Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas

TESIS DOCTORAL

Supersymmetric Vertex Algebras and Killing Spinors

Álgebras de Vértices Supersimétricas y Espinores de Killing

Memoria para optar al grado de doctor presentada por

Andoni De Arriba De La Hera

Directores:

Luis Álvarez Cónsul y Mario García Fernández

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UNIVERSIDAD COMPLUTENSE MADRID

DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

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Álgebras de Vértices Supersimétricas y Espinores de Killing
Supersymmetric Vertex Algebras and Killing Spinors

y dirigida por: Luis Álvarez Cónsul y Mario García Fernández

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Esta DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD debe ser insertada en la primera página de la tesis presentada para la obtención del título de Doctor.

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Eskerrak eman nahi dizkiet nire familiako kide guztiei, oraindik hemen daudenei, eta baita, zoritxarrez, gure ondoan jada ez direnei. Nire betiko adiskideei, eta beti hobetzen lagundu didaten guztiei, bere pazientzia amaitezina ez delako inoiz bukatzen.

And thanks to you reader, for finding a moment to read this work.
Y gracias a ti lector, por encontrar un momento para leer este trabajo.
Eta eskerrik asko zuri irakurle, lan hau irakurtzeko une bat aurkitzeagatik.

Sometimes, it is the people no one imagines anything
who do the things that no one can imagine.

A veces es la gente de la que nadie espera nada
la que hace cosas que nadie puede imaginar.

Batzuetan, ezer ez egiteko gai imajinatzen ditugun pertsonak dira
inor imajinatu ezin dituzten gauzak egiten dituztenak.

Alan Turing (1912-1954).

Contents

Abstract

The goal of the present thesis is to construct embeddings of the $N = 2$ superconformal vertex algebra, motivated by mirror symmetry, into the chiral de Rham complex, provided that we have solutions to the Killing spinor equations.

Our approach to the chiral de Rham complex is based on the universal construction by Bressler and Heluani, which applies to any Courant algebroid over smooth manifolds. In fact, the main results of this document are based on the approach to SUSY vertex algebras studied by Heluani and Kac, and furthermore extend the techniques developed by Heluani and Zabzine to obtain $N = 2$ superconformal structures on the chiral de Rham complex. The Killing spinor equations we consider come from the approach to special holonomy based on Courant algebroids in generalized geometry, and they are inspired by the physics of heterotic supergravity and string theory.

Our embeddings are constructed in two different set-ups. Firstly, for equivariant Courant algebroids over homogeneous manifolds, where the construction reduces to an embedding into the superaffinization of a quadratic Lie algebra, and the Killing spinor equations become purely algebraic conditions that can be checked on explicit examples. Secondly, for string Courant algebroids over complex manifolds, where these equations are equivalent to the Hull-Strominger system, with origins in heterotic σ-models studied by physicists.

The manuscript also includes several examples where our results apply. As an application, we present the first examples of $(0, 2)$ mirror symmetry on compact non-Kähler complex manifolds via the chiral de Rham complex. In fact, this thesis lays the ground to Borisov's vertex algebra approach to $(0, 2)$ mirror symmetry on non-Kähler manifolds.

Resumen en Español

El objetivo de esta tesis es el de construir *embeddings* del álgebra de vértices $N = 2$ superconforme, motivada por la simetría espejo, en el complejo quiral de de Rham, siempre que tengamos soluciones para las ecuaciones de los espinores de Killing.

Se ha seguido la construcción universal de Bressler y Heluani para construir el complejo quiral de de Rham, que se aplica a algebroides de Courant arbitrarios sobre variedades diferenciables. De hecho, los resultados principales de esta tesis están basados en el enfoque de las álgebras de vértices supersimétricas estudiado por Heluani y Kac, y extiende las técnicas desarrolladas por Heluani y Zabzine para construir $N = 2$ estructuras superconformes en el complejo quiral de de Rham. Las ecuaciones de los espinores de Killing consideradas provienen del enfoque de holonomía especial basado en los algebroides de Courant en geometría generalizada, y están inspiradas por la física de supergravedad heterótica y la teoría de cuerdas.

Nuestros *embeddings* se construyen en dos situaciones diferentes. Primero, en algebroides de Courant equivariantes sobre variedades homogéneas, donde la construcción se reduce a *embeddings* de la superafinización de un álgebra de Lie cuadrática, y las ecuaciones de los espinores de Killing vienen dadas por condiciones puramente algebraicas que pueden ser comprobadas en ejemplos explícitos. En segundo lugar, para algebroides de Courant *string* sobre variedades complejas, donde estas ecuaciones son equivalentes al sistema de Hull-Strominger, con orígenes en los modelos σ heteróticos estudiados por los físicos.

Este manuscrito también incluye una gran cantidad de ejemplos en los que se aplican estos resultados. Como aplicación, se presentan los primeros ejemplos de simetría espejo de tipo $(0, 2)$ en variedades complejas compactas no Kähler a través del complejo quiral de de Rham. De hecho, esta tesis sienta las bases del enfoque de álgebras de vértices introducido por Borisov para la simetría espejo de tipo $(0, 2)$ en variedades no Kähler.

Introduction

Vertex algebras provide a surprising bridge between physics and mathematics. On the physical side, they provide a rigorous definition of the chiral part of a 2-dimensional conformal field theory. Indeed, there exists a natural class of operators, called vertex operators, arising from field insertions at points (that is, vertices) and depending holomorphically on the position. The product of these operators admit power series expansions when insertions collide, which satisfy the relations specified in the definition of a vertex algebra. On the mathematical side, vertex algebras were formulated by Borcherds [9] to prove the Monstruous Moonshine Conjecture [10], and they have played an important role in many areas of mathematics, such as representation theory of Kac-Moody algebras, where they were originally discovered in mathematics [29, 67]. In the recent years, vertex algebras have had a strong impact in geometry, first by the construction of the chiral de Rham complex by Malikov-Schechtmann-Vaintrob [73], and also by their applications to mirror symmetry [12], and more recently to gauge theory [47, 59].

The central problem studied in this thesis is motivated by mirror symmetry, and consists of finding appropriate geometric conditions under which the vertex algebra of global sections of the chiral de Rham complex on a smooth manifold admits an embedding of certain superconformal vertex algebras. Here, the chiral de Rham complex is understood as a sheaf of vertex algebras that can be attached to any Courant algebroid, as shown independently by Gorbounov, Malikov and Shechtman [44, 45, 46], and Beilinson and Drinfeld [8], and more explicitly by Bressler and Heluani [15, 53, 54]. The superconformal vertex algebras we will use in this thesis are mainly $N = 2$ supersymmetric extensions of the Virasoro algebra, while the geometric conditions are inspired by the Killing spinor equations in supergravity.

The original construction of the chiral de Rham complex in [73] was carried out locally, gluing the so-called ghost system over coordinate patches, and corresponds to the standard Courant algebroid. The name of chiral de Rham complex was adopted because it carries a grading and a differential such that it is quasisomorphic to the usual de Rham complex. In this seminal work, it was shown that for Calabi-Yau manifolds, the vertex algebra of global sections of the chiral de Rham complex admits an embedding of the $N = 2$ superconformal vertex algebra. Inspired by this work, many other embeddings of different algebras have been obtained in the chiral de Rham complex on special holonomy manifolds. This includes related $N = 2$ superconformal vertex algebra embeddings for Kähler manifolds [53], two commuting $N = 2$ superconformal vertex algebra embeddings for generalized Calabi-Yau metric manifolds [55], two commuting embeddings of the Odake algebra [77] for Calabi-Yau threefolds [24], two commuting embeddings of the Shatashvili-Vafa superconformal algebra [81] for holonomy G_2 manifolds [80], etc. See [56] for more information. Complete descriptions of the space of the global sections of the chiral de Rham complex have been obtained in the special case of a $K3$ surface [84, 85], and a compact Ricci-flat Kähler manifold [86].

Our embeddings are motivated by a conjectural extension of mirror symmetry to general non-Kähler manifolds, a relatively new field of research which is capturing a lot of attention over the last years [1, 65, 79, 90]. Since the works of Candelas, de la Ossa, Green, Parkes [16], Witten [92], Strominger, Yau, Zaslow [88], and many other outstanding researchers in the 90's, the study of mirror symmetry became a highly studied topic in mathematics. Originated by physics, this concept lead us to surprising and deep connections between different areas of mathematics. So far, mirror symmetry was essentially bound to Kähler-Calabi-Yau manifolds, in relation to type IIA and type IIB string theories. There is another version of string theory, called heterotic, that is getting more attention from the mathematical community, and a version of mirror symmetry for it, called $(0, 2)$ *mirror symmetry*, is expected to exist by the mathematical-physic community [74, 75].

A general way to approach mirror symmetry, as formulated geometrically by Borisov [13], is via vertex algebras, following more closely the physics approach to mirror symmetry. As proved by Borisov and Kauffman [14], this is well-suited for understanding certain aspects of $(0, 2)$ mirror symmetry. The basic idea is to construct representations of $N = 2$ superconformal vertex algebras, associated to mirror spaces, and to relate them via an automorphism, called the mirror involution. A general recipe to find such representations is via embeddings of the $N = 2$ superconformal vertex algebra into the chiral de Rham complex, as mentioned earlier. While such embeddings are known to exist for the chiral de Rham complex of (generalized) Calabi-Yau manifolds by the works of Heluani and Zabzine [54, 55], until now nothing was known about the potential embeddings into the chiral de Rham complex of heterotic analogues of Calabi-Yau manifolds. Results in this direction, for the $N = 1$ superconformal vertex algebra, can be found in the physics literature in the works of de la Ossa, Fiset and Galdeano [21, 28, 33].

In the present thesis we give a precise answer to this problem, providing, in much generality, embedded $N = 2$ superconformal vertex algebras into the chiral de Rham complex of a string Courant algebroid over a complex manifold, carrying solutions to the Killing spinor equations. In this set-up, these equations are equivalent to the Hull-Strominger system, which describes supersymmetric compactifications of the heterotic string theory. Furthermore, in the special case of homogeneous Hopf surfaces, we can use these embeddings to obtain the first examples of $(0, 2)$ mirror symmetry on compact non-Kähler complex manifolds.

SUSY Vertex Algebras

In the standard approach (see Definition 1.1.15), a *vertex algebra* consists of a vector space V (space of states), endowed with a non-zero vector $|0\rangle \in V$ (the vacuum), an endomorphism $T\colon V \to V$ (infinitesimal translation operator), and an injective linear map given by $Y\colon V \to (\operatorname{End} V)[[z^{\pm}]]$ (state-field correspondence) mapping vectors into fields. By a *field*, we mean a formal sum

$$A(z) := Y(a,z) = \sum_{n\in\mathbb{Z}} z^{-1-n} a_{(n)}, \quad \text{for } a \in V,$$

in an indeterminate z, with Fourier modes $a_{(n)} \in \operatorname{End} V$, such that $Y(a,z)(b)$ is a formal Laurent series for each $b \in V$, so that the operator product expansions are finite sums of the form

$$A(z)B(w) \sim \sum_{n\in\mathbb{N}} \frac{(a_{(n)}b)(w)}{(z-w)^{n+1}}, \quad \text{for } a, b \in V.$$

This data should satisfy several conditions called vacuum axioms, translation invariance, and locality. One is often interested in vertex algebras that are *conformal*, that is, they carry a so-called conformal vector $\nu \in V$, so the Fourier modes of the corresponding field

$$Y(\nu, z) = L(z) = \sum_{n\in\mathbb{Z}} z^{-2-n} L_n$$

satisfy the *Virasoro commutation relations*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_m^{-n} \frac{m^3 - m}{12} c,$$

where $c \in \mathbb{C}$ is the *central charge*, $L_{-1} = T$, and the operator L_0 is diagonalizable on V, with eigenvalues bounded below. More precisely, we have the following identity

$$[L_0, Y(a,z)] = (\Delta_a + z\partial_z)\, Y(a,z), \quad \text{for } a \in V,$$

where $\Delta_a \in \mathbb{C}$ is the *conformal weight* of $a \in V$.

Sometimes a vertex algebra admits an enhancement of translation symmetry, which is called *supersymmetry*, given by an odd linear map $S\colon V \to V$ (supersymmetry generator) such that $S^2 = T$, or even an enhancement of conformal symmetry, which is given by a *superconformal vector* τ such that the field

$$Y(\tau, z) = G(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} z^{-\frac{3}{2}-n} G_n$$

satisfy the (extra) *Neveu-Schwarz commutation relations*

$$[G_m, L_n] = \left(m - \frac{n}{2}\right) G_{m+n}, \quad [G_m, G_n] = 2L_{m+n} + \frac{c}{3}\left(m^2 - \frac{1}{4}\right)\delta_m^{-n},$$

and $\nu = 1/2G_{-1/2}(\tau)$ recovers the conformal vector with central charge c. This is called the $N=1$ *superconformal vertex algebra*, since the Neveu-Schwarz is the simplest among the superconformal vertex algebras [26].

The previous situation motivates the notion of *supersymmetric vertex algebras*, given as follows. We will work with $\mathbb{Z}_2$-graded vector spaces, simply called *vector superspaces*, and we will use the approach to *SUSY vertex algebras* in terms of superfields developed by Heluani-Kac [52]. In this approach, we define the *superfields*

$$Y(a,z;\theta) = Y(a,z) + \theta Y(Sa,z) = \sum_{\substack{n\in\mathbb{Z}\\ J\in\{0,1\}}} z^{-1-n}\theta^{1-J} a_{(n|J)}, \quad \text{for } a \in V,$$

where θ is an odd Grassmannian indeterminate, commuting with z and anticommuting with S, so $\theta^2 = 0$ and $\theta z = z\theta$. In [52], two equivalent formulations of SUSY vertex algebras are considered. We recall the one that will be important for the present work. This is a SUSY version of a reformulation, studied by Bakalov-Kac [3], of the notion of vertex algebra. Since $S^2 = T$, then V is a supermodule over the translation algebra $\mathcal{H}$, defined as the associative superalgebra with an odd generator S, an even generator T, and the relation $S^2 = T$. This superalgebra can be identified with the parameter algebra, that is, the associative superalgebra $\mathcal{L}$ with an odd generator χ, an even generator λ, and the relation $\chi^2 = -\lambda$. Then, we can introduce the Λ-*bracket*, and the *normally ordered product* as the bilinear maps defined by

$$[a_\Lambda b] = \sum_{\substack{n\in\mathbb{N}\\ J\in\{0,1\}}} \frac{\lambda^n\chi^J}{n!} a_{(n|J)}b, \quad :ab := a_{(-1|1)}b, \quad \text{for } a,b \in V,$$

respectively. The properties of the first operation motivates the definition of *SUSY Lie conformal algebra* (see Definition 2.3.1), while adding the data given by the second one, we obtain the notion of *SUSY vertex algebra* (see Theorem 2.3.6).

Next, we will describe an example of SUSY vertex algebra that plays a fundamental role in this thesis. The $N=2$ *superconformal vertex algebra* of central charge $c \in \mathbb{C}$ is generated by the SUSY Lie conformal algebra, whose underlying $\mathcal{H}$-module is freely generated by two superfields, namely an odd vector H (the Neveu-Schwarz generator), an even vector J (usually known as a *current*), and a scalar c, with non-zero Λ-brackets given by

$$[H_\Lambda H] = (2T + \chi S + 3\lambda)H + \frac{\chi\lambda^2}{3}c,$$

and

$$[J_\Lambda J] = -\left(H + \frac{\lambda\chi}{3}c\right), \quad [H_\Lambda J] = (2T + 2\lambda + \chi S)J.$$

One of the main problems of the present thesis is to construct embeddings from the $N=2$ superconformal vertex algebras into a SUSY vertex algebra naturally associated to any quadratic Lie algebras (that is, Lie algebras with a symmetric and invariant pairing), called the *superaffinization of quadratic Lie algebras.* Given $(\mathfrak{g}, (\cdot|\cdot))$ a finite-dimensional quadratic Lie algebra and $k \in \mathbb{C}$, let $\Pi\mathfrak{g}$ be the corresponding purely odd vector superspace. The *universal superaffine vertex algebra* with *level* k associated to $\mathfrak{g}$ is the SUSY vertex algebra $V^k_{\text{super}}(\mathfrak{g})$ generated by the *supercurrent algebra*, which is the $\mathcal{H}$-module freely generated by $\Pi\mathfrak{g}$ and the level k, with non-zero Λ-brackets given by

$$[\Pi a_\Lambda \Pi b] = \Pi [a, b] + \chi k (a|b), \quad \text{for } a, b \in \mathfrak{g}.$$

The first embeddings of this type were given by Getzler [43], starting with a Manin triple satisfying the technical algebraic condition (3.9). Our embeddings in Theorem 10.1.5 and Theorem 10.1.8 generalizes Getzler's construction, and, furthermore, provide a geometric meaning to Getzler's technical algebraic condition. Moreover, these embeddings reduce to the well-known Kac-Todorov construction [60] under appropriate conditions (see Corolary 10.2.7). The new input for the constructions of these embeddings is a solution of the Killing spinor equations on the quadratic Lie algebra. These equations can be regarded as purely algebraic conditions on real quadratic Lie algebras (see Chapter 7), but in fact they come from geometry and physics, specifically from the approach to special holonomy based on generalized geometry on Courant algebroids [57].

The Killing Spinor Equations

Let M be a $2n$-dimensional spin manifold, and K a compact Lie group. Consider a principal K-bundle $P \longrightarrow M$. For any principal connection A on P, we denote its curvature by $F_A \in \Omega^2(M, \operatorname{ad} P)$. Now, given $H \in \Omega^3(M)$ and g a Riemannian structure, we define the connections

$$\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H, \quad \nabla^{+\frac{1}{3}} = \nabla^g + \frac{1}{6} g^{-1} H,$$

where ∇^g is the Levi-Civita connection of g. The data (g, H, φ, A, η), where $\varphi \in \Omega^1(M)$ and η is a spinor on (TM, g), is a *solution of the Killing spinor equations* on M [35] if

$$F_A \cdot \eta = 0, \quad \nabla^+ \cdot \eta = 0, \quad \left(\not{\nabla}^{+\frac{1}{3}} + \varphi \right) \cdot \eta = 0.$$

When φ is exact, these equations are equivalent in low dimensions to the Killing spinor equations in a compactification of the ten-dimensional heterotic supergravity [27, 30].

Let M be a complex manifold with vanishing first Chern class, and suppose that K is endowed with $\langle \cdot, \cdot \rangle : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{R}$ a bi-invariant non-degenerate pairing. In this situation, the Killing spinor equations are related to the *Hull-Strominger system* [39, 58, 87] when suplemented with the so-called *Bianchi identity*

$$dH + \langle F_A \wedge F_A \rangle = 0.$$

To see this, note that a solution for the Killing spinor equations with φ exact that solves the Bianchi identity gives rise to a solution of the Hull-Strominger system. More generally, when φ is closed, we find the *twisted Hull-Strominger system*, as defined in [39]. We say that a triple (Ψ, ω, A), given by an $\mathrm{SU}(n)$-structure (Ψ, ω) on M, is a solution to the twisted Hull-Strominger system if

$$
\begin{aligned}
F_A^{0,2} = 0, \qquad F_A \wedge \omega^{n-1} &= 0, \\
d\Psi - \theta_\omega \wedge \Psi &= 0, \\
d\theta_\omega &= 0, \\
dd^c\omega - \langle F_A \wedge F_A \rangle &= 0.
\end{aligned}
$$

When $K = \{1\}$ (so, $F_A = 0$), we obtain the *twisted Calabi-Yau equations*

$$
\begin{aligned}
d\Psi - \theta_\omega \wedge \Psi &= 0, \\
d\theta_\omega &= 0, \\
dd^c\omega &= 0.
\end{aligned}
$$

The Killing spinor equations jointly with the Bianchi identity admit a universal formulation in terms of Courant algebroids [36], which is important for the present work. To see this, recall that a *Courant algebroid* is given by a vector bundle $E \longrightarrow M$ together with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and the Dorfman bracket $[\cdot, \cdot]$ on $\Gamma(E)$, with a bundle map $\pi \colon E \longrightarrow TM$ (the anchor) satisfying certain axioms (see Definition 6.1.1). In particular, we obtain a map $\mathcal{D} \colon \mathcal{C}^\infty(M) \longrightarrow \Gamma(E)$ combining π, the de Rham exterior differential d of M and the pairing $\langle \cdot, \cdot \rangle$. In this set-up, the Killing spinor equations have unknowns given by a Riemannian generalized metric $E = C_+ \oplus C_-$, a divergence operator $\operatorname{div} \colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$, and a spinor η (see Definition 6.2.15). They are formulated in terms of natural operators D_-^+ and $\not{D}^+$ associated to C_+ and div, and read as follows:

$$
D_-^+ \eta = 0, \quad \not{D}^+ \eta = 0. \tag{1}
$$

The relation to the twisted Hull-Strominger system above arises when one considers the so-called *string Courant algebroid* over M (see Proposition 6.1.6) associated to a solution of the Bianchi identity [4, 34], that is, the data $(E_{H,A}, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_{H,A}, \pi)$, where we have $E_{H,A} = TM \oplus \operatorname{ad} P \oplus T^*M$ and, for $X + r + \zeta, Y + t + \eta \in E_{H,A}$,

$$
\begin{aligned}
[X + r + \zeta, Y + t + \eta]_{H,A} &= [X, Y] + L_X \eta - \iota_Y d\zeta + \iota_Y \iota_X H \\
&\quad - F_A(X, Y) + 2 \langle \iota_X F_A, t \rangle - 2 \langle \iota_Y F_A, r \rangle \\
&\quad + 2 \langle d^A r, t \rangle + d^A_X t - d^A_Y r - [r, t], \\
\langle X + r + \zeta, Y + t + \eta \rangle &= \frac{1}{2} (\eta(X) + \zeta(Y)) + \langle r, t \rangle, \\
\pi \colon E \longrightarrow TM, &\quad \pi(X + r + \zeta) = X.
\end{aligned}
$$

When the Bianchi identity is suplemented by the Killing spinor equations with $d\varphi = 0$, the string Courant algebroid $E_{H,A}$ is endowed with a solution of (1). As a matter of fact, the formulation of the Killing spinor equations on Courant algebroids provides an unifying framework for metrics with special holonomy, solutions to the Hull-Strominger system, and other interesting canonical geometries.

Embedding Superconformal Vertex Algebras

Surprisingly, the formulation of the Killing spinor equations on Courant algebroids makes sense and has non-trivial solutions for real quadratic Lie algebras, where the Killing spinor equations become purely algebraic conditions. To find these algebraic conditions, we must suppose that the rank of the generalized metric V_+ is even. Then, the Killing spinor equations are described in terms of natural conditions for a decomposition

$$\mathfrak{g}^c := \mathfrak{g} \otimes \mathbb{C} = (V_+ \otimes \mathbb{C}) \oplus (V_- \otimes \mathbb{C}) = l \oplus \bar{l} \oplus (V_- \otimes \mathbb{C}),$$

and a divergence $\varepsilon \in l \oplus \bar{l}$, where $l, \bar{l} \subseteq \mathfrak{g}^c$ are complex isotropic subspaces and we have that $V_- \otimes \mathbb{C} = (l \oplus \bar{l})^\perp$. The resulting conditions

$$[l, l] \subseteq l, \quad [\bar{l}, \bar{l}] \subseteq \bar{l}, \quad \frac{1}{2} \sum_{j=1}^{\dim l} [\epsilon_j, \bar{\epsilon}_j] = \varepsilon_{\bar{l}} - \varepsilon_l \tag{2}$$

make sense over an arbitrary closed field $\mathbb{C}$, where $\epsilon_j \in l, \bar{\epsilon}_j \in \bar{l}$ for $j \in \{1, \ldots, \dim l\}$ defines an isotropic basis of $V_+ \otimes \mathbb{C}$. In this generality, we can state our first results. Let $e_j = \Pi \epsilon_j$, $e^j = \Pi \bar{\epsilon}_j$, and consider the bilinear map $[\cdot, \cdot] : \Pi\mathfrak{g} \times \Pi\mathfrak{g} \longrightarrow \Pi\mathfrak{g}$ given by

$$[\Pi a, \Pi b] := \Pi [a, b], \quad \text{for } a, b \in \mathfrak{g}^c.$$

Firstly, in Theorem 10.1.5, we obtain the following embedding.

Theorem 1. *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a complex quadratic Lie algebra. Assume that $\left(l \oplus \bar{l}, \varepsilon\right)$, with $l, \bar{l} \subseteq \mathfrak{g}$ and $\varepsilon \in l \oplus \bar{l}$, satisfy* (2), *and that*

$$\varepsilon \in [l, l]^\perp \cap \left[\bar{l}, \bar{l}\right]^\perp. \tag{3}$$

Then, the vectors

$$\begin{aligned} J_0 =& \frac{i}{k} \sum_{j=1}^{\dim l} : e^j e_j :, \\ H' =& \frac{1}{k} \sum_{j=1}^{\dim l} \left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) : \right) + \frac{2}{k} T \Pi \varepsilon_+ + \frac{1}{k^2} \sum_{j,k=1}^{\dim l} \Big(: e_j : e^k \left[e^j, e_k\right] :: \\ & + : e^j : e_k \left[e_j, e^k\right] :: - : e_j : e_k \left[e^j, e^k\right] :: - : e^j : e^k \left[e_j, e_k\right] :: \Big), \end{aligned}$$

induce an embedding of the $N = 2$ superconformal vertex algebra with central charge $c_0 = 3 \dim l$ into the universal superaffine vertex algebra $V^k_{\text{super}}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.

Furthermore, in Theorem 10.1.8, we construct a "dilaton correction" of the previous embedding as follows. For this, we require that $\varepsilon \in l \oplus \bar{l}$ is holomorphic (see Definition 7.2.3), which is the geometric version of (3), first appearing in Getzler's construction.

Theorem 2. *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a complex quadratic Lie algebra. Assume that $\left(l \oplus \bar{l}, \varepsilon\right)$, with $l, \bar{l} \subseteq \mathfrak{g}$ and $\varepsilon \in l \oplus \bar{l}$, satisfy (2), and that $\varepsilon \in l \oplus \bar{l}$ is holomorphic. Then, the vectors*

$$J = J_0 - 2\frac{S}{k} i \sum_{j=1}^{\dim l} \left[e^j, e_j\right],$$
$$H = H' - \frac{2}{k} T\Pi\varepsilon_+,$$

induce an embedding of the $N = 2$ superconformal vertex algebra with central charge

$$c = 3\left(\dim l + \frac{4}{k}(\varepsilon|\varepsilon)\right) \in \mathbb{C},$$

into the universal superaffine vertex algebra $V^k_{super}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.

These two results can be applied to the geometric situation, where one has an equivariant Courant algebroid E over a homogeneous manifold (see Proposition 10.1.11). This allows us to obtain two different embeddings of the $N = 2$ superconformal vertex algebra in the vertex algebra of the global sections of the chiral de Rham complex of E. In this thesis, we will follow the construction by Bressler and Heluani of the chiral de Rham complex associated to any Courant algebroid E (see Theorem 9.1.10). This is a sheaf Ω^{ch}_E of SUSY vertex algebras generated by the SUSY Lie conformal algebra, whose underlying $\mathcal{H}$-module is $\mathcal{R} = \mathcal{C}^\infty(M) \oplus (\Gamma(\Pi E) \otimes \mathcal{H})$, with non-zero Λ-brackets given by

$$[\Pi a_\Lambda f] = \langle \Pi\mathcal{D}f, a\rangle, \quad [\Pi a_\Lambda \Pi b] = \Pi[a,b] + 2\chi\langle a,b\rangle, \quad \text{for } a,b \in \Gamma(E), f \in \mathcal{C}^\infty(M).$$

We next turn to consider Killing spinors with closed divergence on a string Courant algebroid $E = E_{H,A}$. For this, we assume that the base manifold is even-dimensional, and we describe the twisted Hull-Strominger system above in terms of a decomposition

$$E \otimes_\mathbb{R} \mathbb{C} = (C_+ \otimes_\mathbb{R} \mathbb{C}) \oplus (C_- \otimes_\mathbb{R} \mathbb{C}) = \ell \oplus \bar{\ell} \oplus (C_- \otimes_\mathbb{R} \mathbb{C}),$$

where $\ell, \bar{\ell} \subseteq E \otimes_\mathbb{R} \mathbb{C}$ are complex isotropic subbundles, and $C_- \otimes_\mathbb{R} \mathbb{C} = (\ell \oplus \bar{\ell})^\perp$. The resulting conditions

$$[\ell, \ell] \subseteq \ell, \quad \left[\bar{\ell}, \bar{\ell}\right] \subseteq \bar{\ell}, \quad \frac{1}{2}\sum_{j=1}^{\dim \ell} [\epsilon_j, \bar{\epsilon}_j] = \varphi_{\bar{\ell}} - \varphi,$$

where $\varphi \in \Omega^1(M)$ is minus the Lee form, make sense over any Courant algebroid, where $\epsilon_j \in \ell, \bar{\epsilon}_j \in \bar{\ell}$ for $j \in \{1, \ldots, \dim \ell\}$ defines an isotropic local frame of $C_+ \otimes_\mathbb{R} \mathbb{C}$ such that

$$[\epsilon_j, \epsilon_k] = 0, \quad \text{for } j,k \in \{1, \ldots, \dim \ell\}.$$

Now, we introduce the *torsion bi-vector*, which is canonically associated to any hermitian structure (see Section 5.3). In holomorphic coordinates, this is given by

$$\sigma_\omega := \sum_{k=1}^{\dim \ell} \left[g^{-1} d\bar{z}_k, \left(g^{-1} \otimes g^{-1}\right)\left(\iota_{\frac{\partial}{\partial \bar{z}_k}} \partial\omega\right)\right]^{0,2} \in \Gamma\left(\Lambda^2 T^{0,1} M\right).$$

In this generality, we state Theorem 10.1.19, writing $e_j = \Pi\epsilon_j, e^j = \Pi\overline{\epsilon}_j$, and defining

$$[\Pi a, \Pi b] := \Pi\,[a, b]\,, \quad \text{for } a, b \in \Gamma\left(E \otimes_{\mathbb{R}} \mathbb{C}\right).$$

Theorem 3. *Let (Ψ, ω, A) be a solution of the twisted Hull-Strominger system. Consider the string Courant algebroid $E_{-d^c\omega,A}$, endowed with the associated decomposition*

$$C_+ \otimes_{\mathbb{R}} \mathbb{C} = \ell \oplus \overline{\ell}$$

for the generalized metric $C_+ \subseteq E_{-d^c\omega,A}$, and $\varphi = -\theta_\omega \in \Gamma(E_{-d^c\omega,A})$. Let $\{\epsilon_j, \overline{\epsilon}_j\}_{j=1}^{\dim\ell}$ be the frames defined by (6.18) *using the atlas given in Lemma 5.2.2. Then, the sections*

$$\begin{aligned}
J =& \frac{i}{2}\sum_{j=1}^{\dim\ell} : e^j e_j : -Si\Pi\varphi, \\
H =& \frac{1}{2}\sum_{j=1}^{\dim\ell}\left(: e_j\left(Se^j\right) : + : e^j\left(Se_j\right) :\right) + \frac{1}{4}\sum_{j,k=1}^{\dim\ell}\Big(: e_j : e^k\left[e^j, e_k\right] :: \\
&+ : e^j : e_k\left[e_j, e^k\right] :: - : e_j : e_k\left[e^j, e^k\right] :: - : e^j : e^k\left[e_j, e_k\right] ::\Big) + T\Pi\varphi_{\overline{\ell}},
\end{aligned}$$

are global. Furthermore, if the torsion bi-vector vanishes identically, then these sections induce an embedding of the $N = 2$ superconformal vertex algebra with $c = 3\dim\ell \in \mathbb{C}$ into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E_{-d^c\omega,A}\otimes_{\mathbb{R}}\mathbb{C}}$.

When $K = \{1\}$, we can study what happens for a *pair of solutions for the Killing spinor equations*, that is, if we have solutions in both C_+ and C_-. We expect that this construction is related with the one given by Heluani-Zabzine [55]. For quadratic Lie algebras, we obtain two Λ-commuting embeddings in Theorem 10.2.2 and Theorem 10.2.4, along with honest $N = 2$ superconformal vertex algebra structures (see Theorem 10.2.6).

The stated embeddings are the main results of this thesis. We should emphasize that the three results are independent, so each one induce, a priori, different embeddings of the $N = 2$ superconformal vertex algebra in examples. In particular, we obtain embeddings for some explicit cases: compact complex surfaces (see Proposition 11.1.9), the Iwasawa manifold (see Proposition 11.1.12, and Proposition 11.1.15), and the Calabi-Yau 3-fold introduced in [78] (see Proposition 11.1.20). Actually, in the Hopf surface, we obtain up to three different embeddings of the $N = 2$ superconformal vertex algebra (see Proposition 11.1.3, and Proposition 11.1.7). Moreover, applying Theorem 10.1.5 to the Hopf surface, we extend these constructions to an embedding of the $N = 4$ superconformal vertex algebra into the superaffine vertex algebra (see Subsection 2.5.5 and Proposition 11.1.22).

Finally, in Section 11.2, we apply Theorem 10.1.8 to obtain the first examples of $(0, 2)$ mirror symmetry on compact non-Kähler complex manifolds via the chiral de Rham complex, following Borisov [13]. We require that the geometric data is homogeneous, so that the construction of the mirror symmetry involution is reduced to the study of the Killing spinor equations on quadratic Lie algebras, to the construction of $N = 2$ superconformal vertex algebra embeddings, and to T-duality applied to the Killing spinor equations. The obtained examples of $(0, 2)$ mirrors are given by pairs of Hopf surfaces endowed with a Bismut-flat pluriclosed metric (see Theorem 11.2.5).

Organization of the Thesis

The thesis is divided in four parts.

Part I is a brief summary of SUSY vertex algebras. Chapter 1 and Chapter 2 contain the background on (SUSY) vertex algebras, including the canonical examples we are going to work with. The main references for this study are [52, 63]. Chapter 3 reviews the previously well-known embeddings of superconformal vertex algebras, the ones constructed by Segal-Sugawara [89], Kac-Todorov [60] and Getzler [43].

Part II is a brief account of geometric structures and Killing spinors. Chapter 4 contains some basic notions about G-structures and spin geometry. Chapter 5 is devoted to Killing spinors on spin manifolds, and the F-term and D-term conditions. Here (see Section 5.3), we also introduce a new tensor, called the torsion bi-vector, which will be fundamental to construct the last embedding in Part III. Chapter 6 is devoted to the study of Killing spinors on Courant algebroids [36], their relation to the Hull-Strominger system, and the links with the F-term and D-term conditions introduced in previous chapter. Chapter 7 contains a complete study of Killing spinors on real quadratic Lie algebras [2]. Chapter 8 includes a summary of generalized Kähler geometry that will be used in Section 9.2.

Part III is devoted to the interplay between vertex algebras and Killing spinors. Chapter 9 contains a brief account of several constructions of the chiral de Rham complex and a quick presentation of two embeddings, constructed by Heluani and Zabzine, of superconformal vertex algebras in the space of global sections of the chiral de Rham complex [55]. Chapter 10 contains the main results of this thesis about embeddings of superconformal vertex algebras. Concretely, there are constructed three new embeddings of the $N = 2$ superconformal vertex algebra. This chapter also includes some conjectures related with the presented constructions. Chapter 11 is devoted to applications of these embeddings, namely the construction of some explicit geometric examples and $(0, 2)$ mirror symmetry on Hopf surfaces.

Finally, part IV contains the most technical aspects to make the reading fluent. This consists of three appendices, respectively devoted to the explanation of some basic identities for (SUSY) Lie conformal algebras and (SUSY) vertex algebras, the technical calculations used in the main computations of (SUSY) vertex algebras, and the proof of Theorem 10.1.1, which contains all the principal technical computations from where we deduce the three results stated before (all these are collected in Theorem 10.1.5, Theorem 10.1.8, and Theorem 10.1.19 in the main text), which are the main results of the present thesis.

Part I

Vertex algebras

Chapter 1

Basics on Vertex Algebras

The goal of this chapter is summarize basic aspects of the theory of vertex algebras. The main examples are in Section 1.4. Here, the adjective "super" applied to vector spaces V, algebras $\mathcal{A}$, modules $\mathcal{R}$, etc. will always mean $\mathbb{Z}/2\mathbb{Z}$-graded. We will omit it sometimes.

1.1 Standard Definition of Vertex Algebras

The most basic structure of a vertex algebra is given by a vector space V, called the *space of states*, a non-vanishing *vacuum vector* $|0\rangle \in V$, and the *state-field correspondence*, which is a linear map from V to endomorphism-valued bilateral series. This will be called the *standard definition*. Firstly, we need some background in formal distributions.

1.1.1 Formal Distributions and Quantum Fields

We will always work with *vector superspaces* in order to include bosons (even elements, with parity 0) and fermions (odd elements, with parity 1). So, we have a vector space decomposed in direct sum of two subspaces $V = V_0 \oplus V_1$, and we shall say that a *homogeneous vector* $v \in V$ has *parity* $|v| \in \{0, 1\}$ if $v \in V_{|v|}$. All computations involving parities are modulo 2. When this term appears, we will understand that all vectors are homogeneous. In addition, all formulas involving parities are only valid for those vectors. So, for the rest of elements, we will extend them by linearity (or other properties we are trying to preserve) in terms of the homogeneous components of that element. Moreover, when we consider bilinear forms on vector superspaces, we will understand that all of them are *consistent*. That is, they are zero on $V_0 \times V_1$ and $V_1 \times V_0$. Finally, when we speak about *superdimension* of a vector superspace, we mean $\operatorname{sdim} V := \dim V_0 - \dim V_1$.

Example 1.1.1 (Superendomorphisms). Let V be a vector superspace. Then, we can endow $\operatorname{End}(V)$ with a vector superspace structure by setting

$$\operatorname{End}(V)_\alpha := \{M \in \operatorname{End}(V) \mid MV_\beta \subseteq V_{\alpha+\beta}, \text{ for } \beta \in \mathbb{Z}_2\}, \quad \text{for } \alpha \in \mathbb{Z}_2.$$

If V is finite-dimensional, for each $M \in \mathrm{End}(V)$, we can take a decomposition such that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) + \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \in \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1.$$

The *supertrace* of M is defined by $\mathrm{str}(M) := \mathrm{tr}(A) - \mathrm{tr}(D)$.

Example 1.1.2 (**Lie Superalgebras**)**.** Let $\mathfrak{a}$ be a vector superspace. We say that $\mathfrak{a}$ is a *superalgebra* if it is endowed with a $\mathbb{Z}_2$-graded *compatible* product $\cdot\colon \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$. This means that if $a, b \in \mathfrak{a}$ are two homogeneous elements, then $a \cdot b \in \mathfrak{a}_{|a|+|b|}$. If the product $\cdot$ is associative, we can define a compatible commutator, the *supercommutator*, by

$$\begin{array}{rccc} [\cdot,\cdot]: & \mathfrak{a} \times \mathfrak{a} & \longrightarrow & \mathfrak{a} \\ & (a,b) & \mapsto & [a,b] := a \cdot b - (-1)^{|a||b|} b \cdot a \end{array}.$$

This is a compatible product (in the sense of Lie algebras), so it satisfies the following:

(1) The *antisymmetry* axiom

$$[a,b] = -(-1)^{|a||b|}\,[b,a]\,, \quad \text{for } a, b \in \mathfrak{a}. \tag{1.1}$$

(2) The *Jacobi identity* axiom

$$[a,[b,c]] = [[a,b]\,,c] + (-1)^{|a||b|}\,[b,[a,c]]\,, \quad \text{for } a, b, c \in \mathfrak{a}. \tag{1.2}$$

This product is known as the *superbracket*, and motivates the notion of *Lie superalgebra*, which is a vector superspace endowed with a superbracket satisfying (1.1) and (1.2).

To introduce quantum fields, we need some background on *formal distributions.* Let V be a vector superspace. Then, we say that a V-valued *formal distribution* in the indeterminates $z_1, \ldots, z_n$ is a formal expression of the form

$$A\,(z_1, z_2, \ldots, z_n) = \sum_{i_1 \in \mathbb{Z}} \sum_{i_2 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} a_{i_1,i_2,\ldots,i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n},$$

where $a_{i_1,i_2,\ldots,i_n} \in V$. We denote by $V\left[\left[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}\right]\right]$ the set of all V-valued formal distributions in $z_1, z_2 \ldots, z_n$. It admits a natural vector superspace structure, since the parity of homogeneous formal distributions is determined by each coefficient.

Remark 1.1.3. Let $\mathfrak{g}$ be a Lie superalgebra. Considering

$$A(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathfrak{g}\left[\left[z^{\pm 1}\right]\right], \quad B(w) = \sum_{m \in \mathbb{Z}} b_m w^m \in \mathfrak{g}\left[\left[w^{\pm 1}\right]\right],$$

it is possible to introduce formally the superbracket

$$[A(z), B(w)] := \sum_{n,m \in \mathbb{Z}} [a_n, b_m]\, z^n w^m \in \mathfrak{g}\left[\left[z^{\pm 1}, w^{\pm 1}\right]\right].$$

This one satisfies both properties (1.1) and (1.2) formally, however it does not induce a Lie superbracket since this is defined for formal distributions in different indeterminates.

We say that a V-valued *Laurent polynomial* in the indeterminates $z_1, \dots, z_n$ is a formal distribution in which almost every coefficient is zero. So, these are expressions

$$A(z_1, z_2, \dots, z_n) = \sum_{i_1=-M_1}^{M_1} \sum_{i_2=-M_2}^{M_2} \cdots \sum_{i_n=-M_n}^{M_n} a_{i_1,i_2,\dots,i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n},$$

where $M_1, \dots, M_n < \infty$. We denote by $V\left[z_1^{\pm 1}, \dots, z_n^{\pm 1}\right]$ the set of all V-valued Laurent polynomials in indeterminates $z_1, \dots, z_n$. It is a subsuperspace of $V\left[\left[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}\right]\right]$.

Remark 1.1.4. For any superalgebra $\mathcal{A}$, the set $\mathcal{A}\left[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}\right]$ has structure of superalgebra. Moreover, the product between any $\mathcal{A}$-valued Laurent polynomial and any $\mathcal{A}$-valued formal distribution is always a well-defined $\mathcal{A}$-valued formal distribution.

We will restrict to the case of one indeterminate, that is, we will focus on V-valued formal distributions of the form

$$A(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in V\left[\left[z^{\pm 1}\right]\right]. \tag{1.3}$$

We say that a V-valued *Laurent series* in the indeterminate z is a formal distribution in which almost every coefficient with negative powers is zero. Hence, these are expressions

$$A(z) = \sum_{n=-M}^{\infty} a_n z^n, \tag{1.4}$$

where $M < \infty$. We denote by $V((z))$ the set of V-valued Laurent series, and it is a vector subsuperspace of $V\left[\left[z^{\pm 1}\right]\right]$. Moreover, it is satisfied that $V\left[z^{\pm 1}\right] \subseteq V((z)) \subseteq V\left[\left[z^{\pm 1}\right]\right]$.

Remark 1.1.5. Given $\mathcal{A}$ superalgebra, then $\mathcal{A}((z))$ has structure of superalgebra.

Example 1.1.6 (Formal Taylor Series). The Laurent series

$$e^z := \sum_{n\in\mathbb{N}} \frac{z^n}{n!} \in \mathbb{C}((z))$$

is called *formal exponential*, and it is a special type of Laurent series. Indeed, a V-valued formal distribution of the form (1.4) is called a *Taylor series* when $M = 0$. The set of all V-valued Taylor series is denoted by $V[[z]]$, and is a supersubspace of $V((z))$. Note that

$$V((z)) = V[[z]]\left[z^{-1}\right].$$

We can introduce the *formal derivative* of $A(z)$ with respect to z as

$$\partial_z A(z) := \sum_{n\in\mathbb{Z}} (n+1) a_{n+1} z^n \in V\left[\left[z^{\pm 1}\right]\right],$$

and the *formal residue* by the usual formula

$$\operatorname{Res}_z A(z) := a_{-1} \in V.$$

It is easily seen that ∂_z is an (even) endomorphism over $V\left[\left[z^{\pm 1}\right]\right]$ satisfying the Lebniz rule, provided that we have a well-defined product, and $\operatorname{Res}_z\colon V\left[\left[z^{\pm 1}\right]\right] \longrightarrow V$ is a linear application that sends even/odd formal distributions to even/odd vectors, for which

$$\operatorname{Res}_z\left(\partial_z A(z)\cdot B(z)\right) = -\operatorname{Res}_z\left(A(z)\cdot \partial_z B(z)\right)$$

follows by Leibniz rule and $\operatorname{Res}_z\left(\partial_z A(z)\right) = 0$. We extend these notions to an arbitrary number of indeterminates, since $V\left[\left[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}\right]\right] \cong \left(V\left[\left[z_2^{\pm 1}, \ldots, z_n^{\pm 1}\right]\right]\right)\left[\left[z_1^{\pm 1}\right]\right]$. Now, note that a V-valued formal distribution $A(z)$ of the form (1.3) induces a V-valued linear form over $\mathbb{C}\left[z^{\pm 1}\right]$ setting the map

$$\begin{array}{rccl} g_{A(z)}\colon & \mathbb{C}\left[z^{\pm 1}\right] & \longrightarrow & V \\ & P(z) & \mapsto & \langle P(z) | A(z)\rangle := \operatorname{Res}_z\left(P(z)\cdot A(z)\right) \end{array}.$$

Notice that any V-valued linear form over $\mathbb{C}\left[z^{\pm 1}\right]$ is obtained in a unique way as above. Indeed, for the existence, given $f\colon \mathbb{C}\left[z^{\pm 1}\right] \longrightarrow V$ any linear form, we define

$$A(z) := \sum_{n\in\mathbb{Z}} f\left(z^{-1-n}\right) z^n \in V\left[\left[z^{\pm 1}\right]\right].$$

Then, $f\left(z^n\right) = \operatorname{Res}_z\left(z^n A(z)\right)$ for $n\in\mathbb{Z}$. This motivates the definition of *Fourier modes.* Given $A(z)$ any V-valued formal distribution of the form (1.3), its Fourier modes are

$$a_{(n)} := \operatorname{Res}_z\left(z^n A(z)\right) \in V, \quad \text{for } n\in\mathbb{Z}.$$

So, formal distributions admit an alternative expression, the one we will use from now,

$$A(z) = \sum_{n\in\mathbb{Z}} z^{-1-n} a_{(n)} \in V\left[\left[z^{\pm 1}\right]\right]. \tag{1.5}$$

We will write in the expressions above the Fourier modes after the indeterminates always, for parity reasons that we will see in future chapters, and because these will be in general endomorphisms. This is the case of (quantum) fields on V that we are ready to introduce.

Definition 1.1.7. [63] We will say that an $\operatorname{End}(V)$-valued formal distribution $A(z)$ as in (1.5) is a (quantum) *field* on V if its Fourier modes satisfy, for each $v\in V$ fixed, that

$$a_{(n)}(v) = 0, \text{ for } \ n \gg 0.$$

The value of n depends on $v\in V$. This is equivalent to the condition $A(z)(v)\in V((z))$, for every $v\in V$. The natural structure of vector superspace for $\operatorname{End}(V)$ implies that the field $A(z)$ has parity $|A|\in\{0,1\}$ if $a_{(n)}\left(V_\alpha\right)\subseteq V_{\alpha+|A|}$ for $\alpha\in\{0,1\}$ and $n\in\mathbb{Z}$. The set of fields on V is denoted by $\mathcal{F}(V)$, and is clearly a vector subsuperspace of $\operatorname{End}(V)\left[\left[z^{\pm 1}\right]\right]$.

An endomorphism is *parity-preserving* when it is even, and otherwise, we will say that is *parity-reversing.* These two notions are naturally extended to general linear and bilinear maps. Indeed, we will say that $f\colon V\times V' \longrightarrow W$ is parity-preserving if $|f(a,b)| = |a|+|b|$ for $a\in V$ and $b\in V'$. Otherwise, it is said that f as before is parity-reversing.

1.1.2 The Notion of Locality

We will work with formal distributions in two indeterminates z and w in this section. Complex rational functions may be expanded in several ways under certain convergence domains. Analogously, we can do the same for formal rational expressions via the notion of *formal expansions*. We will consider the algebra of rational expressions in V with poles only at $z = 0, w = 0$ and $z = w$. Then, we can define the two homomorphisms

$$\begin{aligned} i_{z,w} \colon V\,[[z,w]]\left[z^{-1}, w^{-1}, (z-w)^{-1}\right] &\longrightarrow V((z))((w)), \\ i_{w,z} \colon V\,[[z,w]]\left[z^{-1}, w^{-1}, (z-w)^{-1}\right] &\longrightarrow V((w))((z)), \end{aligned}$$

via *generalized Binomial expansions*. For that, take another formal indeterminate x, and now consider the algebra of formal rational expressions in V with poles only at $x = 0$. For $r \in \mathbb{Z}$, we define the next power series expansion in the domain $|x| < 1$ by

$$V\left[(1-x)^{-1}\right] \ni \frac{1}{(1-x)^r} := \sum_{k\in\mathbb{N}} \binom{r+k-1}{k} x^k \in V[[x]].$$

So, the two homomorphisms $i_{w,z}$ and $i_{w,z}$ above are defined using the linear extensions in V of the formal rational functions as power series expansions in the domains $|z| < |w|$ and $|w| < |z|$, respectively. In particular,

$$i_{z,w}\left(\frac{1}{z-w}\right) := z^{-1}\sum_{n\in\mathbb{N}}\left(\frac{w}{z}\right)^n, \quad i_{w,z}\left(\frac{1}{z-w}\right) := -w^{-1}\sum_{n\in\mathbb{N}}\left(\frac{z}{w}\right)^n$$

Notice that the maps $i_{z,w}$ and $i_{w,z}$ commute with both ∂_z and ∂_w partial derivatives.

Definition 1.1.8. [63] The *delta formal distribution* is

$$\delta(z-w) := (i_{z,w} - i_{w,z})\left(\frac{1}{z-w}\right) \in \mathbb{C}\left[\left[z^{\pm1}, w^{\pm1}\right]\right].$$

The delta formal distribution can be seen as the algebraic abstraction of the usual Dirac's delta distribution (see, for example, [63, Proposition 2.1]). Indeed, for $A(z) \in V\left[\left[z^{\pm1}\right]\right]$, the delta formal distribution is the unique formal distribution such that

$$\operatorname{Res}_z\left(A(z)\delta(z-w)\right) = A(w).$$

Now, by the universal property of the localizations

$$V\,[[z,w]] \hookrightarrow V((z))((w)) \text{ and } V\,[[z,w]] \hookrightarrow V((w))((z)).$$

Moreover, the natural inclusions $V((z))((w)), V((w))((z)) \hookrightarrow V\left[\left[z^{\pm1}, w^{\pm1}\right]\right]$ induce

$$\begin{tikzcd} V\,[[z,w]] \arrow[r, hook] \arrow[d, hook] & V((z))((w)) \arrow[dr, hook] & \\ V\,[[z,w]]\left[z^{-1}, w^{-1}, (z-w)^{-1}\right] \arrow[ur, "i_{z,w}"] \arrow[dr, "i_{w,z}"'] & & V\left[\left[z^{\pm1}, w^{\pm1}\right]\right]. \\ V\,[[z,w]] \arrow[r, hook] \arrow[u, hook] & V((w))((z)) \arrow[ur, hook] & \end{tikzcd}$$

The left-hand side subdiagrams are commutative, but the one in the right-hand side is not, as shown by the delta formal distribution. Now, consider a V-valued formal distribution $A(z,w)$, and suppose there exists $f(z,w) \in V[[z,w]]\left[z^{-1}, w^{-1}, (z-w)^{-1}\right]$ such that

$$A(z,w) = (i_{z,w} - i_{w,z})\, f(z,w) \in V\left[\left[z^{\pm 1}, w^{\pm 1}\right]\right].$$

Let $N \in \mathbb{N}$ be such that $P(z,w) := (z-w)^N f(z,w) \in V[[z,w]]\left[z^{-1}, w^{-1}\right]$. Then,

$$i_{z,w}P(z,w) = i_{w,z}P(z,w) \text{ if and only if } (z-w)^N A(z,w) = 0.$$

Definition 1.1.9. [63] It is said that $A(z,w) \in V\left[\left[z^{\pm 1}, w^{\pm 1}\right]\right]$ is *local* when there exists $N_A \gg 0$ such that

$$(z-w)^{N_A} A(z,w) = 0.$$

Theorem 1.1.10 ([63, Theorem 2.3] **Decomposition Theorem**). *Consider the formal distribution $A(z,w) \in V\left[\left[z^{\pm 1}, w^{\pm 1}\right]\right]$. Then, A is local if and only if there exists $N_A \gg 0$ such that*

$$A(z,w) = \sum_{n=0}^{N_A - 1} C^n(w) \frac{\partial_w^n \delta(z-w)}{n!},$$

where $C^n(w) \in V\left[\left[w^{\pm 1}\right]\right]$ are formal distributions given by

$$C^n(w) := \operatorname{Res}\left((z-w)^n A(z,w)\right), \quad \textit{for } n \in \{0, 1, \ldots, N-1\}.$$

Definition 1.1.11. [63] Let $\mathfrak{g}$ be any Lie algebra. We will say that two $\mathfrak{g}$-valued formal distributions A and B are *mutually local* if $[A(z), B(w)]$ is local.

Remark 1.1.12 (**The Positive n-products in Formal Distributions**)**.** Let $\mathfrak{g}$ be a Lie algebra, and consider A and B two $\mathfrak{g}$-valued mutually local formal distributions. The Decomposition Theorem gives us new $\mathfrak{g}$-valued formal distributions

$$C^n(w) = \left(A_{(n)}B\right)(w) := \operatorname{Res}_z\left((z-w)^n \left[A(z), B(w)\right]\right) \in \mathfrak{g}\left[\left[w^{\pm 1}\right]\right], \quad \text{for } n \in \mathbb{N}.$$

These are known as *the positive n-products* of A and B.

Let $\mathcal{A}$ be any superalgebra. Given A and B two $\mathcal{A}$-valued formal distributions, we can introduce a new $\mathcal{A}$-valued formal distribution in two indeterminates as

$$: A(z)B(w) := A(z)_+ B(w) + (-1)^{|A||B|} B(w) A(z)_- \in \mathcal{A}\left[\left[z^{\pm 1}, w^{\pm 1}\right]\right],$$

where

$$A(z)_+ := \sum_{n<0} z^{-1-n} a_{(n)}, \; A(z)_- := \sum_{n \in \mathbb{N}} z^{-1-n} a_{(n)} \in \mathcal{A}\left[\left[z^{\pm 1}\right]\right].$$

This new operation is very important in the case of fields, as we will see in a moment.

Remark 1.1.13 (Normally Ordered Product and OPE). Let V be a vector space, and consider the fields A and B. The *normally ordered product* of A and B is given by

$$(: AB :)(z) := A(z)_+ B(z) + (-1)^{|A||B|} B(z) A(z)_-.$$

This is a well defined $\mathrm{End}(V)$-valued formal distribution, and is again a field on V (see, for example, [63, Section 3.1]). Moreover, we introduce the *negative j-products* as

$$\left(A_{(-1-j)}B\right)(z) := \frac{:\left(\partial_z^j A(z)\right) B(z):}{j!} \in \mathcal{F}(V), \quad \text{for } j \in \mathbb{N}.$$

Thus, if A and B are mutually local, as a consequence of the Decomposition Theorem,

$$\begin{aligned}
A(z)B(w) &= \; :A(z)B(w): + [A(z)_-, B(w)] \\
&= \; :A(z)B(w): + \sum_{n\in\mathbb{N}} \left(A_{(n)}B\right)(w) i_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right); \\
(-1)^{|A||B|} B(w)A(z) &= \; :A(z)B(w): - [A(z)_+, B(w)] \\
&= \; :A(z)B(w): + \sum_{n\in\mathbb{N}} \left(A_{(n)}B\right)(w) i_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right).
\end{aligned}$$

Here, abusing notation, we write the *operator product expansion* (OPE) of A and B by

$$A(z)B(w) =: A(z)B(w): + \sum_{n\in\mathbb{N}} \frac{\left(A_{(n)}B\right)(w)}{(z-w)^{n+1}} \sim \sum_{n\in\mathbb{N}} \frac{\left(A_{(n)}B\right)(w)}{(z-w)^{n+1}},$$

The *singular part* of the OPE (positive n-products) encodes the information of brackets between coefficients of A and B. Indeed, by the Decomposition Theorem, if $a_{(n)}$, $b_{(m)}$ and $\left(a_{(j)}b\right)_{(k)}$ are the Fourier modes of A, B and $A_{(j)}B$, respectively, then

$$\left[a_{(n)}, b_{(m)}\right] = \sum_{j=0}^{N} \binom{n}{j} \left(a_{(j)}b\right)_{(n+m-j)}, \quad \text{for } n, m \in \mathbb{Z}. \tag{1.6}$$

So, we omit the *regular part* of the OPE. For $j \in \mathbb{Z}$ it satisfies

$$\left(A_{(j)}B\right)(w) = \mathrm{Res}_z \left(\left(A(z)B(w) i_{z,w} - (-1)^{|A||B|} B(w)A(z) i_{w,z}\right)(z-w)^j\right).$$

Then, ∂_w is an even derivation for the j-products, and $\left|A_{(j)}B\right| = |A| + |B|$ for $j \in \mathbb{Z}$.

Remark 1.1.14. Notice that everything we have introduced above for all the positive n-products make perfect sense for formal distributions on Lie algebras. In particular, we can take the singular part of the OPE, and the formulas (1.6) still hold.

1.1.3 The Notion of Vertex Algebra

We are now ready to introduce the first definition of vertex algebra.

Definition 1.1.15. [6] A *vertex algebra* is the data $(V, |0\rangle, T, Y(\cdot, z))$, where

- $V = V_0 \oplus V_1$ is a vector superspace (space of states).
- $|0\rangle \in V_0$ is an even state (vacuum vector).
- $T\colon V \longrightarrow V$ is an even endomorphism (infinitesimal translation operator).
- $Y(\cdot, z)\colon V \longrightarrow \mathcal{F}(V)$ is a parity-preserving linear map from a given state to a field (state-field correspondence) given by

$$Y(a, z) := \sum_{n\in\mathbb{Z}} z^{-1-n} a_{(n)} \in \mathcal{F}(V), \quad \text{for } a \in V,$$

that satisfy the following:

Axiom 1 (*vacuum axioms*) We have that $T|0\rangle = 0$, and $Y(a, z)(|0\rangle)|_{z=0} = a$, for $a \in V$.

Axiom 2 (*translation covariance axiom*) For $a \in V$, we have $[T, Y(a, z)] = \partial_z Y(a, z)$.

Axiom 3 (*locality axiom*) The fields $\mathcal{F} = \{Y(a, z) \mid a \in V\}$ are mutually local. So, for $a, b \in V$, there exist $N \gg 0$, depending on the vectors a and b, such that

$$(z - w)^N [Y(a, z), Y(b, w)] = 0.$$

Remark 1.1.16 (*j*-products in Vertex Algebras). Given $(V, |0\rangle, T, Y(\cdot, z))$ any vertex algebra, we can introduce naturally the following parity-preserving bilinear maps

$$a_{(j)}b = a_{(j)}(b) = \mathrm{Res}_z \left(z^j Y(a, z)(b)\right), \quad \text{for } j \in \mathbb{Z},$$

for any $a, b \in V$, which are known as the *j-products* of the vertex algebra.

Remark 1.1.17 ([6], **Homomorphisms, Subalgebras, Ideals, Quotients in VAs).** Let $(V, |0\rangle, T, Y(\cdot, z))$ be a vertex algebra. We introduce the basic notions for algebraic structures as usual. Indeed, the natural product operations are the j-products. We do not need a separate notion for left and right ideals (see, for example, [63, Section 4.3]).

Proposition 1.1.18 ([22, Corollary 1.7] ***j*-product identities).** *For $(V, |0\rangle, T, Y(\cdot, z))$ a vertex algebra, one has*

$$Y\left(a_{(j)}b, z\right) = Y(a, z)_{(j)} Y(b, z), \quad \textit{for } j \in \mathbb{Z}; a, b \in V.$$

As a consequence,

$$Y(Ta, z) = \partial_z Y(a, z) \tag{1.7}$$

In particular, T is an even derivation for all the j-products.

Remark 1.1.19 (Normally Ordered Product in Vertex Algebras). It is seen that the (-1)-product plays a special role in this theory. Indeed, for $(V, |0\rangle, T, Y(\cdot, z))$ vertex algebra, this is, roughly speaking, the product of the vertex algebra V as an algebra, and it is called the *normally ordered product* of V, and we will write $: ab :$ for $a, b \in V$. It contains the information about all the negative j-products of V thanks to (1.7). Indeed, applying this inductively, we obtain that

$$:\left(\frac{T^n(a)}{n!}\right) b := a_{(-1-n)}b, \quad \text{for } a, b \in V; n \in \mathbb{N}. \tag{1.8}$$

1.2 Bakalov-Kac Characterization of Vertex Algebras

We study now Bakalov-Kac's characterization of vertex algebras from [3], which is the one we will use for the computations in future chapters (see Chapter 3). For this, we need to introduce the notion of *Lie conformal algebras.*

1.2.1 Lie Conformal Algebras

First, we consider the tensor catefgory of $\mathbb{C}[\partial]$-modules. In particular, given $\mathcal{R}$ and $\mathcal{P}$ two $\mathbb{C}[\partial]$-modules, we have that ∂ acts on $\mathcal{R}\otimes\mathcal{P}$ via the Leibniz rule. In this context, we can introduce the following interesting bracket (see [63, Equation 2.7.2]).

Definition 1.2.1. [63] A *Lie conformal algebra* is the data $(\mathcal{R}, [\cdot_\lambda\cdot])$, where

- $\mathcal{R}$ is a $\mathbb{C}[\partial]$-module.
- $[\cdot_\lambda\cdot] : \mathcal{R}\otimes\mathcal{R} \longrightarrow \mathbb{C}[\lambda]\otimes\mathcal{R}$ is a parity-preserving bilinear map, called the λ-*bracket*, where λ is an even formal parameter, satisfying
  - *Sesquilinearity*: this is an equality in $\mathbb{C}[\lambda]\otimes\mathcal{R}$. For $a, b \in \mathcal{R}$,
  $$[\partial a_\lambda b] = -\lambda\, [a_\lambda b]\,,\ [a_\lambda \partial b] = (\partial+\lambda)\, [a_\lambda b]\,. \tag{1.9}$$
  In particular, ∂ is an even derivation for the λ-bracket.
  - *Antisymmetry*: this is an equality in $\mathbb{C}[\lambda]\otimes\mathcal{R}$. For $a, b \in \mathcal{R}$,
  $$[a_\lambda b] = -(-1)^{|a||b|}\,[b_{-\partial-\lambda} a]\,. \tag{1.10}$$
  - *Jacobi identity*: given γ another even formal parameter, this is an equality in $\mathbb{C}[\lambda]\otimes\mathbb{C}[\gamma]\otimes\mathcal{R}$. For $a, b, c \in \mathcal{R}$,
  $$[a_\lambda\,[b_\mu c]] = \Big[[a_\lambda b]_{\lambda+\mu}\, c\Big] + (-1)^{|a||b|}\,[b_\mu\,[a_\lambda c]]\,. \tag{1.11}$$

  The meaning of these expressions are explained in Appendix A.1.

Remark 1.2.2 (Homomorphisms, Subalgebras, Ideals, Quotients in LCAs). We can introduce the basic notions for algebraic structures as usual. Indeed, the natural product operation is the λ-bracket. Thanks to the antisymmetry axiom, we do not need a separate notion for left and right ideals (see, for example, [63, Section 2.7]).

Example 1.2.3 ([63, Section 2.7])**.** Let $\mathcal{R}$ be a Lie conformal algebra, and define
$$\begin{array}{rccc} [\cdot,\cdot]: & \mathcal{R}\otimes\mathcal{R} & \longrightarrow & \mathcal{R} \\ & (a,b) & \mapsto & [a,b] := [a_\lambda b]|_{\lambda=0} \end{array}.$$
It follows from sesquilinearity that $\partial\mathcal{R}\subseteq\mathcal{R}$ is an ideal for this new product. Therefore, defining $\mathfrak{g} := \mathcal{R}/\partial\mathcal{R}$, the previous product descends to a bilinear map $\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$. The product $[\cdot,\cdot]$ endows $\mathfrak{g}$ with Lie superalgebra structure. This is the *Lie algebra associated to* $\mathcal{R}$. Now, let $\widetilde{\mathcal{R}} = L\mathcal{R}$ be the *loop algebra* of $\mathcal{R}$. That is, the Lie conformal algebra of loops $\mathcal{R}[t^{\pm1}]$ (see [76, Section 2.6]) with associated derivation $\widetilde{\partial} := \partial\otimes \mathrm{Id}_t + \mathrm{Id}_{\mathcal{R}}\otimes\partial_t$. The Lie superalgebra associated to $\widetilde{\mathcal{R}}$ is denoted by $\mathrm{Lie}\,(\mathcal{R})$, and it will be useful.

Proposition 1.2.4 ([3, Lemma 5.1])**.** *Let $(V, |0\rangle, T, Y(\cdot, z))$ be a vertex algebra. Then, we obtain a Lie conformal algebra, with $\partial = T$ and the λ-bracket*

$$[a_\lambda b] := \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} a_{(n)} b \in \mathbb{C}[\lambda] \otimes V, \quad \textit{for } a, b \in V.$$

In particular, the λ-bracket above is none other than the OPE of our vertex algebra.

1.2.2 Universal Enveloping Vertex Algebras

We will follow closely the construction explained in [22, Section 1.7]. Let $(\mathcal{R}, [\cdot_\lambda \cdot])$ be a Lie conformal algebra, for which we construct $\mathfrak{g} := \mathrm{Lie}(\mathcal{R})$ a Lie algebra (see Example 1.2.3). Notice that $\widetilde{\partial} \colon \mathcal{R}[t^{\pm 1}] \longrightarrow \mathcal{R}[t^{\pm 1}]$ induces a Lie algebra derivation $-\partial_t \colon \mathfrak{g} \longrightarrow \mathfrak{g}$. Then, we can construct canonically a vertex algebra. Consider the $(-\partial_t)$-invariant Lie subalgebra

$$\mathfrak{g}_- = \mathrm{span}_{\mathbb{C}} \{ [at^n] \in \mathrm{Lie}(\mathcal{R}) \mid n \in \mathbb{N} \} \subseteq \mathfrak{g}.$$

Let $\mathcal{U}(\mathfrak{g})$ be the *universal enveloping Lie algebra* of $\mathfrak{g}$. Then, the derivation $-\partial_t$ extends uniquely to a derivation T of $\mathcal{U}(\mathfrak{g})$, and $T(\mathbb{C}) = 0$. Moreover, the centre $Z(\mathfrak{g})$ of $\mathfrak{g}$ is T-invariant. Now, we are ready to define the data $(V, |0\rangle, T, Y(\cdot, z))$ as follows:

- $V \equiv V(\mathfrak{g}, \mathcal{R}) := \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \mathfrak{g}_-$ is a left $\mathcal{U}(\mathfrak{g})$-module, where $\mathcal{U}(\mathfrak{g}) \mathfrak{g}_- \subseteq \mathcal{U}(\mathfrak{g})$ is the left ideal generated by the T-invariant Lie subalgebra $\mathfrak{g}_- \subseteq \mathfrak{g}$.
- $|0\rangle \in V$ is the element induced by the identity $1 \in \mathcal{U}(\mathfrak{g})$.
- $T \colon V \longrightarrow V$ is the derivation induced by $T \colon \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$.
- $Y(\cdot, z) \colon \mathcal{R} \longrightarrow \mathrm{End}(V)[[z^{\pm}]]$ is a parity-preserving linear map defined, for $a \in \mathcal{R}$, by
$$\begin{array}{cccc} Y(a, z) \colon & V & \longrightarrow & V\left[\left[z^{\pm 1}\right]\right] \\ & \pi(b) & \mapsto & \displaystyle\sum_{n \in \mathbb{Z}} z^{-1-n} \pi([at^n] \cdot b) \end{array},$$
where $\cdot$ is the product on $\mathcal{U}(\mathfrak{g})$, and $\pi \colon \mathcal{U}(\mathfrak{g}) \longrightarrow V$ is the canonical projection.

Theorem 1.2.5 ([22, Theorem 1.17] **Existence Theorem**)**.** *The data $(V, |0\rangle, T, Y(\cdot, z))$ gives a vertex algebra. In particular, the map $Y(\cdot, z)$ induces a state-field correspondence.*

Definition 1.2.6. [22] The vertex algebra $V(\mathcal{R}) \equiv V(\mathfrak{g}, \mathcal{R})$ above is called the *universal enveloping vertex algebra* associated to the Lie conformal algebra $\mathcal{R}$.

Remark 1.2.7. Let V denote the universal enveloping vertex algebra associated to the pair $(\mathfrak{g}, \mathcal{R})$, and consider $\alpha \colon Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ a linear map such that $\alpha(T(Z(\mathfrak{g}))) = 0$. Then,

$$I^\alpha(V) := \mathrm{span}_{\mathbb{C}} \{(C - \alpha(C))V \mid C \in Z(\mathfrak{g})\} \subseteq V$$

is a vertex algebra ideal, and we can introduce the quotient

$$V^\alpha := \frac{V}{I^\alpha(V)},$$

which is going to play a fundamental role to construct our examples in a moment.

Corolary 1.2.8 ([51, Theorem 14.6]). *If $\mathcal{R}$ is a Lie conformal algebra, then there exists a unique vertex algebra $V(\mathcal{R})$ satisfying the following universal property: any homomorphism $\varphi\colon \mathcal{R} \longrightarrow V$ of Lie conformal algebras, where V is a vertex algebra, extends uniquely to an homomorphism of vertex algebras $\overline{\varphi}\colon V(\mathcal{R}) \longrightarrow V$ making commutative the diagram*

$$\begin{tikzcd} V(\mathcal{R}) \arrow[dr, dashed, "\overline{\varphi}"] & \\ \mathcal{R} \arrow[u, hook, "i"] \arrow[r, "\varphi"'] & V. \end{tikzcd}$$

Theorem 1.2.9 ([3, 22]). *A vertex algebra is equivalent to the following data:*

- *A Lie conformal algebra $(V, [\cdot_\lambda \cdot])$, where V is a $\mathbb{C}[T]$-module.*
- *A unital differential algebra $((V, |0\rangle, :\cdot\cdot:), T)$ satisfying quasicommutativity and quasiassociativity axioms, which are respectively given, for $a, b, c \in V$, by*

$$:ab: -(-1)^{|a||b|} :ba: = \int_{-T}^{0} d\lambda\, [a_\lambda b], \tag{1.12}$$

$$::ab:c: - :a:bc:: = :\left(\int_0^T d\lambda\, a\right)[b_\lambda c]: + (-1)^{|a||b|} :\left(\int_0^T d\lambda\, b\right)[a_\lambda c]:. \tag{1.13}$$

- *The non-commutative Wick formula, a quasi-Leibniz rule, is satisfied, which relates the λ-bracket $[\cdot_\lambda \cdot]$ and the product $:\cdot\cdot:$, for $a, b, c \in V$, as follows:*

$$[a_\lambda :bc:] = :[a_\lambda b]\, c: + (-1)^{|a||b|} :b\, [a_\lambda c]: + \int_0^\lambda d\mu \left[[a_\lambda b]_\mu c\right]. \tag{1.14}$$

The meaning of the integrals in these identities are explained in Appendix A.2.

1.3 Conformal Vertex Algebras

Let $(V, |0\rangle, T, Y(\cdot, z))$ be a vertex algebra.

Definition 1.3.1. [3] A diagonalizable operator $H \in \mathrm{End}(V)$ will be called a *Hamiltonian* of V. The state-field correspondence $Y(\cdot, z) : V \longrightarrow \mathcal{F}(V)$ is *graded* by H if

$$[H, Y(a, z)] = Y(Ha, z) + z\partial_z Y(a, z), \quad \text{for } a \in V.$$

If $a \in V$ is homogeneous of degree $\Delta_a \in \mathbb{C}$ (that is, $Ha = \Delta_a a$), then

$$[H, Y(a, z)] = (\Delta_a + z\partial_z)\, Y(a, z).$$

In general, any field $A(z) \in \mathrm{End}(V)\left[\left[z^{\pm 1}\right]\right]$ is said to have *conformal weight* $\Delta \in \mathbb{C}$ if

$$[H, A(z)] = (\Delta + z\partial_z)\, A(z). \tag{1.15}$$

In these cases, we say that $A(z)$ is an *eigenfield* for H of conformal weight $\Delta \in \mathbb{C}$.

Given $A(z)$ eigenfield for $H \in \mathrm{End}(V)$ Hamiltonian, we write, by (1.15),

$$A(z) = \sum_{n \in -\Delta+\mathbb{Z}} z^{-\Delta-n} a_n \in \mathcal{F}(V), \tag{1.16}$$

if it is of conformal weight $\Delta \in \mathbb{C}$. We can shift the subscripts in the coefficients, writing

$$a_n = a_{(n+\Delta-1)}, \text{ for } n \in -\Delta + \mathbb{Z} \text{ or, equivalently, } a_{(n)} = a_{n-\Delta+1}, \text{ for } n \in \mathbb{Z}.$$

Definition 1.3.2. [63] A *conformal vector* of V is $\nu \in V_0$ satisfying:

1. The field associated to ν is a Virasoro field with central charge $c \in \mathbb{C}$. That is,

   $$Y(\nu, z) := L(z) = \sum_{n \in \mathbb{Z}} z^{-2-n} L_n \in \mathcal{F}(V), \tag{1.17}$$

   where the coefficients L_n satisfy, for $C = c\,\mathrm{Id}$ constant field, the relations

   $$[L_m, L_n] = (m-n)L_{m+n} + \delta_m^{-n} \frac{m^3 - m}{12} C. \tag{1.18}$$

2. The infinitesimal translation operator is $T = L_{-1}$.

3. The endomorphism L_0 is diagonizable on V by positive integer eigenvalues. So,

   $$V = \bigoplus_{n \in \mathbb{Z}} V^{(n)},$$

   where

   $$V^{(n)} = \{0\} \text{ if } n < 0, \text{ and } V^{(n)} = \{a \in V \mid L_0(a) = na\} \text{ if } n \in \mathbb{N}.$$

   Sometimes, it may happen that $n \in \frac{1}{2} + \mathbb{N}$ as we will explain in Chapter 2.

The number $c \in \mathbb{C}$ is called the *central charge* of ν. A vertex algebra V endowed with a conformal vector ν is known with the name of *conformal vertex algebra* of rank c.

1.3.1 Virasoro Embeddings

Let $(\mathcal{R}, [\cdot_\lambda \cdot])$ be a Lie conformal algebra, and fix $L \in \mathcal{R}$ an even vector. In what follows, we will embrace the characterization in Theorem 1.2.9 and take this as our definition of vertex algebra. In particular, we will just give the underlying Lie conformal algebras.

Definition 1.3.3. [3] We say that $a \in \mathcal{R}$ has *weight* $\Delta_a \in \mathbb{C}$ with respect to L if

$$[L_\lambda a] = (\partial + \Delta_a \lambda)\, a + O\left(\lambda^2\right).$$

In these cases, we say that a is an *eigenvector* of weight Δ_a with respect to L. Moreover, we will say that $a \in \mathcal{R}$ is a *primary eigenvector* of weight Δ_a with respect to L if

$$[L_\lambda a] = (\partial + \Delta_a \lambda)\, a.$$

We say that $\mathcal{R}$ is $\mathbb{Z}$-*graded* by L if we have a basis of primary eigenvectors with integer weights with respect to L. We say that L is a *Virasoro vector* if

$$[L_\lambda L] = (\partial + 2\lambda)\, L + \frac{\lambda^3}{12} C,$$

where $C \in \mathcal{R}$ is a central element. When L is a Virasoro vector, the weight of any element with respect to L is called the *conformal weight* of such a vector (with respect to L).

As any vertex algebra is in particular a Lie conformal algebra by Proposition 1.2.4, the notions introduced above have perfect sense for vertex algebras.

Proposition 1.3.4 ([63, Corollary 4.10])**.** *Given any vertex algebra $(V, |0\rangle, T, Y(\cdot, z))$ for which there exists an even vector $\nu \in V$ such that $L(z) = Y(\nu, z)$ is the Virasoro field, then, $a \in V$ is an eigenvector of conformal weight $\Delta_a \in \mathbb{C}$ with respect to L if and only if $L_0 a = \Delta_a a$, and $L_{-1} a = Ta$.*

Remark 1.3.5. The previous result can be modified to obtain something similar for primary vectors. Indeed, in that case, the first identity is $L_n a = \delta_n^0 \Delta_a a$ for $n \in \mathbb{N}$.

Corolary 1.3.6. *Let $\mathcal{R}$ be a Lie conformal algebra, and suppose that $V(\mathcal{R})$ contains a Virasoro vector L of central charge $c \in \mathbb{C}$. Then, we have that $(V(\mathcal{R}), L)$ defines a conformal vertex algebra of rank c if and only if we have a basis of eigenvectors of certain conformal weight $\Delta \in \mathbb{Z}$ with respect to L.*

1.4 Examples of Conformal Vertex Algebras

We construct vertex algebras from Lie algebras. We will work with the underlying Lie conformal algebra $\mathcal{R}$, taking the quotient of $V(\mathcal{R})$ by some ideal via Remark 1.2.7.

1.4.1 Virasoro Vertex Algebra

The next example is taken from [6, 63, 64]. The *Virasoro algebra* is the Lie algebra $\mathfrak{g}$, which is the unique non-trivial central extension $0 \longrightarrow \mathbb{C}C \longrightarrow \mathfrak{g} \longrightarrow \mathcal{A} \longrightarrow 0$ of the *Witt algebra* $\mathcal{A}$ by C central element. So,

$$\mathcal{A} = \mathrm{Vect}(\mathring{\mathbb{D}}) := \mathrm{Der}_{\mathbb{C}}\left(\mathbb{C}\left[t^{\pm 1}\right]\right) = \mathbb{C}\left[t^{\pm 1}\right] \partial_t,$$

where $\mathring{\mathbb{D}} := \mathrm{Spec}(\mathbb{C}\left[t^{\pm 1}\right])$ is the punctured line. Fix $L_n := -t^{n+1}\partial_t$, for $n \in \mathbb{Z}$ the basis, for which we have the Lie brackets $[L_n, L_m] := (m-n)L_{n+m}$. Then,

$$\mathfrak{g} = \mathrm{Vect}(\mathring{\mathbb{D}}) \oplus \mathbb{C}C$$

as vector space, with the non-zero commutators (1.18). As a Lie conformal algebra, this is $\mathrm{Vir} := (\mathbb{C}L \otimes \mathbb{C}[T]) \oplus \mathbb{C}C$, where L, C are even, with the non-zero λ-bracket

$$[L_\lambda L] = (T + 2\lambda)\, L + \frac{\lambda^3}{12} C. \tag{1.19}$$

Notice that $\mathrm{Lie}\,(\mathrm{Vir}) = \mathfrak{g}$. Applying the Existence Theorem, we obtain a vertex algebra $V\,(\mathrm{Vir})$, which is known as the *universal Virasoro vertex algebra*. Now, since C is the generator of the centre, we have $\alpha\colon Z\,(\mathfrak{g}) = \mathbb{C}C \longrightarrow \mathbb{C}$ a linear map as in Remark 1.2.7. So, denoting $c := \alpha(C) \in \mathbb{C}$ as the *central charge*, we can take the quotient

$$V^c\,(\mathrm{Vir}) := \frac{V\,(\mathrm{Vir})}{(C-c)V\,(\mathrm{Vir})}.$$

This is known as the *universal Virasoro vertex algebra of central charge* $c \in \mathbb{C}$, and it is seen that has conformal vertex algebra structure, since $L_{-2}\,|0\rangle$ is a Virasoro vector.

1.4.2 Affinization of quadratic Lie algebras

We will work in next two examples with $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ bilinear forms over V vector superspaces that can be symmetric and antisymmetric in this "super" sense. That is,

- we say that $(\cdot,\cdot)$ is *supersymmetric* if $(a,b) = (-1)^{|a||b|}\,(b,a)$ for $a,b \in V$.
- we say that $(\cdot,\cdot)$ is *superantisymmetric* if $(a,b) = -(-1)^{|a||b|}\,(b,a)$ for $a,b \in V$.

Definition 1.4.1. Let $\mathfrak{g}$ be a Lie superalgebra and $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ a non-degenerate, supersymmetric and *invariant* bilinear form. That is, it satisfies $([a,b]|\,c) = (a\,|[b,c])$, for $a,b,c \in \mathfrak{g}$. The pair $(\mathfrak{g}, (\cdot|\cdot))$ is a *quadratic Lie superalgebra*.

The next example is taken from [6, 63, 64]. Let $(\mathfrak{g}, (\cdot|\cdot))$ be a finite-dimensional quadratic Lie superalgebra. The *affinization* of $\mathfrak{g}$ is a complex Lie superalgebra, which is the central extension $0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0$ of the *loop algebra* of $\mathfrak{g}$ via $(\cdot|\cdot)$ by K central element. Here, $L\mathfrak{g}$ is the Lie superalgebra of loops $L\mathfrak{g} := \mathrm{Map}(\mathring{\mathbb{D}}, \mathfrak{g}) = \mathfrak{g}\left[t^{\pm 1}\right]$. Then, $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ as vector superspace, with the non-zero commutators

$$[at^n, bt^m] = [a,b]\,t^{n+m} + n\delta^0_{n+m}\,(a|b)\,K, \quad \text{for } a,b \in \mathfrak{g}, n,m \in \mathbb{Z}. \tag{1.20}$$

If $\mathfrak{g}$ is simple, this is called the *Kac-Moody affinization*, and we have a unique non-trivial central extension as above. In such cases, we can take $h^\vee \in \mathbb{C}^*$ the *dual Coxeter number* of $\mathfrak{g}$ (see [62, Section 6.1]). We can also include the supercommutative case for $h^\vee = 0$. As Lie conformal algebra, this is the *current algebra* $\mathrm{Cur}\mathfrak{g} = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}K$, with the non-zero λ-brackets

$$[a_\lambda b] = [a,b] + \lambda\,(a|b)\,K, \quad \text{for } a,b \in \mathfrak{g}. \tag{1.21}$$

Notice that $\mathrm{Lie}\,(\mathrm{Cur}\mathfrak{g}) = \hat{\mathfrak{g}}$. Applying the Existence Theorem, we obtain a vertex algebra $V\,(\mathrm{Cur}\mathfrak{g})$, which is known as the *universal affine vertex algebra*. Now, since K is the generator of the centre, we have $\alpha\colon Z\,(\widehat{\mathfrak{g}}) = \mathbb{C}K \longrightarrow \mathbb{C}$ a linear map as in Remark 1.2.7. So, denoting $k := \alpha(K)$ as the *level*, we can take the quotient

$$V^k\,(\mathfrak{g}) := \frac{V\,(\mathrm{Cur}\mathfrak{g})}{(K-k)V\,(\mathrm{Cur}\mathfrak{g})}.$$

This is known as the *universal affine vertex algebra of level* $k \in \mathbb{C}$. At last, we can prove that it is possible to endow a conformal vertex algebra structure when $\mathfrak{g}$ is simple or supercommutative to the universal affine vertex algebra of level $k \in \mathbb{C}$ when we are not at the *critical level*, which is $k = -h^\vee$. Finally, we can state the following result.

Theorem 1.4.2 ([89] **The Sugawara Construction**). *If $\mathfrak{g}$ is simple or supercommutative, and $k \neq -h^\vee$, then there exists an embedding $V^c\,(\mathrm{Vir}) \hookrightarrow V^k\,(\mathfrak{g})$, where the vectors $a \in \mathfrak{g}$ are primary of conformal weight* 1*, and $c \in \mathbb{C}$ is given by*

$$c(k) = \frac{k\,\mathrm{sdim}\,\mathfrak{g}}{k + h^\vee} \in \mathbb{C}. \tag{1.22}$$

1.4.3 Clifford Affinization of vector superspaces

We follow [63] to introduce the next example. Let V be a finite-dimensional vector space, which is endowed with $\langle\cdot|\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ a non-degenerate and superantisymmetric bilinear form. The *Clifford affinization* of V is a complex Lie superalgebra, which is the central extension $0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{V} \longrightarrow LV \longrightarrow 0$ of the loop algebra of V (viewed as an abelian Lie algebra) via $\langle\cdot|\cdot\rangle$ by K central element. Then, $\widehat{V} = LV \oplus \mathbb{C}K$ as vector space, with the non-zero commutators

$$[\varphi t^n, \psi t^m] = \delta_m^{-1-n} \langle\varphi|\psi\rangle\, K, \quad \text{for } \varphi, \psi \in V, n, m \in \mathbb{Z}.$$

As Lie conformal algebra, this is $\mathcal{R} = (\mathbb{C}[T] \otimes V) \oplus \mathbb{C}K$, with the non-zero λ-brackets

$$[\varphi_\lambda \psi] = \langle\varphi|\psi\rangle\, K, \quad \text{for } \varphi, \psi \in V.$$

Notice that $\mathrm{Lie}\,(\mathcal{R}) = \widehat{V}$. Applying the Existence Theorem, we obtain a vertex algebra $V(\mathcal{R})$, which is known as the *universal superfermionic vertex algebra*. Now, since K is the generator of the centre, we have $\alpha\colon Z(\widehat{V}) = \mathbb{C}K \longrightarrow \mathbb{C}$ a linear map as in Remark 1.2.7. So, we can take the quotient

$$\mathrm{FF}(V) := \frac{V(\mathcal{R})}{(K-1)V(\mathcal{R})}.$$

This is known as the *free superfermions*. Notice that we are taking level $\alpha(K) = k = 1$ because all the non-zero levels give isomorphic vertex algebras.

Theorem 1.4.3 ([51]). *There exists an embedding $V^c\,(\mathrm{Vir}) \hookrightarrow \mathrm{FF}(V)$, where vectors $\varphi \in \mathfrak{g}$ are primary of conformal weight* $1/2$*, and $c \in \mathbb{C}$ is given by the formula*

$$c(k) = -\frac{\mathrm{sdim}\,V}{2} \in \mathbb{C}.$$

Note that the conformal weights are not integers. We will understand why that happens in the next chapter. In such cases, we will work with *reversed-parities*. That is, consider $\Pi\colon V \longrightarrow \Pi V$ the *parity-reversing* functor, which sends each element $\varphi \in V$ to the same vector, but with reversed parity. Repeating everything exactly as above, but using these parity reversed vectors, we obtain the very same vertex algebra $\mathrm{FF}(V)$ as in the process applied above. This is what it is always done in the literature when we obtain conformal weights in $\mathbb{Z} + 1/2$. Mathematically, there is no difference in taking one or the other (at least, in this moment). The reason comes from conformal field theory, since we use this change of variable so that the obtained conformal weight $\Delta \in \mathbb{Z} + 1/2$ matches with the values that appear in the associated formal distributions written using the form (1.16). So, from now, we will construct our examples keeping this in mind.

Remark 1.4.4. We should be careful, since we will use V and ΠV at the same time in the formulas, and we must be able to distinguish which one is used on each moment. For example, the bilinear form $\langle\cdot|\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ is always defined over V. However, as a vertex algebra, the underlying vector superspace is ΠV. This distinction will be very important to understand correctly the constructions in future chapters.

1.4.4 $\beta\gamma$-system or Symplectic Bosons

If $V_0 = \{0\}$ (so, ΠV is even) in the Clifford affinization, we obtain a symplectic form on V. So, this example is known as the *symplectic bosons* (see, for example, [56, 63, 73]). For the usual property for symplectic forms, V is even dimensional, that is, $n = 2N$. So, denoting by V_N the corresponding Lie conformal algebra, we can consider a symplectic basis $\{\beta^j, \gamma^j\}_{j=1,\dots,N} \subseteq V$, and we obtain the non-zero λ-brackets

$$\left[\beta^j{}_\lambda \gamma^k\right] = \left\langle \beta^j \middle| \gamma^k \right\rangle = \delta_j^k, \quad \text{for } j,k \in \{1,\dots,N\}. \tag{1.23}$$

This is also known as the *$\beta\gamma$-system.* We can endow a conformal vertex algebra structure to the $\beta\gamma$-system giving three different embeddings $V^c(\mathrm{Vir}) \hookrightarrow V(V_N)$. Indeed,

$$\begin{array}{lll} L_1^{\beta\gamma} \mapsto \displaystyle\sum_{j=1}^N :(T\gamma^j)\beta^j:, & L_2^{\beta\gamma} \mapsto -\displaystyle\sum_{j=1}^N :(T\beta^j)\gamma^j:, & L^{\beta\gamma} \mapsto \frac{1}{2}\left(L_1^{\beta\gamma} + L_2^{\beta\gamma}\right); \\ c_1^{\beta\gamma} \mapsto 2N, & c_2^{\beta\gamma} \mapsto 2N, & c^{\beta\gamma} \mapsto -N. \end{array}$$

Moreover, the vectors γ^j for $j \in \{1,\dots,N\}$ are primary of conformal weights $0, 1, 1/2$ with respect to $L_1^{\beta\gamma}, L_2^{\beta\gamma}, L^{\beta\gamma}$, respectively; while the vectors β^j for $j \in \{1,\dots,N\}$ are primary of conformal weights $1, 0, 1/2$ with respect to $L_1^{\beta\gamma}, L_2^{\beta\gamma}, L^{\beta\gamma}$, respectively.

1.4.5 bc-system

We follow [56, 73] to introduce the next example. Let V be a totally odd $2N$-dimensional vector space, and consider $(\cdot|\cdot) : V \times V \longrightarrow \mathbb{C}$ a symmetric and non-degenerate bilinear form. Fix $\{b^j, c^j\}_{j=1,\dots,N} \subseteq V$ a basis, where $\{b^j\}_{j=1,\dots,N}$ and $\{c^j\}_{j=1,\dots,N}$ are dual with respect to $(\cdot|\cdot)$. So, denoting by Λ_N the Lie conformal algebra generated by this basis, with the non-zero λ-brackets

$$\left[b^j{}_\lambda c^k\right] = \left(b^j \middle| c^k\right) = \delta_j^k, \quad \text{for } j,k \in \{1,\dots,N\}. \tag{1.24}$$

This is known as the *bc-system.* We can endow a conformal vertex algebra structure to the bc-system giving three different embeddings $V^c(\mathrm{Vir}) \hookrightarrow V(\Lambda_N)$. Indeed,

$$\begin{array}{lll} L_1^{bc} \mapsto \displaystyle\sum_{j=1}^N :(Tb^j)c^j:, & L_2^{bc} \mapsto \displaystyle\sum_{j=1}^N :(Tc^j)b^j:, & L^{bc} \mapsto \frac{1}{2}\left(L_1^{bc} + L_2^{bc}\right); \\ c_1^{bc} \mapsto -2N, & c_2^{bc} \mapsto -2N, & c^{bc} \mapsto N. \end{array}$$

Moreover, the vectors b^j for $j \in \{1,\dots,N\}$ are primary of conformal weights $0, 1, 1/2$ with respect to $L_1^{bc}, L_2^{bc}, L^{bc}$, respectively; while the vectors c^j for $j \in \{1,\dots,N\}$ are primary of conformal weights $1, 0, 1/2$ with respect to $L_1^{bc}, L_2^{bc}, L^{bc}$, respectively.

Chapter 2

SUSY Vertex Algebras

Our next goal is to introduce $N_K = 1$ *SUSY vertex algebras.* The main reference for that will be [52], where the authors studied N_K and N_W SUSY vertex algebras in the most general case. Since we are just interested in the case of $N_K = 1$, we will drop this qualifier, and we will study this construction from vertex algebras directly. Remember that, for V vector superspace, we denote by $\Pi\colon V \longrightarrow \Pi V$ the parity-reversing functor that sends each vector $v \in V$ to the same one $\Pi v \in \Pi V$, but with reversed parity.

2.1 Superconformal Vertex Algebras

Sometimes, conformal vertex algebras admit what is called a "supersymmetry". We are interested in studying such type of vertex algebras.

Definition 2.1.1. [52] Let (V, ν) be a conformal vertex algebra of rank $c \in \mathbb{C}$, where $Y(\cdot, z)\colon V \longrightarrow \mathcal{F}(V)$ is the state-field correspondence. A *superconformal vector* of (V, ν) (remember that $Y(\nu, z) := L(z) \in \mathcal{F}(V)$ is the Virasoro field given in (1.17)) is an odd element $\tau \in V_1$ satisfying the following:

1. The field associated to τ is a Neveu-Schwarz field with central charge $c \in \mathbb{C}$. That is,
$$Y(\tau, z) := G(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} z^{-\frac{3}{2}-n} G_n \in \mathcal{F}(V),$$
where the coefficients L_n, G_m satisfy the extra commutation relations
$$[G_m, L_n] = \left(m - \frac{n}{2}\right) G_{m+n}, \quad [G_m, G_n] = 2L_{m+n} + \frac{C}{3}\left(m^2 - \frac{1}{4}\right)\delta_m^{-n}. \qquad (2.1)$$
for $C = c\,\mathrm{Id}$ the constant field.

Define the odd endomorphism $S := G_{-\frac{1}{2}}\colon V \longrightarrow V$. Notice that $S^2 = T$ by (2.1).

2. It is satisfied that $S\tau = 2\nu$.

The number $c \in \mathbb{C}$ is also called the *central charge* of τ. Any vertex algebra V endowed with a superconformal vector τ is called a *superconformal vertex algebra* of rank c.

2.1.1 What is a *Supersymmetry*? From VAs to SUSY VAs

Roughly speaking, a *supersymmetry* can be introduced as a mathematical formalism for describing a hypothetical relationship between bosons and fermions. In this set-up, a supersymmetry is just an odd endomorphism $S\colon V \longrightarrow V$ as in a superconformal vertex algebras (see above). We will formalize this point of view using the structure theory of vertex algebras explained in the previous chapter. Let (V, τ) be a superconformal vertex algebra, with $Y(\cdot, z)\colon V \longrightarrow \mathcal{F}(V)$ the state-field correspondence. Note that by (1.6),

$$S\,|0\rangle = G_{-\frac{1}{2}}\,|0\rangle = 0.$$

Furthermore, computing the supercommutator between $S\colon V \longrightarrow V$ and $Y(a,z)$ for each $a \in V$, since $G(z)$ has conformal weight $3/2$, by (1.6),

$$[S, Y(a,z)] = \left[G_{-\frac{1}{2}}, Y(a,z)\right] = Y\left(G_{-\frac{1}{2}}a, z\right) = Y(Sa, z), \quad \text{for } a \in V. \tag{2.2}$$

Moreover, thanks to (1.7), we obtain the extra condition

$$[S, Y(Sa,z)] = [S, Y(Sa,z)] = Y(Ta,z) = \partial_z Y(a,z), \quad \text{for } a \in V. \tag{2.3}$$

So, it is natural to introduce the following notion.

Definition 2.1.2. [5, 52] A *SUSY vertex algebra* is the data $(V, |0\rangle, S, Y(\cdot, z))$ as given in Definition 1.1.15, but where instead of $T\colon V \longrightarrow V$ the infinitesimal translation operator, we have $S\colon V \longrightarrow V$ an odd endomorphism (*supersymmetry generator*), satisfying:

A1 $S\,|0\rangle = 0$ and $Y(a,z)\,(|0\rangle)|_{z=0} = a$ for $a \in V$.

A2 $[S, Y(a,z)] = Y(Sa,z)$ and $[S, Y(Sa,z)] = \partial_z Y(a,z)$ for $a \in V$.

A3 The fields $\mathcal{F} = \{Y(a,z) \mid a \in V\}$ are mutually local.

The notion given above corresponds to $N_K = 1$ *SUSY vertex algebras*, but we will omit the qualifier $N_K = 1$ for simplicity, since we are not going to study the $N_K > 1$ case or the N_W case (for more information about these cases, see [52]). Obviously, any SUSY vertex algebra is going to be a vertex algebra defining $T := S^2$. The advantage of using this type of vertex algebras resides in that we can introduce a new formalism for which we will be able to express certain equations for superconformal theories in a more compact way. This is going to be really useful for our computations.

2.2 Superfield Formalism of SUSY Vertex Algebras

For V a vector superspace, let θ be a *Grassmannian indeterminate* (odd commuting with z such that $\theta^2 = 0$), and consider the pair $Z = (z, \theta)$ for z even indeterminate as usual. We will denote

$$Z^{j|J} := z^j \theta^J, \quad \text{for } j \in \mathbb{Z}; J \in \{0,1\}.$$

We say that a V-valued *formal superdistribution* in the indeterminates $Z = (z, \theta)$ is an element of the vector superspace $V\left[\left[z^{\pm 1}\right]\right][\theta]$, given by

$$\begin{aligned} A(Z) = & \sum_{\substack{j\in\mathbb{Z}\\ J\in\{0,1\}}} Z^{-1-j|1-J} a_{(j|J)} = \sum_{j\in\mathbb{Z}} z^{-1-j} a_{(j|1)} + \theta \sum_{j\in\mathbb{Z}} z^{-1-j} a_{(j|0)} =: \\ & =: A_1(z) + \theta A_0(z), \end{aligned}$$

where A_0 and A_1 are usual formal distributions in z. In particular, coefficients $a_{(j|J)} \in V$ denote the *Fourier supermodes* of A for $j \in \mathbb{Z}$ and $J \in \{0,1\}$. We impose that these are homogeneous elements of the vector superspace $V\left[\left[z^{\pm 1}\right]\right][\theta]$, where $|A| = |A_1| = |a_{(j|1)}|$, while, since θ is odd, $|A| = |A_0| + 1 = |a_{(j|0)}| + 1$ for $j \in \mathbb{Z}$. We could introduce many notions studied in previous chapters in the presence of Grassmannian indeterminates. In particular, the *superesidue* of $A \in V\left[\left[z^{\pm 1}\right]\right][\theta]$ as above is the parity-reversing linear map given by

$$\mathrm{Res}_Z\, A(Z) := a_{(0|0)} \in V.$$

Moreover, if $\mathfrak{g}$ is a Lie superalgebra, we define formally a bracket for $A, B \in \mathfrak{g}\left[\left[z^{\pm 1}\right]\right][\theta]$, where $Z = (z, \theta)$ and $W = (w, \xi)$ are two pairs of different formal indeterminates, by

$$[A(Z), B(W)] := \sum_{\substack{j,k\in\mathbb{Z}\\ J,K\in\{0,1\}}} (-1)^{(1-K)\left|a_{(j|J)}\right|} Z^{-1-j|1-J} W^{-1-k|1-K} \left[a_{(j|J)}, b_{(k|K)}\right].$$

Now, we say that A and B are *mutually local* if there exists $N \gg 0$ such that

$$(z-w)^N [A(Z), B(W)] = 0. \tag{2.4}$$

Let

$$\delta(Z - W) := \left(i_{z,w} - i_{w.z}\right)(Z - W)^{-1|1}$$

be the *delta superdistribution*, where we have used the notation

$$(Z-W)^{j|J} := (z - w - \theta\xi)^j (\theta - \xi)^J, \quad \text{for } j \in \mathbb{Z}; J \in \{0,1\}.$$

As a consequence of the Decomposition Theorem, we have that (2.4) is equivalent to

$$[A(Z), B(W)] = \sum_{\substack{j\in\mathbb{N}\\ J\in\{0,1\}}} (-1)^J \frac{\partial_z^j D_{z,\theta}^J \delta(Z-W)}{j!} C^{j|J}(W),$$

where $D_{z,\theta} := \partial_\theta + \theta\partial_z$ ($D_{z,\theta}^2 = \partial_z$) is the odd action on formal superdistributions, and

$$C^{j|J}(W) := \left(A_{(j|J)}B\right)(W) = \mathrm{Res}_Z\left((Z-W)^{j|J}[A(Z), B(W)]\right)$$

are the $(j|J)$-*products*, for $j \in \mathbb{N}$ and $J \in \{0,1\}$. Now, given $\mathcal{A}$ a superalgebra, we define the *normally ordered product* of A and B by

$$(:AB:)(Z) := A_+(Z)B(Z) + (-1)^{|A||B|} B(Z)A(Z)_-,$$

where

$$A(Z)_+ := \sum_{\substack{n<0 \\ J\in\{0,1\}}} Z^{-1-n|1-J} a_{(n|J)}, \; A(Z)_- := \sum_{\substack{n\in\mathbb{N} \\ J\in\{0,1\}}} Z^{-1-n|1-J} a_{(n|J)} \in \mathcal{A}\left[\left[z^{\pm 1}\right]\right][\theta].$$

When the normally ordered product is well defined, we can introduce

$$\left(A_{(-1-j|1-J)}B\right)(Z) := \frac{:\left(\partial_z^j D_{z,\theta}^J A(Z)\right) B(Z):}{j!}, \quad \text{for } j\in\mathbb{N}; J\in\{0,1\}.$$

The normally ordered product of formal superdistributions is not well defined in general. An $\mathrm{End}(V)$-valued formal superdistribution $A\in\mathrm{End}(V)\left[\left[z^{\pm 1}\right]\right][\theta]$ is a *superfield* when, for $v\in V$, is $A_{(n|J)}(v)=0$ for $n\gg 0$ and $J\in\{0,1\}$. We denote the vector superspace of superfields on V by $\mathcal{F}_{\text{super}}(V)$. In the case of superfields, the normally ordered product is a well-defined superfield, and for $A,B\in\mathcal{F}_{\text{super}}(V)$, $j\in\mathbb{Z}$ and $J\in\{0,1\}$, we have

$$\left(A_{(j|J)}B\right)(W) = \mathrm{Res}_Z\left(\left(A(Z)B(W)i_{z,w} - (-1)^{|A||B|}B(W)A(Z)i_{w,z}\right)(Z-W)^{j|J}\right).$$

Then, $D_{z,w}$ is an odd derivation for the $(j|J)$-products, and $\left|A_{(j|J)}B\right| = |A|+|B|+J+1$. This formalization using even and odd indeterminates was studied by Barron in [5].

Theorem 2.2.1 ([5, 52])**.** *A SUSY vertex algebra is equivalent to the following data:*

- $V=V_0\oplus V_1$ *is a vector superspace.*
- $|0\rangle\in V_0$ *is an even state.*
- $S\colon V\longrightarrow V$ *is an odd endomorphism.*
- $Y(\cdot,Z)\colon V\longrightarrow \mathcal{F}_{super}(V)$ *is a parity-preserving linear map, called state-superfield correspondence, given by*

$$Y(a,Z) := \sum_{\substack{j\in\mathbb{Z} \\ J\in\{0,1\}}} Z^{-1-j|1-J} a_{(j|J)} \in \mathcal{F}_{super}(V), \quad \textit{for } a\in V,$$

where $a_{(j|J)}\in\mathrm{End}(V)$ *for* $j\in\mathbb{Z}$ *and* $J\in\{0,1\}$ *are the Fourier supermodes of the superfield* $Y(a,Z)$*, for each* $a\in V$*, such that satisfy the following:*

Axiom 1 *We have that* $S|0\rangle=0$*, and* $Y(a,Z)(|0\rangle)|_{z=0,\theta=0}=a$*, for* $a\in V$*.*

Axiom 2 *For* $a\in V$*, we have* $[S,Y(a,Z)] = (\partial_\theta - \theta\partial_z)Y(a,Z)$*.*

Axiom 3 *The superfields* $\mathcal{F}=\{Y(a,Z)\mid a\in V\}$ *are mutually local. So, for* $a,b\in V$*, there exists* $N\gg 0$*, depending on the vectors* a *and* b*, such that*

$$(z-w)^N[Y(a,Z),Y(b,W)]=0.$$

Proof. Let $(V, |0\rangle, S, Y(\cdot, z))$ be a SUSY vertex algebra as in Definition 2.1.2. Then, we can introduce a superfield on V by

$$Y(a, Z) := Y(a, z) + \theta Y(Sa, z) \in \mathcal{F}_{\text{super}}(V), \quad \text{for } a \in V.$$

Now, from (2.2) and (2.3), since S and θ are odd,

$$[S, Y(a, Z)] = (\partial_\theta - \theta \partial_z) Y(a, Z), \quad \text{for } a \in V. \tag{2.5}$$

Conversely, if $(V, |0\rangle, S, Y(\cdot, Z))$ is a SUSY vertex algebra as above, we can define a field

$$Y(a, z) := Y(a, z; 0) \in \mathcal{F}(V), \quad \text{for } a \in V.$$

It is easily seen that (2.3) and (2.2) are both satisfied by (2.5). □

Remark 2.2.2 (**$(j|J)$-products in SUSY Vertex Algebras**). Given $(V, |0\rangle, S, Y(\cdot, Z))$ any SUSY vertex algebra, we can introduce naturally the following bilinear maps

$$a_{(j|J)}b = a_{(j|J)}(b) = \operatorname{Res}_Z \left(Z^{j|J} Y(a, Z)(b) \right), \quad \text{for } j \in \mathbb{Z}; J \in \{0, 1\},$$

for any $a, b \in V$, which are known as the $(j|J)$*-products* of the SUSY vertex algebra. In particular, we have that the $(j|1)$-products are the j-products of the underlying vertex algebra, while the $(j|0)$-products are parity-reversing bilinear maps.

Remark 2.2.3. We will compare identity (2.5) with the action of S in the superfields to obtain an identity as in (1.7). Indeed, note that $D_{z,\theta} \neq \partial_\theta - \theta\partial_z$. So, instead of the relation between T and ∂_z obtained in (super)fields, for S and $D_{z,\theta}$ we obtain that

$$Y(Sa, Z) = D_{z,\theta} Y(a, Z), \quad \text{for } a \in V. \tag{2.6}$$

In particular, S is an odd derivation for all the $(j|J)$-products.

Remark 2.2.4 (**Normally Ordered Product in SUSY Vertex Algebras**). Notice that the $(-1|1)$-product is the normally ordered product of the underlying vertex algebra V. We will use the same name and notation for SUSY vertex algebras, calling it the *normally ordered product* of V as SUSY vertex algebra. In this case, it contains the information about all the $(n|J)$-products of V for $n < 0$ thanks to (1.8) and (2.6) as a consequence of all we have mentioned. Indeed, it is easily seen that

$$: \left(\frac{T^n S^J(a)}{n!} \right) b := a_{(-1-n|1-J)} b, \quad \text{for } a, b \in V; n \in \mathbb{N}; J \in \{0, 1\}.$$

2.3 Heluani-Kac Characterization of SUSY VAs

Heluani and Kac introduced an alternative notion for SUSY vertex algebras in [52], which is the one we will use for our computations, similar to the Bakalov-Kac characterization of vertex algebras given in Section 1.2 in terms of Lie conformal algebras. To do that, first of all, we need to introduce the superfield version of Lie conformal algebras.

2.3.1 SUSY Lie Conformal Algebras

Similarly as we have done in Subsection 2.1.1 with vertex algebras, we can just introduce SUSY Lie conformal algebras as Lie conformal algebras $(\mathcal{R}, [\cdot_\lambda \cdot])$ as in Definition 1.2.1, where remember that $\mathcal{R}$ is a $\mathbb{C}[\partial]$-module, for which there exists an odd endomorphism $D\colon \mathcal{R} \longrightarrow \mathcal{R}$ satisfying certain compatibility conditions, which are the following ones:

$$D^2 = \partial, \quad D\,[a_\lambda b] = [Da_\lambda b] + (-1)^{|a|}\,[a_\lambda Db]\,, \quad \text{for } a, b \in \mathcal{R}. \tag{2.7}$$

Let now $\nabla = (\partial, D)$ be the pair formed by the translation operators above, and consider $\mathcal{H}$ the (non-commutative) associative *translation superalgebra* generated by the set ∇, subject to the relations $[\partial, D] = 0$ and $[D, D] = 2\partial$. In particular, $D^2 = \partial$. So, we can see this superalgebra just as $\mathcal{H} = \mathbb{C}[D]$ for which we define the even endomorphism $\partial := D^2$. Then, a SUSY Lie conformal algebra should be a pair $(\mathcal{R}, [\cdot_\lambda \cdot])$, where $\mathcal{R}$ is an $\mathcal{H}$-module and $[\cdot_\lambda \cdot]$ is a λ-bracket satisfying (1.9), (1.10), (1.11), with $D\colon \mathcal{R} \longrightarrow \mathcal{R}$ odd derivation for the λ-bracket. However, this notion does not make sense completely, since we do not have an analogue of the sesquilinearity rule for the λ-bracket with respect to D. So, we must give a new bracket for which we have such a rule. With this purpose, we introduce a new pair $\Lambda = (\lambda, \chi)$ for which we consider χ a new odd parameter, and consider $\mathcal{L}$ the (non-commutative) associative *parameter superalgebra* generated by the set Λ, subject to the relations $[\lambda, \chi] = 0$ and $[\chi, \chi] = -2\lambda$. In particular, $\chi^2 = -\lambda$. So, we can see this superalgebra just as $\mathcal{L} = \mathbb{C}[\chi]$ for which we define the even parameter $\lambda := -\chi^2$. Remember that, for $j \in \mathbb{Z}$ and $J \in \{0, 1\}$, we will write $\nabla^{j|J} := T^j S^J$ and $\Lambda^{j|J} := \lambda^j \chi^J$. In particular, notice that we have an isomorphism $\mathcal{H} \longrightarrow \mathcal{L}$ given by $\nabla \mapsto -\Lambda$. Then, using D, χ and $[\cdot_\lambda \cdot]$, we define

$$\begin{array}{cccc} [\cdot_\Lambda \cdot]: & \mathcal{R} \otimes \mathcal{R} & \longrightarrow & \mathcal{L} \otimes \mathcal{R} \\ & (a, b) & \mapsto & [a_\Lambda b] := [Da_\lambda b] + \chi\,[a_\lambda b] \end{array}.$$

Definition 2.3.1. [52] A *SUSY Lie conformal algebra* is the data $(\mathcal{R}, [\cdot_\Lambda \cdot])$, where

- $\mathcal{R}$ is an $\mathcal{H}$-module.
- $[\cdot_\Lambda \cdot] : \mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{L} \otimes \mathcal{R}$ is a parity-reversing bilinear map, the Λ-*bracket*, satisfying
  - *Sesquilinearity*: this is an equality in $\mathcal{L} \otimes \mathcal{R}$. For $a, b \in \mathcal{R}$,

    $$[Da_\Lambda b] = \chi\,[a_\Lambda b]\,, \; [a_\Lambda Db] = (-1)^{|\Pi a|}(D + \chi)\,[a_\Lambda b]\,. \tag{2.8}$$

    In particular, D is an odd derivation for the Λ-bracket.
  - *Antisymmetry* (understanding this in the odd version): this is an equality in $\mathcal{L} \otimes \mathcal{R}$, where ∇ is as above. For $a, b \in \mathcal{R}$,

    $$[a_\Lambda b] = (-1)^{|a||b|}\,[b_{-\nabla-\Lambda} a]\,. \tag{2.9}$$

  - *Jacobi identity*: given $\mathcal{L}'$ another copy of $\mathcal{L}$ generated by the pair $\Gamma = (\gamma, \eta)$, this is an equality in $\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{R}$. For $a, b, c \in \mathcal{R}$,

    $$[a_\Lambda\,[b_\Gamma c]] = (-1)^{|\Pi a|}\left[[a_\Lambda b]_{\Lambda+\Gamma}\, c\right] + (-1)^{|\Pi a||\Pi b|}\,[b_\Gamma\,[a_\Lambda c]]\,. \tag{2.10}$$

The meaning of these expressions are explained in Appendix A.3.

Proposition 2.3.2 ([52]). *Let $(\mathcal{R}, [\cdot_\lambda \cdot])$ be a Lie conformal algebra, and $D\colon \mathcal{R} \longrightarrow \mathcal{R}$ an odd derivation for the λ-bracket satisfying $D^2 = \partial$. Define*

$$[a_\Lambda b] := [Da_\lambda b] + \chi\, [a_\lambda b]\,, \quad \textit{for } a, b \in \mathcal{R}.$$

Then, $(\mathcal{R}, [\cdot_\Lambda \cdot])$ is a SUSY Lie conformal algebra. Conversely, let $(\mathcal{R}, [\cdot_\Lambda \cdot])$ be a SUSY Lie conformal algebra as in Definition 2.3.1, and define a λ-bracket by

$$[a_\lambda b] := \partial_\chi\, [a_\Lambda b]\,, \quad \textit{for } a, b \in \mathcal{R},$$

where ∂_χ is the partial derivative with respect to the odd indeterminate χ. Then, $(\mathcal{R}, [\cdot_\lambda \cdot])$ is a Lie conformal algebra satisfying (2.7).

Proposition 2.3.3 ([52]). *Let $(V, |0\rangle\,, S, Y(\cdot, Z))$ be a SUSY vertex algebra. Then, we obtain a SUSY Lie conformal algebra, with $D = S$ and the Λ-bracket*

$$[a_\Lambda b] := \sum_{\substack{n\in\mathbb{N} \\ J\in\{0,1\}}} \frac{\Lambda^{n|J}}{n!} a_{(n|J)} b \in \mathcal{L}\otimes V, \quad \textit{for } a, b \in V. \tag{2.11}$$

2.3.2 Universal Enveloping SUSY Vertex Algebras

We can introduce SUSY vertex algebras canonically from SUSY Lie conformal algebras. We can do this via a general construction as in Chapter 1 (see [52][Section 3.4] for details). Furthermore, we can construct a Lie algebra $\operatorname{Lie}\mathcal{R}$ from a SUSY Lie conformal algebra $(\mathcal{R}, [\cdot_\Lambda \cdot])$ in a canonical way (see [52, Lemma 3.2.8] for details). However, we will not do this, and we will give this type of SUSY vertex algebras via universal properties directly.

Theorem 2.3.4 ([52]). *If $\mathcal{R}$ is a SUSY Lie conformal algebra, then there exists a unique SUSY vertex algebra $V(\mathcal{R})$ satisfying the following universal property: any homomorphism $\varphi\colon \mathcal{R} \longrightarrow V$ of SUSY Lie conformal algebras, where V is a SUSY vertex algebra, extends uniquely to an homomorphism of SUSY vertex algebras $\overline{\varphi}\colon V(\mathcal{R}) \longrightarrow V$ making commutative the diagram*

$$\begin{tikzcd} V(\mathcal{R}) \arrow[dr, dashed, "\overline{\varphi}"] & \\ \mathcal{R} \arrow[u, hook, "i"] \arrow[r, "\varphi"'] & V. \end{tikzcd}$$

Definition 2.3.5. [52] The SUSY vertex algebra $V(\mathcal{R})$ from above is called the *universal enveloping SUSY vertex algebra* associated to the SUSY Lie conformal algebra $\mathcal{R}$.

Following Bakalov-Kac characterization of vertex algebras, we obtain that a SUSY vertex algebra should be equivalent to the data $(V, |0\rangle\,, S, [\cdot_\lambda \cdot]\,, :\cdot\cdot:)$, where

- $(V, [\cdot_\lambda \cdot])$ is a Lie conformal algebra for which there exists $S\colon V \longrightarrow V$ odd derivation for the λ-bracket.
- $((V, |0\rangle, : \cdot\cdot :), S)$ is a differential algebra satisfying (1.12) and (1.13).
- The λ-bracket $[\cdot_\lambda \cdot]$ and the product $: \cdot\cdot :$ are related by (1.14).

Then, using Λ-brackets, we arrive at the Heluani-Kac characterization.

Theorem 2.3.6 ([52]). *A SUSY vertex algebra is equivalent to the following data:*

- *A SUSY Lie conformal algebra $(V, [\cdot_\Lambda \cdot])$, where V is an $\mathcal{H}$-module.*
- *A unital differential superalgebra $((V, |0\rangle, : \cdot\cdot :), S)$ satisfying the quasicommutativity and quasiassociativity axioms, which are respectively given, for $a, b, c \in V$, by*

$$: ab : -(-1)^{|a||b|} : ba := \int_{-\nabla}^{0} d\Lambda \, [a_\Lambda b] \,, \tag{2.12}$$

$$:: ab : c : - : a : bc ::=: \left(\int_0^\nabla d^r \Lambda a \right) [b_\Lambda c] : +(-1)^{|a||b|} : \left(\int_0^\nabla d^r \Lambda b \right) [a_\Lambda c] : . \tag{2.13}$$

- *The non-commutative Wick formula, a quasi-Leibniz rule, is satisfied, which relates the Λ-bracket $[\cdot_\Lambda \cdot]$ and the product $: \cdot\cdot :$, for $a, b, c \in V$, as follows:*

$$[a_\Lambda : bc :] =: [a_\Lambda b] \, c : +(-1)^{|\Pi a||b|} : b \, [a_\Lambda c] : + \int_0^\Lambda d\Gamma \, [[a_\Lambda b]_\Gamma \, c] \,. \tag{2.14}$$

The meaning of the integrals in these identities are explained in Appendix A.4.

2.4 Neveu-Schwarz Embeddings

Let $(\mathcal{R}, [\cdot_\Lambda \cdot])$ be a SUSY Lie conformal algebra, and fix $H \in \mathcal{R}$ an odd vector.

Definition 2.4.1. [52] We say that $a \in \mathcal{R}$ has *weight* $\Delta_a \in \mathbb{C}$ with respect to H if

$$[H_\Lambda a] = (2\partial + 2\Delta_a \lambda + \chi D) \, a + O\left(\lambda^2\right) + \chi O(\lambda).$$

In these cases, we say that a is an *eigenvector* of weight Δ_a with respect to H. Moreover, we will say that $a \in \mathcal{R}$ is a *primary eigenvector* of weight Δ_a with respect to H if

$$[H_\Lambda a] = (2\partial + 2\Delta_a \lambda + \chi D) \, a. \tag{2.15}$$

Now, we say that H is a *Neveu-Schwarz vector* if

$$[H_\Lambda H] = (2\partial + 3\lambda + \chi D) \, H + \frac{\chi \lambda^2}{3} C,$$

where $C \in \mathcal{R}$ is a central element. When H is a Neveu-Schwarz vector, the weight of any element with respect to H is the *conformal weight* of such a vector (with respect to H).

As any SUSY vertex algebra is in particular a SUSY Lie conformal algebra by Proposition 2.3.3, the notions introduced above have perfect sense for SUSY vertex algebras.

Proposition 2.4.2 ([52]). *Given any SUSY vertex algebra $(V, |0\rangle, S, Y(\cdot, z))$ for which exists an odd vector $\tau \in V$ such that $G(z) = Y(\tau, z)$ is the Neveu-Schwarz field, then, $a \in V$ is an eigenvector of conformal weight $\Delta_a \in \mathbb{C}$ with respect to τ if and only if the following, for $\nu = S\tau/2 \in V$ so that $L(z) := Y(\nu, z)$, holds:*

$$L_0 a = \Delta_a a, \quad L_{-1} a = Ta, \quad G_{-\frac{1}{2}} a = Sa.$$

Remark 2.4.3. The previous result can be modified to obtain a similar equivalence for primary vectors. Indeed, in that case, the first identity is $L_n a = \delta_n^0 \Delta_a a$ for $n \in \mathbb{N}$, while we add the following one $G_n a = 0$ for $n \in 1/2 + \mathbb{N}$.

Corolary 2.4.4. *Let $\mathcal{R}$ be a SUSY Lie conformal algebra, and suppose that $V(\mathcal{R})$ contains a Neveu-Schwarz vector H of central charge $c \in \mathbb{C}$. Then, we have that $(V(\mathcal{R}), H)$ defines a superconformal vertex algebra of rank $c \in \mathbb{C}$ if and only if we have a basis of eigenvectors of certain conformal weight $\Delta \in 1/2 + \mathbb{Z}$ with respect to H.*

2.5 Examples of Superconformal Vertex Algebras

We construct examples from Lie algebras. We will work with the underlying SUSY Lie conformal algebra, taking the quotient of $V(\mathcal{R})$ by some ideal as we did in Section 1.4.

2.5.1 Neveu-Schwarz SUSY Vertex Algebra

The next example is taken from [5, 52]. The *Neveu-Schwarz algebra* is the Lie superalgebra $\mathfrak{g}$, which is the unique non-trivial central extension $0 \longrightarrow \mathbb{C}C \longrightarrow \mathfrak{g} \longrightarrow \mathcal{A} \longrightarrow 0$ of the complex Lie superalgebra $\mathcal{A}$ by C central element. Here,

$$\mathcal{A} = \mathbb{C}\left[t^{\pm 1}, \zeta\right] D_{t,\zeta} \subseteq \mathrm{Vect}(\mathring{\mathbb{D}}^{1|1}) = \mathrm{Der}_{\mathbb{C}}\left(\mathbb{C}\left[t^{\pm 1}, \zeta\right]\right),$$

where $\mathring{\mathbb{D}}^{1|1} := \mathrm{Spec}\left(\mathbb{C}\left[t^{\pm 1}, \zeta\right]\right)$ is the punctured superline, and $D_{t,\zeta} = \partial_t + \zeta\partial_\zeta$ preserves $\alpha = dt + \zeta d\zeta$ up to multiplication by functions, with ζ a Grassmanian indeterminate. Fix $L_n := -t^{n+1}\partial_t$, $G_m := -\zeta t^{n+1} D_{t,\zeta}$, for $n \in \mathbb{Z}, m \in 1/2 + \mathbb{Z}$, the basis for which we have the Lie superbrackets $[L_n, G_m] := (m-n)L_{m+n}, [L_n, G_m] := (n/2 - m)\, G_{n+m}$ and $[G_n, G_m] := 2L_{n+m}$. Then, $\mathfrak{g} = \mathcal{A} \oplus \mathbb{C}C$ as vector space, with the non-zero commutators (1.18) and (2.1). As Lie conformal algebra, this is $\mathrm{NS} := ((\mathbb{C}L \oplus \mathbb{C}H) \otimes \mathbb{C}[T]) \oplus \mathbb{C}C$, where H is odd, and L, C even, with the non-zero λ-brackets (1.19) and

$$[L_\lambda H] = \left(T + \frac{3}{2}\lambda\right) H, \quad [H_\lambda H] = 2L + \frac{\lambda^2}{3}C. \tag{2.16}$$

Let S be the odd derivation for the λ-bracket defined by $SH := 2L$ and $SL := TH/2$. Then, $S^2 = T$. So, we have a SUSY Lie conformal algebra. Then, $\mathrm{NS} = (\mathbb{C}H \otimes \mathcal{H}) \oplus C\mathbb{C}$ as an $\mathcal{H}$-module, with the non-zero Λ-bracket

$$[H_\Lambda H] = (2T + \chi S + 3\lambda)\, H + \frac{\chi\lambda^2}{3}C. \tag{2.17}$$

The associated universal enveloping SUSY vertex algebra is $V\,(\mathrm{NS})$, which is known as the *universal Neveu-Schwarz SUSY vertex algebra.* Now, since C is the generator of the centre, we can take the quotient

$$V^{c}\,(\mathrm{NS}) := \frac{V\,(\mathrm{NS})}{(C-c)V\,(\mathrm{NS})},$$

where $c \in \mathbb{C}$ is the *central charge.* This is known as the *universal Neveu-Schwarz SUSY vertex algebra of central charge* $c \in \mathbb{C}$, and it is seen that has superconformal vertex algebra structure, since $G_{-3/2}\,|0\rangle$ is a Neveu-Schwartz vector. This is also known as the $N = 1$ *superconformal vertex algebra*, since it is part of a bigger family of vertex algebras.

2.5.2 Superaffinization of quadratic Lie algebras

The next example is taken from [52, 63]. Let $(\mathfrak{g}, (\cdot|\cdot))$ be a finite-dimensional quadratic Lie superalgebra. The *superaffinization* of $(\mathfrak{g}, (\cdot|\cdot))$ is the super extension of the affinization, which is $0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}}_{\text{super}} \longrightarrow SL\mathfrak{g} \longrightarrow 0$ central extension of the *superloop algebra* of $\mathfrak{g}$ via $(\cdot|\cdot)$ by K central element. Here, $SL\mathfrak{g}$ is the Lie superalgebra of loops, so, if ζ is Grassmanian, $SL\mathfrak{g} := \mathfrak{g}\,[t^{\pm}, \zeta]$. Then, $\widehat{\mathfrak{g}}_{\text{super}} = SL\mathfrak{g} \oplus \mathbb{C}K$ as vector space, with the non-zero commutators (1.20) and

$$\begin{aligned}
[at^{n}, bt^{m}\zeta] &= [a,b]\, t^{n+m}\zeta, \quad \text{for } a,b \in \mathfrak{g}, n,m \in \mathbb{Z},\\
[at^{n}\zeta, bt^{m}\zeta] &= (b|a)\, K, \quad \text{for } a,b \in \mathfrak{g}, n,m \in \mathbb{Z}.
\end{aligned}$$

As Lie conformal algebra, this is $\mathrm{SCur}\mathfrak{g} := ((\mathfrak{g} \oplus \Pi\mathfrak{g}) \otimes \mathbb{C}[T]) \oplus \mathbb{C}K$, which is called the *supercurrent algebra*, with the non-zero λ-brackets (1.20) and

$$[a_{\lambda}\Pi b] = \Pi\,[a,b]\,, \quad [\Pi a_{\lambda}\Pi b] = (b|a)\, K, \quad \text{for } a,b \in \mathfrak{g}.$$

Let S be the odd derivation for the λ-bracket defined by $Sa := T\Pi a$, and $S\Pi a := a$, for $a \in \mathfrak{g}$. Then, $S^2 = T$. We have the SUSY Lie conformal algebra $\mathrm{SCur}\mathfrak{g} = (\Pi\mathfrak{g} \otimes \mathcal{H}) \oplus \mathbb{C}K$ as an $\mathcal{H}$-module, with the non-zero Λ-brackets

$$[\Pi a_{\Lambda}\Pi b] = (-1)^{|a|}\left(\Pi\,[a,b] + \chi\,(a|b)\,K\right), \quad \text{for } a,b \in \mathfrak{g}. \tag{2.18}$$

The associated universal enveloping SUSY vertex algebra is $V\,(\mathrm{SCur}(\mathfrak{g}))$, which is known as the *universal superaffine SUSY vertex algebra.* Now, for being C the generator of the centre, we can take the quotient

$$V^{k}_{\text{super}}\,(\mathfrak{g}) := \frac{V\,(\mathrm{SCur}(\mathfrak{g}))}{(K-k)V\,(\mathrm{SCur}(\mathfrak{g}))},$$

where $k \in \mathbb{C}$ is the *level.* This is known as the *universal superaffine vertex algebra of level* $k \in \mathbb{C}$. Now, we can state the supersymmetric version of Theorem 1.4.2.

Theorem 2.5.1 ([60] **The Kac-Todorov Construction**). *If $\mathfrak{g}$ is simple or supercommutative, and $k \neq -h^{\vee}$, then there exists an embedding $V^{c}\,(\mathrm{NS}) \hookrightarrow V^{k+h^{\vee}}_{super}\,(\mathfrak{g})$, where vectors $\Pi a \in \Pi\mathfrak{g}$ are primary of conformal weight $1/2$, and $c \in \mathbb{C}$ is given by*

$$c(k) = \frac{k\,\mathrm{sdim}\,\mathfrak{g}}{k+h^{\vee}} + \frac{\mathrm{sdim}\,\mathfrak{g}}{2} \in \mathbb{C}. \tag{2.19}$$

2.5.3 Ghost or bc-$\beta\gamma$ System

Taking the tensor product between the $\beta\gamma$-system (Subsection 1.4.4), and the bc-system (Subsection 1.4.5), we obtain the *bc-$\beta\gamma$ system* (see, for example, [56, 73]). This is also known as the *ghost system*. For $N \in \mathbb{N}$, this is generated by $\{\beta^1, \dots, \beta^N, \gamma^1, \dots, \gamma^N\}$ even and $\{b^1, \dots, b^N, c^1, \dots, c^N\}$ odd as Lie conformal algebra, with the non-zero λ-brackets (1.23) and (1.24). Let S be the odd derivation for the λ-bracket defined by $S\gamma^j := b^j$, $Sc^j := \beta^j$, $Sb^j := T\gamma^j$ and $S\beta^j := Tc^j$ for $j \in \{1, \dots, N\}$. Then, $S^2 = T$. We have the SUSY Lie conformal algebra

$$\Omega_N := \left(\left(\bigoplus_{j=1}^{N} \mathbb{C}\gamma^j \oplus \bigoplus_{j=1}^{N} \mathbb{C}c^j \right) \otimes \mathcal{H} \right) \oplus \mathbb{C}$$

as an $\mathcal{H}$-module, with the non-zero Λ-bracket

$$\left[\gamma^j{}_\Lambda c^k\right] = \delta_j^k, \quad \text{for } j, k \in \{1, \dots, N\}.$$

The associated universal enveloping SUSY vertex algebra is $V(\Omega_N)$, which is known as the *bc-$\beta\gamma$ system*. We obtain a superconformal structure defining the odd vector

$$G := \sum_{j=1}^{N} \left(:b^j\beta^j: + :(T\gamma^j)c^j: \right) \in V(\Omega_N). \tag{2.20}$$

We will return to this example in Chapter 3, where we will enhance the previous structure in order to have a higher number of supersymmetries. Before ending, we will introduce the examples that have the higher number of supersymetries considered in this thesis. We will use the Heluani-Kac formalism of [52][Examples 5.10 and 5.11].

Remark 2.5.2. The finite simple Lie conformal superalgebras are completely classified [26] (where finite means finitely generated as $\mathbb{C}[\partial]$-modules).

Fix $N \in \mathbb{N}$, and consider the Grassmannian indeterminates $\zeta_1, \dots, \zeta_N$. Let $W(1|N)$ be the Lie superalgebra of all the derivations of $\mathbb{C}\left[t^{\pm 1}, \zeta_1, \dots, \zeta_N\right]$, and $K(1|N) \subseteq W(1|N)$ the Lie subsuperalgebra consisting of vector fields preserving

$$\alpha := dt + \sum_{j=1}^{N} \zeta_j d\zeta_j,$$

up to multiplication by a function. The Lie superalgebras $W(1|N)$ and $K(1|N)$ determine Lie conformal superalgebras $\mathcal{W}_N$ and $\mathcal{K}_N$ (see [26]). Furthermore, the Lie conformal superalgebra $\mathcal{K}_N$ admits a unique non-trivial extension if $N \leq 3$ [26][Proposition 4.17], and no non-trivial central extensions for $N \geq 5$ [26][Proposition 4.19]. In the special case $N = 4$, the Lie superalgebra $K(1|4)$ is not simple, but its derived algebra $K(1|4)'$ is both simple and determines a Lie conformal superalgebra $\mathcal{K}_4'$ (namely, the derived algebra of $\mathcal{K}_4$ [26][Example 3.9]), that has two, up to isomorphism, linearly independent central extensions [26][Proposition 4.18]. The unique central extension of $\mathcal{K}_1$ is the Neveu-Schwarz algebra. Now, we are going to introduce the unique non-trivial central extension of $\mathcal{K}_2$ and a central extension of $\mathcal{K}_4'$ that is in fact a central extension of $\mathcal{K}_4$.

2.5.4 $N = 2$ Superconformal Vertex Algebra

Let $\mathcal{K}_2$ be the associated SUSY Lie conformal algebra to the non-trivial central extension of $K(1|2)$, which is freely generated by H odd, and J, C even (this last one central). That is, $\mathcal{K}_2 := ((\mathbb{C}H \oplus \mathbb{C}J) \otimes \mathcal{H}) \oplus \mathbb{C}C$, with the non-zero Λ-brackets (2.17) and

$$[J_\Lambda J] = -\left(H + \frac{\lambda\chi}{3}C\right), \quad [H_\Lambda J] = (2T + 2\lambda + \chi S)J. \tag{2.21}$$

Now, since C is the generator of the centre, we can take the quotient

$$V^c(\mathcal{K}_2) := \frac{V(\mathcal{K}_2)}{(C-c)V(\mathcal{K}_2)},$$

where $c \in \mathbb{C}$ is the *central charge*. This is known as the $N = 2$ *superconformal vertex algebra of central charge* $c \in \mathbb{C}$. If $Y(\cdot, z)$ and $Y(\cdot, Z)$ denote the state-(super)field correspondences, respectively, and we expand in components the superfields J and H, then

$$Y(J, Z) = -i\left(J(z) + \theta\left(G^-(z) - G^+(z)\right)\right), \quad Y(H, Z) = \left(G^+(z) + G^-(z)\right) + 2\theta L(z),$$

where, from the components of $J(z)$ and $G^\pm(z)$ in coefficients,

$$Y(J, z) = J(z) = \sum_{n\in\mathbb{Z}} z^{-1-n} J_n, \quad Y\left(G^\pm, z\right) = G^\pm(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} z^{-\frac{3}{2}-n} G_n^\pm,$$

since $L(z)$ is given by (1.17), one obtains from (2.17) and (2.21) the extra commutators

$$\begin{gathered}[J_m, J_n] = \frac{m}{3}\delta_m^{-n}c, \quad \left[J_m, G_n^\pm\right] = \pm G_{m+n}^\pm, \quad \left[G_m^\pm, L_n\right] = \left(m - \frac{n}{2}\right) G_{m+n}^\pm, \\ [L_m, J_n] = -nJ_{m+n}, \quad \left[G_m^+, G_n^-\right] = L_{m+n} + \frac{m-n}{2} J_{m+n} + \frac{c}{6}\left(m^2 - \frac{1}{4}\right)\delta_m^{-n}.\end{gathered}$$

Here, it is easily seen that $\mathrm{Lie}(\mathcal{K}_2) = K(1|2) \oplus \mathbb{C}C$ is the underlying Lie superalgebra.

Remark 2.5.3. Notice that we can recover the Neveu-Schwarz vector H from J. More precisely, it is proven using Jacobi identity that (2.21) implies (2.17).

2.5.5 $N = 4$ Superconformal Vertex Algebra

The ("small") $N = 4$ *superconformal vertex algebra of central charge* $c \in \mathbb{C}$ is generated by the SUSY Lie conformal algebra $\mathcal{K}_4^c$ after localizing, whose underlying $\mathcal{H}$-module is freely generated by an odd vector H, three even vectors J^1, J^2, J^3, and $c \in \mathbb{C}$, with the non-zero Λ-brackets (2.17) and (2.21) for $J = J^s$ with $s \in \{1, 2, 3\}$ (each even vector J^s generate an $N = 2$ superconformal vertex algebra of central charge c with same H), and

$$\left[J^s{}_\Lambda J^t\right] = -\varepsilon^{s,t,k}(S + 2\chi)J^k, \quad \text{for } s \neq t; s, t \in \{1, 2, 3\}, \tag{2.22}$$

where $\varepsilon = \left(\varepsilon^{s,t,k}\right)_{s,t,k\in\{1,\dots,N\}}$ is the totally antisymmetric tensor.

Chapter 3

Classical Embeddings

The aim of this chapter is to study classical embeddings of the Virasoro, Neveu-Schwarz, and $N = 2$ superconfomal vertex algebras into the (super)affinization of a quadratic Lie algebra and the bc-$\beta\gamma$ system. The first two constructions have been mentioned before. We will prove both of them using the λ-bracket and Λ-bracket formalisms, respectively, as an example of computations. The other two embeddings can be seen as an introduction for the chiral de Rham complex, and what we are going to do in future chapters.

3.1 The Segal-Sugawara Construction

As we have seen in Chapter 1, Theorem 1.4.2 gives an embedding from the universal Virasoro vertex algebra for certain central charge into the universal affine vertex algebra of a quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$ with non-critical level. So, any module for the affine vertex algebra is also a module for the Virasoro algebra, obtaining infinite dimensional representations of the Virasoro Lie algebra. We need $\mathfrak{g}$ to be simple or supercommutative for representation theory reasons we will see below. Some authors refer to this embedding as the Sugawara construction (since Sugawara was the first proving this result in [89]), but the geometric construction is due to Segal (see [83]), and hence we will refer to it as the Segal-Sugawara construction. To start, we will recall the notion and basic properties of the *Casimir operators* on Lie superalgebras.

3.1.1 (Quadratic) Casimir Operators

Let V be an n-dimensional vector superspace endowed with the non-degenerate bilinear form $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$, and let us consider $\{a_j\}_{j=1,\ldots,n} \subseteq V$ a basis of V.

Definition 3.1.1. The *dual basis* of $\{a_j\}_{j=1,\ldots,n}$ with respect to $(\cdot,\cdot)$ is such that

$$\left(a_j, a^k\right) = \delta_j^k, \quad \text{for } j,k \in \{1,\ldots,n\}.$$

In other words, $\{a^j\}_{j=1,\ldots,n} \subseteq V$ is the unique basis in V given by the canonical bijection $V \cong V^*$ obtained via $(\cdot,\cdot)$ from the dual basis of $\{a_j\}_{j=1,\ldots,n}$ in V^*.

Consider $\{a^j\}_{j=1,\dots,n} \subseteq V$ the dual basis of $\{a_j\}_{j=1,\dots,n} \subseteq V$ with respect to $(\cdot,\cdot)$. Then,

$$x = \sum_{j=1}^{n} (a_j, x)\, a^j = \sum_{j=1}^{n} \left(x, a^j\right) a_j, \quad \text{for } x \in V. \tag{3.1}$$

Remark 3.1.2. Let $\mathcal{A}$ be any superalgebra with $\cdot$ product, and a non-degenerate bilinear form $(\cdot,\cdot) : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}$. Let $\{a_j\}_{j=1,\dots,n} \subseteq \mathcal{A}$ be a basis, such that $\{a^j\}_{j=1,\dots,n} \subseteq \mathcal{A}$ is its dual basis with respect to $(\cdot,\cdot)$. Let $\{b_j\}_{j=1,\dots,n} \subseteq \mathcal{A}$ be now another basis, where $\{b^j\}_{j=1,\dots,n} \subseteq \mathcal{A}$ is its dual basis with respect to $(\cdot,\cdot)$, such that the change of coordinates is

$$b^j = \sum_{k=1}^{n} A^j_k a^k \text{ and } b_j = \sum_{k=1}^{n} B^k_j a_k, \quad \text{for } j \in \{1,\dots,n\}.$$

Then, for $j,k \in \{1,\dots,n\}$,

$$\delta^k_j = \left(b_j, b^k\right) = \left(\sum_{r=1}^{n} B^l_j a_r, \sum_{s=1}^{n} A^k_s a^s\right) = \sum_{r,s=1}^{n} A^k_s B^r_j\, (a_r, a^s) = \sum_{r=1}^{n} A^k_r B^l_j,$$

or, equivalently,

$$\sum_{j=1}^{n} A^j_k B^r_j = \delta^r_k, \quad \text{for } k,r \in \{1,\dots,n\}.$$

So, we have

$$\sum_{j=1}^{n} b^j \cdot b_j = \sum_{k=1}^{n} \left(\sum_{r=1}^{n} \left(\sum_{j=1}^{n} A^j_k B^r_j\right) a^k \cdot a_r\right) = \sum_{j=1}^{n} a^j \cdot a_j.$$

Now, let $\mathfrak{g}$ be a Lie superalgebra, and consider its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Definition 3.1.3. The *Killing form* on $\mathfrak{g}$ is the supersymmetric and invariant bilinear form $k(\cdot,\cdot) : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ defined by

$$k(a,b) := \operatorname{str}\left((\mathrm{ad}_a)(\mathrm{ad}_b)\right), \quad \text{for } a,b \in \mathfrak{g},$$

where $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is the *adjoint representation* $\mathrm{ad}_a(b) := [a,b]$, for $a,b \in \mathfrak{g}$.

Remark 3.1.4. In general, the Killing form is degenerate. By Cartan's criterion, if $\mathfrak{h}$ is a totally even (or odd) Lie algebra, then $\mathfrak{h}$ is semisimple if and only if the Killing form is non-degenerate. This is not true for $\mathfrak{g}$ a general Lie superalgebra (see [61]).

Let $\{a_j\}_{j=1,\dots,n} \subseteq \mathfrak{g}$ be a basis with dual basis $\{a^j\}_{j=1,\dots,n} \subseteq \mathfrak{g}$ with respect to the non-degenerate bilinear form $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$.

Definition 3.1.5. The *Casimir element* for $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is given by

$$\Omega := \sum_{j=1}^{n} a^j \cdot a_j \in \mathcal{U}(\mathfrak{g}),$$

where $\cdot$ is the product in $\mathcal{U}(\mathfrak{g})$. In particular, by Remark 3.1.2, we have that Ω does not dependent on the chosen basis.

Proposition 3.1.6 ([51] **Ω is $\mathfrak{g}$-invariant**)**.** *Let $(\mathfrak{g},(\cdot,\cdot))$ be a Lie superalgebra endowed with an invariant and non-degenerate bilinear form $(\cdot,\cdot):\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathbb{C}$. Then, the associated Casimir element Ω belongs to the center of $\mathcal{U}(\mathfrak{g})$. That is, $[\mathfrak{g},\Omega]=0$ in $\mathcal{U}(\mathfrak{g})$.*

Proof. Suppose that $(\cdot,\cdot)$ is invariant, and let $x\in\mathfrak{g}$. Then, since

$$[a,b\cdot c]=[a,b]\cdot c+(-1)^{|a||b|}b\cdot[a,c]\,,\quad \text{for } a,b,c\in\mathfrak{g},$$

we obtain that $[x,\Omega]=0$ for $x\in\mathfrak{g}$ using (3.1). □

As a consequence of previous result, we have that Ω acts as a scalar in every irreducible representation of $\mathfrak{g}$. Now, notice that in the adjoint representation

$$\Omega(a)\equiv \mathrm{ad}_{\Omega}(a):=\sum_{j=1}^{n}\mathrm{ad}_{a^j}\left(\mathrm{ad}_{a_j}(a)\right)=\sum_{j=1}^{n}\left[a^j,[a_j,a]\right],\quad \text{for } a\in\mathfrak{g}. \tag{3.2}$$

So, since when $\mathfrak{g}$ is simple or supercommutative the adjoint representation is irreducible, then $\Omega\in\mathbb{C}$, and it is related with $h^\vee\in\mathbb{C}$ the *dual Coxeter number* of $\mathfrak{g}$ (we introduced it in Chapter 1, and $h^\vee=0$ when $\mathfrak{g}$ is supercommutative). Indeed, it is satisfied that

$$\Omega(a)=2h^\vee a,\quad \text{for } a\in\mathfrak{g}. \tag{3.3}$$

Remark 3.1.7. By Schur Lemma, when $\mathfrak{h}$ is a totally even (or odd) simple Lie algebra, then any invariant, supersymmetric and non-degenerate bilinear form on $\mathfrak{h}$ is a scalar multiple of the Killing form. However, for $\mathfrak{g}$ Lie superalgebra, this is not true (see [61]).

We are ready for the Segal-Sugawara construction. Let $(\mathfrak{g},(\cdot|\cdot))$ be any n-dimensional quadratic Lie superalgebra, and consider now the basis $\{a_j\}_{j=1,\dots,n}\subseteq\mathfrak{g}$ with dual basis $\{a^j\}_{j=1,\dots,n}\subseteq\mathfrak{g}$ with respect to $(\cdot|\cdot)$. Remember that $V^c(\mathrm{Vir})$ denotes the universal Virasoro vertex algebra of central charge $c\in\mathbb{C}$ generated as a Lie conformal algebra by L with λ-brackets (1.19), while $V^k(\mathfrak{g})$ is the universal affine vertex algebra of level $k\in\mathbb{C}$ generated as a Lie conformal algebra by $\mathfrak{g}$ with λ-brackets (1.21).

Theorem 3.1.8 ([51, Section 7] **The Segal-Sugawara Construction for λ-brackets**)**.** *If $\mathfrak{g}$ is simple or supercommutative, given $k\neq -h^\vee$ as above, then*

$$L=\frac{1}{2(k+h^\vee)}\sum_{j=1}^{n}:a^j a_j:\in V^k(\mathfrak{g}) \tag{3.4}$$

is a Virasoro vector of central charge given by (1.22). Moreover, the vectors $a\in\mathfrak{g}$ are primary of conformal weight 1 with respect to L.

Proof. In order to simplify the computations, we are going to use the Einstein summation convention for repeated indexes. In addition, we will use some technical properties from Appendix B.1 and B.2. First, we define

$$Y:=\frac{1}{2}:a^j a_j:=(k+h^\vee)L\in V^k(\mathfrak{g}).$$

By Remark 3.1.2, notice that Y (and L) above are independent on the chosen basis. Now, for $a \in \mathfrak{g}$, by the non-commutative Wick formula, supersymmetry and invariance,

$$\begin{aligned}
[a_\lambda Y] &= \frac{1}{2}\left[a_\lambda : a^j a_j :\right] \\
&= \frac{1}{2}\left(: \left[a_\lambda a^j\right] a_j : + (-1)^{|a||a^j|} : a^j \left[a_\lambda a_j\right] : + \int_0^\lambda \left[\left[a_\lambda a^j\right]_\mu a_j\right] d\mu \right) \\
&= \frac{1}{2}\left(: \left[a, a^j\right] a_j : + k\lambda \left(a \left| a^j \right.\right) a_j \right) + \frac{1}{2}(-1)^{|a||a^j|}\left(: a^j \left[a, a_j\right] : + k\lambda \left(a \left| a_j \right.\right) a^j \right) \\
&+ \frac{1}{2}\int_0^\lambda \left[\left[a, a^j\right]_\mu a_j\right] d\mu = \frac{1}{2}\left(: \left[a, a^j\right] a_j : + (-1)^{|a||a^j|} : a^j \left[a, a_j\right] : \right) \\
&+ \lambda k a + \frac{1}{2}\int_0^\lambda \left[\left[a, a^j\right], a_j\right] d\mu + \frac{k}{2}\int_0^\lambda \mu \left(\left[a, a^j\right] \middle| a_j \right) d\mu \\
&= \lambda k a + \frac{1}{2}\int_0^\lambda \Omega(a) d\mu + \frac{k}{2}\int_0^\lambda \mu \left(a \middle| \left[a^j, a_j\right] \right) d\mu \\
&= \lambda(k + h^\vee) a,
\end{aligned}$$

thanks to (B.1), (B.2), (3.3) and (B.3). By antisymmetry,

$$\begin{aligned}
[L_\lambda a] :&= \frac{1}{k + h^\vee}\left[Y_\lambda a\right] = -\frac{1}{k + h^\vee}\left[a_{-\lambda - T} Y\right] \\
&= (\lambda + T) a.
\end{aligned}$$

So, the vectors $a \in \mathfrak{g}$ are primary of conformal weight 1 with respect to L provided that L is a Virasoro vector. By the non-commutative Wick formula and sesquilinearity, since T is an even derivation for the normally ordered product,

$$\begin{aligned}
[L_\lambda Y] :&= \frac{1}{2}\left[L_\lambda : a^j a_j :\right] = \frac{1}{2}\left(: \left[L_\lambda a^j\right] a_j : + : a^j \left[L_\lambda a_j\right] : + \int_0^\lambda \left[\left[L_\lambda a^j\right]_\mu a_j\right] d\mu \right) \\
&= \frac{1}{2}\left(: \left((T + \lambda) a^j\right) a_j : + : a^j \left((T + \lambda) a_j\right) : \right) + \frac{1}{2}\int_0^\lambda \left[(T + \lambda) a^j_\mu a_j\right] d\mu \\
&= 2\lambda Y + \frac{1}{2}\left(: \left(T a^j\right) a_j : + : a^j \left(T a_j\right) : \right) + \frac{1}{2}\int_0^\lambda (\lambda - \mu)\left[a^j_\mu a_j\right] d\mu \\
&= 2\lambda Y + \frac{1}{2} T\left(: a^j a_j : \right) + \frac{1}{2}\int_0^\lambda (\lambda - \mu)\left(\left[a^j, a_j\right] + \mu k \left(a^j \middle| a_j\right)\right) d\mu \\
&= (T + 2\lambda) Y + \frac{k \operatorname{sdim} \mathfrak{g}}{2}\int_0^\lambda (\lambda - \mu)\mu d\mu = (T + 2\lambda) Y + \frac{k \operatorname{sdim} \mathfrak{g}}{12}\lambda^3.
\end{aligned}$$

So,

$$[L_\lambda L] := \frac{1}{k + h^\vee}\left[L_\lambda Y\right] = (T + 2\lambda) L + \frac{\lambda^3}{12} c(k),$$

thanks to (B.1). This concludes the proof of the result. □

The vector Y above is known as the *Segal-Sugawara conformal vector* of $\mathfrak{g}$.

Corolary 3.1.9 ([51] **The Segal-Sugawara Modification**). *For $\mathfrak{g}$ simple or supercommutative, if $k \neq -h^\vee$, let L and $c(k)$ be as in* (3.4) *and* (1.22)*, respectively. Define*

$$L^a := L + Ta \in V^k(\mathfrak{g}), \quad c^a(k) := c(k) - 12k\,(a|a) \in \mathbb{C}, \quad \textit{for } a \in \mathfrak{g} \textit{ even.}$$

Then, we have that L^a is another Virasoro vector of central charge $c^a(k)$.

Proof. It is straightforward. Indeed, by sesquilinearity, since $[a,a] = 0$ ($a \in \mathfrak{g}$ is even),

$$\begin{aligned} [L^a_\lambda L^a] :&= [L_\lambda L] + (\lambda + T)\,[L_\lambda a] - \lambda\,[a_\lambda L] - \lambda(\lambda+T)\,[a_\lambda a] \\ &= (T + 2\lambda)(L + Ta) + \frac{\lambda^3}{12}\,(c(k) - 12k\,(a|a))\,. \end{aligned}$$

□

Remark 3.1.10. Notice that previous result may not give us primary vectors, since

$$[L^a{}_\lambda b] = Tb + \lambda\,(b - [a,b]) - \lambda^2\,(a|b)\,k, \quad \text{for } a \in \mathfrak{g}_0, b \in \mathfrak{g}.$$

3.2 The Kac-Todorov Construction

Let $(\mathfrak{g}, (\cdot|\cdot))$ be any n-dimensional quadratic Lie superalgebra, and consider $\Pi\colon \mathfrak{g} \longrightarrow \Pi\mathfrak{g}$ the parity-reversing functor. Now, we are going to take the basis $\{a_j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ with dual basis $\{a^j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ with respect to $(\cdot|\cdot)$. Remember that $V^c\,(\mathrm{NS})$ denotes the universal Neveu-Schwarz SUSY vertex algebra of central charge $c \in \mathbb{C}$ generated as a SUSY Lie conformal algebra by H with Λ-brackets (2.17), while $V^k_{\text{super}}\,(\mathfrak{g})$ is the universal superaffine vertex algebra of level $k \in \mathbb{C}$ generated as a SUSY Lie conformal algebra by $\Pi\mathfrak{g}$ with Λ-brackets (2.18). From now, we will use some usual abuse of notations.

Remark 3.2.1. Abusing notation,

$$[\cdot,\cdot]\colon \Pi\mathfrak{g} \times \Pi\mathfrak{g} \longrightarrow \Pi\mathfrak{g}, \quad (\cdot|\cdot) : \Pi\mathfrak{g} \times \Pi\mathfrak{g} \longrightarrow \mathbb{C},$$

will denote the two bilinear maps corresponding to the Lie superbracket defined on $\mathfrak{g}$, and the supersymmetric, invariant and non-degenerate form on $\mathfrak{g}$, identifying elements $a \in \mathfrak{g}$ with their corresponding odd copies $\Pi a \in \Pi\mathfrak{g}$. Explicitly,

$$[\Pi a, \Pi b] := \Pi\,[a,b]\,, \quad (\Pi a|\Pi b) := (a|b)\,, \qquad \text{for } a, b \in \mathfrak{g}.$$

We write $a_j := \Pi a_j$ and $a^j := \Pi a^j$ for $j \in \{1,\ldots,n\}$. The Kac-Todorov construction we present is more general than the one in Chapter 2 (see Theorem 2.5.1 and Remark 3.2.3).

Theorem 3.2.2 (**The Kac-Todorov Construction for Λ-brackets**). *For $k \neq 0$,*

$$H = \frac{1}{k}\sum_{j=1}^{n}\left((-1)^{|a^j|+1} : \left(Sa^j\right) a_j : -\frac{1}{3k}\sum_{k=1}^{n} : a^j : a_k\left[a^k, a_j\right] :: \right) \in V^k_{super}(\mathfrak{g}) \tag{3.5}$$

is a Neveu-Schwarz vector of central charge

$$c(k) = \frac{3}{2}\left(\operatorname{sdim}\mathfrak{g} - \sum_{j=1}^{n}\frac{\left(\Omega\left(a^j\right)\middle|\, a_j\right)}{3k}\right) \in \mathbb{C}. \tag{3.6}$$

Moreover, the vectors $a \in \Pi\mathfrak{g}$ are primary of conformal weight $1/2$ with respect to H.

Proof. In order to simplify the computations, we are going to use the Einstein summation convention for repeated indexes. In addition, we will use some technical properties from Appendix B.1 and B.3. We will start computing $[a_\Lambda H]$ for $a \in \Pi\mathfrak{g}$. Notice that

$$\begin{aligned}[a_\Lambda H] :&= \frac{1}{k}(-1)^{|a^j|+1}\left[a_\Lambda :\left(Sa^j\right)a_j:\right] - \frac{1}{3k^2}\left[a_\Lambda : a^j : a_k\left[a^k, a_j\right]::\right] \\ &=: \frac{1}{k}\Upsilon_a^1 - \frac{1}{3k^2}\Upsilon_a^2.\end{aligned}$$

Hence, by the non-commutative Wick formula, sesquilinearity, antisymmetry, supersymmetry, invariance, (B.2), (B.4), (B.5) and (B.6),

$$\begin{aligned}\Upsilon_a^1 :&= (-1)^{|a^j|+1}\left[a_\Lambda :\left(Sa^j\right)a_j:\right] \\ &= (-1)^{|a^j|+1}\Big(:\left[a_\Lambda Sa^j\right]a_j: +(-1)^{(|a|+1)|Sa^j|}:\left(Sa^j\right)\left[a_\Lambda a_j\right]: \\ &+ \int_0^\Lambda d\Gamma\left[\left[a_\Lambda Sa^j\right]_\Gamma a_j\right]\Big) = (-1)^{|a^j|+1}\Big((-1)^{|a|+1}:\left((S+\chi)\left[a_\Lambda a^j\right]\right)a_j: \\ &+(-1)^{(|a|+1)(|a^j|+1)}:\left(Sa^j\right)\left[a_\Lambda a_j\right]: + \int_0^\Lambda d\Gamma\left[(-1)^{|a|+1}(S+\chi)\left[a_\Lambda a^j\right]_\Gamma a_j\right]\Big) \\ &= (-1)^{|a^j|+1}:\left(S\left[a,a^j\right]\right)a_j: +(-1)^{|a||a^j|+1}:\left(Sa^j\right)\left[a,a_j\right]: \\ &+ \lambda\Big((-1)^{|a^j|+1}2k\left(a|a^j\right)a_j + (-1)^{|a^j|}k\left(a|a^j\right)a_j + (-1)^{|a|+1}\Omega(a)\Big) \\ &+ \chi\Big((-1)^{|a^j|+1}:\left[a,a^j\right]a_j: +(-1)^{|a^j|(|a|+1)}kS\left(\left(a|a_j\right)a^j\right)\Big) \\ &= (-1)^{|a|+1}\left(\chi : a_j\left[a^j,a\right]: +k\left(\lambda+\chi S\right)a + \lambda\Omega(a)\right), \\ \Upsilon_a^2 :&= \left[a_\Lambda : a^j : a_k\left[a^k,a_j\right]::\right] =: \left[a_\Lambda a^j\right]\left(: a_k\left[a^k,a_j\right]:\right): \\ &+ (-1)^{(|a|+1)|a^j|}: a^j\Big(:\left[a_\Lambda a_k\right]\left[a^k,a_j\right]: +(-1)^{(|a|+1)|a_k|}: a_k\left[a_\Lambda\left[a^k,a_j\right]\right]: \\ &+ \int_0^\Lambda d\Gamma\left[\left[a,a_k\right]_\Gamma\left[a^k,a_j\right]\right]\Big): + \int_0^\Lambda d\Gamma\Big(:\left[\left[a_\Lambda a^j\right]_\Gamma a_k\right]\left[a^k,a_j\right]: \\ &+(-1)^{(|[a_\Lambda a^j]|+1)|a_k|}: a_k\left[\left[a_\Lambda a^j\right]_\Gamma\left[a^k,a_j\right]\right]: + \int_0^\Gamma d\Omega\left[\left[\left[a_\Lambda a^j\right]_\Gamma a_k\right]_\Omega\left[a^k,a_j\right]\right]\Big) \\ &= (-1)^{|a|+1}\Big(:\left[a,a^j\right]: a_k\left[a^k,a_j\right]:: +(-1)^{(|a|+1)|a^j|}: a^j:\left[a,a_k\right]\left[a^k,a_j\right]:: \\ &+(-1)^{(|a|+1)(|a^j|+|a_k|)}: a^j : a_k\left[a,\left[a^k,a_j\right]\right]::\Big) \\ &+ \lambda(-1)^{|a|}k\Big((-1)^{(|a|+1)|a^j|+|a_k|}\left(\left[a,a_k\right]\Big|\left[a^k,a_j\right]\right)a^j \\ &+ (-1)^{|a^j|}\left(\left[a,a^j\right]\big|a_k\right)\left[a^k,a_j\right] + (-1)^{(|a|+|a^j|)|a_k|+|a^j|+|a_k|}\left(\left[a,a^j\right]\Big|\left[a^k,a_j\right]\right)a_k\Big) \\ &+ \chi(-1)^{|a|+1}k\Big(\left(a|a^j\right): a_k\left[a^k,a_j\right]: +(-1)^{|a||a^j|}\left(a|a_k\right): a^j\left[a^k,a_j\right]: \\ &+ (-1)^{(|a^j|+|a_k|)|a|}\left(a\Big|\left[a^k,a_j\right]\right): a^j a_k:\Big) = (-1)^{|a|+1}3k\left(\chi : a_j\left[a^j,a\right]: +\lambda\Omega(a)\right).\end{aligned}$$

By antisymmetry, $[H_\Lambda a] = (\lambda + 2T + \chi S)\, a$. Therefore, the vectors $a \in \Pi\mathfrak{g}$ are primary of conformal weight 1/2 with respect to H provided that H is a Neveu-Schwarz vector. Indeed,

$$\begin{aligned}[][H_\Lambda H] :&= \frac{1}{k}(-1)^{|a^j|+1}\left[H_\Lambda : \left(Sa^j\right) a_j :\right] - \frac{1}{3k^2}\left[H_\Lambda : a^j : a_k \left[a^k, a_j\right] ::\right] \\ &=: \frac{1}{k}\Upsilon_H^1 - \frac{1}{3k^2}\Upsilon_H^2.\end{aligned}$$

So, by the non-commutative Wick formula, sesquilinearity, antisymmetry, supersymmetry, invariance, (B.1), (B.2), (B.4), (B.7), (B.8), and for being S and T both odd and even derivations for the normally ordered product, respectively,

$$\begin{aligned}\Upsilon_H^1 :&= (-1)^{|a^j|+1}\left[H_\Lambda : \left(Sa^j\right) a_j :\right] \\ &= (-1)^{|a^j|+1}\Big(: \left((S+\chi)\left[H_\Lambda a^j\right]\right) a_j : + : \left(Sa^j\right)\left[H_{0\Lambda} a_j\right] : \\ &+ \int_0^\Lambda d\Gamma \left[(S+\chi)\left[H_\Lambda a^j\right]_\Gamma a_j\right]\Big) \\ &= (-1)^{|a^j|+1}\Big(: \left(\left(2ST - \chi S^2 + 2\lambda S + \chi\lambda + 2\chi T\right) a^j\right) a_j : \\ &+ : \left(Sa^j\right)\left((\lambda + 2T + \chi S)\, a_j\right) : + \lambda^2(-1)^{|a^j|+1}\left(\left[a^j, a_j\right] + \chi\frac{k\left(a^j \middle| a_j\right)}{2}\right)\Big) \\ &= (3\lambda + 2T + \chi S)(-1)^{|a^j|+1} : \left(Sa^j\right) a_j : + \chi\lambda(-1)^{|a^j|+1} : a^j a_j : + \chi\lambda^2\frac{k\,\mathrm{sdim}\,\mathfrak{g}}{2} \\ &= (3\lambda + 2T + \chi S)(-1)^{|a^j|+1} : \left(Sa^j\right) a_j : + \chi\lambda^2\frac{k\,\mathrm{sdim}\,\mathfrak{g}}{2}, \\ \Upsilon_H^2 :&= \left[H_\Lambda : a^j : a_k\left[a^k, a_j\right] ::\right] \\ &=: \left[H_\Lambda a^j\right] : a_k \left[a^k, a_j\right] :: + : a^j \Big(: \left[H_\Lambda a_k\right]\left[a^k, a_j\right] : + : a_k \left[H_\Lambda \left[a^k, a_j\right]\right] : \\ &+ \int_0^\Lambda d\Gamma \left[\left[H_\Lambda a_k\right]_\Gamma \left[a^k, a_j\right]\right]\Big) : + \int_0^\Lambda d\Gamma \Big(: \left[\left[H_\Lambda a^j\right]_\Gamma a_k\right]\left[a^k, a_j\right] : \\ &+ (-1)^{\left(|a^j|+1\right)|a_k|} : a_k \left[\left[H_\Lambda a^j\right]_\Gamma \left[a^k, a_j\right]\right] : + \int_0^\Gamma d\Omega \left[\left[\left[H_\Lambda a^j\right]_\Gamma a_k\right]_\Omega \left[a^k, a_j\right]\right]\Big) \\ &= 3\lambda : a^j : a_k \left[a^k, a_j\right] :: + 2\Big(: \left(Ta^j\right) : a_k \left[a^k, a_j\right] :: + : a^j : \left(Ta_k\right)\left[a^k, a_j\right] :: \\ &+ : a^j : a_k \left(T\left[a^k, a_j\right]\right) ::\Big) + \chi\Big(: \left(Sa^j\right) : a_k \left[a^k, a_j\right] :: \\ &+ (-1)^{|a^j|} : a^j : \left(Sa_k\right)\left[a^k, a_j\right] :: + (-1)^{|a^j|+|a_k|} : a^j : a_k \left(S\left[a^k, a_j\right]\right) ::\Big) \\ &+ \lambda\chi\Big((-1)^{|a^j|+|a_k|+1} : a^j \left[a_k, \left[a^k, a_j\right],\right] : + (-1)^{|a^j|+1} : \left[a^j, a_k\right]\left[a^k, a_j\right] : \\ &+ (-1)^{|a_k|+1} : a_k \left[a^j, \left[a^k, a_j\right]\right] :\Big) + (-1)^{|a^j|+1}\chi\lambda^2\frac{k\left(\left[a^j, a_k\right] \middle| \left[a^k, a_j\right]\right)}{2} \\ &= (3\lambda + 2T + \chi S) : a^j : a_k \left[a^k, a_j\right] :: + \chi\lambda^2\frac{k\left(\Omega(a^j) \middle| a_j\right)}{2}.\end{aligned}$$

In conclusion,

$$[H_\Lambda H] = (3\lambda + 2T + \chi S)H + \frac{\chi\lambda^2}{3}c(k),$$

where $c(k)$ is given by formula (3.6), which concludes the proof. Notice that similarly as in Remark 3.1.2, it is easily seen that H above is independent on the chosen basis. □

The vector H above is known as the *Kac-Todorov superconformal vector* of $\mathfrak{g}$.

Remark 3.2.3. In Theorem 2.5.1, which we referred to as the Kac-Todorov construction, we required $\mathfrak{g}$ to be simple or supercommutative. This hypothesis comes from the classical statement, but, as we have seen during the proof, it is not necessary, although we have used it for the Segal-Sugawara construction. Requiring that hypothesis, we obtain a fancier formula for (3.6). Indeed, if (3.3) is satisfied, localizing at $k + h^\vee \in \mathbb{C}$,

$$\begin{aligned} c(k+h^\vee) :&= 3\left(\frac{1}{2} - \frac{h^\vee}{3(k+h^\vee)}\right) \operatorname{sdim}\mathfrak{g} = \frac{(k+h^\vee)+2k}{2(k+h^\vee)} \operatorname{sdim}\mathfrak{g} \\ &= \frac{\operatorname{sdim}\mathfrak{g}}{2} + \frac{k \operatorname{sdim}\mathfrak{g}}{k+h^\vee}, \end{aligned}$$

which is the formula (2.19) given for the central charge in the previous chapter. In fact, we can drop the assumptions on simplicity and supercommutativity, and obtain the same central charge as above, defining a *generalized dual Coxeter number*. Indeed, let $h^\vee \in \mathbb{C}$ be such that

$$\operatorname{str}(\Omega) = 2h^\vee \operatorname{str}(\mathrm{Id}).$$

So, localizing at $k + h^\vee \in \mathbb{C}$, we obtain the previous formula in the non-hypothesis case. If $\mathfrak{g}$ is simple or supercommutative, this coincides with the usual dual Coxeter number.

Remark 3.2.4 ([51] **The Virasoro vector for the Free Superfermions**)**.** For V an n-dimensional vector superspace with $\langle\cdot|\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ non-degenerate and superantisymmetric bilinear form, let $\{\varphi_j\}_{j=1,\dots,n} \subseteq V$ be a basis with $\{\varphi^j\}_{j=1,\dots,n} \subseteq V$ dual basis with respect to $\langle\cdot|\cdot\rangle$. Then, the embedding $V^c(\mathrm{Vir}) \hookrightarrow \mathrm{FF}(V)$ in Theorem 1.4.3 is given by

$$L \mapsto \frac{1}{2}\sum_{j=1}^{n} :(T\varphi^j)\varphi_j:, \quad c \mapsto -\frac{\operatorname{sdim} V}{2}. \tag{3.7}$$

Observe that the Virasoro vector obtained from the Kac-Todorov construction is different from the one which is obtained via the Segal-Sugawara construction. In fact, for commutative Lie superalgebras, this one is the sum of (3.4) and (3.7).

Corolary 3.2.5 ([54] **The Kac-Todorov Modification**)**.** *For $k \neq 0$, let H and $c(k)$ be defined as in* (3.5) *and* (3.6)*, respectively. Define*

$$H^a := H + Ta \in V^k_{\mathrm{super}}(\mathfrak{g}), \quad c^a(k) = c(k) - 3k\,(a|a) \in \mathbb{C}, \quad \textit{for } a \in \Pi\mathfrak{g} \textit{ odd}.$$

Then, we have that H^a is another Neveu-Schwarz vector of central charge $c^a(k)$.

Proof. It is straightforward. Indeed, by sesquilinearity, since $[a,a]=0$ ($a\in\Pi\mathfrak{g}$ is odd),

$$[H^a_\Lambda H^a] := [H_\Lambda H] + \frac{\chi\lambda^2}{3}c(k) + (\lambda+T)\,[H_\Lambda a] - \lambda\,[a_\Lambda H] - \lambda(\lambda+T)\,[a_\Lambda a]$$
$$= (3\lambda + 2T + \chi S)(H + Ta) + \frac{\chi\lambda^2}{3}(c(k) - 3k(a|a)). \qquad \square$$

Remark 3.2.6. Notice that previous result may not give us eigenvectors, since

$$[H^a{}_\Lambda b] = (2T+\chi S)\,b + \lambda(b-[a,b]) - \lambda\chi\,(a|b)\,k, \quad \text{for } a\in\Pi\mathfrak{g}_0, b\in\Pi\mathfrak{g}.$$

Remark 3.2.7. We must notice that without the Λ-bracket formalism this computations will be longer (although maybe easier to understand). Indeed, in [20, Apéndice B] we can find these computations using λ-brackets. Notice that in this case we should compute the double of λ-brackets, since we would do computations for $a\in\mathfrak{g}$ and parity reversed vectors, and, moreover, again the double of λ-brackets, since each one should be done with respect to H and $SH = 2L$ (remember that (2.16) implies (1.19)). So, this new formalism simplifies a lot the computations, at the expense of a more complex procedure.

3.3 $N = 2$ Superconformal VAs from Manin triples

In this section, we are going to take another step in our construction of superconformal vertex algebra embeddings. For instance, we will show that it is possible to extend the Kac-Todorov construction under certain circunstances to obtain $N = 2$ superconformal structures in the universal superaffine vertex algebra. This is very important for us, since it will be the our starting point for the constructions in following chapters. We will work with totally even quadratic Lie algebras.

3.3.1 Manin Triples of Quadratic Lie Algebras

Let $(\mathfrak{g},(\cdot|\cdot))$ be a finite dimensional (even) quadratic Lie algebra. The basic structure we need to construct the $N = 2$ superconformal vertex algebra is known as a *Manin triple*. This ones are related with *Lie bialgebras* (see [25][Chapter 4]).

Definition 3.3.1. [25] A *finite-dimensional Manin triple* of $(\mathfrak{g},(\cdot|\cdot))$ is a triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ of finite-dimensional Lie algebras such that

- Both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_+\oplus\mathfrak{g}_-$.
- Both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are *Lagrangian* (or maximal isotropic) with respect to $(\cdot|\cdot)$. So,

  $$\mathfrak{g}_\pm^\perp = \mathfrak{g}_\pm, \text{ and } \dim\mathfrak{g}_\pm = \frac{1}{2}\dim\mathfrak{g},$$

  where $\mathfrak{g}_\pm^\perp$ denotes the orthogonal subspaces of $\mathfrak{g}_\pm$ with respect to $(\cdot|\cdot)$.

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple of $(\mathfrak{g}, (\cdot|\cdot))$, and consider $\{\epsilon_j\}_{j=1}^n \subseteq \mathfrak{g}_+$ and $\{\epsilon^j\}_{j=1}^n \subseteq \mathfrak{g}_-$ basis such that form an *isotropic basis* of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$, in other words, we have a basis $\{\epsilon_j, \epsilon^j\}_{j=1}^n \subseteq \mathfrak{g}$ such that

$$(\epsilon_j|\epsilon_k) = 0, \quad \left(\epsilon^j \middle| \epsilon_k\right) = \delta^j_k, \quad \left(\epsilon^j \middle| \epsilon^k\right) = 0, \qquad \text{for } j,k \in \{1,\dots,n\}. \tag{3.8}$$

Let $h^\vee \in \mathbb{C}$ be the generalized dual Coxeter number of $\mathfrak{g}$. Then, for $k \neq -h^\vee$, consider $V^{k+h^\vee}_{\text{super}}(\mathfrak{g})$ the universal superaffine vertex algebra associated to $(\mathfrak{g}, (\cdot|\cdot))$. We will write $e_j := \Pi\epsilon_j$ and $e^j := \Pi\epsilon^j$ for $j \in \{1,\dots,n\}$ abusing notation. Define the even vector

$$J := \frac{i}{k+h^\vee} \sum_{j=1}^n :e^j e_j: \in V^{k+h^\vee}_{\text{super}}(\mathfrak{g}).$$

We will work with parity-reversed vectors, so keep in mind Remark 3.2.1. Define now

$$v := \sum_{j=1}^n \left[e^j, e_j\right], \quad w := \sum_{j=1}^n \left(\left[e^j, e_j\right]_+ - \left[e^j, e_j\right]_-\right) \in \Pi\mathfrak{g},$$

where the subscripts denote the canonical projections $\pi_\pm \colon \mathfrak{g} \longrightarrow \mathfrak{g}_\pm$, $a \mapsto a_{\mathfrak{g}_\pm} \equiv a_\pm$.

Remark 3.3.2. The conclusion of Remark 3.1.2 also holds true when we replace a basis and its dual for an isotropic basis. Indeed, the very same proof works. As a consequence, we have that v, w and J are all of them independent on the chosen isotropic basis.

Theorem 3.3.3 ([43, 54] **Getzler's Construction**). *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple of $(\mathfrak{g}, (\cdot|\cdot))$ quadratic Lie algebra. Then,*

(1) The even vector J satisfies $[J_\Lambda J] = -\left(H + \frac{\chi\lambda}{3}c\right)$, where H is given by

$$H := \frac{1}{k+h^\vee} Tw + \frac{1}{k+h^\vee} \sum_{j=1}^n \left(:e_j\left(Se^j\right): + :e^j\left(Se_j\right):\right.$$
$$\left. + \frac{1}{k+h^\vee} \sum_{k=1}^n \left(:e^j:e^k\left[e_j, e_k\right]:: + :e_j:e_k\left[e^j, e^k\right]::\right)\right) \in V^{k+h^\vee}_{super}(\mathfrak{g}),$$

where $c = 3\dim\mathfrak{g}_+ \in \mathbb{C}$. In fact, this vector is obtained applying the Kac-Todorov modification to the vector $w/(k+h^\vee) \in \Pi\mathfrak{g}$. Indeed, we have

$$H := H_0 + \frac{1}{k+h^\vee} Tw \in V^{k+h^\vee}_{super}(\mathfrak{g}),$$

where H_0 is the Kac-Todorov superconformal vector of $\mathfrak{g}$. In particular, we have that this construction may not define a superconformal vertex algebra.

(2) Assuming that

$$w \in \left[\mathfrak{g}_+, \mathfrak{g}_+\right]^\perp \cap \left[\mathfrak{g}_-, \mathfrak{g}_-\right]^\perp, \tag{3.9}$$

the vectors J and H as above generate an $N = 2$ superconformal vertex algebra of central charge $c = 3\dim\mathfrak{g}_+$. That is, the formulas (2.21) are satisfied for $\{J, H, c\}$.

Proof. The proof is given on [54, Proposition 2.14]. We will not repeat it here, since in next chapters we will give a construction that generalizes this one. □

Corolary 3.3.4 ([43, 54], **Freudental's Formula**). *Let* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *be a Manin triple of a quadratic Lie algebra* $(\mathfrak{g}, (\cdot|\cdot))$ *for which* $w \in \Pi\mathfrak{g}$ *satisfies* (3.9). *Then,*

$$(w|w) = -\frac{2}{3} h^\vee \dim \mathfrak{g}_+.$$

Proof. This identity follows after mathing the values for the central charge $c = 3\dim\mathfrak{g}$ and the one from the modification of the Kac-Todorov construction in Theorem 3.3.3. □

3.4 $N = 2$ Superconformal VAs in the bc-$\beta\gamma$ System

We will show now that the bc-$\beta\gamma$ system always admits an $N = 2$ superconformal vertex algebra structure. For that, we will introduce an equivalent notion of this structure.

3.4.1 Topological Vertex Algebras

In [73, Section 2], the notion of *topological vertex algebra* is introduced, which is in fact equivalent to the notion of $N = 2$ superconformal vertex algebras. We will see now that the bc-$\beta\gamma$ system admits this type of structure.

Definition 3.4.1. [73] A *topological vertex algebra* is the structure that is obtained via the universal enveloping vertex algebra of the Lie conformal algebra defined by

$$\mathcal{R} := (\mathbb{C}L \otimes \mathbb{C}[T]) \oplus (\mathbb{C}J \otimes \mathbb{C}[T]) \oplus \left(\mathbb{C}G^+ \otimes \mathbb{C}[T]\right) \oplus \left(\mathbb{C}G^- \otimes \mathbb{C}[T]\right) \oplus \mathbb{C}D,$$

where D is an even central element, J and L are two even vectors, and $G^\pm$ are two odd vectors, all of them related via the non-zero λ-brackets

$$[L_\lambda L] = (T + 2\lambda)L, \quad [J_\lambda J] = D\lambda, \quad [J_\lambda G^\pm] = \pm G^\pm,$$

$$[L_\lambda G^-] = (2\lambda + T)G^-, \quad [L_\lambda G^+] = (\lambda + T)G^+,$$

$$\left[G^+_\lambda G^-\right] = L + \lambda J + D\tfrac{\lambda^2}{2}, \quad [L_\lambda J] = (\lambda + T)J - D\tfrac{\lambda^2}{2}.$$

In the practice, we will work with the quotient

$$\frac{V(\mathcal{R})}{(D-d)\, V(\mathcal{R})},$$

which receives the name of *topological vertex algebra of rank* $d \in \mathbb{C}$.

Theorem 3.4.2 ([73, Section 2.2]). *The bc-*$\beta\gamma$ *system of dimension* $2N \in \mathbb{N}$ *admits a structure of topological vertex algebra of rank* N *for the generators defined by*

$$G^+ := \sum_{j=1}^{N} :b^j \beta^j:, \quad G^- := \sum_{j=1}^{N} :\left(T\gamma^j\right) c^j: \in V(\Omega_N). \tag{3.10}$$

Proof. We leave to the reader this verification using λ-brackets. In particular, computing

$$\left[G^{+}_{\lambda}G^{-}\right] = L + \lambda J + N\frac{\lambda^2}{2},$$

we recover the even vectors

$$L := \sum_{j=1}^{N} :\left(T\gamma^j\right)\beta^j : + :\left(Tb^j\right)c^j :, \quad J := \sum_{j=1}^{N} : b^j c^j : \in V\left(\Omega_N\right). \qquad \square$$

We will show that this structure is equivalent to an $N = 2$ superconformal vertex algebra.

Theorem 3.4.3 ([11])**.** *Let $(L, J, G^{\pm})$ be a set of generators for a topological vertex algebra of rank $d \in \mathbb{C}$. Defining*

$$\widetilde{L} := L - \frac{1}{2}TJ, \tag{3.11}$$

the new set $(\widetilde{L}, J, G^{\pm})$ generate an $N = 2$ superconformal vertex algebra with central charge $c = -3d \in \mathbb{C}$. Conversely, from any $N = 2$ superconformal vertex algebra with central charge $c \in \mathbb{C}$, we can define in a natural way, reversing the previous construction, thanks to formula (3.11)*, a topological vertex algebra of rank $d = -c/3 \in \mathbb{C}$.*

Corolary 3.4.4 ([52, Example 5.10])**.** *The bc-$\beta\gamma$ system of dimension $2N \in \mathbb{N}$ admits a structure of $N = 2$ superconformal vertex algebra with central charge $c = -3N \in \mathbb{C}$ for the generators defined by*

$$J := -i\sum_{j=1}^{N} :\left(S\gamma^j\right)c^j :, \quad H := \sum_{j=1}^{N}\left(:\left(S\gamma^j\right)\left(Sc^j\right): + :\left(T\gamma^j\right)c^j :\right) \in V\left(\Omega_N\right).$$

Although we will not use it, the *bc*-$\beta\gamma$ system admits more supersymmetries.

Remark 3.4.5 ([52, Example 5.11])**.** We can endow the *bc*-$\beta\gamma$ system with a $N = 4$ superconformal vertex algebra structure. Indeed, supposing that this one has dimension $2N \in \mathbb{N}$, let

$$\sigma^s = \left(\sigma^s_{j,k}\right)_{j,k\in\{1,\dots,N\}}, \quad \text{for } s \in \{1,2,3\}$$

be the *Pauli matrices* of rank N. That is, the three $N \times N$ matrices satisfying

$$\sigma^j\sigma^k = i\sum_{l=1}^{N}\varepsilon^{j,k,l}\sigma^l, \quad \left(\sigma^s\right)^2 = \mathrm{Id}, \qquad \text{for } j,k,s \in \{1,2,3\}, j \neq k,$$

where $\varepsilon = \left(\varepsilon^{j,k,l}\right)_{i,j,l\in\{1,\dots,N\}}$ is the totally antisymmetric tensor. Then, the even vectors

$$J^s := -i\sum_{j,k=1}^{N}\sigma^s_{j,k} :\left(S\gamma^j\right)c^k : \in V\left(\Omega_N\right), \quad \text{for } s \in \{1,2,3\},$$

together with the odd vector (2.20), which is obtained adding the two vectors in (3.10), generate an $N = 4$ superconformal vertex algebra with central charge $c = -3N \in \mathbb{C}$. That is, they satisfy the commutation relations (2.22).

The topological vertex algebra structure of the *bc*-$\beta\gamma$ system will be fundamental when we arrive at Chapter 9 to construct the chiral de Rham complex for smooth manifolds.

Part II

Geometric Structures

Chapter 4

Preliminaries

The goal of this chapter is to review some basics on differential geometry. Concretely, we recall and introduce notations in classical and spin geometry, relating them with the viewpoint provided by the *theory of G-structures*. We follow closely [91, Section 2].

4.1 G-structures and Classical Geometry

Let M be any n-dimensional smooth manifold. When n is even, we will write $n = 2m$, and if n is multiple of 4, we will write $n = 4k$. By *classical geometry*, we mean a collection of tensors over TM (*structures*), the tangent bundle of M, satisfying certain algebraic properties. Often, we will require that these tensors satisfy certain *integrability condition*. Now, an alternative viewpoint is provided by the *theory of G-structures*. Remember that, for G a Lie group, a *G-structure* on M is a *reduction* of the $\mathrm{GL}(n)$-principal bundle of the frames of TM to G such that TM is associated to the G-bundle via the monomorphism of Lie groups $G \hookrightarrow \mathrm{GL}(n)$. All the (classical) geometric structures can be described via reductions to a principal G-bundle. In this case, the tangent bundle can be regarded as a vector bundle *associated* to the principal G-bundle given by the G-structure, and refers to the Lie group G as the *structure group* of TM. Locally, choosing some trivializations $U \times \mathbb{R}^n \cong TU$ of TM, a G-structure is a smooth family of Lie group representations $r_p\colon G \longrightarrow \mathrm{GL}\left(T_pM\right)$ parametrized by points $p \in M$, which is well behaved under the transitions of trivializations. We express classical geometric structures via G-structures.

Definition 4.1.1. A *Riemannian structure* on M is a $(2, 0)$-tensor g such that the induced bilinear form on T_pM is positive-definite for each $p \in M$. This endows T_pM the tangent space with a family of representations

$$r_p\colon \mathrm{O}(n) \cong \mathrm{O}\left(T_pM, g_p\right) \hookrightarrow \mathrm{GL}\left(T_pM\right),$$

and, since g is global, they come from a principal $\mathrm{O}(n)$-bundle. In particular, we have that each T_pM is endowed with the structure of an Euclidean space. So, a Riemannian structure is equivalent to having that TM is associated with an $\mathrm{O}(n)$-structure.

Remark 4.1.2. If M is orientable, any Riemannian structure g on M is equivalent to having that TM is associated with a $\mathrm{SO}(n)$-structure. Indeed, for each point $p \in M$, the tangent T_pM is endowed with the structure of an orientable Euclidean space.

Remember that on M we can define *affine connections* ∇ and the associated *torsion* T_∇. If g is a Riemannian structure, the (unique) affine connection ∇^g such that $\nabla^g g = 0$ (it preserves the metric), and is torsion-free receives the name of *Levi-Civita connection*.

Definition 4.1.3. An *almost symplectic structure* on M is an antisymmetric 2-form ω such that the induced bilinear form on T_pM is non-degenerate for each $p \in M$. This endows T_pM the tangent space with a family of representations

$$r_p \colon \mathrm{Sp}(m, \mathbb{R}) \cong \mathrm{Sp}(T_pM, \omega_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since ω is global, they come from a principal $\mathrm{Sp}(m, \mathbb{R})$-bundle. So, an almost symplectic structure is equivalent to having that TM is associated with a $\mathrm{Sp}(m, \mathbb{R})$-structure. Here, M must be even dimensional. The structure is called *integrable*, or *symplectic*, if $d\omega = 0$. By the Darboux Theorem, the integrability condition is equivalent to existence of symplectic coordinates $(q_1, ..., q_n, p_1, ..., p_m)$ such that

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

Definition 4.1.4. An *almost complex structure* on M is an $(1,1)$-tensor J (so, a bundle morphism $J \in \mathrm{End}(TM)$) such that the induced morphism on T_pM satisfies $J_p^2 = -\mathrm{Id}$ for each $p \in M$. This endows T_pM the tangent space with a family of representations

$$r_p \colon \mathrm{GL}(m, \mathbb{C}) \cong \mathrm{GL}(T_pM, J_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since J is global, they come from a principal $\mathrm{GL}(m, \mathbb{C})$-bundle. In particular, we have that each T_pM is endowed with the structure of a complex (orientable) vector space. So, an almost complex structure is equivalent to having that TM is associated with a $\mathrm{GL}(m, \mathbb{C})$-structure. Equivalently, we have a decomposition of the complexification into complex vector bundles by $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, which are conjugate to each other. So, we have that $T^{1,0}M$ and $T^{0,1}M$ are, respectively, the $(\pm i)$-eigenbundles of J. The structure is called *integrable*, or *complex*, if the associated *Nijenhuis tensor*

$$N_J(X, Y) := [X, Y] - [JX, JY] + J\,[JX, JY] + J\,[X, JY]\,, \quad \text{for } X, Y \in \mathfrak{X}(M),$$

vanishes. More geometrically, this is equivalent to the fact that the bundles $T^{1,0}M$ and $T^{0,1}M$ are preserved by the Lie bracket. By the Newlander-Niremberg Theorem, the integrability condition is equivalent to existence of holomorphic coordinates $(z_1, ..., z_n)$ such that J is the multiplication by the imaginary constant i. Let $T^*M \otimes \mathbb{C} = T^{1,0*}M \oplus T^{0,1*}M$ be now the decomposition induced by $J^* \in \mathrm{End}(T^*M)$. Put

$$\Lambda^{p,q}T^*M := \Lambda^p T^{1,0*}M \otimes \Lambda^q T^{0,1*}M,\ \Lambda^k T^*M \bigoplus_{k=p+q} \Lambda^{p,q}T^*M, \quad \text{for } k, p, q \in \{0, 1, \dots, m\}.$$

We define $\partial\colon \Lambda^{p,q}T^*M \longrightarrow \Lambda^{p+1,q}T^*M$, and $\overline{\partial}\colon \Lambda^{p,q}T^*M \longrightarrow \Lambda^{p,q+1}T^*M$ by composing the exterior differential d with the corresponding projection operator. Then, the integrability of J is also equivalent to $d = \partial + \overline{\partial}$. As mentioned above, this is also equivalent to the existence of holomorphic coordinates $z_1 := x_1 + iy_1, \dots, z_m := x_m + iy_m$ on M. That is,

$$J\left(\frac{\partial}{\partial x_j}\right) := \frac{\partial}{\partial y_j} \text{ and } J\left(\frac{\partial}{\partial y_j}\right) := -\frac{\partial}{\partial x_j}, \quad \text{for } j \in \{1, \dots, m\}.$$

The bundles $T^{1,0}M$ and $T^{0,1}M$ are spanned by

$$\frac{\partial}{\partial z_j} := \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} \text{ and } \frac{\partial}{\partial \overline{z}_j} := \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}, \quad \text{for } j \in \{1, \dots, m\},$$

respectively, while the dual bundles $T^{1,0*}M$ and $T^{0,1*}M$ are spanned by the differentials

$$dz_j := dx_j + idy_j \text{ and } d\overline{z}_j := dx_j - idy_j, \quad \text{for } j \in \{1, \dots, m\}.$$

Definition 4.1.5. An *almost special complex structure* on M is given by a global volume form $\Omega \in \Omega^m(M) \otimes \mathbb{C}$ satisfying the following:

(1) $\Omega \wedge \overline{\Omega} \neq 0$.

(2) (*locally decomposable*) $\Omega = \theta_1 \wedge \dots \wedge \theta_m$ locally, for complex 1-forms $\theta_1, \dots, \theta_m$.

For each $p \in M$, this endows T_pM the tangent space with a family of representations

$$r_p\colon \mathrm{SL}(m, \mathbb{C}) \cong \mathrm{SL}(T_pM, \Omega_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since Ω is global, they come from a principal $\mathrm{SL}(m, \mathbb{C})$-bundle. So, a special complex structure is equivalent to having that TM is associated with a $\mathrm{SL}(m, \mathbb{C})$-structure. Note that, by conditions (1) and (2), we obtain that Ω defines an almost complex structure J on M. Indeed, we have the splitting $T^*M \otimes \mathbb{C} = T^{1,0*}M \oplus T^{0,1*}M$, where the $(1,0)$-forms $T^{1,0*}M$ are locally spanned by the 1-forms $\theta_1, \dots, \theta_m$. The structure is called *integrable*, or *special complex*, if

(3) $d\Omega = 0$.

Applying (3), we obtain that J is integrable. Indeed, by the previous decomposition we obtain that $\Omega \wedge \theta = 0$ for any $(1,0)$-form θ. Then, condition (3) implies $\Omega \wedge d\theta = 0$ for any $(1,0)$-form θ. Consequently, we have that the 2-form $d\theta$ has no $(2,0)$-part, which implies the integrability condition of J, the associated almost complex structure, since we have obtained $d = \partial + \overline{\partial}$. In particular, we have $\overline{\partial}\Omega = 0$. So, the volume form Ω defines an holomorphic trivialization of the *canonical bundle* $\mathcal{K}_M = \Lambda^{m,0}T^*M = \Lambda^m T^{1,0*}M$. Note that $T^{1,0}M \cong TM$ as complex vector bundles. Combining the Newlander-Niremberg Theorem with the holomorphic Darboux Theorem, one can prove that $d\Omega = 0$ is equivalent to the existence of holomorphic coordinates $(z_1, ..., z_n)$ with constant determinant of the Jacobian such that

$$\Omega = dz_1 \wedge \dots \wedge dz_n.$$

Definition 4.1.6. Let J be an almost complex structure on M. The pair (J, h) is said an *almost Hermitian structure* on M if h is a complex sesquilinear $(2,0)$-tensor (that is, linear in the first coordinate and antilinear in the second one, for the point-wise complex multiplication induced by J), and such that the induced bilinear form on T_pM is positive definite for each $p \in M$. This endows T_pM the tangent space with a family of representations

$$r_p \colon \mathrm{U}(m) \cong \mathrm{U}(T_pM, h_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since h is global, they come from a principal $\mathrm{U}(m)$-bundle. In particular, we have that each T_pM is endowed with the structure of an hermitian vector space. So, an almost hermitian structure is equivalent to having that TM is associated with a $\mathrm{U}(m)$-structure. An almost hermitian structure on M defines, respectively, an almost symplectic structure ω and a Riemannian metric g by

$$\omega := \mathrm{Im}(h) = \frac{i}{2}\left(h - \overline{h}\right) \text{ and } g := \mathrm{Re}(h) = \frac{i}{2}\left(h + \overline{h}\right).$$

In addition, it follows from the hermitian condition that ω is a $(1,1)$-form. Notice that, if we have any of the pairs (J, ω) or (J, g) consisting on an almost complex structure J, a Riemannian metric g, and an almost symplectic structure ω on M such that the second structure of the pair is invariant under J (so, J acts as an isometry for that structure), requiring also that $\omega(\cdot, J\cdot) > 0$ when ω appears, then we recover an almost hermitian structure, since $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ and $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, so $h := g - i\omega$ defines the almost hermitian structure. The structure is called *integrable*, or *hermitian*, if it is J.

Recall that on (M, h) hermitian manifold we define *hermitian connections* ∇, which are the ones with $\nabla g = 0$ and $\nabla J = 0$. The (unique) hermitian connection ∇^B such that

$$H := g\left(T_{\nabla^B}(\cdot,\cdot),\cdot\right) \in \Lambda T^3 M$$

receives the name of *Bismut connection.* The (unique) hermitian connection ∇^c such that $T^{1,1}_{\nabla^c} = 0$ receives the name of *Chern connection.*

Definition 4.1.7. An *almost Kähler structure* is given by a pair (J, ω) that is an almost hermitian structure on M. The associated $(1,1)$-form ω is known as the *almost Kähler form* of M. Moreover, the structure is called *integrable*, or *Kähler*, if both structures J and ω are integrable. The $(1,1)$-form ω is called the *Kähler form* of M if $d\omega = 0$.

Lemma 4.1.8. *Let (J, ω) be an almost Kähler structure, and consider ∇^g and ∇^B the Levi-Civita and Bismut connections, respectively. Then, $\nabla^g = \nabla^B$ if and only if $d\omega = 0$.*

Definition 4.1.9. An *(almost) hyperkähler structure* on M is given by (g, I), (g, J) and (g, K) three (almost) Kähler structures satisfying $I^2 = J^2 = K^2 = IJK = -1$ (known as the *quaternionic identities*). For each $p \in M$, an almost hyperkähler structure endows T_pM the tangent space with a family of representations

$$r_p \colon \mathrm{UQ}(k) \cong \mathrm{UQ}(T_pM, I_p, J_p, K_p, g_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since I, J, K and g are all of them global, they come from a principal $\mathrm{UQ}(k)$-bundle. So, an almost hyperkähler structure is equivalent to having that TM is associated with a $\mathrm{UQ}(k)$-structure. Here $\mathrm{UQ}(k)$ denotes the k-dimensional *quaternionic unitary group*. The structure is called *integrable*, or *hyperkähler*, if I, J, K are all integrable.

Definition 4.1.10. An *almost Calabi-Yau structure* on M is a pair (Ω, ω) such that ω is an almost symplectic structure and Ω is an almost special complex structure such that

(1) $\Omega \wedge \overline{\Omega} = (-1)^{\frac{m(m-1)}{2}} i^m \omega^m$.

(2) $\Omega \wedge \omega = 0$.

For each $p \in M$, this endows T_pM the tangent space with a family of representations

$$r_p \colon \mathrm{SU}(m) \cong \mathrm{SU}(T_pM, \Omega_p, \omega_p) \hookrightarrow \mathrm{GL}(T_pM),$$

and, since Ω and ω are global, they come from a principal $\mathrm{SU}(m)$-bundle. So, an almost Calabi-Yau structure is equivalent to having that TM is associated with a $\mathrm{SU}(m)$-structure. In particular, the inclusion $\mathrm{SU}(m) \hookrightarrow \mathrm{U}(m)$ induces an almost Hermitian structure. The structure is called *integrable*, or *Calabi-Yau*, if Ω is closed.

4.2 Clifford Bundles and Spin Geometry

To finish this chapter, we include a brief review on the basics of *spin structures*, which we will extensively use in the rest of this work. In particular, we will study their relationship with the previous structures. We will follow closely reference [66]. We will start studying the case given by a finite-dimensional real vector space V with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : V \otimes V \longrightarrow \mathbb{R}$ of arbitrary signature (p, q).

Definition 4.2.1. The *Clifford algebra* $\mathrm{Cl}(V)$ associated to V is an associative algebra with unit defined as follows: we take the quotient of the tensor algebra of V by the ideal generated by the relation

$$v \cdot v = \langle v, v \rangle , \quad \text{for } v \in V. \tag{4.1}$$

We have a $\mathbb{Z}_2$-graded decomposition on $\mathrm{Cl}(V)$ determined by the automorphism

$$\begin{array}{rccc} \alpha\colon & \mathrm{Cl}(V) & \longrightarrow & \mathrm{Cl}(V) \\ & v & \mapsto & -v \end{array} .$$

Indeed, for being $\alpha^2 = \mathrm{Id}$, there exists a decomposition in even and odd eigenspaces

$$\mathrm{Cl}^j(V) := \left\{ \varphi \in \mathrm{Cl}(V) \mid \alpha(\varphi) = (-1)^j \varphi \right\}, \text{ for } j \in \{0, 1\}.$$

The *associated pin and spin spaces* are defined by

$$\mathrm{Pin}(V, \langle \cdot, \cdot \rangle) := \{ v_1 \cdots v_k \in \mathrm{Cl}^*(V) \mid v_j \in V,\ k \in \mathbb{N},\ \langle v_j, v_j \rangle = \pm 1 \text{ for } j \in \{1, \dots, k\} \} ,$$

and

$$\mathrm{Spin}(p,q) \cong \mathrm{Spin}(V, \langle\cdot,\cdot\rangle) := \mathrm{Pin}(V, \langle\cdot,\cdot\rangle) \cap \mathrm{Cl}^0(V).$$

Most of the important applications of Clifford algebras come through a detailed understanding of their representations. We are interested on consider now real representations that we will complexify to obtain a complex representation.

Definition 4.2.2. A *real representation* of the Clifford algebra of V is an algebra homomorphism

$$\rho\colon \mathrm{Cl}(V) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W, W).$$

The vector space W is called a *real Cl*(V)*-module*. We simplify notations by writing

$$\rho(\varphi)(w) \equiv w \cdot \varphi, \quad \text{for } w \in W, \varphi \in \mathrm{Cl}(V).$$

This product is often referred to as *Clifford multiplication* on W.

We now come to the notion or (ir)reducibility of representations.

Definition 4.2.3. A real representation $\rho\colon \mathrm{Cl}(V) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W, W)$ will be said to be *reducible* if the vector space W can be written as a non-trivial real direct sum

$$W = W_1 \oplus W_2$$

such that $\rho(\varphi)(W_j) \subseteq W_j$ for $j \in \{1, 2\}$ for all $\varphi \in \mathrm{Cl}(V)$. Note that in this case we can write

$$\rho = \rho_1 \oplus \rho_2,$$

where

$$\rho_j(\varphi) \equiv \left.\rho(\varphi)\right|_{W_j}, \quad \text{for } j \in \{1, 2\}.$$

A real representation ρ is called *irreducible* if it is not reducible.

The representations of the Clifford algebra $\mathrm{Cl}(V)$ give rise to important representations of certain groups, such as the spin group representations.

Definition 4.2.4. A *real spinor representation* of $\mathrm{Spin}(V, \langle\cdot,\cdot\rangle)$ is an homomorphism

$$\Delta\colon \mathrm{Spin}(V, \langle\cdot,\cdot\rangle) \longrightarrow \mathrm{GL}(S) \tag{4.2}$$

given by the restricting an irreducible real representation $\rho\colon \mathrm{Cl}(V) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W, W)$ to the spin group $\mathrm{Spin}(V, \langle\cdot,\cdot\rangle)$. The real vector space S is called an *irreducible real spinor space*, and its elements $\xi \in S$ are known as *real spinors*.

Now, let $\mathrm{Cl}(V)_{\mathbb{C}} := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Clifford algebra, which has associated irreducible complex spinor space $S_{\mathbb{C}} := S \otimes_{\mathbb{R}} \mathbb{C}$, for S irreducible real spinor space. Then, for each complex spinor $\xi \in S_{\mathbb{C}}$, we consider the $\mathbb{C}$-linear map $j_\xi\colon V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow S_{\mathbb{C}}$ defined by

$$j_\xi(v) := v \cdot \xi, \quad \text{for } v \in V \otimes_{\mathbb{R}} \mathbb{C},$$

where $\cdot$ denotes the $\mathbb{C}$-linear extension of the Clifford multiplication to $\mathrm{Cl}(V)_{\mathbb{C}}$-modules. From this point on, we shall assume that V is an oriented $2n$-dimensional vector space, and that the symmetric bilinear form $\langle\cdot,\cdot\rangle : V \otimes V \longrightarrow \mathbb{R}$ is positive-definite. We will denote the $\mathbb{C}$-linear extension to $V \otimes_{\mathbb{R}} \mathbb{C}$ of this form just by $\langle\cdot,\cdot\rangle$.

Definition 4.2.5. A spinor $\xi \in S_{\mathbb{C}}$ is *pure* if $\mathrm{Ker}(j_\xi)$ is maximal isotropic. That is, if the vector space $\mathrm{Ker}(j_\xi)$ has maximal dimension n.

Moreover, we define $S_{\mathbb{C}}^{\pm} \subseteq S_{\mathbb{C}}$ to be the (± 1)-eigenspaces for the Clifford multiplication by the complex volume form

$$\omega_{\mathbb{C}} := i^n a_1 \cdot \dots \cdot a_{2n} \in \mathrm{Cl}(V)_{\mathbb{C}},$$

for $\{a_1, \dots, a_{2n}\}$ any positively oriented orthonormal basis of V. So, we have the natural decomposition $S_{\mathbb{C}} = S_{\mathbb{C}}^{+} \oplus S_{\mathbb{C}}^{-}$ into two spinor spaces $S_{\mathbb{C}}^{\pm}$ of *positive and negative chirality.*

Lemma 4.2.6 ([66, IV. Lemma 9.6])**.** *If $\xi \in S_{\mathbb{C}}$ is pure, then either $\xi \in S_{\mathbb{C}}^{+}$ or $\xi \in S_{\mathbb{C}}^{-}$.*

For any spinor $\xi \in S_{\mathbb{C}}$, we can define its *isotropy group* by

$$G_\xi := \{\sigma \in \mathrm{Spin}(2n) \mid \sigma \cdot \xi = \xi\}. \tag{4.3}$$

Lemma 4.2.7 ([66, IV. Lemma 9.15])**.** *For $\sigma \in S_{\mathbb{C}}^{\pm}$ pure, then $G_\xi = \mathrm{SU}(n)$.*

Let $\mathrm{PS}_{\mathbb{C}} \subseteq S_{\mathbb{C}}$ be the subset of pure spinors, and let $\mathcal{F}_{2n}$ be the set of maximal isotropic subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$. We consider the natural map $K \colon \mathrm{PS}_{\mathbb{C}} \longrightarrow \mathcal{F}_{2n}$. Now, let $\mathrm{AC}(V)$ be the set of all orthogonal almost complex structures on V. Associated to any $J \in \mathrm{AC}(V)$ there exists a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = E(J) \oplus \overline{E(J)},$$

where $E(J) \subseteq V \otimes_{\mathbb{R}} \mathbb{C}$ denotes the $(-i)$-eigenspace of J. These vector spaces are both maximal isotropic. We can consider the natural map $E \colon \mathrm{AC}(V) \longrightarrow \mathrm{MIS}(V)$ that sends the almost complex structures of V to the associated maximal isotropic subspace. The set $\mathrm{AC}(V)$ of almost complex structures falls into two connected components $\mathcal{C}_n^{\pm}$, where $\mathcal{C}_n^{+}$ consists of those almost complex structures whose canonical orientation agrees with the one given on V. Let $\mathrm{PS}_{\mathbb{C}}$ be the space of pure spinors, which admits a decomposition $\mathrm{PS}^{\pm}$ of pure spinors with, respectively, positive and negative chirality. Let $\mathbb{P}\left(\mathrm{PS}^{\pm}\right)$ be the projectivization of the positive and negative, respectively, pure spinor spaces. We have arrived at the following result that will be useful in the future.

Lemma 4.2.8 ([66, IV. Proposition 9.7])**.** *The maps $\sigma \mapsto K(\sigma)$ and $J \mapsto E(J)$ induce the $\mathrm{SO}(2n)$-equivariant diffeomorphisms*

$$\mathbb{P}\left(\mathrm{PS}^{\pm}\right) \longrightarrow \mathcal{F}_{2n}^{\pm} \longrightarrow \mathcal{C}_n^{\pm}.$$

Now, we will extend the given notions and results to an arbitrary smooth manifold M. Let $E \longrightarrow M$ be a smooth real vector bundle with a non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle : \Gamma(E) \otimes \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$ of arbitrary signature (p, q). Then, we can give the following notions that extends canonically the ones given over a point.

Definition 4.2.9. The *Clifford bundle* $\mathrm{Cl}(E) \longrightarrow M$ associated to E is a smooth real algebra bundle, whose fibers are the *Clifford algebras* generated by each real vector space E_p for $p \in M$. That is, the fiber $\mathrm{Cl}(E)_p$ over $p \in M$ is given by the quotient of the tensor algebra of E_p by the ideal generated by the relation (4.1).

Definition 4.2.10. A *spinor bundle* for E is a pair (S, Υ) consisting of a vector bundle $S \longrightarrow M$, and a morphism of bundles $\Upsilon\colon \mathrm{Cl}(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(S)$ of unital and associative algebras. The sections of S are called *spinors*. If the orthogonal frame bundle of E admits a reduction to $\mathrm{Spin}(p,q)$ (that is, E admits a *spin structure*), then any representation of $\mathrm{Spin}(p,q)$ induces a spinor bundle S. In the special case in which $E = TM$, if it admits a spin structure, then we say that M is a *spin manifold*.

Let (S, Υ) be a spinor bundle for E. Consider the natural inclusion $\sigma\colon \Gamma(E) \longrightarrow \mathrm{Cl}(E)$ and define

$$\Upsilon \circ \sigma\colon \Gamma(E) \longrightarrow \mathrm{End}_{\mathbb{R}}(S),$$

which gives an action for the sections of E on the spinors.

Definition 4.2.11. We can introduce the *Clifford multiplication* on S by the action

$$s \cdot \xi := (\Upsilon \circ \sigma(s))\,(\xi), \quad \text{for } s \in \Gamma(E), \xi \in \Gamma(S).$$

Let ∇ be a connection on E compatible with $\langle \cdot, \cdot \rangle$. Then, we can choose a connection on the spinor bundle given by $\nabla^S\colon \Gamma(S) \longrightarrow \Gamma(E^* \otimes S)$, which is called a *spin connection* such that it satisfies the Leibniz rule with respect to the previous Clifford multiplication. That is,

$$\nabla^S\,(s \cdot \xi) = \nabla s \cdot \xi + s \cdot \nabla^S \xi, \quad \text{for } s \in \Gamma(E), \xi \in \Gamma(S).$$

If the spinor bundle S is such that $\mathrm{End}(S)$ is isomorphic to $\mathrm{Cl}(E)$, two spin connections differ by a 1-form on M with values in the centre of $\mathrm{Cl}(E)$. When the orthogonal frame bundle of E reduces to $\mathrm{Spin}(p,q)$ (for example, when $E = TM$, for M a spin Riemannian manifold), then there exists a canonical choice of spin connection. We assume from now that M is a $2n$-dimensional spin manifold with positive-definite symmetric pairing $\langle \cdot, \cdot \rangle$.

Definition 4.2.12. Let S be a spinor bundle for E of rank r, and suppose that ∇ is a connection on E. The associated *Dirac operator* $\not\nabla\colon \Gamma(S) \longrightarrow \Gamma(S)$ is defined by

$$\not\nabla \xi = \sum_{j=1}^{r} a^j \cdot \nabla^S_{a_j} \xi, \quad \text{for } \xi \in \Gamma(S),$$

for $\left\{a_j, a^j\right\}_{j=1}^r \subseteq \Gamma(E)$ orthogonal dual local frames for E and the Clifford multiplication.

The Dirac operators are independent of the coordinate system. Now, taking $E = TM$, we have that Lemma 4.2.7 and Proposition 4.2.8 generalizes into the following results.

Proposition 4.2.13 ([66, IV. Proposition 9.16])**.** *Each globally defined pure spinor field on M determines a unique reduction of the structure group of M to $\mathrm{SU}(n)$.*

Proposition 4.2.14 ([66, IV. Proposition 9.8])**.** *Let M be an oriented $2n$-dimensional Riemannian manifold. Then, the orthogonal almost complex structures on M, with canonical positive orientation, are in a natural one-to-one correspondence with cross-sections of the projectivized bundle $\mathbb{P}\left(\mathrm{PS}^+\right)$ of positive pure spinors on M.*

We will return in the two following chapters to the concrete study between spinors and geometric structures. This study plays an important role in the present thesis.

Chapter 5

The Killing Spinor Equations

In the present chapter, we introduce the main geometric structure we will work with, the *Killing spinor equations* on smooth spin manifolds. In Section 5.3, we introduce a new tensor that will be fundamental for our purposes. It is noteworthy that, despite being a quantity that can be defined via classical hermitian geometry, it does not seem to have been previously considered in the literature.

5.1 Killing Spinors on Spin Manifolds

Let M be an n-dimensional smooth spin manifold. Let K be a compact Lie group, and consider $p\colon P \longrightarrow M$ principal K-bundle. Given any principal connection A on P, its curvature is $F_A \in \Omega^2(M, \operatorname{ad} P)$. For $H \in \Omega^3(M, \mathbb{R})$, and g a Riemannian structure, we define the connection ∇^+ on TM with skew-symmetric torsion, compatible with g, by

$$\nabla^+ := \nabla^g + \frac{1}{2} g^{-1} H, \tag{5.1}$$

and the connection $\nabla^{+\frac{1}{3}}$ on TM with skew-symmetric torsion by

$$\nabla^{+\frac{1}{3}} := \nabla^g + \frac{1}{6} g^{-1} H, \tag{5.2}$$

where ∇^g denotes the Levi-Civita connection of g. We introduce the following notion.

Definition 5.1.1. [35] We say that the tuple (g, H, φ, A, η), where $\varphi \in \Omega^1(M)$, and η is a spinor on (TM, g), is a *solution of the Killing spinor equations* when, denoting by $\cdot$ the Clifford multiplication,

$$F_A \cdot \eta = 0, \tag{5.3a}$$

$$\nabla^+ \cdot \eta = 0, \tag{5.3b}$$

$$\left(\not{\nabla}^{+\frac{1}{3}} + \varphi \right) \cdot \eta = 0, \tag{5.3c}$$

where $\not{\nabla}^{+\frac{1}{3}}$ denotes the Dirac operator associated to $\nabla^{+\frac{1}{3}}$. We will call $\varphi \in \Omega^1(M)$ the *dilaton 1-form*, although this is the name given to a potential for φ, when φ is exact.

Remark 5.1.2. These equations are known as *Gaugino, gravitino*, and *dilatino equations*, respectively, and they are motivated by physics. If φ is exact, these equations are equivalent in low dimensions to the Killing spinor equations in a compactification of the ten-dimensional heterotic supergravity [27, 30]. Note that there exists a well-stablished notion of Killing spinors for pseudo-Riemannian geometry, which does not agree with the notion considered here. Nonetheless, the name Killing spinor equations refering to (5.3) is now well-stablished in the mathematical physics literature.

We have an interesting equivalence when we are in even dimensions. Recall that a complex spinor induces an $\mathrm{SU}(n)$-structure as we have seen in the previous chapter.

Definition 5.1.3. [41] Let M be a $2n$-dimensional manifold endowed with (J, ω) an almost hermitian structure on M. The associated *Lee form* is defined by

$$\theta_\omega := d^*\omega \circ J, \tag{5.4}$$

where $d^* := - * \circ d \circ *$ (since M is even dimensional), and $*$ is the Hodge star operator.

Remark 5.1.4. Notice that the Lee form (5.4) can also be defined by

$$\theta_\omega := \Lambda_\omega d\omega,$$

where $\Lambda_\omega \colon \Omega^k(M) \longrightarrow \Omega^{k-2}(M), \alpha \mapsto - * (\omega \wedge *\alpha)$ is defined for any $\alpha \in \Omega^2(M)$ by

$$\Lambda_\omega \alpha \frac{\omega^n}{n!} = \alpha \wedge \frac{\omega^{n-1}}{(n-1)!}. \tag{5.5}$$

In particular, the Lee form is the unique 1-form such that $d\omega^{n-1} = \theta_\omega \wedge \omega^{n-1}$.

Let M be any compact almost complex manifold with vanishing first Chern class endowed with a hermitian structure determined by an almost Kähler form ω and a complex structure J. Consider Ψ a smooth global section of the canonical bundle $\mathcal{K}_M$, and the function $\|\Psi\|_\omega$ on M defined via the *point-wise norm* of Ψ, as follows

$$(-1)^{\frac{n(n-1)}{2}} i^n \Psi \wedge \overline{\Psi} = \|\Psi\|_\omega^2 \frac{\omega^n}{n!}.$$

Then, an $\mathrm{SU}(n)$-structure on M is given by a pair (Ψ, ω), as before, satisfying

$$\|\Psi\|_\omega = 1. \tag{5.6}$$

Proposition 5.1.5 ([35, Theorem 5.1],[87, Section 2])**.** *Let M be a $2n$-dimensional spin smooth manifold, and $P \longrightarrow M$ principal K-bundle with K compact Lie group. Consider A principal connection on P with $F_A \in \Omega^2(M, \operatorname{ad} P)$ associated curvature, (J, ω, g) an almost hermitian structure on M, and $H \in \Omega^3(M)$. Then, a solution (g, H, φ, A, η) to the Killing spinor equations* (5.3) *with η pure is equivalent to a tuple $(\Psi, \omega, \varphi, A)$, where (Ψ, ω) is a* $\mathrm{SU}(n)$*-structure, such that*

$$\begin{aligned} F_A \wedge \omega^{n-1} &= 0, & F_A^{0,2} &= 0, \\ \theta_\omega + \varphi &= 0, & d\Psi - \theta_\omega \wedge \Psi &= 0, \\ N_J &= 0, & H + d^c\omega &= 0. \end{aligned}$$

Proof. We assume that there exists a global solution (g, H, φ, A, η) as in Definition 5.1.1. By Proposition 4.2.13, the spinor η determines a reduction of the orthonormal frame bundle of (M, g, J) to $\mathrm{SU}(n)$. So, this is equivalent to a pair (Ψ, ω) given by an $\mathrm{SU}(n)$-structure. Using now the following model for the complex spinor bundle,

$$S_+ \cong \Lambda^{0,\mathrm{even}}(TM \otimes_{\mathbb{R}} \mathbb{C}),$$

where the spinor η can be identified with $\lambda \in \mathbb{C}^*$, one can prove that ω is a $(1,1)$-form, $g = \omega(\cdot, J\cdot)$, and

$$T^{0,1}M = \{X \in TM \otimes \mathbb{C} \mid \iota_X \Psi = 0\}.$$

Now, the gaugino equation (5.3a) is equivalent to $\Lambda_\omega F_A = 0$ and $F_A^{0,2} = 0$. Now, the gravitino equation (5.3b) implies that the connection ∇^+, understood as a connection on the spinor bundle of M, has holonomy contained in $\mathrm{SU}(n)$. This is equivalent to the triple (ω, J, Ψ) be parallel with respect to ∇^+ (as metric connection on TM), and to

$$H = (-d^c\omega)^{(1,2)+(2,1)} + g(N_J\cdot, \cdot).$$

Finally, the dilatino equation (5.3c) implies that J is integrable, that is, $N_J = 0$, and furthermore

$$H = -d^c\omega, \quad \theta_\omega = -\varphi \in \Omega^1(M).$$

At last, by Gauduchon's formula [42, Equation (2.7.6)] for ∇^+ we obtain $d\Psi = \theta_\omega \wedge \Psi$. □

In particular, if we have the trivial fibre bundle with $K = \{1\}$, we obtain the following.

Proposition 5.1.6 ([35, Theorem 5.1],[87, Section 2]). *Let M be a $2n$-dimensional spin smooth manifold. Consider (J, ω, g) almost hermitian structure on M, and $H \in \Omega^3(M)$. Then, a solution (g, H, φ, η) to the Killing spinor equations* (5.3) *with η pure is equivalent to a tuple (Ψ, ω, φ), where (Ψ, ω) is a $\mathrm{SU}(n)$-structure, such that*

$$\begin{aligned} \theta_\omega + \varphi &= 0, & d\Psi - \theta_\omega \wedge \Psi &= 0, \\ N_J &= 0, & H + d^c\omega &= 0. \end{aligned}$$

5.1.1 The F-term and D-term Conditions

Motivated by Proposition 5.1.5, we can distinguish two different type of conditions which appear for solutions of the Killing spinor equations in even dimensions and with η pure.

Definition 5.1.7. We will say that the data given by (g, H, A), where g gives an almost hermitian structure via J, and $H \in \Omega^3(M)$, and A is a connection on P is a solution to the *F-term conditions* if

$$F1)\ F_A^{0,2} = 0, \quad F2)\ H = (-d^c\omega)^{(1,2)+(2,1)} + g(N_J\cdot, \cdot), \quad F3)\ N_J = 0. \tag{5.7}$$

Notice that if J is integrable (it is satisfied $F3$)), then $F2$) is the same as $H = -d^c\omega$.

Definition 5.1.8. We will say that the data given by $(g, H, A, \varphi, \omega, \Psi)$, where (Ψ, ω) is an $\mathrm{SU}(n)$-structure, and $H \in \Omega^3(M)$, is a solution to the *D-term condition* if

$$D1)\ \Lambda_\omega F_A = 0, \quad D2)\ d\Psi = \theta_\omega \wedge \Psi, \quad D3)\ \theta_\omega + \varphi = 0. \tag{5.8}$$

Remark 5.1.9. One has the following equivalences:

- $F1)$ $F_A^{(0,2)} = 0$ and $D1)$ $\Lambda_\omega F_A = 0$ are equivalent to the gaugino equation.
- $F2)$ $H = (-d^c\omega)^{(1,2)+(2,1)} + g(N_J\cdot,\cdot)$ and $D2)$ $d\Psi = \theta_\omega \wedge \Psi$ are equivalent to the gravitino equation.
- $F3)$ $N_J = 0$ and $D3)$ $\theta_\omega + \varphi = 0$ are equivalent to the dilatino equation.

5.2 Special Maximal Holomorphic Atlas

In this work, we are mainly concerned with solutions of the Killing spinor equations with φ closed, in other words, θ_ω closed. Imposing this condition, we will be able to construct new embeddings of SUSY vertex algebras on the chiral de Rham complex. Motivated by this, we can construct a special atlas associated to $\mathrm{SU}(n)$-structures (Ψ, ω) satisfying the following equations:

$$d\Psi - \theta_\omega \wedge \Psi = 0, \quad d\theta_\omega = 0. \tag{5.9}$$

Let (M, J) be any compact complex manifold with vanishing first Chern class.

Lemma 5.2.1 ([39, Lemma 2.2])**.** *If an* $\mathrm{SU}(n)$*-structure* (Ψ, ω) *on* M *satisfies* (5.9)*, then the Bismut connection*

$$\nabla^+ = \nabla^g - \frac{1}{2} g^{-1} d^c\omega, \tag{5.10}$$

where ∇^g *is the Levi-Civita connection of* g*, has holonomy contained in* $\mathrm{SU}(n)$*.*

Proof. By the holonomy principle, it is enough to prove that

$$\nabla^+\Psi = 0. \tag{5.11}$$

Since θ_ω is closed, given $p \in M$ there exists a smooth local function ϕ such that $\theta_\omega = d\phi$ around p. Then, by the first equation in (5.9), we have that $\Omega := e^{-\phi}\Psi$ is closed. Indeed,

$$\begin{aligned} d\Omega &= e^{-\phi}(-d\phi) \wedge \Psi + e^{-\phi} d\Psi \\ &= e^{-\phi}((-d\phi + \theta_\omega) \wedge \Psi) = 0. \end{aligned}$$

Hence, Ω provides an holomorphic trivialization of $\mathcal{K}_M$ around p, and the metric induces by g in $\mathcal{K}_M$ is $\|\Omega\|^2_\omega$. In this trivialization, the Chern connection ∇^c on $\mathcal{K}_M$ induced by ω is given by

$$\nabla^c = d + 2\partial \log \|\Omega\|_\omega = d - 2\partial\phi,$$

since (5.6) implies $\phi = -\log \|\Omega\|_\omega$. Now, the proof follows using Gauduchon's formula [42, Equation (2.7.6)], relating ∇^c with the connection induced by ∇^+ on the canonical bundle

$$\nabla^c\Psi = \nabla^+\Psi + id^*\omega \otimes \Psi,$$

which implies (5.11) around p as desired. $\square$

We want to obtain via (5.9) that M admits a unique maximal holomorphic atlas for which the Jacobian of any change of coordinates has constant determinant.

Lemma 5.2.2. *Let M be a $2n$-dimensional smooth manifold for which we have (J, ω) a hermitian structure, and (Ψ, ω) satisfying* (5.6) *and* (5.9)*. Then, M admits a unique maximal holomorphic atlas such that for $U \subseteq M$ open set we have that*

$$\Psi|_U = e^{\phi_U} dz_1 \wedge \cdots \wedge dz_n,$$

where $\theta_{\omega|_U} = d\phi_U$. Consequently, the holomorphic Jacobian of any change of coordinates in this atlas has constant determinant.

Proof. We will keep the notations used for the proof of Lemma 5.2.1. We will prove that around each $p \in M$ there exists a neighbourhood $U \subseteq M$, and an holomorphic coordinate patch $\varphi \colon U \longrightarrow \mathbb{C}^n$, with functions $z_1, \ldots, z_n$ such that

$$\Psi|_U = e^{\phi_U} dz_1 \wedge \cdots \wedge dz_n,$$

where

$$\theta_\omega|_U = d\phi_U. \tag{5.12}$$

Fix an holomorphic atlas on M, and take $U \subseteq M$ a coordinate domain. By hypothesis, there exists a function $\phi_U \in \mathcal{C}^\infty(U)$ satisfying (5.12). By the first equation in (5.9), the following local $(n, 0)$-form

$$\Omega_U = e^{-\phi_U} \ \Psi|_U \tag{5.13}$$

is holomorphic. So, by the holomorphic Darboux Theorem, we can construct a new atlas associating to each $\varphi \colon U \longrightarrow V \subseteq \mathbb{C}^n$ coordinate patch a new one $\widetilde{\varphi} \colon \widetilde{U} = U \longrightarrow \widetilde{V} \subseteq \mathbb{C}^n$ such that $\Omega_{\widetilde{U}} = d\widetilde{z}_1 \wedge \cdots \wedge d\widetilde{z}_n$, where $\widetilde{z}_1, \ldots, \widetilde{z}_n$ are the new coordinates. We will work with this atlas. Let $\varphi \colon U \longrightarrow V \subseteq \mathbb{C}^n$ and $\varphi' \colon U' \longrightarrow V' \subseteq \mathbb{C}^n$ be two coordinate patches from this atlas, with coordinates $z_1, \ldots, z_n$ and $z'_1, \ldots, z'_n$, respectively, and the transition map

$$\psi = \varphi' \circ \varphi^{-1} \colon \varphi(U \cap U') \longrightarrow \varphi'(U \cap U').$$

The holomorphic volume form satisfies $\Omega_U = dz_1 \wedge \cdots \wedge dz_n$, and $\Omega_{U'} = dz'_1 \wedge \cdots \wedge dz'_n$. Notice that (5.13) still holds in this atlas, for the corresponding local potential ϕ_U of θ_ω. This follows as a consequence that Ψ and θ_ω are globally defined on this atlas. Then,

$$d\left(\phi_U - \phi_{U'}\right) = d\phi_U - d\phi_{U'} = \theta_\omega|_U - \theta_\omega|_{U'} = 0, \ \text{in } U \cap U',$$

so $c := \phi_U - \phi_{U'} \in \mathbb{C}$ in $U \cap U'$. Therefore, in $U \cap U'$ we have the following

$$\begin{aligned}
\Omega_{U'} &= \det\left(d\psi\right) \Omega_U = \det\left(d\psi\right) dz_1 \wedge \cdots \wedge dz_n, \\
\Omega_{U'} &= e^{-\phi_{U'}} \ \Psi|_{U'} = e^{\phi_U - \phi_{U'}} e^{-\phi_U} \ \Psi|_U \\
&= e^c \Omega_U = e^c dz_1 \wedge \cdots \wedge dz_n,
\end{aligned}$$

where $\det\left(d\psi\right) = e^c \in \mathbb{C}$ is the determinant of the Jacobian of the change of coordinates ψ considered above, which is constant as desired. Finally, if we have two atlas as above, the union is again an atlas satisfying the same properties. This ensures the uniqueness. □

The constructed atlas will be fundamental for our purposes.

Remark 5.2.3. Let $(g, H, A, \varphi, \omega, \Psi)$ be a solution to the F-term and D-term conditions. Associated to a solution (ω, Ψ) of (5.9), by Lemma 5.2.2, there exists a unique maximal atlas of holomorphic coordinates such that

$$\Psi|_U = e^{f_\omega} dz_1 \wedge \cdots \wedge dz_n, \quad \text{where } f_\omega := -\log \|\Omega\|_\omega,$$

for $U \subseteq M$ open, with $\{z_j\}_{j=1}^n \subseteq \mathcal{C}^\infty(M)$ local holomorphic coordinates.

5.3 The Torsion Bi-vector

To finish the chapter, we will define a bi-vector, canonically associated to any hermitian structure. This one will play an important role in the construction of our SUSY vertex algebra embeddings in Chapter 10. Let (M, g, J) be a $2n$-dimensional complex manifold with hermitian structure g, fow which the associated almost Kähler form is $\omega = g(J\cdot, \cdot)$. Now, for $v \in \Gamma(T^{0,1}M)$, consider

$$\left(g^{-1} \otimes g^{-1}\right)(\iota_v i\partial\omega) \in \Gamma\left(\Lambda^2 T^{0,1}M\right),$$

which is a bi-vector field of type $(0, 2)$. Then, we can take the Schouten bracket

$$\left[w, \left(g^{-1} \otimes g^{-1}\right)(\iota_v i\partial\omega)\right] \in \Gamma(\Lambda^2 TM \otimes \mathbb{C}),$$

for $w \in \Gamma(T^{1,0}M)$, and its $(0, 2)$ component

$$\left[w, \left(g^{-1} \otimes g^{-1}\right)(\iota_v i\partial\omega)\right]^{0,2} \in \Gamma(\Lambda^2 T^{0,1}M),$$

Given local holomorphic coordinates around a point, we define

$$\sigma_\omega := \sum_{k=1}^n \left[g^{-1} d\overline{z}_k, \left(g^{-1} \otimes g^{-1}\right)\left(\iota_{\frac{\partial}{\partial \overline{z}_k}} \partial\omega\right)\right]^{0,2}. \tag{5.14}$$

Lemma 5.3.1. *The expression* (5.14) *is independent of the choice of local holomorphic coordinates. In consequence, it defines a bi-vector field of type* $(0, 2)$*, that is,*

$$\sigma_\omega \in \Gamma\left(\Lambda^2 T^{0,1}M\right).$$

Furthermore,

$$(\sigma_\omega)_{\overline{ij}} = i \sum_{k=1}^n g^{-1} d\overline{z}_k \left(\partial\omega\left(g^{-1} d\overline{z}_i, g^{-1} d\overline{z}_j, \frac{\partial}{\partial \overline{z}_k}\right)\right), \quad \textit{for } i, j \in \{1, \ldots, n\}. \tag{5.15}$$

Proof. The first part is immediate by change of holomorphic coordinates, since locally

$$\sigma_\omega = \sum_{m,l,p,k} \sum_{i<j} g^{p\overline{k}} \frac{\partial}{\partial z_p} \left(g^{m\overline{i}} g^{l\overline{j}} (\partial\omega)_{ml\overline{k}}\right) \frac{\partial}{\partial \overline{z}_i} \wedge \frac{\partial}{\partial \overline{z}_j},$$

where the sums are taken from 1 to n. The second part follows clearly using (5.14). □

We will prove in Chapter 10 that we are able to construct embeddings of SUSY vertex algebras when the F-term and D-term conditions are satisfied, provided that $\sigma_\omega = 0$.

Chapter 6

The Killing Spinor Equations on Courant Algebroids

Now, we will introduce the *Killing spinor equations in generalized geometry*, following [36]. Generalized geometry is a geometric framework orginally introduced by Hitchin and Gualtieri [50, 57], which puts vectors and covector on a manifold on equal footing. It has been proved to be very useful in understanding field equations in supergravity. Roughly speaking, the Killing spinor equations in generalized geometry amount to suplement (5.3) with a Bianchi identity for H and A, and to prove that the corresponding equations are natural on a *Courant algebroid* associated to (H, A). The Bianchi identity has the effect of rigidifying these equations, and gives rise to the *twisted Hull-Strominger system.*

6.1 Basics on Courant Algebroids

Let M be any n-dimensional smooth manifold. The following notion was first given by Liu-Weinstein-Xu in [71] as an axiomatization of the natural structure on the direct sum of vector fields and the space of 1-forms introduced by Courant in [19].

Definition 6.1.1. [71] A *Courant algebroid* is a vector bundle $E \longrightarrow M$ endowed with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, the *Dorfman bracket* $[\cdot, \cdot]$ on $\Gamma(E)$, and a bundle map $\pi\colon E \longrightarrow TM$ called the *anchor* such that satisfies the following axioms:

(1) (Jacobi identity) $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$ for $a, b, c \in \Gamma(E)$.

(2) (π is an homomorphism) $\pi [a, b] = [\pi(a), \pi(b)]$ for $a, b \in \Gamma(E)$.

(3) ($[\cdot, \cdot]$ is a differential) $[a, fb] = f [a, b] + \pi(a)(f)b$ for $a, b \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$.

(4) (Compatibility) $\pi(a) \langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$ for $a, b, c \in \Gamma(E)$.

(5) (Quasiantisymmetry) $[a, b] + [b, a] = \mathcal{D} \langle a, b\rangle$ for $a, b \in \Gamma(E)$.

Here, the map $\mathcal{D}\colon \mathcal{C}^\infty(M) \longrightarrow \Gamma(E)$ denotes the composition of three maps: the exterior differential $d\colon \Omega(M) \longrightarrow \Omega(M)$ acting on functions, the map $\pi^*\colon T^*M \longrightarrow E^*$ and the isomorphism $E^* \cong E$ provided by the non-degenerate symmetric pairing $\langle\cdot,\cdot\rangle$. So,

$$\langle \mathcal{D}f, a\rangle = \pi(a)(f), \quad \text{for } a \in \Gamma(E); f \in \mathcal{C}^\infty(M).$$

In particular, it is satisfied $\pi \circ \mathcal{D} = 0$. In other words, $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$ for $f, g \in \mathcal{C}^\infty(M)$.

Example 6.1.2 ([19, 38] ***H*-Twisted Courant Algebroids**)**.** Let $H \in \Omega^3(M)$ be a closed 3-form on a manifold M. Then, the data $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ defined, for $X + \zeta, Y + \eta \in TM \oplus T^*M$, by

$$\begin{aligned}
\langle X + \zeta, Y + \eta\rangle &:= \frac{1}{2}\left(\eta(X) + \zeta(Y)\right),\\
[X + \zeta, Y + \eta] &:= [X, Y] + L_X\eta - \iota_Y d\zeta + \iota_Y\iota_X H,\\
\pi\colon TM \oplus T^*M \longrightarrow TM,& \quad \pi\left(X + \zeta\right) := X,
\end{aligned}$$

defines a Courant algebroid on M, usually called the *H-twisted Courant algebroid* (for $H = 0$, this is simply known as the *standard Courant algebroid*). In this explicit situation,

$$\mathcal{D}f = 2df, \quad \text{for } f \in \mathcal{C}^\infty(M).$$

Remark 6.1.3 (**Quadratic Lie Algebras from CAs**)**.** A Courant algebroid over a point (localization) is equivalent to a real (even) quadratic Lie algebra (Definition 1.4.1).

Motivated by the previous examples, we define two special types of Courant algebroids that will play an important role in the present thesis, which firstly appeared in [82].

Definition 6.1.4. [82] A *Courant algebroid* E is said to be *transitive* if the anchor map π is surjective. In addition, E is said *exact* if the kernel of the anchor coincides with the image of π^*. In other words, if it fits into an exact sequence of vector bundles

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0.$$

For E any Courant algebroid, an *isotropic splitting* is a section $\sigma\colon TM \longrightarrow E$ such that the image $\sigma(TM) \subseteq E$ is isotropic with respect to $\langle\cdot,\cdot\rangle$. When E is exact, we can choose σ as above, and define $\left(TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_{H_\sigma}, \pi\right)$ for $H_\sigma \in \Omega^3(M)$ closed defined by

$$H_\sigma(X, Y, Z) := 2\left\langle [\sigma(X), \sigma(Y)], \sigma(Z)\right\rangle, \quad \text{for } X, Y, Z \in T^*M. \tag{6.1}$$

A detailed proof of the next result can be found in [38, Theorem 2.19].

Theorem 6.1.5 ([82])**.** *Given $H \in \Omega^3(M)$ closed, there exists a one-to-one correspondence between exact Courant algebroids and H-twisted Courant algebroids.*

Now, we introduce a special class of transitive Courant algebroids which will play a fundamental role. Let M be a complex manifold, and K compact Lie group with bi-invariant non-degenerate pairing $\langle\cdot,\cdot\rangle : \mathfrak{k} \otimes \mathfrak{k} \longrightarrow \mathbb{R}$. Given $P \longrightarrow M$ a principal K-bundle, let A be any principal connection on P with associated curvature $F_A \in \Omega^2(X, \operatorname{ad} P)$. For such a connection A, we can choose $H \in \Omega^3(M)$ such that satisfies the *Bianchi identity*

$$dH + \langle F_A \wedge F_A\rangle = 0. \tag{6.2}$$

Proposition 6.1.6 ([4, Proposition 3.2],[34, Proposition 2.4]). *Let (H, A) be a pair that solves (6.2). Now, consider the vector bundle $E_{H,A} := TM \oplus \operatorname{ad} P \oplus T^*M$ endowed, for $X + r + \zeta, Y + t + \eta \in E_{H,A}$, with the symmetric pairing*

$$\langle X + r + \zeta, Y + t + \eta\rangle := \frac{1}{2}(\eta(X) + \zeta(Y)) + \langle r, t\rangle\,, \tag{6.3}$$

the bracket

$$\begin{aligned} [X + r + \zeta, Y + t + \eta]_{H,A} :=\; & [X, Y] + L_X\eta - \iota_Y d\zeta + \iota_Y\iota_X H \\ & - F_A(X, Y) + 2\langle \iota_X F_A, t\rangle - 2\langle \iota_Y F_A, r\rangle \\ & + 2\left\langle d^A r, t\right\rangle + d^A_X t - d^A_Y r - [r, t]\,, \end{aligned} \tag{6.4}$$

and the canocical projection

$$\pi\colon E \longrightarrow TM, \quad \pi(X + r + \zeta) := X.$$

Then, the data $(E_{H,A}, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_{H,A}, \pi)$ determines a transitive Courant algebroid.

Definition 6.1.7. [37] Let (H, A) be a pair that solves (6.2). The transitive Courant algebroid $E_{H,A}$ constructed as explained in Proposition 6.1.6 is known as the *string Courant algebroid* associated to the given pair (H, A).

6.1.1 Generalized Metrics on String Courant Algebroids

We recall the notion of generalized metric on Courant algebroids, which generalizes the concept of Riemannian metric on a manifold [48, 57]. Note that reductions of the structure group on the standard Courant algebroid give interesting geometries, since it reduces the structure group to $\mathrm{O}(n, n)$. We want to study a further reduction to $\mathrm{O}(n) \times \mathrm{O}(p, q)$.

Definition 6.1.8. [38] Let E be a general Courant algebroid over M smooth manifold. A *generalized metric* on E is given by an orthogonal decomposition

$$E = C_+ \oplus C_-$$

in subundles $C_\pm \subseteq E$ such that the restriction of $\langle\cdot,\cdot\rangle$ on E to each $C_\pm$ is non-degenerate. We say that a generalized metric is *Riemannian* if $\langle\cdot,\cdot\rangle|_{C_+}$ is positive-definite.

Lemma 6.1.9 ([34], Section 3.2). *A Riemannian generalized metric $C_\pm$ on E any transitive Courant algebroid is equivalent to either:*

(1) a reduction of the frames to a subgroup $\mathrm{O}(n) \times \mathrm{O}(p, q) \subseteq \mathrm{O}(n + p + q)$.

(2) an endomorphism $\mathcal{G} \in \Gamma(\operatorname{End} E)$ satisfying the following:

- *$\langle \mathcal{G}a, \mathcal{G}b\rangle = \langle a, b\rangle$ for $a, b \in \Gamma(E)$.*
- *$\langle \mathcal{G}a, b\rangle = \langle a, \mathcal{G}b\rangle$ for $a, b \in \Gamma(E)$.*
- *The bilinear pairing $\langle \mathcal{G}\cdot,\cdot\rangle$ is positive-definite on $C_+ = \{a \in E \mid \mathcal{G}a = a\}$, and has signature (p, q) on $C_- = \{a \in E \mid \mathcal{G}a = -a\}$.*

We denote by $\mathcal{G}(E)$ the space of generalized metrics for E transitive Courant algebroid.

Remark 6.1.10. Given $C_\pm$ a generalized metric on E exact Courant algebroid,

$$C_\pm = \{X \pm g(X) \mid X \in TM\}.$$

In particular, we have the natural morphisms

$$\begin{aligned}
&\pi_\pm \colon E \longrightarrow C_\pm, \quad \pi_\pm(X+\zeta) = \frac{1}{2}\left(X \pm g^{-1}(\zeta) + \zeta \pm g(X)\right), \\
&\sigma_\pm \colon TM \longrightarrow C_\pm, \quad \sigma_\pm(X) = X \pm g(X),
\end{aligned}$$

inducing isomorphisms $\pi|_{C_\pm} : TM \cong C_\pm$.

Let next study generalized metrics for string Courant algebroids. So, let (H, A) be a pair as above solving (6.2). Fix g a usual Riemannian metric on M. Then, we can construct canonically a generalized metric $C_\pm$ on $E_{H,A}$ by setting

$$\begin{aligned}
C_+ &:= \{X + g(X) \mid X \in TM\}, \\
C_- &:= \{X - g(X) + r \mid X \in TM, r \in \operatorname{ad} P\}.
\end{aligned} \tag{6.5}$$

Remark 6.1.11. More invariantly, if we fix a string Courant algebroid of type $E_{H,A}$ as above, and we consider a generalized metric $E_{H,A} = C_+ \oplus C_-$ such that

$$\pi|_{C_+} : C_+ \cong TM, \quad \langle\cdot,\cdot\rangle|_{C_+} > 0,$$

then this determines a pair (g', σ), where g' denotes a usual Riemannian metric on M and $\sigma \colon TM \longrightarrow E_{H,A}$ is an isotropic splitting such that determines another pair (H', A') as above satisfying (6.2). Even more, the isotropic splitting $\sigma \colon TM \longrightarrow E_{H,A}$ induces an isomorphism $E_{H,A} \cong E_{H',A'}$ for which the generalized metric $C_\pm \subseteq E_{H',A'}$ is (6.5).

6.1.2 Divergence Operators on String Courant Algebroids

Definition 6.1.12. [38] Let E be a Courant algebroid. A *divergence operator* on E is a differential operator $\operatorname{div} \colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$ satisfying the Leibniz rule

$$\operatorname{div}(fe) = \pi(e)(f) + f\operatorname{div}(e), \quad \text{for } e \in \Gamma(E), f \in \mathcal{C}^\infty(M).$$

Remark 6.1.13. By Leibniz rule, it is seen that the divergence operators on E form an affine space modeled on $\Gamma(E)$. Indeed, fixing div and div$'$ divergence operators, then

$$\operatorname{div}' = \operatorname{div} + \langle e, \cdot\rangle, \quad \text{for certain } e \in \Gamma(E).$$

We give a compatibility condition between generalized metrics and divergence operators.

Definition 6.1.14. [38] Given C_+ Riemannian canonical generalized metric on a string Courant algebroid E as in (6.5) such that determines the Riemannian metric g, we define its *Riemannian divergence* as

$$\operatorname{div}_0(e) := \frac{L_{\pi(e)}\mu^g}{\mu^g}, \quad \text{for } e \in \Gamma(E),$$

where μ^g is the associated Riemannian volume form of g Riemannian metric.

Definition 6.1.15. [38] For $\mathrm{div}\colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$ divergence operator on a string Courant algebroid E, we define the *Weyl structure* of div as the map

$$W\colon \begin{array}{ccc} \mathcal{G}(E) & \longrightarrow & \Gamma(E) \\ \mathcal{G} & \mapsto & \mathrm{div} - \mathrm{div}_0 \end{array}.$$

We are ready to introduce the desired condition from the notion of *infinitesimal isometry*. For $\mathcal{G}$ generalized metric on E string Courant algebroid, define, for each $e \in \Gamma(E)$,

$$[e, \mathcal{G}] \in \mathrm{End}\,(\Gamma(E)) \;\text{ by }\; [e,\mathcal{G}]\,(e') := \left[e, \mathcal{G}e'\right] - \mathcal{G}\left[e, e'\right], \quad \text{for } e' \in \Gamma(E).$$

This defines a tensor, and we say that $e \in \Gamma(E)$ is an *infinitesimal isometry* if $[e, \mathcal{G}] = 0$.

Definition 6.1.16. [38] Let E be a string Courant algebroid. A pair $(\mathcal{G}, \mathrm{div})$ is said *compatible* if $W(\mathcal{G})$ is an infinitesimal isometry. Furthermore, it is *closed* if $\pi\left(\mathcal{G}\right) = 0$.

Remark 6.1.17 ([38, Lemma 2.50])**.** For (C_+, div) a compatible pair on a string Courant algebroid E, we have that $e = X + \zeta + r$ in the splitting σ determined by C_+, and the condition infinitesimal isometry reads

$$L_X g = 0, \quad d^A r = -\iota_X F_A, \quad d\zeta = \iota_X H - 2\left\langle F_A, r\right\rangle. \tag{6.6}$$

In particular, being closed corresponds to $X = r = 0$ and $d\zeta = 0$ in the identities above.

6.2 Killing Spinors on Courant Algebroids

We introduce now the equations of our main interest in the most general set-up. These equations were introduced in [36, Definition 5.6] generalizing the ones from Chapter 5.

6.2.1 Generalized Connections and Dirac-type Operators

First, we will start the study of connections in the setting of generalized geometry.

Definition 6.2.1. [38] A *generalized connection* D on E general Courant algebroid is a first-order differential operator $D\colon \Gamma(E) \longrightarrow \Gamma(E^* \otimes E)$ satisfying the Leibniz rule, and certain compatibility condition with the pairing $\langle\cdot,\cdot\rangle$. Explicitly, using the notation

$$D_a b := \langle a, Db\rangle\,, \quad \text{for } a, b \in \Gamma(E),$$

we have, for $a, b, c \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$, that

$$D_a\left(fb\right) = \pi(a)(f)b + fD_a b, \;\text{ and } \pi(a)\left\langle b, c\right\rangle = \left\langle D_a b, c\right\rangle + \left\langle b, D_a c\right\rangle.$$

Remark 6.2.2. If $\mathcal{D}$ denotes the set of generalized connections on E, one can see that it has structure of affine space modelled on the vector space $\Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$, where $\mathfrak{o}(E)$ denotes the bundle of skew-symmetric endomorphisms of E with respect to the bilinear form $\langle\cdot,\cdot\rangle$. This works as follows: for a standard orthogonal connection ∇^E on $(E, \langle\cdot,\cdot\rangle)$, we can construct a generalized connection on E by

$$D_a b := \nabla^E_{\pi(a)} b, \quad \text{for } a, b \in \Gamma(E). \tag{6.7}$$

Moreover, any other generalized connection on E is given by $D+\chi$, for $\chi \in \Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$.

Definition 6.2.3. [38, 49] Given E a Courant algebroid and D a generalized connection on E, the *torsion* T_D of D is defined by

$$T_D(a,b,c) := \langle D_a b - D_b a - [a,b], c\rangle + \langle D_c a, b\rangle, \quad \text{for } a,b,c \in \Gamma(E).$$

Using the axioms of Courant algebroids and connections, then $T_D \in \Gamma\left(\Lambda^3 E^*\right)$.

Fix $T \in \Gamma(\Lambda^3 E^*)$, and consider the set $\mathcal{D}^T$ of generalized connections on E with fixed torsion T. Let $\sigma(a,b,c)$ denote the sum over all the cyclic permutations on $a,b,c \in \Gamma(E)$. Then, for $D \in \mathcal{D}^T$ and $\chi \in \Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$, the condition for $D' = D + \chi$ to be in $\mathcal{D}^T$ is given by

$$\sum_{\sigma(a,b,c)} \langle \chi_a b, c\rangle = 0, \quad \text{for } a,b,c \in \Gamma(E).$$

Lemma 6.2.4 ([38, Lemma 3.6])**.** *Let D be any connection with torsion T. Then, the space $\mathcal{D}^T$ is an affine space modeled on the vector space of mixed symmetric 3-tensors*

$$\Sigma = \left\{ \chi \in \Gamma\left(E^{\otimes 3}\right) \;\middle|\; \chi(a,b,c) = -\chi(a,b,c), \sum_{\sigma(a,b,c)} \chi(a,b,c) = 0 \right\}. \tag{6.8}$$

From this, we obtain that the space Σ admits a canonical splitting $\Sigma = \Sigma_0 \oplus \Gamma(E)$, where $e \in \Gamma(E)$ corresponds to the mixed symmetric tensor χ^e defined by

$$\chi^e(a,b,c) = \langle a,b\rangle \langle e,c\rangle - \langle e,b\rangle \langle a,c\rangle, \quad \text{for } a,b,c \in \Gamma(E),$$

and the orthogonal complement of $\Gamma(E)$ is given by

$$\Sigma_0 = \left\{ \chi \in \Sigma \;\middle|\; \sum_{j=1}^{r_E} \chi\left(a_j, \widetilde{a}_j, \cdot\right) = 0 \right\}. \tag{6.9}$$

Here, the value $r_E \in \mathbb{N}$ denotes the rank of E, and $\{a_j\}_{j=1}^{r_E}$ is an orthogonal local frame for E, where $\{\widetilde{a}_j\}_{j=1}^{r_E}$ are the orthogonal local frame of E defined to be its dual. That is,

$$\langle a_j, \widetilde{a}_k\rangle = \delta_j^k, \quad \text{for } j,k \in \{1,\ldots,r_E\}.$$

More explicitly, given $\chi \in \Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$, there is a unique decomposition

$$\chi = \chi_0 + \chi^e, \tag{6.10}$$

where $\chi_0 \in \Gamma\left(E^* \otimes \mathfrak{o}(E)\right)$ is such that

$$\sum_{j=1}^{r_E} \left(\chi_0\right)_{a_j} \widetilde{a}_j = 0,$$

and

$$\chi^e_a b = \langle a,b\rangle\, e - \langle e,b\rangle\, a, \quad \text{for } a,b \in \Gamma(E)$$

with

$$e = \frac{1}{r_E - 1} \sum_{j=1}^{r_E} \chi_{a_j} \widetilde{a}_j.$$

It follows by construction that $\chi - \chi^e \in \Sigma_0$.

Remark 6.2.5. [36] The E^*-valued skew-symmetric endomorphism χ^e in the decomposition (6.10) is reminiscent of the "1-form valued Weyl endomorphisms" in conformal geometry, which appear in the variation of a metric connection with fixed torsion upon a conformal change of the metric. Similarly, these E^*-valued *Weyl endomorphisms* χ^e enable us to deform a generalized connection D with fixed torsion T inside $\mathcal{D}^T$. It will be important to study the interaction between this space $\mathcal{D}^T$ and divergence operators div. This allows us to "gauge-fix" the Weyl degrees of freedom in the space $\mathcal{D}^T$ corresponding to $\Gamma(E)$ in the splitting $\Sigma = \Sigma_0 \oplus \Gamma(E)$. We will call this procedure *Weyl gauge fixing.*

Definition 6.2.6. [38] The *divergence operator of a generalized connection* D on E is

$$\operatorname{div}_D(a) := \operatorname{tr}(Da) \in \mathcal{C}^\infty(M), \quad \text{for } a \in \Gamma(E).$$

Lemma 6.2.7 ([36, Lemma 2.4])**.** *Fix* $T \in \Gamma(\Lambda^3 E^*)$*, and let* $\operatorname{div}\colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$ *be a divergence operator on* E*. Then,*

$$\mathcal{D}^T(\operatorname{div}) = \{D \in \mathcal{D} \mid T_D = T, \operatorname{div}_D = \operatorname{div}\} \subseteq \mathcal{D}^T$$

is an affine space modelled on Σ_0*, as defined in* (6.9)*.*

We introduce next the natural compatibility condition between generalized connections and generalized metrics.

Definition 6.2.8. [38] Let E be a Courant algebroid, and fix $E = C_+ \oplus C_-$ generalized metric. We say that D generalized connection on E is *compatible* with $C_+ \subseteq E$ if

$$D(\Gamma(C_\pm)) \subseteq \Gamma(E^* \otimes C_\pm).$$

Remark 6.2.9. Let $\mathcal{D}(C_\pm)$ be the space of $\mathcal{G}$-compatible generalized connections. It defines an affine space modeled on $\Gamma(E^* \otimes \mathfrak{o}(C_+)) \oplus \Gamma(E^* \otimes \mathfrak{o}(C_-))$. Consequently, any $D \in \mathcal{D}(C_\pm)$ splits into four first-order differential operators satisfying the Leibniz rule

$$D_-^+ \colon \Gamma(C_+) \longrightarrow \Gamma(C_-^* \otimes C_+), \quad D_+^- \colon \Gamma(C_-) \longrightarrow \Gamma(C_+^* \otimes C_-),$$

$$D_+^+ \colon \Gamma(C_+) \longrightarrow \Gamma(C_+^* \otimes C_+), \quad D_-^- \colon \Gamma(C_-) \longrightarrow \Gamma(C_-^* \otimes C_-).$$

The operators $D_\pm^\pm \colon \Gamma(C_\pm) \longrightarrow \Gamma(C_\pm^* \otimes C_\pm)$ are said of *pure-type*, whereas the operators $D_\mp^\pm \colon \Gamma(C_\pm) \longrightarrow \Gamma(C_\mp^* \otimes C_\pm)$ are called *mixed-type.* Moreover, we will say that the torsion T_D is of *pure-type* when $T_D \in \Gamma\left(\Lambda^3 C_+^* \oplus \Lambda^3 C_-^*\right)$.

Our next result shows that the mixed-type operators are fixed, if we vary a generalized connection D inside $\mathcal{D}(\mathcal{G})$ while preserving the torsion. Furthermore, when the torsion is of pure-type, these operators are uniquely determined by $C_\pm$ and $[\cdot, \cdot]$.

Lemma 6.2.10 ([36, Lemma 3.2])**.** *Fix $D \in \mathcal{D}(C_+)$ with torsion $T_D \in \Gamma(\Lambda^3 E^*)$.*

- *If $D' \in \mathcal{D}(C_+)$ and $T_{D'} = T_D$, then $(D')^{\pm}_{\mp} = D^{\pm}_{\mp}$.*
- *Furthermore, T is of pure-type if and only if the mixed-type operators $D^{\pm}_{\mp}$ are*

$$D_{a_-} b_+ = [a_-, b_+]_+ , \quad D_{a_+} b_- = [a_+, b_-]_- , \quad \textit{for } a, b \in \Gamma(E). \tag{6.11}$$

Now, we must analize the space of torsion-free $\mathcal{G}$-compatible generalized connections

$$\mathcal{D}^0(C_\pm) := \mathcal{D}(C_\pm) \cap \mathcal{D}^0 = \{D \in \mathcal{D}(C_\pm) \mid T_D = 0\} .$$

Proposition 6.2.11 ([36, Proposition 3.3] **Many Levi-Civita GCs**)**.** *Let $C_\pm$ be any generalized metric on E Courant algebroid. Then, we have that $\mathcal{D}^0(C_\pm) \neq \emptyset$.*

Proof. We construct D a C_+-compatible generalized connection of pure-type torsion on E, defining $D^{\pm}_{\mp}$ by (6.11), and $D^{\pm}_{\pm}$ by choosing $\nabla^{\pm}$ metric connections and using (6.7). Finally,

$$D^0 := D - \frac{1}{3} T_D,$$

where we use the metric $\langle \cdot, \cdot \rangle$ to regard $T_D \in \Gamma(V_+^* \otimes \mathfrak{o}(V_+)) \oplus \Gamma(V_-^* \otimes \mathfrak{o}(V_-))$. Crucially, the pure-type condition on T_D implies that D^0 is C_+-compatible. □

The space $\mathcal{D}^0\left(C_\pm\right)$ forms an affine space, modelled on the pure-type mixed symmetric 3-tensors $\Sigma_+ \oplus \Sigma_-$, where $\Sigma^\pm = \Gamma\left(C_\pm^{\otimes 3}\right) \cap \Sigma$, for Σ as in (6.8). So, there are canonical splittings

$$\Sigma^\pm = \Sigma_0^\pm \oplus \Gamma(C_\pm), \tag{6.12}$$

where the first summand corresponds to "trace-free" elements, in analogy with (6.9). To see this, denote by $r_\pm$ the rank of $C_\pm$, and consider the orthogonal dual local frames

$$\left\{ a_j^\pm, \widetilde{a}_j^\pm \right\}_{j=1}^{r_\pm} \subseteq \Gamma(C_\pm).$$

Then, (6.12) corresponds to

$$\chi^\pm = \chi_0^\pm + \chi_\pm^{e_\pm},$$

for a general element $\chi^\pm \in \Gamma\left(C_\pm^* \otimes \mathfrak{o}(C_\pm)\right)$, where $\chi_0^\pm$ is such that

$$\sum_{j=1}^{r_\pm} \left(\chi_0^\pm\right)_{a_j^\pm} \widetilde{a}_j^\pm = 0,$$

and

$$\left(\chi_\pm^{e_\pm}\right)_{a_\pm} b_\pm = \langle a_\pm, b_\pm \rangle\, e_\pm - \langle e_\pm, b_\pm \rangle\, a_\pm, \quad \text{for } a, b \in \Gamma(E) \tag{6.13}$$

with

$$e_\pm = \frac{1}{r_\pm - 1} \sum_{j=1}^{r_\pm} \chi^\pm_{a_j} \widetilde{a}_j^\pm .$$

At last, we will introduce the *space of torsion-free metric connections compatible with a fixed divergence operator* $\operatorname{div}\colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$. So, given $(C_\pm, \operatorname{div})$ a pair formed by a generalized metric and a divergence operator, we define

$$\mathcal{D}^0\left(C_\pm, \operatorname{div}\right) = \left\{D \in \mathcal{D}\left(C_\pm\right) \mid T_D = 0, \operatorname{div}_D = \operatorname{div}\right\}.$$

By Lemma 6.2.7, this space forms an affine space modeled on $\Sigma_0^+ \oplus \Sigma_0^-$.

Lemma 6.2.12 ([38, Lemma 3.17])**.** *Let $C_\pm$ be any generalized metric on E Courant algebroid, and* div *a divergence operator. Then, we have that $\mathcal{D}^0\left(C_\pm, \operatorname{div}\right) \neq \emptyset$. Furthermore, an element $D \in \mathcal{D}^0\left(C_\pm, \operatorname{div}\right)$ is given by the formula*

$$D = D_B - \frac{1}{3} T_{D_B} + \frac{1}{r_\pm - 1}\left(\chi_+^{e_+} + \chi_-^{e_-}\right),$$

where $\operatorname{div}^{\mathcal{G}} - \operatorname{div} = \langle e_+, \cdot\rangle - \langle e_-, \cdot\rangle$*, for $e_\pm \in \Gamma(C_\pm)$, where $\chi_\pm^{e_\pm}$ is as defined in* (6.13).

By Lemma 6.2.12, the freedom in the previous construction of torsion-free generalized connections compatible with $C_\pm$ corresponds to a choice of the pure-type operators

$$D_+^+\colon \Gamma(C_+) \longrightarrow \Gamma(C_+^* \otimes C_+), \quad D_-^-\colon \Gamma(C_-) \longrightarrow \Gamma(C_-^* \otimes C_-).$$

It was proved in [36] that one can define a pair of Dirac-type operators that are independent of these choices, once we have fixed a divergence operator div on the Courant algebroid E (see Lemma 6.2.7). Fixed $C_\pm$ a generalized metric, let $\mathrm{Cl}(C_\pm)$ be the bundles of the Clifford algebras of $C_\pm$. We assume that $\mathrm{Cl}(C_\pm)$ admit irreducible Clifford modules $S_\pm$ such that $\Gamma(\operatorname{End} S_\pm) \cong \Gamma(\mathrm{Cl}(C_\pm))$. Furthermore, we assume that the line bundles

$$\left(\det S_\pm^*\right)^{\frac{1}{r_{S_\pm}}}$$

exist, and let us denote

$$\mathcal{S}_\pm = S_\pm \otimes \left(\det S_\pm^*\right)^{\frac{1}{r_{S_\pm}}},$$

where $r_{S_\pm}$ denotes the rank of $S_\pm$. With these assumptions, for any choice of a connection $D \in \mathcal{D}^0(C_\pm)$, the operators $D_\pm^\pm$ induce canonically the spin connections

$$D_+^{S_+}\colon \Gamma(S_+) \longrightarrow \Gamma(C_+^* \otimes S_+), \quad D_-^{S_-}\colon \Gamma(S_-) \longrightarrow \Gamma(C_-^* \otimes S_-).$$

In these circumstances, we can introduce the desired operators as follows.

Definition 6.2.13. [36] Any generalized connection $D \in \mathcal{D}^0(C_\pm)$ determines a *pair of Dirac-type operators*

$$\not{D}^+\colon \Gamma(\mathcal{S}_+) \longrightarrow \Gamma(\mathcal{S}_+), \quad \not{D}^-\colon \Gamma(\mathcal{S}_-) \longrightarrow \Gamma(\mathcal{S}_-),$$

given explicitly by

$$\not{D}^\pm \alpha = \sum_{j=1}^{r_\pm} \widetilde{a}_j^\pm \cdot D_{a_j^\pm}^{S_\pm} \alpha, \quad \text{for } \alpha \in \Gamma(\mathcal{S}_\pm),$$

where we choose the orthogonal dual local frames $\left\{a_j^\pm, \widetilde{a}_j^\pm\right\}_{j=1}^{r_\pm} \subseteq \Gamma\left(C_\pm\right)$.

Lemma 6.2.14 ([36, Lemma 3.4])**.** *$\not{D}^\pm$ are independent of the chosen $D \in \mathcal{D}^0\left(C_\pm, \operatorname{div}\right)$.*

Before ending, notice that the canonical operators $D^{\pm}_{\mp}$ associated to a generalized metric $C_{\pm} \subseteq E$ (see Lemma 6.2.10) induce the spin connections

$$D^{S_+}_{-} : \Gamma(S_+) \longrightarrow \Gamma(C^*_- \otimes S_+), \quad D^{S_-}_{+} : \Gamma(S_-) \longrightarrow \Gamma(C^*_+ \otimes S_-).$$

We are ready to introduce the following notion of Killing spinors on Courant algebroids.

Definition 6.2.15. [36] Let E be a Courant algebroid over a spin manifold M. We will say that a triple $(C_+, \mathrm{div}, \eta)$ given by $C_+ \subseteq E$ a Riemannian generalized metric with spinor bundle $S_{\pm}$, a divergence operator $\mathrm{div}\colon \Gamma(E) \longrightarrow \mathcal{C}^{\infty}(M)$, and a non-vanishing spinor $\eta_{\pm} \in \Gamma(\mathcal{S}_{\pm})$, is a *solution of the Killing spinor equations*, if

$$D^{S_{\pm}}_{\mp}\eta_{\pm} = 0, \tag{6.14}$$

$$\not{D}^{\pm}\eta_{\pm} = 0, \tag{6.15}$$

where operators $D^{S_{\pm}}_{\mp}$ and $\not{D}^{\pm}$ are defined above. If we have solutions in both C_+ and C_-, we will say that we have *pairs of solutions of the Killing spinor equations* on E.

Notice that, unlike (6.15), the equation (6.14) only depends on the pair (C_+, η). Indeed, by Lemma 6.2.12 and Lemma 6.2.14, the Dirac-type operators depend on div.

6.3 The Hull-Strominger System on Courant Algebroids

Now, we reformulate the Killing spinor equations introduced in Chapter 5 for string Courant algebroids. These will be related with the twisted Hull-Strominger system. Let M be an $2n$-dimensional smooth spin manifold. Let K be a compact Lie group, and fix $\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \longrightarrow \mathbb{R}$ bi-invariant non-degenerate pairing. Consider $p\colon P \longrightarrow M$ principal K-bundle, and A principal connection on P with associated curvature $F_A \in \Omega^2(M, \mathrm{ad}\, P)$ solving (6.2). The Killing spinor equations as introduced in Chapter 5 with these extra integrability conditions are motivated by the *Hull-Strominger system* [58, 87]. Its study was initiated by the works of Fu-Li-Yau [31, 32, 68].

Definition 6.3.1. [39] We say that a triple (Ψ, ω, A), given by an $\mathrm{SU}(n)$-structure (Ψ, ω) on M compact $2n$-dimensional complex manifold, with vanishing first Chern class, and A principal connection on a principal K-bundle $P \longrightarrow M$, is a solution to the *twisted Hull-Strominger system* if

$$\begin{aligned} F_A^{0,2} = 0, \qquad F_A \wedge \omega^{n-1} &= 0, \\ d\Psi - \theta_\omega \wedge \Psi &= 0, \\ d\theta_\omega &= 0, \\ dd^c\omega - \langle F_A \wedge F_A \rangle &= 0. \end{aligned} \tag{6.16}$$

Here θ_ω is the Lee form (5.4) of ω, and $d^c := i\left(\overline{\partial} - \partial\right)$ is the conjugate differential. In particular,

$$d^c\omega\,(\cdot, \cdot, \cdot) = -d\omega\,(J\cdot, J\cdot, J\cdot)\,,$$

where J denotes the complex structure on M. When $K = \{1\}$ (so, $F_A = 0$), we say that an $\mathrm{SU}(n)$-structure (Ψ, ω) on M is a solution to the *twisted Calabi-Yau equations* if

$$\begin{aligned} d\Psi - \theta_\omega \wedge \Psi &= 0, \\ d\theta_\omega &= 0, \\ dd^c\omega &= 0. \end{aligned} \tag{6.17}$$

We say that an almost hermitian structure (J, ω) is *pluriclosed* if $dd^c\omega = 0$,.

Remark 6.3.2. [39] A solution of the twisted Hull-Strominger with $[\theta_\omega] = 0$, is equivalent to a solution of the Hull-Strominger system [39]. If we have a solution to the twisted Calabi-Yau equations with $[\theta_\omega] = 0$, it determines a Kähler-Calabi-Yau structure on M.

These equations are really important for the present work, since these are equivalent to the Killing spinor equations under certain conditions. Let M be a smooth $2n$-dimensional spin manifold, and $P \longrightarrow M$ principal K-bundle with K compact Lie group. Remember that, in even dimensions, a spinor bundle for M decomposes as

$$\mathcal{S}_\pm = \mathcal{S}_\pm^+ \oplus \mathcal{S}_\pm^-,$$

where the factors correspond to irreducible spin representations. In these circunstances, we recover the next result as a consequence of Proposition 5.1.5 and Proposition 5.1.6.

Proposition 6.3.3 ([35, Theorem 1.2])**.** *A solution $(C_+, \mathrm{div}_+, \eta)$ to the Killing spinor equations on a transitive Courant algebroid E over a spin manifold M with (C_+, div_+) closed and $\eta \in \Gamma\left(\mathcal{S}_+^+\right)$ pure is equivalent to a solution (Ψ, ω, A) of the twisted Hull-Strominger system* (6.16)*. In particular, for $K = \{1\}$, a solution $(C_\pm, \mathrm{div}_\pm, \eta_\pm)$ of the Killing spinor equations on E exact Courant algebroid over a spin manifold M for $(C_\pm, \mathrm{div}_\pm)$ closed and $\eta \in \Gamma\left(\mathcal{S}_+^+\right)$ pure is equivalent to a solution for the twisted Calabi-Yau equations* (6.17)*. Here, notice that*

$$\mathrm{div}_\pm := \mathrm{div}|_{C_\pm}, \quad H = -d^c\omega, \quad \mathrm{div}_0 - \mathrm{div} = 2\theta_\omega.$$

Remark 6.3.4. Having a solution to the Killing spinor equations of the form $(C_\pm, \mathrm{div}, \eta)$ on a transitive Courant algebroid E over M is equivalent to Definition 5.1.1 when we add the integrability condition (6.2). Indeed, notice that we can construct generalized connections as above from (5.1) and (5.2) using (6.7), since

$$C_+ = \{X + g(X) \mid X \in TM\} \cong TM$$

in transitive Courant algebroids (see [36, Section 3.3] for details). Note that the condition for C_+ to admit an irreducible Clifford module implies that M admits a spin structure. So, if M is $2n$-dimensional and η is pure, we obtain an $\mathrm{SU}(n)$-structure. Now, suppose that we have pairs of solutions to the Killing spinor equations $(C_\pm, \mathrm{div}, \eta_\pm)$ on a exact Courant algebroid E over a smooth $2n$-dimensional manifold M. In this case, we can characterize pairs of equations as in Definition 5.1.1 doing the same as above, since we have isomorphisms $\pi_\pm \colon C_\pm \longrightarrow TM$, and

$$C_\pm = \{X \pm g(X) \mid X \in TM\} \cong TM, \quad \nabla^\pm = \nabla^g \pm \frac{1}{2}g^{-1}H, \quad \nabla^{\pm\frac{1}{3}} = \nabla^g \pm \frac{1}{6}g^{-1}H$$

makes sense in the exact case. In particular, we obtain an $\mathrm{SU}(n) \times \mathrm{SU}(n)$ structure.

6.3.1 The F-term Conditions on String Courant Algebroids

Let E be a transitive Courant algebroid over a $2n$-dimensional smooth manifold M endowed with a complex structure J. We can consider the complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ of our Courant algebroid, which is the $\mathbb{C}$-linear extension of E. Now, we will see what happens when we introduce an integrability condition for the presented Killing spinor equations.

Definition 6.3.5. [38] An *almost lifting* of $T^{0,1}M$ to $E \otimes_{\mathbb{R}} \mathbb{C}$ is an isotropic subbundle

$$\ell \subseteq E \otimes_{\mathbb{R}} \mathbb{C}$$

mapping isomorphically to $T^{0,1}M$ under the $\mathbb{C}$-linear extension of the anchor map. That is, $\pi(\ell) = T^{0,1}M$. This one is called *integrable*, or *lifting*, if ℓ is involutive for the Dorfman bracket. That is, $[\ell, \ell] \subseteq \ell$. A generalized metric $C_\pm \subseteq E$ is said *compatible* with J if

$$\ell = \left\{ e \in C_+ \otimes_{\mathbb{R}} \mathbb{C} \mid \pi(e) \in T^{0,1}M \right\}$$

is a lifting of $T^{0,1}M$ to $E \otimes_{\mathbb{R}} \mathbb{C}$. So, we have that $\ell \subseteq C_+ \otimes_{\mathbb{R}} \mathbb{C}$ is isotropic and involutive.

Remark 6.3.6. Any solution (ω, Ψ, A) of the twisted Hull-Strominger system (6.16) on M is equivalent to having a string Courant algebroid $E := E_{-d^c\omega, A}$ for which we have solutions $(g, -d^c\omega, A)$ and $(g, -d^c\omega, A, \omega, \Psi)$ to the F-term and D-term conditions, respectively, with $d\varphi = 0$, where $(-d^c\omega, A)$ satisfies (6.2). In particular,

$$C_+ = \{X + g(X) \mid X \in TM\} \text{ and } C_- = \{X - g(X) + r \mid X \in TM, r \in \operatorname{ad} P\}$$

determine a J-compatible generalized metric $C_\pm \subseteq E_{H,A}$.

Proposition 6.3.7 ([38, Theorem 7.56])**.** *Any generalized metric $C_\pm \subseteq E$ of the form (6.5) is compatible with J complex structure on M if and only if the triple (g, H, A) satisfies $H = -d^c\omega$ and $F_A^{0,2} = 0$, where g is compatible with J and $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. That is, if and only if (g, H, A) satisfies the F-term conditions from Definition 5.1.7.*

6.3.2 The D-term Conditions on String Courant Algebroids

Let (g, H, A) be any solution to the F-term conditions on M, and consider the string Courant algebroid $E := E_{H,A}$ over M. In particular, we can consider

$$\overline{\ell} := e^{-i\omega}\left(T^{1,0}M\right) \subseteq C_+ \otimes_{\mathbb{R}} \mathbb{C}.$$

Then, for $\{z_j\}_{j=1}^n$ choice of local holomorphic coordinates on $U \subseteq M$, we can define

$$\epsilon_j := e^{i\omega}\left(\frac{\partial}{\partial \overline{z}_j}\right) = \frac{\partial}{\partial \overline{z}_j} + g\frac{\partial}{\partial \overline{z}_j} \in \Gamma(U, \ell) \text{ and } \overline{\epsilon}_j := g^{-1}d\overline{z}_j + d\overline{z}_j \in \Gamma\left(U, \overline{\ell}\right) \tag{6.18}$$

isotropic local frames, for $j \in \{1, \ldots, n\}$, so $\pi(\overline{\epsilon}_j) = g^{-1}d\overline{z}_j \in T^{1,0}M$, for $j \in \{1, \ldots, n\}$.

Lemma 6.3.8. *The frames* (6.18) *satisfy that*

$$[\epsilon_j, \epsilon_k]_{H,A} = 0, \quad \textit{for } j, k \in \{1, \ldots, n\}. \tag{6.19}$$

Proof. For $j, k \in \{1, \dots, n\}$, using the formula in [35, Proposition 4.3], then

$$\begin{aligned}
[\epsilon_j, \epsilon_k]_{H,A} :&= \left[e^{i\omega}\left(\frac{\partial}{\partial \overline{z}_j}\right), e^{i\omega}\left(\frac{\partial}{\partial \overline{z}_k}\right)\right]_{H,A} = e^{i\omega}\left(\left[\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_k}\right]_{H+id\omega, A}\right) \\
&= e^{i\omega}\left(\left[\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_k}\right] - F_A\left(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_k}\right) + \iota_{\frac{\partial}{\partial \overline{z}_k}} \iota_{\frac{\partial}{\partial \overline{z}_j}} (H + id\omega)\right) \\
&= -F_A^{0,2}\left(\frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_k}\right) + \iota_{\frac{\partial}{\partial \overline{z}_k}} \iota_{\frac{\partial}{\partial \overline{z}_j}} \left(-i\overline{\partial}\omega + i\overline{\partial}\omega\right) = 0,
\end{aligned}$$

by F-term conditions (5.7), since $H = -d^c\omega$ and $F_A^{0,2} = 0$. $\square$

Lemma 6.3.9. *Let E be a Courant algebroid over M endowed with a Riemannian metric g, which is compatible with the complex structure J on M. Moreover, assume that M admits an atlas $\mathcal{A}$ of holomorphic coordinates such that the holomorphic Jacobian of any change of coordinates has constant determinant. In this atlas, we define the isotropic local frame $\{\epsilon_j, \overline{\epsilon}_j\}_{j=1}^n$ as in (6.18) such that satisfies (6.19). Then, the local section*

$$\sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j] \tag{6.20}$$

is global. That is, it does not depend on the frames we have chose.

Proof. For $\left\{z'_j\right\}_{j=1}^n$ other choice of local holomorphic coordinates on $U' \subseteq M$ such that

$$\epsilon'_j := e^{i\omega}\left(\frac{\partial}{\partial \overline{z}'_j}\right) \in\in \Gamma\left(U', \ell\right) \text{ and } \overline{\epsilon}'_j := g^{-1}d\overline{z}'_j + d\overline{z}'_j \in\in \Gamma\left(U', \overline{\ell}\right), \quad \text{for } j \in \{1, \dots, n\}$$

is a new isotropic local frames satisfying the same properties above, we must prove that

$$\sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j] = \sum_{j=1}^{n} \left[\epsilon'_j, \overline{\epsilon}'_j\right].$$

Let us suppose that

$$\epsilon'_j = \sum_{k=1}^{n} A_j^k \epsilon_k \text{ and } \overline{\epsilon}'_j = \sum_{k=1}^{n} B_j^k \overline{\epsilon}_k, \quad \text{for } j \in \{1, \dots, n\},$$

for the matrices

$$A = \left(A_j^k\right)_{j,k\in\{1,\dots n\}}, B = \left(B_j^k\right)_{j,k\in\{1,\dots n\}} \in \mathrm{Mat}_n\left(\mathcal{C}^\infty(M)\right)$$

for the change of coordinates. The isotropy condition implies $B = A^{-1}$. Indeed,

$$\delta_k^j = \left\langle \overline{\epsilon}'_j, \epsilon'_k \right\rangle = \sum_{m,r=1}^{n} A_m^j B_k^r \left\langle \overline{\epsilon}_m, \epsilon_r \right\rangle = \sum_{m,r=1}^{n} A_m^j B_k^r \delta_r^m = \sum_{m=1}^{n} A_m^j B_k^m.$$

Then, by Courant algebroid axioms, (B.10) and Jacobi's formula (see Appendix B.6),

$$\sum_{j=1}^{n}\left[\epsilon'_j,\overline{\epsilon}'_j\right]=\sum_{j=1}^{n}\left(\sum_{k,m=1}^{n}A^j_k\pi\left(\overline{\epsilon}_k\right)\left(B^m_j\right)\epsilon_m-\sum_{k,m=1}^{n}B^m_j\pi\left(\epsilon_m\right)\left(A^j_k\right)\overline{\epsilon}_k+\left[\epsilon_j,\overline{\epsilon}_j\right]\right)$$
$$+\,2d\log\det A.$$

Since $\det A\in\mathbb{C}$ by hypothesis and

$$\left\langle\sum_{j=1}^{n}\left[\epsilon'_j,\overline{\epsilon}'_j\right],\epsilon'_k\right\rangle=0,\quad\text{for } k\in\{1,\dots,n\},$$

by Courant algebroid axioms and (6.19), everything reduces to prove that

$$\pi\left(\overline{\epsilon}_k\right)\left(B^m_j\right)=0,\quad\text{for } j,k,m\in\{1,\dots,n\}$$

because B^m_j is antiholomorphic, and $\pi\left(\overline{\epsilon}_k\right)=\partial_k$ for $j,k,m\in\{1,\dots,n\}$. Indeed,

$$\frac{\partial}{\partial\overline{z}_j}=\sum_{k=1}^{n}\frac{\partial\overline{z}'_k}{\partial\overline{z}_j}\frac{\partial}{\partial\overline{z}'_k}=\sum_{k=1}^{n}\overline{\left(\frac{\partial z'_k}{\partial z_j}\right)}\frac{\partial}{\partial\overline{z}'_k},\quad\text{for } j\in\{1,\dots,n\}$$

for our change of coordinates, so, since $e^{i\omega}$ is $\mathbb{C}$-linear,

$$\epsilon_j=e^{i\omega}\left(\frac{\partial}{\partial\overline{z}_j}\right)=e^{i\omega}\left(\sum_{k=1}^{n}\overline{\left(\frac{\partial z'_k}{\partial z_j}\right)}\frac{\partial}{\partial\overline{z}'_k}\right)=\sum_{k=1}^{n}\overline{\left(\frac{\partial z'_k}{\partial z_j}\right)}e^{i\omega}\left(\frac{\partial}{\partial\overline{z}'_k}\right)=\sum_{k=1}^{n}\overline{\left(\frac{\partial z'_k}{\partial z_j}\right)}\epsilon_k,$$

so

$$\sum_{k=1}^{n}B^j_k\epsilon_k,\quad\text{for } j\in\{1,\dots,n\},\ \text{so } B^j_k=\overline{\left(\frac{\partial z'_k}{\partial z_j}\right)},\quad\text{for } j,k\in\{1,\dots,n\}.$$

Finally, from above identities, we obtain that

$$\pi\left(\overline{\epsilon}_k\right)\left(B^m_j\right)=g^{-1}\left(d\overline{z}_k\right)\left(B^m_j\right)=\sum_{t=1}^{n}g^{kt}\overline{\left(\frac{\partial z'_j}{\partial z_m}\right)}\frac{\partial}{\partial z_t}=\sum_{t=1}^{n}g^{kt}\overline{\frac{\partial z'_j}{\partial z_m}\frac{\partial}{\partial z_t}}=0,$$

for $j,k,m\in\{1,\dots,n\}$, since $\overline{\partial}z'_j=0$ thanks to holomorphicity, for $j\in\{1,\dots,n\}$. □

Remark 6.3.10 (Holomorphicity on CA). By the proof of Lemma 6.3.9, we can give an abstract notion of holomorphicity for a change of frames on any general Courant algebroid E. Indeed, using the same notations as above, a change of frames on E is said *holomorphic* if the matrices A and B satisfy that

$$\pi\left(\overline{\epsilon}_k\right)\left(B^m_j\right)=0=\pi\left(\overline{\epsilon}'_m\right)\left(A^j_k\right),\quad\text{for } j,k,m\in\{1,\dots,n\}.$$

At last, we are ready to prove the following key result, where we compute the value of the section in Lemma 6.3.9. For that, we consider the string Courant algebroid $E_{H,A}$ determined by a solution (g, H, A) to the F-term conditions. We have $\sigma_+ : TM \longrightarrow C_+$ the isomorphism given by the generalized metric (6.5). Even more, define

$$\sigma_-(X) := X - g(X) \in C_-, \quad \text{for } X \in \mathfrak{X}(M).$$

Notice that this only gives an isomorphism when E is exact. We have the following.

Lemma 6.3.11 ([36, Equation (7.5)]). *The Bismut connection* (5.10) *satisfies that*

$$[\sigma_-(X), \sigma_+(Y)]_{H,A} = \sigma_+\left(\nabla^+_X Y - g^{-1}\left\langle \iota_Y F_A, r\right\rangle\right), \quad \textit{for } X, Y \in \mathfrak{X}(M). \tag{6.21}$$

Lemma 6.3.12. *Let M be a complex manifold of complex dimension n. Let (Ψ, ω) be an* $\mathrm{SU}(n)$*-structure on M satisfying* (5.9)*. Assume that (g, H, A) is a solution of the F-term conditions, with $g = \omega(\cdot, J)$, and consider the associated Courant algebroid $E = E_{H,A}$. Now, take the involutive and isotropic frames* (6.18) *constructed via the atlas in Lemma 5.2.2. Then, the global section* (6.20) *is given by*

$$\sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A} = \sigma_+\left(g^{-1}\left(\theta_\omega^{1,0}\right)\right) - \sigma_-\left(g^{-1}\left(\theta_\omega\right)\right) + i\Lambda_\omega F_A \in \Gamma(E \otimes_{\mathbb{R}} \mathbb{C}). \tag{6.22}$$

Proof. Consider $w := \sigma_-(v) \in \Gamma(C_- \otimes_{\mathbb{R}} \mathbb{C})$. By (6.21), using Courant algebroid axioms,

$$\begin{aligned}
\left\langle \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A}, w \right\rangle &= \sum_{j=1}^{n} \left\langle \overline{\epsilon}_j, \left[\sigma_-(v), \sigma_+\left(\frac{\partial}{\partial \overline{z}_j}\right)\right]_{H,A} \right\rangle \\
&= \sum_{j=1}^{n} \left\langle g^{-1} d\overline{z}_j + d\overline{z}_j, \sigma_+\left(\nabla^+_v \frac{\partial}{\partial \overline{z}_j} - g^{-1}\left\langle \iota_{\frac{\partial}{\partial \overline{z}_j}} F_A, r\right\rangle\right)\right\rangle \\
&= \sum_{j=1}^{n} d\overline{z}_j\left(\nabla^+_v \frac{\partial}{\partial \overline{z}_j}\right) - g\left(g^{-1}\left\langle \iota_{\frac{\partial}{\partial \overline{z}_j}} F_A, r\right\rangle, g^{-1} d\overline{z}_j\right) \\
&= \Delta - \sum_{j=1}^{n} \left\langle F_A\left(\frac{\partial}{\partial \overline{z}_j}, g^{-1} d\overline{z}_j\right), r\right\rangle,
\end{aligned}$$

where

$$\Delta = \sum_{j=1}^{n} d\overline{z}_j\left(\nabla^+_v \frac{\partial}{\partial \overline{z}_j}\right).$$

Suppose without loss of generality that v is real. Then,

$$\overline{\Delta} = \sum_{j=1}^{n} dz_j\left(\nabla^+_v \frac{\partial}{\partial z_j}\right).$$

Let $\nabla^{B'}$ be the connection in $\mathcal{K}_M^{-1} = \Lambda^n T^{1,0}M$ induced by ∇^+. Taking traces,

$$\nabla^{B'}(\Phi) = d\Phi + \sum_{i,j=1}^{n} \left(\Gamma_{ji}^{i} dz_j \otimes \Phi + \Gamma_{\overline{j}i}^{i} d\overline{z}_i \otimes \Phi \right), \quad \text{for } \Phi \in \Gamma\left(\mathcal{K}_M^{-1}\right),$$

where the *Christoffel symbols* are defined by

$$\sum_{i=1}^{n} \Gamma_{jk}^{i} \frac{\partial}{\partial z_i} := \nabla^{B'}_{\frac{\partial}{\partial z_j}} \left(\frac{\partial}{\partial z_k} \right), \quad \sum_{i=1}^{n} \Gamma_{\overline{j}k}^{i} \frac{\partial}{\partial z_i} := \nabla^{B'}_{\frac{\partial}{\partial \overline{z}_j}} \left(\frac{\partial}{\partial z_k} \right), \quad \text{for } j, k \in \{1, \ldots, n\}.$$

So, $\nabla_v^B = \iota_v d + \overline{\Delta}$. Let ∇^c be the Chern connection on $\mathcal{K}_M$. By Gauduchon's formula [42, Equation (2.7.6)], since $\Omega = e^{-f_\omega}\Psi$ is holomorphic with $\|\Omega\|_\omega = e^{-f_\omega}$, we have that $\nabla^c = d - 2\partial f_\omega$, which implies that the connection ∇^B in $\mathcal{K}_M$ induced by ∇^+ is given by

$$\nabla^B = d - 2\partial f_\omega - i d^* \omega = d - df_\omega = d - \theta_\omega,$$

so $\nabla^{B'} = d + \theta_\omega$, which is real. In conclusion, we have obtained that $\Delta = \overline{\Delta} = \theta_\omega(v)$. The other quantity is proportional to $\Lambda_\omega F_A$, defined by (5.5). Even more, for any

$$F = \sum_{i,j=1}^{n} F_{i\overline{j}} dz_i \wedge d\overline{z}_j \in \Omega^{1,1}(M),$$

we have that

$$i\Lambda_\omega F = \sum_{j=1}^{n} F\left(\frac{\partial}{\partial \overline{z}_j}, g^{-1} d\overline{z}_j \right).$$

We can compute this for a neighbourhood of $p \in M$ such that

$$\omega|_p = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

In summary, for any $w := \sigma_-(v) \in \Gamma(C_- \otimes \mathbb{C})$ is satisfied that

$$\left\langle \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A}, w \right\rangle = \theta_\omega(v) - i \left\langle \Lambda_\omega F_A, w \right\rangle.$$

Now, notice that by Courant algebroid axioms and (6.19),

$$\left\langle \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A}, \epsilon_k \right\rangle = \sum_{j=1}^{n} \left\langle \overline{\epsilon}_j, [\epsilon_k, \epsilon_j]_{H,A} \right\rangle = 0, \quad \text{for } k \in \{1, \ldots, n\}.$$

At last, since $\overline{\epsilon}_k = 2d\overline{z}_k + \sigma_-\left(g^{-1} d\overline{z}_k\right)$, for $k \in \{1, \ldots, n\}$, by Courant algebroid axioms, holomorphicity and (6.21), since $d\overline{z}_k$ is exact,

$$\begin{aligned}
\left\langle \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A}, \overline{\epsilon}_k \right\rangle &= \sum_{j=1}^{n} \left\langle \overline{\epsilon}_j, [\overline{\epsilon}_k, \epsilon_j]_{H,A} \right\rangle = \sum_{j=1}^{n} \left\langle \overline{\epsilon}_j, \left[\sigma_-\left(g^{-1} d\overline{z}_k\right), \epsilon_j\right]_{H,A} \right\rangle \\
&= \sum_{j=1}^{n} d\overline{z}_j \left(\nabla^+_{g^{-1} d\overline{z}_k} \frac{\partial}{\partial \overline{z}_j} \right) = \theta_\omega^{1,0}\left(g^{-1} d\overline{z}_k\right), \quad \text{for } k \in \{1, \ldots, k\}.
\end{aligned}$$

In summary, we conclude that

$$\sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A} = \theta_\omega + \theta_\omega^{1,0} - g^{-1}\theta_\omega^{0,1} + i\Lambda_\omega F_A. \qquad \square$$

Notice that, from (6.22), we obtain that

$$\left\langle \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A}, \sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A} \right\rangle = -2g\left(\theta_\omega^{0,1}, \theta_\omega^{1,0}\right) =: -\|\theta_\omega\|_g^2. \tag{6.23}$$

In summary, we have arrived at the following result.

Proposition 6.3.13. *With the hypotheses of Lemma 6.3.12, let $\varphi = -\theta_\omega \in \Gamma(E)$ be the section associated to minus the closed Lee form. Then, if (g, H, A, ω, Ψ) is a solution to the D-term conditions from Definition 5.1.8, the identity* (6.22) *is equivalent to*

$$\frac{1}{2}\sum_{j=1}^{n} [\epsilon_j, \overline{\epsilon}_j]_{H,A} = \pi_{\overline{\ell}}\varphi - \varphi, \tag{6.24}$$

where $\pi_{\overline{\ell}} \colon E_{H,A} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \overline{\ell}$ denotes the canonical projection. Moreover, we have that

$$[\varphi, \cdot]_{H,A} = 0 \quad \text{and} \quad \langle \varphi, \varphi \rangle = 0.$$

Proof. The fisrt part of the statement follows directly by Lemma 6.3.12. Indeed, we have that the two equations (6.22) and (6.24) are equivalent when $\varphi = -\theta_\omega$ and $\Lambda_\omega F_A = 0$. The last part of the statement follows from $d\theta_\omega = 0$, by the formula for the pairing (6.3) and the bracket (6.4) corresponding to string Courant algebroids. $\square$

6.3.3 The F-term and D-term Conditions on Complex CAs

We are going to write in a formal set-up the F-term and D-term conditions in Section 5.1.1 for transitive Courant algebroids. Note that the following reformulation makes perfect sense for general Courant algebroids over smooth manifolds. Fix E^c complex (transitive) Courant algebroid over M smooth manifold (see [37] for more information on this). Let

$$\mathcal{L} = \left\{ l \oplus \overline{l} \subseteq E^c \;\middle|\; l, \overline{l} \text{ are isotropic and } \langle \cdot, \cdot \rangle|_{l \oplus \overline{l}} \text{ is non-degenerate} \right\}$$

be the space of non-degenerate isotropic subbundles. By definition, given an element $C_+ := l \oplus \overline{l} \in \mathcal{L}$, we have a canonical identification $l^* \cong \overline{l}$. We will write

$$\pi_+ \colon E^c \longrightarrow C_+, \quad \pi_l \colon E^c \longrightarrow l, \quad \pi_{\overline{l}} \colon E^c \longrightarrow \overline{l}$$

for the orthogonal projections, which exist by assumption. So, when there is no possible confusion, we will use the simplified notation

$$a_+ = \pi_+ a, \quad a_l = \pi_l a, \quad a_{\overline{l}} = \pi_{\overline{l}} a, \quad \text{for } a \in \Gamma(E^c). \tag{6.25}$$

Definition 6.3.14. [2] We say that an element $l\oplus\bar{l}\in\mathcal{L}$ satisfies the (algebraic) *F-term condition* if

$$[l,l]\subseteq l,\qquad [\bar{l},\bar{l}]\subseteq\bar{l}. \tag{6.26}$$

We will use a weaker variant of (6.26). Explicitly, we will refer independently to the condition

$$[l,l]\subseteq l\oplus\bar{l},\qquad [\bar{l},\bar{l}]\subseteq l\oplus\bar{l}, \tag{6.27}$$

Remark 6.3.15. Note that the weaker variant of the F-term condition (6.27) is equivalent to

$$\left[\left(l\oplus\bar{l}\right)^{\perp},l\right]_{+}\subseteq l,\qquad \left[\left(l\oplus\bar{l}\right)^{\perp},\bar{l}\right]_{+}\subseteq\bar{l}.$$

Now, given $l\oplus\bar{l}\in\mathcal{L}$, we fix a dual isotropic frame $\{\epsilon_j,\bar{\epsilon}_j\}_{j=1}^{\dim l}$ of $l\oplus\bar{l}$. So, we have

$$\langle\epsilon_j,\epsilon_k\rangle=0,\quad \langle\epsilon_j,\bar{\epsilon}_k\rangle=\delta_j^k,\quad \langle\bar{\epsilon}_j,\bar{\epsilon}_k\rangle=0,\quad \text{for } j,k\in\{1,\dots,\dim l\}. \tag{6.28}$$

Suppose that M admits an atlas of holomorphic coordinates such that the holomorphic Jacobian of any change of coordinates has constant determinant, for which we can construct the dual isotropic frame $\{\epsilon_j,\bar{\epsilon}_j\}_{j=1}^{\dim l}$ of $l\oplus\bar{l}$ as in (6.18) such that satisfies

$$[\epsilon_j,\epsilon_k]=0,\quad \text{for } j,k\in\{1,\dots,\dim l\}. \tag{6.29}$$

The first part of Proposition 6.3.13 motivates the following notion.

Definition 6.3.16. We say that $l\oplus\bar{l}\in\mathcal{L}$ satisfies the (algebraic) *D-term condition* with Lee form $\varphi\in\Gamma(E^c)$ if, for $\{\epsilon_j,\bar{\epsilon}_j\}_{j=1}^{\dim l}$ of $l\oplus\bar{l}$ any frame as in (6.28),

$$\frac{1}{2}\sum_{j=1}^{\dim l}[\epsilon_j,\bar{\epsilon}_j]=\varphi_{\bar{l}}-\varphi. \tag{6.30}$$

We have that the left-hand side of (6.30) is a well-defined section by Lemma 6.3.9. We will use a weaker variant of (6.30). Explicitly, we will refer independently to the condition

$$\frac{1}{2}\sum_{j=1}^{\dim l}[\epsilon_j,\bar{\epsilon}_j]+\varphi\in l\oplus\bar{l}, \tag{6.31}$$

The following notion will be used independently in the present thesis when we arrive at Chapter 10, although it follows directly by Proposition 6.3.13 in the given conditions.

Definition 6.3.17. Any element $a\in\Gamma(E^c)$ is said to be *closed* if

$$[a,e]=0,\quad \text{for } e\in\Gamma(E^c),\quad \text{and}\quad \langle a,a\rangle=0. \tag{6.32}$$

The presented viewpoint will be useful in Chapter 10 to construct embeddings of SUSY vertex algebras from F-term and D-term conditions in the most general set-up. In other words, we will forget the underlying manifold M to construct embeddings from a general Courant algebroid to which we can endow the presented different conditions.

Chapter 7

The Killing Spinor Equations on Quadratic Lie Algebras

Now, we will study the Killing spinor equations over a real quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$. An important feature of the Killing spinor equations in this case is that they can be regarded as algebraic conditions on the quadratic Lie algebra. We will focus on the equations for pure spinors when the rank of the generalized metric $\mathfrak{g} = V_+ \oplus V_-$ is even (that is, if $\dim V_+ = 2n_+$ for $n_+ \in \mathbb{N}$). This content appeared in [2, Section 2].

7.1 Killing Spinors on Quadratic Lie Algebras

We will study several notions introduced in Chapter 6 when the base manifold is a point. In this case, a Courant algebroid becomes a real quadratic Lie algebra (see Remark 6.1.3), and the main concepts introduced in Chapter 6 become purely algebraic. Let $(\mathfrak{g}, (\cdot|\cdot))$ be a real quadratic Lie algebra (see Definition 1.4.1). We introduce the following notions.

Definition 7.1.1. [2] We have the following:

- A *generalized metric* on $\mathfrak{g}$ is an orthogonal decomposition

$$\mathfrak{g} = V_+ \oplus V_-,$$

  so that the restriction of $(\cdot|\cdot)$ to $V_\pm$ is non-degenerate. We say that the generalized metric is *Riemannian* if $(\cdot|\cdot)|_{V_+}$ is positive definite, and $(\cdot|\cdot)|_{V_-}$ is negative definite.

- A *divergence* on $\mathfrak{g}$ is an element $\alpha \in \mathfrak{g}^*$.

- Let $\mathfrak{g} = V_+ \oplus V_-$ be a generalized metric and $\mathrm{Cl}\,(V_\pm)$ be the complex Clifford algebras of $V_\pm \otimes_{\mathbb{R}} \mathbb{C}$ (see Definition 4.2.1). Fix irreducible representations $S_\pm$ of $\mathrm{Cl}\,(V_\pm)$. Their elements $\eta_\pm \in S_\pm$ are called *spinors*.

Given a generalized metric $V_\pm \subseteq \mathfrak{g}$, the associated orthogonal projections will be denoted by

$$\pi_\pm \colon \mathfrak{g} \longrightarrow V_\pm.$$

When there is no possibility of confusion, we will use the shorter notation

$$a_\pm = \pi_\pm a, \quad \text{for } a \in \mathfrak{g}.$$

In the sequel we will only consider Riemannian metrics, but a similar analysis can be carried out in other signatures. Observe that a generalized metric is uniquely determined by a choice of the positive-definite subspace $V_+ \subseteq \mathfrak{g}$. We will identify divergences with elements $a \in \mathfrak{g}$ via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the bilinear form $(\cdot|\cdot)$.

Definition 7.1.2. [2] Let $V_+ \subseteq \mathfrak{g}$ be a generalized metric. We will say that $a \in \mathfrak{g}$ is an *infinitesimal isometry* of V_+ if

$$[a, V_\pm] \subseteq V_\pm.$$

In this case, we will say that (V_+, a) is a *compatible pair.*

Definition 7.1.3. [2] We have the following:

- A *generalized connection* on $\mathfrak{g}$ is a linear map $D\colon \mathfrak{g} \longrightarrow \mathfrak{g}^* \otimes \mathfrak{g}$ such that

  $$(D_a b|\, c) + (b\,|D_a c) = 0, \quad \text{for } a, b, c \in \mathfrak{g}.$$

  The notation $D_a b$ stands for the elements in $\mathfrak{g}$ obtained from a and Db given above via the natural duality pairing $\mathfrak{g} \otimes \mathfrak{g}^* \longrightarrow \mathbb{R}$. The space of connections is denoted by $\mathcal{D}(\mathfrak{g})$, and can be canonically identified with $\mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}$.

- Given a generalized connection D, its *torsion* $T_D \in \Lambda^3 \mathfrak{g}^*$ is defined by

  $$T_D(a, b, c) = (D_a b - D_b a - [a, b]|\, c) + (D_c a|\, b)\,, \quad \text{for } a, b, c \in \mathfrak{g}.$$

Let $V_+ \subseteq \mathfrak{g}$ be a generalized metric. A generalized connection D is said V_+-*compatible* if $D(V_\pm) \subseteq \mathfrak{g}^* \otimes V_\pm$. The space of V_+-compatible connections is $\mathcal{D}(V_+)$, while the space of torsion-free V_+-compatible connections is

$$\mathcal{D}^0\,(V_+) = \{D \in \mathcal{D}(V_+) \mid T_D = 0\}\,.$$

Now, fix $V_+ \subseteq \mathfrak{g}$ any generalized metric. Notice that there is a canonical identification

$$\mathcal{D}(V_+) \cong \mathfrak{g}^* \otimes \left(\Lambda^2 V_+ \oplus \Lambda^2 V_-\right),$$

so any $D \in \mathcal{D}(V_+)$ splits into four mixed-type and pure-type operators

$$D_-^+ \in V_-^* \otimes \Lambda^2 V_+, \quad D_+^- \in V_+^* \otimes \Lambda^2 V_-,$$

$$D_+^+ \in V_+^* \otimes \Lambda^2 V_+, \quad D_-^- \in V_-^* \otimes \Lambda^2 V_-.$$

By Lemma 6.2.10, both mixed-type operators $D_\mp^\pm$ are uniquely determined by the generalized metric $V_+ \subseteq \mathfrak{g}$ for any element $D \in \mathcal{D}^0(V_+)$. Indeed,

$$D_{a_-} b_+ = [a_-, b_+]_+\,, \quad D_{a_+} b_- = [a_+, b_-]_-\,, \quad \text{for } a, b \in \mathfrak{g}.$$

Now, note that any connection $D \in \mathcal{D}(\mathfrak{g})$ determines a divergence α_D by the formula

$$\alpha_D(a) = -\mathrm{tr} Da, \quad \text{for } a \in \mathfrak{g}.$$

Given $V_+ \subseteq \mathfrak{g}$ a generalized metric and $\varepsilon \in \mathfrak{g}$ a divergence, we denote by

$$\mathcal{D}^0(V_+, \varepsilon) = \left\{ D \in \mathcal{D}^0(V_+) \mid \mathrm{tr} D = -(\varepsilon|\cdot) \right\}$$

the space of torsion-free V_+-compatible generalized connections with a fixed divergence. By Lemma 6.2.12, we can define an element $D \in \mathcal{D}^0(V_+, \varepsilon)$, for $a, b \in \mathfrak{g}$, by

$$D_a b = D_{a_-} b_+ + D_{a_+} b_- + \frac{1}{3}\left([a_+, b_+]_+ + [a_-, b_-]_-\right) + \frac{\phi^{\varepsilon_+}_{a_+} b_+}{n_+ - 1} + \frac{\phi^{\varepsilon_-}_{a_-} b_-}{n_- - 1}, \tag{7.1}$$

where $n_\pm = \dim V_\pm$ and $\phi^{\varepsilon_\pm} \in V_\pm^* \otimes \Lambda^2 V_\pm$ and are defined from $\varepsilon \in \mathfrak{g}$ by

$$\phi^{\varepsilon_\pm}_{a_\pm} b_\pm := (a_\pm | b_\pm)\, \varepsilon_\pm - (\varepsilon_\pm | b_\pm)\, a_\pm, \quad \text{for } a, b \in \mathfrak{g}. \tag{7.2}$$

Consider $\mathrm{Cl}(V_\pm)$ the associated complex Clifford algebras, and fix irreducible representations $S_\pm$ of $\mathrm{Cl}(V_\pm)$. For any choice of $D \in \mathcal{D}(V_+)$, consider the induced spin connections

$$D^{S_+}_+ \in V_+^* \otimes \mathrm{End}(S_+), \quad D^{S_-}_- \in V_-^* \otimes \mathrm{End}(S_-).$$

Fix $\left\{a^\pm_j\right\}_{j=1}^{n_\pm}$ an orthogonal basis of $V_\pm$ with dual basis $\left\{a^j_\pm\right\}_{j=1}^{n_\pm}$.

Definition 7.1.4. [2] Given $D \in \mathcal{D}^0(V_+, \varepsilon)$, we define a *pair of Dirac-type operators*

$$\not{D}^+ \in \mathrm{End}(S_+), \quad \not{D}^- \in \mathrm{End}(S_-),$$

given explicitly by

$$\not{D}^\pm \eta_\pm = \sum_{j=1}^{n_\pm} a^j_\pm \cdot D^{S_\pm}_{a^\pm_j} \eta_\pm, \quad \text{for } \eta_\pm \in S_\pm,$$

where $\cdot$ denotes the Clifford multiplication.

This construction is independent of the chosen basis. Furthermore, these Dirac operators are independent of the choice made for $D \in \mathcal{D}^0(V_+, \varepsilon)$ by Lemma 6.2.14. We are ready to write the desired equations. Notice that the Dirac-type operators depend only on the divergence $\varepsilon \in \mathfrak{g}$ (more explicitly, on the projections $\varepsilon_\pm = \pi_\pm \varepsilon$). The canonical operators $D^\pm_\mp$ associated to $V_+ \subseteq \mathfrak{g}$ generalized metric induce for $S_\pm$ the spin connections

$$D^{S_+}_- \in V_-^* \otimes \mathrm{End}(S_+), \quad D^{S_-}_+ \in V_+^* \otimes \mathrm{End}(S_-).$$

Definition 7.1.5. [2] We say that a triple $(V_+, \varepsilon, \eta_\pm)$ given by $V_+ \subseteq \mathfrak{g}$ generalized metric for which there exists the spinor bundle $S_\pm$, a divergence $\varepsilon_\pm \in \mathfrak{g}$, and a non-vanishing spinor $\eta_\pm \in S_\pm$, is a *solution of the Killing spinor equations*, if

$$D^{S_\pm}_\mp \eta_\pm = 0, \tag{7.3}$$

$$\not{D}^\pm \eta_\pm = 0, \tag{7.4}$$

where the operators $D^{S_\pm}_\mp$ and $\not{D}^\pm$ are defined above. When we have solutions in both V_+ and V_-, we say that we have *pairs of solutions of the Killing spinor equations* on $\mathfrak{g}$.

Now, we will give a more amenable characterization of these equations for the quadratic Lie algebra case, something that we cannot do for general Courant algebroids.

Lemma 7.1.6 ([2, Lemma 2.14])**.** *A pair $(V_+, \eta_\pm)$, given by a generalized metric $V_+ \subseteq \mathfrak{g}$ and a non-vanishing spinor $\eta_\pm \in S_\pm$, is a solution of* (7.3) *if and only if*

$$D^\pm_\mp \in V^*_\mp \otimes \operatorname{Lie} G_{\eta_\pm},$$

where $G_{\eta_\pm} \subseteq Spin(V_\pm)$ is the stabilizer of $\eta_\pm$. More explicitly, (7.3) *is equivalent to*

$$\sum_{j,k=1}^{n_\pm} \left(\left[b_\mp, a_j^\pm \right] \middle| a_\pm^k \right) a_\pm^k a_j^\pm \cdot \eta_\pm = 0,$$

for any choice of orthogonal basis $\left\{ a_j^\pm \right\}_{j=1}^{n_\pm}$ of $V_\pm$ with dual basis $\left\{ a_\pm^j \right\}_{j=1}^{n_\pm}$ and $b_\pm \in V_\pm$.

Proof. The first part of the statement follows simply from the identity (4.3), since

$$\operatorname{Lie} G_\eta = \left\{ B \in \Lambda^2 V_\pm \;\middle|\; B \cdot \eta = 0 \right\}.$$

As for the second part, an endomorphism $A \in \operatorname{End}(V_\pm)$ satisfies

$$A = \sum_{j,k=1}^{n_\pm} \left(A a_j^\pm \middle| a_\pm^k \right) \left(a_\pm^j \middle| \cdot \right) \otimes a_k^\pm.$$

Since

$$\left(a_\pm^j \middle| \cdot \right) \otimes a_k^\pm - \left(a_\pm^k \middle| \cdot \right) \otimes a_j^\pm \in \mathfrak{so}(V_\pm), \quad \text{for } j,k \in \{1, \dots, n_\pm\}$$

embeds as $\frac{1}{2} a_\pm^j a_k^\pm$ in $\operatorname{Cl}(V_\pm)$, an endomorphism $A \in \mathfrak{so}(V_\pm)$ corresponds to

$$A = \frac{1}{4} \sum_{j,k=1}^{n_\pm} \left(A a_j^\pm \middle| a_\pm^k \right) a_\pm^k a_j^\pm \in \operatorname{Cl}(V_\pm).$$

Then, given a spinor $\eta_\pm \in S_\pm$, we have for $b_\pm \in V_\pm$ that

$$D^\pm_{b_\mp} \eta_\pm = \frac{1}{4} \sum_{j,k=1}^{n_\pm} \left(\left[b_\mp, a_j^\pm \right] \middle| a_\pm^k \right) a_\pm^k a_j^\pm \cdot \eta_\pm. \qquad \square$$

Lemma 7.1.7 ([2, Lemma 2.16])**.** *A triple $(V_+, \varepsilon, \eta_\pm)$ as in Definition 7.1.5 is a solution of* (7.4) *if and only if*

$$\frac{1}{6} \sum_{i,j,k=1}^{n_\pm} \left(\left[a_k^\pm, a_i^\pm \right] \middle| a_\pm^j \right) a_\pm^k a_\pm^j a_i^\pm \cdot \eta_\pm = \varepsilon_\pm \cdot \eta_\pm,$$

for any choice of orthogonal basis $\left\{ a_j^\pm \right\}_{j=1}^{n_\pm}$ of $V_\pm$ with dual basis $\left\{ a_\pm^j \right\}_{j=1}^{n_\pm}$.

Proof. Consider $D \in \mathcal{D}(V_+, \varepsilon)$ in (7.1). Arguing as in Lemma 7.1.6, for any $b_\pm \in V_\pm$,

$$
\begin{aligned}
D^{S_\pm}_{b_\pm}\eta &= \frac{1}{12}\sum_{j,k=1}^{n_\pm}\left(\left[b_\pm, a_j^\pm\right] \middle| a_\pm^k\right) a_\pm^k a_j^\pm \cdot \eta_\pm \\
&\quad + \frac{1}{4(n_\pm - 1)}\sum_{j,k=1}^{n_\pm}\left(\left(b_\pm \middle| a_j^\pm\right)\left(\varepsilon_\pm \middle| a_\pm^k\right) - \left(\varepsilon_+ \middle| a_j^\pm\right)\left(b_\pm \middle| a_\pm^k\right)\right) a_\pm^k a_j^\pm \cdot \eta_\pm \\
&= \frac{1}{12}\sum_{j,k=1}^{n_\pm}\left(\left[b_\pm, a_j^\pm\right] \middle| a_\pm^k\right) a_\pm^k a_j^\pm \cdot \eta_\pm + \frac{1}{4(n_\pm - 1)}\left(\varepsilon_\pm b_\pm - b_\pm \varepsilon_\pm\right)\cdot \eta_\pm .
\end{aligned}
$$

Hence, setting

$$
C_\pm := \frac{1}{12}\sum_{i,j,k=1}^{n_\pm}\left(\left[a_k^\pm, a_i^\pm\right] \middle| a_\pm^j\right) a_\pm^k a_\pm^j a_i^\pm \cdot \eta_\pm ,
$$

we have

$$
\begin{aligned}
\not{D}^\pm \eta_\pm &= C_\pm + \frac{1}{4(n_\pm - 1)}\sum_{j=1}^{n_\pm} a_\pm^j\left(\varepsilon_\pm a_j^\pm - a_j^\pm \varepsilon_\pm\right)\cdot \eta_\pm \\
&= C_\pm + \frac{1}{4(n_\pm - 1)}\sum_{j=1}^{n_\pm}\left(2\left(\varepsilon_\pm \middle| a_\pm^j\right) a_j^\pm - 2\varepsilon_\pm\right)\cdot \eta_\pm = C_\pm - \frac{1}{2}\varepsilon_\pm \eta_\pm . \qquad \square
\end{aligned}
$$

We conclude the section by showing that there exists a perfect match between the geometric Definition 6.2.15, and the algebraic Definition 7.1.5, provided that we consider invariant solutions of (6.14) and (6.15) on homogeneous manifolds.

Definition 7.1.8. Let M be a manifold equipped with an action of a Lie group K. A Courant algebroid E over M is *equivariant* if it is equipped with a lift of the K-action on M that preserves its Courant algebroid structure.

Proposition 7.1.9 ([2, Proposition 4.5])**.** *Let M be a smooth oriented spin manifold, endowed with a left-transitive action of the Lie group K. Let E be an exact equivariant Courant algebroid over M. Then, the space of invariant sections*

$$
\mathfrak{g} = \Gamma(E)^K
$$

of E, endowed with the induced bracket and pairing, defines a real quadratic Lie algebra. Furthermore, there is a one-to-one correspondence between the invariant solutions to the equations in Definition 6.2.15, and the solutions to the equations in Definition 7.1.5.

Proof. The first part of the statement is straightforward from the axioms of a Courant algebroid, transitivity of the action, and the equivariance of E. As for the second part, it follows from the natural construction of the operators $D^{S_\pm}_\mp$ and $\not{D}^\pm$ using torsion-free generalized connections done in Chapter 6. Observe here that an invariant generalized connection on E corresponds to an element $D \in \mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}$ as in Definition 7.1.3. Similarly, for an invariant pair $(C_\pm, \mathrm{div}_\pm)$, one has

$$
\mathrm{div}_{0|V_\pm} - \mathrm{div}_\pm = \langle e_\pm, \cdot \rangle \in C_\pm^* . \qquad \square
$$

7.2 The F-term and D-term Conditions on QLAs

Given $(\mathfrak{g}, (\cdot|\cdot))$ real quadratic Lie algebra, we fix $\mathfrak{g} = V_+ \oplus V_-$ a generalized metric and an orientation on $V_\pm$. Let $\mathrm{Cl}(V_\pm)$ be the complex Clifford algebras of $V_\pm$ and fix irreducible representations $S_\pm$. Assuming that $\dim V_\pm = 2n_\pm$ for $n_\pm \in \mathbb{N}$, we have that $S_\pm$ split as irreducible $\mathrm{Spin}(2n_\pm)$-representations by

$$S_\pm = S_\pm^+ \oplus S_\pm^-, \tag{7.5}$$

which corresponds to the (± 1)-eigenspaces for the action of the complex volume form

$$\nu_{\mathbb{C}}^\pm = i^{n_\pm} a_1^\pm \cdots a_{2n_\pm}^\pm,$$

for a choice of an oriented orthonormal basis $\left\{a_j^\pm\right\}_{j=1}^{n_\pm} \subseteq V_\pm$. That is, a basis satisfying

$$\left(a_j^\pm \middle| a_k^\pm\right) = \delta_j^k, \quad \text{for } j,k \in \{1,\ldots,n_\pm\}.$$

Let $\eta_\pm \in S_\pm$ be now a pure spinor. By Lemma 4.2.6, it must have definite chirality. So, either $\eta_\pm \in S_\pm^+$ or $\eta_\pm \in S_\pm^-$. We know that $\eta_\pm \in S_\pm^+$ has isotropy group $G_{\eta_\pm} = \mathrm{SU}(n_\pm)$ in $\mathrm{Spin}(2n_\pm)$ by Lemma 4.2.7. In particular, $\eta_\pm$ determines an almost complex structure J on $V_\pm$ compatible with $(\cdot|\cdot)|_{V_\pm}$ and the orientation, such that the decomposition

$$V_\pm^{\mathbb{C}} := V_\pm \otimes_{\mathbb{R}} \mathbb{C} = V_\pm^{1,0} \oplus V_\pm^{0,1}$$

in $(\pm i)$-eigenspaces is determined by

$$V_+^{1,0} = \left\{a_+ \in V_+^{\mathbb{C}} \;\middle|\; a_+ \cdot \eta_+ = 0\right\} \quad \text{and} \quad V_-^{0,1} = \left\{a_- \in V_-^{\mathbb{C}} \;\middle|\; a_- \cdot \eta_- = 0\right\}.$$

Our goal is to characterize the Killing spinor equations in terms of this $\mathrm{SU}(n_\pm)$-structure. We fix a pure spinor $\eta \in S_+^+$ (the case $\eta \in S_-^+$ is analogue). Then, we have a model

$$S_+ = \Lambda^* V_+^{0,1}$$

in terms of the almost complex structure J on V_+ determined by this spinor, with Clifford action

$$a_+ \cdot \sigma = \sqrt{2}\iota_{\left(a_+^{1,0}|\cdot\right)}\sigma + \sqrt{2}a_+^{0,1} \wedge \sigma, \quad \text{for } \sigma \in S_+, a_+ \in V_+^{1,0}.$$

Here $(\cdot|\cdot)$ denotes the $\mathbb{C}$-linear extension of the pairing to the complexification $V_+^{\mathbb{C}}$, which is a symmetric tensor of type $(1,1)$. With this identification, (7.5) corresponds to

$$\Lambda^* V_+^{1,0} = \Lambda^{\text{even}} V_+^{0,1} \oplus \Lambda^{\text{odd}} V_+^{0,1}.$$

By Lemma 4.2.8, in this model $\eta = \lambda \in \mathbb{C} - \{0\}$. We fix an oriented orthonormal basis

$$\left\{a_1^+, Ja_1^+, \ldots, a_{n_+}^+, Ja_{n_+}^+\right\} \subseteq V_+^{\mathbb{C}}$$

for $(\cdot|\cdot)|_{V_+}$, with associated basis $\left\{\epsilon_j^+\right\}_{j=1}^{n_+} \subseteq V_+^{1,0}$ and $\left\{\overline{\epsilon}_j^+\right\}_{j=1}^{n_+} \subseteq V_+^{0,1}$ defined by

$$\epsilon_j^+ = \frac{1}{\sqrt{2}}\left(a_j^+ - iJa_j^+\right) \quad \text{and} \quad \overline{\epsilon}_j^+ = \overline{\epsilon_j^+} = \frac{1}{\sqrt{2}}\left(a_j^+ + iJa_j^+\right), \quad \text{for } j \in \{1, \dots, n_+\}.$$

Notice that the $\mathbb{C}$-linear extension of our pairing satisfies

$$\left(\epsilon_j^+ \middle| \epsilon_k^+\right) = 0, \quad \left(\epsilon_j^+ \middle| \overline{\epsilon}_k^+\right) = \delta_j^k, \quad \left(\overline{\epsilon}_j^+ \middle| \overline{\epsilon}_k^+\right) = 0, \quad \text{for } j, k \in \{1, \dots, n_+\}.$$

So, we have an isotropic basis. With the previous notation, we have the next result.

Proposition 7.2.1 ([2, Proposition 2.19])**.** *Let $(V_+, \varepsilon_\pm, \eta)$ be a triple, where we have $\dim V_\pm = 2n_\pm$ and $\eta_\pm \in S_\pm^+$ pure. Then, $(V_+, \varepsilon_\pm, \eta)$ is a solution of the Killing spinor equations if and only if*

$$F)\ \left[V_\pm^{0,1}, V_\pm^{0,1}\right] \subseteq V_\pm^{0,1}, \qquad D)\ \frac{i}{2}\sum_{j=1}^{n_\pm}\left[\epsilon_j^\pm, \overline{\epsilon}_j^\pm\right] = \mp J\varepsilon_\pm. \tag{7.6}$$

Proof. We start proving that the pair $(V_+, \eta_\pm)$, with $\dim V_\pm = 2n_\pm$ and $\eta_\pm \in S_\pm^+$ pure, is a solution of the gravitino equation (7.3) if and only if the conditions

$$F1)\ \left[V_\pm^{0,1}, V_\pm^{0,1}\right] \subseteq V_\pm^{\mathbb{C}}, \qquad D1)\ \sum_{j=1}^{n_\pm}\left[\epsilon_j^\pm, \overline{\epsilon}_j^\pm\right] \in V_\pm^{\mathbb{C}} \tag{7.7}$$

are satisfied. Assume $\eta \in S_+^+$. Given $a_- \in V_-$, define $\tau \in \Lambda^2 V_+$ by the formula

$$\tau(b_+, c_+) = \left([a_-, b_+]\,|c_+\right), \quad \text{for } b_+, c_+ \in V_+.$$

We have identified $V_+ \cong V_+^*$ with the isomorphism given by the induced metric on V_+. Then, by Lemma 7.1.6, there exists $\lambda \in \mathbb{C}^*$ such that gravitino equation is equivalent to

$$\tau \cdot \eta = \tau \cdot \lambda = 0.$$

Decompose τ as

$$\tau = \tau^{2,0} + \tau^{1,1} + \tau^{0,2},$$

where $\tau^{0,2} = \overline{\tau^{2,0}}$. Using

$$\begin{aligned} \epsilon_j \cdot 1 &= 0, \\ \overline{\epsilon}_j \cdot 1 &= \sqrt{2}\epsilon_j, \\ \epsilon_j \overline{\epsilon}_k \cdot 1 &= 2\delta_j^k, \end{aligned}$$

we obtain

$$\tau \cdot \eta = 2\lambda\tau^{0,2} + 2\lambda\sum_{j=1}^{n_+}\tau^{1,1}(\epsilon_j, \overline{\epsilon}_j).$$

Thus $\tau \cdot \eta = 0$ holds if and only if

$$0 = \tau\left(b_+^{0,1}, c_+^{0,1}\right) = \left(\left[a_-, b_+^{0,1}\right]\middle| c_+^{0,1}\right) = \left(a_- \middle| \left[b_+^{0,1}, c_+^{0,1}\right]\right),$$
$$0 = \sum_{j=1}^{n_+} \left(a_- \left|[\epsilon_j, \bar{\epsilon}_j]\right.\right),$$

for all $b_+, c_+ \in V_+^{\mathbb{C}}$. The statement follows from the fact that a_- can be chosen arbitrarily. Similarly, $\eta \in S_-^+$ pure is a solution of the gravitino equation if and only if

$$F1)\ \left[V_-^{1,0}, V_-^{1,0}\right] \subseteq V_-^{\mathbb{C}}, \qquad D1)\ \sum_{j=1}^{n_-}\left[\epsilon_j^-, \bar{\epsilon}_j^-\right] \in V_-^{\mathbb{C}}.$$

Now, we will prove that the triple $(V_+, \varepsilon_\pm, \eta_\pm)$, with $\dim V_\pm = 2n_\pm$ and $\eta_\pm \in S_\pm^+$ pure, is a solution of the dilatino equation (7.4) if and only if the conditions

$$F2)\ \left[V_\pm^{0,1}, V_\pm^{0,1}\right]_+ \subseteq V_\pm^{0,1}, \qquad D2)\ \frac{i}{2}\sum_{j=1}^{n_\pm}\left[\epsilon_j^\pm, \bar{\epsilon}_j^\pm\right]_+ = \mp J\varepsilon_\pm \tag{7.8}$$

are satisfied. Assume $\eta \in S_+^+$. Define $H \in \Lambda^3 V_+$ by the formula

$$H(a_+, b_+, c_+) = \left([a_+, b_+]\,|c_+\right), \quad \text{for } a_+, b_+, c_+ \in V_+.$$

Then, by Lemma 7.1.7, the dilatino equation is equivalent to

$$\frac{1}{6} H \cdot \lambda = \varepsilon_+ \cdot \lambda.$$

Decompose H as

$$H = H^{3,0} + H^{2,1} + H^{1,2} + H^{0,3},$$

where $H^{3,0} = \overline{H^{0,3}}$ and $H^{2,1} = \overline{H^{1,2}}$. Using

$$\bar{\epsilon}_j^+ \epsilon_k^+ \bar{\epsilon}_l^+ \cdot 1 = 2\sqrt{2}\delta_k^l \bar{\epsilon}_j^+,$$
$$\epsilon_j^+ \bar{\epsilon}_k^+ \bar{\epsilon}_l^+ \cdot 1 = 2\sqrt{2}\left(\delta_j^k \bar{\epsilon}_l^+ - \delta_j^l \bar{\epsilon}_k^+\right),$$

we obtain

$$(H - 6\varepsilon_+) \cdot \eta = \lambda 2\sqrt{2}\left(H^{0,3} + \frac{3}{2}\sum_{j=1}^{n_+} H^{1,2}\left(\epsilon_j^+, \bar{\epsilon}_j^+\right) - 3\varepsilon_+^{0,1}\right).$$

Thus $(H - 6\varepsilon_+) \cdot \eta = 0$ holds if and only if

$$0 = H\left(a_+^{0,1}, b_+^{0,1}, c_+^{0,1}\right) = \left(\left[a_+^{0,1}, b_+^{0,1}\right]\middle| c_+^{0,1}\right),$$
$$0 = \sum_{j=1}^{n_+}\left(a_+^{1,0}\middle|\left[\epsilon_j^+, \bar{\epsilon}_j^+\right]\right) - 2\left(a_+^{1,0}\middle|\varepsilon_+^{0,1}\right),$$

for $a_+, b_+, c_+ \in V_+^{\mathbb{C}}$. Using the orthogonal decomposition $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = V_+^{\mathbb{C}} \oplus V_-^{\mathbb{C}}$, and the fact that $(\cdot|\cdot)_{|V_+}$ is of type $(1,1)$, we see that the dilatino equation is equivalent to

$$\left[V_+^{0,1}, V_+^{0,1}\right]_+ \subseteq V_+^{0,1} \quad \text{and} \quad \frac{1}{2}\sum_{j=1}^{n_+} \left[\epsilon_j^+, \overline{\epsilon}_j^+\right]_+^{0,1} = \varepsilon_+^{0,1}.$$

Using now

$$\overline{i\left[\epsilon_j^+, \overline{\epsilon}_j^+\right]} = -i\left[\overline{\epsilon}_j^+, \epsilon_j^+\right] = i\left[\epsilon_j^+, \overline{\epsilon}_j^+\right]$$

for $j \in \{1, \ldots, n_+\}$, it follows that

$$-J\varepsilon_+ = i\varepsilon_+^{0,1} + \overline{i\varepsilon_+^{0,1}} = \frac{i}{2}\sum_{j=1}^{n_+}\left(\left[\epsilon_j^+, \overline{\epsilon}_j^+\right]_+^{0,1} + \left[\epsilon_j^+, \overline{\epsilon}_j^+\right]_+^{1,0}\right) = \frac{i}{2}\sum_{j=1}^{n_+}\left[\epsilon_j^+, \overline{\epsilon}_j^+\right]_+.$$

Similarly, a pure spinor $\eta \in S_-^+$ is a solution of the dilatino equation if and only if

$$F2)\ \left[V_-^{1,0}, V_-^{1,0}\right]_- \subseteq V_-^{1,0}, \qquad D2)\ \frac{i}{2}\sum_{j=1}^{n_-}\left[\epsilon_j^-, \overline{\epsilon}_j^-\right]_- = J\varepsilon_-.$$

The desired result follows now by comparing (7.6) with (7.7) and (7.8). $\square$

Remark 7.2.2. Let $(V_+, \varepsilon_\pm, \eta_\pm)$ be any solution of the Killing spinor equations as in Proposition 7.2.1, and let $\eta'_\pm \in S_\pm^-$ be a pure spinor in the line corresponding to $-J$. It follows from (7.6) that $(V_+, \varepsilon_\pm, \eta'_\pm)$ is also a solution of the Killing spinor equations.

7.2.1 The F-term and D-term Conditions on Complex QLAs

The study above suggests a weaker version of the Killing spinor equations, which forgets about the real structure underlying the complex quadratic Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, taking as main object the isotropic subspace $V_+^{1,0} \subseteq \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. This alternative point of view is more flexible, allowing to work over an arbitrary field of characteristic zero. Let $\mathfrak{g}$ be a quadratic Lie algebra over a field $\mathbb{C}$ of characteristic zero. We consider the space

$$\mathcal{L} = \left\{ l \oplus \bar{l} \subseteq \mathfrak{g} \;\middle|\; l, \bar{l} \text{ are isotropic and } (\cdot|\cdot)|_{l \oplus \bar{l}} \text{ is non-degenerate} \right\}$$

of non-degenerate isotropic subspaces. By definition, given an element $V_+ := l \oplus \bar{l} \in \mathcal{L}$, we have a canonical identification $l^* \cong \bar{l}$. We will write

$$\pi_+ \colon \mathfrak{g} \longrightarrow V_+, \quad \pi_l \colon \mathfrak{g} \longrightarrow l, \quad \pi_{\bar{l}} \colon \mathfrak{g} \longrightarrow \bar{l}$$

for the orthogonal projections, which exist by assumption. When there is no possible confusion, we will use the simplified notation

$$a_+ = \pi_+ a, \quad a_l = \pi_l a, \quad a_{\bar{l}} = \pi_{\bar{l}} a, \quad \text{for } a \in \mathfrak{g}.$$

Since the (algebraic) F-term conditions for quadratic Lie algebras coincide with the ones for Courant algebroids in Definition 6.3.14, we will not repeat them here.

Definition 7.2.3. [2] Any element $a \in \mathfrak{g}$ is said to be

1. an *infinitesimal isometry* if
$$\left[a, l \oplus \bar{l}\right] \subseteq l \oplus \bar{l}. \tag{7.9}$$
2. *holomorphic* if
$$[a, l] \subseteq l, \qquad \left[a, \bar{l}\right] \subseteq \bar{l}. \tag{7.10}$$

Note that condition (7.10) implies (7.9). Moreover, the condition (6.32) for quadratic Lie algebras implies (7.10). Notice that the previous notion gives a "holomorphic counterpart" to Definition 7.1.2. Indeed, in the following result we will study a salient feature of holomorphicity for a divergence $\varepsilon \in \mathfrak{g}$ in terms of the derived Lie subalgebras

$$[l, l]\,, \left[\bar{l}, \bar{l}\right] \subseteq \mathfrak{g}.$$

Lemma 7.2.4 ([2, Lemma 2.26])**.** *Assume that $\varepsilon \in \mathfrak{g}$ is a holomorphic divergence. Then, ε is orthogonal to the derived Lie subalgebras of l and $\bar{l}$. That is,*

$$\varepsilon \in [l, l]^{\perp} \cap \left[\bar{l}, \bar{l}\right]^{\perp}. \tag{7.11}$$

Proof. It is a consequence of the invariance for $(\cdot|\cdot)$ and the isotropic condition on $l, \bar{l}$. □

Now, given $l \oplus \bar{l} \in \mathcal{L}$, we fix a dual isotropic basis $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^{\dim l}$ of $l \oplus \bar{l}$. So, we have (3.8).

Definition 7.2.5. [2] We say that an element $l \oplus \bar{l} \in \mathcal{L}$ satisfies the (algebraic) *D-term condition* with divergence $\varepsilon \in l \oplus \bar{l}$ if, for $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^{\dim l}$ of $l \oplus \bar{l}$ any basis as in (3.8),

$$\frac{1}{2}\sum_{j=1}^{\dim l} [\epsilon_j, \bar{\epsilon}_j] = \varepsilon_{\bar{l}} - \varepsilon_l. \tag{7.12}$$

We will use weaker variants of (7.12). Explicitly, we will refer independently to the two conditions

$$\sum_{j=1}^{\dim l} [\epsilon_j, \bar{\epsilon}_j] \in l \oplus \bar{l}, \tag{7.13}$$

and

$$\frac{1}{2}\sum_{j=1}^{\dim l} [\epsilon_j, \bar{\epsilon}_j]_+ = \varepsilon_{\bar{l}} - \varepsilon_l. \tag{7.14}$$

All vectors (7.12), (7.13) and (7.14) are basis independent. Indeed, the same proof as in Remark 3.1.2 for dual basis works here. This formal set-up was applied in [2] to obtain embeddings from the $N = 2$ superconformal vertex algebras into the superaffinization of quadratic Lie algebras, provided that both the F-term and the D-term conditions are satisfied. The main results of this thesis will extend these embeddings to transitive Courant algebroids using the results of Chapter 6. To obtain these new embeddings, we will need some background about generalized Kähler geometry and the chiral de Rham complex.

Chapter 8

Generalized Kähler Geometry

Now, we will review the basics of the theory of generalized Kähler metrics on exact Courant algebroids, following [50]. This will be useful to understand the embeddings of SUSY vertex algebras into the chiral de Rham complex we are going to build on.

8.1 Generalized Complex Geometry

We will give notions for almost generalized complex and Dirac structures. We also need a notion of *integrability*. We describe generalized complex structures in terms of spinors.

Definition 8.1.1. [38] Given E an exact Courant algebroid over M smooth manifold, an *almost generalized complex structure* on E is $\mathcal{J} \in \Gamma(\operatorname{End} E)$ such that

(1) $\mathcal{J}^2 = -\operatorname{Id}$.

(2) $\langle \mathcal{J}a, \mathcal{J}b\rangle = \langle a, b\rangle$ for $a, b \in \Gamma(E)$. That is, $\mathcal{J}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Notice that the manifold M must be $2n$-dimensional to admit such $\mathcal{J} \in \Gamma(\operatorname{End} E)$.

Lemma 8.1.2 ([38, Section 7.1.1])**.** *An almost generalized complex structure on E exact Courant algebroid is equivalent to either:*

(1) a reduction of the frames to a maximal compact subgroup $U(n, n) \subseteq \mathrm{O}(2n, 2n)$.

(2) a maximal isotropic subbundle $L \subseteq E \otimes \mathbb{C}$ satisfying $L \cap \overline{L} = \{0\}$.

Example 8.1.3 ([38] **Examples of Almost Generalized Complex Structures**)**.** Let M be a smooth manifold, with J almost complex structure and $\omega \in \Omega^2(M)$. Then,

$$\mathcal{J}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \in \Gamma(\operatorname{End}(TM \oplus T^*M))$$

define two almost generalized complex structure on the standard Courant algebroid.

8.1.1 Dirac Structures and Integrability

The equivalent condition of having a maximal isotropic subbundle $L \subseteq E \otimes \mathbb{C}$ for which $L \cap \overline{L} = \{0\}$ gives us the integrability condition for generalized complex manifolds.

Definition 8.1.4. [38] Let E be an exact Courant algebroid. An *almost Dirac structure* on E is a subbundle $L \subseteq E$, which is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$. We say that L is an *integrable Dirac structure* if L is *involutive*. That is, $[L, L] \subseteq L$.

Let us consider $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_H, \pi)$ exact Courant algebroid with closed 3-form $H \in \Lambda^3 T^*M$. For $C \subseteq TM$ a subbundle and $\phi \in \Lambda^2 C^*$, define the almost Dirac structure

$$L(C, \phi) := \{X + \zeta \in C \oplus T^*M \ \mid \ \zeta|_U = \phi(X, \cdot)\}\,.$$

Lemma 8.1.5 ([38, Proposition 7.13]). *Every maximal isotropic $L \subseteq TM \oplus T^*M$ with respect to $\langle \cdot, \cdot \rangle$ is of the form $L(C, \phi)$. Moreover, this is integrable if and only if:*

(1) The subbundle C is closed under the Lie bracket between fields.

(2) It is satisfied that $\iota_Y \iota_X (H + d\phi) = 0$ for all $X, Y \in \Gamma(C)$.

Now, we can give the notion of integrability for almost generalized complex structures.

Definition 8.1.6. [38] For E exact Courant algebroid, we say that an almost generalized complex structure $\mathcal{J}$ on E is *integrable* if the $(+i)$-eigenbundle $L \subseteq E \otimes \mathbb{C}$ is an integrable Dirac structure. An integrable $\mathcal{J}$ on E will be called a *generalized complex structure.*

Example 8.1.7 ([38] **Examples of Generalized Complex Structures**). Let M be smooth, with $\mathcal{J}_J, \mathcal{J}_\omega$ the almost generalized complex structures as in Example 8.1.3,

- J is integrable if and only if $\mathcal{J}_J$ is integrable.
- ω is symplectic if and only if $\mathcal{J}_\omega$ is integrable.

8.1.2 Spin Formulation

Let $\mathrm{Cl}(E)$ be the Clifford bundle of E exact Courant algebroid determined by a closed 3-form H, which is defined using the identification (4.1). In particular, remember that we can identify $E = TM \oplus T^*M$ via the H-twisted bracket. Furthermore, consider

$$\mathrm{Spin}(E) = \{v_1 \cdots v_{2k} \in \mathrm{Cl}^*(E) |\ v_j \in E,\ k \in \mathbb{N},\ \langle v_j, v_j \rangle = \pm 1 \text{ for } j \in \{1, \ldots, 2k\}\}\,,$$

the associated spin bundle. Then, the action of E on the space $\Omega(M)$ of polyforms

$$(X + \zeta) \cdot \phi = \iota_X \phi + \zeta \wedge \phi, \quad \text{for } X + \zeta \in \Gamma(E), \phi \in \Omega(M),$$

given by the Clifford multiplication of $\Gamma(E)$, extends to a natural action of the Clifford bundle $\mathrm{Cl}(E)$ on $\Omega(M)$. Now, since TM is $2n$-dimensional, as a spin representation, we have that the space $\Omega(M)$ splits into a direct sum

$$\Omega(M) = \Omega^{\mathrm{ev}}(M) \oplus \Omega^{\mathrm{odd}}(M),$$

corresponding to irreducible representations, of *positive and negative chirality.* We turn next to the study of Dirac structures on TM in terms of spinors in $\Omega^{\mathrm{ev}}(M)$ or $\Omega^{\mathrm{odd}}(M)$.

Definition 8.1.8. [38] For $\rho \in \Omega(M)$ a spinor, we define the *associated annihilator bundle* by

$$L_\rho := \mathrm{Ann}(\rho) = \{e \in \Gamma(E) \mid e \cdot \rho = 0\} \subseteq \Gamma(E).$$

It is clear that L_ρ is always isotropic, since

$$\langle a, b\rangle \cdot \rho = \frac{1}{2}\left(\langle a+b, a+b\rangle - \langle a, a\rangle - \langle b, b\rangle\right) \cdot \rho = 0, \quad \text{for } a, b \in \Gamma(E).$$

We will say that ρ is *pure* if L_ρ is maximal isotropic. That is, it has dimension n.

Lemma 8.1.9 ([38, Proposition 7.18])**.** *Every maximal isotropic subbundle $L = L(C, \phi)$ is completely determined by a pure spinor line subbundle $\mathcal{K}_L \subseteq \Omega(M)$. Given $\{\theta_1, \dots, \theta_k\}$ a frame of* $\mathrm{Ann}(C)$ *and $B \in \Omega^2(M)$ such that $\iota^* B = -\phi$, where $\iota\colon C \longrightarrow TM$ is the inclusion, then $\mathcal{K}_L$ representing $L(C, \phi)$ is generated by the pure spinor*

$$\rho = e^B \theta_1 \wedge \cdots \wedge \theta_k.$$

Consider an almost generalized complex structure $\mathcal{J}$ on E with associated complex Dirac structure $L \subseteq E \otimes \mathbb{C}$. Regarding $\Omega(M) \otimes \mathbb{C}$ as a representation of the complex Clifford algebra bundle of $E \cong TM \oplus T^*M$, the analogue of result above applies, and we obtain a complex pure spinor line subbundle $\mathcal{K}_L \subseteq \Omega(M) \otimes \mathbb{C}$ generated by the pure spinor

$$\rho = e^{B + i\omega} \theta_1 \wedge \cdots \wedge \theta_k,$$

for $B, \omega \in \Omega^2(M)$ and $\theta_1, \dots, \theta_k$ a frame of $L \cap (T^*M \otimes \mathbb{C})$. Further, there is a one-to-one correspondence between the line bundles in $\Omega(M) \otimes \mathbb{C}$, whose local trivializations consist of pure spinors, and maximally isotropic subbundles of $E \otimes \mathbb{C}$.

Definition 8.1.10. [38] Let $\mathcal{J}$ be an almost generalized complex structure on M. The *canonical bundle* of $\mathcal{J}$, denoted by $\mathcal{K}_{\mathcal{J}}$, is the line subbundle in $\Omega(M) \otimes \mathbb{C}$ given via the one-to-one correspondence above by the annihilator subbundle associated to $\mathcal{J}$.

If $C_L := \pi_{TM} L$ is the subbundle associated to an almost Dirac structure L, this yields

$$\Omega(M) = \bigoplus_{k=0}^{n} U_k, \text{ where } U_0 = \mathcal{K}_J, U_k = \Lambda^k \mathrm{Ann}(C)^* \cdot U_0. \tag{8.1}$$

Definition 8.1.11. [38] The *Mukai pairing* is an invariant bilinear form on the spinors of E defined by

$$\begin{array}{rccc} (\cdot,\cdot): & \Omega(M) \otimes \Omega(M) & \longrightarrow & \det T^*M \\ & (\psi, \phi) & \mapsto & \left[\psi^\perp \wedge \phi\right]_{\text{top}} \end{array},$$

where $\alpha \mapsto \alpha^\perp$ is the antiautomorphism of the Clifford bundle $\mathrm{Cl}(E)$ determined by the tensor map $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$, and $[\cdot]_{\text{top}}$ applied to forms indicates taking the top degree component of the form. The Mukai pairing extends $\mathbb{C}$-linearly to $\Omega(M) \otimes \mathbb{C}$.

Lemma 8.1.12 ([38, Lemma 7.19])**.** *Two maximal isotropic subbundles $L, L' \subseteq E \otimes \mathbb{C}$ satisfy $L \cap L' = \{0\}$ if and only if their pure spinor representatives ρ, ρ' satisfy $(\rho, \rho') \neq 0$.*

We arrive at this characterization of what is an almost generalized complex structure.

Lemma 8.1.13 ([38, Proposition 7.20])**.** *An almost generalized complex structure $\mathcal{J}$ on E exact Courant algebroid is equivalent to a choice of a pure spinor line $\mathcal{K}_{\mathcal{J}} \subseteq \Omega(M) \otimes \mathbb{C}$ such that*

$$\mu := (\rho, \overline{\rho}) \neq 0,$$

for ρ a pure spinor representative. Note that $\mu \in \Gamma(\det T^*M)$ is a gobally defined volume form that gives an orientation, which is independent of the choice of ρ, giving a global orientation on the underlying manifold M.

Consider now the H-twisted differential $d_H \colon \Omega(M) \longrightarrow \Omega(M)$ acting on $\Omega(M)$ by

$$d_H \rho = d\rho - H \wedge \rho, \quad \text{for } \rho \in \Omega(M).$$

This is independent of the choice we have done of the isotropic splitting. We will denote by $\not{d}_0 \colon \Omega(M) \longrightarrow \Omega(M)$ the corresponding differential on E.

Lemma 8.1.14 ([38, Proposition 3.8])**.** *Let $\mathcal{J}$ be an almost generalized complex structure on E exact Courant algebroid over M smooth $2n$-dimensional manifold, with associated pure spinor line $\mathcal{K}_{\mathcal{J}} \subseteq \Omega(M) \otimes \mathbb{C}$. Then, $\mathcal{J}$ is integrable if and only if for any local trivialization ρ of $\mathcal{K}_{\mathcal{J}}$ there exists a local section $v \in \Gamma(L^*)$ such that*

$$\not{d}_0 \rho = v \cdot \rho. \tag{8.2}$$

8.2 Generalized Calabi-Yau Geometry

We will recall what we know as generalized Kähler and Calabi-Yau structures.

Definition 8.2.1. [50] For an exact Courant algebroid E, an *almost generalized Kähler structure* is a pair $(\mathcal{J}_+, \mathcal{J}_-)$ of two commutative almost generalized complex structures, for which $\mathcal{G} := -\mathcal{J}_+\mathcal{J}_-$ is Riemannian generalized metric, called the *generalized Kähler metric* of E. If $(\mathcal{J}_+, \mathcal{J}_-)$ are both integrable, we have a *generalized Kähler structure.*

Remark 8.2.2. [50] Note that on any exact Courant algebroid E the presence of a single generalized complex structure, together with $\langle \cdot, \cdot \rangle$ the neutral inner product, reduces the structure group of E to $\mathrm{U}(n, n)$. So, with this new notion, we obtain a further reduction. Indeed, an almost generalized Kähler structure is equivalent to a reduction of the frames to a maximal compact subgroup $\mathrm{U}(n) \times \mathrm{U}(n) \subseteq \mathrm{O}(2n) \times \mathrm{O}(2n)$.

We want to describe now the geometric structures induced on the underlying manifold M by the (almost) generalized Kähler structure $(\mathcal{J}_+, \mathcal{J}_-)$. We need the following notion.

Definition 8.2.3. [50] An almost generalized complex structure $\mathcal{J}$ is said *compatible* with a generalized metric $\mathcal{G}$ if $\mathcal{G}\mathcal{J}$ defines another almost generalized complex structure. This is equivalent to being $\mathcal{J}$ and $\mathcal{G}$ commutative. Moreover, if we have that $E = C_+ \oplus C_-$ is the decomposition given by the metric, this means that $\mathcal{J}$ preserves $C_{\pm}$.

Theorem 8.2.4 ([50, Theorem 2.18])**.** *A generalized Kähler structure $\mathcal{J}_\pm$ on an exact Courant algebroid E is equivalent to having a metric g which is Hermitian with respect to two integrable complex structure $J_\pm$, and such that*

$$\pm d^c_\pm \omega_\pm = H,$$

for $H \in \Omega^3(M)$ closed defined in (6.1)*, and $d^c_\pm = i\left(\overline{\partial}_\pm - \partial_\pm\right)$.*

Let $(g, J_\pm)$ be a data corresponding to a generalized Kähler structure $\mathcal{J}_\pm$ as above. Let $L_\pm$ be the $(+i)$-eigenbundles of $\mathcal{J}_\pm$. Since $\mathcal{J}_\pm$ commute,

$$L_+ = \ell_+ \oplus \ell_-, \quad L_- = \ell_+ \oplus \overline{\ell}_-, \tag{8.3}$$

where $\ell_+ = L_+ \cap L_-$ and $\ell_- = L_+ \cap \overline{L}_-$. Now, since $\mathcal{G} = -\mathcal{J}_+\mathcal{J}_-$ has eigenvalue $+1$ on $\ell_+ \oplus \overline{\ell}_+$, we obtain a decomposition into four n-dimensional isotropic subbundles

$$C_\pm \otimes \mathbb{C} = \ell_\pm \oplus \overline{\ell}_\pm, \quad E \otimes \mathbb{C} = \ell_+ \oplus \ell_- \oplus \overline{\ell}_+ \oplus \overline{\ell}_-. \tag{8.4}$$

Example 8.2.5 ([50] **Example of Generalized Kähler Structure**)**.** Let M be any smooth manifold, with $\mathcal{J}_J, \mathcal{J}_\omega$ generalized complex structures as in Example 8.1.3. Then, J is integrable and ω symplectic. We have that the pair $(\mathcal{J}_J, \mathcal{J}_\omega)$ is commutative, and

$$\mathcal{G} = -\mathcal{J}_J\mathcal{J}_\omega = -\begin{pmatrix} 0 & -J\omega^{-1} \\ -J^*\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

defines a Riemannian generalized metric on $TM \oplus T^*M$ with $g = \omega(\cdot, J\cdot)$.

Example 8.2.6 ([50] **Hyper-Kähler Structures**)**.** Given an hyper-Kähler structure (M, g, I, J, K), we have that the triple (g, I, J) is an almost generalized Kähler structure, and, even more, this one is integrable, since it is satisfied that $d\omega_I = 0 = d\omega_J$. We can reconstruct the associated two generalized complex structures, which are

$$\mathcal{J}_\pm = \frac{1}{2}\begin{pmatrix} I \pm J & -(\omega_I^{-1} \mp \omega_J^{-1}) \\ \omega_I \mp \omega_J & -(I^* \pm J^*) \end{pmatrix}.$$

We are ready to introduce the *generalized Calabi-Yau structures* as appear in [57]. As it is mentioned in [91], these structures mimics the classical relation between the complex and the special complex manifolds, rather than generalizing classical Calabi-Yau manifolds.

Definition 8.2.7. [57] Let E be an exact Courant algebroid over M smooth manifold. An *almost generalized Calabi-Yau structure* on E is $\rho \in \Omega^{\text{ev,odd}}(M) \otimes \mathbb{C}$ with $(\rho, \overline{\rho}) \neq 0$. We say that ρ is *integrable*, or *generalized Calabi-Yau structure*, if $\not{d}_0\rho = 0$.

By definition, the purity of ρ implies that the associated annihilator bundle L_ρ is maximal isotropic. Furthermore, the second condition is equivalent, by Lemma 8.1.12, to having that $L_\rho \cap \overline{L}_\rho = \{0\}$. Now, since $L_{\overline{\rho}} = \overline{L}_\rho$ clearly, an almost generalized Calabi-Yau structure induces a decomposition

$$E \otimes \mathbb{C} = L_\rho \oplus \overline{L}_\rho.$$

In other words, we have obtained an almost generalized complex structure. Now, we want a different formulation for the notion above from $\mathcal{J}$ an almost generalized complex structure. Let $\mathcal{K}_{\mathcal{J}}$ be the canonical bundle of $\mathcal{J}$, and suppose that is *trivial*. Then, we can split $\not{d}_0 = \partial + \overline{\partial}$ having that $\partial\colon U_k \longrightarrow U_{k-1}$ and $\overline{\partial}\colon U_k \longrightarrow U_{k+1}$ for the bundles U_k defined in (8.1). We obtain the following result that gives an equivalent statement.

Lemma 8.2.8 ([91])**.** *An almost generalized Calabi-Yau structure is equivalent to an almost generalized complex structure $\mathcal{J}$, whose canonical bundle $\mathcal{K}_{\mathcal{J}}$ is trivialized by a global section $\rho \in \Gamma(\mathcal{K}_{\mathcal{J}})$ that is generalized holomorphic, in the sense that $\overline{\partial}\rho = 0$ is satisfied. Moreover, we obtain the integrability condition if we also require that $\partial\rho = 0$.*

Example 8.2.9 ([91] **Example of Generalized Calabi-Yau Structure**)**.** Let M be a symplectic $2n$-dimensional manifold with closed non-degenerate $\omega \in \Omega^2(M)$. Then,

$$\rho = e^{i\omega}, \quad (\rho, \overline{\rho}) = c\omega^n,$$

for non-zero $c \in \mathbb{R}$. So, (M, ρ) is a generalized Calabi-Yau manifold. It can be seen that $\mathcal{J}_\omega$ is the correspondent generalized complex structure. Let M be an special complex manifold with Ω holomorphic volume form with associated J complex structure. Then,

$$\rho = \Omega, \quad (\rho, \overline{\rho}) = \pm\Omega \wedge \overline{\Omega} \neq 0,$$

by definition of Ω, and (M, ρ) is a generalized Calabi-Yau manifold. It can be seen that $\mathcal{J}_J$ is the corresponding generalized complex structure.

We are ready for the last definition, which generalizes the classical Calabi-Yau manifolds.

Definition 8.2.10. [48] For E exact Courant algebroid over an $2n$-dimensional smooth manifold M, an *(almost) generalized Calabi-Yau metric structure* is determined by an (almost) generalized Kähler structure $(\mathcal{J}_+, \mathcal{J}_-)$ such that $(M, \mathcal{J}_+)$ and $(M, \mathcal{J}_-)$ are both (almost) generalized Calabi-Yau structures with the corresponding pair of pure spinors $\rho_\pm$ satisfying the normalization condition given by

$$(\rho_+, \overline{\rho_+}) = c\,(\rho_-, \overline{\rho_-})\,,$$

for some non-zero constant $c \in \mathbb{R}$.

Remark 8.2.11. [48] An almost generalized Calabi-Yau metric structure is equivalent to a reduction of the frames to a maximal compact subgroup $\mathrm{SU}(n) \times \mathrm{SU}(n) \subseteq \mathrm{U}(n) \times \mathrm{U}(n)$.

Example 8.2.12 ([55] **Example of Generalized Calabi-Yau Metric Structure**)**.** Let M be a classical $2n$-dimensional Calabi-Yau manifold, determined by the symplectic form ω and the holomorphic volume form Ω with associated J complex structure. Then,

$$\rho_+ = e^{i\omega}, \quad \rho_- = \Omega, \quad (\rho_+, \overline{\rho_+}) = (-1)^{\frac{n(n-1)}{2}}\,(\rho_-, \overline{\rho_-})\,.$$

Further, as we have seen above, we have that $(\mathcal{J}_\omega, \mathcal{J}_J)$ is a generalized Kähler structure.

As a future work, it will be interesting to study the relation between these structures and Killing spinors. This is needed to understand the relation between our new embeddings and the ones known before that we will remember in next chapter.

Part III

Vertex Algebras and Geometric Structures in Interaction

Chapter 9

The Chiral de Rham Complex

We are going to recall the definition of a sheaf of SUSY vertex algebras originally given by Malikov-Schechtman-Vaintrob in [73]. Roughly speaking, this is a complex constructed from $\mathbb{A}^N$ via the topological vertex algebra structure of the bc-$\beta\gamma$ system of dimension $2N$ introduced in Subsection 2.5.3. Furthermore, it is seen that this object is related with the usual de Rham complex, and receives the name of *chiral de Rham complex*. At this point, by gluing these SUSY vertex algebras constructed for each open set, we can construct a sheaf for any smooth manifold M. In fact, this construction works in a coordinate independent way. It was Lian-Linshaw who gave an abstract version for this object in [69, 70]. Bressler and Heluani extended this construction to any Courant algebroid in [15, 53], although the result goes back to Gorbounov-Malikov-Shechtman in [44, 45, 46], and independently by Beilinson-Drinfeld [8][Section 2.8]. We give a quick general view of all these constructions. This allows us to give embeddings from superconformal vertex algebras into the chiral de Rham complex if we have some geometric structures. In Section 9.2, we include the embeddings given by Heluani-Zabzine.

9.1 Classical Constructions of the CDR complex

We review three approaches to the construction of the chiral de Rham complex. First, the classical construction by Malikov-Schechtman-Vaintrob based on the bc-$\beta\gamma$ system. After that, the coordinate independent construction, valid for any smooth manifold, given by Lian-Linshaw. Finally, the generalization by Bressler and Heluani to any Courant algebroid. We will show how these three settings define the same object. Moreover, this last construction gives a relation with the superaffinization of quadratic Lie algebras.

9.1.1 Construction over the Affine Space

We are going to introduce a complex starting with the affine space, which is going to be related with the usual de Rham complex on it. To fix notation, the usual N-dimensional complex affine space will be denoted by

$$\mathbb{A}^N := \{(x_1, \ldots, x_N) \mid x_j \in \mathbb{C},\ \text{for } j \in \{1, \ldots, N\}\}.$$

Then, the de Rham complex is

$$\Omega\left(\mathbb{A}^N\right) := \bigoplus_{|I|\leq N} \mathbb{C}\left[x_1, \ldots, x_N\right] d_{\mathbb{A}^N} x_I = \bigoplus_{p=0}^{N} \Omega^p\left(\mathbb{A}^N\right),$$

where $d_{\mathbb{A}^N} : \Omega^p\left(\mathbb{A}^N\right) \longrightarrow \Omega^{p+1}\left(\mathbb{A}^N\right)$ is the usual de Rham differential defined by

$$d_{\mathbb{A}^N}\varphi := \sum_{|I|\leq N}\left(\sum_{j=1}^{N}\frac{\partial f_I}{\partial x_j} d_{\mathbb{A}^N} x_j\right) \wedge d_{\mathbb{A}^N} x_I, \quad \text{for } \varphi = \sum_{|I|\leq N} f_I d_{\mathbb{A}^N} x_I \in \Omega^p\left(\mathbb{A}^N\right)$$

where $I = \{i_1, \ldots, i_t\}$ is a set such that $1 \leq i_1 \leq \cdots \leq i_t \leq N$, and

$$d_{\mathbb{A}^N} x_I := d_{\mathbb{A}^N} x_{i_j} \wedge \cdots \wedge d_{\mathbb{A}^N} x_{i_t}.$$

Let $\Omega_N = V\left(\Omega_N\right)$ be the *bc*-$\beta\gamma$ system of dimension $2N$ (see Subsection 2.5.3). Recall that by Theorem 3.4.2, Ω_N is a topological vertex algebra of rank N. In particular, it contains the following two generators, along with L and G^-:

$$J = \sum_{j=1}^{N} : b^j c^j :, \quad G^+ = \sum_{j=1}^{N} : b^j \beta^j :\in \Omega_N.$$

Let $V_N = V\left(V_N\right)$ and $\Lambda_N = V\left(\Lambda_N\right)$. Since β^j and γ^j have conformal weight, respectively, 1 and 0 with respect to L,

$$Y\left(\beta^j, z\right) = a^j(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_n^j, Y\left(\gamma^j, z\right) = b^j(z) = \sum_{n\in\mathbb{Z}} z^{-n} b_n^j \in \mathcal{F}(V_N).$$

By the second vacuum axiom, we construct the following isomorphism between the algebra of polynomials in the infinitely many Fourier modes a_n^j, for $n < 0$, and b_m^j, for $m \leq 0$ (viewed as formal even variables), where $j \in \{1, \ldots, N\}$, and V_N:

$$\begin{array}{rcl} \mathbb{C}\left[\left\{a_n^j, b_m^k \;\middle|\; {\scriptstyle 1\leq j,k\leq N \atop n<0, m\leq 0}\right\}\right] & \cong & V_N \\ P\left(a_n^j, b_m^k\right) & \leftrightarrow & P\left(a_n^j, b_m^k\right)|0\rangle. \end{array}$$

Similarly, since b^j and c^j have conformal weight, respectively, 0 and 1 with respect to L,

$$Y\left(b^j, z\right) = \phi^j(z) = \sum_{n\in\mathbb{Z}} z^{-n}\phi_n^j, Y\left(c^j, z\right) = \psi^j(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\psi_n^j \in \mathcal{F}(\Lambda_N).$$

By the second vacuum axiom, we construct the following isomorphism between the exterior algebra of polynomials in the infinitely many Fourier modes ϕ_r^j, for $r < 0$, and ψ_s^j, for $s \leq 0$ (viewed as formal odd variables), where $j \in \{1, \ldots, N\}$, and Λ_N:

$$\begin{array}{rcl} \mathbb{C}\left[\left\{\psi_r^p, \phi_s^q \;\middle|\; {\scriptstyle 1\leq p,q\leq N \atop r<0, s\leq 0}\right\}\right] & \cong & \Lambda_N \\ P\left(\psi_r^p, \phi_s^q\right) & \leftrightarrow & P\left(\psi_r^p, \phi_s^q\right)|0\rangle. \end{array}$$

As a consequence,

$$\Omega_N \cong \mathbb{C}\left[\left\{a_n^j, b_m^k; \psi_r^p, \phi_s^q \;\middle|\; {\scriptstyle 1\leq j,k,p,q\leq N \atop n,r<0; m,s\leq 0}\right\}\right].$$

Definition 9.1.1. [73] The *fermionic charge* is the operator $F := [J_\lambda \cdot]|_{\lambda=0} \in \mathrm{End}(\Omega_N)$.

In particular,

$$F(|0\rangle) = F\left(T^n\beta^j\right) = F\left(T^n\gamma^j\right) = 0, \quad \text{for } n \in \mathbb{N}; j \in \{1, \ldots, N\}, \tag{9.1}$$

while

$$F\left(T^n b^j\right) = T^n b^j \text{ and } F\left(T^n c^j\right) = -T^n c^j, \quad \text{for } n \in \mathbb{N}; j \in \{1, \ldots, N\}. \tag{9.2}$$

Given $p \in \mathbb{Z}$, we can set

$$\Omega_N^p = \{\omega \in \Omega_N \mid F(\omega) = p\omega\},$$

where it is easily seen that $V_N \subseteq \Omega_N^0$ from (9.1), and, obviously,

$$\Omega_N = \bigoplus_{p \in \mathbb{Z}} \Omega_N^p.$$

Definition 9.1.2. [73] The *chiral de Rham differential* is defined by

$$d_{\mathrm{CDR}} := -\left[{G^+}_\lambda \cdot\right]\big|_{\lambda=0} \in \mathrm{End}(\Omega_N).$$

In particular, $d_{\mathrm{CDR}}^2 = 0$ since

$$\left[Y\left(G^+, z\right)_{(0)}, Y\left(G^+, z\right)_{(0)}\right] = 0.$$

The endomorphism d_{CDR} is an odd derivation that increases by 1 the fermionic charge, by (9.1) and (9.2). So, the space Ω_N endowed with the grading given by the fermionic charge F and the chiral de Rham differential d_{CDR} above, define a complex

$$\Omega_N \qquad \ldots \xrightarrow{d_{\mathrm{CDR}}} \Omega_N^{-1} \xrightarrow{d_{\mathrm{CDR}}} \Omega_N^0 \xrightarrow{d_{\mathrm{CDR}}} \Omega_N^1 \xrightarrow{d_{\mathrm{CDR}}} \ldots$$

Definition 9.1.3. [73] The *chiral de Rham complex* of $\mathbb{A}^N$ is the vertex algebra Ω_N with the grading given by the fermionic charge and the chiral de Rham differential.

Identifying the coordinate functions $x_1, \ldots, x_N$ of $\mathbb{A}^N$ with $b_0^1, \ldots, b_0^N$, and their differentials $dx_1, \ldots, dx_N$ with $\phi_0^1, \ldots, \phi_0^N$, we can identify the usual de Rham complex $\Omega(\mathbb{A}^N)$ with the conformal weight zero subspace of the chiral de Rham complex. Indeed, we can consider the subspace $\widetilde{\Omega}_N \subseteq \Omega_N$ obtained via the isomorphism above by

$$\begin{array}{rcl} \mathbb{C}\left[\left\{b_0^j, \phi_0^k \;\middle|\; 1 \leq j, k \leq N\right\}\right] & \cong & \widetilde{\Omega}_N \\ P\left(b_0^j, \phi_0^k\right) & \leftrightarrow & P\left(b_0^j, \phi_0^k\right)|0\rangle \end{array}.$$

Theorem 9.1.4 ([73, Theorem 2.4])**.** *There exists an embedding*

$$i\colon \left(\Omega\left(\mathbb{A}^N\right), d_{\mathbb{A}^N}\right) \hookrightarrow \left(\Omega_N, d_{\mathrm{CDR}}\right),$$

compatible with the differentials, which is a quasisomorphism.

An important step in [73] to define the chiral de Rham complex Ω_M^{ch} of a smooth complex algebraic variety M is the construction, via localization, of a sheaf of vertex algebras on $\mathbb{A}^N$, whose space of global sections is Ω_N. By an explicit calculation of the transformation of the generators of the chiral de Rham complex Ω_N under coordinate changes, these sheaves glue and so determine the required sheaf Ω_M^{ch} over M (see [73, Section 3]). The construction works for smooth manifolds in either the algebraic, complex-analytic or $\mathcal{C}^\infty$ settings. We will just focus on the $\mathcal{C}^\infty$ settings, the one we are interested in.

9.1.2 Construction over Smooth Manifolds

Let M be an n-dimensional smooth manifold. Given an open coordinate patch $U \subseteq M$, with coordinates $(\gamma^1, \ldots, \gamma^n)$, we consider the SUSY Lie conformal algebra $\mathcal{R}(U)$ generated, for $j \in \{1, \ldots, n\}$, by the formal symbols

$$b^j := \Pi d\gamma^j, \quad c^j := \Pi \iota_{\frac{\partial}{\partial \gamma^j}} \cdot, \quad \beta^j := \frac{\partial}{\partial \gamma^j}, \quad f := f\left(\gamma^1, \ldots, \gamma^n\right) \in \mathcal{C}^\infty(U).$$

The non-zero λ-brackets are given by (1.24), and the generalization of (1.23) to

$$\left[\beta^j{}_\lambda f\right] = \frac{\partial f}{\partial \gamma^j}, \quad \text{for } j \in \{1, \ldots, n\}.$$

Notice that the odd derivation S is extended to have that $Sf := \Pi df$, for $f \in \mathcal{C}^\infty(U)$. So, we can endow to $\mathcal{R}(U)$ a structure of SUSY Lie conformal algebra using $f \in \mathcal{C}^\infty(U)$, the vector fields β^j, the 1-forms b^j, and the contractions c^j. Now, consider the universal enveloping SUSY vertex algebra $V(\mathcal{R}(U))$, and define a SUSY vertex algebra $\Omega_M^{\mathrm{ch}}(U)$ taking its quotient by the ideal generated, for $f, g \in \mathcal{C}^\infty(U)$ and $j \in \{1, \ldots, n\}$, by

$$: fb^j : - fb^j, \quad : fc^j : - fc^j, \quad : f\beta^j : - f\beta^j, \quad : fg : - fg, \quad |0\rangle - \mathrm{Id}.$$

Now, using Λ-brackets, it is easily seen that this SUSY Lie conformal algebra is defined, for $f = f\left(\gamma^1, \ldots, \gamma^n\right) \in \mathcal{C}^\infty(U)$, via the unique non-zero Λ-bracket relation

$$\left[c^j{}_\Lambda f\right] = \frac{\partial f}{\partial \gamma^j}, \quad \text{for } j \in \{1, \ldots, n\}.$$

Let us see why we define a sheaf of SUSY vertex algebras over M. For $U' \subseteq M$ other open coordinate patch, let $g := \left(g^1, \ldots, g^n\right) : U \longrightarrow U'$, $f := g^{-1} = \left(f^1, \ldots, f^n\right) : U' \longrightarrow U$ be the corresponding coordinate change, given by

$$\widetilde{\gamma}^j = g^j\left(\gamma^1, \ldots, \gamma^n\right), \quad \gamma^j = f^j\left(\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^n\right), \quad \text{for } j \in \{1, \ldots, n\}.$$

Then, as it is explained in [7, Section 5], we get a unique transformation rule

$$\widetilde{c}^j = \sum_{k=1}^n : \left(\frac{\partial f^j}{\partial \widetilde{\gamma}^k}\left(g\left(\gamma^1, \ldots, \gamma^n\right)\right)\right) c^k :, \quad \text{for } j \in \{1, \ldots, n\}.$$

Indeed, we can recover the relations for the rest of generators using that $Sf = \Pi df$, for $f \in \mathcal{C}^\infty(U)$, and $Sc^j = \beta^j$ for $j \in \{1, \ldots, n\}$. We have arrived at the following result.

Theorem 9.1.5 ([69, Lemma 2.26]). *For M smooth manifold, the change of coordinates $g\colon U \longrightarrow U'$ induces a SUSY vertex algebra isomorphism $\varphi_g\colon \Omega^{\mathrm{ch}}_M(U') \longrightarrow \Omega^{\mathrm{ch}}_M(U)$. In addition, given diffeomorphisms $g\colon U_1 \longrightarrow U_2$, and $h\colon U_2 \longrightarrow U_3$, we get $\varphi_{h\circ g} = \varphi_g \circ \varphi_h$.*

For $U \subseteq M$ open, we can endow to $\Omega^{\mathrm{ch}}_M(U)$ a structure of topological vertex algebra as in Theorem 3.4.2. Thus, we can repeat the process explained in Subsection 9.1.1 to obtain a complex Ω^{ch}_M given by the fermionic charge and the chiral differential d^{ch} defined using $J_{(0)}$ and $G^+_{(0)}$, respectively, which are well-defined operators. Then, denoting by $\Omega(M)$ the usual de Rham complex of M, we obtain that

$$i\colon\ (\Omega(M), d) \hookrightarrow \left(\Omega^{\mathrm{ch}}_M, d^{\mathrm{ch}}\right)$$

is a quasisomorphism compatible with these differentials (see [72, Section 3]).

9.1.2.1 Coordinate Independent Construction

Given M an n-dimensional smooth manifold, let $U \subseteq M$ be open. We will denote by $\mathfrak{X}(U)$ the set of vector fields over U, and by $\Omega^1(U)$ the set of 1-forms over U. Consider

$$\mathcal{R}(U) := \left(\mathcal{C}^\infty(U) \oplus \left(\mathfrak{X}(U) \oplus \Omega^1(U)\right) \oplus \Pi\left(\mathfrak{X}(U) \oplus \Omega^1(U)\right)\right) \otimes \mathbb{C}[T]$$

a vector superspace, with the relation $Tf = df$ for $f \in \mathcal{C}^\infty(U)$.

Proposition 9.1.6 ([72, Theorem 1]). *The non-zero λ-brackets*

$$\begin{aligned}
[X_\lambda f] &= X(f), \quad \textit{for } X \in \mathfrak{X}(U); f \in \mathcal{C}^\infty(U),\\
[X_\lambda \Pi Y] &= \Pi\,[X, Y]\,, \quad \textit{for } X, Y \in \mathfrak{X}(U),\\
[X_\lambda Y] &= [X, Y]\,, \quad \textit{for } X, Y \in \mathfrak{X}(U),\\
[X_\lambda \Pi\eta] &= \Pi L_X\eta, \quad \textit{for } X \in \mathfrak{X}(U); \eta \in \Omega^1(U),\\
[X_\lambda \eta] &= L_X\eta + \lambda\iota_X\eta, \quad \textit{for } X \in \mathfrak{X}(U); \eta \in \Omega^1(U),\\
[\Pi X_\lambda \Pi\eta] &= \iota_X\eta, \quad \textit{for } X \in \mathfrak{X}(U); \eta \in \Omega^1(U),
\end{aligned}$$

endow $\mathcal{R}(U)$ with a structure of Lie conformal algebra.

In addition, we can define naturally an odd derivation $S\colon \mathcal{R}(U) \longrightarrow \mathcal{R}(U)$ as follows:

$$Sf := \Pi df, \quad S\Pi X := X, \quad S\Pi\eta := \eta, \quad \text{for } f \in \mathcal{C}^\infty(U); X \in \mathfrak{X}(U); \eta \in \Omega^1(U).$$

As a consequence, we obtain the isomorphism

$$\mathcal{R}(U) \cong \left(\mathcal{C}^\infty(U) \oplus \Pi\left(\mathfrak{X}(U) \oplus \Omega^1(U)\right)\right) \otimes \mathcal{H}$$

of vector superspaces, with the relation $Sf = \Pi df$ for $f \in \mathcal{C}^\infty(U)$ in the right-hand side.

Proposition 9.1.7 ([52, Example 5.13], [53, Section 5]). *The non-zero Λ-brackets*

$$\begin{aligned}
[\Pi X_\Lambda f] &= X(f), \quad \textit{for } X \in \mathfrak{X}(U); f \in \mathcal{C}^\infty(U),\\
[\Pi X_\Lambda \Pi Y] &= \Pi\,[X, Y]\,, \quad \textit{for } X, Y \in \mathfrak{X}(U),\\
[\Pi X_\Lambda \Pi\eta] &= \Pi L_X\eta + \chi\iota_X\eta, \quad \textit{for } X \in X \in \mathfrak{X}(U); \eta \in \Omega^1(U),
\end{aligned}$$

endow $\mathcal{R}(U)$ with a structure of SUSY Lie conformal algebra.

Remark 9.1.8. Strictly speaking, the version of the construction of the SUSY Lie conformal algebra $\mathcal{R}(U)$ in Proposition 9.1.6 given in [72, Theorem 1] does not involve $\Omega^1(U)$ as generators, and the version of the construction of the SUSY Lie conformal algebra $\mathcal{R}(U)$ in Proposition 9.1.7 given in in [52, Example 5.13] and [53, Section 5] adds $\Omega^1(U)$ as generators. The fact that Proposition 9.1.6 and Proposition 9.1.7 are equivalent to the cited results, respectively, follows from the last part of [53, Proposition 4.6].

Let $V(\mathcal{R}(U))$ be the universal enveloping SUSY vertex algebra, and define a new SUSY vertex algebra $\Omega^{\mathrm{ch}}_M(U)$ taking its quotient by the ideal generated by the relations

$$:fg: - fg, \quad :f\Pi X: - \Pi(fX), \quad :f(\Pi\eta): - \Pi(f\eta), \quad |0\rangle - \mathrm{Id},$$

for $f, g \in \mathcal{C}^\infty(U)$, $X \in \mathfrak{X}(U)$ and $\eta \in \Omega^1(U)$. We have arrived at the following result.

Theorem 9.1.9 ([69, Section 3],[52, Theorem 5.14],[53, Theorem 5.3])**.** *For M a smooth manifold, the assignment $U \mapsto \Omega^{\mathrm{ch}}_M(U)$ defines a sheaf of SUSY vertex algebras Ω^{ch}_M.*

9.1.3 Construction over Courant Algebroids

More generally, one can attach a sheaf of vertex algebras Ω^{ch}_E to any Courant algebroid E, as shown independently by Gorbounov, Malikov and Shechtman [45], and Beilinson and Drinfeld [8], and more explicitly by Bressler and Heluani [15, 53, 54]. To describe Ω^{ch}_E, we will use the superfield formalism, following [53, 54]. Let $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ be a Courant algebroid over a smooth manifold M. Let ΠE be the corresponding purely odd vector superbundle. Abusing notation, we will write

$$[\Pi a, \Pi b] := \Pi[a,b], \quad \langle \Pi a, \Pi b\rangle := \langle a, b\rangle, \qquad \text{for } a, b \in \Gamma(E). \tag{9.3}$$

So, we will write $a := \Pi a$ for $a \in \Gamma(E)$. Similarly, we obtain an odd differential operator $\mathcal{D}\colon \mathcal{C}^\infty(M) \longrightarrow \Gamma(\Pi E)$ from the usual one introduced for Courant algebroids.

Theorem 9.1.10 ([54, Proposition 4.1])**.** *Let E be any Courant algebroid over M smooth manifold. Then, there exists a unique sheaf of SUSY vertex algebras Ω^{ch}_E on M endowed with embeddings of sheaves $i\colon \mathcal{C}^\infty(M) \hookrightarrow \Omega^{\mathrm{ch}}_E$ and $j\colon \Gamma(\Pi E) \hookrightarrow \Omega^{\mathrm{ch}}_E$, satisfying that*

(1) i is an isomorphism of unital commutative algebras onto its image, so

$$i(\mathrm{Id}) = |0\rangle, \quad i(fg) =: i(f)i(g):, \quad \text{for } f, g \in \mathcal{C}^\infty(M).$$

(2) i and j are compatible with the $\mathcal{C}^\infty(M)$-module structure of ΠE and the $\mathcal{H}$-module structure of Ω^{ch}_E. That is, $\mathcal{D}$ and S are compatible and the following identities hold:

$$j(fA) =: i(f)j(a):, \quad 2Si(f) = j(\mathcal{D}f), \quad \text{for } f \in \mathcal{C}^\infty(M); a \in \Gamma(\Pi E).$$

(3) i and j are compatible with the Dorfman bracket and pairing, in the sense that

$$[j(a)_\Lambda j(b)] = j([a,b]) + 2\chi i(\langle a, b\rangle), \quad \text{for } a, b \in \Gamma(\Pi E).$$

(4) i and j are compatible with the action of $\Gamma(\Pi E)$ on $\mathcal{C}^\infty(M)$, in the sense that

$$[j(a)_\Lambda i(f)] = i\pi(a)(f), \quad \text{for } a \in \Gamma(\Pi E); f \in \mathcal{C}^\infty(M).$$

(5) Ω^{ch}_E is universal with all these properties.

Proof. Let $V(\mathcal{R})$ be the universal enveloping SUSY vertex algebra associated to $\mathcal{R}$ the $\mathcal{H}$-module given by $(\mathcal{C}^\infty(M)\oplus\Gamma(\Pi E))\otimes\mathcal{H}$ with the relation $2Sf=\mathcal{D}f$, for $f\in\mathcal{C}^\infty(M)$, which is a SUSY Lie conformal algebra with the non-zero Λ-brackets (see also (B.11))

$$[a_\Lambda f]=\langle\mathcal{D}f,a\rangle\,,\quad \text{for } a\in\Gamma(\Pi E)\,;f\in\mathcal{C}^\infty(M),$$
$$[a_\Lambda b]=[a,b]+2\chi\langle a,b\rangle\,,\quad \text{for } a,b\in\Gamma(\Pi E)\,.$$

Then, the quotient of $V(\mathcal{R})$ by the ideal generated by

$$:fg:-fg,\quad :fa:-fa,\quad \mathrm{Id}-|0\rangle\,,\quad \text{for } f\in\mathcal{C}^\infty(M);a\in\Gamma(\Pi E)$$

satisfies the properties above. This is the unique sheaf satisfying them by construction. The proof of $\mathcal{R}$ being a SUSY Lie conformal algebra is found in [53, Proposition 4.3]. □

Theorem 9.1.11 ([53, Proposition 4.6],[54, Proposition 4.1])**.** *Let E be any complex Courant algebroid over an n-dimensional smooth manifold M such that Ω^{ch}_E is the chiral de Rham complex of E. Then,*

(1) if E is the complexified standard Courant algebroid, then $\Omega^{\mathrm{ch}}_E\cong\Omega^{\mathrm{ch}}_M$, where Ω^{ch}_M is the chiral de Rham complex of M. Given $U\subseteq M$ an open coordinate patch, then $\Gamma\left(\Omega^{\mathrm{ch}}_M(U)\right)$ is isomorphic to the tensor product of ghost systems of dimension $2n$.

(2) if M is a point, then E is a complex quadratic Lie algebra, and, as a consequence, $\Gamma(\Omega^{\mathrm{ch}}_E)$ is isomorphic to the universal superaffine vertex algebra of level $k=2$.

9.2 Embeddings of Superconformal VAs into the CDR

Let M be an n-dimensional smooth orientable manifold, and E an exact Courant algebroid over M. Suppose that $(M,\mathcal{J})$ is a generalized Calabi-Yau manifold (see Definition 8.2.7). Let ρ be a non-vanishing section of the corresponding canonical bundle $\mathcal{K}_{\mathcal{J}}$. If L is the associated Dirac structure, there exists a unique $v\in\Gamma(L^*)$ satisfying (8.2). As a consequence, there exists $\zeta\in\Gamma(\det L^*)$ given by $\overline{\rho}=\zeta\cdot\rho$. Let

$$\{\varepsilon_j\}_{j=1,\dots,n}\subseteq L \text{ and } \{\overline{\varepsilon}_j\}_{j=1,\dots,n}\subseteq\overline{L}\cong L^*$$

be a local isotropic frame. By [55, Section 7], we can write ζ locally for $\eta\in\mathcal{C}^\infty(M)$ as

$$\zeta=e^\eta\overline{\varepsilon}_1\wedge\cdots\wedge\overline{\varepsilon}_n.$$

Moreover, we fix a closed pure spinor, and a corresponding volume form μ such that

$$\operatorname{div}_\mu\varepsilon_j=-\sum_{k=1}^n\langle\overline{\varepsilon}_k,[\varepsilon_j,\varepsilon_k]\rangle\,,\quad \operatorname{div}_\mu\overline{\varepsilon}_j=-\sum_{k=1}^n\langle\varepsilon_k,[\overline{\varepsilon}_j,\overline{\varepsilon}_k]\rangle\,,\quad \text{for } j\in\{1,\dots,n\},$$

where div_μ is the Riemannian divergence with respect to μ, for the frames given above. Remember that we write $e_j=\Pi\varepsilon_j\in\Pi L$ and $e^j=\Pi\overline{\varepsilon}_j\in\Pi\overline{L}$, for $j\in\{1,\dots,n\}$.

Lemma 9.2.1 ([54, Lemma 5.1], [55, Lemma 1])**.** *We have two global sections of Ω^{ch}_E,*

$$J_0:=\frac{i}{2}\sum_{j=1}^n:e^je_j:,\quad J=J_0+iT\eta.$$

Theorem 9.2.2 ([54, Lemma 5.3, Theorem 5.5], [55, Theorem 1])**.** *We obtain that*

(1) the following are global sections of the chiral de Rham complex of E,

$$\begin{aligned} H_0 =& \frac{1}{2}\sum_{j=1}^{n}\big(: e_j(Se^j) : + : e^j(Se_j) :\big) - \frac{i}{2}T\mathcal{J}\sum_{j=1}^{n}\left[e^j, e_j\right] \\ &+\frac{1}{4}\sum_{j,k=1}^{n}\Big(: e^j : e^k\left[e_j, e_k\right] :: + : e_j : e_k\left[e^j, e^k\right] ::\Big), \\ H =& H_0 - iT\mathcal{J}\mathcal{D}\eta. \end{aligned}$$

(2) J_0 and H_0 generate an $N = 2$ superconformal vertex algebra with $c = 3\dim M$.

(3) the functions $f \in \mathcal{C}^\infty(M)$ are primary of conformal weight 0 with respect to H. Moreover, the sections $a \in \Gamma(\Pi E)$ have conformal weight $1/2$ with respect to H.

(4) J and H generate an $N = 2$ superconformal vertex algebra with $c = 3\dim M$.

Let $(M, \mathcal{J}_+, \mathcal{J}_-)$ be a $2n$-dimensional generalized Calabi-Yau metric manifold (see Definition 8.2.10). Since M is generalized Kähler, we have a metric g on M, and we get (8.4). Consider $L_\pm$ the associated Dirac structures of $\mathcal{J}_\pm$, for which it is satisfied (8.3). Now, let $\rho_\pm \in \mathcal{K}_{\mathcal{J}_\pm}$ be the corresponding non-vanishing pure spinors with associated global sections $\zeta_\pm \in \Gamma\left(\det L_\pm^*\right)$. For the holomorphic coordinates $\{z_j^\pm\}_{j=1,\dots,n} \subseteq \mathcal{C}^\infty(M)$, let

$$\left\{\varepsilon_j^\pm := \frac{\partial}{\partial \bar{z}_j^\pm} + g\frac{\partial}{\partial \bar{z}_j^\pm}\right\}_{j=1,\dots,n} \subseteq \ell_\pm \text{ and } \left\{\bar{\varepsilon}_j^\pm = g^{-1}d\bar{z}_j^\pm + d\bar{z}_j^\pm\right\}_{j=1,\dots,n} \subseteq \bar{\ell}_\pm$$

be local isotropic frames. By [55, Section 7], we can write $\zeta_\pm$ locally for $\eta^\pm \in \mathcal{C}^\infty(M)$ as

$$\zeta_+ = e^{\eta^+ + \eta^-}\bar{\varepsilon}_1^+ \wedge \cdots \wedge \bar{\varepsilon}_n^+ \wedge \bar{\varepsilon}_1^- \wedge \cdots \wedge \bar{\varepsilon}_n^-, \quad \zeta_- = e^{\eta^+ - \eta^-}\bar{\varepsilon}_1^+ \wedge \cdots \wedge \bar{\varepsilon}_n^+ \wedge \varepsilon_1^- \wedge \cdots \wedge \varepsilon_n^-.$$

Remember that we write $e_j^\pm = \Pi\varepsilon_j^\pm \in \Pi\ell_\pm$ and $e_\pm^j = \Pi\bar{\varepsilon}_j^\pm \in \Pi\bar{\ell}_\pm$, for $j \in \{1, \dots, n\}$.

Theorem 9.2.3 ([55, Theorem 2])**.** *The global sections*

$$J_\pm := \frac{i}{2}\sum_{j=1}^{n} : e_\pm^j e_j^\pm : + iT\eta^\pm \in \Gamma(\Omega_E^{\mathrm{ch}}), \tag{9.4}$$

$$\begin{aligned} H_\pm :=& \frac{1}{2}\sum_{j=1}^{n}\Big(: e_j^\pm(Se_\pm^j) : + : e_\pm^j(Se_j^\pm) :\Big) - \frac{i}{2}T\mathcal{J}_\pm\sum_{j=1}^{n}\left[e_\pm^j, e_j^\pm\right] - iT\mathcal{J}_\pm\mathcal{D}\eta^\pm \\ &+\frac{1}{4}\sum_{j,k=1}^{n}\Big(: e_\pm^j : e_k^\pm\left[e_j^\pm, e_\pm^k\right] :: - : e_\pm^j : e_\pm^k\left[e_j^\pm, e_k^\pm\right] :: \\ &+ : e_j^\pm : e_\pm^k\left[e_\pm^j, e_k^\pm\right] :: - : e_j^\pm : e_k^\pm\left[e_\pm^j, e_\pm^k\right] ::\Big) \in \Gamma(\Omega_E^{\mathrm{ch}}) \end{aligned} \tag{9.5}$$

generate two Λ-commuting $N = 2$ superconformal vertex algebra with $c = 3/2\dim M$.

The results of this last section are our starting point for the construction of the embeddings from solutions to the Killing spinor equations that we will give in next chapter.

Chapter 10

Embeddings from Killing Spinors

The aim of this chapter is the recollection of the main results of this thesis corresponding to the construction of embeddings from the $N = 2$ superconformal vertex algebra into the superaffinization of quadratic Lie algebras with non-zero level, and the chiral de Rham complex of Courant algebroids. The condition we will require to obtain our embeddings is, essentially, having a solution to the Killing spinor equations. These new embeddings are generalizations of Getzler's Theorem 3.3.3 for Manin triples, while they are based on Heluani-Zabzine's Theorem 9.2.3 for generalized Calabi-Yau metric structures

10.1 Embedding SUSY VAs from F-term and D-term

Let E be the complexification of a real Courant algebroid over a smooth manifold M, for which we can construct the chiral de Rham complex Ω^{ch}_E. Now, fix $E = l \oplus \bar{l} \oplus C_-$ a direct sum decomposition, with $l, \bar{l} \subseteq E$ isotropic n-dimensional subbundles, for which the restriction $\langle \cdot, \cdot \rangle|_{C_\pm}$ is non-degenerate, where $C_+ = l \oplus \bar{l}$, and $C_- = C_+^\perp$. Let

$$\pi_\pm \colon E \longrightarrow C_\pm, \quad \pi_l \colon E \longrightarrow l, \quad \pi_{\bar{l}} \colon E \longrightarrow \bar{l}$$

be the orthogonal projections. So, when there is no possible confusion, we will write

$$a_\pm = \pi_\pm a, \quad a_l = \pi_l a, \quad a_{\bar{l}} = \pi_{\bar{l}} a, \quad \text{for } a \in \Gamma(E). \tag{10.1}$$

Now, fix a frame $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n \subseteq C_+$ satisfying (6.28). Define the associated odd sections

$$e_j = \Pi\epsilon_j, \quad e^j = \Pi\bar{\epsilon}_j, \qquad \text{for } j \in \{1, \ldots, n\}.$$

Remember that we work with parity-reversed sections (see Remark 3.2.1). In particular, for $a, b, c \in l \oplus \bar{l}$ elements of this isotropic frame, it is clearly satisfied that

$$[a, b] = -[a, b], \quad \langle [a, b], c\rangle = -\langle b, [a, c]\rangle, \quad \langle [a, b], c\rangle = \langle a, [b, c]\rangle.$$

We define

$$\begin{array}{rccc} I_+ \colon & C_+ & \longrightarrow & C_+ \\ & a & \mapsto & a_l - a_{\bar{l}} \end{array},$$

writing $I_+\Pi a \equiv \Pi I_+ a$ for $a \in C_+$. Now, define the element of $C_+ \subseteq E$ given by

$$w = \Pi I_+ \left[\bar{\epsilon}_j, \epsilon_j\right]_+ = \left[e^j, e_j\right]_l - \left[e^j, e_j\right]_{\bar{l}} \in \Gamma(\Pi C_+).$$

This also works for $(\mathfrak{g}, (\cdot|\cdot))$ complex quadratic Lie algebra, taking $V^k_{\text{super}}(\mathfrak{g})$ the universal superaffine vertex algebra of $\mathfrak{g}$ with level $0 \neq k \in \mathbb{C}$ by Theorem 9.1.11. So, the following results are written for an arbitrary non-zero level in quadratic Lie algebras, but, for the chiral de Rham complex, we take $k = 2$. Define the local sections

$$\begin{aligned} J_0 :&= \frac{i}{k}\sum_{j=1}^{n} :e^j e_j:, \\ H' :&= \frac{1}{k}\sum_{j=1}^{n}\left(:e_j\left(Se^j\right): + :e^j\left(Se_j\right):\right) + \frac{T}{k}w + \frac{1}{k^2}\sum_{j,k=1}^{n}\Big(:e_j:e^k\left[e^j, e_k\right]:: \\ &\quad + :e^j:e_k\left[e_j, e^k\right]:: - :e_j:e_k\left[e^j, e^k\right]:: - :e^j:e^k\left[e_j, e_k\right]::\Big), \end{aligned} \tag{10.2}$$

and

$$c_0 := 3\dim l \in \mathbb{C}. \tag{10.3}$$

Now, define for each $i, j \in \{1, \ldots, n\}$ the locally defined sections

$$\begin{aligned} R :&= \sum_{j,k=1}^{n}\left(3\left\langle\left[e^j, e_j\right]_-, \left[e^k, e_k\right]_-\right\rangle - \left\langle\left[e^j, e_k\right]_-, \left[e^k, e_j\right]_-\right\rangle\right), \\ F^{ij} :&= \operatorname{tr}|_{\bar{l}}\left(\operatorname{ad}_{[e^i, e^j]}\right) + \sum_{k=1}^{n}\left(\left\langle\mathcal{D}\left\langle e^i, \left[e_k, e^k\right]\right\rangle, e^j\right\rangle - \left\langle\mathcal{D}\left\langle e^j, \left[e_k, e^k\right]\right\rangle, e^i\right\rangle\right), \\ F_{ij} :&= \operatorname{tr}|_{l}\left(\operatorname{ad}_{[e_i, e_j]}\right) + \sum_{k=1}^{n}\left(\left\langle\mathcal{D}\left\langle e_i, \left[e^k, e_k\right]\right\rangle, e_j\right\rangle - \left\langle\mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle\right), \end{aligned}$$

The following result is our starting point for the construction of all our embeddings.

Theorem 10.1.1. *Assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26)*. Then,*

$$\begin{aligned} \left[J_0{}_\Lambda J_0\right] &= -\left(H' + \frac{\lambda\chi}{3}c_0\right), \\ \left[H'{}_\Lambda J_0\right] &= (2\lambda + 2T + \chi S)\left(J_0 - \frac{i}{k}S\sum_{j=1}^{n}\left[e^j, e_j\right]_-\right) \\ &\quad + \frac{i}{2k}TS\mathcal{D}R + \frac{i}{k^2}\lambda\sum_{i,j=1}^{n}\left(:F^{ij}:e_j e_i:: - :F_{ij}:e^j e^i::\right) \\ &\quad - \frac{i}{k}\left(T + \frac{3}{k}\lambda\right)\sum_{i,j=1}^{n}\left(:e^i\left[\left[e^j, e_j\right]_-, e_i\right]_-: + :e_i\left[\left[e^j, e_j\right]_-, e^i\right]_-:\right). \end{aligned} \tag{10.4}$$

Moreover, the sections (10.2) *are global if M admits an atlas of holomorphic coordinates such that the Jacobian of any change of coordinates has constant determinant.*

Proof. The proof is given in Appendix C. $\square$

Let $\alpha \in \Gamma(E)$ be arbitrary, and consider the associated odd section $a = \Pi\alpha \in \Gamma(\Pi E)$. Introduce the locally defined sections

$$\begin{aligned} J &:= J_0 - 2\frac{S}{k} ia, \\ H &:= H' - 2\frac{T}{k} I_+ a_+ + 2\frac{S}{k^2} \sum_{j=1}^{n} \left(: \left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] : -2T \left\langle \left[a, e^j\right], e_j \right\rangle \right), \end{aligned}$$

and

$$c := 3 \left(\dim l - \frac{4}{k} \left(\sum_{j=1}^{n} \left\langle \left[a, e^j\right], e_j \right\rangle - \langle a, a \rangle \right) \right).$$

We obtain the following result for the "corrected" generators above.

Theorem 10.1.2. *Assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26)*. Then,*

$$[J_\Lambda J] = -\left(H + \frac{\lambda\chi}{3} c \right) - \frac{1}{2} (\chi S + \lambda) S (c - c_0). \tag{10.5}$$

Proof. The result follows from Theorem 10.1.1. By sesquilinearity, we have

$$\begin{aligned} [J_\Lambda J] &= [J_{0\Lambda} J_0] - i\frac{2}{k} [J_{0\Lambda} Sa] - i\frac{2}{k} [Sa_\Lambda J_0] - \frac{4}{k^2} [Sa_\Lambda Sa] \\ &= [J_{0\Lambda} J_0] + i\frac{2}{k} (\chi + S) [J_{0\Lambda} a] - i\frac{2}{k} \chi [a_\Lambda J_0] + \frac{4}{k^2} (\lambda - \chi S) [a_\Lambda a]. \end{aligned}$$

The first summand is known from the first identity in (10.4). Moreover, by Lemma C.1.1, we also know the following two summands. We must compute the last one, which is

$$[a_\Lambda a] = [a, a] + 2\chi \langle a, a \rangle = (S + 2\chi) \langle a, a \rangle,$$

using (B.9). So, we obtain the required identity, since

$$\begin{aligned} [J_\Lambda J] &= [J_{0\Lambda} J_0] + i\frac{2}{k} (\chi + S) [J_{0\Lambda} a] - i\frac{2}{k} \chi [a_\Lambda J_0] + \frac{4}{k^2} (\lambda - \chi S) [a_\Lambda a] \\ &= - \left(H' - \frac{2}{k} T I_+ a_+ + \frac{2}{k^2} S \sum_{j=1}^{n} \left(: \left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] : -2T \left\langle \left[a, e^j\right] \middle| e_j \right\rangle \right) \right. \\ &\quad \left. + \frac{\lambda\chi}{3} c + (\chi S + \lambda) S \left(\sum_{j=1}^{n} \left\langle \left[a, e^j\right] \middle| e_j \right\rangle - \langle a, a \rangle \right) \right) \end{aligned}$$ $\square$

Remark 10.1.3. (a) A necessary condition to construct an embedding of the $N = 2$ superconformal vertex algebra generated by J and H as above, is to have

$$\sum_{j=1}^{n} \left\langle \left[a, e^j\right] \middle| e_j \right\rangle - \langle a, a \rangle \in \mathbb{C}. \tag{10.6}$$

This also simplifies the formula (10.5), since the last summand will be zero.

(b) If $\alpha \in \Gamma(E)$ is holomorphic (see Definition 7.2.3), then by (B.38) we obtain that

$$H = H' - 2\frac{T}{k} I_+ a_+. \tag{10.7}$$

We will use this notion in our algebraic results. Note that this follows if $[\alpha, \cdot] = 0$.

(c) Let $\alpha \in \Gamma(E)$ be a section satisfying $[\alpha, \cdot] = 0$ (that is, $\alpha \in \Gamma(E)$ is a *symmetry* of E). Then, the condition (10.6) is equivalent to have that $\langle \alpha, \alpha \rangle \in \mathbb{C}$. In fact, this is always satisfied when $[\alpha, \cdot] = 0$, since $\mathcal{D} \langle \alpha, \alpha \rangle = 2 [\alpha, \alpha] = 0$ implies $\langle \alpha, \alpha \rangle \in \mathbb{C}$. Moreover, this holds provided that α is closed (see Definition 6.3.17). We will use this notion in our geometric results, since (6.32) follows by Proposition 6.3.13.

Remark 10.1.4. In the definitions of J_0 and J, we are considering $l \oplus \bar{l}$ as an ordered pair. If $\overline{J}_0$ and $\overline{J}$ are the sections associated to $\bar{l} \oplus l$, by (B.14), $\overline{J}_0 = -J_0$ and $\overline{J} = -J$. In both cases, the associated candidate to Neveu-Schwarz generator is going to be H' and H, respectively, since the extra minus signs is absorbed by J_0 and J, respectively.

Now, fix $\varphi, \varepsilon \in \Gamma(E)$, related by $I_+ \varepsilon_+ = \varphi_+$, and consider the associated odd sections

$$e = \Pi\varepsilon, \quad u = \Pi\varphi \in \Gamma(\Pi E).$$

From now, to simplify the computations, we are going to use the Einstein summation convention for repeated indices. We study the algebraic and geometric cases independently.

10.1.1 Main Theorems: Algebraic Case

As a first consequence of Theorem 10.1.1, we obtain the embeddings for quadratic Lie algebras [2]. Now, these results are direct consequences of more general computations.

Theorem 10.1.5 ([2, Theorem 3.13])**.** *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a complex quadratic Lie algebra. Assume that $l \oplus \bar{l} \subseteq \mathfrak{g}$ satisfies the F-term condition* (6.26)*, the weaker variant* (7.13) *of the D-term condition, and that*

$$w \in [l, l]^{\perp} \cap \left[\bar{l}, \bar{l}\right]^{\perp}. \tag{10.8}$$

Then, the vectors

$$\begin{aligned} J_0 =& \frac{i}{k} \sum_{j=1}^{n} : e^j e_j : \in V^k_{\text{super}}(\mathfrak{g}), \\ H' =& \frac{1}{k} \sum_{j=1}^{n} \left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) : \right) + \frac{T}{k} w + \frac{1}{k^2} \sum_{j,k=1}^{n} \Big(: e_j : e^k \left[e^j, e_k\right] :: \\ &+ : e^j : e_k \left[e_j, e^k\right] :: - : e_j : e_k \left[e^j, e^k\right] :: - : e^j : e^k \left[e_j, e_k\right] :: \Big) \in V^k_{\text{super}}(\mathfrak{g}), \end{aligned} \tag{10.9}$$

induce an embedding of the $N = 2$ superconformal vertex algebra with central charge $c_0 = 3 \dim l$ into the universal superaffine vertex algebra $V^k_{\text{super}}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.

Proof. Remember that vectors J_0, H and w are well-defined as explained in Chapter 3. Then, the result follows directly from Theorem 10.1.1, since, by the weaker variant of the D-term condition (7.13), we must check $F_{ij} = 0 = F^{ij}$ for $i, j \in \{1, \dots, n\}$. Indeed,

$$\mathrm{tr}|_l \left(\mathrm{ad}_{[e_i,e_j]}\right) = \sum_{k=1}^{n} \left(\left[[e_i, e_j], e^k \right] \middle| e_k \right) = - \left([e_i, e_j]_l \middle| w \right) = 0, \quad \text{for } i, j \in \{1, \dots, n\},$$

and

$$\mathrm{tr}|_{\bar{l}} \left(\mathrm{ad}_{[e^i,e^j]}\right) = \sum_{k=1}^{n} \left(\left[[e^i, e^j], e_k \right] \middle| e^k \right) = - \left([e^i, e^j]_{\bar{l}} \middle| w \right) = 0, \quad \text{for } i, j \in \{1, \dots, n\},$$

by antisymmetry and invariance, using extra condition (10.8). □

The next result can be seen as the "dilaton correction" of previous one, since we correct the supersymmetry generator J_0 by adding $\alpha = \varphi_+ \in l \oplus \bar{l}$ to obtain J. We need to impose holomorphicity condition from Definition 7.2.3. We need some technical results.

Lemma 10.1.6 ([2, Lemma 3.15])**.** *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a complex quadratic Lie algebra. Assume that $l \oplus \bar{l} \subseteq \mathfrak{g}$ satisfies the weaker variant* (7.14) *of the D-term condition, and that $\varphi_+ \in l \oplus \bar{l}$ is holomorphic, so* (7.10) *is satisfied. Recall that $e_+ = \Pi I_+ \varphi_+$. Then,*

$$\begin{aligned} H &= \frac{1}{k} \sum_{j=1}^{n} \left(: e_j \left(S e^j \right) : + : e^j \left(S e_j \right) : \right) + \frac{1}{k^2} \sum_{j,k=1}^{n} \Big(: e_j : e^k \left[e^j, e_k \right] :: \\ &\quad + : e^j : e_k \left[e_j, e^k \right] :: - : e_j : e_k \left[e^j, e^k \right] :: - : e^j : e^k \left[e_j, e_k \right] :: \Big) \in V^k_{\mathrm{super}}(\mathfrak{g}), \\ c &= 3 \left(\dim l + \frac{4}{k} (e_+ | e_+) \right) \in \mathbb{C}. \end{aligned}$$

Proof. The vector H is well defined as explained in Chapter 3. Then, the result follows directly from Theorem 10.1.2 after impossing the extra conditions given in the statement. Indeed, since $\varphi_+ \in l \oplus \bar{l}$ is holomorphic, by (B.38), we obtain that (10.7). Finally, we conclude the identity by the weaker variant of the D-term condition (7.14). □

Remark 10.1.7. Notice that, if $l \oplus \bar{l}$ satisfies the F-term condition (6.26), then $\varepsilon_+ \in l \oplus \bar{l}$ is holomorphic if and only if $\varphi_+ = I_+ \varepsilon_+ \in l \oplus \bar{l}$ is holomorphic.

So, as a consequence of Theorem 10.1.2 and Lemma 10.1.6, we obtain another different embedding for quadratic Lie algebras. Note that we can give a geometric meaning to the condition (10.8) first introduced by Getzler (see Section 3.3) in terms of the previously given notion of holomorphicity (see Definition 7.2.3).

Theorem 10.1.8 ([2, Theorem 3.16])**.** *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a quadratic Lie algebra. Assume that $l \oplus \bar{l} \subseteq \mathfrak{g}$ satisfies the F-term condition* (6.26)*, that $(l \oplus \bar{l}, \varepsilon_+)$ satisfies the D-term condition* (7.12)*, and that $\varepsilon_+ \in l \oplus \bar{l}$ is holomorphic, so* (7.10) *is satisfied. Recall that*

$$e_+ = \Pi \varepsilon_+, \quad u_+ = \Pi I_+ \varepsilon_+.$$

Then, the vectors

$$\begin{aligned} J =&\frac{i}{k}\sum_{j=1}^{n} :e^{j}e_{j}: -2\frac{S}{k}iu_{+} \in V^{k}_{super}(\mathfrak{g}), \\ H =&\frac{1}{k}\sum_{j=1}^{n}\left(:e_{j}\left(Se^{j}\right): + :e^{j}\left(Se_{j}\right):\right) + \frac{1}{k^{2}}\sum_{j,k=1}^{n}\Big(:e_{j}:e^{k}\left[e^{j},e_{k}\right]:: \\ &+ :e^{j}:e_{k}\left[e_{j},e^{k}\right]:: - :e_{j}:e_{k}\left[e^{j},e^{k}\right]:: - :e^{j}:e^{k}\left[e_{j},e_{k}\right]::\Big) \in V^{k}_{super}(\mathfrak{g}), \end{aligned} \tag{10.10}$$

induce an embedding of the $N=2$ superconformal vertex algebra with central charge

$$c = 3\left(\dim l + \frac{4}{k}\left(e_{+}|e_{+}\right)\right) \in \mathbb{C}$$

into the universal superaffine vertex algebra $V^{k}_{super}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.

Proof. The result follows from Proposition 10.1.2 and Lemma 10.1.6. That is, the proof reduces to check the second identity in (2.21). By sesquilinearity, we have

$$\begin{aligned} [H_{\Lambda}J] &= \left[H'{}_{\Lambda}J_{0}\right] - i\frac{2}{k}\left[H'{}_{\Lambda}Su_{+}\right] - \frac{2}{k}\left[Te_{+\Lambda}J_{0}\right] + i\frac{4}{k^{2}}\left[Te_{+\Lambda}Su_{+}\right] \\ &= \left[H'{}_{\Lambda}J_{0}\right] - i\frac{2}{k}\left(S+\chi\right)\left[H'{}_{\Lambda}u_{+}\right] + \frac{2}{k}\lambda\left[e_{+\Lambda}J_{0}\right] - i\frac{4}{k^{2}}\lambda\left(\chi+S\right)\left[e_{+\Lambda}u_{+}\right]. \end{aligned}$$

The first summand is known for having that J_0 and H' are generators for an embedding of an $N=2$ superconformal vertex algebra with central charge c_0. We compute the other summands. So, by sesquilinearity and Remark C.2.4, since $\varepsilon_{+} \in l \oplus \bar{l}$ is an infinitesimal isometry in particular, we have that

$$\left[H'{}_{\Lambda}u_{+}\right] = \left[H'{}_{\Lambda}e_{l}\right] - \left[H'{}_{\Lambda}e_{\bar{l}}\right] = \left(\lambda + 2T + \chi S\right)u_{+} + \frac{\lambda}{k}\left[u_{+},w\right]_{+} - \lambda\chi\left(u_{+}\,|w\right).$$

By (C.2a), using that $\varepsilon_{+} \in l \oplus \bar{l}$ is holomorphic, we have that

$$\begin{aligned} \left[e_{+\Lambda}J_{0}\right] &= \frac{i}{k}\sum_{j=1}^{n}\left(:\left[e_{+},e^{j}\right]_{\bar{l}}e_{j}: + :e^{j}\left[e_{+},e_{j}\right]_{l}: + k\chi e + k\lambda\left(e_{+}\left|\left[e^{j},e_{j}\right]\right.\right)\right) \\ &= i\left(\chi u_{+} - \lambda\left(u_{+}\,|w\right)\right), \end{aligned}$$

where we have used the identity (B.38) for $a = e_{+}$ to obtain the last equality. Now, using again the infinitesimal isometry condition, a simple computation shows that

$$\left[e_{+\Lambda}u_{+}\right] = \left[e_{+},u_{+}\right] + k\chi\left(e_{+}|u_{+}\right) = \left[e_{+},u_{+}\right]_{+}.$$

Applying now the D-term equation, we obtain the required identity, since

$$\begin{aligned} [H_{\Lambda}J] &= \left[H'{}_{\Lambda}J_{0}\right] - i\frac{2}{k}\left(S+\chi\right)\left[H'{}_{\Lambda}u_{+}\right] + \frac{2}{k}\lambda\left[e_{+\Lambda}J_{0}\right] - i\frac{4}{k^{2}}\lambda\left(\chi+S\right)\left[e_{+\Lambda}u_{+}\right] \\ &= -i\frac{4}{k^{2}}\lambda\left(S+\chi\right)\left(\left[u_{+},e_{+}\right]_{+} + \left[e_{+},u_{+}\right]_{+}\right) + \left(2\lambda + 2T + \chi S\right)J \\ &= \left(2\lambda + 2T + \chi S\right)J. \end{aligned}$$

The formula for H and the central charge follow from Lemma 10.1.6. □

Remark 10.1.9. Notice that Theorem 10.1.5 generalizes Getzler's construction. Indeed, it suffices to take $V_- = \{0\}$ to recover Theorem 3.3.3. Moreover, Theorem 10.1.8 gives a dilaton correction of Getzler's construction, where the associated Neveu-Schwarz vector is the one given by the Kac-Todorov construction applied to $\mathfrak{g} = V_+ := l \oplus \bar{l}$.

Remark 10.1.10. The two embeddings of Theorem 10.1.5 and 10.1.8 do not necessarily induce on $V^k_{\text{super}}(\mathfrak{g})$ a superconformal vertex algebra structure. In fact, the Fourier modes L_{-1} and L_0 associated to the underlying Virasoro generator, respectively, may not be the translation operator T of $V^k_{\text{super}}(\mathfrak{g})$, or may not act semisimply on $V^k_{\text{super}}(\mathfrak{g})$.

Notice that these two embeddings induce similar results for the chiral de Rham complex of homogeneous manifolds. Indeed, assume that our manifold is a compact Lie group K, and let E be a left-equivariant Courant algebroid over K. By Proposition 7.1.9, we can associate to E a quadratic Lie algebra $\mathfrak{g}$ given by the invariant sections of E. Applying the universal construction in Theorem 9.1.10, we obtain an embedding of $V^2_{\text{super}}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$.

Proposition 10.1.11 ([2, Proposition 4.17])**.** *Given K compact Lie group, let E be any left-equivariant Courant algebroid over K. Then, there exists*

$$V^2_{\text{super}}\left(\Gamma(E\otimes_{\mathbb{R}}\mathbb{C})^K\right) \hookrightarrow \Gamma\left(\Omega^{\text{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}\right) := H^0\left(K, \Omega^{\text{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}\right)$$

embedding of the universal superaffine vertex algebra of level $k = 2$ into $\Gamma\left(\Omega^{\text{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}\right)$.

Proof. Observe that $H^0(K, \Omega^{\text{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}})$ inherits a natural structure of SUSY vertex algebra. We denote by $\mathcal{R}$ the underlying SUSY Lie conformal algebra. Now, by the superaffine vertex algebra example in Subsection 2.5.2 and Theorem 9.1.10, we have an embedding of the underlying SUSY Lie conformal algebra $\text{SCur}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})$ localized at $k = 2$ into $\mathcal{R}$. Since $V^2_{\text{super}}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})$ is the universal enveloping SUSY vertex algebra of $\text{SCur}(\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C})$ of level $k = 2$, this induces an embedding as in the statement. This follows because any morphism from a SUSY Lie conformal algebra $\mathcal{R}'$ into a SUSY vertex algebra can be extended to a unique SUSY vertex algebra morphism from $V(\mathcal{R}')$. ☐

10.1.2 Main Theorems: Geometric Case

Now, we return to the general case to generalize the embedding given on previous result. We can take $k = 2$ from now, since we will restrict to the chiral de Rham complex. Since we will use closeness condition introduced in Definition 6.3.17 in our conditions, we will work from now with a section $\alpha = \varphi \in \Gamma(E)$ satisfying $[\varphi, \cdot] = 0$.

Lemma 10.1.12. *Assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26) *and that the section $\varphi \in \Gamma(E)$ satisfies that $[\varphi, \cdot] = 0$. Then, setting $u = \Pi\varphi$, for the local sections introduced in* (10.2)*, we obtain that*

$$\begin{aligned}\left[u_\Lambda H'\right] = {}& \chi\sum_{j=1}^n\left(:e_j\left[u_+, e^j\right]_-: + :e^j\left[u_+, e_j\right]_-: - (\lambda + T)\left\langle\left[e_+, e^j\right], e_j\right\rangle\right)\\ &+ \chi\sum_{j,k=1}^n\left(:\left\langle\left[u_+, e^j\right], e^k\right\rangle :e_je_k:: + :\left\langle\left[u_+, e_j\right], e_k\right\rangle :e^je^k::\right) + (\lambda + \chi S)\, u_+,\end{aligned}$$

and

$$\begin{aligned}\left[{H'}_{\Lambda}u\right] &= (\chi+S)\sum_{j=1}^{n}\Big(: e_j\left[u_+,e^j\right]_- : + : e^j\left[u_+,e_j\right]_- : +\lambda\left\langle\left[e_+,e^j\right],e_j\right\rangle\Big)\\ &\quad+(\chi+S)\sum_{j,k=1}^{n}\Big(:\left\langle\left[u_+,e^j\right],e^k\right\rangle : e_je_k :: + :\left\langle\left[u_+,e_j\right],e_k\right\rangle : e^je^k ::\Big)\\ &\quad+(\lambda+\chi S+2T)\,u_+.\end{aligned}$$

Proof. Applying Jacobi identity for the Λ-bracket, thanks to the first identity in (10.4), we obtain by antisymmetry of the Λ-bracket that

$$\left[u_\Lambda H'\right]=\left[u_\Lambda\left(H'+\frac{\gamma\eta}{3}c_0\right)\right]=-\left[u_\Lambda\left[J_{0\Gamma}J_0\right]\right]=-\left[\left[u_\Lambda J_0\right]_{\Lambda+\Gamma}J_0\right]-\left[\left[u_\Lambda J_0\right]_{-\nabla-\Gamma}J_0\right].$$

Now, by (C.2a), since $[\varphi,\cdot]=0$, we obtain that

$$\begin{aligned}\left[\left[u_\Lambda J_0\right]_\Omega J_0\right] &= -i\chi\left[e_{+\Omega}J_0\right]\\ &= \frac{\chi}{2}\left(:\left[e_+,e^j\right]e_j : + : e^j\left[e_+,e_j\right] :\right)+\chi\left(\xi e_+ +\omega\left\langle\left[e_+,e^j\right],e_j\right\rangle\right).\end{aligned}$$

So, we arrive at

$$\begin{aligned}\left[u_\Lambda H'\right] &= -\chi\left(:\left[I_+u_+,e^j\right]e_j : - : e^j\left[I_+u_+,e_j\right] : +(\lambda-T)\left\langle\left[e_+,e^j\right],e_j\right\rangle\right)+(\lambda+\chi S)\,u_+\\ &= -\chi\Big(:\left[u_+,e^j\right]_- e_j : - : e^j\left[u_+,e_j\right]_- : +(\lambda+T)\left\langle\left[e_+,e^j\right],e_j\right\rangle\Big)\\ &\quad+\chi\Big(:\left\langle\left[u_+,e^j\right],e^k\right\rangle : e_je_k :: + :\left\langle\left[u_+,e_j\right],e_k\right\rangle : e^je^k ::\Big)+(\lambda+\chi S)\,u_+,\end{aligned}$$

where last identity follows from (B.39) for $a=u$. So, we have obtained the first desired identity. The last one follows from antisymmetry of the Λ-bracket. $\square$

So, from now, we will work with the local sections

$$J=J_0-Siu,\quad H=H'-Te_+,\tag{10.11}$$

where remember that $e_+=I_+u_+$, and

$$c=3\left(\dim l+2\left\langle\varphi,\varphi\right\rangle\right).\tag{10.12}$$

Proposition 10.1.13. *We assume that $l\oplus\bar{l}$ satisfies the F-term condition* (6.26), *and that $\varphi\in\Gamma(E)$ satisfies that $[\varphi,\cdot]=0$. Then, setting $u=\Pi\varphi$, the locally defined sections* (10.11) *satisfy*

$$\begin{aligned}\left[H_\Lambda J\right] &= (2\lambda+2T+\chi S)\left(J_0-Si\left(u_+ +\frac{1}{2}\sum_{j=1}^{n}\left[e^j,e_j\right]_-\right)\right)\\ &\quad+\frac{i}{4}\left(TS\mathcal{D}R+\lambda\sum_{j,k=1}^{n}\left(: F^{ij} : e_je_i :: - : F_{ij} : e^je^i ::\right)\right)\\ &\quad-i\left(T+\frac{3}{2}\lambda\right)\sum_{j,k=1}^{n}\left(: e_k\left[u_+ +\frac{1}{2}\left[e^j,e_j\right]_-,e^k\right] : + : e^k\left[u_+ +\frac{1}{2}\left[e^j,e_j\right]_-,e_k\right] :\right).\end{aligned}$$

Proof. The result follows from Theorem 10.1.2. First, we start applying sesquilinearity,

$$\begin{aligned}[H_\Lambda J] &= \left[H'{}_\Lambda J_0\right] - i\left[H'{}_\Lambda Su\right] - \left[Te_{+\Lambda} J_0\right] + i\left[Te_{+\Lambda} Su\right] \\ &= \left[H'{}_\Lambda J_0\right] - i\left(\chi + S\right)\left[H'{}_\Lambda u\right] + \lambda\left[e_{+\Lambda} J_0\right] - i\lambda\left(\chi + S\right)\left[e_{+\Lambda} u\right].\end{aligned}$$

The first summand is known from the second identity in (10.4). By Lemma 10.1.12, and Lemma C.1.1 for $a = e_+$, we also known the following two summands. Indeed,

$$\begin{aligned}[e_{+\Lambda} J_0] &= \frac{i}{2}\left(:\left[I_+u_+, e^j\right]e_j : + : e^j\left[I_+u_+, e_j\right]:\right) \\ &\quad + i\left(\chi u_+ + \lambda\left\langle\left[e_+, e^j\right], e_j\right\rangle\right) \\ &= \frac{i}{2}\left(:\left[u_+, e^j\right]_- e_j : - : e^j\left[u_+, e_j\right]_- :\right) \\ &\quad + i(\lambda + T)\left(\left\langle\left[Iu_+, e^j\right], e_j\right\rangle - \chi u_+\right) \\ &\quad + \frac{i}{2}\left(:\left\langle\left[u_+, e^j\right], e^k\right\rangle : e_k e_j :: + : \left\langle\left[u_+, e_j\right], e_k\right\rangle : e^k e^j ::\right),\end{aligned}$$

using (B.39) for $a = u$. We must compute the last one,

$$\begin{aligned}[e_{+\Lambda} u] &= [Iu_+, u] + 2\chi\left\langle Iu_+, u\right\rangle = -\left[u, u_l\right] + \left[u, u_{\bar{l}}\right] + \mathcal{D}\left(\left\langle u, u_l\right\rangle - \left\langle u, u_{\bar{l}}\right\rangle\right) \\ &= 0,\end{aligned}$$

where we have used that $[\varphi, \cdot] = 0$. Applying (6.31), using (B.41) and (B.42) for $a = u$, we obtain the required identity, since

$$\begin{aligned}[H_\Lambda J] &= \left[H'{}_\Lambda J_0\right] - i\left(\chi + S\right)\left[H'{}_\Lambda u\right] + \lambda\left[e_{+\Lambda} J_0\right] \\ &= \left[H'{}_\Lambda J_0\right] - i\left(T + \frac{3}{2}\lambda\right)\left(: e_j\left[u_+, e^j\right]_- : + : e^j\left[u_+, e_j\right]_- :\right) \\ &\quad - \left(2\lambda + 2T + \chi S\right)Su_+ \\ &= \left(2\lambda + 2T + \chi S\right)\left(J_0 - Si\left(u_+ + \frac{1}{2}\sum_{j=1}^n\left[e^j, e_j\right]_-\right)\right) \\ &\quad + \frac{i}{4}\left(TS\mathcal{D}R + \lambda\sum_{j,k=1}^n\left(: F^{ij} : e_j e_i :: - : F_{ij} : e^j e^i ::\right)\right) \\ &\quad - i\left(T + \frac{3}{2}\lambda\right)\sum_{j,k=1}^n\left(: e_k\left[u_+ + \frac{1}{2}\left[e^j, e_j\right]_-, e^k\right] : + : e^k\left[u_+ + \frac{1}{2}\left[e^j, e_j\right]_-, e_k\right] :\right),\end{aligned}$$

for being $[\varphi, \cdot] = 0$ by hypothesis. □

Now, suppose that M admits an atlas of holomorphic coordinates for which the Jacobian of any change of coordinates has constant determinant. Then, notice that, by the weaker variant (6.31) of the D-term condition, there exists a relation between R and the terms with F^{ij}, F_{ij} (for $i, j \in \{1, \ldots, n\}$) thanks to Jacobi identity for the Λ-bracket.

Lemma 10.1.14. *Assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26)*, that $(l \oplus \bar{l}, \varphi)$ satisfies the weaker variant* (6.31) *of the D-term condition, and that $\varphi \in \Gamma(E)$ satisfies that $[\varphi, \cdot] = 0$. Then,*

$$TS\mathcal{D}R = \frac{2}{3} \sum_{i,j=1}^{n} T\left(: F^{ij} : e_j e_i :: - : F_{ij} : e^j e^i ::\right). \tag{10.13}$$

Proof. By Proposition 10.1.13, applying antisymmetry for the Λ-bracket,

$$[J_\Lambda H] = -\left(2\lambda + T + \chi S\right) J + \frac{i}{4}\left(TS\mathcal{D}R - (\lambda + T)\left(: F^{ij} : e_j e_i :: - : F_{ij} : e^j e^i ::\right)\right).$$

Now, by Jacobi identity for the Λ-bracket, since

$$[J_\Lambda J] = -\left(H + \frac{\lambda\chi}{3} c\right),$$

for being $[\varphi, \cdot] = 0$, so $\langle \varphi, \varphi \rangle \in \mathbb{C}$ is satisfied, and then $c \in \mathbb{C}$ (see (10.12)), we obtain

$$\begin{aligned}
[J_\Lambda H] &= \left[J_\Lambda \left(H + \frac{\gamma\eta}{3} c\right)\right] = -\left[J_\Lambda \left[J_\Gamma J\right]\right] \\
&= -\left[\left[J_\Lambda J\right]_{\Lambda+\Gamma} J\right] - \left[J_\Gamma \left[J_\Lambda J\right]\right] \\
&= -\left(\left[\left[J_\Lambda J\right]_{\Lambda+\Gamma} J\right] + \left[\left[J_\Lambda J\right]_{-\nabla-\Gamma} J\right]\right) \\
&= -\left(2\lambda + T + \chi S\right) J - \frac{i}{2} TS\mathcal{D}R - \frac{i}{4}\left(\lambda - T\right)\left(: F^{ij} : e_j e_i :: - : F_{ij} : e^j e^i ::\right),
\end{aligned}$$

so, substrating these two identities, we arrive at (10.13). ∎

In particular, when $F^{ij} = F_{ij} = 0$ for $i, j \in \{1, \ldots, n\}$, then

$$TS\mathcal{D}R = 0.$$

Now, suppose that for the considered atlas we can construct (odd) dual isotropic frames associated to $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n$ of $l \oplus \bar{l}$ satisfying the following condition:

$$[e_j, e_k] = 0, \quad \text{for } j, k \in \{1, \ldots, n\}. \tag{10.14}$$

Proposition 10.1.15. *We assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26)*, that $(l \oplus \bar{l}, \varphi)$ satisfy the D-term condition* (6.30)*, and that $\varphi \in \Gamma(E)$ satisfies that $[\varphi, \cdot] = 0$. Then, setting $u = \Pi\varphi$, the locally defined sections* (10.11) *satisfy*

$$[H_\Lambda J] = \left(2\lambda + 2T + \chi S\right) J + \frac{i}{4}\left(TS\mathcal{D}R + \lambda \sum_{i,j=1}^{n} : F^{ij} : e_j e_i ::\right),$$

where

$$R = 4 \sum_{j=1}^{n} \left\langle \left[e_j, u\right], e^j \right\rangle - 3 \left\langle \sum_{j=1}^{n} \left[e^j, e_j\right], \sum_{k=1}^{n} \left[e^k, e_k\right] \right\rangle,$$

$$F^{ij} = \sum_{k=1}^{n} \left\langle \mathcal{D}\left\langle \left[e^i, e^j\right], e_k \right\rangle, e^k \right\rangle, \quad \textit{for } i, j \in \{1, \ldots, n\}.$$

Proof. The result follows from Proposition 10.1.13. Notice that (10.14) implies $F_{ij} = 0$ by (C.6a). Now, in order to arrive at the desired formula for F^{ij}, we must see that

$$\left\langle \left[\left[e_k, e^k\right], e^i\right], e^j \right\rangle = 0, \quad \text{for } i, j \in \{1, \ldots, n\},$$

which follows by (C.6b). So, by the D-term condition (6.30), since $[\varphi, \cdot] = 0$,

$$\left\langle \left[\left[e_k, e^k\right], e^i\right], e^j \right\rangle = \left\langle \left[2u_{\bar{l}} - 2u, e^i\right], e^j \right\rangle = 2 \left\langle \left[u_{\bar{l}}, e^i\right], e^j \right\rangle = 0.$$

Now, for the last term, note that

$$\left\langle \left[e^j, e_k\right], \left[e^r, e_s\right] \right\rangle = \left\langle \left[e^j, e_k\right]_-, \left[e^r, e_s\right] \right\rangle, \quad \text{for } j, k, r, s \in \{1, \ldots, n\}$$

in our frame (10.14). So,

$$R = 3 \left\langle \left[e^j, e_j\right], \left[e^k, e_k\right] \right\rangle - \left\langle \left[e^j, e_k\right], \left[e^k, e_j\right] \right\rangle.$$

Finally, using Jacobi identity (among others axioms of Courant algebroids), that we are in the frame (10.14), and that it is satisfied the D-term condition (6.30), we arrive at

$$\left\langle \left[e^j, e_k\right], \left[e^k, e_j\right] \right\rangle = 4 \left\langle \left[e_j, u\right], e^j \right\rangle.$$

□

Theorem 10.1.16. *Assume that $l \oplus \bar{l}$ satisfies the F-term condition* (6.26)*, that $\left(l \oplus \bar{l}, \varphi\right)$ satisfies the D-term condition* (6.30)*, that $\varphi \in \Gamma(E)$ satisfies that $[\varphi, \cdot] = 0$, and that*

$$\sum_{i,j=1}^{n} : F^{ij} : e_j e_i :: = 0. \tag{10.15}$$

Then, setting $u = \Pi\varphi$, for the frames satisfying (10.14)*, the local sections*

$$\begin{aligned} J =& \frac{i}{2} \sum_{j=1}^{n} : e^j e_j : - Siu, \\ H =& \frac{1}{2} \sum_{j=1}^{n} \left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) :\right) + \frac{1}{4} \sum_{j,k=1}^{n} \Big(: e_j : e^k \left[e^j, e_k\right] :: \\ & + : e^j : e_k \left[e_j, e^k\right] :: - : e_j : e_k \left[e^j, e^k\right] :: - : e^j : e^k \left[e_j, e_k\right] ::\Big) + T u_{\bar{l}}, \end{aligned}$$

induce an embedding of the $N = 2$ superconformal vertex algebra with central charge given by (10.12) *into the space of local sections of the chiral de Rham complex Ω^{ch}_E.*

Proof. This follows from Proposition 10.1.2, Lemma 10.1.14 and Proposition 10.1.15. □

We are going to state one of the main results of this thesis, which is a direct consequence of the work we have done in Section 6.3, and Theorem 10.1.16 above, about embeddings from the twisted Hull-Strominger system (6.16). We need the next intermediate results.

Lemma 10.1.17. *The above sections J, H, and*

$$\sum_{i,j=1}^{n} :F^{ij}: e_j e_i :: \tag{10.16}$$

are all global. Even more, R is a well-defined global function on M.

Proof. Let $\{f_j, f^j\}_{j=1}^n \subseteq l \oplus \bar{l}$ be a new isotropic frame, for which there exists

$$A = \left(A_j^k\right)_{j,k=1}^n, B = \left(B_j^k\right)_{j,k=1}^n \in \mathrm{Mat}_n\left(\mathcal{C}^\infty(M)\right)$$

matrices for the change of coordinates, such that

$$f_j = \sum_{k=1}^n A_j^k e_k \text{ and } f^j = \sum_{k=1}^n B_j^k e^k, \quad \text{for } j \in \{1, \dots, n\}.$$

By hypothesis,

$$\mathcal{D} \det A = 0, \quad \pi\left(e^k\right)\left(B_j^s\right) = 0 = \pi\left(f^s\right)\left(A_k^j\right), \quad \text{for } j, k, s \in \{1, \dots, n\}. \tag{10.17}$$

Then, thanks to Lemma 6.3.9 and Lemma C.3.1, we must just prove that

$$:\left\langle \mathcal{D}\left\langle \left[e^i, e^j\right], e_k\right\rangle, e^k\right\rangle : e_j e_i ::=: \left\langle \mathcal{D}\left\langle \left[f^i, f^j\right], f_k\right\rangle, f^k\right\rangle : f_j f_i ::,$$

and

$$\left\langle [e_j, u], e^j\right\rangle = \left\langle [f_j, u], f^j\right\rangle.$$

Indeed, for the first term, by (B.19) (B.25) and (B.26), notice that

$$:: A_j^q e_q :: A_i^r e_r :: =: \left(A_j^q A_i^r\right) : e_q e_r ::,$$

while by Courant algebroid axioms and (B.10),

$$\begin{aligned}
\left\langle \mathcal{D}\left\langle \left[B_r^i e^r, B_m^j e^m\right], A_k^s e_s\right\rangle, B_p^k e^p\right\rangle = {} & B_p^k A_m^j B_r^i \left\langle [e^r, e^m], e_s\right\rangle \left\langle \mathcal{D} A_k^s, e^p\right\rangle \\
& + \left\langle [e^r, e^m], e_s\right\rangle \left(B_r^i \left\langle \mathcal{D} B_m^j, e^s\right\rangle + B_m^j \left\langle \mathcal{D} B_r^i, e^s\right\rangle\right) \\
& + B_m^j \left(B_r^i \left\langle \mathcal{D}\left\langle [e^r, e^m], e_s\right\rangle, e^s\right\rangle\right. \\
& \left. + B_p^k \left\langle \mathcal{D} B_s^i, e^m\right\rangle \left\langle \mathcal{D} A_k^s, e^p\right\rangle\right) \\
& + \left\langle \mathcal{D} B_k^i, e^m\right\rangle \left\langle \mathcal{D} A_m^j, e^k\right\rangle + \left\langle \mathcal{D} B_m^j, e^m\right\rangle \left\langle \mathcal{D} B_k^i, e^k\right\rangle \\
& + B_m^j \left\langle \mathcal{D}\left\langle \mathcal{D} B_k^i, e^m\right\rangle, e^k\right\rangle + B_k^i \left\langle \mathcal{D}\left\langle \mathcal{D} B_m^j, e^m\right\rangle, e^k\right\rangle \\
& + B_p^k B_s^i \left\langle \mathcal{D} B_m^j, e^m\right\rangle \left\langle \mathcal{D} A_k^s, e^p\right\rangle,
\end{aligned}$$

from where we obtain the desired identity thanks to second equation in (10.17). Finally, by Courant algebroid axioms and (B.10), using Jacobi's formula from Appendix B.6,

$$\left\langle \left[A_j^k e_k, u\right], B_s^j e^s\right\rangle = \left\langle [e_j, u], e^j\right\rangle - \frac{1}{\det A}\left\langle \mathcal{D} \det A, u\right\rangle + A_s^j \left\langle u, e_k\right\rangle \left\langle \mathcal{D} A_j^k, e^s\right\rangle,$$

from where we obtain the desired identity thanks to (10.17). □

Lemma 10.1.18. *Let σ_ω be the torsion bi-vector from Section 5.3. We have that*

$$(\sigma_\omega)_{\overline{ij}} = F^{ij}, \quad \text{for } i,j,k \in \{1,\dots,n\}.$$

Proof. We must keep in mind Section 6.3. Indeed, by Courant algebroid axioms,

$$\begin{aligned}\left\langle \left[e^i, e^j\right], e_k \right\rangle &= -\left\langle e^j, \left[e_i, e^k\right]_+ \right\rangle = -\left\langle e^j, \sigma_+ \left(\nabla^+_{g^{-1}d\overline{z}_i} \frac{\partial}{\partial \overline{z}_k}\right)\right\rangle \\ &= -d\overline{z}_j \left(\nabla^+_{g^{-1}d\overline{z}_i} \frac{\partial}{\partial \overline{z}_k}\right) = -d\overline{z}_j \left(\nabla^+_{g^{-1}d\overline{z}_i} \frac{\partial}{\partial \overline{z}_k} - \nabla^+_{\frac{\partial}{\partial \overline{z}_k}} \left(g^{-1}d\overline{z}_i\right)\right) \\ &= -d^c\omega \left(g^{-1}d\overline{z}_i, g^{-1}d\overline{z}_j, \frac{\partial}{\partial \overline{z}_k}\right), \quad \text{for } i,j,k \in \{1,\dots,n\},\end{aligned}$$

as desired, since $\nabla^+ g^{-1}d\overline{z}_i \in \Omega^{1,0}(M)$ for $i \in \{1,\dots,n\}$ and

$$\left[g^{-1}d\overline{z}_i, \frac{\partial}{\partial \overline{z}_k}\right]^{0,1} = \overline{\left[g^{-1}dz_i, \frac{\partial}{\partial z_i}\right]^{1,0}} = \overline{\overline{\partial}_{g^{-1}dz_i}\left(\frac{\partial}{\partial z_k}\right)} = 0.$$

The proof follows now from the explicit formula (5.15) combined with

$$\begin{aligned}F^{ij} &= \left\langle \mathcal{D}\left\langle \left[e^i, e^j\right], e_k \right\rangle, e^k \right\rangle = -\pi\left(\overline{\varepsilon}_k\right) \left(d^c\omega \left(g^{-1}d\overline{z}_i, g^{-1}d\overline{z}_j, \frac{\partial}{\partial \overline{z}_k}\right)\right) \\ &= ig^{m\overline{k}} \frac{\partial}{\partial \overline{z}_m} \left(\partial\omega \left(g^{-1}d\overline{z}_i, g^{-1}d\overline{z}_j, \frac{\partial}{\partial \overline{z}_k}\right)\right).\end{aligned}$$

□

Let $P \longrightarrow M$ be a principal K-fibre bundle over M any complex manifold, and fix the bi-invariant non-degenerate pairing $\langle\cdot,\cdot\rangle : \mathfrak{k}\otimes\mathfrak{k} \longrightarrow \mathbb{R}$. At last, we arrive at the following.

Theorem 10.1.19. *Let (ω, Ψ, A) be a solution to the twisted Hull-Strominger system* (6.16)*, and consider the associated string Courant algebroid $E := E_{-d^c\omega,A}$. Then, the following sections, defined for the frames* (6.18) *induced by the atlas in Lemma 5.2.2,*

$$\begin{aligned}J =& \frac{i}{2} \sum_{j=1}^n : e^j e_j : + Si\Pi\theta_\omega, \\ H =& \frac{1}{2} \sum_{j=1}^n \left(: e_j \left(Se^j\right) : + : e^j \left(Se_j\right) :\right) + \frac{1}{4} \sum_{j,k=1}^n \Big(: e_j : e^k \left[e^j, e_k\right] :: \\ &+ : e^j : e_k \left[e_j, e^k\right] :: - : e_j : e_k \left[e^j, e^k\right] :: - : e^j : e^k \left[e_j, e_k\right] ::\Big) - T\Pi g^{-1}\theta_\omega^{0,1},\end{aligned} \tag{10.18}$$

are global. Furthermore, when

$$\sigma_\omega = 0,$$

they induce an embedding of the $N = 2$ superconformal vertex algebra with central charge

$$c = 3 \dim l \in \mathbb{C}$$

into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}$.

Proof. This follows from Proposition 10.1.16, Lemma 10.1.17 and Lemma 10.1.18. □

In particular, this result applies when we have a solution (ω, Ψ) to the twisted Calabi-Yau equations (6.17) for the associated exact Courant algebroid $E_{-d^c\omega}$.

10.2 Pairs of Solutions for the Killing Spinor Equations

Now, we will study what happens when we have pairs of solutions for the Killing spinor equations. In particular, this will give us the analogue of Theorem 9.2.3 for quadratic Lie algebras. This will allow us to construct "an honest" $N = 2$ superconformal vertex algebra structure that will be related with some constructions studied in Chapter 3.

10.2.1 The Λ-Commuting Sectors on Quadratic Lie Algebras

Let $(V_\pm, \varepsilon_\pm)$ be two solutions for the Killing spinor equations on $(\mathfrak{g}, (\cdot|\cdot))$ quadratic Lie algebra, such that they generate a pair of solutions on $\mathfrak{g}$ (see Definition 7.1.5). Following previous section, we fix $\mathfrak{g} = l_+ \oplus l_- \oplus \bar{l}_+ \oplus \bar{l}_-$ direct sum decomposition, with $l_\pm, \bar{l}_\pm \subseteq \mathfrak{g}$ isotropic $n_\pm$-dimensional subspaces, for which we have that the restriction $(\cdot|\cdot)$ to $V_\pm$ is non-degenerate, where $V_\pm := l_\pm \oplus \bar{l}_\pm$, and $V_\pm = V_\mp^\perp$. Let

$$\pi_\pm \colon \mathfrak{g} \longrightarrow V_\pm, \quad \pi_{l_\pm} \colon \mathfrak{g} \longrightarrow l_\pm, \quad \pi_{\bar{l}_\pm} \colon \mathfrak{g} \longrightarrow \bar{l}_\pm$$

be the orthogonal projections. So, when there is no possible confusion, we will write

$$a_\pm = \pi_\pm a, \quad a_{l_\pm} = \pi_{l_\pm} a, \quad a_{\bar{l}_\pm} = \pi_{\bar{l}_\pm} a, \quad \text{for } a \in \mathfrak{g}.$$

Fix $\left\{\epsilon_j^\pm, \bar{\epsilon}_j^\pm\right\}_{j=1}^{n_\pm} \subseteq V_\pm$ basis satisfying (6.28). Let $V^k_{\text{super}}(\mathfrak{g})$ with $0 \neq k \in \mathbb{C}$, and define

$$e_j^\pm = \Pi \epsilon_j^\pm, \quad e^j_\pm = \Pi \bar{\epsilon}_j^\pm, \qquad \text{for } j \in \{1, \dots, n_\pm\}.$$

Remember that we work with parity-reversed vectors (see Remark 3.2.1). We define

$$\begin{array}{rccc} I_\pm \colon & V_\pm & \longrightarrow & V_\pm \\ & a_\pm & \mapsto & a_{l_\pm} - a_{\bar{l}_\pm} \end{array},$$

writing $I_\pm \Pi a \equiv \Pi I_\pm a$ for $a \in V_\pm$. Now, define the element of $V_\pm \subseteq \mathfrak{g}$ given by

$$w_\pm = \Pi I_\pm \left[\bar{\epsilon}_j^\pm, \epsilon_j^\pm\right]_\pm = \left[e^j_\pm, e_j^\pm\right]_{l_\pm} - \left[e^j_\pm, e_j^\pm\right]_{\bar{l}_\pm} \in \Pi V_\pm.$$

Define the well-defined (they do not depend on the chosen basis) vectors

$$\begin{aligned} J_0^\pm :&= \frac{i}{k} \sum_{j=1}^{n_\pm} :e^j_\pm e_j^\pm :\in V^k_{\text{super}}(\mathfrak{g}), \\ H'_\pm :&= \frac{1}{k} \sum_{j=1}^{n_\pm} \left(:e_j^\pm \left(S e^j_\pm\right): + :e^j_\pm \left(S e_j^\pm\right):\right) + \frac{T}{k} w_\pm \\ &+ \frac{1}{k^2} \sum_{j,k=1}^{n_\pm} \left(:e_j^\pm : e^k_\pm \left[e^j_\pm, e_k \pm\right] :: + :e^j_\pm : e_k^\pm \left[e_j^\pm, e^k_\pm\right] :: \right. \\ &\left. - :e_j^\pm : e_k^\pm \left[e^j_\pm, e^k_\pm\right] :: - :e^j_\pm : e^k_\pm \left[e_j^\pm, e_k^\pm\right] ::\right) \in V^k_{\text{super}}(\mathfrak{g}). \end{aligned} \tag{10.19}$$

Moreover, consider

$$c_0^{\pm} := 3n_{\pm} \in \mathbb{C}. \tag{10.20}$$

Remark 10.2.1. Notice that if we have $\{J_1, H_1, c_1\}$ and $\{J_2, H_2, c_2\}$ two different sets of generators for the $N = 2$ superconformal vertex algebra, when $[J_{1\Lambda} J_2] = 0$ is satisfied, then $[H_{1\Lambda} H_2] = 0$. Indeed, by first equation in (2.21),

$$[H_{1\Lambda} H_2] = \left[-\left(H_1 + \frac{\gamma\eta}{3} c_1 \right)_{\Lambda} - \left(H_2 + \frac{\omega\xi}{3} c_2 \right) \right] = \left[[J_{1\Gamma} J_1]_{\Lambda} \, [J_{2\Omega} J_2] \right] = 0.$$

Now, repeating the processes explained in Theorem 10.1.5, we can obtain two embeddings of the $N = 2$ superconformal vertex algebra of central charge (10.20) into $V^k_{\text{super}}(\mathfrak{g})$. In addition, we obtain the next result, where, to simplify computations, we are going to use the Einstein summation convention for repeated indexes, for the mentioned embeddings.

Theorem 10.2.2. *Assume that $l_{\pm} \oplus \bar{l}_{\pm} \subseteq \mathfrak{g}$ satisfies the F-term condition* (6.26)*, that $(l_{\pm} \oplus \bar{l}_{\pm}, \varepsilon_{\pm})$ satisfies the weaker variant* (7.13) *of the D-term condition, that*

$$w_{\pm} \in [l_{\pm}, l_{\pm}]^{\perp} \cap \left[\bar{l}_{\pm}, \bar{l}_{\pm} \right]^{\perp},$$

and that $n := n_+ = n_-$. Then, the vectors (10.19) *induce two Λ-commuting embeddings of the $N = 2$ superconformal vertex algebra with the same central charge $c_0 = 3n$ into the universal superaffine vertex algebra $V^k_{super}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.*

Proof. We have to prove that $\left[J_0^+ {}_{\Lambda} J_0^- \right] = 0$. By the non-commutative Wick formula,

$$\begin{aligned}
\left[J_0^+ {}_{\Lambda} : e_-^j e_j^- : \right] &=: \left[J_0^+ {}_{\Lambda} e_-^j \right] e_j^- : - : e_-^j \left[J_0^+ {}_{\Lambda} e_j^- \right] : + \int_0^{\Lambda} d\Gamma \left[\left[J_0^+ {}_{\Lambda} e_-^j \right]_{\Gamma} e_j^- \right] \\
&= \frac{i}{k} \Big(:: \left[e_-^k, e_+^j \right]_{l_- \oplus \bar{l}_- \oplus l_+} e_j^+ : e_k^- : + :: e_+^j \left[e_-^k, e_j^+ \right]_{\bar{l}_+ \oplus l_- \oplus \bar{l}_+} : e_k^- : \\
&\quad - : e_-^k : \left[e_k^-, e_+^j \right]_{l_+ \oplus l_- \oplus \bar{l}_-} e_j^+ :: - : e_-^k : e_+^j \left[e_k^-, e_j^+ \right]_{\bar{l}_+ \oplus l_- \oplus \bar{l}_-} :: \Big) \\
&\quad + \frac{i}{k} \int_0^{\lambda} d\gamma \, (I_1 + I_2) - i\lambda \left(\left[e_+^j, e_j^+ \right]_{l_-} - \left[e_+^j, e_j^+ \right]_{\bar{l}_-} \right),
\end{aligned}$$

where we have used (C.2b) for $J_0 = J_0^+$ and $a \in V_-$ the basis elements, and after applying the non-commutative Wick formula and antisymmetry for the Λ-bracket,

$$\begin{aligned}
I_1 &:= \partial_{\eta} \left[: \left[e_-^k, e_+^j \right]_{l_+ \oplus l_- \oplus \bar{l}_-} e_j^+ :_{\Gamma} e_k^- \right] = k \left[e_-^k, e_k^- \right]_{l_+}, \\
I_2 &:= \partial_{\eta} \left[: e_+^j \left[e_-^k, e_j^+ \right]_{\bar{l}_+ \oplus l_- \oplus \bar{l}_-} :_{\Gamma} e_k^- \right] = -k \left[e_-^k, e_k^- \right]_{\bar{l}_+}.
\end{aligned}$$

Finally, using basic properties, some of them from Appendix B, we arrive at

$$
\begin{aligned}
[J_{+\Lambda}J_{-}] = -\frac{1}{k^2}\Bigg(&:: \left[e_{-}^{k}, e_{+}^{j}\right]_{l_+\oplus l_-} e_j^{+} : e_k^{-} : + :: e_{+}^{j}\left[e_{-}^{k}, e_j^{+}\right]_{\bar{l}_+\oplus l_-} : e_k^{-} : \\
&- : e_{-}^{k} : \left[e_k^{-}, e_{+}^{j}\right]_{l_+\oplus \bar{l}_-} e_j^{+} :: - : e_{-}^{k} : e_{+}^{j}\left[e_k^{-}, e_j^{+}\right]_{\bar{l}_+\oplus \bar{l}_-} :: \Bigg) \\
&+ i\lambda\left(\left[e_{-}^{j}, e_j^{-}\right]_{l_+} - \left[e_{-}^{j}, e_j^{-}\right]_{\bar{l}_+} - \left[e_{+}^{j}, e_j^{+}\right]_{l_-} + \left[e_{+}^{j}, e_j^{+}\right]_{\bar{l}_-}\right),
\end{aligned}
$$

which is zero by (6.26), (7.13), and Remark 6.3.15. $\square$

Now, fix $\varphi_\pm, \varepsilon_\pm \in V_\pm$, related by $I_\pm \varepsilon_\pm = \varphi_\pm$, and consider the associated odd sections

$$e_\pm = \Pi\varepsilon_\pm, \quad u_\pm = \Pi\varphi_\pm \in \Pi V_\pm.$$

Using (10.19), define the well-defined vectors

$$
\begin{aligned}
J_\pm &:= J_0^\pm \mp 2\frac{S}{k} i u_\pm \in V^k_{\text{super}}(\mathfrak{g}), \\
H_\pm &:= H'_\pm \mp 2\frac{T}{k} e_\pm + 2\frac{S}{k^2}\sum_{j=1}^{n_\pm}\left(: \left[u_\pm, e_\pm^{j}\right] e_j^{\pm} : + : e_\pm^{j}\left[u_\pm, e_j^{\pm}\right] : \right) \in V^k_{\text{super}}(\mathfrak{g}).
\end{aligned}
\tag{10.21}
$$

Moreover, consider

$$c_\pm := 3\left(n_\pm \pm \frac{4}{k}\left(e_\pm \,|\, e_\pm\right)\right) \in \mathbb{C}. \tag{10.22}$$

We also define the vectors $\varepsilon = \varepsilon_+ + \varepsilon_- \in \mathfrak{g}$, and $e = \Pi\varepsilon \in \Pi\mathfrak{g}$.

Remark 10.2.3. Notice that in formula (10.22) above we obtain that $c_+ = c_-$ when $n_+ = n_-$ and $(e\,|e) = 0$. Indeed, this last condition is equivalent to $(e_+|e_+) = -(e_-|e_-)$. We will require in the following result these two conditions. However, strictly speaking, we do not really need the second condition to obtain the Λ-commuting embeddings.

Now, repeating the processes explained in Theorem 10.1.8, we can obtain two embeddings of the $N = 2$ superconformal vertex algebra of central charge (10.22) into $V^k_{\text{super}}(\mathfrak{g})$. In addition, we obtain the following result for the mentioned embeddings.

Theorem 10.2.4. *Assume that $l_\pm \oplus \bar{l}_\pm \subseteq \mathfrak{g}$ satisfies the F-term condition* (6.26)*, that $(l_\pm \oplus \bar{l}_\pm, \varepsilon_\pm)$ satisfies the D-term condition* (7.12)*, that $\varepsilon_\pm \in l_\pm \oplus \bar{l}_\pm$ is holomorphic, that $n := n_+ = n_-$, and that $(e|e) = 0$. Then, the vectors* (10.21) *induce two Λ-commuting embeddings of the $N = 2$ superconformal vertex algebra with the same central charge*

$$c = 3\left(n + \frac{4}{k}\left(e_+ \,|\, e_+\right)\right) \in \mathbb{C}. \tag{10.23}$$

into the universal superaffine vertex algebra $V^k_{super}(\mathfrak{g})$ with level $0 \neq k \in \mathbb{C}$.

Proof. We have to proof $[J_{+\Lambda}J_-]=0$. By sesquilinearity, thanks to Theorem 10.2.2,

$$\begin{aligned}[J_{+\Lambda}J_-] &= \left[J_0^+{}_\Lambda J_0^+\right] + i\frac{2}{k}\left[J_0^+{}_\Lambda Su_-\right] - i\frac{2}{k}\left[Su_{+\Lambda}J_0^-\right] - \frac{4}{k^2}\left[Su_{+\Lambda}Su_-\right] \\ &= -i\frac{2}{k}\left(\chi+S\right)\left[J_0^+{}_\Lambda u_-\right] - i\frac{2}{k}\chi\left[u_{+\Lambda}J_0^-\right] - \frac{4}{k^2}\chi\left(S+\chi\right)\left[u_{+\Lambda}u_-\right].\end{aligned}$$

Now, we have that

$$\begin{aligned}\left[J_0^+{}_\Lambda u_-\right] &= \frac{i}{k}\sum_{j=1}^n\left(:\left[u_-,e_+^j\right]_- e_j^+ : + : e_+^j\left[u_-,e_j^+\right]:\right), \\ \left[u_{+\Lambda}J_0^+\right] &= \frac{i}{k}\sum_{j=1}^n\left(:\left[u_+,e_-^j\right]e_j^- : + : e_-^j\left[u_+,e_j^-\right]:\right),\end{aligned}$$

aplying (C.2b) for $J=J_0^+$ and $a=u_-$, and (C.2a) for $J=J_0^-$ and $a=u_+$. At last,

$$[u_{+\Lambda}u_-] = [u_+,u_-] + k\chi\left(u_+|u_-\right) = [u_+,u_-]_+ - [u_-,u_+]_-\,.$$

In conclusion,

$$\begin{aligned}[J_{+\Lambda}J_-] = &\frac{2}{k^2}(\chi+S)\left(:\left[u_-,e_+^j\right]_- e_j^+ : + : e_+^j\left[u_-,e_j^+\right]_- :\right) \\ &+\frac{2}{k^2}\chi\left(:\left[u_+,e_-^j\right]_+ e_j^- : + : e_-^j\left[u_+,e_j^-\right]_+ :\right) \\ &-\frac{4}{k^2}\left(\chi S-\lambda\right)\left([u_+,u_-]_+ - [u_-,u_+]_-\right),\end{aligned}$$

which is zero for being $u_\pm\in l_\pm\oplus\bar{l}_\pm$ an infinitesimal isometry. □

Remark 10.2.5. Applying the process above for pairs of solutions of the Killing spinor equations, but in exact Courant algebroids, we should recover Theorem 9.2.3. We leave this as a future question, since it requires further analysis.

10.2.2 Honest $N=2$ Superconformal Vertex Algebra Structures

Consider $J_\pm$ two even Λ-commuting elements satisfying, for $c\in\mathbb{C}$ and some $H_\pm$ odd,

$$[J_{\pm\Lambda}J_\pm] = -\left(H_\pm + \frac{\lambda\chi}{3}c\right).$$

Define

$$J_1 := J_+ + J_-,\quad J_2 := J_+ - J_-,\quad H := H_+ + H_-. \tag{10.24}$$

It is proven in [55, Lemma 2] that when $\{J_1,H\}$ and $\{J_2,H\}$ generate the $N=2$ superconformal vertex algebra of central charge $2c$, then the quadruple $\{J_\pm,H_\pm\}$ generates two Λ-commuting copies of the $N=2$ superconformal vertex algebra of central charge c. Now, we will prove the converse, which allows us to obtain honest $N=2$ superconformal vertex algebra structures from pairs of solutions for the Killing spinor equations.

Theorem 10.2.6. *If $\{J_\pm, H_\pm, c\}$ are Λ-commuting $N=2$ superconformal vertex algebra structures, then the even elements J_1, J_2 of* (10.24) *generate two $N=2$ superconformal vertex algebra structures of central charge $2c$ with the same odd element H of* (10.24).

Proof. By Λ-commutativity, we have that

$$[J_1{}_\Lambda J_1] = [J_2{}_\Lambda J_2] = [J_+{}_\Lambda J_+] + [J_-{}_\Lambda J_-] = -\left(H + \frac{\lambda\chi}{3}c\right).$$

Finally, since $[H_+{}_\Lambda J_-] = 0 = [H_-{}_\Lambda J_+]$ clearly, then

$$\begin{aligned}
[H_\Lambda J_1] &= [H_+{}_\Lambda J_+] + [H_+{}_\Lambda J_-] + [H_-{}_\Lambda J_+] + [H_-{}_\Lambda J_-] = (2\lambda + 2T + \chi S)J_1,\\
[H_\Lambda J_2] &= [H_+{}_\Lambda J_+] - [H_+{}_\Lambda J_-] + [H_-{}_\Lambda J_+] - [H_-{}_\Lambda J_-] = (2\lambda + 2T + \chi S)J_2. \qquad \square
\end{aligned}$$

Corolary 10.2.7. *Let $(\mathfrak{g}, (\cdot|\cdot))$ be a quadratic Lie algebra. Then, by Theorem 10.2.2 and Theorem 10.2.4, each J_1 and J_2 defined using $J_\pm$ as given in* (10.19) *and* (10.21) *generate a copy of the $N=2$ superconformal vertex algebra of central charge, respectively,*

$$c_0 = \frac{3}{2}\dim\mathfrak{g} \quad \textit{or} \quad c = 3\left(\frac{\dim\mathfrak{g}}{2} + \frac{8}{k}(e_+|e_+)\right).$$

Furthermore, the vector H defined using $H_\pm$ as in (10.21) *is the Kac-Todorov vector of $\mathfrak{g}$. Consequently, localizing at $k + h^\vee \in \mathbb{C}$ we recover the Freudental's formula*

$$\left(\sum_{j=1}^{n}\left(\left[e_+^j, e_j^+\right] + \left[e_-^j, e_j^-\right]\right)\middle|\sum_{k=1}^{n}\left(\left[e_+^k, e_k^+\right] + \left[e_-^k, e_k^-\right]\right)\right) = \frac{h^\vee}{3}\dim\mathfrak{g}.$$

Remark 10.2.8. The last result above is the true generalization of Getzler's Theorem for pairs of solutions for the Killing spinor equations on quadratic Lie algebras. Indeed, note that we have two Manin triples

$$\left(\mathfrak{g}, l_+ \oplus l_-, \bar{l}_+ \oplus \bar{l}_-\right) \text{ and } \left(\mathfrak{g}, l_+ \oplus \bar{l}_-, \bar{l}_+ \oplus l_-\right),$$

which induce each one two different embeddings. In particular, for the dilaton correction, the odd generator is the Kac-Todorov vector of $\mathfrak{g}$. As a consequence, we have two honest $N=2$ superconformal vertex algebra structures on the superaffinization of $\mathfrak{g}$.

Corolary 10.2.9. *Let E be an exact Courant algebroid over an smooth manifold M. Then, applying Theorem 9.2.3, each J_1 and J_2 defined using $J_\pm$ as given in* (9.4) *generate a copy of the $N=2$ superconformal vertex algebra of central charge $c = 3\dim M$. Note that the odd generator coincide, and it is given by H defined using $H_\pm$ as in* (9.5).

The previos result is a consequence of Theorem 9.2.2, firstly given in [54, Theorem 5.5]. Indeed, the authors used it to prove, precisely, Theorem 9.2.3 given in [55, Theorem 2]. However, since by Remark 10.2.5 we hope that generalized Calabi-Yau metric structures can be rewritten using pairs of solutions for the Killing spinor equations, this last result will be now a consequence of our main Theorem 10.1.19 for exact Courant algebroids.

10.3 Open Questions

We include open problems related to the results we have obtained, for future studies.

10.3.1 Embedding $N = 1$ Superconformal VAs

Originally, when we started the present work, we were expecting to construct embeddings of the $N = 2$ superconformal vertex algebra under much weaker hypothesis, related to the vanishing of the *generalized Ricci tensor*. Even though we have not been able to complete this ambitious programme, our development motivates some open questions related to this initial problem. Let E be a general Courant algebroid over M. In order to define the Ricci tensor of a generalized metric $C_\pm \subseteq E$, we first need to give the following notion for a connection $D \in \mathcal{D}(C_\pm)$, due to Gualtieri.

Definition 10.3.1. [36] For $C_\pm \subseteq E$ generalized metric, the *Ricci tensors* of $D \in \mathcal{D}(C_\pm)$, a compatible generalized connection, are $\mathrm{Ric}_D^\pm \in \Gamma\left(C_\mp^* \otimes C_\pm^*\right)$ defined by

$$\mathrm{Ric}_D^\pm\left(e_\mp, e_\pm\right) = \mathrm{tr}\left(d_\pm \longrightarrow R_D^\pm\left(d_\pm, e_\mp\right) e_\pm\right), \quad \text{for } e_\pm \in \Gamma(C_\pm),$$

where $R_D^\pm \in \Gamma(C_\pm^* \otimes C_\mp^* \otimes \mathfrak{o}(C_\pm))$ are the *curvature operators* associated to D, given by

$$\begin{aligned} R_D^+\left(a_+, b_-\right) c_+ &= D_{a_+} D_{b_-} c_+ - D_{b_-} D_{a_+} c_+ - D_{[a_+, b_-]} c_+, \\ R_D^-\left(a_-, b_+\right) c_- &= D_{a_-} D_{b_+} c_- - D_{b_+} D_{a_-} c_- - D_{[a_-, b_+]} c_-, \end{aligned}$$

for $a_\pm, b_\pm, c_\pm \in \Gamma(C_\pm)$.

If $M \neq \{\cdot\}$, given $\mathrm{div}\colon \Gamma(E) \longrightarrow \mathcal{C}^\infty(M)$ any divergence operator on E, it is proven in [36, Proposition 4.4] that for any choice of generalized connection $D \in \mathcal{D}(C_+, \mathrm{div})$, the induced Ricci tensors $R_D^\pm$ are equal. In such cases, the Ricci tensor $\mathrm{Ric}^\pm \in \Gamma(C_\mp^* \oplus C_{\pm^*})$ of $(C_\pm, \mathrm{div})$ are defined by

$$\mathrm{Ric}^\pm := \mathrm{Ric}_D^\pm$$

for any choice of $D \in \mathcal{D}(C_+, \mathrm{div})$. To see this, assume that $C_\pm \subseteq E$ admits the spinor bundles $S_\pm$, and consider the twisted spinor bundles $\mathcal{S}_\pm$ that are endowed with the canonical Dirac-type operators $\not{D}$ on $\Gamma(\mathcal{S}_\pm)$. Then, with the same notation as in Subsection 6.2.1, for $\alpha \in \Gamma(\mathcal{S}_\pm)$ and $a_\pm \in \Gamma(C_\pm)$, we obtain that

$$\iota_{a^\mp} \mathrm{Ric}^\pm \cdot \alpha = 4\left(\not{D} D_{a^\mp} - D_{e^\mp} \not{D}^\pm - 2\sum_{j=1}^{r_\pm} \widetilde{a}_j^\pm \cdot D_{\left[a_j^\pm, e_\mp\right]_\mp}\right)\alpha. \tag{10.25}$$

Thanks to this property, we are ready to introduce the following notion.

Definition 10.3.2. [36] We say that a pair $(C_\pm, \mathrm{div})$ satisfies the *Ricci-flat condition* if

$$\mathrm{Ric}^\pm = 0. \tag{10.26}$$

Being Ricci flat is a second-order equation, and corresponds to the integrability condition for the first-order equation given by Killing spinor equations in Definition 6.2.15. More precisely, using (10.25), if $(C_\pm, \mathrm{div}, \eta)$ is a solution of the Killing spinor equations, then $\mathrm{Ric}^\pm = 0$. So, we expect that the next conjecture is true, where $(\mathfrak{g}, (\cdot|\cdot))$ is a quadratic Lie algebra for which we obtain equations from (10.25) and (10.26).

Conjecture 1. If $(V_\pm, \mathrm{div})$ satisfies (10.26), then the odd vectors in (10.9) and (10.10) generates two embeddings of the $N=1$ superconformal vertex algebra into $V^k_{\mathrm{super}}(\mathfrak{g})$.

For the case of exact Courant algebroids, one can relate these conditions to the known as *Bismut-Hermite-Einstein metrics.* We can give the following notion.

Definition 10.3.3. [38] Let E be an exact Courant algebroid over a complex manifold M. We say that a J-compatible generalized metric $C_\pm$ is *Bismut-Hermite-Einstein* if we have (g, H, ω) satisfying the following conditions

$$\begin{aligned} 0 =& \mathrm{Ric}_g - \frac{1}{4} H \circ H + \frac{1}{2} L_{g^{-1}\theta_\omega} g, \\ 0 =& d^* H + d\theta_\omega + \frac{1}{2} \iota_{g^{-1}\theta_\omega} H. \end{aligned} \tag{10.27}$$

where Ric_g denotes the Ricci tensor associated to g Riemannian metric, and ω is the almost Kähler form. These conditions are equivalent to $\rho^B(g) = 0$ and $dd^c\omega = 0$.

If a J-compatible generalized metric $C_\pm$ on an exact Courant algebroid E over a complex manifold M is Ricci-flat with respect to the divergence given by the Lee form, $\mathrm{Ric}^+ = 0$, this gives a solution to (10.27). Furthermore, the twisted Calabi-Yau equations (6.17) coming from having a solution to the Killing spinor equations, implies $\rho^B(g) = 0$.

Conjecture 2. Let E be any exact Courant algebroid over any $2n$-dimensional complex manifold M, with (ω, g) hermitian structure such that

$$\rho^B(g) = 0, \quad dd^c\omega = 0, \tag{10.28}$$

for $H = -d^c\omega$. Then, one can prove that there exists $f \in \mathcal{C}^\infty(M)$ such that

$$\varphi := g^{-1}\theta_\omega - \nabla f + \theta_\omega + df \in \Gamma(E)$$

is holomorphic. Then, the section $H \in \Gamma(\Omega^{\mathrm{ch}}_E)$ given in (10.11) is global, and generates an embedding of the $N=1$ superconformal vertex algebra into $\Gamma(\Omega^{\mathrm{ch}}_E)$.

For this conjecture to be true, it is necessary that (10.12) defines a central charge, so one needs to prove that under the given hypothesis

$$\sum_{j=1}^{n} \langle [\varphi, \bar{\epsilon}_j], \epsilon_j \rangle - \langle \varphi, \varphi \rangle \in \mathbb{C},$$

for $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n \subseteq \ell \oplus \bar{\ell}$ the local isotropic frames defined by (6.18).

10.3.2 Superconformal Vertex Algebras from Killing Spinors

All the given constructions does not necessarily induce a superconformal vertex algebra structure (see Remark 10.1.10). Indeed, we need (2.15) for the basis elements. In general, this is not true as we can see in Lemma C.2.3. However, we know the following.

Proposition 10.3.4. *The embeddings in Section 10.1 for quadratic Lie algebras induce an $N = 2$ superconformal vertex algebra structure if*

(1) we have a Manin triple.

(2) we have pairs of solutions for the Killing spinor equations.

Proof. It follows by Theorem 10.1.8 and Theorem 10.2.6. □

It is possible that when stronger conditions on $\mathfrak{g}$ are imposed (but weaker than the two above), then we could obtain the honest $N = 2$ superconformal vertex algebra structure in our constructions. An anonymous referee of [2] suggested the following.

Conjecture 3. When we take BRST cohomology of $\mathfrak{g}$, then we induce an honest $N = 2$ superconformal vertex algebra structure into the cohomology of $V^k_{\text{super}}(\mathfrak{g})$.

10.3.3 More Discussions on Main Theorem 10.1.19

We believe that the bi-vector field given in Section 5.3 vanishes when we have a solution to the twisted Hull-Strominger system (6.16). In other words, we have strong evidence (see examples in Section 11.1) to state the following.

Conjecture 4. Let (M, g) be a complex manifold with an hermitian structure g. If we have a solution (Ψ, ω, A) to the twisted Hull-Strominger system (6.16), then we have that the torsion bi-vector σ_ω identically vanishes. That is,

$$\sigma_\omega = 0$$

is always true in our hypotheses, and it does not give a real obstruction. As a consequence, we have that Theorem 10.1.19 is true without this extra condition. However, maybe, this is a little bit strong, and this is just true for a compact complex threefold. In any case, we have not seen this obstruction given by the torsion bi-vector in the literature, and we have not been able to obtain a counterexample of having solutions for the twisted Hull-Strominger system for which $\sigma_\omega \neq 0$. It seems to be an interesting problem.

The conjecture above is supported by Lemma 10.1.14, since it allows us to obtain that $T^2R = 0$ when $\sigma_\omega = 0$, and that is enough to contruct the desired embedding. However, for a long time, we tried to find a geometric meaning for R, in order to cancel the term T^2R. In the next result, we give an explicit formula for this term R.

Lemma 10.3.5. *Let (M, g) be any complex $2n$-dimensional manifold with an hermitian structure g. If (Ψ, ω) is a solution to the twisted Hull-Strominger system* (6.16) *and θ_ω closed, then*

$$R = 2d^*\theta_\omega - \|\theta_\omega\|^2_g \in \mathcal{C}^\infty(M, \mathbb{C}). \tag{10.29}$$

Proof. We have seen in Proposition 10.1.15 that

$$R = 4\sum_{j=1}^{n} \left\langle [e_j, u], e^j \right\rangle - 3\left\langle \sum_{j=1}^{n} \left[e^j, e_j\right], \sum_{k=1}^{n} \left[e^k, e_k\right] \right\rangle,$$

where by (6.23) we know the value of the second term. Let us see what happens with the first one. By Courant algebroid axioms, since $[\varphi, \cdot] = 0$, in the frames (6.18),

$$\sum_{j=1}^{n} \left\langle [e_j, u], e^j \right\rangle = \sum_{k=1}^{n} g^{-1} d\overline{z}_k \left(\theta_\omega^{0,1}\left(\frac{\partial}{\partial \overline{z}_k}\right)\right).$$

Now, we can assume $\theta_\omega = df_\omega$ as before, and

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$$

locally. Then,

$$\sum_{k=1}^{n} g^{-1} d\overline{z}_k \left(\theta_\omega^{0,1}\left(\frac{\partial}{\partial \overline{z}_k}\right)\right) = -\frac{1}{2}\Delta_{\overline{\partial}} f_\omega,$$

where

$$\Delta_{\overline{\partial}} f_\omega := \Lambda_\omega\left(2i\overline{\partial}\partial f_\omega\right) = \Delta_d f_\omega + \|\theta_\omega\|_g^2. \qquad \square$$

Now, we know that there exists $f \in \mathcal{C}^\infty(M)$ such that

$$\begin{aligned} 0 &= \mathrm{Ric}_g - \frac{1}{4} H \circ H + \nabla^2 f, \\ 0 &= d^* H + \iota_{\nabla f} H. \end{aligned}$$

Using these conditions and the ones in (10.28), after non-trivial computations based on [38, Subsection 4.4.2] about *generalized Ricci solitons*, we obtain that

$$\frac{1}{6}\|d^c\omega\|_g^2 - d^*\theta_\omega - \|\theta_\omega\|_g^2 \in \mathbb{C},$$

which always holds if we have a Bismut-Hermite-Einstein metric (in particular, a solution to the twisted Hull-Strominger system). But, unfortunately, this constant value does not coincide with our value of R. We expect that the value of R is somehow related with the Conjecture 2. So, we can state the following.

Conjecture 5. We can correct the supersymmetry generator J_0 to obtain certain J, so that the associated Neveu-Schwarz section H generates an embedding of the $N = 1$ superconformal vertex algebra into $\Gamma(\Omega_E^{\mathrm{ch}})$, thanks to be R replaced by

$$R' = k\left(\frac{1}{6}\|d^c\omega\|_g^2 - d^*\theta_\omega - \|\theta_\omega\|_g^2\right),$$

where $k \in \mathbb{C}$ is certain constant.

The role played by the quantity R, as computed in (10.29), remains a mistery to us.

Chapter 11

Applications of the Main Results

We collect in this last chapter different applications of the main results from Chapter 10. In particular, we give the first examples of $(0,2)$ mirror symmetry on compact complex non-Kähler manifolds. We include more open questions related with these applications.

11.1 Geometric Examples

First, we will present some geometric examples of solutions to the Killing spinor equations, where we can apply the methods explained in Section 10.1 to construct embeddings.

11.1.1 $N = 2$ from Calabi-Yau Equations on Complex Domains

We will see that a complex domain $X \subseteq \mathbb{C}^2$ endowed with an SU(2)-structure (Ψ, ω) that satisfies the twisted Calabi-Yau equations (6.17) induces an embedding of the $N = 2$ superconformal vertex algebra into the global sections of the chiral de Rham complex using Theorem 10.1.19 for the case of an exact Courant algebroid. So, given $f \in \mathcal{C}^\infty(X)$ a real positive *harmonic function*, that is, such that $f > 0, \Delta_{\omega_0} f = 0$, let

$$\begin{aligned} \Psi : &= f\Omega_0 := f dz_1 \wedge dz_2 \in \Omega^2(X), \\ \omega : &= f\omega_0 := \frac{i}{2} f \left(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right) \in \Omega^2(X). \end{aligned} \tag{11.1}$$

Since $d\omega = d \log f \wedge \omega$, the associated Lee form is

$$\theta_\omega := d \log f \in \Omega^1(X). \tag{11.2}$$

Lemma 11.1.1. (Ψ, ω) *defines a solution of the twisted Calabi-Yau equations* (6.17)*.*

Proof. Notice that

$$\Psi \wedge \omega = 0, \quad \Psi \wedge \overline{\Psi} = \frac{\omega^2}{2}$$

clearly, so we have an SU(2)-structure. Now, since

$$d^c\omega = d^c f \wedge \omega_0 = d^c \log f \wedge \omega \tag{11.3}$$

and $\Delta_{\omega_0} f = 0$, then

$$d\Psi = \theta_\omega \wedge \Psi, \quad d\theta_\omega = 0, \quad dd^c\omega = 0. \qquad \square$$

Let σ_ω be the torsion bi-vector from Section 5.3. We must check that it vanishes using Lemma 10.1.18. Consider for $j \in \{1, 2\}$ the local isotropic frames

$$\epsilon_j := e^{i\omega}\left(\frac{\partial}{\partial \overline{z}_j}\right) = \frac{\partial}{\partial \overline{z}_j} + g\frac{\partial}{\partial \overline{z}_j} \in e^{i\omega}\left(T^{1,0}X\right),$$

$$\overline{\epsilon}_j := g^{-1}d\overline{z}_j + d\overline{z}_j \in e^{-i\omega}\left(T^{1,0}X\right).$$

Lemma 11.1.2. *The torsion bi-vector of ω vanishes, that is,*

$$\sigma_\omega = 0.$$

Proof. Clearly $(\sigma_\omega)_{\overline{11}} = (\sigma_\omega)_{\overline{22}} = 0$ for being $\overline{\ell}$ isotropic. Now, notice that

$$g^{-1}d\overline{z}_j = 2f^{-1}\frac{\partial}{\partial z_j}, \quad \text{for } j \in \{1, 2\}.$$

So, by direct application of Lemma 10.1.18, using (11.3),

$$\begin{aligned}
\langle [\overline{\epsilon}_1, \overline{\epsilon}_2], \epsilon_1\rangle &= -d^c\omega\left(g^{-1}d\overline{z}_1, g^{-1}d\overline{z}_2, \frac{\partial}{\partial \overline{z}_1}\right) = 4f^{-2}d^c\omega\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \overline{z}_1}\right)\\
&= -2if^{-2}d^cf\left(\frac{\partial}{\partial z_2}\right) = -2if^{-2}\left(-i\partial f\left(\frac{\partial}{\partial z_2}\right)\right) = -2\frac{\partial}{\partial z_2}\left(f^{-1}\right),\\
\langle [\overline{\epsilon}_1, \overline{\epsilon}_2], \epsilon_2\rangle &= -d^c\omega\left(g^{-1}d\overline{z}_1, g^{-1}d\overline{z}_2, \frac{\partial}{\partial \overline{z}_2}\right) = 4f^{-2}d^c\omega\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \overline{z}_2}\right)\\
&= 2if^{-2}d^cf\left(\frac{\partial}{\partial z_1}\right) = 2if^{-2}\left(-i\partial f\left(\frac{\partial}{\partial z_1}\right)\right) = 2\frac{\partial}{\partial z_1}\left(f^{-1}\right).
\end{aligned}$$

In summary,

$$\begin{aligned}
(\sigma_\omega)_{\overline{12}} &:= \sum_{j=1}^{2}\langle \mathcal{D}\langle [\overline{\epsilon}_1, \overline{\epsilon}_2], \epsilon_j\rangle, \overline{\epsilon}_j\rangle = 4f^{-1}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\left(f^{-1}\right) - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_1}\left(f^{-1}\right)\right) = 0,\\
(\sigma_\omega)_{\overline{21}} &:= \sum_{j=1}^{2}\langle \mathcal{D}\langle [\overline{\epsilon}_2, \overline{\epsilon}_1], \epsilon_j\rangle, \overline{\epsilon}_j\rangle = -\sum_{j=1}^{2}\langle \mathcal{D}\langle [\overline{\epsilon}_1, \overline{\epsilon}_2], \epsilon_j\rangle, \overline{\epsilon}_j\rangle = 0. \qquad \square
\end{aligned}$$

So, we are in the conditions of Theorem 10.1.19. Let $E := E_{-d^c\omega}$ be the exact Courant algebroid associated to the solution to the twisted Calabi-Yau equations (11.1).

Proposition 11.1.3. *Let $X \subseteq \mathbb{C}^2$ be a complex domain, and consider $f \in \mathcal{C}^\infty(X)$ a harmonic function with respect to the standard flat Kähler metric. Then, the solution of the twisted Calabi-Yau equations* (11.1)*, with associated Lee form* (11.2)*, induces an embedding of the $N = 2$ superconformal vertex algebra of central charge* 6 *into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}$. The generators of this embedding are given by* (10.18) *for the frames above.*

Proof. It is a direct consequence of Theorem 10.1.19 and Lemma 11.1.2. $\square$

11.1.1.1 The (Homogeneous) Hopf Surface

As a particular case of the hypothesis of the previous result, we can take the following:

$$\widetilde{X} := \mathbb{C}^2 - \{(0,0)\} \longrightarrow X = \frac{\widetilde{X}}{\mathbb{Z}}, \quad f := \frac{1}{\|\cdot\|^2}.$$

This corresponds to the universal cover of the *Hopf surface*. Indeed, the solution of the twisted Calabi-Yau equation above is preserved by Deck transformations and descends to a solution (Ψ, ω) of the twisted Calabi-Yau equations in X. Furthermore, the atlas given by the universal covering corresponds in this case to the atlas of X in Lemma 5.2.2. This one corresponds by the classification given in [39] to a locally conformally Kähler metric. As a consequence, if $E := E_{-d^c\omega}$ is the exact Courant algebroid associated to the solution (Ψ, ω), we can apply Proposition 11.1.3 to induce an embedding of the form

$$V^6(\mathcal{K}_2) \hookrightarrow \Gamma\left(\Omega^{\mathrm{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}\right). \tag{11.4}$$

On the other hand, we can obtain a family of solutions for the Killing spinor equations as in Definition 6.2.15 for E exact, such that we have a left-invariant triple $(V_+, \mathrm{div}_+, \eta)$, and Proposition 6.3.3 and Proposition 7.1.9 apply. Then, applying Theorem 10.1.5 and Theorem 10.1.8, we are able to obtain 2 more embeddings (this appears in [2]). We will explain it briefly. Consider the compact 4-dimensional manifold $M = \mathbb{S}^3 \times \mathbb{S}^1$ endowed with the canonical orientation. We use the Lie group structure given by identifying

$$M \cong K = \mathrm{SU}(2) \times \mathrm{U}(1).$$

Consider the next generators for the Lie algebra of left-invariant vector fields

$$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R} = \langle v_1, v_2, v_3, v_4 \rangle,$$

with relations

$$[v_2, v_3] = -v_1, \quad [v_3, v_1] = -v_2, \quad [v_1, v_2] = -v_3, \quad [v_4, \cdot] = 0.$$

Equivalently,

$$dv^1 = v^2 \wedge v^3, \quad dv^2 = v^3 \wedge v^1, \quad dv^3 = v^1 \wedge v^2, \quad dv^4 = 0,$$

for $\{v^j\}_{j\in\{1,2,3,4\}}$ the dual basis. We take $\ell \in \mathbb{R}$ and define a left-invariant 3-form

$$H_\ell = \ell v^{123}. \tag{11.5}$$

This corresponds to a constant multiple of Cartan 3-form on the SU(2) factor, and hence it is bi-invariant and closed. Thus, this defines an exact equivariant Courant algebroid $E_\ell = TK \oplus T^*K$, with usual pairing and Dorfman bracket (where $H = H_\ell$). Our next goal is to define a one-parameter family of left-invariant solutions of the Killing spinor equations on E_ℓ. Given $x, a > 0$ positive real numbers, consider

$$g_{x,a} = \frac{a}{x}(v^1 \otimes v^1 + v^2 \otimes v^2 + v^3 \otimes v^3 + x^2 v^4 \otimes v^4)$$

the bi-invariant metric on K. We define a bi-invariant generalized metric on E_ℓ

$$V_{\pm}^{x,a} = \{v \pm g_{x,a}(v) \mid v \in \mathfrak{k}\}$$

and a bi-invariant divergence $\operatorname{div}_{x,a} = \operatorname{div}_0 - \langle \varepsilon^x, \cdot \rangle$, for div_0 the Riemannian divergence of $g_{x,a}$, and $\varepsilon^x = -xv^4$. We have arrived at the following result.

Lemma 11.1.4 ([2, Lemma 4.7])**.** *For $x, a > 0$ and $\ell \in \mathbb{R}$, the pair $(V_{\pm}^{x,a}, \varepsilon^x)$ is closed.*

Proof. The statement follows trivially from (6.6). □

Now, we must specify a left-invariant spinor line $\langle \eta \rangle \subseteq S_+^+$ corresponding to an almost complex structure on $V_+^{x,a}$. For this, note that the anchor map $\pi\colon E_\ell \to TK$ induces an isomorphism $V_+^{x,a} \cong TK$, and we set $I_x\colon V_+^{x,a} \longrightarrow V_+^{x,a}$ almost complex structure by

$$I_x v_4 := x v_1, \quad I_x v_2 := v_3. \tag{11.6}$$

Now, for

$$\operatorname{div}_+^{x,a} := \operatorname{div}_{x,a|_{V_+^{x,a}}},$$

we have to prove that $(V_+^{x,a}, \operatorname{div}_+^{x,a}, I_x)$ defines a solution of the Killing spinor equations, provided that $\ell = a/x$. For this, we adopt an algebraic approach. By Proposition 7.1.9,

$$\mathfrak{g}_\ell := \Gamma(E_\ell)^K$$

has structure of quadratic Lie algebra, with underlying vector space $\mathfrak{g}_\ell \cong \mathfrak{k} \oplus \mathfrak{k}^*$, where

$$\begin{aligned}
&(v + \alpha | w + \beta) = \frac{1}{2}\left(\alpha(w) + \beta(v)\right), \quad \text{for } v + \alpha, w + \beta \in \mathfrak{g}_\ell,\\
&[v + \alpha, w + \beta] = [v, w] - \beta([v, \cdot]) + \alpha([w, \cdot]) + \ell i_w i_v v^{123}, \quad \text{for } v + \alpha, w + \beta \in \mathfrak{g}_\ell.
\end{aligned}$$

Now, notice that, for fixed $x, a > 0$, we have that the pair $(V_+^{x,a}, \operatorname{div}_{x,a})$ can be regarded as the pair $(V_+^{x,a}, \varepsilon^x)$ of generalized metric and divergence on the quadratic Lie algebra $\mathfrak{g}_\ell$. Denote by $\eta_x \in S_+^+$ a non-vanishing pure spinor in $\langle \eta \rangle \subseteq S_+^+$. We have that

$$I_x(v + g_{x,a}(v)) := I_x v + g_{x,a}(I_x v), \quad \text{for } v \in V_+^{x,a}.$$

Lemma 11.1.5 ([2, Lemma 4.8])**.** *We have that the triple $(V_+^{x,a}, \varepsilon_+^x, \eta_x)$ is a solution of the Killing spinor equations as in Definition 7.1.5 on $\mathfrak{g}_\ell$ if and only if $\ell = a/x$.*

Proof. By Proposition 7.2.1, it suffices to prove that (7.6) holds if and only if $\ell = a/x$. Now, an oriented orthonormal basis for $V_+^{x,a}$ is given by

$$\left\{ \sqrt{\tfrac{x}{a}} v_2 + \sqrt{\tfrac{a}{x}} v^2, \sqrt{\tfrac{x}{a}} v_3 + \sqrt{\tfrac{a}{x}} v^3, \tfrac{1}{\sqrt{xa}} v_4 + \sqrt{xa} v^4, \sqrt{\tfrac{x}{a}} v_1 + \sqrt{\tfrac{a}{x}} v^1 \right\},$$

with associated isotropic basis of $V_+^{1,0}$ and $V_+^{0,1}$ given by

$$\begin{aligned}
\epsilon_1^+ &= \tfrac{1}{\sqrt{2}}\left(\left(\sqrt{\tfrac{x}{a}} v_2 + \sqrt{\tfrac{a}{x}} v^2\right) - i\left(\sqrt{\tfrac{x}{a}} v_3 + \sqrt{\tfrac{a}{x}} v^3\right)\right), \quad & \overline{\epsilon}_1^+ &= \overline{\epsilon_1^+},\\
\epsilon_2^+ &= \tfrac{1}{\sqrt{2}}\left(\left(\tfrac{1}{\sqrt{xa}} v_4 + \sqrt{xa} v^4\right) - i\left(\sqrt{\tfrac{x}{a}} v_1 + \sqrt{\tfrac{a}{x}} v^1\right)\right), \quad & \overline{\epsilon}_2^+ &= \overline{\epsilon_2^+}.
\end{aligned}$$

Now, a direct calculation shows that

$$\left[\epsilon_1^+, \epsilon_2^+\right] = \tfrac{1}{2}\left(\tfrac{x}{a}v_2 + \left(2 - \ell\tfrac{x}{a}\right)v^2\right) - \tfrac{i}{2}\left(\tfrac{x}{a}v_3 + \left(2 - \ell\tfrac{x}{a}\right)v^3\right).$$

So, $\left[\epsilon_1^+, \epsilon_2^+\right] \in V_+^{x,a} \otimes \mathbb{C}$ if and only if $\ell = a/x$ and, assuming this condition, it follows that

$$\left[V_+^{1,0}, V_+^{1,0}\right] \subseteq V_+^{1,0},$$

where $V_+^{1,0} \subseteq V_+^{x,a} \otimes \mathbb{C}$ is the i-eigenbundle of I_x. Similarly,

$$v := \sum_{j=1}^{2}\left[\epsilon_j^+, \overline{\epsilon}_j^+\right] = i\left(-\tfrac{x}{a}v_1 - \left(2 - \ell\tfrac{x}{a}\right)v^1\right).$$

So, $v \in V_+^{x,a} \otimes \mathbb{C}$ if and only if $\ell = a/x$. Finally, assuming this condition and using that

$$I_x(\varepsilon_+^x) = -\tfrac{1}{2}I_x\left(\tfrac{1}{a}v_4 + xv^4\right) = -\tfrac{1}{2}\left(\tfrac{x}{a}v_1 + v^1\right),$$

it follows that the second equation in (7.6) holds. □

Consequently, by direct application of Proposition 7.1.9, the triple $(V_+^{x,a}, \mathrm{div}_+^{x,a}, \eta_x)$ is a left-invariant solution of the Killing spinor equations as in Definition 6.2.15 on E_ℓ if and only if $\ell = a/x$. Our solutions of the Killing spinor equations in the previous result are such that ε^x is an infinitesimal isometry for $V_+^{x,a}$ in the sense of Definition 7.1.2. In the next result we prove that the dual vector field $\pi\varepsilon_+^x$ is holomorphic with respect to I_x.

Lemma 11.1.6 ([2, Lemma 4.10])**.** *The left-invariant vector field $\pi\varepsilon_+^x$ is I_x-holomorphic, that is,*

$$\left[\varepsilon_+^x, V_+^{1,0}\right] \subseteq V_+^{1,0},$$

where $V_+^{1,0} \subseteq V_+^{x,a} \otimes \mathbb{C}$ is the i-eigenbundle of I_x.

Proof. Immediate, since $\varepsilon_+^x = -\frac{1}{2}\left(\frac{1}{a}v_4 + xv^4\right)$ is in the center of $\mathfrak{g}_\ell$. □

We study our family of solutions in terms of complex geometry by means of Proposition 6.3.3 in the exact case, in order to compare with the construction given at the beginning of the present section. Assuming $\ell = a/x$, the family of solutions of (6.17) induced by

$$\left(V_+^x, \mathrm{div}_+^x, \eta_x\right) := \left(V_+^{x,\ell x}, \mathrm{div}_+^{x,\ell x}, \eta_x\right) \tag{11.7}$$

is given by

$$\begin{aligned} \omega_x &= \ell x v^4 \wedge v^1 + \ell v^2 \wedge v^3, \\ \Psi_x &= \tfrac{\ell}{2}\left(iv^1 + xv^4\right) \wedge \left(v^2 + iv^3\right), \end{aligned} \tag{11.8}$$

with Lee form $\theta_x = -xv^4$. To see the relation with the construction we started with, note that the complex manifold (K, I_x) is biholomorphic to the diagonal Hopf surface

$$X_x = \frac{\mathbb{C}^2 - \{(0,0)\}}{\langle\gamma_x\rangle},$$

where $\langle\gamma_x\rangle \cong \mathbb{Z}$ is generated by $\gamma_x(z_1, z_2) = (e^x z_1, e^x z_2)$. So, we arrive at the following.

Proposition 11.1.7. *For any $\ell, x > 0$, the solution of the Killing spinor equations in Lemma 11.1.5 induces embeddings of the $N = 2$ superconformal vertex algebra of central charges* 6 *and* $6 + 6/\ell$ *into the space of global sections of the chiral de Rham complex* $\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}$. *The generators are, on each case,* (10.9) *and* (10.10)*, and in both cases these formulas work for $k = 2$, and the basis is given by formulas in the proof of Lemma 11.1.5.*

Proof. Applying Theorem 10.1.5 and Theorem 10.1.8, and Lemma 11.1.5, we obtain embeddings in the quadratic Lie algebras. So, we have embeddings of the $N = 2$ superconformal vertex algebra of central charges 6 and $6 + 6/\ell$ into the universal superaffine vertex algebra $V^k_{\mathrm{super}}\left(\mathfrak{g}^{\mathbb{C}}_\ell\right)$ with $k \neq 0$. Now, as in Proposition 10.1.11, if $k = 2$, there is an embedding

$$V^2_{\mathrm{super}}\left(\mathfrak{g}^{\mathbb{C}}_\ell\right) \hookrightarrow \Gamma\left(\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}\right)$$

on the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}$. So, we obtain two more embeddings of the $N = 2$ superconformal vertex algebra of central charges 6 and $6 + 6/\ell$ into the global sections of the chiral de Rham complex of $E_\ell \otimes_{\mathbb{R}} \mathbb{C}$. □

Notice that these two constructions can be compared. Indeed, it can be seen that there exists $\ell_0 > 0$ for which $E_{\ell_0} \cong E_{-d^c\omega}$, where $E_{-d^c\omega}$ is the exact Courant algebroid induced by the solution (Ψ, ω) on the quotient X considered to construct the embedding (11.4). So, multipliying ω by $c > 0$, we obtain the family of exact Courant algebroids E_ℓ.

11.1.1.2 Compact Complex Surfaces

A complete classification for the solutions of the twisted Calabi-Yau equations (6.17) on compact complex surfaces was obtained in [39, Proposition 2.10]. As a direct consequence of that classification, we can obtain a more general result for compact complex surfaces. Let X be any compact complex surface, and suppose that (ω_0, Ψ_0, A) is a solution to the twisted Hull-Strominger system (6.16). Now, by Gauduchon's Theorem [41], there exists $\phi \in \mathcal{C}^\infty(X)$ such that

$$\Psi := e^\phi \Psi_0 \in \Omega^2(X),$$
$$\omega := e^\phi \omega_0 \in \Omega^2(X),$$

is an SU(2)-structure satisfying that $dd^c\omega = 0$. Even more, since

$$d\omega = d\phi e^\phi \wedge \omega + e^\phi d\omega = (d\phi + \theta_{\omega_0}) \wedge \omega,$$

clearly, the associated Lee form is by definition $\theta_\omega = \theta_{\omega_0} + d\phi \in \Omega^1(X)$.

Lemma 11.1.8. (Ψ, ω) *is a solution of the twisted Calabi-Yau equations* (6.17).

Proof. Notice that, by hypothesis,

$$\Psi \wedge \omega = 0, \quad \Psi \wedge \overline{\Psi} = \frac{\omega^2}{2}$$

clearly, so we have an SU(2)-structure. Now, since

$$d\Psi = d\phi \wedge \Psi + e^{\phi}\theta_{\omega_0} \wedge \Psi_0$$

and $d\theta_{\omega_0} = 0$ by hypothesis, then

$$d\Psi = \theta_\omega \wedge \Psi, \quad d\theta_\omega = 0, \quad dd^c\omega = 0. \qquad \square$$

In conclusion, we are ready now to state the following important result.

Theorem 11.1.9. *Let X be a compact complex surface for which (ω_0, Ψ_0, A) is a solution to the twisted Hull-Strominger system* (6.16)*, and consider the string Courant algebroid $E := E_{-d^c\omega_0, A}$. Then, there exists an embedding of the $N = 2$ superconformal vertex algebra of central charge* 6 *into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}$. The generators of this embedding are given by* (10.18) *for the frames above.*

Proof. It follows easily for all the work we have done. Indeed, thanks to the classification for solutions of the twisted Calabi-Yau equations (6.17) on compact complex surfaces obtained in [39, Proposition 2.10], we have that either (X, ω, I) is a flat torus $\mathbb{T}^4$ or a K3 surface with a Kähler Ricci-flat metric (and hence $\theta_\omega = 0$), or (X, ω, I) is a quaternionic Hopf surface and $\theta_\omega \neq 0$. In both cases, $\omega_0 = e^{-\phi}\omega$ is, in the corresponding (unique) maximal atlas given by Lemma 5.2.2, locally conformally Kähler. Then, we can repeat the same proof in Lemma 11.1.2 with $f := e^{\phi}$, since we do not use the harmonicity of f. The proof is now a direct consequence of Theorem 10.1.19, since (10.16) is global. $\square$

11.1.2 $N = 2$ from Hull-Strominger System on Complex Dimension 3

We study now two examples whose solutions to the Killing spinor equations come from the Hull-Strominger System. From now, we will use the notations for $m \in \mathbb{N}$,

$$v_1 \wedge \cdots \wedge v_m = v_{1\cdots m}, \quad v^1 \wedge \cdots \wedge v^m = v^{1\cdots m},$$

where $v_1, \cdots, v_m, v^1, \cdots, v^m$ are the 1-forms such that $v^j(v_k) = \delta^j_k$ for $j, k \in \{1, \ldots, m\}$.

11.1.2.1 The (Homogeneous) Iwasawa Manifold

We start discussing an application of Theorem 10.1.5. We show that from an invariant solution of the Hull-Strominger system over a certain Lie group K, we can construct a quadratic Lie algebra $\mathfrak{g}_K$ endowed with a solution of the Killing spinor equations as in Definition 7.1.5, where Theorem 10.1.5 applies. In particular, we will provide an infinite family of examples for which $V_+^{\mathbb{C}} = l \oplus \bar{l}$ does not define a Manin triple (as we mentioned in Remark 10.2.8). We restrict to a family of solutions studied in [40]. We will follow the abstract definition of the equations in [39]. Consider the complex Heisenberg Lie group

$$G := H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z_i \in \mathbb{C} \right\}.$$

The following 1-forms

$$\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - z_2 dz_1$$

define a global left-invariant holomorphic frame of $T^{1,0*}G$, and satisfy the equations

$$d\omega_1 = d\omega_2 = 0, d\omega_3 = \omega_{12},$$

from which all the exterior algebra relations can be easily derived. Now, for any choice of $(m, n, p) \in \mathbb{R}^3 - \{(0,0,0)\}$, we consider the purely imaginary $(1,1)$-form given by

$$F = \pi \left(m \left(\omega_{1\bar{1}} - \omega_{2\bar{2}} \right) + n \left(\omega_{1\bar{2}} + \omega_{2\bar{1}} \right) + ip \left(\omega_{1\bar{2}} - \omega_{2\bar{1}} \right) \right). \tag{11.9}$$

Consider the left-invariant SU(3)-structure on G defined by

$$\Omega = \omega_{123}, \quad \omega = \tfrac{i}{2} \left(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}} \right). \tag{11.10}$$

Notice that $d\Omega = 0$ clearly, while $d\omega \neq 0$ (we have Calabi-Yau non-Kähler structure).

Proposition 11.1.10 ([40, Proposition 4.1])**.** *Fix $\alpha = \left(2\pi^2 \left(m^2 + n^2 + p^2\right)\right)^{-1} \in \mathbb{R}$. Then, the pair (ω, F) is a solution of the Hull-Strominger system. That is,*

$$\begin{aligned} F^{0,2} = 0, \qquad F \wedge \omega^2 &= 0, \\ d(\|\Omega\|_\omega \omega^2) &= 0, \\ dd^c\omega - \alpha F \wedge F &= 0. \end{aligned} \tag{11.11}$$

Starting from any solution of the form (11.11), it is proven in [35] that one can construct solutions of the Killing spinor equations on a transitive Courant algebroid over K. Taking left-invariant sections, one obtains a quadratic Lie algebra endowed with a solution for the Killing spinor equations as in Definition 7.1.5, similarly as in Proposition 7.1.9. Using that the solution in Proposition 11.1.10 is left-invariant, it follows that $\|\Omega\|_\omega$ is constant, and hence ω is balanced. That is, $\theta_\omega = 0$. From this, the induced solution for the Killing spinor equations has zero divergence and Theorem 10.1.5 applies. Rather than given the details of this general argument, we shall provide here an explicit direct proof. Denote by $\mathfrak{h}_\mathbb{C}$ the Lie algebra of G. Then, for any choice of $(m, n, p) \in \mathbb{R}^3 - \{(0,0,0)\}$, one can define a real quadratic Lie algebra with underlying vector space

$$\mathfrak{g}_{m,n,p} = \mathfrak{h}_\mathbb{C} \oplus i\mathbb{R} \oplus \mathfrak{h}_\mathbb{C}^*,$$

pairing

$$(v + r + \eta \,|w + t + \xi\,) = \frac{1}{2}\left(\eta(v) + \xi(w)\right) - \alpha r t, \quad \text{for } v + r + \eta, w + t + \xi \in \mathfrak{g}_{m,n,p},$$

and Lie bracket

$$\begin{aligned} [v + r + \eta, w + t + \xi] = {} & [v, w] - \xi([v, \cdot]) + \eta([w, \cdot]) + i_w i_v \left(d^c\omega\right) - F(v, w) \\ & + 2\alpha(r i_w F - t i_v F), \quad \text{for } v + r + \eta, w + t + \xi \in \mathfrak{g}_{m,n,p}, \end{aligned}$$

where F and α are as in Proposition 11.1.10. The definition of α is necessary for the Lie bracket to satisfy Jacobi identity. More explicitly, taking a real basis of $\mathfrak{h}_{\mathbb{C}}^*$ defined by

$$\omega_1 = v^1 + iv^2, \omega_2 = v^3 + iv^4, \omega_3 = v^4 + iv^6,$$

one has the non-trivial relations

$$dv^5 = v^{13} - v^{24}, dv^6 = v^{14} + v^{23},$$

and the (integrable almost) complex structure on $\mathfrak{h}_{\mathbb{C}}$ reads as

$$Iv_1 = v_2, \quad Iv_3 = v_4, \quad Iv_5 = v_6,$$

where v_j for $j \in \{1, 2, 3, 4, 5, 6\}$ is the dual basis. From this, it follows that

$$d^c\omega = v^{135} + v^{236} + v^{146} - v^{245},$$

and also that

$$F = 2\pi i \left(m\left(v^{34} - v^{12}\right) + n\left(v^{23} - v^{14}\right) + p\left(v^{13} + v^{24}\right)\right).$$

We prove that the previous data determines a solution of the Killing spinor equations on $\mathfrak{g}_{m,n,p}$ with zero divergence. Consider the generalized metric on $\mathfrak{g}_{m,n,p}$ defined by

$$V_+ = \{v + g(v, v) \mid v \in \mathfrak{h}_{\mathbb{C}}\}, \quad V_- = \{v + r - g(v, v) \mid v + r \in \mathfrak{h}_{\mathbb{C}} \oplus i\mathbb{R}\},$$

where

$$g(\cdot, \cdot) := \omega(\cdot, I\cdot) = v^1 \otimes v^1 + v^2 \otimes v^2 + v^3 \otimes v^3 + v^4 \otimes v^4 + v^5 \otimes v^5 + v^6 \otimes v^6.$$

Consider the spinor line $\langle\eta\rangle \subseteq S_+^+$ corresponding to the complex structure on $V_+ \cong \mathfrak{h}_{\mathbb{C}}$.

Lemma 11.1.11 ([2, Lemma 5.5]). *The triple $(V_+, 0, \eta)$ is a solution of the Killing spinor equations on $\mathfrak{g}_{m,n,p}$, and $V_+^{\mathbb{C}} \subseteq \mathfrak{g}_{m,n,p}^{\mathbb{C}}$ is a Lie subalgebra if and only if $m = 0$.*

Proof. It suffices to prove (7.6). An isotropic basis of $V_+^{1,0}$ and $V_+^{0,1}$ is given by

$$\begin{aligned}
\epsilon_1^+ &= \tfrac{1}{\sqrt{2}}\left(\left(v_1 + v^1\right) - i\left(v_2 + v^2\right)\right), & \overline{\epsilon}_1^+ &= \overline{\epsilon_1^+}, \\
\epsilon_2^+ &= \tfrac{1}{\sqrt{2}}\left(\left(v_3 + v^3\right) - i\left(v_3 + v^3\right)\right), & \overline{\epsilon}_2^+ &= \overline{\epsilon_2^+}, \\
\epsilon_3^+ &= \tfrac{1}{\sqrt{2}}\left(\left(v_5 + v^5\right) - i\left(v_6 + v^6\right)\right), & \overline{\epsilon}_1^+ &= \overline{\epsilon_1^+}.
\end{aligned}$$

Now, a direct calculation shows that

$$\left[\epsilon_1^+, \epsilon_2^+\right] = -\sqrt{2}\epsilon_3^+, \qquad \left[\epsilon_1^+, \epsilon_3^+\right] = 0, \qquad \left[\epsilon_2^+, \epsilon_3^+\right] = 0,$$

and therefore $\left[V_+^{1,0}, V_+^{1,0}\right] \in V_+^{1,0}$. Similarly,

$$\left[\epsilon_1^+, \overline{\epsilon}_1^+\right] = -2\pi m, \qquad \left[\epsilon_2^+, \overline{\epsilon}_2^+\right] = 2\pi m, \qquad \left[\epsilon_3^+, \overline{\epsilon}_3^+\right] = 0,$$

and therefore the first part of the statement follows. From previous formula, we conclude that $m \neq 0$ implies $V_+^{\mathbb{C}} \subseteq \mathfrak{g}_{m,n,p}^{\mathbb{C}}$ is not a Lie subalgebra. The other implication is left to the reader. $\square$

So, by Theorem 10.1.5, we obtain an embedding of the $N = 2$ superconformal vertex algebra of central charge $c = 9$ into the universal superaffine vertex algebra associated to $\mathfrak{g}^{\mathbb{C}}_{m,n,p}$, which works for any level $0 \neq k \in \mathbb{C}$. When $m \neq 0$, these embeddings are not associated to Manin triples, showing that Theorem 10.1.5 provides a strict generalization of Getzler's construction mentioned in Theorem 3.3.3. That is, we have the next result.

Proposition 11.1.12 ([2, Proposition 5.6]). *For $(m, n, p) \in \mathbb{R}^3 - \{(0, 0, 0)\}$, the solution of the Killing spinor equations in Lemma 11.1.11 induces an embedding of the $N = 2$ superconformal vertex algebra of central charge 9 into the universal superaffine vertex algebra $V^k_{\text{super}}\left(\mathfrak{g}^{\mathbb{C}}_{m,n,p}\right)$ for $k \neq 0$. The generators are given by (10.9) for the basis above.*

Proof. It is a direct consequence of Theorem 10.1.5 and Lemmma 11.1.11. □

Remark 11.1.13. As it is proven in [35], associated to a solution of the Hull-Strominger system, there exists a transitive Courant algebroid E. Denote by $E_{m,n,p}$ this transitive Courant algebroid on K determined by the solution given in Proposition 11.1.10. Then, similarly as in Proposition 10.1.11, when $k = 2$ there exists an embedding

$$V^2_{\text{super}}\left(\mathfrak{g}^{\mathbb{C}}_{m,n,p}\right) \hookrightarrow \Gamma\left(\Omega^{\text{ch}}_{E_{m,n,p}\otimes_{\mathbb{R}}\mathbb{C}}\right)$$

on the space of sections for the chiral de Rham complex of $E_{m,n,p} \otimes_{\mathbb{R}} \mathbb{C}$. By Proposition 11.1.12, one obtains an embedding of the $N = 2$ superconformal vertex algebra of central charge 9 into the space of global sections of the chiral de Rham complex $\Omega^{\text{ch}}_{E_{m,n,p}\otimes_{\mathbb{R}}\mathbb{C}}$.

Now, we will apply instead the general method for general transitive Courant algebroids provided by Theorem 10.1.19. Indeed, in order to apply Theorem 10.1.19, we must see that the associated torsion bi-vector σ_ω vanishes by Lemma 10.1.18. For $j \in \{1, 2, 3\}$, consider the local isotropic frames

$$\epsilon_j := e^{i\omega}\left(\frac{\partial}{\partial \overline{z}_j}\right) = \frac{\partial}{\partial \overline{z}_j} + g\frac{\partial}{\partial \overline{z}_j} \in e^{i\omega}\left(T^{1,0}G\right),$$

$$\overline{\epsilon}_j := g^{-1}d\overline{z}_j + d\overline{z}_j \in e^{-i\omega}\left(T^{1,0}G\right).$$

Lemma 11.1.14. *The torsion bi-vector of ω vanishes, that is,*

$$\sigma_\omega = 0.$$

Proof. First, we will compute $d^c\omega$ from (11.10). Since $d\omega = \frac{i}{2}\left(\omega_{12\overline{3}} - \omega_{3\overline{1}\overline{2}}\right)$, then

$$d^c\omega = -d\omega(I\cdot, I\cdot, I\cdot) = \frac{1}{2}\left(\omega_{12\overline{3}} + \omega_{3\overline{1}\overline{2}}\right). \tag{11.12}$$

Clearly $(\sigma_\omega)_{11} = (\sigma_\omega)_{22} = (\sigma_\omega)_{33} = 0$, since $\overline{\ell}$ is isotropic. Now, notice that

$$\begin{aligned} g^{-1}d\overline{z}_1 &= 2\left(\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3}\right), \\ g_u^{-1}d\overline{z}_2 &= 2\frac{\partial}{\partial z_2}, \\ g_u^{-1}d\overline{z}_3 &= 2\left(\overline{z}_2\frac{\partial}{\partial z_2} + \left(1 + |z_2|^2\right)\frac{\partial}{\partial z_3}\right). \end{aligned}$$

So, by direct application of Lemma 10.1.18, using (11.12), we obtain the non-zero values

$$\langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_1 \rangle = -d^c\omega \left(g^{-1}d\bar{z}_1, g^{-1}d\bar{z}_2, \frac{\partial}{\partial \bar{z}_1} \right) = -2\bar{z}_2,$$
$$\langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_3 \rangle = -d^c\omega \left(g^{-1}d\bar{z}_1, g^{-1}d\bar{z}_2, \frac{\partial}{\partial \bar{z}_3} \right) = 2.$$

That is, we have that

$$(\sigma_\omega)_{12} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = -2 \left(g^{-1}d\bar{z}_1(\bar{z}_2) - g^{-1}d\bar{z}_3(2) \right) = 0.$$

Again, by direct application of Lemma 10.1.18, using (11.12), we obtain that

$$\langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_1 \rangle = \langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_2 \rangle = \langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_3 \rangle = 0.$$

That is, we have that

$$(\sigma_\omega)_{13} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = 0.$$

By direct application of Lemma 10.1.18, using (11.12), we obtain the non-zero values

$$\langle [\bar{\epsilon}_2, \bar{\epsilon}_3], \epsilon_1 \rangle = -d^c\omega \left(g^{-1}d\bar{z}_2, g^{-1}d\bar{z}_3, \frac{\partial}{\partial \bar{z}_1} \right) = 2\bar{z}_2^2,$$
$$\langle [\bar{\epsilon}_2, \bar{\epsilon}_3], \epsilon_3 \rangle = -d^c\omega \left(g^{-1}d\bar{z}_2, g^{-1}d\bar{z}_3, \frac{\partial}{\partial \bar{z}_3} \right) = -2\bar{z}_2.$$

That is, we have that

$$(\sigma_\omega)_{23} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = 2 \left(g^{-1}d\bar{z}_1(\bar{z}_2^2) + g^{-1}d\bar{z}_3(\bar{z}_2) \right) = 0.$$

In summary, we arrive at the desired result, applying antisymmetry to obtain that

$$(\sigma_\omega)_{21} = (\sigma_\omega)_{31} = (\sigma_\omega)_{32} = 0. \qquad \square$$

Proposition 11.1.15. *For $(m, n, p) \in \mathbb{R}^3 - \{(0, 0, 0)\}$, consider the transitive Courant algebroid $E := E_{m,n,p}$ associated to the solution to the Hull-Strominger system* (11.11)*. Then, we induce an embedding of the $N = 2$ superconformal vertex algebra of central charge* 9 *into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E \otimes_{\mathbb{R}} \mathbb{C}}$. The generators of this embedding are given by* (10.18) *for the frames above.*

Proof. It is a direct consequence of Theorem 10.1.19 and Lemma 11.1.14. $\square$

We expect that the two embeddings we have obtained in Proposition 11.1.12 and Proposition 11.1.15 actually coincide. We hope to prove this in future work.

11.1.2.2 The Picard Calabi-Yau Three-fold

We give our last application of Theorem 10.1.19. We study the non-Kähler Calabi-Yau 3-fold introduced by S. Picard in [78]. Consider the complex 3-dimensional manifold

$$X := \mathbb{C}^3/\sim, \text{ where } (z_1, z_2, z_3) \sim (z_1 + a, z_2 + c, \overline{a}z_2 + b), \text{ for } a, b, c \in \mathbb{Z}[i].$$

Observe that there is an holomorphic fibration structure

$$\begin{array}{rccc} p\colon & \mathbb{T}^2 \hookrightarrow X & \longrightarrow & \mathbb{T}^4 := \mathbb{C}^2/\sim \\ & [(z_1, z_2, z_3)] & \mapsto & [(z_1, z_2)] \end{array}.$$

The following 1-forms

$$\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - \overline{z}_1 dz_2$$

define a global holomorphic frame of $T^{1,0*}M$, and satisfy the equations

$$d\omega_1 = d\omega_2 = 0, d\omega_3 = \omega_{2\overline{1}}.$$

Note that the difference between this new example and the Iwasawa manifold lies in the 1-form ω_3. Indeed, this is qualitatively different, since here $d\omega_3 \in \Omega^{1,1}(X)$. Now, for a choice of $(m, n, p) \in \mathbb{Z}^3 - \{(0, 0, 0)\}$, we can consider the purely imaginary $(1, 1)$-form F given by (11.9). Consider any $u \in \mathcal{C}^\infty(\mathbb{T}^4, \mathbb{R})$, and define on X the forms

$$\Omega = \omega_{123} = dz_{123}, \quad \omega_u = \frac{i}{2}\left(e^u\left(\omega_{1\overline{1}} + \omega_{2\overline{2}}\right) + \omega_{3\overline{3}}\right).$$

Notice that $d\Omega = 0$ clearly, while $d\omega_u \neq 0$ (we have Calabi-Yau non-Kähler structure).

Lemma 11.1.16. *We have that*

$$d\left(\|\Omega\|_{\omega_u}\omega_u^2\right) = 0.$$

Proof. Write

$$\omega_{\mathbb{T}^4} = \frac{i}{2}\left(\omega_{1\overline{1}} + \omega_{2\overline{2}}\right), \omega_u = e^u\omega_{\mathbb{T}^4} + \frac{i}{2}\omega_{3\overline{3}}.$$

Then,

$$\begin{aligned} \|\Omega\|^2_{\omega_u}\frac{\omega_u^3}{6} &= -i\Omega \wedge \overline{\Omega} = -i dz_{1\overline{1}2\overline{2}3\overline{3}}, \\ \frac{\omega_u^3}{6} &= \frac{i}{4}e^{2u}\omega^2_{\mathbb{T}^4} \wedge \omega_{3\overline{3}} = -i\frac{1}{16}e^{2u}dz_{1\overline{1}2\overline{2}3\overline{3}}. \end{aligned}$$

So, $\|\Omega\|^2_{\omega_u} = \lambda^2 e^{-2u}$ for certain $\lambda \in \mathbb{R}$. That is, $\|\Omega\|_{\omega_u} = \lambda e^{-u}$. At last,

$$d\left(\|\Omega\|_{\omega_u}\omega_u^2\right) = \lambda d\left(e^{-u}\left(e^{2u}\omega^2_{\mathbb{T}^4} + i\omega_{3\overline{3}} \wedge e^u\omega_{\mathbb{T}^4}\right)\right) = \lambda d\left(e^u\omega^2_{\mathbb{T}^4}\right) = 0,$$

since

$$d\omega_{3\overline{3}} \wedge \omega_{\mathbb{T}^4} = \omega_3 \wedge \left(d\omega_{\overline{3}} \wedge \omega_{\mathbb{T}^4}\right) - \omega_{\overline{3}} \wedge \left(d\omega_3 \wedge \omega_{\mathbb{T}^4}\right) = 0.$$

Notice that this condition is equivalent to $\theta_\omega = -d\log\|\Omega\|_{\omega_u}$. So, $\theta_{\omega_u} = du$. □

Taking $\Psi_u := \|\Omega\|_{\omega_u}^{-1}\Omega$, we obtain that (Ψ_u, ω_u) define an SU(3)-structure solving (5.9). So, by Lemma 5.2.2, there exists a canonically associated unique maximal atlas on X. Let F be as in (11.9), for $m, n, p \in \mathbb{Z}$. Then, one can prove that

$$\frac{i}{2\pi}[F] \in H^{1,1}\left(\mathbb{T}^4, \mathbb{R}\right) \cap H^2\left(\mathbb{T}^4, \mathbb{R}\right).$$

Hence, there exists an holomorphic line bundle with an hermitian metric (L, h) such that $F_h = F$. Furthermore, consider

$$k(m,n,p) := \int_{\mathbb{T}^4} [F]^2,$$

which is a positive integer. We have arrived at the following result.

Lemma 11.1.17. *There exists an* SU(2)*-bundle* E_1 *over* $\mathbb{T}_4$ *with* $c_2(E_1) = k(m,n,p)$*, and a connection* A_1 *on* E_1*, such that its pull-back to* X *satisfies:*

$$F_{A_1}^{0,2} = 0, \quad F_{A_1} \wedge \omega_u^2 = 0.$$

Proof. By [23, Chapter 4], we know that there exists a Hermitian-Yang-Mills connection on $\mathbb{T}_4$ with respect to $\omega_{\mathbb{T}_4}$ and with second Chern class $k(m,,n,p)$. The fact that A_1 is a Hermitian-Yang-Mills connection for ω_u follows easily from $F \wedge \omega_{\mathbb{T}_4} = 0$. □

Lemma 11.1.18. *For suitable* $\alpha \in \mathbb{R}$*, there exists* $u \in \mathcal{C}^\infty(\mathbb{T}^4, \mathbb{R})$ *such that*

$$dd^c\omega_u = \alpha F \wedge F - \alpha \operatorname{tr} F_{A_1} \wedge F_{A_1}.$$

Proof. First, we will compute $d^c\omega_u$. Since

$$d\omega_u = du \wedge e^u\omega_{\mathbb{T}^4} + \frac{i}{2}\left(\omega_{2\bar{1}\bar{3}} - \omega_{3\bar{2}\bar{1}}\right),$$

then

$$\begin{aligned} d^c\omega_u &= -d\omega_u\left(J\cdot, J\cdot, J\cdot\right) \\ &= d^c u \wedge e^u\omega_{\mathbb{T}^4} - \frac{1}{2}\left(\omega_{2\bar{1}\bar{3}} + \omega_{3\bar{2}\bar{1}}\right). \end{aligned} \tag{11.13}$$

Now, notice that

$$dd^c\omega_u = \frac{1}{2}\Delta_{\omega_{\mathbb{T}^4}}\left(e^u\right)\omega_{\mathbb{T}^4}^2 - \omega_{1\bar{1}2\bar{2}},$$

where

$$\Delta_{\omega_{\mathbb{T}^4}}(\phi) = 1 + \alpha\left(-2\pi^2\left(m^2+n^2+p^2\right)\right) - \alpha\left|F_{A_1}\right|^2, \quad \text{for } \phi \in \mathcal{C}^\infty(\mathbb{T}^4, \mathbb{R}).$$

So,

$$\frac{1}{2}\Delta_{\omega_{\mathbb{T}^4}}(\phi)\,\omega_{\mathbb{T}^4}^2 = \omega_{1\bar{1}2\bar{2}} - \alpha\left(2\pi^2\left(m^2+n^2+p^2\right)\right)\omega_{1\bar{1}2\bar{2}} - \alpha \operatorname{tr} F_{A_1} \wedge F_{A_1}.$$

Now, we can solve this equation [23] for u, α such that $e^u = \phi + C$ taking $C \gg 0$ when

$$0 = \int_{\mathbb{T}^4} \omega_{\mathbb{T}^4}^2 - \alpha\left(2\pi^2\left(m^2+n^2+p^2\right)\right) + 8\pi^2\alpha c_2\left(\mathcal{E}_1\right).$$ □

By proof of Lemma 11.1.17, we have that $\operatorname{ad} P = p^* \left(i\mathbb{R} \oplus \operatorname{End} \mathcal{E}_1\right)$. Now, we consider the transitive Courant algebroid

$$E_{H,A} := TX \oplus \operatorname{ad} P \oplus T^*X,$$

where

$$H := -d^c \omega_u$$

and, for A_h Chern connection of (L, h),

$$A := A_h \oplus A_1.$$

By the previous results, for $\alpha \in \mathbb{R}$ and $u \in \mathcal{C}^\infty(\mathbb{T}^4, \mathbb{R})$ as stated above, the pair (ω_u, F_A), where ω_u is the $(1,1)$-form defined before using this concrete u, and F_A is the curvature associated to this particular connection, is a solution of the Hull-Strominger system

$$\begin{aligned} F_A^{0,2} = 0, \quad F_A \wedge \omega_u^2 &= 0, \\ d\left(\|\Omega\|_{\omega_u} \omega_u^2\right) &= 0, \\ dd^c \omega_u - \alpha \left\langle F_A \wedge F_A \right\rangle &= 0. \end{aligned} \tag{11.14}$$

In this example, the associated Lee form is exact (so, closed). So, we are in the hypothesis of Theorem 10.1.19, and we must see that the associated torsion bi-vector σ_ω from Section 5.3 vanishes. Consider for $j \in \{1,2,3\}$ the local isotropic frames

$$\epsilon_j := e^{i\omega} \left(\frac{\partial}{\partial \overline{z}_j}\right) = \frac{\partial}{\partial \overline{z}_j} + g \frac{\partial}{\partial \overline{z}_j} \in e^{i\omega} \left(T^{1,0}X\right),$$

$$\overline{\epsilon}_j := g^{-1} d\overline{z}_j + d\overline{z}_j \in e^{-i\omega} \left(T^{1,0}X\right).$$

Lemma 11.1.19. *The torsion bi-vector of ω_u vanishes, that is,*

$$\sigma_\omega = 0.$$

Proof. Clearly $(\sigma_\omega)_{11} = (\sigma_\omega)_{22} = (\sigma_\omega)_{33} = 0$, since $\overline{\ell}$ is isotropic. Now, notice that

$$\begin{aligned} g_u^{-1} d\overline{z}_1 &= 2e^{-u} \frac{\partial}{\partial z_1}, \\ g_u^{-1} d\overline{z}_2 &= 2\left(e^{-u} \frac{\partial}{\partial z_2} + e^{-u} \overline{z}_1 \frac{\partial}{\partial z_3}\right), \\ g_u^{-1} d\overline{z}_3 &= 2\left(z_1 e^{-u} \frac{\partial}{\partial z_2} + \left(1 + e^{-u}|z_1|^2\right) \frac{\partial}{\partial z_3}\right). \end{aligned}$$

So, by direct application of Lemma 10.1.18, using (11.13), we obtain the non-zero values

$$\begin{aligned} \left\langle \left[\overline{\epsilon}_1, \overline{\epsilon}_2\right], \epsilon_1 \right\rangle &= -d^c \omega_u \left(g_u^{-1} d\overline{z}_1, g_u^{-1} d\overline{z}_2, \frac{\partial}{\partial \overline{z}_1}\right) = -2 \frac{\partial}{\partial z_2}\left(e^{-u}\right), \\ \left\langle \left[\overline{\epsilon}_1, \overline{\epsilon}_2\right], \epsilon_2 \right\rangle &= -d^c \omega_u \left(g_u^{-1} d\overline{z}_1, g_u^{-1} d\overline{z}_2, \frac{\partial}{\partial \overline{z}_2}\right) = 2 \frac{\partial}{\partial z_1}\left(e^{-u}\right), \end{aligned}$$

That is, we have that

$$(\sigma_\omega)_{12} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = 4e^{-u} \sum_{j=0}^{1} (-1)^j \frac{\partial}{\partial z_{j+1}} \left(\frac{\partial}{\partial z_{2-j}} \left(e^{-u}\right) \right) = 0.$$

Again, by direct application of Lemma 10.1.18, by (11.13), we obtain the non-zero values

$$\begin{aligned}
\langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_1 \rangle &= -d^c\omega_u \left(g_u^{-1} d\bar{z}_1, g_u^{-1} d\bar{z}_3, \frac{\partial}{\partial \bar{z}_1} \right) = -2\frac{\partial}{\partial z_2} \left(z_1 e^{-u}\right), \\
\langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_2 \rangle &= -d^c\omega_u \left(g_u^{-1} d\bar{z}_1, g_u^{-1} d\bar{z}_3, \frac{\partial}{\partial \bar{z}_2} \right) = 2\frac{\partial}{\partial z_1} \left(z_1 e^{-u}\right),
\end{aligned}$$

That is, we have that

$$(\sigma_\omega)_{13} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_3], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = 4e^{-u} \sum_{j=0}^{1} (-1)^j \frac{\partial}{\partial z_{j+1}} \left(\frac{\partial}{\partial z_{2-j}} \left(z_1 e^{-u}\right) \right) = 0.$$

Now, by direct application of Lemma 10.1.18, using (11.13), we obtain that

$$\langle [\bar{\epsilon}_2, \bar{\epsilon}_3], \epsilon_1 \rangle = \langle [\bar{\epsilon}_2, \bar{\epsilon}_3], \epsilon_2 \rangle = \langle [\bar{\epsilon}_2, \bar{\epsilon}_3], \epsilon_3 \rangle = 0.$$

That is, we have that

$$(\sigma_\omega)_{23} := \sum_{j=1}^{3} \langle \mathcal{D} \langle [\bar{\epsilon}_1, \bar{\epsilon}_2], \epsilon_k \rangle, \bar{\epsilon}_k \rangle = 0.$$

In summary, we arrive at the desired result, applying antisymmetry to obtain that

$$(\sigma_\omega)_{21} = (\sigma_\omega)_{31} = (\sigma_\omega)_{32} = 0. \qquad \square$$

So, we can apply Theorem 10.1.19. We have the following.

Proposition 11.1.20. *For $(m,n,p) \in \mathbb{Z}^3 - \{(0,0,0)\}$, consider the string Courant algebroid $E := E_{H,A}$ associated to the solution of the Hull-Strominger system* (11.14)*, with associated Lee form*

$$\theta_{\omega_u} = du, \quad \text{for } u \in \mathcal{C}^\infty(\mathbb{T}^4, \mathbb{R})$$

as in Lemma 11.1.18. Then, we induce an embedding of the $N = 2$ superconformal vertex algebra of central charge 9 into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E\otimes_{\mathbb{R}}\mathbb{C}}$. The generators of this embedding are given by (10.18) *for the frames above.*

Proof. It is a direct consequence of Theorem 10.1.19 and Lemma 11.1.19. $\square$

This is the first (and unique, for the moment) non-homogeneous example with non-zero closed Lee form for which we have constructed an embedding of the $N = 2$ superconformal vertex algebra into the space of global sections of the chiral de Rham complex.

11.1.3 $N = 4$ on Homogeneous Hopf Surfaces

The following discussion is essentially contained in [2, Section 5.1]. We summarize here the main results, and suggest a generalization of the given embeddings. We will show that each element of the constructed family of $N = 2$ superconformal vertex algebra embeddings induced by Theorem 10.1.5 and Lemma 11.1.5 embeds in certain $N = 4$ superconformal vertex algebra with central charge 6. Fix $\ell, x > 0$, and consider the obtained solution of the Killing spinor equations $(V_+^x, \mathrm{div}_+^x, I_x)$ on the equivariant Courant algebroid E_ℓ over $K = \mathrm{SU}(2) \times \mathrm{U}(1)$ from Lemma 11.1.5. Now, notice that g_x is compatible with the left-invariant hyperholomorphic structure (I_x, J_x, K_x) on K, with I_x defined by (11.6), and J_x and K_x defined using (11.8) by

$$2\Psi_x = \omega_{J_x} + i\omega_{K_x},$$

where $g_x = \omega_{J_x}(\cdot, J_x\cdot) = \omega_{K_x}(\cdot, K_x\cdot)$. More explicitly, we have

$$J_x v_4 = x v_2, \qquad J_x v_3 = v_1, \qquad K_x v_1 = v_2, \qquad K_x v_4 = x v_3,$$

and it is straightforward to check that the quaternionic identities hold.

Lemma 11.1.21 ([2, Lemma 5.1])**.** *If $\ell, x > 0$, the triples $(V_+^x, \mathrm{div}_+^x, I_x)$, $(V_+^x, \mathrm{div}_+^x, J_x)$ and $(V_+^x, \mathrm{div}_+^x, K_x)$ are left-invariant solutions of the Killing spinor equations on E_ℓ. Consequently, the triple (I_x, J_x, K_x) is an hyperholomorphic structure compatible with g_x and fixed Lee form*

$$\theta_x = -x v^4.$$

Proof. The claim about the complex structure I_x has been checked in Lemma 11.1.5. We check that $(V_+^x, \mathrm{div}_+^x, K_x)$ is a solution of the Killing spinor equations on E_ℓ, and leave the other case for the reader. It suffices to prove that

$$\omega_{K_x} = \ell v^{12} + \ell x v^{43}, \qquad \Psi_{K_x} = (v^1 + i v^2) \wedge (i v^3 + x v^4)$$

satisfies (6.17) with $d^c_{K_x}\omega_{K_x} = -H_\ell$ and $\theta_{\omega_{K_x}} = -x v^4$. We calculate

$$\begin{aligned} d\Psi_{K_x} &= i x v^{43} \wedge (v^1 + i v^2) = -x v^4 \wedge \Psi_{K_x}, \\ d\omega_{K_x} &= -\ell x v^{412} = -x v^4 \wedge \omega_{K_x}, \\ d^c_{K_x}\omega_{K_x} &= -d\omega_{K_x}(K_x, K_x, K_x) = \ell x v^{412}(K_x, K_x, K_x) = -\ell v^{123} = -H_\ell. \end{aligned}$$

The statement follows from the structure equation for the Lee form on a complex surface, given by $d\omega_{K_x} = \theta_{\omega_{K_x}} \wedge \omega_{K_x}$. □

By Lemma 11.1.21 and Proposition 7.2.1, it follows that

$$w := w^{J_x} = w^{I_x} = w^{K_x} = \varepsilon_+^x,$$

and hence condition (7.11) holds by Lemma 11.1.6.

Proposition 11.1.22 ([2, Proposition 5.2]). *The solutions in Lemma 11.1.21 induce an embedding of the $N=4$ superconformal vertex algebra of central charge 6 into the space of global sections of the chiral de Rham complex $\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}$. If*

$$\begin{aligned} J_0^I &= J_0(V_+^x, \operatorname{div}_+^x, I_x), & H_I' &= H'(V_+^x, \operatorname{div}_+^x, I_x), \\ J_0^J &= J_0(V_+^x, \operatorname{div}_+^x, J_x), & H_J' &= H'(V_+^x, \operatorname{div}_+^x, J_x), \\ J_0^K &= J_0(V_+^x, \operatorname{div}_+^x, K_x), & H_K' &= H'(V_+^x, \operatorname{div}_+^x, K_x), \end{aligned}$$

are all the generators of each $N=2$ superconformal vertex algebra from Theorem 10.1.5, then $H_I' = H_J' = H_K'$, and furthermore there are satisfied

$$\left[J_0^I{}_\Lambda J_0^J\right] = -\left(2\chi + S\right) J_0^K, \quad \left[J_0^J{}_\Lambda J_0^K\right] = -\left(2\chi + S\right) J_0^I, \quad \left[J_0^K{}_\Lambda J_0^I\right] = -\left(2\chi + S\right) J_0^J.$$

Proof. We write $w_j = \Pi v_j$ and $w^j = \Pi v^j$ for $j \in \{1,2,3,4\}$, where the vectors v_j, v^j for $j \in \{1,2,3,4\}$ are as in Paragraph 11.1.1.1. We obtain the explicit formulas

$$\begin{aligned} J_0^I &= \frac{1}{2\ell}\left(:\left(w_2 + \ell w^2\right)\left(w_3 + \ell w^3\right): + :\left(\frac{1}{x}\left(w_4 + \ell x^2 w^4\right)\right)\left(w_1 + \ell w^1\right): \right), \\ J_0^J &= \frac{1}{2\ell}\left(:\left(w_3 + \ell w^3\right)\left(w_1 + \ell w^1\right): + :\left(\frac{1}{x}\left(w_4 + \ell x^2 w^4\right)\right)\left(w_2 + \ell w^2\right): \right), \\ J_0^K &= \frac{1}{2\ell}\left(:\left(w_1 + \ell w^1\right)\left(w_2 + \ell w^2\right): + :\left(\frac{1}{x}\left(w_4 + \ell x^2 w^4\right)\right)\left(w_3 + \ell w^3\right): \right). \end{aligned}$$

We compute $\left[J_0^I{}_\Lambda J_0^J\right]$ By the non-commutative Wick formula, we have

$$\begin{aligned} \left[J_0^I{}_\Lambda J_0^J\right] = \frac{1}{2}\Bigg(\frac{1}{\ell}\Bigg(&\int_0^\Lambda d\Gamma \left[\left[J_0^I{}_\Lambda \left(w_3 + \ell w^3\right)\right]_\Gamma \left(w_1 + \ell w^1\right)\right] \\ &+ :\left[J_0^I{}_\Lambda \left(w_3 + \ell w^3\right)\right]\left(w_1 + \ell w^1\right): - :\left(w_3 + \ell w^3\right)\left[J_0^I{}_\Lambda \left(w_1 + \ell w^1\right)\right]:\Bigg) \\ &+ \Bigg(\frac{1}{\ell x}\int_0^\Lambda d\Gamma \left[\left[J_0^I{}_\Lambda \left(w_4 + \ell x^2 w^4\right)\right]_\Gamma \left(w_2 + \ell w^2\right)\right] \\ &+ :\left[J_0^I{}_\Lambda \left(w_4 + \ell x^2 w^4\right)\right]\left(w_2 + \ell w^2\right): - :\left(w_4 + \ell x^2 w^4\right)\left[J_0^I{}_\Lambda \left(w_2 + \ell w^2\right)\right]:\Bigg)\Bigg). \end{aligned}$$

Now, we compute the following intermediate Λ-brackets,

$$\begin{aligned} \left[\left(w_3 + \ell w^3\right)_\Lambda J_0^I\right] &= -\chi\left(w_2 + \ell w^2\right) + \frac{1}{2\ell}\Bigg(:\left(w_1 + \ell w^1\right)\left(w_3 + \ell w^3\right): \\ &\qquad + :\left(\frac{1}{x}\left(w_4 + \ell x^2 w^4\right)\right)\left(w_2 + \ell w^2\right): \Bigg), \\ \left[\left(w_1 + \ell w^1\right)_\Lambda J_0^I\right] &= -\lambda - \chi\frac{1}{x}\left(w_4 + \ell x^2 w^4\right), \\ \left[\left(w_2 + \ell w^2\right)_\Lambda J_0^I\right] &= \chi\left(w_3 + \ell w^3\right) - \frac{1}{2\ell}\Bigg(:\left(w_2 + \ell w^2\right)\left(w_1 + \ell w^1\right): \\ &\qquad - :\left(\frac{1}{x}\left(w_4 + \ell x^2 w^4\right)\right)\left(w_3 + \ell w^3\right): \Bigg), \\ \left[\left(w_4 + \ell x^2 w^4\right)_\Lambda J_0^I\right] &= x\chi\left(w_1 + \ell w^1\right). \end{aligned}$$

Combining the non-commutative Wick formula, the antisymmetry of Λ-bracket, (1.12), and (B.4), since S is an antiderivation for the normally ordered product, we conclude

$$
\begin{aligned}
\left[J^I_0{}_\Lambda J^J_0\right] = & \frac{1}{2\ell}\left(\chi : \left(w_2 + \ell w^2\right)\left(w_1 + \ell w^1\right) : + : \left(S\left(w_2 + \ell w^2\right)\right)\left(w_1 + \ell w^1\right) :\right. \\
& + \frac{1}{x}\left(\chi : \left(w_3 + \ell w^3\right)\left(w_4 + \ell x^2 w^4\right) : - : \left(w_3 + \ell w^3\right)\left(S\left(w_4 + \ell x^2 w^4\right)\right) :\right) \\
& - (\lambda + T)\left(w_3 + \ell w^3\right) + \lambda\left(w_3 + \ell w^3\right) \\
& - \chi : \left(w_1 + \ell w^1\right)\left(w_2 + \ell w^2\right) : - : \left(S\left(w_1 + \ell w^1\right)\right)\left(w_2 + \ell w^2\right) : \\
& - \frac{1}{x}\left(\chi : \left(w_4 + \ell x^2 w^4\right)\left(w_3 + \ell w^3\right) : - : \left(w_4 + \ell x^2 w^4\right)\left(S\left(w_3 + \ell w^3\right)\right) :\right) \\
= & -(2\chi + S)\, J^K_0 .
\end{aligned}
$$

Finally, the identity $H'_I = H'_J = H'_K$ follows calculating a basis as in (3.8) in each case and substituting in the formula from Lemma C.2.1. □

Remark 11.1.23. Note that the method of Proposition 11.1.22 does not apply to the family of $N = 2$ superconformal structures with central charge $6+6/\ell$ in Theorem 10.1.8. Even more, this value of the central charge 6 for our family of $N = 4$ algebras coincides with the one obtained via Theorem 10.1.19. Maybe the generalizable result is this last one. It would be interesting to compare these two supersymmetry generators.

In conclusion, this Proposition 11.1.22 suggests a generalization of our constructions.

Conjecture 6. When $V_\pm$ is $4k$-dimensional, and we have that $G_\eta = \mathrm{Sp}(k)$ in Lemma 7.1.6, where $\mathrm{Sp}(k)$ denotes the compact symplectic group, then there exists an embedding of the $N = 4$ superconformal vertex algebra into $V^k_{\mathrm{super}}(\mathfrak{g})$.

Remark 11.1.24. It would be interesting to study what happens when we have other type of geometric structures, such as solutions to the *G_2-Strominger system* (see [18] for more information on this). Based on the work by Rodríguez Díaz in [80] for the chiral de Rham complex over a smooth manifold with holonomy G_2, one can expect to induce embeddings of the *Shatashvili-Vafa vertex algebra* [81], generated by a superconformal vector and a non-primary even field, from solutions to the G_2-Strominger system.

11.2 $(0, 2)$ Mirror Symmetry on Hopf Surfaces

The main goal of [2] is to find the first examples of $(0, 2)$ mirror symmetry on compact non-Kähler complex manifolds. For that, we have followed Borisov's approach to mirror symmetry in [13], which needs SUSY vertex algebras and the chiral de Rham complex. The responsible for the mirror symmetry in which we are interested in is our embeddings of the $N = 2$ superconformal vertex algebra. Concretely, we constructed $(0, 2)$ mirror pairs in the Hopf surface, via T-duality for the family of solutions for the Killing spinor equations in Lemma 11.1.5. Thanks to [36, Theorem 6.5], such solutions are preserved under T-duality, and we shall prove that the T-dual of $(V^x_+, \mathrm{div}^x_+, \eta_x)$ is a different element in the same family. We start recalling some background on *topological T-duality* following [38, Chapter 10]. Let $\mathbb{T}^k$ be a k-dimensional torus acting freely and properly on a smooth compact manifold M, so that M is a principal $\mathbb{T}^k$-bundle over the smooth manifold given by $B := M/\mathbb{T}^k$. We endow M with the choice of a $\mathbb{T}^k$-invariant cohomology class

$$\tau \in H^3 \left(M, \mathbb{R}\right)^{\mathbb{T}^k}.$$

Now, fix another pair $(\hat{M}, \hat{\tau})$ consisting of a smooth compact manifold $\hat{M}$ with a proper and free $\mathbb{T}^k$-action such that

$$B = \frac{\hat{M}}{\mathbb{T}^k}, \text{ and } \hat{\tau} \in H^3(\hat{M}, \mathbb{R})^{\hat{\mathbb{T}}^k}.$$

Consider the fibre product $\overline{M} = M \times_B \hat{M}$ and the diagram

$$\begin{tikzcd} & \overline{M} \arrow[dl, "q"'] \arrow[dr, "\hat{q}"] & \\ M \arrow[dr, "p"'] & & \hat{M} \arrow[dl, "\hat{p}"] \\ & B & \end{tikzcd}$$

Definition 11.2.1. We will say that two pairs (M, τ) and $(\hat{M}, \hat{\tau})$ as above are *T-dual* if there exists representatives

$$H \in \Omega^3(M)^{\mathbb{T}^k}, \hat{H} \in \Omega^3(\hat{M})^{\hat{\mathbb{T}}^k}$$

of τ and $\hat{\tau}$ cohomology classes above, respectively, such that $q^*H - \hat{q}^*\hat{H} = d\overline{B}$, where

$$\overline{B} \in \Omega^2(\overline{M})^{\mathbb{T}^k \times \hat{\mathbb{T}}^k}$$

is such that $\overline{B}\colon \operatorname{Ker} dq \otimes \operatorname{Ker} d\hat{q} \to \mathbb{R}$ is non-degenerate.

Given M as above, if $\mathfrak{t} = B \times \text{Lie } \mathbb{T}^k$, we have the natural exact sequence

$$0 \to \mathfrak{t} \to \frac{TM}{\mathbb{T}^k} \to TB \to 0.$$

This one induces a filtration

$$\Omega^*(B) \cong \mathcal{F}^0 \subseteq \mathcal{F}^1 \subseteq \cdots \subseteq \mathcal{F}^\bullet = \Omega^*(M)^{\mathbb{T}^k},$$

where $\mathcal{F}^i = \operatorname{Ann}(\wedge^{i+1}\mathfrak{t})$ for $i \in \mathbb{N}$. Now, given the T-dual pairs (M, τ) and $(\hat{M}, \hat{\tau})$, there exists representatives

$$H \in \tau, \hat{H} \in \hat{\tau}$$

in $\mathcal{F}^1$ of M and $\hat{M}$, respectively, in the conditions of Definition 11.2.1 [38, Lemma 10.5]. Let E be an exact equivariant Courant algebroid over M with respect to the $\mathbb{T}^k$-action.

Recall that such an E has an associated Ševera class $[E] \in H^3(M,\mathbb{R})^{\mathbb{T}^k}$. Consider the vector bundle $E/\mathbb{T}^k \to B$, whose sheaf of sections is given by the invariant section of E. We can endow $E/\mathbb{T}^k$ with a natural structure of Courant algebroid, with pairing and Dorfman bracket given by the restriction of the neutral pairing and Dorfman bracket on E to $\Gamma(E)^{\mathbb{T}^k}$. We will call $E/\mathbb{T}^k$ the *simple reduction* of E by $\mathbb{T}^k$.

Theorem 11.2.2 ([17])**.** *Let $E \longrightarrow M$ and $\hat{E} \longrightarrow \hat{M}$ be two equivariant exact Courant algebroids. Assume that $(M,[E])$ is T-dual to $(\hat{M},[\hat{E}])$. Then, there exists a canonical isomorphism of Courant algebroids between the simple reductions*

$$\psi \colon \frac{E}{\mathbb{T}^k} \to \frac{\hat{E}}{\hat{\mathbb{T}}^k}. \tag{11.15}$$

We briefly describe the construction of (11.15). We will choose the equivariant isotropic splittings of E and $\hat{E}$ such that the corresponding 3-forms H and $\hat{H}$ are in $\mathcal{F}^1$ of their respective fibrations. Given $X+\xi \in \Gamma(TM \oplus T^*M)^{\mathbb{T}^k}$, choose the unique lift $\overline{X}$ of X to the invariant sections of $T\overline{M}$ such that

$$q^*\xi(Y) - \overline{B}(\overline{X},Y) = 0, \quad \text{for } Y \in \mathfrak{t}. \tag{11.16}$$

Due to this condition, the form $q^*\xi - \overline{B}(\overline{X},\cdot)$ is basic for the bundle determined by $\hat{q}$, and, therefore, can be pushed forward to $\hat{M}$. Then, ψ is defined by the explicit formula

$$\psi(X+\xi) = \hat{q}_*(\overline{X} + q^*\xi - \overline{B}(\overline{X},\cdot)).$$

To apply T-duality to the situation of our interest, we regard $K = \mathrm{SU}(2)\times \mathrm{U}(1)$ as a $\mathbb{T}^1$-principal bundle over $\mathbb{S}^3 \cong \mathrm{SU}(2)$, via the natural left action of the central subgroup

$$\mathbb{T}^1 = \mathrm{U}(1) \subseteq \mathrm{SU}(2)\times \mathrm{U}(1)$$

given by the second factor. Given $\ell \in \mathbb{R}$ we consider the closed 3-form H_ℓ in (11.5).

Lemma 11.2.3 ([2, Lemma 4.13])**.** *For any $\ell \in \mathbb{R}$, the pair $(K,[H_\ell])$ is self-T-dual.*

Proof. We regard the correspondence space $\overline{K}$ in Definition 11.2.1 inside the Lie group

$$\iota \colon \overline{K} \hookrightarrow K \times \hat{K},$$

where $\hat{K}$ is a copy of K. Then, define $\overline{B}$ as the pull-back to $\overline{K}$ of a left-invariant 2-form

$$\overline{B} = -\iota^*(v^4 \wedge \hat{v}^4). \tag{11.17}$$

Then, we have $d\overline{B} = 0 = p^*H_\ell - \hat{p}^*\hat{H}_\ell$, where we used that H_ℓ and $\hat{H}_\ell$ are both pull-back of the same 3-form on $B = \mathrm{SU}(2)$. The non-degeneracy condition on $\overline{B}$ follows from the fact that this 2-form is bi-invariant. □

Proposition 11.2.4 ([2, Proposition 4.14])**.** *Given $0 < \ell \in \mathbb{R}$, consider the equivariant exact Courant algebroid E_ℓ over K. Then, two solutions $(V_+^x, \mathrm{div}_+^x, \eta_x)$ and $(V_+^{\hat{x}}, \mathrm{div}_+^{\hat{x}}, \eta_{\hat{x}})$ as in Lemma 11.1.5 are exchanged under T-duality, provided that*

$$\hat{x} = \frac{1}{\ell x}. \tag{11.18}$$

Proof. We calculate first the isomorphism (11.15) in Theorem 11.2.2 corresponding to the 2-form (11.17). Firstly, notice that by (11.16) we have

$$\psi(v_j) = \hat{v}_j, \quad \psi(v^j) = \hat{v}^j, \text{ for } j = 1, 2, 3.$$

A direct calculation also shows that

$$\psi(v_4) = \hat{v}^4, \quad \psi(v^4) = \hat{v}_4.$$

By definition of V_+^x (see (11.7)) we have

$$V_+^x = \langle v_2 + \ell v^2, v_3 + \ell v^3, v_1 + \ell v^1, v_4 + \ell x^2 v^4 \rangle \subseteq TK \oplus T^*K,$$

and therefore

$$\hat{V}_+^x := \psi(V_+^x) = \langle \hat{v}_2 + \ell \hat{v}^2, \hat{v}_3 + \ell \hat{v}^3, \hat{v}_1 + \ell \hat{v}^1, \hat{v}^4 + \ell x^2 \hat{v}_4 \rangle \subseteq TK \oplus T^*K.$$

From this, for $\hat{x}$ defined as in (11.18), the T-dual metric is

$$\hat{g}_x = \ell \left(\hat{v}^1 \otimes \hat{v}^1 + \hat{v}^2 \otimes \hat{v}^2 + \hat{v}^3 \otimes \hat{v}^3 + (\ell x)^{-2} \hat{v}^4 \otimes \hat{v}^4 \right) = g_{\hat{x}}.$$

Similarly, writing $\varepsilon_+^{\hat{x}}$ for the orthogonal projection of $\varepsilon^{\hat{x}}$ onto $\hat{V}_+^x = V_+^{\hat{x}}$, we have that

$$\psi(\varepsilon_+^x) = -\tfrac{1}{2}\psi\left(\tfrac{1}{\ell x} v_4 + x v^4\right) = -\tfrac{1}{2}\left(\tfrac{1}{\ell x}\hat{v}^4 + x \hat{v}_4\right) = \varepsilon_+^{\hat{x}},$$

The T-dual complex structure $\hat{I}_x := \psi I_x \psi^{-1}_{|V_+^x}$ is given by $\hat{I}_x \hat{v}_2 = \hat{v}_3$, and $\hat{I}_x \hat{v}_4 = \frac{1}{\ell x}\hat{v}_1$. □

We are ready to state the main result in [2], which gives the first examples of $(0,2)$ mirror pairs on compact non-Kähler complex manifolds. For that, we fix a non-zero $\ell \in \mathbb{R}$, and we identify a left-invariant solution $(V_+, \mathrm{div}_+, \eta)$ for the Killing spinor equations on the equivariant Courant algebroid E_ℓ over K, as in Lemma 11.1.5, with (V_+, ε_+, I) solution of F-term and D-term equations on $\mathfrak{g}_\ell$ (see Proposition 7.1.9 and Proposition 7.2.1). Firstly, if (V_+, ε_+, I) is a solution of (7.6) with ε_+ holomorphic, then so is $(V_+, \varepsilon_+, -I)$ (see Remark 7.2.2). Secondly, if we denote by

$$J := J(V_+, \varepsilon_+, I), H := H(V_+, \varepsilon_+, I) \in V^2_{\text{super}}\left(\mathfrak{g}_\ell^{\mathbb{C}}\right)$$

the generators of the embeddings constructed in Theorem 10.1.8, then, by Remark 10.1.4,

$$J(V_+, \varepsilon_+, -I) = -J, \qquad H(V_+, \varepsilon_+, -I) = H. \tag{11.19}$$

Finally, by a result of Linshaw-Mathai [72], the T-duality isomorphism (11.15) induces

$$\psi^{\mathrm{ch}} \colon p_*\left(\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}\right)^{\mathbb{T}^1} \to \hat{p}_*\left(\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}\right)^{\mathbb{T}^1} \tag{11.20}$$

isomorphism, where here we have the sheaf of SUSY vertex algebras on K generated by the $\mathbb{T}^1$-invariant sections of E_ℓ and functions on $K/\mathbb{T}^1 \cong \mathrm{SU}(2)$ (for a precise definition, see [72, Section 5.2]). Notice that, by Proposition 10.1.11, we have an embedding

$$V^2_{\text{super}}\left(\mathfrak{g}_\ell^{\mathbb{C}}\right) \hookrightarrow H^0\left(\mathrm{SU}(2), p_*\left(\Omega^{\mathrm{ch}}_{E_\ell \otimes_{\mathbb{R}} \mathbb{C}}\right)^{\mathbb{T}^1}\right).$$

The existence of (11.20) relies on the fact that $(M, [H_\ell])$ is self-T-dual.

Theorem 11.2.5 ([2, Theorem 4.18]). *Given non-zero $\ell \in \mathbb{R}$, consider the one-parameter family of solutions $(V_+^x, \mathrm{div}_+^x, I_x)$ on E_ℓ as in Lemma 11.1.5 parametrized by $x \in \mathbb{R}^+$. Then, $(V_+^x, \mathrm{div}_+^x, I_x)$ and $(V_+^{\hat{x}}, \mathrm{div}_+^{\hat{x}}, -I_{\hat{x}})$ are related by $(0,2)$ mirror symmetry, provided that $\hat{x} = 1/\ell x$. More precisely, if*

$$J = J(V_+^x, \mathrm{div}_+^x, I_x), \qquad H = H(V_+^x, \mathrm{div}_+^x, I_x),$$
$$\hat{J} = J(V_+^{\hat{x}}, \mathrm{div}_+^{\hat{x}}, -I_{\hat{x}}), \qquad \hat{H} = H(V_+^{\hat{x}}, \mathrm{div}_+^{\hat{x}}, -I_{\hat{x}}),$$

are the generators of the embedding with central charge $6 + 6/\ell$ constructed in Theorem 10.1.8, then the Linshaw-Mathai isomorphism (11.20) realises the mirror involution

$$\psi^{\mathrm{ch}}(J) = -\hat{J}, \qquad \psi^{\mathrm{ch}}(H) = \hat{H}.$$

Proof. The solutions of the Killing spinor equations $(V_+^x, \mathrm{div}_+^x, I_x)$ constructed in Lemma 11.1.5 are such that the corresponding divergence $\varepsilon_+ \in V_+^x$ is holomorphic in the sense of Definition 7.2.3 (see Lemma 11.1.6), and hence Theorem 10.1.8 applies. The central charge is given by $c = 6 + 6/\ell$. By (11.19) it suffices to prove the identity

$$\psi^{\mathrm{ch}}(J) = \tilde{J} := J(V_+^{\hat{x}}, \mathrm{div}_+^{\hat{x}}, I_{\hat{x}}).$$

To see this, we write $w_j = \Pi v_j$ and $w^j = \Pi v^j$ for all $j \in \{1,2,3,4\}$, where the v_j, v^j are as in Section 11.1.1.1. By (1.12) and setting $a = \ell x$, a simple calculation shows that

$$\begin{aligned} J = {} & \frac{1}{2}\left(\frac{1}{\ell} : w_2 w_3 : + : w_2 w^3 : + : w^2 w_3 : + \frac{a}{x} : w^2 w^3 : \right. \\ & \left. + \frac{1}{a} : w_4 w_1 : + \frac{1}{x} : w_4 w^1 : + x : w^4 w_1 : + a : w^4 w^1 : \right) - \frac{1}{2} S\left(\frac{1}{\ell} w_1 + w^1\right), \\ \tilde{J} = {} & \frac{1}{2}\left(\frac{1}{\ell} : \widehat{w}_2 \widehat{w}_3 : + : \widehat{w}_2 \widehat{w}^3 : + : \widehat{w}^2 \widehat{w}_3 : + \frac{a}{x} : \widehat{w}^2 \widehat{w}^3 : \right. \\ & \left. + x : \widehat{w}_4 \widehat{w}_1 : + a : \widehat{w}_4 \widehat{w}^1 : + \frac{1}{a} : \widehat{w}^4 \widehat{w}_1 : + \frac{1}{x} : \widehat{w}^4 \widehat{w}^1 : \right) - \frac{1}{2} S\left(\frac{1}{\ell} \hat{w}_1 + \hat{w}^1\right). \end{aligned}$$

Now, using that ψ^{ch} is an isomorphism of SUSY vertex algebras (so, in particular, is an homomorphism for the normally ordered product and $S\psi^{\mathrm{ch}} = \psi^{\mathrm{ch}} S$), we obtain that

$$\begin{aligned} \psi^{\mathrm{ch}}(J) = {} & \frac{1}{2}\left(\frac{1}{\ell} : \widehat{w}_2 \widehat{w}_3 : + : \widehat{w}_2 \widehat{w}^3 : + : \widehat{w}^2 \widehat{w}_3 : + \frac{a}{x} : \widehat{w}^2 \widehat{w}^3 : \right. \\ & \left. + \frac{1}{a} : \widehat{w}^4 \widehat{w}_1 : + \frac{1}{x} : \widehat{w}^4 \widehat{w}^1 : + x : \widehat{w}_4 \widehat{w}_1 : + a : \widehat{w}_4 \widehat{w}^1 : \right) - \frac{1}{2} S\left(\frac{1}{\ell} \hat{w}_1 + \hat{w}^1\right) = \tilde{J}, \end{aligned}$$

which concludes the desired identity. □

Now, since we have a more general result for any string Courant algebroid admitting solutions to the twisted Hull-Strominger system (6.16), we can state the following.

Conjecture 7. It is possible to construct more examples of $(0,2)$ mirror symmetry on (non-homogeneous) compact non-Kähler complex manifolds via T-duality.

We hope to obtain more examples of $(0,2)$ mirror symmetry in future work.

Part IV

Appendices

Appendix A

Rules and Identities in (SUSY) LCAs and VAs

In this appendix, we collect some relevant rules and identities about (SUSY) Lie conformal algebras and (SUSY) vertex algebras in order to clarify their axioms.

A.1 Lie conformal algebras

Let $\mathcal{R}$ be a Lie conformal algebra. Then, we must apply the following rules to define the expressions appearing in Definition 1.2.1:

- *Sesquilinearity.* To obtain $(\partial+\chi)[a_\lambda b]$ as an element of $\mathbb{C}[\mu]\otimes\mathcal{R}$ in (1.9), first calculate the Λ-bracket, and then commute ∂ with λ to the right.
- *Antisymmetry.* To obtain $[b_{-\lambda-\partial}a]$ as element of $\mathbb{C}[\lambda]\otimes\mathcal{R}$ in (1.10), first expand the λ-bracket as follows
$$[b_\mu a] = \sum_{n\in\mathbb{N}} \frac{\mu^n}{n!}c_n$$
in $\mathbb{C}[\mu]\otimes\mathcal{R}$, where μ is a new even formal parameter, and then replace μ by $-\lambda-\partial$ applying ∂ to the coefficients $c_n\in\mathcal{R}$.
- *Jacobi identity.* To obtain
$$\Big[[a_\lambda b]_{\lambda+\mu}\, c\Big] \in \mathbb{C}[\lambda]\otimes\mathbb{C}[\mu]\otimes\mathcal{R}$$
in (1.11), first calculate $[[a_\lambda b]_\omega\, c]\in\mathbb{C}[\lambda]\otimes\mathbb{C}[\omega]\otimes\mathcal{R}$, where ω is another even formal parameter, and apply the identity
$$[p(\mu)a_\lambda b] = p(\mu)[a_\lambda b], \quad \text{for } a,b\in\mathcal{R}, \tag{A.1}$$
for any homogeneous polynomial $p(\mu)$ in μ (the sign follows from the Koszul sign rule applied to the parity-preserving λ-bracket, because μ is even) to obtain an ex-

pansion of the form

$$[[a_\lambda b]_\omega c] = \sum_{n,m\in\mathbb{N}} \frac{\lambda^n \omega^m}{n!m!} c_{n,m}.$$

Then, replace ω by $\lambda+\mu$, and use that λ and μ are commutative formal parameters. To calculate the other two terms in (1.11) as elements of $\mathbb{C}[\lambda]\otimes\mathbb{C}[\mu]\otimes\mathcal{R}$, we proceed as above, using previous relations and

$$[a_\lambda p(\mu) b] = p(\mu)[a_\lambda b], \quad \text{for } a, b \in \mathcal{R},$$

for any homogeneous polynomial $p(\mu)$ in μ (the sign follows from the Koszul sign rule applied to the parity-preserving λ-bracket, because μ is even).

A.2 Vertex Algebras

Let V be a vertex algebra. Then, we apply the following rules to define the expressions appearing in Theorem 1.2.9:

- *Quasicommutativity.* To compute the integral in (1.12), we use the expansion

$$[a_\lambda b] = \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} a_{(n)} b \in \mathbb{C}[\lambda]\otimes V.$$

  Then, on each term $p(\lambda) = \lambda^n a_{(n)} b \in \mathbb{C}[\lambda]\otimes V$, compute the indefinite integral in the formal parameter λ, taking the difference of the values at the limits.

- *Quasiassociativity.* The first integral in (1.13) is computed by expanding $[b_\lambda c]$ as above, putting the powers of λ inside the integral in the left, under the integral sign, and performing the definite integral on each term $p(\lambda) = \lambda^n a \in \mathbb{C}[\lambda]\otimes V$. The second integral in (1.13) is calculated applying the same rules.

- *The non-commutative Wick formula.* To compute the integral in (1.14), expand

$$[[a_\lambda b]_\mu c] = \sum_{n,m\in\mathbb{N}} \frac{\lambda^n \mu^m}{n!m!} \left(a_{(n)} b\right)_{(m)} c \in \mathbb{C}[\lambda]\otimes\mathbb{C}[\mu]\otimes V,$$

  using that λ and μ are commutative formal parameters, and (A.1). Then, perform the definite integral on each term

$$p(\mu) = \lambda^n \mu^m \left(a_{(n)} b\right)_{(m)} c \in \mathbb{C}[\lambda]\otimes\mathbb{C}[\mu]\otimes V.$$

Remark A.2.1. In the quasiassociativity identity (1.13), we have omitted the parenthesis that determines the order for computing the normally ordered products for simplicity, since it is clear thanks to the notation we are using for the normally ordered product.

Remark A.2.2. In the last two identities, that is, (1.13) and (1.14), we must apply the identities

$$:(p(\lambda)a)\, b := p(\lambda) : ab :=: a\,(p(\lambda)b) :, \quad \text{for } a, b \in \mathcal{R},$$

for any homogeneous polynomial $p(\lambda)$ in λ (the sign follows from the Koszul sign rule applied to the parity-preserving normally ordered product, because λ is even).

A.3 SUSY Lie conformal algebras

Let $\mathcal{R}$ be a SUSY Lie conformal algebra. Then, we must apply the following rules to define the expressions appearing in Definition 2.3.1:

- *Sesquilinearity.* To obtain $(D+\chi)[a_\Lambda b]$ as an element of $\mathcal{L}\otimes\mathcal{R}$ in (2.8), first calculate the Λ-bracket, and then commute D with χ and λ to the right using the relations $[D,\lambda]=0$ and $[D,\chi]=2\lambda$.

- *Antisymmetry.* To obtain $[b_{-\Lambda-\nabla}a]$ as element of $\mathcal{L}\otimes\mathcal{R}$ in (2.9), first expand the Λ-bracket as follows
$$[b_\Gamma a]=\sum_{\substack{n\in\mathbb{N}\\ J\in\{0,1\}}}\frac{\Gamma^{n|J}}{n!}c_{n|J}$$
in $\mathcal{L}'\otimes\mathcal{R}$ using the relations $[\gamma,\eta]=0$ and $[\eta,\eta]=-2\gamma$, where $\mathcal{L}'$ is other copy of $\mathcal{L}$ generated by the pair $\Gamma=(\gamma,\eta)$, and then replace Γ by $-\Lambda-\nabla=(-\lambda-\partial,-\chi-D)$ applying ∂ and D to the coefficients $c_{n|J}\in\mathcal{R}$.

- *Jacobi identity.* To obtain $\left[[a_\Lambda b]_{\Lambda+\Gamma}\, c\right]\in\mathcal{L}\otimes\mathcal{L}'\otimes\mathcal{R}$ in (2.10), first calculate $[[a_\Lambda b]_\Omega\, c]\in\mathcal{L}\otimes\mathcal{L}''\otimes\mathcal{R}$, where $\mathcal{L}''$ is other copy of $\mathcal{L}$ generated by the pair $\Omega=(\omega,\xi)$, and apply
$$[p(\Gamma)a_\Lambda b]=(-1)^{|p|}p(\Gamma)[a_\Lambda b],\quad \text{for } a,b\in\mathcal{R}, \tag{A.2}$$
for any homogeneous polynomial $p(\Gamma)$ in Γ (the sign follows from the Koszul sign rule applied to the parity-reversing Λ-bracket) to obtain an expansion of the form
$$[[a_\Lambda b]_\Omega c]=\sum_{\substack{n,m\in\mathbb{N}\\ J,K\in\{0,1\}}}\frac{\Lambda^{n|J}\Omega^{m|K}}{n!m!}c_{n|J,m|K}.$$
Then replace Ω by $\Lambda+\Gamma=(\lambda+\gamma,\chi+\eta)$ and use the following relations between these formal parameters $[\lambda,\gamma]=[\lambda,\eta]=[\chi,\gamma]=[\chi,\eta]=0$. To calculate the other terms in (2.10) as elements of $\mathcal{L}\otimes\mathcal{L}'\otimes\mathcal{R}$, we proceed as above, using the previous relations and
$$[a_\Lambda p(\Gamma)b]=(-1)^{|p|(|a|+1)}p(\Gamma)[a_\Lambda b],\quad \text{for } a,b\in\mathcal{R},$$
for any homogeneous polynomial $p(\Gamma)$ in Γ (the sign follows from the Koszul sign rule applied to the parity-reversing Λ-bracket).

A.4 SUSY Vertex Algebras

Let V be a SUSY vertex algebra. Then, we must apply the following rules to define the expressions appearing in Theorem 2.3.6:

- *Quasicommutativity.* To compute the integral in (2.12), we will use the expansion (2.11). Then on each term $p(\Lambda) = \Lambda^{n|J} a_{(n|J)} b \in \mathcal{L} \otimes V$, apply the formula
$$\int_{-\nabla}^{0} d\Lambda \, p(\Lambda) = \int_{-T}^{0} d\lambda \left(\partial_{\chi} p(\Lambda)\right), \tag{A.3}$$
where we are taking first the (left) partial derivative with respect to the odd formal parameter χ, performing the indefinite integral in the even formal parameter λ, and, finally, taking the difference of the values at the limits. Notice that the (left) partial derivative $\partial_{\chi} p(\Lambda)$ is zero if $J = 0$, while it is given by $\lambda^n b_{(n|1)} a$ if $J = 1$.

- *Quasiassociativity.* The first integral in (2.13) is computed by expanding $[b_{\Lambda} c]$ as above, putting the powers of Λ inside the integral that we have on the left, under the integral sign, and performing the definite integral
$$\int_{0}^{\nabla} d^{r} \Lambda p(\Lambda) = \int_{0}^{T} d\lambda \left(\partial^{r}_{\chi} p(\Lambda)\right)$$
on each term $p(\Lambda) = a\Lambda^{n|J} \in \mathcal{L} \otimes V$. Notice that we perform *right partial derivatives* $\partial^{r}_{\chi} p(\Lambda)$, which is zero if $J = 0$, while it is given by $a\lambda^n$ if $J = 1$. The second integral in (2.13) is calculated applying the same rules.

- *The non-commutative Wick formula.* To compute the integral in (2.14), we expand
$$[[a_{\Lambda} b]_{\Gamma} c] = \sum_{\substack{n,m \in \mathbb{N} \\ J,K \in \{0,1\}}} \frac{\Lambda^{n|J} \Gamma^{m|K}}{n! m!} \left(a_{(n|J)} b\right)_{(m|K)} c \in \mathcal{L} \otimes \mathcal{L}' \otimes V,$$
using the relations $[\lambda, \gamma] = [\lambda, \eta] = [\chi, \gamma] = [\chi, \eta] = 0$ and (A.2). Then, perform the definite integral on each term
$$p(\Gamma) = \Lambda^{n|J} \Gamma^{m|K} \left(a_{(n|J)} b\right)_{(m|K)} c \in \mathcal{L} \otimes \mathcal{L}' \otimes V,$$
applying the formula
$$\int_{0}^{\Lambda} d\Gamma \, p(\Gamma) = \int_{0}^{\Lambda} d\gamma \left(\partial_{\eta} p(\Gamma)\right). \tag{A.4}$$
Notice that the (left) partial derivative $\partial_{\eta} p(\Gamma)$ is zero if $K = 0$, while it is given by $(-1)^J \chi^J \lambda^n \gamma^m \left(a_{(n|J)} b\right)_{(m|1)} c$ if $K = 1$.

Remark A.4.1. In the notations of [52, Equation (3.3.3.2), Theorem 3.3.14, Equation (3.2.6.12)], all the integrals that have the form (A.3) and (A.4) are taken from the right, which means that they take the right partial derivative with respect to the odd formal parameter. This does not change the final result.

Remark A.4.2. In the last two identities, that is, (2.13) and (2.14), we must use that
$$:(p(\Lambda)a)\, b := p(\Lambda) : ab : \quad \text{and} \quad : a\, (p(\Lambda) b) := (-1)^{|a||p|} p(\Lambda) : ab :, \quad \text{for } a, b \in \mathcal{R},$$
for any homomegeneous polynomial $p(\Lambda)$ in Λ (the sign follows from the Koszul sign rule applied to the parity-preserving normally ordered product).

Appendix B

Intermediate Computations

In this appendix, we collect some relevant remarks and various identities about quadratic Lie superalgebras, universal (super)affine vertex algebras, Courant algebroids, the chiral de Rham complex, and basic linear algebra (concretely, Jacobi's Formula). To simplify the computations, we will use the Einstein summation convention for repeated indexes.

B.1 Quadratic Lie Superalgebras

Let $(\mathfrak{g}, (\cdot|\cdot))$ be an n-dimensional quadratic Lie superalgebra, and $\{a_j\}_{j=1,\dots,n} \subseteq \mathfrak{g}$ a basis with $\{a^j\}_{j=1,\dots,n} \subseteq \mathfrak{g}$ dual basis with respect to $(\cdot|\cdot)$.

Lemma B.1.1. *The following identity holds:*

$$\left[a^j, a_j\right] = 0. \tag{B.1}$$

Proof. Firt, notice that $\left[a^j, a_j\right]$ is independent of the choice done of the basis. Therefore we can assume that $\{a_j\}_{j=1,\dots,k} \subseteq \mathfrak{g}_0$, for some $k \leq n$, is formed by even vectors and $\{a_j\}_{j=k+1,\dots,n} \subseteq \mathfrak{g}_1$ is formed by odd vectors. Since the bilinear form$(\cdot|\cdot)$ is even, their dual elements have the same parity. Then

- by the Gram-Schmit method, we can obtain $\{u_j\}_{j=1,\dots,k} \subseteq \mathfrak{g}_0$ an orthogonal basis such that $u^j = \epsilon_j u_j$, where $\epsilon_j \in \{-1, 1\}$ for $j \in \{1, \dots, k\}$. So, given that the bracket is antisymmetric for even elements,

$$\sum_{j=1}^{k} \left[a^j, a_j\right] = \sum_{j=1}^{k} \epsilon_j \left[u_j, u_j\right] = 0.$$

- Since $(\cdot|\cdot)$ is a symplectic form on $\mathfrak{g}_1$, it is even dimensional, say of dimension $2m$, so it has a symplectic basis $\{e_j, f_j\}_{j=1,\dots,m} \subseteq \mathfrak{g}_1$, with dual basis $\{e^j, f^j\}_{j=1,\dots,m} \subseteq \mathfrak{g}_1$ satisfying $e^j = f_j$ and $f^j = -e_j$ for $j \in \{1, \dots, m\}$. Now, since the bracket is symmetric on the odd elements,

$$\sum_{j=k+1}^{n} \left[a^j, a_j\right] = \left[e^j, e_j\right] + \left[f^j, f_j\right] = \left[e^j, e_j\right] - \left[e_j, e^j\right] = 0.$$

In conclusion, we have that $\left[a^j, a_j\right] = \left[u^j, u^j\right] + \left[e^j, e_j\right] + \left[f^j, f_j\right] = 0.$ $\square$

The following result provides a useful formula for the adjoint action of the Casimir element Ω for a quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$ (see Definition 3.1.5 and (3.2)).

Lemma B.1.2. *For $a \in \mathfrak{g}$, the following identity holds:*

$$\Omega(a) = \left[\left[a, a^j\right], a_j\right]. \tag{B.2}$$

Proof. This is a direct consequence of (3.2), because Ω is independent of the choice of basis. Indeed, it is easily checked that the basis $\{(-1)^{|a^j|}a^j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ has dual basis $\{a_j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ with respect to $(\cdot|\cdot)$. Then

$$\begin{aligned}\Omega(a) :&= \left[a^j, [a_j, a]\right] = \left[a_j, \left[(-1)^{|a^j|}a^j, a\right]\right] \\ &= (-1)^{|a^j|}(-1)^{|a_j|+|a||a_j|}(-1)^{|a||a^j|}\left[\left[a, a^j\right], a_j\right] = \left[\left[a, a^j\right], a_j\right],\end{aligned}$$

because $(\cdot|\cdot)$ is even and so, $|a_j| = |a^j|$ for $j \in \{1, \ldots, n\}$. $\square$

B.2 Universal Affine Vertex Algebras

Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra with level $k \in \mathbb{C}$ associated to $(\mathfrak{g}, (\cdot|\cdot))$, and $\{a_j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ a basis, and $\left\{a^j\right\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ the dual basis with respect to $(\cdot|\cdot)$.

Lemma B.2.1. *For $a \in \mathfrak{g}$, the following identity holds:*

$$0 =: \left[a, a^j\right] a_j : +(-1)^{|a||a^j|} : a^j \left[a, a_j\right] : . \tag{B.3}$$

Proof. This follows from supersymmetry and invarince

$$\begin{aligned}: \left[a, a^j\right] a_j : &=: \left(\left(a_k | \left[a, a^j\right]\right) a^k\right) a_j :=: a^k \left(\left(\left[a_k, a\right] | a^j\right) a_j\right) : \\ &= -(-1)^{|a_k||a|} : a^k \left[a, a_k\right] :; \\ 0 &= -(-1)^{|a_k||a|} : a^k \left[a, a_k\right] : +(-1)^{|a||a^j|} : a^j \left[a, a_j\right] : \\ &=: \left[a, a^j\right] a_j : +(-1)^{|a||a^j|} : a^j \left[a, a_j\right] :,\end{aligned}$$

because $(\cdot|\cdot)$ is even, so $|a_j| = |a^j|$ for $j \in \{1, \ldots, n\}$. $\square$

B.3 Universal Superaffine Vertex Algebras

Let $V^k_{\text{super}}(\mathfrak{g})$ be the universal superaffine vertex algebra with level $k \in \mathbb{C}$ associated to $(\mathfrak{g}, (\cdot|\cdot))$, and $\Pi\colon \mathfrak{g} \longrightarrow \Pi\mathfrak{g}$ the parity-reversing functor.

Lemma B.3.1. *For $a, b \in \Pi\mathfrak{g}$, the following identities hold:*

$$\begin{aligned}: ab : &= (-1)^{|a||b|} : ba :, \\ : a(Sb) : &= -(-1)^{|a||\Pi b|} : (Sb)a : +T\left[a, b\right]. \\ : a(Tb) : &= (-1)^{|a||b|} : (Tb)a :,\end{aligned} \tag{B.4}$$

Proof. All the identities are immediate consequences of quasicommutativity. $\square$

Lemma B.3.2. *For $a, b, c \in \Pi\mathfrak{g}$, the following identities hold:*

$$
\begin{aligned}
: a : bc :: &= (-1)^{|a||b|}(-1)^{|a||c|} :: bc : a : + kT\left((a|b)\, c - (a|c)\, b\right),\\
:: ab : c : &=: a : bc :: + kT\left((c|b)\, a + (-1)^{|a||b|}\,(c|a)\, b\right),\\
:: a(Sb) : c : &=: a : (Sb)c :: + : (Ta)\,[b,c] : + (-1)^{|a||b|}(-1)^{|a|} kTS\,(a|c)\, b.
\end{aligned}
$$

Proof. The first identity is an immediate consequence of quasicommutativity, while the last two identities follow from quasiassociativity. $\square$

Now, abusing notation, and keeping in mind Remark 3.2.1, write

$$
a_j := \Pi a_j \quad \text{and} \quad a^j := \Pi a^j, \quad \text{for } j \in \{1, \ldots, n\},
$$

for $\{a_j\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$ basis with dual basis with respect to $(\cdot|\cdot)$ given by $\left\{a^j\right\}_{j=1,\ldots,n} \subseteq \mathfrak{g}$.

Lemma B.3.3. *For $a \in \Pi\mathfrak{g}$, the following identities hold:*

$$
0 = (-1)^{|a^j|+1} : \left(S\left[a, a^j\right]\right) a_j : + (-1)^{|a||a^j|+1} : \left(Sa^j\right)[a, a_j] :, \tag{B.5}
$$

$$
\begin{aligned}
0 =: \left[a, a^j\right] : a_k \left[a^k, a_j\right] :: &+ (-1)^{(|a|+1)|a^j|} : a^j : [a, a_k] \left[a^k, a_j\right] ::\\
&+ (-1)^{(|a|+1)(|a_k|+|a_j|)} : a^j : a_k \left[a, \left[a^k, a_j\right]\right] :: .
\end{aligned} \tag{B.6}
$$

Proof. The identity (B.5) follows because by antisymmetry and invariance,

$$
\begin{aligned}
(-1)^{|a||a^j|+1} : \left(Sa^j\right)[a, a_j] : &= (-1)^{|a||a^j|+1} : \left(Sa^j\right)\left(\left([a, a_j] \middle| a^k\right) a_k\right) :\\
&= (-1)^{|a|+|a^j|+1} : \left(S\left(a_j \middle| \left[a, a^k\right]\right) \overline{a}^j\right) a_k :\\
&= -(-1)^{|a^k|+1} : \left(S\left[a, a^k\right]\right) a_k :,
\end{aligned}
$$

since $|a^j| = |a_j| = |a| + |a^k| + 1$. The identity (B.6) follows because by the Jacobi identity,

$$
\begin{aligned}
(-1)^{(|a|+1)(|a_k|+|a_j|)} : a^j : a_k \left[a, \left[a^k, a_j\right]\right] :: &= (-1)^{(|a|+1)(|a_k|+|a_j|)} : a^j : a_k \left[\left[a, a^k\right], a_j\right]\\
&+ (-1)^{(|a_j|+1)(a+1)} : a^j : a_k \left[a^k, [a, a_j]\right],
\end{aligned}
$$

so, by antisymmetry and invariance,

$$
\begin{aligned}
: \left[a, a^j\right] : a_k \left[a^k, a_j\right] :: &= -(-1)^{(|a|+1)(|a_j|+1)} : a^j : a_k : \left[a^k, [a, a_j]\right],\\
(-1)^{(|a|+1)|a^j|} : a^j : [a, a_k] \left[a^k, a_j\right] :: &= -(-1)^{(|a|+1)(|a_k|+|a_j|)} : a^j : a_k \left[\left[a, a^k\right], a_j\right].
\end{aligned}
$$

$\square$

Lemma B.3.4. *The following identities hold:*

$$
0 = (-1)^{|a^j|+1} : a^j a_j :, \tag{B.7}
$$

$$
0 = (-1)^{|a^j|+1} : \left[a^j, a_k\right] \left[a^k, a_j\right] : . \tag{B.8}
$$

Proof. Both identities follow directly from (B.4). □

We will use other identities for $V^k_{\text{super}}(\mathfrak{g})$ when $\mathfrak{g}$ is even, but we will write them below for the chiral de Rham complex of a general Courant algebroid E over any smooth manifold M (recall that they are isomorphic when $M = \{\cdot\}$ and $k = 2$, by Theorem 9.1.11(2)).

B.4 General Courant Algebroids

Fix E any Courant algebroid over a smooth manifold M (see Definition 6.1.1).

Lemma B.4.1. *For $a \in \Gamma(E)$, the following identity holds:*

$$2\,[a, a] = \mathcal{D}\,\langle a, a\rangle\,. \tag{B.9}$$

Proof. It is straigthforward from quasiantisymmetry axiom for Courant algebroids. □

Lemma B.4.2. *For $f \in \mathcal{C}^\infty(M)$ and $a, b \in \Gamma(E)$, the following identity holds:*

$$[fa, b] = f\,[a, b] - f\mathcal{D}\,\langle a, b\rangle - \langle \mathcal{D}f, b\rangle\, a + \mathcal{D}\,\langle b, fa\rangle\,. \tag{B.10}$$

Proof. It is straigthforward from Courant algebroids axioms. □

B.5 Chiral de Rham Complex

Let E be the complexification of a real Courant algebroid over M smooth manifold, and take the chiral de Rham complex Ω^{ch}_E of E (see Section 9.1.3). Recall that we will work with parity-reversed sections (see (9.3)). We will write some identities for sections of Ω^{ch}_E for $k \in \mathbb{C}$, although they work only for $k = 2$ when $M \neq \{\cdot\}$ (see Theorem 9.1.11(2)).

Lemma B.5.1. *For $f, g \in \mathcal{C}^\infty(M, \mathbb{C})$, the following identity holds:*

$$:(T^m f)\,(T^n g):\;=\;:(T^n g)\,(T^m f):, \quad \textit{for } m, n \in \mathbb{N}.$$

Proof. This identity is immediate from quasicommutativity. □

Since the previous identity is quite obvious and is used very often, we will not refer to it directly in the other results.

Lemma B.5.2. *For $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $a \in \Gamma(\Pi E)$, the following identities hold:*

$$[f_\Lambda a] = \langle \mathcal{D}f, a\rangle\,, \tag{B.11}$$

$$S\,(fa) =: f\,(Sa): + \frac{1}{2} :(\mathcal{D}f)\,a:, \tag{B.12}$$

$$:(T^m a)\,(T^n f):\;=\;:(T^n f)\,(T^m a):, \quad \textit{for } m, n \in \mathbb{N}. \tag{B.13}$$

Proof. The first identity is a direct consequence of the antisymmetry of the Λ-bracket, the second one is for being S an odd derivation for the normally ordered product, while the last one is immediate from quasicommutativity. □

Lemma B.5.3. *For $a, b \in \Gamma(\Pi E)$, the following identities hold:*

$$:ab: + :ba: = 2T\langle a,b\rangle\,, \tag{B.14}$$
$$:(Sa)\,b: =: b\,(Sa): + T\,[a,b]\,, \tag{B.15}$$
$$:a\,(Tb): + :(Tb)\,a: = T^2\langle a,b\rangle\,. \tag{B.16}$$

Proof. All follow from quasicommutativity. Note that these identities are the generalization in the even case of the identities in Lemma B.3.1 to the chiral de Rham complex. □

Lemma B.5.4. *For $f \in \mathcal{C}^\infty(M,\mathbb{C})$ and $a, b \in \Gamma(\Pi E)$, the following identities hold:*

$$::ab:f: =: f:ab::, \tag{B.17}$$
$$:a:fb:: =:: fb:a: + 2T\,(:f\langle a,b\rangle:)\,, \tag{B.18}$$
$$::ab:f: =: a:bf::, \tag{B.19}$$
$$::af:b: =: a:fb:: + 2:(Tf)\langle a,b\rangle:, \tag{B.20}$$
$$::af:(Sb): =: a:f\,(Sb):: - :(Ta)\langle \mathcal{D}f,b\rangle: - :(Tf)\,(\mathcal{D}\langle a,b\rangle - [a,b]):, \tag{B.21}$$
$$::a\,(Tf):b: =: a:(Tf)\,b:: + 2:\left(T^2 f\right)\langle a,b\rangle:, \tag{B.22}$$
$$::(Ta)\,f:b: =: (Ta):fb:: - :\left(T^2 f\right)\langle a,b\rangle:, \tag{B.23}$$
$$:(T\,(:ab:))\,f: =: f\,(T\,(:ab:)):. \tag{B.24}$$

Proof. The first, second and last identities follow directly from quasicommutativity, while the other ones are all immediate consequences of quasiassociativity. Some of the identities need to use the non-commutative Wick formula in their proofs. □

Lemma B.5.5. *For $f, g \in \mathcal{C}^\infty(M,\mathbb{C})$ and $a, b \in \Gamma(\Pi E)$, the following identities hold:*

$$::fg::ab:: =: f:g:ab:::, \tag{B.25}$$
$$::fa::gb:: =: f:a:gb::: + 2:(Tf):g\langle a,b\rangle::. \tag{B.26}$$

Proof. Both identities follow from quasiassociativity. □

Lemma B.5.6. *For $a, b, c \in \Gamma(\Pi E)$, the following identities hold:*

$$:a:bc:: =:: bc:a: + 2T\,(\langle a,b\rangle\,c - \langle a,c\rangle\,b)\,, \tag{B.27}$$
$$::ab:c: =: a:bc:: + 2\,(:(Ta)\langle b,c\rangle: - :(Tb)\langle a,c\rangle:)\,, \tag{B.28}$$
$$::a\,(Sb):c: =: a:(Sb)\,c:: + :(Ta)\,[b,c]: + 2:(TSb)\langle a,c\rangle:, \tag{B.29}$$
$$:a:bc:: =: b:ca:: + 2\,(:c\,(T\langle a,b\rangle): - :b\,(T\langle a,c\rangle):)\,, \tag{B.30}$$
$$:a:bc:: =: c:ab:: + 2\,(:a\,(T\langle b,c\rangle): - :b\,(T\langle a,c\rangle): + :c\,(T\langle a,b\rangle):)\,. \tag{B.31}$$

Proof. The identity (B.27) follows directly from quasicommutativity, while the identities (B.28) and (B.29) follow from quasiassociativity. Note that these three identities are the generalizations in the even case of identities in Lemma B.3.2 to the chiral de Rham complex. The last two identities are obtained combining the first ones in the correct order.

Indeed, the first identity follows applying (B.27) and (B.28), while the second one follows applying (B.28) and (B.27) in that order. Notice that they are valid if $M = \{\cdot\}$ and we have even Lie algebras (in this case the terms multiplied by 2 are zero). □

Lemma B.5.7. *For $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $a, b, c \in \Gamma(\Pi E)$, the following identities hold:*

$$:: fa :: bc ::=: a : b : fc ::: +2\left(: (Tf)\left(: \langle a,b\rangle c : - : b\langle a,c\rangle :\right) :\right), \tag{B.32}$$

$$: a : b : fc ::: = - : b : a : fc ::: +2 : (T\langle a,b\rangle) : fc ::, \tag{B.33}$$

$$: a : b : fc ::: =: a :: bf : c :: -2 : (Tf) : \langle b,c\rangle a :: . \tag{B.34}$$

Proof. The first identity follows from quasiassociativity, and applying identities (B.17), (B.19) and (B.13) in that order. The second one comes from quasiassociativity, (B.14), and quasiassociativity, in that order. The last one follows from quasiassociativity. □

Lemma B.5.8. *For $a, b, c, d \in \Gamma(\Pi E)$, the following identities hold:*

$$:: a : bc :: d : =: a :: bc : d :: +k\left(: (Ta)\left(\langle d,c\rangle b - \langle d,b\rangle c\right) : + : T(: bc :)\langle a,d\rangle :\right), \tag{B.35}$$

$$\begin{aligned} : a :: bc : d :: = - ::: bc : d : a : +k\big(&: \langle a,b\rangle (T(: cd :)) : - : \langle a,c\rangle (T(: bd :)) : \\ &+ : (T(: bc :))\langle a,d\rangle :\big). \end{aligned} \tag{B.36}$$

Proof. The first identity comes from quasiassociativity, while the last one comes from quasicommutativity. In both cases, we use the non-commutative Wick formula. We also have to use (B.17) and (B.24). □

Now, suppose that $E = l \oplus \bar{l} \oplus C_-$, where $C_+ := l \oplus \bar{l} \subseteq E$, for two isotropic subbundles l and $\bar{l}$ such that the restriction of $\langle\cdot,\cdot\rangle$ to $C_\pm$ is non-degenerate, and $C_- = C_+^\perp$. Keep in mind the notations (10.1). Consider $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n \subseteq l \oplus \bar{l}$ isotropic frame, and the associated sections

$$e_j := \Pi\epsilon_j \text{ and } e^j := \Pi\bar{\epsilon}_j, \quad \text{for } j \in \{1, \ldots, n\}.$$

Lemma B.5.9. *The following identity holds:*

$$: \left[e^j, e_k\right]_- \left[e^k, e_j\right]_- := T\left\langle \left[e^j, e_k\right]_-, \left[e^k, e_j\right]_- \right\rangle. \tag{B.37}$$

Proof. It follows from (B.14) directly, using the axioms of Courant algebroids. □

Define

$$\begin{array}{rccc} I_+ : & C_+ & \longrightarrow & C_+ \\ & a & \mapsto & a_l - a_{\bar{l}} \end{array}.$$

Abusing notation, we will write $I_+\Pi a \equiv \Pi I_+ a$ for $a \in C_+$.

Lemma B.5.10. *For $a \in \Gamma(\Pi E)$, the following identities hold:*

$$2T\left\langle \left[a, e^j\right], e_j \right\rangle =: \left[a, e^j\right]_{\bar{l}} e_j : + : e^j \left[a, e_j\right]_l :, \tag{B.38}$$

$$\begin{aligned} : \left[I_+a_+, e^j\right]_+ e_j : + : e^j \left[I_+a_+, e_j\right]_+ : =: \left\langle \left[a_l, e^j\right], e^k \right\rangle : e_k e_j :: + : \left\langle \left[a_{\bar{l}}, e_j\right], e_k \right\rangle : e^k e^j :: \\ + 2T\left\langle \left[I_+a_+, e^j\right], e_j \right\rangle. \end{aligned} \tag{B.39}$$

Proof. The first identity comes from (B.20) and (B.13), while the second one also needs (B.17), (B.18) and (B.19). In both cases, we need the axioms of Courant algebroids. □

Define $w := I_+ \left[e^j, e_j\right]_+ \in \Gamma\left(\Pi C_+\right).$

Lemma B.5.11. *The following identity holds:*

$$\begin{aligned} :e^j\left[e_j, w\right]_l: + :\left[e^j, w\right]_{\bar{l}} e_j: =: e^k\left(\mathcal{D}\left\langle w, e_k\right\rangle\right)_l: + :\left(\mathcal{D}\left\langle w, e^k\right\rangle\right)_{\bar{l}} e_k: \\ -2T\left\langle\left[w, e^j\right], e_j\right\rangle. \end{aligned} \tag{B.40}$$

Proof. It follows using Courant algebroid axioms, (B.13) and (B.20). □

Assume from now that $l \oplus \bar{l}$ satisfies the F-term condition (6.26).

Lemma B.5.12. *For $a \in \Gamma(\Pi E)$ and $j \in \{1, \ldots, n\}$, the following identities hold:*

$$\left[a, e^j\right]_l = \left[a_l, e^j\right]_l \quad \textit{or, equivalently,} \quad \left[e^j, a\right]_l = \left[e^j, a_l\right]_l, \tag{B.41}$$

$$[a, e_j]_{\bar{l}} = \left[a_{\bar{l}}, e_j\right]_{\bar{l}} \quad \textit{or, equivalently,} \quad [e_j, a]_{\bar{l}} = \left[e_j, a_{\bar{l}}\right]_{\bar{l}}. \tag{B.42}$$

Proof. By F-term condition and Courant algebroid axioms

$$\begin{aligned} \left[a_{\bar{l}}, e^j\right]_l = [a_l, e_j]_{\bar{l}} = 0,& \\ \left[a_-, e^j\right]_l &= \left\langle\left[a_-, e^j\right], e^k\right\rangle e_k = \left\langle a_-, \left[e^j, e^k\right]_{\bar{l}}\right\rangle e_k = 0, \\ [a_-, e_j]_{\bar{l}} &= \langle[a_-, e_j], e_k\rangle e^k = \left\langle a_-, [e_j, e_k]_l\right\rangle = 0. \end{aligned}$$

In conclusion, we have obtained the desired identities. □

Now, we collect a result that will be useful in the next Appendix to prove Lemma C.2.3.

Lemma B.5.13. *For each $i \in \{1, \ldots, n\}$, define*

$$\begin{aligned} a_i &:=: e_j : \left[e_k, e^j\right]_l \left[e_i, e^k\right]_l ::, \\ b_i &:=: e_j : \left(\mathcal{D}\left\langle\left[e^k, e_i\right], e^j\right\rangle\right)_l e_k ::, \\ A_i &:=: [e_i, e_j]_l \left(Se^j\right) : + : e_j \left(S\left[e_i, e^j\right]_+\right) : + : \left[e_i, e^j\right]_+ (Se_j) : + : e^j \left(S\left[e_i, e_j\right]_l\right), \\ B_i &:=: [e_i, e_j]_l : e_k \left[e^j, e^k\right]_{\bar{l}} :: + : e_j : [e_i, e_k]_l \left[e^j, e^k\right]_{\bar{l}} :: \\ &\quad + : e_j : e_k \left[e_i, \left[e^j, e^k\right]\right]_+ :: + : \left[e_i, e^j\right]_+ : e^k [e_j, e_k]_l :: \\ &\quad + : e^j : \left[e_i, e^k\right]_+ [e_j, e_k]_l :: + : e^j : e^k [e_i, [e_j, e_k]] ::, \\ C_i &: = 2 : e_j : e_k \left[e^j, \left[e_i, e^k\right]_-\right]_+ :: + : e_j : e_k \left[e_i, \left[e^j, e^k\right]\right]_- :: \\ &\quad + 2 : [e_i, e_j]_l : e^k \left[e^j, e_k\right]_- :: + : \left[e_i, e^j\right]_- : e^k [e_j, e_k]_l :: \\ &\quad + 2 : e_j : \left[e_i, e^k\right] \left[e^j, e_k\right]_- :: + : e^j : \left[e_i, e^k\right]_- [e_j, e_k]_l :: \\ &\quad + 2 : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right] :: . \end{aligned}$$

The following identities hold:

$$
\begin{aligned}
0 &= a_i + b_i, && \text{(B.43)}\\
A_i &= 2 : \left(T\left\langle e^k, [e_i, e_j]\right\rangle\right)\left[e_k, e^j\right] : + : \left(T\left\langle \left[e^k, e_i\right], e^j\right\rangle\right)[e_k, e_j]_l : \\
&+ T\left(: e_j \left(\left\langle \mathcal{D}\left\langle \left[e^k, e_i\right], e_k\right\rangle\right\rangle - \left\langle \mathcal{D}\left\langle e^j, [e_i, e_k]\right\rangle, e^k\right\rangle\right)\right. \\
&\left. + : e^j \left(\left\langle \mathcal{D}\left\langle \left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle\right) :\right) + \frac{1}{k} b_i \\
&+ : e^j : e^k \left(\left\langle \mathcal{D}\left\langle [e_i, e_j], e^m\right\rangle, e_k\right\rangle e_m\right) :: \\
&+ : e_j \left(: e_k \left(\frac{1}{2}\left\langle \mathcal{D}\left\langle \left[e^j, e_i\right], e^k\right\rangle, e_m\right\rangle + \left\langle \mathcal{D}\left\langle \left[e_i, e^j\right], e_m\right\rangle, e^k\right\rangle\right) e^m :\right) : \\
&+ \frac{1}{2} : e_j : \left(\mathcal{D}\left\langle \left[e^k, e_i\right], e^j\right\rangle\right)_- e_k :: + : e_k : \left(\left\langle \mathcal{D}\left[e_i, e^k\right], e_j\right\rangle\right)_- e^j ::, && \text{(B.44)}\\
B_i &= a_i + 2 : e_j : e_k \left[e^j, \left[e_i, e^k\right]_-\right]_+ :: +4 : \left(T\left\langle \left[e^j, e_i\right], e_k\right\rangle\right)\left[e^k, e_j\right]_+ : \\
&+ 2 : \left(T\left\langle \left[e^j, e_i\right], e^k\right\rangle\right)[e_k, e_j]_l : -2 : e^j : e^k \left(\left\langle \mathcal{D}\left\langle e^m, [e_i, e_j]\right\rangle, e_k\right\rangle e_m\right) :: \\
&+ : e_j : e_k \left(\left(\left\langle \mathcal{D}\left\langle e^k, \left[e_i, e^j\right]\right\rangle, e_m\right\rangle - 2\left\langle \mathcal{D}\left\langle e_m, \left[e_i, e^j\right]\right\rangle, e^k\right\rangle\right) e^m\right) :: , && \text{(B.45)}\\
C_i &= 2\left(: e_j : e_k \left[e^j, \left[e_i, e^k\right]_-\right]_- :: + : e_j : e_k \left[e^j, \left[e_i, e^k\right]_-\right]_{\bar{l}} :: \right.\\
&+ : e_j : e^k \left[e_k, \left[e^j, e_i\right]_-\right]_- :: + : e_j : \left[e_i, e^k\right]_- \left[e^j, e_k\right]_- :: \\
&\left. + : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right]_l ::\right) + : e_j : \left(\mathcal{D}\left\langle \left[e_i, e^k\right], e^j\right\rangle\right)_- e_k :: \\
&- 2 : e_k : \left(\mathcal{D}\left\langle \left[e_i, e^k\right], e_j\right\rangle\right)_- e^j :: +4 : \left(T\left\langle \left[e^j, e_i\right], e^k\right\rangle\right)\left[e^k, e_j\right]_- : . && \text{(B.46)}
\end{aligned}
$$

Proof. The first identity is easy. Indeed, by Courant algebroid axioms, (B.13) and (B.34),

$$a_i =: e_j : e_k \left[e_i, \left[e^j, e^k\right]\right]_l :: .$$

So, by Jacobi identity for the Dorfman bracket and (B.41),

$$a_i =: e_j : e_k \left[\left[e_i, e^j\right]_l, e^k\right]_l :: + : e_j : e_k \left[e^j, \left[e_i, e^k\right]_l\right]_l :: .$$

Using Courant algebroid axioms, (B.13), (B.14), (B.30), (B.31), (B.32) and (B.34),

$$: e_j : e_k \left[\left[e_i, e^j\right]_l, e^k\right]_l :: = -a_i - 2b_i.$$

Using Courant algebroid axioms, (B.13), (B.14), (B.32) and (B.34),

$$: e_j : e_k \left[e^j, \left[e_i, e^k\right]_l\right]_l :: = -a_i - b_i.$$

So we have that

$$a_i + b_i = -2\,(a_i + b_i)\,,$$

which gives (B.43). For next identity, by Courant algebroid axioms, (B.12) and (B.21),

$$\begin{aligned}
:e^j\left(S\,[e_i,e_j]_l\right): &=: \left(Te^j\right)\left\langle \mathcal{D}\left\langle \left[e^k,e_i\right],e_j\right\rangle ,e_k\right\rangle : - :\left(T\left\langle \left[e^k,e_i\right],e_j\right\rangle\right)\left[e^j,e_k\right]: \\
&\quad + \frac{1}{2}:e^j:\left(\mathcal{D}\left\langle e^k,[e_i,e_j]\right\rangle\right)e_k:: - :\left[e_i,e^j\right]_{\bar{l}}\left(Se_j\right):, \\
:[e_i,e_j]_l\left(Se^j\right): &=: \left(T\left\langle e^k,[e_i,e_j]\right\rangle\right)\left[e_k,e^j\right]: - :\left(Te_j\right)\left\langle \mathcal{D}\left\langle e^j,[e_i,e_k]\right\rangle ,e^k\right\rangle : \\
&\quad - \frac{1}{2}:e_k:\left(\mathcal{D}\left\langle \left[e^k,e_i\right],e_j\right\rangle\right)e^j:: - :e_j\left(S\left[e_i,e^j\right]_{\bar{l}}\right):, \\
:e_j\left(S\left[e_i,e^j\right]_l\right): &=: \left(Te_j\right)\left\langle \mathcal{D}\left\langle \left[e^k,e_i\right],e^j\right\rangle ,e_k\right\rangle : - :\left(T\left\langle \left[e^k,e_i\right],e^j\right\rangle\right)\left[e_j,e_k\right]: \\
&\quad + \frac{1}{2}:e_j:\left(\mathcal{D}\left\langle e^k,\left[e_i,e^j\right]\right\rangle\right)e_k:: - :\left[e_i,e^j\right]_l\left(Se_j\right):.
\end{aligned}$$

So using Courant algebroid axioms, (B.13), (B.14), (B.30) and (B.32), we obtain (B.44). Now, by Jacobi identity for the Dorfman bracket and (B.41),

$$\begin{aligned}
\left[e_i,\left[e^j,e^k\right]\right]_+ &= \left[\left[e_i,e^j\right],e^k\right]_l + \left[\left[e_i,e^j\right]_l\right]_{\bar{l}} + \left[\left[e_i,e^j\right]_{\bar{l}},e^k\right]_{\bar{l}} + \left[\left[e_i,e^j\right]_-,e^k\right]_+ \\
&\quad + \left[e^k,\left[e_i,e^j\right]\right]_l + \left[e^k,\left[e_i,e^j\right]_l\right]_{\bar{l}} + \left[e^k,\left[e_i,e^j\right]_{\bar{l}}\right]_{\bar{l}} + \left[e^k,\left[e_i,e^j\right]_-\right]_+, \\
\left[e_i,[e_j,e_k]_l\right]_l &= \left[[e_i,e_j]_l,e_k\right]_l + \left[e_j,[e_i,e_k]_l\right]_l.
\end{aligned}$$

By Courant algebroid axioms, (B.10), (B.13), (B.32) and (B.33),

$$\begin{aligned}
:e^j:e^k\left[[e_i,e_j]_l,e_k\right]_l:: &= - :\left[e_i,e^j\right]_{\bar{l}}:e^k[e_j,e_k]_l:: + 2:\left(T\left\langle \left[e^k,e_i\right],e_j\right\rangle\right)\left[e^j,e_k\right]_{\bar{l}}: \\
&\quad + :e^k:e^j\left(\left\langle \mathcal{D}\left\langle e^m,[e_i,e_j]\right\rangle ,e_k\right\rangle e_m\right)::.
\end{aligned}$$

By Courant algebroid axioms, (B.13) and (B.34),

$$\begin{aligned}
:e^j:e^k\left[e_j,[e_i,e_k]_l\right]_l:: &= - :e^j:\left[e_i,e^k\right]_{\bar{l}}[e_j,e_k]_l:: - 2:\left(T\left\langle \left[e^j,e_i\right],e_k\right\rangle\right)\left[e_j,e^k\right]_{\bar{l}}: \\
&\quad + :e^j:e^k\left(\left\langle \mathcal{D}\left\langle e^m,[e_i,e_k]\right\rangle ,e_j\right\rangle e_m\right)::.
\end{aligned}$$

By Courant algebroid axioms, (B.10), (B.13), (B.14), (B.20) and (B.32),

$$\begin{aligned}
:e_j:e_k\left[\left[e_i,e^j\right]_l,e^k\right]_{\bar{l}}:: &= - :\left[e_i,e^j\right]_l:e^k[e_j,e_k]_l:: + 2:\left(T\left\langle e^k,\left[e_i,e^j\right]\right\rangle\right)[e_j,e_k]_l: \\
&\quad + :e_j:e_k\left(\left\langle \mathcal{D}\left\langle e^k,\left[e_i,e^j\right]\right\rangle ,e_m\right\rangle e^m\right)::.
\end{aligned}$$

By Courant algebroid axioms, (B.10), (B.13) and (B.32),

$$\begin{aligned}
:e_j:e_k\left[\left[e_i,e^j\right]_{\bar{l}},e^k\right]_{\bar{l}}:: &= - :[e_i,e_j]_l:e_k\left[e^j,e^k\right]_{\bar{l}}:: + 2:\left(T\left\langle e_k,\left[e_i,e^j\right]\right\rangle\right)\left[e_j,e^k\right]_l: \\
&\quad - :e_j:e_k\left(\left\langle \mathcal{D}\left\langle e_m,\left[e_i,e^j\right]\right\rangle ,e^k\right\rangle e^m\right)::.
\end{aligned}$$

By Courant algebroid axioms, (B.13), (B.31), (B.32) and (B.34),

$$:e_j:e_k\left[e^j,\left[e_i,e^k\right]_l\right]_{\bar{l}}::=-:e^j:\left[e_i,e^k\right]_l\left[e_j,e_k\right]_l::.$$

By Courant algebroid axioms, (B.13) and (B.34),

$$\begin{aligned}:e_j:e_k\left[e^j,\left[e_i,e^k\right]_{\bar{l}}\right]_{\bar{l}}::=-:e_j:\left[e_i,e_k\right]_l\left[e^j,e^k\right]_{\bar{l}}::-2:\left(T\left\langle e_j,\left[e_i,e^k\right]\right\rangle\right)\left[e^j,e_k\right]_l:\\+:e_j:e_k\left(\left\langle\mathcal{D}\left\langle e_m,\left[e_i,e^k\right]\right\rangle,e^j\right\rangle e^m\right)::.\end{aligned}$$

At last, by Courant algebroid axioms and (B.30),

$$:e_j:e_k\left[\left[e_i,e^j\right]_-,e^k\right]_+::+:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]_+::=2:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]_+::.$$

In summary, by Courant algebroid axioms, (B.13) and (B.33), we obtain (B.45). Finally, for the last identity, by Jacobi identity for the Dorfman bracket,

$$\left[e_i,\left[e^j,e^k\right]\right]_-=\left[\left[e_i,e^j\right]_l,e^k\right]_-+\left[\left[e_i,e^j\right]_-,e^k\right]_-+\left[e^j,\left[e_i,e^k\right]_l\right]_-+\left[e^j,\left[e_i,e^k\right]_-\right]_-.$$

By Courant algebroid axioms and (B.30),

$$\begin{aligned}&:\left[e_i,e^j\right]_-:e^k\left[e_j,e_k\right]_l::=:e^j:\left[e_i,e^k\right]_-\left[e_j,e_k\right]_l::,\\&:e_j:e_k\left[\left[e_i,e^j\right]_l,e^k\right]_-::=:e_j:e_k\left[e^j,\left[e_i,e^k\right]_l\right]_-::+:e_j:e_k\left(\mathcal{D}\left\langle\left[e_i,e^j\right],e^k\right\rangle\right)_-::,\\&:e_j:e_k\left[\left[e_i,e^j\right]_-,e^k\right]_-::=:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]_-::,\\&:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]_{\bar{l}}::=:e_j:e_k\left[\left[e_i,e^j\right]_-,e^k\right]_{\bar{l}}::.\end{aligned}$$

By Courant algebroid axioms, (B.13) and (B.34),

$$:e_j:\left[e_i,e^k\right]_l\left[e^j,e_k\right]_-::=-:e_j:e_k\left[e^j,\left[e_i,e^k\right]_l\right]_-::.$$

Since using Jacobi identity for the Dorfman bracket, for being l involutive,

$$\left[e^j,\left[e_k,e_i\right]_l\right]_-=\left[\left[e^j,e_k\right]\right]_-+\left[e_k,\left[e^j,e_i\right]\right]_-,$$

by Courant algebroid axioms, (B.13) and (B.34),

$$\begin{aligned}:e_j:\left[e_i,e^k\right]_{\bar{l}}\left[e^j,e_k\right]_-::=:e_j:e^k\left[\left[e^j,e_k\right]_{\bar{l}},e_i\right]_-::-:e_j:e^k\left[e_i,\left[e^j,e_k\right]_-\right]_-::\\+:e_j:e^k\left[e_k,\left[e^j,e_i\right]_{\bar{l}}\right]_-::+:e_j:e^k\left[e_k,\left[e^j,e_i\right]_-\right]_-::.\end{aligned}$$

So, we have arrived at the next identity.

$$\begin{aligned}C_i = 2\Big(&: [e_i, e^j]_- : e^k [e_j, e_k]_l :: + : e_j : e_k \left[e^j, \left[e_i, e^k\right]_-\right]_- :: \\ &+ : e_j : e_k \left[\left[e_i, e^j\right]_-, e^k\right]_{\bar{l}} :: + : [e_i, e_j]_l : e^k \left[e^j, e_k\right]_- :: + : e_j : e^k \left[\left[e^j, e_k\right]_{\bar{l}}, e_i\right]_- :: \\ &+ : e_j : e^k \left[e_k, \left[e^j, e_i\right]_{\bar{l}}\right]_- :: + : e_j : e^k \left[e_k, \left[e^j, e_i\right]_-\right]_- :: + : e_j \left[e_i, e^k\right]_- \left[e^j, e_k\right]_- :: \\ &+ : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right]_l :: + : e_j : e_k \left(\mathcal{D}\left\langle \left[e_i, e^j\right], e^k\right\rangle\right)_- ::\Big).\end{aligned}$$

By Courant algebroid axioms and (B.31),

$$: \left[e_i, e^j\right]_- : e^k \left[e_j, e_k\right]_l :: = - : \left[e_j, e_k\right]_l : e^k \left[e_i, e^j\right]_- :: +2 : \left[e_i, e^j\right]_- \left(T\left\langle \left[e_j, e_k\right], e^k\right\rangle\right) : .$$

Now, by Courant algebroid axioms (B.10), (B.13) and (B.32),

$$: [e_i, e_j]_l : e^k \left[e^j, e_k\right]_- :: = - : e_j : e^k \left[e_k, \left[e^j, e_i\right]_{\bar{l}}\right]_- :: +2 : \left(T\left\langle e^k, [e_i, e_j]\right\rangle\right) \left[e^j, e_k\right]_- : .$$

At last, by Courant algebroid axioms, (B.10), (B.13), (B.31) and (B.32),

$$\begin{aligned}: e_j : e^k \left[\left[e^j, e_k\right]_{\bar{l}}, e_i\right]_- :: =: [e_i, e_k]_l : e^k \left[e_i, e^j\right]_- :: - : e^k : e_j \left(\mathcal{D}\left\langle e_i, \left[e^j, e_k\right]\right\rangle\right)_- :: \\ - 2 : \left[e_i, e^k\right]_- \left(T\left\langle e^j, [e_k, e_j]\right\rangle\right) : .\end{aligned}$$

In summary, by Courant algebroid axioms, (B.14) and (B.30), we obtain (B.46), which concludes the proof of all the desired identities. □

Remark B.5.14. Analogously, for each $i \in \{1, \ldots, n\}$, define

$$\begin{aligned}a^i &:=: e^j : \left[e^k, e_j\right]_{\bar{l}} \left[e^i, e_k\right]_{\bar{l}} ::, \\ b^i &:=: e^j : \left(\mathcal{D}\left\langle \left[e_k, e^i\right], e_j\right\rangle\right)_{\bar{l}} e^k ::, \\ A^i &:=: \left[e^i, e_j\right]_+ \left(Se^j\right) : + : e_j \left(S\left[e^i, e^j\right]_{\bar{l}}\right) : + : \left[e^i, e^j\right]_{\bar{l}} (Se_j) : + : e^j \left(S\left[e^i, e_j\right]_+\right), \\ B^i &:=: \left[e^i, e_j\right]_+ : e_k \left[e^j, e^k\right]_{\bar{l}} :: + : e_j : \left[e^i, e_k\right]_+ \left[e^j, e^k\right]_{\bar{l}} :: \\ &\quad + : e_j : e_k \left[e^i, \left[e^j, e^k\right]\right] :: + : \left[e^i, e^j\right]_{\bar{l}} : e^k [e_j, e_k]_l :: \\ &\quad + : e^j : \left[e^i, e^k\right]_{\bar{l}} [e_j, e_k]_l :: + : e^j : e^k \left[e^i, [e_j, e_k]\right]_+ ::, \\ C^i &: = 2 : e^j : e^k \left[e_j, \left[e^i, e_k\right]_-\right]_+ :: + : e^j : e^k \left[e^i, [e_j, e_k]\right]_- :: \\ &\quad + 2 : \left[e^i, e^j\right]_{\bar{l}} : e_k \left[e_j, e^k\right]_- :: + : \left[e^i, e_j\right]_- : e_k \left[e^j, e^k\right]_{\bar{l}} :: \\ &\quad + 2 : e^j : \left[e^i, e_k\right] \left[e_j, e^k\right]_- :: + : e_j : \left[e^i, e_k\right]_- \left[e^j, e^k\right]_{\bar{l}} :: \\ &\quad + 2 : e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right] :: .\end{aligned}$$

Then, the following identities hold:

$$
\begin{aligned}
0 &= a^i + b^i, \\
A^i &= 2 : \left(T\left\langle e_k, \left[e^i, e^j\right]\right\rangle\right) \left[e^k, e_j\right] : + : \left(T\left\langle \left[e_k, e^i\right], e_j\right\rangle\right) \left[e^k, e^j\right]_{\bar{l}} : \\
&\quad + T\left(: e^j \left(\left\langle \mathcal{D}\left\langle \left[e_k, e^i\right], e^k\right\rangle\right\rangle - \left\langle \mathcal{D}\left\langle e_j, \left[e^i, e^k\right]\right\rangle, e_k\right\rangle\right)\right. \\
&\quad \left. + : e_j \left(\left\langle \mathcal{D}\left\langle \left[e_k, e^i\right], e^j\right\rangle, e^k\right\rangle\right) : \right) + \frac{1}{k} b^i \\
&\quad + : e_j : e_k \left(\left\langle \mathcal{D}\left\langle \left[e^i, e^j\right], e_m\right\rangle, e^k\right\rangle e^m\right) :: \\
&\quad + : e^j \left(: e^k \left(\frac{1}{2}\left\langle \mathcal{D}\left\langle \left[e_j, e^i\right], e_k\right\rangle, e^m\right\rangle + \left\langle \mathcal{D}\left\langle \left[e^i, e_j\right], e^m\right\rangle, e_k\right\rangle\right) e_m : \right) : \\
&\quad + \frac{1}{2} : e^j : \left(\mathcal{D}\left\langle \left[e_k, e^i\right], e_j\right\rangle\right)_- e^k :: + : e^k : \left(\left\langle \mathcal{D}\left[e^i, e_k\right], e^j\right\rangle\right)_- e_j ::, \\
B^i &= a^i + 2 : e^j : e^k \left[e_j, \left[e^i, e_k\right]_-\right]_+ :: +4 : \left(T\left\langle \left[e_j, e^i\right], e^k\right\rangle\right) \left[e_k, e^j\right]_+ : \\
&\quad + 2 : \left(T\left\langle \left[e_j, e^i\right], e_k\right\rangle\right) \left[e^k, e^j\right]_{\bar{l}} : -2 : e_j : e_k \left(\left\langle \mathcal{D}\left\langle e_m, \left[e^i, e^j\right]\right\rangle, e^k\right\rangle e^m\right) :: \\
&\quad + : e^j : e^k \left(\left(\left\langle \mathcal{D}\left\langle e_k, \left[e^i, e_j\right]\right\rangle, e^m\right\rangle - \left\langle \mathcal{D}\left\langle e^m, \left[e^i, e_j\right]\right\rangle, e_k\right\rangle\right) e_m\right) :: , \\
C^i &= 2\left(: e^j : e^k \left[e_j, \left[e^i, e_k\right]_-\right]_- :: + : e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right]_l :: \right. \\
&\quad + : e^j : e_k \left[e^k, \left[e_j, e^i\right]_-\right]_- :: + : e^j : \left[e^i, e_k\right]_- \left[e_j, e^k\right]_- :: \\
&\quad \left. + : e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right]_{\bar{l}} :: \right) + : e^j : \left(\mathcal{D}\left\langle \left[e^i, e_k\right], e_j\right\rangle\right)_- e^k :: \\
&\quad - 2 : e_j : \left(\mathcal{D}\left\langle \left[e^i, e_k\right], e^j\right\rangle\right)_- e^k :: +4 : \left(T\left\langle \left[e_j, e^i\right], e_k\right\rangle\right) \left[e_k, e^j\right]_- : .
\end{aligned}
$$

B.6 Basic Linear Algebra: Jacobi's Formula

Let M be an n-dimensional smooth manifold, and E a Courant algebroid over M. We will consider the differential algebra $(\mathrm{Mat}_n(\mathcal{C}^\infty(M)), \cdot, \mathcal{D})$, for which

$$
\mathcal{D}A := \left(\mathcal{D}A_j^k\right)_{j,k=1}^n, \quad \text{for } A = \left(A_j^k\right)_{j,k=1}^n \in \mathrm{Mat}_n\left(\mathcal{C}^\infty(M)\right).
$$

We have the following result that will be applied in Chapter 6 and Chapter 10.

Lemma B.6.1. *Given $A \in \mathrm{Mat}_n(\mathcal{C}^\infty(M))$, the following identity holds:*

$$
\mathcal{D}\det A = \mathrm{tr}\left(\mathrm{Adj}(A)^T \cdot \mathcal{D}A\right).
$$

Proof. It is seen by induction on $n \in \mathbb{N}$, using basic properties from linear algebra. □

Remark B.6.2. The previous result is also true for the chiral de Rham complex Ω_E^{ch} if we take the even derivation T and the normally ordered product, using the canonical identification $\Gamma(S\Pi E) \cong \Gamma(E)$ for the elements $2Tf = \mathcal{D}f$ with $f \in \mathcal{C}^\infty(M)$.

Appendix C

Proof of Theorem 10.1.1

In this appendix, we give a complete proof of Theorem 10.1.1, which can be considered the main result of Chapter 10. In order to simplify computations, we will use the Einstein summation convention for repeated indexes.

C.1 Generator of $N = 2$ Supersymmetry

Let E be the complexification of a real Courant algebroid over M smooth manifold, for which we can construct the chiral de Rham complex Ω_E^{ch}. Now, fix $E = l \oplus \bar{l} \oplus C_-$ a direct sum decomposition, with $l, \bar{l} \subseteq E$ isotropic n-dimensional subbundles, for which the restriction $\langle \cdot, \cdot \rangle|_{C_\pm}$ is non-degenerate, where $C_+ = l \oplus \bar{l}$, and $C_- = C_+^\perp$. Let

$$\pi_\pm \colon E \longrightarrow C_\pm, \quad \pi_l \colon E \longrightarrow l, \quad \pi_{\bar{l}} \colon E \longrightarrow \bar{l}$$

be the orthogonal projections. When there is no possible confusion, we use the notation (10.1). Fix a frame $\{\epsilon_j, \bar{\epsilon}_j\}_{j=1}^n \subseteq C_+$ satisfying (6.28). Define the associated odd sections

$$e_j = \Pi \epsilon_j, \quad e^j = \Pi \bar{\epsilon}_j, \qquad \text{for } j \in \{1, \dots, n\}.$$

Remember that we work with parity-reversed sections by (9.3). We define

$$\begin{array}{cccc} I_+ \colon & C_+ & \longrightarrow & C_+ \\ & a & \mapsto & a_l - a_{\bar{l}} \end{array}.$$

Abusing notation, we write $I_+ \Pi a \equiv \Pi I_+ a$ for $a \in C_+$. Now, define

$$w = \Pi I_+ \left[\bar{\epsilon}_j, \epsilon_j\right]_+ = \left[e^j, e_j\right]_l - \left[e^j, e_j\right]_{\bar{l}} \in \Gamma(\Pi C_+).$$

Note that the following computations are also valid for $(\mathfrak{g}, (\cdot|\cdot))$ a complex quadratic Lie algebras, replacing Ω_E^{ch} by $V^k_{\mathrm{super}}(\mathfrak{g})$ the universal superaffine vertex algebra of $\mathfrak{g}$ with non-zero level $k \in \mathbb{C}$ by Theorem 9.1.11. So, the following results are written for arbitrary level $k \neq 0$ to work for complex quadratic Lie algebras, but, for the chiral de Rham complex, we always assume $k = 2$. Define the locally defined section

$$J_0 := \frac{i}{k} : e^j e_j : . \tag{C.1}$$

Lemma C.1.1. *Given $a \in \Gamma(\Pi E)$, the following identities hold:*

$$[a_\Lambda J_0] = \frac{i}{k}\left(:\left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] :\right) + i\left(\chi I_+ a_+ + \lambda \left\langle \left[a, e^j\right] \middle| e_j \right\rangle\right), \tag{C.2a}$$

$$\begin{aligned}[J_{0\Lambda} a] = &\frac{i}{k}\left(:\left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] :\right) \\ &- \left(\left(\chi + S\right) I_+ a_+ + \left(\lambda + T\right)\left\langle \left[a, e^j\right], e_j \right\rangle\right).\end{aligned} \tag{C.2b}$$

Proof. The first identity follows directly by the non-commutative Wick formula, since

$$\begin{aligned}\left[a_\Lambda : e^j e_j :\right] &=: \left[a_\Lambda e^j\right] e_j : + : e^j \left[a_\Lambda e_j\right] : + \int_0^\Lambda d\Gamma\, \left[\left[a_\Lambda e^j\right]_\Gamma e_j\right] \\ &=: \left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] : + k\left(\chi\left(\left\langle a, e^j\right\rangle e_j - \left\langle a, e_j\right\rangle e^j\right) + \lambda\left\langle \left[a, e^j\right], e_j\right\rangle\right) \\ &=: \left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] : + k\chi I_+ a_+ + k\lambda\left\langle \left[a, e^j\right], e_j\right\rangle, \\ [a_\Lambda J_0] &= \frac{i}{k}\left(:\left[a, e^j\right] e_j : + : e^j \left[a, e_j\right] :\right) + i\left(\chi I_+ a_+ + \lambda \left\langle \left[a, e^j\right] \middle| e_j \right\rangle\right).\end{aligned}$$

The other identity is obtained applying antisimmetry of the Λ-bracket. ∎

Define the locally defined sections

$$\begin{aligned} H_0 :&= \frac{1}{k}\left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) :\right) + \frac{1}{k^2}\Big(: e_j : e^k \left[e^j, e_k\right] :: \\ &\quad + : e^j : e_k \left[e_j, e^k\right] :: - : e_j : e_k \left[e^j, e^k\right] :: - : e^j : e^k \left[e_j, e_k\right] ::\Big), \\ H' :&= H_0 + \frac{T}{k} w, \end{aligned} \tag{C.3}$$

and

$$c_0 := 3 \dim l \in \mathbb{C}.$$

Proposition C.1.2. *One has*

$$[J_{0\Lambda} J_0] = -\left(H' + \frac{\lambda\chi}{3} c_0\right).$$

Proof. We start applying the non-commutative Wick formula using (C.2b) to obtain

$$\begin{aligned}\left[J_{0\Lambda} : e^j e_j :\right] &=: \left[J_{0\Lambda} e^j\right] e_j : - : e^j \left[J_{0\Lambda} e_j\right] : + \int_0^\Lambda d\Gamma\, \left[\left[J_{0\Lambda} e^j\right]_\Gamma e_j\right] \\ &= \frac{i}{k}\Big(:: \left[e^j, e^k\right] e_k : e_j : + :: e^k \left[e^j, e_k\right] : e_j : - : e^j : \left[e_j, e^k\right] e_k :: \\ &\quad - : e^j : e^k \left[e_j, e_k\right] ::\Big) + i\lambda\left(\left\langle e_j, \left[e^k, e_k\right]\right\rangle e^j - \left\langle e^j, \left[e^k, e_k\right]\right\rangle e_j\right) \\ &\quad + i\left(: e^j \left(T\left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) : - : \left(T\left\langle e^j, \left[e^k, e_k\right]\right\rangle\right) e_j :\right. \\ &\quad \left. + : e^j \left(S e_j\right) : + : \left(S e^j\right) e_j :\right) + \frac{i}{k}\int_0^\Lambda d\Gamma I_1 + i\int_0^\Lambda d\Gamma\left(I_2^1 + I_2^2\right).\end{aligned}$$

Here, we compute each integral independently using sesquilinearity, antisymmetry and the non-commutative Wick formula. Indeed, we arrive at the following:

$$\begin{aligned}
I_1^1 :&= \left[e_{j\,\Gamma} : \left[e^j, e^k\right] e_k :\right] =: \left[e_{j\,\Gamma} \left[e^j, e^k\right]\right] e_k : + : \left[e^j, e^k\right] \left[e_{j\,\Gamma} e_k\right] : \\
&+ \int_0^\Gamma d\Omega \left[\left[e_{j\,\Gamma} \left[e^j, e^k\right]\right]_\Omega e_k\right] =: \left[e_j, \left[e^j, e^k\right]\right] e_k : + : \left[e^j, e^k\right] \left[e_j, e_k\right] : \\
&+ k\eta \left\langle e_j, \left[e^j, e^k\right]\right\rangle e_k + k\xi \left\langle \left[e_j, \left[e^j, e^k\right]\right], e_k \right\rangle; \\
I_1^2 :&= \left[e_{j\,\Gamma} : e^k \left[e^j, e_k\right] :\right] =: \left[e_{j\,\Gamma} e^k\right] \left[e^j, e_k\right] : + : e^k \left[e_{j\,\Gamma} \left[e^j, e_k\right]\right] : \\
&+ \int_0^\Gamma d\Omega \left[\left[e_{j\,\Gamma} e^k\right]_\Omega \left[e^j, e_k\right]\right] =: \left[e_j, e^k\right] \left[e^j, e_k\right] : + : e^k \left[e_j, \left[e^j, e_k\right]\right] : \\
&+ k\eta \left(\left[e^j, e_j\right] - \left\langle e_j, \left[e^j, e_k\right]\right\rangle e^k\right) + k\xi \left\langle \left[e_j, e^k\right], \left[e^j, e_k\right]\right\rangle; \\
\int_0^\Lambda d\Gamma I_1 :&= \int_0^\Lambda \left[: \left[e^j, e^k\right] e_k : + : e^k \left[e^j, e_k\right] :_\Gamma e_j\right] \\
&= k\lambda \left(\left[e^j, e_j\right]_l - \left[e^j, e_j\right] - \left[e^j, e_j\right]_{\bar{l}}\right) = -2k\lambda \left[e^j, e_j\right]_{\bar{l}},
\end{aligned}$$

and

$$\begin{aligned}
I_2^1 :&= \left[(\chi + S)\, e^j{}_\Gamma e_j\right] = (\eta - \chi) \left(\left[e^j, e_j\right] + k\eta \left\langle e^j, e_j\right\rangle\right) \\
&= \eta \left(\left[e^j, e_j\right] + k\chi \dim l\right) - k\gamma \dim l - \chi \left[e^j, e_j\right]; \\
I_2^2 :&= \left[T \left\langle e^j, \left[e^k, e_k\right]\right\rangle_\Gamma e_j\right] = -\gamma \left[\left\langle e^j, \left[e^k, e_k\right]\right\rangle_\Gamma e_j\right] = -\gamma \left\langle \mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e_j\right\rangle; \\
\int_0^\Lambda d\Gamma I_2 :&= \int_0^\Lambda d\Gamma \left(I_2^1 + I_2^2\right) = \lambda \left[e^j, e_j\right] + k\lambda\chi \dim l.
\end{aligned}$$

Then, we finally obtain that

$$\begin{aligned}
\left[J_{0\Lambda} J_0\right] &= \frac{i}{2} \left[J_{0\Lambda} : e^j e_j :\right] \\
&= -\frac{1}{k^2} \Big(:: \left[e^j, e^k\right] e_k : e_j : + :: e^k \left[e^j, e_k\right] : e_j : - : e^j : \left[e_j, e^k\right] e_k :: \\
&- : e^j : e^k \left[e_j, e_k\right] :: \Big) - \frac{1}{k}\lambda \left(\left[e^j, e_j\right]_{\bar{l}} - \left[e^j, e_j\right]_l\right) \\
&- \frac{1}{k} \Big(: e^j \left(T \left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) : - : \left(T \left\langle e^j, \left[e^k, e_k\right]\right\rangle\right) e_j : \\
&+ : e^j \left(S e_j\right) : + : \left(S e^j\right) e_j : \Big) + \frac{2}{k}\lambda \left[e^j, e_j\right]_{\bar{l}} - \frac{1}{k}\lambda \left[e^j, e_j\right] - \lambda\chi \dim l \\
&= -\frac{1}{k^2} \Big(:: \left[e^j, e^k\right] e_k : e_j : + :: e^k \left[e^j, e_k\right] : e_j : - : e^j : \left[e_j, e^k\right] e_k :: \\
&- : e^j : e^k \left[e_j, e_k\right] :: \Big) - \frac{1}{k} \left(: e^j \left(S e_j\right) : + : \left(S e^j\right) e_j : \right) \\
&- \frac{1}{2} \Big(: e^j \left(T \left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) : - : \left(T \left\langle e^j, \left[e^k, e_k\right]\right\rangle\right) e_j : \Big) - \lambda\chi \dim l.
\end{aligned}$$

Applying properties (B.14), (B.15) and (B.27) we rewrite the above expression, since

$$: \left(Se^j\right) e_j :=: e_j \left(Se^j\right) : +T \left[e^j, e_j\right];$$
$$- : e^j : \left[e_j, e^k\right] e_k :: =: e^j : e_k \left[e_j, e^k\right] :: -2 : e^j \left(T \left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) :;$$
$$:: e^k \left[e^j, e_k\right] : e_j : =: e_j : e^k \left[e^j, e_k\right] :: -kT \left(\left[e^j, e_j\right] + \left[e^j, e_j\right]_{\bar{l}}\right);$$
$$:: \left[e^j, e^k\right] e_k : e_j : = - : e_j : e_k \left[e^j, e^k\right] :: +kT \left[e^j, e_j\right]_l + 2 : \left(T \left\langle e^j, \left[e^k, e_k\right]\right\rangle\right) e_j :,$$

so

$$\begin{aligned}[][J_{0\Lambda} J_0] &= -\frac{1}{k^2}\Big(:: \left[e^j, e^k\right] e_k : e_j : + :: e^k \left[e^j, e_k\right] : e_j : - : e^j : \left[e_j, e^k\right] e_k :: \\ &\quad - : e^j : e^k \left[e_j, e_k\right] ::\Big) - \frac{1}{k}\left(: e^j \left(Se_j\right) : + : \left(Se^j\right) e_j :\right) - \frac{1}{k} Tw - \lambda\chi \dim l, \\ &= -\left(H' + \frac{\lambda\chi}{3} c_0\right),\end{aligned}$$

as desired. □

C.2 Neveu-Schwarz Generator from F-term Condition

Now, we assume that $l \oplus \bar{l}$ satisfies the F-term condition (6.26).

Lemma C.2.1. *The following identity is satisfied:*

$$\begin{aligned} H_0 &= \frac{1}{k}\left(: e_j \left(Se^j\right) : + : e^j \left(Se_j\right) :\right) \\ &\quad + \frac{1}{k^2}\Big(2 : e_j : e^k \left[e^j, e_k\right]_- :: + : e_j : e_k \left[e^j, e^k\right]_{\bar{l}} :: + : e^j : e^k \left[e_j, e_k\right]_l ::\Big).\end{aligned}$$

Proof. First, notice that applying (B.30) we obtain that

$$: e_j : e^k \left[e^j, e_k\right] :: = - : e^k : e_j \left[e^j, e_k\right] ::=: e^k : e_j \left[e_k, e^j\right] :: .$$

Then, using F-term condition, we have that

$$\begin{aligned} H_0 &= \frac{1}{k}\left(: e_j \left(Se^j\right) : + : e^j \left(Se_j\right) :\right) \\ &\quad + \frac{1}{k^2}\Big(2 : e_j : e^k \left[e^j, e_k\right] :: - : e_j : e_k \left[e^j, e^k\right]_{\bar{l}} :: - : e^j : e^k \left[e_j, e_k\right]_l ::\Big).\end{aligned}$$

Moreover, using (B.31) and (B.32), we obtain that

$$\begin{aligned} : e^j : e_k \left[e_j, e^k\right]_l :: &= - : \left[e_j, e^k\right]_l : e_k e^j :: -2 : e_k \left(T \left\langle e^j, \left[e_j, e^k\right]\right\rangle\right) : \\ &= - : \left(\left\langle e^m, \left[e_j, e^k\right]\right\rangle e_m\right) : e_k e^j :: -2 : e_k \left(T \left\langle \left[e^j, e_j\right], e^k\right\rangle\right) : \\ &=: e_m : e_k \left(\left\langle e_j, \left[e^m, e^k\right]\right\rangle e^j\right) :: \\ &\quad + 2\left(: \left(T \left\langle e^j, \left[e_j, e^k\right]\right\rangle\right) e_k : - : e_k \left(T \left\langle \left[e^j, e_j\right], e^k\right\rangle\right) :\right) \\ &=: e_j : e_k \left[e^j, e^k\right]_{\bar{l}} :: .\end{aligned}$$

Analogously, using (B.14), (B.31) and (B.32), we obtain that

$$
\begin{aligned}
: e^j : e_k \left[e_j, e^k\right]_{\bar{l}} :: &= - : \left[e_j, e^k\right]_{\bar{l}} : e_k e^j :: +2 : e^j \left(T \left\langle e_k, \left[e_j, e^k\right]\right\rangle\right) : \\
&=: \left(\left\langle e_m, \left[e_j, e^k\right]\right\rangle e^m\right) : e^j e_k :: +2 : e^j \left(T \left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) : \\
&=: e^m : e^j \left(\left\langle \left[e_m, e_j\right], e^k\right\rangle e_k\right) :: \\
&- 2 \left(: \left(T \left\langle e_k, \left[e_j, e^k\right]\right\rangle\right) e^j : - : e^j \left(T \left\langle e_j, \left[e^k, e_k\right]\right\rangle\right) :\right) \\
&=: e^j : e^k \left[e_j, e_k\right]_l :: .
\end{aligned}
$$

So, we have obtained the required formula for H_0. □

Remark C.2.2. Notice that we can exchange l and $\bar{l}$ in the expression of H_0 and H'. Indeed, in both cases, we obtain the same local sections. So, in particular, the values of $\left[H_{0\Lambda} e_i\right]$ and $\left[H_{0\Lambda} e^i\right]$ (respectively, $\left[H'{}_\Lambda e_i\right]$ and $\left[H'{}_\Lambda e^i\right]$) for $i \in \{1, \ldots, n\}$ are dual. The aim of our next result is to compute the values of the previous Λ-brackets.

We are ready to give the following result, for which we will need some of the properties collected in Lemma B.5.13.

Proposition C.2.3. *For each $i \in \{1, \ldots, n\}$, the following identities hold:*

$$
\begin{aligned}
\left[H_{0\Lambda} e_i\right] = &\frac{\lambda}{2} \left(\left(\left\langle \mathcal{D} \left\langle \left[e^k, e_i\right], e^j\right\rangle, e_k\right\rangle - \left\langle \mathcal{D} \left\langle e^j, \left[e_i, e_k\right]\right\rangle, e^k\right\rangle\right) e_j \right.\\
&\left. - \left\langle \mathcal{D} \left\langle \left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle e^j\right) - \frac{2}{k^2} \left(: e_j : e_k \left[\left[e_i, e^j\right]_-, e^k\right]_{\bar{l}} :: \right.\\
&+ : e_j : e_k \left[\left[e_i, e^j\right]_-, e^k\right]_- :: + : \left[e_i, e^j\right]_- : e_k \left[e_j, e^k\right]_- :: \\
&\left. + : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right]_l :: + : e^j : e_k \left[\left[e^k, e_i\right]_-, e_j\right]_- ::\right) \\
&+ \frac{1}{k} \left(\chi : e_j \left[e_i, e^j\right]_- : -2 : e_j \left(S \left[e_i, e^j\right]_-\right) : +2T \left[e_j, \left[e_i, e^j\right]_-\right]_l \right.\\
&\left. + \lambda \left(\left[e_j, \left[e_i, e^j\right]_-\right]_l + \left[\left[e_i, e^j\right]_-, e_j\right]_-\right)\right) + \left(\lambda + 2T + \chi S\right) e_i,
\end{aligned}
\tag{C.4a}
$$

$$
\begin{aligned}
\left[H_{0\Lambda} e^i\right] = &\frac{\lambda}{2} \left(\left(\left\langle \mathcal{D} \left\langle \left[e_k, e^i\right], e_j\right\rangle, e^k\right\rangle - \left\langle \mathcal{D} \left\langle e_j, \left[e^i, e^k\right]\right\rangle, e_k\right\rangle\right) e^j \right.\\
&\left. - \left\langle \mathcal{D} \left\langle \left[e_k, e^i\right], e^j\right\rangle, e^k\right\rangle e_j\right) - \frac{2}{k^2} \left(: e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right]_l :: \right.\\
&+ : e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right]_- :: + : \left[e^i, e_j\right]_- : e^k \left[e^j, e_k\right]_- :: \\
&\left. + : e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right]_{\bar{l}} :: + : e_j : e^k \left[\left[e_k, e^i\right]_-, e^j\right]_- ::\right) \\
&+ \frac{1}{k} \left(\chi : e^j \left[e^i, e_j\right]_- : -2 : e^j \left(S \left[e^i, e_j\right]_-\right) : +2T \left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}} \right.\\
&\left. + \lambda \left(\left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}} + \left[\left[e^i, e_j\right]_-, e^j\right]_-\right)\right) + \left(\lambda + 2T + \chi S\right) e^i.
\end{aligned}
\tag{C.4b}
$$

$$\begin{aligned}\left[H'{}_\Lambda e_i\right] &= \left[H_{0\Lambda} e_i\right] - \frac{\lambda}{k}\left(\left\langle\left[e^j, e_j\right], e^n\right\rangle\left[e_n, e_i\right]_l - \left\langle\left[e^j, e_j\right], e_n\right\rangle\left[e^n, e_i\right]\right)\\ &+ \frac{\lambda}{2}\left(\left\langle\mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle e_j - \left\langle\mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle e^j\right.\\ &\left.+ \mathcal{D}\left\langle e_i, \left[e^j, e_j\right]\right\rangle\right) + \lambda\chi\left\langle\left[e^j, e_j\right], e_i\right\rangle, \end{aligned} \tag{C.4c}$$

$$\begin{aligned}\left[H'{}_\Lambda e^i\right] &= \left[H_{0\Lambda} e^i\right] + \frac{\lambda}{k}\left(\left\langle\left[e^j, e_j\right], e_n\right\rangle\left[e^n, e^i\right]_{\bar{l}} - \left\langle\left[e^j, e_j\right], e^n\right\rangle\left[e_n, e^i\right]\right)\\ &- \frac{\lambda}{2}\left(\left\langle\mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle e^j - \left\langle\mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle e_j\right)\\ &+ \mathcal{D}\left\langle e^i, \left[e^j, e_j\right]\right\rangle) - \lambda\chi\left\langle\left[e^j, e_j\right], e^i\right\rangle. \end{aligned} \tag{C.4d}$$

Proof. By antisymmetry of the Λ-bracket, we have that

$$\left[H_{0\Lambda} e_i\right] = -\left[e_{i-\Lambda-\nabla} H_0\right], \quad \text{for } i \in \{1, \ldots, n\}.$$

Hence, fixed $i \in \{1, \ldots, n\}$, we need to compute

$$\left[e_{i\Lambda} H_0\right] = \frac{1}{k}\Upsilon_i^1 + \frac{1}{k^2}\Upsilon_i^2,$$

for which we will use the expression for H_0 in Lemma C.2.1. We compute first

$$\Upsilon_i^1 := \left[e_{i\Lambda}\left(: e_j\left(Se^j\right) : + : e^j\left(Se_j\right) :\right)\right] = \Upsilon_i^{1,1} + \Upsilon_i^{1,2}.$$

Applying the non-commutative Wick formula once on each summand, by sesquilinearity,

$$\begin{aligned}\Upsilon_i^{1,1} &:= \left[e_{i\Lambda} : e_j\left(Se^j\right) :\right]\\ &=: \left[e_i, e_j\right]_l\left(Se^j\right) : + : e_j\left(S\left[e_i, e^j\right]\right) : -\chi : e_j\left[e_i, e^j\right] :\\ &+ \lambda\left(ke_i + \left[\left[e_i, e_j\right]_l, e^j\right] + \mathcal{D}\left\langle e_i, \left[e^j, e_j\right]\right\rangle\right),\\ \Upsilon_i^{1,2} &:= \left[e_{i\Lambda} : e^j\left(Se_j\right) :\right]\\ &=: \left[e_i, e^j\right]\left(Se_j\right) : + : e^j\left(S\left[e_i, e_j\right]_l\right) : -\chi : e^j\left[e_i, e_j\right]_l : + k\chi Se_i\\ &+ \lambda\left(\left[\left[e_i, e^j\right], e_j\right] + \mathcal{D}\left\langle e_i, \left[e_j, e^j\right]\right\rangle\right), \end{aligned}$$

where we have used the involutivity of l. Then,

$$\begin{aligned}\Upsilon_i^1 =&: \left[e_i, e^j\right]_-\left(Se_j\right) : + : e_j\left(S\left[e_i, e^j\right]_-\right) : + : e^j : e^k\left(\left\langle\mathcal{D}\left\langle\left[e_i, e_j\right], e^n\right\rangle, e_k\right\rangle e_n\right) ::\\ &+ 2 : \left(T\left\langle e^k, \left[e_i, e_j\right]\right\rangle\right)\left[e_k, e^j\right] : + : \left(T\left\langle\left[e^k, e_i\right], e^j\right\rangle\right)\left[e_k, e_j\right]_l :\\ &+ T\left(: e_j\left(\left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e_k\right\rangle\right\rangle - \left\langle\mathcal{D}\left\langle e^j, \left[e_i, e_k\right]\right\rangle, e^k\right\rangle\right)\right.\\ &\left.+ : e^j\left(\left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle\right) :\right) + \frac{1}{k} : e_j : \left(\mathcal{D}\left\langle\left[e^k, e_i\right], e^j\right\rangle\right)_l e_k ::\\ &+ : e_j\left(: e_k\left(\frac{1}{2}\left\langle\mathcal{D}\left\langle\left[e^j, e_i\right], e^k\right\rangle, e_n\right\rangle + \left\langle\mathcal{D}\left\langle\left[e_i, e^j\right], e_n\right\rangle, e^k\right\rangle\right) e^n :\right) :\\ &+ \frac{1}{2} : e_j : \left(\mathcal{D}\left\langle\left[e^k, e_i\right], e^j\right\rangle\right)_- e_k :: + : e_k : \left(\left\langle\mathcal{D}\left[e_i, e^k\right], e_j\right\rangle\right)_- e^j ::\\ &+ \chi\left(kSe_i - : e_j\left[e_i, e^j\right] : - : e^j\left[e_i, e_j\right] :\right) + \lambda\left(ke_i + \left[\left[e_i, e_j\right], e^j\right] + \left[\left[e_i, e^j\right], e_j\right]\right). \end{aligned}$$

Here, we have used Lemma B.5.13. Concretely, the identity (B.44). Next, we compute

$$\begin{aligned}\Upsilon_i^2 :&= \left[e_{i\Lambda}\left(: e_j : e_k\left[e^j, e^k\right]_{\bar{l}} :: + : e^j : e^k\left[e_j, e_k\right]_l :: +2 : e_j : e^k\left[e^j, e_k\right]_- ::\right)\right] \\ &= \Upsilon_i^{2,1} + \Upsilon_i^{2,2} + \Upsilon_i^{2,3}.\end{aligned}$$

Applying the non-commutative Wick formula twice on each summand,

$$\begin{aligned}\Upsilon_i^{2,1} :&= \left[e_{i\Lambda} : e_j : e_k\left[e^j, e^k\right]_{\bar{l}} ::\right] \\ &=: \left[e_i, e_j\right]_l : e_k\left[e^j, e^k\right]_{\bar{l}} :: + : e_j : \left[e_i, e_k\right]_l\left[e^j, e^k\right]_{\bar{l}} :: + : e_j : e_k\left[e_i, \left[e^j, e^k\right]_{\bar{l}}\right] :: \\ &+ k\chi : e_j\left[e_i, e^j\right]_l : + k\lambda\left(2\left[e^j, \left[e_i, e_j\right]_l\right]_l + 2\left\langle\mathcal{D}\left\langle e^j, \left[e_k, e_i\right]\right\rangle, e^k\right\rangle e_j\right), \\ \Upsilon_i^{2,2} :&= \left[e_{i\Lambda} : e^j : e^k\left[e_j, e_k\right]_l ::\right] \\ &=: \left[e_i, e^j\right] : e^k\left[e_j, e_k\right]_l :: + : e^j : \left[e_i, e^k\right]\left[e_j, e_k\right]_l :: + : e^j : e^k\left[e_i, \left[e_j, e_k\right]_l\right]_l :: \\ &+ 2k\chi : e^j\left[e_i, e_j\right]_l : + k\lambda\left(2\left[e_j, \left[e_i, e^j\right]_{\bar{l}}\right]_{\bar{l}} + \left[e_k, \left[e_i, e^k\right]_l\right]_l\right. \\ &\left.+ 2\left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle e^j + \left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e^j\right\rangle, e_k\right\rangle e_j\right), \\ \Upsilon_i^{2,3} :&= \left[e_{i\Lambda} : e_j : e^k\left[e^j, e_k\right]_- ::\right] \\ &=: \left[e_i, e_j\right]_l : e^k\left[e^j, e_k\right]_- :: + : e_j : \left[e_i, e^k\right]\left[e^j, e_k\right]_- :: + : e_j : e^k\left[e_i, \left[e^j, e_k\right]_-\right] :: \\ &+ k\chi : e_j\left[e_i, e^j\right]_- : + k\lambda\left(\left[e_j, \left[e_i, e^j\right]_-\right]_l + \left[e^j, \left[e_i, e_j\right]_l\right]_-\right),\end{aligned}$$

where we have used the involutivity of $l, \bar{l}$ and Courant algebroid axioms. Then,

$$\begin{aligned}\Upsilon_i^2 =&: e_j : \left[e_k, e^j\right]_l\left[e_i, e^k\right]_l :: +4 : \left(T\left\langle\left[e^j, e_i\right], e_k\right\rangle\right)\left[e^k, e_j\right] : \\ &+ 2 : \left(T\left\langle\left[e^j, e_i\right], e^k\right\rangle\right)\left[e_k, e_j\right]_l : -2 : e^j : e^k\left(\left\langle\mathcal{D}\left\langle e^m, \left[e_i, e_j\right]\right\rangle, e_k\right\rangle e_m\right) :: \\ &+ : e_j : e_k\left(\left(\left\langle\mathcal{D}\left\langle e^k, \left[e_i, e^j\right]\right\rangle, e_m\right\rangle - 2\left\langle\mathcal{D}\left\langle e_m, \left[e_i, e^j\right]\right\rangle, e^k\right\rangle\right)e^m\right) :: \\ &+ : e_j : \left(\mathcal{D}\left\langle\left[e_i, e^k\right], e^j\right\rangle\right)_- e_k :: -2 : e_k : \left(\mathcal{D}\left\langle\left[e_i, e^k\right], e_j\right\rangle\right)_- e^j :: \\ &+ 2\left(: e_j : e_k\left[e^j, \left[e_i, e^k\right]_-\right] :: + : e_j : e^k\left[e_k, \left[e^j, e_i\right]_-\right]_- ::\right. \\ &\left.+ : e_j : \left[e_i, e^k\right]_-\left[e^j, e_k\right]_- :: + : e_j : e^k\left[e_i, \left[e^j, e_k\right]_-\right]_l ::\right) \\ &+ k\chi\left(: e_j\left[e_i, e^j\right]_l : +2 : e^j\left[e_i, e_j\right]_l : + : e_j\left[e_i, e^j\right]_- :\right) \\ &+ k\lambda\left(2\left[e^j, \left[e_i, e_j\right]_l\right]_l + 2\left\langle\mathcal{D}\left\langle e^j, \left[e_k, e_i\right]\right\rangle, e^k\right\rangle e_j + 2\left[e_j, \left[e_i, e^j\right]_{\bar{l}}\right]_{\bar{l}}\right. \\ &+ \left[e_k, \left[e_i, e^k\right]_l\right]_l + 2\left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle e^j + \left\langle\mathcal{D}\left\langle\left[e^k, e_i\right], e^j\right\rangle, e_k\right\rangle e_j \\ &\left.+ \left[e_j, \left[e_i, e^j\right]_-\right]_l + \left[e^j, \left[e_i, e_j\right]_l\right]_-\right).\end{aligned}$$

Here, we have used Lemma B.5.13. Concretely, the identities (B.45) and (B.46). Now, note that using (B.20) and (B.13),

$$k\partial_\chi\left(\frac{1}{k}\Upsilon_i^1+\frac{1}{k^2}\Upsilon_i^2\right)=kSe_i+:e_j\left[e_i,e^j\right]_-:.$$

Analogously, using the Courant algebroid axioms,

$$\begin{aligned}k\partial_\lambda\left(\frac{1}{k}\Upsilon_i^1+\frac{1}{k^2}\Upsilon_i^2\right)&=ke_i+\left[e_j,\left[e_i,e^j\right]_-\right]_l-\left[e_j,\left[e_i,e^j\right]_-\right]_-\\&+\left(\left\langle\mathcal{D}\left\langle e^j,\left[e_k,e_i\right]\right\rangle,e^k\right\rangle+\left\langle\mathcal{D}\left\langle\left[e^k,e_i\right],e^j\right\rangle,e_k\right\rangle\right)e_j\\&+\left\langle\mathcal{D}\left\langle\left[e^k,e_i\right],e_j\right\rangle,e_k\right\rangle e^j.\end{aligned}$$

In summary, using (B.33), (B.43) and the Courant algebroid axioms, we obtain that

$$\begin{aligned}\left[e_{i\Lambda}H_0\right]&=\frac{1}{k}\Upsilon_i^1+\frac{1}{k^2}\Upsilon_i^2\\&=\left(\chi S+\lambda\right)e_i+\frac{\chi}{k}:e_j\left[e_i,e^j\right]_-:+\frac{\lambda}{2}\left(\left[e_j,\left[e_i,e^j\right]_-\right]_l-\left[e_j\left[e_i,e^j\right]_-\right]_-\right)\\&+\frac{(\lambda+T)}{2}\left(\left(\left\langle\mathcal{D}\left\langle\left[e^k,e_i\right],e^j\right\rangle,e_k\right\rangle-\left\langle\mathcal{D}\left\langle e^j,\left[e_i,e_k\right]\right\rangle,e^k\right\rangle\right)e_j\right.\\&\left.+\left\langle\mathcal{D}\left\langle\left[e^k,e_i\right],e_j\right\rangle,e_k\right\rangle e^j\right)+\frac{1}{k}\left(:\left[e_i,e^j\right]_-(Se_j):+:e_j\left(S\left[e_i,e^j\right]_-\right):\right)\\&+\frac{2}{k^2}\left(:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]_-::+:e_j:e_k\left[\left[e_i,e^j\right]_-,e^k\right]_{\bar{l}}::\right.\\&+:e_j:e^k\left[e_k,\left[e^j,e_i\right]_-\right]_-::+:e_j:\left[e_i,e^k\right]_-\left[e^j,e_k\right]::\\&\left.+:e_j:e^k\left[e_i,\left[e^j,e_k\right]_-\right]_l::\right).\end{aligned}$$

Before continuing, using (B.14) and (B.30), notice that

$$\begin{aligned}:e_j:e_k\left[e^j,\left[e_i,e^k\right]_-\right]::&=:e_j:e_k\left[\left[e_i,e^j\right]_-,e^k\right]::,\\:e_j:e^k\left[e_k,\left[e^j,e_i\right]_-\right]_-::&=:e^k:e_j\left[\left[e^k,e_i\right]_-,e_j\right]_-::,\\:e_j:\left[e_i,e^k\right]_-\left[e^j,e_k\right]_-::&=:\left[e_i,e^j\right]_-:e_k\left[e_j,e^k\right]_-::.\end{aligned}$$

So, using Courant algebroid axioms, that S is an odd derivation for the normally ordered product and (B.15), we arrive at (C.4a). Now, by sesquilinearity and (B.10), we compute

$$\begin{aligned}\left[Tw_\Lambda e_i\right]&=-\lambda\left[w_\Lambda e_i\right]\\&=-\lambda\left(\left\langle\left[e^j,e_j\right],e^m\right\rangle\left[e_m,e_i\right]_l-\left\langle\left[e^j,e_j\right],e_m\right\rangle\left[e^m,e_i\right]\right)\\&+\lambda\left(\left\langle\mathcal{D}\left\langle e^j,\left[e^k,e_k\right]\right\rangle,e_i\right\rangle e_j-\left\langle\mathcal{D}\left\langle e_j,\left[e^k,e_k\right]\right\rangle,e_i\right\rangle e^j\right.\\&\left.+\mathcal{D}\left\langle e_i,\left[e^j,e_j\right]\right\rangle\right)+\lambda\chi\left\langle\left[e^j,e_j\right],e_i\right\rangle.\end{aligned}$$

So, we arrive at (C.4c). Since we can exchange the roles played by l and $\bar{l}$, it suffices to check (C.4a) and (C.4c), as we have done, to obtain (C.4b) and (C.4d). Now, note that

$$\begin{aligned}
[e_j, e_i] &= \left\langle e^k, [e_j, e_i] \right\rangle e_k, \\
\left[e^j, e_i\right] &= \left\langle e^k, \left[e^j, e_i\right] \right\rangle e_k + \left\langle e_k, \left[e^j, e_i\right] \right\rangle e^k + \left[e^j, e_i\right]_- , \\
\left[e^j, e^i\right] &= \left\langle e_k, \left[e^j, e^i\right] \right\rangle e^k, \\
\left[e_j, e^i\right] &= \left\langle e^k, \left[e_j, e^i\right] \right\rangle e_k + \left\langle e_k, \left[e_j, e^i\right] \right\rangle e^k + \left[e_j, e^i\right]_- ,
\end{aligned}$$

for $i, j \in \{1, \dots, n\}$, which will be useful in the future. □

Remark C.2.4. If $E = (\mathfrak{g}, (\cdot|\cdot))$ is a quadratic Lie algebra, for any $a = a_l + a_{\bar{l}} \in l \oplus \bar{l}$,

$$\begin{aligned}
[H_{0\Lambda} a_l] = &-\frac{2}{k^2} \Big(: e_j : e_k \left[\left[a_l, e^j\right]_- , e^k \right]_{\bar{l}} :: \\
&+ : e_j : e_k \left[\left[a_l, e^j\right]_- , e^k \right]_- :: + : \left[a_l, e^j\right]_- : e_k \left[e_j, e^k\right]_- :: \\
&+ : e_j : e^k \left[a_l, \left[e^j, e_k\right]_- \right]_l :: + : e^j : e_k \left[\left[e^k, a_l\right]_- , e_j \right]_- :: \Big) \\
&+ \frac{1}{k} \Big(\chi : e_j \left[a_l, e^j\right]_- : -2 : e_j \left(S \left[a_l, e^j\right]_- \right) : +2T \left[e_j, \left[a_l, e^j\right]_- \right]_l \\
&+ \lambda \left(\left[e_j, \left[a_l, e^j\right]_- \right]_l + \left[\left[a_l, e^j\right]_- , e_j \right]_- \right) \Big) + (\lambda + 2T + \chi S) \, a_l, \\
\left[H_{0\Lambda} a_{\bar{l}}\right] = &-\frac{2}{k^2} \Big(: e^j : e^k \left[\left[a_{\bar{l}}, e_j\right]_- , e_k \right]_l :: \\
&+ : e^j : e^k \left[\left[a_{\bar{l}}, e_j\right]_- , e_k \right]_- :: + : \left[a_{\bar{l}}, e_j\right]_- : e^k \left[e^j, e_k\right]_- :: \\
&+ : e^j : e_k \left[a_{\bar{l}}, \left[e_j, e^k\right]_- \right]_{\bar{l}} :: + : e_j : e^k \left[\left[e_k, a_{\bar{l}}\right]_- , e^j \right]_- :: \Big) \\
&+ \frac{1}{k} \Big(\chi : e^j \left[a_{\bar{l}}, e_j\right]_- : -2 : e^j \left(S \left[a_{\bar{l}}, e_j\right]_- \right) : +2T \left[e^j, \left[a_{\bar{l}}, e_j\right]_- \right]_{\bar{l}} \\
&+ \lambda \left(\left[e^j, \left[a_{\bar{l}}, e_j\right]_- \right]_{\bar{l}} + \left[\left[a_{\bar{l}}, e_j\right]_- , e^j \right]_- \right) \Big) + (\lambda + 2T + \chi S) \, a_{\bar{l}}, \\
\left[H'{}_\Lambda a_l\right] = &\, [H_{0\Lambda} a_l] + \frac{\lambda}{k} [a_l, w] - \lambda \chi \, (a_l | w) \, , \\
\left[H'{}_\Lambda a_{\bar{l}}\right] = &\, \left[H_{0\Lambda} a_{\bar{l}}\right] + \frac{\lambda}{k} \left[a_{\bar{l}}, w\right] - \lambda \chi \left(a_{\bar{l}} | w\right) .
\end{aligned}$$

Now, define for each $i, j \in \{1, \dots, n\}$ the locally defined sections

$$\begin{aligned}
R :&= 3 \left\langle \left[e^j, e_j\right]_- , \left[e^k, e_k\right]_- \right\rangle - \left\langle \left[e^j, e_k\right]_- , \left[e^k, e_j\right]_- \right\rangle , \\
F^{ij} :&= \operatorname{tr}|_{\bar{l}} \left(\operatorname{ad}_{[e^i, e^j]} \right) + \left\langle \mathcal{D} \left\langle e^i, \left[e_k, e^k\right] \right\rangle , e^j \right\rangle - \left\langle \mathcal{D} \left\langle e^j, \left[e_k, e^k\right] \right\rangle , e^i \right\rangle , \\
F_{ij} :&= \operatorname{tr}|_l \left(\operatorname{ad}_{[e_i, e_j]} \right) + \left\langle \mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle , e_j \right\rangle - \left\langle \mathcal{D} \left\langle e_j, \left[e^k, e_k\right] \right\rangle , e_i \right\rangle .
\end{aligned}$$

Proposition C.2.5. *One has*

$$\begin{aligned}\left[{H'}_{\Lambda} J_0\right] &= (2\lambda + 2T + \chi S)\left(J_0 - \frac{i}{k} S \sum_{j=1}^{n} \left[e^j, e_j\right]_-\right)\\ &+ \frac{i}{2k} TS\mathcal{D}R + \frac{i}{k^2}\lambda \sum_{j=1}^{n}\sum_{i=1}^{n} \left(: F^{ij} : e_j e_i :: - : F_{ij} : e^j e^i ::\right)\\ &- \frac{i}{k}\left(T + \frac{3}{k}\lambda\right)\sum_{j=1}^{n}\sum_{i=1}^{n}\left(: e^i \left[\left[e^j, e_j\right]_-, e_i\right]_- : + : e_i \left[\left[e^j, e_j\right]_-, e^i\right]_- :\right).\end{aligned}$$

Proof. By the non-commutative Wick formula, it suffices to calculate

$$\begin{aligned}\left[{H'}_{\Lambda} : e^i e_i :\right] &=: \left[{H'}_{\Lambda} e^i\right] e_i : + : e^i \left[{H'}_{\Lambda} e_i\right] : + \int_0^{\Lambda} d\Gamma \left[\left[{H'}_{\Lambda} e^i\right]_{\Gamma} e_i\right]\\ &= P + I,\end{aligned}$$

where I is the third summand of the right-hand side in the first line. Firstly, we compute

$$I := \int_0^{\lambda} d\lambda \left(I_1 + I_2 + I_3 + I_4\right),$$

where

$$\begin{aligned} I_1 &:= \partial_\eta \left[A^i{}_{\Gamma} e_i\right], \quad I_2 := \partial_\eta \left[B^i{}_{\Gamma} e_i\right],\\ I_3 &:= \partial_\eta \left[C^i{}_{\Gamma} e_i\right], \quad I_4 := \partial_\eta \left[D^i{}_{\Gamma} e_i\right],\end{aligned}$$

are given by

$$\begin{aligned} A^i :&= \frac{\chi}{k} : e^j \left[e^i, e_j\right]_- : - : e^j \left(S \left[e^i, e_j\right]_-\right) :\\ &- \frac{2}{k^2}\left(: e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right] :: + : \left[e^i, e_j\right]_- : e^k \left[e^j, e_k\right]_- ::\right.\\ &\left. + : e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right]_{\bar{l}} :: + : e_j : e^k \left[\left[e_k, e^i\right]_-, e^j\right]_- ::\right)\\ B^i :&= \frac{\lambda}{k}\left(\left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}} + \left[\left[e^i, e_j\right]_-, e^j\right]_-\right) + T\left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}},\\ C^i :&= (\lambda + 2T + \chi S)\, e^i,\\ D^i :&= \lambda\chi \left\langle \left[e_j, e^j\right], e^i\right\rangle\\ &+ \frac{\lambda}{k}\left(\left(\left\langle e^m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e_m\right], e^j\right\rangle + \left\langle \mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle\right.\right.\\ &\left. + \left\langle \mathcal{D}\left\langle e_k, \left[e^i, e^j\right]\right\rangle, e^k\right\rangle\right) e_j + \left(\left\langle e^m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e_m\right], e_j\right\rangle\right.\\ &+ \left\langle \mathcal{D}\left\langle e_k, \left[e^i, e_j\right]\right\rangle, e^k\right\rangle - \left\langle \mathcal{D}\left\langle e_j, \left[e^i, e^k\right]\right\rangle, e_k\right\rangle\\ &\left. - \left\langle e_m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e^m\right], e_j\right\rangle - \left\langle \mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle\right) e^j\\ &\left. + \left\langle e^k, \left[e^j, e_j\right]\right\rangle \left[e^i, e_k\right]_- - \mathcal{D}\left\langle e^i, \left[e^j, e_j\right]\right\rangle\right. .\end{aligned}$$

We proceed as follows. By antisymmetry of the Λ-bracket, we can write

$$\begin{aligned} I_1 &= -\partial_\eta \left[e_{i-\Gamma-\nabla} A^i\right] = \partial_\eta \left[e_{i\Gamma} A^i\right] \\ &= \frac{1}{k}\left(I_1^1 - 2I_1^2\right) - \frac{2}{k^2}\left(I_1^3 + I_1^4 + I_1^5 + I_1^6\right), \end{aligned}$$

where, by the non-commutative Wick formula, the Courant algebroid axioms, (B.14) and (B.37),

$$\begin{aligned} I_1^1 &:= \partial_\eta \left[e_{i\Gamma}\left(\chi : e^j \left[e^i, e_j\right]_- :\right)\right] \\ &= k\chi \left[e_j, e^j\right]_-, \\ I_1^2 &:= \partial_\eta \left[e_{i\Gamma} : e^j \left(S \left[e^i, e_j\right]_-\right) :\right] \\ &= k \left[e^j, e_j\right]_- - : e^j \left[e_i, \left[e^i, e_j\right]_-\right] :, \\ I_1^3 &:= \partial_\eta \left[e_{i\Gamma} : e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right] ::\right] \\ &= k\left(: e^k \left[\left[e^j, e_j\right]_-, e_k\right] : - : e^j \left[\left[e^k, e_j\right]_-, e_k\right] :\right), \\ I_1^4 &:= \partial_\eta \left[e_{i\Gamma} : \left[e^i, e_j\right]_- : e^k \left[e^j, e_k\right]_- ::\right] \\ &= -2T \left\langle \left[e^k, e_j\right]_-, \left[e^j, e_k\right]_- \right\rangle, \\ I_1^5 &:= \partial_\eta \left[e_{i\Gamma} : e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right]_{\bar{l}} ::\right] \\ &= k\left(: e_k \left[e^j, \left[e_j, e^k\right]_-\right]_{\bar{l}} : + : e^k \left[e_k, \left[e^j, e_j\right]_-\right]_l :\right), \\ I_1^6 &:= \partial_\eta \left[e_{i\Gamma} : e_j : e^k \left[\left[e_k, e^i\right]_-, e^j\right]_- ::\right] \\ &= k : e_j \left[\left[e^k, e_k\right]_-, e^j\right]_- : . \end{aligned}$$

Combining the previous expressions, using the F-term conditions (in particular, Remark 6.3.15 that is equivalent to the weaker variant of the F-term condition (6.27)), then

$$\begin{aligned} I_1 = T &\left\langle \left[e^k, e_j\right]_-, \left[e^j, e_k\right]_- \right\rangle + (\chi + 2S)\left[e_i, e^j\right]_- \\ &+ : e_k \left[\left[e_j, e^j\right]_-, e^k\right]_- : + : e^k \left[\left[e_j, e^j\right]_-, e_k\right]_- : - : e_k \left[e^j, \left[e_j, e^k\right]_-\right]_{\bar{l}} : . \end{aligned}$$

Using now sesquilinearity, the non-commutative Wick formula and the Courant algebroid axioms,

$$\begin{aligned} I_2 &= (\lambda - 2\gamma)\left\langle \left[e^j, \left[e^i, e_j\right]_-\right], e_i \right\rangle, \\ I_3 &= k(\lambda - 2\gamma)\left\langle e^j, e_j \right\rangle + \chi\left[e^j, e_j\right], \\ I_4 &= 0. \end{aligned}$$

In conclusion, we have arrived at the formula

$$\begin{aligned} I = \lambda T \left\langle \left[e^k, e_j\right]_-, \left[e^j, e_k\right] \right\rangle + \lambda\chi \left[e^j, e_j\right]_+ - \lambda : e_k \left[e^j, \left[e_j, e^k\right]_-\right]_{\bar{l}} : \\ + 2\lambda S \left[e_j, e^j\right]_- + \lambda : e_k \left[\left[e_j, e^j\right]_-, e^k\right]_- : + \lambda : e^k \left[\left[e_j, e^j\right]_-, e_k\right]_- : . \end{aligned}$$

Next, we compute

$$P := P_1 + P_2 + P_3$$

in several steps, where

$$\begin{aligned} P_1 &:=: a^j e_j : + : e^j a_j :, \\ P_2 &:=: b^j e_j : + : e^j b_j :, \\ P_3 &:=: c^j e_j : + : e^j c_j :, \end{aligned}$$

and

$$\begin{aligned} c^j &:= \lambda\partial_\lambda \left[H'{}_\Lambda e^j\right] - \lambda e^j, & c_j &:= \lambda\partial_\lambda \left[H'{}_\Lambda e_j\right] - \lambda e_j, \\ b^j &:= (\lambda + 2T + \chi S)\, e^j, & b_j &:= (\lambda + 2T + \chi S)\, e_j, \\ a^j &:= \left[H'{}_\Lambda e^j\right] - b^j - c^j, & a_j &:= \left[H'{}_\Lambda e_j\right] - b_j - c_j. \end{aligned}$$

First, we can compute directly

$$P_1 = -\frac{2}{k^2} Q + \frac{\chi}{k} W + \frac{\lambda}{k} X + \frac{2}{k} Y - \frac{2}{k} Z,$$

where

$$\begin{aligned} Q &:= Q'_1 + Q'_2 + Q'_3 + Q'_4 + Q'_5, \\ W &:=:: e^j \left[e^i, e_j\right]_- : e_i : - : e^i : e_j \left[e_i, e^j\right]_- ::, \\ X &:=: e^i \left[e_j, \left[e_i, e^j\right]_-\right]_l : + : e^i \left[\left[e_i, e^j\right]_-, e_j\right]_- : \\ &\quad + : \left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}} e_i : + : \left[\left[e^i, e_j\right]_-, e^j\right]_- e_i :, \\ Y &:=: e^i \left(T \left[e_j, \left[e_i, e^j\right]_-\right]_l\right) : + : \left(T \left[e^j, \left[e^i, e_j\right]_-\right]_{\bar{l}}\right) e_i :, \\ Z &:=: e^i : e_j \left(S \left[e_i, e^j\right]_-\right) :: + :: e^j \left(S \left[e^i, e_j\right]_-\right) : e_i :, \end{aligned}$$

and

$$\begin{aligned} Q'_1 &:=:: e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right]_l :: e_i : + :: e^j : e_k \left[e^i, \left[e_j, e^k\right]_-\right]_{\bar{l}} :: e_i :, \\ Q'_2 &:=: e^i : e_j : e_k \left[\left[e_i, e^j\right]_-, e^k\right]_{\bar{l}} ::: + : e^i : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right]_l :::, \\ Q'_3 &:=: e^i : \left[e_i, e^j\right]_- : e_k \left[e_j, e^k\right]_- ::: + :: \left[e^i, e_j\right]_- : e^k \left[e^j, e_k\right]_- :: e_i :, \\ Q'_4 &:=: e^i : e^j : e_k \left[\left[e^k, e_i\right]_-, e_j\right]_- ::: + :: e^j : e^k \left[\left[e^i, e_j\right]_-, e_k\right]_- :: e_i :, \\ Q'_5 &:=: e^i : e_j : e_k \left[\left[e_i, e^j\right]_-, e^k\right]_- ::: + :: e_j : e^k \left[\left[e_k, e^i\right]_-, e^j\right]_- :: e_i : . \end{aligned}$$

Using Courant algebroid axioms, (B.13), (B.14), (B.20) and (B.29), we obtain that

$$
\begin{aligned}
Q_1' :=&:: e_j : e^k \left[e^i, \left[e^j, e_k\right]_-\right]_{\bar{l}} :: e_i : + :: e^k : \left[e^i, \left[e^j, e_k\right]_-\right]_{\bar{l}} e_i :: e_j : \\
&+ 2 :: e^j \left(T \left\langle \left[e_j, e^i\right]_- , \left[e_k, e^k\right]_- \right\rangle\right) : e_i : \\
&=: Q_1 + 2 :: e^j \left(T \left\langle \left[e_j, e^i\right]_- , \left[e_k, e^k\right]_- \right\rangle\right) : e_i :, \\
Q_2' :=&: e^i : e_j : e^k \left[e_i, \left[e^j, e_k\right]_-\right]_l ::: + : e^k : \left[e_i, \left[e^j, e_k\right]_-\right]_l : e_j e^i ::: \\
&+ 2 : e^j : \left(T \left\langle \left[e_k, e^k\right]_- , \left[e_j, e^i\right]_- \right\rangle\right) e_i :: \\
&=: Q_2 + 2 : e^j : \left(T \left\langle \left[e_k, e^k\right]_- , \left[e_j, e^i\right]_- \right\rangle\right) e_i ::, \\
Q_3' :=&:: \left[e^i, e_j\right]_- : e^k \left[e^j, e_k\right]_- :: e_i : + : e^k : \left[e^j, e_k\right]_- : e_i \left[e^i, e_j\right]_- ::: \\
&=: Q_3, \\
Q_4' :=&:: e^j : e^k \left[\left[e^i, e_j\right]_- , e_k\right]_- :: e_i : + : e^j : e^k : e_i \left[\left[e^i, e_j\right]_- , e_k\right]_- ::: \\
&=: Q_4, \\
Q_5' :=&:: e^j : e_k \left[\left[e^i, e_j\right]_- , e^k\right]_- :: e_i : + : e^j : e_i : e_k \left[\left[e_j, e^i\right]_- , e^k\right]_- ::: \\
&=: Q_5.
\end{aligned}
$$

So, applying Courant algebroid axioms and (B.22), we obtain that

$$
\begin{aligned}
Q &= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \\
&+ 4 \left(:: e^j \left(T \left\langle \left[e_j, e^i\right]_- , \left[e_k, e^k\right]_- \right\rangle\right) : e_i : - T^2 \left\langle \left[e_j, e^j\right]_- , \left[e_k, e^k\right]_- \right\rangle \right).
\end{aligned}
$$

We will compute Q_1, Q_2, Q_3, Q_4, Q_5 independently. First, by Courant algebroid axioms, (B.16), (B.28), (B.35), (B.36) and since T is an even derivation for the normally ordered product, we obtain that

$$
\begin{aligned}
Q_1 = k \Big(&:: \left(T \left\langle e_k, \left[e^j, \left[e^i, e_i\right]_-\right]\right\rangle\right) e^k : e_j : \\
&+ T^2 \left(\left\langle \left[e^k, e_j\right]_- , \left[e^j, e_k\right]_- \right\rangle - \left\langle \left[e^j, e_j\right]_- , \left[e^k, e_k\right]_- \right\rangle \right) \Big).
\end{aligned}
$$

Now, by Courant algebroid axioms, (B.14), (B.15), (B.16), (B.20), (B.29), (B.35), (B.36), and since T is an even derivation for the normally ordered product, we obtain that

$$
Q_2 = k :: \left(T \left\langle e_k, \left[e^j, \left[e^i, e_i\right]_-\right]\right\rangle\right) e^k : e_j .
$$

Now, applying Courant algebroid axioms, (B.14), (B.27), (B.28) (B.35) and since T is an even derivation for the normally ordered product, we obtain that

$$
Q_3 = k \left(T^2 \left\langle \left[e^j, e_k\right]_- , \left[e^k, e_j\right]_- \right\rangle + :: \left(T \left\langle \left[e^j, \left[e^i, e_j\right]_-\right], e_k \right\rangle\right) e^k : e_i : \right).
$$

By the Courant algebroid axioms, (B.14), (B.28) and (B.35), we obtain that

$$Q_4 = kT\left(: e^k \left[\left[e^i, e_i \right]_- , e_k \right]_- : - : e^j \left[\left[e^k, e_j \right]_- , e_k \right]_- : \right).$$

Finally, by the Courant algebroid axioms, (B.28), (B.31) and (B.35), we obtain that

$$Q_5 = kT\left(: e_k \left[\left[e^i, e_i \right]_- , e^k \right]_- : \right).$$

In summary, by the Courant algebroid axioms and (B.13), we obtain that

$$\begin{aligned} Q &= 4T^2 \left\langle \left[e^j, e_k \right]_- , \left[e^k, e_j \right]_- \right\rangle - 6T^2 \left\langle \left[e^j, e_j \right]_- , \left[e^k, e_k \right]_- \right\rangle \\ &+ 2 :: \left(T \left\langle \left[e^j, \left[e^i, e_j \right]_- \right], e_k \right\rangle \right) e^k : e_i : \\ &+ 2T \left(: e^k \left[\left[e^i, e_i \right]_- , e_k \right]_- : + : e_k \left[\left[e^i, e_i \right]_- , e^k \right]_- : - : e^j \left[\left[e^k, e_j \right]_- , e_k \right]_- : \right). \end{aligned}$$

We compute now the other terms. By the Courant algebroid axioms, (B.14) and (B.28),

$$W = -kT \left[e^j, e_j \right]_- .$$

Now, by the Courant algebroid axioms, (B.14) and (B.20),

$$\begin{aligned} X &= 2 : e_j \left[e^k, \left[e_k, e^j \right]_- \right]_{\bar{l}} : + : e^i \left[\left[e_i, e^j \right]_- , e_j \right]_- : + : \left[\left[e^i, e_j \right]_- , e^j \right]_- e_i : \\ &+ 2T \left\langle e_j, \left[e^k, \left[e^j, e_k \right]_- \right] \right\rangle . \end{aligned}$$

Now, applying the Courant algebroid axioms, (B.23) and since T is an even derivation for the normally ordered product,

$$\begin{aligned} Y &= T\left(: e^i \left[e_j, \left[e_i, e^j \right]_- \right]_l : \right) + :: \left(T \left\langle e_k, \left[e^j, \left[e^i, e_j \right]_- \right] \right\rangle \right) e^k : e_i : \\ &+ T^2 \left\langle \left[e^j, e_k \right]_- , \left[e^k, e_j \right]_- \right\rangle . \end{aligned}$$

At last, by the Courant algebroid axioms, (B.15) and since T is an even derivation for the normally ordered product,

$$Z = T\left(: e^i \left[\left[e^j, e_i \right]_- , e_j \right] : \right) + 2TS \left[e^j, e_j \right]_- .$$

In conclusion, by the Courant algebroid axioms, we have arrived at

$$\begin{aligned} P_1 &= T\left(2S + \chi\right) \left[e_j, e^j \right]_- + T^2 R - \lambda T \left\langle \left[e^k, e_j \right]_- , \left[e^j, e_k \right]_- \right\rangle \\ &- T\left(: e^k \left[\left[e^j, e_j \right]_- , e_k \right]_- : + : e_k \left[\left[e^j, e_j \right]_- , e^k \right]_- : \right) \\ &+ \frac{\lambda}{k} \left(2 : e_k \left[e^j, \left[e_j, e^k \right]_- \right]_{\bar{l}} : + : e^k \left[\left[e_k, e^j \right]_- , e_j \right]_- : + : \left[\left[e^k, e_j \right]_- , e^j \right]_- e_k : \right). \end{aligned}$$

On the other hand, the calculation of P_2 is immediate because S is an odd derivation for the normally ordered product (so T is an even derivation). Indeed, we have that

$$P_2 = (2\lambda + 2T + \chi S) : e^j e_j : .$$

To finish, we must compute the last term. We have that

$$P_3 = \lambda\chi A + \frac{\lambda}{k} B,$$

where

$$\begin{aligned} A &: = - : e^i \left(\left\langle \left[e^j, e_j\right], e_i\right\rangle\right) : - : \left(\left\langle \left[e^j, e_j\right], e^i\right\rangle\right) e_i :, \\ B &: = B_1 + B_1' + B_2 + B_2' + B_3 + B_3' + B_4 + B_4' - B_5 - B_5', \end{aligned}$$

and

$$\begin{aligned} B_1 &: =: \left(\left\langle \mathcal{D}\left\langle e_k, \left[e^i, e_j\right]\right\rangle, e^k\right\rangle e^j\right) e_i : - : e^i \left(\left\langle \mathcal{D}\left\langle e^j, \left[e_i, e_k\right]\right\rangle, e^k\right\rangle e_j\right) :, \\ B_1' &: =: e^i \left(\left\langle \mathcal{D}\left\langle \left[e^k, e_i\right], e^j\right\rangle, e_k\right\rangle e_j\right) : - : \left(\left\langle \mathcal{D}\left\langle e_j, \left[e^i, e^k\right]\right\rangle, e_k\right\rangle e^j\right) e_i :, \\ B_2 &: =: e^i \left(\mathcal{D}\left\langle e_i, \left[e^j, e_j\right]\right\rangle\right) : \\ &\quad - : \left(\left\langle \mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle e^j\right) e_i : - : e^i \left(\left\langle \mathcal{D}\left\langle e_j, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle e^j\right) :, \\ B_2' &: =: e^i \left(\left\langle \mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle e_j\right) : + : \left(\left\langle \mathcal{D}\left\langle e^j, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle e_j\right) e_i : \\ &\quad - : \left(\mathcal{D}\left\langle e^i, \left[e^j, e_j\right]\right\rangle\right) e_i :, \\ B_3 &: =: e^i \left(\left\langle \mathcal{D}\left\langle \left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle e^j\right) :, \\ B_3' &: =: \left(\left\langle \mathcal{D}\left\langle \left[e_k, e^i\right], e^j\right\rangle, e^k\right\rangle e_j\right) e_i :, \\ B_4 &: =: e^i \left(\left\langle e_m, \left[e^k, e_k\right]\right\rangle \left\langle e^j, \left[e^m, e_i\right]\right\rangle e_j\right) : + : e^i \left(\left\langle e_m, \left[e^k, e_k\right]\right\rangle \left\langle e_j, \left[e^m, e_i\right]\right\rangle e^j\right) : \\ &\quad + : e^i \left(\left\langle e_j, \left[e^k, e_k\right]\right\rangle \left[e^j, e_i\right]_-\right) :, \\ B_4' &: =: \left(\left\langle e^m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e_m\right], e^j\right\rangle e_j\right) e_i : + : \left(\left\langle e^m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e_m\right], e_j\right\rangle e^j\right) e_i : \\ &\quad + : \left(\left\langle e^j, \left[e^k, e_k\right]\right\rangle \left[e^i, e_j\right]_-\right) e_i :, \\ B_5 &: =: e^i \left(\left\langle e^m, \left[e^k, e_k\right]\right\rangle \left\langle e^j, \left[e_m, e_i\right]\right\rangle e_j\right) :, \\ B_5' &: =: \left(\left\langle e_m, \left[e^k, e_k\right]\right\rangle \left\langle \left[e^i, e^m\right], e_j\right\rangle e^j\right) e_i : . \end{aligned}$$

First, notice that clearly

$$A = -\left[e^j, e_j\right]_+ .$$

Now, by the Courant algebroid axioms, (B.13) and (B.20),

$$B_1 = 2T\left\langle \mathcal{D}\left\langle e_k, \left[e^j, e_j\right]\right\rangle, e^k\right\rangle .$$

Analogously,

$$B_1' = 2T \left\langle \mathcal{D} \left\langle e^k, [e_j, e^j] \right\rangle, e_k \right\rangle.$$

Now, by the Courant algebroid axioms, (B.13), (B.14) and (B.20),

$$B_2 = 2 : e^i \left(\mathcal{D} \left\langle e_i, [e^j, e_j] \right\rangle\right)_{\bar{l}} : + : e^i \left(\mathcal{D} \left\langle e_i, [e^j, e_j] \right\rangle\right)_{-} : -2T \left\langle \mathcal{D} \left\langle e_k, [e^j, e_j] \right\rangle, e^k \right\rangle.$$

Analogously,

$$B_2' = -2 : \left(\mathcal{D} \left\langle e^i, [e^j, e_j] \right\rangle\right)_{l} e_i : - : \left(\mathcal{D} \left\langle e^i, [e^j, e_j] \right\rangle\right)_{-} e_i : +2T \left\langle \mathcal{D} \left\langle e^k, [e^j, e_j] \right\rangle, e_k \right\rangle.$$

Now, by the Courant algebroid axioms, together with (B.10), (B.13), (B.14) and (B.20),

$$B_4 =: \left(\mathcal{D} \left\langle e_i, [e^j, e_j] \right\rangle\right)_{\bar{l}} e^i : - : e^i \left[e_i, \left[e^k, e_k\right]_{\bar{l}}\right] : .$$

Analogously,

$$B_4' =: \left(\mathcal{D} \left\langle e^i, [e^j, e_j] \right\rangle\right)_{l} e_i : + : \left[e^i, \left[e^k, e_k\right]_{l}\right] e_i : .$$

At last, by the Courant algebroid axioms, (B.13) and (B.20),

$$B_5 = 2T \left\langle \mathcal{D} \left\langle e^i, \left[e^k, e_k\right] \right\rangle, e_i \right\rangle + : e^i \left[e_i, \left[e^k, e_k\right]_{l}\right] : - : \left(\mathcal{D} \left\langle e^i, \left[e^k, e_k\right] \right\rangle\right)_{\bar{l}} e_i : .$$

Analogously,

$$B_5' = 2T \left\langle \mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle, e^i \right\rangle - : \left[e^i, \left[e^k, e_k\right]_{\bar{l}}\right] e_i : + : e^i \left(\mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle\right)_{l} : .$$

Consequently, applying (B.40), we arrive at

$$\begin{aligned}
P_3 = &-\lambda\chi \left[e^j, e_j\right]_{+} - \lambda T \left\langle [w, e^j], e_j \right\rangle \\
&+ \frac{\lambda}{k} \Big(: e^k [e_k, w]_{\bar{l}} : + : \left[e^k, w\right]_{l} e_k : + : e^k [e_k, w]_{-} : + : \left[e^k, w\right]_{-} e_k : \\
&+2T \left(\left\langle \mathcal{D} \left\langle e^i, \left[e^k, e_k\right] \right\rangle, e_i \right\rangle + \left\langle \mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle, e^i \right\rangle \right) \\
&+ : e^i \left(\left(\mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle\right)_{\bar{l}} + \left(\mathcal{D} \left\langle e_i, \left[e^k, e_k\right] \right\rangle\right)_{-} \right) : \\
&- : \left(\left(\mathcal{D} \left\langle e^i, \left[e^k, e_k\right] \right\rangle\right)_{l} + \left(\mathcal{D} \left\langle e^i, \left[e^k, e_k\right] \right\rangle\right)_{-} \right) e_i : \\
&+ : e^i \left(\left\langle \mathcal{D} \left\langle \left[e^k, e_i\right], e_j \right\rangle, e_k \right\rangle e^j \right) : + : \left(\left\langle \mathcal{D} \left\langle [e_k, e^i], e^j \right\rangle, e^k \right\rangle e_j \right) e_i : \Big).
\end{aligned}$$

In conclusion, we are ready to compute

$$\begin{aligned}
\left[H'{}_\Lambda : e^i e_i :\right] &=: \left[H'{}_\Lambda e^i\right] e_i : + : e^i \left[H'{}_\Lambda e_i\right] : + \int_0^\Lambda d\Gamma \left[\left[H'{}_\Lambda e^i\right]_\Gamma e_i\right] \\
&= I + P_1 + P_2 + P_3
\end{aligned}$$

Indeed, using that $T = S^2$ we get the following

$$\begin{aligned}
\left[H'_\Lambda : e^i e_i :\right] &= (2\lambda + 2T + \chi S)\left(: e^j e_j : -S\left[e^j, e_j\right]_-\right) \\
&- T\left(: e^k\left[\left[e^j, e_j\right]_-, e_k\right]_- : + : e_k\left[\left[e^j, e_j\right]_-, e^k\right]_- :\right) \\
&+ T^2 R + \frac{\lambda}{k} V,
\end{aligned}$$

where

$$\begin{aligned}
V :=&: \left[e^k, \left[e^j, e_j\right]_l\right]_l e_k : - : e^k\left[e_k, \left[e^j, e_j\right]_{\bar{l}}\right]_{\bar{l}} : \\
&+ 3\left(: e^k\left[e_k, \left[e^j, e_j\right]_-\right]_- : - : \left[e^k, \left[e^j, e_j\right]_-\right]_- e_k :\right) \\
&+ : \left[e^k, \left[e^j, e_j\right]\right]_- e_k : - : e^k\left[e_k, \left[e^j, e_j\right]\right]_- : \\
&+ : e^k\left[\left[e_k, e^j\right]_-, e_j\right]_- : + : \left[\left[e^k, e_j\right]_-, e^j\right]_- e_k : \\
&+ 2T\left(\left\langle \mathcal{D}\left\langle e^i, \left[e^k, e_k\right]\right\rangle, e_i\right\rangle + \left\langle \mathcal{D}\left\langle e_i, \left[e^k, e_k\right]\right\rangle, e^i\right\rangle - \left\langle \left[w, e^j\right], e_j\right\rangle\right) \\
&+ : e^i\left(\left(\mathcal{D}\left\langle e_i, \left[e^k, e_k\right]\right\rangle\right)_{\bar{l}} + \left(\mathcal{D}\left\langle e_i, \left[e^k, e_k\right]\right\rangle\right)_-\right) : \\
&+ : \left(\left(\mathcal{D}\left\langle e^i, \left[e^k, e_k\right]\right\rangle\right)_l + \left(\mathcal{D}\left\langle e^i, \left[e^k, e_k\right]\right\rangle\right)_-\right) e_i : \\
&+ : e^i\left(\left\langle \mathcal{D}\left\langle \left[e^k, e_i\right], e_j\right\rangle, e_k\right\rangle e^j\right) : + : \left(\left\langle \mathcal{D}\left\langle \left[e_k, e^i\right], e^j\right\rangle, e^k\right\rangle\right) e_i : .
\end{aligned}$$

Now, we are going to apply Jacobi identity of Dorfman bracket to the brackets appearing in the third line above. Using that l and $\bar{l}$ are involutive and Courant algebroid axioms,

$$\begin{aligned}
\left[e^k, \left[e^j, e_j\right]\right]_- &= -\left[\left[e^k, e_j\right]_-, e^j\right]_- + \left(\mathcal{D}\left\langle e^k, \left[e^j, e_j\right]\right\rangle\right)_-, \\
\left[e_k, \left[e^j, e_j\right]\right]_- &= \left[\left[e_k, e^j\right]_-, e_j\right]_- + \left(\mathcal{D}\left\langle e_k, \left[e^j, e_j\right]\right\rangle\right)_- .
\end{aligned}$$

Using the Courant algebroid axioms, (B.13), (B.14), (B.19), (B.20) and Lemma B.5.12,

$$\begin{aligned}
V = &-3\left(: e^k\left[\left[e^j, e_j\right]_-, e_k\right]_- : + : e_k\left[\left[e^j, e_j\right]_-, e^k\right]_- :\right) \\
&+ : \left(\left\langle \left[\left[e_k, e^k\right], e^i\right], e^j\right\rangle + \left\langle \left[e_k, \left[e^i, e^j\right]_{\bar{l}}\right], e^k\right\rangle + \left\langle \left[\left[e^i, e^j\right]_{\bar{l}}, e_k\right], e^k\right\rangle\right) : e_j e_i :: \\
&- : \left(\left\langle \left[\left[e^k, e_k\right], e_i\right], e_j\right\rangle + \left\langle \left[e^k, \left[e_i, e_j\right]_l\right], e_k\right\rangle + \left\langle \left[\left[e_i, e_j\right]_l, e^k\right], e_k\right\rangle\right) : e^j e^i :: .
\end{aligned}$$

Here, notice that

$$\begin{aligned}
\left\langle \mathcal{D}\left\langle e_k, \left[e^i, e^j\right]\right\rangle, e^k\right\rangle &= \left\langle \left[e_k, \left[e^i, e^j\right]\right], e^k\right\rangle + \left\langle \left[\left[e^i, e^j\right], e_k\right], e^k\right\rangle, \\
\left\langle \mathcal{D}\left\langle e^k, \left[e_i, e_j\right]\right\rangle, e_k\right\rangle &= \left\langle \left[e^k, \left[e_i, e_j\right]\right], e_k\right\rangle + \left\langle \left[\left[e_i, e_j\right], e^k\right], e_k\right\rangle .
\end{aligned}$$

As a consequence, we can prove using Courant algebroid axioms that

$$F_{ij} = \left\langle \left[\left[e^k, e_k \right], e_i \right], e_j \right\rangle + \left\langle \mathcal{D} \left\langle [e_i, e_j], e^k \right\rangle, e_k \right\rangle. \tag{C.6a}$$

$$F^{ij} = \left\langle \left[\left[e_k, e^k \right], e^i \right], e^j \right\rangle + \left\langle \mathcal{D} \left\langle [e^i, e^j], e_k \right\rangle, e^k \right\rangle, \tag{C.6b}$$

In conclusion, we have arrived at the desired formula, which concludes the proof. □

C.3 Global Sections from Constant Determinant Atlas

Finally, we will find when the sections J_0 and H' are global.

Lemma C.3.1. *Let $\{f_j, f^j\}_{j=1}^n \subseteq l \oplus \bar{l}$ be a new isotropic frame, for which there exists*

$$A = \left(A_j^k \right)_{j,k=1}^n, B = \left(B_j^k \right)_{j,k=1}^n \in \mathrm{Mat}_n \left(\mathcal{C}^\infty(M) \right)$$

matrices for the change of coordinates, such that

$$f_j = \sum_{k=1}^n A_j^k e_k \text{ and } f^j = \sum_{k=1}^n B_j^k e^k, \quad \text{for } j \in \{1, \ldots, n\}.$$

In addition, suppose that the change of frames has constant determinant, that is,

$$\mathcal{D} \det A = 0.$$

Then, we have that

$$\sum_{j=1}^n : f^j f_j := \sum_{j=1}^n : e^j e_j : \text{ and } \sum_{j=1}^n \left[f^j, f_j \right]_- = \sum_{j=1}^n \left[e^j, e_j \right]_-.$$

In conclusion, the sections (C.1) *and* (C.3) *are global if the change of frames has constant determinant, and the same happens for the projection of $[\bar{\epsilon}_j, \epsilon_j]$ to C_-.*

Proof. This follows by a direct computation. Indeed, we can obtain that

$$: f^j f_j : =: e^j e_j : + : B_k^j \left(T A_j^k \right) : \text{ and } \left[f^j, f_j \right]_- = \left[e^j, e_j \right]_- + B_k^j \left(\mathcal{D} A_j^k \right)_-,$$

by the Courant algebroid axioms, (B.10), (B.13), (B.17), (B.19) (B.25) and (B.26). Note that the identity $A^{-1} = B$ is satisfied because our frames are isotropic. Now,

$$: B_k^j \left(T A_j^k \right) := \mathrm{tr} \left(: A^{-1} (TA) : \right) \text{ and } B_k^j \left(\mathcal{D} A_j^k \right) = \mathrm{tr} \left(A^{-1} \cdot \mathcal{D} A \right),$$

so, since by Jacobi's formula (see Appendix B.6) this coincides, respectively, with

$$T \log \det A = \frac{T \det A}{\det A} \text{ and } \mathcal{D} \log \det A = \frac{\mathcal{D} \det A}{\det A},$$

we obtain the identities since $T \det A = \mathcal{D} \det A = 0$ by hypothesis. So, as a consequence, both J_0 and H' are also global, since H' is the Λ-bracket between J_0 by itself. □

Once we have done all these computations, we are ready to return to Section 10.1.

Bibliography

[1] M. Aldi, R. Heluani: *On a Complex-Symplectic Mirror Pair*, Int. Math. Res. Not. **2018** (22) (2018) 6934–6960.

[2] L. Alvarez-Consul, A. De Arriba De La Hera, M. Garcia-Fernandez: *(0,2) mirror symmetry on homogeneous Hopf surfaces*, Int. Math. Res. Notices (2023), DOI https://doi.org/10.1093/imrn/rnad016, preprint arXiv:2012.01851 (2020).

[3] R. E. Bakalov, V. G. Kac: *Field algebras*, Int. Math. Res. Not. **3** (2003) 123–159.

[4] D. Baraglia, P. Hekmati: *Transitive Courant Algebroids, String Structures and T-duality*, Adv. Theor. Math. Phys. **19**, No. **5** (2015) 613–672.

[5] K. Barron: $N = 1$ *Neveu–Schwarz vertex operator superalgebras over Grassmann algebras and with odd formal variables*, In: Representations and Quantizations (Shanghai, 1998). China High. Educ. Press, Beijing (2000) 9–35.

[6] D. Ben-Zvi, E. Frenkel: *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, Vol. **88**. RI: Amer. Math. Soc. 2001. Second Edition, 2004.

[7] D. Ben-Zvi, R. Heluani, M. Szczesny: *Supersymmetry of the Chiral de Rham Complex*, Compositio Math. **144** (2008) 503–521.

[8] A. Beilinson, V. G. Drinfeld: *Chiral Algebras*. Colloquium Publications, Vol. **51**. Chicago, RI: Amer. Math. Soc. 2004.

[9] R. E. Borcherds: *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA, Vol. **83** (1986) 3068–3071.

[10] R. E. Borcherds: *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992) 405–444.

[11] L. A. Borisov: *Introduction to the vertex algebra approach to mirror symmetry*, preprint, arXiv:9912195 (2000).

[12] L. A. Borisov, A. Libgober: *Elliptic genera of toric varieties and applications to mirror symmetry*, Invent. Math. **140** (2000), No. **2**, 453–485. MR 1757003.

[13] L. A. Borisov: *Vertex algebras and mirror symmetry*, Comm. Math. Phys. **215** (2001) 517–557.

[14] L. A. Borisov, R. Kaufmann: *On CY-LG correspondence for (0,2) toric mirror models*, Adv. Math. **230** (2012) 531–551.

[15] P. Bressler: *Vertex Algebroids I*, preprint arXiv:math/0202185 (2002).

[16] P. Candelas, X. de la Ossa, P. S. Green, L. Parkes: *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B **359** (1991) 21–74.

[17] G. Cavalcanti, M. Gualtieri: *Generalized complex geometry and T-duality*, In: A Celebration of the Mathematical Legacy of Raoul Bott. CRM Proceedings & Lecture Notes. Amer. Math. Soc. (2010) 341–366.

[18] A. Clarke, M. Garcia-Fernandez, C. Tipler: *T-Dual solutions and infinitesimal moduli of the G_2-Strominger system*, to appear in Advances in Theoretical and Mathematical Physics, preprint arXiv:2005.09977 (2020).

[19] T. J. Courant: *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), 631–661.

[20] A. De Arriba De La Hera: *Álgebras de Vértices y la Construcción de Segal-Sugawara*, Trabajo Final de Máster, Universidad Complutense de Madrid (2017).

[21] X. de la Ossa, M.-A. Fiset: *G-structure symmetries and anomalies in* $(1,0)$ *non-linear σ-models*, J. High Energ. Phys. **2019**, 62 (2019).

[22] A. De Sole, V. G. Kac: *Finite vs. Affine W-algebras*, Mathe. Soc. Japan, Springer--Verlag (2006) 137–261.

[23] S. K. Donaldson, P. B. Kronheimer: *The Geometry of Four-Manifolds* (Oxford Mathematical Monographs), Oxford University Press, 1990.

[24] J. Ekstrand, R. Heluani, J. Kallen, M. Zabzine: *Chiral de Rham complex on Riemannian manifolds and special holonomy*, Comm. Math. Phys. **318** (2013) 575–613.

[25] P. Etingof, O. Schiffmann: *Lectures on Quantum Groups*, International Press of Boston Inc, 1998, Second Edition, 2010.

[26] D. Fattori, V. G. Kac: *Classification of finite simple Lie conformal superalgebras*, Journal of Algebra **258** (2002) 23–59.

[27] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa: Non-Kähler Heterotic String Compactifications with Non-Zero Fluxes and Constant Dilaton, Comm. Math. Phys. **288** (2009) 677–697.

[28] M.-A. Fiset, M. Galdeano: *Superconformal algebras for generalized Spin(7) and G_2 connected sums*, J. High Energ. Phys. **2021**, 137 (2021).

[29] I. B. Frenkel, V. G. Kac: *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980) 23–66.

[30] Th. Friedrich, S. Ivanov: *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian Journ. Math. **6** (2002), 303–336.

[31] J. X. Fu, S.-T. Yau: *A Monge-Ampère type equation motivated by string theory*, Comm. Anal. Geom. **15** (2007) 29–76.

[32] J. X. Fu, S.-T. Yau: *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ámpere*, J. Diff. Geom. **78** (2008) 369–428.

[33] M. Galdeano: *The Geometry and Superconformal Algebras of String Compactifications with a G-structure*, PhD Thesis (2022).

[34] M. Garcia-Fernandez: *Torsion-free generalized connections and Heterotic Supergravity*, Comm. Math. Phys. **332** (2014) 89-115.

[35] M. Garcia-Fernandez, R. Rubio, C. Tipler: *Infinitesimal Moduli for the Strominger System and Killing Spinors in Generalized Geometry*, Math. Ann. **369** (2017) 539–595.

[36] M. Garcia-Fernandez: *Ricci flow, Killing spinors, and T-duality in generalized geometry*, Adv. Math. **350** (2019) 1059–1108.

[37] M. Garcia-Fernandez, R. Rubio, C. Tipler: *Gauge theory for string algebroids*, to appear in J.Diff. Geom., preprint arXiv:2004.11399 (2020).

[38] M. Garcia-Fernandez, J. Streets: *Generalized Ricci flow.* University Lecture series, Vol. **76**. Providence. RI: Amer. Math. Soc. 2021.

[39] M. Garcia-Fernandez, R. Rubio, C. Shahbazi, C. Tipler: *Canonical metrics on holomorphic Courant algebroids*, Proc. London Math. Soc. **125** (3) (2022) 700–758.

[40] M. Garcia-Fernandez, R. Gonzalez-Molina: *Harmonic metrics for the Hull-Strominger system and stability*, preprint, arXiv:2301.08236 (2023).

[41] P. Gauduchon: *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518.

[42] P. Gauduchon: *Hermitian connections and Dirac operators*, Bolletino U.M.I. (7) **11-B** (1997), 257–288.

[43] E. Getzler: *Manin Pairs and Topological Field Theory*, Annals of Physics **237** (1995) 161–201.

[44] V. Gorbounov, F. Malikov, V. Schechtman: *Gerbes of chiral differential operators*, Math. Res. Lett. **7** (2000), No. **1**, 55–66. MR 1748287

[45] V. Gorbounov, F. Malikov, V. Schechtman: *Gerbes of chiral differential operators II*, preprint, arXiv:math/0003170v2 (2000).

[46] V. Gorbounov, F. Malikov, V. Schechtman: *Gerbes of chiral differential operators III*, Part of the Progress in Mathematics book series (PM, volume **213**), 73–100.

[47] J. Gross, D. Joyce, Y. Tanaka: *Universal structures in $\mathbb{C}$-linear enumerative invariant theories*, SIGMA **18** (2022), 068.

[48] M. Gualtieri: *Generalized Complex Geometry*, PhD Thesis (2004).

[49] M. Gualtieri: *Branes on Poisson varieties, the many facets of geometry*, Oxford Univ. Press (2010) 368–394.

[50] M. Gualtieri: *Generalized Kähler Geometry*, Comm. Math. Phys. **331** (2014) 297–331.

[51] R. Heluani: *Vertex Algebras*, lecture notes written by Reimundo Heluani for the course on vertex algebras given by Victor Kac during the spring of 2003 at MIT. Available at http://w3.impa.br/~heluani/files/lect.pdf.

[52] R. Heluani, V. G. Kac: *Supersymmetric Vertex Algebras*, Comm. Math. Phys. **271** (2007) 103–178.

[53] R. Heluani: *Supersymmetry of the Chiral de Rham Complex II: Commuting Sectors*, Int. Math. Res. Notices **6** (2009) 953–987.

[54] R. Heluani, M. Zabzine: *Generalized Calabi-Yau Manifolds and the Chiral de Rham Complex*, Adv. Math. **223** (2010) 1815–1844.

[55] R. Heluani, M. Zabzine: *Superconformal Structures on Generalized Calabi-Yau Metric Manifolds*, Comm. Math. Phys. **306** (2011) 333–364.

[56] R. Heluani: *Recent advances and open questions on the SUSY structure of the Chiral de Rham Complex*, J. Phys. A: Math. Theor. **50** (2017) 423002.

[57] N. Hitchin: *Generalized Calabi-Yau manifolds*, Q. J. Math. **54**, No. **3** (2003) 281–308. MR 2013140.

[58] C. Hull: *Superstring compactifications with torsion and space-time supersymmetry*, In: Turin (1985), Proceedings "Superunification and Extra Dimensions" (1986) 347–375.

[59] D. Joyce: *Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces*, work in progress, incomplete draft, preliminary version, 2020. Available at https://people.maths.ox.ac.uk/ joyce/hall.pdf.

[60] V. G. Kac, I. Todorov: *Superconformal current algebras and their unitary representations*, Comm. Math. Phys. **102** (1985) 337–347.

[61] V. G. Kac: *Lie superalgebras*, Adv. in Math. **26** (1977) 8–96.

[62] V. G. Kac: *Infinite-Dimensional Lie Algebras*, Progress in Mathematics, Vol. **44**. Basel; Boston; Stuttgart: Birkhiiuser, 1984.

[63] V. G. Kac: *Vertex Algebras for Beginners*, University Lecture series, Vol. **10**. Providence, RI: Amer. Math. Soc. 1996. Second Edition, 1998.

[64] V. G. Kac, A. K. Raina, N. Rozhkovskaya: *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, Adv. Series in Math. Phys. Vol. **29**. 1983. Second Edition, 2013.

[65] S.-C. Lau, L.-S. Tseng, S.-T. Yau: *Non-Kähler SYZ mirror symmetry*, Comm. Math. Phys. **340** (2015) 145–170.

[66] H. Lawson, M. Michelsohn: *Spin Geometry*, Princenton Mathematical Series **38**. Princenton University Press, Princenton, N.J., 1989.

[67] J. Lepowsky, R. L. Wilson: *Construction of the Affine Lie Algebra* $A_1^{(1)}$, Comm. Math. Phys. **62** (1978) 43–53.

[68] J. Li, S.-T. Yau: *The existence supersymmetric string theory with torsion*, J. Diff. Geom. **70** (2005) 143–181.

[69] B. H. Lian, A. R. Linshaw: *Chiral Equivariant Cohomology I*, Adv. Math. **209** (2007) 99–161.

[70] B. H. Lian, A. R. Linshaw, B. Song: *Chiral Equivariant Cohomology II*, Trans. Amer. Math. Soc. **360** (2008) 4739–4776.

[71] Z. J. Liu, A. Weinstein, P. Xu: *Manin triples for Lie bialgebroids*, J. Diff. Geom. **45**, No. **3** (1997) 547–574,

[72] A. R. Linshaw, V. Mathai: *Twisted Chiral de Rham Complex, Generalized Geometry, and T-duality*, Comm. Math. Phys. **339** (2015) 663–697.

[73] F. Malikov, V. Schechtman, A. Vaintrob: *Chiral de Rham Complex*, Comm. Math. Phys. **204** (1999) 439–473.

[74] J. McOrist: *The revival of (0,2) linear σ-models*, Int. J. Mod. Phys. A **26** (2011) 1–41.

[75] I. Melnikov, S. Sethi, E. Sharpe: *Recent developments in (0,2) mirror symmetry*, SIGMA **8** (2012) 068.

[76] C. Nozaradan: *Introduction to Vertex Algebras*, Université Catholique de Louvain, Mathematics Department. Louvain-la-Neuve, Belgium (2008).

[77] S. Odake: *Extension of $N = 2$ superconformal algebra and Calabi–Yau compactification*, Mod. Phys. Lett. **A** (1989), Vol. **4**, No. **6**, 557–68.

[78] S. Picard: *Calabi-Yau manifolds with torsion and geometric flows*, Complex non--Kähler geometry, 57–120, Lecture Notes in Math., 2246, Fond. CIME/CIME Found. Subser., Springer, Cham 2019.

[79] D. Popovici: *Non-Kähler Mirror Symmetry of the Iwasawa Manifold*, Int. Math. Res. Notices **2020** (2018), No. **23**, 9471–9538.

[80] L. O. Rodríguez Díaz: *G_2 holonomy manifolds are superconformal*, Comm. Math. Phys. **364** (2018) 385–440.

[81] S. L. Shatashvili, C. Vafa: *Superstring and manifolds of exceptional holonomy*, Selecta Mathematica, New Series, Vol. **1**, No. **2** (1995).

[82] P. Severa: *Letters to Alan Weinstein about Courant algebroids*, Unpublished (1998-2000), preprint, arXiv:1707.00265v2 (2017).

[83] G. Segal: *Unitary Representations of some Infinite Dimensional Groups*, Comm. Math. Phys. **80** (1981) 301–342.

[84] B. Song: *The global sections of the chiral de Rham complex on a Kummer surface*, Int. Math. Res. Not. **14** (2016), 4271–4296.

[85] B. Song: *Vector bundles induced from jet schemes*, Trans. Amer. Math. Soc. **374**, No. **4** (2021) 2661–2685.

[86] B. Song: *The global sections of chiral de Rham complexes on compact Ricci-flat Kähler manifolds*, Comm. Math. Phys. **382** (2021) 351–379.

[87] A. Strominger: *Superstrings with torsion*, Nucl. Phys. B **274** (1986) 253–284.

[88] A. Strominger, S.-T. Yau, E. Zaslow: *Mirror Symmetry is T-duality*, Nucl. Phys. B **479** (1996) 243–259.

[89] H. Sugawara: *A field theory of currents*, Phys. Rev. **176** (1968) 2019–2025.

[90] A. Ward: *Homological mirror symmetry for elliptic Hopf surfaces*, preprint arXiv:2101.11546 (2021).

[91] F. Witt: *Calabi-Yau manifolds with B-fields*, Rend. Sem. Mat. Univ. Pol. Torino, Vol. **66**, **1** (2008).

[92] E. Witten: *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. B **403** (1993) 159–222.