Contact Hamiltonian systems

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Abstract

Contact Hamiltonian systems are a generalization of the Hamiltonian systems of classical mechanics. The action is added as an extra variable in phase space, and symplectic geometry is changed by contact geometry. In this way, we are able to model a large new class of Hamiltonian systems. Indeed, symplectic geometry is unable to deal with dynamics where there is energy dissipation. Nonetheless, this is possible in the contact world.

In the recent years, there is a growing interest in the applications of contact Hamiltonian systems, expanding the classical ones in equilibrium thermodynamics. Several problems in physics, from areas including dissipative mechanical systems, electromagnetism, non-equilibrium thermodynamics, geometric optics, celestial mechanics, cosmology and quantum mechanics can be studied in the framework of contact Hamiltonian systems. Applications do not end here, but they cover other fields such as information geometry, control theory and optimization.

The aim of this PhD thesis is to provide a comprehensive theory of contact Hamiltonian systems, hoping that this will aid other scientists and mathematicians doing research in this topic. This dissertation incorporates the Hamiltonian and Lagrangian formalism, including the variational formulation through the Herglotz principle. We study and classify the symmetries and the dissipated quantities they are related to. Means to deal with different kinds of constrained systems are provided. We develop alternative formulations of the dynamics, such as the Hamilton-Jacobi theory and the Tulczyjew triples. A theory of variational integrators adapted to contact systems is also included.

Apart from this theoretical research, we also discuss novel applications in control theory and non-equilibrium thermodynamics, using the recently introduced evolution vector field.

Resumen

Los sistemas hamiltonianos de contacto son una generalización de los sistemas hamiltonianos de la mecánica clásica, en los que la acción se añade como una variable extra al espacio de fases. De esta manera es posible modelizar un amplio abanico de nuevos sistemas hamiltonianos. La geometría simpléctica es incapaz de tratar situaciones en las que hay disipación de energía. Sin embargo, esto sí es posible usando la geometría de contacto.

En los últimos años hay un interés creciente en las aplicaciones de los sistemas hamiltonianos de contacto que va mas allá de su clásico uso en termodinámica de equilibrio. Estas incluyen problemas en física, de áreas tan diversas como los sistemas mecánicos disipativos, el electromagnetismo, la termodinámica de no equilibrio, la óptica geométrica, la mecánica celeste, la cosmología y la mecánica cuántica. Los usos de los sistemas de contacto no acaban aquí, sino que también aparecen en otros campos, entre los que se encuentran la geometría de la información la teoría de control y la optimización.

El objetivo de esta tesis doctoral es construir una teoría exhaustiva de los sistemas hamiltonianos de contacto, esperando que sirva de apoyo a otros científicos y matemáticos en cuya investigación participen estos sistemas. En este trabajo se incluyen la formulación hamiltoniana y la lagrangiana, incorporando el principio variacional de Herglotz. Se estudian y clasifican las simetrías y las cantidades disipadas con las que están relacionadas. Se aportan métodos para tratar con sistemas con varios tipos de ligaduras. Se desarrollan formulaciones alternativas de la dinámica, como la teoría de Hamilton-Jacobi y los triples de Tulczyjew. También se constuyen integradores discretos variacionales adaptados a los sistemas de contacto.

Además de la investigación teórica, se encuentran nuevas aplicaciones dentro de la teoría de control óptimo y la termodinámica de no equilibrio, en la que se usa el campo de evolución.

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Contents

Scientific articles of the thesis

The majority of the work present on this dissertation has been developed and published on the series of articles that we list here:

- M. de León and M. Lainz, "Singular Lagrangians and precontact Hamiltonian systems", *Int. J. Geom. Methods Mod. Phys.*, vol. 16, no. 10, p. 1 950 158, Aug. 7, 2019, ISSN: 0219-8878. DOI: `10.1142/S0219887819501585`
- M. de León and M. Lainz, "Contact hamiltonian systems", *Journal of Mathematical Physics*, vol. 60, no. 10, p. 102 902, Sep. 2019. DOI: `10.1063/1.5096475`
- M. de León and M. Lainz, "Infinitesimal symmetries in contact Hamiltonian systems", *Journal of Geometry and Physics*, vol. 153, p. 103 651, Jul. 2020, ISSN: 03930440. DOI: `10.1016/j.geomphys.2020.103651`
- M. de León, J. Gaset, M. Lainz, X. Rivas, and N. Román-Roy, "Unified Lagrangian-Hamiltonian Formalism for Contact Systems", *Fortschr. Phys.*, vol. 68, no. 8, p. 2 000 045, Aug. 2020, ISSN: 0015-8208, 1521-3978. DOI: `10.1002/prop.202000045`
- A. Anahory Simoes, M. de León, M. Lainz, and D. Martín de Diego, "Contact geometry for simple thermodynamical systems with friction", *Proc. R. Soc. A.*, vol. 476, no. 2241, p. 20 200 244, Sep. 2020, ISSN: 1364-5021, 1471-2946. DOI: `10.1098/rspa.2020.0244`
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- M. de León, J. Gaset, M. Lainz, M. C. Muñoz-Lecanda, and N. Román-Roy, "Higher-order contact mechanics", *Annals of Physics*, vol. 425, p. 168 396, Feb. 1, 2021, ISSN: 0003-4916. DOI: `10.1016/j.aop.2021.168396`

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- M. de León, M. Lainz, and M. C. Muñoz-Lecanda, "The Herglotz Principle and Vakonomic Dynamics", in *Geometric Science of Information*, ser. Lecture Notes in Computer Science, F. Nielsen and F. Barbaresco, Eds., vol. 12829, Cham: Springer International Publishing, 2021, pp. 183–190, ISBN: 978-3-030-80208-0 978-3-030-80209-7. DOI: `10.1007/978-3-030-80209-7_21`
- M. de León, M. Lainz, and Á. Muñiz-Brea, "The Hamilton–Jacobi Theory for Contact Hamiltonian Systems", *Mathematics*, vol. 9, no. 16, p. 1993, Aug. 20, 2021, ISSN: 2227-7390. DOI: `10.3390/math9161993`
- M. de León, M. Laínz, M. C. Muñoz-Lecanda, and N. Román-Roy, "Constrained Lagrangian dissipative contact dynamics", *J. Math. Phys.*, vol. 62, no. 12, p. 122 902, Dec. 1, 2021, ISSN: 0022-2488, 1089-7658. DOI: `10.1063/5.0071236`
- M. de León, M. Lainz, and M. C. Muñoz-Lecanda, "Optimal control, contact dynamics and Herglotz variational problem", Jun. 25, 2020. arXiv: `2006.14326 [math-ph]`
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Notation

We will assume that all our manifolds and functions are infinitely smooth unless otherwise stated.

In the following diagram we picture the names of the natural projections we will frequently be using

$$
\begin{tikzcd}
 & & \mathbb{R} & & \\
TQ \times \mathbb{R} \arrow[urr, "z"] \arrow[rr, "\tau_Q^1"] \arrow[d, "\mathrm{pr}_{TQ}"'] \arrow[ddrr, "\tau_Q^0"] & & Q \times \mathbb{R} \arrow[u] \arrow[dd, "\mathrm{pr}_Q"] & & T^*Q \times \mathbb{R} \arrow[ull, "z"'] \arrow[ll, "\pi_Q^1"'] \arrow[d, "\mathrm{pr}_{T^*Q}"] \arrow[ddll, "\pi_Q^0"'] \\
TQ \arrow[drr, "\tau_Q"'] & & & & T^*Q \arrow[dll, "\pi_Q"] \\
 & & Q & &
\end{tikzcd}
\tag{0.1}
$$

If (x^i) are local coordinates for M, we will denote by $(x^i, \dot{x}^i)$ the induced coordinates for TM, and by (x^i, p_i) or, in more complex situations (x^i, p_{x^i}) the induced coordinates for T^*M.

The following symbols will be frequently used. They are listed together with the page in which they are defined, if applicable.

id_M Identity map on the set M.

$\Omega(M)$ Differential forms on the manifold M.

$\mathfrak{X}(M)$ Vector fields on the manifold M.

$\iota_X\theta$ Contraction of the vector field X with the form θ.

$\mathcal{L}_X\theta$ Lie derivative of the form θ by the vector field X.

$\Gamma(E)$ Sections of the bundle E.

J^1Q Manifold of 1-jets on Q.

j^1f 1-jets of the function f.

η Contact form. , 17

1. Introduction

For more than a century, the relationship of differential geometry and physics is manifest. With the advent of General Relativity, elaborated by A. Einstein, the structure of space-time is a dynamical variable, a solution of what we call nowadays the Einstein equations. These equations show that gravity can be explained by the curvature of space-time caused by massive objects. In order to have a better description of the laws of physics, one is forced to abandon flat Euclidean space and model space-time as a Riemannian manifold.

However, differential geometry was already present on the classical mechanics of Newton, Lagrange and Hamilton, but in a subtler way. One does not have to look only at the geometry of the physical space, but to the internal geometry of the system, which is much richer. The *phase space* is the set of possible states of the system. In classical mechanics it is given by cotangent bundle T^*Q, the space of positions and momenta of the system, where Q is the configuration space, i.e., the set of positions of the system. The study of this space was pioneered by Henry Poincaré on his work on the stability of the three body problem. Part of the novelty of these methods is that, instead of studying the properties of individual trajectories that solve the equations of motion, they are able to study all these solutions at the same time by considering the flow of the Hamiltonian vector field. This added geometrical and topological methods to the mathematical toolbox for mechanics, which supplemented the already established analytical methods.

Nevertheless, the study of the geometry of the phase space was far from complete. Another important development was due to Élie Cartan [59]. With the language of differential forms and the power of Cartan calculus, one is able to work *intrinsically*, that is, without specifying a particular system of coordinates.

The full picture came together at the middle of the 20-th century. According to [2], the first published modern exposition of mechanics was due to Reeb [220], thought it was already known in some mathematical circles. The key ingredient was to acknowledge that the crucial part of geometrical structure held by the cotangent bundle T^*Q was its symplectic form ω_Q. Only a closed and nondegenerate 2-form is sufficient information to obtain the equations of motion for a Hamiltonian function H defined on the manifold.

Symplectic geometry has been the most relevant driver in the study of Hamiltonian systems, as well as the natural extension to Poisson geometry, largely motivated by the study of singular Lagrangian systems and the rich Lie-Poisson structure of the duals of Lie algebras (or time-dependent systems, with the cosymplectic geometry, which is again Poisson). In fact, there has been a realignment between Hamiltonian mechanics and symplectic geometry, enriching both areas of research over the last sixty years.

The symplectic structure of phase space has allowed to formulate in a very efficient way such relevant facts as Noether's Theorem, and in general the notion of moment map (the bridge between Hamiltonian systems and algebra, with the concept of polytope), the reduction via symmetries due to Marsden and Weinstein, the study of singularities, equilibria, bifurcation theory, etc.

These facts are based on the energy conservation properties of the so-called symplectic Hamiltonian systems, which is no more than the manifestation of the skew-symmetry of the symplectic form.

If we look at classical mechanical systems (those whose phase space is the cotangent bundle, or in its Lagrangian formulation, the tangent bundle of the configuration space), the Hamilton or Euler-Lagrange equations can be obtained from a variational principle, Hamilton's Principle. As it is well known, Hamilton's Principle is equivalent in this case to the symplectic formulation, and the invariance of the form of the Euler-Lagrange equations before a change of coordinates has its basis in this equivalence.

But there are other types of Hamiltonian systems whose essence is dissipative rather than conservative, so we need an alternative geometry to describe them properly. This geometry is contact geometry, and it is developed in differentiable manifolds of odd dimension, so that if this is $2n + 1$, there are $2n$ variables corresponding to the positions and moments, and an extra one that accounts for the action. Contact geometry makes it possible to obtain Hamilton's equations, which now provide an account of this dissipative character, similarly to the symplectic one. Moreover, the corresponding brackets can be obtained, which are now not Poisson but Jacobi, since they do not satisfy Leibniz's rule.

In the case of mechanical systems, i.e., those developed in cotangent and tangent bundles extended by incorporating an extra variable, Hamilton's Principle is no longer valid. Fortunately, even before thinking of this application to Hamiltonian dynamics, Gustav Herglotz introduced in 1930 [159] a generalization (which we now call the Herglotz Principle) and which, in an almost magical way, provides the generalized Euler-Lagrange equations (now called Herglotz equations) which correspond to Hamilton's with additional dissipative terms. In this case, it can be easily seen that the extra variable corresponds to the action of the Lagrangian, which is why they are usually called action-dependent in physics.

We must say that there are many other dissipative Hamiltonian systems apart from the contact ones, but that the latter are of indisputable interest, both from the more theoretical aspects as well as for their important applications in many areas of science.

Recent interest in contact Hamiltonian systems is mainly due to the publication of several papers by Alessandro Bravetti and collaborators [33], and Manuel de León et al. The paper [88] can be considered as a first step in which the contact Hamiltonian systems are systematically studied, connecting them with their description as Jacobi manifolds, studying the different types of submanifolds (in particular those of Legendre, the contact counterpart to Lagrangian submanifolds in symplectic geometry), proving that contact dynamics can always be interpreted as a Legendre submanifold, establishing a coisotropic reduction theorem, and contact reduction via a Lie group of symmetries.

Since that first paper, our goal has been to develop a systematic study of contact Hamiltonian systems as well as to explore various applications, especially in the field of thermodynamics. Most of the results obtained are contained in the memory of this doctoral thesis.

Symplectic and contact systems in mechanics

A symplectic manifold is, thus, a natural context for studying mechanics [2, 106]. Indeed, given a symplectic manifold (M, ω) and a Hamiltonian function H on M, then the Hamiltonian vector field is provided by the equation

$$\iota_{X_H}\omega = \mathrm{d}H. \tag{1.1}$$

Notice that we lose the structure of a cotangent bundle T^*Q, blurring the distinction between position and momenta, treating both on equal footing. This can be an advantage. For example, one can consider symmetries that transform them independently which are not visible if we only look as lifts of transformations on the configuration space.

Nevertheless, there exist Darboux coordinates (q^i, p_i), that ensure that the local geometry is as the one in the cotangent bundle. But one must remember that those coordinates are non-canonical. The symplectic form is given by $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$, and the Hamiltonian vector field is

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{1.2}$$

so that the integral curves $(q^i(t), p_i(t))$ of X_H satisfy the Hamilton equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} \tag{1.3}$$

In classical mechanics, the phase space is just the cotangent bundle T^*Q of the configuration manifold Q, equipped with its canonical symplectic form ω_Q. However, we remark exist physical systems that are not modelled by a cotangent bundle, but a general symplectic manifold [237]. For example, the phase space of the classical spin is the sphere.

In addition to the Hamiltonian formalism, there is also a Lagrangian counterpart, which is formulated on the tangent bundle TQ, in terms of the velocities $\dot{q}^i$ instead of the momenta p_i. Here, the dynamics is dictated by the Lagrangian function $L : TQ \to \mathbb{R}$. The equations of motion can be obtained through Hamilton's Variational principle. The curves which are solutions of the equations of motion $q : [T_0, T_1] \to Q$ are those that are critical points of the action map

$$\mathcal{A}(q) = \int_{T_0}^{T_1} L(q(t), \dot{q}(t))\mathrm{d}t, \tag{1.4}$$

among the curves with the same endpoints $q_0 = q(T_0), q_1 = q(T_1)$. The curves satisfying this principle are the ones that fulfill the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial \dot{q}} = 0. \tag{1.5}$$

This formulation and the Hamiltonian formulation are related through the Legendre transformation

$$\begin{aligned} \mathrm{F}L : TQ &\to T^*Q, \\ (q^i, p_i) &\mapsto \left(q^i, \frac{\partial L}{\partial \dot{q}^i}\right). \end{aligned} \tag{1.6}$$

Provided some regularity conditions, this map transforms solutions of the Euler-Lagrange equations onto solutions of the Hamilton equations.

In order to be able to model more systems, one can extend this formalism by allowing explicit time dependence. The variational interpretation is straightforward. Now the Lagrangian $L : \mathbb{R} \times TQ \to \mathbb{R}$ depends also on time t. The action is now given by

$$\int_{T_0}^{T_1} L(t, q(t), \dot{q}(t))\mathrm{d}t. \tag{1.7}$$

The critical points of this action map among the curves with fixed endpoints are also the solutions to the Euler-Lagrange equations, which in coordinates read as the time independent case (1.5), but now the Lagrangian depends explicitly on time.

The geometric perspective of this formalism is related to cosymplectic geometry. Indeed, a cosymplectic structure on an odd-dimensional manifold is given by a pair (Ω, η) where Ω is a closed 2-form and η is a closed 1-form such that $\eta \wedge \Omega^n \neq 0$ where M has dimension $2n + 1$. Given a function H on M, one defines its Hamiltonian vector field X_H as the one satisfying the equations

$$\iota_{X_H}\omega = \mathrm{d}H, \quad \iota_{X_H}\eta = 0. \tag{1.8}$$

The Reeb vector field $\mathcal{R}$, which is defined as

$$\iota_{\mathcal{R}}\omega = 0, \quad \iota_{\mathcal{R}}\eta = 1. \tag{1.9}$$

From this, we define the evolution vector field as

$$E_H = X_H + \mathcal{R}. \tag{1.10}$$

There are also Darboux coordinates (t, q^i, p_i) for this geometry such that

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i, \quad \eta = \mathrm{d}t, \quad \mathcal{R} = \frac{\partial}{\partial t}. \tag{1.11}$$

The integral curves of E_H obey the same equations as in (1.3), but now the Hamiltonian is time-dependent [3, 43, 106].

Nevertheless, both of this geometric models have limitations. For example, they are unable to deal with situations in which there is energy dissipation. Contact Hamiltonian systems provide a setting in which new classes of systems can be modeled.

From the variational perspective, the idea is the following. We again add an extra variable z on the tangent bundle $TQ \times \mathbb{R}$, but now it will not represent time, but action. We now consider an *action-dependent Lagrangian* $L : TQ \times \mathbb{R} \to \mathbb{R}$. Hence, the action should be computed as follows

$$z(T_1) = \int_{T_0}^{T_1} L(q(t), \dot{q}(t), z(t))\mathrm{d}t. \tag{1.12}$$

Taking derivatives, we obtain the following non-autonomous ODE.

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= L(q(t), \dot{q}(t), z(t)), \\ z(T_0) &= z_0, \end{aligned} \tag{1.13}$$

for some fixed initial action $z_0 \in \mathbb{R}$. The action of a curve $q : [T_0, T_1] \to Q$ is, thus, the value of $z(T_1)$, where $z(t)$ is the solution to (1.13). The critical points of this action are the solutions to the Herglotz equations [159]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}. \tag{1.14}$$

We note that both the variational principle and the equations of motion reduce to the usual Euler-Lagrange case if the Lagrangian does not depend on the action z.

The geometric setting takes place on a contact manifold (M, η) with contact form η. This means that η is a 1-form such that $\eta \wedge \mathrm{d}\eta^n \neq 0$ and M has odd dimension $2n+1$. There exists a unique vector field $\mathcal{R}$ (also called the Reeb vector field) such that

$$\iota_{\mathcal{R}}\mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}}\eta = 1. \tag{1.15}$$

We are augmenting the phase space by adding the action as an extra dimension. Physically, a contact manifold is an *extended phase space* of the system. Now, we are not only jointly considering the position and the momenta of the system, but also the action. Of course, this extra flexibility also allows us to find geometric properties, such as symmetries that do not exist on the usual phase space.

There is also a Darboux theorem for contact manifolds, so that around each point in M one can find local coordinates (called Darboux coordinates) (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i, \quad \mathcal{R} = \frac{\partial}{\partial z}. \tag{1.16}$$

Then, given a Hamiltonian function H on M we obtain the Hamiltonian vector field X_H, which is the one satisfying

$$\mathcal{L}_{X_H}\eta = -\mathcal{R}(H)\eta, \quad \iota_{X_H}\eta = -H \tag{1.17}$$

In Darboux coordinates we get this local expression,

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}. \tag{1.18}$$

Therefore, an integral curve $(q^i(t), p_i(t), z(t))$ of X_H satisfies the contact Hamilton equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = p_i\frac{\partial H}{\partial p_i} - H. \tag{1.19}$$

Let us give an example of how these equations can model some dissipative systems. Consider a Hamiltonian system given by the Hamiltonian

$$H(q, p, z) = \frac{p^2}{2m} + V(q) + \gamma z, \tag{1.20}$$

where $\gamma \in \mathbb{R}$ is a constant. This Hamiltonian corresponds to a system with a friction force that depends linearly on the velocity (in our case, on the momenta). Indeed, its contact Hamilton equations are the following

$$\dot{q} = \frac{p}{m}, \tag{1.21a}$$

$$\dot{p} = -\frac{\partial V}{\partial q} - \gamma p, \tag{1.21b}$$

$$\dot{z} = \frac{p^2}{2m} - V(q) - \gamma z. \tag{1.21c}$$

That are just the damped Newtonian equation, which has a dissipative behavior.

Indeed, as we will later see, along a solution $\chi(t) = (q(t), p(t), z(t))$ of the equations of motion for any Hamiltonian H, the following is satisfied

$$\frac{\mathrm{d}H(\chi(t))}{\mathrm{d}t} = -\mathcal{R}(H)(\chi(t))H(\chi(t)). \tag{1.22}$$

Hence, loss of energy is present in these dynamics for non-trivial Hamiltonians.

As one can easily see, the contact Hamilton equations Equation (1.19) are far to be considered as a simple odd-dimensional counterpart of the symplectic ones: both geometries show different properties, symplectic and cosymplectic manifolds are Poisson, but a contact manifold is strictly a Jacobi manifold.

Applications of contact systems

In recent years, several applications of contact Hamiltonian dynamics have been found in many areas of science [32]. We provide a non-exhaustive list of recent applications.

- One of its earlier applications was on the geometrization Constantin Carathéodory's theory of thermodynamics. The relationship of contact geometry with thermodynamics was known at least since [12, 160]. The use of contact Hamiltonian systems to describe thermodynamic processes was latter introduced [14, 36, 141, 203, 244]. More recently, this description has been extended to some non-equilibrium thermodynamic systems [7, 118, 122, 148, 156].
- Applications have been found in many areas of physics [169], such as quantum mechanics [64, 157, 219], geometric optics [57], electromagnetism [134], celestial mechanics [37], cosmology [235, 236], and statistical mechanics [31, 253], and dissipative mechanical systems [53, 129, 200].
- There are applications in control theory [93, 208, 216, 246].
- The relationship between contact systems, stochastic processes and information geometry has also been explored [149].
- Contact Hamiltonian systems may also be used in optimization [34] and neural networks [128].

- Several kinds of numerical integrators have been developed [6, 9, 37, 189, 248].
- The problem of complete integrability has also been discussed [28, 232, 250]

The aim of this dissertation and of my research on the last years is to provide tools which can be useful for other mathematicians, scientist or engineers which encounter problems that can be modelled using contact Hamiltonian systems in their work. On the one hand, part of our work is theoretical. It consists on improving our understanding of contact systems, proving theorems about their behavior, developing alternate formalisms in order to express the dynamics (as the Tulczyjew triples or the Hamilton–Jacobi theory), etc. On the other hand, we are also interested on finding potential new applications of contact Hamiltonian systems. Indeed, we directly worked on thermodynamics and control theory. Now we present a brief overview of the content of this thesis.

1.1. Outline of this thesis

This thesis has two parts. The first part is devoted to the theory of contact Hamiltonian systems, which is the major part of this work. In the second part, we cover applications on thermodynamics and control theory. The first part is composed by the following chapters.

In Chapter 2 we introduce the elementary results and definitions on contact geometry and contact Hamiltonian systems, and put them in the more general context of Jacobi manifolds. At the end of the chapter, we define the evolution vector field, introduced in [7] which provides an alternative dynamics for contact systems. These results are part of the core of our work, and will be used repeatedly on the thesis. Apart from the evolution vector field section, this chapter is mostly based in [88], but some results have been expanded in order to provide support for the rest of the thesis sections.

Chapter 3 deals with the Lagrangian formalism, which is expressed both geometrically, where the dynamics is constructed using the structure of the tangent bundle, and through the Herglotz variational principle. This formalism will be used and extended throughout the thesis. This work was published on [89], except from the part concerning the evolution vector field, which was also introduced in [7].

In Chapter 4 we study the transformations which preserve the dynamics of the system. In our geometric theory, these transformations will consist of diffeomorphisms and vector fields, their infinitesimal counterparts. We cover both the Hamiltonian and Lagrangian formalisms, where we obtain a generalization of the Noether's Theorem. Unlike in the symplectic case, here symmetries are not related to conserved quantities, but to *dissipated* quantities that dissipate at the same rate as the energy [90]. Also, we introduce the concept of equivalent Lagrangians, which is important to understand how the equations of motion are modified when we change the way of measuring the action, as in the case of gauge transformations in electromagnetism [84]. Last of all, we cover the reduction of these systems. We prove a general theorem on coisotropic reduction and use it to obtain a Marsden-Weinstein-like reduction theorem for contact systems on presence of symmetries [88].

In Chapter 5 we deal with the singular Lagrangians, that is, those whose Hessian matrix with respect to the velocities is singular. We present a geometric model for these systems, which is not contact geometry, but *precontact geometry*. The dynamics of those systems may not be defined on the whole manifold, but just on a submanifold. We provide an algorithm, similar to the one introduced by Gotay and Nester, which finds the largest submanifold in which the dynamics can be defined. These results were first obtained in [89].

Chapter 6 concerns about action-dependent Lagrangian systems and constraints, in two different ways. On the one hand, we can interpret action-dependent Lagrangian systems as Lagrangian systems in the usual sense with constraints [95]. On the other hand, we can add additional constraints to an action-dependent Lagrangian system. Moreover, these constrained systems also come in two flavors. There is a *nonholonomic* principle [87], which are usually useful in applications to mechanics, and also a vakonomic principle [94], which has applications in control theory.

In Chapter 7, we introduce a set of tools that relate objects in contact geometry with homogeneous objects on symplectic geometry. This process, which is called *symplectization*, sometimes allows converting problems in contact geometry into problems of symplectic geometry, which are usually better studied. This chapter is based on some unpublished notes, and it tries to give a general theory expanding some tricks that we have used on some of our articles.

In Chapter 8 we provide an analog of Tulczyjew triples for contact systems in which both Hamiltonian and Lagrangian formalism can be interpreted as Legendrian submanifolds of contact manifolds. These triples were initially introduced in [127], but here we present them differently, using symplectization.

In Chapter 9, we introduce the Hamilton–Jacobi theory for contact systems [92], which provides an alternative method to solve the equations of motion by solving a PDE instead of a system of ODEs.

Now we enter on the part of our thesis related to the applications of contact systems, where we focus on two areas.

In Chapter 10 we discuss some applications to thermodynamics. The first section of this chapter deals with contact geometry in equilibrium thermodynamics, which is a classical theory tracing back to Arnold [12]. Here we also include examples of applications of some of the tools developed on the first chapter. On the second section we present a dynamical theory to describe non-equilibrium thermodynamic contact systems based on the evolution vector field that we introduced in [7, 10].

In Chapter 11 we formulate a generalization of the optimal control problem, which we call the *Herglotz control problem*, in which the cost is given by a non-autonomous ODE instead of a definite integral. We solve the problem in two ways: by developing a contact version of the Pontryagin maximum principle [93], and by applying the theory of vakonomic constraints on action-dependent Lagrangian systems [94].

At the end of this dissertation, in Chapter 12 we include the highlights of articles in which I have worked during my time as a predoctoral researcher, but have not been discussed in this dissertation. Here we include the following items:

- The introduction of a unified formalism [86] (similar to the Skinner-Rusk formalism) in which we are able to study the Hamiltonian and Lagrangian formulation at the same time.
- A theory of action-dependent Lagrangians in which they are able to depend on derivative of the position of higher order than the velocity [85].
- Finally, in [9] we defined what a discrete contact system is and proved that one can construct an exact discrete Lagrangian from the continuous counterpart, providing numerical methods to approximate their solutions which preserve geometric properties of the system, based on a discrete version of the Herglotz principle.

We finish in Chapter 13 by presenting some open problems in contact Hamiltonian systems.

Part I.

Theory

2. Contact Hamiltonian systems

This chapter is devoted to the introduction of the basic concepts of contact geometry and contact Hamiltonian systems. Some reference textbooks are [2, 13, 21, 22, 106, 142].

In Section 2.1 we provide the elementary definitions of contact geometry and give some examples of contact manifolds.

Next, in Section 2.2, we add the dynamics to the picture. Hamiltonian vector fields are introduced, and their elementary properties are studied.

In Section 2.3 contact manifolds are depicted on the more general context of Jacobi manifolds. Those manifolds are equipped with a bracket which provides a Lie algebra structure on the space of functions. We compare contact manifolds with other Jacobi manifolds, including symplectic, cosymplectic and locally conformally symplectic.

In Section 2.4 we deal with the properties of special types of submanifolds of contact manifolds, such as Legendrian, isotropic and coisotropic submanifolds, and we explain their relationship with Hamiltonian dynamics.

Up to now, most of the content of this article is an expanded version of the introductory chapters of [88]. We end in Section 2.5 where we present the evolution vector field, which was introduced by us in [7] in order to model some non-equilibrium thermodynamic systems. This vector field provides an alternative dynamics which is not Jacobi, but almost-Poisson.

2.1. Contact geometry

In this section we introduce some basic definitions and results of contact geometry.

Definition 2.1. A *contact manifold* is a pair (M, η), where M is a $(2n + 1)$-dimensional manifold and $\eta \in \Omega^1(M)$ is a *contact form*, that is, a nondegenerate 1-form such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form, i.e., it is non-zero at each point of M.

Definition 2.2. A *contact distribution* $\mathcal{H}$ is a rank $2n$ distribution on a $(2n+1)$-dimensional manifold M such that $\mathcal{H}$ is locally the kernel of a contact form. That is, at any point $p \in M$ there exist an open set U and a contact form η_U on U such that $\mathcal{H}_{|U} = \ker \eta_U$. We do not require $\mathcal{H}$ to be globally the kernel of a contact form. Equivalently, we may require $\mathcal{H}$ to be a maximally non-integrable rank $2n$ distribution [137].

Remark 2.1 (Another definition of contact manifold). Some authors [13, 182, 190, 257] define a contact manifolds $(M, \mathcal{H})$ as an odd-dimensional manifolds M with a contact distribution $\mathcal{H}$. However, we will not use this definition for the reasons that we explain below.

Although part of the concepts we will be working with only depend on the contact distribution, such as those of isotropic and Legendrian submanifold, the choice of contact forms plays a crucial role on the dynamics of the contact Hamiltonian system. A contact distribution does not provide enough structure to define the Reeb vector field, or to set up the correspondence between functions and their contact Hamiltonian vector fields.

We acknowledge that part of this theory can be carried away to the setting of a manifold with a contact distribution $(M, \mathcal{H})$. As it is explained in [189], one denotes by $L = M/\mathcal{H}$ the quotient line bundle. Then, it would be possible to define an L-valued "contact form" $\theta : TM \to L$ such that $\theta(X) = X \pmod{\mathcal{H}}$. There is a natural bijection between Hamiltonian vector fields and sections of the line bundle (but not for functions on M).

What we refer to as contact manifolds is called by some authors *exact contact manifolds* [190] or *co-oriented contact manifolds* [257].

Remark 2.2 (Contact forms vs contact distributions). Given a contact form η, $\mathcal{H} = \ker \eta$ is clearly a contact distribution. Conversely, a contact distribution $\mathcal{H}$ is globally the kernel of contact form if and only if $\mathcal{H}$ is co-orientable [138].

Notice that in the co-orientable case, the contact form defining $\mathcal{H}$ is not unique. Indeed, if $\mathcal{H} = \ker \eta$ it will also be the kernel of the contact forms $f\eta$ for any nonzero $f \in C^\infty(M)$. Hence, there is no canonical way to obtain a contact manifold (M, η) from a manifold with a contact distribution $(M, \mathcal{H})$.

Given a contact manifold (M, η), in addition to the contact distribution $\ker \eta$ we can define the complementary *Reeb distribution*

$$\mathcal{V} = \ker \mathrm{d}\eta. \tag{2.1}$$

By the conditions on the contact form, the following is a Whitney sum decomposition:

$$TM = \mathcal{H} \oplus \mathcal{V}, \tag{2.2}$$

that is, we have the direct sum decomposition of the tangent space at each point $x \in M$:

$$T_xM = \mathcal{H}_x \oplus \mathcal{V}_x. \tag{2.3}$$

We will denote by $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ the projections on these subspaces.

We notice that $\dim \mathcal{H} = 2m$ and $\dim \mathcal{V} = 1$, and that $\mathrm{d}\eta|_{\mathcal{H}}$ is nondegenerate, giving $(\mathcal{H}, \mathrm{d}\eta|_{\mathcal{H}})$ the structure of a symplectic vector bundle over M.

The contact structure of (M, η) gives rise to an isomorphism between tangent vectors and covectors. For each $x \in M$,

$$\begin{aligned} \flat : T_xM &\to T_x^*M \\ v &\mapsto \iota_v \mathrm{d}\eta + \eta(v)\eta. \end{aligned} \tag{2.4}$$

In fact, the previous map is an isomorphism if and only if η is a contact form [3]. Similarly, we obtain a vector bundle isomorphism

$$\begin{tikzcd} TM \arrow[rr, "\flat"] \arrow[dr, "\tau_M"'] & & T^*M \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd} \tag{2.5}$$

where $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$ are the canonical projections.

We will also denote by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ the corresponding isomorphism of $C^\infty(M)$-modules of vector fields and 1-forms on M. We denote $\sharp$ to the inverse of $\flat$, that is $\sharp = \flat^{-1}$.

Definition 2.3. From the definition of the contact form and the dimensions of the Reeb and contact distributions, we can easily prove that there exists a unique vector field $\mathcal{R}$, named the *Reeb vector field*, such that

$$\iota_{\mathcal{R}}\eta = 1, \quad \iota_{\mathcal{R}}\mathrm{d}\eta = 0. \tag{2.6}$$

This is equivalent to say that

$$\flat(\mathcal{R}) = \eta, \tag{2.7}$$

so that, in this sense, $\mathcal{R}$ is the dual object of η.

This vector field generates the Reeb distribution, $\mathcal{V} = \langle \mathcal{R} \rangle$.

There are some interesting classes of maps between contact manifolds.

Definition 2.4. A diffeomorphism between two contact manifolds $F : (M, \eta) \to (N, \tau)$ is a *contactomorphism* if

$$F^*\tau = \eta. \tag{2.8}$$

A diffeomorphism $F : (M, \eta) \to (N, \tau)$ is a *conformal contactomorphism* if there exist a nowhere zero function $f \in C^\infty(M)$ such that

$$F^*\tau = f\eta. \tag{2.9}$$

That is, F preserves their corresponding contact distributions, i.e. $F_*(\ker\eta) = \ker\tau$.

A vector field $X \in \mathfrak{X}(M)$ is an *infinitesimal contactomorphism* (respectively *infinitesimal conformal contactomorphism*) if its flow ϕ_t consists of contactomorphisms (resp. conformal contactomorphisms).

Proposition 2.3. *A vector field X on a contact manifold (M, η) is an infinitesimal contactomorphism if and only if*

$$\mathcal{L}_X\eta = 0. \tag{2.10}$$

Furthermore, X is an infinitesimal conformal contactomorphism if and only if there exists $a \in C^\infty(M)$ such that

$$\mathcal{L}_X\eta = a\eta. \tag{2.11}$$

Proof. Let $X \in \mathfrak{X}(M)$ and let ϕ_t be the corresponding flow. The proof of both statements follows from the following fact [2]

$$\frac{\partial}{\partial t}\phi_t^*\eta = \phi_t^*\mathcal{L}_X\eta. \tag{2.12}$$

□

In what it follows, we say Z an infinitesimal conformal contactomorphism with *conformal factor a* if $\mathcal{L}_X(\eta) = a\eta$.

Every pair of contact manifolds are locally contactomorphic. Thus, we can use a canonical set of coordinates for any contact manifold. This is implied by Darboux Theorem [2, Thm. 5.1.5] or [142]:

Theorem 2.4 (Darboux theorem). *Let (M, η) be a $(2n+1)$-dimensional contact manifold. Around any point $x \in M$ there is a chart with coordinates $(q^1, \dots, q^n, p_1 \dots, p_n, z)$ such that:*

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i.$$

In these coordinates,

$$\begin{aligned} \mathrm{d}\eta &= \mathrm{d}p_i \wedge \mathrm{d}q^i, \\ \mathcal{R} &= \frac{\partial}{\partial z}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \mathcal{V} &= \left\langle \frac{\partial}{\partial z} \right\rangle, \\ \mathcal{H} &= \langle \{A_i, B^i\}_{i=1}^n \rangle, \end{aligned} \tag{2.14}$$

where

$$A_i = \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \tag{2.15}$$

$$B^i = \frac{\partial}{\partial p_i}. \tag{2.16}$$

The local vector fields A_i and B^i have the following property:

$$\mathrm{d}\eta(A_i, A_j) = \mathrm{d}\eta(B^i, B^j) = 0, \quad \mathrm{d}\eta(A_i, B^j) = \delta^i_j. \tag{2.17}$$

Furthermore, $\{A_1, B^1, \dots, A_n, B^n, \mathcal{R}\}$ and $\{\mathrm{d}x^1, \mathrm{d}y_1, \dots, \mathrm{d}x^n, \mathrm{d}y_n, \eta\}$ are dual basis.

This basis is not a coordinate basis of any chart, since the form η is not closed. The Lie brackets do not vanish:

$$[A_i, B^i] = -\mathcal{R}, \ \forall i \in \{1, \dots n\}. \tag{2.18}$$

Indeed, these vector fields form a basis for the Heisenberg Lie algebra [35].

We finish this section by providing some examples of contact manifolds.

Example 2.1 (Extended cotangent bundle). The extended cotangent bundle of a manifold Q is the manifold $T^*Q \times \mathbb{R}$. The contact structure is constructed using by the pullback of the Liouville 1-form θ_Q and the differential of the $\mathbb{R}$ coordinate. That is,

$$\eta_Q = \mathrm{d}z - p_i \mathrm{d}q^i. \tag{2.19}$$

In physical applications, Q is the *configuration manifold*, which parametrizes the positions of the system. The points of $T^*Q \times \mathbb{R}$ parametrize the positions, momenta, and the extra variable is the action.

In some situations it is useful to think about this manifold as the manifold of 1-jets of functions on Q, J^1Q [230], which is naturally isomorphic to $T^*Q \times \mathbb{R}$, by the identification $(\mathrm{d}f, f) \simeq j^1 f$, where $j^1 f$ is the 1-jet of a function $f : Q \to \mathbb{R}$.

Example 2.2 (Projective cotangent bundle)**.** Given an $n+1$-dimensional manifold $\bar{Q}$, its projective cotangent bundle $\mathbb{P}(T^*\bar{Q})$ is the $(2n-1)$-dimensional manifold whose fibers at $q \in \bar{Q}$ are the projective space over $T_q^*\bar{Q}$ (the set of 1-dimensional subspaces of $T_q^*\bar{Q}$).

A nonzero covector $\alpha_q \in T_q^*\bar{Q}$ it determines an element $\mathbb{P}(v_q) = \langle v_q \rangle \in \mathbb{P}(T^*\bar{Q})$. Two covectors determine the same element on $\mathbb{P}(T^*\bar{Q})$ if and only if they are proportional.

Another way to think about this space is as the *space of contact elements* on $T\bar{Q}$, which is the set of hyperplanes. Indeed, $\langle \alpha \rangle \in T^*\bar{Q}$ corresponds with the hyperplane $\ker \alpha$. Since two forms define the same hyperplane if and only if they are proportional, this correspondence is well-defined and bijective.

Points on the projective cotangent bundle can be described using homogeneous coordinates

$$(q^0, \dots, q^n, [p_0 : \dots : p_n]) = \mathscr{P}((q^0, \dots, q^n, p_0, \dots, p_n)). \tag{2.20}$$

Another way to describe the points of $\mathbb{P}(T^*\bar{Q})$ is through an affine chart. For this, we start with a set of projective coordinates, and we assign to each line the point it intersects[1] with $\{p_a = -1\} \subseteq \mathbb{P}(T^*\bar{Q})$ for some $a \in \{0, \dots n\}$. This is defined on the set $U_a = \{p_a \neq 0\} \subseteq \mathbb{P}(T^*\bar{Q})$. Its complement $\{p_a = 0\} \subseteq \mathbb{P}(T^*\bar{Q})$ is referred as the *hyperplane at infinity* of this affine chart. The correspondence is given by

$$(q^0, \dots, q^n, [p_0 : \dots : p_n]) \mapsto \left(q^0, \dots, q^n, -\frac{p_0}{p_a} \dots, -\frac{p_{a-1}}{p_a}, -\frac{p_{a+1}}{p_a}, \dots - \frac{p_n}{p_a} \right) \tag{2.21a}$$

$$(q^0, \dots, q^n, [P_0 : \dots : P_{a-1} : -1 : P_{a+1} : P_n]) \leftarrow (q^0, \dots, q^n, P_1 \dots P_n). \tag{2.21b}$$

The projective cotangent bundle has a natural contact distribution

$$\mathcal{H} = \mathbb{P}(\ker \theta_{\bar{Q}}) = \mathbb{P}(\ker(q^\alpha \mathrm{d}p_\alpha)). \tag{2.22}$$

In an affine chart we can define a contact form η^a such that $\mathcal{H} = \ker \eta^a$ in U_a, by taking

$$\eta^a = \mathrm{d}q^a + \sum_{i \neq a} \frac{p_i}{p_a} \mathrm{d}p_i = \mathrm{d}q^a - \sum_{i \neq a} P_i \mathrm{d}q^i. \tag{2.23}$$

Since the sets U_a cover $\mathbb{P}(T^*\bar{Q})$, we see that $\mathcal{H}$ is a contact distribution. However, the contact form η^a and η^b do not agree at $U_a \cap U_b$. Indeed,

$$\eta^a = \frac{p_b}{p_a} \eta^b, \tag{2.24}$$

hence they do not define a global contact form. In fact, this is an example of a manifold which has a contact distribution but not a contact form, as in Remark 2.2. This is the case

[1] Usually, one chooses the plane $p_a = 1$, but we choose $p_a = -1$ so that the affine chart gives us Darboux coordinates.

because $\mathcal{H}$ is not co-orientable: there exist no distribution $\mathcal{V}$ such that $T\mathbb{P}(\ker\theta_{\bar{Q}}) = \mathcal{H}\oplus\mathcal{V}$. A proof of this fact can be found in [138, Proposition 2.1.13].

During this text, the projective cotangent bundle of $\bar{Q} = Q \times \mathbb{R}$ will appear. On this case, we note that the affine chart on $\{p_z = -1\}$ provides a contactomorphism with the extended cotangent bundle of Q, $\bar{\psi} : T^*Q \times \mathbb{R} \to \mathbb{P}(T^*(Q \times \mathbb{R})) \setminus \{p_z = -1\}$, which is given by

$$\bar{\psi}(q,p,z) = (q,z,[p:-1]), \tag{2.25a}$$
$$\bar{\psi}^{-1}(q,z,[p:p_z]) = (q,z,-p/p_z). \tag{2.25b}$$

Symplectic manifolds with a Liouville vector field are closely related to contact manifolds. Indeed, generic hypersurfaces of those manifolds inherit a contact structure. In Chapter 7 we will elaborate on this relationship, but now we give an example. More information will be given about those manifolds in Section 7.1.

Example 2.3 (Hypersurface transverse to the Liouville vector field)**.** Given a hypersurface $i : N \hookrightarrow (M,\omega)$ in a symplectic manifold transverse to a Liouville vector field Δ (that is, a vector field satisfying $\mathcal{L}_\Delta\omega = \omega$) of a $(2n)$-dimensional exact symplectic manifold, the form $\eta = i^*\theta$, where $\theta = \iota_\Delta\omega$, is a contact form for M [142, Theorem 5.9].

Indeed, using Cartan's formula we can proof that $\mathcal{L}\Delta\omega = \omega$. Hence,

$$\eta \wedge (\mathrm{d}\eta)^n = i^*((\iota_\Delta\theta)\wedge\omega^n) = \frac{1}{n+1} i^*(\iota_\Delta\omega^n). \tag{2.26}$$

Since Δ is transverse to N, at $x \in N$ we have that $T_xM = T_xN \oplus \langle\Delta_x\rangle$. Thus, as ω^n is non-degenerated $(\iota_\Delta\omega^n)_{|T_xN}$ is also non-degenerated. Hence, η is a contact form.

2.2. Contact Hamiltonian systems

Contact Hamiltonian systems are the main subject of study in this work. In this section we will introduce them and explain their main properties.

As in the case of symplectic manifolds, the contact structure provides a correspondence between functions and vector fields. Indeed, given a *Hamiltonian function* the contact structure will produce dynamical equations.

Definition 2.5. Given a smooth real function $H : M \to \mathbb{R}$ on a contact manifold (M,η), we define its *contact Hamiltonian vector field* (or just *Hamiltonian vector field*) as the vector field X_H satisfying

$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}(H) + H)\eta. \tag{2.27}$$

We note that this vector field exists and is unique, since $\flat$ is an isomorphism.

In Darboux coordinates, this is written as follows

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}. \tag{2.28}$$

An integral curve of this vector field satisfies the contact Hamilton equations:

$$\dot{q}^i = \frac{\partial H}{\partial q^i}, \tag{2.29a}$$

$$\dot{p}_i = -\left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z} \right), \tag{2.29b}$$

$$\dot{z} = p_i \frac{\partial H}{\partial p_i} - H. \tag{2.29c}$$

These equations are a generalization of the conservative Hamilton equations. We recover this particular case when $\mathcal{R}(H) = 0$. That is, when H does not depend on the *action* z.

The equation for the *positions* q^i (2.29a) has the same form that the usual Hamilton equation. However, the equation for the *momenta* p_i (2.29b) has an extra term proportional to the momenta and the derivative of H with respect to z, which will create dissipative effects on some situations.

In the last equation (2.29c), we see that $\dot{z}$ coincides with the expression of the action on classical mechanics. This will play an important role in the Lagrangian formalism.

We remark that the Reeb vector field is just the Hamiltonian vector field for the Hamiltonian $H = -1$, i.e.,

$$X_{-1} = \mathcal{R}. \tag{2.30}$$

There are some equivalent definitions of the Hamiltonian vector field, which we will often use.

Proposition 2.5. *Let $H : M \to \mathbb{R}$ be a Hamiltonian function. The following statements are equivalent:*

1. *X_H is the Hamiltonian vector field of H.*
2. *The vector field X_H is given by*

$$X_H = \sharp(\mathrm{d}H) - (\mathcal{R}(H) + H)\mathcal{R}. \tag{2.31}$$

3. *The vector field X_H satisfies that*

$$\eta(X_H) = -H, \tag{2.32a}$$

$$\iota_{X_H}\mathrm{d}\eta = -\mathcal{R}(H)\eta + \mathrm{d}H. \tag{2.32b}$$

4. *The vector field X_H satisfies that*

$$\eta(X_H) = -H, \tag{2.33a}$$

$$\mathcal{L}_{X_H}\eta = -\mathcal{R}(H)\eta. \tag{2.33b}$$

5. *The vector field X_H is an infinitesimal conformal contactomorphism satisfying*

$$\eta(X_H) = -H. \tag{2.34a}$$

6. *The vector field X_H satisfies that*

$$\eta(X_H) = -H, \tag{2.35a}$$
$$\iota_{X_H}\mathrm{d}\eta_{|\mathcal{H}} = \mathrm{d}H_{|\mathcal{H}}. \tag{2.35b}$$

Equivalently, the projection of X_H onto the Reeb distribution is $-H\mathcal{R}$ and its projection onto the contact distribution, which is a symplectic vector bundle, $(\mathcal{H}, \mathrm{d}\eta_{|\mathcal{H}})$ is the Hamiltonian vector field of H.

Proof. Items 1 and 2 can be seen to be equivalent because $\flat$ and $\sharp$ are inverses, and $\flat(\mathcal{R}) = \eta$.

Now, we will see that 1 is equivalent to 3. Expanding $\flat$ on 1, we obtain

$$\iota_{X_H}\mathrm{d}\eta + \eta(X_H)\eta = \mathrm{d}H - (\mathcal{R}(H) + H)\eta. \tag{2.36}$$

Contracting both sides of the previous equation with $\mathcal{R}$, we find that $\eta(X_H) = -H$. Substituting this on the previous Equation (2.36), we obtain Equation (2.32b), proving item 3. Conversely, we can prove Equation (2.36) using the identities in Equation (2.32) to expand the left hand side of Equation (2.36).

The equivalence of items 3 and 4 follows from Cartan's formula $\mathcal{L}_{X_H}\eta = \iota_{X_H}\mathrm{d}\eta + \mathrm{d}\iota_{X_H}\eta$.

Now, we turn to item 5. From Proposition 2.3 we that X is a Hamiltonian vector field if it satisfies

$$\eta(X_H) = -H, \tag{2.37a}$$
$$\mathcal{L}_{X_H}\eta = a\eta \tag{2.37b}$$

for some function a. In order to see that this is equivalent to item 4 we just need to prove that $a = -\mathcal{R}(H)$. Indeed, contracting both sides of the second equation with $\mathcal{R}$ and using Cartan's formula for the Lie derivative $\mathcal{L}_{X_H}\eta$, we obtain

$$a = \iota_{\mathcal{R}}\mathcal{L}_{X_H}\eta = \iota_{\mathcal{R}}(\iota_{X_H}\mathrm{d}\eta - \mathrm{d}H) = -\mathcal{R}(H). \tag{2.38}$$

Last of all, we need to see that item 6 is equivalent to item 3. Clearly, if we restrict Equation (2.33b) to the contact distribution $\mathcal{H}$, we obtain Equation (2.35b). Conversely, from Equation (2.35b) we obtain

$$\iota_{X_H}\mathrm{d}\eta = \mathrm{d}H + f\eta, \tag{2.39}$$

for some function f. Noticing that $-\mathrm{d}H = \mathrm{d}\iota_{X_H}\eta$ and using Cartan's formula, we obtain that $f = -\mathcal{R}(H)\eta$. Thus, we retrieve Equation (2.33b).

□

We will now define the main object of study of this work, which is no more that a contact manifold equipped with a function, and the dynamical equations of its Hamiltonian vector field.

Definition 2.6. A *contact Hamiltonian system* is a triple (M, η, H), where (M, η) is a contact manifold and H is a smooth real function on M that we will refer to as the *action dependent Hamiltonian* or just the *Hamiltonian*.

The contact Hamiltonian vector fields model the dynamics of dissipative systems. As opposed to the case of symplectic Hamiltonian systems, the evolution does not preserve the energy, the contact form or the natural volume form.

Theorem 2.6 (Energy and volume dissipation)**.** *Let (M, η, H) be a Hamiltonian system. The flow of the Hamiltonian vector field X_H does not preserve the energy H. In fact*

$$\mathcal{L}_{X_H} H = -\mathcal{R}(H) H. \tag{2.40}$$

The contact form is also not preserved:

$$\mathcal{L}_{X_H} \eta = -\mathcal{R}(H) \eta. \tag{2.41}$$

As well, the contact volume element $\Omega = \eta \wedge (\mathrm{d}\eta)^n$ is not preserved.

$$\mathcal{L}_{X_H} \Omega = -(n+1) \mathcal{R}(H) \Omega. \tag{2.42}$$

However, if H and $\mathcal{R}(H)$ are nowhere zero, there is a unique volume form depending on the Hamiltonian[2] *(up to multiplication by a constant) that is preserved [30]. By this, we mean that there is a unique form $\tilde{\Omega} = (g \circ H)\Omega$, where $g : H(\mathbb{R}) \to \mathbb{R}$ is a smooth function, which is given by*[3]

$$\tilde{\Omega} = H^{-(n+1)} \Omega. \tag{2.43}$$

Proof. The first claim follows from contracting Equation (2.32b) with X_H.

The second claim was already proved in Proposition 2.5

We proceed with the third claim. We compute the derivative using the product rule and Equation (2.33b)

$$\begin{aligned} \mathcal{L}_{X_H}(\eta \wedge (\mathrm{d}\eta)^n) &= -\mathcal{R}(H)\eta \wedge (\mathrm{d}\eta)^n \\ &\quad + n\eta \wedge (\mathrm{d}\eta)^{n-1} \wedge (-(\mathrm{d}\mathcal{R}(H))\eta - \mathcal{R}(H)\mathrm{d}\eta) \\ &= -(n+1)\mathcal{R}(H)\eta \wedge (\mathrm{d}\eta)^n, \end{aligned} \tag{2.44}$$

as we wanted to show.

Last of all, consider a volume form $\tilde{\Omega} = (g \circ H)\Omega$. Then

$$\begin{aligned} \mathcal{L}_{X_H} \tilde{\Omega} &= \mathcal{L}_{X_H}(g \circ H)\Omega + (g \circ H)\mathcal{L}_{X_H}\Omega \\ &= -((g' \circ H) H \mathcal{R}(H) + (n+1)(g \circ H)\mathcal{R}(H))\Omega. \end{aligned} \tag{2.45}$$

[2] This is useful for applications in statistical mechanics, as can be read on the article [30]. There might be other invariant volume forms.

[3] Indeed, the form Ω is just a multiple of the volume form of $\tilde{\eta} = \eta/H$, which is the contact form inducing the same contact distribution as η, and such that $\tilde{\mathcal{R}} = X_H$ is its Reeb vector field. This is an example of a conformal equivalence, which will be studied on Section 4.1.

Since Ω is a volume form and $\mathcal{R}(H)$ is non-zero, this Lie derivative vanishes if and only if

$$g'(h)h + (n+1)g(h) = 0, \tag{2.46}$$

for $h = H(x) \in \mathbb{R}$. Hence, g is the solution to this linear ODE, which is unique up to multiplication by a constant, and it is given by

$$g(h) = Ch^{-(n+1)}, \tag{2.47}$$

where $C \in \mathbb{R}$ is a constant. Therefore, $\tilde{\Omega}$ is preserved if and only if it is of the form

$$\tilde{\Omega} = CH^{-(n+1)}\Omega. \tag{2.48}$$

□

The question of existence of an invariant measure on a Hamiltonian system in which the Hamiltonian is allowed to vanish is more subtle and the answer is not always affirmative. In the case that $\mathcal{R}$ is complete $\mathcal{R}(H)$ is a non-zero constant on the zero set of H we can link the existence of an invariant measure to the existence of a geometric construction called a *symplectic sandwich with contact bread* [29, Theorem 4.14]. The behavior of the restriction of the dynamics to the zero level set has, indeed, special properties. The dynamics outside the zero set are those of a Liouville vector field and inside they are those of a reparametrization of a Liouville vector field.

Theorem 2.7 (Reeb-Liouville dynamics of contact Hamiltonian systems [29, Section 3])**.** *Let (M, η, H) be a contact Hamiltonian system. Then:*

- *The Hamiltonian vector field X_H is the Reeb vector field of $\tilde{\eta} = \eta/H$ on $U = \{x \in M \mid H \neq 0\}$.*
- *Assuming that $\mathcal{R}(H) \neq 0$ in the zero level set $\mathcal{L}_0 = H^{-1}(0)$ then, the form*

  $$\omega_0 = -\mathrm{d}i_0^*\eta \tag{2.49}$$

  is an exact symplectic form on $\mathcal{L}_0$, where $i_0 : \mathcal{L}_0 \to M$ is the canonical inclusion. Its Liouville vector field,

  $$\iota_{\Delta_0}\omega_0 = i_0^*\eta, \tag{2.50}$$

  satisfies

  $$X_H|_{\mathcal{L}_0} = \mathcal{R}(H)|_{\mathcal{L}_0}\Delta_0. \tag{2.51}$$

  That is, the restriction of the Hamiltonian vector field to the zero level set is a reparametrization of the Liouville vector field.

Proof. First we prove that X_H is the Reeb vector field $\tilde{\mathcal{R}}$ of $\tilde{\eta}$. The contact distribution $\mathcal{H} = \ker \eta = \ker \tilde{\eta}$ is the same for both forms. Hence, X_H is a conformal contactomorphism and, thus, a Hamiltonian vector field $\tilde{X}_{\tilde{H}}$ for $\tilde{\eta}$, with Hamiltonian $\tilde{H} = -\iota_{X_H}\tilde{\eta} = -H/H = -1$, by Item 5 of Proposition 2.5. Thus, $X_H = \tilde{X}_{-1} = \tilde{\mathcal{R}}$, by Equation (2.30).

Now, we continue with the second item. The form ω_0 is closed since it is exact. Then, it is a symplectic form if it is non-degenerate. Let $p \in \mathcal{L}_0$. Notice that, at that point, $\omega_0 = -d\eta|_{T_p\mathcal{L}_0}$. By the condition $\mathcal{R}(H) \neq 0$, we have that $\mathcal{R}_p$ (and, hence $\ker \mathrm{d}\eta = \langle \mathcal{R} \rangle$) is transverse to $T_p\mathcal{L}_0$. But since $\eta_p \wedge d\eta_p^n \neq 0$,then $d\eta|_V$ is non-degenerate for every subspace V transverse to $\ker \mathrm{d}\eta$. Therefore, ω_0 is also non-degenerated.

Finally, we first remark that $X_H(H) = -\mathcal{R}(H)H$ vanishes when restricted to $\mathcal{L}_0$, hence $(i_0)_* X_H = X_H|_{\mathcal{L}_0}$ is a well-defined vector field. By Item 3 of Theorem 2.8,

$$\iota_{X_H}\mathrm{d}\eta = -\mathcal{R}(H)\eta + \mathrm{d}H.$$

Pulling back by i_0, we get

$$\iota_{(i_0)_* X_H} i_0^* \mathrm{d}\eta = -(\mathcal{R}(H) \circ i_0) i_0^* \eta + \mathrm{d} i_0^* H = -(\mathcal{R}(H) \circ i_0) i_0^* \eta,$$

dividing by $-(\mathcal{R}(H) \circ i_0)$,

$$-\iota_{(i_0)_* X_H / \mathcal{R}(H)} i_0^* \mathrm{d}\eta = i_{(i_0)_* X_H / \mathcal{R}(H)} \omega_0 = i_c^* \eta.$$

Thus, $(i_0)_* (X_H / \mathcal{R}(H)) = \Delta_0$, as we wanted to show. □

We also provide some simple examples of contact Hamiltonian systems.

Example 2.4 (System with homogeneous Rayleight dissipation)**.** Let Q be the configuration space of a mechanical system and consider is extended cotangent bundle with its natural contact structure $(T^*Q \times \mathbb{R}, \eta_Q)$ as in Example 2.1. Let $H_0 : T^*Q \to \mathbb{R}$ be a Hamiltonian function (in the usual sense) and let $\gamma \in \mathbb{R}$. We now consider a Hamiltonian

$$\begin{aligned} H : T^*Q \times \mathbb{R} &\to \mathbb{R} \\ (q, p, z) &\mapsto H_0(q, p) + \gamma z. \end{aligned} \tag{2.52}$$

Hence, $(T^*Q \times \mathbb{R}, \eta_Q, H)$ is a Hamiltonian system. Its Hamiltonian vector field is given by

$$X_H = \frac{\partial H_0}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H_0}{\partial q^i} + \gamma p_i \right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H_0}{\partial p_i} - H_0 - \gamma z \right) \frac{\partial}{\partial z}. \tag{2.53}$$

An integral curve of the Hamiltonian vector field satisfies the following equations of motion.

$$\dot{q}^i = \frac{\partial H_0}{\partial p_i}, \tag{2.54a}$$

$$\dot{p}_i = -\frac{\partial H_0}{\partial q^i} - \gamma p, \tag{2.54b}$$

$$\dot{z} = p_i \frac{\partial H_0}{\partial p_i} - H_0 - \gamma z. \tag{2.54c}$$

We can see that the first two equations are the usual Hamilton equations but with an extra term that causes homogeneous Rayleigh dissipation [143]. That is, a dissipation term linear on the momenta. The extra equation describes the behavior of the action and, in this case, the first two equations are uncoupled from the first one.

A simple particular case is to take $Q = \mathbb{R}$ and $H_0 = \frac{1}{2}(p^2 + kq^2)$, for a constant $k \in \mathbb{R}$. That is, the Hamiltonian of a harmonic oscillator of mass 1. In this case, the equations of motion are those of a damped harmonic oscillator

$$\dot{q} = p, \tag{2.55a}$$

$$\dot{p} = -kq - \gamma p, \tag{2.55b}$$

$$\dot{z} = p_i \frac{\partial H_0}{\partial p_i} - H_0 - \gamma z. \tag{2.55c}$$

Example 2.5 (Mechanical system)**.** Most examples in mechanics and thermodynamics have the following structure. Let (Q, g) be a pseudo-Riemannian manifold, and let

$$\begin{aligned} H : T^*Q \times \mathbb{R} &\to \mathbb{R} \\ (q, p, z) &\mapsto \frac{1}{2} g^{-1}(p, p) + V(q, z) = \frac{1}{2} g^{ij}(p, p) + V(q, z), \end{aligned} \tag{2.56}$$

where g^{-1}, with components g^{ij}, is the inverse of the metric g, with components g_{ij}. The first term $g^{-1}(p, p)$ is the kinetic energy and the second one, $V(q, z)$ the potential, which is allowed to depend on the action. The equations of motion are

$$\dot{q}^i = g^{ij} p_j, \tag{2.57a}$$

$$\dot{p}_i = -\frac{1}{2} \frac{\partial g^{jk}}{\partial q^i} q_j q_k - p_i \frac{\partial V}{\partial z}, \tag{2.57b}$$

$$\dot{z} = \frac{1}{2} g^{ij}(p, p) + V(q, z). \tag{2.57c}$$

This generalizes the well-known case when the potential does not depend on the action [2, Section 3.7].

2.3. Contact manifolds as Jacobi manifolds

Contact manifolds are examples of a more general kind of geometric structures introduced in [170, 188], the so-called Jacobi manifolds. Those manifolds generalize many known geometric structures and are characterized by having a bracket which provides a Lie algebra structure on the space of functions. From a dynamical perspective, this is a vast generalization of the Poisson bracket of classical mechanics, and it provides an abstract context in which we can talk about Hamiltonian vector fields. Jacobi brackets are characterized as being the most general local Lie brackets we can write on functions. It is necessarily given by a bivector field and a vector field having suitable compatibility conditions.

Definition 2.7. A *Jacobi manifold* is a triple (M, Λ, E), where Λ is a bivector field (a skew-symmetric contravariant 2-tensor field) and $E \in \mathfrak{X}(M)$ is a vector field, so that the following identities are satisfied:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \tag{2.58}$$

$$\mathcal{L}_E \Lambda = [E, \Lambda] = 0, \tag{2.59}$$

where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket [205, 231].

The Jacobi structure can be characterized in terms of a Lie bracket on the space of functions $C^\infty(M)$, the so-called *Jacobi bracket*.

Definition 2.8. A *Jacobi bracket* $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on a manifold M is a map that satisfies

1. $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra. That is, $\{\cdot, \cdot\}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{2.60}$$

   for arbitrary $f, g, h \in C^\infty(M)$.

2. It satisfies the following locality condition: for any $f, g \in C^\infty(M)$,

$$\operatorname{supp}(\{f, g\}) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g), \tag{2.61}$$

   where $\operatorname{supp}(f)$ is the topological support of f, i.e., the closure of the set in which f is non-zero.

This means that $(C^\infty(M), \{\cdot, \cdot\})$ is a local Lie algebra in the sense of Kirillov [170].

Given a Jacobi manifold (M, Λ, E) we can define a Jacobi bracket by setting

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f). \tag{2.62}$$

In fact, every Jacobi bracket arises in this way.

Theorem 2.8. *Given a manifold M and a $\mathbb{R}$-bilinear map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, the following are equivalent.*

1. *The map $\{\cdot, \cdot\}$ is a Jacobi bracket.*

2. *$(M, \{\cdot, \cdot\})$ is a Lie algebra which satisfies the generalized Leibniz rule*

$$\{f, gh\} = g\{f, h\} + h\{f, g\} + ghE(f), \tag{2.63}$$

   where E is a vector field on M.

3. *There is a bivector field Λ and a vector field E such that (M, Λ, E) is a Jacobi manifold and $\{\cdot, \cdot\}$ is given as in Equation (2.62).*

Proof. By a straightforward computation, (3) implies (2).

The statement (1) follows from (2) by noticing that the generalized Leibniz rule implies that the map $X_f : C^\infty(M) \to C^\infty(M)$ such that $X_f(g) = \{f, g\} + gE(f)$ is a $\mathbb{R}$-linear derivation on $C^\infty(M)$, hence it defines a smooth vector field. Therefore, if g vanishes on a neighborhood of $p \in M$ then $X(g)$ and $gE(f)$ also vanish and, consequently, so does $\{f, g\}$. Hence, $\operatorname{supp}(\{f, g\}) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g)$.

In [170, Section 2], it was proven that every local Lie algebra on the space of functions is provided by a Jacobi structure. That is, (1) implies (3). □

The Jacobi structure also induces a morphism between covectors and vectors.

Definition 2.9. Let (M, Λ, E) be a Jacobi manifold. We define the following morphism of vector bundles:

$$\begin{aligned} \sharp_\Lambda : TM^* &\to TM \\ \alpha &\mapsto \Lambda(\alpha, \cdot), \end{aligned} \tag{2.64}$$

which also induces a morphism of $C^\infty(M)$-modules between the covector and vector fields, as in Equation (2.4).

We will now discuss the main examples of Jacobi manifolds.

One important particular case of Jacobi manifolds are Poisson manifolds, such as symplectic manifolds. A Poisson manifold is a manifold M equipped with a Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ that satisfies the following Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g. \tag{2.65}$$

This can be seen to imply the weak Leibniz rule, giving a local Lie algebra structure on $C^\infty(M)$. In terms of the Jacobi structure (M, Λ, E), Equation (2.63) shows that a Jacobi bracket is Poisson if and only if $E = 0$, hence a Poisson manifold will be denoted (M, Λ).

Of course, the simplest example of a Poisson manifold is a symplectic manifold (M, η), where Λ is just the inverse of the symplectic form ω.

Another example of Poisson manifolds are cosymplectic manifolds.

Example 2.6 (Cosymplectic manifold)**.** A cosymplectic manifold [3, 43, 46] is given by a triple (M, Ω, η) where M is a $(2n+1)$-dimensional manifold, Ω a closed 2-form and η is a closed 1-form, such that $\eta \wedge \Omega^n$ is a volume form on M.

We consider the isomorphism

$$\begin{aligned} \bar{\flat} : T_x M &\to T_x^* M \\ X &\mapsto \iota_X \Omega + \eta(X)\eta. \end{aligned} \tag{2.66}$$

If we denote its inverse by $\bar{\sharp} = \bar{\flat}^{-1}$, then

$$\Lambda(\alpha, \beta) = \Omega(\bar{\sharp}\alpha, \bar{\sharp}\beta),$$

is a Poisson tensor on M.

Not every Jacobi manifold is Poisson. Indeed, given a contact manifold (M, η) we can define a Jacobi structure (M, Λ, E) by taking[4]

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\sharp\alpha, \sharp\beta), \quad E = -\mathcal{R}, \tag{2.67}$$

where $\sharp$ is defined as the inverse of $\flat$, given in Equation (2.4). A simple computation shows that Λ and E satisfy the conditions of Definition 2.7. In coordinates, Λ is given by

$$\Lambda = \frac{\partial}{\partial p_i} \wedge \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z} \right) \tag{2.68}$$

Other important examples of non-Poisson Jacobi manifolds are locally conformally symplectic manifolds.

Example 2.7 (Locally conformal symplectic manifolds)**.** Let (M, Ω) be an *almost symplectic manifold*. That is, a manifold M equipped with a nondegenerate and antisymmetric, but not necessarily closed two-form $\Omega \in \Omega^2(M)$.

(M, Ω) is said to be *locally conformally symplectic* if for each point $x \in M$ there is an open neighborhood U such that $d(e^\sigma \Omega) = 0$, for some $\sigma \in C^\infty(U)$ so that $\mathrm{d}\sigma$ so $(U, e^\sigma \Omega)$ is a symplectic manifold. If $U = M$, then it is said to be globally conformally symplectic. An almost symplectic manifold is a locally (globally) conformally symplectic if there exists a one-form γ that is closed $\mathrm{d}\gamma = 0$ (respectively exact $\gamma = \mathrm{d}\sigma$) and

$$\mathrm{d}\Omega = \gamma \wedge \Omega.$$

The one-form γ is called the *Lee one-form*. Locally conformally symplectic manifolds with Lee form $\gamma = 0$ are symplectic manifolds. We define a bivector Λ on M and a vector field E given by

$$\Lambda(\alpha, \beta) = \Omega(\tilde{\flat}^{-1}(\alpha), \tilde{\flat}^{-1}(\beta)) = \Omega(\tilde{\sharp}(\alpha), \tilde{\sharp}(\beta)), \quad E = \tilde{\flat}^{-1}(\gamma),$$

with $\alpha, \beta \in \Omega^1(M)$ and $\tilde{\flat} : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M)$ modules defined by $\tilde{\flat}(X) = \iota_X \Omega$. Here $\tilde{\sharp} = \tilde{\flat}^{-1}$. In this case, we also have $\tilde{\sharp}_\Lambda = \tilde{\sharp}$. The vector field E satisfies $\iota_E \gamma = 0$ and $\mathcal{L}_E \Omega = 0$, $\mathcal{L}_E \gamma = 0$. Then, (M, Λ, E) is an even dimensional Jacobi manifold.

We will now state the Characteristic Foliation theorem for Jacobi manifolds [75, 170], which shows that Jacobi manifolds can be decomposed in contact and locally conformally symplectic leafs. Before that, we introduce some terminology.

Hamiltonian vector fields associated with functions f on the algebra of smooth functions $C^\infty(M)$ are defined as

$$X_f = \sharp_\Lambda(\mathrm{d}f) + fE, \tag{2.69}$$

[4]The minus signs on the definition are chosen so that the equations of motion and the brackets coincide with the symplectic ones in case of functions independent on z.

The *characteristic distribution* $\mathcal{C}$ of (M, Λ, E) is generated by the values of all the vector fields X_f:

$$\mathcal{C}_x = \langle \{X_f(x) \mid f \in \mathcal{C}^\infty(M)\} \rangle,\ x \in M, \tag{2.70}$$

$$\mathcal{C} = \bigsqcup_{x \in M} \mathcal{C}_x. \tag{2.71}$$

This characteristic distribution $\mathcal{C}$ is defined in terms of Λ and E as follows

$$\mathcal{C}_x = (\sharp_\Lambda)_x (T_x^* M) + \langle E_x \rangle, \quad \forall x \in M$$

where $(\sharp_\Lambda)_x : T_x^* M \to T_x M$ is the restriction of $\sharp_\Lambda$ to $T_x^* M$ for every $p \in M$. Then, $\mathcal{C}_x = \mathcal{C} \cap T_p M$ is the vector subspace of $T_x M$ generated by E_x and the image of the linear mapping $\sharp_x$.

A Jacobi structure (or Jacobi manifold) is said to be *transitive* if its characteristic distribution is the whole tangent bundle TM.

Theorem 2.9 (Characteristic Foliation theorem for Jacobi manifolds [188])**.** *The characteristic distribution of a Jacobi manifold (M, Λ, E) is completely integrable in the sense of Stefan–Sussmann, thus M is equipped with a foliation whose leaves are not necessarily of the same dimension, and it is called the* characteristic foliation. *Each leaf has a unique transitive Jacobi structure such that its canonical injection into M is a Jacobi map (that is, it preserves the Jacobi brackets). Each leaf can be*

1. *A locally conformally symplectic manifold (including symplectic manifolds) if the dimension is even.*

2. *A manifold equipped with a contact one-form if its dimension is odd.*

Remark 2.10. A *completely integrable distribution in the sense of Stefan–Sussmann* is involutive but not necessarily of constant rank, therefore it defines a singular foliation in which leaves are allowed to have different dimensions. A rigorous and more complete statement of this result, which is a generalization of Frobenius Theorem, can be read in [238, Thm. 4.2].

This theorem generalizes the Symplectic Foliation Theorem [256] for Poisson manifolds. In the Poisson case, all the leaves are symplectic manifolds.

2.3.1. Contact manifolds as Jacobi manifolds

We end this section by explaining how some concepts related to contact manifolds and Hamiltonian systems fields fit in this more general framework.

The map $\sharp_\Lambda$ (Definition 2.9), in the case of a contact manifold, is given by

$$\sharp_\Lambda(\alpha) = \sharp(\alpha) - \alpha(\mathcal{R})\mathcal{R}, \tag{2.72}$$

where the equality follows from this computation:

$$\begin{aligned} \Lambda(\alpha, \beta) &= -\iota_{\sharp\beta}\iota_{\sharp\alpha} \mathrm{d}\eta \\ &= \iota_{\sharp\alpha}\iota_{\sharp\beta} \mathrm{d}\eta \\ &= \iota_{\sharp\alpha}(\beta - \eta(\sharp\beta)\eta) = \beta(\sharp\alpha) - \alpha(\mathcal{R})\beta(\mathcal{R}), \end{aligned} \tag{2.73}$$

where we have used that

$$\beta = \flat\sharp\beta = \iota_{\sharp\beta}\mathrm{d}\eta + \eta(\sharp\beta)\eta = \iota_{\sharp\beta}\mathrm{d}\eta + \beta(\mathcal{R})\eta. \tag{2.74}$$

For a contact manifold, $\sharp_\Lambda$ is not an isomorphism. In fact,

$$\ker \sharp_\Lambda = \langle \eta \rangle, \quad \operatorname{im} \sharp_\Lambda = \mathcal{H}. \tag{2.75}$$

The concepts of Hamiltonian vector fields for Jacobi manifolds (Equation (2.69)) also generalizes the one introduced for contact manifolds (Equation (2.27)). This can easily be seen using the characterization on Equation (2.31) and changing $\sharp$ by $\sharp_\Lambda$ using Equation (2.72).

The Jacobi brackets can also be expressed in terms of the Hamiltonian vector field and the contact form.

Proposition 2.11. *On a contact manifold, the Jacobi bracket can be expressed in the following ways.*

$$\{f,g\} = X_f(g) - g\mathcal{R}(f) = -X_g(f) + f\mathcal{R}(g) = \eta([X_g, X_f]). \tag{2.76}$$

In Darboux coordinates, the brackets are given by

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial z}\left(p_i\frac{\partial g}{\partial p_i} - g\right) + \frac{\partial g}{\partial z}\left(p_i\frac{\partial f}{\partial p_i} - f\right) \tag{2.77}$$

Moreover, the map $f \to X_f$ is an antiisomorphism of the Lie algebras.

Proposition 2.12. *Given a contact manifold (M, η), the map $f \to X_f$ is a Lie algebra antiisomorphism between the set of smooth functions with the Jacobi bracket and the set of infinitesimal conformal contactomorphisms with the Lie bracket. Its inverse is given by $X \to -\iota_X\eta$.*

Furthermore, X_f is an infinitesimal contactomorphism if and only if $\mathcal{R}(f) = 0$.

Proof. Let $f \in C^\infty(M)$. We will see that the map is well-defined. Since $\mathcal{L}_{X_f}\eta = -\mathcal{R}(f)\eta$, X_f is an infinitesimal conformal contactomorphism, and it is an infinitesimal contactomorphism if and only if $\mathcal{R}(f) = 0$.

By contracting X_f with the contact form, we can recover f, so the Hamiltonian map is a bijection.

$$-\eta(X_f) = -\eta(\sharp_\Lambda \mathrm{d}f) + \eta(f\mathcal{R}) = f. \tag{2.78}$$

Last of all, we will show that $X \to -\iota_X\eta$ is an antiisomomorphism. Since it is a bijection because $f \mapsto X_f$ is its inverse we only have to show that it is an antihomomorphism. That is, if $f, g \in C^\infty(M)$,

$$\iota_{[X_f,X_g]}\eta = \{f,g\}. \tag{2.79}$$

For this, we will use again Cartan's formula. We first notice that $\mathrm{d}\eta(X_f, X_g) = -\Lambda(\mathrm{d}f, \mathrm{d}g)$. Indeed,

$$\begin{aligned}
\mathrm{d}\eta(X_f, X_g) &= X_f(\eta(X_g)) - X_g(\eta(X_f)) - \iota_{[X_f,X_g]}\eta \\
&= -X_f(g) + X_g(f) - \iota_{[X_f,X_g]}\eta \\
&= -\sharp_\Lambda(\mathrm{d}f)(g) + f\mathcal{R}(g) + \sharp_\Lambda(\mathrm{d}g)(f) - g\mathcal{R}(f) - \iota_{[X_f,X_g]}\eta \\
&= -2\Lambda(\mathrm{d}f, \mathrm{d}g) + f\mathcal{R}(g) - g\mathcal{R}(f) - \iota_{[X_f,X_g]}\eta,
\end{aligned} \tag{2.80}$$

since $\sharp_\Lambda(\mathrm{d}f)(g) = -\sharp_\Lambda(\mathrm{d}g)(f) = \Lambda(\mathrm{d}f, \mathrm{d}g)$, due to the antisymmetry of Λ. From this, we get

$$-\{f,g\} = -2\{f,g\} - \iota_{[X_f,X_g]}\eta, \tag{2.81}$$

hence $\{f,g\} = \iota_{[X_f,X_g]}\eta$. □

2.4. Submanifolds of a contact manifold

As in the case of symplectic manifolds, we can consider several interesting types of submanifolds of a contact manifold (M,η).

To define them, we will use the following notion of *complement* given by the contact structure:

Definition 2.10 (Contact complement)**.** Let (M,η) be a contact manifold and $x \in M$. Let $\Delta_x \subset T_xM$ be a linear subspace. We define the *Jacobi complement* of Δ_x

$$\Delta_x^{\perp_\Lambda} = \sharp_\Lambda(\Delta_x{}^\circ), \tag{2.82}$$

where $\Delta_x{}^\circ = \{\alpha_x \in T_x^*M \mid \alpha_x(\Delta_x) = 0\}$ is the annihilator.

We extend this definition for distributions $\Delta \subseteq TM$ by taking the complement pointwise in each tangent space.

Definition 2.11. Let $N \subseteq M$ be a submanifold. We say that N is[5]:

- *Isotropic* if $TN \subseteq TN^{\perp_\Lambda}$.
- *Coisotropic* if $TN \supseteq TN^{\perp_\Lambda}$.
- *Legendrian* if $TN = TN^{\perp_\Lambda}$.

Indeed, this definition makes sense for arbitrary Jacobi manifolds, in this context, if $TN = TN^{\perp_\Lambda}$ the submanifolds are called *Legendre-Lagrangian*, and they also generalize Lagrangian submanifolds of symplectic manifolds [162].

The coisotropic condition can be written in local coordinates as follows.

Proposition 2.13. *Assume that (M,η) is a $(2n+1)$-dimensional contact manifold. Let $N \subseteq M$ be a k-dimensional submanifold given locally by the zero set of functions $\phi_a : U \to \mathbb{R}$, with $a \in \{1,\dots,2n+1-k\}$. We use Darboux coordinates (Theorem 2.4). We have that*

$$TN^{\perp_\Lambda} = \langle \{Z_a\}_{a=1}^{2n+1-k} \rangle,$$

where,

$$\begin{aligned} Z_a = \sharp_\Lambda(\mathrm{d}\phi_a) &= A_i(\phi_a)B^i - B^i(\phi_a)A_i \\ &= \left(\frac{\partial\phi_a}{\partial q^i} + p_i\frac{\partial\phi_a}{\partial z}\right)\frac{\partial}{\partial p_i} - \frac{\partial\phi_a}{\partial p_i}\left(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial z}\right). \end{aligned}$$

[5] These are the usual definitions for Jacobi manifolds. In the context of contact geometry [138] isotropic (resp. Legendrian) submanifolds are defined as integrable (maximally integrable) submanifolds of the contact distribution. We will prove that both definitions are equivalent in Proposition 2.16.

Therefore, N is coisotropic if and only if, $Z_a(\phi_b) = 0$ for all a, b. In coordinates:

$$\left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right)\frac{\partial \phi_b}{\partial p_i} - \frac{\partial \phi_a}{\partial p_i}\left(\frac{\partial \phi_b}{\partial q^i} + p_i \frac{\partial \phi_b}{\partial z}\right) = 0. \tag{2.83}$$

For studying the properties of these submanifolds we need to analyze the orthogonal complement $\perp_\Lambda$ with some detail.

Proposition 2.14. *Let $\Delta, \Gamma \subseteq TM$ be distributions. The contact complement has the following properties:*

- $(\Delta \cap \Gamma)^{\perp_\Lambda} = \Delta^{\perp_\Lambda} + \Gamma^{\perp_\Lambda}$.
- $(\Delta + \Gamma)^{\perp_\Lambda} = \Delta^{\perp_\Lambda} \cap \Gamma^{\perp_\Lambda}$.

Proof. This is due to the fact that the annihilator interchanges intersections and sums, while the linear map $\sharp_\Lambda$ preserves them. □

We note that the contact distribution $(\mathcal{H}, \mathrm{d}\eta|_{\mathcal{H}})$ is symplectic. Let $\Delta \subseteq \mathcal{H}$. We denote by $\perp_{\mathrm{d}\eta}$ the symplectic orthogonal component:

$$\Delta^{\perp_{\mathrm{d}\eta}} = \{v \in TM \mid \mathrm{d}\eta(v, \Delta) = 0\}, \tag{2.84}$$

We remark that $\mathcal{R} \in \Delta^{\perp_{\mathrm{d}\eta}}$ for any distribution Δ. There is a simple relationship between both notions of orthogonal complement:

Proposition 2.15. *Let $\Delta \subseteq TM$ be a distribution, then*

$$\Delta^{\perp_\Lambda} = \pi_{\mathcal{H}}(\Delta). \tag{2.85}$$

Proof. Let $v \in \Delta^{\perp_{\mathrm{d}\eta}} \cap \mathcal{H}$. Equivalently, $\mathrm{d}\eta(v, \Delta) = 0$ and v is horizontal. We will see that $v \in \Delta^{\perp_\Lambda}$. Indeed, we can easily check that

$$\sharp_\Lambda(\iota_v \mathrm{d}\eta) = \flat^{-1}(\iota_v \mathrm{d}\eta) - \mathrm{d}\eta(v, \mathcal{R}) = \flat^{-1}(\iota_v \mathrm{d}\eta) = v, \tag{2.86}$$

since,

$$\flat(v) = \iota_v \mathrm{d}\eta + \eta(v)\eta = \iota_v \mathrm{d}\eta, \tag{2.87}$$

because v is horizontal. Therefore, $\Delta^{\perp_{\mathrm{d}\eta}} \cap \mathcal{H} \subseteq \Delta^{\perp_\Lambda}$.

To prove the other inclusion, we just count the dimensions. Let $k = \dim \Delta$, so that $\dim \Delta^\circ = 2n + 1 - k$. Since $\Delta^{\perp_\Lambda} = \sharp_\Lambda(\Delta^\circ)$, and $\ker(\sharp_\Lambda) = \langle \eta \rangle$, we find out that if $\eta \in \Delta^\circ$ (i.e., Δ is horizontal), and then $\dim \sharp_\Lambda(\Delta^\circ) = 2n - k$. Otherwise, $\dim \sharp_\Lambda(\Delta^\circ) = 2n - k + 1$. This trivially coincides with the dimension of the right hand side. □

We have the following possibilities regarding the relative position of a distribution Δ in a contact manifold and the contact and Reeb distributions.

Definition 2.12. Let $\Delta \subseteq TM$ be a rank k distribution. We say that a point $x \in M$ is

1. *Horizontal* if $\Delta_x \subseteq \mathcal{H}_x$.

2. *Vertical* if $\Delta_x = (\Delta_x \cap \mathcal{H}_x) \oplus \langle \mathcal{R}_x \rangle$.

3. *Oblique* if $\Delta_x = (\Delta_x \cap \mathcal{H}_x) \oplus \langle \mathcal{R}_x + v_x \rangle$, with $v_x \in \mathcal{H}_x \setminus \Delta_x$.

If x is horizontal, then $\dim \Delta^{\perp_\Lambda} = 2n - k$. Otherwise, $\dim \Delta^{\perp_\Lambda} = 2n + 1 - k$.

We say that a point x in a submanifold $N \subseteq M$ is horizontal, vertical or oblique, respectively, if $x \in T_x N$ is horizontal, vertical or oblique.

There is a characterization of the concept of isotropic/Legendrian submanifolds as integral submanifolds of $\mathcal{H}$ which is frequently used as a definition [138].

Proposition 2.16. *A submanifold $i : N \to M$ of a $(2n+1)$-dimensional contact manifold (M, η) is isotropic if and only if $i^*\eta = 0$, that it, N is an integral submanifold of $\mathcal{H}$, or N is horizontal. Furthermore, it is Legendrian if and only if it is isotropic and of maximal dimension $(\dim(N) = n)$.*

Proof. If N is isotropic, by Proposition 2.15 then $TN \subseteq TN^{\perp_{\mathrm{d}\eta}} \cap \mathcal{H} \subseteq \mathcal{H}$. Thus, N is clearly horizontal.

Conversely, if N is horizontal, we already know that $TN \subseteq \mathcal{H}$. Hence, if $TN \subseteq TN^{\perp_{\mathrm{d}\eta}}$, then it is coisotropic. But $i^*\mathrm{d}\eta = \mathrm{d}i^*\eta = 0$. Hence, $\mathrm{d}\eta$ will vanish on every pair $v, w \in T_x N$. Thus, $TN \subseteq TN^{\perp_{\mathrm{d}\eta}} \cap \mathcal{H}$ and, again by Proposition 2.15, H is isotropic.

If L is isotropic, or, equivalently, horizontal, we can count the dimensions of the orthogonal to the tangent bundle using Definition 2.12 and see that it is Lagrangian precisely when its dimension is n. □

2.4.1. Some facts about Legendrian submanifolds

Legendrian submanifolds are of great importance on contact geometry. According to Alan Weinstein's *Lagrangian creed*, "everything is a Lagrangian submanifold" [255] (in the context of symplectic geometry). In many respects, Legendrian submanifolds are the contact analog of contact submanifolds. Indeed, they also turn out to be "everything" in contact geometry. We provide a couple of examples.

First consider the extended cotangent space. We can characterize which

Proposition 2.17 (1-jets are Legendrian submanifolds)**.** *Let γ be a section of the 1-jet bundle Example 2.1 $J^1 Q = T^*Q \times \mathbb{R} \to \mathbb{R}$. Then $\operatorname{im}\gamma \subseteq T^*Q \times \mathbb{R}$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$ if and only if γ is the 1-jet of a function $f : Q \to \mathbb{R}$. That is, $\gamma = j^1 f = (\mathrm{d}f, f)$.*

Proof. Let $\gamma = (\alpha, f) : Q \to T^*Q \times \mathbb{R}$. Then, either by a coordinate computation or by using the properties of the Liouville 1-form $(\alpha^*\theta_Q = \alpha)$ we can compute

$$\gamma^*(\alpha) = \mathrm{d}f - \alpha^*\theta_Q = \mathrm{d}f - \alpha. \tag{2.88}$$

By Proposition 2.16 $\operatorname{im}\gamma$ is a Legendrian submanifold if and only if (2.88) vanishes. That is, $\alpha = \mathrm{d}f$. □

This theorem is analogous to the fact that, in symplectic geometry, the image of a section $\alpha : Q \to T^*Q$ is Lagrangian in $(T^*Q, \omega_Q = -\mathrm{d}\theta_Q)$ if and only if α is closed (hence, locally the differential of a function) [2].

We can also study when the sections of the extended tangent bundle are Legendrian. In symplectic geometry, the image of a Hamiltonian vector field is a Lagrangian submanifold of the tangent bundle, with the appropriate symplectic structure [242]. The following two theorem ins [162] will show that an analogous result holds for contact manifolds.

Proposition 2.18. *Let (M, η) be a contact manifold. Let $\bar{\eta}$ be the 1-form on $TM \times \mathbb{R}$ given by*

$$\eta^T = \eta^C + t\eta^V, \tag{2.89}$$

where t is the usual coordinate on $\mathbb{R}$ and η^C and η^V are the complete and vertical lifts Appendix A of η to TM.

Then, $(TM \times \mathbb{R}, \bar{\eta})$ is a contact manifold with Reeb vector field $\bar{\mathcal{R}} = \mathcal{R}^V$, that will be called the contact tangent *of M.*

Proof. We denote by $\bar{\flat}$ the $C^\infty(TM \times \mathbb{R})$-module morphism given by

$$\begin{aligned} \bar{\flat} : \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta^T + \eta^T(X)\bar{\eta}. \end{aligned} \tag{2.90}$$

The map $\flat$ denotes the contact isomorphism of (M, η) (see Equation (2.4)). Let X be such that $\eta(X) = 0$ and let t be the coordinate corresponding to $\mathbb{R}$ in $M \times \mathbb{R}$. Then, it follows from a straightforward computation that

$$\begin{aligned} \bar{\flat}(X^V) &= \flat(X)^V, \\ \bar{\flat}(X^C) &= \flat(X)^C + t\flat(X)^V, \\ \bar{\flat}(\mathcal{R}^V) &= \eta^T, \\ \bar{\flat}(\mathcal{R}^C) &= -\mathrm{d}t + t\eta^T, \\ \bar{\flat}\left(\frac{\partial}{\partial s}\right) &= \eta^V. \end{aligned} \tag{2.91}$$

Hence, $\bar{\flat}$ is an isomorphism and η^T is a contact form (we recall that vertical and complete lifts are linearly independent). □

Theorem 2.19. *Let (M, η) be a contact manifold, and let $X \in \mathfrak{X}(M), f \in C^\infty(M)$. We denote*

$$\begin{aligned} X \times f : M &\to TM \times \mathbb{R} \\ x &\mapsto (X_x, f(x)), \end{aligned} \tag{2.92}$$

Then, X is a conformal contactomorphism with factor $-f$ if and only if $\operatorname{im}(X \times f) \subseteq (TM \times \mathbb{R}, \bar{\eta})$ *is a Legendrian submanifold.*

Proof. Let $L = (X \times f)(M)$. Take $x \in M$ and $z = (X_x, f(x)) \in L$. Then

$$\begin{aligned} T_z L = \{w_v = ((X_x)_*(v), v(f)) \mid v \in T_x M\} &\subseteq T_z(M \times \mathbb{R}) \\ &\simeq T_{X_x} M \oplus T_{f(x)} \mathbb{R}. \end{aligned} \tag{2.93}$$

On the other hand, by the properties of lifts [258],

$$X^*(\eta^C) = \mathcal{L}_X \eta, \quad (\eta_{X_x})^V \circ (X_x)_* = \eta_x. \tag{2.94}$$

Notice that L is n-dimensional, hence, by Proposition 2.16, it is Legendrian if and only if η vanishes on TL. Let $v \in TM$,

$$\bar{\eta}(w_v) = (\mathcal{L}_X \eta)_x(v) + f(x)\eta_x(v). \tag{2.95}$$

Hence, η^T vanishes on TL precisely when $(X, -f)$ is an infinitesimal conformal contactomorphism. □

Since a Hamiltonian vector field X_H is a conformal contactomorphism with factor $-\mathcal{R}(f)$, we obtain the following result.

Corollary 2.20. *Let X_H be a Hamiltonian vector field of (M, η, H). Then,* $\operatorname{im}(X_H, \mathcal{R}(H))$ *is a Legendrian submanifold of $(TM \times \mathbb{R}, \eta^T)$.*

This result states that we can understand the dynamics of a contact system as a Legendrian submanifold. This is also a translation for the contact setting of the result obtained by W. M. Tulczyjew in the symplectic setting [241]. We will further develop this theory in Chapter 8.

Legendrian submanifolds are also of great importance on applications in thermodynamics. They represent equilibrium states of the system [203] (see Section 10.1).

We end this chapter stating a theorem [191] [178, Proposition 43.18] that describes the semilocal structure of a Legendrian submanifold. Indeed, the neighborhood of a Legendrian submanifold is like a neighborhood of the zero section of its jet bundle.

Theorem 2.21. *Let L be a Legendrian submanifold of a contact manifold (M, η). Then, there exists an open neighborhood U of L on M and an open neighborhood V of the zero section of $J^1 L = T^*L \times \mathbb{R}$ such that (U, η) and (V, η_L) are strictly contactomorphic.*

This can be seen as a stronger version of Darboux theorem, since the open set might be taken to contain a Legendrian submanifold. This is also a contact version of Weinstein's Lagrangian Neighborhood Theorem [254].

2.5. The evolution vector field

As we have shown on the previous sections, given a function f on a contact manifold (M, η), one can naturally define its Hamiltonian vector field and study its dynamics. Nevertheless, another vector can be defined: the evolution vector field $\mathcal{E}_f$. This vector was introduced in [7] and its properties were further described in [9]. This work was motivated by the study of some non-equilibrium thermodynamic systems, but the evolution vector field can be introduced from a purely geometric perspective.

Definition 2.13. Given a function H on a contact manifold (M, η), we define its *evolution vector field* $\mathcal{E}_H$ by

$$\mathcal{E}_H = \sharp_\Lambda(\mathrm{d}H) = X_H + H\mathcal{R} \tag{2.96}$$

or, equivalently

$$\flat(\mathcal{E}_f) = \mathrm{d}f - \mathcal{R}(f)\,\eta. \tag{2.97}$$

In Darboux coordinates, it is given by:

$$\mathcal{E}_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + p_i\frac{\partial f}{\partial p_i}\frac{\partial}{\partial z} \tag{2.98}$$

As it is shown on the previous sections, a contact manifold naturally carries a Jacobi structure such that the brackets are related to the contact Hamiltonian dynamics. On the other hand, the dynamics of the evolution vector field is given by an almost-Poisson bi-vector Λ (2.67), which is part of the Jacobi structure (Λ, E). From this bi-vector, we can define the *Cartan bracket*

$$\begin{aligned}[f, g] &= \Lambda(\mathrm{d}f, \mathrm{d}g) = \mathcal{E}_f(g) = -\mathcal{E}_g(f) \\ &= \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial z}\left(p_i\frac{\partial g}{\partial p_i}\right) + \frac{\partial g}{\partial z}\left(p_i\frac{\partial f}{\partial p_i}\right),\end{aligned} \tag{2.99}$$

which is an *almost-Poisson* bracket. This means that it is bilinear, antisymmetric and it fulfills the Leibniz rule, but it does not satisfy the Jacobi identity (hence it is not Poisson), since $[\Lambda, \Lambda] = 2E \wedge \Lambda \neq 0$.

Remark 2.22. It is important to notice that, that unlike the case of the contact Hamiltonian vector field and like the symplectic case, the dynamics depend on $\mathrm{d}H$. This implies that the dynamical vector field is unnafected by adding constants to the Hamiltonian.

The dynamical properties of the evolution vector field differ with those of the Hamiltonian vector field. Indeed, unlike the Hamiltonian vector field, the evolution vector field preserves the energy of the system, but it does not preserve the contact distribution.

Proposition 2.23. *The flow of evolution vector field preserves the Hamiltonian of the system. Indeed,*

$$\mathcal{E}_H(H) = 0. \tag{2.100}$$

The Lie derivative of the contact form η with respect to the evolution vector field $\mathcal{E}_H$ associated to the Hamiltonian function f satisfies the following relation

$$\mathcal{L}_{\mathcal{E}_H}\eta = -\mathcal{R}(H)\eta + dH. \tag{2.101}$$

The evolution vector field is tangent to the contact distribution. That is,

$$\iota_{\mathcal{E}_H}\eta = 0. \tag{2.102}$$

Proof. The first statement follows from the antisymmetry of Λ. Indeed,

$$\mathcal{E}_H(H) = \Lambda(\mathrm{d}H, \mathrm{d}H) = 0. \tag{2.103}$$

The second one follows from the properties of the Lie derivative and those of the Hamiltonian vector field (Proposition 2.5):

$$\begin{aligned} \mathcal{L}_{\mathcal{E}_H}\eta &= \mathcal{L}_{X_H + H\mathcal{R}}\eta = \mathcal{L}_{X_H}\eta + \mathcal{L}_{H\mathcal{R}}\eta \\ &= -\mathcal{R}(f)\eta + (\iota_{\mathcal{R}}\eta)\mathrm{d}f = -\mathcal{R}(f)\eta + \mathrm{d}f. \end{aligned}$$

The last claim follows from the fact that $\operatorname{im}\sharp_\Lambda = \mathcal{H}$ (Equation (2.75)). □

However, the evolution vector field is not a conformal contactomorphism.

Proposition 2.24. *The Lie derivative of the contact form η with respect to the evolution vector field $\mathcal{E}_f$ associated to the Hamiltonian function f satisfies the following relation*

$$\mathcal{L}_{\mathcal{E}_f}\eta = -R(f)\eta + \mathrm{d}f. \tag{2.104}$$

Proof. The proof is a trivial consequence of the properties of the Lie derivative and those of the Hamiltonian vector field:

$$\begin{aligned} \mathcal{L}_{\mathcal{E}_f}\eta &= \mathcal{L}_{X_f + f\mathcal{R}}\eta = \mathcal{L}_{X_f}\eta + \mathcal{L}_{f\mathcal{R}}\eta \\ &= -\mathcal{R}(f)\eta + (i_{\mathcal{R}}\eta)\mathrm{d}f = -\mathcal{R}(f)\eta + \mathrm{d}f \end{aligned}$$

□

An interesting remark is that, on the level sets of the Hamiltonian, the evolution vector field behaves as a reparametrization of a Liouville vector field. The following theorem should be compared to Theorem 2.7.

Theorem 2.25. *Let $\mathcal{L}_c = H^{-1}(c)$ be a level set of $H : M \to \mathbb{R}$ where $c \in \mathbb{R}$. We assume that $\mathcal{R}(H)(x) \neq 0$ for all $x \in \mathcal{L}_c$. Then*

1. *The 2-form $\omega_c \in \Omega^2(\mathcal{L}_c)$ defined by*

$$\omega_c = -\mathrm{d}i_c^*\eta$$

   is an exact symplectic structure. Here $i_c : \mathcal{L}_c H \hookrightarrow M$ denotes the canonical inclusion

2. *If Δ_c is the Liouville vector field, that is,*

$$\iota_{\Delta_c}\omega_c = i_c^*\eta$$

   then the restriction of $\mathcal{E}_H$ to $\mathcal{L}_c$ verifies that

$$\mathcal{E}_H|_{\mathcal{L}_c(H)} = \mathcal{R}(H)|_{\mathcal{L}_c}\Delta_c. \tag{2.105}$$

   This just means that the evolution vector field restricted to the level sets of the Hamiltonian is a reparametrization of the Liouville vector field.

Proof. The form ω_c is trivially closed. To see that it is a symplectic form, we just need to check that is non-degenerate. Let $p \in \mathcal{L}_c(H)$. Notice that, at that point, $\omega_c = -d\eta|_{T_p\mathcal{L}_c(H)}$. By the condition $\mathcal{R}(H) \neq 0$, we have that $\mathcal{R}_p$ (and, hence $\ker \mathrm{d}\eta = \langle \mathcal{R} \rangle$) is transverse to $T_p\mathcal{L}_c(H)$. But since $\eta_p \wedge d\eta_p^n \neq 0$,then $\mathrm{d}\eta|_V$ is non-degenerate for every subspace V transverse to $\ker \mathrm{d}\eta$. Therefore, ω_c is also non-degenerate.

For the second part, we first remark that $\mathcal{E}_H = 0$, hence $(i_c)_*\mathcal{E}_H = \mathcal{E}_H|_{\mathcal{L}_c(H)}$ is a well-defined vector field. By Proposition 2.24 and Cartan's identity

$$\iota_{\mathcal{E}_H}\mathrm{d}\eta = -\mathcal{R}(H)\eta + \mathrm{d}H.$$

Pulling back by i_c, we get

$$\iota_{(i_c)_*\mathcal{E}_H} i_c^*\mathrm{d}\eta = -(\mathcal{R}(H) \circ i_c) i_c^*\eta + d i_c^* H = -(\mathcal{R}(H) \circ i_c) i_c^*\eta,$$

dividing by $-(\mathcal{R}(H) \circ i_c)$,

$$-\iota_{(i_c)_*\mathcal{E}_H/\mathcal{R}(H)} i_c^*\mathrm{d}\eta = i_{(i_c)_*\mathcal{E}_H/\mathcal{R}(H)}\omega_c = i_c^*\eta.$$

Thus, $(i_c)_*(\mathcal{E}_H/\mathcal{R}(H)) = \Delta_c$, as we wanted to show. □

3. Contact Lagrangian systems

The theory of Hamiltonian mechanics, in its geometric interpretation is placed on the cotangent bundle T^*Q of a configuration manifold Q, equipped with its canonical symplectic form ω_Q. Given a Hamiltonian function H on T^*Q, we obtain a Hamiltonian system (T^*Q, ω_Q, H) such that the integral curves of its Hamiltonian vector field are solutions to the Hamilton equations.

On the other hand, the Lagrangian formalism takes place on the tangent bundle TQ. Unlike, the cotangent bundle, the tangent bundle does not carry a canonical symplectic structure. However, given a *regular* Lagrangian function $L : TQ \to \mathbb{R}$ (i.e., the Hessian matrix L with respect to the velocities is a regular matrix), we can use the structure of the tangent bundle to construct a symplectic form ω_L. Then we obtain the usual Euler-Lagrange equations (in the latter case, the vector field providing the dynamics is the solution ξ_L of the equation $\iota_{\xi_L}\omega_L = \mathrm{d}E_L$, where E_L is the energy of the system; ξ_L is a second order differential equation on TQ whose solutions are the ones of the Euler-Lagrange equations). This is the so-called Klein or Cartan geometric formalism of Lagrangian mechanics [73, 74, 171, 226], which produces the same equations of motion as the variational formalism through the least action principle. The Lagrangian and Hamiltonian formalism are related by the Legendre transformation.

This situation is completely mirrored on the contact case. As we have seen in the previous chapter, the extended cotangent bundle $T^*Q \times \mathbb{R}$ has a natural contact structure. Hence, given a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$, we can define the dynamics through the Hamiltonian vector field X_H.

On the contrary, the *extended tangent bundle* $TQ \times \mathbb{R}$ has no natural contact structure. However, an *action-dependent Lagrangian function* $L : TQ \times \mathbb{R} \to \mathbb{R}$ satisfying some regularity conditions induces a contact structure η_L and some dynamics. We remark that the $\mathbb{R}$ factor on the extended tangent bundle represents the action, not the time.

This dynamics is given by the Herglotz equations and can be constructed in two equivalent ways. One comes from the Herglotz [159] variational principle, in which the Lagrangian depends on the action. The connection of the Herglotz principle with contact geometry was developed in [139, 252]. There is also a geometric, which uses the structure of the tangent bundle was introduced by us in [89].

This chapter will have three sections. In the first two, Sections 3.1 and 3.2, we explain the geometric and the variational theory of action-dependent Lagrangians, and are based on the first part of our article [89]. The last part deals with the geometric theory of Lagrangian mechanics for the evolution vector field, which was introduced in [7]. We need new tools to formulate a variational principle for the evolution vector field, so it will be postponed until Section 6.3.

3.1. The geometric theory

Consider an *action dependent Lagrangian* function $L : TQ \times \mathbb{R} \to \mathbb{R}$ (which we will often just call a *Lagrangian*).

We will assume that L is regular. That is, its Hessian matrix with respect to the velocities (W_{ij}) is regular, where

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}. \tag{3.1}$$

We denote by S the vertical endomorphism on the tangent bundle TQ and by Δ, the Liouville vector field. We will denote with the same symbols their extension to the product $TQ \times \mathbb{R}$ by zeros. That is, in local bundle coordinates $(q^i, \dot{q}^i, z)$ we have that

$$S = \mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i}, \tag{3.2a}$$

$$\Delta = \frac{\partial}{\partial \dot{q}^i}. \tag{3.2b}$$

Using these structures and the Lagrangian, we are able to define the *Lagrangian 1-form*

$$\eta_L = \mathrm{d}z - \theta_L = \mathrm{d}z - S^*(\mathrm{d}L) = \mathrm{d}z - \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i, \tag{3.3}$$

which is a contact form on $TQ \times \mathbb{R}$ if and only if L is regular. Indeed,

$$\eta_L \wedge (\mathrm{d}\eta_L)^n = \det(W_{ij}) \mathrm{d}^n q \wedge \mathrm{d}^n \dot{q} \wedge \mathrm{d}z \neq 0, \tag{3.4}$$

The corresponding Reeb vector field is,

$$\mathcal{R}_L = \frac{\partial}{\partial z} - W^{ij} \frac{\partial^2 L}{\partial \dot{q}^i \partial z} \frac{\partial}{\partial \dot{q}^j}, \tag{3.5}$$

where (W^{ij}) is the inverse of the Hessian matrix of L with respect to the velocities.

The *Lagrangian energy* of the system is defined as

$$E_L = \Delta(L) - L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L. \tag{3.6}$$

From a straightforward computation, we obtain

$$\mathcal{R}_L(E_L) = -\frac{\partial L}{\partial z}. \tag{3.7}$$

We denote by $\flat_L$ the vector bundle isomorphism given by Equation (2.4) for the contact form η_L on $TQ \times \mathbb{R}$. That is,

$$\begin{aligned} \flat_L : T(TQ \times \mathbb{R}) &\to T^*(TQ \times \mathbb{R}) \\ v &\mapsto \iota_v(\mathrm{d}\eta_L) + (\iota_v \eta_L)\, \eta_L. \end{aligned} \tag{3.8}$$

The dynamics of the system is given by the *Herglotz vector field* ξ_L, which is just the Hamiltonian vector field of the system $(TQ \times \mathbb{R}, \eta_L, E_L)$. That is, it is the unique vector field satisfying

$$\flat_L(\xi_L) = \mathrm{d}E_L - (\mathcal{R}(E_L) + E_L)\,\eta_L \tag{3.9}$$

A direct computation from Equation (3.9) shows that ξ_L is locally given by

$$\xi_L = \dot{q}^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i} + L\frac{\partial}{\partial z}, \tag{3.10}$$

where the components b^i satisfy the equation

$$b^i \frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + \dot{q}^i \frac{\partial}{\partial q^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + L\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = \frac{\partial L}{\partial \dot{q}^j}\frac{\partial L}{\partial z}. \tag{3.11}$$

Then, if $(q^i(t), \dot{q}^i(t), z(t))$ is an integral curve of ξ_L and substituting its values in Equation (3.11), we obtain

$$\ddot{q}^i \frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + \dot{q}^i \frac{\partial}{\partial q^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + \dot{z}\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}, \tag{3.12}$$

which corresponds to the generalized Euler-Lagrange equations considered by G. Herglotz in 1930 [159] (see also [139, 140])

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z} \tag{3.13}$$

$$\dot{z} = L \tag{3.14}$$

Remark 3.1. The vector field ξ_L is a so-called a *second order differential equation* (*SODE*) or a *semispray* [106].

A vector field X is said to be a SODE if, in bundle coordinates $(q^i, \dot{q}^i, z)$, it has the form

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i} + c\frac{\partial}{\partial z}. \tag{3.15}$$

This condition can be written in algebraic terms as follows

$$S(X) = \Delta. \tag{3.16}$$

Also equivalently, X is a SODE if and only if its integral curves are of the form $(c, \dot{c})$ in bundle coordinates, where c is a curve in Q and $\dot{c}$ its derivative.

3.1.1. The Legendre transformation

In classical mechanics the Lagrangian formulation is connected to the Hamiltonian formulation through the Legendre transformation, which is a map from the tangent to the cotangent bundle. In the contact framework the Hamiltonian formulation takes place on the contact manifold $(TQ \times \mathbb{R}, \eta_Q)$ (see Example 2.1).

The *Legendre transformation* is the fiber derivative of the Lagrangian $\mathrm{F}L : TQ \times \mathbb{R} \to \mathbb{R}$ on the vector bundle $TQ \times \mathbb{R} \to Q \times \mathbb{R}$. That is, is the map given by

$$\begin{aligned} \mathrm{F}L(q^i, \dot{q}^i, z) &= (q^i, \hat{p}_i, z) \\ \hat{p}_i &= \frac{\partial L}{\partial \dot{q}^i} \end{aligned} \tag{3.17}$$

We say that L is *hyperregular* if the Legendre transformation is a diffeomorphism. In that case, the generalized Euler-Lagrange equations are transformed into the contact Hamilton equations.

Indeed, a direct computation shows that

$$(\mathrm{F}L)^* \eta_Q = \eta_L, \tag{3.18}$$

and then we have

$$(\mathrm{F}L)_* (\xi_L) = X_H, \tag{3.19}$$

where $H = E_L \circ (\mathrm{F}L)^{-1}$.

$$\begin{tikzcd} & \mathbb{R} & \\ TQ \times \mathbb{R} \arrow[ur, "E_L"] \arrow[rr, "\mathrm{F}L"] \arrow[dr, "\tau_M"'] & & T^*Q \times \mathbb{R} \arrow[ul, "H"'] \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd} \tag{3.20}$$

We note that Equation (3.19) implies that $\gamma : [t_0, t_1] \to TQ \times \mathbb{R}$ is an integral curve of ξ_L if and only if $\mathrm{F}L \circ \gamma$ is an integral curve of X_H.

3.1.2. Examples of action-dependent Lagrangians

The following Lagrangian is the counterpart of Example 2.5.

Example 3.1 (Mechanical system)**.** Let (Q, g) be a pseudo-Riemannian manifold, and let

$$\begin{aligned} &L : TQ \times \mathbb{R} \to \mathbb{R}, \\ &(q, p, z) \mapsto \frac{1}{2} g(\dot{q}, \dot{q}) - V(q, z) = \frac{1}{2} g_{ij}(\dot{q}, \dot{q}) - V(q, z). \end{aligned} \tag{3.21}$$

Again, the first term is the kinetic energy and the second one, $V(q, z)$, the potential, which is allowed to depend on the action. The Herglotz equations are

$$\left(\frac{1}{2} \frac{\partial g^{jk}}{\partial q^i} - \frac{\partial g^{ij}}{\partial q^k} \right) \dot{q}^j \dot{q}^k - \frac{\partial V}{\partial q^i} - g_{ij} \ddot{q}^j = -g_{ij} \dot{q}^j \frac{\partial V}{\partial z}, \tag{3.22a}$$

$$\dot{z} = L. \tag{3.22b}$$

This can be rewritten as

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = \frac{\partial V}{\partial \dot{q}} - \dot{q}^j\frac{\partial V}{\partial z}, \tag{3.23a}$$

$$\dot{z} = L, \tag{3.23b}$$

where

$$\Gamma^i_{jk} = -g^{li}\left(\frac{1}{2}\frac{\partial g^{jk}}{\partial q^l} - \frac{\partial g^{lj}}{\partial q^k}\right) \tag{3.23c}$$

are the Christoffel symbols of g (see [2, Section 3.7]).

The Legendre transformation is given by

$$\mathrm{F}L(q^i, \dot{q}^i, z) = (q^i, g_{ij}\dot{q}^i, z), \tag{3.23d}$$

which maps this system to Example 2.5

The following example was introduced in [131], and all the computations are performed in detail in [221]. This Lagrangian Has a different structure that the others that we have seen so far, and it is able to provide a friction force quadratic in the velocities.

Example 3.2 (Parachute equation)**.** The parachute equation models a falling object under the action of constant gravity with drag proportional to the square of the velocity. It is the Herglotz equation of the Lagrangian

$$\begin{aligned} L : T\mathbb{R} \times \mathbb{R} &\to \mathbb{R}, \\ (q, p, z) &\mapsto \frac{1}{2}\dot{y}^2 - \frac{mg}{2\gamma}(e^{2\gamma y} - 1) + 2\gamma\dot{y}z. \end{aligned} \tag{3.24}$$

Indeed, the equation of motion is

$$\ddot{y} - \gamma\dot{y}^2 + g = 0. \tag{3.25}$$

3.2. The Herglotz variational principle

In this chapter we present a variational principle for contact Lagrangian systems. Since the Lagrangian depends on the action, the action will no longer be defined through a definite integral, but through a non-autonomous ODE.

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function. In this section we will recall the so-called Herglotz's principle, a modification of Hamilton's principle that allows us to obtain Herglotz's equations (Equation (3.13)), sometimes called generalized Euler-Lagrange equations. See [159], or [189] for a recent discussion.

Fix $q_0, q_1 \in Q$ and an interval $[t_0, t_1] \subset \mathbb{R}$. We denote by $\Omega(q_0, q_1, [t_0, t_1]) \subseteq (C^\infty([t_0, t_1] \to Q))$ the space of smooth curves c such that $c(t_0) = q_0$ and $c(t_1) = q_1$. This space has the structure of an infinite dimensional smooth manifold whose tangent space at c is given by the set of vector fields over c that vanish at the endpoints [2, Proposition 3.8.2], that is,

$$\begin{aligned}T_c\Omega(q_0, q_1, [t_0, t_1]) = \{&\delta c \in \mathcal{C}^\infty([t_0, t_1] \to TQ) \mid \\ &\tau_Q \circ \delta c = c,\ \delta c(t_0) = 0,\ \delta c(t_1) = 0\}.\end{aligned} \tag{3.26}$$

Curves c_s, $s \in \mathbb{R}$ on $\Omega(q_0, q_1, [t_0, t_1])$ are often called *variations* of the curve $c = c_0$ on the physics literature. Tangent vectors at c are *infinitesimal variations*.

We will define the action in two steps. First, we fix an *initial action* $z_0 \in \mathbb{R}$. Consider the operator

$$\mathcal{Z} : \Omega(q_0, q_1, [t_0, t_1]) \to \mathcal{C}^\infty([t_0, t_1] \to \mathbb{R}) \tag{3.27}$$

that assigns to each curve c the function $\mathcal{Z}(c)$ that is the solution of the following Cauchy problem:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Z}(c)(t)}{\mathrm{d}t} &= L(c(t), \dot{c}(t), \mathcal{Z}(c)(t)),\\ \mathcal{Z}(c)(t_0) &= z_0.\end{aligned} \tag{3.28}$$

The quantity $Z(c)(t)$ can be interpreted as the action of the curve $c(t)$ at time t.

Now we define the *Herglotz action functional* as the map which assigns to each curve its action increment:

$$\begin{aligned}\mathcal{A} : \Omega(q_0, q_1, [t_0, t_1]) &\to \mathbb{R},\\ c &\mapsto \mathcal{Z}(c)(t_1) - z_0 = \int_{t_0}^{t_1} L \circ (c, \dot{c}, \mathcal{Z}(c)),\end{aligned} \tag{3.29}$$

that is, $\mathcal{A} = \mathrm{ev}_{t_1} \circ \mathcal{Z} - \mathrm{ev}_{t_0} \circ \mathcal{Z}$, where $\mathrm{ev}_t : \zeta \mapsto \zeta(t_1)$ is the evaluation map at t.

Remark 3.2. This theorem generalizes Hamilton's Variational Principle [2, Theorem 3.8.3]. Indeed, in the case that the Lagrangian is independent of the action (i.e., $L(q^i, \dot{q}^i, z) = \hat{L}(q^i, \dot{q}^i)$) the Herglotz action reduces to the usual Euler-Lagrange action.

Remark 3.3. The action functional and the operator $\mathcal{Z}$ do not only depend on the Lagrangian, like in the case of Hamilton's principle, but also on the initial action z_0.

Remark 3.4. The action is sometimes defined [89, 140] as the final value of $\mathcal{Z}(c)$ instead of the increment

$$\mathcal{A}_0(c) = \mathcal{Z}(c)(t_1) = \mathcal{A}(c) + z_0. \tag{3.30}$$

This two action functionals only differ by a constant. In particular, they have the same critical points. We prefer the former one because it coincides with the Euler-Lagrange action in the case that the Lagrangian is independent on z.

Last of all, we will prove that the critical points of this action functional correspond to solutions of Herglotz's equations (3.13).

Theorem 3.5 (Herglotz's variational principle)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $c \in \Omega(q_0, q_1, [t_0, t_1])$. Then, c is a critical point of $\mathcal{A}$ if and only if $(c, \dot{c}, \mathcal{Z}(c))$*

satisfies Herglotz's equations:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}(c(t),\dot{c}(t),\mathcal{Z}(c)(t))\right) - \frac{\partial L}{\partial q^i}(c(t),\dot{c}(t),\mathcal{Z}(c)(t)) \\
= \frac{\partial L}{\partial \dot{q}^i}(c(t),\dot{c}(t),\mathcal{Z}(c)(t))\frac{\partial L}{\partial z}(c(t),\dot{c}(t),\mathcal{Z}(c)(t)).
\end{aligned}
\tag{3.31}
$$

Proof. We let $\delta c \in T_c\Omega(q_0,q_1,[t_0,t_1])$ be an arbitrary infinitesimal variation such that $\delta c = \frac{\mathrm{d}c_s}{\mathrm{d}s}|_{t=0}$.

In order to simplify the notation, let $\psi = T_c\mathcal{Z}(\delta v)$. It can be naturally identified with a function $\psi : [t_0,t_1] \to \mathbb{R}$, such that $\psi(t_0) = 0$ since $\mathcal{Z}(c_s)(0) = z_0$ is constant. We remind that our aim is to compute $T_c\mathcal{A}$, but, since $T_c\mathcal{A}(\delta c) = T_c\mathcal{Z}(\delta c)(t_1) = \psi(t_1)$, we will first compute ψ. For this, we will take the derivative of the ODE (3.31) defining $\mathcal{Z}$:

$$
\begin{aligned}
\dot{\psi}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{Z}(c_s(t))|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}(c_s(t))|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}L(c_s(t),\dot{c}_s(t),\mathcal{Z}(c_s)(t))|_{s=0} \\
&= \frac{\partial L}{\partial q^i}(\chi(t))\delta c^i(t) + \frac{\partial L}{\partial \dot{q}^i}(\chi(t))\delta\dot{c}^i(t) + \frac{\partial L}{\partial z}(\chi(t))\psi(t).
\end{aligned}
$$

Hence, the function ψ is the solution to the first order ODE above. Explicitly

$$
\psi(t) = \frac{1}{\sigma(t)}\int_{t_0}^{t}\sigma(\tau)\left(\frac{\partial L}{\partial q^i}(\chi(\tau))\delta c^i(\tau) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\delta\dot{c}^i(\tau)\right)\mathrm{d}\tau, \tag{3.32}
$$

where

$$
\sigma(t) = \exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}(\chi(\tau))\mathrm{d}\tau\right) > 0. \tag{3.33}
$$

Note that

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) = -\frac{\partial L}{\partial z}(\chi(t))\sigma(t). \tag{3.34}
$$

Integrating by parts the last term and using that the infinitesimal variation δc vanishes at the endpoints, we get the following expression:

$$
\begin{aligned}
T_c\mathcal{A}(\delta c) &= \psi(t_1) \\
&= \frac{1}{\sigma(t_1)}\int_{t_0}^{t_1}\delta c^i(t)\left(\sigma(t)\frac{\partial L}{\partial q^i}(\chi(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma(t)\frac{\partial L}{\partial \dot{q}^i}(\chi(t))\right)\right)\mathrm{d}t \\
&= \frac{1}{\sigma(t_1)}\int_{t_0}^{t_1}\sigma(t)\left(\frac{\partial L}{\partial q^i}(\chi(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}(\chi(t)) + \frac{\partial L}{\partial \dot{q}^i}(\chi(t))\frac{\partial L}{\partial z}(\chi(t))\right)\mathrm{d}t.
\end{aligned}
$$

Using the fundamental lemma of calculus of variations, we can conclude that c is a critical point of $\mathcal{A}$ if and only if

$$
\sigma(t)\left(\frac{\partial L}{\partial q^i}(\chi(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}(\chi(t)) + \frac{\partial L}{\partial \dot{q}^i}(\chi(t))\frac{\partial L}{\partial z}(\chi(t))\right) = 0, \tag{3.35}
$$

or, reordering terms and dividing by σ, if and only if c satisfies Herglotz's equations. □

Remark 3.6. By the results of the previous section, if the Lagrangian is regular, then Herglotz's equations, (and, therefore, the variational problem) is equivalent to a contact Hamiltonian system. The situation for singular Lagrangians is more subtle. It will be studied on Section 5.1.

3.3. The Lagrangian formalism for the evolution vector field

In [7], we introduced a Lagrangian formalism for the evolution vector field. The geometric formulation is completely analogous to the one for the Hamiltonian vector field.

Indeed, given a regular Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ we again consider the induced contact Hamiltonian system structure $(TQ \times \mathbb{R}, \eta_L, \mathcal{E}_L)$, and we consider the *Herglotz-evolution vector field* Ξ_L. This vector field is just the evolution vector field of this contact Hamiltonian system. That is, the one fulfilling

$$\flat_L(\Xi_L) = \mathrm{d}E_L - \mathcal{R}_L(E_L)\eta_L. \tag{3.36}$$

We note that Ξ_L is also a SODE:

$$S(\Xi_L) = \Delta. \tag{3.37}$$

From a similar argument to the one we used for the Herglotz vector field, we see that the integral curves of Ξ_L are those of the form $(c, \dot{c}, c_z)$, where (c, c_z) is a solution of the *Herglotz-evolution* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z} \tag{3.38a}$$

$$\dot{z} = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}. \tag{3.38b}$$

We note from Equation (3.38b) that the z variable can no longer be interpreted as the action. In some situations, it can be interpreted as a thermodynamic potential (see Section 10.2).

One can also check that, in the case that the Lagrangian is hyperregular and $(\mathrm{F}L)^*H = E_L$, then the evolution and Herglotz-evolution vector fields are $(\mathrm{F}L)$-related. That is,

$$(\mathrm{F}L)_*(\Xi_L) = \xi_H. \tag{3.39}$$

In [9] we introduced an analogous to the Herglotz variational principle for the evolution vector field. This is a nonlinear nonholonomic principle which we will be explained in Section 6.3.

4. Symmetries and equivalences of Lagrangian systems

In this chapter we study transformations which preserve the dynamics. These transformations come in different flavors: similarities, symmetries and equivalences, and each of them is also classified according how much of the geometry of the system it keeps invariant (e.g. the contact form, the contact distribution or just the dynamics). Similarities are diffeomorphisms between two different manifolds, symmetries are transformations that map a system to itself. An equivalence occurs when two different geometric structures produce the same dynamics. Although an equivalence is not given by a diffeomorphism, we can formally think about it as a symmetry in which the transformation is the identity map.

Infinitesimal symmetries and dissipated quantities

Symmetries also have their infinitesimal counterparts that are given by vector fields. Indeed, X is an infinitesimal symmetry (of a given type) if its flow consists of symmetries (of the same type). On classical mechanics, these symmetries are related to conserved quantities. Indeed, Noether's Theorem is one of the most relevant results relating symmetries of a Lagrangian system and conserved quantities of the corresponding Euler-Lagrange equations. In the simplest view, the existence of a *cyclic* coordinate implies the conservation of the corresponding momentum. Indeed, if $L = L(q^i, \dot{q}^i)$ does not depend on the coordinate q^j, then, using the Euler-Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0, \tag{4.1}$$

we deduce that (see [13])

$$\dot{p}_j = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = 0. \tag{4.2}$$

Noether's Theorem can be described on a geometric framework [4, 5, 44, 47, 48, 54, 55, 58, 67, 73, 99, 100, 213, 214, 227–229]. In that framework, L is a function on the tangent bundle TQ of the configuration manifold Q and X be a vector field on Q. We denote by X^C and X^V the complete and vertical lifts of X to the tangent bundle TQ. Then (see [106]):

Theorem 4.1 (Noether). $X^C(L) = 0$ *if and only if* $X^V(L)$ *is a conserved quantity.*

This approach has permitted a deep investigation on other possible infinitesimal symmetries, relating them with the corresponding conserved quantities. A first distinction with the Hamiltonian framework is that we can consider point-base symmetries and symmetries on the phase space of velocities.

The literature about this subject is indeed very extensive. See for example Cantrijn and Sarlet [44], Sarlet [228] and Sarlet and Cantrijn [229], Prince [213, 214], Crampin [73], Marmo and Mukunda [192], Cicogna and Gaeta [67], Aldaya and de Azcárraga [4, 5], even with more general symmetries, Sarlet, Cantrijn, and Crampin [227], or, for the time dependent case, Cariñena *et al.* [47, 48, 54, 55, 167], or singular Lagrangian systems [58], and for higher order Lagrangian systems in de León and Martín de Diego [99–101].

Nonetheless, when we try to extend this result to the case of action-dependent Lagrangians, something unexpected happens. Now, our equations of motion are the Herglotz equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}. \tag{4.3}$$

Assume that q^j is a cyclic quantity, that is, L does not depend on q^j. Then, by the Herglotz equations (4.3), we obtain

$$\dot{p}_j = \frac{\partial L}{\partial z} p_j, \tag{4.4}$$

As we know, we also have

$$\dot{E}_L = \frac{\partial L}{\partial z} E_L. \tag{4.5}$$

Note that, assuming that E_L is nonzero, then p_j/E_L is a conserved quantity.

A relevant comment here is that in contact Lagrangian systems we do not obtain conserved quantities, but quantities that dissipate at the same rate as the energy of the system E_L. Those quantities will be called *dissipated quantities*. They have a nice characterization using the Jacobi bracket as functions that commute with the Hamiltonian.

On this chapter we will provide a geometric formalism on which this case of a cyclic variable can be interpreted and generalized.

We note that in [131], the authors describe the concept of infinitesimal symmetries and dissipated quantities on the Hamiltonian framework. Their results and definitions are particular cases of ours. Also, we acknowledge that symmetries and integrability has been studied for contact systems of the Reeb type [28, 168].

4.0.1. Equivalent Lagrangians

After discussing symmetries and dissipated quantities, we turn to equivalences. More precisely, to the problem of *equivalent Lagrangians* [217] which can be stated as follows in the case of classical mechanics. Given a Lagrangian L, find every Lagrangian $\tilde{L}$ that have the same Euler-Lagrange equations. A sufficient and necessary condition is that their difference is a *total derivative*. In coordinates, this means that

$$L(q^i, \dot{q}^i) - \tilde{L}(q^i, \dot{q}^i) = \dot{q}^i \frac{\partial f}{\partial \dot{q}^i}(q^i, \dot{q}^i), \tag{4.6}$$

for some local function $f : U \subseteq TQ \to \mathbb{R}$. On some physical theories such as in electromagnetism [134], there are so-called *gauge symmetries*[1] that preserve the dynamics but change the Lagrangian to an equivalent one. These are understood as changes of the mathematical formulation of the problem (i.e., the Lagrangian) which leave the physics invariant. When trying to develop a theory of electromagnetism with dissipation on a contact framework, we are interested in preserving these symmetries. And hence, a theory of equivalent action-dependent Lagrangian systems is needed.

At a first glance, we encounter a problem. Unlike on the classical case, action-dependent Lagrangians are in one to one correspondence with their Herglotz vector fields, or with the Herglotz equation. Just note that the last equation is

$$\dot{z} = L(q, \dot{q}, z). \tag{4.7}$$

Hence, we trivially recover the Lagrangian from the last equation. In other words, any two different Lagrangians give rise to different equations of motion, and hence to two different dynamics. Hence, the only equivalent Lagrangian to L would be L itself.

However, we argue that we should be more flexible on what we mean by "having the same dynamics". We will give two reasons, one from a physical perspective and another variational one.

From a physical point of view, we are interested on when two dynamics are physically distinguishable. But the variable z represents the action (or a thermodynamic potential such as the entropy or the internal energy), which are not directly measurable. If the change on the dynamics only changes the action z to $\bar{z}$ but preserves the physical motion of the system, we would be perfectly satisfied. This is the case of the example on electromagnetism [134].

From the variational perspective, on the classical setting theorem of calculus the total derivative of f, the action of a curve $q : [t_0, t_1] \to Q$ changes from

$$\mathcal{A}(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \mathrm{d}t, \tag{4.8}$$

into

$$\begin{aligned} \bar{\mathcal{A}}(q) &= \int_{t_0}^{t_1} \bar{L}(q(t), \dot{q}(t) \mathrm{d}t \\ &= \int_{t_0}^{t_1} (L(q(t), \dot{q}(t)) - \dot{q}^i \frac{\partial f}{\partial \dot{q}^i}(q^i, \dot{q}^i))) = \mathcal{A}(q) + f(t_0) - f(t_1), \end{aligned} \tag{4.9}$$

by the fundamental theorem of calculus. Hence, the action changes by an overall constant independent on the path q (thus, it has the same critical points and induces the same dynamics). This also points out that if we want to understand these equivalences on the more general context of action dependent Lagrangians, we should allow some change on the action, which is just the variable z on the action-dependent formalism.

These arguments lead us to think that, in order to obtain a more useful statement of the equivalent Lagrangians' problem in the action-dependent setting, we need to consider of equivalence of the dynamics "up to change of variables in z". In [84] we formalize

[1]This is the name in the physics language, in our language those are Lagrangian equivalences.

this notion in a geometric with the introduction of *extended contact systems*, and analyze their consequences.

Finally, we come back to symmetries. But now, they are given by the action of a Lie group. On some situations we may be able to construct a lower dimensional Hamiltonian system preserving the major features of the dynamics, by taking some kind of quotient. This process is called reduction, and it has been widely studied in geometric mechanics. The main result of our paper [88] is a proof of the contact version of the famous result due to A. Weinstein, the coisotropic reduction theorem [196]. This result provides a reduced contact quotient manifold such that a Legendre submanifold of the original contact manifold with clean intersection with the coisotropic submanifold is projected in a reduced Legendre submanifold. This result is used to give a simple proof of the contact reduction theorem in presence of symmetries (i.e, there is a Lie group acting on the contact manifold by contactomorphisms and a moment map), an extension of the well-known symplectic reduction theorem proved by J.E. Marsden and A. Weinstein [196] (see also [202] for a previous version of this result). Even if the contact reduction theorem is known in the literature, we are interested in its dynamical implications when a Hamiltonian function is also invariant by the group of symmetries.

We also notice that there are some results on coisotropic reduction [27, 239], but from a different point of view, since we are interested in the applications of the reduction to the contact Hamiltonian dynamics.

This chapter is structured as follows. In Section 4.1 we introduce the language of similarities, symmetries and equivalences and their infinitesimal counterpart and study some of their properties on the context of contact Hamiltonian systems (M, η, H). Later, in Section 4.2 we review the relationship between dissipated quantities and infinitesimal symmetries on contact Hamiltonian systems. Next, Section 4.3 also deals with symmetries and infinitesimal quantities, but this time in the Lagrangian formalism. Section 4.4 is about extended Lagrangian systems and equivalent Lagrangians. Finally, in Section 4.5 we deal with reduction and moment maps on contact systems.

The results of this chapter are mostly published in [84, 88, 90]. From [90] we take the results on the correspondence of action symmetries and dissipated quantities. The part of on the Lagrangian equivalence and extended tangent bundles comes from [84]. The results on reduction were published on [88].

However, there are some changes. For the sake of having a consistent terminology, we have adopted the general framework of [90], which has forced us to rename some symmetries. Also, whenever possible, we try to relate symmetries with their infinitesimal counterpart, which made us introduce the *generalized dynamical symmetry* and the *action symmetry*, which is the infinitesimal version of a Lagrangian symmetry. Both do not appear on the published articles.

4.1. Similarities, symmetries and equivalences in contact Hamiltonian systems

In order to have a consistent terminology, we will introduce the concepts of equivalence, similarity and symmetry, as we defined them in [84]. Assume that M and $\bar{M}$, are smooth manifolds equipped some geometric structure and some dynamics (of course, in our case we will be dealing with contact Hamiltonian system) and let $F : M \to \bar{M}$ be a diffeomorphism preserving the dynamics and, perhaps, part of the geometric structure. Then we will say that F is an *equivalence*, and that M and $\bar{M}$ are equivalent. In the case that M and $\bar{M}$ are the same manifold we say that F is a symmetry. Furthermore, if $F = \mathrm{id}_M$, we say that the geometric structures on M and $\bar{M}$ are *equivalent*. We now proceed to explain what we mean explicitly from this idea.

In our case (M, η, H) and $(\bar{M}, \bar{\eta}, \bar{H})$ are contact Hamiltonian systems. We denote by X_H the Hamiltonian vector field of $H : M \to \mathbb{R}$ with respect to η and by $\bar{X}_{\bar{H}}$ the Hamiltonian vector field of $\bar{H} : N \to \mathbb{R}$ with respect to $\bar{\eta}$. We also denote by $\mathcal{R}$ and $\bar{\mathcal{R}}$ the Reeb vector fields of η and $\bar{\eta}$, respectively, and $\mathcal{H} = \ker \eta$ and $\bar{\mathcal{H}} = \ker \bar{\eta}$ to the contact distributions.

Similarities and their related symmetries and equivalences are defined according to how much of the contact structure they preserve (either the contact form, the contact distribution or just the dynamics).

Definition 4.1. Let (M, η, H) and $(\bar{M}, \bar{\eta}, \bar{H})$ be contact Hamiltonian systems. We say that F is a:

- *Generalized dynamical similarity* if it preserves the vertical component of the dynamics. That is, $\bar{\eta}(F_* X_H) = \bar{\eta}(\bar{X}_{\bar{H}})$.
- *Dynamical similarity* if it preserves the dynamics. That is, $F_* X_H = \bar{X}_{\bar{H}}$.
- *Conformal similarity* if it preserves the dynamics and the contact distribution. That is, F a dynamical similarity ($F_* X_H = \bar{X}_{\bar{H}}$) and a conformal contactomorphism ($F^* \bar{\eta} = f \eta$ for some $f : M \to \mathbb{R}$ or, equivalently, $F_* \mathcal{H} = \bar{\mathcal{H}}$).
- *Strict similarity* if it preserves the dynamics and the contact form. That is, F a dynamical similarity and a contactomorphism ($F^* \bar{\eta} = \eta$).

In the case that $M = \bar{M}$, we say that F is a *generalized dynamical*, *dynamical*, *conformal* or *strict symmetry*.

When $F = \mathrm{id}_M$ we say that the systems are equivalent. That is, (M, η, H) and $(M, \bar{\eta}, \bar{H})$ are

- *Generalized dynamically equivalent* if $\eta(X_H) = \eta(\bar{X}_{\bar{H}})$.
- *Dynamically equivalent* if $X_H = \bar{X}_{\bar{H}}$.
- *Conformally equivalent* if they are dynamically equivalent ($F_* X_H = \bar{X}_{\bar{H}}$) and id_M is a conformal contactomorphism ($\bar{\eta} = f \eta$ for some $f : M \to \mathbb{R}$ or, equivalently, $\mathcal{H} = \bar{\mathcal{H}}$).

Remark 4.2. We have not listed the definition of *strictly equivalent* systems. Indeed, assume that (M, η, H) and $(M, \bar{\eta}, \bar{H})$ are *strictly equivalent*, that is, $X_H = \bar{X}_{\bar{H}}$ and $\eta = \bar{\eta}$. Then

$$H = -\iota_{X_H}\eta = -\iota_{\bar{X}_{\bar{H}}}\bar{\eta} = \bar{H}. \tag{4.10}$$

Hence, two contact Hamiltonian systems can only be strictly equivalent if they are equal.

Proposition 4.3. *Let $F : M \to \bar{M}$ be a strict (resp. conformal with factor f) contactomorphism. Then, it is a strict (resp. conformal) similarity if and only if $F^*H = \bar{H}$ (resp. $F^*H = f\bar{H}$).*

Proof. We prove it in the conformal case. The strict case follows by setting $f = 1$.

Since F is a conformal contactomorphism, $F^*\bar{\eta} = f\eta$ for some non-vanishing $f : M \to \mathbb{R}$. Pulling back by F the equation

$$\bar{\eta}(\bar{X}_{\bar{H}}) = -\bar{H}\,, \tag{4.11}$$

we obtain

$$F^*\bar{\eta}(\bar{X}_{\bar{H}}) = f\eta((F^{-1})_*(X_H)) = -F^*\bar{H}\,. \tag{4.12}$$

Assume $F^*\bar{H} = fH$. Since $\bar{X}_{\bar{H}}$ is a conformal contactomorphism, $(F^{-1})_*\bar{X}_{\bar{H}}$ is an infinitesimal conformal contactomorphism. Moreover, dividing by f on (4.12), we obtain $\eta((F^{-1})_*\bar{X}_{\bar{H}}) = -H$. Thus, by Item 5 in Proposition 2.5 we conclude that $(F^{-1})_*\bar{X}_{\bar{H}}$ is the Hamiltonian vector field of H. Thus, F is a conformal similarity.

Conversely, if F is a conformal similarity, then $(F^{-1})_*\bar{X}_{\bar{H}} = X_H$. Thus, $F^*\bar{H} = -f\eta(X_H) = fH$. ☐

Demanding that a generalized dynamical similarity preserves the contact distribution forces it to preserve the dynamics. Indeed:

Proposition 4.4. *Let $F : M \to \bar{M}$ be a generalized conformal similarity and a conformal contactomorphism. Then F is a conformal similarity.*

Proof. Assume that $F^*\bar{\eta} = f\eta$. We let

$$\begin{aligned}
-F^*\bar{H} &= F^*(\bar{\eta}(\bar{X}_{\bar{H}})) \\
&= F^*(\eta)((F_*)^{-1}\bar{X}_{\bar{H}}) \\
&= f\eta((F_*)^{-1}\bar{X}_{\bar{H}}) \\
&= f\eta(X_H) = -fH.
\end{aligned} \tag{4.13}$$

Thus, by Proposition 4.3, it is a conformal similarity. ☐

Conformal equivalences preserve many geometric properties of the system. Nevertheless, they are able to convert any Hamiltonian vector field of a non-vanishing Hamiltonian into a Reeb vector field of a modified contact form.

Proposition 4.5. *Every contact Hamiltonian system (M, η, H) such that H does not vanish, is conformally equivalent to $(M, \bar{\eta} = -\frac{\eta}{H}, \bar{H} = -1)$, so that $X_H = \bar{X}_{\bar{H}} = \bar{\mathcal{R}}$.*

However, not every Hamiltonian vector field can be turned onto a Reeb vector field. Indeed, since the conformal factor f has to be non-vanishing, we obtain the following restriction.

Proposition 4.6. *A conformal similarity of contact Hamiltonian systems preserves the zero set of H. That is $F : M \to N$ maps the zero set of H to the zero set of $\bar{H}$.*

The zero set of the Hamiltonian has interesting geometric properties. For example, the Hamiltonian vector field can only be tangent to Legendrian manifolds whenever $H = 0$ [93]. This property is important in the thermodynamic formalism, where the equilibrium states are represented by a Legendrian submanifold of the zero set of H.

This result by [29], Theorem 2.7, also indicates that the Hamiltonian vector field has a special behavior at the points where H vanishes. The Hamiltonian is a Reeb vector field in the set $\{H \neq 0\}$, and it is a reparametrization of the Liouville vector field in $H^{-1}(0)$.

Infinitesimal symmetries

In many situations is easier to deal with vector fields than with smooth maps. We will define "infinitesimal" versions of the symmetries introduced above.

Definition 4.2. Let (M, η, H) be a contact Hamiltonian system. We say that a vector field $X \in \mathfrak{X}(M)$ is an *infinitesimal dynamical* (resp. *generalized dynamical*, *conformal* or *strict*) *symmetry* if its flow $\phi_t : M \to M$ consists of *dynamical* (resp. *generalized dynamical*, *conformal* or *strict*) symmetries.

The following characterization can be obtained using the formula Equation (2.12), which relates the Lie derivative by X with the time derive along $(\phi_t)^*$.

Proposition 4.7. *Let (M, η, H) be a contact Hamiltonian system. $X \in \mathfrak{X}(M)$ is an:*

- *Infinitesimal generalized dynamical symmetry if and only if $\eta([X, X_H]) = 0$.*
- *Infinitesimal dynamical symmetry if and only if $[X, X_H] = 0$.*
- *Infinitesimal conformal symmetry if and only if X is an infinitesimal conformal contactomorphism and an infinitesimal dynamical symmetry.*
- *Infinitesimal strict symmetry if and only if X is an infinitesimal contactomorphism and an infinitesimal dynamical symmetry*

We can also obtain infinitesimal versions of Propositions 4.3 and 4.4.

Proposition 4.8. *Let $X \in \mathfrak{X}(M)$ be a strict (resp. conformal with factor a) infinitesimal contactomorphism. Then, it is a strict (resp. conformal) similarity if and only if $H = \bar{H}$ (resp. $H = a\bar{H}$).*

Proposition 4.9. *A generalized infinitesimal dynamical symmetry X that is also an infinitesimal conformal contactomorphism if and only if it is an infinitesimal conformal symmetry.*

4.2. Infinitesimal symmetries and dissipated quantities

As it is well-known, in Lagrangian and Hamiltonian mechanics, infinitesimal symmetries can be associated to conserved quantities. In the Lagrangian formalism this can be done through Noether's theorem and its generalizations, which we will cover on the following sections. In the case of Hamiltonian mechanics, this is usually studied through Poisson brackets. Indeed, on a symplectic manifold, f is a conserved quantity if and only if $\{H,f\} = X_H F = -X_f(H) = 0$.

On contact manifolds this is no longer the case. Indeed, using Proposition 2.11, one has that $\{H,f\} = 0$ if and only if

$$X_H(f) = -\mathcal{R}(H)f. \tag{4.14}$$

Because of this, we make the following definition.

Definition 4.3. On a Hamiltonian system (M, η, H), we say that a function f is *H-dissipated* if $\{H,f\} = 0$. Equivalently, f dissipates at the same rate as the Hamiltonian, i.e., $X_H(f) = -\mathcal{R}(H)f$.

We note that the set of H-dissipated functions is a Lie subalgebra of $(C^\infty(M), \{\cdot,\cdot\})$. Indeed, $\mathbb{R}$-linear combinations of H-dissipated functions are H-dissipated, and, because of the Jacobi identity, the Jacobi bracket of two H-dissipated functions is H-dissipated. Moreover, it is an algebra over the set of conserved quantities; this is, if f is a H-dissipated quantity and g is a conserved quantity, then fg is H-dissipated:

$$X_H(fg) = gX_H(f) = -\mathcal{R}(H)fg. \tag{4.15}$$

If we assume that H has no zeros, we can relate H-dissipated functions to conserved functions. Assume that f is H-dissipated, then f/H is a conserved quantity. Indeed:

$$X_H\left(\frac{f}{H}\right) = \frac{X_H(f)H - fX_H(H)}{H^2} = \frac{-\mathcal{R}(H)fH + \mathcal{R}(H)fH}{H^2} = 0. \tag{4.16}$$

In general, if f_1, f_2 commutes with H, then f_1/f_2 is a conserved quantity, assuming f_2 has no zeros.

The conclusion is that, in order to obtain conserved quantities, one should find quantities that dissipate at the same rate as the Hamiltonian.

Remark 4.10. In the particular case where $\mathcal{R}(H) = 0$, then the H-dissipated quantities are precisely the conserved quantities. That is, $\{H,f\} = 0$ if and only if $X_H(f) = 0$.

In the case that H has no zeros there is a correspondence between sets of m independent conserved quantities and sets of m H-dissipated quantities by taking the quotients. Explicitly, if $f_1, \dots f_m$ commute with H, then

$$g_i = \frac{f_i}{H} \tag{4.17}$$

are conserved quantities. Conversely, if $g_1, \dots g_m$ are conserved quantities, then

$$f_i = g_i H \tag{4.18}$$

are H-dissipated quantities.

We will now study the relationship of infinitesimal symmetries for a contact Hamiltonian system (M, η, H) and H-dissipated quantities. First we prove the following result, which will help us to compute Jacobi brackets.

Proposition 4.11. *Let X be a vector field such that $\eta(X) = -f$, then*

$$\{H, f\} = -\eta([X_H, X]) = (\mathcal{L}_X \eta)(X_H) + X(H). \tag{4.19}$$

Proof. If $\eta(X) = -f$, then $\eta(X - X_f) = 0$, so that $X - X_f$ is in the kernel of η.

Since

$$\mathcal{L}_{X_H} \eta = -\mathcal{R}(H)\eta,$$

we deduce that

$$(\mathcal{L}_{X_H} \eta)(X_f) = (\mathcal{L}_{X_H} \eta)(X).$$

Therefore, using Proposition 2.11 and Cartan's formula twice, one finds

$$\begin{aligned}
\{H, f\} &= -\eta([X_H, X_f]) \\
&= (\mathcal{L}_{X_H} \eta)(X_f) - X_H(\eta(X_f)) \\
&= (\mathcal{L}_{X_H} \eta)(X) - X_H(\eta(X)) \\
&= -\eta([X_H, X]).
\end{aligned}$$

From the second equality, we have

$$\begin{aligned}
-\eta([X_H, X]) &= (\mathcal{L}_X \eta)(X_H) - X(\eta(X_H)) \\
&= (\mathcal{L}_X \eta)(X_H) + X(H),
\end{aligned}$$

applying again Cartan's formula. □

Using Proposition 4.11, we deduce the following.

Theorem 4.12. *Let X be a vector field on M. Then, X is a generalized infinitesimal dynamical symmetry of (M, η, H) if and only if $\eta(X)$ is an H-dissipated quantity.*

Remark 4.13. The natural correspondence between generalized infinitesimal dynamical symmetries and H-dissipated quantities f is not one-to-one. Indeed, given a generalized infinitesimal dynamical symmetry X such that $-\eta(X) = f$, the set of vector fields $\mathcal{F} = \{X_f + Y \mid Y \in \ker \eta\}$ are the generalized infinitesimal dynamical symmetries corresponding to the quantity f. As one easily sees from Item 4 of Proposition 2.5, X_f is the only one which is a Hamiltonian vector field.

Thus, the map $f \to X_f$ provides a bijection between infinitesimal conformal symmetries and H-dissipated quantities.

There is another concept of symmetry on this setting: *Cartan symmetries*, which generalizes the concept of Cartan symmetries for symplectic Hamiltonian systems.

Definition 4.4. We say that $X \in \mathfrak{X}(M)$ is a *Cartan symmetry* for (M, η, H) if $\mathscr{L}_X \eta = a\eta + \mathrm{d}g$ for some functions $a, g \in C^\infty(M)$ and $X(H) = aH + g\mathcal{R}(H)$.

Theorem 4.14. *Let X be a Cartan symmetry such that $\mathscr{L}_X \eta = \mathrm{d}g + a\eta$. Then $f = \eta(X) - g$ is an H-dissipated quantity.*

Proof. From Proposition 4.11, we have

$$\begin{aligned}
\{H, f\} &= \{H, \eta(X)\} - \{H, g\} \\
&= (\mathscr{L}_X \eta)(X_H) + X(H) - X_H(g) - g\mathcal{R}(H) \\
&= a\eta(X_H) - \mathrm{d}g(X_H) + X(H) - X_H(g) - g\mathcal{R}(H) \\
&= -aH + X(H) - g\mathcal{R}(H) = 0.
\end{aligned}$$

□

Remark 4.15. A Cartan symmetry such that $\mathscr{L}_X \eta = \mathrm{d}g + a\eta$ is a generalized infinitesimal dynamical symmetry when $\mathrm{d}g = 0$.

A generalized infinitesimal dynamical symmetry X is a Cartan symmetry when it is an infinitesimal contactomorphism.

4.3. Symmetries and conserved quantities in contact Lagrangian systems

We let Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a regular Lagrangian and consider the associated contact Hamiltonian system $(TQ \times \mathbb{R}, \eta_L, E_L)$ defined in Section 3.1 and its Herglotz vector field ξ_L.

We will not only consider symmetries of the configuration space Q, but also symmetries that mix the configuration variables and the action, that is, that act on $Q \times \mathbb{R}$. Several kinds of symmetries are related to their associated dissipated quantities in several degrees of generality.

Indeed, we introduce configuration space symmetries of the Lagrangian, generalized configuration space symmetries of the Lagrangian, Noether symmetries, Lie symmetries and action symmetries. We also add examples of systems with some of those symmetries.

4.3.1. Lifts of vector fields on Q and $Q \times \mathbb{R}$

A quick overview of the lifts can be seen in Appendix A. The natural extensions of Y^V and Y^C to $TQ \times \mathbb{R}$ will be denoted by the same symbols.

One can consider more general vector fields. Indeed, let Y be a vector field on $Q \times \mathbb{R}$ given by

$$Y = Y^i \frac{\partial}{\partial q^i} + Z \frac{\partial}{\partial z}, \tag{4.20}$$

that is, Z does not depend on the positions q. In order to obtain a *partial complete lift* $\bar{Y}^C$ on $TQ \times \mathbb{R}$, we can do as follows. We first consider its complete lift to $T(Q \times \mathbb{R})$, which is

$$\begin{aligned} Y^C =& Y^i \frac{\partial}{\partial q^i} + Z \frac{\partial}{\partial z} + \dot{q}^j \frac{\partial Y^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i} \\ &+ \dot{q}^j \frac{\partial Z}{\partial q^j} \frac{\partial}{\partial \dot{z}} + \dot{z} \frac{\partial Y^i}{\partial z} \frac{\partial}{\partial \dot{q}^i} + \dot{z} \frac{\partial Z}{\partial z} \frac{\partial}{\partial \dot{z}}, \end{aligned}$$

Thus, $\bar{Y}^C$ is such that

$$\bar{Y}^C = Y^i \frac{\partial}{\partial q^i} + Z \frac{\partial}{\partial z} + \dot{q}^j \frac{\partial Y^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}. \tag{4.21}$$

In such a case, we will denote by $\bar{Y}^V$ the vertical lift of the projection of Y to Q, say,

$$\bar{Y}^V = Y^i \frac{\partial}{\partial \dot{q}^i}, \tag{4.22}$$

which is obviously tangent to $TQ \times \mathbb{R}$.

4.3.2. Configuration space symmetries

We will now try to understand the symmetries of a Lagrangian system which are lifts of vector fields on the configuration space.

Definition 4.5. We say that a vector field $Y \in \mathfrak{X}(Q)$ is a *configuration space symmetry*[2] of L if $Y^C(L) = 0$.

Theorem 4.16. *Let (M, η_L, E_L) be a contact Lagrangian system and let $Y \in \mathfrak{X}(Q)$. Then, Y is a configuration space symmetry of L if and only if $f = Y^V(L)$ is an E_L-dissipated quantity, that is, it commutes with E_L, or,*

$$\xi_L(f) = -\mathcal{R}_L(E_L) f = \frac{\partial L}{\partial z} f. \tag{4.23}$$

For the proof of Theorem 4.16, we will use the identities listed on the following lemma, that can be proved by a direct computation.

Lemma 4.17. *Let $Y \in \mathfrak{X}(Q)$. The following identities hold:*

$$\eta_L(Y^C) = -Y^V(L), \tag{4.24}$$

$$\mathcal{L}_{Y^C} \eta_L = -\theta_{Y^C(L)} = -\frac{\partial Y^C(L)}{\partial \dot{q}^i} \mathrm{d} q^i, \tag{4.25}$$

$$\mathcal{R}_L(f) = 0, \tag{4.26}$$

$$[Y^C, \Delta] = [Y^C, S] = 0, \tag{4.27}$$

$$S(Y^C) = Y^V. \tag{4.28}$$

[2]These symmetries were called *infinitesimal symmetries* on [90], but we change its name here so that they are not confused with the infinitesimal symmetries presented on Section 4.1.

Proof of Theorem 4.16. We can now compute the Jacobi brackets using the identities form the previous lemma and Proposition 4.11. Let $f = Y^V(L)$ so that $\eta(Y^C) = -f$. Then we have,

$$\begin{aligned}
\{E_L, f\} &= \mathcal{L}_{Y^C}\eta_L(\xi_L) + Y^C(E_L) \\
&= -\theta_{Y^C(L)}(\xi_L) + Y^C(\Delta(L) - L) \\
&= (S(\xi_L))(Y^C(L)) - Y^C(\Delta(L) - L) \\
&= \Delta(Y^C(L)) - Y^C(\Delta(L)) - Y^C(L) = -Y^C(L).
\end{aligned}$$

Therefore, the result follows. □

Remark 4.18. We notice that whenever Y is a configuration space symmetry of L, then Y^C is the Hamiltonian vector field of $Y^V(L)$. Indeed, Y^C is an infinitesimal conformal equivalence.

Remark 4.19. This result should be compared with the *First Noether Theorem* from [140] in the case that L does not depend explicitly on time. The conserved quantity obtained from a symmetry in [140] is not a function on M, but a functional that depends on the chosen integral curve γ of ξ_L. We can recover the result by noticing that the conserved quantity in [140] is given by

$$\begin{aligned}
G[\gamma](t) &= \exp\left(-\int_0^t \frac{\partial L}{\partial z}\mathrm{d}\gamma\right)(Y^V(L)\circ\gamma)(t) \\
&= \exp\left(-\int_0^t \frac{\dot f}{f}\mathrm{d}\gamma\right)(f\circ\gamma)(t),
\end{aligned} \tag{4.29}$$

hence, along an integral curve γ of ξ_L, we have:

$$\dot G[\gamma](t) = -\dot f \exp\left(-\int_0^t \frac{\dot f}{f}\mathrm{d}\gamma\right) + \exp\left(-\int_0^t \frac{\dot f}{f}\mathrm{d}\gamma\right)\dot f = 0. \tag{4.30}$$

We now consider vector fields $Y \in \bar{\mathfrak{X}}(Q\times\mathcal{R})$ of the form

$$Y = Y^i\frac{\partial}{\partial q^i} + Z\frac{\partial}{\partial z}. \tag{4.31}$$

To go further in this case, we need to extend the computations in Lemma 4.17.

Lemma 4.20. *Let $Y \in \bar{\mathfrak{X}}(Q\times\mathbb{R})$ with z component Z. Let $f = -\eta_L(\bar Y^C)$. Then, we have:*

$$\eta_L(\bar Y^C) = -(\bar Y^V(L) - Z), \tag{4.32}$$

$$\mathcal{L}_{\bar Y^C}\eta_L = -\mathcal{R}_L(f)\mathrm{d}z - \theta_{\bar Y^C(L)}, \tag{4.33}$$

$$S(\bar Y^C) = \bar Y^V. \tag{4.34}$$

In this setting, we can provide generalization of the concept of configuration space symmetry and the corresponding dissipated quantities.

Definition 4.6. We say that a vector field $Y \in \bar{\mathfrak{X}}(Q \times \mathbb{R})$ is an *extended configuration space symmetry of* L if $Y^C(L) = -\mathcal{R}_L(f)L$, where $f = -\eta_L(\bar{Y}^C)$

Theorem 4.21. *Let* $Y \in \mathfrak{X}(Q \times \mathbb{R})$*. Then* $f = \bar{Y}^V(L) - Z$ *is an* E_L*-dissipated quantity if and only if* Y *is an extended configuration space symmetry.*

Proof of Theorem 4.21. We proceed as in Theorem 4.16. Let $f = \bar{Y}^V(L) - Z = -\eta(\bar{Y}^C)$.

$$\begin{aligned}
\{E_L, f\} &= (\mathcal{L}_{\bar{Y}^C}\eta_L)(\xi_L) + \bar{Y}^C(E_L) \\
&= -\mathcal{R}_L(f)\mathrm{d}z(\xi_L) - \theta_{\bar{Y}^C(L)}(\xi_L) + \bar{Y}^C(\Delta(L) - L) \\
&= -\mathcal{R}_L(f)L + (S(\xi_L))(\bar{Y}^C(L)) - \bar{Y}^C(\Delta(L) - L) \\
&= -\mathcal{R}_L(f)L + \Delta(\bar{Y}^C(L)) - \bar{Y}^C(\Delta(L)) - \bar{Y}^C(L) \\
&= -\mathcal{R}_L(f)L - \bar{Y}^C(L).
\end{aligned}$$

Therefore, the result follows. □

Remark 4.22. In this case, the fact that $\bar{Y}^C(L) = -\mathcal{R}_L(f)L$ does not ensure that $\bar{Y}^C$ is a Hamiltonian vector field. Indeed, from Equation (4.33), we compute $\mathcal{L}_{\bar{Y}^C}\eta_L = -\mathcal{R}_L(f)\eta_L + L\theta_{\mathcal{R}_L(f)}$, hence $\bar{Y}^C$ is Hamiltonian if and only if $\theta_{\mathcal{R}_L(f)} = 0$. Otherwise, $\bar{Y}^C$ is only an infinitesimal generalized dynamical symmetry.

4.3.3. Noether symmetries

Definition 4.7. We say that $Y \in \bar{\mathfrak{X}}(Q \times \mathbb{R})$ with z component Z is a *Noether symmetry* if $\bar{Y}^C$ is a Cartan symmetry.

From the conservation theorem for Cartan symmetries (Theorem 4.14), we can deduce a new one for Noether symmetries.

Theorem 4.23. *Let* Y *be a Noether symmetry such that* $\mathcal{L}_{\bar{Y}^C}(\eta_L) = \mathrm{d}g + a\eta_L$*. Then* $f = \bar{Y}^V(L) - Z - g$ *is an* E_L*-dissipated quantity.*

Proof. Apply Theorem 4.14 to $\bar{Y}^C$, so, using Lemma 4.20, $\eta(\bar{Y}^C) - g = \bar{Y}^V(L) - Z - g = f$ commutes with E_L. □

Remark 4.24. Infinitesimal symmetries on Q are Noether symmetries for $g = 0$. However, general infinitesimal symmetries on $Q \times \mathbb{R}$ can fail to be Noether symmetries because its complete lift is not a Hamiltonian vector field. See Remark 4.15

4.3.4. Lie symmetries

Definition 4.8. A Lie symmetry is a vector field $Y \in \bar{\mathfrak{X}}(Q \times \mathbb{R})$ such that $\bar{Y}^C$ is a dynamical symmetry.

As a consequence of Theorem 4.12, we obtain the following E_L-dissipated quantity.

Theorem 4.25. *Let* Y *be a Lie symmetry. Then* $f = -\eta_L(\bar{Y}^C) = \bar{Y}^V(L) - Z$ *is an* E_L*-dissipated quantity.*

Remark 4.26. Any infinitesimal symmetry of the Lagrangian L is a Lie symmetry.

4.3.5. Action symmetries

We can define another kind of symmetries that can be thought as infinitesimal reparametrizations of the action. We will expand on this point of view on the next chapter.

Definition 4.9. Let Z_0 be a vector field on $TQ \times \mathbb{R}$ satisfying $(\mathrm{pr}_{TQ})_* Z_0 = 0$. That is,

$$Z_0 = Z(q, \dot{q}, z) \frac{\partial}{\partial z}. \tag{4.35}$$

We say that Z_0 is an *action symmetry* if it is a dynamical symmetry.

Furthermore, Z it is a *strong action symmetry* if, in addition, it is projectable by $\tau_Q^1 : TQ \times \mathbb{R} \to Q \times \mathbb{R}$. That is,

$$Z_0 = Z(q, z) \frac{\partial}{\partial z}. \tag{4.36}$$

Theorem 4.27. *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a regular Lagrangian and let Z_0 be of the form* (4.35). *The following are equivalent:*

1. *Z_0 is a strong action symmetry.*
2. *Z_0 is an infinitesimal conformal symmetry.*
3. *Z does not depend on $\dot{q}$ and*

$$\frac{\partial Z}{\partial z} L + \dot{q}^i \frac{\partial Z}{\partial q^i} = Z \frac{\partial L}{\partial z}. \tag{4.37}$$

Proof. Z_0 is a strong action symmetry if and only if

$$[Z_0, \xi_L] = \left(-Z \frac{\partial L}{\partial z} + \dot{q}^i \frac{\partial Z}{\partial \dot{q}^i} + L \frac{\partial Z}{\partial z} \right) \frac{\partial}{\partial z} = 0. \tag{4.38}$$

This proves that Items 1 and 3 are equivalent.

Now we assume Item 2. Since an infinitesimal conformal symmetry is also an infinitesimal dynamical symmetry, we only need to prove that Z does not depend on the velocities to see that it is an strong action symmetry. Indeed, since Z_0 is a contactomorphism,

$$\mathcal{L}_{Z_0} \eta_L = a \eta_L, \tag{4.39}$$

for some $a : TQ \times \mathbb{R} \to \mathbb{R}$. Or, in bundle coordinates,

$$\mathrm{d}z - \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i = a \left(\mathrm{d}Z - Z \frac{\partial^2 L}{\partial \dot{q}^i \partial z} \mathrm{d}q^i \right). \tag{4.40}$$

Contracting the previous equation by $\partial / \partial \dot{q}^i$, we obtain

$$\frac{\partial Z}{\partial \dot{q}^i} = 0. \tag{4.41}$$

Hence, Item 1 follows.

Finally, we assume Item 1 and Item 3 (which we have already proven equivalent), and try to obtain Item 2. Since Z_0 is a dynamical symmetry, it last to show that it is a conformal contactomorphism. That is, we need to show that Equation (4.40) is satisfied.

Since Z does not depend on the velocities, the $\dot{q}^i$ component vanish at both sides of the equation.

From the z component, we obtain that

$$a = \frac{\partial Z}{\partial z}. \tag{4.42}$$

The q^i components, are equal if and only if

$$-\frac{\partial Z}{\partial z}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial Z}{\partial q^i} - Z\frac{\partial^2 L}{\partial \dot{q}^i \partial z}. \tag{4.43}$$

This expression is deduced by taking the partial derivative to Equation (4.37) by $\dot{q}^i$. Thus, Z_0 is a conformal contactomorphism and Item 2 follows. □

Strong action symmetries are the infinitesimal counterpart of strong Lagrangian symmetries (which will be defined in Section 4.4), and this theorem can be seen as an infinitesimal version of Theorem 4.35.

4.3.6. Examples

We will provide two examples of infinitesimal symmetries for contact Lagrangian systems. The first one is similar to the classical angular momentum, but the second one shows that more symmetries appear when we consider vector fields that also depend on the action.

Example 4.1. We will consider a Lagrangian of a 2D particle with a rotationally symmetric potential. Let $Q = \mathbb{R}^2$ and consider the Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ defined by

$$L(q^1, q^2, \dot{q}^1, \dot{q}^2, z) = \frac{(\dot{q}^1)^2 + (\dot{q}^2)^2}{2} - V((q^1)^2 + (q^2)^2, z), \tag{4.44}$$

with a quadratic kinetic energy and a potential energy V. Let the vector field $Y \in \mathfrak{X}(Q)$ be given by

$$Y = -q^2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial q^2}. \tag{4.45}$$

We compute its complete lift according to Equation (A.9):

$$Y^C = -q^2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial q^2} - \dot{q}^2 \frac{\partial}{\partial \dot{q}^1} + \dot{q}^1 \frac{\partial}{\partial \dot{q}^2}. \tag{4.46}$$

We notice that $Y^C(L) = 0$, that is, Y^C is a configuration space symmetry. Hence, by Theorem 4.16 the quantity f is dissipated:

$$f = X^V(L) = -q^1 \dot{q}^2 + q^2 \dot{q}^1. \tag{4.47}$$

This quantity is the well-known angular momentum. Unlike on the symplectic case, it is not conserved but dissipated:

$$\dot{f} = \frac{\partial L}{\partial z} f = -\frac{\partial V}{\partial z} f. \tag{4.48}$$

Example 4.2 (Extended configuration space symmetry: 1D particle)**.** We will study for which potentials does a 1D particle admit configuration space symmetries. Let $L : T\mathbb{R} \to \mathbb{R}$ be given by

$$L(q, \dot{q}, z) = \frac{1}{2}\dot{q} - V(q, z). \tag{4.49}$$

This kind of Lagrangians are inspired in those considered in [235] to describe systems with scaling symmetry in cosmology.

We consider a generic vector field $Y \in \bar{\mathfrak{X}}(\mathbb{R})$ given by

$$Y = y(q, z)\frac{\partial}{\partial q} + Z(z)\frac{\partial}{\partial z}. \tag{4.50}$$

Y is a generalized configuration space symmetry (Definition 4.6) if and only if

$$\bar{Y}^C(L) - \mathcal{R}_L(\eta_L(\bar{Y}^C))L = 0. \tag{4.51}$$

After some computations we find the coordinate expression for Equation (4.51):

$$\dot{q}^2 \frac{\partial y}{\partial q} + \left(\frac{1}{2}\dot{q}^3 - \dot{q}V\right)\frac{\partial y}{\partial z} + \left(\frac{1}{2}\dot{q}^2 - V\right)\frac{\partial Z}{\partial z} + \frac{\partial V}{\partial q} y + \frac{\partial V}{\partial z} Z = 0. \tag{4.52}$$

Taking the third derivative on Equation (4.52) with respect to $\dot{q}$, we find

$$\frac{\partial y}{\partial z} = 0. \tag{4.53}$$

Moreover, if we now take the second derivative with respect to $\dot{q}$, we get

$$2\frac{\partial y}{\partial q} = \frac{\partial Z}{\partial z}. \tag{4.54}$$

Since y depends only on q and Z depends only on z, one has that

$$\frac{\partial^2 y}{\partial q \partial x} = \frac{\partial^2 Z}{\partial q \partial x} = 0, \tag{4.55}$$

hence both y and z are affine functions of the form

$$\begin{cases} y &= aq + b \\ Z &= 2a + c, \end{cases} \tag{4.56}$$

where $a, b, c \in \mathbb{R}$. Substituting this on Equation (4.52) we obtain

$$(aq + b)\frac{\partial V}{\partial q} + (2az + c)\frac{\partial V}{\partial z} - 2aV = 0. \tag{4.57}$$

The solutions of this PDE for V are

$$V(q,z) = \begin{cases} \phi(bz - cq), & \text{if } a = 0 \\ (aq+b)^{-2}\phi\left(\dfrac{2az+c}{(aq+b)^2}\right), & \text{if } a \neq 0, \end{cases} \tag{4.58}$$

for some arbitrary function ϕ. The corresponding H-dissipated quantities are

$$f = (aq+b)\dot{q} - (2az+c). \tag{4.59}$$

In particular, notice that in the cases that Y is a vector field on Q (that is $a = c = 0$), the only potentials that admit a symmetry are the translation invariant ones: those of the form $V = \phi(z)$. We conclude that there are 1D systems that have extended configuration space symmetries but no configuration space symmetries.

4.4. Extended Lagrangian systems and equivalent Lagrangians

When we are given an infinitesimal symmetry on a Lagrangian system, we might be interested on how the flow of the symmetry modifies the system. Unfortunately, the tangent bundle structure is not preserved by its flow. In [84] we introduced the concept of extended Lagrangian systems. Those systems were defined in order to give a formulation of the inverse problem and to study equivalent Lagrangians in the contact setting.

In order to do that, we will allow extra flexibility on how to measure the action. We will "forget" about the projection $z : TQ \times \mathbb{R} \to \mathbb{R}$. We formalize this notion with the *extended tangent bundle* (Definition 4.10). The following sections are devoted to present and explore the extended tangent bundle and how to define a contact Lagrangian formalism and its corresponding contact Hamiltonian formalism. Later, we will introduce the problem of equivalent Lagrangians and explain how it is related to symmetries.

4.4.1. Extended tangent bundles

The main object that we introduce in this section is the *extended tangent bundle* over a manifold Q:

Definition 4.10. An *extended tangent bundle* P of Q is a line bundle $\rho : P \to TQ$. ρ is called the *mechanical state function*. We also denote by $\rho^0 : P \to Q$ to the map $\rho^0 = \tau_Q \circ \rho$, where $\tau_Q : TQ \to Q$ is the canonical projection.

We can think of the extended tangent bundle as the contact phase space. The projection ρ provides the mechanical variables (the positions and velocities). The extra degree of freedom on the bundle represents the action. However, we do not prescribe how the action is measured.

Definition 4.11. An *action function* of the extended tangent bundle $\rho : P \to TQ$ is a surjective map $\zeta : P \to \mathbb{R}$ such that $TP = \ker T\rho \oplus \ker T\zeta$.

$$
\begin{tikzcd}
TP \arrow[r, "\tau_P"] \arrow[d, "T\rho"'] & P \arrow[r, "\zeta"] \arrow[d, "\rho"] \arrow[dr, "\rho^0"] & \mathbb{R} \\
T(TQ) \arrow[r, "\tau_{TQ}"] & TQ \arrow[r] & Q
\end{tikzcd}
\tag{4.60}
$$

Given a natural coordinate system $(q^i, \dot{q}^i)$, we can construct a coordinate system $(\rho^* q^i, \rho^* \dot{q}^i, \zeta)$ on P. From now on, we will abuse notation and omit the ρ^* when using the coordinates on P. Along the text we will use two action functions z and ζ, with the respective coordinate systems $(q^i, \dot{q}^i, z)$ and $(q^i, \dot{q}^i, \zeta)$. Notice that the coordinate basis of vector fields on P depend not only on the coordinates $(q^i, \dot{q}^i)$ on TQ, but also on the action functions. To make this clear, we will denote them as $\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial z}\right)$ and $\left(\left(\frac{\partial}{\partial q^i}\right)_\zeta, \left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta, \frac{\partial}{\partial \zeta}\right)$ respectively. They are related by:

$$
\left(\frac{\partial}{\partial q^i}\right)_\zeta = \frac{\partial}{\partial q^i} - \frac{\frac{\partial \zeta}{\partial q^i}}{\frac{\partial \zeta}{\partial z}} \frac{\partial}{\partial z}; \quad \left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta = \frac{\partial}{\partial \dot{q}^i} - \frac{\frac{\partial \zeta}{\partial \dot{q}^i}}{\frac{\partial \zeta}{\partial z}} \frac{\partial}{\partial z}; \quad \frac{\partial \zeta}{\partial z} \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z}. \tag{4.61}
$$

We proceed to study the geometric structure of the extended tangent bundle given an action function ζ. We have an isomorphism $(\rho, \zeta) : P \to TQ \times \mathbb{R}$. However, only the projection onto the first factor is independent of the choice of the action function. The decomposition $TP = \ker \rho \oplus \ker \zeta$ induces a section λ^ζ of $T\rho$ as follows: given an element $y \in T(TQ)$, $\lambda^\zeta(y) \in TP$ is the unique element such that $T\rho(\lambda^\zeta(y)) = y$ and $T\zeta(\lambda^\zeta(y)) = 0$. With this section we can lift the canonical elements of TQ to P. In local coordinates we have that

$$
\lambda^\zeta = \mathrm{d}q^i \otimes \left(\frac{\partial}{\partial q^i}\right)_\zeta + \mathrm{d}\dot{q}^i \otimes \left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta. \tag{4.62}
$$

Definition 4.12. The *extended almost tangent structure* on P by the action function ρ is the $(1,1)$ tensor field $S^\zeta = \lambda^\zeta \circ S \circ T\rho$.

The *extended Liouville vector field* on P by the action function ρ is $\Delta^\zeta = \lambda^\zeta \circ \Delta \circ \rho$.

In local coordinates, S^ζ is given by

$$
S^\zeta = \mathrm{d}q^i \otimes \left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta = \mathrm{d}q^i \otimes \left(\frac{\partial}{\partial \dot{q}^i} - \frac{\frac{\partial \zeta}{\partial q^i}}{\frac{\partial \zeta}{\partial z}} \frac{\partial}{\partial z}\right). \tag{4.63}
$$

Note that $S^\zeta = S^{\bar{\xi}}$ if and only if $\zeta - \bar{\xi}$ does not depend on $\dot{q}^i$. Also, $\operatorname{im} S^\zeta = \ker T\rho^0 \subseteq \ker S^\zeta$, and $\ker S^\zeta$ is an integrable rank $n+1$ distribution. The extended Liouville vector field is given by

$$
\Delta^\zeta = \dot{q}^i \left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta = \dot{q}^i \left(\frac{\partial}{\partial \dot{q}^i} - \frac{\frac{\partial \zeta}{\partial q^i}}{\frac{\partial \zeta}{\partial z}} \frac{\partial}{\partial z}\right), \tag{4.64}
$$

which is precisely the infinitesimal generator of the $\mathbb{R}$-action $\chi(a)v = \exp(a)v$ on the vector bundle $(\rho^0, \zeta) : P \to Q \times \mathbb{R}$. This allows us to define SODEs on TP.

Definition 4.13. A vector field ξ of P is an *extended SODE* (Second Order Differential Equation) if $S^\zeta(\xi) = \Delta^\zeta$ for some action function ζ.

An extended SODE ξ has the local expression

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + a^i \frac{\partial}{\partial \dot{q}^i} + b \frac{\partial}{\partial z}, \tag{4.65}$$

for $a^i, b \in C^\infty(P)$.

Lemma 4.28. *ξ is an extended SODE if, and only if, $S(\rho_* \xi_p) = \Delta_{\rho(p)}$ for all $p \in P$.*

Proof. Since λ^ζ is a section of $T\rho$:

$$S^\zeta(\xi) = \Delta^\zeta \Leftrightarrow \lambda^\zeta(S(T\rho(\xi))) = \lambda(\Delta \circ \rho) \Leftrightarrow S(T\rho(\xi)) = \Delta \circ \rho\,. \tag{4.66}$$

□

Thus, the concept of extended SODE is independent on the choice of action function.

Extended contact Lagrangian systems

In this section we will define a generalization of contact Lagrangian systems, which will provide us an adequate formulation of the problem of equivalent Lagrangians.

Definition 4.14. An *extended Lagrangian system* on an extended tangent bundle $\rho : P \to TQ$ consists of a Lagrangian function $L : P \to \mathbb{R}$ and an action function $\zeta : P \to \mathbb{R}$.

From an extended Lagrangian system, we can define the *ζ-Lagrangian form*

$$\eta_L^\zeta = \mathrm{d}\zeta - (S^\zeta)^* \mathrm{d}L = \mathrm{d}\zeta - \left(\frac{\partial L}{\partial \dot{q}^i}\right)_\zeta \mathrm{d}q^i. \tag{4.67}$$

This form will be a contact form if and only if L is *ζ-regular*, that is, the matrix (W_{ij}^ζ) defined by

$$W_{ij}^\zeta = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)_\zeta \tag{4.68}$$

is not degenerate. The regularity depends on the Lagrangian and on the action function. For instance, the Lagrangian $L = \frac{1}{2}v^2 - \gamma z$ is regular by the action function $\zeta = z$, but it is not regular by the action function $\zeta = \frac{1}{2}v^2 - \gamma z$.

The pair (P, η_L^ζ) is a contact manifold. We denote its Reeb vector field by

$$\mathcal{R}^\zeta = \frac{\partial}{\partial \zeta} + (W^\zeta)^{ij} \left(\frac{\partial^2 L}{\partial q^i \partial \zeta}\right)_\zeta \left(\frac{\partial}{\partial \dot{q}^j}\right)_\zeta \tag{4.69}$$

Given the *ζ-Liouvill*e vector field

$$\Delta^{\zeta} = \dot{q}^i \left(\frac{\partial}{\partial \dot{q}^i} \right)_{\zeta}, \tag{4.70}$$

we define the *ζ-energy*

$$E_L^{\zeta} = \Delta^{\zeta}(L) - L. \tag{4.71}$$

$(P, \eta_{L,\zeta}, E_{L,\zeta})$ is a Hamiltonian contact system. The corresponding Hamiltonian vector field, that we call the *ζ-Herglotz vector field* $\xi_{L,\zeta}$ is,

$$\iota_{\xi_{L,\zeta}} \eta_{L,\zeta} = -E_L^{\zeta}, \tag{4.72a}$$

$$\mathcal{L}_{\xi_{L,\zeta}} \eta_{L,\zeta} = -\mathcal{R}_L(E_L^{\zeta}) \eta_{L,\zeta} = \frac{\partial L}{\partial \zeta} \eta_{L,\zeta}. \tag{4.72b}$$

We remark that if $\zeta = z$, this is just the usual Herglotz vector field ξ_L.

A variational principle may also be written for an extended Lagrangian system (L, ζ). Given the space $\Omega(q_0, q_1)$ of paths $\gamma : [t_0, t_1] \to Q$ such that $\gamma(t_0) = q_0$ and $\gamma(t_1) = q_1$, and given $\zeta_0 \in \mathbb{R}$, we define

$$\mathcal{X}_{L,\zeta,\zeta_0} : \Omega(q_0, q_1) \to C^{\infty}([t_0, t_1] \to P) \tag{4.73}$$

such that for each curve $\gamma \in \Omega(q_0, q_1)$, $\mathcal{X}_{L,\zeta,\zeta_0}(\gamma)$ is the curve that satisfies $\rho \circ \mathcal{X}_{L,\zeta,\zeta_0}(\gamma) = \dot{\gamma}$, and its ζ component, which we denote by $\mathcal{Z}_{L,\zeta,\zeta_0}(\gamma) = \zeta \circ \mathcal{X}_{L,\zeta,\zeta_0}(\gamma)$ solves the initial value problem

$$\frac{\mathrm{d}\mathcal{Z}^{\zeta}(\gamma)}{\mathrm{d}t} = L \circ \mathcal{X}_{L,\zeta,\zeta_0}(\gamma), \tag{4.74a}$$

$$\mathcal{Z}^{\zeta}(\gamma)(t_0) = \zeta_0. \tag{4.74b}$$

We now define the action

$$\begin{aligned} \mathcal{A}^{\zeta} &: \Omega(q_0, q_1) \to \mathbb{R}, \\ &\gamma \to \mathcal{Z}^{\zeta}(\gamma)(t_1) - \zeta_0 = \int_{t_0}^{t_1} L(q^i(t), \dot{q}^i(t), \zeta(t)) \mathrm{d}t. \end{aligned} \tag{4.75}$$

By repeating the usual computation, but this time with the coordinates $(q^i, \dot{q}^i, \zeta)$, we obtain

Theorem 4.29. *$\gamma \in \Omega(q_0, q_1)$ is a critical point of $\mathcal{A}$ if and only if $(\gamma, \dot{\gamma}, \mathcal{Z}^{\zeta}(\gamma))$ are solutions to the ζ-Herglotz equations:*

$$\left(\frac{\partial L}{\partial q^i} \right)_{\zeta} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right)_{\zeta} = \left(\frac{\partial L}{\partial \dot{q}^i} \right)_{\zeta} \frac{\partial L}{\partial \zeta}, \tag{4.76a}$$

$$\dot{\zeta} = L. \tag{4.76b}$$

Extended contact Hamiltonian systems

Now we define the Hamiltonian counterpart an extended Lagrangian system.

An extended cotangent bundle is a line bundle $\tilde{\rho} : \tilde{P} \to T^*Q$. A *Hamiltonian action function* $\tilde{\zeta} : P \to \mathbb{R}$ is a surjective map such that $T\tilde{P} = \ker T\tilde{\rho} \oplus T\tilde{\zeta}$.

$$
\begin{tikzcd}
T\tilde{P} \arrow[r, "\tau_{\tilde{P}}"] \arrow[d, "T\tilde{\rho}"'] & \tilde{P} \arrow[r, "\tilde{\rho}^0"] \arrow[d, "\tilde{\rho}"] \arrow[dr, "\tilde{\rho}^0"] & \mathbb{R} \\
T(T^*Q) \arrow[r, "\tau_{T^*Q}"] & T^*Q \arrow[r] & Q
\end{tikzcd}
\tag{4.77}
$$

$$
\eta_{Q,\tilde{\zeta}} = \mathrm{d}\tilde{\zeta} - \tilde{\rho}^*\theta_Q = \mathrm{d}\tilde{\zeta} - p_i \mathrm{d}q^i. \tag{4.78}
$$

Now, given a Hamiltonian function $H : \tilde{P} \to \mathbb{R}$ we compute its Hamiltonian vector field $X_{H,\tilde{\zeta}}$ with respect to $\eta_{Q,\tilde{\zeta}}$, which is given in coordinates by

$$
X_H = \frac{\partial H}{\partial p_i}\left(\frac{\partial}{\partial q^i}\right)_{\tilde{\zeta}} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\left(\frac{\partial}{\partial p_i}\right)_{\tilde{\zeta}} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{4.79}
$$

The maps and spaces that will be used on this section are summarized on the following commutative diagram:

$$
\begin{tikzcd}
 & & \mathbb{R} & & \\
P \arrow[urr, "\zeta"] \arrow[rr, "(\rho^0,\zeta)"'] \arrow[d, "\rho"'] \arrow[ddrr, "\rho^0"] & & Q \times \mathbb{R} \arrow[dd, "\mathrm{pr}_Q"] & & \tilde{P} \arrow[ull, "\tilde{\zeta}"'] \arrow[ll, "(\tilde{\rho}^0,\tilde{\zeta})"] \arrow[d, "\tilde{\rho}"] \arrow[ddll, "\tilde{\rho}^0"'] \\
TQ \arrow[drr, "\tau_Q"'] & & & & T^*Q \arrow[dll, "\pi_Q"] \\
 & & Q & &
\end{tikzcd}
\tag{4.80}
$$

The ζ-Legendre transformation

Given an extended Lagrangian system (L, ζ) on the extended tangent bundle $\rho : P \to TQ$, we can define its ζ-Legendre transformation, which maps it to an extended Hamiltonian system.

First, we need to construct the dual Hamiltonian bundle $\tilde{P}$ of P. Indeed, given the action function ζ, we will be able to construct $\tilde{\rho} : \tilde{P} \to T^*Q$ and the action function ζ.

We let $\rho^0 = \rho \circ \tau_Q : P \to Q$. Note that the bundle $(\rho^0, \zeta) : P \to Q \times \mathbb{R} \to Q \times \mathbb{R}$ has a unique structure of a vector bundle such that the map $(\rho, \zeta) : P \to TQ \times \mathbb{R}$ is a vector bundle isomorphism. We can then construct the linear dual bundle $(\tilde{\rho}, \tilde{\zeta}) : \tilde{P} \to Q \times \mathbb{R}$. Now we define the map $\tilde{\rho} : \tilde{P} \to T^*Q$, such that, for any $\alpha_{q_0,\zeta_0} \in \tilde{P}$ and $v_{q_0} \in TQ$, we have that

$$
\tilde{\rho}(\alpha_{q_0,\zeta_0})(v_{q_0}) = \alpha_{q_0,\zeta_0}(v_{q_0,\zeta_0}), \tag{4.81}
$$

where $v_{q_0,\zeta_0} \in P$ is the unique element satisfying $\rho(v_{q_0,\zeta_0}) = v_{q_0}$ and $\zeta(v_{q_0,\zeta_0}) = \zeta_0$.

Now, we can define the ζ-Legendre transform of L as the fiber derivative $\mathrm{F}^{\zeta}L : P \to \tilde{P}$ over the bundle $(\tilde{\rho}_0, \tilde{\zeta}) : \tilde{P} \to Q \times \mathbb{R}$. That is,

$$\mathrm{F}^{\zeta}L(v_{q,\zeta})(w_{q,\zeta}) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} L(v_{q,\zeta} + t w_{q,\zeta}), \tag{4.82}$$

where $v_{q,\zeta}, w_{q,\zeta} \in (\tau, \zeta)^{-1}(q, \zeta)$. In local coordinates,

$$\tilde{\rho}(\mathrm{F}^{\zeta}L(q, \dot{q}, \zeta)) = \left(q, \left(\frac{\partial L}{\partial \dot{q}^i} \right)_{\zeta} \right), \tag{4.83}$$

where we used coordinates $(q^i, \dot{q}^i, \zeta)$ on the right and the dual coordinates $(q^i, p_i^{\zeta}, \zeta)$ on the left. The map $\mathrm{F}^{\zeta}L$ is a local diffeomorphism if and only if L is ζ-regular. If there exist H is such that $\mathrm{F}^{\zeta}L_* H = E_{L,\zeta}$, then $(\mathrm{F}^{\zeta}L)^* \eta_{Q,\zeta^*} = \eta_{L,\zeta}$. Hence, $\mathrm{F}^{\zeta}L$ is a strict similarity for the contact systems $(TQ \times \mathbb{R}, E_L^{\zeta}, \eta_L^{\zeta})$ and $(T^*Q \times \mathbb{R}, H, \eta_Q^{\zeta^*})$.

4.4.2. Equivalent extended contact systems

A smooth change in the z variable corresponds to a change in the action function, which are realized by fiber bundle automorphisms on P.

Definition 4.15. A *horizontal diffeomorphism* is a fiber bundle automorphism ϕ of $\rho : P \to TQ$.

Given two action functions z, ζ, there exists a unique horizontal diffeomorphism that satisfies the commutative diagram

$$\begin{tikzcd}
 & \mathbb{R} & \\
P \arrow[ur, "z"] \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & P \arrow[ul, "\zeta"'] \arrow[dl, "\rho"] \\
 & TQ &
\end{tikzcd} \tag{4.84}$$

and it is given by $\phi = (\mathrm{id}_{TQ}, \zeta)$ on the trivialization provided by (ρ, z).

In the case that ζ does not depend on the velocities, that is $\zeta = \tau^* \zeta_0$, where $\zeta_0 : Q \times \mathbb{R} \to \mathbb{R}$ we say that ϕ is a *strong horizontal diffeomorphism*.

Subsequently, we analyze how the horizontal transformation acts on the structures of the extended tangent bundle and how they can be used to study equivalent Lagrangians.

Equivalence in extended tangent bundles

A horizontal diffeomorphism ϕ acts on the action function by pullback $z = \phi^* \zeta = \zeta \circ \phi$, as we can see in diagram (4.84). The corresponding extended almost tangent structures and Liouville vector fields are not preserved by ϕ, in general. More precisely, we have the following result.

Lemma 4.30. *If ϕ is a strong horizontal diffeomorphism, then*

$$\phi_* S^{\phi^*\zeta} = S^\zeta, \quad \phi_* \Delta^{\phi^*\zeta} = \Delta^\zeta. \tag{4.85}$$

Now we will study the action of horizontal diffeomorphisms on extended SODEs. First we must see that it is well-behaved. It turns out that preserving extended SODEs actually characterizes horizontal diffeomorphisms.

Proposition 4.31. *A vector bundle automorphism ϕ of $\rho^0 : P \to Q$ preserves extended SODEs (that is $\phi_*\xi$ is a SODE whenever ξ is a SODE) if and only if it is a horizontal diffeomorphism.*

Proof. Let, ξ be a SODE and let $\phi(q^i, \dot{q}^i, z) = (q^i, v^i, \zeta)$. Then, using the characterization given by Lemma 4.28,

$$S(T\rho(\phi_*\xi)) = v^j \frac{\partial}{\partial \dot{q}^i}. \tag{4.86}$$

ϕ is a horizontal diffeomorphism if and only if $v^i = \dot{q}^i$. Clearly, this is the case if and only if $\phi_*\xi$ is a SODE.

□

Since horizontal diffeomorphisms preserve SODEs, we can classify SODEs by these transformations.

Definition 4.16. We say that two extended SODEs ξ and $\bar{\xi}$ on P are *horizontally similar* if there exists a horizontal diffeomorphism ϕ such that $\phi_*\xi = \bar{\xi}$. If ϕ is a strong horizontal diffeomorphism, then we say that ξ and $\bar{\xi}$ are *strongly horizontally similar*.

A direct computation shows that two SODEs, ξ and $\bar{\xi}$, are horizontally similar if, and only if, there exists a function $\phi = (\mathrm{id}_{TQ}, \zeta)$ that satisfies

$$a^i = \phi^* \bar{a}^i, \tag{4.87a}$$

$$\frac{\partial \zeta}{\partial q^i}\dot{q}^i + a^i \frac{\partial \zeta}{\partial \dot{q}^i} + b\frac{\partial \zeta}{\partial z} = \phi^* \bar{b}, \quad \frac{\partial \zeta}{\partial z} \neq 0, \tag{4.87b}$$

where

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + a^i \frac{\partial}{\partial \dot{q}^i} + b\frac{\partial}{\partial z}, \tag{4.88a}$$

$$\bar{\xi} = \dot{q}^i \frac{\partial}{\partial q^i} + \bar{a}^i \frac{\partial}{\partial \dot{q}^i} + \bar{b}\frac{\partial}{\partial z}. \tag{4.88b}$$

An interesting particular case is when a SODE ξ in P is *ρ-projectable*. In the coordinates (4.88), this means that a^i does not depend on z. That is, the Herglotz equations for the accelerations of the coordinates q^i are uncoupled form z. Horizontal equivalences preserve this property.

Proposition 4.32. *Let $\xi, \bar{\xi}$ be extended SODEs on P and let ξ be ρ-projectable. Then $\xi, \bar{\xi}$ are horizontally equivalent if and only if $\bar{\xi}$ is also ρ-projectable and $\rho_*\xi = \rho_*\bar{\xi}$.*

Proof. Assume that ξ is ρ-projectable and horizontally equivalent to $\bar{\xi}$ coordinates (4.88a), then, by (4.87a), by taking the inverse of ϕ, we obtain $(\phi^{1-})^* a^i = a^i = \bar{a}^i$, hence $\bar{\xi}$ is τ_1-projectable and $\tau_{1_*}\xi = \rho_* \bar{\xi}$.

For the converse, we will see if ξ is projectable, then it is horizontally equivalent to

$$\hat{\xi} = \dot{q}^i \frac{\partial}{\partial q^i} + \bar{a}^i \frac{\partial}{\partial \dot{q}^i}. \tag{4.89}$$

By transitivity of the equivalence relation, this will imply that ξ is horizontally equivalent to any other SODE with the same projection.

Using (4.87), we see that ξ and $\hat{\xi}$ are equivalent if and only if there exists a solution for the following equation

$$\frac{\partial \zeta}{\partial q^i}\dot{q}^i + a^i \frac{\partial \zeta}{\partial \dot{q}^i} = b, \quad \frac{\partial \zeta}{\partial z} \neq 0. \tag{4.90}$$

Since this is a linear, first order PDE, there exist local solutions. Since the equation only involves partial derivatives of ζ with respect to q^i and $\dot{q}^i$ adding a function of z to the solution, we can obtain a new one so that $\frac{\partial \zeta}{\partial z} \neq 0$ does not vanish.

□

Equivalent contact Lagrangian systems

Notice that extended Lagrangian systems are pullbacks by horizontal diffeomorphisms of usual Lagrangian systems. That is, $\phi = (\mathrm{id}_{TQ}, \zeta)$ is a strict similarity for the contact systems $(TQ \times \mathbb{R}, \eta_L, E_L)$ and $(TQ \times \mathbb{R}, \eta_{\phi^* L, \zeta} E_{\phi^* L, \zeta})$. Indeed, we have

$$\begin{aligned}
\phi^* E_L &= \phi^*(\Delta(L)) - \phi^*(L) = (\phi_* \Delta)(\phi^* L) - \phi^*(L) = \eta^{\zeta}_{\phi^* L}, \\
\phi^* \eta_L &= \phi^* \mathrm{d}\zeta - \phi^* \left(\frac{\partial L}{\partial \dot{q}^i}\right) \mathrm{d}q^i = \mathrm{d}\zeta - \left(\frac{\partial L}{\partial \dot{q}^i}\right)_{\xi} \mathrm{d}q^i,
\end{aligned}$$

by Proposition 4.3. Note that given two Lagrangian systems (L, z) and $(\bar{L}, \zeta)$ the map ϕ^{-1} is a dynamical equivalence between the systems $(TQ \times \mathbb{R}, E_L, \eta_L)$ and $(TQ \times \mathbb{R}, E^{\zeta}_L, \eta^{\zeta}_L)$.

As a consequence, L and $\bar{L}$ are horizontally equivalent if and only if $\xi_L = \xi_{\bar{L},\zeta}$ for some action function ζ. Hence, in order to study the problem of equivalent Lagrangians, we can equally study the following problem.

Problem 4.1 (Equivalent Lagrangians). Find which extended Lagrangian systems (L, z) $(\bar{L}, \zeta)$ have the same dynamics.

Definition 4.17. Two extended Lagrangian systems (L, z) $(\bar{L}, \zeta)$ are *equivalent* if $\xi_L = \xi_{\bar{L},\zeta}$. If ζ does not depend on $\dot{q}^i$, we say that they are *strongly* equivalent.

Remark 4.33. Note that we can also define that a map is a (*strong*) *Lagrangian symmetry* whenever a horizontal diffeomorphism $\phi : P \to P$ is a dynamical symmetry for the contact systems induced by the extended Lagrangian systems (L, z) and $(\bar{L}, \zeta)$. We say that it is *strong* whenever the diffeomorphism does not depend on the velocities (that is, it is a bundle morphism over $(\rho^0, \zeta) : P \to Q \times \mathbb{R}$).

In Section 4.3.5 we introduced the concept of (strong) action symmetry. This concept is the infinitesimal version of (strong) Lagrangian symmetries. Indeed, the flow of a (strong) action symmetry, as can be easily seen from its coordinate expression, is a (strong) horizontal equivalence. By definition, an action symmetry is also an infinitesimal dynamical symmetry. Hence, the flow of a (strong) action symmetry consist (strong) Lagrangian equivalences.

Theorem 4.34. *Let (L, z) and $(\bar{L}, \zeta)$ be regular extended Lagrangian systems. Both systems are equivalent if and only if*

$$\bar{L} = \mathcal{L}_{\xi_L}(\zeta) = \dot{q}^i \frac{\partial \zeta}{\partial q^i} + h_L^i \frac{\partial \zeta}{\partial \dot{q}^i} + \frac{\partial \zeta}{\partial z} L \tag{4.91a}$$

$$\xi_L(p_i^{\bar{L},\zeta}) - \left(\frac{\partial \bar{L}}{\partial q^i}\right)_\zeta = \left(\frac{\partial \bar{L}}{\partial \zeta}\right)_\zeta p_i^{\bar{L},\zeta}, \tag{4.91b}$$

where

$$p_i^{\bar{L},\zeta} = \left(\frac{\partial \bar{L}}{\partial \dot{q}^i}\right)_\zeta. \tag{4.92}$$

Proof. Assume that both systems are equivalent. By definition $\xi_{L,z} = \xi_{\bar{L},\zeta}$, hence $\bar{L} = \xi_{\bar{L},\zeta} = \mathcal{L}_{\xi_L}(\zeta)$. Also, (4.91b), after changing $\xi_{L,z}$ by $\xi_{\bar{L},\zeta}$, are just the Herglotz equations for $(\bar{L}, \zeta)$.

Conversely, assume that conditions (4.91) hold. Thus, we need to prove that both Herglotz vector fields are equal. We can do that by proving that $\xi_{L,z}$ is the Hamiltonian vector field of $\eta_{\bar{L},\zeta}$ with respect to the energy function $E_{\bar{L}}$. That is,

$$\begin{aligned} \eta_{\bar{L},\zeta}(\xi_{L,z}) &= -E_{\bar{L},\zeta}, \\ \mathcal{L}_{\xi_{L,z}} \eta_{\bar{L},\zeta} &= \frac{\partial \bar{L}}{\partial \zeta} \eta_{\bar{L},\zeta}. \end{aligned}$$

Expanding the first equation, we obtain

$$\mathcal{L}_{\xi_{L,\zeta}}(\zeta) - p_i^{\bar{L},\zeta} \dot{q}^i = -(p_i^{\bar{L},\zeta} \dot{q}^i - \bar{L}),$$

hence it is equivalent to Equation (4.91a). Assuming that the first condition holds, the second condition yields

$$\mathrm{d}\bar{L} - \mathcal{L}_{\xi_{L,z}}(p_i^{\bar{L},\zeta}) \mathrm{d}q^i - p_i^{\bar{L},\zeta} \mathrm{d}\dot{q}^i = \frac{\partial \bar{L}}{\partial \zeta}(\mathrm{d}\zeta - p_i^{\bar{L},\zeta} \mathrm{d}q^i).$$

Contracting with $(\partial/\partial \dot{q}^i)_\zeta$ and $\partial/\partial \zeta$ we obtain 0 on both sides of the equation. If we contract with $(\partial/\partial q^i)_\zeta$, we obtain

$$\left(\frac{\partial \bar{L}}{\partial q^i}\right)_\zeta - \mathcal{L}_{\xi_{L,z}}(p_i^{\bar{L},\zeta}) = -\frac{\partial L}{\partial \zeta} p_i^{\bar{L},\zeta}, \tag{4.93}$$

which is Equation (4.91b). □

The notion of strong equivalence has a nice characterization: it coincides with that of conformal equivalence. Moreover, it is easy to find a closed form for these Lagrangians.

Theorem 4.35. *(L, z) and $(\bar{L}, \zeta)$ be regular extended Lagrangian systems. Then, the following are equivalent*

1. *(L, z) and $(\bar{L}, \zeta)$ are strongly equivalent.*

2. *The Hamiltonian systems $(TQ \times \mathbb{R}, \eta_L, E_L)$ and $(TQ \times \mathbb{R}, \eta_{\bar{L}}^{\zeta}, E_{\bar{L},\zeta})$ are conformally equivalent.*

3. *We have*
$$\frac{\partial \zeta}{\partial z} L + \dot{q}^i \frac{\partial \zeta}{\partial q^i} = \bar{L}, \tag{4.94}$$
and ζ is independent of $\dot{q}^i$.

Proof. We will proof that $3 \implies 2 \implies 1 \implies 3$.

Assume that (4.94) holds. Then,

$$\eta_{\bar{L}}^{\xi} = \mathrm{d}\zeta - \left(\frac{\partial \bar{L}}{\partial \dot{q}^i}\right)_{\zeta} \mathrm{d}q^i = \frac{\partial \zeta}{\partial q^i}\mathrm{d}q^i + \frac{\partial \zeta}{\partial z}\mathrm{d}z - \left(\frac{\partial \zeta}{\partial z}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial \zeta}{\partial q^i}\right)\mathrm{d}q^i = \frac{\partial \zeta}{\partial z}\eta_L,$$
$$E_{\bar{L}}^{\xi} = \dot{q}^i \left(\frac{\partial \bar{L}}{\partial \dot{q}^i}\right)_{\zeta} - \bar{L} = \dot{q}^i \left(\frac{\partial \zeta}{\partial z}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial \zeta}{\partial q^i}\right) - \frac{\partial \zeta}{\partial z} L - \dot{q}^j \frac{\partial \zeta}{\partial q^i} = \frac{\partial \zeta}{\partial z} E_L.$$

Therefore, both systems are conformally equivalent.

Now, assume that both systems are conformally equivalent. Thus, $\eta_L = f \eta_L$ for a non-vanishing f. We now take the contraction of the previous expression with $\partial / \partial \dot{q}^i$. We obtain
$$0 = f \frac{\partial \bar{\zeta}}{\partial \dot{q}^i}. \tag{4.95}$$

Hence, both extended Lagrangian systems are strongly equivalent.

Last of all, assume that (L, z) and $(\bar{L}, \zeta)$ are strongly equivalent. Then $\xi_L(\zeta) = \xi_{\bar{L},\zeta}(\zeta)$, thus
$$\frac{\partial \zeta}{\partial z} L + \dot{q}^i \frac{\partial \zeta}{\partial q^i} = \bar{L}. \tag{4.96}$$
$\square$

Hence, the set of strongly equivalent Lagrangians is parametrized by a function $\zeta_0 : Q \times \mathbb{R} \to \mathbb{R}$.

Remark 4.36. This result includes the symplectic Lagrangian equivalence [193, 217]. Consider Lagrangians that do not depend on z and take $\zeta = cz + \nu(q^i)$, with c a non-zero constant. Then $\bar{L}(q^i, \dot{q}^i) = cL(q^i, \dot{q}^i) + \dot{q}^i \frac{\partial \nu}{\partial q^i}$. In particular, we have $\theta_{\bar{L}}^{\zeta} = \theta_{\bar{L}} = c\theta_L + \mathrm{d}\nu$.

Variational formulation

We will now analyze the problem from a variational perspective.

Let (L, z) and (L, ζ) be extended Lagrangian systems. We remind that, by Theorem 4.29, the critical points of the action functional are the projections onto Q of the integral curves of their Herglotz vector fields. Thus, if the two systems are equivalent, their corresponding action functionals must have the same critical points. But this is not sufficient, the curves on Q have to be lifted to P, on the same way through the operator $\mathcal{X}$.

Proposition 4.37. *Let (L, z) and $(\bar{L}, \zeta)$ be Lagrangian systems. Both systems are equivalent if and only if*

1. *$\gamma : [t_0, t_1] \to Q$ is a critical point of $\mathcal{Z}_{L,z,z_0}$ if and only if it is a critical point of $\mathcal{Z}_{\bar{L},\zeta,\zeta_0}$, where, $\zeta_0 = \zeta(\gamma(t_0), \dot{\gamma}(t_0), z_0)$.*
2. *For every critical point γ of one has that $\mathcal{X}_{L,z,z_0}(\gamma) = \mathcal{X}_{\bar{L},\zeta,\zeta_0}(\gamma)$.*

It will be useful for our purposes to have a geometric characterization of the operator $\mathcal{X}$.

Proposition 4.38. *Let $L : P \to \mathbb{R}$ be a Lagrangian function and let ξ be an extended SODE. Then, for every integral curve $\delta : [t_0, t_1] \to P$ of ξ, if we let $\gamma = (\rho^0)^* \delta : [t_0, t_1] \to Q$ and $\zeta(\delta(t_0)) = \zeta_0$, we have*

$$\mathcal{X}_{L,\zeta,\zeta_0}(\gamma) = \delta \tag{4.97}$$

if and only if

$$\mathcal{L}_\xi \zeta = L. \tag{4.98}$$

Proof. We note that (4.97) holds if and only if for every integral curve δ of ξ satisfies

$$\frac{\mathrm{d}(\zeta \circ \delta)}{\mathrm{d}t} = L \circ \delta, \tag{4.99}$$

while

$$\frac{\mathrm{d}(\zeta \circ \mathcal{X}_{L,z,z_0})(\gamma)}{\mathrm{d}t} = L \circ \mathcal{X}_{L,z,z_0}(\gamma). \tag{4.100}$$

Since $\zeta(\delta(t_0)) = \zeta_0$, by uniqueness of solution of the above ODE, we conclude that $\mathcal{X}_{L,\zeta,\zeta_0}(\delta) = \xi$. □

We can assume a stronger hypothesis regarding the action; $\mathcal{X}_{L,z,z_0} = \mathcal{X}_{\bar{L},\zeta,\zeta_0}$ not only for the critical points of the action, but for every curve. We then obtain the following.

Theorem 4.39. *Let (L, z) and $(\bar{L}, \zeta)$ be Lagrangian systems. Both systems are strongly equivalent if and only if for every curve $\gamma : [t_0, t_1] \to Q$, we have*

$$\mathcal{X}_{L,z,z_0}(\gamma) = \mathcal{X}_{\bar{L},\zeta,\zeta_0}(\gamma), \tag{4.101}$$

where $\zeta_0 = \zeta(\gamma(t_0), \dot{\gamma}(t_0), z_0)$.

Proof. Assume that both systems have the same $\mathcal{X}$ operators. By Proposition 4.38 we have that both for every extended SODE ξ such that $\mathcal{L}_\xi z = L$, that is, of the form

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + a^i \frac{\partial}{\partial \dot{q}^i} + L \frac{\partial}{\partial z} \tag{4.102}$$

one has that

$$\mathcal{L}_\xi \zeta = \dot{q}^i \frac{\partial \zeta}{\partial q^i} + a^i \frac{\partial \zeta}{\partial \dot{q}^i} + \frac{\partial \zeta}{\partial z} L = \bar{L}. \tag{4.103}$$

Since this must hold for any extended SODE of this form and the accelerations a^i are arbitrary, then it is necessary that ζ does not depend on $\dot{q}^i$. Moreover, since

$$\dot{q}^i \frac{\partial \zeta}{\partial q^i} + \frac{\partial \zeta}{\partial z} L = \bar{L}, \tag{4.104}$$

by Theorem 4.35, both systems are strongly equivalent.

Conversely, if both systems are strongly equivalent, again, by Theorem 4.35 we know that ζ does not depend on the velocities and that (4.104) holds. Thus, for every SODE satisfying $\mathcal{L}_\xi z = L$ one has that $\mathcal{L}_\xi \zeta = \bar{L}$ and vice versa. By Proposition 4.38, both systems have the same $\mathcal{X}$ operators. □

4.4.3. Examples

The next examples are some applications of the previous results in equivalent Lagrangians, exploring their limits and implications.

Example 4.3 (Total time derivative)**.** In the action-independent case, two Lagrangians $L_0, \bar{L}_0 : TQ \to \mathbb{R}$ are independent if their difference is a *total derivative*, that is, a function the form

$$\frac{\partial h_i(q^i)}{\partial q^i} \dot{q}^i, \tag{4.105}$$

for some function $h : TQ \to \mathbb{R}$. We would like to check that if we extend this Lagrangians to $TQ \times \mathbb{R}$ they are still equivalent. In contact geometry one has to be careful, because the contact equations are not linear on the Lagrangian. Nevertheless, one can proceed in a similar fashion by considering the transformation $\zeta = z + h$, resulting in the Lagrangian:

$$\bar{L}(q^i, \dot{q}^i, \zeta) = L(q^i, \dot{q}^i, z) + \frac{\partial h}{\partial q^i} \dot{q}^i, \tag{4.106}$$

where $L(q, \dot{q}, z) = L_0(q, \dot{q})$, $\bar{L}(q, \dot{q}, z) = \bar{L}_0(q, \dot{q})$ are the extensions of $L_0, \bar{L}_0$ to $TQ \times \mathbb{R}$.

The extended Lagrangian systems (L, z) and $(\bar{L}, \zeta)$ are equivalent by Theorem 4.35. When the Lagrangian L does not depend on z we recover the usual result of the symplectic case, as explained in Remark 4.36.

Example 4.4. Lorentz force The classical Lagrangian to describe the motion of a particle under the Lorentz force is

$$L = \sum_{i=1}^{3} \left(\frac{m}{2} (\dot{q}^i)^2 + kA^i \dot{q}^i \right) - k\phi \, .$$

If one performs a change of gauge by a function $h(q^i)$, then the new Lagrangian is $L + \frac{\partial h}{\partial q^i} \dot{q}^i$. Since the difference is a total derivative both Lagrangians have the same dynamical equations. This is a classical case of a "gauge transformation" in physics, where we one can change the vector potential A to a different one by adding the gradient of a function, so one is able to choose the most convenient one in a process called "gauge fixing" [155].

In [134] a contact version of the previous Lagrangian is considered, which adds a dissipation term

$$\tilde{L} = \sum_{i=1}^{3} \left(\frac{m}{2} (\dot{q}^i)^2 + kA^i \dot{q}^i \right) - k\phi - \gamma z. \tag{4.107}$$

Here, when trying to perform a gauge transformation $A^i \mapsto A^i + \frac{\partial h}{\partial q^i}$, where the equations of motion are derived, and they turn out not to change [134].

In order to find a gauge invariant Lagrangian description of the Lorentz force in the contact setting, in [134] is proposed a generalized description of the gauge given by a triple (ϕ, A, f), from this triple we construct the Lagrangian

$$L_{(\phi,A,f)} = \sum_{i=1}^{3} \left(\frac{m}{2} (\dot{q}^i)^2 + k\dot{q}^i \left(A^i + \frac{\partial f}{\partial q^i} \right) \right) - k\phi - \gamma z. \tag{4.108}$$

The gauge transformation would be given by a function h as $(\phi - \frac{\partial h}{\partial t}, A + \nabla h, f - h)$. This Lagrangian is invariant under a gauge transformation, therefore, there is no need to invoke equivalence results. However, we can interpret it as a Lagrangian equivalence in our framework. If we perform a gauge transformation in the usual way

$$L = \sum_{i=1}^{3} \left(\frac{m}{2} (\dot{q}^i)^2 + k\dot{q}^i A^i \right) - k\phi - \gamma (z + kf), \tag{4.109}$$

which, after a gauge change by h (and renaming z by ζ) transforms into

$$\bar{L}(q^i, \dot{q}^i, \zeta) = \sum_{i=1}^{3} \left(\frac{m}{2} (\dot{q}^i)^2 + k\dot{q}^i A^i \right) - k\phi - \gamma (\zeta - kh + kf) + k \frac{\partial h}{\partial q^i} \dot{q}^i. \tag{4.110}$$

Using Theorem 4.35 one can check that (L, z) and $(\bar{L}, \zeta = z + kh)$ are strongly equivalent. Hence, the results of [134] fit in our formulation.

Example 4.5 (Parachute equation)**.** The Lagrangian of the parachute equation (Example 3.2) is

$$L = \frac{1}{2} \dot{y}^2 - \frac{mg}{2\gamma} (e^{2\gamma y} - 1) + 2\gamma \dot{y} z \, .$$

One wonders if there exists a strongly equivalent Lagrangian with a simpler expression. Theorem 4.35 allows us to explore all possible Lagrangians providing the same dynamics.

We want to find a strongly equivalent Lagrangian which conserve the kinetic energy term, that is, with the structure:

$$\bar{L} = \frac{1}{2}\dot{y}^2 + a(y,\zeta)\dot{y} + b(y,\zeta)\,.$$

From the Theorem 4.35, ζ has to satisfy the identity

$$\frac{1}{2}\dot{y}^2 + a(y,\zeta)\dot{y} + b(y,\zeta) = \dot{y}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial z}\left(\frac{1}{2}\dot{y}^2 - \frac{mg}{2\gamma}(e^{2\gamma y} - 1) + 2\gamma\dot{y}z\right).$$

This implies that $\frac{\partial \zeta}{\partial z} = 1$, thus $\zeta = z + f(y)$. Then $a(y,\zeta) = \frac{\partial \zeta}{\partial y} + 2\gamma z = f' - 2\gamma f + 2\gamma\zeta$ and $b(y,\zeta) = -\frac{mg}{2\gamma}(e^{2\gamma y} - 1)$. Therefore, the possible Lagrangians are:

$$\bar{L} = \frac{1}{2}\dot{y}^2 + (f' - 2\gamma f + 2\gamma\zeta)\,\dot{y} - \frac{mg}{2\gamma}(e^{2\gamma y} - 1)\,.$$

Thus, the exponential and the term proportional to $\dot{y}z$ are necessary for a contact Lagrangian of this type to describe the parachute equation.

Example 4.6 (Non-strong equivalent Lagrangians)**.** In general, (L,z) and $(\bar{L}, \zeta = z + \dot{q}^n)$ are not equivalent. First, we need to check (4.91b), which in this case imposes $(n-1)^2 = n-1$, therefore n can only be 1 or 2. This leaves us with the Lagrangians

$$\bar{L}_1 = \frac{1}{2}\dot{q}^2 - \gamma\zeta; \quad \bar{L}_2 = \left(\frac{1}{2} - \gamma\right)\dot{q}^2 - \gamma\zeta.$$

$\bar{L}_1$ is a ζ-regular Lagrangian, but $\bar{L}_2$ is only ζ-regular if $\gamma \neq \frac{1}{2}$. In this case, they are equivalent to (L,z) in virtue of Theorem 4.34. We can check this explicitly by computing the Herglotz vector field of $(\bar{L}, \zeta = z + \dot{q}^n)$.

$$\xi_{\bar{L},\zeta} = \dot{q}\left(\frac{\partial}{\partial q^i}\right)_\zeta + \bar{a}\left(\frac{\partial}{\partial \dot{q}^i}\right)_\zeta + g\frac{\partial}{\partial \zeta}.$$

Equation (4.76b) tell us that $g = \bar{L}$. Equation (4.76a) is

$$-\bar{a}(1 - \gamma n(n-1)^2\dot{q}^{n-2}) = \gamma\dot{q}(1 - \gamma n(n-1)\dot{q}^{n-2})\,. \tag{4.111}$$

Since $a = -\gamma\dot{q}$ (the component of the vector field ξ_L corresponding to $\frac{\partial}{\partial \dot{q}^i}$) does not depend on z, from Proposition 4.32 we know that $\bar{a} = a$. Thus, (4.111) becomes

$$(1 - \gamma n(n-1)^2\dot{q}^{n-2}) = (1 - \gamma n(n-1)\dot{q}^{n-2})\,,$$

which is only satisfied if $n = 1, 2$. For $n = 2$ and $\gamma = \frac{1}{2}$, (4.111) becomes $0 = 0$ and any function $\bar{a}$ is a possible solution, a sign that the system is singular and, in particular, not equivalent to (L,z).

General equivalence of extended Lagrangian systems is given by Theorem 4.34, where the regularity hypothesis and 4.91b condition are important, as we will see in this example. Consider the Lagrangian ($\gamma \neq 0$)

$$L(q, \dot{q}, z) = \frac{1}{2}\dot{q}^2 - \gamma z\,,$$

whose Herglotz vector field is

$$\xi_L = \dot{q}\frac{\partial}{\partial q^i} - \gamma \dot{q}\frac{\partial}{\partial \dot{q}^i} + L\frac{\partial}{\partial z}\,.$$

Given an action function $\zeta = z + \dot{q}^n$ (with $n \neq 0$), we can use condition (4.91a) to compute the potential equivalent Lagrangian:

$$\bar{L}(q, \dot{q}, \zeta) = -\gamma n \dot{q}^n + \frac{1}{2}\dot{q}^2 - \gamma\zeta + \gamma\dot{q}^n\,.$$

4.5. Reduction and moment maps

In this section we present a result of reduction in the context of contact geometry, which is analogous to the well-known coisotropic reduction in symplectic geometry. This theorem is not true in more general contexts, such as Jacobi manifolds, where more structure is needed to perform the reduction [197, 206, 207].

This will be followed by an application of this theorem to prove a reduction theorem via the moment map. We relate the symmetries in this context with dissipated quantities.

4.5.1. Coisotropic reduction in contact geometry

Proposition 4.40. *Given a coisotropic submanifold $\iota : N \hookrightarrow M$, we define*

$$\begin{aligned} \eta_0 &= \iota^*\eta = \eta|_{TN}, \\ \mathrm{d}\eta_0 &= \iota^*(\mathrm{d}\eta) = \mathrm{d}(\iota^*\eta). \end{aligned}$$

Then,

$$TN^{\perp_\Lambda} = \ker(\eta_0) \cap \ker(\mathrm{d}\eta_0),$$

which will be called the characteristic distribution *of N.*

Proof. We shall prove the last equality:

$$\begin{aligned} TN^{\perp_\Lambda} &= \{v \in TM \cap \mathcal{H} \mid \mathrm{d}\eta(v, TN) = 0\} \\ &= \{v \in TN \mid \eta(v) = \eta_0(v) = 0, \mathrm{d}\eta(v, TN) = \mathrm{d}\eta_0(v, TN) = 0\} \\ &= \ker \eta_0 \cap \ker \mathrm{d}\eta_0, \end{aligned}$$

where the first equality is due to Proposition 2.15 and the second one to the fact that N is coisotropic, which ensures that all orthogonal vectors are in TN. □

The following theorem is very related to [239, Proposition 4.2]. Indeed, this result provides a coisotropic reduction theorem for *regular coisotropic submanifolds* [189, Definition 5.8], which coincides with our notion of *coisotropic submanifolds without horizontal points*. The reduction, however, is carried in the context of manifolds with a contact distribution explained in Remark 2.1.

Theorem 4.41 (Coisotropic reduction in contact manifolds)**.** *Let $\iota : N \hookrightarrow M$ be a coisotropic submanifold. Then $TN^{\perp_\Lambda}$ is an involutive distribution.*

Assume that the leaf space of the characteristic distribution $TN^{\perp_\Lambda}$ on N, denoted $\tilde{N} = N/TN^{\perp_\Lambda}$, is a manifold and that N does not have horizontal points. Let $\pi : N \to \tilde{N}$ be the projection. Then there is a unique 1-form $\tilde{\eta} \in \Omega^1(\tilde{N})$ such that

$$\pi^* \tilde{\eta} = \iota^* \eta. \tag{4.112}$$

Moreover, $(N, \tilde{\eta})$ is a contact manifold.

Furthermore, if N consists of vertical points, $\tilde{\mathcal{R}} = \pi_* \mathcal{R}$ is well-defined and is the corresponding Reeb vector field.

Proof. First, we will prove that the distribution $TN^{\perp_\Lambda}$ is involutive. Let $X, Y \in TN^{\perp_\Lambda} = \ker(\eta_0) \cap \ker(\mathrm{d}\eta_0)$. We will show that $[X, Y] \in TN^{\perp_\Lambda}$. By Cartan's formula:

$$0 = \mathrm{d}\eta_0(X, Y) = X(\eta_0(Y)) - Y(\eta_0(X)) - \eta_0([X, Y]) = -\eta_0([X, Y]), \tag{4.113}$$

thus $[X, Y] \in \ker(\eta_0)$. Now, we will use Cartan's formula with $\mathrm{d}\eta_0$. Let Z be a vector field tangent to N,

$$\begin{aligned} 0 = \mathrm{d}\mathrm{d}\eta_0(X, Y, Z) &= X(\mathrm{d}\eta_0(Y, Z)) - Y(\mathrm{d}\eta_0(X, Z)) + Z(\mathrm{d}\eta_0(X, Y)) \\ -\mathrm{d}\eta_0([X, Y], Z) &+ \mathrm{d}\eta_0([X, Z], Y) - \mathrm{d}\eta_0([Y, Z], X) = -\mathrm{d}\eta_0([X, Y], Z), \end{aligned} \tag{4.114}$$

from which we conclude that $[X, Y] \in \Gamma(TN^{\perp_\Lambda})$.

Now we will check that there is a unique 1-form $\tilde{\eta}$ such that $\pi^* \tilde{\eta} = \iota^* \eta$. We note that it is enough to show this locally (on open subsets of $\tilde{N}$).

For proving the existence, we take a smooth section $\mu : \tilde{N} \to N$ of π (that is, $\pi \circ \mu = \mathrm{id}_{\tilde{M}}$), which always exists locally because π is a submersion. Let $\tilde{\eta} = \mu^* \eta_0$.

We check the uniqueness in the tangent space of each $x \in N$. We know that $\ker(\eta_0)_x \supseteq TN_x^{\perp_\Lambda} = \ker(T\pi)_x$. Thus, $\tilde{\eta}_{[x]}$ does not depend on the chosen element of the preimage of $T_x \pi$. The following diagram illustrates this situation.

$$\begin{tikzcd} T_x N \arrow[r, "T_x \iota"] \arrow[d, "T_x \pi"'] \arrow[dr, "(\eta_0)_x"] & T_x M \arrow[d, "\eta_x"] \\ T_{[x]} \tilde{N} \arrow[r, dashed, "\tilde{\eta}_{[x]}"] & \mathbb{R}. \end{tikzcd} \tag{4.115}$$

We also have to prove that this projection does not depend on the base point of the fiber $\pi^{-1}(\{p\}) \supseteq N$. We compute the Lie derivative of η_0 in the direction $X \in TN^{\perp_\Lambda}$ using, again, Cartan's formula:

$$\mathcal{L}_X \eta_0 = \mathrm{d}\iota_X \eta_0 + \iota_X \mathrm{d}\eta_0 = 0,$$

hence $\tilde{\eta}$ is well-defined. Likewise, we can check that $\mathcal{R}$ projects to $\tilde{\mathcal{R}}$ on the vertical points.

On the horizontal points, we obtain $\tilde{\eta} = 0$, thus we do not get a contact form.

On non-horizontal points, $\tilde{\eta}$ is nondegenerate because $\tilde{\eta}(\pi_*(\mathcal{R}+v)) = 1$. Given that we have taken the quotient by $\ker(\mathrm{d}\eta_0) \cap \ker(\eta_0)$, $\mathrm{d}\tilde{\eta}_{|\ker \tilde{\eta}}$ is obviously nondegenerate. □

Corollary 4.42. *With the notations from previous theorem, assume that $L \subseteq M$ is Legendrian, N does not have horizontal points, and N and L have clean intersection (that is, $N \cap L$ is a submanifold and $T(N \cap L) = TN \cap TL$). Then $\tilde{L} = \pi(L) \subseteq \tilde{N}$ is Legendrian.*

Proof. Let $n + k + 1$ be the dimension of N, then, $TN^{\perp_\Lambda}$ has dimension $n - k$ by Definition 2.12. Hence,

$$\dim \tilde{N} = \dim N - \dim(TN^{\perp_\Lambda}) = 2k + 1. \tag{4.116}$$

Since $\tilde{L}$ is trivially horizontal, we only need to show that

$$\dim \tilde{L} = \dim L \cap N - \dim TL \cap (TN^{\perp_\Lambda}) = k. \tag{4.117}$$

Since by Definition 2.12,

$$\dim (L \cap N)^{\perp_\Lambda} + \dim(L \cap N) = 2n + 1, \tag{4.118}$$

and by Proposition 2.14,

$$(L + N)^{\perp_\Lambda} = L^{\perp_\Lambda} \cap N^{\perp_\Lambda} = L \cap N^{\perp_\Lambda},$$

using the incidence formula,

$$\dim(L + N^{\perp_\Lambda}) = \dim L + \dim N - \dim(L \cap N) = 2n + 1 + k + \dim(L \cap N), \tag{4.119}$$

and substituting in (4.117) concludes the proof. □

4.5.2. Moment maps

The moment map is well-known in symplectic geometry. There is a contact analogue [3, 137, 190, 257] which has been used to prove reduction theorems via this map. In our proof of this theorem we can see that it can be interpreted as a coisotropic reduction of the level set of the moment map.

We remind that given a Lie group G, we denote its Lie algebra by $\mathfrak{g}$ and the dual of its Lie algebra by $\mathfrak{g}^*$.

Definition 4.18. Let (M, η) be a contact manifold and let G be a Lie group acting on M by contactomorphisms. In analogy to the exact symplectic case, we define the moment map $J : M \to \mathfrak{g}^*$ such that

$$J(x)(\xi) = -\eta(\xi_M(x)), \tag{4.120}$$

where $x \in M$, $\xi \in \mathfrak{g}$ and $\xi_M \in \mathfrak{X}(M)$ is defined by

$$\xi_M(x) = \frac{\partial}{\partial t}(\exp(t\xi) \cdot x)|_{t=0} \tag{4.121}$$

is the infinitesimal generator of the action corresponding to ξ.

The moment map has the following properties:

Proposition 4.43. *Let G be a Lie group acting by contactomorphisms on a contact manifold (M,η). If we let*

$$\begin{aligned} \hat{J} : \mathfrak{g} &\to C^\infty(M) \\ \xi &\to -\iota_{\xi_M}\eta, \end{aligned} \tag{4.122}$$

so that $\hat{J}(\xi)(x) = J(x)(\xi)$. We obtain that the so-called moment condition*:*

$$\mathrm{d}\hat{J}(\xi) = \iota_{\xi_M}\mathrm{d}\eta. \tag{4.123}$$

Furthermore,

$$X_{\hat{J}(\xi)} = \xi_M. \tag{4.124}$$

Proof. The fact that G acts by contactomorphisms implies that

$$\mathcal{L}_{\xi_M}\eta = 0. \tag{4.125}$$

Thus, by Cartan's formula

$$\mathrm{d}\hat{J}(\xi) = -\mathrm{d}\iota_{\xi_M}\eta = \iota_{\xi_M}\mathrm{d}\eta. \tag{4.126}$$

The other equality is a consequence of Proposition 2.12. □

Proposition 4.44. *The moment map defined as above is equivariant under the coadjoint action. That is, for every $g \in G$, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "J"] & M \arrow[d, "J"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g^{-1}}"] & \mathfrak{g}^* \end{tikzcd} \tag{4.127}$$

where $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is the coadjoint representation, that is, if $g \in G$, $\alpha \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$

$$\mathrm{Ad}^*_g(\alpha)(\xi) = \alpha(\mathrm{Ad}_g(\xi)) = \alpha(T(R_{g^{-1}}L_g)\xi), \tag{4.128}$$

and $L_g, R_g : G \to G$ are, respectively, left and right multiplication by g.

Proof. We must show

$$\hat{J}(\xi)(gx) = \hat{J}(\mathrm{Ad}_{g^{-1}}\xi)(x), \tag{4.129}$$

that is,

$$(\iota_{\xi_M}\eta)(gx) = (\iota_{(\mathrm{Ad}_{g^{-1}}\xi)_M}\eta)(x). \tag{4.130}$$

The proof follows from the following identity [2, Prop. 4.1.26], which is true for any smooth action

$$(\mathrm{Ad}_{g^{-1}}\xi)_M = g^*\xi_M, \tag{4.131}$$

together with the fact that g preserves the contact form. □

Lemma 4.45. *Let (M, η) be a contact manifold on which a Lie group G acts by contactomorphisms. Let $\mu \in \mathfrak{g}^*$ be a regular value of the moment map J. Then, for all $x \in J^{-1}(\mu)$*

$$T_x(G_\mu x) = T_x(Gx) \cap T_x(J^{-1}(\mu)), \tag{4.132}$$

where $G_\mu = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}} \mu = \mu\}$ is the isotropy group of μ with respect to the coadjoint action.

It is also true that

$$T_x(J^{-1}(\mu)) = T_x(Gx)^{\perp_{\mathrm{d}\eta}}. \tag{4.133}$$

In particular, if $G = G_\mu$, then $T_x(Gx) \subseteq T_x(J^{-1}(\mu))$ and $T_x(J^{-1}(\mu))$ is coisotropic and consists of vertical points. Furthermore,

$$T_x(J^{-1}(\mu))^{\perp_\Lambda} = T_x(Gx). \tag{4.134}$$

Proof. In [2, Cor. 4.1.22] we see that

$$T_x(Gx) = \{\xi_M(x) \mid \xi \in \mathfrak{g}\}. \tag{4.135}$$

If $\mathfrak{g}_\mu \subseteq \mathfrak{g}$ denotes the Lie subalgebra corresponding to the Lie subgroup $G_\mu \subseteq G$, we conclude that $\xi_M(x) \in T_x(G_\mu x)$ if and only if $\xi \in \mathfrak{g}_\mu$. By Ad^*-equivariance, one deduces that

$$(T_x J)(\xi_M(x)) = \xi_{\mathfrak{g}^*}(\mu), \tag{4.136}$$

thus $\xi_M \in T_x(J^{-1}(\mu)) = \ker T_x J$ if and only if $\xi_{\mathfrak{g}^*}(\mu) = 0$, which means that μ is a fixed point of $\mathrm{Ad}^*_{\exp(-t\xi)}$ or, equivalently, $\exp(\xi) \in G_\mu$ which, by basic Lie group theory, is the same as $\xi \in \mathfrak{g}_\mu$.

For the second part, remember (Proposition 4.44) that if $\xi \in \mathfrak{g}$ and $v \in T_x M$, then

$$\mathrm{d}\eta(\xi_M(x), v) = \mathrm{d}\hat{J}(\xi)(x)(v) = T_x J(v)(\xi). \tag{4.137}$$

Thus, $v \in T_x(J^{-1}(\mu)) = \ker T_x J$ if and only if $\mathrm{d}\eta(\xi_M(x), v) = 0$ for all $\xi \in \mathfrak{g}$. That is, $T_x(J^{-1}(\mu)) = T_x(Gx)^{\perp_{\mathrm{d}\eta}} = \{\xi_M(x) \mid \xi \in \mathfrak{g}\}^{\perp_{\mathrm{d}\eta}}$.

In the case $G = G_\mu$, we note that, because G acts by contactomorphisms, $T_x(Gx) \subseteq \mathcal{H}$, thus, by Proposition 2.15 we see that

$$T_x(J^{-1}(\mu))^{\perp_\Lambda} = T_x(Gx) \subseteq T_x(J^{-1}(\mu)). \tag{4.138}$$

□

Theorem 4.46 (Reduction via moment map)**.** *Let (M, η) be a contact manifold on which a connected Lie group G acts by contactomorphisms and let J be the moment map. Let $\mu \in \mathfrak{g}$ be a regular value of J which is a fixed point of G under the coadjoint action and such that the action of G is free and proper on $J^{-1}(\mu)$. Then, $M_\mu = J^{-1}(\mu)/G$ has a unique contact form η_μ such that*

$$\pi_\mu^* \eta_\mu = \iota_\mu^* \eta, \tag{4.139}$$

where $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $\iota_\mu : J^{-1}(\mu) \to M$ is the inclusion.

Also, the Reeb vector field of the quotient $\mathcal{R}_\mu = \pi_\mu^* \mathcal{R}$ is the projection of the Reeb vector field of (M, η).

A similar result was proved in [137, Theorem 6]. We present a new proof that clarifies the role of coisotropic submanifolds and coisotropic reduction.

Proof. This follows from combining Theorem 4.41 and Lemma 4.45. Since $J^{-1}(\mu)$ is a coisotropic manifold (since μ is a regular value, its preimage is a manifold) the quotient by its characteristic distribution has a unique contact structure projected from M. Also, the quotient of a manifold by a free and proper group action is again a manifold. Both quotients coincide because $T_x(J^{-1}(\mu))^{\perp_\Lambda} = T_x(Gx)$, so the leaves of the characteristic distribution of $J^{-1}(\mu)$ coincide with the orbits of G. □

Theorem 4.47 (Contact Hamiltonian system reduction)**.** *Let G be a connected Lie group acting by contactomorphisms on (M, η, H) such that H is G-invariant (that is $H \circ g = H$ for all $g \in G$). Then, with the notations of previous theorem and assuming that the action is free and proper on $J^{-1}(\mu)$, (M_μ, η_μ, H_μ) is a Hamiltonian system where H_μ is projection of H by the action of G. This situation is illustrated by the following diagram,*

$$\begin{tikzcd}
M \arrow[dr, "H"] & \\
J^{-1}(\mu) \arrow[u, "\iota_\mu"] \arrow[d, "\pi_\mu"'] & \mathbb{R} \\
M_\mu \arrow[ur, "H_\mu"] &
\end{tikzcd} \quad , \tag{4.140}$$

Furthermore ${\pi_\mu}_* X_H|_{J^{-1}(\mu)} = X_{H_\mu}$.

Proof. The fact that (M_μ, η_μ, H_μ) is a Hamiltonian system is a consequence of Theorem 4.46. We note that H_μ is well-defined because H is G-invariant.

Now we need to see that $X_H|_{J^{-1}(\mu)} \in \mathfrak{X}(J^{-1}(\mu))$. Since H is G-invariant, for all $g \in G$ we have

$$\xi_M(H) = \iota_{\xi_M} \mathrm{d}H = 0, \tag{4.141}$$

that is, $\mathrm{d}H_x \in (T_x Gx)^\circ$, for all $x \in J^{-1}(\mu)$ or, equivalently, $\sharp_\Lambda(\mathrm{d}H) \in \sharp_\Lambda(T_x Gx)^\circ = (T_x Gx)^{\perp_\Lambda}$. Hence,

$$(X_H)_x = \sharp_\Lambda \mathrm{d}H_x - H(x)\mathcal{R}_x \in (T_x Gx)^{\perp_\Lambda} \oplus \mathcal{V} = (T_x Gx)^{\perp_{\mathrm{d}\eta}} = T_x(J^{-1}(\mu)), \tag{4.142}$$

where the last equality is due to Lemma 4.45.

We remark that ${\pi_\mu}_* X_H|_{J^{-1}(\mu)}$ is well-defined, since both H and η are preserved by the action of G. We now will show that ${\pi_\mu}_* X_H|_{J^{-1}(\mu)}$ equals X_{H_μ}. The $\flat$ isomorphism (Equation (2.4)) corresponding to the contact structure in the quotient is denoted by $\flat_\mu$.

$$\begin{aligned}
\flat_\mu({\pi_\mu}_* X_H|_{J^{-1}(\mu)}) &= \iota_{{\pi_\mu}_* X_H|_{J^{-1}(\mu)}} \mathrm{d}\eta_\mu + (\iota_{{\pi_\mu}_* X_H|_{J^{-1}(\mu)}} \eta_\mu)\eta_\mu \\
&= \mathrm{d}H_\mu - (\mathcal{R}_\mu(H_\mu) + H_\mu)\eta_\mu,
\end{aligned}$$

since $\pi_\mu^* \eta_\mu = \eta|_{J^{-1}(\mu)}$ by Theorem 4.46. Hence, ${\pi_\mu}_* X_H|_{J^{-1}(\mu)}$ is the Hamiltonian vector field for H_μ. □

Remark 4.48 (Lifting solutions). A solution to the reduced problem can be lifted to a solution of the initial system [196]. That is, any integral curve $[c(t)]$ for X_{H_μ} is the projection of a unique integral curve $c(t)$ for X_H after choosing a base point $c(0) = x \in J^{-1}(\mu)$. To see that, we pick a curve $d(t)$ such that $d(0) = x$, $[d(t)] = [c(t)]$, that is, $c(t) = g(t)d(t)$ with $g(t) \in G$. We can find that $g(t)$ by solving the following equation

$$X_H(d(t)) = d'(t) + \left((T_{g(t)}R^{-1}_{g(t)})\dot{g}(t)\right)_M(d(t)), \tag{4.143}$$

which can be seen to have a unique solution by solving

$$(\xi(t))_M(d(t)) = X_H(d(t)) - d'(t), \tag{4.144}$$

for $\xi(t) \in \mathfrak{g}$ and then, we solve

$$\begin{cases} \dot{g}(t) & = TL_{g(t)}\xi(t), \\ g(0) & = e, \end{cases} \tag{4.145}$$

for $g(t)$.

Example 4.7 (Angular momentum). Consider $Q = \mathbb{R}^3$, a Hamiltonian function H on $M = T^*Q \times \mathbb{R}$ that is spherically symmetric, say $H(q,p,z) = \bar{H}(\|q\|, \|p\|)$. The manifold M is naturally equipped with a contact structure

$$\eta_Q = \mathrm{d}z - \alpha_Q = \mathrm{d}z - p_i \mathrm{d}q^i, \tag{4.146}$$

where $\alpha_Q = p_i \mathrm{d}q^i$ is the Liouville one-form.

Consider the Lie group $G = \mathrm{SO}(3) = \{O \in \mathbb{R}^{3\times 3} \mid O^T O = \mathrm{id}, \det(O) = 1\}$ acting by rotations on Q. The action can be lifted to $TQ \times \mathbb{R}$ by extending the cotangent lift with the identity. Explicitly, for $O \in \mathrm{SO}(3)$ we let

$$\begin{aligned} g_O : T^*Q \times \mathbb{R} &\to T^*Q \times \mathbb{R} \\ (q,p,z) &\mapsto (O \cdot q, p \cdot O^T, z). \end{aligned} \tag{4.147}$$

The group $\mathrm{SO}(3)$ acts freely and properly by contactomorphisms away from 0.

The Lie algebra of the group is given by $\mathfrak{g} = \mathfrak{so}(3) = \{o \in \mathbb{R}^{3\times 3} \mid o^T + o = 0\}$. This algebra will be identified with the algebra of 3-dimensional vectors with the cross product by taking

$$\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \tag{4.148}$$

The infinitesimal generator of $\xi \in \mathfrak{g}$ is given by

$$\xi_M(q,p,z) = (\xi \times q, p \times \xi, 0). \tag{4.149}$$

One can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the inner product on $\mathbb{R}^3$. The moment map is then given by [2, Example 4.2.15]

$$J(q,p,z) = q \times p. \tag{4.150}$$

Identifying $\mathfrak{g}^* \simeq \mathbb{R}^3$ one sees that the coadjoint actions of G is the usual one (by rotations). Let $\mu \in \mathfrak{g}^*$, $\mu \neq 0$ then the isotropy group $G_\mu \simeq S^1$ of μ under the coadjoint action, which are the rotations around the axis μ.

Without loss of generality, one can take $\mu = (0, 0, \mu_0)$. Hence, if $(q, p) \in J^{-1}(\mu)$, both q and p lie on the xy-plane. Moreover, they must satisfy the equation $q^1 p_2 - p_1 q^2 = \mu_0$.

We can apply Theorem 4.47, to our system and find out that (M_μ, η_μ, H_μ), which is a Hamiltonian system over a 3-dimensional manifold. This manifold is similar to the one obtained on the symplectic case [2, Example 4.3.4], but multiplied by $\mathbb{R}$ because of the extra coordinate z.

4.5.3. Moment maps dissipated quantities

Lie groups acting by contactomorphisms on contact Hamiltonian systems also produce dissipated quantities. Here we explain how to construct them using the moment map.

Lie group of symmetries on a contact Hamiltonian system

An important case of symmetries for contact Hamiltonian or Lagrangian systems appears when a Lie group preserving the geometric structure and the energy. That is, let G be a Lie group acting on a contact Hamiltonian system (M, η, H) by contactomorphisms and preserving H. Since

$$H \circ \Phi_g = H, \ \forall g \in G. \tag{4.151}$$

Then we deduce that

$$\mathcal{L}_{\xi_M} H = 0. \tag{4.152}$$

Using Proposition 4.11, we can deduce that

$$\{H, \hat{J}(\xi)\} = X_H(\hat{J}(\xi)) + (\hat{J}(\xi))\mathcal{R}(H). \tag{4.153}$$

But,

$$\begin{aligned}\{H, \hat{J}(\xi)\} &= -\{\hat{J}(\xi), H\} \\ &= -X_{\hat{J}(\xi_M)}(H) - H\mathcal{R}(\hat{J}(\xi)) \\ &= -\xi_M(H) - H\mathcal{R}(\hat{J}(\xi)) = 0.\end{aligned}$$

Therefore,

$$X_H(\hat{J}(\xi)) = -\mathcal{R}(H)\hat{J}(\xi). \tag{4.154}$$

That is, $\hat{J}(\xi)$ is an H-dissipated quantity. Therefore, we have obtained the following:

Theorem 4.49. *ξ_M is a dynamical symmetry for (M, η, ξ) and $\hat{J}(\xi)$ is an H-dissipated quantity.*

Lie groups acting on contact Lagrangian systems

Assume that a Lie group G acts on Q

$$\Phi : G \times Q \to Q, \tag{4.155}$$

such that the action preserves a (regular) Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$. This means that the lifted action to $TQ \times \mathbb{R}$,

$$\tilde{\Phi} : G \times TQ \times \mathbb{R} \to TQ \times \mathbb{R}, \tag{4.156}$$

given by $\tilde{\Phi} = (T\Phi, \mathrm{id}_{\mathbb{R}})$ preserves L. As a direct consequence, G preserves the contact form η_L. In other words, G acts by contactomorphisms on $(TQ \times \mathbb{R}, \eta_L)$.

Consider the corresponding moment maps:

$$\begin{aligned} J_L &: TQ \times \mathbb{R} \to \mathfrak{g}^*, \\ J_L(v_q, z)(v_q, \xi) &= -\eta_L(\xi_{TQ\times\mathbb{R}}). \end{aligned} \tag{4.157}$$

Notice that

$$\xi_{TQ\times\mathbb{R}} = {\xi_Q}^C \tag{4.158}$$

Using the results of Section 4.3.2, we conclude that ξ_Q is an infinitesimal symmetry of L and the function

$$f = {\xi_Q}^V(L) \tag{4.159}$$

is an H-dissipated quantity.

5. Singular Lagrangians and precontact manifolds

As we saw in Chapter 3, from a regular Lagrangian function $L : TQ \to \mathbb{R}$ we can construct a symplectic form ω_L, and we obtain a Hamiltonian system (TQ, ω_L, E_L) with well-defined dynamics.

The situation is more subtle when the Lagrangian function is not regular, that is, its Hessian matrix with respect to the velocities is singular. Hence, the 2-form ω_L is not symplectic because it is degenerate, and the equation $\iota_X\omega_L = \mathrm{d}E_L$ has no solution in general. Indeed, at each point of the manifold this is a liner equation where the linear operator is singular, so the solution either does not exist or is non-unique. It can be the case that the equation can only be solved on part of the manifold. In order to deal with singular Lagrangians, and motivated for the need to study the quantization of electromagnetism, P.A.M. Dirac developed a constraint algorithm (which was developed independently by P. G. Bergmann in order to quantize general relativity and is now called Dirac-Bergmann algorithm) that allows us to construct the dynamics of the system [19, 116], if possible. This constraint algorithm has been later geometrized by M.J. Gotay and J.M. Nester [147].

The geometric version of the algorithm relies on the concept of presymplectic systems, that is, a closed 2-form ω on a manifold M which is not symplectic but has constant rank. So we analyze the Hamilton equations

$$\iota_X\omega = \mathrm{d}H \tag{5.1}$$

for a Hamiltonian function H on M. We consider the points where there is a solution of the above equation, and so we obtain a *primary constraint submanifold* M_1 along which there is a solution. But the dynamics should be tangent to M_1, so we have to restrict ourselves to those points in M_1 where a solution exists and it is tangent to M_1, obtaining a *secondary constraint submanifold* M_2. The algorithm continues and, in the favorable cases, it stabilizes at some level, $M_{i+1} = M_i$ which is called the final constraint submanifold.

The above algorithm can be applied to the case of singular Lagrangian systems, but, when the Lagrangian satisfies some weak regularity condition, we can also develop a Hamiltonian counterpart and the corresponding constraint algorithm. Both algorithms are conveniently related by the Legendre transformation.

One issue on the Lagrangian part of the algorithm, the so-called *second order equation problem*, is that the solutions of the equations of motion are not necessarily integral curves of a SODE. Indeed, they might be solutions of the equations of motion $(q(t), v(t))$ such that $v(t) \neq \dot{q}(t)$. This problem was solved in [146]. They showed that along a further submanifold of the final constraint submanifold, solutions that satisfy $(v(t) = \dot{q}(t))$ may be found.

In addition, Dirac identified two kinds of constraints, first and second class. The second class constraints allow to define a Poisson bracket (called Dirac bracket) that gives the dynamics of the constrained system just as in the classical case with the canonical Poisson bracket.

The aim of this chapter is to find an analog of the Gotay–Nester algorithm for contact manifolds. In Section 5.1 we present precontact geometry, which is the geometric model for precontact systems (the counterpart of presymplectic geometry). In Section 5.2 we add dynamics to the picture, defining what a contact Hamiltonian system is and constructing a constraint algorithm that finds in which submanifold the equations of motion can be solved. Next, in Section 5.3 we study the constraint algorithm in the context of Lagrangian mechanics and prove that it commutes with the Legendre transformation. Later, in Section 5.4 we introduce a generalization of the Dirac bracket: the so-called Dirac-Jacobi bracket, which is Jacobi but not Poisson. We are also able to classify the constraint functions in first or second class, depending on weather they carry dynamical information or not. Then, in Section 5.5 we construct explicitly a submanifold S of the final constraint manifold, such that there is a unique solution to the equations of motion that satisfy the second order differential equation condition along S. Finally, in Section 5.6 we provide examples with explicit computations of the constraints and the Dirac-Jacobi brackets.

Most of the work in this chapter was published in [89]. We have also introduced a characterization of a Hamiltonian vector field on a precontact system, and the definition of morphisms of precontact system which simplifies proving the equivalence of the Lagrangian and Hamiltonian formalisms. These last ideas come from [93].

5.1. Precontact manifolds

The theory presented on Section 3.1 provides well-defined dynamics for regular Lagrangian systems and there is a satisfactory correspondence between the Lagrangian and Hamiltonian formalisms on the hyperregular case. However, we would like to treat more general kinds of systems in which Lagrangians are allowed to be singular. For that, we will need to introduce a geometric model that generalizes contact geometry: precontact geometry. This geometry plays a similar role than presymplectic geometry for singular symplectic Lagrangian systems.

Let η be a 1-form in an m-dimensional manifold M. We define the *characteristic distribution* of η as

$$\mathcal{C} = \ker \eta \cap \ker \mathrm{d}\eta \subseteq TM, \tag{5.2}$$

which we suppose to be regular, that is, of constant rank. We say that η is a *1–form of class c* if the rank of the distribution $\mathcal{C}$ is $m - c$. There exist some characterizations of this notion for a 1–form given in [142].

Proposition 5.1. *Let η be a one-form on an m-dimensional manifold M. Then it is equivalent:*

1. *The form η is of class* $2r+1$.

2. *At every point of M,*
$$\eta \wedge (\mathrm{d}\eta)^r \neq 0, \quad (\mathrm{d}\eta)^{r+1} = 0. \tag{5.3}$$

3. *Around any point of M, there exist local* Darboux *coordinates* $q^1, \dots q^r, p_1, \dots p_r, z, u_1, \dots u_s$, *where* $2r+s+1 = m$, *such that*
$$\eta = \mathrm{d}z - \sum_{i=1}^{r} p_i \mathrm{d}q^i. \tag{5.4}$$

In these Darboux coordinates, the characteristic distribution of η is given by
$$\mathcal{C} = \left\langle \left\{ \frac{\partial}{\partial u_a} \right\}_{a=1,\dots,s} \right\rangle. \tag{5.5}$$

Remark 5.2. The distribution $\mathcal{C}$ is involutive, and it gives rise to a foliation of M. If the quotient $\pi : M \to M/\mathcal{C}$ has a manifold structure, then there is a unique 1-form $\tilde{\eta}$ such that $\pi^*\tilde{\eta} = \eta$. From a direct computation, $\tilde{\eta}$ is a contact form on $M/\mathcal{C}$. This justifies the name of *precontact form*.

A pair (M, η) of a manifold equipped with a precontact form will be called a *precontact manifold*.

We define the following morphism of vector bundles over M, generalizing (2.4):
$$\begin{aligned} \flat : TM &\to TM^* \\ v &\mapsto \iota_v \mathrm{d}\eta + \eta(v)\eta. \end{aligned} \tag{5.6}$$

The following 2-tensors are associated to $\flat$ and its transpose
$$\omega = \mathrm{d}\eta + \eta \otimes \eta, \quad \bar{\omega} = -\mathrm{d}\eta + \eta \otimes \eta. \tag{5.7}$$

In other words, $\flat(X) = \omega(X, \cdot) = \bar{\omega}(\cdot, X)$. Therefore, $\omega(X,Y) = \bar{\omega}(Y,X)$.

A *Reeb vector field* for (M, η) is a vector field $\mathcal{R} \in \mathfrak{X}(M)$ such that
$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \eta(\mathcal{R}) = 1. \tag{5.8}$$

We note that there exists Reeb vector fields in every precontact manifold. Indeed, we can define local vector fields $\mathcal{R} = \frac{\partial}{\partial z}$ in Darboux coordinates and can extend them using partitions of unity.

Proposition 5.3. *Let (M, η) be a precontact manifold. Then,*
$$\mathcal{C} = \ker \eta \cap \ker \mathrm{d}\eta = \ker \flat = (\operatorname{im} \flat)^\circ. \tag{5.9}$$

Proof. We will prove the previous equalities. We first focus on $\ker\eta \cap \ker \mathrm{d}\eta = \ker \flat$. In order to see that $\ker\eta \cap \ker \mathrm{d}\eta \subseteq \ker \flat$, we let $\flat(X) = 0$. Then $\iota_X \mathrm{d}\eta + \eta(X)\eta = 0$. If we contract the previous expression with a Reeb vector field $\mathcal{R}$ we obtain that $\eta(X) = 0$. Thus, $\iota_X \mathrm{d}\eta$ also vanishes. The other inclusion is trivial.

Now we will see that $(\operatorname{im} \flat)^\circ = \ker \flat$. Let $X \in \ker \flat$. By the first equality, $\iota_X \mathrm{d}\eta = 0$ and $\iota_X \eta = 0$. Then, for any vector field Y

$$\iota_X \flat(Y) = \iota_X \iota_Y \mathrm{d}\eta + \eta(Y)\eta(X) = -\iota_Y \iota_X \mathrm{d}\eta = 0, \tag{5.10}$$

hence $(\operatorname{im} \flat)^\circ \supseteq \ker \flat$. By noticing that at each point $p \in M$ both subspaces of T_pM have the same dimensions, we conclude that both distributions are equal. □

On a precontact manifold, the Reeb vector field is not unique, but there is a family of them. They are sections of an affine subbundle of TM

Proposition 5.4. *A vector field X is a Reeb vector field for (M, η) if and only if $\flat(X) = \eta$. That is, the set of Reeb vector fields is $\mathcal{R} + \Gamma(\mathcal{C})$, where $\mathcal{R}$ is an arbitrary Reeb vector field and $\Gamma(\mathcal{C})$ is the set of vector fields tangent to $\mathcal{C}$.*

Proof. Let $\mathcal{R}$ be a Reeb vector field. Then, X is also a Reeb vector field if and only if $\eta(X) = \eta(\mathcal{R}) = 1$ and $\iota_X \mathrm{d}\eta = \iota_{\mathcal{R}} \mathrm{d}\eta = 0$. That is, if and only if $\mathcal{R} - X$ is tangent to $\mathcal{C}$. Equivalently $\flat(\mathcal{R} - X) = 0$ or $\flat(X) = \eta$. □

For a distribution $\Delta \subseteq TM$, we define the following notion of complement with respect to ω. Since ω is neither symmetric nor antisymmetric, we need to distinguish between right and left complements:

$$\begin{aligned} \Delta^\perp &= \{X \in TM \mid \omega(Z, X) = \flat(Z)(X) = 0,\ \forall Z \in \Delta\} = (\flat(\Delta))^\circ, \\ {}^\perp\Delta &= \{X \in TM \mid \omega(X, Z) = 0, \forall Z \in \Delta\}. \end{aligned} \tag{5.11}$$

These complements have the following relationship

$${}^\perp(\Delta^\perp) = ({}^\perp\Delta)^\perp = \Delta + \mathcal{C}. \tag{5.12}$$

We remark that these complements interchange sums and intersections, since the annihilator interchanges them and the linear map $\flat$ preserves them. Consequently, if Δ, Γ are distributions, we have

$$\begin{aligned} (\Delta \cap \Gamma)^\perp &= \Delta^\perp + \Gamma^\perp \\ (\Delta + \Gamma)^\perp &= \Delta^\perp \cap \Gamma^\perp \end{aligned} \tag{5.13}$$

A triple (M, η, H), where (M, η) is a precontact manifold and $H \in \mathcal{C}^\infty(M)$ is the *Hamiltonian function* will be called a *precontact Hamiltonian system*, which is the main object of study of this chapter.

5.2. Precontact Hamiltonian systems and the constraint algorithm

We aim to solve Hamilton equations on a precontact Hamiltonian system (M, η, H). In order to do that, we will introduce an algorithm similar to the one introduced on [147] for presymplectic systems and that was extended in [63, 187] to the cosymplectic case.

Let $\gamma_H = \mathrm{d}H - (H + \mathcal{R}(H))\eta$ where $\mathcal{R}$ is a Reeb vector field (we will later see that the algorithm is independent on the choice of the Reeb vector field) and consider the equation

$$\flat(X) = \gamma_H. \tag{5.14}$$

This equation might not have solution, so we will consider the subset $M_1 \subseteq M_0 = M$ of the points in which a solution exists. That is,

$$M_1 = \{p \in M_0 \mid (\gamma_H)_p \in \flat(T_p M_0)\}. \tag{5.15}$$

We note that this condition is equivalent to the following

$$M_1 = \{p \in M_0 \mid \langle (\gamma_H)_p, TM_0{}^{\perp} \rangle = 0\}, \tag{5.16}$$

since $\flat(TM_0) = (\flat(TM_0)^\circ)^\circ = \left(TM_0{}^{\perp}\right)^\circ$.

If we choose a local basis $\{X_a\}_{j=a}^{k_1}$ of $TM_0{}^{\perp}$, we can easily compute the so-called *primary constraint* functions $\phi^a(p) = \langle \mathrm{d}H_p - (\mathcal{R}(H) + H)\eta_p, X_a \rangle$, whose zero set is the manifold M_1. We note that $TM_0{}^{\perp} = (\operatorname{im} \flat)^\circ = \ker \flat = \mathcal{C}$ by Equation (5.9). Hence,

$$\langle \mathrm{d}H_p - (\mathcal{R}(H) + H)\eta_p, TM_0{}^{\perp} \rangle = \{Z_p(H) = 0 \mid Z_p \in \mathcal{C}_p\}. \tag{5.17}$$

Therefore, in Darboux coordinates,

$$\phi^a = \frac{\partial H}{\partial s^a}. \tag{5.18}$$

We note that this implies that $\mathcal{R} = \tilde{\mathcal{R}}(H)$ along M_1 for every Reeb vector field $\tilde{\mathcal{R}}$, since $\mathcal{R}_p - \tilde{\mathcal{R}}_p \in \mathcal{C}_p$. Consequently, $\gamma_H|_{M_1}$ is independent on the choice of the Reeb vector field. Therefore, the election of $\mathcal{R}$ doesn't affect the constraints produced by the algorithm.

Now we can solve Hamilton equations, but, in order to have meaningful dynamics, the solution X should be tangent to the constraint submanifold. Otherwise, a solution of the equations of motion might escape from M_1. This tangency condition is equivalent to demand that $\flat(X_p) \in \flat(TM_p)$ since $\flat$ is an isomorphism modulo $\mathcal{C}_p$:

$$M_2 = \{p \in M_1 \mid \langle (\gamma_H)_p, TM_1{}^{\perp} \rangle = 0\}, \tag{5.19}$$

providing a second constraint submanifold, with its corresponding constraint functions. However, it is not enough. We must again require that the vector field is tangent to the new submanifold.A sequence of submanifolds is produced

$$\begin{aligned} M_{i+1} &= \{p \in M_i \mid (\gamma_H)_p \in \flat(T_p M_i)\} \\ &= \{p \in M_i \mid \langle (\gamma_H)_p, T_p M_i{}^{\perp} \rangle = 0\} \end{aligned} \tag{5.20}$$

which eventually stabilizes, that is, there exist some i_f such that $M_{i_f} = M_{i_f+1}$. We call this manifold the *final constraint submanifold* and denote it by M_f. This submanifold is locally described by the zero set of some constraint functions $\{\phi^j\}_{j=1}^{k_f}$.

If M_f has positive dimension, there will exist Hamiltonian vector fields along M_f. The pair (M_f, X) will be called a Hamiltonian vector field solution to the Hamiltonian precontact system (M, η, H).

A useful characterization of such pairs is given by the following

Proposition 5.5. *X is a Hamiltonian vector field along M' for (M, η, H) if and only if, at the points of M',*

$$\eta(X) = -H, \tag{5.21a}$$
$$\mathscr{L}_X \eta = g\eta, \tag{5.21b}$$

where $g : M' \to \mathbb{R}$. Moreover, if this holds, then $g = -\mathcal{R}(H)$ for any Reeb vector field $\mathcal{R}$.

Proof. Let X be a Hamiltonian vector field along M'. By the definition of $\flat$, at the points of M', becomes

$$\iota_X \mathrm{d}\eta + \eta(X)\eta = \mathrm{d}H - (H + \mathcal{R}(H))\eta, \tag{5.22}$$

and, by contraction with $\mathcal{R}$, we obtain

$$\eta(X) = -H. \tag{5.23}$$

Combining (5.22) and (5.23), we deduce

$$\iota_X \mathrm{d}\eta + \mathrm{d}\iota_X \eta = -\mathcal{R}(H)\eta, \tag{5.24}$$

but the left-hand side of this equation equals $\mathscr{L}_X \eta$ by Cartan's formula, hence X fulfills (5.21) at the points of M'.

Now assume that X satisfies (5.21) on the points of M'. Once again, by contraction of (5.21b) with a Reeb vector field $\mathcal{R}$, we have

$$g = \iota_{\mathcal{R}} \mathscr{L}_X(\eta) = \iota_{\mathcal{R}}(\iota_X \mathrm{d}\eta + d(\eta(X))) = -\iota_{\mathcal{R}}(\mathrm{d}H) = -\mathcal{R}(H). \tag{5.25}$$

Combining this with (5.21), we can easily retrieve (5.22). $\square$

5.2.1. Tangency of the Reeb vector field

Next, we will discuss when there is a Reeb vector field tangent to the final constraint submanifold. We can guarantee it in some situations, like in the case of Rayleigh dissipation (as in the example of Example 5.1), in which $\mathcal{R}(H)$ is constant. However, this is not true in general, as can be seen in the example of Example 5.2.

Lemma 5.6. *Let N be a submanifold of a precontact manifold (M, η). Then, there exists a Reeb vector field on M tangent to N if and only if $TN^\perp \subseteq \ker \eta$*

Proof. Let $\mathcal{R}$ be a Reeb vector field on M tangent to N. Let X be tangent to $TN^{\perp}$. That is, for all $Y_p \in TN^{\perp}$ and $p \in N$,

$$\flat(Y_p)(X_p) = 0. \tag{5.26}$$

In particular, if we let $Y = \mathcal{R}$,

$$\flat(\mathcal{R}_p)(X_p) = \mathrm{d}\eta(\mathcal{R}_p, X_p) + \eta(\mathcal{R}_p)\eta(X_p) = \eta(X_p) = 0, \tag{5.27}$$

hence $TN^{\perp} \subseteq \ker \eta$.

For the converse, $TN^{\perp} \subseteq \ker \eta$ implies $\eta \in \left(TN^{\perp}\right)^{\circ} = \flat(TN)$. So there $\eta = \flat(Y)$ with Y tangent to TN. Y is a Reeb vector field by Proposition 5.4. □

Proposition 5.7. *Let (M, η, H) be a precontact Hamiltonian system. Then, there is a Reeb vector field $\mathcal{R}$ tangent to the final constraint submanifold if and only if $Z(\mathcal{R}(H)) = 0$ for all $Z \in TM_f{}^{\perp}$. In particular, if $\mathcal{R}(H)$ is constant, then $\mathcal{R}$ is tangent to M_f.*

Proof. We will prove the result by induction. Let $p \in TM_f$ and let $\mathcal{R}$ be a Reeb vector field tangent to M_i. Notice that $TM_i{}^{\perp} \subseteq (\ker \eta)$ by Lemma 5.6. $\mathcal{R}$ will be tangent to M_{i+1} at p if the Lie derivative of the $(i+1)$-th constraint functions vanish. That is, for every Z tangent to $TM_i{}^{\perp}$ in a neighborhood of p,

$$(\mathcal{L}_{\mathcal{R}}\langle \gamma_H, Z\rangle)_p = \langle \mathcal{L}_{\mathcal{R}}\gamma_H, Z\rangle_p + \langle \gamma_E, \mathcal{L}_X Z\rangle_p = 0. \tag{5.28}$$

We compute the first term. Since $\mathcal{L}_{\mathcal{R}}\eta = 0$, we have that

$$\mathcal{L}_{\mathcal{R}}\gamma_H = \mathcal{L}_{\mathcal{R}}\mathrm{d}H - \mathcal{L}_X(E + \mathcal{R}(E))\eta.$$

Therefore, because $\eta(Z) = 0$, we deduce

$$\langle \mathcal{L}_{\mathcal{R}}\gamma_H, Z\rangle = Z(\mathcal{R}(H)) \tag{5.29}$$

We will now see that $[R, Z]_p \in T_pM_i$. Let W be any vector field on M_i. Then, along M_i,

$$\omega(W, [\mathcal{R}, Z]) = -\mathcal{L}_{\mathcal{R}}\omega(W, Z) + \mathcal{L}_{\mathcal{R}}(\omega(W, Z)) + \omega([\mathcal{R}, W], Z) = 0.$$

The first term vanishes since $\mathcal{L}_{\mathcal{R}}\omega = 0$ because $\mathcal{L}_{\mathcal{R}}\eta = 0$. The second and third terms are also zero because $Z \in TM_i{}^{\perp}$. Hence, the last term of Equation (5.28) vanishes along M_{i+1}. Therefore, $\mathcal{R}$ is tangent to M_{i+1} if and only if

$$(\mathcal{L}_{\mathcal{R}}\langle \gamma_H, Z\rangle)_p = Z(\mathcal{R}(H))_p = 0, \tag{5.30}$$

for all $Z_p \in (TM_i)_p{}^{\perp} \subseteq (TM_f)_p{}^{\perp}$.

Notice that if $\mathcal{R}$ is not tangent to T_pM_i it will not be tangent to $TM_f \subseteq T_pM_i$, so the converse follows. □

As we have proven in Proposition 5.7, the Reeb vector field is not necessarily tangent to the constraint submanifold M_f. A modification of the previous algorithm guarantees this fact, just by requiring that a chosen Reeb vector field $\mathcal{R}$ is tangent to the constraint submanifold after each step. This will produce a new sequence of submanifolds. Explicitly, $\bar{M}_0 = \hat{M}_0 = M$, and for $i \geq 1$ we define recursively:

$$\begin{aligned} \bar{M}_i &= \{p \in \hat{M}_{i-1} \mid (\gamma_H)_p \in \flat(T_p\hat{M}_{i-1})\} \\ &= \{p \in \hat{M}_{i-1} \mid \langle (\gamma_H)_p, T_p\hat{M}_{i-1}{}^{\perp} \rangle = 0\} \\ \hat{M}_i &= \{p \in \bar{M}_i \mid \mathcal{R}_p \in T_p\bar{M}_i\} \\ &= \{p \in \bar{M}_i \mid \mathcal{L}_{\mathcal{R}} \langle (\gamma_H)_p, T_p\hat{M}_{i-1}{}^{\perp} \rangle = 0\}. \end{aligned} \tag{5.31}$$

Locally, in terms of constraint functions, if $\bar{M}_i$ is described as the zero set of functions $(\phi_k)_k$, then $\hat{M}_i$ would be the zero set of $(\phi_k, \mathcal{R}(\phi_k))_k$. We get a sequence of constraint submanifolds as follows:

$$\cdots \hookrightarrow \hat{M}_{i+1} \hookrightarrow \bar{M}_i \hookrightarrow \hat{M}_i \hookrightarrow \cdots \hookrightarrow \hat{M}_2 \hookrightarrow \bar{M}_1 \hookrightarrow \hat{M}_1 \hookrightarrow M, \tag{5.32}$$

The algorithm stops when we reach submanifold such that none of the two steps produces new constraints. That is: $\bar{M}_{j_f} = \hat{M}_{j_f} = \bar{M}_{j_f+1}$.

Remark 5.8. By construction, the first algorithm will produce the largest submanifold M_f in which there is a solution to the equations of motion. The second algorithm produces a final constraint submanifold $\bar{M}_f$ in which there is a solution to the equations of motion and a Reeb vector field is tangent, hence $\bar{M}_f \subseteq M_f$. Apart from this, no much more about the relationship between M_f and $\bar{M}_f$ seems possible to state. It can be the case that $\bar{M}_f = M_f$, such as in the example of Example 5.1, or that $\bar{M}_f = \emptyset$ and M_f is nonempty for any choice of Reeb vector field, as in the example of Example 5.2.

5.2.2. Morphisms of precontact Hamiltonian systems

Let (M, η, H) and $(\bar{M}, \bar{\eta}, \bar{H})$ be precontact Hamiltonian systems. A map $F : M \to \bar{M}$ is said to be a *conformal morphism of precontact systems* if $F^*\bar{\eta} = f\eta$ and $F^*\bar{H} = fH$ for some non-vanishing function $f : M \to \mathbb{R}$. If $f = 1$, we say that F is a *strict morphism of precontact systems*.

Theorem 5.9. *Let $F : M \to \bar{M}$ be a strict morphism of precontact systems. Assume that $X, \bar{X}$ are F-related vector fields defined along submanifolds $M' \subseteq M$ and $\bar{M}' = F(M') \subseteq \bar{M}$, respectively. Therefore, if $\bar{X}$ is a Hamiltonian vector field along $\bar{M}'$, then X is also a Hamiltonian vector field along M'.*

Proof. Since $\bar{X}$ is a Hamiltonian vector field, its satisfies (5.21) along $\bar{M}'$

$$\bar{\eta}(\bar{X}) = -\bar{H}, \tag{5.33a}$$

$$\mathcal{L}_{\bar{X}}\bar{\eta} = \bar{g}\bar{\eta}. \tag{5.33b}$$

Pulling back by F, we obtain

$$f\eta(X) = -fH, \tag{5.34a}$$
$$\mathcal{L}_X(f\eta) = (\bar{g} \circ F)f\eta. \tag{5.34b}$$

From this expression, we obtain

$$\eta(X) = -H, \tag{5.35a}$$
$$\mathcal{L}_X(\eta) = g\eta, \tag{5.35b}$$

where $g = \bar{g} \circ F - (\mathcal{L}_X f)/f$. Hence, X is a Hamiltonian vector field. □

Observe that if F is a diffeomorphism, then we have a bijective correspondence between pairs of Hamiltonian vector fields along submanifolds. Moreover, if F is a submersion onto each image, every vector defined along the image if F is F-related to some vector field along M. Thus, we have

Theorem 5.10. *Let $F : M \to \bar{M}$ be a conformal morphism of precontact systems such that F is a submersion onto its image. Denote by M_f and $\bar{M}_f$ the corresponding final constraint submanifolds. Then,*

- *For every F-projectable Hamiltonian vector field X along M_f, $(F)_*(X)$ is a Hamiltonian vector field along $\bar{M}_f$.*
- *For every Hamiltonian vector field Y along $\bar{M}_f$, every $X \in \mathfrak{X}(M)$ such that $(F)_*(X) = Y$ is a Hamiltonian vector field along M_f.*

Moreover, the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "F"] \arrow[dr, "F_1"] & M \\
M_2 \arrow[u, hook, "j_2"] \arrow[dr, "F_2"] & \bar{M}_1 \arrow[u, hook, "g_1"'] \\
\vdots \arrow[u, hook, "j_3"] & \bar{M}_2 \arrow[u, hook, "g_2"'] \\
M_f \arrow[u, hook, "j_f"] \arrow[dr, "F_f"] & \vdots \arrow[u, hook, "g_3"'] \\
 & \bar{M}_f \arrow[u, hook, "g_f"']
\end{tikzcd} \tag{5.36}$$

where M_i and $\bar{M}_i$ are the i-th constraint submanifolds obtained in the constraint algorithm to $M_1 = M$ and to $\bar{M}_1 = F(F)$ respectively, and $j_i : M_i \to M_{i-1}$, $g_i : \bar{M}_i \to \bar{M}_{i-1}$ are the canonical inclusions. The submersions $F_i : M_i \to \bar{M}_i$ are the restrictions of F to the corresponding constraint submanifolds.

Proof. In order to proof the first claims, we can use Theorem 5.9 and the fact that F is a submersion onto its image. In order to do that, we need to prove that $F(M_f) = \bar{M}_f$. We will do this by proving the commutativity of the diagram by induction.

We assume that $\bar{M}_{i-1} = F(M_i)$. Since the algorithm is independent of the choice of Reeb vector fields $\mathcal{R}$ and $\bar{\mathcal{R}}$, we choose Reeb vector fields which are FL-related.

First, let $\omega = \mathrm{d}\eta + \eta \otimes \eta$ and $\bar{\omega} = \mathrm{d}\bar{\eta} + \bar{\eta} \otimes \bar{\eta}$. Since $(F)^*(\bar{\eta}) = \eta_L$, then $(F)^*(\bar{\omega}) = \omega$. From this and the fact that F_* is surjective, it easily follows that F_* maps $TM_i{}^\perp$ onto $T\bar{M}^\perp$.

By taking F-related Reeb vector fields, from a straightforward computation we find that $(F)^*(\gamma_{\bar{H}}) = \gamma_H$. With this, we have that for any $Y \in T\bar{M}_i{}^\perp$ and any $X \in TM_i{}^\perp$ such that $(F)_* X = Y$, $(F)^*(\gamma_{\bar{H}}(Y)) = \gamma_H(X)$. Hence, $F_i(M_i) = \bar{M}_i$ because their constraints are related by F. □

5.3. Singular Lagrangians and the Legendre transformation

In this section, we will apply the previous constraint algorithm to singular Lagrangian systems. We will also study the Legendre transformation and the existence of a Hamiltonian formulation.

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a singular Lagrangian and assume that $\eta_L = \mathrm{d}z - S^*\mathrm{d}L$ is a precontact form of class $2r+1$. We will use the results and notation of Section 3.1.

Let $E_L = \Delta(L) - L$ be the energy and $\gamma_{E_L} = \mathrm{d}E_L - (\mathcal{R}(E_L) + E_L)\eta_L$. We remark that $(TQ \times \mathbb{R}, \eta_L, E_L)$ is a precontact Hamiltonian system. Hence, we can apply the constraint algorithm developed in Section 5.2 to the equation $\flat_L(X) = \gamma_{E_L}$.

If we denote $P_1 = TQ \times \mathbb{R}$, we will obtain a sequence of constraint submanifolds

$$\cdots \hookrightarrow P_i \hookrightarrow \cdots \hookrightarrow P_2 \hookrightarrow P_1, \tag{5.37}$$

where

$$P_{i+1} = \{p \in P_i \mid \langle (\gamma_H)_p, T_p P_i{}^\perp \rangle = 0\}, \tag{5.38}$$

and P_f is the final constraint submanifold. If it has positive dimension, then there would exist a vector field X tangent to P_f that solves the equations of motion along P_f.

Of course, this solution will not be unique in general. We would get a new solution by adding a section of $\mathcal{C}_L \cap TP_f$, where $\mathcal{C}_L = \ker \flat_L$ is the characteristic distribution.

5.3.1. The Hamiltonian side and the equivalence problem

Now we will develop a Hamiltonian counterpart of this theory. This problem was addressed in [145] for singular Lagrangians in the presymplectic case and by [63] for the time dependent case. We will require the following additional regularity conditions on L to make sure we get a precontact Hamiltonian system which is amenable to the constraint algorithm:

Definition 5.1. We say that a contact Lagrangian $L \in \mathcal{C}^\infty(TQ \times \mathbb{R})$ is *almost regular* if

- η_L is precontact.

- $\mathrm{F}L$ is a submersion onto its image.
- For every $p \in T^*Q \times \mathbb{R}$, the fibers $(\mathrm{F}L)^{-1}(p)$ are connected submanifolds.

We denote by M_1 be the image of $\mathrm{F}L$, which will be called the *primary constraint submanifold*. Let $\mathrm{F}L_1$ denote the restriction of $\mathrm{F}L$ to M_1, that is

$$\begin{tikzcd} TQ\times\mathbb{R} \arrow[r, "\mathrm{F}L"] \arrow[dr, "\mathrm{F}L_1"'] & T^*Q\times\mathbb{R} \\ & M_1 \arrow[u, hook, "g_1"'] \end{tikzcd} \tag{5.39}$$

where $g_1 : M_1 \hookrightarrow TQ \times \mathbb{R}$ is the canonical inclusion.

The submanifold M_1 is equipped with the form $\eta_1 = {g_1}^*(\eta_Q)$, where η_Q is the canonical contact form in $T^*Q \times \mathbb{R}$. By the commutativity of the diagram in Equation (5.39), we deduce

$$(\mathrm{F}L_1)^*(\eta_1) = (\mathrm{F}L)^*(\eta_Q) = \eta_L \tag{5.40}$$

Proposition 5.11. *Let $L : P_1 = TQ \times \mathbb{R} \to \mathbb{R}$ be an almost regular Lagrangian. Then $\eta_1 = {g_1}^*(\eta_Q)$ is a precontact form of the same class as η_L.*

Proof. Assume η_L is of class $2r+1$. Then, $\eta_1 \wedge \mathrm{d}\eta_1^r$ is nowhere zero because its image by $(\mathrm{F}L)^*$ is nowhere zero.

Also, $\eta_1 \wedge \mathrm{d}\eta_1^{r+1}$ is everywhere zero. Let $p \in M_1$. Since $\mathrm{F}L_1 : P_1 \to M_1$ is a submersion, there are smooth local sections $G : U \to P_1$, where $p \in U \subseteq M_1$ such that $\mathrm{F}L_1 \circ G = \mathrm{id}_U$. Then,

$$0 = G^*(\eta_L \wedge \mathrm{d}{\eta_L}^{r+1}) = G^*((\mathrm{F}L_1)^*(\eta_1 \wedge \mathrm{d}\eta_1^{r+1})) = \eta_1 \wedge \mathrm{d}\eta_1^{r+1}. \tag{5.41}$$

Therefore, η_1 is a precontact form of class $2r+1$. □

The last ingredient for setting up a precontact Hamiltonian system on M_1 is a Hamiltonian function $H_1 : M_1 \to \mathbb{R}$. By requiring that $\mathrm{F}L$ has connected fibers we obtain the following result:

Proposition 5.12. *Let $L : \mathbb{R} \times TQ \to \mathbb{R}$, be an almost regular Lagrangian, then, there is a unique function $H_1 : M_1 \to \mathbb{R}$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{R} \\ TQ\times\mathbb{R} \arrow[ur, "E_L"] \arrow[r, "\mathrm{F}L"] \arrow[dr, "\mathrm{F}L_1"'] & T^*Q\times\mathbb{R} \\ & M_1 \arrow[u, hook, "g_1"'] \arrow[uu, bend right=60, "H_1"'] \end{tikzcd} \tag{5.42}$$

That is,

$$H_1 \circ \mathrm{F}L = E_L, \tag{5.43}$$

Proof. We will prove that E_L is constant along the fibers of $\mathrm{F}L$, so H_1 is well-defined. Since the fibers are connected, it is enough to see that $\mathcal{L}_Z E = 0$ for every $Z \in \ker(\mathrm{F}L)_*$.

One can compute (see [63, page 3424])

$$\ker(\mathrm{F}L)_* = \ker \mathrm{d}\theta_L \cap \operatorname{im} S = \ker \mathrm{d}\eta_L \cap \operatorname{im} S = \mathcal{C} \cap \operatorname{im} S. \tag{5.44}$$

In bundle coordinates, one can see that $X \in \ker(\mathrm{F}L)_*$ if and only if

$$X = b^j \frac{\partial}{\partial \dot{q}^j}, \tag{5.45}$$

where, for all i,

$$b^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0. \tag{5.46}$$

By using the coordinate expression of the energy (3.6) we find that

$$X(E_L) = \dot{q}^i b^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0. \tag{5.47}$$

□

By the results of this chapter we conclude that if the Lagrangian is almost regular, then (M_1, η_1, H_1) is a precontact Hamiltonian system. Thus, we apply the constraint algorithm (Section 5.2) to the equation $\flat_1(Y) = \gamma_{H_1}$, where $\flat_1$ is the mapping defined by η_1. Thus, we obtain a sequence of constraint submanifolds

$$\cdots \hookrightarrow M_i \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1, \tag{5.48}$$

where M_f is the final constraint submanifold.

We note that if L is almost-regular, then $\mathrm{F}L$ is a strict morphism of the precontact systems (P_1, η_L, E_L) and (M_1, η_1, H_1) that is a submersion onto its image. Thus, applying Theorem 5.10 we obtain the following results.

Proposition 5.13. *The following diagram commutes*

$$\begin{tikzcd}
TQ \times \mathbb{R} \arrow[r, "\mathrm{F}L"] \arrow[dr, "\mathrm{F}L_1"] & T^*Q \times \mathbb{R} \\
P_2 \arrow[u, hook, "j_2"] \arrow[dr, "\mathrm{F}L_2"] & M_1 \arrow[u, hook, "g_1"'] \\
\vdots \arrow[u, hook, "j_3"] & M_2 \arrow[u, hook, "g_2"'] \\
P_f \arrow[u, hook, "j_f"] \arrow[dr, "\mathrm{F}L_f"] & \vdots \arrow[u, hook, "g_3"'] \\
 & M_f \arrow[u, hook, "g_f"']
\end{tikzcd} \tag{5.49}$$

where P_i and M_i are the i-th constraint submanifolds obtained in the constraint algorithm to $P_1 = TQ \times \mathbb{R}$ and to M_1 respectively, and $j_i : P_i \to P_{i-1}$, $g_i : M_i \to M_{i-1}$ are the canonical inclusions. The submersions $\mathrm{FL}_i : P_i \to M_i$ are the restrictions of the Legendre transformation FL to the corresponding constraint submanifolds.

From the commutativity of the diagram, we get the following result.

Theorem 5.14 (Equivalence Theorem). *Let $L : P \times \mathbb{R} \to \mathbb{R}$ be an almost regular Lagrangian, let (P, η_L, E_L) be the corresponding precontact system, and let (M_1, η_1, H_1) be its Hamiltonian counterpart. We denote the final constraint submanifolds by P_f and M_f, respectively. Then*

- *For every FL-projectable solution X of the equations of motion along P_f, $(\mathrm{FL})_*(X)$ is a solution of Hamilton equations of motion along M_f.*
- *For every solution Y of Hamilton equations of motion along M_f, every $X \in \mathfrak{X}(TQ \times \mathbb{R})$ such that $(\mathrm{FL})_*(X) = Y$ solves the equations of motion along P_f.*

5.4. The Dirac-Jacobi bracket

The aim of this section is to develop a local version of the constraint algorithm based on the Jacobi bracket of the contact manifold $T^*Q \times \mathbb{R}$, similar to the Dirac-Bergmann algorithm for the presymplectic case [20, 115]. This bracket also has some global geometric descriptions [164, 186]. It has been extended to the time-dependent case in [63].

The bracket formalism will allow us to classify the constraints produced by the algorithm depending on whether they provide dynamical information (*first class*) or not (*second class*). Furthermore, we will define a modified bracket, the *Dirac-Jacobi bracket* which will provide us expressions for the evolution of the observables which are manifestly independent on the second class constraints.

As we have explained in Section 2.3, a contact manifold (M, η) is a particular case of a Jacobi manifold, with Jacobi structure $(\Lambda, -\mathcal{R})$. We remind that the Jacobi bracket is given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) - f\mathcal{R}(g) + g\mathcal{R}(f), \tag{5.50}$$

for $f, g \in C^\infty(M)$. We recall that these brackets are not Poisson. Instead, they satisfy the following generalized Leibniz rule:

$$\{fg, h\} = f\{g, h\} + g\{f, h\} + fg\mathcal{R}(h), \tag{5.51}$$

for arbitrary functions $f, g, h \in C^\infty(M)$.

The evolution of an observable $f \in C^\infty(T^*Q)$ can be written in terms of its bracket with the Hamiltonian H,

$$\dot{f} = X_H(f) = \{H, f\} - f\mathcal{R}(H), \tag{5.52}$$

where H is an arbitrary extension of H_1.

In this section we will be working with the Hamiltonian formulation of a system that is given by an almost regular Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ as in Section 5.3. Assume that we obtain the *primary constraint submanifold* $M_1 = \mathrm{F}L(TQ) \subseteq T^*Q$, with a Hamiltonian function $H_1 : M_1 \to \mathbb{R}$. We can extend H_1 to $T^*Q \times \mathbb{R}$ as follows:

$$H_T = H + u_a \phi^a, \tag{5.53}$$

where H is an arbitrary extension of H_1, ϕ^a are a set of constraints defining M_1 and u_a are Lagrange multipliers. Hence, we can compute the evolution of an observable f with respect to H_T:

$$\begin{aligned}
\dot{f} &= \{H_T, f\} - f\mathcal{R}(H_T) \\
&= \{H, f\} + u_a\{\phi^a, f\} - f\mathcal{R}(H) - f u_a \mathcal{R}(\phi^a) \\
&\quad + \phi^a(\{u_a, f\} + u_a\mathcal{R}(f) - f\mathcal{R}(u_a)) \\
&= \{H, f\} - f\mathcal{R}(H) + u_a(\{\phi^a, f\} - f\mathcal{R}(\phi^a)) + \phi^a \Lambda(\mathrm{d}u_a, \mathrm{d}f) \\
&= (X_H + u_a X_{\phi^a})(f) + \phi^a \Lambda(\mathrm{d}u_a, \mathrm{d}f),
\end{aligned} \tag{5.54}$$

where we have used the generalized Leibniz rule Equation (5.51).

The constraint algorithm can be locally interpreted in terms of this bracket, similarly to the Dirac algorithm for the symplectic case [147].

Remark 5.15. A local version of the constraint algorithm for constraints on the extended phase $M_1 \subseteq T^*Q \times \mathbb{R}$ can be given in terms of the Jacobi bracket as follows.

First, we demand that the primary constraints should be preserved along the evolution of the system. Geometrically, this means that X_{H_T} should be tangent to M_1, that is:

$$(0 = \dot{\phi}^a = X_{H_T}(\phi^a) = \{H, \phi_a\} + u_b\{\phi^b, \phi^a\})|_{M_1}, \tag{5.55}$$

since $\phi_b = 0$ on M_1. We should demand this condition for all linear combinations of the constraints. Some will be satisfied trivially, others will fix the multipliers u_b, and the remaining ones will be independent on the multiples u_b. The later take the form $f_a^\alpha \dot{\phi}^a$, where

$$(f_a^\alpha\{\phi^a, \phi^b\} = 0)|_{M_1}. \tag{5.56}$$

If we let $\psi^\alpha = f_a^\alpha \dot{\phi}^a$, then

$$(\psi^\alpha = f_a^\alpha\{H, \phi_a\})|_{M_1}. \tag{5.57}$$

These new constraints may define a *secondary constraint submanifold*, M_2 (that we assume that it is indeed a submanifold). We can now modify the Hamiltonian by adding the new constraints $H_T' = H_T + v_\alpha \psi^\alpha$ and iterate this procedure until it stabilizes and we get not additional constraints.

Let M_f be the final constraint submanifold. We say that a function $f \in C^\infty(T^*Q \times \mathbb{R})$ is *first class* if $\{f, \phi\}|_{M_f} = 0$. Denote by $\mathcal{F} \subseteq C^\infty(M)$ the set of first class functions, which is a subalgebra with respect to the Jacobi bracket since, by the Jacobi identity, if $\psi, \chi \in \mathcal{F}$ and ϕ is a constraint, then, along M_f,

$$\{\{\psi, \chi\}, \phi\} = \{\{\psi, \phi\}, \chi\} + \{\psi, \{\chi, \phi\}\} = 0.$$

The Hamiltonian H_T is an example of a first class function because of the constraint preservation condition given in Equation (5.55).

We say that a function is *second class* if it is not first class.

We will show that family of independent constraints ϕ^α defining M_f (by independent, we mean that their differentials are linearly independent) we may exstract a maximal subfamily of second class constraints such that the matrix of their pairwise Jacobi brackets is invertible at each point of the final submanifold.

Consider the matrix $(\langle \phi^\alpha, \phi^\beta \rangle)_{\alpha,\beta}$. Assume that it has constant rank k in a neighborhood of M_f, that is, up to reordering, the first k rows are linearly independent. Denote by ϕ^a (with latin indices) those functions and $\phi^{\bar{a}}$ (with overlined latin indices) the rest of them. We use greek indices when we want to refer to every constraint. Then the rest of the rows are linear combinations of the first k, that is

$$\{\phi^{\bar{a}}, \phi^\beta\}_{DJ} = B^{\bar{a}}_a \{\phi^a, \phi^\beta\}_{DJ}. \tag{5.58}$$

Define

$$\bar{\phi}^{\bar{a}} = \phi^{\bar{a}} - B^{\bar{a}}_a \phi^a. \tag{5.59}$$

Using the generalized Leibniz rule (Equation (5.51)) we can check that these new constraints are first class, so $\phi^a, \bar{\phi}^{\bar{a}}$ is a basis of the constraints with the desired properties.

Now let $C^{ab} = \{\phi^a, \phi^b\}$ and let C_{ab} denote the inverse matrix. We define the *Dirac-Jacobi* bracket such that

$$\{f, g\}_{DJ} = \{f, g\} - \{f, \phi^a\} C_{ab} \{\phi^b, g\}. \tag{5.60}$$

Proposition 5.16. *The Dirac-Jacobi bracket has the following properties:*

1. *It is a Jacobi bracket (Definition 2.8) which satisfies the generalized Leibniz rule*

$$\{fg, h\}_{DJ} = f\{g, h\}_{DJ} + g\{f, h\}_{DJ} + fg\mathcal{R}_{DJ}(h), \tag{5.61}$$

where

$$\mathcal{R}_{DJ} = \mathcal{R} + C_{ab}\mathcal{R}(\phi^b)(\sharp_\Lambda(\mathrm{d}\phi^a) + \phi^a \mathcal{R}). \tag{5.62}$$

2. *The second class constraints ϕ^a are Casimir functions for the Dirac-Jacobi bracket.*

3. *For any first class function F,*

$$\begin{aligned} (\{F, \cdot\}_{DJ} &= \{F, \cdot\})|_{M_f}, \\ (\mathcal{R}_{DJ}(F) &= \mathcal{R}(F))|_{M_f}. \end{aligned} \tag{5.63}$$

4. *The evolution of observables is given by*

$$\begin{aligned} (\dot{f} &= \{H, f\}_{DJ} - f\mathcal{R}_{DJ}(H) + \bar{u}_{\bar{a}}(\{\bar{\phi}^{\bar{a}}, f\}_{DJ} - f\mathcal{R}_{DJ}(\bar{\phi}^{\bar{a}})) \\ &= (X_H + \bar{u}_{\bar{a}} X_{\bar{\phi}^{\bar{a}}})(f))|_{M_f}, \end{aligned} \tag{5.64}$$

where $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ is an arbitrary extension of the Hamiltonian H_1.

We remark that the motion depends on the multipliers of the first class constraints $\bar{u}_{\bar{a}}$, but it is independent on the multipliers of the second class constraints u_a.

Proof. It is clear that the brackets are bilinear and antisymmetric. The Jacobi identity follows from a computation as the one performed by Dirac in [116] for the symplectic case. Moreover, the locality of the Dirac-Jacobi bracket follows from the locality of the bracket associated to the natural Jacobi structure of $T^*Q \times \mathbb{R}$. Therefore, by Theorem 2.8, there is another Jacobi structure $(\Lambda_{DJ}, \mathcal{R}_{DJ})$ on $T^*Q \times \mathbb{R}$ such that

$$\{f, g\}_{DJ} = \Lambda_{DJ}(\mathrm{d}f, \mathrm{d}g) - f\mathcal{R}_{DJ}(g) + g\mathcal{R}_{DJ}(f). \tag{5.65}$$

The vector field $\mathcal{R}_{DJ}$ can be computed by taking into account that

$$\begin{aligned}
\mathcal{R}_{DJ}(f) &= \{f, 1\}_{DJ} \\
&= \{f, 1\} - \{(f, \phi^a)\}C_{ab}\{\phi^b, 1\} \\
&= \mathcal{R}(f) - C_{ab}(\Lambda(\mathrm{d}f, \mathrm{d}\phi^a) - f\mathcal{R}(\phi^a) + \phi^a\mathcal{R}(f))\mathcal{R}(\phi^b) \\
&= (\mathcal{R} + C_{ab}\mathcal{R}(\phi^b)(\sharp_\Lambda(\mathrm{d}\phi^a) + \phi^a\mathcal{R}))(f) - fC_{ab}\mathcal{R}(\phi^a)\mathcal{R}(\phi^b),
\end{aligned}$$

where $-C_{ab}\mathcal{R}(\phi^a)\mathcal{R}(\phi^b) = \{1, 1\}_{DJ} = 0$, by the antisymmetry of the bracket.

The fact that ϕ^a are Casimir functions follows from a straightforward calculation from the definition of the brackets.

For proving the statement (3), if F is first class it is clear that both brackets coincide along M_f since they differ by multiples of the Jacobi brackets of F with constraints. For the second part, notice that, along M_f,

$$\mathcal{R}(F) = \{1, F\} = \{1, F\}_{DJ} = \mathcal{R}_{DJ}(F). \tag{5.66}$$

The last claim follows from the combination of the formula for the evolution of observables Equation (5.54) and Item (3). Since the second class constraints ϕ^a are Casimir functions, their brackets, including $\mathcal{R}_{DJ}(\phi^a) = \{1, \phi^a\}_{DJ}$, will vanish, so the terms with the corresponding multipliers u^a will not affect the evolution of the observable. □

5.5. The second order problem

Using the theory developed on the previous section, given an almost regular Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ we are able to develop the constraint algorithm on the Lagrangian side, as well as on the Hamiltonian counterpart starting with the image $M_1 = \mathrm{F}L(\mathbb{R} \times TQ)$, the precontact form η_1 and the restricted Hamiltonian H_1. The following diagram summarizes the situation:

$$\begin{tikzcd}
TQ \times \mathbb{R} \arrow[r, "\mathrm{F}L"] \arrow[dr, "\mathrm{F}L_1"] & T^*Q \times \mathbb{R} \\
P_f \arrow[u, hook] \arrow[dr, "\mathrm{F}L_f"] & M_1 \arrow[u, hook] \\
 & M_f \arrow[u, hook]
\end{tikzcd} \tag{5.67}$$

where P_f and M_f are the final constraint submanifolds on the Lagrangian and Hamiltonian sides, which are the maximal submanifolds in which solutions to the equations of motion

$$\flat_L(X) = \gamma_{E_L}, \tag{5.68a}$$
$$\flat(Y) = \gamma_{H_1} \tag{5.68b}$$

exist and are tangent to the respective submanifolds. Both submanifolds are connected by the Legendre transformation $\mathrm{F}L_f : P_f \to M_f$, which is a surjective submersion.

Remark 5.17. Notice that in order to get a solution X on the Lagrangian side we can start with a solution Y and use that $\mathrm{F}L_f : P_f \to M_f$ is a fibration to construct X such that $(\mathrm{F}L)_* X = Y$.

As we know, if the Lagrangian is regular, the Euler-Lagrange equations are of second order. That is, the solution X is a so-called a SODE (see Remark 3.1).

However, this is not the case for singular Lagrangians. We are interested on finding a submanifold R of P_f and a solution X tangent to R that *satisfies the second order condition along* R. That is $S(X)_p = \Delta_p$ at every $p \in R$. This is the so-called *second order problem*, which was studied for presymplectic Lagrangian systems in [146] and in [63] for time dependent Lagrangians.

The connection with Herglotz's equations and the related variational problem is apparent from the next result, which parallels [2, Theorem 3.5.17] in the symplectic case.

Proposition 5.18. *Let X be a vector field on $TQ \times \mathbb{R}$ that verifies the second order equation condition along a submanifold $R \subseteq TQ \times \mathbb{R}$, and let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian. Then, along R, X solves the equations of motion for L if and only if it solves Hertglotz's equations.*

Proof. Indeed, if X satisfies the second order equation condition along R, then, along R

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i} + c \frac{\partial}{\partial z}. \tag{5.69}$$

If it solves the equations of motion, necessarily $\eta_L(X) = -E_L$. Substituting the coordinate expression of X, we find out that $(c = L)|_R$. Hence, we can perform the same computation as in the regular case (Equation (3.10)). That is, along R, the coefficients b^i must satisfy the equation

$$b^i \frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + \dot{q}^i \frac{\partial}{\partial q^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + L \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = \frac{\partial L}{\partial \dot{q}^j}\frac{\partial L}{\partial z}. \tag{5.70}$$

Hence, an integral curve $(q^i, \dot{q}^i, z)$ satisfies Herglotz's equation along R:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}. \tag{5.71}$$

The converse follows by reversing the computation. □

In this section we will construct a submanifold $R \subseteq P_f$ along which the equations of motion have a unique solution which is a SODE. The first observation is that $\ker{(\mathrm{F}L_f)}_*$ is an involutive distribution. Indeed, it is the vertical distribution of the fibration $\mathrm{F}L_f : P_f \to M_f$. By the construction of the constraint submanifolds, we can see that for $x \in P_f$ $\ker{(\mathrm{F}L)}_*(x) = \ker{(\mathrm{F}L_f)}_*(x) \subseteq T_xP_f$.

Let X be a vector field on $TQ \times \mathbb{R}$. We define the *deviation* of X as

$$X^* = S(X) - \Delta. \tag{5.72}$$

We note that $X^* = 0$ if and only if X is a second order equation. The next step in the construction of the solution to the second order problem is the following.

Lemma 5.19. *If X is a solution of the equations of motion along P_f, then $X^* \in \ker{(\mathrm{F}L_f)}_*$.*

Proof. Assume that X is written in bundle coordinates $(q^i, \dot{q}^i, z)$ on $TQ \times \mathbb{R}$ by

$$X = a^i \frac{\partial}{\partial q^i} + b^i \frac{\partial}{\partial \dot{q}^i} + c\frac{\partial}{\partial z}. \tag{5.73}$$

Then

$$X^* = S(X) - \Delta = (a^i - \dot{q}^i)\frac{\partial}{\partial \dot{q}^i}. \tag{5.74}$$

If we contract both sides of the equation of motion $\flat_L(X) = \gamma_H$ by $\frac{\partial}{\partial \dot{q}^i}$ we get

$$(a^i - \dot{q}^i)\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0. \tag{5.75}$$

Next, we compute

$$\mathrm{F}L_*(X^*) = (a^j - \dot{q}^j)\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0. \tag{5.76}$$

Hence, X^* is tangent to the leaves of the fibration determined by $\ker{(\mathrm{F}L_f)}_*$, or, in other words to the fibers of the fibration $\mathrm{F}L_f : P_f \to M_f$. □

Next we will construct the submanifold R. Fix a point $y \in M_f$ and let x be an arbitrary point on the leaf over y, say $\mathrm{F}L_f(x) = y$. Assume that $x = (q_0^i, \dot{q}_0^i, z_0)$ in bundle coordinates.

Notice that X is projectable, hence along a leaf it can only vary from point to point in a direction tangent to $\ker(\mathrm{F}L)_*$. Since $\ker(\mathrm{F}L)_* \subseteq \operatorname{im} S = \langle\{\frac{\partial}{\partial \dot{q}^i}\}_i\rangle$, this implies that a^i and c are constant functions along the leafs and only b^i might change.

Consider the vector field

$$-X^* = (\dot{q}^i - a^i)\frac{\partial}{\partial \dot{q}^i}, \tag{5.77}$$

and compute the integral curve of $-X^*$ passing through x, say

$$\sigma(t) = (q^i(t), \dot{q}^i(t), z(t)). \tag{5.78}$$

Therefore,

$$\sigma(0) = (q^i(0), \dot{q}^i(0), z(0)) = (q_0^i, \dot{q}_0^i, z_0). \tag{5.79}$$

This integral curve has to satisfy the system of differential equations:

$$\frac{\mathrm{d}\dot{q}^i}{\mathrm{d}t} = \dot{q}^i(t) - a^i. \tag{5.80}$$

Consequently, the solution passing through x is just

$$\sigma(t) = (q_0^i, a^i + \exp(t)(q_0^i - a^i), z_0), \tag{5.81}$$

which is entirely contained on the fiber over y. In addition, the limit point as $t \to -\infty$,

$$\tilde{x} = \lim_{t\to-\infty} \sigma(t), \tag{5.82}$$

is also on the same fiber, since fibers are closed. A direct computation shows that

$$\tilde{x} = (q_0^i, a^i, z_0), \tag{5.83}$$

and that

$$S(X)_{\tilde{x}} = \Delta_{\tilde{x}}. \tag{5.84}$$

Summarizing, we have constructed a smooth section $\alpha : M_f \to P_f$ of the fibration $\mathrm{F}L_f : P_f \to M_f$, by taking $\alpha(y) = \tilde{x}$ for some x on the fiber over y (notice that $\tilde{x}$ only depends on $\mathrm{F}L(x)$). By taking $R = \alpha(M_f)$ we have the following.

Theorem 5.20 (Second order differential equation)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be an almost regular Lagrangian and let P_f be the final constraint embedded submanifold. Then, there exists a submanifold $R \subseteq P_f$ such that the equations of motion have a unique solution $X \in \mathfrak{X}(TM \times \mathbb{R})$ satisfying the SODE condition. That is, along R,*

$$\flat_L(X) = \gamma_{E_L}, \quad S(X) = \Delta. \tag{5.85}$$

5.6. Examples

We end the chapter by providing examples of singular Lagrangians where the algorithm can be applied.

Example 5.1. Example 1: Cawley's Lagrangian The Lagrangian considered by Cawley [60] can be modified by adding a linear dissipative term γz, where γ is a real number. Let $Q = \mathbb{R}^3$, $P_1 = TQ \times \mathbb{R}$ and consider the Lagrangian function $L : P_1 \to \mathbb{R}$ such that

$$L(q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3, z) = \frac{m}{2}(\dot{q}^1 + \dot{q}^2)^2 + \frac{\mu}{2}(\dot{q}^3)^2 + V(q^1, q^2, q^3) + \gamma z, \tag{5.86}$$

for some potential function V and some real nonzero constants m, μ. This Lagrangian induces the following precontact structure on P_1

$$\eta_L = \mathrm{d}z - m(\dot{q}^1 + \dot{q}^2)(\mathrm{d}q^1 + \mathrm{d}q^2) - \mu\dot{q}^3\mathrm{d}q^3 \tag{5.87}$$

$$\mathrm{d}\eta_L = m(\mathrm{d}q^1 + \mathrm{d}q^2) \wedge (\mathrm{d}\dot{q}^1 + \mathrm{d}\dot{q}^2) + \mu(\mathrm{d}q^3 \wedge \mathrm{d}\dot{q}^3), \tag{5.88}$$

One can check that $\eta_L \wedge (\mathrm{d}\eta_L)^2$ is nowhere zero and $\eta_L \wedge (\mathrm{d}\eta_L)^3 = 0$, hence (P_1, η_L) is a precontact manifold of class 5, with the corresponding energy function $E_L = \Delta(L) - L$ given by

$$E_L = \frac{m}{2}(\dot{q}^1 + \dot{q}^2)^2 + \frac{\mu}{2}(\dot{q}^3)^2 - V(q^1, q^2, q^3) - \gamma z. \tag{5.89}$$

We now apply the constraint algorithm to the precontact Hamiltonian system (P_1, η_L, E_L), choosing the following Reeb vector field:

$$\mathcal{R} = \frac{\partial}{\partial z}. \tag{5.90}$$

In order to compute the constraints, we find the complement of the tangent bundle of P_1,

$$TP_1{}^{\perp} = \ker \eta_L \cap \ker \mathrm{d}\eta_L = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2} \right\rangle. \tag{5.91}$$

By imposing $\gamma_{E_L}(X) = 0$ for $X \in TP_1{}^{\perp}$, we get the following constraint,

$$\phi^1 = -\frac{\partial V}{\partial q^1} + \frac{\partial V}{\partial q^2}, \tag{5.92}$$

which defines the submanifold $P_2 = \{p \in M \mid \phi^1(p) = 0\}$. Its tangent space is given by

$$TP_2 = \left\langle \frac{\partial}{\partial \dot{q}^1}, \frac{\partial}{\partial \dot{q}^2}, \frac{\partial}{\partial \dot{q}^3}, \frac{\partial}{\partial z}, \frac{\partial \phi^1}{\partial q^2}\frac{\partial}{\partial q^1} - \frac{\partial \phi^1}{\partial q^1}\frac{\partial}{\partial q^2}, \frac{\partial \phi^1}{\partial q^3}\frac{\partial}{\partial q^1} - \frac{\partial \phi^1}{\partial q^1}\frac{\partial}{\partial q^3} \right\rangle. \tag{5.93}$$

The complement is given by

$$TP_2{}^{\perp} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2} \right\rangle. \tag{5.94}$$

Demanding $\gamma_{E_L}(X) = 0$ for $X \in TP_2{}^{\perp}$ produces no new constraints, hence $P_2 = P_3 = P_f$ and the algorithm ends.

Notice that the $\mathcal{R}$ vector field is already tangent to the submanifold, so we would get the same result by using the modified version of the algorithm which imposes the tangency of $\mathcal{R}$.

Hamiltonian formulation and the Legendre transformation

For this Lagrangian system, the Legendre transformation is given by $\mathrm{F}L : TQ \times \mathbb{R} \to T^*Q \times \mathbb{R}$,

$$\mathrm{F}L(q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3, z) = (q^1, q^2, q^3, m(\dot{q}^1 + \dot{q}^2), m(\dot{q}^1 + \dot{q}^2), \mu\dot{q}^3, z) \tag{5.95}$$

We obtain that

$$\ker(\mathrm{F}L)_* = \left\langle \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2} \right\rangle, \tag{5.96}$$

hence it is a submersion onto its image and its fibers are connected, so the Lagrangian system is almost regular. By the Equivalence theorem (Theorem 5.14), there is a Hamiltonian formulation of the problem. The primary constraint submanifold is given by $M_1 = \mathrm{F}L(P)$ and can be described by the following constraint function

$$\psi^1 = p_1 - p_2. \tag{5.97}$$

The unique Hamiltonian function $H_1 : M_1 \to \mathbb{R}$ such that $H_1 \circ \mathrm{F}L = E_L$ is given by

$$H_1 = \frac{1}{2m}{p_1}^2 + \frac{1}{2\mu}{p_3}^2 - V(q_1, q_2, q_3) - \gamma z. \tag{5.98}$$

Let $\eta_1 = g_1^*(\eta)$, where $g_1 : M_1 \hookrightarrow T^*Q \times \mathbb{R}$ is the inclusion and η is the canonical contact form on $T^*Q \times \mathbb{R}$, then (M_1, η_1, H_1) is a precontact Hamiltonian system. We can apply the algorithm to compute the secondary constraints or use the commutativity of the diagram on Theorem 5.14. We now obtain a secondary constraint submanifold given by $M_2 = \{p \in M_1 \mid \psi^2(p) = 0\}$, where

$$\psi^2 = \frac{\partial V}{\partial q^1} - \frac{\partial V}{\partial q^2}. \tag{5.99}$$

The algorithm now ends, as $M_3 = M_2 = M_f$.

The Dirac-Jacobi bracket

We compute the bracket

$$\{\psi^1, \psi^2\} = \frac{\partial^2 V}{\partial (q^1)^2} - 2\frac{\partial^2 V}{\partial q^1 \partial q^2} + \frac{\partial^2 V}{\partial (q^2)^2}, \tag{5.100}$$

which is the determinant of the Hessian matrix of V with respect to (q^1, q^2). We will assume that this bracket does not vanish along M_f, hence both constraints are second class.

We will call $F = \{\psi^1, \psi^2\}$. The Dirac-Jacobi bracket is given by

$$\{f, g\}_{DJ} = \{f, g\} + \frac{\{f, \psi^1\}\{\psi^2, g\} - \{f, \psi^2\}\{\psi^1, g\}}{F} \tag{5.101}$$

The non-zero Dirac-Jacobi brackets of the coordinate functions are

$$\{q^1, p_1\}_{DJ} = \{q^1, p_2\}_{DJ} = \frac{\frac{\partial \psi^2}{\partial q^2}}{F} \tag{5.102a}$$

$$\{q^2, p_1\}_{DJ} = \{q^2, p_2\}_{DJ} = -\frac{\frac{\partial \psi^2}{\partial q^1}}{F} \tag{5.102b}$$

$$\{q^1, p_3\}_{DJ} = -\{q^2, p_3\}_{DJ} = \frac{\frac{\partial \psi^2}{\partial q^3}}{F} \tag{5.102c}$$

$$\{q^3, p_3\}_{DJ} = -1 \tag{5.102d}$$

$$\{q^1, z\}_{DJ} = -q^1 + \frac{\psi^2}{F} = -q^1 \text{ along } M_f \tag{5.102e}$$

$$\{q^2, z\}_{DJ} = -q^2 - \frac{\psi^2}{F} = -q^2 \text{ along } M_f \tag{5.102f}$$

$$\{q^3, z\}_{DJ} = -q^3. \tag{5.102g}$$

With those brackets, we can easily compute the equations of motion along the constrained submanifold M_f,

$$\dot{q}^1 = -\frac{\frac{p_1}{m}\frac{\partial \psi^2}{\partial q^2} + \frac{p_3}{\mu}\frac{\partial \psi^2}{\partial q^2}}{F} \tag{5.103a}$$

$$\dot{q}^2 = \frac{p_1}{m} + \frac{\frac{p_1}{m}\frac{\partial \psi^2}{\partial q^2} + \frac{p_3}{\mu}\frac{\partial \psi^2}{\partial q^2}}{F} \tag{5.103b}$$

$$\dot{q}^3 = \frac{p_3}{\mu} \tag{5.103c}$$

$$\dot{p}_i = \frac{\partial V}{\partial q^i} + \gamma p_i \tag{5.103d}$$

$$\dot{z} = -\frac{1}{2m}{p_1}^2 - \frac{1}{2\mu}{p_3}^2 - V(q_1, q_2, q_3) + \gamma z \tag{5.103e}$$

The second order problem

Consider the vector field Y associated to the equations Equation (5.103). That is,

$$Y = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}^i \frac{\partial}{\partial p^i} + \dot{z}\frac{\partial}{\partial z}, \tag{5.104}$$

where $(\dot{q}^i, \dot{p}^i, \dot{x})$ are those from Equation (5.103). The vector field X is a solution to the equations of motion on $TQ \times \mathbb{R}$ that satisfies $(\mathrm{F}L)_* X = Y$ and is given by

$$\begin{aligned}
X = &-\frac{2\dot{q}^1 \frac{\partial \psi^2}{\partial q^2} + \dot{q}^3 \frac{\partial \psi^2}{\partial q^2}}{F} \frac{\partial}{\partial q^1} \\
&+ \left(2\dot{q}^1 + \frac{2\dot{q}^1 \frac{\partial \psi^2}{\partial q^2} + \dot{q}^2 \frac{\partial \psi^2}{\partial q^3}}{F} \right) \frac{\partial}{\partial q^2} \\
&+ \dot{q}^3 \frac{\partial}{\partial q^3} \\
&+ \left(\frac{\partial V}{\partial q^1} + 2m\gamma \dot{q}^1 \right) \frac{\partial}{\partial \dot{q}^1} \\
&+ \left(\frac{\partial V}{\partial q^3} + \mu\gamma \dot{q}^3 \right) \frac{\partial}{\partial \dot{q}^3} \\
&+ \left(2m(\dot{q}^1)^2 + \mu(\dot{q}^1)^2 - V + \gamma z \right) \frac{\partial}{\partial z}.
\end{aligned} \tag{5.105}$$

We will construct the section α of $\mathrm{F}L_f$. Notice that, by the first constraint $p_1 = p_2$ on M_f, hence any point on M_f has the form $y = (q^1, q^2, q^3, p_1, p_1, p_3, z)$. Take $x = (q^1, q^2, q^3, p_1/m, 0, p_3/\mu, z) \in P_f$ so that $\mathrm{F}L(x) = y$. We set $\alpha(y) = \tilde{x}$, that is,

$$\begin{aligned}
\alpha(q^1, q^2, q^3, p^1, p^2, p^3, z) = \Bigg(& q^1, q^2, q^3, \\
& -\frac{2\frac{p_1}{m}\frac{\partial \psi^2}{\partial q^2} + \frac{p_3}{\mu}\frac{\partial \psi^2}{\partial q^2}}{F}, 2\frac{p_1}{m} + \frac{2\frac{p_1}{m}^1 \frac{\partial \psi^2}{\partial q^2} + \frac{p_3}{\mu}\frac{\partial \psi^2}{\partial q^2}}{F}, \frac{p_u}{\mu}, z \Bigg).
\end{aligned} \tag{5.106}$$

Hence, X satisfies the SODE condition along $\operatorname{im} \alpha$.

Example 5.2. Let $Q = \mathbb{R}^2$, $P_1 = TQ \times \mathbb{R}$ and consider the Lagrangian function $L : P_1 \to \mathbb{R}$ defined by

$$L(q^1, q^2, \dot{q}^1, \dot{q}^2, z) = \frac{1}{2}(\dot{q}^1 + \dot{q}^2)^2 + q^1 + q^2 z. \tag{5.107}$$

This Lagrangian induces the following precontact structure of class 3

$$\eta_L = \mathrm{d}z - (\dot{q}^1 + \dot{q}^2)(\mathrm{d}q^1 + \mathrm{d}q^2) \tag{5.108}$$

$$\mathrm{d}\eta_L = (\mathrm{d}q^1 + \mathrm{d}q^2) \wedge (\mathrm{d}\dot{q}^1 + \mathrm{d}\dot{q}^2) \tag{5.109}$$

$$E_L = \frac{1}{2}(\dot{q}^1 + \dot{q}^2)^2 - q^1 - q^2 z \tag{5.110}$$

We choose the following Reeb vector field,

$$\mathcal{R} = \frac{\partial}{\partial z}. \tag{5.111}$$

As in the previous example, we apply the algorithm, obtaining the following constrained submanifolds. Since

$$TP_1{}^{\perp} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2} \right\rangle, \tag{5.112}$$

then $P_2 = \{p \in M \mid \phi^1(p) = 0\}$, where

$$\phi^1 = z - 1. \tag{5.113}$$

$$TP_2 = \left\langle \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^2}, \frac{\partial}{\partial \dot{q}^2} \right\rangle \tag{5.114}$$

$$TP_2{}^{\perp} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2}, (\dot{q}^1 + \dot{q}^2)\frac{\partial}{\partial \dot{q}^1} + \frac{\partial}{\partial z} \right\rangle \tag{5.115}$$

$P_3 = \{p \in M \mid \phi^2(p) = \phi^2(p) = 0\}$,

$$\phi_2 = L - 2q^2 = \frac{1}{2}(\dot{q}^1 + \dot{q}^2)^2 + q^1 + q^2(z - 2) \tag{5.116}$$

$$TP_3 = \left\langle (2 - z)\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial q^2}, (\dot{q}^1 + \dot{q}^2)\frac{\partial}{\partial q^1} - \frac{\partial}{\partial \dot{q}^1} \right\rangle, \tag{5.117}$$

$$TP_3{}^{\perp} = \left\langle \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}, \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2}, (\dot{q}^1 + \dot{q}^2)\frac{\partial}{\partial \dot{q}^1} + \frac{\partial}{\partial z} \right\rangle, \tag{5.118}$$

so we get no new constraints and the algorithm ends.

We remark that any Reeb vector field $\mathcal{R}$ satisfies $\mathcal{R}(\phi^1) = 1$, hence if we imposed the tangency of $\mathcal{R}$, we would get the empty set.

Hamiltonian formulation and the Legendre transformation

The Legendre transformation is given by

$$\mathrm{F}L(q^1, q^2, \dot{q}^1, \dot{q}^2, z) = (q^1, q^2, \dot{q}^1 + \dot{q}^2, \dot{q}^1 + \dot{q}^2, z), \tag{5.119}$$

and then

$$\ker{(\mathrm{F}L)_*} = \left\langle \frac{\partial}{\partial \dot{q}^1} - \frac{\partial}{\partial \dot{q}^2} \right\rangle. \tag{5.120}$$

In this case, the primary constraint submanifold $M_1 = \mathrm{F}L(P)$ is described by the constraint

$$\psi^1 = p_1 - p_2. \tag{5.121}$$

The corresponding Hamiltonian $H_1 : M_1 \to \mathbb{R}$ is given by

$$H_1 = \frac{1}{2}(p_1)^2 - q^1 - q^2 z. \tag{5.122}$$

By the correspondence with the Lagrangian formulation, there will be two constraint submanifolds, $M_3 \hookrightarrow M_2 \hookrightarrow M_1$, defined by the constraint functions

$$\psi^2 = z - 1, \tag{5.123}$$

$$\psi^3 = \frac{1}{2}(p_1)^2 + q^1 + q^2(z-2), \tag{5.124}$$

respectively.

The Dirac-Jacobi bracket

The constraints have the following Dirac brackets.

$$\{\psi^1, \psi^2\}_{DJ} = 0 \tag{5.125a}$$

$$\{\psi^1, \psi^3\}_{DJ} = 3 - z = 2 \text{ along } M_f \tag{5.125b}$$

$$\{\psi^1, \psi^3\}_{DJ} = -\frac{1}{2}(p^1)^2 + q^1 - q^2 = -2(q^2 - q^1) \text{ along } M_f \tag{5.125c}$$

The rank of the matrix $(\{\psi^\alpha, \psi^\beta\}_{DJ})_{\alpha,\beta}$ is 2, so we can extract one first class constraint as a C^∞-linear combination. We set

$$\bar{\chi} = \psi^2 - (q^2 - q^1)\psi^0 = (p^1 - p^2)(q^2 - q^1) + z - 1, \tag{5.126}$$

which is a first class constraint, and the other two are second class, which we will relabel $\chi^1 = \psi^1, \chi^2 = \psi^3$.

The Dirac-Jacobi bracket is given by

$$\begin{aligned}\{f, g\}_{DJ} &= \{f, g\} + \frac{\{f, \psi^1\}\{\psi^2, g\} - \{f, \psi^2\}\{\psi^1, g\}}{-\frac{1}{2}(p^1)^2 + q^1 - q^2} \\ &= \{f, g\} - \frac{\{f, \psi^1\}\{\psi^2, g\} - \{f, \psi^2\}\{\psi^1, g\}}{2(q^2 - q^1)},\end{aligned} \tag{5.127}$$

along M_f. Notice that the denominators do not vanish along the submanifold.

The non-zero Dirac-Jacobi brackets of the coordinate functions, along M_f are the following

$$\{q^1, q_2\}_{DJ} = -\frac{1}{2}(q^1 q^2 + (q^2)^2 + p_1) \tag{5.128a}$$

$$\{q^1, p_1\}_{DJ} = \{q^1, p_2\}_{DJ} = \frac{1}{2} \tag{5.128b}$$

$$\{q^2, p_1\}_{DJ} = \{q^2, p_2\}_{DJ} = -\frac{1}{2} \tag{5.128c}$$

$$\{q^1, z\}_{DJ} = -\frac{3}{2}q^2 \tag{5.128d}$$

$$\{q^2, z\}_{DJ} = -q^1 + \frac{1}{2}q^2. \tag{5.128e}$$

Now consider the total Hamiltonian $H_T : T^*Q \times \mathbb{R} \to \mathbb{R}$

$$H_T = \frac{1}{2}(p_1)^2 - q^1 - q^2 z + \bar{u}\bar{\chi} = H_0 + \bar{u}\bar{\chi}, \tag{5.129}$$

where $\bar{u}$ is an unspecified Lagrange multiplier and

$$H_0 = \frac{1}{2}(p_1)^2 - q^1 - q^2 z. \tag{5.130}$$

The other two Lagrange multipliers are irrelevant for the motion, so we can eliminate them. We can compute the equations of motion. By Equation (5.64), for any observable f along M_f

$$\begin{aligned}\dot{f} &= \{H_0, f\}_{DJ} - f\mathcal{R}_{DJ}(H_0) + (\{\bar{\chi}, f\}_{DJ} - f\mathcal{R}_{DJ}(\bar{\chi}))\bar{u} \\ &= \{H_0, f\}_{DJ} - q_2 f + (\{\bar{\chi}, f\}_{DJ} - f)\bar{u}.\end{aligned} \tag{5.131}$$

Below we compute the equations of motion along M_f,

$$\dot{q}^1 = (q^1 - q^2)q^1 q^2 - 2(q^2)^2 + (q_2 - q_1)\bar{u} \tag{5.132a}$$

$$\dot{q}^2 = \frac{1}{2}(p_1)^2 q^2 + (q^2)^2 + p_1 + (q_1 - q_2)\bar{u} \tag{5.132b}$$

$$\dot{p}_1 = \dot{p}_2 = p_1 q^2 + 1 - (p_1)\bar{u} \tag{5.132c}$$

$$\dot{z} = 0. \tag{5.132d}$$

The second order problem

Consider the vector field Y associated to the equations Equation (5.132) with $\bar{u} = 0$. That is,

$$\begin{aligned}Y =& (q^1 - q^2)q^1 q^2 - 2(q^2)^2 \frac{\partial}{\partial q^1} \\ &+ \left(\frac{1}{2}(p_1)^2 q^2 + (q^2)^2 + p_1\right)\frac{\partial}{\partial q^2} \\ &+ (p_1 q^2 + 1)\left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}\right).\end{aligned} \tag{5.133}$$

The vector field X is a solution to the equations of motion on $TQ \times \mathbb{R}$ that satisfies $(\mathrm{F}L)_* X = Y$ and is given by

$$\begin{aligned}X =& \left((q^1 - q^2)q^1 q^2 - 2(q^2)^2\right)\frac{\partial}{\partial q^1} \\ &+ \left(2(\dot{q}^1)^2 q^2 + (q^2)^2 + 2\dot{q}^1\right)\frac{\partial}{\partial q^2} \\ &+ (2\dot{q}^1 q^2 + 1)\frac{\partial}{\partial \dot{q}^1}.\end{aligned} \tag{5.134}$$

We will construct the section α of $\mathrm{F}L_f$. Notice that, by the first constraint $p_1 = p_2$ on M_f, hence any point on M_f has the form $y = (q_1, q_2, p_1, p_1, z)$. Take $x = (q_1, q_2, p_1, 0, z) \in P_f$ so that $\mathrm{F}(x) = y$. We set $\alpha(y) = \tilde{x}$, that is,

$$\alpha(q^1, q^2, p^1, p^2, z) = (q^1, q^2, (q^1 - q^2)q^1 q^2 - 2{(q^2)}^2, 2{(\dot{q}^1)}^2 q^2 + {(q^2)}^2 + 2\dot{q}^1, z). \quad (5.135)$$

Hence, X satisfies the SODE condition along $\operatorname{im}\alpha$.

6. Contact systems and constraints

In this chapter, we study the relationship between contact Lagrangian dynamics and constrained systems.

In the Herglotz variational principle, as it is presented in Section 3.2, the action is defined implicitly (through an ODE, instead of as an integral) for arbitrary paths on Q. We show that it can also be interpreted differently. We can consider instead curves on $Q \times \mathbb{R}$ and define the action explicitly. Nevertheless, in order to obtain equivalent dynamics, we need to implement constraints through a submanifold $TQ \times \mathbb{R} \simeq N_L = \{\dot{z} = L(q^i, \dot{q}, z)\} \subseteq T(Q \times \mathbb{R})$, where L is the Lagrangian of the original system.

The constraint submanifold N_L can be interpreted in two ways, which are, in principle, not equivalent. From a variational point of view, one possibility is to use a *vakonomic principle* [13, 23, 70, 79, 121, 198], where we minimize the action restricted to the paths that are tangent to N_L. This principle has applications in control theory.

Another possibility is to employ a *nonholonomic principle* [13, 70, 102, 249]. Now, we restrict the *admissible variations* of the paths so that they are tangent to N_L. We find the paths such that the differential of the action vanishes when contracted with all the admissible variations. This principle is often useful in engineering and mechanics. It can be interpreted as the limit of a friction force.

Even if nonholonomic mechanics is an old subject [89], it was in the middle of the nineties that it received a decisive boost due to the geometric description by several independent teams: Bloch *et al.* [24], de León *et al.* [96, 102–104, 107, 163] and Bates and Śniatycki [18], based on the seminal paper by J. Koiller in 1992 [172]. Another relevant but not so well known work is due to Vershik and Faddeev [249] (see also [26, 174, 175, 177, 180, 222, 223] for other developments in the field).

Apart from applications in mechanics [62, 204], nonholonomic systems appear in the study of thermodynamics [135, 136].

Once we have established this result, we can take advantage of it in order to add additional constraints to the action-dependent Lagrangian dynamics. Indeed, we can easily consider vakonomic or nonholonomic dynamics on a contact Lagrangian system by taking a further submanifold $N \subseteq N_L \simeq TQ \times \mathbb{R}$.

In addition, we will also explain how the dynamics of the evolution vector field can be obtained from a nonholonomic variational principle.

This chapter contains three sections. In Sections 6.1 and 6.2 we cover the vakonomic and the nonholonomic theory, respectively. On each of these section, we first derive the action-dependent Lagrangian dynamics as a constrained action-independent system. On a second subsection of each section, we introduce the vakonomic/nonholonomic dynamics for action dependent Lagrangians. Finally, in Section 6.3, we propose a variational principle for the evolution vector field.

The vakonomic mechanics for contact systems comes from [94] and the nonholonomic part was published in [87] for the linear case and [95] for the nonlinear case. Also, in [95] we made a formulation of the problem based on the Euler-Lagrange and Herglotz operators that we do not include in this work.

6.1. Action dependent Lagrangian systems and vakonomic constraints

6.1.1. The Herglotz principle as a vakonomic principle

We will work on the manifold $\bar{\Omega}(q_0, q_1, z_0)$ of curves $\bar{c} = (c, c_z) : [t_0, t_1] \to Q \times \mathbb{R}$ such that $c(t_0) = q_0$, $c(t_1) = q_1$, $c_z(t_0) = z_0$. We do not constraint $c_z(t_1)$. The tangent space at the curve c is given by

$$\begin{aligned} T_c\bar{\Omega}(q_0, q_1, z_0) = \{&\delta\bar{c} = (\delta c, \delta c_z) : [t_0, t_1] \to T(Q \times \mathbb{R}) \mid \\ &\delta\bar{c}(t) \in T_{c(t)}(Q \times \mathbb{R}) \text{ for all t} \in [t_0, t_1], \delta c(t_0) = 0,\ \delta c(t_1) = 0, \delta c_z(t_0) = 0\}. \end{aligned} \tag{6.1}$$

In this space, the action functional $\bar{\mathcal{A}}$ can be defined as an integral

$$\begin{aligned} \bar{\mathcal{A}} : \bar{\Omega}(q_0, q_1, z_0) &\to \mathbb{R}, \\ \bar{c} = (c, c_z) &\mapsto c_z(t_1) - z_0 = \int_{t_0}^{t_1} \dot{c}_z(t) \mathrm{d}t. \end{aligned} \tag{6.2}$$

We will restrict this action to the set of paths that satisfy $\dot{c}_z = L$. For this, consider the hypersurface $N_L \subseteq T(Q \times \mathbb{R})$, which is the zero set of the constraint function ϕ:

$$\phi(q, \dot{q}, z, \dot{z}) = \dot{z} - L(q, \dot{q}, z). \tag{6.3}$$

We consider the submanifold of curves tangent to N_L

$$\bar{\Omega}_L(q_0, q_1, z_0) = \{\bar{c} \in \bar{\Omega}(q_0, q_1, z_0) \mid \dot{\bar{c}}(t) \in N_L \text{ for all t}\} \tag{6.4}$$

Notice that the map $\mathrm{id} \times \mathcal{Z} : \Omega(q_0, q_1) \to \bar{\Omega}_N(q_0, q_1, z_0)$ given by $(\mathrm{id} \times \mathcal{Z})(c) = (c, \mathcal{Z}(c))$ is a bijection, with inverse $\mathrm{pr}_Q(c, c_z) = c$. Here, $\mathcal{Z}$, is defined on (3.27). Moreover, the following diagram commutes

$$\begin{tikzcd} & \mathbb{R} & \\ \Omega(q_0, q_1) \arrow[ur, "\mathcal{A}"] \arrow[rr, "\mathrm{id} \times \mathcal{Z}"] & & \bar{\Omega}_N(q_0, q_1, z_0) \arrow[ul, "\bar{\mathcal{A}}"'] \end{tikzcd} \tag{6.5}$$

Hence $\bar{c} \in \bar{\Omega}_N(q_0, q_1, z_0)$ is a critical point of $\bar{\mathcal{A}}$ if and only if c is a critical point of $\mathcal{A}$. So the critical points of $\mathcal{A}$ restricted to $\bar{\Omega}(q_0, q_1, z_0)$ are precisely the curves that satisfy the Herglotz equations.

We will also provide direct proof. We find the critical points of $\bar{\mathcal{A}}$ restricted to $\bar{\Omega}_N(q_0, q_1, z_0) \subseteq \bar{\Omega}(q_0, q_1, z_0)$ using the following infinite-dimensional version of the Lagrange multiplier theorem [1, p. 3.5.29].

Theorem 6.1 (Lagrange multiplier Theorem)**.** *Let M be a smooth manifold and let E be a Banach space such that $g : M \to E$ is a smooth submersion, so that $A = g^{-1}(\{0\})$ is a smooth submanifold. Let $f : M \to \mathbb{R}$ be a smooth function. Then $p \in A$ is a critical point of $f|_A$ if and only if there exists $\hat{\lambda} \in E^*$ such that p is a critical point of $f + \hat{\lambda} \circ g$.*

We will apply this result to our situation. In the notation of this last theorem, $M = \bar{\Omega}(q_0, q_1, z_0)$ is the smooth manifold. We pick the Banach space $E = L^2([t_0, t_1] \to \mathbb{R})$ of square integrable functions. This space is, indeed, a Hilbert space with inner product

$$\langle \alpha, \beta \rangle = \int_{t_0}^{t_1} \alpha(t)\beta(t)\mathrm{d}t. \tag{6.6}$$

We remind that, by the Riesz representation theorem, there is a bijection between $L^2([t_0, t_1] \to \mathbb{R})$ and its dual such that for each $\hat{\alpha} \in L^2([t_0, t_1] \to \mathbb{R})^*$ there exists $\alpha \in L^2([t_0, t_1] \to \mathbb{R})$ with $\hat{\alpha}(\beta) = \langle \alpha, \beta \rangle$ for all $\beta \in L^2([t_0, t_1] \to \mathbb{R})$.

Our constraint function is

$$\begin{aligned} g : \bar{\Omega}(q_0, q_1, z_0) &\to L^2([t_0, t_1] \to \mathbb{R}), \\ \bar{c} &\mapsto (\phi) \circ (\bar{c}, \dot{\bar{c}}), \end{aligned} \tag{6.7}$$

where ϕ is a constraint locally defining N. Note that $A = g^{-1}(0) = \bar{\Omega}_L(q_0, q_1, z_0)$.

By Theorem 6.1, c is a critical point of $f = \bar{\mathcal{A}}$ restricted to $\bar{\Omega}_L(q_0, q_1, z_0)$ if and only if there exists $\hat{\lambda} \in L^2([t_0, t_1] \to \mathbb{R})^*$ (which is represented by $\lambda \in L^2([t_0, t_1] \to \mathbb{R})$) such that c is a critical point of $\bar{\mathcal{A}}_\lambda = \bar{\mathcal{A}} + \hat{\lambda} \circ g$.

Indeed,

$$\bar{\mathcal{A}}_\lambda = \int_{t_0}^{t_1} \mathcal{L}_\lambda(\bar{c}(t), \dot{\bar{c}}(t))\mathrm{d}t, \tag{6.8}$$

where

$$\mathcal{L}_\lambda(q, z, \dot{q}, \dot{z}) = \dot{z} - \lambda\phi(q, z, \dot{q}, \dot{z}). \tag{6.9}$$

Since the endpoint of c_z is not fixed, the critical points of this functional $\bar{\mathcal{A}}_\lambda$ are the solutions of the Euler-Lagrange equations for $\mathcal{L}_\lambda$ that satisfy the natural boundary condition:

$$\frac{\partial \mathcal{L}_\lambda}{\partial \dot{z}}(\bar{c}(t_1), \dot{\bar{c}}(t_1)) = 1 - \lambda(t_1)\frac{\partial \phi}{\partial \dot{z}}(\bar{c}(t_1), \dot{\bar{c}}(t_1)) = 0. \tag{6.10}$$

For $\phi = \dot{z} - L$, this condition reduces to $\lambda(t_1) = 1$.

The Euler-Lagrange equations of $\mathcal{L}_\lambda$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t)\frac{\partial\phi(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{q}^i}\right) - \lambda(t)\frac{\partial\phi(\bar{c}(t), \dot{\bar{c}}(t))}{\partial q^i} = 0 \tag{6.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t)\frac{\partial\phi(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{z}}\right) - \lambda(t)\frac{\partial\phi(\bar{c}(t), \dot{\bar{c}}(t))}{\partial z} = 0, \tag{6.11b}$$

since $\phi = \dot{z} - L$, the equation (6.11b) for z is just

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -\lambda(t)\frac{\partial L}{\partial z}, \tag{6.12}$$

substituting on (6.11a) and dividing by λ, we obtain Herglotz equations.

Theorem 6.2 (Herglotz's variational principle, vakonomic interpretation)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $\bar{c} = (c, c_z) \in \bar{\Omega}_L(q_0, q_1, z_0)$. Then, $\bar{c}$ is a critical point of $\mathcal{A}|_{\bar{\Omega}_L(q_0,q_1,z_0)}$ if and only if $(c, \dot{c}, c_z)$ satisfies Herglotz's equations:*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} &= \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}, \\
\dot{z} &= L.
\end{aligned}
\tag{6.13}
$$

Remark 6.3. We could instead have used the action

$$
\begin{aligned}
\bar{\mathcal{A}}_L : \bar{\Omega}(q_0, q_1, z_0) &\to \mathbb{R}, \\
(c, c_z) &\mapsto= \int_{t_0}^{t_1} L \circ (c, \dot{c}, c_z),
\end{aligned}
\tag{6.14}
$$

since it coincides with $\bar{\mathcal{A}}$ when restricted to $\bar{\Omega}_L(q_0, q_1, z_0)$, hence they give rise to the same vakonomic principle. This approach was followed in [95].

Remark 6.4. Given any hypersurface N transverse to the $\dot{z}$-parametric curves, by the implicit function theorem there exists locally a function L such that N is given by the equation $\dot{z} = L$. Hence, this procedure provides dynamics for any hypersurface $N \subseteq T(Q \times \mathbb{R})$.

6.1.2. The Herglotz principle with vakonomic constraints

So far we have interpreted the Herglotz principle as a vakonomic principle. We will now study how we can add further constraints to the Herglotz principle. Formally, we can obtain vakonomic dynamics just by changing ϕ by ϕ^a and λ by λ_a on (6.11), where a ranges from 0 to the number of constraints k. Indeed, we restrict our path space to the ones tangent to submanifold $N \subseteq N_L \subseteq T(Q \times \mathbb{R})$, where N is the zero set of ϕ^0, given by $\phi^0 = \dot{z} - L$. We also define the space of paths tangent to N:

$$
\bar{\Omega}_N(q_0, q_1, z_0) = \{\bar{c} \in \bar{\Omega}(q_0, q_1, z_0) \mid \dot{\bar{c}}(t) \in N \,\text{for all t}\}
\tag{6.15}
$$

Repeating the similar computations, we would find that the critical points of $\mathcal{A}|_{\Omega_N(q_0,q_1,z_0)}$ are the solutions of

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_a(t)\frac{\partial \phi^a(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{q}^i}\right) - \lambda_a(t)\frac{\partial \phi^a(\bar{c}(t), \dot{\bar{c}}(t))}{\partial q^i} = 0 \tag{6.16a}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_a(t)\frac{\partial \phi^a(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{z}}\right) - \lambda_a(t)\frac{\partial \phi^a(\bar{c}(t), \dot{\bar{c}}(t))}{\partial z} = 0, \tag{6.16b}
$$

$$
\phi^a(\bar{c}(t), \dot{\bar{c}}(t)) = 0, \tag{6.16c}
$$

where $(\phi^a)_{a=0}^k$ are constraints defining $\tilde{N}$ as a submanifold of $TQ \times \mathbb{R}$. Since $\frac{\partial \phi^0}{\partial \dot{z}} = 0$, the rest of the constraints can be chosen to be independent of $\dot{z}$. We denote

$$\psi^\alpha(q,\dot{q},z) = \phi^\alpha(q,\dot{q},z,L(q,\dot{q},z)), \tag{6.17}$$

$$\mu_\alpha = \frac{\lambda_\alpha}{\lambda_0} \tag{6.18}$$

$$\mathcal{L}_\mu(q,\dot{q},z,t) = L(q,\dot{q},z) - \mu_\alpha(t)\psi^\alpha(q,\dot{q},z) \tag{6.19}$$

for $\alpha \in \{1,\dots k\}$, provided that $\lambda_0 \neq 0$.

From this, we can write the equations (6.16) as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_0(t)\frac{\partial \mathcal{L}_\mu}{\partial \dot{q}^i}\right) + \lambda_0(t)\frac{\partial \mathcal{L}_\mu}{\partial q^i} = 0 \tag{6.20a}$$

$$\frac{\mathrm{d}\lambda_0(t)}{\mathrm{d}t} = \lambda_0(t)\frac{\partial \mathcal{L}_\mu}{\partial z} = 0, \tag{6.20b}$$

$$\psi^\alpha(\bar{c}(t),\dot{c}(t)) = 0, \tag{6.20c}$$

$$\dot{c_z}(t) = \mathcal{L}_\mu(\bar{c}(t),\dot{c}(t),t). \tag{6.20d}$$

Substituting (6.20b) onto (6.20a), dividing by λ_0 and reordering terms, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_\mu}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}_\mu}{\partial q^i} = \frac{\partial \mathcal{L}_\mu}{\partial \dot{q}^i}\frac{\partial \mathcal{L}_\mu}{\partial z} \tag{6.21a}$$

$$\psi^\alpha(\bar{c}(t),\dot{c}(t)) = 0, \tag{6.21b}$$

$$\dot{c_z}(t) = \mathcal{L}_\mu(\bar{c}(t),\dot{c}(t),t). \tag{6.21c}$$

Theorem 6.5 (Herglotz's vakonomic variational principle)**.** *Let $L : TQ\times\mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $\bar{c} = (c,c_z) \in \bar{\Omega}_N(q_0,q_1,z_0)$ and $\mu : [t_0,t_1] \to \mathbb{R}^k$. Then, $\bar{c}$ is a critical point of $\mathcal{A}|_{\bar{\Omega}_N(q_0,q_1,z_0)}$ if and only if $(c,\dot{c},c_z,\mu^a)$ satisfies Herglotz's vakonomic equations* (6.21)*.*

A more geometric approach to these equations can be obtained by considering the dynamics of an extended action-dependent Lagrangian, assuming that L is regular. This extended Lagrangian is singular, so it will be a precontact Hamilton system.

Corollary 6.6. *Let $L : TQ\times\mathbb{R} \to \mathbb{R}$ be a regular[1] Lagrangian function and let $\bar{c} = (c,c_z) \in \bar{\Omega}_N(q_0,q_1,z_0)$ and $\mu : [t_0,t_1] \to \mathbb{R}^k$. Then, (c,c_z,μ) are solution of the vakonomic Herglotz equations* (6.21) *if and only if$(c,\mu,\dot{c},\dot{\mu},c_z)$ are the integral curves of a Herglotz vector field $\xi_{\mathcal{L}}$ for the singular Lagrangian*

$$\begin{aligned}\mathcal{L} : T(Q\times\mathbb{R}^k)\times\mathbb{R} &\to \mathbb{R}\\ (q,\mu,\dot{q},\dot{\mu},z) &\mapsto L(q,\dot{q},z) - \mu_\alpha\psi^\alpha(q,\dot{q},z)\end{aligned} \tag{6.22}$$

Proof. This follows from applying the constraint algorithm explained in *Section* 5.2 to the singular Lagrangian $\mathcal{L}$. Equation (6.21b) are the initial constraints. Using the holonomy condition we fix term in $\frac{\partial}{\partial\dot{\mu}}$. Equations (6.21a) and (6.21c) come from the terms on $\frac{\partial}{\partial\dot{q}}$ and $\frac{\partial}{\partial\dot{z}}$, respectively. These computations are performed with detail in [95]. □

[1]This condition could be omitted, but then we should keep adding constraints until the algorithm stops.

6.2. Action dependent Lagrangians and nonholonomic constraints

In this section, we will discuss the relationship between nonholonomic principle and contact mechanics. Nonholonomic principles, are understood in mechanics as principles for which the constraints depend on the velocities (unlike holonomic constraints, which only depend on the position). However, the mathematical details are subtle. Nonholonomic principles, unlike vakonomic principles, are not truly variational (they are sometimes called *pseudo-variational*). Instead of restricting the space of admissible curves $\Omega_N \subseteq \Omega$ and find the critical points on this submanifold, we directly restrict the space of admissible variations $\mathcal{D} \subseteq T\Omega$, so that the differential of the action evaluated at the admissible variations has to vanish only on $\mathcal{D}$. Hence, the solutions of these principles are not necessarily critical points of the action function restricted to any space.

These setting can be studied in several degrees of generality. We will use three different ones, form the most general to the most particular: the generalized Chetaev principle, the Chetaev principle and the linear nonholonomic principle. The first one, described in [61] will be necessary to obtain a principle for the Herglotz-evolution dynamics. The second one is needed to obtain a principle equivalent to the Herglotz principle. In the linear case we have a deeper understanding of the geometric properties of the dynamics (see [87]).

We now proceed to prove the principle on the most general case. The other cases are simple corollaries of this one. In this most general case, we will have some kinematic constraint D_K and variational constraints D_V, both of which are *semi-basic* distributions on the tangent bundle of a manifold M. That is $D_K, D_V \subseteq (S(TTM))$. More concretely, distributions are can be described locally as the annihilator of a set of *semi-basic* 1-forms on TM, which are sections of $S^*(T^*TM)$. A form α is semi-basic if and only if in natural bundle coordinates has the expression

$$\alpha(q, \dot{q}) = \alpha_a(q, \dot{q}) dq^a. \tag{6.23}$$

We will assume that the ${D_K}^\circ$ is generated by forms α^a and ${D_V}^\circ$ is generated by forms β^a. We now define the spaces of admissible paths and admissible variations as the spaces of paths and variations which are tangent to the respective distributions.

$$\mathcal{D}_K(q_0, q_1) = \{c \in \Omega(q_0, q_1) \mid \dot{c}(t) \in D_K \text{ for all t} \in [t_0, t_1]\} \tag{6.24}$$

$$\mathcal{D}_V(q_0, q_1) = \{\delta c \in \Omega(q_0, q_1) \mid \delta c(t) \in D_V \text{ for all t} \in [t_0, t_1]\}. \tag{6.25}$$

We define the action map given a Lagrangian $L : TM \to \mathcal{R}$ as usual:

$$\begin{aligned} \mathcal{A} : \Omega(q_0, q_1) &\to \mathbb{R}, \\ c &\mapsto \int L \circ (c, \dot{c}). \end{aligned} \tag{6.26}$$

Then we can formulate the generalized Chetaev principle

Theorem 6.7 (Generalized Chetaev principle). *A path $c \in \mathcal{D}_K(q_0, q_1)$ satisfies $T_c\mathcal{A}(\delta c)$ for all $\delta c \in (\mathcal{D}_V(q_0, q_1))_c$ if and only if there exist $\lambda : [t_0, t_1] \to \mathbb{R}^k$ such that (c, λ) satisfies the equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \beta_i^a \tag{6.27a}$$

$$\alpha_i^a \dot{q}^i = 0. \tag{6.27b}$$

Proof. The second equation is just equivalent to the statement that the path c is admissible (i.e., $c \in \mathcal{D}_K(q_0, q_1)$).

In order to obtain the first equation, we use the standard arguments from calculus of variations to deduce that

$$T_c\mathcal{A}(\delta c) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right) \delta c^i \mathrm{d}t, \tag{6.28}$$

If the integral vanishes for arbitrary variations satisfying the variational constraints, it must be that

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \in \mathcal{D}_V.$$

Since $\mathcal{D}_V$ is generated by the forms α^a this is equivalent to Equation (6.27a). □

In the case of the Chetaev principle, the kinematic constraints are given by $D_K = TN$, where $N \subseteq TM$, which is locally given by constraints $\phi^a(q, \dot{q})$. The variational constraints D_V are such that its annihilator is generated by

$$S^*(\mathrm{d}\phi^a) = \frac{\partial \phi^a}{\partial \dot{q}^i} \mathrm{d}q^i. \tag{6.29}$$

Thus, Chetaev equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^i} \tag{6.30a}$$

$$\phi^a(q, \dot{q}) = 0. \tag{6.30b}$$

In the linear case follows from the Chetaev case when $N = D$ is a distribution on Q. If D° is generated by forms

$$\alpha^a = \alpha^a(q)\mathrm{d}q, \tag{6.31}$$

then the nonholonomic equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \alpha^a(q) \tag{6.32a}$$

$$\alpha^a(q)\dot{q} = 0. \tag{6.32b}$$

6.2.1. The Herglotz principle as a nonholonomic principle

We are now ready to deduce the Herglotz principle as a particular case of the Chetaev principle. Indeed, we take

- our manifold M to be $Q \times \mathbb{R}$,
- the Lagrangian to be the pullback of an action dependent Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$,
- The constraint $\phi(q, \dot{q}, z, \dot{q}) = \dot{q} - L(q, \dot{q}, z)$, which defines the submanifold N_L.

We can take the space of admissible curves as $\Omega_L(q_0, q_1, z_0)$. In order to follow strictly the setting of the beginning of the section we should also fix the value of $c_z(t_1)$, but this is only necessary to remove the boundary terms when computing $T\mathcal{A}$ in Equation (6.28). Since $\frac{\partial L}{\partial \dot{z}} = 0$, these boundary terms do not appear anyway.

We also note that $\pi_0^* \eta_L = S^* \mathrm{d}\phi$, where $\pi_0 : T(Q \times \mathbb{R}) \to TQ \times \mathbb{R}$ is the natural projection. Thus, the admissible curves are the ones whose projection onto $TQ \times \mathbb{R}$ is contained in the contact distribution:

$$(\mathcal{D})_c = \{\delta c \in T_c \bar{\Omega} \mid \pi_0^* \eta_L(\delta c) = 0\}. \tag{6.33}$$

We have the following.

Theorem 6.8 (Herglotz's variational principle, nonholonomic interpretation). *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $\bar{c} = (c, c_z) \in \bar{\Omega}_L(q_0, q_1, z_0)$. Then $T_{\bar{c}}\mathcal{A}|_{(\mathcal{D})_c} = 0$ if and only if $(c, \dot{c}, c_z)$ satisfies Herglotz's equations:*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} &= \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}, \\ \dot{z} &= L. \end{aligned} \tag{6.34}$$

Proof. Using Chetaev's principle (6.30), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda \frac{\partial \phi}{\partial \dot{q}^i} = -\lambda \frac{\partial L}{\partial \dot{q}^i} \tag{6.35a}$$

$$-\frac{\partial L}{\partial z} = \lambda \frac{\partial \phi}{\partial z} = \lambda \tag{6.35b}$$

$$\dot{z} = L. \tag{6.35c}$$

If we substitute the value of λ from the second equation into the first one, we obtain the Herglotz equations. □

6.2.2. The Herglotz principle with nonholonomic constraints

As we did with the vakonomic case, we can obtain a variational principle for nonlinear dynamics by considering a submanifold $N \subseteq N_L \simeq TQ \times \mathbb{R}$ given by constraints ϕ^a, where $\phi^0 = \dot{z} - L$. We let a range from 0 to k and α range from 1 to k.

Now we take again $D_K = \bar{\Omega}(q_0, q_1, z_0)$ as the space of admissible paths and Δ_V as defined for the Chetaev principle (6.29). As in the case of the unconstrained Herglotz principle, we deduce the equations of motion from Equation (6.30). We obtain the following:

Theorem 6.9 (Herglotz's variational principle with nonholonomic constraints). *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $\bar{c} = (c, c_z) \in \bar{\Omega}_N(q_0, q_1, z_0)$. Then $T_{\bar{c}}\mathcal{A}|_{(\mathcal{D})_c} = 0$ if and only if there exist $\lambda : [t_0, t_1] \to \mathbb{R}^k$ such that $(c, \dot{c}, c_z, \lambda)$ satisfies Herglotz's equations with nonholonomic constraints:*

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z} + \lambda_\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^i}, \\ \dot{z} = L.\end{gathered} \tag{6.36}$$

In the linear case, the constraints are given by a non-integrable distribution $\mathcal{D}$ on TQ extended naturally to $TQ \times \mathbb{R}$. In this situation we obtain

Corollary 6.10. *A path $\xi \in \Omega(q_1, q_2, [t_0, t_1])$ satisfies the nonholonomic Herglotz variational principle with linear constraints $\mathcal{D}$ if and only if*

$$\begin{cases}\left(\dfrac{\partial L}{\partial q^i} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}^i} + \dfrac{\partial L}{\partial \dot{q}^i}\dfrac{\partial L}{\partial z}\right) dq^i \in \mathcal{D}^\circ_{\xi(t)}, \\ \dot{\xi}(t) \in \mathcal{D}_{\xi(t)}.\end{cases} \tag{6.37}$$

The linear theory is developed in [87]. The results include the interpretation of the constrained dynamics as an orthogonal projection of the unconstrained dynamics onto the constraint distribution, and the construction of the nonholonomic bracket, which is an almost Jacobi bracket on the space of observables and provides the nonholonomic dynamics. It also includes Chaplygin's sleigh as an example.

6.3. A nonholonomic principle for the evolution vector field

We end this chapter by explaining the nonholonomic variational principle for the Herglotz-evolution vector field. The integral curves of the Lagrangian evolution vector field satisfy a nonholonomic variational principle with nonlinear constraints. Indeed, the solutions of the equations of motion are critical points of the action with a condition of tangency to the contact distribution. This is also a particular case of the generalized Chetaev principle in which $D_K = D_V = (\pi_0)^{-1}(\mathcal{H})$, where $\pi_0 : T(Q \times \mathbb{R}) \to TQ \times \mathbb{R}$ is the natural projection. This is also similar to the one introduced in [135], where they also consider principles in which they take $D_K = D_V$.

We also take M to be the manifold $Q \times \mathbb{R}$ and consider the pullback of an action dependent Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$. Our spaces of admissible curves and variations are

$$\mathcal{D}_K = \{c \in T_c\bar{\Omega}(q_0, q_1, z_0) \mid \pi_0^* \eta_L(\dot{c}) = 0\} \tag{6.38a}$$

$$(\mathcal{D}_V)_c = \{\delta c \in T_c\bar{\Omega}(q_0, q_1, z_0) \mid \pi_0^* \eta_L(\delta c) = 0\}. \tag{6.38b}$$

Theorem 6.11 (Nonholonomic variational principle for the evolution vector field)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be an action-dependent Lagrangian function and let $\bar{c} = (c, c_z) \in \mathcal{D}_K$. Then $T_{\bar{c}}\mathcal{A}|_{(\mathcal{D}_V)_c} = 0$ if and only if $(c, \dot{c}, c_z)$ is an integral curve of the Herglotz-evolution vector field Ξ_L.*

Proof. Indeed, the solution of the generalized Chetaev principle satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\lambda \frac{\partial L}{\partial \dot{q}^i}, \tag{6.39a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}^i}\right) - \frac{\partial L}{\partial z^i} = \lambda, \tag{6.39b}$$

$$\dot{z} = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}. \tag{6.39c}$$

Since L does not depend on $\dot{z}$, the second equation is reduced to

$$\frac{\partial L}{\partial z} = -\lambda, \tag{6.40}$$

So we retrieve the Herglotz-evolution equations (3.38a).

□

7. Symplectization and contactization

Symplectic and contact Hamiltonian systems are closely related [28, 162, 245]. Indeed, given a contact manifold (M, η), one can define its symplectization $(M^s, \theta = \eta^s)$, which is an exact symplectic manifold of one dimension more than M and symplectic form $\omega = -\mathrm{d}\theta$. Conversely, given an exact symplectic manifold (M, θ), one can define its contactization $(M^{co}, \eta = \theta^{co})$, which is a contact manifold of one dimension more. One is not only able to simplectize and contactize the forms, but also the symplectomorphisms/contactomorphisms, the Lagrangian/Legendrian submanifolds, the functions, the Hamiltonian vector fields and the brackets. Moreover, these procedures can be interpreted as functors between the categories of exact symplectic manifolds and homogeneous symplectomorphisms and the category of contact manifolds and (conformal) contactomorphism.

Although symplectization is a powerful tool for studying contact dynamics, it is sometimes problematic to use when more geometric structures than the contact distribution forms are involved, as in the case of tangent or cotangent bundles. For example, as we will see in this chapter, it is not trivial to relate the symplectization of the 1-jet bundle $(TQ \times \mathbb{R})^s$ with the cotangent bundle $T^*(Q \times \mathbb{R})$ preserving their natural structures. We will relate the constructions in Proposition 2.17 and Theorem 2.19 to the corresponding ones in symplectic geometry. In order to facilitate working on these contexts, we will generalize the concept and define a symplectization as a map $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ from an exact symplectic manifold to the contact manifold satisfying certain properties. In this way we can work with a different symplectization in each context, but also to prove general results for all of them.

One issue that will limit the applicability of these techniques on many situations is that we have been unable to symplectize a contact Lagrangian system. Indeed, we have not been able to equip its symplectization with a compatible tangent bundle structure. However, we can relate it to constrained systems in one dimension more (see Chapter 6).

Another important remark is that symplectization does not relate contact manifolds with symplectic manifolds, but with exact symplectic ones. In those manifolds a symplectic potential is chosen and one is restricted to consider homogeneous objects. This is called *Liouville geometry* by some authors [245]. We will review their properties in Section 7.1, before discussing symplectization (Section 7.2) and contactization (Section 7.3).

This chapter is made up of unpublished work. Our main objective was trying to formalize some tricks that we used on some of our articles [92, 93, 127], and fit them on a more general theoretical framework.

7.1. Exact symplectic manifolds and homogeneous Hamiltonian Systems

Exact symplectic manifolds are symplectic manifolds in which a symplectic potential is chosen. Equivalently, it can be defined as a symplectic manifold with a Liouville vector field.

Definition 7.1. An *exact symplectic manifold* is a pair (M, θ) such that $\omega = -\mathrm{d}\theta$ is a symplectic form. We call θ the *symplectic potential*. We note that two symplectic potentials give rise to the same symplectic form if and only if they differ by a closed one-form.

Given an exact symplectic manifold (M, θ), we can define a vector field Δ dual to the symplectic potential. That is,

$$\theta = -\iota_\Delta \omega. \tag{7.1}$$

The field Δ is a Liouville vector field for (M, ω), meaning that

$$\mathcal{L}_\Delta \omega = \omega. \tag{7.2}$$

Conversely, given a Liouville vector field Δ for (M, ω), it follows from Cartan's formula that $\theta = \iota_\Delta \omega$ is a symplectic potential.

There exist (symplectic) Darboux coordinates (q^i, p_i) such that

$$\theta = p_i \mathrm{d}q^i, \tag{7.3a}$$

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i, \tag{7.3b}$$

$$\Delta = p_i \frac{\partial}{\partial p_i}. \tag{7.3c}$$

During this section, we will assume that our exact symplectic manifolds have no *singular points*, that is, those in which Δ (equivalently θ) vanish. Notice that those points form a closed subset which is contained on a submanifold of dimension at most half of the original manifold. In the case that $M = T^*Q$ is a cotangent bundle, we would remove the zero section.

We notice that,

$$\iota_\Delta \theta = -\iota_\Delta \iota_\Delta \omega = 0, \tag{7.4a}$$

$$\mathcal{L}_\Delta \theta = \iota_\Delta \mathrm{d}\theta + \mathrm{d}\iota_\Delta \theta = \theta. \tag{7.4b}$$

We now define the maps that preserve the geometry of this system.

Definition 7.2. A diffeomorphism between two exact symplectic manifolds $F : (M_1, \theta_1) \to (M_2, \theta_2)$ is a *homogeneous symplectomorphism* if and only if $F^*\theta_2 = \theta_1$.

We also say that a vector field X is an *infinitesimal homogeneous symplectomorphism* for (M, θ) if and only if its flow consists of symplectomorphisms. By a similar argument to the one in the proof of Proposition 2.3, X is an infinitesimal homogeneous symplectomorphism if and only if $\mathcal{L}_X \theta = 0$.

The flow ϕ^{Δ} of the Liouville vector field Δ can be thought as homogeneous dilations of our geometry. Indeed, given a tensor τ, we say that it is *homogeneous* of degree n if

$$\mathcal{L}_{\Delta}\tau = n\tau. \tag{7.5}$$

Equivalently,

$$\phi_t^*\tau = \exp(nt)\tau. \tag{7.6}$$

We also say that a map $F : (M_1, \theta_1) \to (M_2, \theta_2)$ is *homogeneous* if and only if the corresponding Liouville vector fields Δ_1, Δ_2 are F-related. Equivalently,

$$F \circ \phi_t^{\Delta_1} = \phi_t^{\Delta_2} \circ F, \tag{7.7}$$

where $\phi_t^{\Delta_i}$ are the flows of Δ_i at time t.

Homogeneous symplectomorphisms are indeed homogeneous and symplectomorphisms.

Proposition 7.1. *A map $F : (M_1, \theta_1) \to (M_2, \theta_2)$, with Liouville vector fields Δ_1, Δ_2 is a homogeneous symplectomorphism if and only if it is a symplectomorphism ($F^*\omega_2 = \omega_1$), and it is homogeneous ($F_*\Delta_1 = \Delta_2$).*

A vector field X on (M, θ) is an infinitesimal homogeneous symplectomorphism if and only if it is an infinitesimal symplectomorphism $\mathcal{L}_{\Delta}\omega = 0$, and it is homogeneous of degree 0 *($\mathcal{L}_{\Delta}X = 0$).*

Proof. Assume that F preserves the symplectic potentials. Then it is obviously a symplectomorphism, since

$$F^*(-\mathrm{d}\theta_2) = -\mathrm{d}F^*\theta_2 = -\mathrm{d}\theta_1. \tag{7.8}$$

One can check that a symplectomorphism preserves the symplectic potential if and only if preserves the Liouville vector field by computing

$$0 = F^*(\iota_{\Delta_2}\omega_2 - \theta_2) = \iota_{(F_*)^{-1}\Delta_2}\omega_1 - F^*\theta_2. \tag{7.9}$$

Since ω_1 is nondegenerate, $F^*\theta_2 = \theta_1$ if and only $(F_*)^{-1}\Delta_2 = \Delta_1$.

For the vector fields the proof follows form Cartan calculus. Indeed, if $\mathcal{L}_X\theta = 0$, then $\mathcal{L}_X\omega = -\mathrm{d}\mathcal{L}_X\theta = 0$. We now let X be a symplectomorphism and compute

$$0 = \mathcal{L}_X(\iota_{\Delta}\omega - \theta) = -\iota_{\mathcal{L}_{\Delta}X}\omega - \mathcal{L}_X\theta. \tag{7.10}$$

Thus, $\mathcal{L}_X\theta$ vanishes if and only if $\mathcal{L}_{\Delta}X$ vanishes. □

We now discuss the Hamiltonian vector fields of the system, but we only focus on the ones which are homogeneous. Indeed, homogeneous Hamiltonian of degree one are in natural correspondence with infinitesimal homogeneous symplectomorphisms.

Proposition 7.2. *Let $H : (M, \theta) \to \mathbb{R}$ be a function. If H is homogeneous of degree* 1 *($\Delta(H) = H$), then X_H is a homogeneous of degree* 0*.*

Moreover, there is a bijection between infinitesimal homogeneous symplectomorphisms X and homogeneous functions H of degree 1 *given by*

$$\begin{aligned} H &\mapsto X_H, \\ \theta(X) &\leftarrow X. \end{aligned} \tag{7.11}$$

Proof. Let X_H be the Hamiltonian vector field of H. Now,

$$0 = \mathcal{L}_\Delta(\iota_{X_H}\omega - \mathrm{d}H) = \iota_{\mathcal{L}_\Delta X_H}\omega - \mathrm{d}\Delta(H) = \iota_{\mathcal{L}_\Delta X_H}\omega, \tag{7.12}$$

Thus, $\mathcal{L}_\Delta X_H = 0$ and X_H is homogeneous of degree 0.

In order to proof that the map above is a well-defined bijection, we need to see that, for any infinitesimal homogeneous symplectomorphism X, $\theta(X)$ is homogeneous of degree 1 and that $X_{\theta(X)} = X$. The first claim follows form,

$$\mathcal{L}_\Delta\theta(X) = (\mathcal{L}_\Delta\theta)(X) = \theta(X). \tag{7.13}$$

The second one by noting,

$$\iota_X\omega = -\iota_X\mathrm{d}\iota_\Delta\theta = \mathrm{d}\iota_X\iota_\Delta\theta = \mathrm{d}(\theta(X)). \tag{7.14}$$

Thus, X is the Hamiltonian vector field of $\theta(X)$. □

We note that in coordinates $\theta(X_H) = H$ reads as

$$p_i\frac{\partial H}{\partial p_i} = H. \tag{7.15}$$

Hence, this can be understood as a geometric version of Euler's theorem for homogeneous functions.

We now make the central definition of this section.

Definition 7.3. A *homogeneous Hamiltonian system* is a triple (M, θ, H) where (M, θ) is an exact contact manifold and $H : M \to \mathbb{R}$ is a Hamiltonian function which is homogeneous of degree 1.

Its dynamics is given by the homogeneous (of degree 0) Hamiltonian vector field X_H.

We turn to study the submanifolds of these manifolds. First we notice that these spaces carry a natural distribution.

Definition 7.4. Given a $(2n)-$ *dimensional* exact manifold (M, θ), we define its *Liouville distribution*

$$\mathcal{L} = \ker(\theta). \tag{7.16}$$

This distribution is of dimension $2n - 1$.

We note that, since

$$\omega^{n+1} = -\mathrm{d}\theta \wedge \omega^n - \theta wedge \mathrm{d}\omega^n = -\mathrm{d}(\theta \wedge \omega^n) \neq 0, \tag{7.17}$$

the form $(\theta \wedge \mathrm{d}\theta^n)$ does not vanish. By Frobenius theorem, this distribution is maximally non-integrable and its maximal integrable submanifolds are of dimension n.

Definition 7.5. We say that an n-dimensional submanifold $i : N \hookrightarrow (M, \theta)$ of a $(2n)$-dimensional exact symplectic manifold is *homogeneous Lagrangian*[1] if it is an integral submanifold of the Liouville distribution. Equivalently, $i^*\theta = 0$.

[1] Also called Liouville submanifolds in [245].

Again a homogeneous Lagrangian submanifold is precisely a submanifold that is homogeneous and Lagrangian.

Proposition 7.3. *The submanifold $i : N \hookrightarrow (M, \theta)$ is homogeneous Lagrangian if and only if $i^*\omega = 0$ and Δ is tangent to N.*

Proof. Let N be such that $i^*\theta = 0$. Then $i^*\omega = -i^*\mathrm{d}\theta = -\mathrm{d}i^*\theta = 0$, thus N is Lagrangian. Now, let $x \in N$ and $X_x \in T_xN$, then

$$\omega(\Delta_x, X_x) = \theta(X_x) = 0, \tag{7.18}$$

thus $\omega(T_xN, \Delta_x) = 0$, hence $\Delta_x \in TN$ because N is Lagrangian, thus $\omega(TN, \cdot)^\circ = TN$.

Conversely, if N is Lagrangian and Δ is tangent to N, then for any X tangent to N

$$0 = \omega(\Delta, X) = -\theta(X) = 0. \tag{7.19}$$

Thus, $i^*\eta = 0$. □

Remark 7.4. Notice that at some points Δ and θ vanish. The Liouville distribution will be singular at those points. By the Darboux theorem these points are at most an n-dimension submanifold, so they can be largely ignored.

Homogeneous Lagrangian submanifolds seem to be as ubiquitous as Lagrangian and Legendrian submanifolds, when we consider homogeneous objects. These are the analogs for Theorem 2.19 and Proposition 2.17 on symplectic and contact systems.

Theorem 7.5. *Let (M, θ) be an exact symplectic manifold. Then, (TM, θ^C) is also an exact symplectic manifold. Moreover, a section $X : M \to TM$ is such that $\operatorname{im} X \subseteq (M, \eta)$ is homogeneous Lagrangian if and only if X is a homogeneous symplectomorphism.*

Proof. It is well-known that $\mathrm{d}\theta^C = (\mathrm{d}\theta)^C$ is a symplectic form [2]. Now, $\operatorname{im} X$ is homogeneous Lagrangian if and only if

$$0 = X^*\theta^C = \mathcal{L}_X\theta, \tag{7.20}$$

which is precisely when X is a homogeneous symplectomorphism. □

If we try to find a counterpart of Proposition 2.17, naively one can ask when a section $\gamma : Q \to T^*Q$ is such that $\operatorname{im}\gamma \subseteq (T^*Q, \theta_Q)$ is homogeneous Lagrangian. Unfortunately,

$$\gamma^*\theta_Q = \gamma, \tag{7.21}$$

hence it is only homogeneous Lagrangian if and only if γ is the zero section.

A possible problem is that it is not clear what it means for a section $\gamma : Q \to T^*Q$ to be homogeneous without more geometric structure. One possibility is that Q is itself a homogeneous symplectic manifold. We can now modify the symplectic potential without changing the symplectic form and obtain the following result.

Theorem 7.6. *Let (M, θ) be an exact symplectic manifold and Δ its Liouville vector field. We consider the exact symplectic manifold (T^*M, θ_Δ), where*

$$\theta_\Delta = \theta_M - \mathrm{d}\hat{\Delta}, \tag{7.22}$$

and $\hat{\Delta} : T^*M \to M$ is just the function $\hat{\Delta}(\alpha_x) = \iota_{\Delta_x}(\alpha_x)$.

Then a section $\gamma : M \to T^*M$ is such that $\operatorname{im}\gamma \subseteq (T^*M, \theta_\Delta)$ is Legendrian if and only if $\gamma = \mathrm{d}f$ for some homogeneous function of degree one $f : M \to \mathbb{R}$. That is, $\Delta(f) = f$.

Proof. Again, we compute

$$\gamma^*(\theta_\Delta) = \gamma^*\theta_M - \gamma^*\mathrm{d}\hat{\Delta} = \gamma - \mathrm{d}(\iota_\Delta\gamma). \tag{7.23}$$

Hence, we have that $\operatorname{im}\gamma$ is Legendrian if and only if $\gamma - \mathrm{d}(\iota_\Delta\gamma) = 0$. Thus, $\gamma = \mathrm{d}f_0$ where $f_0 = \iota_\Delta\gamma$. Moreover, it satisfies

$$\mathrm{d}(\Delta(f_0)) = \mathrm{d}\iota_\Delta\mathrm{d}f_0 = \mathrm{d}f_0. \tag{7.24}$$

Thus, f_0 is homogeneous function plus a locally constant term. We can obtain a homogeneous function with the same differential by taking

$$f = \theta(X_{f_0}). \tag{7.25}$$

Indeed,

$$\mathrm{d}f = \mathrm{d}\theta(X_{f_0}) = -\mathrm{d}\omega(\Delta, X_{f_0}) = \mathrm{d}\iota_\Delta\iota_{X_{f_0}}\omega = \mathrm{d}\iota_\Delta\mathrm{d}f_0 = \mathrm{d}f_0. \tag{7.26}$$

Thus, $\operatorname{im}\gamma$ is homogeneous Lagrangian if and only if one can write $\gamma = \mathrm{d}f$ for a homogeneous function f. □

7.2. Symplectization

On most references [28, 162, 245] the symplectization of a contact manifold (M, η) is defined as a specific symplectic manifold M^s which is constructed form M and allows to convert the relevant objects from contact geometry into their symplectic analogs. We will take a different approach. Instead of considering "the" symplectization of M, we will define what "a" symplectization is, abstracting the properties of other symplectizations which are useful for us. In this way, we will be able to choose the symplectization that suits our needs better on each situation.

Definition 7.6. A *symplectization* of a manifold with a contact distribution $(M, \mathcal{H})$ is fiber bundle $\Sigma : (M^\Sigma, \theta) \to (M, \mathcal{H})$, where (M, θ) is an exact symplectic manifold with Liouville vector field Δ and Liouville distribution $\mathcal{L} = \ker\theta$ and a manifold with a contact distribution $(M, \mathcal{H})$. The map Σ satisfies

$$\Sigma_*\mathcal{L} = \mathcal{H}, \tag{7.27}$$

outside the singular points[2] on which $\theta = 0$.

[2]During this section we will be ignoring those points. Indeed, one can remove those points and latter extend the maps by continuity if necessary.

An important property of the symplectization is the following one.

Theorem 7.7. *A symplectization $\Sigma : (M^\Sigma, \theta) \to (M, \mathcal{H})$ satisfies* $\ker T\Sigma = \langle \Delta \rangle$. *Thus, homogeneous objects of degree* 0 *in M^Σ project onto M.*

Proof. Since $\operatorname{im} T\Sigma \supseteq \mathcal{H}$ and $\operatorname{im} T\Sigma$ is involutive, then necessarily $\operatorname{im} T\Sigma \supseteq [\mathcal{H}, \mathcal{H}] = TM$. Thus, since the rank of the image is $(2n+1)$, then the rank of the kernel is 1. Now we need to see that Δ is in the kernel.

Let $y \in M^\Sigma$ and $x = \Sigma(y)$. Now, let N be any Legendrian submanifold passing through x. Consider the submanifold $N^\Sigma = \Sigma^{-1}(N)$. Since $TN \subseteq \mathcal{H}$, $TN^\Sigma = (T\Sigma)^{-1}(TN) \subseteq \mathcal{L}$, so N^Σ, is an integral submanifold of $\mathcal{L}$. From the first isomorphism theorem in linear algebra, it follows that since $T_{y_0}M^\Sigma$ is surjective, for any y_0 in M^Σ, then $\dim(T_{y_0}M^\Sigma) = \dim(T_{\Sigma(y_0)}M) + \dim(\ker T_{y_0}\Sigma) = n+1$. Hence, N^Σ is indeed a homogeneous Lagrangian submanifold.

Now, by Proposition 7.3, $\Delta_y \in T_yM^\Sigma$. Hence, $T\Sigma_y(\Delta_y)$ is tangent to any Legendrian submanifold N passing through x. If we can show that the only vector tangent to all such submanifolds is the 0 vector, we would conclude that $T\Sigma_y(\Delta_y)$. Thus, $\ker T_y\Sigma = \langle \Delta_y \rangle$, finishing the proof of the theorem.

For this last part of the argument we take a Darboux chart around x, so that x corresponds to the point $q^i = p_i = z = 0$. We consider the submanifolds $i_1 : N_1 \to M$, $i_2 : N_2 \to M$ defined by the equations

$$N_1 = \{p_i = 0, z = 0\}, \quad N_2 = \{q_i = 0, z = 0\}.$$

Clearly, x is in both submanifolds and the two are Legendrian, since

$$i_1^*\eta = 0, \quad i_2^*\eta 0.$$

The tangent bundles at x are thus given by

$$T_xN_1 = \langle \mathrm{d}q^i, \mathrm{d}z \rangle^\circ, \quad T_xN_2 = \langle \mathrm{d}p_i, \mathrm{d}z \rangle^\circ. \tag{7.28}$$

A simple calculation shows that $T_xN_1 \cap T_xN_2 = \{0\}$, hence we have proved that the only vector tangent to all Legendrian submanifolds is the zero vector. □

By counting the dimensions of the distributions, it is enough for a symplectization Σ to satisfy $\Sigma_*\mathcal{L} \subseteq \mathcal{H}$. By taking duals, this condition is equivalent to

$$\Sigma^*(\mathcal{H}^\circ) \subseteq \mathcal{L}^\circ = \langle \theta \rangle. \tag{7.29}$$

If $\mathcal{H} = \ker(\eta)$ is a contact form, we have the following characterization.

Proposition 7.8. *A map $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ is a symplectization if and only if there exists a function $\sigma : M^\Sigma \to \mathbb{R}$, which we call the* factor *of Σ, such that*

$$\sigma\Sigma^*\eta = \theta. \tag{7.30}$$

The factor σ is necessarily homogeneous, hence the level sets $\{\sigma = s\}$ give a foliation of M^Σ into contact manifolds $(M_s, i^*\theta = s\eta)$. This also induces a local trivialization (Σ, σ). Indeed, given a coordinate chart (x^i) on M, $(\Sigma^* x^i, \sigma)$ is a coordinate chart on M^Σ. In that chart,

$$\Delta = \sigma \frac{\partial}{\partial \sigma}. \tag{7.31}$$

Proof. Since $\Sigma_* \Delta = 0$, one has:

$$\theta = \mathcal{L}_\Delta(\theta) = \mathcal{L}_\Delta(\sigma \Sigma^* \eta) = \Delta(\sigma)\Sigma^*\eta + \sigma\Sigma^*(\mathcal{L}_{\Sigma_*\Delta}\eta) = \Delta(\sigma)\Sigma^*\eta. \tag{7.32}$$

Hence, $\Delta(\sigma)\Sigma^*\eta = \theta = \sigma\Sigma^*\eta$. So, $\Delta(\sigma) = \sigma$.

Note that the pullback of functions f on M satisfies $\Delta(\Sigma^*(f)) = 0$. Hence, σ is functionally independent to those pullbacks, and it provides the local trivialization mentioned on the theorem. Thus, given a set of coordinates x^i, we have that $\Delta(\Sigma^* x^i) = 0$ and $\Delta(\sigma) = \sigma$. Therefore, Δ takes the coordinate expression (7.31). □

Now that we have established some basic results, we will explain how to relate contact objects to the symplectic counterparts.

Theorem 7.9 (Simplectization of conformal contactomorphisms)**.** *A pair of symplectizations $\Sigma_1 : (M_1^\Sigma, \theta_1) \to (M_1, \mathcal{H}_1)$, $\Sigma_2 : (M_2^\Sigma, \theta_2) \to (M_2, \mathcal{H}_2)$ provides a bijection between conformal contactomorphisms $F : M_1 \to M_2$ and homogeneous symplectomorphisms $F^\Sigma : M_1^\Sigma \to M_2^\Sigma$. A pair (F, F^σ) are related by the bijection if and only if the following diagram commutes:*

$$\begin{array}{ccc} M^\Sigma & \xrightarrow{F^\Sigma} & M^\Sigma \\ \big\downarrow{\scriptstyle \Sigma_1} & & \big\downarrow{\scriptstyle \Sigma_2} \\ M & \xrightarrow{F} & M \end{array} \tag{7.33}$$

We say that F^Σ is the symplectization *of F.*

Given a pair of contact forms $\mathcal{H}_1 = \ker(\eta_1)$, $\mathcal{H}_2 = \ker(\eta_2)$ with factors σ_1 and σ_2, F^Σ has the expression

$$F^\Sigma = \left(F, \frac{1}{\sigma_1 \Sigma_1^*(f)}\right) \tag{7.34}$$

on the trivialization induced by the contact forms. Here, f is the conformal factor of F.

Moreover, the symplectization is functorial, meaning that if F and G are conformal contactomorphism on the appropriate spaces, $(G \circ F)^\Sigma = G^\Sigma \circ F^\Sigma$.

Proof. Let F^Σ be a homogeneous symplectomorphism. Since $(F^\Sigma)_* \Delta_1 = \Delta_2$, and $\ker{(\Sigma_i)}_* = \langle \Delta_i \rangle$, F^Σ projects onto a function F such that the diagram Equation (7.33) commutes. Now, using the commutative diagram, we obtain

$$F_* \mathcal{H}_1 = (\Sigma_2)_* \circ (F^\Sigma)_* \circ ((\Sigma_1)_*)^{-1}(\mathcal{H}_1) = (\Sigma_2)_* \circ (F^\Sigma)_* \mathcal{L}_1 = (\Sigma_2)_* \mathcal{L}_2 = \mathcal{H}_2. \tag{7.35}$$

Thus, F is a conformal contactomorphism.

Now assume that F is a conformal contactomorphisms. We take forms η_1 and η_2, generating locally both contact distributions. On the trivialization induced by the forms, we let $F^\Sigma = (F, \bar{F})$ be a function that satisfies Equation (7.33). We will see that the unique function of this form that is a homogeneous symplectomorphism is the one in Equation (7.34). Indeed,

$$\begin{aligned}(F^\Sigma)^* \theta_2 &= (F^\Sigma)^* (\sigma_2 (\Sigma_2)^* \eta_2)\\ &= (F^\Sigma)^* (\sigma_2 (F)^* \eta_2)\\ &= (F^\Sigma)^* (\sigma_2)(\Sigma_1^* F^* \eta_2)\\ &= (F^\Sigma)^* (\sigma_2)(\Sigma_1^*(f \eta_1))\\ &= (F^\Sigma)^* (\sigma_2) \Sigma_1^*(f) \sigma_1 \theta_1 = \theta_1.\end{aligned} \tag{7.36}$$

Thus,

$$\bar{F} = (F^\Sigma)^* (\sigma_2) = \frac{1}{\Sigma_1^*(f)\sigma_1}. \tag{7.37}$$

Then, we have proved that the correspondence $F \to F^\Sigma$ is well-defined and bijective.

In order to show the functorial character of the symplectization, we consider the following diagram:

$$\begin{tikzcd} M_1^\Sigma \arrow[r, "F^\Sigma"'] \arrow[d, "\Sigma"] \arrow[rr, bend left, "(G\circ F)^\Sigma"] & M_2^\Sigma \arrow[r, "G^\Sigma"'] \arrow[d, "\Sigma"] & M_3^\Sigma \arrow[d] \\ M_1 \arrow[r, "F"] \arrow[rr, bend right, "G\circ F"'] & M_2 \arrow[r, "G"] & M_3 \end{tikzcd} \tag{7.38}$$

Since the symplectization is unique and $G^\Sigma \circ F^\Sigma$ is a homogeneous symplectomorphism that projects onto $G \circ F$, we can conclude that it is its symplectization. That is, $G^\Sigma \circ F^\Sigma = (G \circ F)^\Sigma$. □

The fact that the symplectization preserves the composition of morphisms allows us to simplectize the vector fields. Let X be a conformal contactomorphism, and let ϕ_t be its flow. Notice that

$$\phi_{t+r}^\Sigma = (\phi_r \circ \phi_t)^\Sigma = \phi_t^\Sigma \circ \phi_r^\Sigma. \tag{7.39}$$

Hence, ϕ_t^Σ is also a flow. Since it is the symplectization of contactomorphisms, it is a flow of homogeneous symplectomorphisms. Hence, its infinitesimal generator, X^Σ is an infinitesimal homogeneous symplectomorphism. Thus, we have the following:

Theorem 7.10 (Symplectization of vector fields). *A symplectization $\Sigma : (M^\Sigma, \theta) \to (M, \mathcal{H})$ provides a bijection between infinitesimal conformal contactomorphisms X on M and infinitesimal homogeneous symplectomorphisms X^Σ on M^Σ such that X and X^Σ are related by the bijection if and only if*

$$\Sigma^* X^\Sigma = X. \tag{7.40}$$

If the contact distribution is generated by a form η, this vector field can also be characterized as the one that projects onto X and satisfies

$$X(\sigma) = -a_X \sigma, \tag{7.41}$$

where a_X is the conformal factor of X. We can also write it as

$$X^\Sigma = X^\sigma - a_X \Delta, \tag{7.42}$$

where X^σ is the vector field that projects onto X and satisfies $X^\sigma(\sigma) = 0$. In particular, for the Reeb vector field,

$$\mathcal{R}^\Sigma = \mathcal{R}^\sigma. \tag{7.43}$$

Moreover, for any pair of infinitesimal conformal contactomorphisms X, Y

$$[X, Y]^\Sigma = [X^\Sigma, Y^\Sigma]. \tag{7.44}$$

Proof. In order to prove that the one-to-one correspondence is well-defined, we only need to apply Theorem 7.9 to their flows. Thus, given the flow ϕ of X there exist a unique flow ϕ^Σ which projects onto ϕ, thus there exist a unique infinitesimal homogeneous symplectomorphisms X^Σ, which is the infinitesimal generator of ϕ^Σ. Of course, reversing the argument we see that this correspondence works on the opposite direction.

We now consider the contact form η with factor σ and let X be an infinitesimal conformal contactomorphism with factor a_X. Then, since X^Σ is a homogeneous symplectomorphism,

$$0 = \mathcal{L}_{X^\Sigma}\theta = \mathcal{L}_{X^\Sigma}(\sigma \Sigma^* \eta) = (X^\Sigma(\sigma) + \sigma a_X)\eta. \tag{7.45}$$

Thus. $X^\Sigma(\sigma) = -\sigma a_X$.

Since $\ker \Sigma = \langle \Delta \rangle$, X is of the form $X^\sigma + b\Delta$ for some function b. Thus,

$$- a_X \sigma = X^\Sigma(\sigma) = b\Delta(\sigma) = b\sigma, \tag{7.46}$$

since σ is homogeneous of degree 1. Hence, $b = -a_X b$.

Finally, we also obtain that the symplectization commutes with the Lie brackets as a consequence of the fact that it preserves composition. Indeed, the Lie bracket is the second derivative of the commutator of the flows and the commutator is preserved by symplectization ([173, Lemma 6.19.2]). $\square$

Since the contact Hamiltonian vector fields and the functions on (M, η) on one side, and the homogeneous Hamiltonian vector fields and the homogeneous functions on (M^Σ, θ), are in one-to-one correspondence, we can obtain a bijection between both sets of functions. Indeed,

Theorem 7.11. *Given a symplectization $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ with factor σ, there is a bijection between functions f on M and homogeneous functions f^Σ on M^Σ such that*

$$X_{f^\Sigma} = (X_f)^\Sigma. \tag{7.47}$$

This bijection is given by

$$f^{\Sigma} = \sigma\Sigma^* f. \tag{7.48}$$

Moreover, one has that

$$\{f^{\Sigma}, g^{\Sigma}\} = \{f, g\}^{\Sigma}. \tag{7.49}$$

Proof. In order to obtain f^{Σ}, we compute

$$f^{\Sigma} = \theta(X_{f^{\Sigma}}) = \theta((X_f)^{\Sigma}) = \sigma\Sigma^*(\eta)((X_f)^{\Sigma}) = \sigma\Sigma^*(\eta(X_f)) = \sigma\Sigma^* f. \tag{7.50}$$

The other claims follow from Proposition 2.12, the equivalent theorem for symplectic geometry, that is $X_{\{f,g\}} = [X_g, X_f]$, and the result that symplectization preserve the Lie brackets. □

Remark 7.12. The last theorem shows that

$$\sigma\Sigma^* f, \sigma\Sigma^* g = \sigma\Sigma^*\{f, g\}. \tag{7.51}$$

This implies that (Σ, σ) is a conformal Jacobi morphism [162, 188].

We say that $\Sigma : (M^{\Sigma}, \theta, H^{\Sigma}) \to (M, \eta, H)$ is the *symplectization of a contact Hamiltonian system* if $\Sigma : (M^{\Sigma}, \theta) \to (M, \eta)$ is a symplectization with factor σ and $H^{\Sigma} = \sigma\Sigma^* H$..

Finally, from the definition of symplectization and taking into account that homogeneous Lagrangian and Legendrian submanifolds are integral submanifolds of their corresponding distributions, we can also symplectize them.

Theorem 7.13. *Let $\Sigma : (M^{\Sigma}, \theta) \to (M, \eta)$ be a symplectization and $N \hookrightarrow M$ be a submanifold. Then, $\Sigma^{-1}(N)$ is homogeneous Lagrangian if and only if N is Legendrian. On that situation, we denote $N^{\Sigma} = \Sigma^{-1}(N)$ and call it the* symplectization *of N.*

Now we provide examples of symplectizations.

Definition 7.7. The *natural symplectization* of a manifold with a contact distribution $(M, \mathcal{H})$, where $\mathcal{H} = \ker \eta$, and M^s is the exact symplectic manifold (M^s, η^s), that can be defined intrinsically as

$$M^s = \ker(\eta)^{\circ} \setminus 0_{T^*M} = \langle\eta\rangle \setminus 0_{T^*M} = \{(x, r\eta_x) \subseteq T^*M \mid r \in \mathbb{R} \setminus \{0\}\}, \tag{7.52a}$$
$$\theta^s = i^*\theta_M, \tag{7.52b}$$

where 0_{T^*M} is the zero section of $T^*M \to M$ and $i : \langle\eta\rangle \setminus 0_{T^*M} \hookrightarrow T^*M$ is the canonical inclusion.

The symplectization map is $\Sigma^s = \pi_{M|M^s}$, where $\pi_M : T^*M \to M$ is the canonical projection.

Proposition 7.14. *The natural symplectization is indeed a symplectization. Its conformal factor is*

$$\begin{aligned} M^s &\to \mathbb{R}, \\ r\eta_x &\mapsto r. \end{aligned} \tag{7.53}$$

Moreover, the symplectization of a function $F : M_1 \to M_2$, which we denote $F^s : M_1^s \to M_2$, is given by the restriction of the inverse of the cotangent lift:

$$
\begin{tikzcd}
T^*M_1 & T^*M_2 \arrow[l, "T^*F"'] \\
M_1^s \arrow[u, hook, "i_1"] \arrow[r, "F^s"] \arrow[d, "\Sigma_1^s"] & M_2^s \arrow[u, hook, "i_2"'] \arrow[d, "\Sigma_2^s"'] \\
M_1 \arrow[r, "F"] & M_2
\arrow[from=1-1, to=3-1, bend right, "\pi_{M_1}"']
\arrow[from=1-2, to=3-2, bend left, "\pi_{M_2}"]
\end{tikzcd}
\tag{7.54}
$$

Proof. First we prove that θ^s, which is given by

$$\theta^s_{(x, r\eta_x)} = r\eta_x, \tag{7.55}$$

is a symplectic potential. Indeed,

$$\mathrm{d}\theta^s = \mathrm{d}r \wedge \eta + r\mathrm{d}\eta, \tag{7.56}$$

and hence,

$$(\mathrm{d}\theta^s)^n = r^n \mathrm{d}r \wedge \eta \wedge \mathrm{d}\eta \tag{7.57}$$

does not vanish outside the zero section. Thus, (M^s, θ^s) is an exact symplectic manifold.

One has that

$$r\Sigma^s(\eta_x) = r\eta_x = \theta^s_{(x, r\eta_x)}, \tag{7.58}$$

thus Σ^s is a symplectization with conformal factor r.

We will proof directly that F^s is a homogeneous symplectomorphism, using that F^* is a homogeneous symplectomorphism (see [2, Thm. 3.2.12]),

$$
\begin{aligned}
(F^s)^*(\theta_2^s) &= (F^s)^*(i_2^*\theta_2) = (i_2 \circ F^s)^*(\theta_2) \\
&= (F_* \circ i_1)^*(\theta_2) = i_1^*((F_*)^*(\theta_1)) = i_1^*\theta_1 = \theta_1^s.
\end{aligned}
\tag{7.59}
$$

□

Definition 7.8 (An equivalent symplectization)**.** Let (M, η) be a contact manifold. Its symplectization is defined in [162] as the exact symplectic manifold $(\bar{M}, \exp(s)\eta)$, where

$$\bar{M} = M \times \mathbb{R} \tag{7.60}$$

$$\bar{\eta} = \exp(s)\eta, \tag{7.61}$$

where s is the $\mathbb{R}$-coordinate. The symplectization is just $\bar{\Sigma} = \mathrm{pr}_M : M \times \mathbb{R} \to M$, which is the canonical projection. Here, the factor is $\bar{\sigma} = \exp(s)$.

Moreover, let $F : (M_1, \eta_1) \to (M_2, \eta_2)$ be a diffeomorphism [162, Thm 3.16]. Then, the map

$$
\begin{aligned}
\bar{F}_a : M_1 \times \mathbb{R} &\to M_2 \times \mathbb{R}, \\
(x, a) &\to (x, a + \log(a))
\end{aligned}
\tag{7.62}
$$

is a symplectomorphism (with respect to the symplectized structures) if and only if $F^*\eta_2 = a\eta_1$.

This symplectization is equivalent to the natural symplectization. Indeed, the map

$$\begin{aligned} \Phi : \bar{M} &\to M^s, \\ (x, a) &\mapsto (x, \exp(a)\eta_x) \end{aligned} \tag{7.63}$$

is a symplectomorphism onto its image. Moreover, it allows us to translate the lifts of contactomorphisms $F : M_1 \to M_2$ with conformal factor $a > 0$ as follows

$$\begin{tikzcd} & \bar{M}_1 \arrow[r, "\bar{F}_a"] \arrow[dl, "\bar{\Sigma}_1"'] \arrow[dd, "\Phi_1"'] & \bar{M}_2 \arrow[dr, "\bar{\Sigma}_2"] \arrow[dd, "\Phi_2"] & \\ M_1 & & & M_2 \\ & M_1^s \arrow[r, "F^s"] \arrow[ul, "\Sigma_1^s"] & M_2^s \arrow[ur, "\Sigma_s^2"'] & \end{tikzcd} \tag{7.64}$$

where Φ_i are the maps (7.63) for the symplectization of M_i.

7.2.1. Symplectization of a Hamiltonian system on a cotangent bundle

The extended cotangent bundle, $T^*Q \times \mathbb{R}$ has a natural contact structure $\eta_Q = dz - \theta_Q$. In the presence of a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$, it is the natural framework for Hamiltonian mechanics on contact manifolds. Its natural symplectization $((T^*Q \times \mathbb{R})^s, \theta^s, H^s)$, is a symplectic Hamiltonian system. However, we loose the cotangent bundle structure. We would like to obtain a symplectization $\Sigma_Q : (T^*(Q \times \mathbb{R}), \theta_Q) \to (T^*Q \times \mathbb{R}, \eta_Q)$.

One way to approach the problem is to start with the natural symplectization and try to find a symplectomorphism $(T^*Q \times \mathbb{R})^s \to T^*(Q \times \mathbb{R})$. However, this identification is not completely trivial. There is a natural map $\psi_0 : (T^*Q \times \mathbb{R})^s \to T^*(Q \times \mathbb{R})$ given by

$$\begin{tikzcd} & T^*(T^*Q \times \mathbb{R}) \arrow[dr, "\pi_{T^*(Q\times\mathbb{R})}"] & \\ (T^*Q \times \mathbb{R})^s \arrow[ur, hook, "i"] \arrow[rr, "\psi_0"] & & T^*(Q \times \mathbb{R}) \end{tikzcd} \tag{7.65}$$

In coordinates, the map ψ_0 is given by

$$\psi_0(q, p, z, r) = (q, p, z, r). \tag{7.66}$$

Unfortunately, a quick computation shows that this map is not a homogeneous symplectomorphism. However, we can construct another map ψ using the canonical involution of a double vector bundle $\kappa_Q : T^*T^*Q \to T^*T^*Q$ [242], such that $\kappa_Q(q, p, \alpha, \beta) = (q, \alpha, p, \beta)$.

Let ψ be defined by

$$\begin{tikzcd}
T^*(T^*Q \times \mathbb{R}) \arrow[r, "\kappa_Q \times \mathrm{id}_{T^*\mathbb{R}}"] & T^*(T^*Q \times \mathbb{R}) \arrow[d, "\pi_{T^*(Q\times\mathbb{R})}"] \\
(T^*Q \times \mathbb{R})^s \arrow[u, hook, "i"] \arrow[r, "\psi"] & T^*(Q \times \mathbb{R}),
\end{tikzcd} \tag{7.67}$$

where $T^*(T^*Q \times \mathbb{R})$ is identified with $T^*T^*Q \oplus T^*\mathbb{R}$. In coordinates, ψ satisfies

$$\psi(q,p,z,r) = (q,z,-rp,r), \tag{7.68a}$$

$$\psi^{-1}(q,z,p,p_z) = (q,z,-p/p_z,p_z) \tag{7.68b}$$

A simple computation, shows that $\psi^*(\theta_{Q\times\mathbb{R}}) = \eta_Q^s$, hence it is a homogeneous symplectomorphism.

A nice interpretation of the projection $\Sigma^s : (T^*Q\times\mathbb{R})^s \to T^*Q\times\mathbb{R}$ is the projectivization of the cotangent bundle. Indeed, since ϕ preserves the symplectic potential, it also maps the Liouville vector field of $(T^*Q\times\mathbb{R})^s$ onto the canonical Liouville vector field on $T^*(Q\times\mathbb{R})$. Now, notice that the quotient of $(T^*Q\times\mathbb{R})^s$ by this vector field is naturally equivalent to $T^*Q\times\mathbb{R}$. Moreover, the quotient of $T^*(Q\times\mathbb{R})$ by its Liouville vector field is the projectivization of the cotangent bundle $\mathbb{P}(T^*(Q\times\mathbb{R}))$. Thus, ψ descends to the quotient and induces a map $\bar{\psi}$, which is precisely the one in Example 2.2 and Equation (2.25).

$$\begin{tikzcd}
(T^*Q \times \mathbb{R})^s \arrow[rr, "\psi"] \arrow[d, "\Sigma^s"] & & T^*(Q \times \mathbb{R}) \arrow[d, "\mathbb{P}"] \arrow[dll, "\Sigma_Q"'] \\
T^*Q \times \mathbb{R} \arrow[rr, "\bar{\psi}"] \arrow[dr] & & \mathbb{P}(T^*(Q \times \mathbb{R})). \arrow[dl] \\
& Q \times \mathbb{R} &
\end{tikzcd} \tag{7.69}$$

We define the map $\Sigma_Q = \pi^s \circ \psi^{-1} = \bar{\psi} \circ \mathbb{P}$. Then, we have:

Theorem 7.15. *The map $\Sigma_Q : (T^*(Q\times\mathbb{R}), \theta_Q) \to (T^*Q\times\mathbb{R}, \eta_Q)$, given by*

$$\begin{aligned}
\Sigma_Q : T^*(Q\times\mathbb{R}) &\to T^*Q\times\mathbb{R}, \\
(q,z,p,p_z) &\mapsto (q,-p/p_z,z),
\end{aligned} \tag{7.70}$$

is a symplectization with conformal factor p_z. Moreover, it is a fiber bundle morphism over $Q\times\mathbb{R}$. That is, it satisfies the following commutative diagram:

$$\begin{tikzcd}
& T^*(Q \times \mathbb{R}) \arrow[dl, "\Sigma"'] \arrow[dd, "\pi_{Q\times\mathbb{R}}"] \\
T^*Q \times \mathbb{R} \arrow[dr, "\pi_Q^1"'] & \\
& Q \times \mathbb{R}
\end{tikzcd} \tag{7.71}$$

We will try to use symplectization in order to give a proof of Proposition 2.17.

Let $\gamma = (\alpha, f)$ be a section of the 1-jet bundle $T^*Q \times \mathbb{R} \to Q$. We would like to relate it to a section of $T^*(Q \times \mathbb{R})$ and apply the theorem that the image of a form is Lagrangian if and only if it is closed. Nevertheless, by (7.21), the only section of $(T^*(Q \times \mathbb{R}), \theta_{Q\times\mathbb{R}})$ having homogeneous Lagrangian image is the zero section, we will be forced to modify the symplectic potential.

We start again from the natural symplectization. Indeed, by the above results, $\operatorname{im}\gamma$ is Legendrian if and only if $(\Sigma^s)^{-1}(\operatorname{im}\gamma)$ is Lagrangian. Also notice that $(\Sigma^s)^{-1}(\operatorname{im}\gamma) = \operatorname{im}\gamma^s$, where

$$\begin{aligned} \gamma^s : Q \times \mathbb{R} &\to (T^*Q \times \mathbb{R})^s, \\ (q, r) &\mapsto (q, \alpha(q), f(q), r) \end{aligned} \tag{7.72}$$

Notice that γ^s is a section over the bundle

$$\begin{aligned} \tilde{\pi} : (T^*Q \times \mathbb{R})^s &\to Q \times \mathbb{R}, \\ (q, p, z, r) &\mapsto (q, r). \end{aligned} \tag{7.73}$$

We would like to construct a symplectomorphism that maps $(T^*Q \times \mathbb{R})^s$ onto $T^*(Q\times\mathbb{R})$ and maps γ^s onto a 1-form. Unfortunately, no symplectomorphism can map sections to sections and preserve the Liouville 1-form θ_Q.

In order to fix this, we need to exchange the position and momenta variables on the $T^*\mathbb{R}$ component. Consider the following map

$$\begin{aligned} \tilde{\psi} : (T^*Q \times \mathbb{R})^s &\to T(Q \times \mathbb{R}), \\ \tilde{\psi}(q, p, z, r) &= (q, r, -rp, -z), \\ \tilde{\psi}^{-1}(q, z, p, p_z) &= (q, -p/z, -p_z, z), \end{aligned} \tag{7.74}$$

which is given by $(\mathrm{id}_{T^*Q}, \chi) \circ \phi$, where $T^*(Q \times \mathbb{R}) \simeq T^*(Q) \times \mathbb{R} \times \mathbb{R}$ and

$$\begin{aligned} \chi : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (z, p_z) &\mapsto (p_z, -z). \end{aligned} \tag{7.75}$$

The map $\tilde{\psi}$ is a (non-strict) symplectomorphism and a fiber bundle morphism. Indeed, $\tilde{\psi}^*\tilde{\theta}_Q = \theta^s$, where

$$\tilde{\theta}_{Q\times\mathbb{R}} = \theta_{Q\times\mathbb{R}} - \mathrm{d}(p_z z) = p_i \mathrm{d}q^i - z\mathrm{d}p_z. \tag{7.76}$$

Continuing our discussion, we know have that $\tilde{\psi}(\operatorname{im}\gamma^s) = \operatorname{im}\tilde{\gamma}$ is Lagrangian if and only if $(\operatorname{im}\gamma^s)$ is and that $\tilde{\gamma} = \tilde{\psi} \circ \gamma : Q \times \mathbb{R}$ is one-form.

Since the image of a one-form is Lagrangian if and only if it is closed, we can conclude that $\operatorname{im}(\gamma)$ is Legendrian if and only if $\tilde{\gamma}$ is closed. This is the situation:

$$\begin{tikzcd}
 & (T^*Q \times \mathbb{R})^s \arrow[dl, "\Sigma^s"'] \arrow[dr, "\tilde{\pi}"] \arrow[rr, "\tilde{\psi}"] & & T^*(Q \times \mathbb{R}) \arrow[dl, "\pi_{Q\times\mathbb{R}}"] \\
T^*Q \times \mathbb{R} \arrow[dr] & & Q \times \mathbb{R} \arrow[ul, bend left, "\gamma^s"] \arrow[ur, bend right, "\tilde{\gamma}"'] \arrow[dl] & \\
 & Q \arrow[ul, bend left, "\gamma"] & &
\end{tikzcd} \tag{7.77}$$

We just need to compute $\tilde{\gamma}$ and see when it is closed. Indeed,

$$\tilde{\gamma}(q,r) = \tilde{\psi}(q, \alpha(q), f(q), r) = (q, r, -r\alpha(q), -f(q)). \tag{7.78}$$

That is,

$$\tilde{\gamma} = -r\alpha - f\mathrm{d}r. \tag{7.79}$$

Thus,

$$\mathrm{d}\tilde{\gamma} = -r\mathrm{d}\alpha - \mathrm{d}r \wedge \alpha - \mathrm{d}f \wedge \mathrm{d}r = 0 \iff \mathrm{d}f = \alpha. \tag{7.80}$$

Hence, we have proved that $\operatorname{im}(\gamma)$ is Legendrian if and only if $\gamma = j^1 f$. Moreover, in that situation,

$$\tilde{\gamma} = -r\mathrm{d}f - f\mathrm{d}r = -\mathrm{d}(rf) = \mathrm{d}(f^s). \tag{7.81}$$

We now consider the map $\tilde{\Sigma}_Q = \Sigma^s \circ \tilde{\psi}^{-1}$. We have proved the following.

Theorem 7.16. *The map $\tilde{\Sigma}_Q : T^*(Q \times \mathbb{R}, \tilde{\theta}_Q) \to (T^*Q \times \mathbb{R}, \eta_Q)$, where $\tilde{\theta}_Q$ is given in* (7.76)*, and the map is given by*

$$\begin{aligned} \tilde{\Sigma}_Q : T^*(Q \times \mathbb{R}) &\to T^*Q \times \mathbb{R}, \\ (q, z, p, p_z) &\mapsto (q, -p/z, -p_z), \end{aligned} \tag{7.82}$$

is a symplectization with conformal factor $-z$. Moreover, given a section σ, the section $\tilde{\sigma}(q,s) = s\sigma$ satisfies

$$\begin{tikzcd}
 & T^*(Q \times \mathbb{R}) \arrow[dl, "\tilde{\Sigma}_Q"'] \arrow[dr, "\pi_{Q\times\mathbb{R}}"] & \\
T^*Q \times \mathbb{R} \arrow[ur, bend left, "\sigma"] \arrow[dr, "\pi_Q^0"'] & & Q \times \mathbb{R} \arrow[ul, bend right, "\tilde{\sigma}"'] \arrow[dl, "\mathrm{pr}_Q"] \\
 & Q &
\end{tikzcd} \tag{7.83}$$

We will introduce one last way to symplectize cotangent bundles. In some situations, the base M might be a contact manifold itself, with a symplectization $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$. We can lift it to a symplectization $\Sigma : T^*M^\Sigma \to T^*M$. However, we will need to modify our symplectic form as in Theorem 7.6.

Theorem 7.17 (Cotanget lift of a symplectization)**.** *Let $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ be a symplectization with factor σ. Then, there exist a symplectization $\Sigma^{T^*} : (T^*M^\Sigma, \theta_\Delta) \to (T^*M \times \mathbb{R}, \eta_M)$, where*

$$\theta_\Delta = \theta_\Sigma + \mathrm{d}\hat{\Delta}, \tag{7.84}$$

Δ is the Liouville vector field of M, and $\hat{\Delta} : T^*M \to M$ is just the function $\hat{\Delta}(\alpha_x) = \iota_{\Delta_x}(\alpha_x)$, which makes the following diagram commutative

$$\begin{tikzcd} & T^*M^\Sigma \arrow[dl, "\Sigma^{T^*}"'] \arrow[dr, "\pi_{M^\Sigma}"] & \\ T^*M \times \mathbb{R} \arrow[dr, "\pi_M^0"] & & M^\Sigma \arrow[dl, "\Sigma"'] \\ & M & \end{tikzcd} \tag{7.85}$$

The symplectization is given by $\Sigma^{T^*} = \tilde{\Sigma}^M \circ (T^*\Sigma, \sigma)^{-1}$. In local coordinates, it is

$$\begin{aligned} \Sigma^{T^*} &: T^*M^\Sigma \to T^*M \times \mathbb{R}, \\ &(x, \sigma, p, p_\sigma) \mapsto \left(q, -\frac{p}{\sigma}, -p_\sigma\right). \end{aligned} \tag{7.86}$$

The conformal factor of Σ^{T^*} is $\pi^*_{M^\Sigma}\sigma$.

Proof. We start with the symplectization in Theorem 7.16 $\tilde{\Sigma}^M : (T^*M, \tilde{\theta}_{M\times\mathbb{R}}) \to (T^*M, \eta_M)$, and we will choose a symplectization so that the following diagram commutes

$$\begin{tikzcd} T^*M \times \mathbb{R} \arrow[ddd, "\pi_M^0"'] & & T^*(M \times \mathbb{R}) \arrow[ll, "\tilde{\Sigma}^M"'] \arrow[dl, "T^*(\Sigma,\sigma)"] \arrow[ddd, "\pi_{M\times\mathbb{R}}"] \\ & T^*M^\Sigma \arrow[ul, "\Sigma^{T^*}"] \arrow[d, "\pi_{M^\Sigma}"] & \\ & M^\Sigma \arrow[dl, "\Sigma"'] \arrow[dr, "(\Sigma,\sigma)"] & \\ M & & M \times \mathbb{R} \arrow[ll, "\mathrm{pr}_M"] \end{tikzcd} \tag{7.87}$$

First, we need to see that $T^*(\tilde{\Sigma}^M, \sigma)$ is a homogeneous symplectomorphism. We choose a chart (x^i) of M, so that (x^i, σ) is a chart of M^Σ and $(x^i, \sigma, p_i, p_\sigma)$ is a chart of T^*M^Σ. By Equation (7.31), $\hat{\Delta} = \sigma p_\sigma$. Thus, in this chart,

$$\begin{aligned} \tilde{\theta}_{M\times\mathbb{R}} &= \theta_{M^\Sigma} - \mathrm{d}\hat{\Delta} \\ &= p_i \mathrm{d}x^i + p_\sigma + \mathrm{d}\sigma - \mathrm{d}(p_\sigma \sigma) = p_i \mathrm{d}x^i - \sigma \mathrm{d}p_\sigma \\ &= T\Sigma^* \theta_{TM} - T\sigma^*(z\mathrm{d}p_z) = T(\Sigma, \sigma)^* \tilde{\theta}_M. \end{aligned} \tag{7.88}$$

We now just compute Σ^{T^*} so that the diagram Equation (7.87) commutes, obtaining Equation (7.86). $\square$

This construction relates Proposition 2.17 and Theorem 7.6. Indeed, section $\gamma : M \to T^*M \times \mathbb{R}$, are in one-to-one correspondence with homogeneous sections $\gamma^\Sigma = \sigma\Sigma^*\gamma : M^\Sigma \to T^*M^\Sigma$, such that the following diagram commutes:

$$\begin{tikzcd}
 & T^*M^\Sigma \arrow[dl, "\Sigma^{T^*}"'] \arrow[dr, "\pi_{M^\Sigma}"] & \\
T^*M \times \mathbb{R} \arrow[dr, "\pi^0_M"] & & M^\Sigma \arrow[ul, bend right, "\gamma^\Sigma"'] \arrow[dl, "\Sigma"] \\
 & M \arrow[ul, bend left, "\gamma"] &
\end{tikzcd} \tag{7.89}$$

In this situation, it is equivalent

- $\operatorname{im}\gamma$ is Legendrian,
- $\operatorname{im}\gamma$ is homogeneous Lagrangian,
- $\gamma = j^1 f$,
- $\gamma = \mathrm{d}f^\Sigma$,

for some $f : M \to \mathbb{R}$.

7.2.2. The contact tangent and symplectization

Using symplectization we can give an alternate construction of the contact tangent (Proposition 2.18). Indeed, let $\Sigma : (M^\sigma, \theta) \to (M, \eta)$ be a symplectization. Consider the pullback bundle $i_\eta^* TM^\Sigma$ given by,

$$\begin{tikzcd}
i_\eta^* TM^\Sigma \arrow[r, "j_\eta"] \arrow[d, "\tau_M"] & TM^\Sigma \arrow[d, "\tau_{M^\Sigma}"] \\
M \arrow[r, "i_\eta"] & M^\Sigma
\end{tikzcd} \tag{7.90}$$

where $i_\eta(x) = \eta_x$.

We choose a local trivialization such that $x \in M$, $(x, r) \in M^\Sigma$, $(x, r, \dot{x}, \dot{r}) \in TM^\Sigma$ and $(x, \dot{x}, \dot{r}) \in i_\eta^* TM^\Sigma$. We note that the manifold $i_\eta^* TM^\Sigma$ can be identified with $TM \times \mathbb{R}$ just by taking $\dot{r}$ as the $\mathbb{R}$ coordinate.

It is well known that the tangent bundle of an exact symplectic manifold (M, θ) is also an exact symplectic manifold [2], with the symplectic potential θ^C. In our case, the form is given by

$$\theta^C = (r\eta)^C = r^V \eta^C + r^C \eta^V = r\eta^C + \dot{r}\eta^V. \tag{7.91}$$

Here, we used the identity Equation (A.12). The map j_η is given by

$$\begin{aligned} j_\eta : TM \times \mathbb{R} &\to TM^\Sigma \\ (x, \dot{x}, t) &\mapsto (x, 1, \dot{x}, t), \end{aligned} \tag{7.92}$$

Hence,

$$j_\eta^* \theta^C = \eta^C + t\eta^C = \eta^T. \tag{7.93}$$

Therefore, we retrieve the construction of the contact tangent that was made in Proposition 2.18.

We can also recover Theorem 2.19. This will come as a corollary of the following theorem.

Theorem 7.18 (Tangent lift of a symplectization)**.** *Let $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ be a symplectization with factor σ. Then, there exist a unique symplectization $\Sigma^T : (TM^\Sigma, \theta^C) \to (TM, \eta^T)$ that makes the following diagram commute:*

$$\begin{tikzcd}
 & TM^\Sigma \arrow[dl, "\Sigma^T"'] \arrow[d, "T\Sigma"] \arrow[dr] & \\
TM \times \mathbb{R} \arrow[r, "\mathrm{pr}_{TM}"] \arrow[dr, "\pi_M^0"'] & TM \arrow[d, "\tau_M"] & M^\Sigma \arrow[dl, "\Sigma"] \\
 & M &
\end{tikzcd} \tag{7.94}$$

This map is given by

$$\Sigma^T = \left(T\Sigma, \frac{\sigma^C}{\sigma^V} \right), \tag{7.95}$$

and the factor of the symplectization is σ^V.

Proof. Since Σ^T fibers over TM, it is of the form $\Sigma^T = (T\Sigma, \Sigma_t)$. We compute Σ_t so that Σ^T is a symplectization with factor σ^T, using the properties of the lifts (see Appendix A),

$$\sigma^T (\Sigma^T)^* (\eta^T) = \sigma^T ((T\Sigma)^* (\eta^C) + \Sigma_t (T\Sigma)^* (\eta^V) = \sigma^T ((\Sigma^* \eta)^C + \Sigma_t \Sigma^* \eta^V). \tag{7.96}$$

This must equal θ^C. Indeed,

$$\theta^C = \sigma \Sigma^* \eta^C = \sigma^C \Sigma^* \eta^V + \sigma^V \Sigma^* \eta^C. \tag{7.97}$$

Equating the last two expressions, we obtain $\sigma^T = \sigma^V$, and $\Sigma_t = \sigma^C / \sigma^V$. $\square$

As in the case of the cotangent lift of the symplectization, we can use this construction to relate Theorem 2.19 and Theorem 7.5. Now, let $X : M \to TM \times \mathbb{R}$ be a section that is Σ-related with homogeneous vector field, $X^\Sigma = M^\Sigma \to TM^\Sigma$. That is, such that the following diagram commutes:

$$\begin{tikzcd}
 & TM^\Sigma \arrow[dl, "\Sigma^T"'] \arrow[dr, "\pi_{M^\Sigma}"] & \\
TM \times \mathbb{R} \arrow[dr, "\pi_M^0"] & & M^\Sigma \arrow[dl, "\Sigma"] \arrow[ul, bend right, "X^\Sigma"'] \\
 & M \arrow[ul, bend left, "X"] &
\end{tikzcd} \tag{7.98}$$

Under these circumstances, the following statements are equivalent:

- $\operatorname{im} X$ is Legendrian,
- $\operatorname{im} X^{\Sigma}$ is homogeneous Lagrangian,
- $X = (X_f, \mathcal{R}(f))$,
- $X^{\Sigma} = X_{f^{\Sigma}}$,

for some $f : M \to \mathbb{R}$.

7.3. Contactization

Although contactization not as useful for us as symplectization for our interests, since it is mostly a tool to understand symplectic dynamics from the contact point of view, we include it for completeness. Given an exact symplectic manifold (M, θ), one can easily create a contact manifold[3] of one dimension more (M^{co}, θ^{co}), which is just given by

$$M^{co} = M \times \mathbb{R} \tag{7.99a}$$

$$\theta^{co} = \mathrm{d}z - (\pi^{co})^* \theta, \tag{7.99b}$$

where $\pi^{co} : M \times \mathbb{R} \to M$ is the canonical projection and z is, as usual, the $\mathbb{R}$ coordinate.

All the relevant geometric objects are easy to contactize. Indeed, a map $F : M_1 \to M_2$ is a symplectomorphism if and only if $F^{co} = (F, \mathrm{id})$ is a strict contactomorphism.

Similarly, a vector field X is an infinitesimal symplectomorphism if and only if X^{co} (which is no more that X extended by 0 using the product structure on $Q \times \mathbb{R}$) is a conformal infinitesimal contactomorphism.

Finally, we can define $H^{co} = (\pi^{co})^* H$ for a function $H : M \to \mathbb{R}$ and immediately see that $(X_H)^{co} = X_{H^{co}}$.

Nevertheless, the contactized dynamics is not very interesting. All the contactized objects preserve not only the contact distribution, but the contact form. In local coordinates this means that the objects do not depend on z. Hence, there will be no dissipation and the equations on motion will look just as Hamilton's.

[3] Alternatively, one can also create a contact manifold from an S^1-bundle satisfying certain topological properties [25]. This construction has the advantage of preserving compactness.

8. Tulczyjew triples for contact systems

Lagrangian Dynamics is generated by a Lagrangian function defined on the tangent bundle TQ (or $TQ \times \mathbb{R}$ in the contact case) of the configuration space of a physical system whereas Hamiltonian Dynamics is governed by a Hamiltonian function on the cotangent bundle T^*Q which is canonically symplectic, (or the extended cotangent bundle $T^*Q \times \mathbb{R}$, which is canonically contact). If a Lagrangian function is regular then the Legendre transformation provides an equivalence between the Hamiltonian and Lagrangian formulations of the system.

In the case that the Lagrangian is degenerate, we obtain then the Legendre transformation fails to be a local diffeomorphism, and we are forced to apply the constraint algorithm Section 5.1, obtaining a presymplectic/precontact Hamiltonian system on a submanifold, possibly smaller than the image of the Legendre transformation. A related approach is the Skinner-Rusk unified formalism [86, 233, 234], in which Lagrangian and Hamiltonian formalisms are united on the Whitney sum of tangent and cotangent bundles. In this chapter we will review a different approach, due to Tulczyjew. First, we will review the Tulczyjew triple for symplectic systems, and then we will introduce its contact counterpart.

Tulczyjew's triple is a commutative diagram linking three symplectic bundles namely TT^*Q, T^*T^*Q and T^*TQ via symplectomorphisms. This geometrization enables one to recast Lagrangian and Hamiltonian dynamical equations as Lagrangian submanifolds of the Tulczyjew symplectic space TT^*Q [241] as Lagrangian submanifolds. In this picture, we can understand the Legendre transformation as a change of generating function [240, 242]. This definition is free of non-degeneracy conditions. Evidently, this approach is in harmony with the creed by Weinstein "everything is a Lagrangian submanifold" [255].

Tulczyjew's triple has been generalized for more geometric frameworks: higher order Lagrangian dynamics [76], physical theories where the configuration space is a Lie group [124, 151]. Also, for principal fiber bundles [11] and vector bundles [152]. There has been extensive work on studying Tulzyjew's formalism for field theories [40, 45, 77, 81], including the higher order case [150].

The aim of this chapter to give a construction of the Tulczyjew's triple for the case of contact manifolds. Here, in the contact case, we will change the role of Tulczyjew's symplectic spaces (TT^*Q, T^*T^*Q and T^*TQ) by their extended counterparts ($T(T^*Q\times\mathbb{R})\times\mathbb{R}$, $T^*(T^*Q \times \mathbb{R}) \times \mathbb{R}$ and $T^*(TQ \times \mathbb{R}) \times \mathbb{R}$); the symplectomorphism by contactomorphism, and the Lagrangian by Legendrian submanifolds. This picture enables us to understand Lagrangian and Hamiltonian functions as generating functions of the same Legendrian submanifold, even in the case of singular systems.

Instead of following the presentation of our results from [127] we take a different route. We connect the contact with the symplectic dynamics through symplectization, and we obtain the contact version of Tulczyjew isomorphism α, β in this fashion. We note that on [127] are result on Morse families, the Legendre transformation and the triples for the evolution vector field which are not present in this chapter. These tools are specially interesting for equilibrium thermodynamics systems, since the Hamiltonians usually are not regular and there is no Lagrangian formulation in the usual sense.

This chapter has two sections. After recalling the symplectic Tulczyjew triples in Section 8.1, we introduce contact special submanifolds, show how to symplectize them in order to obtain the contact triples in Section 8.2.2.

8.1. The symplectic Tulczyjew's triple

8.1.1. Special Symplectic manifolds.

The concept of special symplectic manifold is central for Tulczyjew's approach to dynamics. It is just a fiber bundle which is equivalent to the cotangent bundle.

Definition 8.1. A *special symplectic manifold* is a quintuple $(P, \rho, Q, \theta, \Phi)$ where:

- $\rho : P \to Q$ is a fiber bundle.
- (P, θ) is an exact symplectic manifold.
- $\Phi : (P, \theta) \to (T^*Q, \theta_Q)$ is a fiber bundle morphism over Q and a homogeneous symplectomorphism.

A special symplectic manifold can be drawn in a commutative diagram as

$$\begin{tikzcd}
(T^*Q, \theta_Q) \arrow[dr, "\pi_Q"'] & & (P, \theta) \arrow[ll, "\Phi"'] \arrow[dl, "\rho"] \\
& Q &
\end{tikzcd} \tag{8.1}$$

The symplectic diffeomorphism ϕ can be characterized by the following pairing [241]

$$\langle \phi(x), T\pi \circ X_x \rangle = \langle \theta_x, X_x \rangle \tag{8.2}$$

for a vector field X on P, for any point x in P.

8.1.2. The Tulczyjew triples

We now construct the special symplectic manifolds concerning the Hamiltonian and Lagrangian formulations of mechanics.

The Hamiltonian part of the triple was developed in [241].

Theorem 8.1. *Given an exact symplectic manifold (M, θ), the quintuple $(TM, \tau_M, M, \theta_1, \beta)$ is a special symplectic manifold, where*

- *θ_1 is the symplectic potential given by the canonical pairing, that is, for $X \in TTM$, $\theta(X) = \langle \tau_M X, T\tau_M(X)\rangle$. In local coordinates it is given by*

$$\theta_1 = \dot{p}_i \mathrm{d}q^i - \dot{q}^i \mathrm{d}p_i. \tag{8.3}$$

- *The isomorphism $\beta : TM \to T^*M$ is given by contraction with the symplectic form, i.e., $\flat(X) = \iota_X \omega = -\iota_X \mathrm{d}\theta$. In local coordinates, it is given by*

$$\begin{aligned} \beta : TM &\to T^*M, \\ (q^i, p_i, \dot{q}^i, \dot{p}_i) &\mapsto (q^i, p_i, \dot{p}_i, -\dot{q}^i). \end{aligned} \tag{8.4}$$

Moreover, for any $H : M \to \mathbb{R}$, $\operatorname{im} \mathrm{d}H$ *and* $\operatorname{im} X_H$ *are Lagrangian submanifolds which are related by β. This construction is illustrated on the following commutative diagram.*

$$\begin{tikzcd} TM \arrow[rr, "\beta"] \arrow[dr, "\tau_M"] & & T^*M \arrow[dl, "\pi_M"'] \\ & M \arrow[ul, bend left, "X_H"] \arrow[ur, bend right, "\mathrm{d}H"'] & \end{tikzcd} \tag{8.5}$$

The Lagrangian part was also introduced by Tulczyjew in his follow-up article [242].

Theorem 8.2. *The quintuple $(TT^*Q, T\pi_Q, TQ, {\theta_Q}^C, \alpha)$ is a special symplectic manifold.*

Here, the isomorphism α was defined by Tulczyjew in [242] and its coordinate expression is

$$\begin{aligned} \alpha : TT^*Q &\to T^*TQ, \\ (q^i, p_i, \dot{q}^i, \dot{p}_i) &\mapsto (q^i, p_i, \dot{p}_i, \dot{q}^i). \end{aligned} \tag{8.6}$$

Moreover, for any $L : TQ \to \mathbb{R}$, $\operatorname{im} \mathrm{d}L$ *is a Lagrangian submanifold such that the Lagrangian submanifold $N_L = \alpha^{-1}\mathrm{d}L$ is given by the local expression*

$$\dot{p}_i = \frac{\partial L}{\partial q^i}, \quad p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{8.7}$$

which are the Euler-Lagrange equations.

This construction is illustrated on the following commutative diagram.

$$\begin{tikzcd} T^*TQ \arrow[dr, "\pi_{TQ}"] & & TT^*Q \arrow[ll, "\alpha"'] \arrow[dl, "T\pi_Q"'] \\ & TQ \arrow[ul, bend left, "\mathrm{d}L"] & N_L \arrow[u, hook] \end{tikzcd} \tag{8.8}$$

Connecting the Hamiltonian part for $(M, \theta) = (T^*Q, \theta_Q)$ and Lagrangian part, we obtain the Tulczyjew triple

$$\begin{tikzcd} T^*TQ \arrow[dr, "\pi_{T^*Q}"] & & TT^*Q \arrow[ll, "\alpha"'] \arrow[rr, "\beta"] \arrow[dl, "T\pi_Q"'] \arrow[dr, "\tau_{T^*Q}"] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"'] \\ & TQ \arrow[ul, bend left, "\mathrm{d}L"] \arrow[dr, "\tau_Q"] & N_L \arrow[u, hook] & T^*Q \arrow[ul, bend right, "X_H"'] \arrow[ur, bend right, "\mathrm{d}H"'] \arrow[dl, "\pi_Q"'] & \\ & & Q & & \end{tikzcd} \tag{8.9}$$

8.2. Tulczyjew triples for contact systems

8.2.1. Special contact manifolds and its symplectization

Now we will introduce the analog of the Tulczyjew triples for contact systems. We start by defining special contact submanifolds.

Definition 8.2. A *special contact manifold* is a quintuple (P, ρ, Q, η, Φ) where:

- $\rho : P \to Q$ is a fiber bundle.
- (P, η) is a contact manifold.
- $\Phi : P \to T^*Q \times \mathbb{R}$ is a fiber bundle morphism over Q and a contactomorphism.

A special contact manifold can be drawn in a commutative diagram as

$$\begin{tikzcd} (T^*Q \times \mathbb{R}, \eta_Q) \arrow[dr, "\pi_Q^0"'] & & (P, \eta) \arrow[ll, "\Phi"'] \arrow[dl, "\rho"] \\ & Q & \end{tikzcd} \tag{8.10}$$

We will use symplectization in order to obtain the contact geometric counterparts of the Tulczyjew isomorphism α and β. When developing mechanics on a symplectic manifold, the symplectic potential does not play an important role. Instead, the symplectic form is the important piece of geometric information. Nevertheless, for homogeneous symplectic systems, the symplectic potential is crucial. Indeed, on some situations taking θ_Q as the form on Q does not lead to the desired homogeneous structure. We will extend the definition of special symplectic structure in order to take that into account

Definition 8.3. A *special homogeneous symplectic manifold* is a 6-tuple $(P, \rho, Q, \theta, \tilde{\theta}_Q, \Phi)$ where:

- $\rho : P \to Q$ is a fiber bundle.
- (P, θ) is an exact symplectic manifold.
- $\tilde{\theta}_Q$ is a symplectic potential for (T^*Q, ω_Q). That is, $\omega_Q = -\mathrm{d}\tilde{\theta}_Q = -\mathrm{d}\theta_Q$.
- $\Phi : (P, \theta) \to (T^*Q, \tilde{\theta}_Q)$ is a fiber bundle morphism over Q and a homogeneous symplectomorphism.

The corresponding diagram is obtained by changing θ_Q by $\tilde{\theta}_Q$ on the diagram for a special symplectic manifold

$$\begin{tikzcd} (T^*Q, \tilde{\theta}_Q) \arrow[dr, "\pi_Q"'] & & (P, \theta) \arrow[ll, "\Phi"'] \arrow[dl, "\rho"] \\ & Q & \end{tikzcd} \tag{8.11}$$

We remark that $\alpha = \tilde{\theta}_Q - \theta_Q$ is a closed form and that $(P, \rho, Q, \theta, \tilde{\theta}_Q, \Phi)$ is a special homogeneous symplectic manifold if and only if $(P, \rho, Q, \theta - \Phi^*\alpha, \Phi)$ is a special symplectic manifold. Conversely, given a closed 1-form $\alpha \in \Omega(T^*Q)$, $(P, \rho, Q, \theta + \Phi^*\alpha, \tilde{\theta}_Q + \alpha, \Phi)$ is a special homogeneous symplectic manifold.

Now we can define the symplectization of a Tulczyjew triple as follows.

Definition 8.4. A *symplectization of a Tulczyjew triple* is given by the data $(P, \rho, Q, \eta, \Phi, P^\Sigma, \rho^\Sigma, \tilde{Q}, \theta, \tilde{\theta}_{\tilde{Q}}, \tilde{\rho}, \Phi^\Sigma, \tilde{\Sigma}_Q, \Sigma)$ such that

- (P, ρ, Q, η, Φ) is a special contact manifold.
- $(P^\Sigma, \rho^\Sigma, \tilde{Q}, \theta, \tilde{\theta}_{\tilde{Q}}, \Phi^\Sigma)$ is a special homogeneous symplectic manifold.
- $\tilde{\rho} : \tilde{Q} \to Q$ is a fiber bundle with 1-dimensional fibers.
- $\tilde{\Sigma}_Q : (T^*\tilde{Q}, \tilde{\theta}_{\tilde{Q}}) \to (TQ \times \mathbb{R}, \eta_Q)$ is a symplectization and a fiber bundle morphism over $\tilde{\rho} : Q \times \mathbb{R} \to Q$.
- Φ^Σ is the symplectization of Φ.
- $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ is a symplectization and a fiber bundle morphism over $\tilde{\rho} : \tilde{Q} \to Q$.

  This is depicted on the following diagram.

$$
\begin{tikzcd}
(T^*\tilde{Q}, \tilde{\theta}_{\tilde{Q}}) \arrow[dd, "\tilde{\Sigma}_Q"'] \arrow[dr, "\pi_{\tilde{Q}}"] & & (M^\Sigma, \theta) \arrow[ll, "\Phi^\Sigma"'] \arrow[dl, "\rho^s"'] \arrow[dd, "\Sigma"] \\
& \tilde{Q} \arrow[d, "\tilde{\rho}"] & \\
& Q & \\
(T^*Q \times \mathbb{R}, \eta_Q) \arrow[ur, "\pi^0_Q"'] & & (M, \eta) \arrow[ll, "\Phi"] \arrow[ul, "\rho"]
\end{tikzcd}
\tag{8.12}
$$

Part of the information on this definition is redundant. For example, we can use Theorem 7.9 in order to obtain Φ^Σ from Φ or vice-versa. Indeed, we will obtain the Tulczyjew isomorphisms for the contact systems in this fashion.

8.2.2. The contact Tulczyjew Triples

First we start with the Hamiltonian part of the triple:

Theorem 8.3. *Let $\Sigma : (M^\Sigma, \theta) \to (M, \eta)$ be a symplectization, and let Δ be the Liouville vector field of M^Σ. Then, the following diagram represents a symplectization of a Tulczyjew triple*

$$
\begin{tikzcd}
(TM^\Sigma, \theta^C) \arrow[rr, "\beta"] \arrow[dr, "\tau_{M^\Sigma}"] \arrow[ddd, "\Sigma^T"'] & & (T^*M^\Sigma, \theta_\Delta) \arrow[dl, "\pi_{M^\Sigma}"'] \arrow[ddd, "\Sigma^{T^*}"] \\
& (M^\Sigma, \theta) \arrow[ul, bend left, "X_{H^\Sigma}"] \arrow[ur, bend right, "\mathrm{d}H^\Sigma"'] \arrow[d, "\Sigma"] & \\
& (M, \eta) \arrow[dl, bend right, "\hat{X}_H"'] \arrow[dr, bend left, "j^1 H"] & \\
(TM \times \mathbb{R}, \eta^T) \arrow[ur, "\tau^0_M"'] \arrow[rr, "\beta_c"'] & & (T^*M \times \mathbb{R}, \eta_M) \arrow[ul, "\pi^0_M"]
\end{tikzcd}
\tag{8.13}
$$

where the corresponding forms and symplectizations are defined in Theorems 7.16 and 7.18, and $\hat{X}_H = (X_H, \mathcal{R}(H))$. The map β_c is given explicitly by

$$
\begin{aligned}
\beta_c : TM \times \mathbb{R} &\to T^*M \times \mathbb{R}, \\
(x, \dot{x}, t) &\mapsto (x, t\eta + \iota_{\dot{x}_x} \mathrm{d}\eta, -\eta(\dot{x}_x)).
\end{aligned}
\tag{8.14}
$$

Moreover, the differentials, one jets and Hamiltonian vector fields are related as depicted on the diagram.

Proof. We need to proof that the map β is still a homogeneous symplectomorphism with this new choice of forms. If this holds, the commutativity of the diagram then follows from Theorems 7.16 and 7.18.

In order to do that, we let $\beta^*\theta_\Delta = \beta^*(\theta_Q) - \mathrm{d}\beta^*\hat{\Delta} = \theta_1 - \mathrm{d}\beta^*\hat{\Delta}$.

We take a Darboux chart (q^i, p_i) for M^Σ, and in the bundle charts induced on its tangent and cotangent bundles we have

$$
\beta^*\hat{\Delta} = \beta^*(p_i \mathrm{d}p_{p_i}) = -p_i \dot{q}^i. \tag{8.15}
$$

Hence,

$$
\beta^*\theta_\Delta = \dot{p}_i \mathrm{d}q^i - \dot{q}^i \mathrm{d}p_i + \mathrm{d}(p_i \dot{q}^i) = \dot{p}_i \mathrm{d}q^i + p_i \dot{q}^i = \theta^c_Q \tag{8.16}
$$

Thus β is a symplectomorphism.

Now, we only need to compute β_c using the commutativity of the diagram. Let (x^i) be a chart for M, and we denote by $(\hat{x}^i = \Sigma^* x^i, \sigma)$ the induced coordinates on M^Σ. In those coordinates,

$$
\Sigma^T(\hat{x}^i, \sigma, \dot{\hat{x}}^i, \dot{\sigma}) = (x^i, \dot{x}^i, \dot{\sigma}/\sigma). \tag{8.17}
$$

Thus, if $y = (x^i, \dot{x}^i, t)$ is a point of $TM \times \mathbb{R}$, the point $\hat{y} = (\hat{x}^i, 1, \dot{\hat{x}}^i, t) \in TM^\Sigma$ satisfies

$\Sigma_Q^T(\hat{y}) = y$. Now, we compute

$$\begin{aligned}
\beta(\hat{y}) &= \iota_{(\dot{x},t)_{(\hat{x},1)}} \mathrm{d}\omega \\
&= -\iota_{(\dot{x},t)_{(\hat{x},1)}} \mathrm{d}(\sigma \Sigma^* \eta) \\
&= -\iota_{(\dot{x},t)_{(\hat{x},1)}} (\mathrm{d}\sigma \wedge \Sigma^* \eta - \sigma \mathrm{d}\Sigma^* \mathrm{d}\eta) \\
&= -t\Sigma^* \eta_x + \Sigma^* \eta(\dot{x}_x)\mathrm{d}\sigma - \Sigma^*(\iota_{\dot{x}_x} \mathrm{d}\eta) \\
&= (-t\Sigma^* \eta_x - \Sigma^*(\iota_{\dot{x}_x} \mathrm{d}\eta), \Sigma^* \eta(\dot{x}_x))).
\end{aligned} \tag{8.18}$$

Finally, we project $\beta(\hat{y})$ onto $T^*M \times \mathbb{R}$ using $\Sigma_Q^{T^*}$, and β_c is computed using the commutativity of the diagram

$$\begin{aligned}
\beta_c(y) &= \beta_c(\Sigma^T(\hat{y})) = \Sigma^{T^*}(\beta(\hat{y})) \\
&= \Sigma^{T^*}(-t\Sigma^* \eta_x - \Sigma^*(\iota_{\dot{x}_x} \mathrm{d}\eta, \Sigma^* \eta(\dot{x}_x)) \\
&= (x, t\eta + \iota_{\dot{x}_x} \mathrm{d}\eta, -\eta(\dot{x}_x)).
\end{aligned} \tag{8.19}$$

□

As a corollary of this construction, we obtain the analog of Theorem 8.1.

Corollary 8.4. *Given a contact manifold (M, η), the quintuple $(TM \times \mathbb{R}, \tau_M^0, M, \eta^T, \beta_c)$ is a special contact manifold. Moreover, for any $H : M \to \mathbb{R}$, $\operatorname{im} j^1 H$ and $\operatorname{im} X_H$ are Legendrian submanifolds which are related by β_c. This construction is illustrated on the following commutative diagram.*

$$\begin{tikzcd}
TM \times \mathbb{R} \arrow[rr, "\beta"] \arrow[dr, "\tau_M^0"] & & T^*M \times \mathbb{R} \arrow[dl, "\pi_M"'] \\
& M \arrow[ul, bend left, "X_H"] \arrow[ur, bend right, "j^1 H"'] &
\end{tikzcd} \tag{8.20}$$

Now we turn to the Lagrangian part of the triple.

Theorem 8.5. *Then, the following diagram represents a symplectization of a Tulczyjew triple,*

$$\begin{tikzcd}
T^*T(Q \times \mathbb{R}) \arrow[dr, "\pi_{T(Q\times\mathbb{R})}"] \arrow[ddd, "\Sigma_{TQ\times\mathbb{R}}"'] & & TT^*(Q \times \mathbb{R}) \arrow[ll, "\alpha"'] \arrow[dl, "T\pi_{Q\times\mathbb{R}}"'] \arrow[ddd, "\Sigma_Q^T"] \\
& T(Q \times \mathbb{R}) \arrow[d, "(\mathrm{pr}_{TQ}, z)"] & \\
& TQ \times \mathbb{R} & \\
T^*(TQ \times \mathbb{R}) \times \mathbb{R} \arrow[ur, "\pi^0_{TQ\times\mathbb{R}}"'] & & T(T^*Q \times \mathbb{R}) \times \mathbb{R} \arrow[ll, "\alpha_c"] \arrow[ul, "(T\pi_Q^0, z)"]
\end{tikzcd} \tag{8.21}$$

where the map

$$\begin{aligned}\Sigma_{TQ\times\mathbb{R}} : T^*T(Q\times\mathbb{R}) \simeq T^*((TQ\times\mathbb{R})\times\mathbb{R}) &\to T^*(TQ\times\mathbb{R})\times\mathbb{R}\\ (q,z,\dot q,\dot z,p_q,p_z,p_{\dot q},p_{\dot z}) &\mapsto \left(q,\dot q,z,-\frac{p_q}{p_{\dot z}},-\frac{p_{\dot q}}{p_{\dot z}},-\frac{p_z}{p_{\dot z}},\dot z\right)\end{aligned} \tag{8.22}$$

is the symplectization in Theorem 7.15 and

$$\begin{aligned}\Sigma_Q^T : TT^*(Q\times\mathbb{R}) &\to T(T^*Q\times\mathbb{R}),\times\mathbb{R}\\ (q,z,p,p_z,\dot q,\dot z,\dot p,\dot p_z) &\mapsto \left(q,z,-\frac{p}{p_z},\dot q,\dot z,-\frac{\dot p}{p_z}+\frac{\dot p_z p}{p_z^2},\frac{\dot p_z}{p_z}\right),\end{aligned} \tag{8.23}$$

is the tangent lift (Theorem 7.18) of the symplectization of Theorem 7.15.

The map α_c is given by

$$\begin{aligned}\alpha_c : T^*(TQ\times\mathbb{R})\times\mathbb{R} &\to T(T^*Q\times\mathbb{R})\times\mathbb{R}\\ (q,p,z,\dot q,\dot p,\dot z,u) &\to (q,z,\dot q,\dot p+up,p,-u,\dot z).\end{aligned} \tag{8.24}$$

Proof. By the choice of the maps, it is clear that this is a symplectization of a Tulczyjew triple. We only need to find α_c. We let $y=(q,p,z,\dot q,\dot p,\dot z,u)\in T(T^*Q\times\mathbb{R})\times\mathbb{R}$. Then, $\hat y=(q,z,p,-1,\dot q,\dot z,\dot p+up,-u)$ satisfies $\Sigma_Q^T(\hat y)=y$. Now we compute its image through α using Equation (8.6).

$$\alpha(\hat y)=\alpha(q,z,p,-1,\dot q,\dot z,\dot p+up,-u)=(q,z,\dot q,\dot z,\dot p+up,-u,p,-1). \tag{8.25}$$

Thus, we obtain

$$\alpha_c(y)=\Sigma_{TQ\times\mathbb{R}}(\alpha(\hat y))=\Sigma(q,z,\dot q,\dot z,\dot p+up,-u,p,-1)=(q,z,\dot q,\dot p+up,p,-u,\dot z), \tag{8.26}$$

which is the map we obtained in [92]. □

Using this map, we construct the Lagrangian part of the triples.

Theorem 8.6. *The quintuple $(T(T^*Q\times\mathbb{R})\times\mathbb{R},(T\pi_Q^0,z),TQ\times\mathbb{R},{\eta_Q}^T,\alpha_c)$ is a special contact manifold.*

Moreover, for any $L : TQ\times\mathbb{R}\to\mathbb{R}$, $\operatorname{im}\mathrm{d}L$ is a Lagrangian submanifold such that the Lagrangian submanifold $N_L=\alpha_{-1}j^1L$ is given by the local expression

$$\begin{aligned}p_i &= \frac{\partial L}{\partial q^i},\\ \dot p_i &= \frac{\partial L}{\partial q^i}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot q^i},\\ \dot z &= L,\\ u &= -\frac{\partial L}{\partial z},\end{aligned} \tag{8.27}$$

which are the Herglotz equations.

This construction is illustrated on the following commutative diagram.

$$
\begin{tikzcd}
T^*(TQ\times\mathbb{R})\times\mathbb{R} \arrow[dr, "\pi_{TQ\times\mathbb{R}}"] & & T(T^*Q\times\mathbb{R})\times\mathbb{R} \arrow[ll, "\alpha_c"'] \arrow[dl, "T\pi_Q"'] \\
& TQ\times\mathbb{R} \arrow[ul, bend left, "j^1L"] & N_L \arrow[u, hook]
\end{tikzcd}
\tag{8.28}
$$

The following diagram shows the symplectization of the triples.

$$
\begin{tikzcd}
T^*T(Q\times\mathbb{R}) \arrow[dr, "\pi_{T(Q\times\mathbb{R})}"] \arrow[ddd] & & TT^*(Q\times\mathbb{R}) \arrow[ll, "\alpha"'] \arrow[rr, "\beta"] \arrow[dl, "T\pi_{Q\times\mathbb{R}}"'] \arrow[dr, "\tau_{T^*(Q\times\mathbb{R})}"] \arrow[ddd, "\Sigma_Q^T"'] & & T^*T^*(Q\times\mathbb{R}) \arrow[dl, "\pi_{T^*(Q\times\mathbb{R})}"'] \arrow[ddd, "\Sigma_Q^{T^*}"] \\
& T(Q\times\mathbb{R}) \arrow[d] & & T^*(Q\times\mathbb{R}) \arrow[d, "\Sigma_Q"] & \\
& TQ\times\mathbb{R} & & T^*Q\times\mathbb{R} & \\
T^*(TQ\times\mathbb{R})\times\mathbb{R} \arrow[ur, "\pi^0_{TQ\times\mathbb{R}}"] & & T(T^*Q\times\mathbb{R})\times\mathbb{R} \arrow[ll, "\alpha_c"] \arrow[rr, "\beta_c"'] \arrow[ul, "(T\pi_Q{,}z)"'] \arrow[ur, "\tau^0_{T^*Q\times\mathbb{R}}"] & & T^*(T^*Q\times\mathbb{R})\times\mathbb{R} \arrow[ul, "\pi^0_{T^*Q\times\mathbb{R}}"']
\end{tikzcd}
\tag{8.29}
$$

The complete contact version of the triples is depicted bellow.

$$
\begin{tikzcd}
T^*(TQ\times\mathbb{R})\times\mathbb{R} \arrow[dr, "\pi^0_{TQ\times\mathbb{R}}"] & & T(T^*Q\times\mathbb{R})\times\mathbb{R} \arrow[ll, "\alpha_c"'] \arrow[rr, "\beta_c"] \arrow[dl, "(T\pi^0_Q{,}z)"'] \arrow[dr, "\tau^0_{T^*Q\times\mathbb{R}}"] & & T^*(T^*Q\times\mathbb{R})\times\mathbb{R} \arrow[dl, "\pi^0_{T^*Q\times\mathbb{R}}"'] \\
& TQ\times\mathbb{R} \arrow[ul, bend left, "\mathrm{d}L"] \arrow[dr, "\tau_0"] & N_L \arrow[u, hook] & T^*Q\times\mathbb{R} \arrow[ul, bend left, "\hat{X}_H"] \arrow[ur, bend right, "j^1H"'] \arrow[dl, "\pi_0"'] & \\
& & Q & &
\end{tikzcd}
\tag{8.30}
$$

9. Hamilton–Jacobi theory

The Hamilton–Jacobi equation [2, 13, 144] is an alternative formulation of classical mechanics, equivalent to other formulations such as Hamiltonian mechanics, and Lagrangian mechanics for regular systems. It is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely.

The Hamilton–Jacobi equation has been extensively studied in the case of symplectic Hamiltonian systems, more specifically, for Hamiltonian functions H defined in the cotangent bundle T^*Q of the configuration space Q.

The Hamilton–Jacobi problem consists in finding a function $S : Q \longrightarrow \mathbb{R}$ (called the *generating function*) such that

$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E, \tag{9.1}$$

for some $E \in \mathbb{R}$. The above equation (9.1) is called the Hamilton–Jacobi equation for H. Of course, one can easily see that (9.1) can be written as follows

$$\mathrm{d}(H \circ dS) = 0, \tag{9.2}$$

which opens the possibility to consider general 1-forms on Q (considered as sections of the cotangent bundle $\pi_Q : T^*Q \to Q$), instead of just differentials of a function.

Recently, the observation that given such a section $\gamma : Q \longrightarrow T^*Q$ permits to relate the Hamiltonian vector field X_H with its projection X_H^γ via γ onto Q, in the sense that X_H^γ and X_H are γ-related if and only if (9.2) holds, provided that γ is closed (or, equivalently, its image is a Lagrangian submanifold of (T^*Q, ω_Q)) has opened the possibility to discuss the Hamilton–Jacobi problem in many other scenarios, such as nonholonomic systems [49, 50, 97, 165, 209], singular Lagrangian systems [83, 98, 184], higher-order systems [68], and field theories [39, 41, 105, 110, 113, 251, 259]. A unifying Hamilton-Jacobi theory for almost-Poisson manifolds was developed in reference [112]. The Hamilton-Jacobi theory has also been generalized to Hamiltonian systems with non-canonical symplectic structures [199], non-Hamiltonian systems [218] locally conformally symplectic manifolds [126], Nambu-Poisson and Nambu-Jacobi manifolds [111], Lie algebroids [185] and implicit differential systems [125], mechanical systems with external forces [91], as well as more general dynamical systems [51]. The applications of Hamilton-Jacobi theory include the relation between classical and quantum mechanics [38, 52, 56, 194], information geometry [65, 66], control theory [224] and the study of phase transitions [176].

In this section, we study the Hamilton–Jacobi theory for the contact Hamiltonian vector field and the evolution vector field. Moreover, we will consider sections of π_Q^0 :

$T^*Q \times \mathbb{R} \to Q$, what we call *approach I*, and also sections of $\pi_Q^1 : TQ \times \mathbb{R} \to Q \times \mathbb{R}$ (*approach II*).

We notice that the Hamilton–Jacobi problem has been treated by other authors [42, 166], who establish a relationship between the Herglotz variational principle and the Hamilton–Jacobi equation, although their interests are analytical rather than geometrical.

This chapter is composed by three sections. In Section 9.1, we introduce the Hamilton–Jacobi equation for both vector fields and both approaches. The relationship of this problem with its symplectic counterpart is treated in Section 9.2. An example is given in Section 9.3. One more example is given in Section 10.1.2, related to thermodynamics.

The results on this chapter were published in [92, 109]. Nonetheless, in this dissertation we introduce the concept of *pseudolegendrian submanifold*, which generalizes the properties of the submanifolds of $TQ \times \mathbb{R}$ that appeared on the *approach II* to Hamilton–Jacobi theory to arbitrary contact manifolds, and give a simpler definition from the geometric point of view.

9.1. The Hamilton–Jacobi equations

9.1.1. The Hamilton–Jacobi equations for a Hamiltonian vector field

Approach I

Let $(T^*Q \times \mathbb{R}, \eta_Q, H)$ be a Hamiltonian system on the extended cotangent bundle. We will consider a section γ of the canonical projection $\pi_Q^0 : T^*Q \times \mathbb{R} \to Q$. In local coordinates it reads

$$\begin{aligned} \gamma : Q &\to T^*Q \times \mathbb{R}, \\ q &\mapsto (q, \gamma_i(q), \gamma^z(q)). \end{aligned} \tag{9.3}$$

We can use γ to project X_H onto Q by defining

$$X_H^\gamma = T\pi_Q^0 \circ X_H \circ \gamma, \tag{9.4}$$

as it is depicted on the following diagram

$$\begin{tikzcd} T^*Q \times \mathbb{R} \arrow[r, "X_H"] \arrow[d, "\pi_Q^0"'] & T(T^*Q \times \mathbb{R}) \arrow[d, "T\pi_Q^0"'] \\ Q \arrow[u, bend left, "\gamma"] \arrow[r, "X_H^\gamma"] & TQ \arrow[u, bend right, dashed, "T\gamma"'] \end{tikzcd} \tag{9.5}$$

But this diagram does not necessarily commute. Indeed, X_H and X_H^γ are not necessarily γ-related. That is,

$$X_H \circ \gamma = T\gamma \circ X_H^\gamma \tag{9.6}$$

does not necessarily hold. The local expression of X_H^γ is given by

$$X_H^\gamma = \left(\frac{\partial H}{\partial p_i} \circ \gamma \right) \frac{\partial}{\partial q^i}. \tag{9.7}$$

Thus, Equation (9.6) in local coordinates is given by

$$-\left(\gamma_i \frac{\partial H}{\partial z} + \frac{\partial H}{\partial q_i}\right) = \frac{\partial H}{\partial p_j} \frac{\partial \gamma_i}{\partial q^j}, \tag{9.8a}$$

$$\gamma_i \frac{\partial H}{\partial p_i} - H = \frac{\partial H}{\partial p_i} \frac{\partial \gamma^z}{\partial q^i}, \tag{9.8b}$$

where we have abused notation and the Hamiltonian and its derivatives are actually evaluated at $\gamma(q)$.

By Proposition 2.17, $\operatorname{im}\gamma$ is a Legendrian submanifold if and only if it is the 1-jet of a function, namely $\gamma = j^1\gamma^z$. Performing the substitution

$$\gamma_i = \frac{\partial \gamma^z}{\partial q^i}, \tag{9.9}$$

we can see that Equation (9.8) transform into

$$\mathrm{d}(H \circ \gamma) = 0, \tag{9.10a}$$

$$H \circ \gamma = 0. \tag{9.10b}$$

Obviously, the first one is redundant. Hence, we have proved the following.

Theorem 9.1. *Let γ be a section of $TQ \times \mathbb{R} \to Q$ such that* $\operatorname{im}\gamma$ *is a Legendrian submanifold of* $(T^*Q \times \mathbb{R}, \eta_Q)$.

Then, X_H^γ and X_H are γ-related if and only if the Hamilton–Jacobi equation *holds:*

$$H \circ \gamma = 0. \tag{9.11}$$

Under the conditions of the above theorem, we say that γ is a *solution of the Hamilton–Jacobi problem*. In this case, we can compute integral curves of X_H by computing those of X_H^γ. Indeed, since both vector fields are γ-related, given $c : [0, T] \to Q$, we have that $\gamma \circ c$ is an integral curve of X_H if and only if c is an integral curve of X_H^γ.

Approach II

Instead of taking a section γ of $\pi_Q^0 : TQ \times \mathbb{R} \to Q$, we can take a section $\bar{\gamma}$ of the bundle $\pi_Q^1 : T^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ and repeat the computations on the previous case. The section $\bar{\gamma}$ takes the local expression

$$\begin{aligned} \bar{\gamma} : Q \times \mathbb{R} &\to T^*Q \times \mathbb{R}, \\ (q, z) &\mapsto (q, \gamma(q, z), z). \end{aligned} \tag{9.12}$$

First, we will give a couple of definitions. Given section $\alpha : Q \times \mathbb{R} \to \Lambda^k(Q) \times \mathbb{R}$, (which we call a *z-dependent k-form*) and $z \in \mathbb{R}$, we let

$$\begin{aligned} \alpha_z : Q &\to \Lambda^k(Q) \\ q &\to \operatorname{pr}_{\Lambda^k(Q)}(\alpha(q, z)), \end{aligned} \tag{9.13}$$

where $\mathrm{pr}_{\Lambda^k(Q)} : \Lambda^k(Q) \times \mathbb{R} \to \Lambda^k(Q)$ is the canonical projection.

Furthermore, we define the *exterior derivative at fixed* z, which is a section of $\Lambda^{k+1}(Q) \times \mathbb{R} \to Q \times \mathbb{R}$ given by

$$\mathrm{d}_Q \alpha(q, z) = (\mathrm{d}\alpha_z(q), z). \tag{9.14}$$

In local coordinates, for $f : Q \times \mathbb{R} \to \mathbb{R}$ and $\alpha : Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$, the local expressions are

$$\begin{aligned} \mathrm{d}_Q f &= \frac{\partial f}{\partial q^i} dq^i, \\ \mathrm{d}_Q(\alpha_i dq^i) &= \frac{\partial \alpha_j}{\partial q^i} dq^i \wedge dq^j. \end{aligned} \tag{9.15}$$

Again, we project X_H through $\bar{\gamma}$ and define

$$X_H^{\bar{\gamma}} = T_\pi \circ X_H \circ \bar{\gamma}. \tag{9.16}$$

The corresponding diagram is now

$$\begin{tikzcd} T^*Q \times \mathbb{R} \arrow[r, "X_H"] \arrow[d, shift left, "\pi_Q^1"] & T(T^*Q \times \mathbb{R}) \arrow[d, shift right, "T\pi_Q^1"'] \\ Q \times \mathbb{R} \arrow[r, "X_H^{\bar{\gamma}}"] \arrow[u, bend left, "\bar{\gamma}"] & T(Q \times \mathbb{R}) \arrow[u, dashed, bend right, "T\bar{\gamma}"'] \end{tikzcd} \tag{9.17}$$

Note that $\mathrm{im}(\gamma)$ is $(n+1)$-dimensional, so it no longer makes sense to require it to be Legendrian. Demanding it to be coisotropic is not enough to obtain a satisfactory Hamilton–Jacobi equation. Our submanifold will need to have the following properties.

Definition 9.1. We say that an $(n+1)$-dimensional submanifold $i : N \hookrightarrow M$ of a $(2n+1)$-dimensional contact manifold (M, η) is *quasilegendrian* if and only if

1. N is coisotropic.
2. $i^*\mathrm{d}\eta = 0$.

We make a small detour and analyze the properties of quasilegendrian submanifolds.

Proposition 9.2. *Assume that an $(n+1)$-dimensional submanifold N of a $(2n+1)$-dimensional contact submanifold (M, η) is locally the zero set of the constraint functions $\{\phi_a\}_{a=1}^n$. Then, N is quasilegendrian if and only if the following equations hold in Darboux coordinates*

$$\frac{\partial \phi_a}{\partial q^i} \frac{\partial \phi_b}{\partial p_i} = \frac{\partial \phi_b}{\partial q^i} \frac{\partial \phi_a}{\partial p_i}, \tag{9.18a}$$

$$p_i \frac{\partial \phi_a}{\partial z} \frac{\partial \phi_b}{\partial p_i} = p_i \frac{\partial \phi_b}{\partial z} \frac{\partial \phi_a}{\partial p_i}. \tag{9.18b}$$

Proof. Equation (9.18b) is obtained by requiring that $i^* \mathrm{d}\eta = 0$. In order to obtain Equation (9.18a), we use the equations for a coisotropic submanifold (Equation (2.83)) and simplify them using Equation (9.18b). $\square$

For our case, we let $N = \operatorname{im} \bar{\gamma}$, which is locally defined by the constraints $\phi_i = p_i - \gamma_i$. Applying the previous result we obtain the following:

Corollary 9.3. *Let $\bar{\gamma}$ be a section of $T^*Q \times \mathbb{R}$ over $Q \times \mathbb{R}$. Then $\operatorname{im} \bar{\gamma}$ is a quasilegendrian submanifold if and only if*

$$\frac{\partial \gamma_i}{\partial q^j} = \frac{\partial \gamma_j}{\partial q^i}, \tag{9.19a}$$

$$\gamma_j \frac{\partial \gamma_i}{\partial z} = \gamma_i \frac{\partial \gamma_j}{\partial z}. \tag{9.19b}$$

A coordinate-free characterization of when $\operatorname{im} \bar{\gamma}$ is coisotropic can be given as follows.

Theorem 9.4. *Let $\bar{\gamma}$ be a section of $T^*Q \times \mathbb{R}$ over $Q \times \mathbb{R}$. Then, $\operatorname{im} \bar{\gamma}$ is a quasilegendrian submanifold if and only if $\mathrm{d}_Q \bar{\gamma} = 0$ and $\mathcal{L}_{\partial/\partial z} \bar{\gamma} = a\bar{\gamma}$ for some function $a : Q \times \mathbb{R} \to \mathbb{R}$. That is, there exists locally a function $f : Q \times \mathbb{R} \to \mathbb{R}$ such that $\mathrm{d}_Q f = \bar{\gamma}$ and $\mathrm{d}_Q \frac{\partial f}{\partial z} = a \mathrm{d}_Q f$.*

Proof. $\operatorname{im} \bar{\gamma}$ is coisotropic if and only if $\bar{\gamma}$ fulfills Equation (9.19).

Equation (9.19a) can be written as

$$\mathrm{d}_Q \bar{\gamma} = 0. \tag{9.20}$$

We can also write Equation (9.19b) as

$$\bar{\gamma} \wedge \mathcal{L}_{\partial/\partial z} \bar{\gamma} = 0, \tag{9.21}$$

or, equivalently, that $\bar{\gamma}$ and $\mathcal{L}_{\partial/\partial z} \bar{\gamma}$ are linearly dependent.

Locally, by the Poincaré Lemma, we obtain that $\bar{\gamma}_z$ is exact for every $z \in \mathbb{R}$, thus $\bar{\gamma} = \mathrm{d}_Q f$. The other condition is just $\mathrm{d}_Q \frac{\partial f}{\partial z} = a \mathrm{d}_Q f$. $\square$

We continue deriving the Hamilton–Jacobi equation. X_H and X_H^γ are γ-related, that is,

$$X_H \circ \bar{\gamma} = T\bar{\gamma}(X_H^{\bar{\gamma}}), \tag{9.22}$$

if and only if the following local expression holds:

$$\frac{\partial H}{\partial q^j} + \frac{\partial \gamma_j}{\partial q^i} \frac{\partial H}{\partial p_i} + \gamma_j \frac{\partial H}{\partial z} + \gamma_i \frac{\partial \gamma_j}{\partial z} \frac{\partial H}{\partial p_i} - H \frac{\partial \gamma_j}{\partial z} = 0. \tag{9.23}$$

Assuming that $\operatorname{im} \bar{\gamma}$ is quasilegendrian and using Equation (9.19), we obtain the following expression.

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i} \frac{\partial \gamma_i}{\partial q^j} + \gamma_j \left(\frac{\partial H}{\partial z} + \frac{\partial H}{\partial p_i} \frac{\partial \gamma_i}{\partial z} \right) - H \frac{\partial \gamma_j}{\partial z} = 0. \tag{9.24}$$

This can be written globally, though the resulting equation is more involved than on the case of sections of $TQ \times \mathbb{R} \to Q$.

Theorem 9.5. *Let $\bar{\gamma}$ be a section of $\pi^1_Q : TQ \times \mathbb{R} \to Q \times \mathbb{R}$ such that* $\operatorname{im}\bar{\gamma}$ *is a quasilegendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$.*

Then, $X_H^{\bar{\gamma}}$ and X_H are $\bar{\gamma}$-related if and only if the Hamilton–Jacobi equation *holds:*

$$\mathrm{d}_Q(H \circ \bar{\gamma}) + \mathcal{L}_{\frac{\partial}{\partial z}}(H \circ \bar{\gamma})\bar{\gamma} = (H \circ \bar{\gamma})\mathcal{L}_{\frac{\partial}{\partial z}}\bar{\gamma}. \tag{9.25}$$

Whenever $\bar{\gamma}$ satisfies the conditions of the above theorem, we say that it is a *solution of the Hamilton–Jacobi problem*.

Remark 9.6. Notice that if $\bar{\gamma}$ is a solution of the Hamilton–Jacobi problem for H, then X_H is tangent to the coisotropic submanifold $\bar{\gamma}$, but not necesarily to the Lagrangian submanifolds $\operatorname{im}\bar{\gamma}$, for $z \in \mathbb{R}$. This occurs when

$$X_H(z - z_0) = 0 \tag{9.26}$$

for any z_0, that is, if and only if

$$H \circ \bar{\gamma}_{z_0} = \gamma_i \frac{\partial H}{\partial p_i} \tag{9.27}$$

In such a case, we call $\bar{\gamma}$ a *strong solution* of the Hamilton–Jacobi problem.

Complete solutions

Next, we shall discuss the notion of complete solutions of the Hamilton–Jacobi problem for a Hamiltonian H in approach II.

Definition 9.2. A *complete solution* of the Hamilton–Jacobi equation for a Hamiltonian H is a diffeomorphism $\Phi : Q \times \mathbb{R} \times \mathbb{R}^n \to T^*Q \times \mathbb{R}$ such that for any set of parameters $\lambda \in \mathbb{R}^n, \lambda = (\lambda_1, \dots, \lambda_n)$, the map

$$\begin{aligned} \Phi_\lambda : Q \times \mathbb{R} &\to T^*Q \times \mathbb{R} \\ (q^i, z) &\mapsto \Phi_\lambda(q^i, z) = \Phi(q^i, z, \lambda) \end{aligned} \tag{9.28}$$

is a solution of the Hamilton–Jacobi equation. If, in addition, any Φ_λ is strong, then the complete solution is called a strong complete solution.

We have the following diagram

$$\begin{array}{ccc} Q \times \mathbb{R} \times \mathbb{R}^n & \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} & T^*Q \times \mathbb{R} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f_i} \\ \mathbb{R}^n & \xrightarrow{\quad \pi_i \quad} & \mathbb{R} \end{array}$$

where we define functions f_i such that for a point $p \in T^*Q \times \mathbb{R}$, it is satisfied

$$f_i(p) = \pi_i \circ \alpha \circ \Phi^{-1}(p), \tag{9.29}$$

and $\alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection.

The first result is that

$$\operatorname{im} \Phi_\lambda = \cap_{i=1}^{n} f_i^{-1}(\lambda_i),$$

where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$. In other words,

$$\operatorname{im} \Phi_\lambda = \{x \in T^*Q \times \mathbb{R} \mid f_i(x) = \lambda_i, i = 1, \dots, n\}.$$

Therefore, since X_H is tangent to any of the submanifolds $\operatorname{im} \Phi_\lambda$, we deduce that

$$X_H(f_i) = 0.$$

Hence, these functions are conserved quantities.

Moreover, we can compute

$$\{f_i, f_j\} = \Lambda(df_i, df_j) - f_i \mathcal{R}(f_j) + f_j \mathcal{R}(f_i).$$

But

$$\Lambda(df_i, df_j) = \sharp_\Lambda(df_i)(f_j) = 0$$

since $(T \operatorname{im} \Phi_\lambda)^\perp = \sharp_\Lambda((T \operatorname{im} \Phi_\lambda)^o) \subset T \operatorname{im} \Phi_\lambda$, so

$$\{f_i, f_j\} = -f_i \mathcal{R}(f_j) + f_j \mathcal{R}(f_i) \tag{9.30}$$

Theorem 9.7. *There exists no linearly independent commuting set of first-integrals in involution* (9.29) *for a complete strong solution of the Hamilton–Jacobi equation.*

Proof. If all the particular solutions are strong, then the Reeb vector field $\mathcal{R}$ will be transverse to the coisotropic submanifold $\Phi_\lambda(Q \times \mathbb{R})$. Indeed, if $\mathcal{R}$ is tangent to that submanifold, we would have

$$\mathcal{R}(p_i - (\Phi_\lambda)_i) = -\frac{\partial (\Phi_\lambda)_i}{\partial z},$$

where $\Phi_\lambda(q^i, z) = (q^i, (\Phi_\lambda)_i, z)$. So, Φ_λ does not depend on z, hence it cannot be a diffeomorphism.

Therefore, if the brackets $\{f_i, f_j\}$ vanish, then we would obtain that the functions f_i cannot be linearly independent. Indeed, we should have

$$f_i \mathcal{R}(f_j) = f_j \mathcal{R}(f_i),$$

for all i, j. But this would imply that f_i and f_j are linearly dependent in the case $\lambda = (0, \dots, 0)$.

□

9.1.2. The Hamilton–Jacobi equations for the evolution vector field

In this section we will repeat the computations of the Hamilton–Jacobi equation, but this time we will use the evolution vector field, $\mathcal{E}_H$ instead of the Hamiltonian vector field X_H, of a contact Hamiltonian system (M, η_Q, H).

Approach I

We use the notation from Section 9.1.1. Let γ be a section of the canonical projection $\pi_Q^0 : T^*Q \times \mathbb{R} \to Q$

This time we project vector field $\mathcal{E}_H$, obtaining

$$\mathcal{E}_H^\gamma = T\pi_Q^1 \circ \mathcal{E}_H \circ \gamma. \tag{9.31}$$

Both vector fields are γ-related, that is

$$\mathcal{E}_H \circ \gamma = T\gamma \circ \mathcal{E}_H^\gamma, \tag{9.32}$$

if and only if

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial q^i} + \gamma_j \frac{\partial H}{\partial z} = 0, \tag{9.33a}$$

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial \gamma_z}{\partial q^i} - \gamma_i\right) = 0. \tag{9.33b}$$

If we assume that $\gamma = j^1\gamma^z$, (or, equivalently, $\operatorname{im}\gamma$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$), then Equation (9.33b) is automatically satisfied and (9.33a) just states that $H \circ \gamma$ is closed. Therefore, we have the following.

Theorem 9.8. *Let γ be a section of $TQ \times \mathbb{R} \to Q$ such that* $\operatorname{im}\gamma$ *is a Legendrian submanifold of* $(T^*Q \times \mathbb{R}, \eta_Q)$.

Then, $\mathcal{E}_H^\gamma$ and $\mathcal{E}_H$ are γ-related if and only if the Hamilton–Jacobi equation for the evolution vector field *holds:*

$$\mathrm{d}(H \circ \gamma) = 0. \tag{9.34}$$

A section γ fulfilling the assumptions of the theorem and the Hamilton–Jacobi equation will be called a *solution of the Hamilton–Jacobi problem for the evolution vector field*.

Approach II

As in Section 9.1.1, we take a section $\bar{\gamma}$ of the bundle $\pi_Q^1 : T^*Q \times \mathbb{R} \to Q \times \mathbb{R}$. As usual, we define the projected evolution vector field

$$\mathcal{E}_H^{\bar{\gamma}} = T\pi_Q^1 \circ \mathcal{E}_H \circ \bar{\gamma}.$$

We have that those vector fields are $\bar{\gamma}$-related, that is $\mathcal{E}_H \circ \gamma = T\gamma(\mathcal{E}_H^\gamma)$ if and only if

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial q^i} + \gamma_i \frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial z} + \gamma_j \frac{\partial H}{\partial z} = 0 \tag{9.35}$$

Assuming that $\operatorname{im}\bar{\gamma}$ is quasilegendrian and using Equation (9.19), we can arrive to the following global expression.

Theorem 9.9. *Let $\bar{\gamma}$ be a section of $TQ \times \mathbb{R} \to Q \times \mathbb{R}$ such that* $\operatorname{im}\bar{\gamma}$ *is a quasilegendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$.*

Then, $\mathcal{E}_H^{\bar{\gamma}}$ and $\mathcal{E}_H$ are $\bar{\gamma}$-related if and only if the Hamilton–Jacobi equation *holds:*

$$\mathrm{d}_Q(H \circ \bar{\gamma}) + (H \circ \bar{\gamma})\mathcal{L}_{\frac{\partial}{\partial z}}\bar{\gamma} + \mathcal{L}_{\frac{\partial}{\partial z}}(H \circ \bar{\gamma})\bar{\gamma} = 0. \tag{9.36}$$

Whenever the conditions of the above theorem hold, we say that $\bar{\gamma}$ is a *solution of the Hamilton–Jacobi problem for the evolution vector field*.

Complete solutions

As in the case of the Hamiltonian vector field, we can consider complete solutions for the evolution vector field.

Definition 9.3. A *complete solution* of the Hamilton–Jacobi equation for the evolution vector field $\mathcal{E}_H$ of a Hamiltonian H on a contact manifold (M, η) is a diffeomorphism $\Phi : Q \times \mathbb{R} \times \mathbb{R}^n \to T^*Q \times \mathbb{R}$ such that for any set of parameters $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R} \times \mathbb{R}^n$, the mapping

$$\begin{array}{rcc} \Phi_\lambda : Q & \to & T^*Q \times \mathbb{R} \\ (q^i) & \mapsto & \Phi_\lambda(q^i) = \Phi(q^i, \lambda_0, \lambda_1, \dots, \lambda_n) \end{array} \tag{9.37}$$

is a solution of the Hamilton–Jacobi equation.

For simplicity, we will use the notation $(\lambda_\alpha\ ,\ \alpha = 0, 1, \dots, n)$.

As in the previous case, we define functions f_α such that for a point $p \in T^*Q \times \mathbb{R}$, it is satisfied

$$f_\alpha(p) = \pi_\alpha \circ \Phi^{-1}(p), \tag{9.38}$$

where $\pi_\alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the canonical projection onto the α factor.

A direct computation shows that

$$\operatorname{im}\Phi_\lambda = \cap_{\alpha=0}^n f_\alpha^{-1}(\lambda_\alpha).$$

In other words,

$$\operatorname{im}\Phi_\lambda = \{x \in T^*Q \times \mathbb{R} \mid f_\alpha(x) = \lambda_\alpha, \alpha = 0, \dots, n\}$$

Therefore, since under our hypothesis, $\mathcal{E}_H$ is tangent to any of the submanifolds $\operatorname{im}\Phi_\lambda$, we deduce that

$$\mathcal{E}_H(f_\alpha) = 0.$$

Hence, these functions are conserved quantities for the evolution vector field.

Moreover, we can compute

$$\{f_\alpha, f_\beta\} = \Lambda(df_\alpha, df_\beta) - f_\alpha \mathcal{R}(f_\beta) + f_\beta \mathcal{R}(f_\alpha).$$

But

$$\Lambda(df_\alpha, df_\beta) = \sharp_\Lambda(df_\alpha)(f_\beta) = 0,$$

since $(T\operatorname{im}\Phi_\lambda)^\perp = T\operatorname{im}\Phi_\lambda$, so

$$\{f_\alpha, f_\beta\} = -f_\alpha \mathcal{R}(f_\beta) + f_\beta \mathcal{R}(f_\alpha). \tag{9.39}$$

Theorem 9.10. *There exist no linearly independent commuting set of first-integrals in involution* (9.38) *for a complete solution of the Hamilton–Jacobi equation for the evolution vector field.*

Proof. Since the images of the sections are Legendrian then they are integral submanifolds of $\ker \eta_Q$. So, the Reeb vector field $\mathcal{R}$ will be transverse to them, and consequently, there is at least some index α_0 such that

$$\mathcal{R}(f_{\alpha_0}) \neq 0$$

Therefore, if all the brackets $\{f_\alpha, f_\beta\}$ vanish, then we would obtain that the functions f_α cannot be linearly independent.

□

9.2. Symplectization of the Hamilton–Jacobi equation

We will try to understand the relationship of the Hamilton–Jacobi problem for the contact Hamiltonian system $(TQ \times \mathbb{R}, \eta_Q, H)$ and the Hamilton–Jacobi problem for the symplectization of this system $(T(Q \times \mathbb{R}), \omega_Q, H^\Sigma)$. For this, we will use the result and notation of Chapter 7, specially, those of Section 7.2.1.

Now we will establish a relationship between solutions to the Hamilton–Jacobi problem in both scenarios. Suppose that

$$\begin{aligned} \hat{\hat{\gamma}}\colon Q \times \mathbb{R} &\to T^*(Q \times \mathbb{R}) \\ (q^i, z) &\mapsto (q^i, \hat{\hat{\gamma}}_j(q^i, z), z, \hat{\hat{\gamma}}_t(q^i, z)) \end{aligned}$$

is a solution of the symplectic Hamilton–Jacobi equation, i.e., $\hat{\hat{\gamma}}(Q \times \mathbb{R})$ is Lagrangian and

$$d(\hat{H} \circ \hat{\hat{\gamma}}) = 0,$$

or, equivalently,

$$T\hat{\hat{\gamma}} \circ X_{\hat{H}}^{\hat{\hat{\gamma}}} = X_{\hat{H}} \circ \hat{\hat{\gamma}},$$

where $X_{\hat{H}}^{\hat{\hat{\gamma}}} = Tp \circ X_{\hat{H}} \circ \hat{\hat{\gamma}}$ is the projected vector field and $p\colon T^*(Q \times \mathbb{R}) \to Q \times \mathbb{R}$ the canonical projection. We want to use the solution $\hat{\hat{\gamma}}$ of the Hamilton–Jacobi problem in the symplectization (which we will often refer to as "symplectic solution") to obtain a section that is a solution in the contact setting ("contact solution", for simplicity).

9.2.1. Approach I

For each z, we have sections $\gamma = (T\pi_Q, \dot{z}) \circ \hat{\bar{\gamma}}^z : Q \to T^*Q \times \mathbb{R}$ of the form $(q^i) \mapsto (q^i, \hat{\bar{\gamma}}_j(q^i, z), \hat{\bar{\gamma}}_t(q^i, z))$, being $(T\pi_Q, \dot{z}) : (q^i, p_i, z, t) \mapsto (q^i, p_i, t)$. We know that γ is a solution of the contact Hamilton–Jacobi problem if and only if $\gamma(Q)$ is Legendrian and

$$H \circ \gamma = 0.$$

The condition that $\gamma(Q)$ is Legendrian is equivalent to

$$\gamma_i = \frac{\partial \gamma_z}{\partial q^i},$$

where we write $\gamma(q^i) = (q^i, \gamma_j(q^i), \gamma_z(q^i))$, which by definition of γ and using that $\gamma(Q \times \mathbb{R})$ is Lagrangian reads

$$\hat{\bar{\gamma}}_i = \frac{\partial \hat{\bar{\gamma}}_t}{\partial q^i} = \frac{\partial \hat{\bar{\gamma}}_i}{\partial z},$$

and therefore $\hat{\bar{\gamma}}_i = e^z g(q^i)$, with g_i functions depending only on the (q^i). This can be summarized as follows:

Theorem 9.11. *Suppose $\hat{\bar{\gamma}} \colon Q \times \mathbb{R} \to T^*(Q \times \mathbb{R})$ is a solution of the symplectized Hamilton–Jacobi problem. Then,*

$$\begin{aligned} \gamma \colon Q &\to T^*Q \times \mathbb{R} \\ (q^i) &\mapsto (q^i, \hat{\bar{\gamma}}_j(q^i, z), \hat{\bar{\gamma}}_t(q^i, z)) \end{aligned}$$

is a solution of the contact Hamilton–Jacobi problem if and only if $H \circ \gamma = 0$ and $\hat{\bar{\gamma}}_i = e^z g_i$.

9.2.2. Approach II

In this approach we can relate the contact and the symplectic solution using the symplectization on Theorem 7.15. We will use the map Σ^Q defined in Equation (7.70). That is,

$$\begin{aligned} \Sigma^Q : T^*(Q \times \mathbb{R}) &\to T^*Q \times \mathbb{R}, \\ (q, z, p, p_z) &\mapsto (q, z, -p/p_z). \end{aligned} \tag{9.40}$$

We also denote by $\hat{H} = H^\Sigma = p_z (\Sigma_Q)^* H$ the symplectized Hamiltonian

$$\begin{aligned} \hat{H} : T^*(Q \times \mathbb{R}) &\to \mathbb{R}, \\ (q, z, p, p_z) &\to p_z H(q, -p/p_z, z). \end{aligned} \tag{9.41}$$

We assume $\hat{\bar{\gamma}}_t(q^i, z) \neq 0$ and take $\bar{\gamma} = \Sigma^Q \circ \hat{\bar{\gamma}} \colon Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$. In local coordinates

$$\begin{aligned} \bar{\gamma} \colon Q \times \mathbb{R} &\to T^*Q \times \mathbb{R} \\ (q^i, z) &\mapsto \left(q^i, \gamma_j(q^i, z) = -\frac{\hat{\bar{\gamma}}_j(q^i, z)}{\hat{\bar{\gamma}}_t(q^i, z)}, z \right) \end{aligned}$$

We can summarize the situation in the following commutative diagram:

$$
\begin{tikzcd}
Q \times \mathbb{R} \arrow[r, "\hat{\bar{\gamma}}"] \arrow[rr, bend left=50, "\bar{\gamma}"] \arrow[d, "X_{\hat{H}}^{\hat{\bar{\gamma}}} = X_H^{\bar{\gamma}}"'] & T^*(Q \times \mathbb{R}) \arrow[r, "\Sigma^Q"] \arrow[l, bend left, "\pi_{Q\times\mathbb{R}}"] \arrow[d, "X_{\hat{H}}"] & T^*Q \times \mathbb{R} \arrow[d, "X_H"] \arrow[ll, bend right=30, "\pi^1_Q"'] \\
T(Q \times \mathbb{R}) \arrow[r, dashed, "T\hat{\bar{\gamma}}"] \arrow[rr, dashed, bend right=50, "T\bar{\gamma}"'] & T(T^*(Q \times \mathbb{R})) \arrow[r, "T\Sigma^Q"] \arrow[l, bend right, "T\pi_{Q\times\mathbb{R}}"'] & T(T^*Q \times \mathbb{R}) \arrow[ll, bend left=30, "T\pi^1_Q"]
\end{tikzcd}
\tag{9.42}
$$

We note that the projected vector fields $X_{\hat{H}}^{\hat{\bar{\gamma}}}$ and $X_H^{\bar{\gamma}}$ coincide. The dashed lines of $T\hat{\bar{\gamma}}$ (resp. $T\bar{\gamma}$) commute if and only if $\hat{\bar{\gamma}}$ is a symplectic solution (resp. $\bar{\gamma}$ is a contact solution) of the Hamilton–Jacobi problem.

Lemma 9.12. *Let H be a Hamiltonian and $\hat{H}$ its symplectized version. Assume $\hat{\bar{\gamma}}_t(q^i, z) \neq 0$. Then, $X_{\hat{H}}^{\hat{\bar{\gamma}}}$ and $X_{\hat{H}}$ are $\hat{\bar{\gamma}}$-related if and only if $X_H^{\bar{\gamma}}$ and X_H are $\bar{\gamma}$-related.*

Proof. Assume that $X_{\hat{H}}^{\hat{\bar{\gamma}}}$ and $X_{\hat{H}}$ are $\hat{\bar{\gamma}}$-related. Then, by the commutativity of the diagram (9.42) we see that $X_H^{\bar{\gamma}}$ and X_H are $\bar{\gamma}$-related.

Conversely, assume that $X_H^{\bar{\gamma}}$ and X_H are $\bar{\gamma}$-related. Let $P_z \in \mathbb{R} \setminus \{0\}$ and let

$$
\begin{aligned}
\xi : T^*Q \times \mathbb{R} &\to T^*(Q \times \mathbb{R}), \\
(q^i, P_i, z) &\mapsto (q^i, z, -P_i \hat{\bar{\gamma}}_t(q^i, z)).
\end{aligned}
\tag{9.43}
$$

We note that ξ_P is the inverse of Σ^Q along the submanifold $\{P_z = \hat{\bar{\gamma}}_t\} \subseteq T^*(Q \times \mathbb{R})$. In particular $\hat{\bar{\gamma}} = \xi \circ \bar{\gamma}$. Looking at the diagram (9.42), this implies that $X_{\hat{H}}^{\hat{\bar{\gamma}}}$ and $X_{\hat{H}}$ are $\hat{\bar{\gamma}}$-related. □

Lemma 9.13. *Assume that the image of $\hat{\bar{\gamma}} = (\hat{\bar{\gamma}}_Q, \hat{\bar{\gamma}}_t)$ is Lagrangian. Then, the image of $\bar{\gamma}$ is quasilegendrian if and only if $\mathrm{d}_Q \tilde{\gamma}_Q = \tau \gamma_Q$ for some function $\tau : Q \times \mathbb{R} \to \mathbb{R}$.*

Conversely, if the image of $\bar{\gamma}$ is quasilegendrian, then we can choose $\tilde{\gamma}_t$ so that the image of $\hat{\bar{\gamma}}$ is coisotropic. Indeed, it is given by either $\hat{\bar{\gamma}}_t = \pm \exp(g)$, where g is a solution to the PDE

$$
\mathrm{d}_Q g + \bar{\gamma} \mathcal{L}_{\partial/\partial z} g = -\mathcal{L}_{\partial/\partial z} \bar{\gamma}. \tag{9.44}
$$

Proof. Let $\hat{\bar{\gamma}} = (\hat{\bar{\gamma}}_Q, \hat{\bar{\gamma}}_t)$ be such that its image is Lagrangian. That is, $\mathrm{d}\hat{\bar{\gamma}} = 0$. Splitting the part in Q and in $\mathbb{R}$, we see that this is equivalent to

$$
\mathcal{L}_{\partial/\partial z} \hat{\bar{\gamma}}_Q = \mathrm{d}_Q \hat{\bar{\gamma}}_t, \quad \mathrm{d}_Q \hat{\bar{\gamma}}_Q = 0. \tag{9.45}
$$

Now, $\bar{\gamma} = -\hat{\bar{\gamma}}_Q/\hat{\bar{\gamma}}_t$. By Theorem 9.4, it is necessary that $\mathrm{d}_Q\bar{\gamma} = 0$ and $(\mathcal{L}_{\partial/\partial z}\bar{\gamma}) \wedge \bar{\gamma} = 0$. We compute

$$\mathrm{d}_Q\bar{\gamma} = \frac{(\mathrm{d}_Q\hat{\bar{\gamma}}_t) \wedge \hat{\bar{\gamma}}_Q}{\hat{\bar{\gamma}}_t^2} = \frac{(\mathcal{L}_{\partial/\partial z}\hat{\bar{\gamma}}_Q) \wedge \hat{\bar{\gamma}}_Q}{\hat{\bar{\gamma}}_t^2}. \tag{9.46}$$

But

$$\begin{aligned}(\mathcal{L}_{\partial/\partial z}\bar{\gamma}) \wedge \bar{\gamma} &= -\frac{(\mathcal{L}_{\partial/\partial z}\hat{\bar{\gamma}}_t)\hat{\bar{\gamma}}_Q - (\mathcal{L}_{\partial/\partial z}\hat{\bar{\gamma}}_Q)\hat{\bar{\gamma}}_t}{\hat{\bar{\gamma}}_t^2} \wedge \frac{\tilde{\gamma}_Q}{\hat{\bar{\gamma}}_t} \\ &= \frac{(\mathcal{L}_{\partial/\partial z}\hat{\bar{\gamma}}_Q) \wedge \hat{\bar{\gamma}}_Q}{\hat{\bar{\gamma}}_t^2}.\end{aligned} \tag{9.47}$$

Hence, $\operatorname{im}\bar{\gamma}$ is pseudolegendrian if and only if $\mathcal{L}_{\partial/\partial z}(\hat{\bar{\gamma}}_Q)$ is proportional to $\hat{\bar{\gamma}}_Q$.

Conversely, assume that $\bar{\gamma}$ satisfies $\mathrm{d}_Q\bar{\gamma} = 0$ and $\mathcal{L}_{\partial/\partial z}\bar{\gamma} = a\bar{\gamma}$. We must find $\hat{\bar{\gamma}}_t$ so that (9.45) are satisfied. Since $\hat{\bar{\gamma}}_Q = -\hat{\bar{\gamma}}_t\bar{\gamma}$, we have that (9.45) are equivalent to

$$\mathcal{L}_{\partial/\partial z}(\hat{\bar{\gamma}}_Q) = -(\mathcal{L}_{\partial/\partial z}\hat{\bar{\gamma}}_t + a\hat{\bar{\gamma}}_t)\bar{\gamma} = \mathrm{d}_Q\hat{\bar{\gamma}}_t \tag{9.48}$$

$$\mathrm{d}_Q(\hat{\bar{\gamma}}_Q) = -\mathrm{d}_Q\hat{\bar{\gamma}}_t \wedge \hat{\gamma} = 0. \tag{9.49}$$

A solution for $\hat{\bar{\gamma}}_t$ on the first equation above clearly solves the second one. Since we look for nonvanishing $\hat{\bar{\gamma}}_t$, we let $g = \log \circ |\hat{\bar{\gamma}}_t|$, so that (9.48) is just

$$\mathrm{d}_Q g + \hat{\gamma}\mathcal{L}_{\partial/\partial z} g = -a\hat{\gamma} = -\mathcal{L}_{\partial/\partial z}\hat{\gamma}. \tag{9.50}$$

If we let

$$A_i = \frac{\partial}{\partial q^i} + \gamma^i \frac{\partial}{\partial z}, \tag{9.51}$$

this equation can be written as

$$A_i(g) = -\frac{\partial \gamma_i}{\partial z}, \tag{9.52}$$

and we note that this vector fields commute, indeed

$$[A_i, A_j] = \gamma_i \frac{\partial \gamma_j}{\partial z} - \gamma_j \frac{\partial \gamma_i}{\partial z} = a\gamma_i\gamma_j - a\gamma_j\gamma_i = 0. \tag{9.53}$$

If this PDE has local solutions, operating with the equations above, one has

$$A_i\left(\frac{\partial \gamma_j}{\partial z}\right) - A_j\left(\frac{\partial \gamma_i}{\partial z}\right) = 0.$$

This condition is clearly necessary, and it is also sufficient by [183, Thm. 19.27]. We have that

$$A_i\left(\frac{\partial \gamma_j}{\partial z}\right) - A_j\left(\frac{\partial \gamma_i}{\partial z}\right) = \gamma_i \frac{\partial^2 \gamma_j}{(\partial z)^2} - \gamma_j \frac{\partial^2 \gamma_i}{(\partial z)^2} = \gamma_i \frac{\partial (a\gamma_j)}{\partial z} - \gamma_j \frac{\partial (a\gamma_i)}{\partial z} = 0. \tag{9.54}$$

□

Combining the last two results, we obtain a correspondence between symplectic and contact solutions to the Hamilton–Jacobi problem.

Theorem 9.14. *Let H be a Hamiltonian and $\hat{H}$ its symplectized version. Then, $\hat{\bar{\gamma}}\colon Q \times \mathbb{R} \to T^*(Q \times \mathbb{R})$ is a solution of the symplectic Hamilton–Jacobi problem for $\hat{H}$, if and only if $\hat{\gamma} = \Sigma^Q \circ \hat{\bar{\gamma}}\colon Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$ is a solution of the contact Hamilton–Jacobi problem for H and $\mathrm{d}_Q \hat{\bar{\gamma}}_Q = \tau \gamma_Q$ for some function $\tau : Q \times \mathbb{R} \to \mathbb{R}$.*

Conversely, given a contact solution $\hat{\gamma}$ of the Hamilton–Jacobi equation, there exists a symplectic solutions $\hat{\bar{\gamma}}_t$ such that $|\hat{\bar{\gamma}}| = \exp(g)$, where g is a solution to the PDE

$$\mathrm{d}_Q g + \gamma \mathscr{L}_{\partial/\partial z} g = -\mathscr{L}_{\partial/\partial z} \gamma. \tag{9.55}$$

9.3. Example: Particle with linear dissipation

Consider the Hamiltonian H

$$H(q,p,z) = \frac{p^2}{2m} + V(q) + \lambda z, \tag{9.56}$$

where $\lambda \in \mathbb{R}$ is a constant. The extended phase space is $T^*Q \times \mathbb{R} \simeq \mathbb{R}^3$.

The Hamiltonian and evolution vector field are given by

$$X_H = \frac{p}{m}\frac{\partial}{\partial q} - \left(\frac{\partial V}{\partial q} + \lambda z\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m} - V(q) - \lambda z\right)\frac{\partial}{\partial z}, \tag{9.57}$$

$$\mathcal{E}_H = \frac{p}{m}\frac{\partial}{\partial q} - \left(\frac{\partial V}{\partial q} + \lambda z\right)\frac{\partial}{\partial p} + \frac{p^2}{m}\frac{\partial}{\partial z}. \tag{9.58}$$

Assume that $\gamma : Q \to T^*Q \times \mathbb{R}$ is a section of the canonical projection $T^*Q \times \mathbb{R} \to Q$, that is,

$$\gamma(q) = (q, \gamma_p(q), \gamma_z(q)). \tag{9.59}$$

We assume that $\gamma(Q)$ is a Legendrian submanifold of $T^*Q \times \mathbb{R}$ as in Section 9.1.2; then,

$$\gamma_p(q) = \frac{d\gamma_z}{dq}, \tag{9.60}$$

and $\mathcal{E}_H$ and $\mathcal{E}_H^\gamma$ are γ-related if and only if

$$H \circ \gamma = k, \tag{9.61}$$

for a constant $k \in \mathbb{R}$. Then, the Hamilton–Jacobi equation becomes

$$H(\gamma(q)) = \frac{\gamma_p^2}{2m} + V(q) + \lambda \gamma_z = k, \tag{9.62}$$

or, equivalently,

$$\frac{1}{2m}\left(\frac{d\gamma_z}{dq}\right)^2 + V(q) + \lambda \gamma_z = k, \tag{9.63}$$

which is a non-linear ordinary differential equation.

A general solution of the Hamilton–Jacobi equation (9.63) is then

$$\gamma_p(q) = \exp(-2m\lambda q) \int (2mk - 2mV(q)) \exp(2m\lambda q) dq. \tag{9.64}$$

Part II.

Applications

10. Thermodynamics

In this chapter we study the relationship between contact geometry and thermodynamics.

Section 10.1 deals with the so-called classical or equilibrium thermodynamics, and its relationship with contact geometry and contact Hamiltonian systems has been acknowledged for the last decades [203]. We add and example that we used in [92]. Another example is present in Section 11.3. Also, an application of Tulczyjew's triples to thermodynamics, including a Lagrangian formalism, can be found on [127].

In Section 10.2 we show how some non-equilibrium systems can be studied through the evolution vector field. These are novel results developed in [7, 10].

10.1. Equilibrium thermodynamics

According to Arnold [12], "Every mathematician knows it is impossible to understand an elementary course in thermodynamics. The reason is that thermodynamics is based—as Gibbs has explicitly proclaimed—on a rather complicated mathematical theory, on the contact geometry. Contact geometry is one of the few 'simple geometries' of the so-called Cartan's list, but it is still mostly unknown to the physicist."

Nevertheless, thermodynamics has been studied extensively in the framework of contact geometry during the last decades. For some recent work directly related with the present discussions, we cite [32, 36, 141, 203, 215].

The relation between symplectic and contact manifolds via the symplectization procedure has permitted to go deeper in the geometric description of thermodynamic systems. This way has been explored in [14].

10.1.1. Thermodynamic systems and contact geometry

Equilibrium thermodynamics deals with processes in systems in which thermal effects are taken into account. The aim is not to describe the time evolution of the system, but to analyze which of these processes are possible, and how to go from one state to another in the most efficient way.

The state of the system can be described by a finite number of variables (indeed, the *thermodynamic phase space* will be a finite-dimensional manifold M). Those variables are typically denoted (U, T, S, P_i, V^i), where U is the *internal energy*, T is the *temperature*, S is the *entropy*, P_i are the *generalized pressures* and V^i are the *generalized volumes*. This last two quantities, in the case of gases are physical pressures and volumes, but, for example, in the case of magnets, they are the magnetic field and the magnetization. These variables are divided in *extensive* (U, S, V^i), and *intensive variables* (T, P_i). The

Thermodynamics	Contact geometry
Thermodynamic phase space	Manifold with a contact distribution $(M, \mathcal{H})$
Equilibrium states	Legendrian submanifold $\mathbb{L} \subseteq (M, \mathcal{H})$
Thermodynamic system	Triple $(M, \mathcal{H}, \mathbb{L})$
Process	isotropic curve $c : I \to (M, \mathcal{H})$
Quasistatic process	Curve $c : I \to \mathbb{L}$

Table 10.1.: Correspondence between some thermodynamic and geometric concepts.

first ones are proportional to the quantity of matter in the system and the last ones are independent on it.

A *thermodynamic process* is just a curve $c : I = [t_0, t_1] \to M$ (here, the variable t is not the physical time, but just an evolution parameter). According to the laws of thermodynamics, along this curve it must be satisfied

$$\mathrm{d}U - T\mathrm{d}S - P_i \mathrm{d}V^i = 0. \tag{10.1}$$

That is, it must be tangent to the contact distribution $\mathcal{H} = \ker(\eta_U)$, where η_U is the so-called *Gibbs one-form*,

$$\eta_U = \mathrm{d}U - T\mathrm{d}S - P_i \mathrm{d}V^i. \tag{10.2}$$

In other words, $\gamma(I) \subseteq (M, \mathcal{H})$ is an isotropic submanifold (see Section 2.4).

The Gibbs form η_U is written in *energy representation*, but we can choose another contact form that generates the same contact distribution. For example, we can obtain the *entropy representation* by taking

$$\eta_S = \frac{1}{T}\eta_U = \mathrm{d}S - \beta \mathrm{d}U - p_i \mathrm{d}V, \tag{10.3}$$

where $\beta = 1/T$ is the inverse temperature and $p_i = P_i/T$.

When unperturbed, a thermodynamic system always lies on the equilibrium submanifold $\mathbb{L} \subseteq M$, which is described by the *equations of state*. Since equilibrium processes fulfill the laws of thermodynamics, they are forced to be integral submanifolds of the contact distribution. Indeed, $\mathbb{L}$ is a Legendrian submanifold. The triple $(M, \mathcal{H}, \mathbb{L})$ is a *thermodynamic system*. A process $c : I \to \mathbb{L}$ that is contained on the Legendrian submanifold (it only passes through equilibrium processes) is called a *quasistatic process*.

On a thermodynamic system $(M, \mathcal{H}, \mathbb{L})$, one can consider the dynamics generated by a Hamiltonian vector field X_H associated to a Hamiltonian H. If this dynamics represents quasistatic processes, that is, its evolution states remain in the submanifold $\mathbb{L}$, it is required for the contact Hamiltonian vector field XHh to be tangent to $\mathbb{L}$. This happens if and only if H vanishes on $\mathbb{L}$. On Table 10.1 we summarize the correspondence between some thermodynamic and geometric concepts.

Equivalently, using symplectization $\Sigma : (M^\Sigma, \theta) \to (M, \mathcal{H})$ (Section 7.2), one can consider the *extended thermodynamic phase space*

$$\omega_{Q\times\mathbb{R}} = \mathrm{d}P^i \wedge \mathrm{d}P_i + \mathrm{d}S \wedge \mathrm{d}P_S + P_U \mathrm{d}U \tag{10.4}$$

In this formulation a thermodynamic system is a triple $(M^\sigma, \theta, \mathcal{L})$, where $\mathcal{L}$ is a homogeneous Lagrangian submanifold. Dynamics are given by a homogeneous Hamiltonian K. Indeed, in [14, 247], they use the projective cotangent bundle $\mathbb{P}(T^*Q \times \mathbb{R})$ as the thermodynamic phase space and the cotangent bundle $T^*(Q \times \mathbb{R})$ as the extended thermodynamic phase.

10.1.2. Hamilton-Jacobi: the classical ideal gas

This example is fully described in [141]. In [92] we applied the Hamilton-Jacobi theory to it.

The open classical ideal gas is described by the following variables.

- U: internal energy,
- T: temperature,
- S: entropy,
- P: pressure,
- V: volume,
- μ: chemical potential,
- N: mole number.

Thus, the thermodynamic phase space is $T^*\mathbb{R}^3 \times \mathbb{R}$ and the contact 1-form is

$$\eta = dU - TdS + PdV - \mu dN \tag{10.5}$$

The Hamiltonian function is

$$H = TS - RNT + \mu N - U, \tag{10.6}$$

where R is the constant of ideal gases. The Reeb vector field is $\mathcal{R} = \frac{\partial}{\partial U}$.

The Hamiltonian and evolution vector fields are just

$$X_H = (S - RN)\frac{\partial}{\partial S} + N\frac{\partial}{\partial N} + P\frac{\partial}{\partial P} + RT\frac{\partial}{\partial \mu} + U\frac{\partial}{\partial U} \tag{10.7}$$

$$\mathcal{E}_H = (S - RN)\frac{\partial}{\partial S} + N\frac{\partial}{\partial N} + P\frac{\partial}{\partial P} + RT\frac{\partial}{\partial \mu} + (TS - RNT + \mu N)\frac{\partial}{\partial U}. \tag{10.8}$$

The Hamiltonian vector field here represents an isochoric and isothermal process on the ideal gas.

Assume that $\gamma : \mathbb{R}^3 \to T^*\mathbb{R}^3 \times \mathbb{R}$ is the section locally given by

$$\gamma(S, V, N) = (S, V, N, \gamma_T, \gamma_P, \gamma_\mu \gamma_U). \tag{10.9}$$

we know that $\gamma(\mathbb{R}^3)$ is a Legendrian submanifold of $(T^*\mathbb{R}^3 \times \mathbb{R}, \eta)$ if and only if,

$$\begin{aligned}\gamma_T &= \frac{\partial \gamma_U}{\partial S},\\ \gamma_P &= -\frac{\partial \gamma_U}{\partial V},\\ \gamma_\mu &= \frac{\partial \gamma_U}{\partial N}.\end{aligned}$$

The Hamilton–Jacobi equation is

$$(H \circ \gamma)(S, V, N) = (S - RN)\gamma_T + N\gamma_\mu - \gamma_U = k, \tag{10.10}$$

for some $k \in \mathbb{R}$. That is,

$$(H \circ \gamma)(S, V, N) = (S - RN)\frac{\partial \gamma_U}{\partial S} + N\frac{\partial \gamma_U}{\partial N} - \gamma_U = k. \tag{10.11}$$

This is a first order linear PDE, whose solution is given by

$$\gamma_U(S, V, N) = k \operatorname{arcsinh}\left(\frac{S}{\sqrt{-S^2 + (-7N + S)^2}}\right) + F(-S^2 + (RN - S)^2, V), \tag{10.12}$$

with $F : \mathbb{R}^2 \to \mathbb{R}$ an arbitrary function. The case $k = 0$, which is the one relevant for the thermodynamic interpretation, is given by

$$\gamma_U(S, V, N) = F(-S^2 + (RN - S)^2, V). \tag{10.13}$$

Using Hamilton–Jacobi theory, one sees that a section σ satisfied $H \circ \sigma = 0$ if and only if X_H^σ and X_H are σ-related.

10.2. Non-equilibrium thermodynamics of simple systems

There are several nonequivalent definitions of what a non-equilibrium thermodynamic system is in the literature. We will be dealing with systems which can be described by a finite number of variables[1] (we will suppose that phase space is a finite dimensional manifold). We are interested in describing how the state of the system changes with time. Its evolution will be given by the flow of a vector field satisfying the laws of thermodynamics. A thermodynamic process will be an integral curve of this vector field. The aim of this section is to describe how we can use the evolution vector field (Section 2.5) to model systems satisfying these properties.

[1] There are also situations in which the thermodynamic variables, such as the temperature and the pressure are not uniform along the system. This would require a field theory, which is outside the scope of this work.

We will restrict ourselves to *simple systems*[2]. That is, thermodynamic systems whose configuration space is composed by just one scalar thermal variable (in our case the entropy) and a finite set of mechanical variables (position and momenta). We will assume that the system is *isolated*, that is, there is not any transfer of work, matter or heat.

The thermodynamic phase space is naturally equipped with two linearly independent one forms: the work $\delta \mathcal{W}$ and the heat $\delta \mathcal{Q}$ one-forms. The *energy* of the system is given by an energy function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$. For a *closed* system (one that does not exchange matter, but may exchange energy), the first law can be written as follows. Along any process, χ,

$$dH = \delta \mathcal{Q} - \delta \mathcal{W}. \tag{10.14}$$

Since our system is simple, the form $\delta \mathcal{Q}$ needs to have rank one. Therefore, it can be written as

$$\delta \mathcal{Q} = T \mathrm{d} S, \tag{10.15}$$

for some functions T and S (which will be the temperature and the entropy). Furthermore, $\delta \mathcal{W}$ can be written locally as follows

$$\delta \mathcal{W} = P_i \mathrm{d} q^i, \tag{10.16}$$

which as many functions P_i and q^i as the rank of $\delta \mathcal{W}$. Moreover, q^i and S are functionally independent. In the physical interpretation, P_i is the pressure.

Thus, the first law of thermodynamics in this setting states that along χ, the following is satisfied

$$\mathrm{d} H = T dS - P_i dq^i. \tag{10.17}$$

From this, by contracting with $\partial / \partial S$, we obtain the relationship

$$T = \frac{\partial H}{\partial S}. \tag{10.18}$$

Furthermore, for an isolated system, the energy must be constant along χ. Hence, we must have the relationship

$$0 = T dS - P_i dq^i, \tag{10.19}$$

or, dividing by the temperature, and identifying

$$p_i = P_i / T, \tag{10.20}$$

we obtain

$$\eta_Q = dS - p_i dq^i = 0 \tag{10.21}$$

Hence, χ satisfies the first law of thermodynamics for an isolated system if and only if the energy is constant along χ and χ is tangent to $\ker \eta_Q$.

We now assume that χ is an integral curve of the evolution vector field $\mathcal{E}_H$. By Proposition 2.23 we can extract the following conclusion.

[2]In [10] we proposed a model for non-simple systems, but it is outside the realm of contact geometry.

Proposition 10.1. *The integral curves of $\mathcal{E}_H$ describe an isolated system, that is*

$$\frac{dH}{dt} = 0.$$

Moreover, the time evolution of the entropy is locally given by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = p_i \frac{\mathrm{d}q^i}{\mathrm{d}t},$$

which is exactly the first law of thermodynamics with $p_i = P_i/T$.

Remark 10.2. Note that the first law of thermodynamics for an isolated system may be geometrically written as a tangency condition, that is,

$$\iota_{\mathcal{E}_H}\eta = 0. \tag{10.22}$$

The second law of thermodynamics follows from the expression of the evolution vector fields (2.98), or (3.38a) for the Lagrangian formalism, and it depends on the choice of Hamiltonian function.

Proposition 10.3. *The integral curves of $\mathcal{E}_H$ (respectively Ξ_L) satisfy the Second law of thermodynamics, that is,*

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq 0 \tag{10.23}$$

if and only if $\Delta_Q(H) \geq 0$ (respectively, $\Delta(L) \geq 0$), where

$$\Delta_Q = p_i \frac{\partial}{\partial p_i}, \quad \Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i} \tag{10.24}$$

are the Liouville vector fields of the tangent and cotangent bundle, respectively.

Remark 10.4. The variational principle satisfied by the evolution vector field (Theorem 6.11) might have some physical significance. Indeed, it states that the integral curves extremize the entropy with the nonholonomic constraint given by the first law of thermodynamics (tangency to the contact distribution).

10.2.1. An example

Let the Hamiltonian H be given by

$$H(q^i, p_i, S) = \frac{1}{2} g^{ij} p_i p_j + V(q, S) \tag{10.25}$$

where (g^{ij}) is a symmetric bilinear tensor on Q. Note that all[3] the integral curves this system satisfies the second law of thermodynamics if and only if

$$\Delta_Q(H) = 2 g^{ij} p_i p_j \geq 0, \tag{10.26}$$

that is, if g_{ij} is a positive semi-definite metric.

[3] if the metric is pseudo-Riemannian, the light-like and space-like curves still satisfy the second law of thermodynamics.

Example 10.1 (Linearly damped system)**.** Consider a linearly damped system [7] described by coordinates (q, p, S), where q represents the position, p the momentum of the particle and S is the entropy of the surrounding thermal bath. We assume that the system is subjected to a viscous friction force, proportional to the minus velocity of the particle. The system is described by the Hamiltonian

$$H(q,p,S) = \frac{p^2}{2m} + V(q) + \gamma S, \quad \gamma > 0 \tag{10.27}$$

and $T = \frac{\partial H}{\partial S} = \gamma > 0$ represents the temperature of the thermal bath.

Therefore, the equations of motion for $\mathcal{E}_H = \sharp_\Lambda(\mathrm{d}H)$ are:

$$\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -V'(q) - \gamma p \\ \dot{S} &= \frac{p^2}{m} \end{aligned} \tag{10.28}$$

Obviously, the system is isolated since $\dot{H} = 0$, and it is also clear from the equation for $\dot{S}$ that the first and second laws are satisfied since $\dot{S} \geq 0$.

In the Lagrangian side we obtain the system given by

$$\begin{aligned} m\ddot{q} &= -V'(q) - \gamma m\dot{q} \\ \dot{S} &= m\dot{q}^2. \end{aligned} \tag{10.29}$$

Observe that in this system the friction force is given by the map $F_{fr} : TQ \to T^*Q$ given by

$$F_{fr}(q, \dot{q}) = \gamma \dot{q}^i dq^i. \tag{10.30}$$

Therefore, the equation of entropy production can be rewritten in terms of the friction force as follows

$$T\dot{S} = -\langle F_{fr}(q, \dot{q}), \dot{q} \rangle$$

These equations coincide with the set of equations proposed in [135, 136] for this particular choice of Lagrangian L and friction force F_{fr}.

11. Optimal control and contact systems: the Herglotz control problem

The Herglotz optimal control problem was introduced in [93]. It is a generalization of an optimal control problem in which the cost equation is given by a non-autonomous differential equation, instead of by a definite integral like on the usual optimal control problems. We will see that the solutions are integral curves of a precontact Hamiltonian system. This is a generalization of the presymplectic approach to the optimal control problems, which we now explain.

After recalling the presymplectic theory of control systems in Section 11.1, we introduce the Herglotz control problem and solve it in two ways in Section 11.2. One, published in [93], through a presymplectic optimal control principle. The other one, introduced in [94], is through vakonomic dynamics. Finally, in Section 11.3, we provide some examples of control problems in contact mechanics and equilibrium thermodynamics.

11.1. The presymplectic approach to optimal control problems: a review

A control problem is given by the data $(M_0, \rho_0, W_0, F, X, I, x_0, x_1)$. That is:

- A *state space* M_0, which is a smooth manifold.
- A *control bundle*[1] $\rho_0 : W_0 \to M_0$, with local coordinates (x^i, u^a), which we refer as the *variables* x^i, and the *controls* u^a.
- A *cost function* $F : W_0 \to \mathbb{R}$.
- A *control equation*, which is a vector field along ρ_0, that is:

$$\begin{array}{ccc} & & TM_0 \\ & \overset{X}{\nearrow} & \big\downarrow{\scriptstyle \tau_{M_0}} \\ W_0 & \xrightarrow{\rho_0} & M_0 \end{array} \tag{11.1}$$

In local bundle coordinates it is given by

$$X = X^i(q, u) \frac{\partial}{\partial x^i}. \tag{11.2}$$

[1]In [93] the control bundle is trivial $W = M \times U$. However, considering an arbitrary fiber bundle is useful in some situations, as the control of mechanical systems with non-parallelizable configuration manifolds.

- A time interval $I = [t_0, t_1] \subseteq \mathbb{R}$ and the *initial and final values of the variables* $x_0, x_1 \in M_0$.

Problem 11.1 (Optimal control problem). Find the curves $c : I = [t_0, t_1] \to W_0$, (in a local trivialization we write $c = (c_{M_0}, c_U)$), which satisfy the following conditions:

1. *End points conditions*: $c_{M_0}(t_0) = x_0, c_{M_0}(t_1) = x_1$.
2. *Control equation*: c_{M_0} is an integral curve of X_0: $\dot{c}_M = X_0 \circ c$.
3. *Maximal condition*: $\mathcal{A}(c) = \int_{t_0}^{t_1} F \circ c$ is maximum over all curves satisfying 1. and 2.

For our purposes, it will be simpler to consider the *extended optimal control problem*. This problem is equivalent to the previous optimal control problem, but now we include the cost as an extra variable. Explicitly, form a control problem, we construct an extended optimal control problem, which is given by the data $(M, \rho, W, X, I, x_0, x_1)$ where:

- The *extended state space* is $M = \mathbb{R} \times M_0$, which has coordinates (x^0, x^i).
- The *extended control bundle* is $\rho = (\mathrm{id}_{\mathbb{R}}, \rho_0) : W = \mathbb{R} \times W_0 \to \mathbb{R} \times M_0 = M$. This bundle has coordinates (x^0, x^i, u^a)
- The *extended control equation* X is given by

$$X = F \frac{\partial}{\partial x_0} + X_0. \tag{11.3}$$

This is the statement of the problem.

Problem 11.2 (Extended optimal control problem). Find the curves $c : I = [t_0, t_1] \to W$, (in a local trivialization we write $c = (c_M, c_U)$, $c_M = (c^0, c_{M_0})$), which satisfy the following conditions:

1. *End points conditions*: $c_{M_0}(t_0) = x_0, c_{M_0}(t_1) = x_1, c_0(t_0) = 0$.
2. *Control equation*: c_M is an integral curve of X: $\dot{c}_M = X \circ c$.
3. *Maximal condition*: $c^0(t_1)$ is maximum over all curves satisfying 1. – 2.

Remark 11.1. This problem is clearly equivalent to the optimal control problem. Indeed, the part of the control equation on $\dot{c}^0$ can be integrated to obtain

$$c^0(t) = \int_0^t F \circ (c_{M_0}, c_U), \tag{11.4}$$

thus $x^0(t_1) = \mathcal{A}(c_{M_0}, c_U)$.

A necessary condition for this problem is given by the Pontryagin maximum principle. A detailed proof can be found on [15]. However, now we are interest on its presymplectic formulation [114, 119] which is as follows. Given a control problem, we construct the following presymplectic Hamiltonian system $(\tilde{W}, \omega, H)$, where

- The bundle $\tilde{\rho} : \tilde{W} \to T^*M$ is just $\tilde{W} = W \times_M T^*M$

$$
\begin{tikzcd}
 & \tilde{W} \arrow[dl, "\tilde{\pi}_W"'] \arrow[dr, "\tilde{\rho}"] & \\
W \arrow[dr, "\rho"'] & & T^*M \arrow[dl, "\pi_M"] \\
 & M &
\end{tikzcd}
\tag{11.5}
$$

  which has local coordinates $(x^0, x^i, p_0, p_i, u^a)$.

- The presymplectic form $\omega = \tilde{\rho}^* \omega_M$, which in local coordinates is

$$\omega = \mathrm{d}z \wedge p_z + \mathrm{d}x^i \wedge \mathrm{d}p_i. \tag{11.6}$$

- The Hamiltonian is given by contraction with the control equation: $H(\alpha_q, u) = \alpha_x(X_{(x,u)})$, or, in local coordinates

$$H(x^0, x^i, p_0, p_i, u^a) = p_0 F(x^i, u^a) + p_i X^i(x^i, u^a). \tag{11.7}$$

Theorem 11.2 (Presymplectic version of Pontryagin's maximum principle)**.** *Given the extended optimal control problem $(M, \rho, W, F, X, I, x_0, x_1)$, we let $\hat{c} : I \to W$ be a solution, $\hat{c} = (c_M, c_U)$. Then, there exists $\hat{\sigma} : I \to \tilde{W}$, $\hat{\sigma} = (\sigma_{T^*M}, \sigma_U)$ such that*

1. *It is a solution to the Hamiltonian presymplectic problem $(\tilde{W}, \omega, H)$. That is, it is an integral curve of Y, solution to the equation $\iota_Y \omega = \mathrm{d}H$.*
2. *The projection $\hat{c} = \tilde{\pi}_W \circ \hat{\sigma}$ onto W satisfies the end points condition.*

Using the constraint algorithm [147], we can see that Hamiltonian vector fields of the presymplectic system are of the form

$$Y = F\frac{\partial}{\partial x^0} + X^i \frac{\partial}{\partial x^i} - \left(\lambda_o \frac{\partial F}{\partial x^i} + p_j \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial p_i} + A_a \frac{\partial}{\partial u^a}, \tag{11.8}$$

where $\lambda_0 \in \mathbb{R}$. Moreover, the first step of the algorithm produces the constraint

$$\frac{\partial H}{\partial u^a} = 0, \tag{11.9}$$

which is called the *compatibility equation*. In the case that this equation determine the controls u^a, we say that the problem is *regular*. In this situation, the terms A_a of the Hamiltonian vector fields are determined on the first step of the algorithm, and we obtain a first constraint submanifold symplectomorphic to T^*M. Otherwise, the problem is *singular*, and we need to keep applying the constraint algorithm.

Here, some contact dynamics are visible. We divide the solutions $\hat{c}$ in two cases. If the constant $\lambda_0 = p_0$ present in (11.8) is non-zero, then we say that it is a *normal* *solution*. Otherwise, if $\lambda_0 = 0$, it is called an *abnormal solution*. In [93] we showed that $N_\lambda = \{p_0 = \lambda\} \subseteq \tilde{W}$ inherits the structure of a precontact Hamiltonian system for the normal case ($\lambda \neq 0$) and a presymplectic one in the abnormal case ($\lambda = 0$). Moreover, on the regular and normal case, one can obtain a contact system by fixing the constraints. Also, as proposed by [208] one can give a unified formulation by using the projective cotangent bundle. We will discuss these constructions later, in the context of the more general Herglotz control problem.

11.2. The Herglotz control problem

A Herglotz control problem is given by the following elements $(M_0, \rho, W, F, X_0, I, x_0, x_1, z_0)$. That is,

- A *state space* M_0, and the corresponding *extended state space* $M = M_0 \times \mathbb{R}$.
- A *control bundle* $\rho : W \to M_0 \times \mathbb{R}$, with local coordinates (x^i, z, u^a), which we refer as the *variables* x^i, the *action* or *cost* z, and the *controls* u^a.

$$\begin{tikzcd}
 & W \arrow[dl, "\rho_0"'] \arrow[d, "\rho"] \arrow[dr, "z"] & \\
M_0 & M_0 \times \mathbb{R} \arrow[l, "\mathrm{pr}_{M_0}"] \arrow[r, "z"'] & \mathbb{R}
\end{tikzcd} \tag{11.10}$$

- A *cost function* $F : W \to \mathbb{R}$.
- A *control equation*, which is a vector field along ρ_0, that is:

$$\begin{tikzcd}
 & TM_0 \arrow[d, "\tau_{M_0}"] \\
W \arrow[ur, "X_0"] \arrow[r, "\rho_0"] & M_0
\end{tikzcd} \tag{11.11}$$

  In local bundle coordinates it is given by

$$X_0 = X^i(q, z, u)\frac{\partial}{\partial x^i}. \tag{11.12}$$

- A time interval $I = [t_0, t_1] \subseteq \mathbb{R}$, *initial and final values of the variables* $x_0, x_1 \in M_0$, and the *initial action or cost* $z_0 \in \mathbb{R}$.

The Herglotz control problem can be formulated as follows.

Problem 11.3 (Herglotz control problem). Find the curves $c : I = [t_0, t_1] \to W$, (in a local trivialization we write $c = (c_{M_0}, c_z, c_U)$), which satisfy the following conditions:

1. *End points conditions*: $c_{M_0}(t_0) = x_a, c_{M_0}(b) = x_b, c_z(t_0) = z_0$.

2. *Control equation*: c_{M_0} is an integral curve of X_0: $\dot{c}_{M_0} = X \circ c$. That is,

$$\dot{c}^i_M = X^i \circ c. \tag{11.13}$$

3. *Cost equation* c_z satisfies the differential equation $\dot{c}_z = F \circ c$, and

4. *Maximal condition*: $c_z(b)$ is maximum over all curves satisfying 1. – 3.

Remark 11.3. In the case that the both cost function and the control equation do not depend on z, this problem reduces to the usual optimal control problem for (c_{M_0}, c_U). Indeed, one has

$$c_z(t_1) = \int_{t_0}^{t_1} F \circ (c_{M_0}, c_Q), \tag{11.14}$$

and the control equation does not depend on c_z.

There are two approaches to find necessary conditions to solve this problem. In [93] we added extra variables in order to convert this problem in a usual optimal control problem. Then we reduce the system eliminating the extra variables. We obtain a precontact formulation of the optimal control principle.

Another possibility, which we explored in [94] is to consider a related Herglotz vakonomic principle, and then we can directly obtain the equations of motion. However, we do not obtain the abnormal solutions (those that do not depend on the cost function).

11.2.1. The Pontryagin maximum principle approach

We first consider the extended version of the Herglotz control problem. In order to which is given by $(M, \rho, W, X, I, x_0, x_1, z_0)$ where

- The *extended state space* is $M = M_0 \times \mathbb{R}$, which has coordinates (x^i, z).
- The *control bundle* is $\rho = (\mathrm{id}_{\mathbb{R}}, \rho_0) : W = W_0 \times \mathbb{R} \to M_0 \times \mathbb{R} = M$. This bundle has coordinates (x^i, z, u^a)
- The *extended control equation* X is a vector field along ρ

$$\begin{tikzcd} & TM \arrow[d, "\tau_M"] \\ W \arrow[ur, "X"] \arrow[r, "\rho"] & M \end{tikzcd} \tag{11.15}$$

The problem is stated as follows.

Problem 11.4 (Extended Herglotz optimal control problem). Find the curves $c : I = [t_0, t_1] \to W$, (in a local trivialization we write $c = (c_M, c_U)$, $c_M = (c_{M_0}, c_z)$ which satisfy the following conditions:

1. *End points conditions*: $c_{M_0}(t_0) = x_0, c_{M_0}(t_1) = x_1, c_z(t_0) = 0$.

2. *Control equation*: c_M is an integral curve of X: $\dot{c}_M = X \circ c$.
3. *Maximal condition*: $c_z(t_1)$ is maximum over all curves satisfying 1. – 2.

Again, given a Herglotz control problem, $(M = M_0 \times \mathbb{R}, \rho, W, F, X_0, I, x_0, x_1, z_0)$ we can construct an extended Herglotz control problem $(M_0, \rho, W, X, I, x_0, x_1, z_0)$, where

$$X = F\frac{\partial}{\partial z} + X_0, \tag{11.16}$$

and vice versa.

We construct a similar precontact system to the one in Theorem 11.2, that is, consider the system $(\tilde{W}, \omega, H)$, where, again:

- The bundle $\tilde{\rho} : \tilde{W} \to T^*M$ is just $\tilde{W} = W \times_M T^*M$ which has local coordinates $(x^i, z, p_i, p_z u^a)$.
- The presymplectic form $\omega = \tilde{\rho}^*\omega_M$.
$$\omega = \mathrm{d}p^0 \wedge p_z + \mathrm{d}x^i \wedge \mathrm{d}p^i. \tag{11.17}$$
- The Hamiltonian is given by contraction with the control equation: $H(\alpha_q, u) = \alpha_{(x,z)}(X_{(x,z,u)})$, or, in local coordinates
$$H(x^i, z, p_i, p_z, u^a) = p_z F(x^i, z, u^a) + p_i X^i(x^i, z, u^a). \tag{11.18}$$

Although the following result looks identical to Theorem 11.2, it is a generalization. Indeed, now the control equation and the cost function are allowed to depend on z.

Theorem 11.4 (Presymplectic version of Herglotz-Pontryagin's maximum principle)**.** *Given the extended optimal control problem $(M, \rho, W, X, I, x_0, x_1, z_0)$, we let $\hat{c} : I \to W$ be a solution, $\hat{c} = (c_M, c_U)$. Then, there exists $\hat{\sigma} : I \to \tilde{W}$, $\hat{\sigma} = (\sigma_{T^*M}, \sigma_U)$ such that*

1. *It is a solution to the Hamiltonian presymplectic problem $(\tilde{W}, \omega, H)$. That is, it is an integral curve of Y, solution to the equation $\iota_Y\omega = \mathrm{d}H$.*
2. *The projection $\hat{c} = \tilde{\pi}_W \circ \hat{\sigma}$ onto W satisfies the end points condition.*

Proof. We cannot apply Theorem 11.2 directly since our control equation depends on z. Although this could be proved directly, we will do the following trick: adding new variable x^0 that duplicates z. That is, we let $\bar{M} = \mathbb{R} \times M$, $\bar{W} = W \times_M T^*\bar{M}$ and $\bar{\rho} = \mathrm{id}_{\mathbb{R}} \times \rho$. We also let

$$\bar{X} = F\frac{\partial}{\partial x^0} + X = F\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial z}\right) + X_0. \tag{11.19}$$

Note that (c_{M_0}, c_z) fulfills the control equation X if and only if (c_z, c_{M_0}, c_z) fulfills the control equation for $\bar{X}$, so the solutions of this extended control problem for which $c^0 = c_z$ are precisely the solutions of the extended Herglotz control problem. Nevertheless, now F does not depend on x^0, and we are on the situation of Theorem 11.2.

Thus, we obtain the corresponding presymplectic system $(\tilde{\tilde{W}}, \bar{\theta}, \bar{H})$. Here,

$$\bar{H} = p_0 F + p_z F + p_i X^i. \tag{11.20}$$

Since we are only interested in solutions with $z = x^0$, we consider the submanifold $N_0 = \{z = x_0\} \subseteq \tilde{\tilde{W}}$, and we let $i : N \hookrightarrow \tilde{\tilde{W}}$ be the canonical inclusion. The Hamiltonians vector fields of $(\tilde{\tilde{W}}, \bar{\theta}, \bar{H})$ tangent to N_0 are thus Hamiltonian vector fields of $(N_0, i^*\bar{\omega}, i^*H)$. In local coordinates $(x^i, z, p_0, p_i, p_z, u^a)$

$$\begin{aligned} i^*\bar{\omega} &= \mathrm{d}z \wedge (p_0 + p_z) + \mathrm{d}q^i \wedge \mathrm{d}p_i, \\ i^*\bar{H} &= (p_0 + p_z)F + p_i X^i. \end{aligned} \tag{11.21}$$

Thus, $(p_0 + p_z)$ plays the role of the "momentum for z". We consider the following projection

$$\begin{aligned} \pi : N_0 &\to \tilde{W}, \\ (x^i, z, p_0, p_i, p_z, u^a) &\mapsto (x^i, z, p_i, p_0 + p_z, u^a). \end{aligned} \tag{11.22}$$

One then checks that π projects the solutions of Hamilton's equation for $(N, i^*\bar{\omega}, i^*H)$ onto those of $(\tilde{W}, \omega, H)$. Thus, the integral curves of the system $(\tilde{W}, \omega, H)$ are the projections of the integral curves of the system $(\tilde{\tilde{W}}, \bar{\theta}, \bar{H})$ satisfying $\sigma_0 = \sigma_z$. □

Again, using the constraint algorithm, we obtain solutions of this form

$$Y = F\frac{\partial}{\partial z} + X^i \frac{\partial}{\partial x^i} - \left(\lambda_0 \frac{\partial F}{\partial x^i} + p_j \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial p_i} + A_a \frac{\partial}{\partial u^a}, \tag{11.23}$$

where $\lambda_0 \in \mathbb{R}$. Moreover, we also obtain a similar *compatibility equation*

$$\frac{\partial H}{\partial u^a} = 0, \tag{11.24}$$

In the case that this equation determine the controls u^a, we say that the problem is *regular*. In this case, the values of A_a are determined, and we can obtain a formulation on T^*M by substituting the values of u_a. Otherwise, the problem is *singular*, and we need to keep applying the constraint algorithm. Again, we say that solutions are *normal* if $\lambda_0 \neq 0$ and abnormal if $\lambda_0 = 0$.

We obtained a precontact formulation of the Pontryagin maximum principle for our problem. Now, we will obtain a precontact one by nothing that our system is homogeneous, thus it is the symplectization of a contact system on the projective cotangent bundle. On the case that the solutions are normal, one can obtain a precontact system. The abnormal solutions will take place on the hyperplane at infinity $\{p_z = 0\}$.

Indeed, the system $(\tilde{W}_0, \eta, H_0)$ is constructed as follows

- The bundle $\tilde{\rho}_0 : \tilde{W}_0 \to T^*M_0 \times \mathbb{R}$ is constructed as $\tilde{W} = W \times_M T^*M_0 \times \mathbb{R}$

$$\begin{tikzcd}
 & \tilde{W}_0 \arrow[dl, "\tilde{\pi}_{W_0}"'] \arrow[dr, "\tilde{\rho}"] & \\
W_0 \arrow[dr, "\rho"] & & T^*M \arrow[dl, "\pi^1_{M_0}"] \\
 & M_0 \times \mathbb{R} &
\end{tikzcd} \tag{11.25}$$

  which has local coordinates (x^i, p_i, z, u^a).

- The precontact form $\eta = \tilde{\rho}^* \omega_M$, which in local coordinates is

$$\eta = \mathrm{d}z - p^i \wedge \mathrm{d}x^i. \tag{11.26}$$

- The Hamiltonian is given by the cost function minus the contraction with the control equation $H(\alpha_{(x,z)}, u) = \alpha_{(x,z)}(X_{(x,z,u)}) - F(x,z,u)$, or, in local coordinates

$$H(x^0, x^i, p_0, p_i, u^a) = F(x^i, u^a) - p_i X^i(x^i, u^a). \tag{11.27}$$

Theorem 11.5 (Contact approach to the Pontryagin maximum principle)**.** *Given the extended optimal control problem $(M, \rho, W, X, I, x_0, x_1, z_0)$, we let $\hat{c} : I \to W$ be a normal solution, $\hat{c} = (c_M, c_U)$. Then, there exists $\hat{\sigma}^0 : I \to W_0$, $\hat{\sigma}^0 = ({\sigma^0}_{T^*M_0 \times \mathbb{R}}, {\sigma^0}_U)$ such that*

1. *It is a solution to the Hamiltonian precontact problem (W_0, η, H_0). That is, it is an integral curve of a Hamiltonian vector field Y.*
2. *The projection $\hat{c} = \tilde{\pi}_{W_0} \circ \hat{\sigma}^0$ onto W satisfies the end points condition.*

Proof. We start from the presymplectic Hamiltonian system $(\tilde{W}, \omega, H)$ defined in Theorem 11.4. Moreover, the system is exact by taking $\theta = \tilde{\rho}^* \theta_M$ and the Hamiltonian H is homogeneous. We just need to project its solutions onto a solution of the system (W_0, η, H_0). In order to do that, we use the map $\Sigma^{M_0} : T^*(M_0 \times \mathbb{R}) \to T^*M_0 \times \mathbb{R}$ defined in Theorem 7.15 induces a map $\tilde{\Sigma}^{M_0} : \tilde{W} \to \tilde{W}_0$

$$\begin{tikzcd}
 & \tilde{W} \arrow[ddl, "\tilde{\rho}"'] \arrow[d, "\mathbb{P}"] \arrow[dr, "\tilde{\Sigma}^{M_0}"] & \\
 & \mathbb{P}(\tilde{W}) \arrow[d, "\mathbb{P}(\tilde{\rho})"] & \tilde{W}_0 \arrow[l, "\tilde{\psi}"] \arrow[ddl, "\tilde{\rho}_0"] \\
T^*M \arrow[r, "\mathbb{P}"] \arrow[dr, "\Sigma^{M_0}"'] & \mathbb{P}(T^*M) & \\
 & T^*M_0 \times \mathbb{R} \arrow[u, "\bar{\psi}"] &
\end{tikzcd} \tag{11.28}$$

In local coordinates, this map is given by

$$
\begin{aligned}
\tilde{\Sigma}^{M_0} &: \tilde{W} \to \tilde{W}_0, \\
(q,z,p,p_z,u) &\mapsto (\Sigma^{M_0}(q,z,p,p_z),u) = (q,z,-p/p_z).
\end{aligned}
\tag{11.29}
$$

This map is well-defined whenever $p_z \neq 0$, which is the case of the normal solutions that we are interested in.

One now checks that

$$
H = p_z(\Sigma^{M_0})^* H_0. \tag{11.30}
$$

Nevertheless, the map $\tilde{\Sigma}^{M_0} : (\tilde{W},\theta,H) \to (\tilde{W}_0,\eta,H_0)$ is not a symplectization, because these are precontact systems, not contact. Instead of developing a theory of "*presymplectization*", we can do the following trick. For any section γ_0 and its pullback section γ, so that

$$
\begin{tikzcd}
 & \tilde{W} \arrow[dl, "\tilde{\rho}"'] \arrow[dr, "\tilde{\Sigma}^{M_0}"] & \\
T^*M \arrow[ur, bend right, "\gamma"'] \arrow[dr, "\Sigma^{M_0}"'] & & \tilde{W}_0 \arrow[dl, "\tilde{\rho}_0"] \\
 & T^*M_0 \times \mathbb{R} \arrow[ur, bend left, "\gamma_0"] &
\end{tikzcd}
\tag{11.31}
$$

one notices that $\gamma_0 : (TM_0 \times \mathbb{R}, \eta_{M_0}, \gamma_0^* H) \to (\tilde{W}_0, \eta, H_0)$ is a morphism of precontact systems (Theorem 5.9), and $\gamma : (TM, \omega_M, \gamma^* H) \to (\tilde{W}, \omega, H)$ is, which we might call a *morphism of presymplectic systems*. Anyway, from a direct calculation it is easy to see that $(\operatorname{im}\gamma, \gamma^* X)$ and $(\operatorname{im}\gamma_0, \gamma_0^* X_0)$ are Hamiltonian vector fields for their respective presymplectic/precontact systems if and only if X and X_0 are. Moreover, $\gamma^* X$ is homogeneous if and only if X is. Since the map $\Sigma^Q : (TM_0 \times \mathbb{R}, \eta_{M_0}, \gamma_0^* H) \to (\tilde{W}, \theta, H)$ is a symplectization, there is a bijection between Hamiltonian vector fields tangent to $\operatorname{im}\gamma_0$ and Hamiltonian vector fields tangent to $\operatorname{im}\gamma$.

Since every non-trivial solution $\hat{\sigma}$ or $\hat{\sigma}^0$ are contained to the image of a section, then solutions of the precontact system are the projections of the solutions of presymplectic systems. Indeed, if a solution is not tangent to the image of any section then they project onto the constant path. □

The equations of motion of the aforementioned Hamiltonian problem are

$$
\dot{x}^i = X^i, \tag{11.32a}
$$

$$
\dot{p}_i = p_i \frac{\partial F}{\partial z} - p_j \frac{\partial X^j}{\partial x^i} + \frac{\partial F}{\partial x^i} - \frac{\partial X^j}{\partial z} p_i p_j, \tag{11.32b}
$$

$$
\dot{z} = F, \tag{11.32c}
$$

subjected to the constraints

$$
\frac{\partial H}{\partial u^a} = \frac{\partial F}{\partial u^a} - p_j \frac{\partial X^j}{\partial u^a} = 0. \tag{11.32d}
$$

Remark 11.6. Some authors [208] prefer to work on $\mathbb{P}(T^*M)$ and $\mathbb{P}(W\tilde{W})$ instead. The advantage of those manifolds is that we do not lose the abnormal solutions by taking out the points at infinity (see Example 2.2). The disadvantage is that those manifolds do not possess a global (pre)contact form, just a (pre)contact distribution, hence we cannot construct a Hamiltonian system directly. We should project the Hamiltonian vector fields from the (pre)symplectic systems, or map them from T^*M_0 and $\tilde{W}_0$ and extend them to the plane at infinity by continuity.

11.2.2. The vakonomic approach

In [94] we found an alternate way to obtain this equations by proposing the vakonomic Herglotz principle. We will derive Equation (11.32) from Theorem 6.5. First, from a Herglotz control problem $(M_0, \rho, W, F, X_0, I, x_0, x_1, z_0)$ we construct the following vakonomic problem:

- The configuration space is W_0 (the extended configuration space is $W = W_0 \times \mathbb{R}$).

- The Lagrangian is the pullback of cost function $L = (\tau^1_{W_0})^* F : TW_0 \times \mathbb{R} \to \mathbb{R}$, where $\tau^1_{W_0} : TW_0 \times \mathbb{R} \to W_0 \times \mathbb{R} = W$ is the canonical projection.

- The constraints are given by the control equation $\phi^i = X^i(x, u, z) - \dot{q}^i$ and N is the submanifold defined by those constraints and the one given by the Lagrangian $\phi^0 = \dot{z} - L$.

- The initial and final points are x_0 and x_1, respectively. The initial action is z_0.

Thus, a path $c = (c_{M_0}, c_z, c_u) : I \to W$ satisfies the Herglotz vakonomic variational principle if:

- $c \in \Omega_N(q_0, q_1, z_0)$ which implies that c satisfies the boundary conditions and the initial action condition, thus it satisfies Item 1 of Problem 11.3. It also implies that c is tangent to the submanifold N. By our choice of constraints ϕ^i, this implies that c satisfies the control equation (Item 2). The addition constraint ϕ^0 makes c fulfill the cost equation (Item 3).

- c is a critical point of $\mathcal{A}|_{\Omega_N(q_0,q_1,z_0)}(c) = c_z(t_1)$. Item 4 demands $c_z(t_1)$ to be maximal among the paths satisfying Items 1 to 3. That is, maximal among the paths in $\Omega_N(q_0, q_1, z_0)$. This means c is a maximum of $\mathcal{A}|_{\Omega_N(q_0,q_1,z_0)}$, hence it implies that c is a critical point of the action restricted to the space of admissible paths.

We have just proved that a solution of the Herglotz control problem must satisfy the Herglotz vakonomic variational principle, which, by Theorem 6.5, implies that c solves the equations

$$\dot{q}^i = X^i, \tag{11.33a}$$

$$\begin{aligned}\dot{\mu}_i &= \frac{\partial F}{\partial x^i} - \mu_j \frac{\partial X^j}{\partial x^i} - \mu_j \left(\frac{\partial F}{\partial z} - \mu_i \frac{\partial X^j}{\partial z} \right) \\ &= \mu_i \frac{\partial F}{\partial z} - \mu_j \frac{\partial X^j}{\partial x^i} + \frac{\partial F}{\partial x^i} - \frac{\partial X^j}{\partial z} \mu_i \mu_j,\end{aligned} \tag{11.33b}$$

$$\dot{z} = F, \tag{11.33c}$$

subjected to the constraints

$$\frac{\partial F}{\partial u^a} - \mu_j \frac{\partial X^j}{\partial u^a} = 0. \tag{11.33d}$$

These equations are identical to Equation (11.32).

11.3. Applications

11.3.1. Herglotz variational principle

One of the simplest applications we can think of is to obtain a new derivation of Herglotz's variational principle for a Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$. Notice that this problem is a particular case of the Herglotz optimal control problem, where

- Controls are the velocities $u^a = v^i$.
- The cost function is the Lagrangian $F = L$.
- The control equation is $X = v^i \frac{\partial}{\partial x^i}$, demanding that the solution is a SODE.

The solutions to this problem are given by Theorem 11.32:

$$\dot{q}^i = v^i, \tag{11.34}$$

$$\dot{p}_i = p_i \frac{\partial L}{\partial z} + \frac{\partial L}{\partial q^i} \tag{11.35}$$

$$\dot{z} = L, \tag{11.36}$$

with the constraints

$$\frac{\partial L}{\partial v^i} = p_i, \tag{11.37}$$

which are precisely the Herglotz equations.

11.3.2. Control of contact systems

We let $M = T^*Q \times \mathbb{R}$ be our extended state space, with coordinates (q^i, p_i, z), and assume that we are given a parametrized family of Hamiltonians $H : W \to \mathbb{R}$, where $\rho : W \to M = T^*Q \times \mathbb{R}$ is the control bundle. In a local trivialization of W, we consider the Hamiltonian contact vector fields X_{H_u}, where $H_u(q,z,p) = H(q,z,p,u)$, u being the coordinate of the fiber of the control bundle. Then, we can define the extended control $Z(q,p,z,u) = X_{H_u}(q,p,z)$. A curve $c = (c_M, c_u) : I = [t_0, t_1] \to W$ satisfies the extended control equation if and only if c_M satisfies the contact Hamilton equations for $H_{c_u(t)}$:

$$
\begin{aligned}
\frac{\mathrm{d}q^i}{\mathrm{d}t} &= \frac{\partial H_u}{\partial p_i},\\
\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\partial H_u}{\partial q^i} - p_i \frac{\partial H_u}{\partial z},\\
\frac{\mathrm{d}z}{\mathrm{d}t} &= p_i \frac{\partial H_u}{\partial p_i} - H_u.
\end{aligned}
$$

One can consider the extended Herglotz control problem given by these elements. Then, by Theorem 11.5, we know that the normal solutions are the projections of the solutions to the precontact system $\tilde{W}_0, H_0)$, where

$$
H_0 = p_{q^i} \frac{\partial H}{\partial p_i} - p_{p_i} \frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial z} - p_i \frac{\partial H}{\partial p_i} + H. \tag{11.38}
$$

11.3.3. Application: Optimal control on thermodynamic systems

Port-thermodynamic systems were introduced in [247], but in a homogeneous symplectic formalism.

Definition 11.1 (Port-thermodynamic system)**.** A *port-thermodynamic system* on $T^*(Q \times \mathbb{R})$ is defined as a pair $(\mathcal{L}, K)$, where the homogeneous Lagrangian submanifold $\mathcal{L} \subset T^*(Q \times \mathbb{R})$ specifies the state properties. The dynamics is given by the homogeneous Hamiltonian dynamics with parametrized homogeneous Hamiltonian $K := K^a + {K^c}_a u^a : T^*(Q \times \mathbb{R}) \to \mathbb{R}$, $u \in \mathbb{R}^k$, $K^c : T^*(Q \times \mathbb{R}) \to \mathbb{R}^k$, with K^a, K^c both equal to zero on the points of $\mathcal{L}$, and K^a as the internal Hamiltonian. One needs the additional condition

$$
\frac{\partial K}{\partial S}|_{\mathcal{L}} \geq 0, \tag{11.39}
$$

so that the second law of thermodynamics holds.

Using the results of Sections 7.2 and 10.1, we could instead consider the following contact formulation.

Definition 11.2 (Port-thermodynamic system, contact formalism)**.** A port-thermodynamic system on $(T^*Q \times \mathbb{R}, \eta_Q)$ is defined as a pair $(\mathbb{L}, h)$, where the Legendrian submanifold

$\mathbb{L} \subset T^*Q \times \mathbb{R}$ specifies the state properties. The dynamics is given by the contact Hamiltonian dynamics with parametrized contact Hamiltonian $h = h^a + h^c_a u^a : T^*Q \times \mathbb{R} \to \mathbb{R}$, $u \in \mathbb{R}^m$, $h^c : T^*Q \times \mathbb{R} \to \mathbb{R}^k$, with h^a, h^c zero on $\mathbb{L}$, and the internal Hamiltonian h^a satisfying

$$\frac{\partial h}{\partial S}|_{\mathbb{L}} \geq 0, \tag{11.40}$$

so that the second law of thermodynamics holds.

Our theory provides tools to understand which of the available thermodynamic processes minimize the entropy production of the system. Observe that we can consider processes that maximize or minimize other thermodynamic variables, such as the energy, via a Legendre transform. We end this section with an explicit example which can be found in [247].

Example 11.1 (Gas-Piston-Damper system)**.** Consider an adiabatically isolated cylinder closed by a piston containing a gas with internal energy $U(V, S)$.

The extended phase space has the following extensive variables

- the momentum of the piston π,
- the volume of the gas V,
- the energy E,
- the entropy S.

They correspond to $Q \times \mathbb{R}$ with local coordinates (V, π, E, S). The Legendrian submanifold is given by

$$\begin{aligned} \mathbb{L} = \{(V, \pi, E, p_V, p_\pi, p_E, S) | E = \frac{\pi^2}{2m} + U(S, V), \\ p_V = -p_E \frac{\partial U}{\partial V}, p_\pi = -p_E \frac{\pi}{m}, p_E = 1/\frac{\partial U}{\partial S}\}. \end{aligned} \tag{11.41}$$

The energy is then given by

$$h = p_V \frac{\pi}{m} + p_\pi \left(-\frac{\partial U}{\partial V} - d\frac{\pi}{m} \right) - \frac{d(\frac{\pi}{m})^2}{\frac{\partial U}{\partial S}} + \left(p_\pi + p_E \frac{\pi}{m} \right) u, \tag{11.42}$$

where d is the diameter of the piston and m is its mass.

The Hamiltonian vector field is given by

$$
\begin{aligned}
X_h = {} & \frac{\pi}{m}\frac{\partial}{\partial V} + \left(-\frac{\pi d}{m} + u - \frac{\partial U}{\partial V}\right)\frac{\partial}{\partial \pi} + \frac{\pi u}{m}\frac{\partial}{\partial E} \\
& + \left(\left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2\left(\frac{\partial U}{\partial S}\right)^2}\right)p_V + p_\pi \frac{\partial^2 U}{\partial V^2} - \frac{\pi^2 d \frac{\partial^2 U}{\partial V \partial S}}{m^2\left(\frac{\partial U}{\partial S}\right)^2}\right)\frac{\partial}{\partial p_V} \\
& + \left(\left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2\left(\frac{\partial U}{\partial S}\right)^2}\right)p_\pi + \frac{d p_\pi}{m} - \frac{p_E u}{m} - \frac{p_V}{m} + \frac{2\pi d}{m^2 \frac{\partial U}{\partial S}}\right)\frac{\partial}{\partial p_\pi} \\
& + \left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2\left(\frac{\partial U}{\partial S}\right)^2}\right)p_E \frac{\partial}{\partial p_E} \\
& + \left(\frac{\pi^2 d}{m^2 \frac{\partial U}{\partial S}}\right)\frac{\partial}{\partial S}.
\end{aligned}
\tag{11.43}
$$

We construct the contact Hamiltonian system $(T^*(T^*Q) \times \mathbb{R}, \eta_{T^*Q}, H)$ as in (11.38):

$$
\begin{aligned}
H = {} & -\left(\frac{d p_\pi}{m} - \frac{p_E u}{m} - \frac{p_V}{m} + \frac{2\pi d}{m^2 \frac{\partial U}{\partial S}}\right)P_\pi \\
& -\left(p_\pi \frac{\partial^2}{(\partial V)^2} U(V,S) - \frac{\pi^2 d \frac{\partial^2}{\partial V \partial S} U(V,S)}{m^2 \frac{\partial U}{\partial S}^2}\right)P_V \\
& -\left(\frac{\pi d}{m} - u + \frac{\partial}{\partial V} U(V,S)\right)P_{p_\pi} + \frac{\pi P_{p_E} u}{m} + \frac{\pi P_{p_V}}{m} - \frac{\pi^2 d}{m^2 \frac{\partial U}{\partial S}},
\end{aligned}
\tag{11.44}
$$

where we denote by $q^i, p_{q^i}, \Pi_{q^i}, \Pi_{p_{q^i}}$ the natural coordinates on T^*T^*Q, where q^i runs through V, π, E, and $\Pi_{q^i}, \Pi_{p_{q^i}}$ are the corresponding moments to q^i, p_i respectively.

The solutions to the control problem are then the integral curves of the Hamiltonian vector field of this system, which are the following

$$
\begin{aligned}
\dot{V} &= \frac{\pi}{m} \\
\dot{\pi} &= -\frac{\pi d}{m} + u - \frac{\partial U}{\partial V} \\
\dot{E} &= \frac{\pi u}{m} \\
\dot{p_V} &= \left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \right) p_V + p_\pi \frac{\partial^2 U}{\partial V^2} - \frac{\pi^2 d \frac{\partial^2 U}{\partial V \partial S}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \\
\dot{p_\pi} &= \left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \right) p_\pi + \frac{d p_\pi}{m} - \frac{p_E u}{m} - \frac{p_V}{m} + \frac{2\,\pi d}{m^2 \frac{\partial U}{\partial S}} \\
\dot{p_E} &= \left(p_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \right) p_E \\
\dot{S} &= \frac{\pi^2 d}{m^2 \frac{\partial U}{\partial S}} \\
\dot{\Pi_V} &= \alpha \Pi_V - \frac{\Pi_\pi}{m} \\
\dot{\Pi_\pi} &= \alpha \Pi_E - \frac{\Pi_\pi u}{m} \\
\dot{\Pi_{p_V}} &= -p_\pi p_V \Pi_V \frac{\partial^3 U}{\partial V^2 \partial S} - p_\pi \Pi_V \frac{\partial^3 U}{\partial V^3} - p_V \Pi_{p_\pi} \frac{\partial^2 U}{\partial V \partial S} - p_\pi \Pi_{p_V} \frac{\partial^2 U}{\partial V \partial S} \\
&\quad + \alpha \Pi_{p_V} - \Pi_{p_\pi} \frac{\partial^2 U}{\partial V^2} + \frac{\pi^2 d p_V \Pi_V \frac{\partial^3 U}{\partial V \partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} - \frac{2\,\pi^2 d p_V \Pi_V \frac{\partial^2 U}{\partial V \partial S} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} \\
&\quad - \frac{2\,\pi^2 d \Pi_V {\frac{\partial^2 U}{\partial V \partial S}}^2}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} + \frac{\pi^2 d \Pi_V \frac{\partial^3 U}{\partial V^2 \partial S}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{2\,\pi d p_V \Pi_\pi \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\pi^2 d \Pi_{p_V} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{2\,\pi d \Pi_\pi \frac{\partial^2 U}{\partial V \partial S}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \\
\dot{\Pi_{p_\pi}} &= -{p_\pi}^2 \Pi_V \frac{\partial^3 U}{\partial V^2 \partial S} - 2\, p_\pi \Pi_{p_\pi} \frac{\partial^2 U}{\partial V \partial S} + \alpha \Pi_{p_\pi} + \frac{\pi^2 d p_\pi \Pi_V \frac{\partial^3 U}{\partial V \partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} - \frac{2\,\pi^2 d p_\pi \Pi_V \frac{\partial^2 U}{\partial V \partial S} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} \\
&\quad - \frac{d \Pi_{p_\pi}}{m} + \frac{\Pi_{p_E} u}{m} + \frac{2\,\pi d p_\pi \Pi_\pi \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\pi^2 d \Pi_{p_\pi} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\Pi_{p_V}}{m} + \frac{2\,\pi d \Pi_V \frac{\partial^2 U}{\partial V \partial S}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} - \frac{2\, d \Pi_\pi}{m^2 \frac{\partial U}{\partial S}} \\
\dot{\Pi_{p_E}} &= -p_E p_\pi \Pi_V \frac{\partial^3 U}{\partial V^2 \partial S} - p_\pi \Pi_{p_E} \frac{\partial^2 U}{\partial V \partial S} - p_E \Pi_{p_\pi} \frac{\partial^2 U}{\partial V \partial S} + \alpha \Pi_{p_E} + \frac{\pi^2 d p_E \Pi_V \frac{\partial^3 U}{\partial V \partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \\
&\quad - \frac{2\,\pi^2 d p_E \Pi_V \frac{\partial^2 U}{\partial V \partial S} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} + \frac{2\,\pi d p_E \Pi_\pi \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\pi^2 d \Pi_{p_E} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \frac{\partial F}{\partial S} - \Pi_j \frac{\partial X_j}{\partial S} \\
&= -p_E p_\pi \Pi_{p_E} \frac{\partial^3 U}{\partial V \partial S^2} - {p_\pi}^2 \Pi_{p_\pi} \frac{\partial^3 U}{\partial V \partial S^2} - p_\pi p_V \Pi_{p_V} \frac{\partial^3 U}{\partial V \partial S^2} - p_\pi \Pi_{p_V} \frac{\partial^3 U}{\partial V^2 \partial S} + \Pi_\pi \frac{\partial^2 U}{\partial V \partial S} - \frac{2\pi^2 d p_E \Pi_{p_E} \left(\frac{\partial^2 U}{\partial S^2}\right)^2}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} \\
&\quad - \frac{2\pi^2 d p_\pi \Pi_{p_\pi} \left(\frac{\partial^2 U}{\partial S^2}\right)^2}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} - \frac{2\pi^2 d p_V \Pi_{p_V} \left(\frac{\partial^2 U}{\partial S^2}\right)^2}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} + \frac{\pi^2 d p_E \Pi_{p_E} \frac{\partial^3 U}{\partial S^3}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\pi^2 d p_\pi \Pi_{p_\pi} \frac{\partial^3 U}{\partial S^3}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} \\
&\quad + \frac{\pi^2 d p_V \Pi_{p_V} \frac{\partial^3 U}{\partial S^3}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{\pi^2 d \Pi_{p_V} \frac{\partial^3 U}{\partial V \partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} - \frac{2\pi^2 d \Pi_{p_V} \frac{\partial^2 U}{\partial V \partial S} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^3} - \frac{\pi^2 d \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2} + \frac{2\pi d \Pi_{p_\pi} \frac{\partial^2 U}{\partial S^2}}{m^2 \left(\frac{\partial U}{\partial S}\right)^2},
\end{aligned}
$$

and they are subject to the constraint

$$
\frac{p_E \Pi_\pi}{m} + \frac{\pi \Pi_{p_E}}{m} + \Pi_{p_\pi} = 0. \tag{11.45}
$$

Part III.

Conclusions and further work

12. Other topics in contact Hamiltonian systems

12.1. Discrete contact mechanics

Discrete Lagrangian mechanics and variational integrators provide a tool to construct numerical methods that preserve the geometric properties of the system. One replaces the tangent bundle TQ, which represent initial positions and velocities, by the discrete version $Q \times Q$, which represents initial and final positions. The article [195] is a major reference for this theory. Given the wide range of applications of contact Hamiltonian mechanics, it is necessary to adapt discrete mechanics to this setting.

To our knowledge, the first attempt give a discrete version of contact mechanics is in [248], where the authors present geometric numerical integrators for contact flows that stem from a discretization of Herglotz' variational principle. On [9] we made some contributions to this theory.

First, we argued that the choice of $Q \times Q \times \mathbb{R}$ instead of $Q \times Q \times \mathbb{R}^2$, as in [248] made more sense from a geometric point of view. Then, we were able to formulate a discrete version of the Herglotz principle and obtain some discrete equations for arbitrary regular discrete Lagrangians $L_d : Q \times Q \times \mathbb{R} \to \mathbb{R}$. We provided a geometric formulation in the spirit of [195]. A theorem providing a relationship between the discrete version symmetries and dissipated quantities.

Finally, we construct the exact discrete Lagrangian of a continuous Lagrangian, for which we are lead to define the contact exponential map and prove that it is a local diffeomorphism.

In the following pages we give an overview of [8]. We invite the reader to check it for more results, the missing proofs and examples.

12.1.1. The discrete Herglotz principle

On discrete mechanics, instead of trajectories, one uses their discrete counterpart. The *discrete path space* is just the set of sequences of some fixed length $N + 1 \in \mathbb{N}$.

$$C_d^N(Q) = \{(q_0, q_1, \dots, q_N) | q_k \in Q\}. \tag{12.1}$$

Given a discrete Lagrangian $L : Q \times Q \times \mathbb{R} \to \mathbb{R}$, for each $q_d \in C_d^N(Q)$ and some *initial discrete action* $z_0 \in \mathbb{R}$, we can construct a sequence $z_d = (z_k)_{k=1}^n \in C_d^N(\mathbb{R})$, that we call the *discrete action* defined by the following difference equation

$$z_{k+1} - z_k = L_d(q_k, q_k + 1, z_k), \quad k \in \{0, \dots, N-1\}. \tag{12.2}$$

Thus, we define the *Herglotz discrete action map* as

$$\begin{aligned} \mathcal{A}_d : C_d^N(Q) \times \mathbb{R} &\to \mathbb{R}, \\ (q_d, z_0) &\mapsto z_N. \end{aligned} \tag{12.3}$$

We say that a path satisfies the *discrete Herglotz principle* if it is a critical point of $\mathcal{A}_d$.

Theorem 12.1. *Let L_d be a discrete Lagrangian function such that $1 + D_z L_d$ is non-vanishing. Given $z_0 \in \mathbb{R}$, a discrete path $q_d \in C_d^N(Q)$ satisfies the discrete Herglotz principle if and only if it satisfies the* discrete Herglotz equations

$$D_1 L_d(q_k, q_{k+1}, z_k) + (1 + D_z L_d(q_k, q_{k+1}, z_k)) D_2 L_d(q_{k-1}, q_k, z_{k-1}) = 0, \tag{12.4a}$$

$$z_{j+1} - z_j = L_d(q_j, q_{j+1}, z_j) \tag{12.4b}$$

for $k \in \{1, \dots, N-1\}$ and $j \in \{0, \dots, N-1\}$.

12.1.2. Discrete Lagrangian flows

Given the discrete Lagrangian, we can define the *discrete Legendre transforms*

$$\mathrm{F}^+ L_d(q_0, q_1, z_0) = (q_1, D_2 L_d(q_0, q_1, z_0), z_1), \tag{12.5a}$$

$$\mathrm{F}^- L_d(q_0, q_1, z_0) = \left(q_0, -\frac{D_1 L_d(q_0, q_1, z_0)}{1 + D_z L_d(q_0, q_1, z_0)}, z_0 \right). \tag{12.5b}$$

The discrete Herglotz equations can be written as

$$\mathrm{F}^+ L_d(q_0, q_1, z_0) = \mathrm{F}^- L_d(q_1, q_2, z_1). \tag{12.6}$$

We say that the Lagrangian is *regular* whenever $\mathrm{F}^- L$ is a local diffeomorphism. In this situation, there is a well-defined local discrete flow given by

$$\begin{aligned} \Phi_d : Q \times Q \times \mathbb{R} &\to Q \times Q \times \mathbb{R}, \\ (q_0, q_1, z_0) &\to (q_1, q_2, z_1), \end{aligned} \tag{12.7}$$

where (q_0, q_1, q_2) is solution of the discrete Herglotz equations for L_d with initial action z_0.

We can also locally define the *Hamiltonian discrete flow* $\tilde{\Phi}_d : T^*Q \times \mathbb{R} \to Q \times Q \times \mathbb{R}$ as the one fulfilling the following commutative diagram.

$$\begin{tikzcd} & Q \times Q \times \mathbb{R} \arrow[rr, "\Phi_d"] \arrow[dl, "\mathrm{F}^- L_d"] \arrow[dr, "\mathrm{F}^+ L_d"] & & Q \times Q \times \mathbb{R} \arrow[dl, "\mathrm{F}^- L_d"] \arrow[dr, "\mathrm{F}^+ L_d"] & \\ T^*Q \times \mathbb{R} \arrow[rr, "\widetilde{\Phi_d}"'] & & T^*Q \times \mathbb{R} \arrow[rr, "\widetilde{\Phi_d}"'] & & T^*Q \times \mathbb{R} \end{tikzcd} \tag{12.8}$$

The discrete Hamiltonian flow is a conformal contactomorphism. We can obtain two contact forms on $Q \times Q \times \mathbb{R}$ using the Legendre transforms, i.e.,

$$\eta^+_{L_d} = (\mathrm{F}^+ L_d)^* \eta, \quad \eta^-_{L_d} = (\mathrm{F}^- L_d)^* \eta. \tag{12.9}$$

These forms define the same *discrete Herglotz distribution* $\mathcal{H}_{L_d} = \ker \eta^+_{L_d} = \eta^-_{L_d}$. Indeed, they are related by the *discrete conformal factor* $\eta^+_{L_d} = \sigma \eta^-_{L_d}$, where

$$\sigma(q_0, q_1, z_0) = 1 + D_z L_d(q_0, q_1, z_0). \tag{12.10}$$

The discrete flow is a conformal contactomorphism with respect to these forms. Indeed,

$$(\Phi_d)^* \eta^+ = (\sigma \circ \Phi_d)\eta^+, \quad (\Phi_d)^* \eta^- = \sigma \eta^-. \tag{12.11}$$

12.1.3. Discrete symmetries and dissipated quantities

We proved a discrete version of the results on Section 4.5.3. Indeed, assume that a Lie group G acts on Q, by diffeomorphisms $\Phi_g : Q \times \mathbb{R} \to Q \times \mathbb{R}$ for $g \in G$. We can lift it to an action on $Q \times Q \times \mathbb{R}$, by setting $\tilde{\Phi}_g = (\Phi_g, \Phi_g, \mathrm{id}_{\mathbb{R}}) : Q \times Q \times \mathbb{R} \to Q \times Q \times \mathbb{R}$. We denote by ξ_g the infinitesimal generator of the action on Q, and by $\tilde{\xi}_g = (\xi, \xi, 0)$ is the lifted action.

Now consider the discrete moment map J_d given by

$$\begin{aligned} &J_d : Q \times Q \times \mathbb{R} \to \mathfrak{g}^*, \\ &J_d(q_0, q_1, z_0)(\xi) = \eta^-_{L_d}(\tilde{\xi}(q_0, q_1, z_0)). \end{aligned} \tag{12.12}$$

We say that $f : Q \times Q \times \mathbb{R} \to \mathbb{R}$ is a *dissipated quantity* for a discrete Lagrangian $L_d : Q \times Q \times \mathbb{R} \to \mathbb{R}$ if it satisfies $f \circ \Phi_d = \sigma f$, where σ is the discrete conformal factor Equation (12.10).

Theorem 12.2. *Let L_d be an invariant discrete Lagrangian function for the lifted action $\widetilde{\Phi}$. Then $\widetilde{\Phi}$ acts by contactomorphisms on $Q \times Q \times \mathbb{R}$ and the function $\hat{J}_d(\xi) : Q \times Q \times \mathbb{R} \to \mathbb{R}$ given by*

$$\hat{J}_d(\xi)(q_0, q_1, z_0) = J_d(q_0, q_1, z_0)(\xi)$$

is dissipated for L_d, where $\xi \in \mathfrak{g}$.

12.1.4. The exponential map and the exact discrete Lagrangian

So far, we studied discrete Lagrangians on $Q \times Q \times \mathbb{R}$ without making reference to a continuous one. Nevertheless, the most important application of discrete mechanics is creating integrators for continuous systems. There is a tool to construct a discrete Lagrangian that perfectly matches the continuous dynamics, the so-called *exact Lagrangian*. In order to construct this, we need to overcome a technical prerequisite.

Define the open subset U_h of $TQ \times \mathbb{R}$ given by

$$U_h = \{(q_0, \dot{q}_0, z_0) \in TQ \times \mathbb{R} \mid \phi_t^{\xi_L} \text{ is defined for } t \in [0, h)\}$$

and let the *contact exponential map* be defined by

$$\begin{aligned} \exp_h^{\xi_L} :\ & U_h \subseteq TQ \times \mathbb{R} \to Q \times Q \times \mathbb{R} \\ & (q_0, \dot{q}_0, z_0) \mapsto (q_0, q_1, z_0), \end{aligned} \tag{12.13}$$

where $q_1 = \tau_Q^0 \circ \phi_h^{\xi_L}(q_0, \dot{q}_0, z_0)$. That is, this map sends the initial position, initial velocity and initial action of the system to the initial position, final position and initial action. In order to construct the exact Lagrangian we need the following result.

Theorem 12.3. *The contact exponential map* $\exp_h^{\xi_L}$ *is a local diffeomorphism.*

The proof of this theorem is somewhat involved. We use the construction on Section 6.2.1 to lift our contact Lagrangian system to one on $T(Q \times \mathbb{R})$ with nonholonomic constraints. Then we use the results on [8] to prove that the exponential map restricted to the constraint submanifold is a local diffeomorphism. Finally, we use the Herglotz principle to proof that one can project back the image of the lifted exponential map.

Since the contact exponential map is a local diffeomorphism we can define a local inverse called the *exact retraction* and denote it by $R^{e-} : Q \times Q \times \mathbb{R} \to TQ \times \mathbb{R}$. The discrete Lagrangian is thus defined as

$$L_h^e(q_0, q_1, z_0) = \int_0^h L \circ \phi_t^{\xi_L} \circ R^{e-}(q_0, q_1, z_0) \mathrm{d}t. \tag{12.14}$$

The dynamics of this Lagrangian is the discrete version of the contact Lagrangian dynamics. Among other results, we have the following one.

Theorem 12.4. *Take a regular Lagrangian* L *and fix a time step* $h > 0$. *Then we have that:*

1. L_h^e *is a regular discrete Lagrangian function.*

2. *If* H *is the Hamiltonian function corresponding to* L, *and* $\phi_t^{X_H}$ *is its contact Hamiltonian flow, we have that*
$$\mathrm{F}L_h^{e+} = \phi_h^{X_H} \circ \mathrm{F}L_h^{e-}. \tag{12.15}$$

3. *If* $(q, z) : [0, Nh] \to Q \times \mathbb{R}$ *is a solution of the Herglotz's equations, then it is related to the solution of the discrete Herglotz's equations* $\{(q_0, z_0), (q_1, z_1), \dots, (q_N, z_N)\}$ *for the corresponding exact discrete Lagrangian with* $(q(0), q(h), z(0))$ *as initial conditions in the following way:*
$$q_k = q(kh), \quad z_k = z(kh) \quad \text{for } k = 0, \dots, N. \tag{12.16}$$

We remark that computing the discrete Lagrangian is not possible on many applications, since in order to compute the retraction one needs to solve the equations of motion. However, one is usually able to approximate this Lagrangian. On the symplectic case there is a Variational Error Theorem [195], which relates the error on the discrete Lagrangian and the error on the solutions. We hope that a similar result holds in this situation, but it remains to be proven.

On [9] we provide examples of explicit computations of the discrete Lagrangian and examples of other discrete Lagrangians produced with more ad-hoc discretizations where we see the numerical behavior of the discrete Herglotz equations.

12.1.5. Unified formalism for contact mechanics

In the seminal paper [233], Skinner and Rusk introduced a new formalism which unifies the Lagrangian and Hamiltonian formalism in a common framework. In this so-called *unified* or *Skinner-Rusk* formalism takes place on the *unified* or *Pontryagin bundle* $TQ \times_Q T^*Q$. This formalism was extended to many situations, including time-dependent systems [71], general non-autonomous system [17], higher order systems [154], control theory [16, 69], and fields [39, 82, 120].

In [86] we adapted this formalism for contact systems. We will summarize the main results below. The reader will find more details and examples in the aforementioned article.

12.1.6. The unified bundle

We define the *extended unified bundle* (also called the *extended Pontryagin bundle*)

$$W = TQ \times_Q T^*Q \times \mathbb{R}\,, \tag{12.17}$$

which is endowed with the natural submersions that we show on the following diagram.

$$\begin{tikzcd}
 & & \mathbb{R} & & \\
TQ \times \mathbb{R} \arrow[urr, "z"] \arrow[d, "\mathrm{pr}_{TQ}"'] \arrow[ddrr, "\tau_Q^0"] & & W \arrow[ll, "\rho_1"'] \arrow[rr, "\rho_2"] \arrow[u, "z"] \arrow[dd, "\rho_0"] & & T^*Q \times \mathbb{R} \arrow[ull, "z"'] \arrow[d, "\mathrm{pr}_{T^*Q}"] \arrow[ddll, "\pi_Q^0"'] \\
TQ \arrow[drr, "\tau_Q"'] & & & & T^*Q \arrow[dll, "\pi_Q"] \\
 & & Q & &
\end{tikzcd} \tag{12.18}$$

It has natural coordinates (q^i, v^i, p_i, z), We say that a vector field X on W is a *SODE* if its integral curves $(q^i(t), v^i(t), p_i(t), z(t))$ satisfy $v(t) = \dot{q}(t)$, or, equivalently, it has the following expression

$$X = v^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial v^i} + G_i \frac{\partial}{\partial p_i} + f \frac{\partial}{\partial z}. \tag{12.19}$$

The bundle W has the following canonical structures.

- The *coupling function* defined as

$$\begin{aligned} \mathcal{C} : W &\to \mathbb{R}, \\ w = (v_q, p_q, z) &\mapsto \langle p_q, v_q \rangle. \end{aligned} \tag{12.20}$$

- The *canonical precontact form* $\eta = \rho_1^*(\eta_Q)$.

Given a Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$, we define the Hamiltonian $H : W \to \mathbb{R}$ by

$$H = \mathcal{C} - L = p_i v^i - L. \tag{12.21}$$

Thus, (W, η, H) is a precontact system that can be studied through the constraint algorithm.

12.1.7. The unified equations of motion

By applying the constraint algorithm (see Chapter 5) to an arbitrary vector field of the form (12.19), we obtain the following constraints

- $f^i = v^i$, which ensures that the Hamiltonian vector fields are SODEs.
- $p_i = \frac{\partial H}{\partial \dot{q}^i}$ defining for the first constraint submanifold W_1, which is the graph of the Legendre transformation.
- $G_i = \frac{\partial L}{\partial q^i} + p_i \frac{\partial L}{\partial z}$ provides dynamical equations.

We remark that this formalism automatically solves the second order problem (see Section 5.5), since the solutions are necessarily SODEs. Also, the algorithm itself provides the definition of the Legendre transformation.

If the Lagrangian is regular, then we are able to determine the functions F^i and the algorithm ends. In case that the Lagrangian is singular, we will obtain further submanifolds $\cdots W_2 \hookrightarrow W_1 \hookrightarrow W_0$ by applying the algorithm which might stop on a final constraint submanifold W_f.

Even on the singular case, the constraint submanifolds, the Lagrangian solutions which are SODEs and the Hamiltonian solutions can be retrieved through the projections ρ^1, ρ^2.

12.2. Higher order Lagrangian systems

The Hamiltonian formulation of systems described by higher order regular Lagrangians (depending on derivatives of order greater than 1) was first developed by Ostrogradsky [210] in 1850, providing the corresponding Euler-Lagrange and Hamiltonian mechanics. Nevertheless, the geometrization of took a long time, compared with the theory of first order Lagrangians and Hamiltonian systems. Nowadays, there exist geometric versions for both autonomous and non-autonomous mechanics and field theory [80, 108, 153, 154, 211, 212], and with application in control theory [16, 69].

The geometric setting for this theory is the *tangent bundle of order k*, T^kQ. Indeed, if the Lagrangian L depends on derivatives up to order k, then it is a function on T^kQ. Its dynamics occurs on the bundle of order $2k-1$, $T^{2k-1}Q$, giving the Euler-Lagrange equations of order $2k$. In addition, the Hamiltonian description takes place on the cotangent bundle $T^*(T^{k-1}Q)$, which is canonically a symplectic manifold and (for regular Lagrangians) both are connected by the *Legendre transform* $\mathrm{F}L \colon T^{2k-1}Q \to T^*(T^{k-1}Q)$.

In [85] we developed the theory of higher order action-dependent Lagrangian. We obtain the dynamics on two ways: through a variational approach generalizing the Herglotz principle, and through a unified formalism. Both dynamics coincide. Below, we present the variational formalism and obtain the equations of motion, which have a similar proof than the first order case. The reader can find the full geometric theory and some examples on the paper.

12.2.1. The higher order Herglotz principle

Assume that L is a Lagrangian on the tangent bundle of order k, $T^kQ\times\mathbb{R}$. This bundle has local coordinates (q^i_α), for $\alpha\in\{0,\dots,k\}$, representing the k-th derivative of the position on the i-th direction.

For a curve $c:[t_0,t_1]\to Q$, we define its canonical lift $\bar{c}^k$ to T^kQ by

$$(\bar{c}^k)^i_\alpha = \frac{\mathrm{d}^\alpha c^i}{\mathrm{d}t^\alpha}. \tag{12.22}$$

If $q_0,q_1\in T^kQ$, we denote by $\Omega(q_0,q_1)$ the space of smooth curves $c:[t_0,t_1]\to Q$ such that their k-jet lifts fulfill $\bar{c}^k_\alpha(0)=q_{0,\alpha}, \bar{c}^k_\alpha(t_1)=q_{1,\alpha}$, for $0\leq\alpha\leq k-1$.

Given a Lagrangian of order k, $L\colon T^kQ\times\mathbb{R}\to\mathbb{R}$. We fix $z_0\in\mathbb{R}$, and define the operator

$$\mathcal{Z}_{L,z_0}\colon \Omega(q_0,q_1)\to\mathcal{C}^\infty([t_0,t_1]) \tag{12.23}$$

which assigns to each curve c the function $\mathcal{Z}_{L,z_0}(c)$ that solves the following differential equation

$$\begin{aligned}\frac{\mathrm{d}\mathcal{Z}_{L,z_0}(c)}{\mathrm{d}t} &= L(\bar{c}^k,\mathcal{Z}_{L,z_0}(c)),\\ \mathcal{Z}_{L,z_0}(c)(0) &= z_0.\end{aligned} \tag{12.24}$$

Definition 12.1. The *higher-order contact action functional* associated with a Lagrangian function L is the map which assigns to each curve c, the increment of the solution to the equation (12.24):

$$\begin{aligned}\mathcal{A}_{L,z_0}:\Omega(q_0,q_1)&\to\mathbb{R},\\ c&\mapsto\mathcal{Z}_{L,z_0}(c)(t_1)-z_0=\int_{t_0}^{t_1}L(\bar{c}^k(t))\mathrm{d}t.\end{aligned} \tag{12.25}$$

We remark that the Euler-Lagrange action of order k is retrieved from this principle in case that the Lagrangian does not depend on the action z. We also obtain the Herglotz action in the case that $k=1$.

Theorem 12.5 (Higher-order Herglotz variational principle)**.** *Let $L\colon T^kQ\times\mathbb{R}\to\mathbb{R}$ be a Lagrangian function and let $c\in\Omega(q_0,q_1)$ and $z_0\in\mathbb{R}$. Then, c is a critical point of $\mathcal{A}_{L,z_0}$ if, and only if, $(\bar{c},\mathcal{Z}_{L,z_0}(c))$ satisfies the following equations*

$$\sum_{\alpha=0}^{k}(-1)^\alpha\mathcal{D}^\alpha_{L,z_0,c}\left(\frac{\partial L}{\partial q^i_\alpha}(\bar{c}^k(t),\mathcal{Z}_{L,z_0}(c)(t))\right)=0\,; \tag{12.26}$$

where, for $f\colon \mathbb{R} \to \mathbb{R}$*, the* Herglotz operator $\mathcal{D}_{L,z_0,c}$ *is defined as*

$$(\mathcal{D}_{L,z_0,c}f)(t) := \frac{\mathrm{d}f}{\mathrm{d}t}(t) - \frac{\partial L}{\partial z}(\bar{c}^k(t), \mathcal{Z}_{L,z_0}(c)(t))f(t)\ ; \tag{12.27}$$

Equations (12.26) *are called the* $(2k)$*-th order Herglotz equations.*

Proof. Let $c \in \Omega(q_0, q_1)$. Then c is a critical curve of the map $\mathcal{A}_{L,z_0}$, if and only if $(T_c\mathcal{A}_{L,z_0})(\delta c) = 0$ for every $\delta c \in T_c\Omega(q_0, q_1)$. This tangent space is given by

$$\begin{aligned} T_c\Omega(q_0, q_1) = \{\delta c = (c; \delta c_0, \delta c_1, \dots, \delta c_{k-1}) : [t_0, t_1] \to T^kQ \mid \\ \rho_0^k \circ \delta c = c, \delta c_\alpha(t_0) = 0,\ \delta c_\alpha(t_1) = 0 \text{ for } 0 \leq \alpha < k-1\}. \end{aligned} \tag{12.28}$$

First, we compute $(T_c\mathcal{Z}_{L,z_0})(\delta c)$. Let $c_\lambda \in \Omega(q_0, q_1)$, $\lambda \in (-\delta, \delta) \subseteq \mathbb{R}$, be a smoothly parametrized family of curves giving δc and their k-jets with respect to λ at $\lambda = 0$. That is:

$$\delta c_0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0} c_\lambda, \quad \delta c_\alpha = \frac{\mathrm{d}^\alpha}{\mathrm{d}\lambda^\alpha}|_{\lambda=0}\dot{c}_\lambda. \tag{12.29}$$

We also denote

$$\delta c_k = \frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0}\frac{\mathrm{d}^k}{\mathrm{d}t^k}c_\lambda\ .$$

In order to simplify the notation, we write $\psi = (T_c\mathcal{Z}_{L,z_0})(\delta c)$. Since $\mathcal{Z}_{L,z_0}(c_\lambda)(t_0) = z_0$ for all λ, then $\psi(t_0) = 0$. Also notice that $\psi(t_1) = (T_c\mathcal{Z}_{L,z_0})(\delta c)(t_1) = (T_c\mathcal{A}_{L,z_0})(\delta c)$. We have that:

$$\psi = (T_c\mathcal{Z}_{L,z_0})(\delta c) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{Z}_{L,z_0}(c_\lambda)|_{\lambda=0}\ .$$

Then, computing the derivative of ψ with respect to t, interchanging the derivatives, we have:

$$\begin{aligned} \dot{\psi}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{Z}_{L,z_0}(c_\lambda)(t)|_{\lambda=0} = \frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_{L,z_0}(c_\lambda)(t)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}|_{\lambda=0}L(\bar{c}_\lambda^k(t), \mathcal{Z}_{L,z_0}(c_\lambda)(t)) \\ &= \sum_{\alpha=0}^{k}\frac{\partial L}{\partial q_\alpha^i}(\bar{c}^k(t), \mathcal{Z}_{L,z_0}(c)(t))\delta c_\alpha^i(t) + \frac{\partial L}{\partial z}(\bar{c}^k(t), \mathcal{Z}_{L,z_0}(c)(t))\psi(t) \\ &= A(t) + B(t)\psi(t)\ . \end{aligned}$$

Hence, ψ is the solution to the Cauchy condition problem:

$$\begin{aligned} \dot{\psi}(t) &= A(t) + B(t)\psi(t), \\ \psi(t_0) &= 0. \end{aligned} \tag{12.30}$$

The solution is given by

$$\psi(t) = \exp\left(\int_{t_0}^t B(s)\mathrm{d}s\right)\int_{t_0}^t \mathrm{d}u\, A(u)\exp\left(-\int_{t_0}^t B(s)\mathrm{d}s\right), \tag{12.31}$$

and writing

$$\sigma_{L,z_0}(t) = \exp\left(- \int_{t_0}^{t} \frac{\partial L}{\partial z}(\tilde{c}^k(\tau), \mathcal{Z}_{L,z_0}(c)(\tau))\mathrm{d}\tau \right) > 0, \tag{12.32}$$

we have that

$$\psi(t) = \frac{1}{\sigma_{L,z_0}(t)} \int_{t_0}^{t} \sigma_{L,z_0}(\tau) \left(\sum_{\alpha=0}^{k} \frac{\partial L}{\partial q^i_\alpha}(\tilde{c}^k(\tau), \mathcal{Z}_{L,z_0}(c)(\tau))\delta c^i_\alpha(\tau) \right) \mathrm{d}\tau. \tag{12.33}$$

Now integrating by parts, and taking into account that variations vanish at the endpoints, we have that:

$$\begin{aligned}\psi(t_1) &= (T_c\mathcal{A}_{L,z_0})(\delta c)\\ &= \frac{1}{\sigma_{L,z_0}(t)} \int_{t_0}^{t_1} \sum_{\alpha=0}^{k} \delta c^i_0(\tau)(-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha} \left(\sigma(\tau) \frac{\partial L}{\partial q^i_\alpha}(\tilde{c}^k(\tau), \mathcal{Z}_{L,z_0}(c)(\tau)) \right) \mathrm{d}\tau.\end{aligned}$$

We know that ψ vanishes if the curve c is a critical point. Then as the variations δc_0 are arbitrary, by the fundamental theorem of calculus of variations, we obtain

$$\frac{1}{\sigma_{L,z_0}(t)} \sum_{\alpha=0}^{k} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \left(\sigma(t) \frac{\partial L}{\partial q^i_\alpha}(\tilde{c}^k(t), \mathcal{Z}_{L,z_0}(c)(t)) \right) = 0\,,$$

where we have used that $\sigma_{L,z_0} : [t_0, t_1] \to \mathbb{R}$, as it has been defined, satisfies the conditions

$$\frac{\mathrm{d}\sigma_{L,z_0}}{\mathrm{d}t} = -\frac{\partial L}{\partial z}\sigma_{L,z_0}\ , \quad \sigma_{L,z_0}(t_0) = z_0\,.$$

Then, introducing the operator $\mathcal{D}_{L,z_0,c}$ defined in (12.27), this equation reduces to (12.26). □

Although the Herglotz operator makes the equations look simpler, written explicitly for the second order case the equations are as follows

$$
\begin{aligned}
0 = {} & \frac{\partial L}{\partial q_0^i} - \mathcal{D}_L \frac{\partial L}{\partial q_1^i} + \mathcal{D}_L \mathcal{D}_L \frac{\partial L}{\partial q_2^i} = q_4^j \frac{\partial^2 L}{\partial q_2^j \partial q_2^i} + q_3^j q_3^k \frac{\partial^3 L}{\partial q_2^k \partial q_2^j \partial q_2^i} \\
& + q_3^k \Big(-\frac{\partial^2 L}{\partial q_2^k \partial q_1^i} + \frac{\partial^2 L}{\partial q_1^k \partial q_2^i} + 2 q_2^j \frac{\partial^3 L}{\partial q_2^k \partial q_1^j \partial q_2^i} + 2 q_1^j \frac{\partial^3 L}{\partial q_2^k \partial q_0^j \partial q_2^i} + \frac{\partial L}{\partial q_2^k} \frac{\partial^2 L}{\partial z \partial q_2^i} \\
& + 2L \frac{\partial^3 L}{\partial q_2^k \partial z \partial q_2^i} - \frac{\partial^2 L}{\partial q_2^k \partial z} \frac{\partial L}{\partial q_2^i} - 2 \frac{\partial L}{\partial z} \frac{\partial^2 L}{\partial q_2^k \partial q_2^i} \Big) \\
& + q_2^k \Big(-\frac{\partial^2 L}{\partial q_1^k \partial q_1^i} + q_2^j \frac{\partial^3 L}{\partial q_1^k \partial q_1^j \partial q_2^i} + \frac{\partial^2 L}{\partial q_0^k \partial q_2^i} + 2 q_1^j \frac{\partial^3 L}{\partial q_1^k \partial q_0^j \partial q_2^i} + \frac{\partial L}{\partial q_1^k} \frac{\partial^2 L}{\partial z \partial q_2^i} \\
& + 2L \frac{\partial^3 L}{\partial q_1^k \partial z \partial q_2^i} - \frac{\partial^2 L}{\partial q_1^k \partial z} \frac{\partial L}{\partial q_2^i} - 2 \frac{\partial L}{\partial z} \frac{\partial^2 L}{\partial q_1^k \partial q_2^i} \Big) \\
& + q_1^k \Big(-\frac{\partial^2 L}{\partial q_0^k \partial q_1^i} + q_1^j \frac{\partial^3 L}{\partial q_0^k \partial q_0^j \partial q_2^i} + \frac{\partial L}{\partial q_0^k} \frac{\partial^2 L}{\partial z \partial q_2^i} \\
& + 2L \frac{\partial^3 L}{\partial q_0^k \partial z \partial q_2^i} - \frac{\partial^2 L}{\partial q_0^k \partial z} \frac{\partial L}{\partial q_2^i} - 2 \frac{\partial L}{\partial z} \frac{\partial^2 L}{\partial q_0^k \partial q_2^i} \Big) \\
& + L^2 \frac{\partial^3 L}{\partial z \partial z \partial q_2^i} - L \frac{\partial^2 L}{\partial z \partial z} \frac{\partial L}{\partial q_2^i} - L \frac{\partial L}{\partial z} \frac{\partial^2 L}{\partial z \partial q_2^i} - \frac{\partial L}{\partial z} \frac{\partial L}{\partial z} \frac{\partial L}{\partial q_2^i} - L \frac{\partial^2 L}{\partial z \partial q_1^i} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial q_1^i} + \frac{\partial L}{\partial q_0^i}.
\end{aligned}
$$

Nevertheless, in cases of interest, most of the higher order derivatives vanish.

13. Open problems and further work on contact systems

Several major problems are still open in the field of contact Hamiltonian systems. Here we explain in detail two of them.

13.1. The inverse problem for the Herglotz principle

The inverse problem of mechanics has a long history since its establishment by Helmholtz [158] and Hirsch [161] at the end of the 19th century. It was solved it in two dimensions by Douglas [117], who received the Fields medal for it. This problem has been studied extensively from numerous viewpoints and generalized to many other situations [54, 78, 179, 201, 225, 243].

Given a Lagrangian function $L : TQ \to \mathbb{R}$ one obtains the *Euler-Lagrange* vector field ξ_L, which is a second order differential equation such that its integral curves $c(t)$ are solutions of the Euler-Lagrange equation for L. As its name suggests, the inverse problem asks the inverse question. Consider a SODE ξ on TQ such that its integral curves $c(t)$ satisfy the equation

$$\ddot{c}^i(t) = f^i(c(t), \dot{c}(t)) \tag{13.1}$$

for some local functions $f^i : U \subseteq TQ \to \mathbb{R}$. The so-called *inverse problem of calculus of variations* asks if the equation above can be derived through a Lagrangian L. Namely, is $\xi = \xi_L$ for some Lagrangian L. A partial solution to this problem is given by Helmholtz conditions [72]. The inverse problem can be solved if and only if there exist functions g_{ij} such that

$$\det g_{ij} = 0 \tag{13.2a}$$

$$g_{ij} = g_{ji} \tag{13.2b}$$

$$\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial g_{ik}}{\partial q^j} \tag{13.2c}$$

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} + \frac{1}{2}\frac{\partial f^k}{\partial \dot{q}^j} g_{ik} + \frac{1}{2}\frac{\partial f^k}{\partial \dot{q}^i} g_{kj} = 0 \tag{13.2d}$$

$$g_{ik}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f^k}{\partial \dot{q}^j}\right) - 2\frac{\partial f^k}{\partial q^j} - \frac{1}{2}\frac{\partial f^l}{\partial \dot{q}^j}\frac{\partial f^k}{\partial \dot{q}^l}\right) = g_{jk}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f^k}{\partial \dot{q}^i}\right) - 2\frac{\partial f^k}{\partial q^i} - \frac{1}{2}\frac{\partial f^l}{\partial \dot{q}^i}\frac{\partial f^k}{\partial \dot{q}^l}\right). \tag{13.2e}$$

Nevertheless, there are systems that cannot be described through the Euler-Lagrange equations, but they can be obtained form the Herglotz principle, such as the mechanical system with action-dependent potential (Example 3.1), including systems with homogeneous Rayleigh dissipation as a particular case, or the parachute equation (Example 3.2). Something like the Helmholtz conditions form the Herglotz principle would be of great help for recognizing Herglotz vector fields from ODEs, and enlarging the number of applications of contact Hamiltonian systems. Nonetheless, the naive extension to the contact case is problematic.

Problem 13.1 (Inverse Herglotz problem, first attempt). Given a SODE ξ on $TQ \times \mathbb{R}$, determine if there exists a contact Lagrangian $L : TQ \times \mathbb{R} \to \mathbb{R}$ such that $\xi = \xi_L$.

Indeed, the local form of the Herglotz vector field is

$$\xi_L = \dot{q}^i \frac{\partial}{\partial q} + h_L^i \frac{\partial}{\partial \dot{q}} + L \frac{\partial}{\partial z}, \tag{13.3}$$

where h_L^i is the unique solution to the equation

$$\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} h_L^i + \dot{q}^i \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} + L \frac{\partial^2 L}{\partial \dot{q}^j \partial z} \frac{\partial L}{\partial q^j} = -\frac{\partial L}{\partial \dot{q}^j} \frac{\partial L}{\partial z}. \tag{13.4}$$

So, if ξ is a SODE on $TQ \times \mathbb{R}$ with a coordinate expression

$$\xi = \dot{q}^i \frac{\partial}{\partial q} + a^i \frac{\partial}{\partial \dot{q}} + b \frac{\partial}{\partial z}. \tag{13.5}$$

Hence, comparing coefficients, if $\xi = \xi_L$, then $L = b$. In order to solve this problem, it would suffice to compute the Herglotz equations for $L = b$ and see if they match those of ξ, which makes the problem trivial.

Nonetheless, the language of equivalent Lagrangians (Section 4.1) suggests a formulation of the inverse problem which is both interesting from the mathematical point of view and can be useful for the applications. As it was discussed in Section 4.1, the variable z is not measurable on many practical situations. What can be directly observed is an equivalence class of Lagrangians (Definition 4.17). Taking this into account, we would obtain the following problem.

Problem 13.2 (Inverse Herglotz problem). Given a SODE ξ on $TQ \times \mathbb{R}$, determine weather it is an extended Herglotz vector field, that is, weather there exists an extended Lagrangian system (L, ζ) such that $\xi = \xi_L^\zeta$.

Now we work on an extended Lagrangian system, and we are able to choose independently the action function ζ and the Lagrangian L. That is, we are asking if ξ is horizontally equivalent to a Lagrangian system.

For some Lagrangians, such as the one for the parachute equation (Example 3.2), the Herglotz equations for the position are uncoupled from the z variables. Geometrically, the Herglotz vector field ξ_L is ρ-projectable (defined before Proposition 4.32), where $\rho P \to TQ$ is an extended tangent bundle. The equations of motion for the positions would be represented geometrically by $\xi_0 = \rho_* \xi$, removing the information about the equation $\dot{\zeta} = L$. Given second order differential equations on Q, we might ask if it is possible to add an extra action variable ζ and an action-dependent Lagrangian L so that the Herglotz equations of L for the positions (removing the equation $\dot{\zeta} = L$) coincide with the aforementioned second order equations. Geometrically, this can be expressed as follows.

Problem 13.3 (Inverse Herglotz problem, projectable version). Given a SODE ξ_0 on TQ, determine weather there exists an extended Herglotz vector field ξ_L^ζ such that $\rho_* \xi_L^\zeta = \xi_0$.

By Proposition 4.32, given an extended Lagrangian (L, ζ) with a ρ-projectable Herglotz vector field, all the extended Lagrangians equivalent to (L, ζ) are also ρ-projectable and have the same projection. This allows as to see that the following problem is completely equivalent to the previous one.

Problem 13.4 (Inverse Herglotz problem, projectable version). Given a SODE ξ_0 on TQ, determine weather there exists a Herglotz vector field ξ_L such that $\left(\mathrm{pr}_{TQ}\right)_* \xi_L = \xi_0$, where $\mathrm{pr}_{TQ} : TQ \times \mathbb{R} \to TQ$ is the canonical projection.

These problems were formulated and studied in [84]. In that paper we provide a geometric characterization in the spirit of [225] and a partial solution for the strong case, but the problem is largely unsolved.

13.2. The Herglotz principle for fields

There are several approaches to generalize both classical field theory and contact systems. One of them is a geometric framework introduced in [130, 132] (see also [221]) called the k-contact formalism. This theory is inspired by the k-symplectic formalism, combining it with the contact theory. The phase space of this theory is the k-tangent $\oplus^k TQ \times \mathbb{R}$ (or the k-cotangent $\oplus^k TQ \times \mathbb{R}$ bundle). Many important PDEs which are not Lagrangian, such as the damped wave equation or the Burger's equation fit on this framework. Some disadvantages of this theory is that it is not covariant and that arbitrary jet bundles cannot be chosen as phase space.

From a variational point of view, there has been several attempts [140, 181] to generalize the Herglotz principle to field theories.

In [133] we propose a variational theory that works on arbitrary fiber bundles without extra structure and generalizes the previous variational principles available on the literature. It is inspired on the vakonomic approach to the Herglotz principle that we described in Section 6.1.1.

We let M be a spacetime manifold and the fields $\sigma^a(x_\mu)$ are local sections of the fiber bundle $E \to M$. Let J^1E be the first jet bundle of E, with natural coordinates (x^μ, u^a, u^a_μ).

Now, instead of adding an extra coordinate to the phase space for the action, we will add an *action density* ζ, which is a $(m-1)$-form on M. Hence, our extended phase space will be the product of the bundle of $(m-1)$-forms with the jet bundle J^1E over M, that is, $J^1E \times_M \Lambda^{m-1}$. The Lagrangian density will be allowed to depend on the action density in addition to the spacetime position, the fields, and their first derivatives. Hence, $\mathcal{L} : J^1E \times_M \Lambda^{m-1}M \to \Lambda^m M$ is a fiber bundle morphism over M.

Now we need to generalize the fact that the time derivative of the action is the Lagrangian. From a geometrical viewpoint, a good candidate is to change the Lagrangian by the Lagrangian density and the time derivative of the action by the exterior derivative of the action density:

$$\mathrm{d}\zeta = \mathcal{L} \circ (j^1\sigma, \zeta). \tag{13.6}$$

The situation is described by the following commutative diagram

$$\begin{tikzcd}
J^1E \arrow[d] & J^1E \times_M \Lambda^{m-1}M \arrow[l] \arrow[d] \arrow[dr, "\mathcal{L}"] & \\
E \arrow[dr] & \Lambda^{m-1} \arrow[d] & \Lambda^m M \\
& M \arrow[ul, bend left, "\sigma"] \arrow[uul, "j^1\sigma"'] \arrow[u, bend right, "\zeta"'] \arrow[uu, bend right, "(j^1\sigma,\zeta)"'] \arrow[ur, "d\zeta"'] &
\end{tikzcd}$$

Given a set of local coordinates x^μ of M, we can construct a basis $*(\mathrm{d}x)_\mu = \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge \dots \wedge \mathrm{d}x^n$ of the $(m-1)$-forms. This basis induces a set of coordinates z^μ on Λ^{m-1}. If, in local coordinates, $\mathcal{L}(x^\mu, u^a, u^a_\mu, z^\mu) = L(x^\mu, u^a, u^a_\mu, z^a)\mathrm{d}^n x$, and $\zeta = \zeta^\mu * (\mathrm{d}x)_\mu$ is an m-form. If $\sigma \in \Gamma_M E$, its 1-jet is $j^1\sigma = (\sigma^a, \partial_\mu \sigma^a)$. Equation (13.6) is written as follows

$$\partial_\mu \zeta^\mu(x^\mu) = L(x^\mu, \sigma^a, \partial_\mu \sigma^a, z_a)\,.$$

Now, let $D \subseteq M$ be a compact manifold with boundary diffeomorphic to the n-dimensional closed ball. We fix a section $\bar\rho = (\rho, \tau) : \partial D \to E \oplus_M \Lambda^{m-1}M$ of boundary values. We consider the space $\bar\Omega(\bar\rho)$ of sections $\bar\sigma = (\sigma, \zeta)$ of $E \oplus_M \Lambda^{m-1}M$ satisfying the boundary condition $\bar\sigma|_{\partial D} = \bar\rho$.

The action $\bar{\mathcal{A}} : \bar\Omega(\bar\rho) \to \mathbb{R}$ is defined as

$$\bar{\mathcal{A}}(\sigma, \zeta) = \int_D \mathrm{d}\zeta = \int_{\partial D} \zeta, \tag{13.7}$$

where the equality follows from Stokes theorem.

We consider the set of *admissible sections* $\bar\sigma = (\sigma, \zeta)$, with vakonominc constraints given by Equation (13.6). We denote it by $\bar\Omega_{\mathcal{L}}(\bar\rho) \subseteq \bar\Omega_{\mathcal{L}}(\bar\rho)$. For those admissible sections, the action equals

$$\bar{\mathcal{A}}(\sigma, \zeta) = \int_{\partial D} \mathcal{L} \circ (j^1\sigma, \zeta). \tag{13.8}$$

In particular, it coincides with the usual action for fields in the case that the Lagrangian density does not depend on ζ.

With this action we can obtain the following vakonomic principle.

Theorem 13.1 (Herglotz variational principle, constrained version)**.** *Let $\mathcal{L} : J^1E \oplus_M \Lambda^{m-1}M \to \Lambda^m M$ be a Lagrangian density and let $(\sigma, \zeta) \in \bar\Omega_{\mathcal{L}}(\bar\rho)$. Then, $(j^1\sigma, \zeta)$ satisfies the*

Herglotz equations for fields:

$$\partial_\mu \left(\frac{\partial L}{\partial u^a_\mu} \right) - \frac{\partial L}{\partial u^a} = \frac{\partial L}{\partial u^a} \frac{\partial L}{\partial z^\mu}\,, \tag{13.9}$$

$$\partial_\nu \frac{\partial L}{\partial z^\mu} = \partial_\mu \frac{\partial L}{\partial z^\nu}, \tag{13.10}$$

if and only if (σ, ζ) is a critical point of the action restricted[1] *to the space of admissible sections.*

The missing ingredient of this theory is a geometric formulation. On the action-independent case, this is achieved through *multisymplectic geometry*. Here we hope to find something that we might call *multicontact geometry*. This theory is almost completely undeveloped, and studying its properties and applications is a fascinating research project.

[1]By technical reasons we need to allow the admissible variations $\bar{\sigma}_s$ to violate the constraints up to first order in s,

Appendix

A. Lifts of functions, vector fields and forms to the tangent bundle

Given a form α or a vector field X on a manifold M, one is able to lift it to a form or a vector field on the tangent bundle TM. In order to clarify this, we will explain the definition vertical and complete lifts of differential forms, which can be found in [106, Sec. 2.5]. For a comprehensive reference on lifts of tensor fields, one can read [258].

Definition A.1 (Vertical lift)**.** Let M be an m-dimensional differentiable manifold with a set of coordinates $(x_1, \ldots, x_m)$, let TM be its tangent bundle, with induced coordinates $(x_1, \ldots, x_m, \dot{x}_1, \ldots, \dot{x}_m)$, and let $\tau_M : TM \to M$ be the canonical projection.

Let $\alpha \in \Omega^k(M)$ be a form, which can be expressed in coordinates as

$$\sum_{(i_1,\ldots,i_k)} a_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}. \tag{A.1}$$

The *vertical lift* of α is the form $\alpha^v \in \Omega(TM)$ such that,

$$\alpha^V = \tau_M^* \alpha. \tag{A.2}$$

In coordinates, this is written as follows

$$\sum_{(i_1,\ldots,i_k)} a_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}, \tag{A.3}$$

where we remark that we are using the coordinates from TM, and not the ones form M.

Let X be a vector field on M. Its *vertical lift* X^V is the unique vector field on TM satisfying [258, Proposition 2.1]

$$X^V(\hat{\alpha}) = (\alpha(X))^V, \tag{A.4}$$

where $\aleph \in \Omega^1(M)$ and $\hat{\alpha} : TM \to \mathbb{R}$ is just the function $\hat{\alpha}(X_x) = \alpha_x(X_x)$. If X has the coordinate expression

$$\mathrm{X} = X^i \frac{\partial}{\partial x^i}, \tag{A.5}$$

then

$$X^V = X^i \frac{\partial}{\partial \dot{x}^i}, \tag{A.6}$$

Definition A.2 (Complete lift)**.** Using the same notations as in the previous definition, let $f \in C^\infty(M)$. We define the *complete lift of the function f*, which we will denote f^C:

$$\begin{aligned} f^C : TM &\to \mathbb{R} \\ v_x &\mapsto \mathrm{d}_x f(v). \end{aligned} \tag{A.7}$$

It has the coordinate expression

$$f^C = \frac{\partial f}{\partial x^i}\dot{x}^i \tag{A.8}$$

The *complete lift of a vector field* $X \in \mathfrak{X}(M)$ is denoted $X^C \in \mathfrak{X}(TM)$ and it is defined as the infinitesimal generator of $T\phi_t$, where ϕ_t is the flow of X. In coordinates it is given by

$$X^C = X^i \frac{\partial}{\partial x^i} + \dot{x}^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial \dot{x}^i}. \tag{A.9}$$

The *complete lift of a form* $\omega \in \Omega^k(M)$ is denoted $\omega^C \in \Omega^k(TM)$, and it is defined as the unique (see [106, Sec. 2.5]) form such that

$$\omega^C({X_1}^C, \dots, {X_k}^C) = (\omega(X_1, \dots, X_k))^C, \tag{A.10}$$

where $X_1, \dots, X_k \in \mathfrak{X}(M)$. For a 1-form $\alpha = \alpha_i \mathrm{d}x^i$ it is given by

$$\alpha^C = (\alpha^C)^i \mathrm{d}x^i + (\alpha^V)^i \mathrm{d}\dot{x}^i \tag{A.11}$$

The complete lift of forms has the following property [258, Eq. 3.30]:

$$(\omega \wedge \rho)^C = \omega^C \wedge \rho^V + \omega^V \wedge \rho^C. \tag{A.12}$$

Other properties that we will use are [106, 258]

$$X^*(\eta^C) = \mathcal{L}_X \eta, \quad X^*(\eta)^V = \eta. \tag{A.13}$$

$$\mathrm{d}(\alpha^C) = (\mathrm{d}\alpha)^C, \quad \mathrm{d}(\alpha^V) = (\mathrm{d}\alpha)^V \tag{A.14}$$

Also, given a map $F : TM \to M_2$ and a form $\alpha \in \Omega(M_2)$,

$$TF^*({\alpha_1}^C) = (F^*\alpha)^C. \tag{A.15}$$

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