grad}_1 d^2(q,p), X \right\rangle &= \frac{d}{ds}\Big|_{s=0} d^2(\alpha(s),p) = \frac{d}{ds}\Big|_{s=0} \left(\int_0^1 \left\| \frac{\partial\gamma}{\partial t} \right\| dt \right)^2 \\
&= 2\int_0^1 \left\langle D_s \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s} \right\rangle \Big|_{s=0} dt = 2\int_0^1 \frac{d}{dt} \left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle \Big|_{s=0} dt \\
&= \left\langle -2\exp_q^{-1}(p), X \right\rangle,
\end{aligned}$$

□

6.1.2 General Obstacles

We now focus our attention to avoiding more general obstacle avoidance problem $\mathcal{P} = (P, r, r^*, R)$, where $P \subset Q$. For points $m, p \in Q$, it is easy to see that the triangle inequality holds as $d(m,P) \leq d(m,p) + d(p,P)$. From this, it is clear that the analysis carried out in Proposition 17 and Corollary 6 follows identically, with all occurrences of $d(p,x)$ replaced by $d(P,x)$.

Corollary 7. *Suppose that the obstacle avoidance problem $\mathcal{P} = (P, D)$ is feasible. Then for all $r < D$, there exists $\tau^* \in \mathbb{R}^+$, $k^* \in \mathbb{N}$ such that for all $\tau > \tau^*$ and $k > k^*$, every minimizer of J with $V = V^k_{\tau,D}$ avoids P with tolerance r, where $V = V^k_{\tau,D}$ is an avoidance family of artificial potentials for the obstacle $P \subset Q$.*

This corollary poses an issue for practical purposes, as calculating $d(P,x)$ is in generally computationally intensive (if possible at all). Even in the case where the boundary points $q_a, q_b \in Q$ for Ω are sufficiently close so that the distance between points may be represented via the inverse of the exponential map as $d(x,p) = \left|\left|\exp_x^{-1}(p)\right|\right|$, we would still need to calculate the infimum of this term over all of P to find $d(P,x)$. Moreover, the gradient $\text{grad}_1 d(P,x)$ may not have a closed form (and so neither will $\text{grad}V^k_{\tau,R}(x)$).

We seek an alternative characterization which relies only on the distance between points. In particular, we suppose that the obstacle P is totally bounded, so that we may cover it in a finite collection of open balls—the centers of which will be treated as point-obstacles. The potential will then be taken as a sum of the point-obstacle avoidance potentials that result in the minimizers of J avoiding these balls. Before stating the result, we introduce a lemma describing some basic facts of totally bounded sets that will be useful in the subsequent proof.

Lemma 6.1.3. *Suppose $P \subset Q$ is totally bounded. Then,*

(1) $\overline{P}$ is totally bounded.

(2) Any subset $A \subset P$ is totally bounded.

(3) $B_r(P)$ is totally bounded for any $r > 0$.

Proposition 18. *Suppose that the obstacle avoidance problem $\mathcal{P} = (P, D)$ is feasible. Then for all $r < D$, there exists $\tau^* > 0$, $k^* \in \mathbb{N}$, and a finite collection of points $\{p_i\} \subset P$ such that for all $\tau > \tau^*$ and $k > k^*$, any minimizer of J corresponding to the potential $V(x) = \sum_i V_i(x)$ avoids P with tolerance r, where $V_i(x) = V^k_{\tau,D}(x)$ is an avoidance potential for p_i.*

Proof. Fix $r < r^* < D$ and $\delta, \epsilon > 0$ such that $\delta < r^* - r$ and $\epsilon < D - r^*$. By lemma 6.1.3, the set $\overline{B_r}(P)$ is totally bounded and hence $\partial B_r(P) \subset \overline{B_r}(P)$ is totally bounded, where $\partial B_r(P)$ denotes the boundary of $B_r(P)$. Let $\{p_i^*\}_{i=1}^N \subset \partial B_r(P)$ be a finite collection of points such that $\partial B_r(P) \subset \bigcup_{i=1}^N B_\delta(p_i^*)$, and $\{p_i\}_{i=1}^N \subset P$ be a finite collection of points such that $d(p_i, p_i^*) = r$ for $1 \leq i \leq N$. We further define $P_0 := \bigcup_{i=1}^N B_{r^*}(p_i)$. We aim to show the chain on inclusions

$$\partial B_r(P) \subset P_0 \subset B_\epsilon(P_0) \subset \bigcup_{i=1}^N B_D(p_i) \subset B_D(P).$$

(1) $\partial B_r(P) \subset P_0$: Let $x \in \partial B_r(P)$ and choose $p_j^* \in \{p_i^*\}_{i=1}^N$ such that $x \in B_\delta(p_j^*)$. Then $d(x, p_j) \leq d(x, p_j^*) + d(p_j^*, p_j) \leq \delta + r < r^*$. Hence $x \in B_{r^*}(p_j)$.

(2) $P_0 \subset B_\epsilon(P_0)$ is trivial.

(3) $B_\epsilon(P_0) \subset \bigcup_{i=1}^N B_D(p_i)$: Let $x \in B_\epsilon(P_0)$. Then there exists a $p \in P_0$ with $d(p,x) \leq \epsilon$. Suppose that $p_j \in \{p_i\}_{i=1}^N$ is such that $d(p,p_j) < r^*$ (such a p_j must exist). Then we have $d(p_j,x) \leq d(p,x) + d(p,p_j) < \epsilon + r^* < D$, so that $x \in B_D(p_j) \subset \bigcup_{i=1}^N B_D(p_i)$.

(4) $\bigcup_{i=1}^N B_D(p_i) \subset B_D(P)$: Consider $x \in \bigcup_{i=1}^N B_D(p_i)$. Then there exists a $p_j \in \{p_i\}_{i=1}^N$ with $x \in B_D(p_j)$, and $d(P,x) \leq d(p_j,x) < D$ so that $x \in B_D(P)$.

Consider the obstacle avoidance problem $\mathcal{P}_0 = (P_0, \epsilon)$. Since $\mathcal{P}$ is feasible and $B_\epsilon(P_0) \subset B_D(P)$, it follows that $\mathcal{P}_0$ is feasible. From Corollary 6, it is clear that we may make the potential V_i arbitrarily large on $B_D(p_i)$ by choosing τ, k sufficiently large. Since every potential of this form is positive-definite, it follows that the sum $V = \sum_{i=1}^N V_i$ can be made arbitrarily large on $\bigcup_{i=1}^N B_D(p_i) \supset B_\epsilon(P_0)$. Moreover, $V = \sum_{i=1}^N V_i$ can be made arbitrarily small on $Q \setminus \bigcup_{i=1}^N B_D(p_i)$. Using arguments similar to those given in Proposition 17 and Corollary 6, it follows that every minimizer of J with the potential $V = \sum_{i=1}^N V_i$ avoids P_0 with tolerance ϵ, and hence avoids $\partial B_r(P)$ by inclusion. Since any trajectory must cross $\partial B_r(P)$ to enter $B_r(P)$, the conclusion follows. □

Remark 7. Proposition 18 implies that we can avoid a finite collection of point-obstacles (when feasible) by summing over an appropriate family of potentials—each of which correspond to the avoidance of one of the point-obstacles. Similarly, we can avoid a finite collection of totally bounded obstacles (when feasible) by summing over appropriate potentials.

Example 9. We conduct numerical simulations of the obstacle avoidance problem in the case of $Q = \mathbb{R}^3$ with the Euclidean metric. For our obstacle, we choose the section of the unit sphere given by $P = \{(\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), \cos(\phi)) \in \mathbb{R}^3 \ : \ 0 < \phi < \pi/4, \ 0 < \theta < \pi/2\}$. We consider the boundary conditions

$$q(0) = (0,0,0), \qquad \dot{q}(0) = (0.125, 0.125, 0.45),$$

$$q(1) = (0.2, 0.5, 1.8), \qquad \dot{q}(1) = (0.3, 0.25, 0.5),$$

which were chosen so that the minimizing Riemannian cubic satisfying the boundary conditions collides with the obstacle (thus prompting the need for the artificial potential).

Motivated by Proposition 18, we choose three point obstacles p_1, p_2, p_3 of the form $(x, y, z) = (\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), \cos(\phi))$ with $(\phi, \theta) = (\frac{\pi}{12}, \frac{\pi}{4}),\ (\frac{\pi}{5}, \frac{\pi}{9}),\ (\frac{\pi}{5}, \frac{\pi}{3})$ so that the balls of radius $R = 0.3$ centered at these points completely cover the obstacle P. We consider the artificial potentials $V_i : \mathbb{R}^3 \to \mathbb{R}$ (6.1) with $D = 0.3$, $k = 4$, and $\tau = 100/e$. That is,

$$V_i(q) = \begin{cases} 100 \exp\left(-\frac{1}{1-(\|p_i - q\|/0.3)^8}\right) & \|p_i - q\| < R \\ 0 & \text{else,} \end{cases}$$

and construct the artificial potential $V : \mathbb{R}^3 \to \mathbb{R}$ defined by $V(q) = V_1(q) + V_2(q) + V_3(q)$ for all $q \in \mathbb{R}^3$. In Figure 6.2, we plot the solutions to the necessary conditions (5.2) with the artificial potential V as defined above, from which it is clear that the agent avoids the obstacle P while solving the boundary value problem. Numerical integration was conducted via the Euler method with a time step of $h = 0.001$. A shooting method based on the downhill simplex algorithm was used to find the initial accelerations and jerks that led to solutions to the boundary value problem.

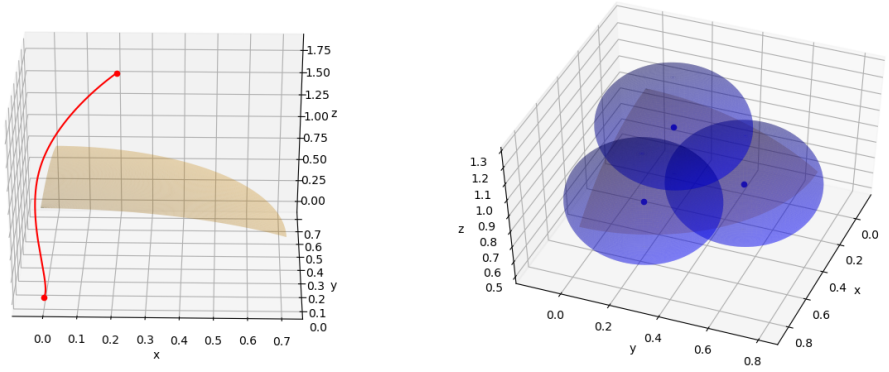


Figure 6.2: Numerical solutions to equation (5.2) with the potential V as defined above (left), and the covering of the obstacle P by balls of radius $R = 0.3$ centered at the points p_1, p_2, and p_3 (right).

6.2 The Collision Avoidance Task

We now switch our attention to multi-agent systems and the collision avoidance task. Consider a set $\mathcal{V}$ consisting of $s \geq 2$ agents on Q, a complete and connected Riemannian manifold. The configuration of each agent at any given time is determined by the element $q_i(t) \in Q$, $i = 1, \ldots, s$. The neighboring relationships are described by an undirected time-invariant graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors $\mathcal{N}_i$ for the agent $i \in \mathcal{V}$ is given by $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. An agent $i \in \mathcal{V}$ can measure its Riemannian distance from other agents in the subset $\mathcal{N}_i \subseteq \mathcal{V}$. The assumptions that Q is complete and connected and $\mathbb{G}$ is undirected and time-invariant will remain for the remainder of the section.

For $i = 1, ..., s$, and points $\xi_i, \eta_i \in TQ$, consider the sets $\Omega_i := \Omega^{(2),[a,b]}_{\xi_i,\eta_i}$ as in (2.18) and the functional J_{coll} on $\Omega := \Omega_1 \times \cdots \times \Omega_s$ defined by:

$$J_{\text{coll}}(q_1, q_2, \ldots, q_s) = \frac{1}{2} \sum_{i=1}^{s} \int_a^b \Big(\Big|\Big| \frac{D\dot{q}_i}{dt}(t) \Big|\Big|^2 + \sum_{(i,j)\in\mathcal{E}} V_{ij}(q_i(t), q_j(t)) \Big) dt. \tag{6.3}$$

where $V_{ij} : Q \times Q \to \mathbb{R}$ is a smooth non-negative function called an *artificial potential* satisfying the symmetry relations $V_{ij} = V_{ji}$ and $V_{ij}(p,x) = V_{ij}(x,p)$ for all $(i,j) \in \mathcal{E}$ and $(p,x) \in Q \times Q$.

As discussed in Section 2.1, the sets $Q^s := Q \times Q \times \ldots Q$ (s times) and Ω are (complete) Riemannian manifolds when equipped with the respective product metrics. Moreover, Ω is an infinite-dimensional Hilbert manifold with model space $(H^2([a,b], \mathbb{R}^n))^s$, which may be identified with the path space $\Omega^{(2)}$ (2.18) of Q^s. With this identification, it is clear that J_{coll} may be identified with $J : \Omega^{(2)} \to \mathbb{R}$ as in (5.1), where $V : Q^s \to \mathbb{R}$ is given by $V(q) = V(q^1, \ldots, q^s) := \sum_{j\in\mathcal{N}_i} V_{ij}(q_i(t), q_j(t))$. It follows that the analysis done in Chapter 5 also applies to J_{coll} and its minimizers. In particular, the necessary conditions for optimality take the following form:

Proposition 19. *A curve $q = (q_1, ..., q_s) \in \Omega$ is a critical point of J_{coll} if and only if for each $i \in \mathcal{V}$, $q_i \in \Omega_i$ is smooth and for all $t \in [0,T]$ satisfies*

$$D_t^3 \dot{q}_i + R\Big(D_t \dot{q}_i, \dot{q}_i\Big)\dot{q}_i = -\sum_{j\in\mathcal{N}_i} \text{grad}_1\, V_{ij}(q_i(t), q_j(t)), \tag{6.4}$$

where R denotes the curvature endomorphism (2.2) *on Q and* $\text{grad}_1 V_{ij}$ *denotes the gradient vector field of V_{ij} with respect to the first argument (as discussed in Section 2.1).*

We now consider the collision avoidance problem. That is, we seek to design the artificial potential so that the agents (whose trajectories are given by the BVP (6.4) on Ω) avoid one another within some desired tolerance. The methodology will mirror that of the point-obstacle avoidance task discussed in Section 6.1.1. We define a *collision avoidance problem* $\mathcal{P} = (r_{ij})$ as a $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrix (r_{ij}) whose coordinate elements $r_{ij} > 0$ for all $(i,j) \in \mathcal{E}$ and otherwise are zero are called the *tolerances* for $\mathcal{P}$ (in particular, r_{ij} denotes the tolerance between the i^{th} and j^{th} agents). We further define an *extended collision avoidance problem* as a 3-tuple $\mathcal{P}_{\text{ext}} = (r_{ij}, r^*_{ij}, R_{ij})$, where $(r_{ij}), (r^*_{ij}), (R_{ij})$ are all $|\mathcal{V}| \times |\mathcal{V}|$ symmetric matrices whose coordinate elements satisfy $0 < r_{ij} < r^*_{ij} \leq R_{ij}$ for all $(i,j) \in \mathcal{E}$ and otherwise are all zero. As for obstacle avoidance problems, we define the following sets for each $(i,j) \in \mathcal{E}$:

$$C_{ij} := \{(p,q) \in Q \times Q \ : \ d(p,q) < r_{ij}\}, \qquad \textit{the collision region} \text{ for the edge } (i,j) \in \mathcal{E},$$

$$C^*_{ij} := \{(p,q) \in Q \times Q \ : \ d(p,q) < r^*_{ij}\} \qquad \textit{the risk region} \text{ for the edge } (i,j) \in \mathcal{E},$$

$$S_{ij} := \{(p,q) \in Q \times Q \ : \ d(p,q) \geq R_{ij}\} \qquad \textit{the safety region} \text{ for the edge } (i,j) \in \mathcal{E},$$

Since the matrices $(r_{ij}), (r^*_{ij}), (R_{ij})$ are symmetric, it follows immediately that each of these regions is symmetric in (i,j). That is, $C_{ij} = C_{ji}$ and equivalent statements for C^*_{ij} and S_{ij}. We say that $q \in \Omega$ *avoids collision with tolerances* r_{ij} if $(q_i(t), q_j(t)) \notin C_{ij}$ for all $(i.j) \in \mathcal{E}$ and $t \in [a,b]$. Moreover, we say that the collision avoidance task is *feasible with tolerances* r_{ij} if there exists such a $q \in \Omega$ which avoids collision with tolerances r_{ij}. In Figure 6.3, we depict these regions with respect to the agent q_i.

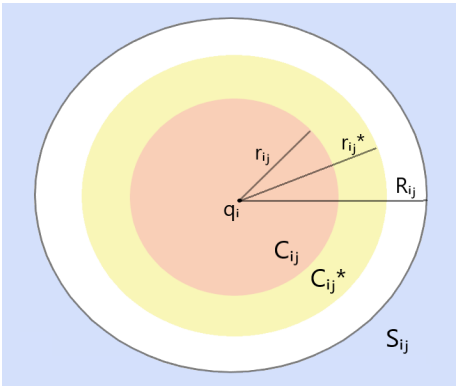


Figure 6.3: The regions C_{ij}, C^*_{ij}, and S_{ij} for $(i,j) \in \mathcal{E}$ taken with respect to the position of the agent q_i.

We call a curve $q = (q_1, ..., q_s) \in \Omega$ such that $(q_i(t), q_j(t)) \in S_{ij}$ for each $(i,j) \in \mathcal{E}$ and $t \in [a,b]$ a *reference trajectory*. Observe that a reference trajectory exists if and only if collision avoidance is feasible with the tolerances R_{ij}. Moreover, this existence is independent of the chosen artificial potentials V_{ij}, and requires $R_{ij} < \min\left\{d(q_i^a, q_j^a),\ d(q_i^b, q_j^b)\right\}$. We now seek to prove that the minimizers of J_{coll} over Ω avoid collision under some conditions on the artificial potential expressed in terms of a reference trajectory. The conditions and proof follow analogously to those in Proposition 17.

Proposition 20. *Suppose that the extended collision avoidance problem $\mathcal{P}_{\text{ext}} = (r_{ij}, r_{ij}^*, R_{ij})$ is feasible, and that there exists a reference trajectory $q \in \Omega$. For all $i \in \mathcal{V}$, define the non-negative real numbers:*

$$A_i := \sup_{t\in[a,b]} \{||D_t\dot{q}_i(t)||\}, \quad c := (b-a)\sum_{i=1}^{s}\left[A_i^2 + \sum_{j\in\mathcal{N}_i} V_{ij}^-\right], \quad v_i := \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_i^a||^2}.$$

If $V_{ij}^* > \frac{c(v_i+v_j)}{2(r_{ij}^*-r_{ij})}$ for all $(i,j) \in \mathcal{E}$, then every minimizer $q^* = (q_1^*, ..., q_s^*) \in \Omega$ of J avoids collision with tolerances r_{ij}.

Proof. Observe that, since each pair of agents in our reference trajectory remains in the Safety Region for all $t \in [a,b]$,

$$\begin{aligned} J(q) &= \sum_{i=1}^{s}\int_a^b \left[||D_t\dot{q}_i(t)||^2 + \sum_{j\in\mathcal{N}_i} V_{ij}(q_i(t), q_j(t))\right] dt \\ &\leq (b-a)\sum_{i=1}^{s}\sup_{t\in[a,b]}\left[||D_t\dot{q}_i(t)||^2 + \sum_{j\in\mathcal{N}_i} V_{ij}(q_i(t), q_j(t))\right] \\ &\leq (b-a)\sum_{i=1}^{s}\left[A_i^2 + \sum_{j\in\mathcal{N}_i} V_{ij}^-\right] =: c. \end{aligned}$$

For all $i \in \mathcal{V}$, let $v_i^* = \sup_{t\in[a,b]} \{||\dot{q}_i^*(t)||\}$, so that

$$\begin{aligned}
||\dot{q}_i^*(t)||^2 &= ||v_i^a||^2 + \int_a^b \frac{d}{dt}||\dot{q}_i^*(t)||^2 dt \\
&= ||v_i^a||^2 + 2\int_a^b g\left(D_t\dot{q}_i^*(t), \dot{q}_i^*(t)\right) dt \\
&\leq ||v_i^a||^2 + 2\left[\int_a^b ||D_t\dot{q}_i^*(t)||^2\, dt\right]^{1/2}\left[\int_a^b ||\dot{q}_i^*(t)||^2\, dt\right]^{1/2} \\
&\leq ||v_i^a||^2 + 2v_i^*\sqrt{J(q^*)(b-a)} \\
&\leq ||v_i^a||^2 + 2v_i^*\sqrt{c(b-a)},
\end{aligned}$$

since $J(q^*) \leq J(q) \leq c$. Taking the supremum over $t \in [a,b]$ on both sides of the inequality, we have $(v_i^*)^2 \leq ||v_i^a||^2 + 2v_i^*\sqrt{c(b-a)}$, so that $v_i^* \leq \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_i^a||^2} =: v_i$.

Now assume towards contradiction that q^* does not avoid collision. Then for some $(i,j) \in \mathcal{E}$ and $\tau \in [a,b]$, $(q_i^*(\tau), q_j^*(\tau)) \in C_{ij}$, and so for $t > \tau$ we have by repeated applications of the triangular inequality that

$$\begin{aligned}
d(q_i^*(t), q_j^*(t)) &\leq d(q_i^*(t), q_i^*(\tau)) + d(q_i^*(\tau), q_j^*(t)) \\
&\leq d(q_i^*(t), q_i^*(\tau)) + d(q_j^*(\tau), q_j^*(t)) + d(q_i^*(\tau), q_j^*(\tau)) \\
&\leq L\left(q_i^*|_{(\tau,t)}\right) + L\left(q_j^*|_{(\tau,t)}\right) + r_{ij} \\
&= \int_\tau^t ||\dot{q}_i^*(t)||dt + \int_\tau^t ||\dot{q}_j^*(t)||dt + r_{ij} \\
&\leq (v_i^* + v_j^*)(t-\tau) + r_{ij} \\
&\leq (v_i + v_j)(t-\tau) + r_{ij},
\end{aligned}$$

and so for all $t \in \left[\tau, \min\{b, \tau + \frac{r_{ij}^* - r_{ij}}{v_i + v_j}\}\right)$ then $((q_i^*(t), q_j^*(t)) \in C_{ij}^*$. It can be seen by repeating the argument with $t < \tau$ that in fact $((q_i^*(t), q_j^*(t)) \in C_{ij}^*$ for all $t \in \left(\max\left\{a, \tau - \frac{r_{ij}^* - r_{ij}}{v_i + v_j}\right\}, \min\left\{b, \tau + \frac{r_{ij}^* - r_{ij}}{v_i + v_j}\right\}\right)$. Therefore,

$$2V_{ij}^*\left(\frac{r_{ij}^*-r_{ij}}{v_i+v_j}\right) \leq \int_{\max\left\{a,\tau-\frac{r_{ij}^*-r_{ij}}{v_i+v_j}\right\}}^{\min\left\{b,\tau+\frac{r_{ij}^*-r_{ij}}{v_i+v_j}\right\}} \sum_{j\in\mathcal{N}_i} V_{ij}(q_i^*(t),q_j^*(t))dt$$
$$\leq \int_a^b \sum_{j\in\mathcal{N}_i} V_{ij}(q_i^*(t),q_j^*(t))dt$$
$$\leq J(q^*) \leq J(q) \leq c.$$

So that $V_{ij}^* \leq \frac{c(v_i+v_j)}{2(r_{ij}^*-r_{ij})}$. By contradiction, q^* avoids collision with tolerances r_{ij}. □

Example 10. Consider 4 agents on $\mathbb{R}^3$ with the Euclidean metric, and the following neighboring relations between agents $\mathcal{N}_1=\{2,3,4\}$, $\mathcal{N}_2=\{1,3\}$, $\mathcal{N}_3=\{1,2,4\}$, $\mathcal{N}_4=\{1,3\}$. We choose $[a,b]=[0,4]$ and the boundary conditions as:

$$q_1(0)=(-1,-1,0), \quad q_1(4)=(1,1,0), \quad \dot{q}_1(0)=\dot{q}_1(4)=(0.5,0.5,0)$$
$$q_2(0)=(-1,1,0), \quad q_2(4)=(1,-1,0), \quad \dot{q}_2(0)=\dot{q}_2(4)=(0.5,-0.5,0)$$
$$q_3(0)=(1,1,0), \quad q_3(4)=(-1,-1,0), \quad \dot{q}_3(0)=\dot{q}_3(4)=(0.5,0.5,0)$$
$$q_4(0)=(1,-1,0), \quad q_4(4)=(-1,1,0), \quad \dot{q}_4(0)=\dot{q}_4(4)=(-0.5,0.5,0)$$

For our reference trajectories, we choose piecewise defined curves consisting of cubic polynomials, with constant speed arcs along the circle of radius 2 centered at the origin. For example, the reference trajectory γ_1 for q_1 consists of the cubic polynomial verifying the relevant initial conditions and $\gamma(1.5)=(2,0,0)$, $\dot{\gamma}(1.5)=(0,-\pi,0)$, followed by the circular arc $\gamma(t)=(2\cos(\frac{\pi}{2}(t-1.5)),\ -2\sin(\frac{\pi}{2}(t-1.5),\ 0)$ for $1.5<t<2.5$, followed by the cubic polynomial starting from the circular arcs end points in position and velocity to the final points in the boundary value problem. Note that, in this manner $\gamma_1 \in \Omega_1$. The other reference trajectories were chosen similarly, so that $\gamma=(\gamma_1,...,\gamma_4)\in\Omega$. It can be seen that $||\gamma_i(t)-\gamma_j(t)||>2$ for all $(i,j)\in\mathcal{E}$ and $t\in[0,4]$. We consider the avoidance family of artificial potentials given by $V_{ij}(p,q)=\frac{\tau}{1+(||p-q||/D)^{2k}}$. With $D=2$, we clearly have $V_{ij}^-=1$ in the context of Proposition 20.

Moreover, it can be shown numerically that $A_i<3.22$, so that $c<185.9$ and $v_i<54.57$ for $i=1,...,4$. We choose our tolerances for the Collision regions and Risk regions as $r_{ij}=\frac{1}{2}$ and $r_{ij}^*=1$ for

all $(i,j) \in \mathcal{E}$, so that by Proposition 20, collision is avoided if we choose k and τ such that $\frac{\tau}{1+2^{-2k}} > 20,290$. One such solution is $\epsilon = 100,000$ and $k = 8$, with which the necessary conditions for extrema (19) take the form

$$\frac{d^4 q_i}{dt^4} = \sum_{j \in \mathcal{N}_i} \frac{6400000||q_i - q_j||^{14}}{(1 + ||q_i - q_j||^{16})^2}(q_i - q_j), \qquad i = 1, \ldots, 4. \tag{6.5}$$

In Figure 6.4, we numerically integrate equations (6.5) with the above initial conditions. For the purposes of comparison, we also include the numerical solutions to the cubic polynomials satisfying the boundary conditions (that is, the solutions to the necessary conditions for extrema in the case that $V \equiv 0$). Note that the

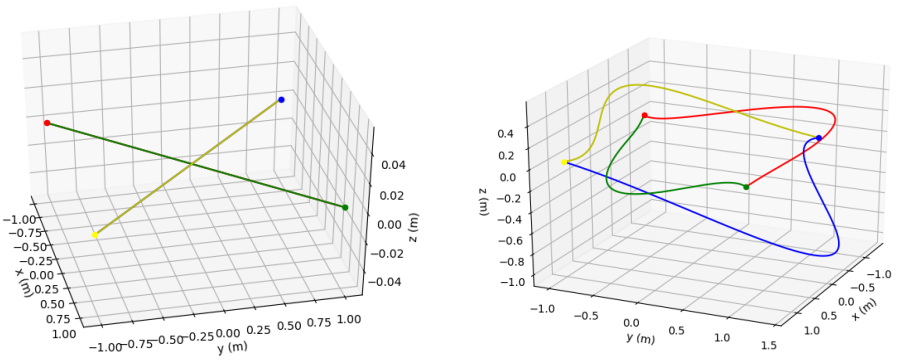


Figure 6.4: The Riemannian cubics (left) and solutions to the necessary conditions for extrema (right) satisfying the boundary conditions in positions and velocities as above. In the case of Riemannian cubics, all four agents collide at the center. With the potential, it is easily verified that collision is avoided within the desired tolerance given by $r_{ij} = \frac{1}{2}$.

As for the obstacle avoidance problem, we wish to remove the dependency of the reference trajectory on our conditions for collision avoidance. We can do this by using the same family $f^k_{\tau,D} : \mathbb{R} \to \mathbb{R}$ of smooth non-negative functions and then defining a *collision avoidance family* of artificial potentials via $V^k_{\tau,D}(x,p) = f^k_{\tau,D}(d(x,p))$ for all $(x,p) \in Q \times Q$. This leads to the following corollary of Proposition 20:

Corollary 8. *Suppose that the collision avoidance problem $\mathcal{P} = (D_{ij})$ is feasible. Then, if $r_{ij} < D_{ij}$ for all $(i,j) \in \mathcal{E}$, there exists $\tau_{ij} \in \mathbb{R}^+$, $k_{ij} \in \mathbb{N}$ such that every minimizer of J_{coll} with $V_{ij} = V^{k_{ij}}_{\tau_{ij},D_{ij}}$ avoids collision with tolerances r_{ij}, where $V^k_{\tau,D}$ is a collision avoidance family of artificial potentials*

Proof. Fix $r_{ij} < D_{ij}$ for all $(i,j) \in \mathcal{E}$ and consider the extended collision avoidance problem $\mathcal{P}_{\text{ext}} = (r_{ij}, D_{ij}, D_{ij})$. Since $\mathcal{P}$ is feasible, there exists a reference trajectory $q = (q^1, \ldots, q^s) \in \Omega$ of $\mathcal{P}_{\text{ext}}$ such that $(q^i(t), q^j(t)) \in S_{ij}$ for all $(i,j) \in \mathcal{E}$ and $t \in [a,b]$. Fix $\epsilon_{ij} > 0$ for $(i,j) \in \mathcal{E}$ and define $A_i := \sup_{t\in[a,b]} \{||D_t\dot{q}_i(t)||\}$, $c := (b-a)\sum_{i=1}^{s}\left[A_i^2 + \sum_{j\in\mathcal{N}_i}\epsilon_{ij}\right]$, $v_i := \sqrt{c(b-a)} + \sqrt{c(b-a) + ||v_i^a||^2}$ for all $1 \leq i \leq s$. Take $\tau_{ij}^* > c + \epsilon_{ij}$ for all $(i,j) \in \mathcal{E}$.

For all $(i,j) \in \mathcal{E}$, since $\lim_{k_{ij}\to\infty} V^{k_{ij}}_{\tau_{ij},D_{ij}}(p,x) = 0$ whenever $d(p,x) > D_{ij}$ and $\lim_{k_{ij}\to\infty} V^{k_{ij}}_{\tau_{ij},D_{ij}}(p,x) = \tau$ whenever $d(p,x) < D_{ij}$, it follows that there exists k_{ij}^* such that $k_{ij} > k_{ij}^*$ implies $V^{k_{ij}}_{\tau_{ij}^*,D_{ij}}(p,x) > \tau_{ij}^* - \epsilon_{ij}$ for $d(p,x) < D_{ij}$ and $V^{k_{ij}}_{\tau_{ij}^*,D_{ij}}(p,x) < \epsilon_{ij}$ for $d(p,x) > D_{ij}$ (that is, for $(p,x) \in S_{ij}$). Hence, for k_{ij} sufficiently large, we may choose $V_{ij}^* = \tau_{ij}^* - \epsilon_{ij}$ and $V_{ij}^- = \epsilon_{ij}$. It follows immediately by Proposition 20 that any minimizer of J_{coll} avoids collision with tolerance r. Moreover, it is clear that the same result holds for any $\tau_{ij} > \tau_{ij}^*$. □

In certain robotic applications, restrictions may be placed (either for security or due to physical restrictions) on the magnitudes of the derivatives of the trajectories (see [61] and [82] for instance). While the existence of minimizers is not guaranteed in such a case, we still find its analysis instructive. In particular, we provide alternate conditions for V_{ij}^*—expressed in terms of the uniform bounds and initial conditions—which ensures the collision avoidance of minimizers. We do so by taking advantage of an invariant for the flow of equations (6.4). In fact, the analysis will guarantee the collision avoidance for all critical points of J_{coll} rather than just the minimizing trajectories (as was the case in Proposition 20 and Corollary 8)—provided that the derivatives of these solutions satisfy the given bounds.

For each $i \in \mathcal{V}$, we define the domain

$$\Omega_i^{\max} = \left\{ q \in \Omega_i \mid ||\dot{q}(t)|| \leq v_i^{\max},\ ||D_t\dot{q}(t)|| \leq a_i^{\max},\ \left|\left|D_t^2\dot{q}(t)\right|\right| \leq \eta_i^{\max},\ \ \forall t \in [a,b] \right\},$$

where $v_i^{\max}, a_i^{\max}, \eta_i^{\max}$ are non-negative real numbers with $v_i^{\max} > \max\left\{||v_i^a||,\ \left|\left|v_i^b\right|\right|\right\}$ and further define $\Omega^{\max} := \Omega_1^{\max} \times \cdots \times \Omega_s^{\max}$ and $\eta_i^a := D_t^2\dot{q}(a)$.

Proposition 21. *Suppose that the collision avoidance problem $\mathcal{P} = (r_{ij})$ is feasible, and that $q = (q_1, ..., q_s) \in$*

$\Omega^{\max}$ *is such that* q_i *satisfies equation* (6.4) *for all* $1 \leq i \leq s$. *If*

$$V_{ij}^* > \sum_{i=1}^{s} \left[(a_i^{\max})^2 + v_i^{\max}\eta_i^{\max} + v_i^a \eta_i^a + \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}^- \right]$$

for all $(i,j) \in \mathcal{E}$, *then* q *avoids collision with tolerances* r_{ij}.

Proof. We define the function H by:

$$H\left(q, \dot{q}, D_t\dot{q}, D_t^2\dot{q}\right) = \sum_{i=1}^{s} \left[g\left(\dot{q}_i, D_t^2\dot{q}_i\right) - \frac{1}{2}\left|\left|D_t\dot{q}_i\right|\right|^2 + \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}(q_i, q_j) \right].$$

Using the symmetry of the potential and since $\mathbb{G}$ is undirected, we have that the time evolution of H satisfies

$$\begin{aligned}
\frac{d}{dt}H &= \sum_{i=1}^{s} \Big[- g\Big(\dot{q}_i, R\left(\dot{q}_i, D_t\dot{q}_i\right)\dot{q}_i + \sum_{j \in \mathcal{N}_i} \text{grad}_1 V_{ij}(q_i, q_j)\Big) \\
&\qquad + \frac{1}{2} \sum_{j \in \mathcal{N}_i} \left(g\left(\text{grad}_1 V_{ij}(q_i, q_j), \dot{q}_i\right) + g\left(\text{grad}_j V_{ij}(q_i, q_j), \dot{q}_j\right)\right) \Big] \\
&= \sum_{i=1}^{s} \Big[- g\left(\dot{q}_i, R\left(\dot{q}_i, D_t\dot{q}_i\right)\dot{q}_i\right) - \sum_{j \in \mathcal{N}_i} g\left(\dot{q}_i, \text{grad}_1 V_{ij}(q_i, q_j)\right) + \sum_{j \in \mathcal{N}_i} g\left(\text{grad}_1 V_{ij}(q_i, q_j), \dot{q}_j\right) \Big] \\
&= -\sum_{i=1}^{s} Rm\left(D_t\dot{q}_i, \dot{q}_i, \dot{q}_i, \dot{q}_i\right),
\end{aligned}$$

which vanishes identically due to the symmetries of the curvature tensor (2.11). Suppose towards contradiction that q does not avoid collision. Then, there exists some $(i,j) \in \mathcal{E}$ and $\tau \in [a,b]$ such that $d(q_i(\tau), q_j(\tau)) \leq r_{ij}$, so that $V_{ij}(q_i(\tau), q_j(\tau)) > V_{ij}^*$. But, since $\dot{H} \equiv 0$, we have:

$$\begin{aligned}
V_{ij}^* - \sum_{i=1}^{s} \left[\frac{1}{2}(a_i^{\max})^2 + v_i^{\max}\eta_i^{\max} \right] \leq H(\tau) = H(0) &\leq \sum_{i=1}^{s} \left[v_i^a \eta_i^a + \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}^- \right] \\
\implies V_{ij}^* &\leq \sum_{i=1}^{s} \left[(a_i^{\max})^2 + v_i^{\max}\eta_i^{\max} + v_i^a \eta_i^a + \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_{ij}^- \right],
\end{aligned}$$

which contradicts the assumption on V_{ij}^*. □

Observe that Proposition 21 applies almost identically in the case of the obstacle avoidance task by removing the subscripts corresponding to the agents. For that reason, we do not prove the result here.

6.3 Obstacle Avoidance in Systems with Impulse Effects (SIEs)

In recent years, *hybrid dynamics systems* have become increasingly ubiquitous in control engineering and robotics due to their versatility in applications and ability to encapsulate complex behaviors with relatively simple descriptions. At their core, these systems combine continuous and discrete dynamical systems into a single theory—allowing for instantaneous changes between continuous dynamical systems when certain conditions are met. In this section, we view interpolation in a common class of hybrid dynamical systems known as *systems with impulse effects* as an application of the obstacle avoidance problem discussed in Section 6.1.2. We are motivated by robotic applications (such as bipedal walkers [58]), which make extensive use of hybrid periodic orbits to generate favorable behavior—and which can be viewed as a particular case of the interpolation problem considered here.

6.3.1 Multiple-Domain Hybrid Dynamical Systems

A *Dynamical system with impulse effects* (SIEs) is a class of dynamical system that exhibits both discrete and continuous behaviors [41], [46], [51]. The transition from one to the other occurs when the continuous-time flow reaches a co-dimension one submanifold of the state space and re-initializes via a C^1 map on this submanifold.

Definition: A k-domain SIEs is a tuple $\mathscr{H} = (\Gamma, Q, S, \Delta, X)$, where:

(i) $\Gamma = (\mathcal{V}, \mathcal{E})$ is a connected, directed graph such that $\mathcal{V} = \{y_1, ..., y_k\}$ is a set of k vertices, and $\mathcal{E} \subset Q \times Q$ is the set of edges. We further define the set of neighbors of a vertex $i \in \mathcal{V}$ by $\mathcal{N}_i = \{j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}$.

(ii) $Q = \{Q_y\}_{y \in \mathcal{V}}$ is a collection of smooth complete, connected Riemannian manifolds with Riemannian metric g_y.

(iii) $S = \{S_e\}_{e \in \mathcal{E}}$ is a collection of guards, where for $e = (y_i, y_j) \in \mathcal{E}$, S_e is assumed to be a totally bounded embedded open section of Q_{y_i}.

(iv) $\Delta = \{\Delta_e\}_{e \in \mathcal{E}}$ is a collection of reset maps, which are C^1 mappings where, for $e = (y_i, y_j) \in \mathcal{E}$,

$\Delta_e : S_e \to M_{y_j}$.

(v) $X = \{X_y\}_{y \in \mathcal{V}}$ is a collection of smooth vector fields $X_y : Q_y \to TQ_y$.

The underlying dynamical system with impulse effects is then defined by

$$\begin{cases} \dot{q} = X_i(q) & \text{if } q \in Q_i \setminus \bigcup_{j \in \mathcal{N}_i} S_{ij} \\ q^+ = \Delta_{ij}(q^-) & \text{if } q^- \in S_{ij} \text{ for some } j \in \mathcal{N}_i, \end{cases}$$

where it is understood that $X_i = X_{y_i}$, $Q_i = Q_{y_i}$, and $S_{ij} = S_{e_{ij}}$, $\Delta_{ij} = \Delta_{e_{ij}}$, and where q^- and q^+ denote the left-limit and right-limit, respectively, of the trajectory as it intersects S_i (and is correspondingly reset by Δ). In general, $q^- \neq q^+$, so that there may be a point of discontinuity here. However, as in [46], we are given the choice in deciding whether the trajectory will be left-continuous or right-continuous at this point. That is, whether $q^- \in S_{ij}$ or $q^+ \in \Delta_{ij}(S_{ij})$ belong to our trajectory. We will choose the former. Note that the results that follow in this work hold regardless of this choice [51].

Remark 8. The above definition may in principle lead to a phenomenon known as *Zeno behavior*, characterized by an infinite number of resets in finite time—which is particularly problematic in applications where numerical work is used, as computation time grows infinitely large at these Zeno points. There are two primary modes through which Zeno behavior can occur:

(1) A trajectory is continuously reset back to a guard. To exclude this type of behavior, we require that $\left(\bigcup_{k \in \mathcal{N}_j} S_{jk}\right) \cap \overline{\Delta_{ij}}(S_{ij}) = \emptyset$. This ensures that the trajectory will always be reset to a point with positive distance from the subsequent guard.

(2) The set of times when a solution to our system reaches the guard (called the set of *impact times*) has a limit point (as happens, for instance, in the case of the bouncing ball with coefficient of restitution less than 1). To exclude this type of situation, we require that the set of impact times be closed and discrete, as in [46].

The above two assumptions will be assumed implicitly throughout the remainder of this chapter.

6.3.2 Interpolation on Hybrid Systems

SIEs appear in a variety of robotic applications, typically with some control parameters in the dynamics. It is common in practice to develop a *trajectory tracking controller*, which causes the dynamics to converge to a desired trajectory exponentially fast from some set of initial conditions. Once such a controller is designed, one need only select the trajectory to track. For example, one may wish to design this trajectory so that it solves some interpolation problem. In general, however, the energy consumed along the task will depend on the particular choice in trajectory.

In continuous-time dynamical systems, cubic splines are a ubiquitous choice, because they are piecewise smooth and low energy or even energy-optimal interpolants. In SIEs, however, cubic polynomials may intersect a guard along its trajectory by chance and fail to reach the desired boundary condition. This can be rectified, for instance, by using the minimizers of J with an appropriate potential used to avoid the guard as needed. For this to be possible, we will assume that obstacle avoidance is feasible whenever necessary.

For some $s \in \mathbb{N}$, consider a collection of times $a = t_1 < t_2 < \cdots < t_s = b$ and knot points $\{ \xi_n^{\sigma(n)}\}$, where $\sigma : \{1, \ldots, s\} \to \mathcal{V}$ such that for each $n \in \{1, \ldots, s\}$, $\xi_n^{\sigma(n)} = (q_n^{\sigma(n)}, v_n^{\sigma(n)}) \in TQ_{\sigma(n)}$ and $q_n^{\sigma(n)} \notin S_{(\sigma(n),j)}$ for any $j \in \mathcal{N}_{\sigma(n)}$.

Interpolation Problem: Find a piecewise smooth curve $\gamma : [a, b] \to Q$ such that $\gamma(t_n) = q_n^{\sigma(n)}$ and $\dot{\gamma}(t_n) = v_n^{\sigma(n)}$.

Procedure: We will construct a sequence of piecewise smooth curves, each of which solve some boundary conditions of the form $\gamma_n(t_n) = \xi_n^{\sigma(n)}$ and $\gamma_n(t_{n+1}) = \xi_{n+1}^{\sigma(n+1)}$. We then need only "glue" the curves together to get the desired result. We consider the two operators as follows: For any $\xi, \eta \in TQ_i$, denote by $\mathbb{P}^\tau(\xi, \eta; \cdot) : [0, \tau] \to Q_i$ a minimizer of J (equation (5.1)) on $\Omega^\tau_{\xi,\eta}$ corresponding to some avoidance potential used to avoid $\bigcup_{j \in \mathcal{N}_i} S_{ik}$ (as, for example, was designed in Section 6.1.2). Denote by $\mathbb{P}^\tau_{ij}(\xi, \eta; \cdot) : [0, \tau] \to Q_i$ a minimizer corresponding to some potential used to avoid $\bigcup_{j \in \mathcal{N}_i, k \neq j} S_{ik}$. In other words, $\mathbb{P}^\tau$ avoids all guards in Q_i, while $\mathbb{P}^\tau_{ij}$ avoids all guards except for S_{ij}.

Case 1: $\sigma(n) = \sigma(n + 1)$. In this case, we wish to remain in $Q_{\sigma(n)}$ along the trajectory, and so we must avoid each guard. Clearly it suffices to consider

$\gamma_n(t) = \mathbb{P}^{t_{n+1}-t_n}(\xi_n^{\sigma(n)}, \xi_{n+1}^{\sigma(n+1)}; t - t_n)$ for $t \in [t_n, t_{n+1})$.

Case 2: $\sigma(n) \neq \sigma(n+1)$. Here, we must move to a different component of the SIEs by passing through the guard(s). In general, it may happen that there is no edge connecting the vertices $y_{\sigma(n)}$ and $y_{\sigma(n+1)}$. However, since the graph is connected by assumption, we can always find a path through the graph connecting $y_{\sigma(n)}$ to $y_{\sigma(n+1)}$. We denote such a path by $y_{\sigma(n)} = y_1^* \to y_2^* \to ... \to y_m^* = y_{\sigma(n+1)}$, and further define the edge $S_{i^*} := S_{(y_i^*, y_{i+1}^*)}$ and similarly for Δ_{i^*} and $\mathbb{P}_{i^*}^{\tau}$. Set $\alpha_n = \frac{t_{n+1}-t_n}{m}$ and choose a point $\eta_1 \in S_{1^*}$. We consider the curve $\gamma_n^1(t) := \mathbb{P}_{1^*}^{\alpha_n}(\xi_{\sigma(n)}, \eta_1; t - t_n)$ for $t \in [t_n, t_n + \alpha_n]$. Let $t_n + \tau_1$ denote the first time for which $\gamma(t_n + \tau_1) \in S_{1^*}$ (it may happen by chance that the curve intersects the guard before reaching η_1). Denote this point of contact by $\xi_1^* := \gamma(t_n + \tau_1)$.

We now define the curve γ_n^j recursively for $j = 2, ..., m-1$. Choose $\eta_j \in S_{j^*}$, and let $\tau_j^* := t_n + \sum_{k=1}^{j-1} \tau_k$. Define $\gamma_n^j(t) := \mathbb{P}_{j^*}^{\alpha_n}(\Delta_{j^*}(\xi_{j-1}^*), \eta_j; t - \tau_j^*)$ for $t \in [\tau_j^*, t_n + j\alpha_m]$. Further let τ_j and ξ_j^* be defined by the relation $\xi_j^* := \gamma_n^j(\tau_j^* + \tau_j) \in S_{j^*}$ and $\gamma_n^j(t) \notin S_{j^*}$ for any $t < \tau_j^* + \tau_j$ in its domain.

For $j = m$, we choose

$\gamma_n^m(t) := \tilde{\mathbb{P}}^{t_{n+1}-\tau_{m-1}^*}(\Delta_{m-1^*}(\xi_{m-1}^*), \xi_{\sigma(n+1)}^{n+1}; t - \tau_{m-1}^*)$

for $t \in [\tau_{m-1}^*, t_{n+1}]$. Finally, we set

$$\gamma_n(t) = \begin{cases} \gamma_n^1(t) & \text{for } t \in [t_n, \tau_1^*) \\ \gamma_n^2(t) & \text{for } t \in [\tau_1^*, \tau_2^*) \\ \quad \vdots & \\ \gamma_n^m(t) & \text{for } t \in [\tau_{m-1}^*, t_{n+1}] \end{cases}$$

From which it is clear that γ_n is a piecewise smooth curve satisfying $\gamma_n(t_n) = q_n^{\sigma(n)}$, $\dot{\gamma}_n(t_n) = v_n^{\sigma(n)}$, $\gamma_n(t_{n+1}) = q_{n+1}^{\sigma(n+1)}$, $\dot{\gamma}_{n+1}(t_{n+1}) = v_{n+1}^{\sigma(n+1)}$ for $n = 1, ..., s$.

This strategy, when feasible, produces a solution to the interpolation problem. However, it is worth noting that the procedure (particularly along Case 2) requires some choice in point on the guard. This choice can be made arbitrarily, but one may desire to understand if an "optimal" choice can be made. We can find necessary conditions for the final point by redoing the derivation of Proposition 5.2, but without the assumption that the variational vector fields vanishing at the end point. The effect of this is that the boundary

terms picked up when integrating by parts will no longer vanish at $t = b$, from which necessary conditions can be derived with appropriate choices in the variational vector field. In particular, we have

Lemma 6.3.1. *Suppose $S \subset Q$. A critical point q of J satisfying the initial conditions $q(a) = q_a$ and $\dot{q}(a) = v_a$, and the unilateral constraint $q(b) \in S$ is a smooth curve satisfying*

$$D_t^3\dot{q}(t) + R\Big(D_t\dot{q}(t), \dot{q}(t)\Big)\dot{q}(t) = -\text{grad}\, V(q(t)), \tag{6.6}$$

$$D_t\dot{q}(b) = D_t^2\dot{q}(b) = 0. \tag{6.7}$$

We see that equation 6.6 is identical to the necessary conditions 5.2 where the end points were both fixed. On the other hand, (6.7) tells us that J is minimized among the modified Riemannian cubics whose final (covariant) acceleration and jerk vanish. Note, however, that when closed forms for the solutions to (6.6) cannot be found (which in general cannot be expected), solving (6.7) may be difficult. Moreover, it is not guaranteed that the trajectory will not intersect the guard at some point before reaching the desired final point (though we may get arbitrarily close by choosing a point near the desired point that is not on the guard and then solving the boundary value problem with the avoidance potential—however this may lead to a very costly trajectory). Note also that condition (6.7) is different from the one imposed for velocities to reach a final submanifold in [4].

Remark 9. Analogous to continuous-time dynamical systems, periodic orbits can be defined in a SIEs by a continuous curve $\gamma : \mathbb{R} \to Q$ and a real number $T \in R$ such that $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$. The prevailing difference in a SIEs is that such orbits will in general not be closed. Note that the solution to the interpolation problem can be used to construct periodic orbits in a SIEs by choosing the knot points such that $\xi_1^{\sigma(1)} = \xi_s^{\sigma(s)}$.

Chapter 7

Reduction by Symmetry in the Obstacle Avoidance Problem

The major drawback of utilizing variationally defined curves in path planning strategies is that they are typically impossible to compute explicitly, and often computationally expensive to approximate numerically. However, in many applications, the configuration manifold is a Lie group or a Riemannian homogeneous space, which possess special symmetries that allow to express the necessary conditions as some lower-order ODE evolving on the Lie algebra of the underlying Lie group (together with a reconstruction equation). The Lie algebra, in turn, carries a natural vector space structure, and therefore is well-suited for classical numerical integration schemes. Such a process is known as *reduction by symmetry* (or Euler-Poincaré reduction in the Lagrangian formalism). Moreover, the symmetries of such spaces allow for explicit computations of the Levi-Civita connection, Riemannian curvature, and artificial potentials that were designed in Chapter 6.

The inclusion of potentials in the action functional further complicates the reduction process. In general, the potential will not share all the symmetries of the kinetic component of the Lagrangian, but instead will be invariant under the action of some isotropy subgroup of the underlying Lie group (with respect to its action on some parameter space Q). That is, the potential must admit partial symmetry to allow for reduction (as discussed in Section 3.4). Such problems are typically referred to as reduction with symmetry breaking. Reduction with symmetry breaking on Lie groups has been studied in [36] using a Lagrangian formalism, for variational principles of arbitrary order and where the parameter space is some smooth manifold. For homogeneous spaces, reduction with symmetry breaking has only been studied using a Lagrangian formalism for first order variational principles [109].

In this chapter, we consider reduction by symmetry in the obstacle avoidance problem. Reduction by

symmetry for Riemannian cubic polynomials on Lie groups has been studied in [1, 114] using a Riemannian formalism. Reduction by symmetry for modified Riemannian cubics on $\mathrm{SO}(3)$ and S^2 were studied in [7] using the Riemannian formalism (but without symmetry breaking), and may be viewed as an application of the results derived in [36]. In this chapter, we will adopt the Riemannian formalism, and our strategy will parallel that which was implemented in Section 3.4 for mechanical Lagrangians. On homogeneous spaces, reduction by symmetry for modified Riemannian cubics has not been studied in the literature, neither from the Riemannian nor the Lagrangian perspective. The strategy we use here differs significantly from that proposed by [114] for Riemannian cubics, as we will be using the horizontal connection defined in Section 4.1 to define a constrained variational principle on the underlying Lie group whose critical points project to modified Riemannian cubics on the associated homogeneous space, rather than lift the modified Riemannian cubics to the Lie group. In particular, our process allows for reduction with broken symmetry, provides a different form for the necessary conditions on the Lie group which directly parallels the necessary conditions for modified Riemannian cubics (5.2) (but with a different connection and tensor field), and allows understanding which curves on the Lie group project to modified Riemannian cubics (which is more useful for the purposes of applications). Reduction of the sufficient conditions for optimality (framed in terms of the reduction of the bi-Jacobi equation (2.29)) on Lie groups will also be considered—which has neither been addressed in the literature for modified Riemannian cubics nor for Riemannian cubic polynomials.

In the particular case of reduction with symmetry breaking for the obstacle avoidance problem, a challenge arises from the fact that the potential is typically a function of the Riemannian distance between the agent and the obstacle, which is often impossible to compute explicitly outside certain special cases (such as Lie groups with bi-invariant metrics), and many applications (rigid body motions, for example) are not modelled using metrics which are only left-invariant. Even numerical approximations are very costly, rendering the strategy undesirable for the purpose of path-planning. A principal goal of this chapter is to extend the notion of reduction with symmetry breaking in such a way that the generated trajectories may be found in a cost-effective manner—amplifying considerably the strategy's utility in applications.

7.1 Reduction by Symmetry on Lie Groups

In this section, we reduce the necessary conditions (5.2) by symmetry on a Lie group G equipped with a left-invariant Riemannian metric. This process amounts to left-translating the necessary conditions on G to some equivalent set of equations on the Lie algebra $\mathfrak{g}$, together with a reconstruction equation. In Section 7.1.2, we will additionally consider the special case that G admits a bi-invariant metric, and show how the gradient vector field of the artificial potential can be calculated explicitly in the obstacle avoidance problem (as discussed in Chapter 6). Reduced Riemannian cubics for rigid body motions in $\mathrm{SO}(3)$ are considered as an example. Finally, in Section 7.1.3, we reduce the bi-Jacobi equation (5.7) and find an equivalent statement to Corollary 3 in terms of the invertibility of some curve in the set of $2n \times 2n$ matrices $\mathcal{M}(2n, \mathbb{R})$.

7.1.1 Reduction of Necessary Conditions

We wish to obtain Euler-Poincaré equations corresponding to (5.2) under the following assumption:

G1: $Q = G$ is a connected Lie group endowed with a left-invariant Riemannian metric and corresponding Levi-Civita connection ∇.

To do this, we first must understand the forms that $D_t^3 \dot{g}$ and $R(D_t\dot{g}, \dot{g})\dot{g}$ take when left-translated to curves in the Lie algebra $\mathfrak{g}$. This is summarized in the following lemma:

Lemma 7.1.1. *Let $g : [a, b] \to G$ be a smooth curve and set $\xi^{(0)} := g^{-1}\dot{g}$. Recursively define $\xi^{(i)} = \dot{\xi}^{(i-1)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(i-1)}$ for $i = 1, 2$. Then,*

$$D_t^3 \dot{g} = g\Big(\dot{\xi}^{(2)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(2)}\Big), \tag{7.1}$$

$$R\left(D_t\dot{g}, \dot{g}\right)\dot{g} = gR\big(\xi^{(1)}, \xi^{(0)}\big)\xi^{(0)}. \tag{7.2}$$

Proof. It's clear from Lemma 3.3.4 and Lemma 3.3.3 that $\xi^{(i)} = g^{-1}D_t^i\dot{g}$ for $i = 0, 1, 2$. One more application of Lemma 3.3.4 to $D_t^2\dot{g} = g\xi^{(2)}$ yields equation (7.1). Equation (7.2) follows immediately by the fact that R is a left-invariant tensor field and observing that $R\left(D_t\dot{g}, \dot{g}\right)\dot{g} = R\big(g\xi^{(1)}, g\xi^{(0)}\big)g\xi^{(0)} = gR\big(\xi^{(1)}, \xi^{(0)}\big)\xi^{(0)}$. □

The quantities calculated in Lemma 7.1.1 may be substituted directly into equation (5.2). If g were a Riemannian cubic polynomial (i.e., in the case that $V \equiv 0$), we would immediately obtain reduced equations on the Lie algebra $\mathfrak{g}$. However, we still must handle the artificial potential. There are two strategies that we may employ. First, we may left-translate the gradient potential directly to the Lie algebra. That is, we write $\text{grad}V(g) = L_{g^*} \circ L_{g^{-1}*}\text{grad}V(g)$, and substitute this directly into (5.2) along with equations (7.1) and (7.2) to obtain the reduced equations. Alternatively, we may consider the *extended artificial potential at* $g_0 \in G$, $V_{\text{ext}} : G \times G \to \mathbb{R}$, defined by $V_{\text{ext}}(g, h) = V(g_0 h^{-1} g)$ for all $g, h \in G$. The extended artificial potential is left-invariant (that is, $V_{\text{ext}}(rg, rh) = V_{\text{ext}}(g, h)$ for all $g, h \in G$), and satisfies $V_{\text{ext}}(g, g_0) = V(g)$ for all $g \in G$. Moreover, it is easily seen that V_{ext}, as we've defined it, is the unique scalar field on $G \times G$ satisfying these two properties. Observe that

$$L_{g^*}\text{grad}_1 V_{\text{ext}}(e, g^{-1}g_0) = \text{grad}_1 V_{\text{ext}}(g, g_0) = \text{grad}V(g). \tag{7.3}$$

While not strictly necessary, we choose to work with the extended artificial potential formalism. This is due to two major reasons: $i)$ It is more consistent with the literature, where extended Lagrangians (which follow similar definitions) are necessarily considered for the purposes of Euler-Poincaré reduction. $ii)$ The left-invariance of the extended artificial potential is convenient in the analysis that follows related to the obstacle avoidance problem. Moreover, the role of the obstacle is more easily seen: if we wish to avoid the point-obstacle $g_0 \in G$, we consider the extended artificial potential at g_0.

Using (7.1), (7.2), and (7.3), we may reduce equation (5.2) directly. However, for consistency with the Euler-Poincaré reduction literature, we will reduce the equations via variational analysis.

Proposition 22. *Suppose that G satisfies assumption **G1**. Then $g \in \Omega$ is a modified Riemannian cubic if and only if $\xi^{(0)} := g^{-1}\dot{g}$ and $h := g^{-1}g_0$ solve:*

$$\dot{h} = -\xi^{(0)} h, \tag{7.4}$$

$$\dot{\xi}^{(i)} = \xi^{(i+1)} - \nabla^{\mathfrak{g}}_{\xi^{(0)}} \xi^{(i)}, \qquad i = 0, 1 \tag{7.5}$$

$$\dot{\xi}^{(2)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}} \xi^{(2)} + R\big(\xi^{(1)}, \xi^{(0)}\big)\xi^{(0)} = -\text{grad}_1 V_{\text{ext}}(e, h). \tag{7.6}$$

on $[a,b]$*, together with the boundary conditions* $\xi^{(0)}(a) = g_a^{-1}v_a$, $\xi^{(0)}(b) = g_b^{-1}v_b$, *and* $h(a) = g_a^{-1}g_0$, $h(b) = g_b^{-1}g_0$.

Proof. Suppose that $g \in \Omega$ satisfies equation (5.2). Then, it is clear from Proposition 7 and the definition of the extended artificial potential that

$$\delta J[g] = \int_a^b \left\langle D_t^3\dot{g} + R\big(D_t\dot{g},\dot{g}\big)\dot{g}, \delta g\right\rangle dt + \delta\int_a^b V_{\text{ext}}(g,g_0)dt. \tag{7.7}$$

Let g_s be an admissible variation of g with variational vector field $\dfrac{\partial}{\partial s}\Big|_{s=0} g_s = \delta g$, and define $\eta := g^{-1}\delta g$. Then,

$$\begin{aligned}\delta\int_a^b V_{\text{ext}}(g,g_0)dt &= \int_a^b \frac{\partial}{\partial s}\Big|_{s=0} V_{\text{ext}}(g_s,g_0)dt = \int_a^b \frac{\partial}{\partial s}\Big|_{s=0} V_{\text{ext}}(g^{-1}g_s,h)dt \\ &= \int_a^b \langle \text{grad}_1 V_{\text{ext}}(e,h),\eta\rangle\, dt.\end{aligned}$$

Plugging this into (7.7) and using Lemma 7.1.1, we obtain

$$\int_a^b \left\langle \dot{\xi}^{(2)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(2)} + R\big(\xi^{(1)},\xi^{(0)}\big)\xi^{(0)} + \text{grad}_1 V_{\text{ext}}(e,h),\ \eta\right\rangle dt = 0.$$

The conclusion follows upon applying the fundamental lemma of the calculus of variations, together with the observation that $\dot{h} = -g^{-1}\dot{g}g^{-1}g_0 = -\xi^{(0)}h$. □

Remark 10. As in the first order case discussed in Remark 4, Proposition 22 can be considered as a special case of Euler-Poincaré reduction for second order Lagrangians (that is, Lagrangians defined on the second order tangent bundle T^2G). This was studied on Lie groups in [36], where the corresponding higher order Euler-Poincaré equations were obtained. Using the Riemannian formalism, we bypass the necessity to work with higher-order tangent bundles, and obtain equations evolving Lie algebra $\mathfrak{g}$ rather than its dual $\mathfrak{g}^*$. Furthermore, equations (7.4) -(7.6) can be shown to be equivalent to those obtained in [9], where instead of considering the extended artificial potential V_{ext}, the gradient vector field grad$V$ was left-translated to the Lie algebra.

We now switch our attention to the obstacle avoidance task, as discussed in Chapter 6. In particular, following the strategy implemented in Section 6.1.1, we define an artificial potential $V : G \to \mathbb{R}$ by

$V(g) = f(d^2(g, g_0))$ for all $g \in G$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function, $d^2 : G \times G \to \mathbb{R}$ is the square of the Riemannian distance on G, and $g_0 \in G$ is the obstacle that we wish to avoid. The extended artificial potential is then given by $V_{\text{ext}}(g, h) = V(g_0 h^{-1} g) = f(d^2(g_0 h^{-1} g, g_0))$ for all $g, h \in G$. We now show that the Riemannian distance d is invariant under left-translation, which follows immediately by the left-invariance of the metric:

Lemma 7.1.2. *$d(gq, gp) = d(q, p)$ for all $g, q, p \in G$.*

Proof. Since G is complete as a Riemannian manifold, there exists a geodesic $\gamma : [0, 1] \to G$ which minimizes the length functional $L(c) = \int_0^1 \|\dot{c}(t)\| dt$ among all smooth curves $c : [0, 1] \to G$ satisfying $c(0) = p,\ c(1) = q$. Moreover, we have $d(p, q) = L(\gamma)$ by equation (2.24). By left-invariance of the metric, we then have that $d(p, q) = L(\gamma) = L(g\gamma) \geq d(gp, gq)$, since in particular $g\gamma$ is a smooth curve such that $g\gamma(0) = gp,\ g\gamma(1) = gq$. On the other hand, there exists some geodesic γ^* such that $L(\gamma^*) = d(gp, gq)$, and so $d(gp, gq) = L(\gamma^*) = L(g^{-1}\gamma^*) \geq d(p, q)$. It follows that $d(p, q) = d(gp, gq)$. □

In particular, we find that $V_{\text{ext}}(g, h) = f(d^2(g, h))$, so that $\text{grad}_1 V_{\text{ext}}(g, h) = f'(d^2(g, h))\text{grad}_1 d^2(g, h)$. Evaluating this quantity in practice is highly non-trivial due to the fact that, in general, there is no closed form for the Riemannian distance function nor its gradient. However, we have seen in Lemma 6.1.2 that the Riemannian distance and its gradient can be understood in terms of the inverse of the Riemannian exponential map in the case that this inverse is well-defined. This leads to the following additional assumption:

> **DIST:** The curve $h : [a, b] \to G$ is contained within a geodesically convex neighborhood of the identity $e \in G$ on which the Riemannian exponential map is a diffeomorphism.

Under assumption **DIST**, we obtain the following expression for the gradient of the extended artificial potential using Lemma 6.1.2:

$$\text{grad}_1 V_{\text{ext}}(e, h(t)) = -2f'(\|\exp_e^{-1}(h(t))\|^2) \exp_e^{-1}(h(t)). \tag{7.8}$$

Unfortunately, equation (7.8) suffers from the same basic issue that the previous expression did. Namely, in general, there is no closed form for the Riemannian exponential map or its inverse, due to a lack of closed

form expressions for the geodesics on G. In the next section, we will study a special case where such closed form expressions may be obtained.

7.1.2 Reduction on Lie Groups with Bi-invariant Metrics

Equations (7.4) - (7.6) can be simplified dramatically in the case that the metric $\langle \cdot, \cdot \rangle$ is bi-invariant, as we saw in Section 3.4 for mechanical Lagrangians. In particular, we view equipping G with such a bi-invariant metric as a strengthening of assumption **G1**:

> **G2 :** G is a connected Lie group equipped with a bi-invariant Riemannian metric and corresponding Levi-Civita connection ∇.

Using Lemma 3.3.7, we obtain the following corollary to Proposition 22.

Corollary 9. *Suppose that G satisfies assumption **G2**. Then $g \in \Omega$ is a modified Riemannian cubic if and only if $\xi := g^{-1}\dot{g}$ and $h := g^{-1}g_0$ solve:*

$$\dot{h} = -\xi h, \tag{7.9}$$

$$\dddot{\xi} + \big[\xi, \ddot{\xi}\big] = -\text{grad}_1 V_{\text{ext}}(e, h), \tag{7.10}$$

together with the boundary conditions $\xi(a) = g_a^{-1}v_a,\ \xi(b) = g_b^{-1}v_b$, and $h(a) = g_a^{-1}g_0,\ h(b) = g_b^{-1}g_0$.

Proof. From Proposition 22, it follows that if we take $\xi^{(0)} = \xi$, then $g \in \Omega$ solves (5.2) if and only if:

$$\begin{aligned} \dot{h} &= -\xi h, \\ \dot{\xi}^{(i)} &= \xi^{(i+1)} - \frac{1}{2}\big[\xi, \xi^{(i)}\big], \qquad i = 0, 1 \\ \dot{\xi}^{(2)} + \frac{1}{2}\big[\xi, \xi^{(2)}\big] - \frac{1}{4}\big[\big[\xi^{(1)}, \xi\big], \xi\big] &= -\text{grad}_1 V_{\text{ext}}(e, h). \end{aligned}$$

Now observe that, from Lemmas 7.1.1 and 3.3.7:

$$\begin{aligned} \dot{\xi} &= \xi^{(1)} - \frac{1}{2}\big[\xi, \xi\big] = \xi^{(1)}, \\ \ddot{\xi} &= \dot{\xi}^{(1)} = \xi^{(2)} - \frac{1}{2}\big[\xi, \dot{\xi}\big], \\ \dddot{\xi} &= \dot{\xi}^{(2)} - \frac{1}{2}\big[\xi, \ddot{\xi}\big]. \end{aligned}$$

Substituting these into the necessary conditions, we obtain (7.9) and (7.10). □

Perhaps even more important than the simplified form that the reduced equations take under assumption **G2**, we have from Theorem 3.3.8 that the restriction of the Riemannian exponential map to the Lie algebra, $\exp_e$, agrees with the Lie exponential map $\text{Exp} : \mathfrak{g} \to G$. In particular, under assumption **DIST**, equation (7.8) takes the form

$$\text{grad}_1 V_{\text{ext}}(e, h(t)) = -2f'(\|\text{Log}(h(t))\|^2)\text{Log}(h(t)), \tag{7.11}$$

where $\text{Log} : \mathfrak{g} \to G$ is the Logarithmic map, as discussed in Section 3.1. As seen in Lemma 3.3.9, the logarithmic map can be calculated explicitly in certain special cases, such as $G = \text{SO}(3)$.

In Example 4, we saw that the left-invariant metric for a rigid body is in-fact bi-invariant in the case that the coefficient of inertia matrix $\mathbb{M}$ (or equivalently that the moment of inertia tensor $\mathbb{J}$) is a scalar multiple of the identity. This is akin to the rigid body being symmetric about all of its axes, which is a strong assumption that typically does not hold in application. Hence, even in the case that G admits a bi-invariant metric, this metric may not correspond to the kinetic energy of the physical system that we are interested in studying, and a left-invariant metric must be used anyway. However, as we will see, we may still take advantage of the bi-invariant metric to design our artificial potential. Consider the following alternative assumption to **G1**:

> **G3 :** G is a connected Lie group equipped with a left-invariant metric $\langle \cdot, \cdot \rangle$, with corresponding Levi-Civita connection ∇. Moreover, G admits a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle_{\text{Bi}}$, and $\beta : \mathfrak{g} \to \mathfrak{g}$ is the unique linear endomorphism such that $\langle \xi, \eta \rangle_{\text{Bi}} = \langle \beta(\xi), \eta \rangle$ for all $\xi, \eta \in \mathfrak{g}$.

Let $h \in G$ and $\eta \in \mathfrak{g}$. Observe that:

$$\begin{aligned}
\langle \text{grad}_1 V_{\text{ext}}(e, h), \eta \rangle &= \frac{d}{ds}\Big|_{s=0} V_{\text{ext}}(\text{Exp}(s\eta), h) \\
&= \left\langle \text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h), \eta \right\rangle_{\text{Bi}} \\
&= \left\langle \beta(\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h)), \eta \right\rangle,
\end{aligned}$$

for all $s \in (-\epsilon, \epsilon)$, where $\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h)$ denotes the gradient vector field of V_{ext} with respect to its first argument and $\langle \cdot, \cdot \rangle_{\text{Bi}}$. It follows that $\text{grad}_1 V_{\text{ext}}(e, h) = \beta(\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h))$ for all $h \in G$. Hence, we have the following corollary to Proposition 22 in the case that assumption **G3** is satisfied.

Corollary 10. *Suppose that G satisfies assumption **G3**. Then $g \in \Omega$ is a modified Riemannian cubic if and only if $\xi^{(0)} := g^{-1}\dot{g}$ and $h := g^{-1}g_0$ solve:*

$$\dot{h} = -\xi^{(0)}h, \tag{7.12}$$

$$\dot{\xi}^{(i)} = \xi^{(i+1)} - \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(i)}, \qquad i = 0, 1, \tag{7.13}$$

$$\dot{\xi}^{(2)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(2)} + R\big(\xi^{(1)}, \xi^{(0)}\big)\xi^{(0)} = -\beta(\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h)), \tag{7.14}$$

on $[a, b]$, together with the boundary conditions $\xi^{(0)}(a) = g_a^{-1}v_a$, $\xi^{(0)}(b) = g_b^{-1}v_b$, and $h(a) = g_a^{-1}g_0$, $h(b) = g_b^{-1}g_0$.

In the context of Corollary 10, we now design the artificial potential to be of the form $V(g) = f(d^2_{\text{Bi}}(g, g_0))$, where $d_{\text{Bi}} : G \times G \to \mathbb{R}$ is the Riemannian distance function corresponding to the bi-invariant metric $\langle\cdot,\cdot\rangle_{\text{Bi}}$. The extended potential is similarly defined by $V_{\text{ext}}(g, h) = f(d^2_{\text{Bi}}(g, h))$ for all $g, h \in G$. Under assumption **DIST**, equation (7.11) applies to $\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h(t))$. That is,

$$\beta(\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, h(t))) = -2f'(\|\text{Log}(h(t))\|^2)\beta(\text{Log}(h(t))), \tag{7.15}$$

Example 11. We return to the example of the rigid body modelled on $\text{SO}(3)$. As in Example 4, we equip $\text{SO}(3)$ with the left-invariant metric $\left\langle \dot{R}_1, \dot{R}_2 \right\rangle = \text{tr}(\dot{R}_1 \mathbb{M} \dot{R}_2^T)$ for all $R \in \text{SO}(3), \dot{R}_1, \dot{R}_2 \in T_R\text{SO}(3)$. Using the hat map (3.4), this takes the form of $\left\langle \hat{\Omega}, \hat{\eta} \right\rangle = \Omega^T \mathbb{J}\eta$ (3.19) on the Lie algebra $\mathfrak{so}(3)$. we see the following:

$$\left\langle \text{ad}_{\hat{\Omega}}\hat{\eta}, \hat{\sigma} \right\rangle = \mathbb{J}(\Omega \times \eta) \cdot \sigma$$

$$\left\langle \text{ad}^{\dagger}_{\hat{\Omega}}\hat{\eta}, \hat{\sigma} \right\rangle = (\mathbb{J}\eta \times \Omega) \cdot \sigma$$

for all $\Omega, \eta, \sigma \in \mathbb{R}^3$. It follows from Lemma 3.3.3 that the Riemannian $\mathfrak{g}$-connections can be decomposed as:

$$\left\langle \nabla^{\mathfrak{g}}_{\hat{\Omega}}\hat{\eta}, \hat{\sigma} \right\rangle = \frac{1}{2}\left(\mathbb{J}(\Omega \times \eta) - \mathbb{J}\eta \times \Omega - \mathbb{J}\Omega \times \eta\right) \cdot \sigma. \tag{7.16}$$

We also consider the bi-invariant metric $\langle\cdot,\cdot\rangle_{\text{Bi}}$ defined by $\left\langle \dot{R}_1, \dot{R}_2 \right\rangle_{\text{Bi}} = \text{tr}(\dot{R}_1 \dot{R}_2^T)$ for all $R \in \text{SO}(3), \dot{R}_1, \dot{R}_2 \in T_R\text{SO}(3)$. Through the hat isomorphism, we find that $\left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle_{\text{Bi}} = \Omega_1^T \Omega_2$, which is just the standard inner

product on $\mathbb{R}^3$. From this, it is clear that $\beta(\hat{\Omega}) = \hat{\Omega}\mathbb{M}^{-1}$ for all $\hat{\Omega} \in \mathfrak{so}(3)$, since

$$\left\langle \beta(\hat{\Omega}_1), \hat{\Omega}_2 \right\rangle = \text{tr}((\hat{\Omega}_1\mathbb{M}^{-1})\mathbb{M}\hat{\Omega}_2^T) = \text{tr}(\hat{\Omega}_1\hat{\Omega}_2^T) = \left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle_{\text{Bi}}$$

for all $\hat{\Omega}_1, \hat{\Omega}_2 \in \mathfrak{so}(3)$. Moreover, considering the restriction of β to $\mathfrak{so}(3)$, we see that

$$\Omega \cdot \sigma = \left\langle \hat{\Omega}, \hat{\sigma} \right\rangle_{\text{Bi}} = \left\langle \beta(\hat{\Omega}), \hat{\sigma} \right\rangle = (\beta(\hat{\Omega})^\vee)^T\mathbb{J}\sigma = (\mathbb{J}\beta(\hat{\Omega})^\vee) \cdot \sigma = \left\langle \widehat{\mathbb{J}\beta(\Omega)^\vee}, \hat{\sigma} \right\rangle_{\text{Bi}},$$

from which it follows that $\beta(\hat{\Omega}) = \widehat{\mathbb{J}^{-1}\Omega}$ for all $\Omega \in \mathbb{R}^3$.

Consider a point-obstacle $R_0 \in \text{SO}(3)$, and the artificial potential $V(R) = f(d^2_{\text{Bi}}(R, R_0))$, where $f : \mathbb{R} \to \mathbb{R}$ is smooth and non-negative. The extended artificial potential satisfies $V_{\text{ext}}(R_1, R_2) = f(d^2_{\text{Bi}}(R_1, R_2))$ for all $R_1, R_2 \in \text{SO}(3)$. It follows from Lemma 3.3.9 and Equation (7.11) that

$$\text{grad}_1^{\text{Bi}} V_{\text{ext}}(e, H) = -\frac{2f'(\phi(H)^2)\phi(H)}{\sin(\phi(H))}(H - H^T). \tag{7.17}$$

for all $H \in \text{SO}(3)$ such that $\text{tr}(H) \neq -1$. Hence, from Corollary 10, we have the following Proposition:

Proposition 23. *Consider a rigid body with moment of inertia tensor $\mathbb{J}$ and attitude $R \in \Omega^{[a,b]}_{(R_a,\hat{\Omega}_a),(R_b,\hat{\Omega}_b)}(\text{SO}(3))$. Then, R is a modified Riemannian cubic if and only if $\hat{\Omega}^{(0)} := R^T\dot{R}$ and $H := R^TR_0$ solve:*

$$\dot{H} = -\hat{\Omega}^{(0)}H,$$

$$\mathbb{J}\dot{\Omega}^{(i)} = \mathbb{J}\Omega^{(i+1)} - \frac{1}{2}\Big(\mathbb{J}(\Omega^{(0)} \times \Omega^{(i)}) - \mathbb{J}\Omega^{(0)} \times \Omega^{(i)} - \mathbb{J}\Omega^{(i)} \times \Omega^{(0)}\Big), \qquad i = 0, 1,$$

$$\mathbb{J}\dot{\Omega}^{(2)} + \frac{1}{2}\Big(\mathbb{J}(\Omega^{(0)} \times \Omega^{(2)}) - \mathbb{J}\Omega^{(0)} \times \Omega^{(2)} - \mathbb{J}\Omega^{(2)} \times \Omega^{(0)}\Big) + \mathbb{J}\left(R(\hat{\Omega}^{(1)}, \hat{\Omega}^{(0)})\hat{\Omega}^{(0)}\right)^\vee = \frac{2f'(\phi(H)^2)\phi(H)}{\sin(\phi(H))}(H - H^T)^\vee,$$

on $[a, b]$, together with the boundary conditions $\Omega^{(0)}(a) = R_a^T\hat{\Omega}_a$, $\Omega^{(0)}(b) = R_b^T\hat{\Omega}_b$ and $H(a) = R_a^TR_0$, $H(b) = R_b^TR_0$, and as long as $\text{tr}(H) \neq -1$.

Observe that the curvature term $\left(R(\hat{\Omega}^{(1)}, \hat{\Omega}^{(0)})\hat{\Omega}^{(0)}\right)^\vee$ can be written in terms of $\Omega^{(0)}$ and $\Omega^{(1)}$ via the relation $R(\hat{\Omega}^{(1)}, \hat{\Omega}^{(0)})\hat{\Omega}^{(0)} = \nabla^{\mathfrak{g}}_{\hat{\Omega}^{(1)}}\nabla^{\mathfrak{g}}_{\hat{\Omega}^{(0)}}\hat{\Omega}^{(0)} - \nabla^{\mathfrak{g}}_{\hat{\Omega}^{(0)}}\nabla^{\mathfrak{g}}_{\hat{\Omega}^{(1)}}\hat{\Omega}^{(0)} - \nabla^{\mathfrak{g}}_{\widehat{\Omega^{(1)} \times \Omega^{(0)}}}\hat{\Omega}^{(0)}$ together with equation (7.16). Moreover, it follows from Corollary 6 that (when feasible) the point-obstacle R_0 can be avoided within some desired tolerance by choosing f appropriately. For example, using an avoidance family, as in Definition 6.1.1.

In the case that $\mathbb{J} = I$, we obtain the following corollary from Corollary 9:

Corollary 11. *Consider a rigid body with moment of inertia tensor $\mathbb{J} = I$ and attitude $R \in \Omega^{[a,b]}_{(R_a,\hat{\Omega}_a),(R_b,\hat{\Omega}_b)}(\mathrm{SO}(3))$. Then, R is a modified Riemannian cubic if and only if $\hat{\Omega} := R^T\dot{R}$ and $H := R^T R_0$ solve:*

$$\dot{H} = -\hat{\Omega}H,$$
$$\dddot{\Omega} + \Omega \times \ddot{\Omega} = \frac{2f'(\phi(H)^2)\phi(H)}{\sin(\phi(H))}(H - H^T)^\vee,$$

on $[a,b]$, together with the boundary conditions $\Omega(a) = R_a^T\hat{\Omega}_a$, $\Omega(b) = R_b^T\hat{\Omega}_b$ and $H(a) = R_a^T R_0$, $H(b) = R_b^T R_0$, and as long as $\mathrm{tr}(H) \neq -1$.

Remark 11. Recall that in the collision avoidance task (as described in Section 6.2), the artificial potential takes the particular form $V(g_1,\ldots,g_s) = \sum_{(j,k)\in\mathcal{E}} V_{jk}(g_j,g_k)$, and the corresponding modified Riemannian cubics satisfy (6.4). For the purposes of collision avoidance, the particular sub-potentials V_{jk} will take the form $V_{jk}(g_j,g_k) = f(d^2(g_j,g_k))$, and as such there is no symmetry breaking. That is, we have $V_{jk}(hg_j,hg_k) = V_{jk}(g_j,g_k)$ for all $h \in G$, $(j,k) \in \mathcal{E}$, from which it is clear that V is left-invariant with respect to left-translation on G^s. As such, there is no extended artificial potential to be considered, and we instead left-translate (6.4) directly to the Lie algebra.

Observe that $g_j^{-1}\mathrm{grad}_1 V_{jk}(g_j,g_k) = \mathrm{grad}_1 V_{jk}(e,g_j^{-1}g_k)$. As in Proposition 22, this motivates the definition $h_{jk} = g_j^{-1}g_k$, from which we find that $\dot{h}_{jk} = -g_j^{-1}\dot{g}_j g_j^{-1}g_k + g_j^{-1}\dot{g}_k = -\xi_j h_{jk} + h_{jk}\xi_k$, where $\xi_i := g_i^{-1}\dot{g}_i$ for all $i = 1,\ldots,s$. This leads to the following result:

Proposition 24. *Suppose that $Q = G^s$, where G satisfies assumption **G1**. Then $g = (g_1,\ldots,g_s) \in C^\infty([a,b],G)^s$ satisfies* (6.4) *if and only if $\xi_j^{(0)} := g_j^{-1}\dot{g}_j$ and $h_{jk} := g_j^{-1}g_k$ solve:*

$$\dot{h}_{jk} = -\xi_j^{(0)}h_{jk} + h_{jk}\xi_k^{(0)}, \tag{7.18}$$
$$\dot{\xi}_j^{(i)} = \xi_j^{(i+1)} - \nabla^{\mathfrak{g}}_{\xi_j^{(0)}}\xi_j^{(i)}, \qquad i = 0,1, \tag{7.19}$$
$$\dot{\xi}_j^{(2)} + \nabla^{\mathfrak{g}}_{\xi_j^{(0)}}\xi_j^{(2)} + R\big(\xi_j^{(1)},\xi_j^{(0)}\big)\xi_j^{(0)} = -\sum_{r\in\mathcal{N}_j}\mathrm{grad}_1 V_{jr}(e,h_{jr}), \tag{7.20}$$

for all $j = 1,\ldots,s$ and $k \in \mathcal{N}_j$.

Analogues to Corollaries 9 and 10 follow similarly in the case that G admits a bi-invariant metric.

7.1.3 Reduction of Sufficient Conditions on Lie Groups

As we did in the previous section for the necessary conditions for optimality (5.2), we can also apply reduction by symmetry to the sufficient conditions for optimality on a Lie group equipped with a left-invariant metric. In the end, this amounts to left-translating the Bi-Jacobi fields described by equation (5.7) to the Lie algebra $\mathfrak{g}$, and studying the corresponding bi-conjugate points so that we may apply Corollary 3.

To that end, suppose that G is a Lie group satisfying condition **G1**, and $g \in \Omega$ is a modified Riemannian cubic. Let $\xi^{(0)} := g^{-1}\dot{g}$, and recursively define $\xi^{(i+1)} = \dot{\xi}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(i)}$ for $i = 0, 1, 2$. Consider a vector field $X \in \Gamma(g)$. From Lemma 3.3.4, it is clear that if we recursively define $\mathcal{X}^{(i+1)} := \dot{\mathcal{X}}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(i)}$ with $\mathcal{X}^{(0)} := g^{-1}X$, then $g^{-1}D_t^i X = \mathcal{X}^{(i)}$ for all $i \in \mathbb{N}$. In particular,

$$g^{-1}D_t^4 X = \dot{\mathcal{X}}^{(3)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(3)}. \tag{7.21}$$

We now translate each term of $F(X, \dot{g})$, defined in (5.6). Since the Riemannian curvature R is a tensor field, it follows that

$$R(R(X,\dot{g})\dot{g},\dot{g})\dot{g} = gR(R(\mathcal{X}^{(0)},\xi^{(0)})\xi^{(0)},\xi^{(0)})\xi^{(0)} \tag{7.22}$$

$$R(X, D_t^2\dot{g})\dot{g} = gR(\mathcal{X}^{(0)},\xi^{(2)})\xi^{(0)} \tag{7.23}$$

$$R(D_t^2 X,\dot{g})\dot{g} = gR(\mathcal{X}^{(2)},\xi^{(0)})\xi^{(0)} \tag{7.24}$$

$$R(X,\dot{g})D_t^2\dot{g} = gR(\mathcal{X}^{(0)},\xi^{(0)})\xi^{(2)} \tag{7.25}$$

$$R(X, D_t\dot{g})D_t\dot{g} = gR(\mathcal{X}^{(0)},\mathcal{X}^{(1)})\mathcal{X}^{(1)} \tag{7.26}$$

$$R(D_t X,\dot{g})D_t\dot{g} = gR(\mathcal{X}^{(1)},\xi^{(0)})\xi^{(1)}. \tag{7.27}$$

The remaining terms of $F(X, \dot{q})$ involve covariant derivatives of the Riemannian curvature. Note that for

$X, Y, Z \in \Gamma(g)$, we have

$$\begin{aligned}(D_tR)(X,Y)Z &= D_t(R(X,Y)Z) - R(D_tX,Y)Z - R(X,D_tY)Z - R(X,Y)D_tZ \\ &= D_t(gR(\mathcal{X},\mathcal{Y})\mathcal{Z}) - gR(\dot{\mathcal{X}} + \nabla^{\mathfrak{g}}_{\xi}\mathcal{X},\mathcal{Y})\mathcal{Z} - gR(\mathcal{X},\dot{\mathcal{Y}} + \nabla^{\mathfrak{g}}_{\xi}\mathcal{Y})\mathcal{Z} - gR(\mathcal{X},\mathcal{Y})(\dot{\mathcal{Z}} + \nabla^{\mathfrak{g}}_{\xi}\mathcal{Z}) \\ &= D_t(gR(\mathcal{X},\mathcal{Y})\mathcal{Z}) - g\left(\frac{d}{dt}R(\mathcal{X},\mathcal{Y})\mathcal{Z} - R(\nabla^{\mathfrak{g}}_{\xi}\mathcal{X},\mathcal{Y})\mathcal{Z} - R(\mathcal{X},\nabla^{\mathfrak{g}}_{\xi}\mathcal{Y})\mathcal{Z} - R(\mathcal{X},\mathcal{Y})\nabla^{\mathfrak{g}}_{\xi}\mathcal{Z}\right).\end{aligned}$$

Moreover,

$$D_t(gR(\mathcal{X},\mathcal{Y})\mathcal{Z}) = g\left(\frac{d}{dt}R(\mathcal{X},\mathcal{Y})\mathcal{Z} + \nabla^{\mathfrak{g}}_{\xi}\left(R(\mathcal{X},\mathcal{Y})\mathcal{Z}\right)\right).$$

Hence, if we let $\mathcal{X} = g^{-1}X,\ \mathcal{Y} = g^{-1}Y,\ \mathcal{Z} = g^{-1}Z$, we find that

$$(D_tR)(X,Y)Z = g\left(\nabla^{\mathfrak{g}}_{\xi}(R(\mathcal{X},\mathcal{Y})\mathcal{Z}) - R(\nabla^{\mathfrak{g}}_{\xi}\mathcal{X},\mathcal{Y})\mathcal{Z} - R(\mathcal{X},\nabla^{\mathfrak{g}}_{\xi}\mathcal{Y})\mathcal{Z} - R(\mathcal{X},\mathcal{Y})\nabla^{\mathfrak{g}}_{\xi}\mathcal{Z}\right). \tag{7.28}$$

Therefore,

$$\begin{aligned}g^{-1}(D_tR)(D_tX,\dot{g})\dot{g} &= \nabla^{\mathfrak{g}}_{\xi^{(0)}}\left(R(\mathcal{X}^{(1)},\xi^{(0)})\xi^{(0)}\right) - R(\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(1)},\xi^{(0)})\xi^{(0)} \\ &\quad - R(\mathcal{X}^{(1)},\nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(0)})\xi^{(0)} - R(\mathcal{X}^{(1)},\xi^{(0)})\nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(0)}, \qquad (7.29) \\ g^{-1}(D_tR)(X,D_t\dot{g})\dot{g} &= \nabla^{\mathfrak{g}}_{\xi^{(0)}}\left(R(\mathcal{X}^{(0)},\xi^{(1)})\xi^{(0)}\right) - R(\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(0)},\xi^{(1)})\xi^{(0)} \\ &\quad - R(\mathcal{X}^{(0)},\nabla^{\mathfrak{g}}_{\xi^{(1)}}\xi^{(0)})\xi^{(0)} - R(\mathcal{X}^{(0)},\xi^{(1)})\nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(0)}, \qquad (7.30) \\ g^{-1}(D_tR)(X,\dot{g})D_t\dot{g} &= \nabla^{\mathfrak{g}}_{\xi^{(0)}}\left(R(\mathcal{X}^{(0)},\xi^{(0)})\xi^{(1)}\right) - R(\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(0)},\xi^{(0)})\xi^{(1)} \\ &\quad - R(\mathcal{X}^{(0)},\nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(0)})\xi^{(1)} - R(\mathcal{X}^{(0)},\xi^{(0)})\nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(1)}. \qquad (7.31)\end{aligned}$$

By a similar argument,

$$g^{-1}(D_t^2R)(X,Y)Z = \nabla^{\mathfrak{g}}_{\xi^{(0)}}((D_tR)(\mathcal{X},\mathcal{Y})\mathcal{Z}) - (D_tR)(\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X},\mathcal{Y})\mathcal{Z} - (D_tR)(\mathcal{X},\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{Y})\mathcal{Z} - (D_tR)(\mathcal{X},\mathcal{Y})\nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{Z},$$

which may then be used to calculate $g^{-1}(D_t^2R)(X,\dot{g})\dot{g}$ as a function of $\mathcal{X}^{(0)}$ and $\xi^{(0)}$ and written in terms of the Riemannian curvature R by applying (7.28) to each term.

The remaining term of $F(X,\dot{q})$ is $(\nabla_X R)(D_t\dot{g},\dot{g})\dot{g}$, which differs from the rest because we are now taking the covariant derivative of the curvature with respect to X. To address this, suppose that $\Gamma(s,t)$ is a

two-parameter variation of g, and define $T = \partial_t \Gamma(s,t)$ and $S := \partial_s \Gamma(s,t)\big|$. Further define $\mathcal{T}(s,t) := \Gamma^{-1} T$ and $\mathcal{S}(s,t) := \Gamma^{-1} S$, and suppose Γ is defined such that $T(0,t) = \dot{g}(t)$ and $S(0,t) = X(t)$. Then, it can be seen by applying equation (7.28) that:

$$\begin{aligned}(D_s R)(D_t T, T)T &= D_s(R(D_t T, T)T) - R(D_s D_t T, T)T - R(D_t T, D_s T)T - R(D_t T, T) D_s T \\ &= \Gamma\big[\nabla^{\mathfrak{g}}_{\mathcal{S}}\big(R(\dot{\mathcal{T}} + \nabla^{\mathfrak{g}}_{\mathcal{T}}\mathcal{T}, \mathcal{T})\mathcal{T}\big) - R(\nabla^{\mathfrak{g}}_{\mathcal{S}}(\nabla^{\mathfrak{g}}_{\mathcal{T}} + \dot{\mathcal{T}}), \mathcal{T})\mathcal{T} - R(\nabla^{\mathfrak{g}}_{\mathcal{T}} + \dot{\mathcal{T}}, \nabla^{\mathfrak{g}}_{\mathcal{S}}\mathcal{T})\mathcal{T} - R(\nabla^{\mathfrak{g}}_{\mathcal{T}} + \dot{\mathcal{T}}, \mathcal{T})\nabla^{\mathfrak{g}}_{\mathcal{S}}\mathcal{T}\big]\end{aligned}$$

Setting $s = 0$, we obtain

$$\begin{aligned}g^{-1}(\nabla_X R)(D_t\dot{g}, \dot{g})\dot{g} = \nabla_{\mathcal{X}^{(0)}}(R(\xi^{(1)}, \xi^{(0)})\xi^{(0)}) - R(\nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}}\xi^{(1)}, \xi^{(0)})\xi^{(0)} \\ - R(\xi^{(1)}, \nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}}\xi^{(0)})\xi^{(0)} - R(\xi^{(1)}, \xi^{(0)})\nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}}\xi^{(0)}.\end{aligned} \tag{7.32}$$

From the previous analysis, it is clear that $F(X, \dot{q})$ satisfies

$$g^{-1}F(X, \dot{q}) = \mathcal{F}(\mathcal{X}^{(0)}, \mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \xi^{(0)}, \xi^{(1)}, \xi^{(2)}),$$

for some smooth multilinear function $\mathcal{F} : \mathfrak{g}^6 \to \mathfrak{g}$.

Finally, we must translate $\nabla_X \text{grad} V(g)$ to $\mathfrak{g}$. To that end, let α be a variation of g such that $\partial_s \alpha|_{s=0} = X$. Moreover, let $\{A_i\}$ be a basis for $\mathfrak{g}$ and write $g_0 \text{grad}_1 V_{\text{ext}}(g, g_0) = V^i(g, g_0) g A_i$ for $V^i : G \times G \to \mathbb{R}$ for $i = 1, \ldots, \dim(G)$, we have

$$\begin{aligned}\nabla_X \text{grad} V(g) &= D_s \text{grad} V(\alpha)\big|_{s=0} \\ &= D_s \text{grad}_1 V_{\text{ext}}(\alpha, g_0)\big|_{s=0} \\ &= D_s \alpha \left(V^i(\alpha, g_0) A_i\right)\big|_{s=0} \\ &= g\left(\frac{\partial}{\partial s} V^i(\alpha, g_0)\Big|_{s=0} A_i + \nabla^{\mathfrak{g}}_{\mathcal{X}} g^{-1} \text{grad}_1 V_{\text{ext}}(g, g_0)\right).\end{aligned}$$

Observe that, due to the symmetry of the extended potential,

$$\langle \text{grad}_1 V_{\text{ext}}(g, g_0), X\rangle = \frac{\partial}{\partial s}\Big|_{s=0} V_{\text{ext}}(\alpha, g_0) = \frac{\partial}{\partial s}\Big|_{s=0} V_{\text{ext}}(g^{-1}\alpha, h) = \langle g \text{grad}_1 V_{\text{ext}}(e, h), X\rangle,$$

so that $g^{-1}\text{grad}_1 V_{\text{ext}}(g, g_0) = \text{grad}_1 V_{\text{ext}}(e, h)$. Moreover, this implies that $V^i(\alpha, g_0) = V^i(g^{-1}\alpha, h)$, so that $\frac{\partial}{\partial s} V^i(\alpha(r), g_0)\Big|_{s=0} A_i = \frac{\partial}{\partial s} V^i(g^{-1}\alpha(r), h^{-1})\Big|_{s=0} A_i = d_1 V^i_{(e,h)}(\mathcal{X}) A_i$. Furthermore, this expression

is independent of the chosen basis for $\mathfrak{g}$, which allows to define a linear operator $D_{\mathcal{X}} : \mathfrak{g} \to \mathfrak{g}$ such that $D_{\mathcal{X}}\text{grad}_1 V_{\text{ext}}(e,h) = d_1 V^i_{(e,h)}(\mathcal{X})A_i$ when written with respect to any basis $\{A_i\}$ for $\mathfrak{g}$. In all, we see that

$$\nabla_X \text{grad} V(g) = g(D_{\mathcal{X}^{(0)}} + \nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}})\text{grad}_1 V_{\text{ext}}(e,h). \tag{7.33}$$

Hence, we have proven the following proposition:

Proposition 25. *Suppose that G satisfies assumption **G1**, and let $g \in \Omega$ solve* (5.2)*. Define $\xi^{(i+1)} = \dot{\xi}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(i)}$ for $i = 0,1,2$, with $\xi^{(0)} := g^{-1}\dot{g}$, and let $h := g^{-1}g_0$ for some $g_0 \in G$. Then X is a bi-Jacobi field along g if and only if $\mathcal{X}^{(0)} := g^{-1}X$ solves:*

$$\mathcal{X}^{(i+1)} = \dot{\mathcal{X}}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(i)}, \qquad \text{for } i = 0,1,2, \tag{7.34}$$

$$\dot{\mathcal{X}}^{(3)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(3)} + \mathcal{F}(\mathcal{X}^{(0)}, \mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \xi^{(0)}, \xi^{(1)}, \xi^{(2)}) + \left(D_{\mathcal{X}^{(0)}} + \nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}}\right)\text{grad}_1 V_{\text{ext}}(e,h) = 0, \tag{7.35}$$

We call a smooth solution $\mathcal{X}^{(0)}$ to (7.34)-(7.35) *a reduced bi-Jacobi field.*

Note that, contrary to the case of modified Riemannian cubics, where the reduction process reduces the order of the governing ODE by 1 (as seen in Proposition 22), the resulting reduced equations in Proposition 25 are of the same order as the original equation describing bi-Jacobi fields. Ultimately, equations (7.34)-(7.35) are simply a translation of (5.7) to the Lie algebra $\mathfrak{g}$. Despite not reducing the order, there are numerous advantage to this. As we will see in Proposition 26, the vector space structure of $\mathfrak{g}$ allows us to express the sufficient conditions for optimality (as outlined in Corollary 3) succinctly in terms of the determinant of a matrix which depends only on the solutions to a series of initial value problems. Moreover, in many applications where G admits a bi-invariant metric, the curvature tensor R (and thus $\mathcal{F}$) may be calculated explicitly on $\mathfrak{g}$ (see Lemma 3.3.7 and Example 11), in addition to many obstacle avoidance artificial potentials, which greatly simplifies the solution of the initial value problems.

Observe that left-translation provides an isomorphism $T_g\Omega \cong \mathring{H}^2([a,b],\mathfrak{g})$, where $\mathring{H}^2([a,b],\mathfrak{g})$ denotes the space of Sobolev class H^2 curves $\eta : [a,b] \to \mathfrak{g}$ such that

$$\eta(a) = \eta(b) = 0, \quad \dot{\eta}(a) = -\nabla^{\mathfrak{g}}_{\xi^{(0)}(a)}\eta(a), \quad \dot{\eta}(b) = -\nabla^{\mathfrak{g}}_{\xi^{(0)}(b)}\eta(b),$$

where $\xi := g^{-1}\dot{g}$. Hence, we are able to reinterpret the index form (5.5) along a modified Riemannian cubic $g \in \Omega$ as the bilinear form $\mathcal{I} : \mathring{H}^2([a,b],\mathfrak{g}) \times \mathring{H}^2([a,b],\mathfrak{g}) \to \mathbb{R}$ defined by:

$$\mathcal{I}(\mathcal{X}^{(0)},\mathcal{Y}^{(0)}) = \int_a^b \left[\left\langle \mathcal{X}^{(2)},\mathcal{Y}^{(2)}\right\rangle + \left\langle \mathcal{Y}^{(0)},\ \mathcal{F}(\mathcal{X},\xi) + \left(D_{\mathcal{X}^{(0)}} + \nabla^{\mathfrak{g}}_{\mathcal{X}^{(0)}}\right)\operatorname{grad}_1 V_{\text{ext}}(e,g^{-1}g_0)\right\rangle\right] dt, \tag{7.36}$$

where we have used the notation $\xi = (\xi^{(1)},\xi^{(2)},\xi^{(3)})$ and $\mathcal{X} = (\mathcal{X}^{(0)},\mathcal{X}^{(1)},\mathcal{X}^{(2)})$, and recursively defined $\xi^{(i+1)} = \dot{\xi}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\xi^{(i)}$ and $\mathcal{X}^{(i+1)} = \dot{\mathcal{X}}^{(i)} + \nabla^{\mathfrak{g}}_{\xi^{(0)}}\mathcal{X}^{(i)}$ for $i = 0,1,2$, with $\xi^{(0)} := g^{-1}\dot{g}$. We call $\mathcal{I}$ the *reduced index form*. It is clear that $\mathcal{I}$ is equivalent to I, in the sense that for all $X, Y \in T_g\Omega$, we have that $\mathcal{X} = g^{-1}X$ and $\mathcal{Y} = g^{-1}Y$ satisfy $I(X,Y) = \mathcal{I}(\mathcal{X},\mathcal{X})$, and vice versa. In particular, we have $\ker(\mathcal{I}) = \{g^{-1}X \in \mathring{H}^2([a,b],\mathfrak{g}) \mid X \in T_g\Omega \text{ is a bi-Jacobi field }\}$. From Proposition 25, it follows that the kernel of $\mathcal{I}$ is precisely the set of reduced bi-Jacobi fields. Moreover, $t = t_0$ and $t = t_1$ are biconjugate along g if and only if there exists a reduced bi-Jacobi field satisfying $\mathcal{X}^{(0)}(t_0) = \mathcal{X}^{(0)}(t_1) = \mathcal{X}^{(1)}(t_0) = \mathcal{X}^{(1)}(t_1) = 0$. This leads to the following proposition:

Proposition 26. *Let $\{A_i\}$ be a basis for $\mathfrak{g}$, and suppose that $\mathcal{X}_i^{(0)}$ is a reduced bi-Jacobi fields satisfying the initial conditions*

$$\begin{aligned}
&\mathcal{X}_i^{(0)}(a) = 0, \quad \mathcal{X}_i^{(1)}(a) = 0, \quad \mathcal{X}_i^{(2)}(a) = A_i, \quad \mathcal{X}_i^{(3)}(a) = 0, && \text{for } i = 1,\dots,n,\\
&\mathcal{X}_i^{(0)}(a) = 0, \quad \mathcal{X}_i^{(1)}(a) = 0, \quad \mathcal{X}_i^{(2)}(a) = 0, \quad \mathcal{X}_i^{(3)}(a) = A_{i-n}, && \text{for } i = n+1,\dots,2n.
\end{aligned}$$

Let $\alpha_i^k, \beta_i^k \in \mathbb{R}$ be such that $\mathcal{X}_i^{(0)}(t) = \alpha_i^k(t)A_k$ and $\mathcal{X}_i^{(1)}(t) = \beta_i^k(t)A_k$ for $i = 1,\dots,2n$ and define $A(t) = \begin{bmatrix} \alpha_1^1 & \dots & \alpha_1^n & \beta_1^1 & \dots & \beta_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{2n}^1 & \dots & \alpha_{2n}^n & \beta_{2n}^1 & \dots & \beta_{2n}^n \end{bmatrix}$. Then g is an Ω-local minimizer of J if and only if $\det(A(t)) \neq 0$ for all $t \in (a,b]$.

Proof. Let $X_i := g\mathcal{X}_i^{(0)}$ for $i = 1,\dots,2n$. From Proposition 25, it is clear that each X_i is a Bi-Jacobi field. Moreover, since $\{A_i\}$ is a basis for $\mathfrak{g}$, $\{X_i\}_{i=1}^{2n}$ is a basis for the space $J_g(0)$ of Bi-Jacobi fields along g which vanish at $t = a$ along with their first covariant derivatives. Hence, for all $Z \in J_g(0)$, there exists $2n$

constants $a^i \in \mathbb{R}$ such that

$$Z = \sum_{i=1}^{2n} a^i X_i = g \sum_{i=1}^{2n} a^i \mathcal{X}_i^{(0)},$$
$$D_t Z = \sum_{i=1}^{2n} a^i D_t X_i = g \sum_{i=1}^{2n} a^i \mathcal{X}_i^{(1)},$$

for all $t \in [a, b]$. In particular, this implies that $(Z, D_t Z) = g \sum_{i=1}^{2n} a^i (\mathcal{X}_i^{(0)}, \mathcal{X}_i^{(1)})$, where we define the left-action $G \times (\mathfrak{g} \times \mathfrak{g}) \to \mathfrak{g} \times \mathfrak{g}$ by $(g, (\xi, \eta)) \mapsto (L_{g^*}\xi, L_{g^*}\eta)$.

Let $t_0 \in (a, b]$ and suppose that g is an Ω-local minimizer of J. Then the only bi-Jacobi field $Z \in J_g(0)$ such that $Z(t_0) = D_t Z(t_0) = 0$ is the zero vector field $Z \equiv 0$ by Corollary 3. This implies that $\sum_{i=1}^{2n} a^i (\mathcal{X}_i^{(0)}(t_0), \mathcal{X}_i^{(1)}(t_0)) = 0$ if and only if $a^i = 0$ for $i = 1, \ldots, 2n$. It follows immediately that $\left\{ (\mathcal{X}_i^{(0)}(t_0), \mathcal{X}_i^{(1)}(t_0)) \right\}_{i=1}^{2n}$ is a basis for $\mathfrak{g} \times \mathfrak{g}$, which implies that $\det(A(t_0)) \neq 0$. Since this holds for all $t_0 \in (a, b]$, the result holds.

Now suppose that g is not an Ω-local minimizer. Then there is a time $t = t_0 \in (a, b]$ which is biconjugate to $t = a$ by Proposition 12. Hence, by definition, there exists a non-trivial bi-Jacobi field $Z \in J_g(0)$ such that $Z(t_0) = D_t Z(t_0) = 0$, and so there is a non-trivial solution to $\sum_{i=1}^{2n} a^i (\mathcal{X}_i^{(0)}(t_0), \mathcal{X}_i^{(1)}(t_0)) = 0$. In particular, the vectors $(\mathcal{X}_i^{(0)}(t_0), \mathcal{X}_i^{(1)}(t_0))$ are linearly dependent, so that $\det(A(t_0)) = 0$. □

Remark 12. As we did in Section 7.1.2 for the reduction of necessary conditions, we too may consider Proposition 26 and the corresponding equations for the reduced bi-Jacobi fields (7.34)-(7.35) in the case that G satisfies conditions **G2** or **G3**, as well as the artificial potential design for the obstacle avoidance task in both cases under assumption **DIST**. These results follow from exactly the same reasoning as used in Section 7.1.2 (namely by applying Lemma (3.3.7) to each term in the reduced Bi-Jacobi equations), and so we do not repeat the analysis here.

7.2 Reduction by Symmetry on Riemannian Homogeneous Spaces

In this section, we reduce the necessary conditions (5.2) by symmetry in the case that $Q = H$ is a Riemannian homogeneous space of some Lie group equipped with a left-invariant Riemannian metric,

under the assumption that the lift of the artificial potential to G admits a partial symmetry. In particular, we will find some set of equations on G whose solutions project to modified Riemannian cubics on H, and subsequently left-translate these equations to some system of equations on the Lie algebra $\mathfrak{g}$ (together with a reconstruction equation). As in the previous section, we will additionally consider the special case that G admits a bi-invariant metric, and show how the gradient vector field of the artificial potential can be calculated explicitly in the obstacle avoidance problem (as discussed in Chapter 6). Moreover, the case that H is a Riemannian symmetric space of G (and the subsequent analysis of the artificial potential) will be considered. Reduction by symmetry on the unit sphere S^2 is considered as an example.

7.2.1 Reduction of Necessary Conditions

Suppose that H is a homogeneous space of some Lie group G. If H is equipped with a Riemannian metric, then it is a Riemannian manifold in its own right. Hence, we may consider the functional $J : \Omega \to \mathbb{R}$, as defined in (5.1), and the necessary conditions for optimality (5.2) derived in Proposition 7 will hold. As in Section 7.1, however, the goal of this section is to reduce these equations by symmetry. In this case, there is no explicit Lie group symmetry on H. However, we may nonetheless take advantage of the Lie group symmetry on G provided that $\pi : G \to H$ is a Riemannian submersion (that is, H is a Riemannian homogeneous space of G). In this case, Lemmas 4.1.2 and 4.1.4 apply. With that in mind, the assumptions that we may employ throughout the section are as follows.

> **H1** (respectively **H2**): $H := G/K$ is a Riemannian homogeneous space, where G satisfies assumption **G1** (respectively **G2**).
>
> **H3**: $H := G/K$ is a Riemannian homogeneous space with respect to the left-invariant metric $\langle \cdot, \cdot \rangle_G$ on G. We denote the Levi-Civita connection on G with respect to $\langle \cdot, \cdot \rangle_G$ by $\tilde{\nabla}$, and the corresponding Riemannian $\mathfrak{g}$-connection and curvature tensor are denoted by $\tilde{\nabla}^{\mathfrak{g}}$ and $\tilde{R}$, respectively. G also admits a bi-invariant metric, denoted by $\langle \cdot, \cdot \rangle_G^{\text{Bi}}$, and we let $\beta : \mathfrak{g} \to \mathfrak{g}$ be the linear endomorphism such that $\langle \xi, \eta \rangle_G^{\text{Bi}} = \langle \beta(\eta), \eta \rangle_G$ for all $\xi, \eta \in \mathfrak{g}$.
>
> **S1** (respectively **S2**, **S3**): $H := G/K$ is a Riemannian symmetric space satisfying assumption **H1** (respectively **H2**, **H3**).

For the remainder of the section, we assume that assumption **H1** is satisfied. In [114], the necessary conditions (5.2) on H were horizontally lifted to a set of equations on G in the case that $g = \tilde{q}$ and $V \equiv 0$.

However, these equations are not derived from a variational principle, and as such are more limited in analysis. Moreover, they do not lend themselves to the reverse (and often more practical) problem of finding curves on G which project to modified Riemannian cubics on H under $\pi : G \to H$. To that end, we now seek a variational principle defined on G which is equivalent to $\delta J[q] = 0$. The methodology employed here will in many ways mirror that used in Section 4.1.1.

Denote the admissible path space on H by Ω (as defined in (2.18)) and the admissible path space on G by $\tilde{\Omega}$. We suppose that the boundary conditions on $\tilde{\Omega}$ are chosen so that $g_a \in \pi^{-1}(\{q_a\})$ and $g_b \in \pi^{-1}(\{q_b\})$. Consider some $q \in \Omega$ and let $g \in \tilde{\Omega}$ be any curve satisfying $\pi \circ g = q$. Then, it follows that $\|D_t(\pi_* \dot{g})\|_H = \|D_t \dot{q}\|_H$. This motivates the functional $\tilde{J} : \tilde{\Omega} \to \mathbb{R}$ defined by

$$\tilde{J}[g] = \int_a^b \left(\frac{1}{2} \|D_t(\pi_* \dot{g})\|_H + \tilde{V}(g(t)) \right) dt, \tag{7.37}$$

where $\tilde{V} : G \to \mathbb{R}$ is some smooth and non-negative artificial potential on G. Following the same argument presented in Proposition 3, we have the following proposition.

Proposition 27. *Suppose that H satisfies assumption **H1**. Let $q \in \Omega$ and $g \in \tilde{\Omega}$ satisfy $\pi \circ g = q$ on $[a, b]$. If $\tilde{V} = V \circ \pi$, then q solves $\delta J[q] = 0$ if and only if g solves $\delta \tilde{J}[g] = 0$.*

While Proposition 27 demonstrates that $\delta J = 0 \iff \delta \tilde{J} = 0$ as variational principles, deriving the necessary conditions for optimality that correspond to $\delta \tilde{J} = 0$ is complicated. First, in order to eventually apply reduction by symmetry, we will need to express the integrand of $\tilde{J}$ in terms of the norm $\| \cdot \|_G$ on G. This is straight-forward, as $\|D_t(\pi_* \dot{g})\|_H = \|\widetilde{D_t(\pi_* \dot{g})}\|_G$. However, unless $g \in \tilde{\Omega}$ is a basic curve, we have no clear way to express this quantity in terms of the Levi-Civita connection $\tilde{D}_t$ on G (or equivalently, the horizontal connection $\tilde{D}_t^{\mathcal{H}}$ defined by (4.4)). Even in the case that we assume g to be basic (so that it is the horizontal lift of some $q \in \Omega$), the variations taken along g need not be. Hence, in order to derive necessary conditions for optimality, we will need to consider a *constrained* variational principle which aligns with $\delta \tilde{J} = 0$ in the case that g is basic. Consider $\tilde{J}^{\mathcal{H}} : \tilde{\Omega} \to \mathbb{R}$ defined by

$$\tilde{J}^{\mathcal{H}}[g] = \int_a^b \left(\frac{1}{2} \|\tilde{D}_t^{\mathcal{H}} \dot{g}\|_G^2 + \tilde{V}(g) \right) dt, \tag{7.38}$$

where $\tilde{V} : G \to \mathbb{R}$ is smooth and non-negative. It is easily seen by Lemma 4.1.5 that $\tilde{J}^{\mathcal{H}}[g] = \tilde{J}[g]$ for all basic curves $g \in \Omega$ if we take $\tilde{V} = V \circ \pi$. Moreover, the integrand of $\tilde{J}^{\mathcal{H}}$ is completely analogous to that of the functional J, with the horizontal connection taking place of the Levi-Civita connection. We now would like to consider the variational principle $\delta\tilde{J}^{\mathcal{H}} = 0$ along all *basic* variations (see Definition 4.1.10). For simplicity, we denote this variational principle by $\delta\tilde{J}^{\mathcal{H}}[g] = 0$. This leads to the following proposition.

Proposition 28. *Suppose that $g \in \tilde{\Omega}$ is basic and solves $\delta\tilde{J}[g] = 0$. Then $\delta_{\mathcal{H}}\tilde{J}^{\mathcal{H}}[g] = 0$.*

The remainder of the section is dedicated to deriving necessary conditions corresponding to $\delta_{\mathcal{H}}\tilde{J}^{\mathcal{H}}[g] = 0$, and then using the symmetry of G and partial symmetry of V to reduce them to some set of equations on $\mathfrak{h}$.

Proposition 29. *Suppose that H satisfies assumption **H1**. Then, a basic curve $g \in \tilde{\Omega}$ solves $\delta_{\mathcal{H}}\tilde{J}^{\mathcal{H}}[g] = 0$ if and only if it is smooth and satisfies*

$$\left(\tilde{D}_t^{\mathcal{H}}\right)^3 \dot{g} + \tilde{Q}\left(\tilde{D}_t^{\mathcal{H}}\dot{g}, \dot{g}\right)\dot{g} + \mathcal{H}(\text{grad}\tilde{V}(g)) = 0, \tag{7.39}$$

on the full interval $[a, b]$.

Proof. Consider a basic variation g_s of g with variational vector field δg, and let $T = \frac{\partial g_s}{\partial t}$ and $S = \frac{\partial g_s}{\partial s}$. Then, from Lemmas 4.1.5, 4.1.11, and the symmetries of the curvature tensor 2.10-2.14,

$$\begin{aligned}
0 = \delta\tilde{J}^{\mathcal{H}}[g] &= \frac{d}{ds}\Big|_{s=0}\int_a^b \left(\frac{1}{2}\left\|\tilde{D}_t^{\mathcal{H}}\dot{g}_s\right\|_G^2 + \tilde{V}(g_s)\right)dt = \int_a^b \left(\left\langle \tilde{D}_s^{\mathcal{H}}\tilde{D}_t^{\mathcal{H}}T, \tilde{D}_t^{\mathcal{H}}T\right\rangle_G + \left\langle \text{grad}\tilde{V}(g_s), S\right\rangle_G\right)\Big|_{s=0}dt \\
&= \int_a^b \left(\left\langle \tilde{D}_t^{\mathcal{H}}\tilde{D}_s^{\mathcal{H}}T, \tilde{D}_t^{\mathcal{H}}T\right\rangle_G + \left\langle \tilde{Q}(S,T)T, \tilde{D}_t^{\mathcal{H}}T\right\rangle_G + \left\langle \text{grad}\tilde{V}(g_s), S\right\rangle_G\right)\Big|_{s=0}dt \\
&= \int_a^b \left(\left\langle (\tilde{D}_t^{\mathcal{H}})^2 S, \tilde{D}_t^{\mathcal{H}}T\right\rangle_G + \left\langle \tilde{Q}(\tilde{D}_t^{\mathcal{H}}T, T)T + \text{grad}\tilde{V}(g_s), S\right\rangle_G\right)\Big|_{s=0}dt \\
&= \int_a^b \left(\left\langle (\tilde{D}_t^{\mathcal{H}})^2\delta g, \tilde{D}_t^{\mathcal{H}}\dot{g}\right\rangle_G + \left\langle \tilde{Q}(\tilde{D}_t^{\mathcal{H}}\dot{g}, \dot{g})\dot{g} + \mathcal{H}(\text{grad}\tilde{V}(g)), \delta g\right\rangle_G\right)dt
\end{aligned}$$

where in the last line, we have used the fact that $\langle X, Y\rangle_G = \langle \mathcal{H}(X), Y\rangle_G$ for all $X \in \Gamma(TG), Y \in \Gamma(HG)$. The smoothness of g can be proven using a bootstrap argument (as was done, for example, in the proof of Proposition 7). As a result, we can integrate by parts and use the fact that $\delta g(a) = \delta g(b) = \tilde{D}_t^{\mathcal{H}}\delta g(a) = \tilde{D}_t^{\mathcal{H}}\delta g(b) = 0$ to see that

$$\int_a^b \left\langle (\tilde{D}_t^{\mathcal{H}})^3\dot{g} + \tilde{Q}(\tilde{D}_t^{\mathcal{H}}\dot{g}, \dot{g})\dot{g} + \mathcal{H}(\text{grad}\tilde{V}(g)), \delta g\right\rangle_G dt = 0.$$

As a consequence of Lemma 4.0.2 and the definitions of $\tilde{D}^{\mathcal{H}}$ and $\tilde{Q}$, the left-hand side of (7.39) is a basic vector field, so that we may consider $\delta g = f(t)\left[\left(\tilde{D}_t^{\mathcal{H}}\right)^3 \dot{g} + \tilde{Q}(\tilde{D}_t^{\mathcal{H}}\dot{g},\dot{g})\dot{g} + \mathcal{H}(\text{grad}\tilde{V}(g))\right]$, where $f : [a,b] \to \mathbb{R}$ is smooth and satisfies $f(t) > 0$ for $t \in (a,b)$ and $f(a) = f(b) = \dot{f}(a) = \dot{f}(b) = 0$. Then,

$$\int_a^b f(t)\left\| \left(\tilde{D}_t^{\mathcal{H}}\right)^3 \dot{g} + \tilde{Q}(\tilde{D}_t^{\mathcal{H}}\dot{g},\dot{g})\dot{g} + \mathcal{H}(\text{grad}\tilde{V}(g))\right\|_G^2 dt = 0,$$

from which equation (7.39) follows immediately. □

Note that, since $g \in \tilde{\Omega}^{\mathcal{H}}$, it is the horizontal lift at g_a of some $q \in \Omega$. It is clear from their respective definitions that $\left(\tilde{D}_t^{\mathcal{H}}\right)^3 \dot{g} = \widetilde{D_t^3 \dot{q}}$ and $\tilde{Q}(\tilde{D}_t^{\mathcal{H}}\dot{g},\dot{g})\dot{g} = \widetilde{R(D_t\dot{q},\dot{q})\dot{q}}$. Moreover, from Lemma 4.0.2, there exists some vector field $X \in \Gamma(q)$ whose tangent lift satisfies $\tilde{X}(t) = \mathcal{H}(\text{grad}\tilde{V}(g(t)))$ for all $t \in [a,b]$. Observe that X is locally extendable. That is, for any $t_0 \in [a,b]$, there exists some subinterval $(a^*,b^*) \subset [a,b]$ containing t_0, a neighborhood $U \subset H$ containing $q((a^*,b^*))$, and a smooth vector field $Y : U \to TH$ such that $Y(q(t)) = X(t)$ for all $t \in (a^*,b^*)$. Consequently, the horizontal lift of Y satisfies $Y(g(t)) = \mathcal{H}(\text{grad}\tilde{V}(g(t))$ for all $t \in (a^*,b^*)$. On the other hand, consider the 1-form $Y^\flat$. If H is simply connected (this follows for instance if G is simply connected and K is connected), then there must exist a scalar field $V : U \to \mathbb{R}$ such that $dV = Y^\flat$. Hence, $\text{grad}V = Y$, and so $\widetilde{\text{grad}V}(g(t)) = \mathcal{H}(\tilde{V}(g(t)))$ for all $t \in (a^*,b^*)$. Observe that these vector fields need not agree away from the curve g, however we obtain immediately that g is locally the horizontal lift of a modified Riemannian cubic with respect to V. This, together with Proposition 27, leads to the following corollary to Proposition 29.

Corollary 12. *Suppose that $g : [a,b] \to G$ is a basic curve satisfying equation* (7.39) *for some smooth $\tilde{V} : G \to \mathbb{R}$. Then for each $t_0 \in [a,b]$, there exists some $(a^*,b^*) \subset [a,b]$ containing t_0, a neighborhood $U \subset H$ containing $q((a^*,b^*))$, and a scalar field $V : U \to \mathbb{R}$ such that the curve $q|_{(a^*,b^*)} := \pi \circ g|_{(a^*,b^*)}$ is a modified Riemannian cubic with respect to V.*

If we take $\tilde{V} := V \circ \pi$, the above corollary is strengthened significantly. Namely, it follows that $\text{grad}\tilde{V}(g)$ must be horizontal, since $\left\langle \text{grad}\tilde{V}(h), U\right\rangle_G = d\tilde{V}(U) = dV \circ \pi_*(U) = 0$ for all $h \in G, U \in V_h$. Hence $\mathcal{H}(\text{grad}\tilde{V}) = \text{grad}\tilde{V}$. Moreover, if we consider any $\tilde{X} \in \text{Hor}_h$, then:

$$\left\langle \text{grad}\tilde{V}(h), \tilde{X}\right\rangle_G = d\tilde{V}(\tilde{X}) = dV(X) = \langle \text{grad}V(\pi(h)), X\rangle_H = \left\langle \widetilde{\text{grad}V(\pi(h))}, \tilde{X}\right\rangle_G,$$

where $X = \pi_* \tilde{X}$. Therefore, grad$\tilde{V}$ is precisely the horizontal lift of grad$V$. Moreover, we see that equation (7.39) is the horizontal lift of equation (5.2) on H. From Lemma 4.0.2, it follows that the projection of any basic curve satisfying equation (7.39) onto H is a modified Riemannian cubic. Together with Propositions 28 and 27, we have the following equivalency:

Proposition 30. *Suppose that H satisfies assumption **H1**, and $\tilde{V} = V \circ \pi$. If $q \in \Omega$ is a modified Riemannian cubic, then any horizontal lift g of q satisfies equation* (7.39)*. Moreover, if g is a basic curve and solves* (7.39)*, then its projection $q := \pi \circ g$ is a modified Riemannian cubic.*

We now seek to derive the Euler-Poincaré equations corresponding to (7.39). For this, we will make use of the Riemannian $\mathfrak{h}$-connection, as defined by equation (4.8). Moreover, following the nomenclature of Section 7.1, we let the extended artificial potential on G be defined by $\tilde{V}_{\text{ext}}(g, h) := V(\pi(g_0 h^{-1} g))$ for all $g, h \in G$. Following the same strategy employed in the proof of Proposition 22, and utilizing equation (4.9), Lemma 4.1.6, and Proposition 29, we obtain the following result:

Proposition 31. *Suppose that H satisfies assumption **H1**, and $\tilde{V} = V \circ \pi$. Then $g \in \tilde{\Omega}$ satisfies equation* (7.39) *if and only if $\xi^{(0)} := g^{-1}\dot{g}$ and $h := g^{-1} g_0$ solve*

$$\dot{h} = -\xi^{(0)} h, \tag{7.40}$$

$$\dot{\xi}^{(i)} = \xi^{(i+1)} - \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}} \xi^{(i)}, \qquad i = 0, 1 \tag{7.41}$$

$$\dot{\xi}^{(2)} + \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}} \xi^{(2)} + \tilde{Q}\big(\xi^{(1)},\, \xi^{(0)}\big)\xi^{(0)} = -\text{grad}_1 \tilde{V}_{\text{ext}}(e, h), \tag{7.42}$$

on $[a, b]$, subject to the boundary conditions $\xi^{(0)}(a) = g_a^{-1} v_a,\ \xi^{(0)}(b) = g_b^{-1} v_b$, and $h(a) = g_a^{-1} g_0,\ h(b) = g_b^{-1} g_0$.

From Propositions 27, 28, and 31, we know that the horizontal lift g of any modified Riemannian cubic q on H satisfies the reduced equations (7.40)-(7.42). In application, we are interested in the reverse problem. That is, we wish to find modified Riemannian cubics on H by solving some equations on the Lie algebra $\mathfrak{g}$ on G. Proposition 30 tells us that any solution g to (7.39) that is basic projects to a modified Riemannian cubic in H. In the context of Proposition 31, this means that $\xi^{(0)}(t) \in \mathfrak{h}$ for all $t \in [a, b]$. The following proposition gives the criteria for this to occur.

Proposition 32. *Suppose that H satisfies assumption **H1**, and $\xi^{(i)} : [a,b] \to \mathfrak{g}$ for $i = 1,2,3$ and $h : [a,b] \to G$ solve* (7.40)-(7.42)*. Then, $g := g_0 h^{-1}$ is basic if and only if $\xi^{(0)}(a),\ \xi^{(1)}(a),\ \xi^{(2)}(a) \in \mathfrak{h}$.*

Proof. Observe that $g = g_0 h^{-1}$ is equivalent to $\dot{g} = g\xi^{(0)}$, so that g is basic if and only if $\xi^{(0)}(t) \in \mathfrak{h}$ for all $t \in [a,b]$. Now, the system (7.40)-(7.42) may be written in the form

$$\begin{aligned}
\dot{\xi}^{(0)} &= X^{(0)}(\xi^{(0)}, \xi^{(1)}) \\
\dot{\xi}^{(1)} &= X^{(1)}(\xi^{(1)}, \xi^{(2)}) \\
\dot{\xi}^{(2)} &= X^{(2)}(\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, h) \\
\dot{h} &= -\xi h,
\end{aligned}$$

where $X^{(i)}$ is smooth for each $i = 0,1,2$. Moreover, it is clear from (7.41) that $X^{(0)}(\xi^{(0)}, \xi^{(1)}) \in \mathfrak{h}$ if and only if $\xi^{(1)} \in \mathfrak{h}$. Similarly, $X^{(1)}(\xi^{(1)}, \xi^{(2)}) \in \mathfrak{h}$ if and only if $\xi^{(2)} \in \mathfrak{h}$. Finally, by the definition of $\tilde{Q}$ (4.24) and the fact that $\tilde{V} = V \circ \pi$, we have that $X^{(2)}(\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, h) \in \mathfrak{h}$ independently of its arguments. From the decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, it is clear that we may write $X^{(2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & X^{(2)}_{m+1} & \cdots & X^{(2)}_n \end{bmatrix}^T$, where each $X^{(2)}_i : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times G \to \mathfrak{g}$ is smooth, $n = \dim(G)$, and $m = \dim(H)$. Hence, we find that $\xi^{(2)} = \begin{bmatrix} \xi^{(2)}_1 & \cdots & \xi^{(2)}_n \end{bmatrix}^T$ satisfies $\xi^{(2)}_j(t) = \xi^{(2)}_j(a)$ for $j = m+1, \ldots, n$ and for all $t \in [a,b]$. It follows immediately that $\xi^{(2)}(t) \in \mathfrak{h}$ for all $t \in [a,b]$ if and only if $\xi^{(2)}(a) \in \mathfrak{h}$—which in turn implies that $X^{(1)} \in h$. Repeating this argument inductively, the conclusion follows. □

There is a one-to-one correspondence between the modified Riemannian cubics on H and the basic solutions to (7.39) on G, with the artificial potential $\tilde{V} = V \circ \pi$. Note, however, that there is no reason to suspect that basic solutions exist on $\tilde{\Omega}$. This is because, if q is a modified Riemannian cubic on H and $\tilde{q}$ is its horizontal lift at the initial point $g_a \in \pi^{-1}(\{q(a)\})$ specified by $\tilde{\Omega}$, then $\tilde{q}(b)$ need only be some element of $\pi^{-1}(\{q_b\})$ and not necessarily the point $g_b \in \pi^{-1}(\{q_b\})$ specified by $\tilde{\Omega}$. However, in application, this is of no consequence to us.

In general, we will be supplied with some boundary conditions $(q_a, w_a), (q_b, w_b) \in TH$, and will be tasked with finding a modified Riemannian cubic satisfying them. Due to Propositions 30, 31, and 32, this

can be accomplished as follows: Fix some $g_a \in \pi^{-1}(\{q_a\})$ and let $v_a := \pi_*\big|^{-1}_{T_{g_a}G}(w_b)$. Solve equations (7.40) - (7.42) for $h(a) = g_a^{-1}g_0$, $\xi^{(0)}(a) = g_a^{-1}v_a$, and $\xi^{(1)},\ \xi^{(2)} \in \mathfrak{h}$. Apply a shooting method over $\xi^{(1)},\ \xi^{(2)}$ until a solution satisfying $g_0h(b)^{-1} \in \pi^{-1}(\{q_b\})$ with $\pi_*\xi^{(0)}(b) = w_b$ is found (such a solution exists by Proposition 8). Then, $q := \pi(g_0h^{-1})$ is the desired modified Riemannian cubic.

7.2.2 Reduction on Symmetric Spaces and Homogeneous Spaces with bi-invariant Metrics

Analogously to Section 7.1.2, we are interested in further simplifying equations (7.40)-(7.42) in the case that G is endowed with a bi-invariant metric. We obtain the following result by combining Proposition 31 with Lemma 4.2.1.

Proposition 33. *Suppose that H satisfies assumption **H2**, and $\tilde{V} = V \circ \pi$. Then g satisfies* (7.39) *if and only if $\xi := g^{-1}\dot{g}$ and $h := g_0^{-1}g$ satisfy:*

$$\dot{h} = -\xi h., \tag{7.43}$$

$$\dddot{\xi} + \mathcal{H}([\xi, \ddot{\xi}]) - \frac{1}{4}\mathcal{H}\big([\xi, \mathcal{H}([\dot{\xi}, \xi])]\big) = -\text{grad}_1\tilde{V}_{\text{ext}}(e, h), \tag{7.44}$$

*Similarly, if H satisfies assumption **H3**, then g satisfies* (7.39) *if and only if $\xi^{(0)} := g^{-1}\dot{g}$ and $h := g^{-1}g_0$ solve*

$$\dot{h}(t) = -\xi^{(0)}(t)h(t), \tag{7.45}$$

$$\dot{\xi}^{(i)} = \xi^{(i+1)} - \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}}\xi^{(i)}, \qquad i = 0, 1 \tag{7.46}$$

$$\dot{\xi}^{(2)} + \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}}\xi^{(2)} + \tilde{Q}\big(\xi^{(1)},\ \xi^{(0)}\big)\xi^{(0)} = -\beta(\text{grad}_1^{\text{Bi}}\tilde{V}_{\text{ext}}(e, h)), \tag{7.47}$$

where $\text{grad}_1^{\text{Bi}}\tilde{V}_{\text{ext}}$ *denotes the gradient vector field of* $\tilde{V}_{\text{ext}}$ *with respect to its first argument and* $\langle\cdot,\cdot\rangle_G^{\text{Bi}}$.

Proof. Using Lemma 4.2.1,

$$\begin{aligned}
\xi^{(1)} &= \dot{\xi} \\
\xi^{(2)} &= \dot{\xi}^{(1)} + \frac{1}{2}\mathcal{H}([\xi, \xi^{(1)}]) \\
&= \ddot{\xi} + \frac{1}{2}\mathcal{H}([\xi, \dot{\xi}]) \\
\dot{\xi}^{(2)} &= \dddot{\xi} + \frac{1}{2}\mathcal{H}([\xi, \ddot{\xi}]) \\
\tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi^{(2)} &= \tilde{\nabla}^{\mathfrak{h}}_{\xi}(\ddot{\xi} + \frac{1}{2}\mathcal{H}([\xi, \dot{\xi}])) \\
&= \frac{1}{2}\mathcal{H}\big([\xi, \ddot{\xi}]\big) - \frac{1}{4}\mathcal{H}\big([\xi, \mathcal{H}([\dot{\xi}, \xi])]\big) \\
\tilde{Q}(\dot{\xi}, \xi)\xi &= \frac{1}{4}\mathcal{H}\Big([\xi, [\dot{\xi}, \xi]] + 3[\xi, \mathcal{V}\big([\dot{\xi}, \xi]\big)\Big) \\
&= \frac{1}{4}\mathcal{H}\Big([\xi, \mathcal{H}\big([\dot{\xi}, \xi]\big)] + 4[\xi, \mathcal{V}\big([\dot{\xi}, \xi]\big)\Big)
\end{aligned}$$

Substituting these quantities into equation (7.42) yields (7.44). Equations (7.45)-(7.47) follow directly from Proposition 31, together with the identity $\beta(\text{grad}_1^{\text{Bi}}\tilde{V}_{\text{ext}}(e,h)) = \text{grad}_1\tilde{V}_{\text{ext}}(e,h)$ (as shown in Section 7.1.2).

□

Another special class of spaces that appears frequently in applications is Riemannian symmetric spaces (as discussed in Section 4.2. As an immediate consequence of Lemma 4.2.2 and Proposition 31, we obtain the following result:

Proposition 34. *Suppose that H satisfies assumption **S1** or **S2**, and $\tilde{V} = V \circ \pi$. Then g satisfies equation* (7.39) *if and only if $\xi := g^{-1}\dot{g}$ and $h := g_0^{-1}g$ satisfy:*

$$\dot{h}(t) = -\xi(t)h(t), \tag{7.48}$$

$$\dddot{\xi} + [\xi, [\dot{\xi}, \xi]] = -\text{grad}_1\tilde{V}_{\text{ext}}(e,h). \tag{7.49}$$

*If instead H satisfies assumption **S3**, then g satisfies equation* (7.39) *if and only if $\xi := g^{-1}\dot{g}$ and $h := g_0^{-1}g$ satisfy:*

$$\dot{h}(t) = -\xi(t)h(t), \tag{7.50}$$

$$\dddot{\xi} + [\xi, [\dot{\xi}, \xi]] = -\beta(\text{grad}_1^{\text{Bi}}\tilde{V}_{\text{ext}}(e,h)). \tag{7.51}$$

We now return to the obstacle avoidance task. Let $q_0 \in H$ be a point-obstacle, and consider the artifical potential $V(q) = f(d_H^2(q, q_0))$, where $d_H^2 : H \to H$ is the square of the Riemannian distance on H corresponding to $\langle \cdot, \cdot \rangle_H$. As in Lemma 7.1.2, the G-invariance of d_H follows immediately from the G-invariance of $\| \cdot \|_H$:

Lemma 7.2.1. *$d(gq, gp) = d(q, p)$ for all $g \in G, q, p \in H$.*

Proof. Since any Riemannian homogeneous space is complete as a Riemannian manifold, there exists a geodesic $\gamma : [0, 1] \to H$ which minimizes the length functional $L(c) = \int_0^1 \|\dot{c}(t)\|_H dt$ among all smooth curves $c : [0, 1] \to H$ satisfying $c(0) = p,\ c(1) = q$. Moreover, we have $d(p, q) = L(\gamma)$. From Lemma 4.1.2, we obtain $d(p, q) = L(\gamma) = L(g\gamma) \geq d(gp, gq)$, since in particular $g\gamma$ is a smooth curve such that $g\gamma(0) = gp,\ g\gamma(1) = gq$. On the other hand, there exists some geodesic γ^* such that $L(\gamma^*) = d(gp, gq)$, and so $d(gp, gq) = L(\gamma^*) = L(g^{-1}\gamma^*) \geq d(p, q)$. It follows that $d(p, q) = d(gp, gq)$. □

Fix some $g_0 \in \pi^{-1}(\{q_0\})$, and consider the artificial potential $\tilde{V} : G \to \mathbb{R}$ given by $\tilde{V}(g) = f(d_H^2(\pi(g), \pi(g_0)))$ for all $g \in G$, from which it is clear that $\tilde{V} = V \circ \pi$. From Lemma 7.2.1, the extended artificial potential is then given by $\tilde{V}_{\text{ext}}(g_1, g_2) = f(d_H^2(\pi(g_1), \pi(g_2)))$ for all $g_1, g_2 \in G$, which clearly satisfied $\tilde{V}_{\text{ext}}(g, g_0) = \tilde{V}(g)$ for all $g \in G$, and is invariant under left-translation by Lemma 7.2.1.

Observe, however, that in order to take advantage of a bi-invariant metric on G to calculate the potential through the Lie exponential map (as we did in Section 7.1.2), we must be able to write $\tilde{V}_{\text{ext}}$ with respect to the Riemannian distance d_G on G. Unfortunately, it is not necessarily the case $d_G(g, g_0) = d_H(\pi(g), \pi(g_0))$. In fact, if we let $c : [0, 1] \to G$ be the length-minimizing geodesic such that $d_G(g, g_0) = L(c)$ (here we assume that $g, g_0 \in G$ are contained in some geodesically convex ball), then $\bar{c} := \pi \circ c$ is a smooth curve connecting $\pi(g)$ and $\pi(g_0)$, and

$$d_G(g, g_0) = \int_0^1 \|\dot{c}\|_G dt \geq \int_0^1 \|\mathcal{H}(\dot{c})\|_G dt = \int_0^1 \|\dot{\bar{c}}\|_H dt \geq d_H(\pi(g), \pi(g_0)).$$

In particular, equality holds if and only if the geodesic c is basic, at which point $\bar{c}$ is also a geodesic by Proposition 5. On the other hand, the horizontal lift of a geodesic on H is in turn a geodesic on G, so that in particular the curve $c^* := \widetilde{\pi \circ c}$ with $c^*(0) = g$ is the length-minimizing geodesic connecting its endpoints.

However, it will not generally be the case that $c^*(1) = g_0$; we need only have $c^*(1) \in \pi^{-1}(\{\pi(g_0)\})$. This leads to the following lemma.

Lemma 7.2.2. *Let $g \in G$. Then, for all h in a geodesically convex neighborhood of g, there exists a unique $h^* \in \pi^{-1}(\{\pi(h)\})$ such that $d_G(g, h^*) = d_H(\pi(g), \pi(h^*))$. Moreover, the function $\theta : G \times G \to G$ defined implicitly by $\theta(g, h) = h^*$ is smooth and left-invariant.*

Proof. We have already seen that such an h^* exists, as we need only consider the end point of the horizontal lift of the length-minimizing geodesic on H connecting $\pi(g)$ and $\pi(h)$ with initial point g. The uniqueness of h^* follows by the uniqueness of geodesics satisfying boundary conditions in a geodesically convex neighborhood.

To see that θ is smooth, first let $\theta_g(\cdot) := \theta(g, \cdot)$ and observe that $d_G(g, \theta_g(h)) = d_H(\pi(g), \pi(h))$ for all h in a geodesically convex neighborhood of G. Moreover, it is clear that we have $\min_{r \in \pi^{-1}(\{\pi(h)\})} d_G(g, r) = d_H(\pi(g), \pi(h)) = d_G(g, h^*)$. In particular, by uniqueness of the length-minimizing geodesic, we find that $\theta_g(h) = \operatorname{arg\,min}_{r \in \pi^{-1}(\{\pi(h)\})} d_G(g, r)$. Since d_G is smooth and $r \mapsto d_G(g, r)$ has a unique global minimum for each $g \in G$, we find that θ_g is smooth by an application of the implicit function theorem. A similar argument then shows that θ is smooth.

Let $\gamma_1 : [0, 1] \to G$ be the length-minimizing geodesic from $\theta(g, g_0)$ to g_0. Then for any $h \in G$, the curve $h\gamma_1$ satisfies $h\gamma_1(0) = hg$ and $h\gamma_1(1) = h\theta(g, g_0)$, and we have $d(g, \theta(g, g_0)) = d(hg, h\theta(g, g_0))$ by Lemma 7.1.2. Let $\gamma_2 : [0, 1] \to G$ be the length-minimizing geodesic from hg to $\theta(hg, hg_0)$. By definition of θ, we must have $d(hg, \theta(hg, hg_0)) \leq d(hg, h\theta(g, g_0))$, and by another application of Lemma 7.1.2, we find that $d(g, h^{-1}\theta(hg, hg_0)) \leq d(g, \theta(g, g_0))$, which is only possible if in fact they are equal. By the uniqueness of length-minimizing curves, we then find $h\theta(g, g_0) = \theta(hg, hg_0)$. □

We have shown that the previously chosen artificial potential $\tilde{V}$ in fact satisfies

$$\tilde{V}(g) = f(d_H^2(\pi(g), \pi(g_0))) = f(d_G^2(g, \theta(g, g_0))).$$

Due to the left-invariance of θ, we are able to employ the same strategy used in 7.1.2 to express $\operatorname{grad}_1 \tilde{V}_{\text{ext}}$ in terms of the Lie logarithmic map in the case that G possesses a bi-invariant metric. However, we still don't

know how to calculate θ, and so the advantage of this strategy is somewhat superficial. To that end, we now seek to express θ in terms of some more well-known quantities.

Lemma 7.2.3. *Suppose that $g, h \in G$ are contained within a goedesically convex neighborhood on which the exponential map is a diffeomorphism. Then,*

$$\theta(g,h) = \exp_g(\mathcal{H}(\exp_g^{-1}(h))). \tag{7.52}$$

Proof. Since g and h are contained within a geodesically convex neighborhood, it follows that, for $X := \exp_g^{-1}(h)$, the curve $c(t) = \exp_g(tX)$ is the unique geodesic satisfying $c(0) = g$ and $\dot{c}(0) = X$, and we have $c(1) = h$. Let $\tilde{c}$ be the curve defined by $\tilde{c}(t) = \pi \circ \widetilde{\exp_g}(tX)$ with $\tilde{c}(0) = g$. It follows from Proposition 5 is a geodesic, and $\dot{\tilde{c}}(0) = \mathcal{H}(X)$. Moreover, the geodesic c^* defined by $c^*(t) = \exp_g(t\mathcal{H}(X))$ satisfies $c^*(0) = g$ and $\dot{c}^*(0) = \mathcal{H}(X)$. Hence $c^* = \tilde{c}$. It follows that $\pi \circ \exp_g(tX) = \pi \circ \exp_g(t\mathcal{H}(X))$, and so $\exp_g(\mathcal{H}(X)) := \exp_g(\mathcal{H}(\exp_g^{-1}(h))) \in \pi^{-1}(\{\pi(h)\})$. □

Supposing that $h : [a,b] \to G$ remains within a geodesically convex neighborhood of e on which the Riemannian exponential map is a diffeomorphism, we see from Lemma 7.2.3 that $\exp_e^{-1}(\theta(h(t))) = \mathcal{H}(\exp_g^{-1}(h(t))$ for all $t \in [a,b]$. Hence, from Lemma 6.1.2, and equations (7.8) and (2.26), it follows that:

$$\operatorname{grad}_1 \tilde{V}_{\text{ext}}(e, h(t)) = -2f'(\|\mathcal{H}(\exp_g^{-1}(h(t))\|_G^2)\mathcal{H}(\exp_g^{-1}(h(t)). \tag{7.53}$$

Moreover, if G is equipped with a bi-invariant metric, then Theorem 3.3.8 yields:

$$\operatorname{grad}_1 \tilde{V}_{\text{ext}}(e, h(t)) = -2f'(\|\mathcal{H}(\operatorname{Log}(h(t))\|_G^2)\mathcal{H}(\operatorname{Log}(h(t))) \tag{7.54}$$

Finally, in the case that G is equipped with a left-invariant and a bi-invariant metric, and we take d_G to be the Riemannian distance with respect to the bi-invariant metric, we have

$$\operatorname{grad}_1 \tilde{V}_{\text{ext}}(e, h(t))) = -2f'((\|\mathcal{H}_{\text{Bi}}(\operatorname{Log}(h(t)))\|_G^{\text{Bi}})^2)\beta(\mathcal{H}_{\text{Bi}}(\operatorname{Log}(\theta(h(t)))))), \tag{7.55}$$

where the gradient vector field is taken with respect to the left-invariant metric, $\beta : \mathfrak{g} \to \mathfrak{g}$ is the linear endomorphism such that $\langle \xi, \eta \rangle_{\text{Bi}} = \langle \beta(\xi), \eta \rangle$ for all $\xi, \eta \in \mathfrak{g}$, and $\mathcal{H}_{\text{Bi}} : \mathfrak{g} \to \mathfrak{h}_{\text{Bi}}$ is the projection onto $\mathfrak{h}_{\text{Bi}} = \mathfrak{s}^{\perp}$, where the orthogonal complement is taken with respect to the bi-invariant metric. Note that in

general, it is not the case that $\mathfrak{h}_{\text{Bi}} = \mathfrak{h}$, so that the orthogonal projections that they induce are not equivalent maps. It need only be true that $\xi \in \mathfrak{h}_{\text{Bi}}$ if and only if $\beta(\xi) \in \mathfrak{h}$.

Remark 13. Following Corollary 12, we may alternatively consider the potential

$$\tilde{V}(g) = f(d_G^2(g, g_0)),$$

whose left-invariant extension is given by $\tilde{V}_{\text{ext}}(g_1, g_2) = f(d_G^2(g_1, g_2))$. In this case, equations (7.40)-(7.42) take the form:

$$\dot{\xi}^{(i)} = \xi^{(i+1)} - \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}} \xi^{(i)}, \qquad i = 0, 1 \tag{7.56}$$

$$\dot{\xi}^{(2)} + \tilde{\nabla}^{\mathfrak{h}}_{\xi^{(0)}} \xi^{(2)} + \tilde{Q}\big(\xi^{(1)}, \xi^{(0)}\big)\xi^{(0)} + \mathcal{H}(\text{grad}_1 \tilde{V}_{\text{ext}}(e, h)) = 0, \tag{7.57}$$

$$\dot{h} = -\xi^{(0)} h. \tag{7.58}$$

The gradient potential may then be calculated exactly as in (7.11) if G is equipped with a bi-invariant metric. In general, the solutions to (7.56)-(7.58) will only correspond locally to horizontal lifts of modified Riemannian cubics on H. Moreover, the artificial potential V corresponding to these modified Riemannian cubics may vary among the curve segments, and we will not in general know its exact form.

However, for the purpose of the obstacle avoidance application, we only care that $q := g \circ \pi$ is well-behaved in the sense that it is smooth, avoids obstacles, and is sufficiently close to a Riemannian cubic. This, of course, still holds true. Namely, we can design f so that q avoids the given obstacle and approximateas a Riemannian cubic. Moreover, we may guarantee that q avoids the given obstacle by choosing τ, D, N appropriately (in particular, τ will determine how "far" from a Riemannian cubic q is in a global sense). Hence, we see that, morally, q has all the relevant features that we desire from modified Riemannian cubics, despite not globally being one in the strict sense of the definition.

Example 12. Consider the Lie group $G = \text{SO}(3)$ equipped with the bi-invariant metric defined by $\left\langle \hat{\Omega}_1, \hat{\Omega}_2 \right\rangle_{\text{SO}(3)} = \Omega_1^T \Omega_2$ for all $\Omega_1, \Omega_2 \in \mathbb{R}^3$. We consider the unit sphere $S^2 \cong \text{SO}(3)/\text{SO}(2)$, where $q_0 = e_3$ is the obstacle we wish to avoid, and $\text{SO}(2) \cong \text{Stab}(e_3)$. As was shown in Example 8, S^2 becomes a Riemannian symmetric space of $\text{SO}(3)$ when equipped with the Riemannian metric defined by (4.37).

Using the orthonormal basis (4.36) for $\mathfrak{g}$, it is easily seen that

$$\|\mathcal{H}(\hat{\Omega})\|_G = \|\hat{\Omega}e_3\| = \|\Omega \times e_3\| = \|\Omega\||\sin(\theta)| = \|\hat{\Omega}\|_G|\sin(\theta)|,$$

where $\theta = \arccos\left(\frac{\Omega \cdot e_3}{\|\Omega\|}\right)$ is the angle between Ω and e_3. In particular, $\sin(\theta) = \sqrt{1 - \left(\frac{\Omega \cdot e_3}{\|\Omega\|}\right)^2}$. Moreover, we have $\mathcal{H}(\hat{\Omega}) = \widehat{\Omega \times e_3}$. Fix $R_0 \in \mathrm{SO}(3)$ such that $R_0 e_3 = q_0$, and let $\tilde{V}_{\text{ext}}(R_1, R_2) = f(d^2_{\mathrm{SO}(3)}(R_1, R_2))$. From Lemma 3.3.9 and equation (7.54), we have

$$\text{grad}_1 \tilde{V}_{\text{ext}}(e, H) = -\frac{2f'(\phi(H)^2 \sin^2(\theta))\phi(H)}{\sin(\phi(H))}((H - H^T)^\vee \times e_3)^\wedge,$$

where $\phi(H) := \arccos(\frac{1}{2}(\text{tr}(H) - 1))$. From Proposition 34, we have the following result:

Proposition 35. *Suppose that $\Omega \in C^\infty([a, b], \mathbb{R}^3)$ and $H \in C^\infty([a, b], \mathrm{SO}(3))$ solve*

$$\dddot{\Omega} + \Omega \times (\dot{\Omega} \times \Omega) = \frac{2f'(\phi(H)^2 \sin^2(\theta))\phi(H)}{\sin(\phi(H))}(H - H^T)^\vee \times e_3, \tag{7.59}$$

$$\dot{H} = -\hat{\Omega}H, \tag{7.60}$$

with $\Omega(a),\ \dot{\Omega}(a),\ \ddot{\Omega}(a)$ horizontal. Then, the curve $q : [a, b] \to S^2$ defined by $q(t) = R_0 H(t)^{-1} e_3$ is a modified Riemannian cubic.

Note that if we wish to use Proposition 18 to ensure obstacle avoidance, we must in fact avoid the entire fiber $\pi^{-1}(\{q_0\})$. Such a problem was handled in Section 6.1.2.

Chapter 8

Quadrotor Load Transportation

8.1 Background

The use of aerial robots has become increasingly popular in the last years due to their superior mobility and versatility in individual and cooperative tasks (see [59, 31, 113, 3, 33, 93, 80, 107, 97, 104], and references therein). For instance, aerial robots can be utilized for monitoring [6], mapping [25], agriculture tasks [96], and many other applications. These vehicles are typically underactuated by construction, which poses several challenges from the control perspective [95, 105]. Most of the control approaches for individual UAVs are based on feedback linearization [71] and backstepping methods [94] which are also analyzed in terms of stability, e.g. in [32, 77].

Multiple aerial robots can be used to transport heavier payloads, thus expanding the capabilities of a single aerial robot [35, 108, 87, 84, 66, 99]. In the aerial transportation task with multiple quadrotor UAVs considered in this work, a cable establishes a physical connection between the UAV and the cargo. Geometric control of single and multiple quadrotors with a suspended point-mass load with inelastic cables has been studied in [101, 42] and with a rigid body load in [112, 75]. In [43], the authors model the cables as flexible chains comprised of inflexible links with mass. While these works have considered the cable to be inelastic, we are instead motivated by applications where the elasticity in the cable tethers cannot be ignored without compromising the validity of the estimations [62]. In this chapter, we study the problem of two quadrotors transporting an inflexible rigid rod suspended through inflexible, elastic cables. While the use of elastic cables provides the benefit of reducing impulsive forces on the load, large or rapid oscillations of the load can produce undesired aggressive movements, compromising the load. Therefore,

we propose to use elastic cables with high stiffness and damping to guarantee the safety of the rigid load in the transportation task inspired by the fact that most industrial springs are quite stiff and have high damping effects, meaning that larger forces are required to notice the stretching, and oscillations decay quickly. By employing techniques from singular perturbation theory, these assumptions will additionally allow us to reduce the model to that of the corresponding model with inelastic cables, which is differentially flat [100]. Such techniques have been applied in the case of a single quadrotor carrying a point-mass load via an inflexible elastic cable in [70].

Methods for trajectory tracking and estimation algorithms for underactuated robotic systems evolving on differentiable manifolds such as Lie groups are commonly employed for improving accuracy in simulations. In addition, these methods avoid singularities by working with coordinate-free expressions of controllers [67, 78, 60]. The use of geometric controllers in the UAVs literature has been extensively developed in the last years (see, for instance, [77, 42, 73, 75, 79], and references therein). In this chapter, we propose a coordinate-free form of the equations of motion for the cooperative task, which are derived according to Lagrangian mechanics on manifolds—in particular, via the Lagrange d-Alembert principle for forced systems [13]. Working directly on the manifold allows us to avoid potential singularities of local parameterizations (e.g. Euler Angles), generating agile maneuvers of the payload in a uniform manner. In particular, a geometric control scheme for an underactuated multi-body system taking the form of a geometric PD controller (with additional feedback terms) is designed such that both the load position and attitude track a given desired trajectory exponentially fast. We suppose a familiarity with the basic notions of control theory throughout the chapter.

8.2 Two Quadrotors Transporting a Rigid Rod via Elastic Cables

In this section, we model the cooperative task of two quadrotor UAVs transporting a rigid rod via inflexible, elastic cables. Consider two identical quadrotor UAVs of mass m_Q transporting a rigid rod of length $2L_r$ and total mass m_r as it is shown in Figure 8.1. The rod is considered inflexible and of uniform mass density. Each endpoint of the rod is connected to the center of mass of one quadrotor via a massless inflexible elastic cable of rest length L_c. Note that we consider the rest lengths of both cables to be equal,

however this in general will not affect the analysis in the remainder of the section and can be easily modified to consider cables with different rest lengths. The basic notation and methodology is fairly standard within the literature, and we have attempted to use traditional symbols and definitions wherever feasible. Table 8.1 provides the symbols and geometric spaces that are used frequently throughout the section.

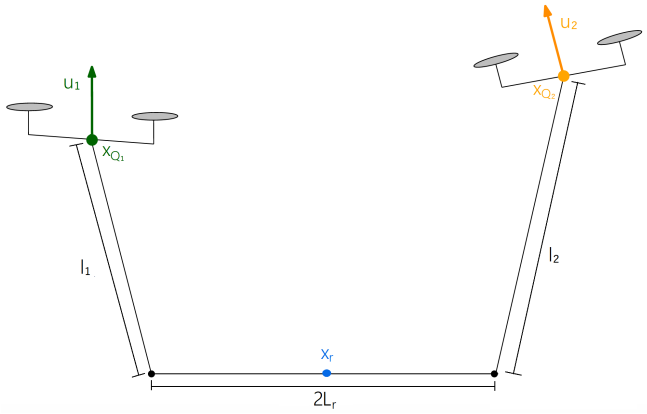


Figure 8.1: Modeling of the mechanical system describing the cooperative transportation task.

The configuration space of the mechanical system is given by $Q = \underbrace{(S^2 \times \mathbb{R}^3)}_{\text{Rigid rod}} \times \underbrace{(S^2 \times \mathbb{R})^2}_{\text{Cables}} \times \underbrace{(\text{SO}(3) \times \text{SO}(3))}_{\text{Quadrotor attitudes}}$. The system has 17 degrees of freedom—5 degrees corresponding to the rod, 3 degrees for each cable, and 3 degrees for each quadrotor. Observe that the positions of the quadrotors do not appear in the configuration space, as they are uniquely defined in terms of the other state variables due to the holonomic constraint $x_{Q_j} = x_r + (-1)^j L_r q_r - l_j q_j$ for $j = 1, 2$. Meanwhile, we have 8 inputs to the system in the form of two thrust controls $f_1, f_2 \in \mathbb{R}$, corresponding to the total lift force exerted on the quadrotors by the spinning propellers, and two moment controls $M_1, M_2 \in \mathbb{R}^3$, which are related to the torques induced on the quadrotors by the rotating propellers. Thus, the complete system has 9 degrees of under-actuation. Note also that, upon fixing a body frame to each quadrotor such that the vector $e_3 = [0, 0, 1]^T$ points in the direction of the applied thrust, we may additionally express the thrust controls f_j via the vectors $u_j = f_j R_j e_3 \in \mathbb{R}^3$ in the inertial frame for $j = 1, 2$. We will utilize this representation frequently throughout the section. Observe that only the quadrotor attitudes and positions are directly actuated by the control inputs. Hence, we will control the load position and attitude (the primary goal of our

Variable	Space	Description
m_Q	$\mathbb{R}$	Mass of quadrotor.
m_L	$\mathbb{R}$	Mass of the load.
$2L_r$	$\mathbb{R}$	Length of rod.
L_c	$\mathbb{R}$	Rest length of the elastic cables.
l_j	$\mathbb{R}$	Length of elastic cable attached to quadrotor j.
x_{Q_j}	$\mathbb{R}^3$	Position of quadrotor j in inertial frame.
x_r	$\mathbb{R}^3$	Position of center of mass of rod in inertial frame.
q_r	S^2	Attitude of rod in inertial frame.
ω_r	TS^2	Angular velocity of rod in inertial frame.
q_j	S^2	Direction vector of cable suspended from quadrotor j.
ω_j	TS^2	Angular velocity of cable j in the inertial frame.
R_j	$\mathrm{SO}(3)$	Attitude of quadrotor j.
$\hat{\Omega}_j$	$\mathfrak{so}(3)$	Angular velocity of quadrotor j in body frame.
$\mathbb{J}_Q$	$\mathrm{Sym}_{\succ 0}(\mathbb{R})$	Moment of inertia of quadrotor.
$\mathbb{J}_{\text{rod}}$	$\mathrm{Sym}_{\succ 0}(\mathbb{R})$	Moment of inertia of rod.
u_j	$\mathbb{R}^3$	Net thrust applied vertically in the body frame of quadrotor j.
$u^{\perp_j}$	$\mathbb{R}^3$	The component of u that is perpendicular to q_j.
$u^{\parallel_j}$	$\mathbb{R}^3$	The component of u that is parallel to q_j.
M_j	$\mathbb{R}^3$	Moment vector in the body frame of quadrotor j.

Table 8.1: Nomenclature in the problem of two quadrotor UAVs transporting a rigid rod.

trajectory tracking problem) indirectly by finding the quadrotor positions and attitudes corresponding to our desired load trajectory.

Remark 14. Instead of controlling the total thrust and moment, we may choose to control the total thrust of each propeller individually. Such an approach is equivalent under certain assumptions. In particular, we have a correspondence of the form $\begin{bmatrix} f_j & M_j^T \end{bmatrix}^T = K \begin{bmatrix} f_j^1 & f_j^2 & f_j^3 & f_j^4 \end{bmatrix}^T$, where f_j^k is the thrust exerted by the k^{th} propeller of the j^{th} quadrotor for $j = 1, 2$, and K is a 4×4 matrix [77]. Under the assumption that the propellers lie in a plane and adjacent propellers spin in opposite directions, K is invertible and so the approaches are equivalent. More general UAVs, such as those with different numbers of propellers (e.g. hexarotors) and those with alternative propeller orientations and attitudes, may be considered. In such cases, the results obtained throughout the remainder of the chapter will apply as long as a similar one-to-one correspondence can be established between the total thrust and moment control pair and the individual thrusts of the propellers. That said, we choose to work with the total thrust control and moment control

because they lead to the (very convenient) decoupling of the quadrotor attitude dynamics from the rest of the system dynamics, as will be seen in Proposition 40.

8.2.1 Dynamics

We now seek to derive the dynamics corresponding to the model derived in the previous section. We will do this using a Lagrangian formalism (see 4). In particular, we must construct the total kinetic and potential energies of the mechanical system describing the cooperative task—in addition to the virtual work done by non-conservative forces—and subsequently apply the *Lagrange d'Alembert Principle* [13]. The translational kinetic energy of each quadrotor can be described by $\frac{1}{2}m_Q\|\dot{x}_{Q_j}\|^2$, where m_Q denotes the mass of the quadrotor. Similarly, the translational kinetic energy of the rod is $\frac{1}{2}m_r\|\dot{x}_r\|^2$. As the quadrotors and the rod are rigid bodies, we further have rotational kinetic energy components in the total kinetic energy. Fixing a body frame to each quadrotor and denoting the angular velocity in this body frame by $\Omega_j \in \mathbb{R}^3$ through the hat map (3.4), the angular kinetic energy is given by $\frac{1}{2}\|\dot{R}_j\|^2_{\mathrm{SO}(3)} = \frac{1}{2}\Omega_j^T\mathbb{J}_Q\Omega_j$, where $\mathbb{J}_Q$ is the moment of inertia tensor and $\hat{\Omega}_j := R_j^T\dot{R}_j$.

The angular kinetic energy of the rod is similarly given by $\frac{1}{2}\Omega_r^T\mathbb{J}_{\mathrm{rod}}\Omega_r$, however the symmetry of the rod allows us to reinterpret the expression. In particular, we will fix a body frame to one of the endpoints of the rod with $e_1 := q_r$ as one of the (orthonormal) basis vectors. In this frame, the component of the (body) angular velocity along e_1 is identically zero. Moreover, we can describe the inertia tensor explicitly by $\mathbb{J}_{\mathrm{rod}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}m_rL_r^2 & 0 \\ 0 & 0 & \frac{4}{3}m_rL_r^2 \end{bmatrix}$. From this is clear that $\frac{1}{2}\Omega_r^T\mathbb{J}_{\mathrm{rod}}\Omega_r = \frac{2}{3}m_rL_r^2\|\Omega_r\|^2$. The angular velocity of the rod in the inertial frame is defined implicitly by the kinematic relation $\dot{q}_r = \omega_r \times q_r$ together with the condition that $\omega_r^T q_r = 0$, from which it is obvious that $\|\dot{q}_r\| = \|\omega_r\|$. Moreover, for some rotation $R_r \in \mathrm{SO}(3)$, we have $\Omega_r = R_r\omega_r$, and since $\|\cdot\|_{S^2}$ is $\mathrm{SO}(3)$-invariant, we see that $\|\Omega_r\| = \|\dot{q}_r\|$. Altogether, the angular kinetic energy of the rod is given by $\frac{2}{3}m_rL_r^2\|\dot{q}_r\|^2$.

Recall that, for all $n \in \mathbb{N}$, we have $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ since $\mathbb{R}^n$ is a vector space. Moreover, we may make the identification $T\mathrm{SO}(3) \cong \mathrm{SO}(3) \times \mathfrak{so}(3)$ by a left-trivialization. That is, the map $(R, \dot{R}) \in$

$T\mathrm{SO}(3) \mapsto (R, R^{-1}\dot{R}) =: (R, \hat{\Omega}) \in SO(3) \times \mathfrak{so}(3)$ is a diffeomorphism. Therefore, the tangent bundle of Q can be identified as

$$TQ \cong (TS^2 \times \mathbb{R}^3 \times \mathbb{R}^3) \times (TS^2 \times \mathbb{R} \times \mathbb{R})^2 \times (\mathrm{SO}(3) \times \mathfrak{so}(3))^2.$$

Finally, the total kinetic energy $K : TQ \to \mathbb{R}$ of the system is given by summing the respective translational and angular kinetic energies of the quadrotors and rigid rod:

$$K = \underbrace{\frac{1}{2}m_r\|\dot{x}_r\|^2 + \sum_{j=1}^{2}\frac{1}{2}m_Q\|\dot{x}_{Q_j}\|^2}_{\text{Translational K.E.}} + \underbrace{\frac{2}{3}m_r L_r^2\|\dot{q}_r\|^2 + \sum_{j=1}^{2}\frac{1}{2}\Omega_j^T\mathbb{J}_Q\Omega_j}_{\text{Angular K.E.}}.$$

Next, we fix a basis for the inertial frame such that e_3 is oriented opposite to the direction of gravitational acceleration, from which the total potential energy $U : Q \to \mathbb{R}$ of the system is given by

$$U = \sum_{j=1}^{2}\underbrace{m_Q g e_3^T x_{Q_j} + m_r g e_3^T x_r}_{\text{Gravitational P.E.}} + \sum_{j=1}^{2}\underbrace{\frac{1}{2}k(L_c - l_j)^2}_{\text{Elastic P.E.}},$$

which corresponds to the gravitational potential energies of the quadrotors and the rigid rod, as well as the elastic potential of the cables. As usual, the Lagrangian of the system $L : TQ \to \mathbb{R}$ is defined by $L := K - U$.

The controls take the form of non-conservative external forces, so that we must use the Lagrange d'Alembert Variational Principle (see [13] for instance) – where the controls play the role of the virtual forces in our system – to obtain our system dynamics from the Lagrangian L. We further wish to add a non-conservative force corresponding to a damping in the elastic cables. In particular, we use the standard model of a damped harmonic oscillator [62], used for instance in surgical robotic applications [106], wherein the damping force is proportional to the rate of oscillation of the spring, and is applied in the direction which serves to reduce the amplitude of the oscillations. It should be noted that this model may fail in the case of high oscillation rates—in which non-linear velocity dependent terms may begin to dominate. Throughout the section, we will assume implicitly that we remain in the low-velocity regime. This in part is justified by the assumption that the spring constant k and damping coefficient c are large, which naturally prevents this extreme behavior.

Denote by $C^\infty(Q, q_a, q_b)$ the space of smooth function from $[a, b]$ to Q with fixed endpoints, denoted by q_a and q_b, respectively. Consider the action functional $\mathcal{A} : C^\infty(Q, q_a, q_b) \to \mathbb{R}$ given by

$$\mathcal{A}(c(t)) = \int_a^b L(c(t), \dot{c}(t))\, dt + \sum_{j=1}^{2} \int_a^b \left(\|f_j R_j e_3\|^2_{\mathbb{R}^3} + \|\hat{M}_j\|^2_{\mathfrak{so}(3)} - c\dot{l}_j \right)\, dt = 0, \tag{8.1}$$

where $\|\hat{M}_j\|_{\mathfrak{so}(3)} := \langle \hat{M}_j, \hat{M}_j \rangle^{1/2} = \sqrt{\text{Tr}(\hat{M}_j^T \hat{M}_j)}$, with $c(t) := (q_r(t), x_r(t), q_j(t), l_j(t), R_j(t))) \in C^\infty(Q, q_a, q_b)$. In order to use the aforementioned Lagrange d'Alembert Variational Principle, we must describe the variations of our state variables. In particular, we choose $\delta x_r \in \mathbb{R}^3$ and $\delta l_j \in \mathbb{R}$ arbitrary, $\delta q_j = \frac{d}{d\epsilon} \mid_{\epsilon=0} \exp(\epsilon \hat{\xi}_j) q_j = \xi_j \times q_j \in T_{q_j} S^2$, $\delta q_r = \frac{d}{d\epsilon} \mid_{\epsilon=0} \exp(\epsilon \hat{\xi}_r) q_r = \xi_r \times q_r \in T_{q_r} S^2$, satisfying $\xi_j \cdot q_j = 0$ and $\xi_r \cdot q_r = 0$, for arbitrary vectors $\xi_j,\ \xi_r \in \mathbb{R}^3$ and $j = 1, 2$. By defining the curve on the Lie algebra $\mathfrak{so}(3)$ given by $\hat{\eta}_j = R_j^T \delta R_j \in \mathfrak{so}(3)$, it can be shown that $\widehat{\delta\Omega}_j = \hat{\Omega}_j \hat{\eta}_j + \dot{\hat{\eta}}_j$ with $\hat{\eta}_j$ satisfying $\hat{\eta}_j(a) = \hat{\eta}_j(b) = 0$ (see for instance [78], Chapter 6). Moreover, we have the following relations

$$\begin{aligned}
\delta x_{Q_j} &= \delta x_r + (-1)^j L_r \delta q_r - (\delta l_j) q_j - l_j(\delta q_j), \\
\delta \dot{x}_{Q_j} &= \delta \dot{x}_r + (-1)^j L_r \delta \dot{q}_r - (\delta \dot{l}_j) q_j - (\delta l_j) \dot{q}_j - \dot{l}_j(\delta q_j) - l_j(\delta \dot{q}_j).
\end{aligned}$$

Proposition 36. *A smooth curve $q = (x_r, q_r, q_1, l_1, q_2, l_2, R_1, R_2) \in Q$ is a critical point of $\mathcal{A}$ if and only if it satisfies the controlled Euler-Lagrange equations*

$$\dot{x}_r = v_r, \tag{8.2}$$

$$m_{\text{eff}}(\dot{v}_r + g e_3) = u_1 + u_2 + m_Q(\ddot{\zeta}_1 + \ddot{\zeta}_2), \tag{8.3}$$

$$\dot{q}_r = \omega_r \times q_r, \tag{8.4}$$

$$I_{\text{eff}} \dot{\omega}_r = q_r \times \left[u_2 - u_1 + m_Q(\ddot{\zeta}_2 - \ddot{\zeta}_1) \right], \tag{8.5}$$

$$m_Q q_j^T \ddot{\zeta}_j = m_Q q_j^T (\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3 - \frac{1}{m_Q} u_j) - c\dot{l}_j + k(L_c - l_j), \tag{8.6}$$

$$q_j \times \ddot{\zeta}_j = q_j \times (\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3 - \frac{1}{m_Q} u_j), \tag{8.7}$$

$$\dot{R}_j = R_j \hat{\Omega}_j, \tag{8.8}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \quad \textit{for } j = 1, 2, \tag{8.9}$$

where $m_{\text{eff}} := 2m_Q + m_r,\ I_{\text{eff}} := (2m_Q + \frac{2}{3} m_r) L_r$, *and* $\zeta_j := l_j q_j$.

Remark 15. Equations (8.2)-(8.3) describe the kinematics and dynamics of the rod's position, respectively. Similarly, equations (8.4)-(8.5) describe the kinematics and dynamics of the rod's attitude, and equations (8.8)-(8.9), corresponds with the kinematics and dynamics of the attitudes of each quadrotor, respectively.

Equation (8.6), indexed for $j = 1, 2$, describes the dynamics of the lengths of the elastic cables. This can be understood by observing that the projection of $\ddot{\zeta}_j$ onto q_j preserves the acceleration of the length (which is inherently oriented along the cable), while removing the acceleration of the attitude from consideration with the identity $q_j^T \ddot{q}_j = -\|\dot{q}_j\|^2$. Conversely, equation (8.7), indexed for $j = 1, 2$, describes the dynamics of the attitudes of the elastic cables, as the cross-product with q_j preserves the acceleration of the cable attitude while annihilating the acceleration of the cable length.

Proof of Proposition 40: We wish to apply Lagrange- d'Alembert Variational Principle. Therefore, our system dynamics must satisfy

$$\delta \int_a^b L(c(t), \dot{c}(t))\, dt + \sum_{j=1}^{2} \int_a^b \left(\delta x_{Q_j}^T u_j + \left\langle R_j^T \delta R_j, \hat{M}_j \right\rangle - c\dot{l}_j \delta l_j \right) dt = 0, \tag{8.10}$$

where the integral on the right represents the virtual work done by the thrust controls u_j, the moment controls $M_j \in \mathbb{R}^3$, and the spring damping, respectively.

Expanding the variations within (8.10), substituting the corresponding infinitesimal variations, and grouping like terms, we obtain

$$\begin{aligned}
0 = &\int_a^b \left[\delta \dot{x}_r^T (m_{\text{eff}} \dot{x}_r - m_Q(\dot{\zeta}_1 + \dot{\zeta}_2)) + \delta x_r^T (-m_{\text{eff}} g e_3 + u_1 + u_2) \right] dt \\
&+ L_r \int_a^b \left[\dot{\xi}_r^T (q_r \times (I_{\text{eff}} \dot{q}_r + m_Q(\dot{\zeta}_2 - \dot{\zeta}_1)) + \xi_r^T (\dot{q}_r \times m_Q(\dot{\zeta}_2 - \dot{\zeta}_1) + q_r \times (u_2 - u_1)) \right] dt \\
&- \sum_{j=1}^{2} \int_a^b \left[m_Q(\delta \dot{l}_j) q_j^T \dot{x}_{Q_j} + \delta l_j (m_Q \dot{q}_j^T \dot{x}_{Q_j} - m_Q g e_3^T q_j - k(L_c - l_j) + c\dot{l}_j + q_j^T u_j) \right] dt \\
&- \sum_{j=1}^{2} \int_a^b \left[\xi_j^T \left(q_j \times (m_Q \dot{l}_j \dot{x}_{Q_j} - m_Q g l_j e_3 + l_j u_j) + \dot{q}_j \times m_Q l_j \dot{x}_{Q_j} \right) + \dot{\xi}_j^T (q_j \times m_Q l_j \dot{x}_{Q_j}) \right] dt \\
&+ \sum_{j=1}^{2} \int_a^b \left[\dot{\eta}_j^T \mathbb{J}_Q \Omega_j + \eta_j^T \left(\mathbb{J}_Q \Omega_j \times \Omega_j + M_j \right) \right] dt,
\end{aligned}$$

where $\zeta_j := l_j q_j$, $m_{\text{eff}} := 2m_Q + m_r$, and $I_{\text{eff}} := (2m_Q + \frac{2}{3} m_r) L_r$. Integrating by parts and applying the

equality of mixed partial derivatives and the fact that variations vanish on the endpoints, we obtain

$$
\begin{aligned}
0 = &\int_a^b \delta x_r \left[m_{\text{eff}}(\ddot{x}_r + g e_3) - m_Q(\ddot{\zeta}_1 + \ddot{\zeta}_2) - (u_1 + u_2) \right] dt + \sum_{j=1}^{2} \int_a^b \eta_j^T \left[\mathbb{J}_Q \dot{\Omega}_j - \mathbb{J}_Q \Omega_j \times \Omega_j - M_j \right] dt \\
&+ \; L_r \int_a^b \xi_r^T \left[I_{\text{eff}}(q_r \times \ddot{q}_r) - q_r \times \left(u_2 - u_1 + m_Q(\ddot{\zeta}_2 - \ddot{\zeta}_1) \right) \right] dt \\
&- \sum_{j=1}^{2} \int_a^b \delta l_j \left[m_Q q_j^T (\ddot{x}_r + (-1)^j L_r \ddot{q}_r - \ddot{\zeta}_j + g e_3) - c\dot{l}_j + k(L_c - l_j) - q_j^T u_j \right] dt \\
&- l_j \sum_{j=1}^{2} \int_a^b \xi_j^T \left[m_Q q_j \times (\ddot{x}_r + (-1)^j L_r \ddot{q}_r - \ddot{\zeta}_j + g e_3) - q_j \times u_j \right] dt.
\end{aligned}
$$

Each of these integrals can be treated independently, as their respective variations are independent. That is, for the above equation to be satisfied, we necessarily have that each integral vanish identically. Applying the Fundamental Lemma of the Calculus of Variations [38] to each integral yields the dynamical system:

$$
\begin{aligned}
m_{\text{eff}}(\ddot{x}_r + g e_3) &= u_1 + u_2 + m_Q(\ddot{\zeta}_1 + \ddot{\zeta}_2), \\
I_{\text{eff}}(q_r \times \ddot{q}_r) &= q_r \times \left[u_2 - u_1 + m_Q(\ddot{\zeta}_2 - \ddot{\zeta}_1) \right], \\
m_Q q_j^T (\ddot{x}_r + (-1)^j L_r \ddot{q}_r - \ddot{\zeta}_j + g e_3) &= c\dot{l}_j - k(L_c - l_j) + q_j^T u_j, \\
m_Q q_j \times (\ddot{x}_r + (-1)^j L_r \ddot{q}_r - \ddot{\zeta}_j + g e_3) &= q_j \times u_j, \\
\dot{R}_j &= R_j \hat{\Omega}_j, \\
\mathbb{J}_Q \dot{\Omega}_j &= \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \qquad \text{for } j = 1, 2.
\end{aligned}
$$

where we have made the assumption that $l_j \neq 0$. After implicitly defining the translational and angular velocities of the load with the kinematic equations $\dot{v}_r = x_r$ and $\dot{q}_r = \omega_r \times q_r$, and rearranging terms, we have the desired dynamical control system. □

8.2.2 Reduction via Singular Perturbations

While the use of elastic cables provide the benefit of reducing impulsive forces on the rod, large or rapid oscillations of the rod can produce undesired aggressive movements, compromising the safety of the cooperative task. Therefore, the strategy is to use elastic cables with high stiffness and damping to

guarantee the safety of the rod in the transportation task. To this end, we will employ techniques from singular perturbation theory [56] to study such a situation.

In particular, we will consider the case that $k = \dfrac{\bar{k}}{\epsilon^2}$ and $c = \dfrac{\bar{c}}{\epsilon}$ with $\bar{k}, \bar{c} > 0$ and $\epsilon > 0$ sufficiently small, and we will show that the dynamics approach that of the same model with inelastic cables (that is, with $l \equiv L_c$) as $\epsilon \to 0$. We further consider a change of variables of the form $l_j = \epsilon^2 y_j + L_c$ and $\dot{l}_j = \epsilon z_j$, which is motivated by observing that $k(L_c - l_j) = -\bar{k} y_j$ and $c\dot{l}_j = \bar{c} z_j$. From this, we can see that $\zeta_j = (\epsilon^2 y_j + L_c) q_j$. Therefore, $\ddot{\zeta}_j = L_c \ddot{q}_j + \epsilon(\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j$. Making these substitutions into the dynamics described in Proposition 40, in addition to defining the angular velocity of the cables ω_j by $\dot{q}_j = \omega_j \times q_j$ and $\omega_j^T q_j = 0$, we obtain

$$\dot{x}_r = v_r,$$

$$m_{\text{eff}}(\dot{v}_r + g e_3) = u_1 + u_2 + m_Q L_c(\ddot{q}_1 + \ddot{q}_2) + m_Q \epsilon \left(\dot{z}_1 q_1 + \dot{z}_2 q_2 + z_1 \dot{q}_1 + z_2 \dot{q}_2\right) + m_Q \epsilon^2 (y_1 \ddot{q}_1 + y_2 \ddot{q}_2),$$

$$\dot{q}_r = \omega_r \times q_r,$$

$$I_{\text{eff}} \dot{\omega}_r = -q_r \times \left[u_2 - u_1 + m_Q L_c(\ddot{q}_2 - \ddot{q}_1) + m_Q \epsilon \left(\dot{z}_2 q_2 - \dot{z}_1 q_1 + z_2 \dot{q}_2 - z_1 \dot{q}_1\right) + m_Q \epsilon^2 (y_2 \ddot{q}_2 - y_1 \ddot{q}_1)\right],$$

$$\epsilon \dot{y}_j = z_j,$$

$$\epsilon \dot{z}_j = \frac{1}{m_Q} q_j^T \left[-\bar{c} z_j q_j - \bar{k} y_j q_j - u_j + m_Q(L_c + \epsilon^2 y_j)\ddot{q}_j + m_Q(\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3)\right],$$

$$\dot{q}_j = \omega_j \times q_j,$$

$$\dot{\omega}_j = \frac{1}{L_c} q_j \times \left[\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3 - \frac{1}{m_Q} u_j - \epsilon(\dot{z}_j q_j + z_j \omega_j) - \epsilon^2 y_j \ddot{q}_j\right],$$

$$\dot{R}_j = R_j \hat{\Omega}_j,$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \qquad \text{for } j = 1, 2.$$

The previous system of differential equations can be written as

$$\dot{x} = f(t, x, z; \epsilon), \tag{8.11}$$

$$\epsilon \dot{z} = g(t, x, z; \epsilon), \tag{8.12}$$

where f and g are smooth functions, x is the vector representing $(x_r, v_r, q_r, \omega_r, q_j, \omega_j, R_j, \Omega_j)$, and z is the vector representing (y_j, z_j), for $j = 1, 2$. The above dynamical system is known as a *singular perturbation*

model [64], with (8.11) describing the slow dynamics and (8.12) describing the fast dynamics.

Evaluating at $\epsilon = 0$, the fast dynamics provide us with algebraic equations that can be solved to obtain $z = h(t, x)$. In particular,

$$y_j = \frac{1}{\bar{k}} q_j^T \left[-u_j + m_Q L_c \ddot{q}_j + m_Q (\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3) \right], \tag{8.13}$$

$$z_j = 0. \tag{8.14}$$

Substituting these equations back into the slow dynamics, we obtain the reduced (slow) model of the control system describing the cooperative task, given by $\dot{x} = f(t, x, h(t, x), 0)$. That is,

$$\dot{x}_r = v_r, \tag{8.15}$$

$$m_{\text{eff}}(\dot{v}_r + g e_3) = u_1 + u_2 + m_Q L_c (\ddot{q}_1 + \ddot{q}_2), \tag{8.16}$$

$$\dot{q}_r = \omega_r \times q_r, \tag{8.17}$$

$$I_{\text{eff}} \omega_r = -q_r \times \left[u_2 - u_1 + m_Q L_r (\ddot{q}_2 - \ddot{q}_1) \right], \tag{8.18}$$

$$\dot{q}_j = \omega_j \times q_j, \tag{8.19}$$

$$m_Q L_c \dot{\omega}_j = m_Q q_j \times (\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3) - q_j \times u_j, \tag{8.20}$$

$$\dot{R}_j = R_j \hat{\Omega}_j, \tag{8.21}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \quad \text{for } j = 1, 2. \tag{8.22}$$

Note that the previous dynamical control system is equivalent to the model with inelastic cables (that is, where $l \equiv L$). Achieving exponentially stable tracking of the reduced model on some set of initial conditions will guarantee exponentially stable tracking in some subset of those initial conditions – whose relative size depends on ϵ. This claim will be formalized in the subsequent section, and will make direct use of Proposition 11.2 in [64]. We will work within this reduced model to design geometric controllers towards the end of tracking the position and attitude of the rod.

Remark 16. As discussed in [101], it is more physically realistic to model our system by a hybrid dynamical system which transitions between the cases of a taut cable (with positive tension magnitude) and a cable with zero tension. However, such a consideration will not play a role in the control design and subsequent analysis, so we will omit such a development.

Observe that the reduced model (8.15) - (8.22) preserves the original 8 inputs given by f_1, f_2, M_1, M_2, but two degrees of freedom (namely the cable lengths) have been removed from the configuration space. Hence, the reduced model has 7 degrees of under-actuation. We now show that it is in fact *differentially flat* [85]. In other words, its states and inputs can be written as algebraic functions of 8 *flat outputs* and their derivatives. Such a notion dramatically reduces the difficulty involved in generating dynamically feasible trajectories for under-actuated systems.

The differential flatness of the under-actuated system comprised of one quadrotor carrying a cable-suspended point-mass load is discussed for instance in [101]. In that case, the 4 flat outputs are given by $(x_L, \psi) \in \mathbb{R}^3 \times \mathbb{R}$, where x_L is the load position and ψ is the *yaw angle* of the quadrotor. The strategy for the rod is similar, as summarized in the following lemma.

Lemma 8.2.1. *$\mathcal{Y} = (x_{L_1}, x_{L_2}, \psi_1, \psi_2) \in (\mathbb{R}^3)^2 \times (\mathbb{R})^2$ is a set of flat outputs for the reduced model* (8.15) - (8.22)*, where $\psi_j \in \mathbb{R}$ denotes the yaw angle of the j^{th} quadrotor.*

Proof: We may follow the same analysis as in [101] to show that the state variables and inputs $q_j, R_j, \Omega_j, u_j, M_j$ may be recovered from the outputs $(x_{L_j}, \psi_j) \in \mathbb{R}^3 \times \mathbb{R}$ for $j = 1, 2$. It remains to be seen that the load position x_r and load attitude q_r can additionally be recovered. This follows immediately from the fact that $x_{L_j} = x_r + (-1)^j L_r q_r$ for $j = 1, 2$, as we obtain $\frac{1}{2}(x_{L_1} + x_{L_2}) = x_r$ and $\frac{1}{2L_r}(x_{L_2} - x_{L_1}) = q_r$. □

8.2.3 Control Design for the Reduced Model

In this section, we will design a thrust controller $u_j \in \mathbb{R}^3$ such that the position and attitude of the rigid rod reach a desired position $\tilde{x}_r \in \mathbb{R}^3$ and attitude $\tilde{q}_r \in S^2$. The strategy is to split u_j into its components which are parallel and perpendicular to the cable attitudes q_j. This is motivated by the fact that these components appear independently of one another within the system, and as such can be considered as decoupled controllers.

Notice that equations (8.21)-(8.22) describing the quadrotor attitudes are decoupled from the rest of the dynamical system, and the moment controllers M_j appear exclusively within them. Moreover, these

equations are identical to those that appear in [74], for which M_j was designed to attain almost-global exponential tracking of desired trajectories. We will use the same controller for the attitude of the quadrotors.

We will also introduce configuration error functions for each state variable, from which we may derive our error dynamics. The controls will be selected such that the origin is an exponentially stable equilibrium point of the error dynamics. In particular, the controls will take the form of a geometric PD controller with some additional feedback terms. We begin by further simplifying the dynamical system (8.15)-(8.20). In particular, we find an equation for $\ddot{q}_j$ that we will substitute into equations (8.16) and (8.20). By differentiating (8.19) and expanding it with the vector triple product identity, it can be shown that $\ddot{q}_j = \dot{\omega}_j \times q_j - \|\omega_j\|^2 q_j$. Now we may substitute (8.20) in for $\dot{\omega}_j$ to find that

$$\begin{aligned} m_Q L_c \ddot{q}_j &= -q_j \times (m_Q L_c \dot{\omega}_j) - m_Q L_c \|\omega_j\|^2 q_j \\ &= -q_j \times \left[m_Q q_j \times (\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3) - q_j \times u_j \right] - m_Q L_c \|\omega_j\|^2 q_j \\ &= -m_Q \hat{q}_j^2 \left[\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3 \right] + \hat{q}_j^2 u_j - m_Q L_c \|\omega_j\|^2 q_j \\ &= m_Q (\dot{v}_r^{\perp_j} + (-1)^j L_r \ddot{q}_r^{\perp_j} + g e_3^{\perp_j}) - u_j^{\perp_j} - m_Q L_c \|\omega_j\|^2 q_j, \end{aligned}$$

where "$\perp_j$" stands for the component of the vector that is perpendicular to the cable attitude $q_j \in S^2$. Similarly, for every vector $v \in \mathbb{R}^3$, we define the component $v^{\|_j}$ parallel to q_j such that $v = v^{\|_j} + v^{\perp_j}$. Notice that $v^{\|_j} = (v^T q_j) q_j$ and $v^{\perp_j} = -\hat{q}_j^2 v$. Substituting the above equation for $m_Q L_c \ddot{q}_j$ into (8.16) and making use of the fact that $m_{\text{eff}} = 2m_Q + m_r$, we obtain

$$\begin{aligned} m_r(\dot{v}_r + g e_3) &= \left[u_1 - m_Q(\dot{v}_r + g e_3) + m_Q L_c \ddot{q}_1 \right] + \left[u_2 - m_Q(\dot{v}_r + g e_3) + m_Q L_c \ddot{q}_2 \right], \\ &= \left[u_1^{\|_1} - m_Q(\dot{v}_r^{\|_1} + g e_3^{\|_1} + L_c \|\omega_1\|^2 q_1 - L_r \ddot{q}_r^{\perp_1}) \right] \\ &\quad + \left[u_2^{\|_2} - m_Q(\dot{v}_r^{\|_2} + g e_3^{\|_2} + L_c \|\omega_2\|^2 q_2 + L_r \ddot{q}_r^{\perp_2}) \right]. \end{aligned}$$

Define the parallel component of the thrust controllers as

$$u_j^{\|_j} := \mu_j + m_Q(\ddot{x}_r^{\|_j} + g e_3^{\|_j} + L_c \|\omega_j\|^2 q_j + (-1)^j L_r \ddot{q}_r^{\|_j}), \tag{8.23}$$

where μ_j is an additional control to be designed later – and which is constrained to be parallel to q_j. Re-

peating this procedure with (8.18) yields

$$\begin{aligned} I_{\text{eff}}\omega_r &= -q_r \times [u_2 + m_Q L_c \ddot{q}_2] - q_r \times [u_1 + m_Q L_c \ddot{q}_1], \\ &= -q_r \times \left[u_2^{\parallel_2} + m_Q(\dot{v}_r^{\perp_2} + g e_3^{\perp_2} - L_c \|\omega_2\|^2 q_2 + L_r \ddot{q}_r^{\perp_2}) \right] \\ &+ q_r \times \left[u_1^{\parallel_1} + m_Q(\dot{v}_r^{\perp_1} + g e_3^{\perp_1} - L_c \|\omega_1\|^2 q_1 - L_r \ddot{q}_r^{\perp_1}) \right] \\ &= -q_r \times [\mu_2 - \mu_1 + 2 m_Q L_r \ddot{q}_r]. \end{aligned}$$

Similarly, we see from (8.20) that $m_Q L_c(q_j \times \omega_j) = -m_Q(\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3)^{\perp_j} + u_j^{\perp_j}$, which motivates the definition

$$u_j^{\perp_j} = \nu_j + m_Q(\dot{v}_r + (-1)^j L_r \ddot{q}_r + g e_3)^{\perp_j}, \tag{8.24}$$

where ν_j is a control to be later designed, and is constrained to be perpendicular to q_j. Further making use of the fact that $I_{\text{eff}} = 2 m_Q L_r + \frac{2}{3} m_r L_r$, we see that the dynamical system (8.15)-(8.22) is equivalent to the following dynamical system

$$\dot{x}_r = v_r, \tag{8.25}$$

$$m_r(\dot{v}_r + g e_3) = \mu_1 + \mu_2, \tag{8.26}$$

$$\dot{q}_r = \omega_r \times q_r, \tag{8.27}$$

$$\frac{2}{3} m_r L_r \dot{\omega}_r = -q_r \times (\mu_2 - \mu_1), \tag{8.28}$$

$$\dot{q}_j = \omega_j \times q_j, \tag{8.29}$$

$$m_Q L_c \dot{\omega}_j = -q_j \times \nu_j. \tag{8.30}$$

$$\dot{R}_j = R_j \hat{\Omega}_j, \tag{8.31}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \quad \text{for } j = 1, 2. \tag{8.32}$$

Equations (8.25), (8.27), and (8.29) are the kinematic equations for the load position, load attitude, and cable attitude, respectively, while equations (8.26), (8.28), and (8.30) describe the dynamics. We now introduce the tracking errors for the position and velocity of the rod as $e_{x_r} := x_r - \tilde{x}_r$ and $e_{v_r} := v_r - \tilde{v}_r = \dot{e}_x$, respectively. These clearly have the desirable property that $e_{x_r} = e_{v_r} = 0$ implies $x_r = \tilde{x}_r$ and $v_r = \tilde{v}_r$.

We define the tracking error for the attitude of the rod as $e_{q_r} := \tilde{q}_r \times q_r$, which is a tangent vector in $T_{q_r}S^2$ with the property that if $e_{q_r} = 0$ then $q_r = \tilde{q}_r$ or $q_r = -\tilde{q}_r$. Note that ω_r and $\tilde{\omega}_r$ will in general belong to different tangent spaces, and so to compare them, we must first translate $\tilde{\omega}_r$ to $T_{q_r}S^2$. After which, they are vectors belonging to the same tangent space and this be can be compared using their difference, just as we did with e_{x_r} and e_{v_r}. That is, we can define the tracking error for the angular velocity of the rod as $e_{\omega_r} = \omega_r + \hat{q}_r^2\tilde{\omega}_r$. This analysis is not specific to q_r or ω_r, but instead relies only on the manifold structure of S^2. Hence, we may use the same configuration errors to compare q_j, ω_j with their respective desired trajectories for $j = 1, 2$.

Similarly, we define the configuration error function on S^2 for q_j as $\Psi_j = 1 - \tilde{q}_j^T q_j$, for $j = 1, 2, r$, which clearly satisfies $0 \leq \Psi_j \leq 2$, with $\Psi_j = 0$ if and only if $q_j = \tilde{q}_j$. Observe that for some $\theta \in [0, 2\pi)$, we may write $\Psi_j = 1 - \cos(\theta)$ and $\|e_{q_r}\|^2 = \sin^2(\theta)$. With this representation, is it easy to see that whenever $\Psi_j \leq \psi_j < 1$, we have $\frac{1}{2}\|e_{q_j}\|^2 \leq \Psi_j \leq \frac{1}{2-\psi_j}\|e_{q_j}\|^2$.

Note that, from (8.26), the error dynamics for the load position are given by $m_r(\dot{e}_{v_r} + \dot{\tilde{v}}_r + ge_3) = \mu_1 + \mu_2$, while the error dynamics for the load attitude are described by

$$\begin{aligned}
\frac{2}{3}m_r L_r \dot{e}_{\omega_r} &= \frac{2}{3}m_r L_r \left[\dot{\omega}_r + \dot{q}_r \times (q_r \times \tilde{\omega}_r) + q_r \times (\dot{q}_r \times \tilde{\omega}_r) + \hat{q}_r^2 \dot{\tilde{\omega}}_r\right] \\
&= \frac{2}{3}m_r L_r \left[\dot{\omega}_r + \hat{q}_r^2 \dot{\tilde{\omega}}_r + (\dot{q}_r^T \tilde{\omega}_r) q_r + (q_r^T \tilde{\omega}_r)\dot{q}_r\right] \\
&= \hat{q}_r \left[-(\mu_2 - \mu_1) + \frac{2}{3}m_r L_r \hat{q}_r \dot{\tilde{\omega}}_r - \frac{2}{3}m_r L_r (q_r^T \tilde{\omega}_r)(e_{\omega_r} - \hat{q}_r^2 \tilde{\omega}_r)\right] + \frac{2}{3}m_r L_r (\dot{q}_r^T \tilde{\omega}_r) q_r.
\end{aligned}$$

This motivates us to choose our "desired" controls $\tilde{\mu}_1, \tilde{\mu}_2$ such that their sum is a feedback linearization plus a PD controller for the above error dynamics. That is

$$\tilde{\mu}_1 + \tilde{\mu}_2 = m_r(\dot{\tilde{v}}_r + ge_3 - k_{v_r}e_{v_r} - k_{x_r}e_{x_r}), \tag{8.33}$$

$$\tilde{\mu}_2 - \tilde{\mu}_1 = -\frac{2}{3}m_r L_r \left[-\hat{q}_r \dot{\tilde{\omega}}_r + (q_r^T \tilde{\omega}_r)\hat{q}_r^2 \tilde{\omega}_r - (q_r^T \tilde{\omega}_r)e_{\omega_r} + k_{\omega_r}\hat{q}_r e_{\omega_r} + k_{q_r}\hat{q}_r e_{q_r}\right]. \tag{8.34}$$

Note, however, that we can not simply take $\mu_j = \tilde{\mu}_j$, as $\tilde{\mu}_j$ is not guaranteed to satisfy the constraint of parallelism to q_j. Therefore, we take μ_j as the projection of $\tilde{\mu}_j$ onto the corresponding cable attitude, given explicitly by $\mu_j = (I + \hat{q}_j^2)\tilde{\mu}_j$. We further select our desired cable attitudes with the goal that $q_j \to \tilde{q}_j$ as $\tilde{\mu}_j \to \mu_j$. In particular, we define $\tilde{q}_j = -\dfrac{\tilde{\mu}_j}{\|\tilde{\mu}_j\|}$, so that $\tilde{q}_j$ is parallel to $\tilde{\mu}_j$. With such a choice, the error

dynamics of the translational and angular velocities of the rod become

$$\dot{e}_{v_r} = -k_{v_r} e_{v_r} - k_{x_r} e_{x_r} + \frac{1}{m_r}\|\tilde{\mu}_1\|\hat{q}_1 e_{q_1} + \frac{1}{m_r}\|\tilde{\mu}_2\|\hat{q}_2 e_{q_2},$$
$$\dot{e}_{\omega_r} = (\dot{q}_r^T \tilde{\omega}_r) q_r - k_{\omega_r} e_{\omega_r} - k_{q_r} e_{q_r} + \frac{3}{2m_r L_r}\hat{q}_r(\|\tilde{\mu}_2\|\hat{q}_2 e_{q_2} - \|\tilde{\mu}_1\|\hat{q}_1 e_{q_1}).$$

Next, we design ν_j in such a way q_j approaches $\tilde{q}_j$ asymptotically. Repeating the procedure that we used to find e_{ω_r}, we find from (8.29)-(8.30) that the error dynamics for the angular velocities of the cables are given by

$$\begin{aligned} m_Q L_c \dot{e}_{\omega_j} &= m_Q L_c \dot{\omega}_j + m_Q L_c \left[\dot{q}_j \times (q_j \times \tilde{\omega}_j) + q_j \times (\dot{q}_j \times \tilde{\omega}_j) + \hat{q}_j^2 \dot{\tilde{\omega}}_j)\right] \\ &= m_Q L_c \left[\hat{q}_j^2 \dot{\tilde{\omega}}_j - (q_j^T \tilde{\omega}_j)\hat{q}_j \omega_j + (\dot{q}_j^T \tilde{\omega}_j) q_j\right] - \hat{q}_j \nu_j. \end{aligned}$$

We design ν_j as follows:

$$\begin{aligned} \hat{q}_j \nu_j &= m_Q L_c \left[\hat{q}_j^2 \dot{\tilde{\omega}}_j - (q_j^T \tilde{\omega}_j)\hat{q}_j \omega_j + k_{q_j} e_{q_j} + k_{\omega_j} e_{\omega_j}\right] \\ \nu_j &= -m_Q L_c \left[(q_j^T \tilde{\omega}_j)\omega_j - \hat{q}_j \dot{\tilde{\omega}}_j + k_{q_j}\hat{q}_j e_{q_j} + k_{\omega_j}\hat{q}_j e_{\omega_j}\right], \end{aligned} \tag{8.35}$$

which clearly satisfies the orthogonality constraints and leads to the following error dynamics for ω_j

$$\dot{e}_{\omega_j} = (\dot{q}_j^T \tilde{\omega}_j) q_j - k_{q_j} e_{q_j} - k_{\omega_j} e_{\omega_j}. \tag{8.36}$$

8.2.4 Stability Analysis for the Reduced Model

We now show that an appropriate choice of gains will cause the origin of the error dynamics for the reduced model to be exponentially stable, where the control inputs are defined as above. In particular, we use a Lyapunov candidate to find sufficient conditions under which the origin of the error dynamics is exponentially stable. These conditions are written in terms of the gains and some arbitrary Lyapunov parameters. Then, we show that these conditions can be satisfied, provided that the initial errors in the cable attitudes are sufficiently small.

Consider the indexing set $\mathcal{I} = 1, 2, r$, the domain

$$\mathcal{D} = \{(e_{x_r}, e_{v_r}, e_{q_r}, e_{\omega_r}, e_{q_1}, e_{\omega_1}, e_{q_2}, e_{\omega_2}) |\ \|e_{x_r}\| \leq \bar{e}_{x_r}, \|e_{v_r}\| \leq \bar{e}_{v_r},\ \|e_{\omega_j}\| \leq \bar{e}_{\omega_j},\ \Psi_j \leq \psi_j \ \text{ for } \ j \in \mathcal{I}\},$$

and define a Lyapunov candidate function on $\mathcal{D}$ given by

$$V = \frac{1}{2}\|e_{v_r}\|^2 + \frac{1}{2}k_{x_r}\|e_{x_r}\|^2 + c_{x_r}e_{x_r}^T e_{v_r} + \sum_{j\in\mathcal{I}}\left[\frac{1}{2}\|e_{\omega_j}\|^2 + k_{q_j}\Psi_j + c_{q_j}e_{q_j}^T e_{\omega_j}\right],$$

where c_{x_r} and c_{q_j} are positive real numbers for $j \in \mathcal{I}$. Observe that V_x defined as $V_x := \frac{1}{2}\|e_{v_r}\|^2 + c_{x_r}e_{x_r}^T e_{v_r} + \frac{1}{2}k_{x_r}\|e_{x_r}\|^2$ can be bounded from above and below as $\frac{1}{2}z_x^T \underline{\mathrm{P}}_x z_x \leq V_x \leq \frac{1}{2}z_x^T \bar{P}_x z_x$ where $\underline{\mathrm{P}}_x = \begin{bmatrix} k_{x_r} & -c_{x_r} \\ -c_{x_r} & 1 \end{bmatrix}, \bar{P}_x = \begin{bmatrix} k_{x_r} & c_{x_r} \\ c_{x_r} & 1 \end{bmatrix}$, and $z_x = \begin{bmatrix} \|e_{x_r}\| & \|e_{v_r}\| \end{bmatrix}^T$. Further note that both $\underline{\mathrm{P}}_x$ and $\bar{P}_x$ are positive-definite provided that $c_{x_r} < \sqrt{k_{x_r}}$.

Similarly, for $j \in \mathcal{I}$, we define $V_{q_j} := \frac{1}{2}\|e_{\omega_j}\|^2 + c_{q_j}e_{q_j}^T e_{\omega_j} + k_{q_j}\Psi_{q_j}$, which is bounded as $\frac{1}{2}z_{q_j}^T \underline{\mathrm{P}}_{q_j} z_{q_j} \leq V_{q_j} \leq \frac{1}{2}z_{q_j}^T \bar{P}_{q_j} z_{q_j}$, where $\underline{\mathrm{P}}_{q_j} = \begin{bmatrix} k_{q_j} & -c_{q_j} \\ -c_{q_j} & 1 \end{bmatrix}, \bar{P}_{q_j} = \begin{bmatrix} \frac{2k_{q_j}}{2-\psi_{q_j}} & c_{q_j} \\ c_{q_j} & 1 \end{bmatrix}$, and $z_{q_j} = \begin{bmatrix} \|e_{q_j}\| & \|e_{\omega_j}\| \end{bmatrix}^T$. As before, $\underline{\mathrm{P}}_{q_j}$ and $\bar{P}_{q_j}$ are positive-definite when $c_{q_j} < \sqrt{k_{q_j}}$.

Observing that $V = V_x + \sum_{j\in\mathcal{I}} V_{q_j}$, we then have that our Lyapunov candidate is bounded as $\frac{1}{2}z^T \underline{\mathrm{P}} z \leq V \leq \frac{1}{2}z^T \bar{P} z$, where $z = \begin{bmatrix} \|e_{x_r}\| & \|e_{v_r}\| & \|e_{q_r}\| & \|e_{\omega_r}\| & \|e_{q_1}\| & \|e_{\omega_1}\| & \|e_{q_2}\| & \|e_{\omega_2}\| \end{bmatrix}^T$,

$$\underline{\mathrm{P}} = \begin{bmatrix} \underline{\mathrm{P}}_x & 0 & 0 & 0 \\ 0 & \underline{\mathrm{P}}_{q_r} & 0 & 0 \\ 0 & 0 & \underline{\mathrm{P}}_{q_1} & 0 \\ 0 & 0 & 0 & \underline{\mathrm{P}}_{q_2} \end{bmatrix}, \quad \text{and} \quad \bar{P} = \begin{bmatrix} \bar{P}_x & 0 & 0 & 0 \\ 0 & \bar{P}_{q_r} & 0 & 0 \\ 0 & 0 & \bar{P}_{q_1} & 0 \\ 0 & 0 & 0 & \bar{P}_{q_2} \end{bmatrix}.$$

where $\underline{\mathrm{P}}, \bar{P}$ are positive-definite for $c_{x_r} < \sqrt{k_{x_r}}$ and $c_{q_j} < \sqrt{k_{q_j}}$. Next, note that by the invariance of circular shifts of the scalar triple product and the fact that $q_j^T e_{q_j} = 0$, we have:

$$\begin{aligned} \frac{d}{dt}\Psi_{q_j} &= -\tilde{q}_j^T \dot{q}_j - q_j^T \dot{\tilde{q}}_j = -\tilde{q}_j^T(\omega_j \times q_j) - q_j^T(\tilde{\omega}_j \times \tilde{q}_j) \\ &= \omega_j^T(\tilde{q}_j \times q_j) - \tilde{\omega}_j^T(\tilde{q}_j \times q_j) \\ &= (\omega_j - \tilde{\omega}_j)^T e_{q_j} = (\omega_j + (q_j^T\tilde{\omega})q_j - \tilde{\omega}_j)^T e_{q_j} = e_{\omega_j}^T e_{q_j}. \end{aligned}$$

Additionally, from the vector triple product, we see

$$\begin{aligned}\dot{e}_{q_j} &= (\dot{\tilde{q}}_j \times q_j) + (\tilde{q}_j \times \dot{q}_j) = (\tilde{\omega}_j \times \tilde{q}_j) \times q_j - (\omega_j \times q_j) \times \tilde{q}_j \\ &= \tilde{\omega}_j \times (\tilde{q}_j \times q_j) - \tilde{q}_j \times (\tilde{\omega}_j \times q_j) - \omega_j \times (q_j \times \tilde{q}_j) + q_j \times (\omega_j \times \tilde{q}_j) \\ &= (\omega_j + \tilde{\omega}_j) \times e_{q_j} + (\tilde{q}_j^T q_j) e_{\omega_j} - (\tilde{q}_j^T q_j)(q_j^T \tilde{\omega}_j) q_j \\ &= e_{\omega_j} \times e_{q_j} + (\tilde{q}_j^T q_j) e_{\omega_j} + 2\tilde{\omega}_j \times e_{q_j} - (\tilde{q}_j^T q_j)(q_j^T \tilde{\omega}_j) q_j,\end{aligned}$$

so that

$$\dot{e}_{q_j}^T e_{\omega_j} = (\tilde{q}_j^T q_j)\|e_{\omega_j}\|^2 + 2(\tilde{\omega}_j \times e_{q_j})^T e_{\omega_j} \leq \|e_{\omega_j}\|^2 + C_{q_j}\|e_{q_j}\|\|e_{\omega_j}\|,$$

where $C_{q_j} \leq 2 \sup \|\tilde{\omega}_j\|$ is a non-negative constant depending on $\bar{e}_{x_r},\ \bar{e}_{v_r},\ \bar{e}_{\omega_j}$, and ψ_j for $j \in \mathcal{I}$, as well as the desired trajectories $\tilde{x}_r$ and $\tilde{q}_r$. Therefore, the time derivative of the proposed Lyapunov function is bounded as

$$\begin{aligned}\dot{V} \leq &- (k_{v_r} - c_{x_r})\|e_{v_r}\|^2 + c_{x_r} k_{v_r} \|e_{v_r}\|\|e_{x_r}\| - c_{x_r} k_{x_r} \|e_{x_r}\|^2 \\ &+ \frac{1}{m_r}(\|e_{v_r}\| + c_{x_r}\|e_{x_r}\|)\|Y\| - (k_{\omega_r} - c_{q_r})\|e_{\omega_r}\|^2 \\ &+ c_{q_r}(k_{\omega_r} + C_{q_r})\|e_{\omega_r}\|\|e_{q_r}\| - c_{q_r} k_{q_r}\|e_{q_r}\|^2 - c_{q_j} k_{q_j}\|e_{q_j}\|^2 \\ &+ \frac{2}{3 m_r L_r}(\|e_{\omega_r}\| + c_{q_r}\|e_{q_r}\|)\|Y\| - (k_{\omega_j} - c_{q_j})\|e_{\omega_j}\|^2 \\ &+ c_{q_j}(k_{\omega_j} + C_{q_j})\|e_{q_j}\|\|e_{\omega_j}\|,\end{aligned}$$

where Y satisfies the inequality

$$\|Y\| \leq \left[m_r(k_{v_r}\|e_{v_r}\| + k_{x_r}\|e_{x_r}\|) + \frac{2}{3} m_r L_r((C_{q_r}^2 + k_{\omega_r})\|e_{\omega_r}\| + k_{q_r}\|e_{q_r}\|) + C \right] (\|e_{q_1}\| + \|e_{q_2}\|)$$

for a non-negative constant $C \leq m_r \sup \|\dot{\tilde{v}}_r\| + \frac{2}{3} m_r L_r \sup \|\dot{\tilde{\omega}}_r\|$. Furthermore, within $\mathcal{D}$, we have $\|e_{q_j}\| \leq \sqrt{\psi_{q_j}(2 - \psi_{q_j})} := \alpha_j$. With $\alpha = 2 \max\{\alpha_1, \alpha_2\}$ and $I_r = \frac{2}{3} m_r L_r$, we have

$$\begin{aligned}\frac{1}{m_r}(\|e_{v_r}\| + c_{x_r}\|e_{x_r}\|)\|Y\| \leq &\alpha(k_{v_r}\|e_{v_r}\|^2 + c_{x_r} k_{v_r}\|e_{x_r}\|\|e_{v_r}\| + c_{x_r}\|e_{x_r}\|^2) \\ &+ \frac{I_r}{m_r}\alpha(\|e_{v_r}\| + c_{x_r}\|e_{x_r}\|)((C_{q_r}^2 + k_{\omega_r})\|e_{\omega_r}\| + k_{q_r}\|e_{q_r}\|) \\ &+ \frac{1}{m_r}((C + m_r k_{x_r}\bar{e}_{x_r})\|e_{v_r}\| + c_{x_r} C\|e_{x_r}\|)(\|e_{q_1}\| + \|e_{q_2}\|),\end{aligned}$$

and,

$$\begin{aligned}(\|e_{\omega_r}\| + c_{q_r}\|e_{q_r}\|)\|Y\| \leq & I_r\alpha((C_{q_r}^2 + k_{\omega_r})\|e_{\omega_r}\|^2 \\ & + c_{q_r}(C_{q_r}^2 + k_{\omega_r})\|e_{q_r}\|\|e_{\omega_r}\| + c_{q_r}k_{q_r}\|e_{q_r}\|^2) \\ & + m_r\alpha(\|e_{\omega_r}\| + c_{q_r}\|e_{q_r}\|)(k_{v_r}\|e_{v_r}\| + k_{x_r}\|e_{x_r}\|) \\ & + ((C + I_rk_{q_r}\alpha_r)\|e_{\omega_r}\| + c_{q_r}C\|e_{q_r}\|)(\|e_{q_1}\| + \|e_{q_2}\|).\end{aligned}$$

Applying this inequality directly to the above bound for $\dot{V}$, we find that $\dot{V} \leq -z^T\mathcal{W}z$, where $\mathcal{W}$ is the 8×8 matrix defined as

$$\mathcal{W}_j = \begin{bmatrix} 2W_{x_r} & -W_{x_r,q_r} & -\frac{1}{2}W_{x_r,q_1} & -\frac{1}{2}W_{x_r,q_2} \\ -W_{x_r,q_r} & 2W_{q_r} & -\frac{1}{2}W_{q_r,q_1} & -\frac{1}{2}W_{q_r,q_1} \\ -\frac{1}{2}W_{x_r,q_1} & -\frac{1}{2}W_{q_r,q_1} & W_{q_1} & 0 \\ -\frac{1}{2}W_{x_r,q_2} & -\frac{1}{2}W_{q_r,q_2} & 0 & W_{q_2} \end{bmatrix}$$

for 2×2 sub-matrices given by

$$W_{x_r} = \frac{1}{2}\begin{bmatrix} (1-\alpha)c_{x_r}k_{x_r} & -\frac{1}{2}(1+\alpha)c_{x_r}k_{v_r} \\ -\frac{1}{2}(1+\alpha)c_{x_r}k_{v_r} & (1-\alpha)k_{v_r} - c_{x_r} \end{bmatrix},$$

$$W_{q_j} = \begin{bmatrix} c_{q_j}k_{q_j} & -\frac{1}{2}c_{q_j}(k_{\omega_j} + C_{q_j}) \\ -\frac{1}{2}c_{q_j}(k_{\omega_j} + C_{q_j}) & k_{\omega_j} - c_{q_j} \end{bmatrix},$$

$$W_{q_r} = \frac{1}{2}\begin{bmatrix} (1-\alpha)c_{q_r}k_{q_r} & -\frac{1}{2}c_{q_r}((1+\alpha)k_{\omega_r} + C_{q_r} + \alpha C_{q_r}^2) \\ -\frac{1}{2}c_{q_r}((1+\alpha)k_{\omega_r} + C_{q_r} + \alpha C_{q_r}^2) & (1-\alpha)k_{\omega_r} - c_{q_r} - \alpha C_{q_r}^2 \end{bmatrix}$$

$$W_{x_r,q_r} = \alpha\begin{bmatrix} \frac{I_r}{m_r}c_{x_r}k_{q_r} + \frac{m_r}{I_r}c_{q_r}k_{x_r} & \frac{I_r}{m_r}c_{x_r}(C_{q_r}^2 + k_{\omega_r}) + \frac{m_r}{I_r}k_{x_r} \\ \frac{I_r}{m_r}k_{q_r} + \frac{m_r}{I_r}c_{q_r}k_{v_r} & \frac{I_r}{m_r}(C_{q_r}^2 + k_{\omega_r}) + \frac{m_r}{I_r}k_{v_r} \end{bmatrix},$$

$$W_{x_r,q_j} = \frac{1}{m_r}\begin{bmatrix} c_{x_r}C & 0 \\ C + m_rk_{x_r}\bar{e}_{x_r} & 0 \end{bmatrix},\ W_{q_r,q_j} = \frac{1}{I_r}\begin{bmatrix} c_{q_r}C & 0 \\ C + I_rk_{q_r}\alpha_r & 0 \end{bmatrix}.$$

Note that $\mathcal{W}$ is not necessarily symmetric, so the quadratic form defined by $z^T\mathcal{W}z$ is positive definite if and only if the symmetric part of $\mathcal{W}$—that is $\frac{1}{2}(\mathcal{W} + \mathcal{W}^T)$—is a positive definite matrix. The following proposition states, in essence, that the gains and Lyapunov constants can be chosen such that $\underline{P}$, $\bar{P}$, and $\frac{1}{2}(\mathcal{W} + \mathcal{W}^T)$

are simultaneously positive-definite—thus ensuring that the origin is exponentially stable—provided that the initial errors in the cable attitudes are sufficiently small.

Proposition 37. *Consider the control system defined by equations 8.25 - 8.30 with control inputs 8.33, 8.34, and 8.35. For sufficiently small α, there exists control gains k_{x_r}, k_{v_r}, k_{q_r}, k_{ω_r}, k_{q_1}, k_{ω_1}, k_{q_2}, and k_{ω_2} such that the zero equilibrium of the tracking errors e_{x_r}, e_{v_r}, e_{q_r}, e_{ω_r}, e_{q_1}, e_{ω_1}, e_{q_2} and e_{ω_2} is exponentially stable.*

Before proving Proposition 37, we state some general facts about positive-definite matrices that will be used multiple times in the proof.

Lemma 8.2.2. *(Facts about positive-definite matrices) Let $A \succ 0$ and $B \succeq 0$ with $\lambda_{\min}(A) > \lambda_{\max}(B)$, and M arbitrary, be $n \times n$ matrices. Then, the following characterizations hold*

(1) $\|x\|\|y\|\lambda_{\min}(A) \leq x^T A y \leq \|x\|\|y\|\lambda_{\max}(A)$,

(2) $M^T A M \succeq 0$,

(3) $A - B \succ 0$ *with* $\lambda_{\min}(A-B) \geq \lambda_{\min}(A) - \lambda_{\max}(B)$ *and* $\lambda_{\max}(A-B) \leq \lambda_{\max}(A)$.

(4) $A + B \succ 0$ *with* $\lambda_{\min}(A+B) \geq \lambda_{\min}(A) + \lambda_{\min}(B)$.

Proof of Proposition 37. Observe that the condition $\dot{V} \leq z^T \mathcal{W} z$ is equivalent to $\dot{V} \leq \sum_{j=1,2} z_j^T \mathcal{W}_j z_j$, where

$z = \begin{bmatrix} \|e_{x_r}\| & \|e_{v_r}\| & \|e_{q_r}\| & \|e_{\omega_r}\| & \|e_{q_j}\| & \|e_{\omega_j}\| \end{bmatrix}^T$ and $\mathcal{W}_j = \begin{bmatrix} W_{x_r} & -\frac{1}{2}W_{x_r,q_r} & -\frac{1}{2}W_{x_r,q_j} \\ -\frac{1}{2}W_{x_r,q_r} & W_{q_r} & -\frac{1}{2}W_{q_r,q_j} \\ -\frac{1}{2}W_{x_r,q_j} & -\frac{1}{2}W_{q_r,q_j} & W_{q_j} \end{bmatrix}$. By Lemma 8.2.2, it suffices to show that each $\mathcal{W}_j$ is positive-definite. Denote the symmetric part of $\mathcal{W}_j$ by $\bar{\mathcal{W}}_j = \frac{1}{2}(\mathcal{W}_j + \mathcal{W}_j^T)$ and similarly define the symmetric parts of the submatrices by $\bar{W}_{x_r,q_r}, \bar{W}_{x_r,q_j}$, and $\bar{W}_{q_r.q_j}$. It is clear that $\bar{\mathcal{W}}_j$ can be expressed in the form $\bar{\mathcal{W}}_j = \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix}$, where $P = \begin{bmatrix} W_{x_r} & -\frac{1}{2}\bar{W}_{x_r,q_r} \\ -\frac{1}{2}\bar{W}_{x_r,q_r} & W_{q_r} \end{bmatrix}$, $S = -\frac{1}{2}\begin{bmatrix} \bar{W}_{x_r,q_j} & \bar{W}_{q_r,q_j} \end{bmatrix}^T$, and $Q = W_{q_j}$.

Now, observe that $\mathcal{W}$ can be decomposed as:

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} = \begin{bmatrix} I & SQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P - SQ^{-1}S^T & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & SQ^{-1} \\ 0 & I \end{bmatrix}^T \tag{8.37}$$

Where $P - SQ^{-1}S^T$ is often referred to as the *Schur complement* of Q. From Lemma 8.2.2, it then follows that $\bar{\mathcal{W}}_j \succ 0$ if and only if $P - SQ^{-1}S^T \succ 0$ and $Q \succ 0$. Note that $P - SQ^{-1}S^T$ can itself be expressed in form of a 4×4 block matrix given by

$$P - SQ^{-1}S^T = \begin{bmatrix} W_{x_r} - \frac{1}{4}\bar{W}_{x_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} & -\frac{1}{2}\bar{W}_{x_r,q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} \\ -\frac{1}{2}\bar{W}_{x_r,q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} & W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j} \end{bmatrix}.$$

Repeating the previous analysis, but now on $P - SQ^{-1}S^T$, we find that $\bar{\mathcal{W}}_j \succ 0$ if and only if the following three conditions hold:

(1) $W_{q_j} \succ 0$, (2) $W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j} \succ 0$,

(3)

$$\begin{aligned} 0 \prec W_{x_r} &- \frac{1}{4}\bar{W}_{x_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} - \frac{1}{4}\bar{W}_{x_r,q_r}(W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1}\bar{W}_{x_r,q_r} \\ &- \frac{1}{8}\bar{W}_{x_r,q_r}(W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} \\ &- \frac{1}{8}\bar{W}_{x_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j}(W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1}\bar{W}_{x_r,q_r} \\ &- \frac{1}{16}\bar{W}_{x_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j}(W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j}. \end{aligned}$$

Moreover, the minimum (maximum) eigenvalue of $\mathcal{W}_j$ is exactly the smallest (largest) of the minimum (maximum) eigenvalues of the three matrices in the above conditions. We now seek to verify that appropriate choices in the gains and constants can be made to satisfy the above conditions. First, by looking at the characteristic equation of W_{q_j}, it can be seen that

$$2\lambda_{\min}(W_{q_j}) = (k_{\omega_j} - c_{q_j} + c_{q_j}k_{q_j}) - \sqrt{(k_{\omega_j} - c_{q_j} - c_{q_j}k_{q_j}) + c_{q_j}^2(k_{\omega_j} + C_{q_j})^2}.$$

By taking c_{q_j} sufficiently small, k_{ω_j} sufficiently large, and $k_{q_j} = \frac{k_{\omega_j}}{c_{q_j}} - 1$, $\lambda_{\min}(W_{q_j})$ can be made arbitrarily large. Consequently, $\lambda_{\max}(W_{q_j}^{-1}) = \lambda_{\min}(W_{q_j})^{-1}$ can be made arbitrarily small (and positive).

From Lemma 8.2.2, we have that $\frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j} \succeq 0$ and, since $\bar{W}_{q_r,q_j}$ and W_{q_j} are independent, we can shrink the maximum eigenvalue of $\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j}$ arbitrarily by shrinking the maximum eigenvalue of $W_{q_j}^{-1}$.

Similarly, from the characteristic equation of W_{q_r}, we see that the eigenvalues satisfy

$$\begin{aligned}2\lambda_{\pm} &= ((1-\alpha)c_{q_r}k_{q_r} + (1-\alpha)k_{\omega_r} - c_{q_r} - \alpha C_{q_r}^2) \\ &\pm \sqrt{((1-\alpha)c_{q_r}k_{q_r} - (1-\alpha)k_{\omega_r} + c_{q_r} + \alpha C_{q_r}^2)^2 + c_{q_r}^2((1+\alpha)k_{\omega_r} + C_{q_r} + \alpha C_{q_r}^2)^2}.\end{aligned}$$

Now choose $k_{\omega_r} = \frac{\alpha}{1-\alpha}C_{q_r}^2 + c_{q_r}k_{q_r} + \frac{1}{1-\alpha}c_{q_r} > 0$ and define $\bar{k}_{q_r} = c_{q_r}k_{q_r}$. Then,

$$2\lambda_{\min}(W_{q_r}) = 2(1-\alpha)\bar{k}_{q_r} - c_{q_r}\left(\frac{2\alpha}{1-\alpha}C_{q_r}^2 + C_{q_r} + (1+\alpha)\bar{k}_{q_r} + \frac{1+\alpha}{1-\alpha}c_{q_r}\right),$$

from which it is clear that $\lambda_{\min}(W_{q_r})$ can be made arbitrarily large (and positive) by choosing c_{q_r} and $\bar{k}_{q_r}$ appropriately. Another application of Lemma 8.2.2 then shows that $W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j} \succ 0$, and its minimum eigenvalue can be made arbitrarily large with appropriate choices of $k_{q_j}, k_{\omega_j}, c_{q_j}$ for $j \in \mathcal{I}$.

Now we look at condition 3. First, choose $k_{x_r}, k_{v_r}, c_{x_r}$ such that $W_{x_r} \succ 0$ (this can always be done by, for instance, choosing c_{x_r} sufficiently small). We now wish to show that the remaining subtractive terms can be shrunk arbitrarily. Observe that $\bar{W}_{x_r,q_j}$ and $W_{q_j}^{-1}$ are independent, so that we may force the maximum eigenvalue of $\bar{W}_{x_r,q_j}W_{q_j}^{-1}\bar{W}_{x_r,q_j} \succeq 0$ to be arbitrarily small after (potentially) further shrinking the maximum eigenvalue of $W_{q_j}^{-1}$. Observe that we may write the third term, $\frac{1}{4}\bar{W}_{x_r,q_r}(W_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}W_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1}\bar{W}_{x_r,q_r}$, in the form $\alpha^2 M^T A M$, where M is independent of α and the terms of A are at most of $O(\frac{1}{\alpha})$. Hence, we may shrink this term arbitrarily by shrinking α. The fourth and fifth terms are transposes of each other and therefore may be handled simultaneously. Note that find that the maximum eigenvalue is bounded above by

$$\begin{aligned}&\lambda_{\max}((\bar{W}_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1})\|\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{x_r,q_j}\|\,\|\bar{W}_{x_r,q_r}\| \\ &\quad\leq \sqrt{\lambda_{\max}(\bar{W}_{q_j}^{-1})}\lambda_{\max}((\bar{W}_{q_r} - \frac{1}{4}\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1})\|\bar{W}_{q_r,q_j}\|\,\|\bar{W}_{x_r,q_j}\|\,\|\bar{W}_{x_r,q_r}\|.\end{aligned}$$

This term therefore can be arbitrarily shrunk by shrinking the maximum eigenvalue of $\bar{W}_{q_j}$. Moreover, the presence of the norm $\|W_{x_r,q_r}\|$ also gives us control of the size of the term via α. The final term is handled

similarly—we find that the maximum eigenvalue is bounded by

$$\begin{aligned}\lambda_{\max}((\bar{W}_{q_r}-\frac{1}{4}\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1})&\|\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{x_r,q_j}\|^2\\ \leq&\lambda_{\max}(\bar{W}_{q_j}^{-1})\lambda_{\max}((\bar{W}_{q_r}-\frac{1}{4}\bar{W}_{q_r,q_j}\bar{W}_{q_j}^{-1}\bar{W}_{q_r,q_j})^{-1})\|\bar{W}_{q_r,q_j}\|^2\ \|\bar{W}_{x_r,q_j}\|^2,\end{aligned}$$

which again may be shrunk arbitrarily by shrinking the maximum eigenvalue of $\bar{W}_{q_j}^{-1}$. In summation, for sufficiently small α and $\lambda_{\max}(W_{q_j})$ sufficiently large, condition 3 is satisfied and $\bar{\mathcal{W}}_j \succ 0$.

Now, letting $x \in S^6$ such that $x = (x_1, x_2) \in \mathbb{R}^4 \times \mathbb{R}^2$, we get that $\lambda_{\min}(\bar{\mathcal{W}}_j) \geq \min\{\lambda_{\min}(Q),\ \lambda_{\min}(P - SQ^{-1}S^T)\}\|x + SQ^{-1}x_2\|^2$. Note that $\|SQ^{-1}x_2\|$ can be made arbitrarily small by increasing the maximum eigenvalue of Q, so that $\|x+SQ^{-1}x_2\|$ can be made arbitrarily close to 1. Further decomposing $P - SQ^{-1}S^T$, we find that the minimum eigenvalue of $\bar{\mathcal{W}}_j$ is bounded below by a quantity that can be made arbitrarily close to the minimum of the minimum eigenvalue of the matrices in conditions (1), (2), and (3) above—all of which can be made arbitrarily large. Hence, the minimum eigenvalue of $\bar{\mathcal{W}}_j$ can be made arbitrarily large. Hence, the minimum eigenvalue of $\bar{\mathcal{W}}$ can also be made arbitrarily large. □

Remark 17. Observe that, in the proof of Proposition 37, it is not important that the minimum eigenvalue can be made arbitrarily large—we need only have that it is positive. However, in Proposition 39 of Section 8.2.6, we will introduce unstructured bounded disturbances to the problem. In such a case, this fact will be crucial. Moreover, while we have no exact estimates on the gains needed to guarantee exponential tracking, the proof does provide us with a strategy which may be used to decrease the complexity in numerically calculating/verifying gains. In particular, for any gains $k_{x_r} > 0$ and $k_{v_r} > 0$, we have that $W_{x_r}, W_{q_r}, W_{q_j}, \underline{\mathrm{P}}_{x_r}, \underline{\mathrm{P}}_{q_r}, \underline{\mathrm{P}}_{q_j} \succ 0$ if the following 7 conditions are satisfied:

(I) $c_{x_r} < \min\left\{\frac{(1-\alpha)^2 k_{x_r}k_{v_r}}{(1-\alpha)k_{x_r}+\frac{1}{4}(1+\alpha)^2k_{v_r}^2},\ \sqrt{k_{x_r}}\right\}$,

(II) $k_{q_r} > \frac{\alpha}{(1-\alpha)^2}C_{q_r}^2 + \frac{1}{2(1-\alpha)}C_{q_r}$,

(III) $c_{q_r} < \min\left\{\frac{2(1-\alpha)k_{q_r}-\frac{2\alpha}{1-\alpha}C_{q_r}^2-C_{q_r}}{(1+\alpha)k_{q_r}+\frac{1+\alpha}{1-\alpha}},\ \sqrt{k_{q_r}}\right\}$,

(IV) $k_{\omega_r} = \frac{\alpha}{1-\alpha}C_{q_r}^2 + c_{q_r}k_{q_r} + \frac{1}{1-\alpha}c_{q_r}$,

(V) $k_{\omega_j} > C_{q_j} + 2$,

(VI) $1 < c_{q_j} < \frac{2k_{\omega_j}}{k_{\omega_j}+C_{q_j}+2}$,

(VII) $k_{q_j} = \frac{k_{\omega_j}}{c_{q_j}} + 1$.

It is easily verified that these conditions are consistent. The proof of Proposition 37 tells us exactly that exponential tracking is guaranteed so long as α is sufficiently small and $\lambda_{\min}(W_{q_j})$ is sufficiently large. The latter of which can always be satisfied for instance by fixing $c_{q_j} = 1 + \epsilon$ for some arbitrary small $0 < \epsilon < 1$ and taking k_{ω_j} sufficiently large. In particular, this reduces the 9 degrees of freedom involved in satisfying $\mathcal{W}_j \succ 0$ to a single degree of freedom—namely k_{ω_j}—while simultaneously guaranteeing that $\underline{\mathrm{P}}_{x_r}, \underline{\mathrm{P}}_{q_r}, \underline{\mathrm{P}}_{q_j} \succ 0$. Finding an appropriate k_{ω_j} can then be solved numerically, provided that α is sufficiently small to guarantee a solution.

Remark 18. Due to the constraint $x_{Q_j} = x_r + (-1)^j L_r q_r - L_c q_j$, and the fact that the desired cable attitudes are pre-selected as $\tilde{q}_j = \tilde{\mu}_j / \|\tilde{\mu}_j\|$, we are not able to choose the desired trajectories of the quadrotors while also choosing the desired trajectories for the load position and attitude. In particular, such choices will force the "desired" quadrotor positions $\tilde{x}_{Q_j}$ to similarly obey $\tilde{x}_{Q_j} = \tilde{x}_r + (-1)^j L_r \tilde{q}_r - L_c \tilde{\mu}_j / \|\tilde{\mu}_j\|$. This leads to the concern that the quadrotors may stretch apart and play "tug of war" with the rod, or in the case that $2L_c > L_r$, may even collide with one another. To find conditions under which this behavior is avoided, note that

$$\tilde{x}_{Q_2} - \tilde{x}_{Q_1} = L_c \left(\tilde{\mu}_1 / \|\tilde{\mu}_1\| - \tilde{\mu}_2 / \|\tilde{\mu}_2\| \right) - 2L_r \tilde{q}_r.$$

$$\tilde{\mu}_j = \frac{1}{2} m_r \left(\dot{\tilde{v}}_r + g e_3 \right) + (-1)^j \frac{1}{3} m_r L_r \left(-\hat{q}_r \dot{\tilde{\omega}}_r + (q_r^T \tilde{\omega}_r) \hat{q}_r^2 \tilde{\omega}_r \right) + O(e_{x_r}, e_{v_r}, e_{q_r}, e_{\omega_r}).$$

Suppose now that $\| \frac{1}{3} m_r L_r \left(-\hat{q}_r \dot{\tilde{\omega}}_r + (q_r^T \tilde{\omega}_r) \hat{q}_r^2 \tilde{\omega}_r \right) \|$ is small compared to $\| \frac{1}{2} m_r \left(\dot{\tilde{v}}_r + g e_3 \right) \|$. This can be accomplished for instance by choosing $\tilde{q}_r$ such that $\|\tilde{\omega}_r\|$ and $\|\dot{\tilde{\omega}}_r\|$ are sufficiently small (that is, that the desired angular velocity and angular acceleration of the rod's orientation are sufficiently small). Further suppose that the initial errors $e_{x_r}(0), e_{v_r}(0), e_{q_r}(0)$, and $e_{\omega_r}(0)$ are small, and that the gains are chosen so that the origin of the error dynamics is an exponentially stable equilibrium point (in accordance with Proposition 37). It follows that both $\tilde{\mu}_1$ and $\tilde{\mu}_2$ will remain close to the vector $\frac{1}{2} m_r \left(\dot{\tilde{v}}_r + g e_3 \right)$, so

that $\tilde{x}_{Q_2} - \tilde{x}_{Q_1}$ will remain close to $2L_r\tilde{q}_r$. In particular, this implies that $\|\tilde{x}_{Q_2} - \tilde{x}_{Q_1}\|$ remains close to $2L_r$. Moreover, since the gains have been chosen to guarantee exponential tracking, we further have that $\|x_{Q_2} - x_{Q_1}\|$ will remain within some small neighborhood of $2L_r$, from which it is clear that the quadrotors will neither converge to nor diverge from one another to an undesirable degree.

8.2.5 Control Design for the Full Model

We begin by incorporating the attitude dynamics for the quadrotors (8.31)-(8.32) back into the reduced model. Note that, in the design of the geometric controllers u_j, we assumed that each quadrotor can generate a thrust in any direction. However, in reality, the direction of the total thrust is always parallel to its third body-fixed axis via the relation $u_j = f_j R_j e_3$. Only the magnitude of the thrust f_j can be changed arbitrarily. The strategy we implement, following [77], is to choose the desired trajectories for the quadrotor attitudes $\tilde{R}_j$ such that the third body-fixed axis becomes parallel to the direction of the control force u_j designed by (8.23) and (8.35). Moreover, we will design the total thrust f_j as the magnitude of the u_j. Using the moment control M_j designed in [101], we then guarantee that the *actual* thrust control $f_j R_j e_3$ converges exponentially fast to the u_j that we previously designed.

The desired direction of the third body-fixed axis for each quadrotor, denoted by $b_j^3 \in S^2$ is given by $b_j^3 = \frac{u_j}{\|u_j\|}$. Usually, such an expression is considered as a constraint on the desired attitude of each quadrotor. Therefore, to solve the dimensionality problem arising after introduce the constraint, and to be able to solve the system, the desired direction of the first body-fixed axis $b_j^1(t) \in S^2$ is introduced as a smooth function of time. Since the first body-fixed axis is normal to the third body-fixed axis, one can not reach the arbitrary body axis b_j^1 exactly. The usual strategy in this situation is to project b_j^1 onto the plane normal to b_j^3, and the desired direction of the second body-fixed axis is chosen to obtain an orthonormal frame. That is, the desired attitude for each quadrotor are given by $R_{j,d} = \left[\frac{(\hat{b}_j^3)^2 b_j^1}{\|(\hat{b}_j^3)^2 b_j^1\|}, \quad -\frac{\hat{b}_j^3 b_j^1}{\|\hat{b}_j^3 b_j^1\|}, \quad -b_j^3\right] \in \mathrm{SO}(3)$.

Using (8.21), the desired angular velocity for each quadrotor is given by $\tilde{\Omega}_j = (\tilde{R}_j^T \dot{\tilde{R}}_j)^\vee \in \mathbb{R}^3$, where $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ is the vee map. Define the tracking error vectors for the attitude of each quadrotors $e_{R_j} = \frac{1}{2}(\tilde{R}_j^T R_j - R_j^T \tilde{R}_j)^\vee$, and the tracking error vectors for the angular velocity of each quadrotor as $e_{\Omega_j} = \Omega_j - R_j^T \tilde{R}_j \tilde{\Omega}_j$. The moment control is chosen as a tracking controller on $\mathrm{SO}(3)$ defined in [77]. In

particular,

$$f_j = u_j \cdot R_j e_3, \tag{8.38}$$

$$M_j = -\frac{k_R}{\epsilon^2} e_{R_j} - \frac{k_\Omega}{\epsilon} e_{\Omega_j} + \Omega_j \times \mathbb{J}_Q \Omega_j - \mathbb{J}_Q (\hat{\Omega}_j R_j^T \tilde{R}_j \tilde{\Omega}_j - R_j^T \tilde{R}_j \dot{\tilde{\Omega}}_j), \qquad j = 1, 2, \tag{8.39}$$

where ϵ, k_R, k_Ω are positive constants.

Stability of the corresponding controlled systems for the unreduced model can be studied by using singular perturbation theory for the attitude dynamics of quadrotors as in [101]. In particular, as a direct application of [73], in the context of Proposition 37 for our particular cooperative transportation task lead to the following result.

Corollary 13. *Consider the control system defined by* (8.15)-(8.22) *and the control inputs designed by* (8.38) *and* (8.39)*. Then, there exists $\delta > 0$, such that for all $\epsilon < \delta$, the zero equilibrium of the tracking errors $(e_{x_r}, e_{v_r}, e_{q_r}, e_{\omega_r}, e_{q_1}, e_{\omega_1}, e_{q_2}, e_{\omega_2}, e_{R_1}, e_{\Omega_1}, e_{R_2}, e_{\Omega_2})$ is exponentially stable.*

Now that we have established the exponential tracking of the full reduced model, we wish to connect this back to our original model with elastic cables. This can be done by showing that our system is under the conditions of Theorem 11.2 in [64]. The first condition is a regularity condition for f, g, h (as defined by (8.11)-(8.12) and (8.13)-(8.14)), and the continuity and boundedness of their partial derivatives. Both the original and reduced model have smooth dynamics, and h is clearly smooth as well, so that the regularity follows. Moreover, we consider our control system as evolving on the bounded domain $\mathcal{D}$, so that boundedness follows immediately. The second condition requires that the reduced model have an exponentially stable equilibrium point at the origin (note that we must consider the error dynamics instead of the dynamical model (8.15)-(8.22) for this to make sense, but the result follows all the same since the configuration errors are smooth functions of the state variables). This follows immediately by Proposition 37. The third condition in Theorem 11.2 in [64] is that the *boundary layer system* for (8.11)-(8.12) have an exponentially stable equilibrium point at the origin—which we define now.

Definition 8.2.3. The *boundary layer system* for the singular perturbation problem given by (8.11)-(8.12) is

defined as:

$$\frac{\partial r}{\partial \tau} = g(t, x, r + h(t, x), 0),$$

where $r := z - h(t, x)$ with $h(t, x)$ as defined by (8.13)-(8.14) and $\tau := \frac{t-t_0}{\epsilon}$ for t_0 the value of time from which we obtain our initial data.

Corollary 14. *The boundary layer system for* (8.11)-(8.12) *with control inputs u_j and M_j as defined above has an exponentially stable equilibrium point at the origin.*

Proof: The analysis follows almost immediately from the case of a single quadrotor transporting a point mass load with an elastic cable (see Lemma 2 in [70]). □

With all conditions of Theorem 11.2 in [64] satisfied, we obtain the following proposition relating the dynamics of the original model to the dynamics of the reduced model.

Proposition 38. *Let the control inputs u_j and M_j be defined as above. Denote by $x(t)$ a trajectory of the reduced model* (8.15)-(8.22) *which converges exponentially to the desired trajectory. Denote by $r(t)$ a trajectory of the boundary layer system which converges exponentially to the origin. Then, there exists a positive constant ϵ^* such that for all $t \geq t_0$ and $0 < \epsilon < \epsilon^*$, there exists a unique solution $x(t, \epsilon)$, $z(t, \epsilon)$ of the singular perturbation problem* (8.11)-(8.12) *on $[t_0, \infty)$ satisfying*

$$\begin{aligned} x(t, \epsilon) - x(t) &= \mathcal{O}(\epsilon), \\ z(t, \epsilon) - h(t, x(t)) - r\left(\frac{t - t_0}{\epsilon}\right) &= \mathcal{O}(\epsilon), \end{aligned}$$

uniformly on $t \in [t_0, \infty)$. Moreover, for $t_1 > t_0$, we have

$$z(t, \epsilon) - h(t, x(t)) = O(\epsilon),$$

uniformly on (t_1, ∞) for $\epsilon < \epsilon^{**} < \epsilon^*$.

8.2.6 Stability in the Presence of Bounded Unstructured Disturbances

While the control scheme developed in Section 8.2.3 is theoretically sound, in practice there may be disturbances and measurement errors which cause its failure. This in part is handled by the well-known fact

that exponential stability is robust to small disturbances [65]. However, it would additionally be beneficial to understand how the proposed controller behaves when it is subject to larger disturbances. Note that disturbances in the attitude of the quadrotors under the proposed control (8.39) has been discussed already in [76]. Since the attitude dynamics are decoupled from the rest of the system, we temporarily omit them from our analysis.

We introduce bounded unstructured disturbances to the reduced dynamical model (8.25)-(8.30). That is, for some real numbers $\delta x_r, \delta q_j$ for $j \in \mathcal{I}$, we consider the dynamical system

$$\dot{x}_r = v_r, \tag{8.40}$$

$$m_r(\dot{v}_r + ge_3) = \mu_1 + \mu_2 + \Delta x_r, \tag{8.41}$$

$$\dot{q}_r = \omega_r \times q_r, \tag{8.42}$$

$$\frac{2}{3}m_r L_r \dot{\omega}_r = -q_r \times (\mu_2 - \mu_1) + \Delta q_r, \tag{8.43}$$

$$\dot{q}_j = \omega_j \times q_j, \tag{8.44}$$

$$m_Q L_c \dot{\omega}_j = -q_j \times \nu_j + \Delta q_j, \tag{8.45}$$

where for $j \in \mathcal{I}$, $\Delta x_r, \Delta q_r, \Delta q_j$ are unstructured disturbances satisfying $\|\Delta x_r\| \leq \delta x_r$ and $\|\Delta q_j\| \leq \delta q_j$. Defining the controllers, configuration errors, and Lyapunov candidate V as in section (8.2.4), the bounds on the Lyapunov function remain the same. That is, we have $\frac{1}{2} z_{q_j}^T \underline{\mathrm{P}}_{q_j} z_{q_j} \leq V_{q_j} \leq \frac{1}{2} z_{q_j}^T \bar{P}_{q_j} z_{q_j}$. However, some additional terms appear in its derivative (last three terms in the following inequality):

$$\begin{aligned}
\dot{V} \leq & -(k_{v_r} - c_{x_r})\|e_{v_r}\|^2 + c_{x_r} k_{v_r} \|e_{v_r}\| \|e_{x_r}\| - c_{x_r} k_{x_r} \|e_{x_r}\|^2 + \frac{1}{m_r}(\|e_{v_r}\| + c_{x_r}\|e_{x_r}\|)\|Y\| \\
& - (k_{\omega_r} - c_{q_r})\|e_{\omega_r}\|^2 + c_{q_r}(k_{\omega_r} + C_{q_r})\|e_{\omega_r}\|\|e_{q_r}\| - c_{q_r} k_{q_r}\|e_{q_r}\|^2 - c_{q_j} k_{q_j}\|e_{q_j}\|^2 \\
& + \frac{2}{3m_r L_r}(\|e_{\omega_r}\| + c_{q_r}\|e_{q_r}\|)\|Y\| - (k_{\omega_j} - c_{q_j})\|e_{\omega_j}\|^2 + c_{q_j}(k_{\omega_j} + C_{q_j})\|e_{q_j}\|\|e_{\omega_j}\| \\
& + E_{x_r}^T(e_{v_r} + c_{x_r} e_{x_r}) + E_{q_r}^T(e_{\omega_r} + c_{q_r} e_{q_r}) + E_{q_j}^T(e_{\omega_j} + c_{q_j} e_{q_j}),
\end{aligned}$$

where E_{x_r}, E_{q_r}, and E_{q_j} are defined as $E_{x_r} = \frac{\Delta x_r}{m_r}, E_{q_r} = \frac{3\Delta q_r}{2m_r L_r}, E_{q_j} = \frac{\Delta q_j}{m_Q L_c}$. We may then write this is the form $\dot{V} \leq -z^T \mathcal{W} z + E^T z$, where $E := \left[\frac{c_{x_r}\delta x_r}{m_r}, \frac{\delta x_r}{m_r}, \frac{c_{q_r} 3\delta q_r}{2m_r L_r}, \frac{3\delta q_r}{2m_r L_r}, \frac{c_{q_1}\delta q_1}{m_Q L_c}, \frac{\delta q_1}{m_Q L_c}, \frac{c_{q_2}\delta q_2}{m_Q L_c}, \frac{\delta q_2}{m_Q L_c} \right]$ and

$\mathcal{W}$ is as in section 8.2.4. From Young's Inequality, we have $E^T z \leq \frac{\|E\|^2}{4\epsilon} + \epsilon\|z\|^2$ for any $\epsilon > 0$. Hence,

$$\dot{V} \leq -z^T(\mathcal{W} - \epsilon I)z + \frac{\|E\|^2}{4\epsilon}.$$

Fix $\epsilon > 0$. Analogous to before, we replace the matrix $\mathcal{W} - \epsilon I$ with its symmetric part $\bar{\mathcal{W}}^* := \frac{1}{2}(\mathcal{W} + \mathcal{W}^T) - \epsilon I$. Observe that:

$$\begin{aligned}\lambda_{\min}(\bar{\mathcal{W}}^*) &= \min_{x \in S^6}\{x^T(\bar{\mathcal{W}} - \epsilon I)x\} = \min_{x \in S^6}\{x^T\bar{\mathcal{W}}x - \epsilon\|x\|^2\} = \min_{x \in S^6}\{x^T\bar{\mathcal{W}}x\} - \epsilon \\ &= \lambda_{\min}(\bar{\mathcal{W}}) - \epsilon\end{aligned}$$

From the proof of Proposition 37, $\lambda_{\min}(\bar{\mathcal{W}})$ can be made arbitrarily large by choosing gains appropriately. Hence, we can choose them so that $\bar{\mathcal{W}}^* \succ 0$, with an arbitrarily large minimum eigenvalue. We then have the following inequalities:

$$\lambda_{\min}(\underline{\mathbf{P}})\|z\|^2 \leq V \leq \lambda_{\max}(\bar{P})\|z\|^2,$$
$$\dot{V} \leq -\lambda_{\min}(\bar{\mathcal{W}}^*)\|z\|^2 + \frac{\|E\|^2}{4\epsilon}.$$

This implies that $\dot{V} \leq -\frac{\lambda_{\min}(\bar{\mathcal{W}}^*)}{\lambda_{\max}(\bar{P})}V + \frac{\|E\|^2}{4\epsilon}$, so that $\dot{V} < 0$ when $V > \frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{\mathcal{W}}^*)}\frac{\|E\|^2}{4\epsilon} := d_1 > 0$. Clearly, d_1 can be arbitrarily small by making $\lambda_{\min}(\bar{\mathcal{W}}^*)$ sufficiently large. If we now define the set $S_r := \{z \in \mathcal{D} : V(z) < r\}$, where r is some real number, then any trajectory starting in the open set $\mathcal{D} \setminus \bar{S}_{d_1}$ will converge exponentially to the region $\bar{S}_{d_1}$, where $\bar{S}_{d_1}$ denotes the topological closure of S_{d_1}. Since V is continuous and positive, $\bar{S}_{d_1}$ is some closed neighborhood of the origin that can be made arbitrarily small (by shrinking d_1). We formalize this result with the following proposition:

Proposition 39. *Consider the control system with disturbances defined by equations 8.40 - 8.45 with control inputs 8.33, 8.34, and 8.35. For sufficiently small α, there exists control gains k_{x_r}, k_{v_r}, k_{q_r}, k_{ω_r}, k_{q_1}, k_{ω_1}, k_{q_2}, and k_{ω_2} such that the zero equilibrium of the tracking errors e_{x_r}, e_{v_r}, e_{q_r}, e_{ω_r}, e_{q_1}, e_{ω_1}, e_{q_2} and e_{ω_2} are uniformly ultimately bounded with an arbitrarily small ultimate bound.*

Remark 19. We will not extend this result to include the quadrotor attitude kinematics and dynamics here, nor relate it back to the original elastic model. Such a task follows by essentially the same strategy as employed in Section 8.2.5, just replacing exponential stability with uniform ultimate boundedness where appropriate.

8.2.7 Energy-Optimal Path Planning with Obstacle Avoidance

Now that we have designed a control scheme which allows the two quadrotor system to converge exponential fast to a desired trajectory, the natural question arises of how one should design the trajectory to be tracked. In application, it is often the case that we wish for the trajectories to interpolate some collection of knot points (for example initial and final positions and velocities). Moreover, it may be desirable that some quantity is minimized along the trajectory—be it the time taken to connect knot points, the jerk (for example, if the payload is delicate), or the energy consumption. In this section, we will choose to minimize the latter. In particular, we seek to find trajectories which minimize the magnitude of the controls, as these quantities are directly related to energy consumption in most applications. That is, we are interested in solving the *optimal control problem*

$$\min_{q,u} \sum_{j=1}^{2} \int_a^b \left(\|u_j\|^2 + \|M_j\|^2 \right) dt, \tag{8.46}$$

over all curves $q = (x_r, q_r, q_1, q_2, R_1, R_2) \in \Omega(Q)$ and controls $u = (u_1, u_2, M_1, M_2) \in C^\infty([a,b], \mathbb{R}^3)^4$ subject to the constraints (8.15)-(8.22). Note that we have chosen the state dynamics of the reduced model as our constraints, but this will still lead to desirable behavior in the original elastic model by Proposition 38. As discussed in Section 8.2.5, the quadrotor attitude dynamics are not completely decoupled from the rest of the system dynamics due to the physical constraint $u_j = f_j R_j e_3$. However, due to the fact that we choose $\tilde{R}_j$ and the controls so that $f_j R_j e_3 \to u_j$ exponentially fast, we may effectively treat the system as if it were decoupled. In particular, we split (8.46) into the following three optimal control problems:

$$\min_{q,u} \int_a^b \left(\|u_1\|^2 + \|u_2\|^2 \right) dt, \tag{8.47}$$

over all curves $q = (x_r, q_r, q_1, q_2) \in C^\infty([a,b], \mathbb{R}^3 \times (S^2)^3)$ and controls $u = (u_1, u_2) \in C^\infty([a,b], \mathbb{R}^3)^2$ subject to the constraints (8.15)-(8.20), and

$$\min_{R_j, M_j} \int_a^b \|M_j\|^2 dt, \tag{8.48}$$

over all curves $R_j \in C^\infty([a,b], \mathrm{SO}(3))$ and controls $M_j \in C^\infty([a,b], \mathbb{R}^3)$ subject to the constraints (8.21)-(8.22) for $j = 1, 2$.

We equip each copy of SO(3) with the left-invariant metric $\langle \cdot, \cdot \rangle_{\mathrm{SO}(3)}$ induced by (3.19) with $\mathbb{J} = \mathbb{J}_Q$. It follows by (8.21)-(8.22) and Example (5) that (8.48) is equivalent to the variational principle

$$\delta \int_a^b \|D_t \dot{R}_j\|^2_{\mathrm{SO}(3)} dt = 0 \tag{8.49}$$

for $j = 1, 2$, taken over all admissible variations of R_j. Clearly, the solutions to (8.49) are just Riemannian cubic polynomials, so that the moment controls M_j are minimized when the desired trajectories $\tilde{R}_j$ are chosen to be Riemannian cubic polynomials for $j = 1, 2$. In the presence of an obstacle, which may in this case be interpreted as undesirable quadrotor attitudes, we may augment the action functional in (8.49) with an artificial potential designed for the obstacle avoidance task (as in Section 6.1). In such a case, the optimal desired trajectories are modified Riemannian cubics. Hence, upon applying reduction by symmetry, $\tilde{R}_j$ satisfies the necessary conditions given by Proposition 23 for $j = 1, 2$.

We now seek to handle (8.47). First, note that from (8.23) and (8.24), we have

$$u_j = \mu_j + \nu_j + m_Q(\ddot{x}_r + ge_3 + L_c\|\omega_j\|^2 q_j + (-1)^j L_r \ddot{q}_r). \tag{8.50}$$

Since the control scheme has been designed for exponential tracking, we see that in the limit that the state variables approach their desired trajectories (and the configuration errors approach 0), the controls (8.50) approach

$$u_j = \tilde{\mu}_j + \nu_j + m_Q(\ddot{\tilde{x}}_r + ge_3 + L_c\|\tilde{\omega}_j\|^2 \tilde{q}_j + (-1)^j L_r \ddot{\tilde{q}}_r), \tag{8.51}$$

where, from (8.33), (8.34), and (8.35), $\tilde{\mu}_j$ and ν_j approach

$$\tilde{\mu}_1 + \tilde{\mu}_2 = m_r(\ddot{\tilde{x}}_r + ge_3) \tag{8.52}$$

$$\tilde{\mu}_2 - \tilde{\mu}_1 = \frac{2}{3} m_r L_r \tilde{q}_r \times \dot{\tilde{\omega}}_r \tag{8.53}$$

$$\nu_j = m_Q L_c \tilde{q}_j \times \dot{\tilde{\omega}}_j. \tag{8.54}$$

We now equip the configuration space for the reduced system (excluding the quadrotor attitudes), given by $Q = \underbrace{(S^2 \times \mathbb{R}^3)}_{\text{Rigid rod}} \times \underbrace{(S^2 \times S^2)}_{\text{Cables}}$, with a Riemannian metric as follows: Equip $\mathbb{R}^3$ with the Riemannian metric $\langle X, Y \rangle_{\mathbb{R}^3} = m_{\text{eff}}(X \cdot Y)$ for all $X, Y \in \mathbb{R}^3$. Equip the copy of S^2 corresponding to the rigid rod attitude with the Riemannian metric $\langle \cdot, \cdot \rangle_{S^2_r} = I_{\text{eff}} \langle \cdot, \cdot \rangle_{S^2}$, where $\langle \cdot, \cdot \rangle_{S^2}$ is the SO(3)-invariant metric defined in

(4.37). Similarly, equip the copies of S^2 corresponding to the cable attitudes with the Riemannian metric $\langle \cdot, \cdot \rangle_{S_c^2} = m_Q L_c \langle \cdot, \cdot \rangle_{S^2}$, where $\langle \cdot, \cdot \rangle_{S^2}$ is the SO(3)-invariant metric defined in (4.37).

It is easily seen that scalar multiples of a Riemannian metric induce the same Levi-Civita connection, so that in particular, the Levi-Civita connection on $(\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{R}^3})$ is given by the ordinary directional derivative. On S^2, the covariant derivative induced by the Levi-Civita connection is given by (4.38). Hence, with $\omega = \dot{q} \times q$, we find that $D_t \dot{q} = q \times \dot{\omega}$. It follows that the system (8.52)-(8.54) can be written in the form

$$\tilde{\mu}_1 = \frac{1}{2} m_r (D_t \dot{\tilde{x}}_r + g e_3) + \frac{1}{3} m_r L_r D_t \dot{\tilde{q}}_r \tag{8.55}$$

$$\tilde{\mu}_2 = \frac{1}{2} m_r (D_t \dot{\tilde{x}}_r + g e_3) - \frac{1}{3} m_r L_r D_t \dot{\tilde{q}}_r \tag{8.56}$$

$$\nu_j = m_Q L_c q_j D_t \dot{\tilde{q}}_j. \tag{8.57}$$

In particular, the control (8.51) takes the form

$$u_j = \frac{1}{2} m_{\text{eff}} (D_t \dot{\tilde{x}}_r + g e_3) + \frac{(-1)^j}{2} I_{\text{eff}} D_t \dot{\tilde{q}}_r + m_Q L_c D_t \dot{\tilde{q}}_j + m_Q L_c \|\tilde{\omega}_j\|^2 \tilde{q}_j + (-1)^j m_Q L_r \|\tilde{\omega}_r\|^2 \tilde{q}_r \tag{8.58}$$

The final two terms, $m_Q L_c D_t \dot{\tilde{q}}_j + m_Q L_c \|\tilde{\omega}_j\|^2 \tilde{q}_j + (-1)^j m_Q L_r \|\tilde{\omega}_r\|^2 \tilde{q}_r$, correspond to lift forces exerted on the cables and load in the direction of their respective attitudes. We see that their magnitudes are modulated by the norms of their respective angular velocities (which in turn are equivalent to the norms of their respective linear velocities). In all, we see that the norms of the controls (8.58) are minimized provided that the desired trajectories satisfy the following variational principles:

$$\delta \int_a^b \left\| D_t \dot{\tilde{x}}_r + g e_3 \right\|_{\mathbb{R}^3}^2 dt = 0 \tag{8.59}$$

$$\delta \int_a^b \left(\frac{1}{2} \|D_t \dot{q}_r\|_{S_r^2}^2 + \tau_r^2 \|\dot{q}_r\|_{S_r^2}^2 \right) dt = 0 \tag{8.60}$$

$$\delta \int_a^b \left(\frac{1}{2} \|D_t \dot{q}_j\|_{S_c^2}^2 + \tau_j^2 \|\dot{q}_j\|_{S_c^2}^2 \right) dt = 0, \quad \text{for } j = 1, 2, \tag{8.61}$$

where $\tau_r, \tau_1, \tau_2 \in \mathbb{R}$ are called *elastic parameters*, which serve to control how closely the resulting critical points resemble geodesics or Riemannian cubic polynomials (that is, when the elastic parameters vanish identically, we obtain Riemannian cubics polynomials, and when the elastic parameters are large, we obtain curves that approximate geodesics). The variational principles (8.60) and (8.122) yield solutions known as *elastica*, which have been studied for instance in [17]. In general, the values for the elastic parameters

will depend on the desired behavior in the particular application. For our purposes, we choose $\tau_r = \tau_1 = \tau_2 = 0$, so that the corresponding critical points are Riemannian cubic polynomials. This assumption is partially justified by the desire to minimize the torques applied to the load and cables (which correspond to minimizing the respective covariant accelerations with respect to our Riemannian structure), as large torques can lead to erratic behavior that compromises the load and the mechanical features of the cables and quadrotors.

As for the quadrotor attitudes, we may augment the action functionals in (8.59)-(8.122) with artificial potentials to avoid obstacles (for example, physical obstacles in phase space and undesirable cable/load attitudes). In particular, $\tilde{q}_r,\ \tilde{q}_1$, and $\tilde{q}_2$ are chosen to be modified Riemannian cubics, which may be calculated as in Proposition 35, whereas $\tilde{x}_r$ is easily seen to satisfy

$$\frac{d^4\tilde{x}_r}{dt^4} = -ge_3 - \nabla V_x(\tilde{x}_r), \tag{8.62}$$

where $V_x : R^3 \to \mathbb{R}$ is some artificial potential, and $\nabla V : \mathbb{R}^3 \to \mathbb{R}^3$ is its gradient vector field. In summary, with respect to the trajectory tracking controls u_1, u_2, M_1, M_2 designed in Section 8.2.3, the reduced two quadrotor system is energy-optimal (while avoiding obstacles) when the desired trajectories for the state variables are chosen to be modified Riemannian cubics.

8.2.8 Simulation Results

We now test the results of Proposition 37 with numerical simulations. Simulations of the dynamics for the reduced model were conducted using the proposed controller.

We implement an Euler method for the numerical integration of equations (8.25)-(8.30) with time step $h = 0.002$ and $T = 20$ seconds. The system parameters were chosen as $m_Q = 0.755$, $m_r = 0.5$, $L_c = L_r = 1.0$, and $\mathbb{J}_Q = \begin{bmatrix} 0.082 & 0 & 0 \\ 0 & 0.0845 & 0 \\ 0 & 0 & 0.1377 \end{bmatrix}$. The gains we consider are $k_{x_r} = 9$, $k_{v_r} = 6$, $k_{q_r} = 2$, $k_{w_r} = 2\sqrt{2}$, $k_{q_1} = 36$, $k_{w_1} = 12$, $k_{q_2} = 36$, $k_{w_2} = 12$. It should be noted that both smaller and larger gain-sets were found that yielded convergence. This particular set was chosen because it offered an appropriate balance between performance and feasibility, in particular with regards

to the convergence of the load position, which was chosen as the *Lissajous curve* described by $\tilde{x}(t) = (1.2\sin(0.4\pi t),\ 4.2\cos(0.2\pi t),\ -0.5)$, together with a constant desired load attitude $\tilde{q}_r(t) = (0, 1, 0)$ (in accordance with Remark 18 to guarantee that the quadrotors do not collide nor stretch apart). The initial conditions were chosen as $x_r(0) = (1, 4.9, -1)$, $v_r(0) = (1.2, 0.55, 0.15)$, $q_r(0) = (0.24, 0.97, -0.1)$, $\omega_r(0) = (0.1, -0.1, 0)$, $q_1 = (0.53, 0.63, -0.56)$, $\omega_1(0) = (0, 0, 0)$, $q_2(0) = (0.48, 0.67, -0.56)$, $\omega_2(0) = (0, 0, 0)$.

Figure 8.2 shows the magnitude of the error functions versus times. In each case, the blue curve represents the error in the position/attitude, and red curve represents the error in the velocity/angular velocity. Additionally, we plot the magnitude of the controllers $\|u_1\|$ and $\|u_2\|$ versus time. For illustrative purposes, we also include Figure 8.2, which plots the load position (blue curve), the desired trajectory for the load (red curve), and the positions of the quadrotors (green and yellow curves) in $3D$ space. The highlighted points along the yellow and green curves mark the final positions of the two quadrotors.

Note that, as expected, the error asymptotically converge to zero (or a small neighborhood of zero) in all cases, while the controls approach steady state solutions.

8.3 Arbitrarily Large Teams of Quadrotors Transporting a Rigid Body Load via Elastic Cables

In this section, we repeat the procedure of the previous sections, but instead for the task of an arbitrarily large team of quadrotors transporting a rigid body load via inflexible elastic cables. A majority of the analysis will follow similarly to that performed in 8.2, so that we will skip many of details. It should be noted that the same model with inelastic cables was considered in [75], and we will utilize the same control scheme here.

We begin by modelling the control system describing the transportation task between the quadrotors. As before, this will done by constructing the total kinetic K and potential U energies of the system, in addition to the virtual work done by non-conservative forces and, subsequently, using the Lagrange d'Alembert Principle on the Lagrangian $L := K - U$.

Consider N identical quadrotor UAVs transporting a rigid body of total mass $m_L \in \mathbb{R}_{>0}$ and positive-

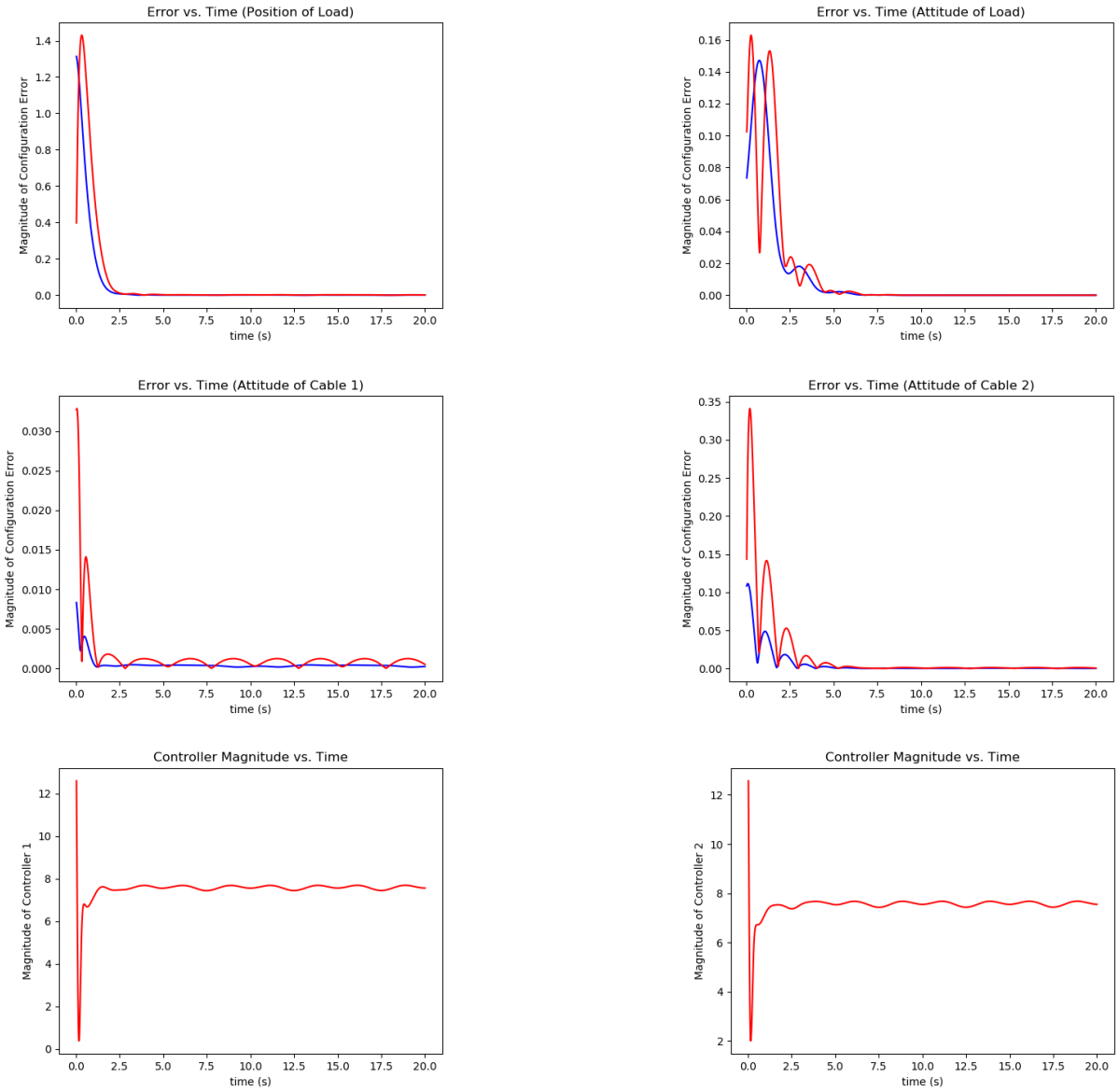


Figure 8.2: Configuration errors vs. time and control magnitude vs. time.

definite inertia matrix $\mathbb{J}_L \in \mathbb{R}^{3\times3}$. The load is considered rigid and of uniform mass density. Massless inflexible elastic cables with of rest length L connect the center of mass of each quadrotor to the load. A graphical depiction of the proposed model is given in Figure 8.3, and Table 8.2 provides the symbols and geometric spaces that are used frequently throughout the section.

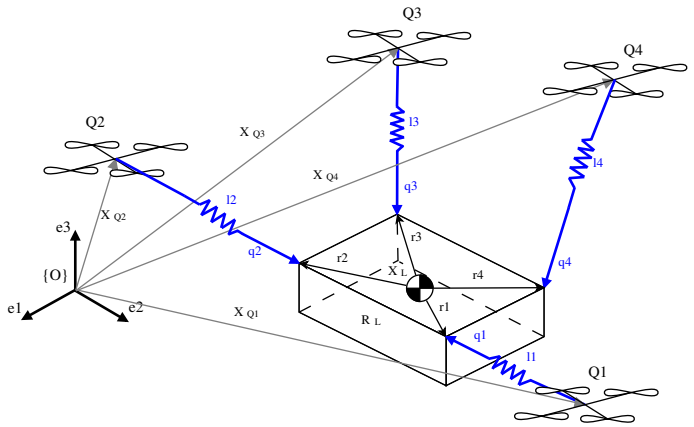


Figure 8.3: Proposed model for the transportation task with quadrotor UAVs.

Variable	Space	Description
m_Q	$\mathbb{R}$	Mass of quadrotor.
m_L	$\mathbb{R}$	Mass of the load.
k	$\mathbb{R}$	Spring constant of the elastic cables.
c	$\mathbb{R}$	Damping coefficient of the elastic cables.
L	$\mathbb{R}$	Rest length of the elastic cables.
l_j	$\mathbb{R}$	Length of elastic cable attached to quadrotor j.
x_{Q_j}	$\mathbb{R}^3$	Position of quadrotor j in inertial frame.
x_L	$\mathbb{R}^3$	Position of center of mass of load in inertial frame.
R_L	$\mathrm{SO}(3)$	Attitude of load in body frame.
$\hat{\Omega}_L$	$\mathfrak{so}(3)$	Angular velocity of the load in body frame.
q_j	S^2	Direction vector of cable suspended from quadrotor j.
ω_j	TS^2	Angular velocity of cable j in the inertial frame.
r_j	$\mathbb{R}^3$	Connection point of elastic cable q_j to the load as measured in the body frame of the load.
R_j	$\mathrm{SO}(3)$	Attitude of quadrotor j.
$\hat{\Omega}_j$	$\mathfrak{so}(3)$	Angular velocity of quadrotor j in body frame.
$\mathbb{J}_Q$	$\mathrm{Sym}_{\succ 0}(\mathbb{R})$	Moment of inertia of quadrotor.
$\mathbb{J}_L$	$\mathrm{Sym}_{\succ 0}(\mathbb{R})$	Moment of inertia of the load.
u_j	$\mathbb{R}^3$	Net thrust applied vertically in the body frame of quadrotor j.
$u^{\perp_j}$	$\mathbb{R}^3$	The component of u that is perpendicular to q_j.
$u^{\parallel_j}$	$\mathbb{R}^3$	The component of u that is parallel to q_j.
M_j	$\mathbb{R}^3$	Moment vector in the body frame of quadrotor j.

Table 8.2: Nomenclature for arbitrarily large teams of quadrotor UAVs transporting rigid body load.

The configuration space of the mechanical system describing the cooperative task is given by

$$Q = \underbrace{(\mathrm{SO}(3) \times \mathbb{R}^3)}_{\text{Rigid load}} \times \underbrace{(S^2 \times \mathbb{R})^N}_{\text{Cables}} \times \underbrace{(\mathrm{SO}(3))^N}_{\text{Quadrotor attitudes}} .$$

The system has $6(N+1)$ degrees of freedom—6 degrees corresponding to the load, 3 degrees for each cable, and 3 degrees for each quadrotor. As in the two quadrotor system system described in Section 8.2, the positions of the quadrotors do not appear in the configuration space, as they are uniquely defined in terms of the other state variables due to the constraints $x_{Q_j} = x_L + R_L r_j - l_j q_j$ for $j \in \{1, \ldots, N\}$. Meanwhile, we have $4N$ inputs to the system in the form of N thrust controls $f_j \in \mathbb{R}$, corresponding to the total lift force exerted on the quadrotors by the spinning propellers, and N moment controls $M_j \in \mathbb{R}^3$, which are related to the torques induced on the quadrotors by the rotating propellers—for each $j \in \{1, \ldots, N\}$. Hence, the complete systems has $2N + 6$ degrees of under-actuation, with the quadrotor positions and attitudes being directly actuated. Note also that upon fixing a body frame to each quadrotor such that the vector $\bar{e}_3 = [0, 0, 1]^T$ points in the direction of the applied thrust, we may alternatively express the thrust controls f_j via the vectors $u_j = f_j R_j e_3 \in \mathbb{R}^3$ in the inertial frame for $j \in \{1, \ldots, N\}$. We will utilize this representation frequently throughout the section. Alternatively, one may choose to control the total thrust of *each* propeller individually. However we opt for the former approach both because it is more pervasive in the literature, and because it leads nicely to the separation of the quadrotor's attitude dynamics from the rest of the system's dynamics. The thrust generated by the i^{th} propeller along the e_3-axis can be determined by the total thrust and the moment controller [77].

8.3.1 Dynamics

Following [75] and the analysis conducted in Section 8.2, it is easily seen that the total kinetic energy $K : TQ \to \mathbb{R}$ of the system is given by

$$K = \underbrace{\frac{1}{2} m_L \|\dot{x}_L\|^2 + \sum_{j=1}^{N} \frac{1}{2} m_Q \|\dot{x}_{Q_j}\|^2}_{\text{Translational K.E.}} + \underbrace{\frac{1}{2} \Omega_L^T \mathbb{J}_L \Omega_L + \sum_{j=1}^{N} \frac{1}{2} \Omega_j^T \mathbb{J}_Q \Omega_j}_{\text{Angular K.E.}},$$

while the total potential energy of the system, $U : Q \to \mathbb{R}$, is given by

$$U = \sum_{j=1}^{N} \left[\underbrace{m_Q g e_3^T x_{Q_j} + m_L g e_3^T x_L}_{\text{Gravitational P.E.}} + \underbrace{\frac{1}{2} k (L - l_j)^2}_{\text{Elastic P.E.}} \right],$$

As usual, the Lagrangian of the system $\mathbf{L} : TQ \to \mathbb{R}$ is defined by $\mathbf{L} := K - U$. As in Section 8.2, the control inputs take the form of non-conservative external forces, so that we must use Lagrange d'Alembert Principle to obtain our system dynamics from the Lagrangian, and we further wish to add a non-conservative force corresponding to a damping in the elastic cables. In particular, we wish to minimize the action

$$\mathcal{A}(c(t)) = \int_a^b L(c(t), \dot{c}(t))\, dt + \sum_{j=1}^{N} \int_a^b \left(\|f_j R_j e_3\|^2_{\mathbb{R}^3} + \|\hat{M}_j\|^2_{\mathfrak{so}(3)} - c\dot{l}_j \right)\, dt,$$

over $C^\infty(Q, q_a, q_b)$. Following similar arguments as in the proof of Proposition 40, and in [75], we obtain the following system dynamics:

Proposition 40. *Critical points of the action functional $\mathcal{A}$ for variations with fixed endpoints corresponds to solutions of the controlled Euler-Lagrange equations*

$$\dot{x}_L = v_L \tag{8.63}$$

$$m_{\text{eff}}(\dot{v}_L + ge_3) = \sum_{j=1}^{N} \left[u_j + m_Q \ddot{\zeta}_j - m_Q R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j \right], \tag{8.64}$$

$$\dot{R}_L = R_L \hat{\Omega}_L \tag{8.65}$$

$$\mathbb{J}_{\text{eff}} \dot{\Omega}_L + \hat{\Omega}_L \mathbb{J}_{\text{eff}} \Omega_L = \sum_{j=1}^{N} m_Q \hat{r}_j R_L^T \left(-ge_3 - \dot{v}_L + \ddot{\zeta}_j + \frac{1}{m_Q} u_j \right), \tag{8.66}$$

$$m_Q q_j^T \ddot{\zeta}_j = m_Q q_j^T \left(\dot{v}_L + R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + ge_3 - \frac{1}{m_Q} u_j \right) - c\dot{l}_j + k(L - l_j), \tag{8.67}$$

$$q_j \times \ddot{\zeta}_j = q_j \times \left(\dot{v}_L + R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + ge_3 - \frac{1}{m_Q} u_j \right) \tag{8.68}$$

$$\dot{R}_j = R_j \hat{\Omega}_j \tag{8.69}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \qquad j = 1, \ldots, N \tag{8.70}$$

where $m_{\text{eff}} := Nm_Q + m_L$, $\mathbb{J}_{\text{eff}} := \mathbb{J}_L - \sum_{j=1}^{N} m_Q \hat{r}_j^2$, *and* $\zeta_j := l_j q_j$.

Remark 20. Equations (8.63) and (8.65) describe the kinematics of the load's position and attitude, respectively. Similarly, equations (8.64) and (8.66) describe the dynamics of the load's position and attitude, respectively. Equations (8.67), indexed for $j = 1, \ldots, N$, describe the dynamics of the lengths of the elastic cables. This can be understood by observing that the projection of $\ddot{\zeta}_j$ onto q_j preserves the acceleration of the

length (which is inherently oriented along the cable), while removing the acceleration of the attitude from consideration with the identity $q_j^T \ddot{q}_j = -\|\dot{q}_j\|^2$. Conversely, equations (8.68), indexed for $j = 1, \ldots, N$, describe the dynamics of the attitudes of the elastic cables, as the cross-product with q_j preserves the acceleration of the cable attitude while annihilating the acceleration of the cable length. Finally, the kinematics and dynamics of the quadrotor attitudes are described by equations (8.69) and (8.70), respectively, with each indexed for $j = 1, \ldots, N$.

8.3.2 Reduction via Singular Perturbations

As in Section 8.2.2, we now wish to reduce the dynamical system (8.63)-(8.70) using singular perturbation theory based on the premise that, in many applications, elastic cables have large spring constants and damping coefficients. As we will see, this results in the model described in [75] with inelastic cables (that is, by setting $l_j = L$ in the original model). The advantage of using singular perturbation theory is that it allows to obtain results in the elastic model from results in the inelastic model, as was done in Section 8.2.5 for the two quadrotor model. In particular, any control scheme that guarantees exponential tracking in the reduced model will provide near exponential tracking in the elastic model, provided that k and c are sufficiently large.

Consider the case that $k = \dfrac{\bar{k}}{\epsilon^2}$ and $c = \dfrac{\bar{c}}{\epsilon}$ with $\bar{k}, \bar{c} > 0$ and $\epsilon > 0$ sufficiently small. We show that the dynamics approach that of the same model with inelastic cables (that is, with $l_j \equiv L$) as $\epsilon \to 0$. Consider the change of variables $l_j = \epsilon^2 y_j + L$ and $\dot{l}_j = \epsilon z_j$, which is motivated by observing that $k(L - l_j) = \bar{k} y_j$ and $c\dot{l}_j = \bar{c} z_j$. From this, it is clear that $\zeta_j = (\epsilon^2 y_j + L) q_j$. Therefore, $\ddot{\zeta}_j = L\ddot{q}_j + \epsilon(\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j$. Making these substitutions into the dynamics described in Proposition 40, in addition to employing the

equation $\dot{q}_j = \omega_j \times q_j$, we obtain

$$\dot{x}_L = v_L$$

$$\dot{R}_L = R_L \hat{\Omega}_L,$$

$$\epsilon \dot{y}_j = z_j$$

$$\dot{q}_j = \omega_j \times q_j,$$

$$\dot{R}_j = R_j \hat{\Omega}_j$$

$$m_{\text{eff}}(\dot{v}_L + g e_3) = \sum_{j=1}^{N} \Big(u_j - m_Q R_L (\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + m_Q (L \ddot{q}_j + \epsilon(\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j) \Big),$$

$$\mathbb{J}_{\text{eff}} \dot{\Omega}_L + \hat{\Omega}_L \mathbb{J}_{\text{eff}} \Omega_L = \sum_{j=1}^{N} m_Q \hat{r}_j R_L^T \Big(- g e_3 - \dot{v}_L + L \ddot{q}_j + \epsilon(\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j + \frac{1}{m_Q} u_j \Big),$$

$$m_Q L \epsilon \dot{z}_j = q_j^T \Big(\bar{c} z_j q_j + \bar{k} y_j q_j + m_Q u_j + m_Q L (1 + \epsilon^2 y_j) \ddot{q}_j - m_Q (\dot{v}_L - m_Q R_L (\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + g e_3) \Big),$$

$$\dot{\omega}_j = \frac{1}{L} q_j \times \Big(\dot{v}_L - R_L (\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + g e_3 - \frac{1}{m_Q} u_j - \epsilon(\dot{z}_j q_j + z_j \omega_j) - \epsilon^2 y_j \ddot{q}_j \Big),$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \qquad j = 1, ..., N.$$

The previous system can be written in the form

$$\dot{x} = f(t, x, z; \epsilon) \tag{8.71}$$

$$\epsilon \dot{z} = g(t, x, z; \epsilon), \tag{8.72}$$

where f and g are smooth functions, x is the vector representing $(x_L, v_L, R_L, \Omega_L, q_j, \omega_j, R_j, \Omega_j)$, and z is the vector representing (y_j, z_j), for $j = 1, \ldots, N$. Note that the singular perturbation model (8.71)-(8.72) is equivalent to the elastic model (8.63)-(8.70), just condensed and with a change of variables. Evaluating at $\epsilon = 0$, the fast dynamics (8.72) provide us with algebraic equations that can be solved to obtain $z = h(t, x)$. In particular,

$$z_j = 0, \tag{8.73}$$

$$y_j = -\frac{m_Q}{\bar{k}} q_j^T \left[u_j + L \ddot{q}_j - \dot{v}_L + R_L (\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j - g e_3 \right]. \tag{8.74}$$

Substituting these equations back into the slow dynamics, we obtain the reduced (slow) model of the control system describing the cooperative task, given by $\dot{x} = f(t, x, h(t, x), 0)$. That is,

$$\dot{x}_L = v_L \tag{8.75}$$

$$m_{\text{eff}}(\dot{v}_L + ge_3) = \sum_{j=1}^{N} \Big(u_j - m_Q R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + m_Q L \ddot{q}_j \Big), \tag{8.76}$$

$$\dot{q}_j = \omega_j \times q_j, \tag{8.77}$$

$$\dot{\omega}_j = \frac{1}{L} \hat{q}_j \Big(\dot{v}_L - R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + ge_3 - \frac{1}{m_Q} u_j \Big), \tag{8.78}$$

$$\dot{R}_L = R_L \hat{\Omega}_L, \tag{8.79}$$

$$\mathbb{J}_{\text{eff}} \dot{\Omega}_L + \hat{\Omega}_L \mathbb{J}_{\text{eff}} \Omega_L = \sum_{j=1}^{N} m_Q \hat{r}_j R_L^T \Big(- ge_3 - \dot{v}_L + L \ddot{q}_j + \frac{1}{m_Q} u_j \Big), \tag{8.80}$$

$$\dot{R}_j = R_j \hat{\Omega}_j, \tag{8.81}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j, \qquad j = 1, ..., N. \tag{8.82}$$

Remark 21. Observe that the reduced model (8.75) - (8.82) preserves the original $4N$ inputs of the system, but the configuration space has lost N degrees of freedom (namely the cable lengths). Hence, the reduced model has $N + 6$ degrees of under-actuation. As previously mentioned, the reduced model is equivalent to the original model with inelastic cables (that is, where $l_j \equiv L$ for $j = 1, \ldots, N$). This system was shown to be differentially flat in [100].

8.3.3 Control Design

In this Section, a control design for the reduced model (8.75)-(8.82) is provided. That is, we provide a set of controllers $u_j \in \mathbb{R}^3$ such that the position and attitude of the load reach a desired position $\tilde{x}_L \in \mathbb{R}^3$ and attitude $\tilde{R}_L \in \text{SO}(3)$ exponentially fast. As in the two quadrotor task, we first treat the attitude dynamics as if they were decoupled from the rest of the system dynamics, and design u_j under the assumption that thrust can be generated in any direction. We will later design the total thrust controls f_j, the moment controls M_j, and the desired quadrotor attitudes $\tilde{R}_j$, so that the quantity $f_j R_j e_3$ converges exponentially fast to the u_j that we design. The strategy again will be to decompose u_j into components which are parallel and

perpendicular to the cable attitudes q_j via $u_j = u_j^{\parallel} + u_j^{\perp}$, where $u_j^{\parallel} = (q_j^T u_j) q_j$ and $u_j^{\perp} = (I - q_j q_j^T) u_j$. This is motivated by the fact that only the $u_j^{\parallel}$ components influence the dynamics of the load, while only the $u_j^{\perp}$ components influence the cables' dynamics. Its graphical description is shown in Figure 8.4.

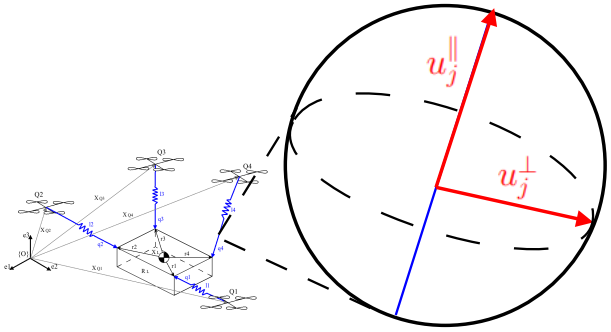


Figure 8.4: Graphical description for the decomposition of the control input viewed on each cable.

We begin by further simplifying the dynamical system (8.75)-(8.82). From equation (8.78) and the identity $\ddot{q}_j = \dot{\omega}_j \times q_j - \|\omega_j\|^2 q_j$, we find that

$$\begin{aligned} L\ddot{q}_j &= -q_j \times (L\dot{\omega}_j) - L\|\omega_j\|^2 q_j \\ &= (I - q_j q_j^T)\Big[\dot{v}_L - R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + g e_3 - \frac{1}{m_Q} u_j\Big] - L\,\|\omega_j\|^2\, q_j. \end{aligned}$$

Substituting this equation for into (8.76) and making use of the fact that $m_{\text{eff}} = N m_Q + m_L$, we obtain

$$M_L(\dot{v}_L + g e_3) = \sum_{j=1}^{N} \Big[m_Q q_j q_j^T R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + u_j^{\parallel} - m_Q L\,\|\omega_j\|^2\, q_j \Big], \tag{8.83}$$

where $M_L = m_L I + \sum_{j=1}^{N} m_Q q_j q_j^T$. Repeating this procedure with (8.80), and making use of the fact that $\mathbb{J}_{\text{eff}} = \mathbb{J}_L - \sum_{j=1}^{N} m_Q \hat{r}_j^2$ yields

$$\mathbb{J}_L \dot{\Omega}_L + \hat{\Omega}_L \mathbb{J}_L \Omega_L = \sum_{j=1}^{N} m_Q \hat{r}_j R_L^T \Big[q_j q_j^T R_L(\hat{r}_j \dot{\Omega}_L + \hat{\Omega}_L^2 r_j) - q_j q_j^T(\dot{v}_L + g e_3) - L\|\omega_j\|^2 q_j + \frac{1}{m_Q} u_j^{\parallel} \Big]. \tag{8.84}$$

With equation (8.83) in place of (8.75), and (8.84) in place of (8.80), the reduced model is exactly the control system discussed in [75]. As such, we may adopt the control scheme used there. In particular, if we denote

our desired trajectories by $\tilde{x}_L, \tilde{v}_L, \tilde{R}_L, \tilde{\Omega}_L, \tilde{q}_j, \tilde{\omega}_j$ and define tracking errors as:

$$
\begin{aligned}
e_{x_L} &= x_L - \tilde{x}_L, & e_{v_L} &= v_L - \tilde{v}_L, \\
e_{R_L} &= \frac{1}{2}\left(\tilde{R}_L^T R_L - R_L^T \tilde{R}_L\right)^\vee, & e_{\Omega_L} &= \Omega_L - R_L^T \tilde{R}_L \tilde{\Omega}_L, \\
e_{q_j} &= \tilde{q}_j \times q_j, & e_{\omega_j} &= \omega_j + \hat{q}_j^2 \tilde{\omega}_j,
\end{aligned}
$$

then for some choice of gains $k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}, k_q, k_\omega \in \mathbb{R}$, the controllers take the form:

$$u_j^{\parallel} = \mu_j + m_Q L \|\omega_j\|^2 q_j + m_Q (q_j^T a_j) q_j, \tag{8.85}$$

$$u_j^{\perp} = m_Q L \hat{q}_j \left(-k_q e_{q_j} - k_\omega e_{\omega_j} - (q_j^T \tilde{\omega}_j)\dot{q}_j - \hat{q}_j^2 \dot{\tilde{\omega}}_j\right) - m_Q \hat{q}_j^2 a_j, \tag{8.86}$$

where $a_j = \dot{v}_L + g e_3 + R_L \hat{\Omega}_L^2 r_j - R_L \hat{r}_j \dot{\Omega}_L$, and μ_j is an additional controller which satisfies:

$$\mu_j = (q_j^T \tilde{\mu}_j) q_j \tag{8.87}$$

$$\sum_{j=1}^{N} \tilde{\mu}_j = m_L \left(-k_{x_L} e_{x_L} - k_{v_L} e_{v_L} + \dot{\tilde{v}}_L + g e_3\right) \tag{8.88}$$

$$\sum_{j=1}^{N} \hat{r}_j R_L^T \tilde{\mu}_j = -k_{R_L} e_{R_L} - k_{\Omega_L} e_{\Omega_L} + \left(R_L^T \tilde{R}_L \tilde{\Omega}_L\right)^\wedge \mathbb{J}_L R_L^T \tilde{R}_L \tilde{\Omega}_L + \mathbb{J}_L R_L^T \tilde{R}_L \dot{\tilde{\Omega}}_L. \tag{8.89}$$

We further define the desired cable attitudes as $\tilde{q}_j = -\frac{\tilde{\mu}_j}{\|\tilde{\mu}_j\|}$, so that $\mu_j \to \tilde{\mu}_j$ as $\tilde{q}_j \to q_j$. Notice that equations (8.88) and (8.89) can also be written in the form

$$\mathcal{P}\text{diag}(R_L^T, ..., R_L^T) \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix} = \begin{bmatrix} R_L^T F_L \\ M_L \end{bmatrix}$$

where

$$\mathcal{P} = \begin{bmatrix} I_{3\times 3} & \cdots & I_{3\times 3} \\ \hat{r}_1 & \cdots & \hat{r}_N \end{bmatrix} \in \mathbb{R}^{6\times 3N}, \tag{8.90}$$

$$F_L = m_L \left(-k_{x_L} e_{x_L} - k_{v_L} e_{v_L} + \dot{\tilde{v}}_L + g e_3\right), \tag{8.91}$$

$$M_L = -k_{R_L} e_{R_L} - k_{\Omega_L} e_{\Omega_L} + \left(R_L^T \tilde{R}_L \tilde{\Omega}_L\right)^\wedge \mathbb{J}_L R_L^T \tilde{R}_L \tilde{\Omega}_L + \mathbb{J}_L R_L^T \tilde{R}_L \dot{\tilde{\Omega}}_L. \tag{8.92}$$

Then, if we assume that rank$(\mathcal{P}) = 6$ (which depends on the physical connection points of the cables to the load), there is guaranteed to exist a solution $\begin{bmatrix} \tilde{\mu}_1 & \dots & \tilde{\mu}_N \end{bmatrix}^T$. However, in general there will exist multiple solutions, so we choose the solution with minimal (Euclidean) norm. This is given by

$$\begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix} = \text{diag}(R_L, ..., R_L)\mathcal{P}^T(\mathcal{P}\mathcal{P}^T)^{-1} \begin{bmatrix} R_L^T F_L \\ M_L \end{bmatrix}. \tag{8.93}$$

With these controllers, we have the following error dynamics:

$$\dot{e}_{v_L} = -k_{v_L} e_{v_L} - k_{x_L} e_{x_L} - \frac{1}{m_Q} \sum_{j=1}^{N} (\tilde{q}_j^T \tilde{\mu}_j) \hat{q}_j e_{q_j}, \tag{8.94}$$

$$\mathbb{J}_L \dot{e}_{\Omega_L} = -k_{\Omega_L} e_{\Omega_L} - k_{R_L} e_{R_L} + [\mathbb{J}_L e_{\Omega_L} + (2\mathbb{J}_L - \text{tr}(\mathbb{J}_L) I) R_L^T \tilde{R}_L \tilde{\Omega}_L]^\vee e_{\Omega_L} - \sum_{j=1}^{N} \hat{r}_j R_L^T (\tilde{q}_j^T \tilde{\mu}_j) \hat{q}_j e_{q_j}, \tag{8.95}$$

$$e_{\omega_j} = (\dot{q}_j^T \tilde{\omega}_j) q_j - k_\omega e_{\omega_j} - k_q e_{q_j}. \tag{8.96}$$

8.3.4 Stability Analysis

The goal now is to prove that the gains can be chosen in such a way that the error dynamics (8.94)-(8.96) have an exponentially stable equilibrium point at the origin. This will be accomplished via Lyapunov analysis, and the procedure will follow similarly to that implemented in Section 8.2.4. Before this, however, we will introduce two configuration errors—Ψ_{q_j} and Ψ_{R_L}. These are real-valued functions which are positive-definite about the points $q_j = \tilde{q}_j$ and $R_L = \tilde{R}_L$, respectively, and are defined explicitly as $\Psi_{q_j} = 1 - \tilde{q}_j^T q_j$ and $\Psi_{R_L} = \frac{1}{2}\text{tr}\left(I - \tilde{R}_L^T R_L\right)$.

We now define a Lyapunov candidate $V : \mathcal{D} \to \mathbb{R}$ defined as

$$\begin{aligned} V =& \frac{1}{2}\|e_{v_L}\|^2 + \frac{1}{2} k_{x_L} \|e_{x_L}\|^2 + c_x e_{x_L}^T e_{v_L} + \frac{1}{2} e_{\Omega_L}^T \mathbb{J}_L \Omega_L + k_{R_L} \Psi_{R_L} + c_R e_{R_L}^T \mathbb{J}_L e_{\Omega_L} \\ &+ \sum_{j=1}^{N} \left[\frac{1}{2}\|e_{\omega_j}\|^2 + k_q \Psi_{q_j} + c_q e_{q_j}^T e_{\omega_j} \right], \end{aligned}$$

which is defined on the domain

$$\mathcal{D} = \{(e_{x_L}, e_{v_L}, e_{R_L}, e_{\Omega_L}, e_{q_j}, e_{\omega_j}) \ : \ \|e_{x_L}\| < e_{x_{max}}, \ \Psi_{R_L} < \psi_{R_L} < 1, \ \Psi_{q_j} < \psi_{q_j} < 1\},$$

where $c_x, c_q, c_R, \psi_{q_j}, \psi_{R_L}$ are positive real numbers. Before establishing sufficient conditions for the gains which ensure exponential stability of the zero equilibrium for the tracking error, we state the following Lemma, which follows from [75].

$$\underline{\mathrm{P}}_{x_L} = \frac{1}{2}\begin{bmatrix} k_{x_L} & -c_x \\ -c_x & 1 \end{bmatrix}, \quad \bar{P}_{x_L} = \frac{1}{2}\begin{bmatrix} k_{x_L} & c_x \\ c_x & 1 \end{bmatrix}, \quad \underline{\mathrm{P}}_{R_L} = \frac{1}{2}\begin{bmatrix} k_{R_L} & -c_R\lambda_{\max}(\mathbb{J}_L) \\ -c_R\lambda_{\max}(\mathbb{J}_L) & \lambda_{\min}(\mathbb{J}_L) \end{bmatrix}, \tag{8.97}$$

$$\bar{P}_{R_L} = \frac{1}{2}\begin{bmatrix} \frac{2k_{R_L}}{2-\psi_{R_L}} & c_R\lambda_{\max}(\mathbb{J}_L) \\ c_R\lambda_{\max}(\mathbb{J}_L) & \lambda_{\max}(\mathbb{J}_L) \end{bmatrix}, \underline{\mathrm{P}}_{q_j} = \frac{1}{2}\begin{bmatrix} k_q & -c_q \\ -c_q & 1 \end{bmatrix}, \bar{P}_{q_j} = \frac{1}{2}\begin{bmatrix} \frac{2k_q}{2-\psi_{q_j}} & c_q \\ c_q & 1 \end{bmatrix}, \tag{8.98}$$

$$W_j = \begin{bmatrix} W_{x_j} & -\frac{1}{2}W_{xR_j} & -\frac{1}{2}W_{xq_j} \\ -\frac{1}{2}W_{xR_j} & W_{R_j} & -\frac{1}{2}W_{Rq_j} \\ -\frac{1}{2}W_{xq_j} & -\frac{1}{2}W_{Rq_j} & W_{q_j} \end{bmatrix}, \quad W_{x_j} = \frac{1}{4}\begin{bmatrix} c_x k_{x_L}(1-4\alpha_j\beta) & -\frac{1}{2}c_x k_{v_L}(1+4\alpha_j\beta) \\ -\frac{1}{2}c_x k_{v_L}(1+4\alpha_j\beta) & k_{v_L}(1-4\alpha_j\beta) - c_x \end{bmatrix}, \tag{8.99}$$

$$W_{R_j} = \frac{1}{4}\begin{bmatrix} c_R k_{R_L}(1-4\alpha_j\sigma_j) & -\frac{c_R}{2}(k_{\Omega_L} + B + 4\alpha_j\sigma_j) \\ -\frac{c_R}{2}(k_{\Omega_L} + B + 4\alpha_j\sigma_j) & k_{\Omega_L}(1-4\alpha_j\sigma_j) - 2c_R\lambda_{\max}(\mathbb{J}_L) \end{bmatrix}, \quad W_{xq_j} = \begin{bmatrix} c_x B & 0 \\ \beta k_{x_L} e_{x_{\max}} + B & 0 \end{bmatrix}, \tag{8.100}$$

$$W_{q_j} = \begin{bmatrix} c_q k_q & -\frac{1}{2}c_q(k_\omega + C_{q_j}) \\ -\frac{1}{2}c_q(k_\omega + C_{q_j}) & k_\omega - c_q \end{bmatrix}, \quad W_{xR_j} = \alpha_j \begin{bmatrix} \gamma c_x k_{R_L} + \delta_j c_R k_{x_L} & \gamma c_x k_{\Omega_L} + \delta_j k_{x_L} \\ \gamma k_{R_L} + \delta_j c_R k_{v_L} & \gamma k_{\Omega_L} + \delta_j k_{v_L} \end{bmatrix}, \tag{8.101}$$

$$W_{qR_j} = \begin{bmatrix} c_R B & 0 \\ \alpha_L \sigma_j k_{R_L} + B & 0 \end{bmatrix}. \tag{8.102}$$

Lemma 8.3.1. *Consider the set of matrices* (8.97)-(8.102) *for* $\alpha_j := \sqrt{\psi_{q_j}(2-\psi_{q_j})}$, $\alpha_L := \sqrt{\psi_{R_L}(2-\psi_{R_L})}$, $\gamma = \frac{1}{m_L\gamma_{\min}(\mathcal{P}\mathcal{P}^T)}$, $\beta = m_L\gamma$, $\delta_j = m_L\frac{\|\hat{r}_j\|}{\sqrt{\lambda_{\min}(\mathcal{P}\mathcal{P}^T)}}$, $\sigma_j = \frac{\delta_j}{m_L}$, C_{q_j} *is a positive constant that can be obtained from* $\tilde{q}_j$, *and* B *is a positive constant which can be obtained from* $\tilde{x}_L$ *and* $\tilde{R}_L$. *If* $\bar{P}_{x_L}, \underline{P}_{x_L}, \bar{P}_{R_L}, \underline{P}_{R_L}, \bar{P}_{q_j}, \underline{P}_{q_j}$, *and* $\frac{1}{2}(W_j + W_j^T)$ *are positive-definite matrices for* $j = 1, ..., N$, *then the origin of the error dynamics* (8.94)-(8.96) *is an exponentially stable equilibrium point.*

While Lemma 8.3.1 was obtained in [77] and [75], no rigorous study guaranteeing the existence of stabilizing gains, much less strategies to obtain said gains, were ever obtained. That is what we seek to do in the following Theorem, which adopts a similar strategy to that used in Proposition 37.

Proposition 41. *Consider the control system defined by* (8.75)-(8.80) *and control inputs determined by* (8.87)-(8.89), *with* $u_j^{\parallel}$ *and* $u_j^{\perp}$ *given by equation* (8.85)-(8.86). *For sufficiently small* α_j, *there exists gains* $k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}, k_q, k_\omega$, *such that the zero equilibrium of the tracking error* $(e_{x_L}, e_{v_L}, e_{R_L}, e_{\Omega_L}, e_{q_j}, e_{\omega_j})$

is exponentially stable, for $j = 1, \ldots, N$.

Proof. It is easy to see that the matrices $\bar{P}_{x_L}, \underline{\mathrm{P}}_{x_L}, \bar{P}_{R_L}, \underline{\mathrm{P}}_{R_L}, \bar{P}_{q_j}$, and $\underline{\mathrm{P}}_{q_j}$ are positive-definite for c_x, c_R, c_q sufficiently small. In particular, the matrices are simultaneously positive-definite for any choice of gains if and only if we take

$$c_x < \sqrt{k_x}, \quad c_q < \sqrt{k_q}, \quad c_R < \sqrt{\frac{k_R \lambda_{\min}(\mathbb{J}_L)}{\lambda_{\max}(\mathbb{J}_L)}}$$

Now denote the symmetric part of W_j by $\mathcal{W}_j = \frac{1}{2}(W_j + W_j^T)$ and similarly define the symmetric parts of the submatrices by $\mathcal{W}_{xR_j}, \mathcal{W}_{xq_j}$, and $\mathcal{W}_{qR_j}$. It is clear that $\mathcal{W}_j$ can be expressed in the form $\mathcal{W}_j = \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix}$, where $P = \begin{bmatrix} W_{x_j} & -\frac{1}{2}\mathcal{W}_{xR_j} \\ -\frac{1}{2}\mathcal{W}_{xR_j} & W_{R_j} \end{bmatrix}$, $S = -\frac{1}{2}\begin{bmatrix} \mathcal{W}_{xq_j} & \mathcal{W}_{qR_j} \end{bmatrix}^T$, and $Q = W_{q_j}$. Now, observe that $\mathcal{W}$ can be decomposed as:

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} = \begin{bmatrix} I & SQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P - SQ^{-1}S^T & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & SQ^{-1} \\ 0 & I \end{bmatrix}^T$$

where $P - SQ^{-1}S^T$ is often referred to as the *Schur complement* of Q. It then follows that $\mathcal{W} \succ 0$ if and only if $P - SQ^{-1}S^T \succ 0$ and $Q \succ 0$. Note that $P - SQ^{-1}S^T$ can itself be expressed in form of a 4×4 block matrix given by

$$P - SQ^{-1}S^T = \begin{bmatrix} W_{x_j} - \frac{1}{4}\mathcal{W}_{xq_j}W_{q_j}^{-1}\mathcal{W}_{xq_j} & -\frac{1}{2}\mathcal{W}_{xR_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{xq_j} \\ -\frac{1}{2}\mathcal{W}_{xR_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{xq_j} & W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j} \end{bmatrix}.$$

Repeating the previous analysis, but now on $P - SQ^{-1}S^T$, we find that $\mathcal{W}_j \succ 0$ if and only if the following three conditions hold: (**1**) $W_{q_j} \succ 0$, (**2**)$W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j} \succ 0$ and (**3**)

$$\begin{aligned} 0 \prec\ & W_{x_j} - \frac{1}{4}\mathcal{W}_{xq_j}W_{q_j}^{-1}\mathcal{W}_{xq_j} \\ & - \frac{1}{4}\mathcal{W}_{xR_j}(W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1}\mathcal{W}_{xR_j} \\ & - \frac{1}{8}\mathcal{W}_{xR_j}(W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{xq_j} \\ & - \frac{1}{8}\mathcal{W}_{xq_j}W_{q_j}^{-1}\mathcal{W}_{qR_j}(W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1}\mathcal{W}_{xR_j} \\ & - \frac{1}{16}\mathcal{W}_{xq_j}W_{q_j}^{-1}\mathcal{W}_{qR_j}(W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{xq_j}. \end{aligned}$$

Moreover, the minimum (maximum) eigenvalue of $\mathcal{W}$ is precisely the smallest (largest) of the minimum (maximum) eigenvalues of the three matrices in the above conditions. We now seek to verify that appropriate choices in the gains and constants can be made to satisfy the above conditions. For (**1**), observe that the characteristic polynomial of W_{q_j} yields

$$2\lambda_{\min}(W_{q_j}) =(k_\omega - c_q + c_q k_q) - \sqrt{(k_\omega - c_q - c_q k_q)^2 + c_q^2(k_\omega + C_{q_j})^2}.$$

This motivates the choice $k_\omega = c_q k_q + c_q$, from which we obtain $\lambda_{\min}(W_{q_j}) = c_q\left(2k_q - c_q(k_q+1) - C_{q_j}\right)$. Now let $C_q := \max_{j=1,\dots,4} C_{q_j}$ and choose $c_q = \frac{k_q - C_q}{1+k_q}$. For some fixed $\epsilon > 0$, we then require $k_q > C_q + \epsilon := k_{q^*}$, from which it follows that $c_q^* := \frac{\epsilon}{1+\epsilon+C_q} < c_q < \sqrt{k_q}$, and $\lambda_{\min}(W_{q_j}) \geq c_q^* k_q > 0$. In summary, $\underline{P}_q \succ 0$, and $W_{q_j} \succ 0$ provided that $k_q > k_q^*$. Consequently, $\lambda_{\max}(W_{q_j}^{-1}) = \lambda_{\min}(W_{q_j})^{-1}$ can be made arbitrarily small (but positive). Hence, For (2), we have $\frac{1}{4}\mathcal{W}_{qR_j} W_{q_j}^{-1} \mathcal{W}_{qR_j} \succeq 0$ and, since $\mathcal{W}_{qR_j}$ and W_{q_j} are independent, we can shrink the maximum eigenvalue of $\mathcal{W}_{qR_j} W_{q_j}^{-1} \mathcal{W}_{qR_j}$ arbitrarily by shrinking the maximum eigenvalue of $W_{q_j}^{-1}$. Moreover, observe that

$$2\lambda_{\min}(W_{R_j}) = (c_R k_{R_L} + k_{\Omega_L})\nu_j - 2c_R\lambda_{\max}(\mathbb{J}_L) - \sqrt{\left((c_R k_{R_L} - k_{\Omega_L})\nu_j + 2c_R\lambda_{\max}(\mathbb{J}_L)\right)^2 + c_R^2\left(k_{\Omega_L} + B + 1 - \nu_j\right)^2}.$$

Now choose $k_{\Omega_L} = c_R k_{R_L} + \dfrac{2c_R\lambda_{\max}(\mathbb{J}_L)}{\nu}$, where $\underline{\nu} = \min\limits_{j=1,\dots,4} \nu_j$. We will also need to similarly define $\bar{\nu} = \max\limits_{j=1,\dots,4} \nu_j$. In all,

$$\lambda_{\min}(W_{R_j}) \geq c_R\left[2k_{R_L}\nu_j - \sqrt{E^2 + (c_R F + G)^2}\right],$$

for $E = 2\left(\dfrac{\bar{\nu}}{\underline{\nu}} - 1\right)\lambda_{\max}(\mathbb{J}_L)$, $F = \dfrac{k_{\Omega_L}}{c_R}$, and $G = B + 1 - \underline{\nu}$. Furthermore, we assume that $\underline{\nu}_j > \frac{2}{3}$ (which amounts to requiring the initial cable attitude errors to be sufficiently small). It can be shown that with

$$k_{R_L} \geq \frac{-E + \sqrt{E^2 + (G+\epsilon)^2}}{\underline{\nu}} := k_{R_L}^*,$$
$$c_R = \min\left\{\sqrt{\frac{k_{R_L}\lambda_{\min}(\mathbb{J}_L)}{\lambda_{\max}(\mathbb{J}_L)}}, \frac{\underline{\nu}\left(\sqrt{k_{R_L}^2\underline{\nu}^2 + 2k_{R_L}E\underline{\nu}} - G\right)}{k_{R_L}\underline{\nu} + 2\lambda_{\max}(\mathbb{J}_L)}\right\},$$

we have $c_R > c_R^*$, where

$$c_R^* = \min\left\{\sqrt{\frac{B + 1 - \underline{\nu} + \epsilon}{\underline{\nu}}}, \frac{\underline{\nu}\epsilon}{-E + \sqrt{E^2 + (G+\epsilon)^2} + 2\lambda_{\max}(\mathbb{J}_L)}\right\}.$$

Hence, $\underline{P}_{W_L} \succ 0$, and $\lambda_{\min}(W_{R_j}) \geq c_R^* \nu_j k_{R_L}$. Therefore, $W_{R_j} \succ 0$ for $j = 1, \ldots, 4$, and $\lambda_{\min}(W_{R_j})$ can be made arbitrarily large by taking k_{R_L} sufficiently large. In total, condition (2) can easily be satisfied by taking the appropriate combination of $\lambda_{\min}(W_{q_j})$ and $\lambda_{\min}(W_{R_j})$ (say, but taking each to be sufficiently large).

Now, we look at condition (3). First notice that, by a similar argument to those provided for $\lambda_{\min}(W_{q_j})$ and $\lambda_{\min}(W_{R_j})$, it can be shown that there exists positive constants $k_{x_L}^*, c_x^*$ bounded away from zero which depend only on the initial cable attitude errors, the desired trajectory chosen for $\tilde{x}_L$, the connection points of the cable to the load, and the parameter $\epsilon > 0$, such that, for $k_{v_L} = c_x k_{x_L} + \frac{c_x}{\eta_j}$ and $k_{x_L} > k_{x_L}^*$, we have $\underline{P}_{x_L} \succ 0$, $c_x > c_x^*$, and $\lambda_{\min}(W_{x_j}) > c_x^* \eta_j k_{x_L}$, where $\eta_j := 1 - 4\alpha_j \beta$. In particular, $W_{x_j} \succ 0$ for $j = 1, \ldots, 4$, and $\lambda_{\min}(W_{x_j})$ can be made arbitrarily large by taking k_{x_L} sufficiently large. Now suppose that such a choice is made, so that $W_{x_j} \succ 0$ for $j = 1, \ldots, 4$. We wish to show that the remaining subtractive terms in condition (3) can be shrunk arbitrarily, independent of W_{x_j}. Observe that $\mathcal{W}_{xq_j}$ and $W_{q_j}^{-1}$ are independent, so that we may force the maximum eigenvalue of $\mathcal{W}_{xq_j} W_{q_j}^{-1} \mathcal{W}_{xq_j} \succeq 0$ to be arbitrarily small after (potentially) further shrinking the maximum eigenvalue of $W_{q_j}^{-1}$.

The fourth and fifth terms are transposes of each other and therefore may be handled simultaneously. Note that find that the maximum eigenvalue is bounded above by

$$\begin{aligned}
&\lambda_{\max}((W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1})\|\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{xq_j}\| \, \|\mathcal{W}_{xR_j}\| \\
&\leq \sqrt{\lambda_{\max}(\mathcal{W}_{q_j}^{-1})}\lambda_{\max}((W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1})\|\mathcal{W}_{qR_j}\| \, \|\mathcal{W}_{xq_j}\| \, \|\mathcal{W}_{xR_j}\|.
\end{aligned}$$

Consequently, this term can be made arbitrarily small by making the maximum eigenvalue of $\mathcal{W}_{q_j}$ sufficiently large. The final term is handled similarly—we find that the maximum eigenvalue is bounded by

$$\begin{aligned}
&\lambda_{\max}((W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1})\|\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{xq_j}\|^2 \\
&\leq \lambda_{\max}(\mathcal{W}_{q_j}^{-1})\lambda_{\max}((W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}\mathcal{W}_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1})\|\mathcal{W}_{qR_j}\|^2 \, \|\mathcal{W}_{xq_j}\|^2,
\end{aligned}$$

which again may be shrunk arbitrarily by shrinking the maximum eigenvalue of $\mathcal{W}_{q_j}^{-1}$. Finally, the third term $\frac{1}{4}\mathcal{W}_{xR_j}(W_{R_j} - \frac{1}{4}\mathcal{W}_{qR_j}W_{q_j}^{-1}\mathcal{W}_{qR_j})^{-1}\mathcal{W}_{xR_j}$ may be written in the form $\alpha^2 M^T A M$, where M is independent of α and the terms of A are at most of $\mathcal{O}(\frac{1}{\alpha})$. It follows that if α is sufficiently small and $\lambda_{\max}(W_{q_j})$ is

sufficiently large, condition (3) holds and $\mathcal{W}_j \succ 0$. $\square$

Remark 22. We now wish to address the issue of control saturation, which is marked by the control magnitude becoming larger than is permitted by the system's physical constraints. From the control design conducted in Section 8.3.3, it is clear that saturation depends in part on the chosen desired trajectories. This is intuitively sensible, as trajectories with very high velocities, accelerations, and jerks will be impossible to track effectively. Beyond this, the control depends linearly on the gains. Assuming that sufficiently "nice" desired trajectories are chosen, and initial configuration errors are sufficiently small to allow exponential convergence to said trajectories, it is reasonable to assume that control saturation may be avoided provided that the gains can be chosen sufficiently small. For simplicity, we assume that the initial configuration errors are zero, from which the result will immediately follow for sufficiently small initial errors.

For simplicity, let $x = \lambda_{\min}(W_{q_j})$, $y = \lambda_{\min}(W_{R_j})$, and $z = \lambda_{\min}(W_{x_j})$, and choose the gains and Lyapunov constants so that they satisfy the conditions previously outlined in the proof of Theorem 41. That is, we have $k_q > k_{q^*}, k_{R_L} > k^*_{R_L}$, and $k_{x_L} > k^*_{x_L}$, which implies that $c_q > c^*_q$, $c_R > c^*_R$, and $c_x > c^*_x$, where $k^*_q, k^*_{R_L}, k^*_{x_L}, c^*_q, c^*_R$, and c^*_x are positive constants which are bounded away from zero and depend only on the norms of the velocities, accelerations, and jerks of the chosen desired trajectories, the initial configuration errors (which we assume to be zero), and some fixed parameter $\epsilon > 0$. Moreover, as $B, C_q \to 0$ (which depends only on the chosen desired trajectories), $k_q \to \epsilon$, $k_{R_L} \to (1+\epsilon)^2$, and k_{x_L} tends to a quadratic polynomial in ϵ. With these choices, we have

$$
\begin{aligned}
k_q &< \frac{x}{c^*_q}, & k_\omega &< k_q + 1, \\
k_{R_L} &< \frac{y}{c^*_{R_L}}, & k_{\Omega_L} &< k_{R_L} + 2\lambda_{\max}(\mathbb{J}_L), \\
k_{x_L} &< \frac{z}{c^*_x}, & k_{V_L} &< k_{x_L} + 1.
\end{aligned}
$$

Condition (2) from the proof of Theorem 41 will be satisfied if $y > \dfrac{B^2(1+2c^*_R)^2}{16x}$ (such a choice can always be made). Moreover, we see that as $B \to 0$ (which depends only on the desired trajectories chosen for x_L and R_L), we need only take $x, y > 0$. Similarly, condition (3) from the proof of Theorem 41

will be satisfied if

$$z > \frac{B^2(1+2c_x^*)^2}{16x} - \frac{B^4(1+2c_x^*)^2(1+2c_R^*)^2}{16x(16xy - B^2(1+2c_R^*)^2}.$$

As before, we see that if the desired trajectory data B tends to zero, we need only take $x, y, z > 0$. Moreover, since the gains are bounded above by linear functions of the corresponding eigenvalues x, y, or z, and below by polynomials in ϵ (which is a free parameter), it follows that the exponential convergence of the state variables to their desired trajectories can be obtained while also choosing the gains small enough to avoid control saturation provided that the initial errors are sufficiently small and the desired trajectories are chosen sufficiently "nice," as expected.

8.3.5 Control Design for the Full Model

Along the control design discussed in Section 8.2.3, we assumed that the thrust u_j could be generated in any direction. In reality, however, the constraint $u_j = f_j R_j e_3$ requires the thrust to always point in the direction of the third fixed-body axis. That is, u_j cannot be designed to point in any arbitrary direction. However, as in the two quadrotor task, may design the thrust controls f_j, the moment controls M_j, and the desired quadrotor attitudes $\tilde{R}_j$, so that the quantity $f_j R_j e_3$ converges exponentially fast to the u_j that we designed in Section 8.2.3. We again use the controls (8.38) and (8.39), and design $\tilde{R}_j$ as we did in Section 8.2.5, which leads to the following corollary to Proposition 41:

Corollary 15. *Consider the control system defined by* (8.75)-(8.82) *and control inputs determined by* (8.87)-(8.89)*, with* $u_j^{\parallel}$ *and* $u_j^{\perp}$ *given by equation* (8.85)-(8.86)*, and moment control* (8.39)*. There exists* $\delta > 0$*, such that for all* $\epsilon < \delta$ *and for sufficiently small* α_j*, there exists gains* k_{x_L}*,* k_{v_L}*,* k_{R_L}*,* k_{Ω_L}*,* k_q*,* k_ω*,* k_{R_j}*,* k_{Ω_j}*, such that the state variables* x_L, R_L, q_j, R_j *approach their desired trajectories* $\tilde{x}_L, \tilde{R}_L, \tilde{q}_j, \tilde{R}_j$ *exponentially fast for* $j = 1, \ldots, N$.

Moreover, as in Section 8.2.5, we may make the following conclusion about the tracking problem in the original (elastic) using Corollary 15, Theorem 11.2 in [64], and the analysis conducted in Lemma 2 of [70]):

Proposition 42. *Let the control inputs u_j and M_j be defined as above. Denote by $x(t)$ a trajectory of the reduced model* (8.75)-(8.82) *which converges exponentially to the desired trajectory. Denote by $r(t)$ a trajectory of the boundary layer system which converges exponentially to the origin.*

There exists a positive constant ϵ^* such that for all $t \geq t_0$ and $0 < \epsilon < \epsilon^*$, there exists a unique solution $x(t,\epsilon)$, $z(t,\epsilon)$ of the singular perturbation problem (8.71)-(8.72) *on $[t_0, \infty)$ satisfying $x(t,\epsilon) - x(t) = \mathcal{O}(\epsilon)$ and $z(t,\epsilon) - h(t,x(t)) - r\left(\frac{t-t_0}{\epsilon}\right) = \mathcal{O}(\epsilon)$, uniformly on $t \in [t_0, \infty)$. Moreover, for $t_1 > t_0$, we have $z(t,\epsilon) - h(t,x(t)) = \mathcal{O}(\epsilon)$, uniformly on (t_1, ∞) for $\epsilon < \epsilon^{**} < \epsilon^*$.*

Finally, for completion, we consider the case of unbounded unstructured disturbances. In particular, consider the dynamical system

$$\dot{x}_L = v_L \tag{8.103}$$

$$m_{\text{eff}}(\dot{v}_L + ge_3) = \sum_{j=1}^{N} \Big(u_j - m_Q R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + m_Q L \ddot{q}_j \Big) + \Delta x_L, \tag{8.104}$$

$$\dot{q}_j = \omega_j \times q_j, \tag{8.105}$$

$$\dot{\omega}_j = \frac{1}{L}\hat{q}_j \Big(\dot{v}_L - R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_j + ge_3 - \frac{1}{m_Q} u_j \Big) + \Delta q_j, \tag{8.106}$$

$$\dot{R}_L = R_L \hat{\Omega}_L, \tag{8.107}$$

$$\mathbb{J}_{\text{eff}} \dot{\Omega}_L + \hat{\Omega}_L \mathbb{J}_{\text{eff}} \Omega_L = \sum_{j=1}^{N} m_Q \hat{r}_j R_L^T \Big(- ge_3 - \dot{v}_L + L\ddot{q}_j + \frac{1}{m_Q} u_j \Big) + \Delta R_L, \tag{8.108}$$

$$\dot{R}_j = R_j \hat{\Omega}_j, \tag{8.109}$$

$$\mathbb{J}_Q \dot{\Omega}_j = \mathbb{J}_Q \Omega_j \times \Omega_j + M_j + \Delta R_j, \qquad j = 1, ..., N, \tag{8.110}$$

where $\Delta x_L, \Delta q_j, \Delta R_L, \Delta R_j$ are unstructured disturbances satisfying the bounds $\|\Delta x_L\| \leq \bar{x}_L$, $\|\Delta q_j\| \leq \bar{q}_j$, $\|\Delta R_L\| \leq \bar{R}_L$, $\|\Delta R_j\| \leq \bar{R}_j$ for some real numbers $\bar{x}_L, \bar{q}_j, \bar{R}_L, \bar{R}_j$, for $j = 1, \ldots, N$.

From the proof of Proposition 41 and a near identical analysis to that conducted in Section 8.2.6, we obtain the following result:

Proposition 43. *Consider the system with disturbances defined by equations* (8.103)-(8.110) *with control inputs determined by* (8.87)-(8.89)*, with $u_j^{\parallel}$ and $u_j^{\perp}$ given by equation* (8.85)-(8.86)*, and moment controller*

(8.39). *There exists $\delta > 0$, such that for all $\epsilon < \delta$ and for sufficiently small α_j, there exists gains k_{x_L}, k_{v_L}, k_{R_L}, k_{Ω_L}, k_q, k_ω, k_{R_j}, k_{Ω_j} $j = 1, \ldots, N$, such that the zero equilibrium of the tracking errors e_{x_L}, e_{v_L}, e_{R_L}, e_{Ω_L}, e_{q_j}, e_{ω_j}, e_{R_j}, e_{Ω_j}, are uniformly ultimately bounded.*

8.3.6 Energy-Optimal Path Planning with Obstacle Avoidance

As in Section 8.2.7, we now wish to design our desired trajectories so that the state variables are energy-optimal (that is, that the magnitude of the controls u_j, M_j are minimized for $j = 1, \ldots, N$) under the control scheme designed in Section 8.3.3. In particular, we are interested in solving the optimal control problem

$$\min_{q,u} \sum_{j=1}^{N} \int_a^b \left(\|u_j\|^2 + \|M_j\|^2 \right) dt, \tag{8.111}$$

over all curves $q \in \Omega(Q)$ and controls $u = (u_1, \ldots, u_N, M_1, \ldots, M_N) \in C^\infty([a,b], \mathbb{R}^3)^{2N}$ subject to the constraints (8.75)-(8.82). As before, we have chosen the state dynamics of the reduced model as our constraints, since we will nevertheless expect favorable energy behavior in the original model as long as the cables are sufficiently stiff and have high damping by Proposition 42. As in Section 8.2.7, we will treat the quadrotor attitude dynamics as if they were decoupled from the rest of the system dynamics, which allows us to split (8.111) into the following $N + 1$ optimal control problems:

$$\min_{q,u} \sum_{j=1}^{N} \int_a^b \|u_j\|^2 dt, \tag{8.112}$$

over all curves $q = (x_L, R_L, q_1, \ldots, q_N) \in C^\infty([a,b], \mathbb{R}^3 \times \mathrm{SO}(3) \times (S^2)^N)$ and controls $u = (u_1, \ldots, u_N) \in C^\infty([a,b], \mathbb{R}^3)^N$ subject to the constraints (8.75)-(8.78), and

$$\min_{R_j, M_j} \int_a^b \|M_j\|^2 dt, \tag{8.113}$$

over all curves $R_j \in C^\infty([a,b], \mathrm{SO}(3))$ and controls $M_j \in C^\infty([a,b], \mathbb{R}^3)$ subject to the constraints (8.81)-(8.82) for $j = 1, \ldots, N$. Clearly, (8.113) is equivalent to (8.48), which was already solved in Section 8.2.7. In particular, upon equipping each copy of $\mathrm{SO}(3)$ corresponding to the quadrotor attitudes with the left-invariant metric $\langle \cdot, \cdot \rangle_{\mathrm{SO}(3)}$ induced by (3.19) with $\mathbb{J} = \mathbb{J}_Q$, we find that the energy-optimal trajectories are given when the desired trajectories $\tilde{R}_j$ are chosen to be Riemannian cubic polynomials for $j = 1, \ldots, N$.

Moreover, in the presence of an obstacle, we may instead utilize modified Riemannian cubics, whose necessary conditions for optimality are provided by Proposition 23, and with the artificial potential as designed in Section 6.1.

We now handle the remaining optimal control problem (8.112). From (8.85) and (8.86), the control u_j approaches

$$u_j = \tilde{\mu}_j + m_Q(\ddot{\tilde{x}}_L + ge_3 + L\|\tilde{\omega}_j\|^2\tilde{q}_j + L\tilde{q}_j \times \dot{\tilde{\omega}}_j + \tilde{R}_L\hat{\tilde{\Omega}}^2 r_j - \tilde{R}_L\hat{r}_j\dot{\tilde{\Omega}}_L) \tag{8.114}$$

as the state variables approach their desired trajectories, where $\tilde{\mu}_j$ is given by (8.93). Moreover, we see that

$$F_L \to m_L(\ddot{\tilde{x}}_L + ge_3) \tag{8.115}$$

$$M_L \to \mathbb{J}_L\dot{\tilde{\Omega}}_L - \mathbb{J}_L\tilde{\Omega}_L \times \tilde{\Omega}_L \tag{8.116}$$

as the state variables approach their desired trajectories. We now equip the configuration space for the reduced system (excluding the quadrotor attitudes), given by $Q_L = \mathbb{R}^3 \times \mathrm{SO}(3) \times (S^2)^N$ with a Riemannian metric as follows: Equip $\mathbb{R}^3$ with the Riemannian metric $\langle X, Y\rangle_{\mathbb{R}^3} = m_L(X \cdot Y)$ for all $X, Y \in \mathbb{R}^3$. Equip $\mathrm{SO}(3)$ with the left-invariant metric $\langle \cdot, \cdot \rangle_{\mathrm{SO}(3)}$ induced by (3.19) with $\mathbb{J} = \mathbb{J}_L$. Finally, equip S^2 with the Riemannian metric $\langle \cdot, \cdot \rangle_{S^2_c} = m_Q L \langle \cdot, \cdot \rangle_{S^2}$, where $\langle \cdot, \cdot \rangle_{S^2}$ is the $\mathrm{SO}(3)$-invariant metric defined in (4.37). We may consider Q_L as a Riemannian manifold equipped with the product metric. It follows that the limit of the control (8.114) can be written as:

$$u_j = \tilde{\mu}_j + m_Q(D_t\dot{\tilde{x}}_L + ge_3) + m_Q L D_t\dot{\tilde{q}}_j + m_Q(L\|\tilde{\omega}_j\|^2\tilde{q}_j + \tilde{R}_L\hat{\tilde{\Omega}}^2 r_j - \tilde{R}_L\hat{r}_j\dot{\tilde{\Omega}}_L), \tag{8.117}$$

and we similarly see that

$$F_L \to m_L(D_t\dot{\tilde{x}}_L + ge_3) \tag{8.118}$$

$$M_L \to \tilde{R}_L^T D_t\dot{\tilde{R}}_L \tag{8.119}$$

From (8.93), it is clear that $\tilde{\mu}_j$ will approach some linear combination of $D_t\dot{\tilde{x}}_L + ge_3$ and $\tilde{R}_L^T D_t\dot{\tilde{R}}_L$, also involving the physical connection points $\hat{r}_j$ for $j = 1, \ldots, N$. Moreover, the terms $\tilde{R}_L\hat{\tilde{\Omega}}^2 r_j - \tilde{R}_L\hat{r}_j\dot{\tilde{\Omega}}_L$ in (8.117) depend on the physical connection points of the cables to the load, which will be determined by the

shape of the load and the particular configuration that we have in the application. For that reason, we will neglect these terms as they appear in the optimal control problem 8.112. As in Section 8.2.7, we see that the norms of the controls (8.58) are minimized provided that the desired trajectories satisfy the following variational principles:

$$\delta \int_a^b \left\| D_t \dot{\tilde{x}}_r + g e_3 \right\|_{\mathbb{R}^3}^2 dt = 0 \tag{8.120}$$

$$\delta \int_a^b \left\| D_t \dot{\tilde{R}}_L \right\|_{\mathrm{SO}(3)}^2 dt = 0 \tag{8.121}$$

$$\delta \int_a^b \left(\frac{1}{2} \| D_t \dot{q}_j \|_{S^2}^2 + \tau_j^2 \| \dot{q}_j \|_{S^2}^2 \right) dt = 0, \quad \text{for } j = 1, \ldots, N, \tag{8.122}$$

where $\tau_j \in \mathbb{R}$ are the *elastic parameters* for $j = 1, \ldots, N$. As before, we choose $\tau_j = 0$, so that the corresponding solutions to (8.122) are Riemannian cubic polynomials. As we did with the quadrotor attitudes, we may augment the action functionals in (8.120)-(8.122) with artificial potentials to avoid obstacles. In particular, $\tilde{q}_j$ is chosen to be a modified Riemannian cubic for $j = 1, \ldots, N$, which may be calculated as in Proposition 35, $\tilde{R}_L$ is chosen to be the modified Riemannian cubic as calculated in Proposition 23, and $\tilde{x}_r$ is chosen to satisfy

$$\frac{d^4 x_r}{dt^4} = -g e_3 - \nabla V_x(x_r), \tag{8.123}$$

where $V_x : R^3 \to \mathbb{R}$ is some artificial potential, and $\nabla V : \mathbb{R}^3 \to \mathbb{R}^3$ is its gradient vector field. In summary, with respect to the trajectory tracking controls $u_1, \ldots, u_N, M_1, \ldots, M_N$ designed in Section 8.3.3, the reduced system is energy-optimal (while avoiding obstacles) when the desired trajectories for the state variables are chosen to be modified Riemannian cubics.

8.3.7 Simulation Results

To validate the controller, we consider a numerical simulation with four quadrotors carrying a load with a mass of $1.5\,\mathrm{kg}$, and rigid body shape of rectangular box with length, width and height are $1.0\,\mathrm{m}$, $0.8\,\mathrm{m}$ of $0.2\,\mathrm{m}$, respectively[1] . The mass of each quadrotor is $m_Q = 0.5\,\mathrm{kg}$. The rest length of the cables are set to $L = 1.0\,\mathrm{m}$ and the load is attached in a symmetric manner to the cables. All further parameters of the configuration, initial values, and controller parameters are presented in Table 8.3.

[1] The video of the simulation can be found here: `https://youtu.be/0Pc_1ATqDlg`

Symbol	Value	Symbol	Value
m_Q	$0.755\,\mathrm{kg}$	m_L	$2\,\mathrm{kg}$
g	$9.81\,\mathrm{m/s^2}$	$\mathbb{J}_L$	$[1.04, 5.0, 4.04]$
r_1	$[0.4, 0.5, 0.1]$	r_2	$[-0.5, 0.4, 0.1]$
r_3	$[0.5, -0.4, 0.1]$	r_4	$[-0.5, -0.4, 0.1]$
Initial Conditions			
$x_L(0)$	$[0, 0, 0.3]^T\,\mathrm{m}$	$v_L(0)$	$[-2, -3.5, 0]^T\,\mathrm{m/s}$
$R_L(0)$	$\begin{bmatrix} -0.49 & 0.87 & 0 \\ -0.87 & -0.49 & 0 \\ 0.03 & 0 & 1 \end{bmatrix}$	$\Omega_L(0)$	$[0, 0, 0]^T$
$q_{1,\ldots,4}(0)$	$[0, 0, -1]^T$	$\omega_{1,..,4}(0)$	$[0, 0, 0]^T$
Controller			
k_{x_L}	8	k_{v_L}	5.2
k_{R_L}	10	k_{Ω_L}	2
k_q	10	k_ω	3
Cables			
L	$1.0\,\mathrm{m}$	$l_{1,2,3,4}(0)$	$\{1, 1, 1, 1\}\,\mathrm{m}$
$k_{\text{run}=1,2,3}$	$\{10, 8, 20\}$	$c_{\text{run}=1,2,3}$	$\{5, 30, 3.3\}$

Table 8.3: Simulation parameters

The numerical simulation was realized with $20\,\mathrm{s}$ of simulation, with time step of $0.1\,\mathrm{ms}$ implemented in Python on a computer with an Apple M1 Pro processor and $32\,\mathrm{GB}$ of RAM memory. The desired trajectory for position of the load used for this simulation is given by $x_{d,1}(t) = 1.2\sin(0.6\pi(t + 1.66))$, $x_{d,2}(t) = 4.2\cos(0.3\pi(t + 1.66))$, $x_{d,3}(t) = 0.5\tanh(t - 1.5) + 2.5$, and the desired attitude by $R_d(t) = \left[\frac{\tilde{v}}{|\tilde{v}|}, \frac{\hat{e_3}\tilde{v}}{|\hat{e_3}\tilde{v}|}, e_3\right]$, where $\tilde{v}$ is the desired velocity. As discussed in Section 8.3.3, the quadrotor attitudes are independent of the rest of the dynamics and almost-global exponential stabilizing controllers for the attitude have been presented in [77] and [75]. Thus, we use the same controller for the attitude of the quadrotors, and focus here on the discussion on the position results.

8.3.7.1 Reduced Model with Inelastic cables

We begin by conducting a simulation of the reduced model (8.75) - (8.82) with the controller (8.85), (8.86). That is, the cables are considered inelastic, each with length L. The values for the control gains, as shown in Table 8.3, were selected to guarantee the global exponentially stability for the system by using conditions (**1**), (**2**) and (**3**) in the proof of Proposition 41. After $10\,\mathrm{s}$, the system is perturbed by an unknown

disturbance. The disturbance, as introduced in Section 8.2.6, is set to

$$\Delta x_L(t) = \begin{bmatrix} 0.5\sin(0.43t) \\ 0.5\cos(0.21t) \\ 0.2\sin(0.75t) - e^{-t} \end{bmatrix}, \qquad \Delta R_L(t) = \begin{bmatrix} 0.2 + 0.45\sin(3.0t) \\ 0.3 - 0.65\cos(1.4t) \\ 0.05\sin(2.1t) \end{bmatrix}$$

The positions for the desired and simulated trajectories are given in Figure 8.5, from which the following two observations can be made:

- Before the introduction of the disturbance in the time-frame ($0\,\mathrm{s}$-$10\,\mathrm{s}$), the actual trajectory converges to the desired trajectory as proposed in Proposition 41.
- In the presence of the disturbance ($10\,\mathrm{s}$-$20\,\mathrm{s}$), the tracking error is increased in contrast to the undisturbed case, but remains ultimately bounded as expected from Proposition 43.

Note that the magnitude of the tracking error bound depends on the value of the feedback gains, which can be increased to reduce the error. In practice, the limit is given by input constraints on the thrust and moment. However, such an analysis is out of the scope of this thesis.

8.3.7.2 Original Model with Elastic Cables

Next, we evaluate the performance of the controller on the original model with elastic cables, as described in Proposition 40. The setting and system parameters of the load and quadrotors are the same as they were in the previous simulation for the reduced model. In particular, all simulation parameters are again provided in Table 8.3. We consider three alternative sets of values for the stiffness and damping of the cables, namely $k_{\text{run}=1,2,3} = \{10, 8, 20\}$ and $c_{\text{run}=1,2,3} = \{5, 30, 3.3\}$, respectively.

The response of the controlled system with elastic cables is visualized in Figure 8.6. Each column belongs to a different set of stiffnesses and dampings, as indicated by the corresponding titles. The top row shows the tracking error of the load. It can be observed that the tracking error is similar among the settings with oscillating behavior. Note that the frequencies and damping of the oscillations are impacted by i) the desired trajectory ii) the properties of the cables and iii) the selected feedback gains of the controller. By comparing the tracking error in Figure 8.6 with the desired position in Figure 8.5, it can be seen that

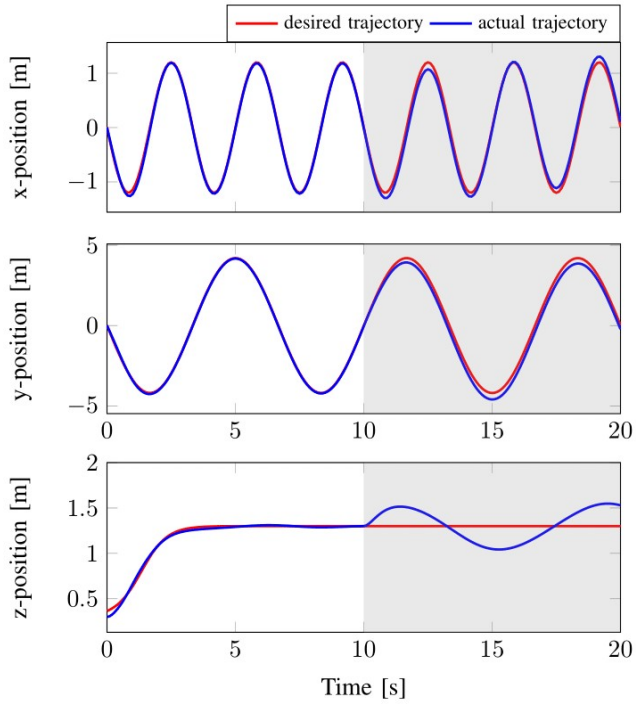


Figure 8.5: Desired position and actual position of the transported load for the controlled reduced model. After 10 s the system is perturbed by an external disturbance. As shown in Theorem 41 and Theorem 39, the tracking error is asymptotically stable (in case of no disturbance) and ultimately bounded (in case of a disturbance).

the frequency of the oscillation in the x and y tracking error is synchronous to the desired trajectory. In contrast, the amplitude is impacted by the properties of the cables, see e.g., the x-error and the length of the cables in the middle column of Figure 8.6. As stated in Proposition 38, the tracking error remains bounded for all time. The second row visualizes the length of the four cables l_1 to l_4 connected to the load and quadcopters. As expected, the dynamics of the cables mainly depends on their stiffness and damping properties. With decreased damping, more oscillations can be observed, as seen in the third column. Note, that the controller (8.85)-(8.86) does not actively reduce the oscillation but guarantees boundedness of the tracking error. Finally, the bottom row displays the trajectory of the load (black) and the quadcopters (red, blue, green, yellow) in the $3d$-space. It can be observed that the four quadcopters anticipate the oscillation to reduce the tracking error of the load.

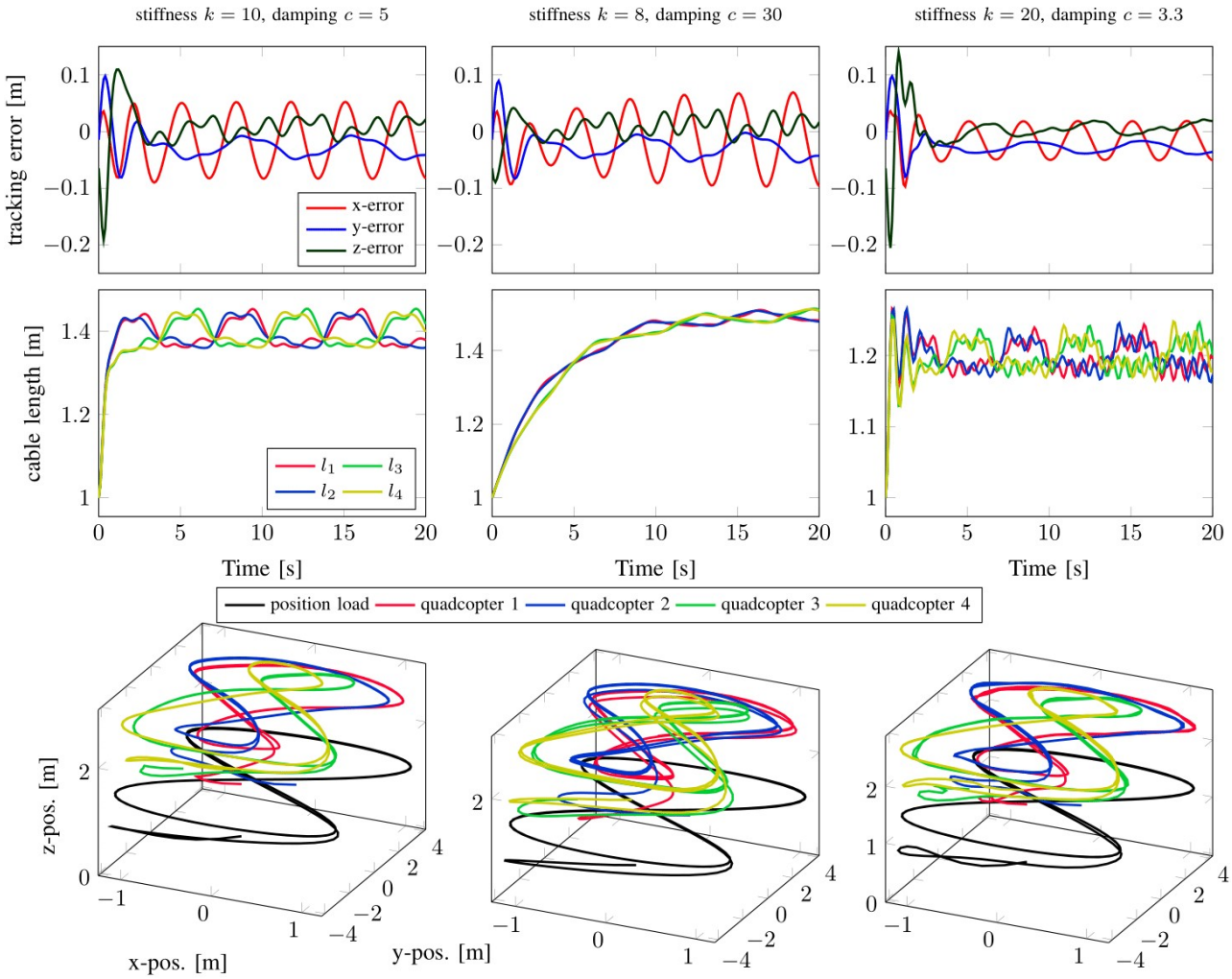


Figure 8.6: Dynamics of the controlled model with elastic cables for different stiffness and damping values of the cables. Top row: Tracking error of the load is similar among the different setting and remains bounded as proposed. Second row: The oscillations of the cables mainly depend on their properties. Bottom row: Trajectory of the load and the quadcopters. Video: `https://youtu.be/0Pc_1ATqDlg`

Chapter 9

Concluding Remarks and Future Work

On the whole, this thesis is dedicated to the study of the so-called modified Riemannian cubics on Riemannian manifolds, and how they can be applied to the tasks of obstacle and collision avoidance. Among their many applications, such curves appear naturally in a wide variety of trajectory tracking problems for robotic systems (such as the task of teams of quadrotors transporting a payload via elastic cables, as we saw in Chapter 8) when searching for energy-optimal trajectories that also allow for prescribed obstacles to be avoided.

In Chapter 5, the equations governing modified Riemannian cubics were derived as the critical points to a smooth functional on a closed submanifold of the Hilbert manifold $H^2([a,b],Q)$. This, in a sense, is the most general framework under which the modified Riemannian cubics can be described, as any lower degree of regularity would not allow for the functional (5.1) to be well-defined. As a future work, I would be interested in seeing if the potential can somehow be incorporated into the manifold geometry, giving rise to normal Riemannian cubic polynomials over some appropriate Lorentzian manifold. Using the techniques of global analysis, the existence of global minimizers was established. We concluded the chapter by conducting a study of local minimizers, which were divided into two categories: Ω-local minimizers, and Q-local minimizers. It was shown that the Q-local minimizers were equivalent to modified Riemannian cubics, which is analogous to the result that all geodesics are locally length-minimizing. For the Ω-local minimizers, sufficient conditions for optimality were derived in terms of the index form. This led naturally to the definitions of bi-Jacobi fields and biconjugate points, from which more tractable sufficient conditions could be derived. As a future work, I would be interested in further exploring the properties of the biexpo-

nential map. Moreover, it would be interesting to find some analogue of Riemannian normal coordinates for Riemannian cubic polynomials, with which analysis could be conducted more easily.

In Chapter 6, we developed sufficient conditions for safety guarantees in the obstacle and collision avoidance tasks, and designed *avoidance families* of artificial potentials that can be used to guarantee avoidance. There are many "potential" ways to extend the results here. First, I would like to consider a more diverse class of obstacles, such as moving obstacles. Moreover, I would like to make the sufficient conditions more practically useful. As can be seen in Example 10, using the conditions outlines in Proposition 20 often lead to very large parameter values for the obstacle avoidance potential. In such a case, the resulting modified Riemannian cubics are not good approximations of the corresponding Riemannian cubic polynomials, and so may not be good options in applications where energy-optimal curves are highly desirable. In reality, much smaller parameter values are likely to work, and I believe that the configuration of the boundary values and the obstacle, together with the particular manifold geometry (such as its curvature) could play a role in obtaining more optimized conditions. Moreover, this further begs the question: how far is a modified Riemannian cubic from its corresponding Riemannian cubic polynomial in general? Answering such a question could prove extremely useful in applications. Finally, I would like to apply the extension of the obstacle avoidance task to interpolation on SIEs to actual hybrid dynamical systems, such as the bipedal walker.

In Chapter 7, the necessary and sufficient conditions for optimality were reduced on Lie groups with left-invariant metrics, on Riemannian homogeneous spaces, and a number of special cases. As a future work, I would like to develop artificial potentials which can be practically applied on Riemannian homogeneous spaces whose underlying Lie group do not possess a bi-invariant metric. As it stands, there is no simple way to calculate the Riemannian distance in such cases, so that the types of artificial potentials that we utilize would not be useful in applications. This could either be rectified by finding easy-to-manage approximations to the (gradient of the) Riemannian distance, or by using an alternative type of artificial potential altogether.

Finally, in Chapter 8, we considered the task of teams of quadrotor UAVs transporting a payload via elastic cables. Dynamics for the control system were derived, which were subsequently reduced via singular perturbations under the assumption that the cable stiffnesses and dampings were large (a physically

reasonable assumption). Controls were designed to guarantee the exponential convergences of the state variables to some desired trajectories in the reduced model, and the uniform ultimate boundedness in the original model. Robustness results were obtained in the case that the system is perturbed by bounded, unstructured disturbances. Riemannian cubic polynomials were shown to be the energy-optimal trajectories for the control system, at which point modified Riemannian cubics may be used in the case that obstacle avoidance is needed. As a future work, I would like to produce better simulations—particularly towards the end of making detailed comparisons between modified Riemannian cubics and other trajectories that solve the interpolation tasks while avoiding obstacles. I would also like to expand on the treatment of the physical connection points between the cables and the payload in the task of an arbitrarily large team of quadrotors carrying a rigid body load. These terms were neglected in the optimal control analysis conducted in Section 8.3.6, but it would be useful to better understand the role that these terms play. Finally, I would like to develop variational integrators which can be applied in such cases to further improve simulation results.

Chapter 10

Conclusiones e Investigación Futura

En general, esta tesis está dedicada al estudio de las denominadas cúbicas Riemannianas modificadas sobre variedades Riemannianas, y cómo se pueden aplicar a las tareas de evasión de obstáculos y colisiones. Entre sus muchas aplicaciones, tales curvas aparecen naturalmente en una amplia variedad de problemas de seguimiento de trayectoria para sistemas robóticos (como la tarea de equipos de cuadrotores que llevan una carga a través de cables elásticos, como vimos en el Capítulo 8) cuando buscando trayectorias energéticamente óptimas que también permitan evitar los obstáculos prescritos.

En el Capítulo 5, las ecuaciones que gobiernan las cúbicas Riemannianas modificadas se derivaron como los puntos críticos de un funcional de clase C^∞ en una subvariedad cerrada de la variedad de Hilbert $H^2([a,b],Q)$. Este, en cierto sentido, es el marco más general bajo el cual se pueden describir las cúbicas Riemannianas modificadas, ya que cualquier menor grado de regularidad no permitiría que el funcional (5.1) esté bien definido. Como trabajo futuro, me interesaría ver si el potencial puede incorporarse de alguna manera a la geometría de la variedad, dando lugar a polinomios cúbicos Riemannianos normales sobre alguna variedad Lorentziana apropiada. Utilizando las técnicas de análisis global, se estableció la existencia de minimizadores globales. Concluimos el capítulo realizando un estudio de los minimizadores locales, que se dividieron en dos categorías: Ω-minimizadores locales y Q-minimizadores locales. Se demostró que los minimizadores locales Q eran equivalentes a las cúbicas Riemannianas modificadas, lo que es análogo al resultado que todas las geodésicas minimizan la longitud localmente. Para los minimizadores Ω-locales, se derivaron condiciones suficientes para la optimización en términos de la forma del índice. Esto condujo naturalmente a las definiciones de campos bi-Jacobi y puntos biconjugados, a partir de los cuales se podrían

derivar condiciones suficientes más manejables. Como trabajo futuro, me interesaría seguir explorando las propiedades del mapa biexponencial. Además, sería interesante encontrar algún análogo de las coordenadas normales de Riemann para los polinomios cúbicos Riemannianos, con los que se podría realizar el análisis más fácilmente.

En el Capítulo 6, desarrollamos condiciones suficientes para garantizar la seguridad en las tareas de evasión de obstáculos y colisiones, y diseñamos *familias de evasión* de potenciales artificiales que pueden usarse para garantizar la evasión. Hay muchas formas ”potenciales” de extender los resultados aquí. Primero, me gustaría considerar una clase más diversa de obstáculos, como los obstáculos en movimiento. Además, me gustaría hacer que las condiciones suficientes sean más útiles en la práctica. Como se puede ver en el Ejemplo 10, el uso de las condiciones delineadas en la Proposición **??** a menudo conduce a valores de parámetros muy grandes para el potencial de evasión de obstáculos. En tal caso, las cúbicas Riemannianas modificadas resultantes no son buenas aproximaciones de los polinomios cúbicos Riemannianos correspondientes y, por lo tanto, pueden no ser buenas opciones en aplicaciones donde las curvas de energía óptima son muy deseables. En realidad, es probable que funcionen valores de parámetros mucho más pequeños, y creo que la configuración de los extremos de la trayectoria y la posición del obstáculo, junto con la geometría particular de la variedad (como su curvatura), podría desempeñar un papel en la obtención de condiciones más optimizadas. Además, esto plantea la pregunta: ¿qué tan lejos está una cúbica Riemanniana modificada de su polinomio cúbico Riemanniano correspondiente en general? Responder a esa pregunta podría resultar extremadamente útil en las aplicaciones. Finalmente, me gustaría aplicar la extensión de la tarea de evasión de obstáculos a la interpolación en SEIs a sistemas dinámicos híbridos reales, como el andador bípedo.

En el Capítulo 7, las condiciones necesarias y suficientes para la optimización se redujeron en grupos de Lie con métricas invariantes a la izquierda, en espacios homogéneos Riemannianos y en varios casos especiales. Como trabajo futuro, me gustaría desarrollar potenciales artificiales que puedan aplicarse prácticamente en espacios homogéneos Riemannianos cuyo grupo de Lie subyacente no posea una métrica bi-invariante. Tal como está, no existe una forma sencilla de calcular la distancia Riemanniana en tales casos, por lo que los tipos de potenciales artificiales que utilizamos no serían útiles en las aplicaciones correspondientes. Esto podría corregirse encontrando aproximaciones fáciles de manejar al (gradiente de) la

distancia Riemanniana, o usando un tipo alternativo de potencial artificial por completo.

Finalmente, en el Capítulo 8, consideramos la tarea de equipos de VANTs que llevan una carga a través de cables elásticos. Se derivó la dinámica para el sistema de control, que posteriormente se redujo a través de perturbaciones singulares bajo el supuesto de que las rigideces y los amortiguamientos del cable eran grandes (un supuesto físicamente razonable). Se diseñaron controles para garantizar las convergencias exponenciales de las variables de estado a algunas trayectorias deseadas en el modelo reducido, y la acotación última uniforme en el modelo original. Los resultados de robustez se obtuvieron en el caso de que el sistema sea perturbado por perturbaciones no estructuradas y acotadas. Se demostró que los polinomios cúbicos Riemannianos son las trayectorias energéticamente óptimas para el sistema de control, momento en el cual se pueden usar las cúbicas Riemannianas modificadas en el caso de que sea necesario evitar obstáculos. Como trabajo futuro, me gustaría producir mejores simulaciones, particularmente hacia el final de hacer comparaciones detalladas entre cúbicas Riemannianas modificadas y otras trayectorias que resuelven las tareas de interpolación mientras evitan obstáculos. También me gustaría ampliar el tratamiento de los puntos de conexión física entre los cables y la carga en la tarea de un equipo arbitrariamente grande de cuadrotores que transportan una carga de cuerpo rígido. Estos términos se ignoraron en el análisis de control óptimo realizado en la Sección 8.3.6, pero sería útil comprender mejor el papel que desempeñan estos términos. Finalmente, me gustaría desarrollar integradores variacionales que puedan aplicarse en tales casos para mejorar aún más los resultados de la simulación.

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