

Department of Mathematics and Applications

Curriculum in Algebra and Geometry

A doctoral Dissertation

Results on Artin and twisted Artin groups

PhD candidate: **Islam Foniqi**

Tutor/supervisor: **Prof. Thomas Weigel**

Registration number: **847971**

Second supervisor: **Prof. Yago Antolín**

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Coordinator: **Prof. Pierluigi Colli**

Abstract

This thesis consists of three main chapters, and they all revolve around Artin groups. Proving results for all Artin groups is a serious challenge, so one usually focuses on particular subclasses. Among the most well understood subfamilies of Artin groups is the family of right-angled Artin groups (RAAGs shortly). One can define them using simplicial graphs, which determine the group up to isomorphism. They are also interesting as there are a variety of methods for studying them, coming from different viewpoints, such as geometry, algebra, and combinatorics. This has resulted in the understanding of many problems in RAAGs, like the word problem, the spherical growth, intersections of parabolic subgroups, etc.

In Chapter [2](#) we focus on the geodesic growth of RAAGs, over link-regular graphs, and we extend a result in that direction, by providing a formula of the growth over link-regular graphs without tetrahedra.

In Chapter [3](#) we work with slightly different groups, the class of twisted right-angled Artin groups (tRAAGs shortly). They are defined using mixed graphs, which are simplicial graphs where edges are allowed to be directed edges. We find a normal form for presenting the elements in a tRAAG. If we forget about directions of edges, we obtain an underlying undirected graph, which we call the naïve graph. Over the naïve graph, which is simplicial, one can define a RAAG, which corresponds naturally to our tRAAG. We will discuss some algebraic and geometric similarities and differences between tRAAGs and RAAGs. Using the normal form theorem we are able to conclude that the spherical and geodesic growth of a tRAAG agrees with the respective growth of the underlying RAAG.

Chapter [4](#) has a different theme, and it consists of the study of parabolic subgroups in even Artin groups. The work is motivated by the corresponding results in RAAGs, and we generalize some of these results to certain subclasses of even Artin groups.

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Chapter 1

Introduction and Preliminaries

In this chapter we present the core results of the thesis, and we provide an overview of the topics that motivate these results. Here is also where we set up the notation and where we give the most important definitions that we use throughout this dissertation. We also introduce key concepts and some examples which allow us to give the statements of the main results.

1.1 Introduction

Artin groups present one of the most celebrated families of groups in contemporary research, where the interplay from both algebraic and geometric techniques have given rewarding results. They are also known as Artin–Tits groups, named after Emil Artin, and Jacques Tits, due to their work on braid groups (in [4]), and on extensions of Coxeter groups (in [37]) respectively.

One defines *Artin groups* as finitely presented groups by giving a set S of generators called *Artin generators*, and a set of relations (*Artin relations*) defined for distinct pairs a, b of elements of S in the form:

$$\underbrace{abab\dots}_{k\text{-factors}} = \underbrace{baba\dots}_{k\text{-factors}} \quad (1.1)$$

for some $k \geq 2$, such that any distinct pair a, b can have at most one Artin relation associated to it.

We allow $k = \infty$ in Equation (1.1), to mean that the corresponding pair a, b has no Artin relation; in this case we say that the pair a, b is *free of relations*. We use the notation (A, S) to mean that A is an Artin group with Artin generators S , which gives preference to the generating set and it will be useful when defining parabolic subgroups.

Equivalently we can encode the definition for Artin groups on a finite simplicial graph Γ (see Definition 1.2.1) whose edges are labeled by integers. The vertices of Γ present

the Artin generators, while an edge joining two vertices a, b , and labeled by $k \geq 2$, presents the corresponding Artin relation appearing in Equation (1.1). The graph Γ is called the *defining graph* of the corresponding Artin group, which we denote by A_Γ , and furthermore we refer to A_Γ as the Artin group based on Γ .

If (A, S) is an Artin group, then adding the relations $a^2 = 1$ for any $a \in S$, we obtain a quotient group W , which is called a *Coxeter group*. There is a natural quotient morphism:

$$p: A \longrightarrow W \quad \text{defined by} \quad p(a) = a \text{ for any } a \in S, \quad (1.2)$$

and we refer to (W, S) as the *Coxeter group corresponding to the Artin group (A, S)* . Coxeter groups are known quite well and they provide a lot of motivation for achieving analogous results in Artin groups.

Although defining Artin groups is quite easy, understanding them well is difficult. For example, the word problem, which asks if a given word on generators represents the identity, is not solved for the whole class. In general, it is challenging to obtain good results for all Artin groups, and often one works on certain subfamilies.

Some of the well known classes among Artin groups are:

- *free groups*, where any two generators are free of relations,
- *free abelian groups*, where any two generators a, b are related by the commutative relation $ab = ba$,
- *right-angled Artin groups*, where any two generators a, b are either related by a commutative relation $ab = ba$, or they are free of relations,
- *Artin groups of spherical type*, where the corresponding Coxeter group is finite.

So the subclass of RAAGs (short for right-angled Artin groups) is the one where the relations among Artin generators in Equation (1.1) have $k = 2$, or $k = \infty$, i.e. the only possible relations are commutations. By definition they also include free groups and free abelian groups.

Using the language of the defining graphs we have that two RAAGs are isomorphic if and only if they have isomorphic defining graphs (see [19]).

The Coxeter groups corresponding to RAAGs are called *right-angled Coxeter groups* (shortly RACGs). In fact, any RAAG can be seen as a finite index subgroup of some RACG (see the theorem of Section 1 in [17]).

RAAGs and RACGs have attracted much attention in recent years, as they can be studied from different points of view. They possess algorithmic properties, which will be used for several combinatorial computations in Chapter 2. One associates to them a variety of complexes which serve as important tools for many applications (see [10] for a survey on RAAGs and RACGs).

Another interesting class of groups appearing in Chapter 3 of the thesis is the class of *twisted Artin groups*. They are finitely presented and are similarly defined by a canonical

presentation which looks similar to the one of Artin groups. For any two generators a, b the corresponding relation is either an Artin relation (as in equation (1.1)), or of the form:

$$\underbrace{abab\dots}_{(k+1)\text{-factors}} = \underbrace{babab\dots}_{(k-1)\text{-factors}} \quad (1.3)$$

for some $k \geq 2$. Again, we allow $k = \infty$ and the corresponding generators are free of relations. Note that for $k = 2$ in Equation (1.3) we get $aba = b$, which also can be written as $abab^{-1} = 1$, and it expresses the so-called *Klein relation* (see Example 3.1.1).

The main focus on twisted Artin groups is going to be the subclass of *twisted right-angled Artin groups* (tRAAGs shortly). This class is obtained when we get only commutations or Klein relations between generators, but not both relations for the same pair of generators.

Similarly to RAAGs, one can define tRAAGs using graphs. In this case the graphs we use are called *mixed graphs*, which are like simplicial graphs, but we allow some edges to be directed edges. The vertices represent the generators of the group, and edges give rise to the relations. An undirected edge, connecting generators a, b , gives rise to the commutation $ab = ba$, while a directed edge with origin at a and terminus at b defines the relation $aba = b$. If generators a, b are not connected by an edge, they are free of relations.

If we forget about directions of edges, we obtain an underlying undirected graph, which we call the *underlying naïve graph*, and one can define a RAAG over it, which is called the *underlying RAAG* of our tRAAG.

Despite having similar presentations, studying tRAAGs one can notice big differences with RAAGs. For example, one can have isomorphic tRAAGs based on non-isomorphic mixed graphs (see Example 3.1.3 in Chapter 3). Also one can have torsion in tRAAGs (see Section 3.6.3), which is not the case in RAAGs.

Both Chapter 2, and Chapter 3 of the thesis deal with the notion of growth in groups. This topic is fascinating as it connects algebra, geometry, analysis, and combinatorics to obtain profound results. When we discuss growth in a finitely presented group G we use a preferred generating set T to express the elements of G . For example, in the case of Artin groups (A, S) as a preferred generating set serves $T = S \sqcup S^{-1}$.

The *spherical growth function* counts the number of group elements in a sphere of a given radius with respect to the word metric (the alphabet used is the preferred generating set of the group). One can also view this function as counting the group elements on its Cayley Graph which are a given distance far from the identity element. One important feature of the spherical growth is that its asymptotic behaviour does not depend on the generating set of the group and it is quasi-isometry invariant.

Another growth function encountered often in the literature is the *standard growth function*. It is defined similarly as the spherical one, but using closed balls instead of

spheres in the word metric. Its value, for a given number n , is the cumulative sum of the values of the spherical growth function on integers from 0 to n .

One of the most celebrated results related to growth is Gromov's theorem on polynomial growth, which was proved by Gromov (see Section 8 in [23]), and it states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, i.e. has a finite index subgroup which is nilpotent. This provides an equivalence between polynomial standard growth, a geometric property, and virtual nilpotency, an algebraic property.

The *geodesic growth function* of a finitely generated group, with respect to a preferred generating set, counts the number of geodesics (shortest paths) of a given length, starting from the identity vertex in the Cayley graph of the group. This represents the word growth of the language of geodesics of the group.

To any growth function $f: \mathbb{N} \rightarrow \mathbb{N}$ we can associate a generating growth series, which is a formal power series of the form

$$\mathcal{S}(z) = \sum_{n=0}^{\infty} f(n)z^n \in \mathbb{Z}[[z]].$$

The series is called *rational* if it can be expressed as a quotient of two polynomials with coefficients in \mathbb{Z} .

The series associated to the spherical and geodesic growth functions are called spherical and geodesic growth series respectively. They encode geometric information of the Cayley graph of G (with respect to the generating set X).

Gromov studied the notion of geodesics on hyperbolic groups (see [24]). Theorem 3.4.5 in [21] shows that the language of geodesics of hyperbolic groups, with respect to any finite generating set, is regular (see Definition 1.2.18) which in particular implies that the geodesic growth series is rational (Remark 1.2.19). The geodesic growth has been less studied compared with standard growth, and moreover it is much more sensitive to the change of generating sets. The asymptotic properties of geodesic growth depend on the generating set and hence it is less intuitive to make connections with the algebraic properties of the group.

In Chapter 2 we refer to the dimension of RAAGs which comes from the general definition of the dimension of Artin groups as in Definition 1.1.1. The main result of that Chapter is Corollary 2.3.6, where we compute the growth series of RAAGs based on link-regular graphs (Definition 2.2.2) which do not contain tetrahedra.

Corollary 2.3.6 (Geodesic growth in some 3-dimensional RAAGs). Let $\Gamma = (V, E)$ be a simplicial graph with n vertices, and without tetrahedra. Assume that any vertex belongs to l edges, and any edge belongs to q triangles. Let $A = A_\Gamma$ be the RAAG based on Γ , and let $\mathcal{A}(z)$ denote the geodesic growth series of A with respect to the

generating set $V \cup V^{-1}$. Then we have the equality:

$$\mathcal{A}(z) = 1 + \frac{a(n, l, q, z)}{b(n, l, q, z)},$$

where the polynomials a, b are given as:

$$\begin{aligned} a(n, l, q, z) &= 2nz[1 + (5 - 2l - 2q)z + (4lq - 6l + 6)z^2], \text{ and} \\ b(n, l, q, z) &= 1 + (6 - 2n - 2l - 2q)z + (4nl + 4lq + 4qn - 10n - 6l - 2q + 11)z^2 \\ &\quad + (12nl + 6 - 8nlq - 12n)z^3. \end{aligned}$$

We see that over link-regular graphs without tetrahedra, the geodesic growth series is rational and it only depends on the numbers n, l, q . These numbers determine the clique-polynomial (Definition 2.2.3) of Γ uniquely (see 5), hence in our case the growth series is determined by the clique-polynomial of Γ and its coefficients (see also 1).

With a similar theme in mind, in Chapter 3 we find a normal form theorem for tRAAGs (see Section 3.4), and to state it we use the notion of *reduced sequence of syllables*. By *syllable* we mean an element $g \neq 1$ which is a power of a single generator. A sequence of syllables (g_1, \dots, g_n) is called reduced if any two members which are powers of the same generator, cannot be brought in adjacent positions using *shufflings* (which are moves that change the order of syllables, sometimes by affecting the sign of the powers). The most basic examples of shufflings are $ab \longleftrightarrow ba$ when a, b commute, and $ab \longleftrightarrow ba^{-1}$ when $aba = b$.

Theorem 3.4.13 (Normal form theorem for tRAAGs). Let G_Γ be a tRAAG. Each element $g \in G$ can be expressed uniquely (up to shuffling) as a product $g = g_1 \cdots g_n$, where (g_1, \dots, g_n) is a reduced sequence of syllables in G_Γ .

The unique reduced sequence representing the identity $1 \in G_\Gamma$ is the empty sequence.

An application of the normal form theorem in tRAAGs is the solution of the word problem in tRAAGs. Moreover, by comparing the normal forms of elements in tRAAGs and RAAGs we show that the spherical growth and the geodesic growth of a tRAAG agree with the corresponding growth of the RAAG based on the underlying naïve graph.

When studying Artin groups, objects that come often into play are parabolic subgroups. Let (A, S) be an Artin group. For $X \subset S$ consider the subgroup P generated by X . By a theorem of Van der Lek in 27 the subgroup P is isomorphic to the Artin group with X as its set of Artin generators, and the relations the ones induced by the relations in A . These subgroups P are called *standard parabolic subgroups*. Their conjugates gPg^{-1} , for some $g \in A$, are called *parabolic subgroups* of A . A morphism $\rho : A \rightarrow P$ is called a *retraction* if $\rho(p) = p$ for any $p \in P$.

Using parabolic subgroups, one can define the dimension of an Artin group.

Definition 1.1.1. The *dimension of an Artin group* (A, S) is the maximal cardinality of a subset $X \subseteq S$ such that the standard parabolic subgroup P generated by X is of spherical type.

The importance of parabolic subgroups can be explained from their use to study Artin groups from a geometric viewpoint. There are several simplicial complexes associated to an Artin group, and usually one uses parabolic subgroups to build them. As a general example serves the Deligne complex which is the flag complex associated to the partially ordered set of cosets of standard parabolic subgroups of spherical type (see [18] and [11]). This complex is used to help deduce results about Artin groups, such as towards the $K(\pi, 1)$ -conjecture (see [11]), or Tits alternative for Artin Groups of FC-type (see [30]) among other applications.

Another complex associated to Artin groups is the Salvetti complex, which is a flag complex, and for building it, one uses the standard parabolic subgroups of spherical type in the corresponding Coxeter group.

Recently, there have been defined complexes using not only the standard parabolic subgroups and their cosets, but also using the general parabolic subgroups. For example, Cumplido et al. [15], 2019, define a complex (called the complex of parabolic subgroups), associated to the Artin groups of spherical type. This was later generalized for Artin groups of FC-type in [31], 2021. They use these complexes to show that the intersection of parabolic subgroups of spherical type in these groups is again a parabolic subgroup.

It is still an open question, for general Artin groups, whether the set of parabolic subgroups is stable under intersections. The same problem has a positive answer in the class of Coxeter groups (see [33] and the references within).

Several articles have dealt with this question for subfamilies of Artin groups. For example, one gets a positive answer for the case of braid groups, and later generalized to all Artin groups of spherical type ([15]) using Garside theory. More recently, Cumplido et al. [14], showed the result for the class of large-type Artin groups.

In Chapter 4 we discuss properties of parabolic subgroups in even Artin groups, where *even Artin groups* is a subclass of Artin groups which generalizes RAAGs. In even Artin groups, the relations coming from Equation (1.1) have k even, or $k = \infty$, i.e. the only possible relations among any two Artin generators a, b have the form $(ab)^l = (ba)^l$, for some $l \in \mathbb{N}$ (here $l = k/2$).

Even Artin groups share many properties with RAAGs, and we will use results on the latter class as a motivation for providing similar results to this new broader class.

In this thesis, we will give some contributions to RAAGs and the family of even Artin groups. One useful property that even Artin groups possess is the presence of retractions for any parabolic subgroup.

We also show that the set of parabolic subgroups, in a certain subclass \mathcal{C} , is stable under intersections. The class \mathcal{C} consists of those even Artin where for any vertex v belonging to a triangle in Γ , all the edges containing v are labeled by 2's. In particular,

the class \mathcal{C} contains RAAGs, and 2-dimensional even Artin groups.

Theorem 4.3.4 (Intersections of parabolic subgroups). Let G be an even Artin group belonging to the class \mathcal{C} . Then the intersection of any collection of parabolic subgroups in G is again a parabolic subgroup.

If instead of \mathcal{C} in the theorem above, we consider RAAGs (which are contained in \mathcal{C}), we obtain a new proof of the same result for the class of RAAGs.

1.2 Preliminaries

In this part we will give the main definitions and we cite most of the useful results that we use in the upcoming chapters.

1.2.1 Graphs

Whenever we define classes of groups in this thesis, we use graphs to give a canonical presentation for the group. There are a range of definitions for graphs, and for most of our purposes we will use simplicial graphs.

Definition 1.2.1. A *simplicial graph* Γ is a pair $\Gamma = (V, E)$, where $V = V\Gamma$ is a set whose elements are called vertices, and $E = E\Gamma \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\}$ is a set of paired distinct vertices, whose elements are called edges.

If $e = \{x, y\}$ is an edge, then both $\{x, y\}$ and $\{y, x\}$ represent it.

An *empty graph* is a graph that has an empty set of vertices, and thus an empty set of edges as well.

Definition 1.2.2. Let $\Gamma = (V, E)$ be a simplicial graph.

- Two vertices x and y are called *adjacent* if $\{x, y\} \in E$, and in this case x and y are called the endpoints of the edge $e = \{x, y\}$.
- A vertex may belong to no edge, in which case it is not adjacent to any other vertex; such vertices are called *isolated*.
- The graph Γ is called *totally disconnected* if all vertices are isolated.
- The graph Γ is called *complete* if any two distinct vertices form an edge. We call the complete graph on n vertices an *n -clique*.

Remark 1.2.3. In simplicial graphs there can be at most one edge between two distinct vertices. Also, there are no loops (edges whose endpoints are equal).

Definition 1.2.4. Let $\Gamma = (V, E)$ be a graph.

- A *path* p between two vertices x and y is a sequence of pairwise distinct edges $p = (e_1, e_2, \dots, e_n)$, where $e_i = \{z_i, z_{i+1}\}$ with $z_1 = x, z_{n+1} = y$.

- The graph Γ is called *connected* if there is a path between any two distinct vertices.
- The length of the path is the number of edges in the sequence.
- A path between two vertices x and y is called a *geodesic* if it is a path of shortest length between x and y .

Definition 1.2.5. Given a graph $\Gamma = (V, E)$, the *double graph* associated to Γ , denoted by $\Gamma^{[2]}$, is defined as follows. Its vertex set is $V\Gamma^{[2]} = V \sqcup V'$ with $V = V\Gamma$, and V' is a disjoint copy of V . Denote the vertices in V' as $\{a' \mid a \in V\}$. For any edge $\{a, b\}$ in Γ there are exactly four corresponding edges $\{a, b\}, \{a', b\}, \{a, b'\}, \{a', b'\}$ in $\Gamma^{[2]}$. See Figure 1.1 for an example.

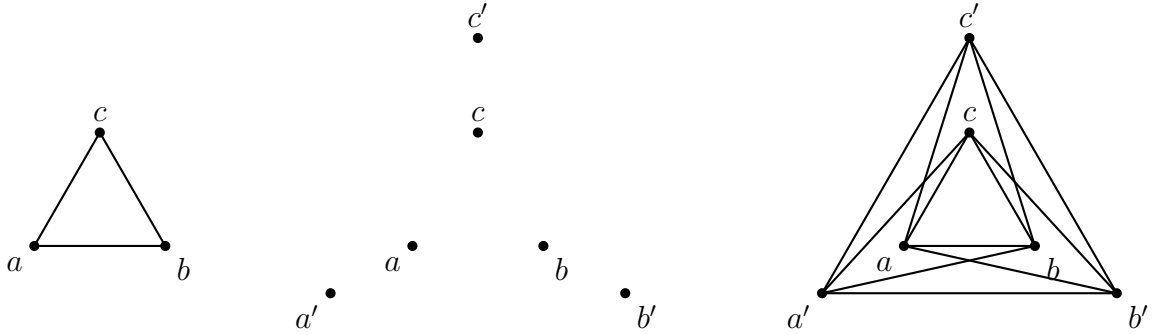


Figure 1.1: Construction of the double of a Graph: $\Gamma, V\Gamma^{[2]}$ and $\Gamma^{[2]}$.

The main reason for introducing the double graph is to bring some computations done for RACGs to RAAGs as well. We will use it in Chapter 2 to give a formula for the geodesic growth of some RAAGs (see Corollary 2.3.6).

Definition 1.2.6. A *mixed graph* $\Gamma = (V, E, D, o, t)$ consists of a simplicial graph (V, E) , a set of directed edges $D \subseteq E$, and two maps

$$o, t: D \longrightarrow V.$$

For an edge $e = \{x, y\} \in D$, the maps o, t satisfy $o(e) \neq t(e)$, and $o(e), t(e) \in \{x, y\}$. Refer to $o(e), t(e)$ as the origin and the terminus of edge e respectively.

Notation 1.2.7. Let $\Gamma = (V, E, D, o, t)$ be a mixed graph.

- (i) If $e = \{a, b\}$ is an undirected edge, i.e. $e \in E \setminus D$, we will write $e = [a, b]$ (note that also $e = [b, a]$).
- (ii) Instead, if $e = \{a, b\}$ is a directed edge, i.e. $e \in D$, we will write $e = [o(e), t(e)]$. In this case, either $e = [a, b]$, or $e = [b, a]$.

Geometrically, we present the cases (i) $e = [a, b]$, and (ii) $e = [a, b]$ as follows:

Figure 1.2: Types of edges in $E\Gamma$

Since every edge in $E = E\Gamma$ has exactly one of the types (i), (ii) as in Figure 1.2, one can express the set of edges in Γ as

$$E\Gamma = \overline{E\Gamma} \sqcup \overrightarrow{E\Gamma},$$

where $\overline{E\Gamma} = E \setminus D$ consists of undirected edges (type (i)), and $\overrightarrow{E\Gamma} = D$ consists of directed edges (type (ii)).

1.2.2 Artin Groups

Let $X = \{a_1, \dots, a_n\}$ be a finite set. A *Coxeter Matrix* $M = (m_{ij})$ over X is a square symmetrical matrix of order n , with entries in $\mathbb{N} \cup \{\infty\}$ and 1's in the diagonal. In other words, $m_{ii} = 1$, for all i , and $m_{jk} = m_{kj}$ for any k, j with $j \neq k$.

M corresponds to a *Coxeter graph* Γ via $V\Gamma = X$, and $\{a_i, a_j\}$ is an edge in Γ labeled by m_{ij} , if and only if $m_{ij} \geq 2$ and $m_{ij} \neq \infty$.

One defines Artin Groups via a *standard presentation* which can be read from the Coxeter matrix or the Coxeter graph. The presentation of the Artin Group G_Γ , based on the graph Γ is:

$$G_\Gamma = \langle a_1, \dots, a_n \mid \forall i < j : \text{prod}(a_i, a_j; m_{ij}) = \text{prod}(a_j, a_i; m_{ji}) \rangle$$

where $\text{prod}(a, b; k) = \underbrace{abab\dots}_{k\text{-factors}}$

Remark 1.2.8. If $m_{ij} = \infty$, then there is no edge in the defining graph Γ joining the vertices a_i, a_j . This implies that there is no relation between a_i, a_j in the presentation.

1.2.3 Twisted Artin groups

Twisted Artin groups (see [13]) represent a generalization of Artin groups. To define them we use a twisted Coxeter Matrix, which does not need to be symmetric.

Let $X = \{a_1, \dots, a_n\}$ be a finite set. A *twisted Coxeter Matrix* $T = (t_{ij})$ over X is a square matrix of order n , with entries in $\mathbb{N} \cup \{\infty\}$ and 1's in the diagonal such that for all $1 \leq j, k \leq n$ one has $|t_{jk} - t_{kj}| \in \{0, 2\}$.

Definition 1.2.9. A *twisted Artin group* $G = G_T$, based on a twisted Coxeter Matrix T , is a finitely presented group given by the presentation:

$$G_T = \langle a_1, \dots, a_n \mid \forall i < j : \text{prod}(a_i, a_j; t_{ij}) = \text{prod}(a_j, a_i; t_{ji}) \rangle$$

where $prod(a, b; k) = \underbrace{abab\dots}_{k\text{-factors}}$

All Artin groups are also twisted Artin groups. Another simple example of a twisted Artin group is the fundamental group of the Klein bottle, given as:

$$K = \langle a, b \mid aba = b \rangle.$$

As for Artin groups, one can associate a graph Γ to a twisted Coxeter Matrix T . T corresponds to a *twisted Coxeter graph* Γ via $V\Gamma = X$, and $\{a_i, a_j\}$ is an edge in Γ , labeled by t_{ij} , if and only if $t_{ij} = t_{ji} \geq 2$ and $t_{ij} \neq \infty$; or $\{a_i, a_j\}$ is a directed edge in Γ , labeled by $(t_{ij} + t_{ji})/2$, and directed from a_i to a_j , if and only if $t_{ij} - t_{ji} = 2$;

Example 1.2.10. The following edge:

$$a \bullet \xrightarrow{m} \blacktriangleright b$$

with $m \geq 2$ corresponds to the relation

$$\underbrace{abab\dots}_{m+1 \text{ factors}} = \underbrace{baba\dots}_{m-1 \text{ factors}}$$

Notice that for $m = 2$ we obtain the Klein relation.

Definition 1.2.11. The underlying graph of the twisted Artin group, when we treat the directed edges as regular edges, but we keep the labels, is called the *naïve graph*.

Any twisted Artin group gives rise to an Artin group over the naïve graph. This will serve to make some connections between tRAAGs and the corresponding RAAGs over the naïve graph. Now we provide an example of a twisted Artin group (not a tRAAG), for the corresponding dihedral case.

Example 1.2.12. Consider the following graph Γ :

$$a \bullet \xrightarrow{m} \blacktriangleright b$$

with $m \geq 2$. Corresponding to this graph we get the twisted Artin group:

$$G_\Gamma = \langle a, b \mid \underbrace{abab\dots}_{m+1 \text{ factors}} = \underbrace{baba\dots}_{m-1 \text{ factors}} \rangle.$$

We can classify these groups by distinguishing the parity of m (see also [13] for a different treatment when m is even).

- (i) If $m = 2k + 1$ for some positive integer k , then the relation can also be written as $a(ba)^{k+1}a^{-1} = (ba)^k$, and substituting $s = ba$ we get the group presentation:

$$G_\Gamma = \langle a, s | as^{k+1}a^{-1} = s^k \rangle.$$

This is an equivalent presentation because $a, b \mapsto a, sa^{-1}$ and $a, s \mapsto a, ba$ give two morphisms whose compositions are equal the identity morphisms.

This is the Baumslag–Solitar group denoted usually by $BS(k+1, k)$.

- (ii) If $m = 2k$ for some $k \in \mathbb{N}$, then rewrite the relation in the form $a(ba)^k = (ba)^ka^{-1}$, and substitute $s = ba$ to obtain the group presentation:

$$G_\Gamma = \langle a, s | as^k = s^ka^{-1} \rangle.$$

Let C_k be the cyclic group of order k , written in additive notation. Define a morphism

$$\alpha: G_\Gamma \rightarrow C_k, \quad a \mapsto 0, s \mapsto 1.$$

We want to compare G_Γ and $K = \ker(\alpha)$. To find a presentation of K we use the Reidemeister–Schreier procedure (see Appendix [A.3](#)) with

$$G = G_\Gamma = \langle X, R \rangle \text{ where } X = \{a, s\}, \text{ and } R = \{as^kas^{-k}\}.$$

The set $T = \{1, s, s^2, \dots, s^{k-1}\}$ gives a Schreier transversal for K in G .

The set of generators for K is $Y = \{tx(\overline{tx})^{-1} \mid t \in T, x \in X, tx \notin T\}$ where \overline{w} is the representative of w in T . Any $t \in T$ can be written as s^i for $0 \leq i \leq k-1$, so:

$$tx(\overline{tx})^{-1} = s^i x(\overline{s^i x})^{-1} = \begin{cases} s^i a(\overline{s^i a})^{-1} & \text{if } x = a \\ s^i s(\overline{s^i s})^{-1} & \text{if } x = s \end{cases}.$$

As $s^i a \notin T$ for all $0 \leq i \leq k-1$ and $\overline{s^i a} = s^i$ we get $b_i = s^i a s^{-i}$ as generators of K . On the other hand $s^i s \notin T$ if and only if $i = k-1$ in which case $\overline{s^{k-1} s} = 1$ in T , which gives $b_k = s^k$ as a generator of K . To recap, the set

$$Y = \{b_i = s^i a s^{-i}, 0 \leq i \leq k-1, b_k = s^k\},$$

gives a set of generators for K .

To get relations for K , rewrite each trt^{-1} for $t \in T$ and $r \in R$, using generators in Y . Write any $t \in T$ as s^i for some $0 \leq i \leq k-1$. The only relation in R is $r = as^kas^{-k}$. So, for $0 \leq i \leq k-1$ we obtain:

$$trt^{-1} = s^i (as^kas^{-k}) s^{-i} = (s^i a s^{-i})(s^i s^k s^{-i})(s^i a s^{-i})(s^i s^{-k} s^{-i}) = b_i b_k b_i b_k^{-1}.$$

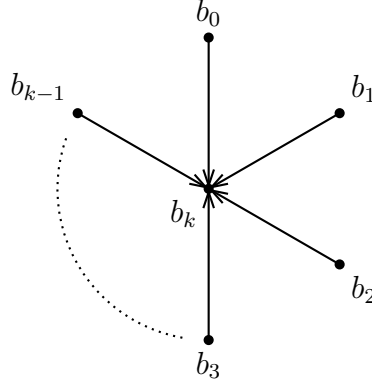
So, $S = \{b_i b_k b_i b_k^{-1} \mid 0 \leq i \leq k-1\}$ gives the set of relations in K . Ultimately, the presentation for K is given as:

$$K = \langle Y \mid S \rangle = \langle b_0, b_1, \dots, b_{k-1}, b_k \mid b_i b_k b_i b_k^{-1} = 1, 0 \leq i \leq k-1 \rangle.$$

In this case our group fits into an exact sequence:

$$1 \longrightarrow K \xrightarrow{i} G_\Gamma \xrightarrow{\alpha} C_k \longrightarrow 1.$$

Moreover $K \simeq G_{\Gamma'}$ is a tRAAG over the graph Γ' given below, so in this case, the group G_Γ is virtually a tRAAG.



1.2.4 Languages

Definition 1.2.13. Let S be a set, equipped with a binary operation

$$\cdot : S \times S \rightarrow S.$$

The tuple (S, \cdot) is called a *monoid* if the following axioms are satisfied:

- (Associativity) For any $a, b, c \in S$ one has:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- (Identity) There is an element $\varepsilon \in S$, such that for any $a \in S$ the following equation holds

$$a \cdot \varepsilon = \varepsilon \cdot a = a$$

One particular example of a monoid, that will be very useful to us, is the free monoid over a set S denoted by S^* .

Definition 1.2.14. The *free monoid* on a set S is the monoid S^* whose elements are all the finite sequences (or words) of zero or more elements from S , with word concatenation as the monoid operation, and with the empty word (unique sequence of zero elements) denoted by ε as the identity element.

One way to encounter monoids is in the context of formal languages. In that context one has an *alphabet* S which can be any set. The elements of an alphabet are called its *letters*. In all our work, alphabets are going to be finite sets. Using the elements of the alphabet, one obtains *words*, which can be any finite sequence of letters. We will denote by S^* the set of all words over an alphabet S .

To any word w one can associate a number $\ell(w)$, the *length of the word*, which is the number of letters it is composed of. For any alphabet, there is a unique word of length 0, the empty word, denoted by ε .

There is also a way to combine two words, by concatenating them, and form a new word. In this case, the length of the new word is equal to the sum of the lengths of the original words. The result of concatenating a word with the empty word is the original word.

Definition 1.2.15. A *language* L over a finite set S is any subset of S^* .

In other words, a language L is a set of words over the alphabet S .

For a given language L over S we have the notion of growth. The *growth function*, associated to L is defined as:

$$\sigma = \sigma_L : \mathbb{N} \rightarrow \mathbb{N}; \sigma(n) = \#\{w \in L : \ell(w) = n\}.$$

One defines the *growth series* associated to L as the formal power series

$$\mathcal{S}_L(z) = \sum_{n=0}^{\infty} \sigma(n)z^n \in \mathbb{Z}[[z]].$$

Definition 1.2.16. A *finite state automaton* is a 5-tuple $M = (Q, S, q_0, A, \delta)$, where S is a finite alphabet, Q is a finite set of states, $q_0 \in Q$ is the start state, $A \subseteq Q$ is the set of accepting states, and $\delta : Q \times S \rightarrow Q$ is the transition function.

Let $w = a_1a_2 \dots a_n$ be a word with letters in S . An automaton M *accepts the word* w if there is a sequence of states t_0, t_1, \dots, t_n in Q with

- $t_0 = q_0$,
- $t_{i+1} = \delta(t_i, a_{i+1})$ for any $0 \leq i \leq n-1$, and
- $t_n \in A$.

Definition 1.2.17. Let $M = (Q, S, q_0, A, \delta)$ be a finite state automaton. Denote by $L(M)$ the set of all words over S accepted by M , and call it *the language accepted by the automaton* M .

Definition 1.2.18. A language L is called *regular* if there is a finite state automaton accepting it, i.e. $L = L(M)$ for some finite state automaton M .

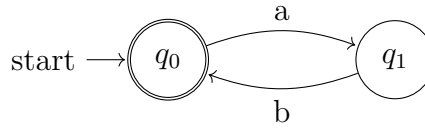
Automata will be useful for recognizing subgroups of free groups, in Chapter [4](#) (see their connections with digraphs as well, in Appendix [A.2](#)).

Remark 1.2.19. Regular languages have rational growth series.

Example 1.2.20. The language $L = \{(ab)^k | k \in \mathbb{N}_0\}$ is the language of the finite state automaton $M = (Q = \{q_0, q_1, F\}, S = \{a, b\}, q_0, A = \{q_0\}, \delta)$, where

$$\delta(q_0, a) = q_1, \delta(q_1, b) = q_0, \text{ and for other combinations } \delta(q, s) = F.$$

This example serves to portray the graphical description of an automaton, as given below:



1.2.5 Growth in groups

Let G be a finitely generated group, and X a *finite monoid generating set* for G , that is, there exists a monoid morphism $\pi: X^* \rightarrow G$ that is surjective. An element $g \in G$ is represented by a word $w \in X^*$, if $\pi(w) = g$.

One obtains a length function

$$|\cdot|: G \longrightarrow \mathbb{N}$$

for the pair (G, X) as well, by defining

$$|g| = \min_{w \in X^*} \{\ell(w) : \pi(w) = g\},$$

where $\ell(w)$ is the length of w as a word in X^* .

The *spherical growth function*, associated to (G, X) is defined as:

$$\sigma = \sigma_{(G, X)} : \mathbb{N} \rightarrow \mathbb{N}; \sigma(n) = \#\{g \in G : |g| = n\}.$$

One defines the *standard growth series* associated to (G, X) as the formal power series

$$\mathcal{S}_{(G, X)}(z) = \sum_{i=0}^{\infty} \sigma(i) z^i \in \mathbb{Z}[[z]].$$

Similarly, one can define the geodesic growth functions and the geodesic growth series.

A word $w \in X^*$ is a *geodesic* if $\ell(w) = \min_{u \in X^*} \{\ell(u) : \pi(u) = \pi(w)\}$.

The *geodesic growth function*, associated to (G, X) is defined as:

$$\rho = \rho_{(G, X)} : \mathbb{N} \rightarrow \mathbb{N}; \rho(n) = \#\{w \in X^n : w \text{ geodesic}\}.$$

The *geodesic growth series* associated to (G, X) is the formal power series

$$\mathcal{G}_{(G, X)}(z) = \sum_{i=0}^{\infty} \#\{w \in X^n : w \text{ geodesic}\} z^n = \sum_{i=0}^{\infty} \rho(i) z^i \in \mathbb{Z}[[z]].$$

Remark 1.2.21. One can define the spherical growth and the geodesic growth of a finitely generated group G , with a finite generating set X , in terms of languages. Indeed, let \mathcal{G} be the language of geodesics in G with respect to the generating set X , and \mathcal{E} a language of elements (one can take a unique representative for any element from the language of geodesics). Then the spherical growth and the geodesic growth of G are equal to the growth of languages \mathcal{E} , \mathcal{G} respectively.

Chapter 2

Geodesic growth of some 3-dimensional RACGs

In this chapter we treat the geodesic growth of RAAGs based on link-regular graphs without tetrahedra. The work is motivated by [1], where the authors compute the geodesic growth of RAAGs based on link-regular graphs without triangles. The approach for these computations is combinatorial in nature. Firstly, we give explicit formulae for the geodesic growth series of a right-angled Coxeter group, and then using a result on Cayley graphs we can bring similar formulae also to RAAGs. This chapter is based on the article [3].

2.1 Introduction

Given a simplicial graph $\Gamma = (V, E)$, one associates to it the right-angled Coxeter group C_Γ defined by the following presentation:

$$C_\Gamma = \langle V \mid v^2 = 1 \forall v \in V, uv = vu \forall \{u, v\} \in E \rangle.$$

One calls V the *standard generating set* of C_Γ . We can also associate to Γ the *right-angled Artin group* A_Γ given by the presentation:

$$A_\Gamma = \langle V \mid uv = vu \forall \{u, v\} \in E \rangle.$$

One calls $V \sqcup V^{-1}$ the *standard generating set* of A_Γ .

The languages of geodesics and shortlex representatives of a RACG with respect to its standard generating sets are regular [6, 9, 28], and thus the corresponding standard and geodesic growth series are rational functions. Concrete formulae for the standard growth series of Coxeter groups, proved without the use of automata theory, can be found in [32, 36]. Recently, it was shown that the growth rates of the geodesic and the

standard growth functions (i.e. $\lim \sqrt[n]{s_n}$ and $\lim \sqrt[n]{g_n}$) are either 1 or Perron numbers (see [26]).

Moreover, it is well understood how the geometry of the defining graph reflects on the standard growth of C_Γ with respect to V : it only depends on the cliques of Γ of each size (see [16, Proposition 17.4.2.] or [32, 36]). For example, in the case of RACGs based on trees, this implies that the growth only depends on the number of vertices and edges of the tree.

Geodesic growth is still a very mysterious object compared to the standard growth and it is not clear which properties of the defining graph are reflected into the geodesic growth function. In [12], Ciobanu and Kolpakov showed that there exist infinitely many pairs of non-isomorphic RACGs based on trees with the same geodesic growth series with respect to the standard generators. These examples were based on co-spectral defining graphs, but then they gave infinitely many pairs of non-isomorphic RACGs with co-spectral defining graphs and different geodesic growth series with respect to the standard generating set.

On the other hand, if the defining graph possesses enough symmetry (the graph is link-regular), then the main theorem of [1] states that the geodesic growth only depends on the number of cliques of each size and the isomorphism types of the links of the cliques. A simplicial graph is link-regular, if the number of elements of the link of a clique only depends on the size of the clique (See Definition 2.2.2).

For example, if Γ is a totally disconnected graph with n vertices, then it is link regular, and C_Γ is a free product of cyclic groups of order 2. It is well-known that

$$\mathcal{G}_{(C_\Gamma, V)}(z) - 1 = \frac{nz}{1 - (n-1)z}.$$

Moreover, in [1] it is computed explicitly the geodesic growth series of right-angled Coxeter groups based on link-regular graphs with n vertices, vertices of degree l , and without triangles. For such cases, one obtains the following formula for the growth series.

$$\mathcal{G}_{(C_\Gamma, V)}(z) - 1 = \frac{nz(1 + (2-l)z)}{1 + (-n-l+3)z + (-2n+2+nl)z^2}.$$

Note that if $l = 0$, one recovers the formula for a totally disconnected graph.

In this chapter we continue to explore the geodesic growth of RACGs based on link-regular graphs. Our main result is to provide an explicit formula for the geodesic growth, if the graph does not contain 4-cliques.

Theorem 2.1.1. Let Γ be a link-regular graph with n vertices, l -regular and let q be the link-number of an edge (which is the same for any edge), and without 4-cliques. Then,

$$\mathcal{G}_{(C_\Gamma, V)}(z) - 1 = \frac{a(n, l, q, z)}{b(n, l, q, z)}$$

where the polynomials a, b are given as:

$$a(n, l, q, z) = nz(1 + (5 - l - q)z + (lq - 3l + 6)z^2)$$

$$b(n, l, q, z) = 1 + (6 - n - l - q)z + (nl + lq + qn - 5n - 3l - q + 11)z^2 + (3nl + 6 - nlq - 6n)z^3$$

One can check that if one lets $q = 0$, then one gets the previous formula for triangle-free link-regular graphs.

2.2 Definitions and notation

Let Γ be a finite simplicial graph. For a vertex a in Γ , define the *star of a* to be the set:

$$\text{St}(a) = \{b \in V\Gamma \mid \{a, b\} \in E\Gamma\} \cup \{a\}.$$

Let $\sigma \subseteq V\Gamma$ be such that the vertices of σ span a complete subgraph of Γ , then σ is called a *clique*. If σ is a clique with k -vertices, then we call it a k -clique. Sometimes we refer to 3-cliques and 4-cliques by triangles and tetrahedra respectively.

The *link of a clique σ* , denoted by $\text{Lk}(\sigma)$, is the set of vertices in $V\Gamma \setminus \sigma$ that are connected with every vertex in σ . That is,

$$\text{Lk}(\sigma) = \{v \in V\Gamma \setminus \sigma : \{v\} \cup \sigma \text{ spans a clique}\}.$$

The *star of σ* , denoted by $\text{St}(\sigma)$, is the set of vertices in Γ that are connected with every vertex in σ . That is,

$$\text{St}(\sigma) = \{v \in V\Gamma : \{v\} \cup \sigma \text{ spans a clique}\}.$$

These sets satisfy $\sigma \cup \text{Lk}(\sigma) = \text{St}(\sigma)$.

Definition 2.2.1. A graph Γ is called *l -regular*, if any vertex belongs to exactly l edges. In other words $|\text{Lk}(v)| = l$ for any vertex v in Γ . We also use the term *regular*, when we do not use l .

We use link-regular graphs, defined below, which possess regularity for cliques of higher sizes as well.

Definition 2.2.2. A graph Γ is called *link-regular* if for any clique $\sigma \in \Gamma$, $|\text{Lk}(\sigma)|$ depends on $|\sigma|$ and not on σ itself, i.e. if σ_1, σ_2 are cliques with $|\sigma_1| = |\sigma_2|$ then $|\text{Lk}(\sigma_1)| = |\text{Lk}(\sigma_2)|$.

In this chapter we will consider graphs which do not contain tetrahedra. As the graph does not have k -cliques with $k \geq 4$ it is link-regular if there are numbers l and q such that the graph is l -regular, and any edge is contained in q triangles.

Definition 2.2.3. The *f-polynomial* associated to Γ , or the clique-polynomial of Γ , is the polynomial

$$f_\Gamma(z) = \sum_{n=0}^{|V|} \#\{\Delta \subseteq \Gamma : \Delta \text{ is an } n\text{-clique}\} z^n,$$

and essentially records the number of cliques of each size.

Theorem 2.2.4 (Main theorem of [1]). Let C_Γ be the RACG based on a link-regular graph Γ . The geodesic growth of G is fully determined by the *f*-polynomial of Γ and the set of pairs $\{(|\sigma|, |\text{Lk}(\sigma)|) : \sigma \text{ a clique in } \Gamma\}$.

Remark 2.2.5. As noted in [5], there is a relationship between the sizes of cliques and the coefficients of the *f*-polynomial. So one has that the geodesic growth of a RACG based on a link-regular graph Γ is fully determined by the *f*-polynomial of Γ .

One can use the notion of the double graph to bring results about geodesic growth of RACGs also to RAAGs.

Let $\Gamma = (V, E)$ be a simplicial graph, and $\Gamma^{[2]}$ its double. By definition, $|\text{V}\Gamma^{[2]}| = 2|\text{V}\Gamma|$, and $|\text{E}\Gamma^{[2]}| = 4|\text{E}\Gamma|$. One has also a projection

$$\rho : \Gamma^{[2]} \longrightarrow \Gamma,$$

which identifies naturally the two copies of the vertices of $\Gamma^{[2]}$. The map ρ is 2-to-1 on vertices and 4-to-1 on edges. Moreover, by the construction of $\Gamma^{[2]}$ and the definition of ρ , we get $|\text{Lk}_{\Gamma^{[2]}}(v)| = 2|\text{Lk}_\Gamma(\rho(v))|$, i.e. ρ is 2-to-1 on links of vertices.

Lemma 2.2.6. If Γ is link-regular without tetrahedra, then so is $\Gamma^{[2]}$.

Proof. We already have $|\text{Lk}_{\Gamma^{[2]}}(v)| = 2|\text{Lk}_\Gamma(\rho(v))|$.

For any edge $e = \{u, v\}$ in $\Gamma^{[2]}$ consider the edge $\rho(e)$ in Γ , and all the triangles $(\rho(u), \rho(v), c)$ over it. Triangles over $e = \{u, v\}$ are (u, v, w) where $w \in \rho^{-1}(c)$. This means that $|\text{Lk}_{\Gamma^{[2]}}(e)| = 2|\text{Lk}_\Gamma(\rho(e))|$.

If there was a tetrahedron $\{x_1, x_2, x_3, x_4\}$ in $\Gamma^{[2]}$, then $\{\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)\}$ would be a tetrahedron in Γ . Indeed $\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)$ are all different since $\{x, x'\}$ cannot be an edge in $\Gamma^{[2]}$, and all the edges of the tetrahedron in $\{x_1, x_2, x_3, x_4\}$ induce edges for a tetrahedron over $\rho(x_1), \rho(x_2), \rho(x_3), \rho(x_4)$. Since there are no tetrahedra in Γ , we conclude that there are no tetrahedra in $\Gamma^{[2]}$. \square

Remark 2.2.7. An important application of the double graph is provided in [20, Lemma 2]: one has that the Cayley graph of the RAAG based on Γ is isomorphic as an undirected graph to the Cayley graph of the RACG based on $\Gamma^{[2]}$.

Using the remark above we get the following:

Corollary 2.2.8. Let $G = A_\Gamma$ be a RAAG based on a link-regular graph Γ . The geodesic growth of G is equal to the geodesic growth of $C_{\Gamma^{[2]}}$.

2.2.1 Geodesics in RACGs

In this section we give characterizations of geodesics in RACGs. Using Theorem 3.9 in [22], and the characterization of a reduced sequence for graph products, we get the following result for RACGs.

Theorem 2.2.9. Let C_Γ be a right-angled Coxeter group based on Γ and $V = V\Gamma$ the standard generating set. Let $w = s_1 \dots s_n$ be a word over V . Then w is not a geodesic if and only if there are indices $1 \leq i < j \leq n$ such that $s_i = s_j$ and $[s_i, s_k] = 1$ for all k satisfying $i < k < j$.

Notation 2.2.10. Let C_Γ be a RACG associated to Γ , with generating set $V = V\Gamma$. If $w \in V^*$ is a word, we denote by E_w the set of geodesics ending in w , and by $E_w(z)$ the generating growth series of E_w . That is

$$E_w(z) = \sum_{n=0}^{\infty} \#(E_w \cap V^n) z^n.$$

Theorem 2.2.11. Let w_1, w_2 be words over V and $x \in V$, such that $w_1 w_2 x$ is a geodesic. Assume $bx = xb$ for all letters b in w_2 , and x does not commute with any letter a in w_1 (so, w_1 is not the empty word in particular). Then:

$$E_{w_1 w_2 x} = E_{w_1 w_2} x, \text{ and also } E_{w_1 w_2 x}(z) = E_{w_1 w_2}(z) \cdot z.$$

Proof. Obviously, one has the inclusion $E_{w_1 w_2 x} \subseteq E_{w_1 w_2} x$. To show $E_{w_1 w_2 x} \supseteq E_{w_1 w_2} x$, we take a geodesic word $w \in E_{w_1 w_2}$. We suppose that wx is not geodesic and derive a contradiction. As $w_1 w_2 x$ is geodesic, there exist a shortest suffix of wx , say $w_0 w_1 w_2 x$ that is not geodesic. Clearly, w_0 is non-empty, and we can write it as yu , with y a letter and u a word (maybe empty). As $y u w_1 w_2 x$ is not geodesic, but every proper subword is, we get by the Theorem 2.2.9 that $x = y$ and x commutes with every letter of u, w_1 and w_2 which is the desired contradiction. \square

Notation 2.2.12. Let A, B be two subsets of a finitely generated free monoid V^* . If there is a bijection $f: A \rightarrow B$ that is length preserving (i.e. $\ell(f(a)) = \ell(a)$ for all $a \in A$) we write $A \equiv B$.

In particular, if $A \equiv B$ then the corresponding growth series $A(z)$ and $B(z)$ are equal.

For example, with Notation 2.2.10, if $a, b \in V$ commute, then $E_{ab} \equiv E_{ba}$ and $E_{ab}(z) = E_{ba}(z)$.

2.3 Main Theorem

Throughout the rest of the chapter, Γ will be a link-regular finite simplicial graph (with n vertices) which does not contain tetrahedra. We denote by l the number of edges meeting at any vertex, and q the number of triangles containing a fixed edge. Denote by G the right-angled Coxeter group C_Γ defined by Γ , with generating set $V = V\Gamma$.

Notation 2.3.1. We use $\Delta\Gamma$ to denote the set of 3-cliques of Γ . In the main theorem we use the following notation:

$$\mathcal{E}_v(z) = \sum_{a \in V\Gamma} E_a(z), \quad \mathcal{E}_e(z) = \sum_{\substack{(a,b) \in (V\Gamma)^2 \\ \{a,b\} \in E\Gamma}} E_{ab}(z), \quad \mathcal{E}_\Delta(z) = \sum_{\substack{(a,b,c) \in (V\Gamma)^3 \\ \{a,b,c\} \in \Delta\Gamma}} E_{abc}(z).$$

Theorem 2.3.2. Let Γ be a link-regular graph with n vertices, l -regular and let q be the link-number of an edge (which is the same for any edge), and without 4-cliques. Let G be the corresponding right-angled Coxeter group, and $\mathcal{G}(z)$ the geodesic growth series of G with respect to the standard generators. Then, there are polynomials p_v, p_e, p_Δ (given below) such that the following relations hold:

$$\mathcal{E}_v(z) = \mathcal{G}(z) - 1, \tag{2.1}$$

$$\mathcal{E}_e(z) = [\mathcal{G}(z) - 1](1 - (n - l - 1)z) - nz, \tag{2.2}$$

$$\mathcal{E}_\Delta(z) = [\mathcal{G}(z) - 1]p_\Delta(z, n, l, q) - nz + n(l - 2q - 2)z^2, \tag{2.3}$$

$$\sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z) = [\mathcal{G}(z) - 1] - p_4(n, l, z), \tag{2.4}$$

and

$$\begin{aligned} \sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z) &= [n + l + q - 6]z \cdot \mathcal{E}_\Delta(z) \\ &\quad + p_e(n, l, q)z^2 \cdot \mathcal{E}_e(z) + p_v(n, l, q)z^3 \cdot \mathcal{E}_v(z). \end{aligned} \tag{2.5}$$

Note than one can find $\mathcal{G}(z)$ by substituting the Equations (2.1), (2.2), (2.3) and (2.4) into (2.5). Moreover, the polynomials p_v, p_e, p_Δ are given by:

$$\begin{aligned} p_\Delta(z, n, l, q) &= 1 - (n + l - 2q - 3)z + (2(n - l - 1)(l - q - 1) - l(n - 2l + q))z^2, \\ p_e(n, l, q) &= n^2 + l^2 - 2q^2 + nl - 2nq - 2lq - 4n - 6l + 10q + 7, \\ p_v(n, l, q) &= (n - l - 1)^3 + 2l(n - 2l + q)(n - q - 2) + lq(n - 3l + 3q), \\ p_3(n, l, z) &= nz + n(n - 1)z^2 + [n(n - 1)(n - 2) + n(n - l - 1)]z^3 \end{aligned}$$

Remark 2.3.3. Note that the Equations (2.1), (2.2) and (2.4) are obtained by subtracting from $\mathcal{G}(z)$ the generating growth series of geodesics on those of length at most 0, 1 and 3 respectively.

Note 2.3.4. Theorem 2.3.2 provides a way to calculate $\mathcal{G}(z)$ using a system of linear equations. The coefficients of the system are polynomials on z (and n, l, q) of degree at most 3. Theorem 2.1.1 was obtained by solving this linear system of equations with the help of Sage.

In the following example we calculate $\mathcal{G}(z)$ on a particular family. The general case can be computed similarly.

Example 2.3.5. Let us compute the geodesic growth of an infinite family Γ_m , defined inductively by

$$\begin{aligned}\Gamma_0 &= (\{a, b, c\}, \{\{a, b\}, \{a, c\}, \{b, c\}\}) = \text{triangle}, \\ \Gamma_{m+1} &= \Gamma_m^{[2]} = \text{the double of } \Gamma_m.\end{aligned}$$

In terms of (n, l, q) we have $(n_0, l_0, q_0) = (3, 2, 1)$, and therefore, by the double construction:

$$(n_m, l_m, q_m) = (3 \cdot 2^m, 2 \cdot 2^m, 2^m).$$

Taking $2^m = k$ and substituting $3k, 2k, k$ on the polynomials of Theorem 2.3.2 for n, l, q respectively, we find:

$$\begin{aligned}p_\Delta(z, 3k, 2k, k) &= 2(k-1)^2 z^2 - 3(k-1)z + 1, \\ p_e(3k, 2k, k) &= 7(k-1)^2, \\ p_v(3k, 2k, k) &= (k-1)^3, \\ p_3(3k, 2k, z) &= 3kz + 3k(3k-1)z^2 + [3k(3k-1)(3k-2) + 3k(k-1)]z^3.\end{aligned}$$

Now substitute $3k, 2k, k$ on the other equations of Theorem 2.3.2 for n, l, q respectively. Use also the polynomials above, and we can find $\mathcal{G}(z)$ as a solution of the following system:

$$\mathcal{E}_v(z) = \mathcal{G}(z) - 1, \tag{2.6}$$

$$\mathcal{E}_e(z) = (\mathcal{G}(z) - 1)[1 - (k-1)z] - 3kz, \tag{2.7}$$

$$\mathcal{E}_\Delta(z) = (\mathcal{G}(z) - 1)[2(k-1)^2 z^2 - 3(k-1)z + 1] - 3kz - 6kz^2, \tag{2.8}$$

$$\sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z) = 6(k-1)z\mathcal{E}_\Delta(z) + [7(k-1)^2]z^2\mathcal{E}_e(z) + [(k-1)^3]z^3\mathcal{E}_v(z), \tag{2.9}$$

$$\mathcal{G}(z) = 1 + p_3(3k, 2k, z) + \sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z). \tag{2.10}$$

Finally, solving for $\mathcal{G}(z)$, we find:

$$\mathcal{G}(z) = -\frac{6z^3 + (2k^2 - 7k + 11)z^2 - 3(k-2)z + 1}{(z(k-1) - 1)(2z(k-1) - 1)(3z(k-1) - 1)},$$

which agrees with the formula provided in Theorem 2.1.1, for $(n, l, q) = (3k, 2k, k)$.

Now using Theorem 2.1.1, and Corollary 2.2.8 we get the following:

Corollary 2.3.6. Let Γ be a graph as in the hypothesis of Theorem 2.1.1. One can find the geodesic growth series $\mathcal{A}(z)$ for the right-angled Artin group based on Γ with respect to the generating set $V\Gamma \cup V\Gamma^{-1}$ by substituting $2n, 2l, 2q$ for n, q, l respectively in the formula of Theorem 2.1.1 and we get:

$$\mathcal{A}(z) - 1 = \frac{a(n, l, q, z)}{b(n, l, q, z)}$$

where the polynomials a, b are given as:

$$\begin{aligned} a(n, l, q, z) &= 2nz[1 + (5 - 2l - 2q)z + (4lq - 6l + 6)z^2] \\ b(n, l, q, z) &= 1 + (6 - 2n - 2l - 2q)z + (4nl + 4lq + 4qn - 10n - 6l - 2q + 11)z^2 \\ &\quad + (12nl + 6 - 8nlq - 12n)z^3 \end{aligned}$$

2.4 Proof of the main theorem

Throughout this section Γ is a link-regular graph without tetrahedra. The graph Γ has n vertices, the link of each vertex has l vertices, and the link of each edge has q vertices. Let $G = C_\Gamma$ be the associated RACG.

Note that there is 1 geodesic word of length 0, n geodesic words of length 1, $n(n-1)$ geodesics of length 2. A word of length 3 is geodesic in G if all its 3 letters are different or if it is of the form aba with $b \notin \text{St}(a)$. Thus there are $n(n-1)(n-2) + n(n-l-1)$ geodesic words of length 3.

With Notation [2.2.10](#), one can write the geodesic growth series $\mathcal{G}(z)$ in any of the following forms:

$$\mathcal{G}(z) = 1 + \sum_{a \in V\Gamma} E_a(z), \quad (2.11)$$

$$\mathcal{G}(z) = 1 + nz + \sum_{(a,b) \in (V\Gamma)^2} E_{ab}(z), \quad (2.12)$$

$$\mathcal{G}(z) = 1 + nz + n(n-1)z^2 + \sum_{(a,b,c) \in (V\Gamma)^3} E_{abc}(z), \quad (2.13)$$

$$\begin{aligned} \mathcal{G}(z) &= 1 + nz + n(n-1)z^2 + [n(n-1)(n-2) + n(n-l-1)]z^3 \\ &\quad + \sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z). \end{aligned} \quad (2.14)$$

We get [\(2.1\)](#) and [\(2.4\)](#) of Theorem [2.3.2](#) from Equations [\(2.11\)](#) and [\(2.14\)](#), respectively.

We derive Equation [\(2.2\)](#) of Theorem [2.3.2](#) from [\(2.12\)](#) by expanding $\sum_{(a,b) \in (V\Gamma)^2} E_{ab}(z)$. Given a word $ab \in V^*$, we distinguish three cases: $a = b$, $b \in \text{Lk}(a)$ and when $b \notin \text{St}(a)$. We can describe these cases geometrically as in the Figure [2.1](#) (omitting the case $a = b$).

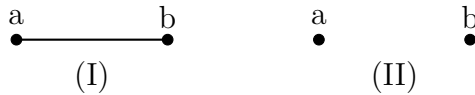


Figure 2.1: Configurations of 2 generators.

The case $a = b$ is impossible, since no geodesic ends with aa . In the case when $b \notin \text{St}(a)$

we can write $E_{ab} = E_a \cdot b$. Hence $E_{ab}(z) = E_a(z) \cdot z$ and we have $n - l - 1$ choices for b .

$$\begin{aligned} \sum_{(a,b) \in (V\Gamma)^2} E_{ab}(z) &= \sum_{a \in V\Gamma} \left(\sum_{b \in \text{Lk}(a)} E_{ab}(z) + \sum_{b \notin \text{St}(a)} E_{ab}(z) \right) \\ &= \sum_{a \in V\Gamma} \left(\sum_{b \in \text{Lk}(a)} E_{ab}(z) \right) + \sum_{a \in V\Gamma} (n - l - 1) z E_a(z) \\ &= \mathcal{E}_e(z) + (n - l - 1) z \mathcal{E}_v(z). \end{aligned} \quad (2.15)$$

So, from (2.12), (2.11), and (2.15) we get:

$$\mathcal{E}_e(z) = (1 - (n - l - 1)z)[\mathcal{G}(z) - 1] - nz$$

which appears in the main theorem as the Equation (2.2).

We can work similarly to get (2.3) of the main theorem from the Equation (2.13). Consider the word $w = abc$. Since we will consider the geodesics that end in w , one needs w itself to be a geodesic, and this implies that $a \neq b$ and $b \neq c$. The generators of the geodesic abc , lie in one of the following disjoint cases:

- | | |
|---------------------------------------|--|
| (I) $\{a, b\} \subseteq \text{St}(c)$ | (II) $\{a, b\} \not\subseteq \text{St}(c)$ |
| (I.1) $a \in \text{Lk}(b)$ | (II.1) $a \in \text{Lk}(b)$ |
| (I.2) $a \notin \text{Lk}(b)$ | (II.2) $a \notin \text{Lk}(b)$ |

We can express these cases using the configurations of generators as in Figure 2.2. In the first three cases, the generators a, b, c are all distinct, as explained in the respective cases. The generators in cases (I.1), and (I.2) appear in the defining graph Γ exactly as they appear in Figure 2.2. In the case (II.2) one can have $a = c$ thought, and the way that a, b, c appear in the defining graph Γ , in cases (II.1) and (II.2), depends on a subcase study.

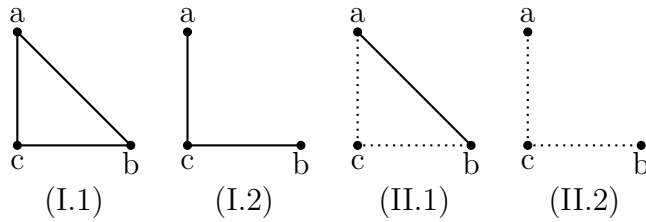


Figure 2.2: Configurations of 3 generators when $a \neq b \neq c$. Dashed edges might or might not appear in the configuration. Two vertices connected by a dashed edge might be the same vertex in Γ (e.g. $a = c$ in (II.2)). No two dashed edges can be edges of the configuration simultaneously.

We can express $\sum_{(a,b,c) \in (V\Gamma)^3} E_{abc}(z)$ as a sum over the 4 disjoint subcases given above. For convenience, we will write \sum_X , where X is a case, to denote the summation over all triples $(a, b, c) \in (V\Gamma)^3$ satisfying the hypothesis of case X .

In (I.1) we also have $a \neq c$, because otherwise we would get $abc = aba = a^2b$ which would not be a geodesic. So, in this case the generators a, b, c form a triangle and we get $\sum_{(I.1)} E_{abc}(z) = \mathcal{E}_\Delta(z)$.

In (I.2) one gets $a \neq c$ as well, as $a = c$ would imply $\{a, b\} \subseteq \text{St}(a)$ and hence $a \in \text{St}(b)$ which cannot happen since $a \neq b$ and $a \notin \text{Lk}(b)$. By Theorem [2.2.11](#) we have $E_{abc} \equiv E_{acb} = E_{ac} \cdot b$. Now we get $E_{abc}(z) = E_{acb}(z) = E_{ac}(z) \cdot z$, for a fixed b . Starting by fixing the edge $e = \{a, c\}$ we get $l - 1 - q$ choices for b , so in this case we have

$$\begin{aligned} \sum_{(I.2)} E_{abc}(z) &= \sum_{a \in V} \sum_{c \in \text{Lk}(a)} \sum_{\substack{b \in \text{Lk}(c) \\ b \notin \text{Lk}(a)}} E_{abc}(z) \\ &= \sum_{a \in V} \sum_{c \in \text{Lk}(a)} (l - 1 - q) z E_{ac}(z) = (l - 1 - q) z \mathcal{E}_e(z). \end{aligned} \quad (2.16)$$

In (II.1) once again $a \neq c$, as $a = c$ would imply $\{a, b\} \not\subseteq \text{St}(a)$ and hence $a \notin \text{St}(b)$ which cannot happen since $a \in \text{Lk}(b)$. We count by first fixing the edge $\{a, b\}$, and then letting c be any vertex different to a and b that does not form a triangle with $\{a, b\}$. One has $n - 2 - q$ choices for c . As both of $\{a, c\}, \{b, c\}$ cannot be edges, using Theorem [2.2.11](#) and considering all the subcases, we get the formula $E_{abc} = E_{ab} \cdot c$. Now, arguing as in (I.2) we get

$$\sum_{(II.1)} E_{abc}(z) = (n - q - 2) z \mathcal{E}_e(z).$$

In (II.2) we count by first fixing the vertex a and then considering the choices when $c = a, c \in \text{Lk}(a)$, and $c \notin \text{St}(a)$. In this case, $b \notin \text{St}(a) \cup \{c\}$ and moreover, when $c \in \text{Lk}(a)$ then b is not linked to c .

For $c = a$ we have $(n - l - 1)$ possible choices for b ; for $c \in \text{Lk}(a)$ we have $l \cdot (n - 2l + q)$ possible choices for c, b ; finally for $c \notin \text{St}(a)$ we get $(n - l - 1)(n - l - 2)$ possible choices for c, b . Using Theorem [2.2.11](#) we get the formula $E_{abc} = E_{ab} \cdot c = E_a \cdot b \cdot c = E_a \cdot bc$, so

$$\sum_{(II.2)} E_{abc}(z) = \sum_{a \in V} \left(\sum_{c=a} \sum_{b \notin \text{Lk}(a)} E_{abc}(z) + \sum_{c \in \text{Lk}(a)} \sum_{\substack{b \notin \text{Lk}(a) \\ b \notin \text{Lk}(c)}} E_{abc}(z) + \sum_{c \notin \text{St}(a)} \sum_{\substack{b \notin \text{Lk}(a) \\ b \neq c}} E_{abc}(z) \right)$$

Considering the possibilities, that we found above, for our vertices, this is equal to:

$$\sum_{a \in V} \left((n - l - 1) z^2 E_a(z) + l(n - 2l + q) z^2 E_a(z) + (n - l - 1)(n - l - 2) z^2 E_a(z) \right).$$

Putting the parts together, we get the formula:

$$\sum_{(II.2)} E_{abc}(z) = [(n - l - 1)^2 + l(n - 2l + q)] z^2 \mathcal{E}_v(z)$$

Now summing everything up we get:

$$\begin{aligned} \sum_{(a,b,c) \in (V\Gamma)^3} E_{abc}(z) &= \mathcal{E}_\Delta(z) + (n+l-2q-3)z\mathcal{E}_e(z) \\ &\quad + [(n-l-1)^2 + l(n-2l+q)]z^2\mathcal{E}_v(z) \end{aligned} \quad (2.17)$$

Substituting (2.1), (2.2), (2.13) into (2.17), we get

$$\begin{aligned} \mathcal{G}(z) - (1 + nz + n(n-1)z^2) &= \mathcal{E}_\Delta(z) \\ &\quad + z \cdot (n+l-2q-3) ((1 - (n-l-1)z)[\mathcal{G}(z) - 1] - nz) \\ &\quad + z^2 \cdot [(n-l-1)^2 + l(n-2l+q)](\mathcal{G}(z) - 1) \end{aligned}$$

And one gets a formula for $\mathcal{E}_\Delta(z)$:

$$\begin{aligned} \mathcal{E}_\Delta(z) &= (\mathcal{G}(z) - 1)[1 - (n+l-2q-3)z + (2(n-l-1)(l-q-1) - l(n-2l+q))z^2] \\ &\quad - nz + n(l-2q-2)z^2 \end{aligned}$$

which appears in the main theorem as Equation (2.3).

To finish the proof, we need to show that (2.5) holds. As in the previous cases, we proceed to rewrite $\sum_{(a,b,c,d) \in (V\Gamma)^4} E_{abcd}(z)$ depending on different cases for the word $abcd$. Since we consider the geodesics that end in $abcd$, we want $abcd$ to be a geodesic itself, and this implies that $a \neq b$, $b \neq c$, and $c \neq d$.

We distinguish the following disjoint cases:

(I) $\{a, b, c\} \subseteq \text{St}(d)$	(II) $\{a, b, c\} \not\subseteq \text{St}(d)$
(I.1) $\{a, b\} \subseteq \text{St}(c)$	(II.1) $\{a, b\} \subseteq \text{St}(c)$
(I.1.1) $a \in \text{Lk}(b)$	(II.1.1) $a \in \text{Lk}(b)$
(I.1.2) $a \notin \text{Lk}(b)$	(II.1.2) $a \notin \text{Lk}(b)$
(I.2) $\{a, b\} \not\subseteq \text{St}(c)$	(II.2) $\{a, b\} \not\subseteq \text{St}(c)$
(I.2.1) $a \in \text{Lk}(b)$	(II.2.1) $a \in \text{Lk}(b)$
(I.2.2) $a \notin \text{Lk}(b)$	(II.2.2) $a \notin \text{Lk}(b)$

We can express them geometrically as configurations of 4 points in Figure 2.3. Each individual figure will be considered in detail, as most of them represent a family of subcases, and not necessarily an actual configuration of 4 generators in Γ .

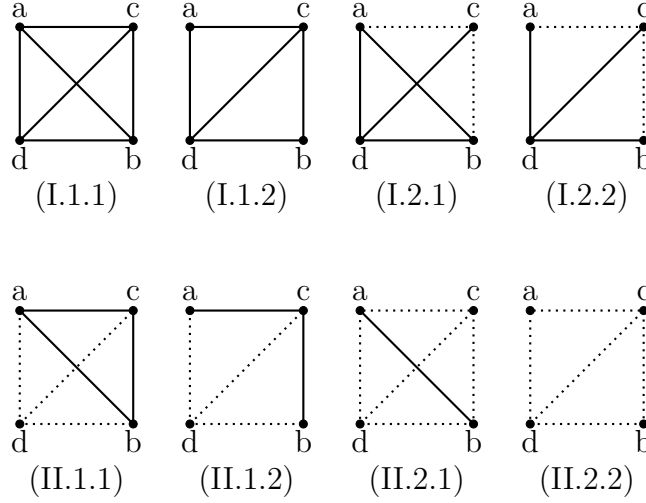


Figure 2.3: Configurations of 4 generators with $a \neq b \neq c \neq d$. Dashed edges represent pairs of vertices that might or might not be in the star of each other (including the possibility of being equal). In cases (I.2) at least one dashed edge is not an edge in Γ (i.e. the vertices are different and do not commute). In cases (II) at least one of the dashed edges incident to d is not an edge in Γ . Moreover, in cases (II.2), at least one of a or b does not form an edge with c .

As above, we will write \sum_X , where X is one of the cases above, to denote the summation over all quadruples $(a, b, c, d) \in (V\Gamma)^4$ satisfying the hypothesis of case X .

In (I.1.1) any two of the vertices a, b, c, d would commute, so they would have to be pairwise distinct and hence form a tetrahedron; however in Γ there are no 4-cliques, so we get $\sum_{(I.1.1)} E_{abcd}(z) = 0$.

In (I.1.2), except the pair (a, b) any two other pairs of vertices among a, b, c, d commute, so all of them have to be pairwise distinct, as otherwise $abcd$ would not be a geodesic. Fixing the triangle $\{a, c, d\}$, we have $q - 1$ choices for b . Also, for a, b, c, d in this case, $E_{abcd} \equiv E_{acdb} = E_{acd} \cdot b$ (by Theorem 2.2.11, as b does not commute with a), hence $E_{abcd}(z) = E_{acd}(z) \cdot z$. Therefore

$$\sum_{(I.1.2)} E_{abcd}(z) = (q - 1)z\mathcal{E}_\Delta(z).$$

In (I.2.1), the vertex d commutes with any of the letters of the word $abcd$ so it should be distinct from any of a, b, c . The only equal pair could be (a, c) ; but $a = c$ implies $\{a, b\} \not\subseteq \text{St}(a)$ which contradicts the assumption $a \in \text{Lk}(b)$. So, once again all the vertices are pairwise distinct. Fixing the triangle $\{a, b, d\}$, we have $l - 2$ choices for c . Also, for a, b, c, d in this case, we have $E_{abcd} \equiv E_{abdc} = E_{abd} \cdot c$, as c does not commute with at least one of a, b . So, $E_{abcd}(z) = E_{abd}(z) \cdot z$, and now we get

$$\sum_{(I.2.1)} E_{abcd}(z) = (l - 2)z\mathcal{E}_\Delta(z).$$

In (I.2.2), as in (I.2.1), the vertex d commutes with any of the letters of the word $abcd$ so it should be distinct from any of a, b, c . Except the pair (a, c) , which could be equal, any other two vertices among a, b, c, d are different. So $\{a, d\}$ forms an edge and we start by fixing it. As, $b, c \in \text{Lk}(d)$, and as both b and c do not commute with at least one of a, d we have $E_{abcd} \equiv E_{adb}c = E_{adb} \cdot c = E_{ad} \cdot b \cdot c$. Now one gets $E_{abcd}(z) = E_{ad}(z) \cdot z^2$. We need to count the possibilities for b and c ; consider the following disjoint subcases for c :

- (1) $c = a$: Here we have $l - 1 - q$ choices for b .
- (2) $c \in \text{Lk}(a)$: Here we have q choices for c . Note that as $\{a, b\} \not\subseteq \text{St}(c)$, we get that $b \notin \text{St}(c)$. Therefore, b is the link of d but b is not in the link of the edge $\{a, d\}$ and neither in the link of the edge $\{c, d\}$. Note that the $\text{Lk}(\{a, d\}) \cap \text{Lk}(\{c, d\})$ is empty, as if not, Γ will contain a tetrahedron. Therefore, there are $l - 2q$ choices for b in this subcase.
- (3) $c \notin \text{St}(a)$: we have $l - 1 - q$ choices for c and $l - 2 - q$ choices for b , since on top we have that $b \neq c$, as there is no geodesic of the form $accd$.

At the end one gets

$$\sum_{(I.2.2)} E_{abcd}(z) = \sum_{a \in V} \sum_{d \in \text{Lk}(a)} \left(\sum_{c=a} E_{abcd}(z) + \sum_{c \in \text{Lk}(a)} E_{abcd}(z) + \sum_{c \notin \text{St}(a)} E_{abcd}(z) \right)$$

which is equal to the sum:

$$\sum_{a \in V} \sum_{d \in \text{Lk}(a)} \left((l - 1 - q)z^2 E_{ad}(z) + q(l - 2q)z^2 E_{ad}(z) + (l - 1 - q)(l - 2 - q)z^2 E_{ad}(z) \right)$$

and as a final expression we get:

$$\sum_{(I.2.2)} E_{abcd}(z) = [(l - 1 - q)^2 + q(l - 2q)]z^2 \mathcal{E}_e(z).$$

We now consider the case (II).

In (II.1.1), all the vertices should be distinct, indeed, any pair among a, b, c commute, so $a \neq c$, and any of them is distinct from d as one can permute the letters of the subword abc in $abcd$. We count by fixing the triangle abc . We have $E_{abcd} = E_{abc} \cdot d$, as d does not commute with at least one of a, b, c . One gets also $E_{abcd}(z) = E_{abc}(z) \cdot z$. Since Γ does not have tetrahedra, there is no condition on d except that $d \notin \{a, b, c\}$. We have $n - 3$ choices for d , so we get

$$\sum_{(II.1.1)} E_{abcd}(z) = (n - 3)z \mathcal{E}_\Delta(z).$$

In (II.1.2), a, b, c are distinct as for $a = c$ we get $a \in \text{Lk}(b)$. As $abc = acb$, the only equal pair could be (a, d) , which will be discussed below. Here we count by first fixing the edge $\{a, c\}$ and then considering the different cases for d : $d \in \{a, c\}$, $d \in \text{Lk}(\{a, c\})$ or $d \notin \text{St}(\{a, c\})$. As c commutes with b , b does not commute with at least one of a, c , and d does not commute with at least one of a, c, b , we have $E_{abcd} \equiv E_{acbd} = E_{acb} \cdot d = E_{ac} \cdot bd$. Hence, also $E_{abcd}(z) = E_{ac}(z) \cdot z^2$. We now count the choices for b, d :

- (1) $d \in \{a, c\}$: The case $d = c$ is impossible. The case $d = a$, we have that $b \in \text{Lk}(c) \setminus \text{St}(a)$, and this gives $l - 1 - q$ choices for b .
- (2) $d \in \text{Lk}(\{a, c\})$: there are q choices for d . In this case we have that $b \in \text{Lk}(c)$. Since $\{a, b, c\} \not\subseteq \text{St}(d)$ and $d \in \text{Lk}(\{a, c\})$, we get $b \notin \text{Lk}(d)$. Also from the hypothesis, we get $b \notin \text{Lk}(a)$. Therefore b is in $\text{Lk}(c) \setminus (\text{Lk}(\{a, d\}) \cup \text{Lk}(\{c, d\}))$. Note that the links of two edges in a triangle are disjoint as Γ has no tetrahedra. Thus, there are $l - 2q$ choices for b .
- (3) $d \notin \text{St}(\{a, c\})$: We subdivide this case into two subcases:
 - (3.1) $d \in \text{St}(c)$: In this case, d is in $\text{St}(c) \setminus \text{St}(\{a, c\})$ and there are $(l+1) - (q+2) = l - q - 1$ possibilities for d .
Note that in this case $b \neq d$ since otherwise $abcd$ is not geodesic. Thus we have that b is in $\text{St}(c) \setminus (\text{St}(\{a, c\}) \cup \{d\})$, and we have $l - 2 - q$ possibilities for b .
 - (3.2) $d \notin \text{St}(c)$: As $d \in V \setminus \text{St}(c)$, we have $n - l - 1$ choices for d . As $b \in \text{St}(c) \setminus \text{St}(\{a, c\})$, we have $l - 1 - q$ choices for b .

Ultimately, we get

$$\sum_{(II.1.2)} E_{abcd}(z) = \sum_{a \in V} \sum_{c \in \text{Lk}(a)} \left(\sum_{d \in \{a, c\}} E_{abcd}(z) + \sum_{d \in \text{Lk}(\{a, c\})} E_{abcd}(z) + \sum_{d \notin \text{St}(\{a, c\})} E_{abcd}(z) \right)$$

which is equal to the sum:

$$\sum_{a \in V} \sum_{c \in \text{Lk}(a)} \left[1 \cdot (l - 1 - q) z^2 E_{ac}(z) + q(l - 2q) z^2 E_{ac}(z) + (l - q - 1)(n - q - 3) z^2 E_{ac}(z) \right]$$

and this gives us:

$$\sum_{(II.1.2)} E_{abcd}(z) = [(l - 1 - q)(n - q - 2) + q(l - 2q)] z^2 \mathcal{E}_e(z).$$

In (II.2.1), we start by fixing the edge $\{a, b\}$ and then considering the different cases for d : $d \in \{a, b\}$, $d \in \text{Lk}(\{a, b\})$ or $d \notin \text{St}(\{a, b\})$. The vertices a, b, c are all different, as $abc = bac$. Since $\{a, b, c\} \not\subseteq \text{St}(d)$ we get $E_{abcd} = E_{abc} \cdot d$. Moreover, $\{a, b\} \not\subseteq \text{St}(c)$, so $E_{abc} = E_{ab} \cdot c$. Putting these together, we get $E_{abcd} = E_{ab} \cdot cd$, and hence $E_{abcd}(z) = E_{ab}(z) \cdot z^2$. We count the choices for c and d .

- (1) $d \in \{a, b\}$: If $d = a$, we have $\{a, b\} \subseteq \text{St}(d)$ as $a \in \text{Lk}(b)$. Since $\{a, b, c\} \not\subseteq \text{St}(d)$ we get $c \notin \text{St}(d)$. Thus c can be any vertex outside of $\text{St}(d)$, which means $n - l - 1$ choices for c . If $d = b$, the discussion is analogous and we can take for c any vertex outside $\text{St}(b) = \text{St}(d)$ and we have $n - l - 1$ choices for c yet again.
- (2) $d \in \text{Lk}(\{a, b\})$: Here we have q choices for d . Since $\{a, b, c\} \not\subseteq \text{St}(d)$, c can not be in $\text{Lk}(d)$. Since $\{a, b\} \not\subseteq \text{St}(c)$, c is not in $\text{Lk}(\{a, b\})$. Since a, b, d form a triangle, and we do not have tetrahedra, these two links are disjoint. There are $n - q - l$ choices for c .
- (3) $d \notin \text{St}(\{a, b\})$: Here both c and d are not in $\text{St}(\{a, b\})$ and moreover $c \neq d$ to have $abcd$ a geodesic. We have $d \in V \setminus \text{St}(\{a, b\})$ that gives $n - q - 2$ choices for d and we have $c \in V \setminus (\text{St}(\{a, b\}) \cup \{d\})$ that gives $n - q - 3$ for c .

Ultimately, we get

$$\sum_{(\text{II.2.1})} E_{abcd}(z) = [2(n - l - 1) + q(n - q - l) + (n - q - 2)(n - q - 3)]z^2 \mathcal{E}_e(z).$$

In (II.2.2), we first fix a and then we consider different cases for d : $d = a, d \in \text{Lk}(a)$ or $d \notin \text{St}(a)$. Note that since $\{a, b, c\} \not\subseteq \text{St}(d)$ we have: $E_{abcd} = E_{abc} \cdot d$. Similarly since $\{a, b\} \not\subseteq \text{St}(c)$ we have $E_{abc} = E_{ab} \cdot c$, and ultimately, since $a \notin \text{Lk}(b)$ we get $E_{ab} = E_a \cdot b$. Putting everything together we get $E_{abcd} = E_a \cdot bcd$ and hence $E_{abcd}(z) = E_a(z) \cdot z^3$. We count the choices for b, c, d .

- (1) $d = a$: Here we have 1 choice for d . We split this case into the following disjoint subcases:
 - (1.1) $c = a$: this is impossible, since $abaa$ is not a geodesic.
 - (1.2) $c \in \text{Lk}(a)$: Here we have l choices for c . Since $\{a, b, c\} \not\subseteq \text{St}(d)$, $d = a$ and $c \in \text{Lk}(a) = \text{Lk}(d)$, we have $b \notin \text{St}(a) = \text{St}(d)$. Further, since $\{a, b\} \not\subseteq \text{St}(c)$ and $a \in \text{Lk}(c)$ it must be that $b \notin \text{Lk}(c)$. In this case, b can be any vertex of $V \setminus (\text{St}(a) \cup \text{St}(c))$. As $|\text{St}(a) \cap \text{St}(c)| = q + 2$, we have $n - 2(l + 1) + (q + 2) = n - 2l + q$ possibilities for b .
 - (1.3) $c \notin \text{St}(a)$: Here $b \notin \text{St}(a) \cup \{c\}$. So we have $n - l - 1$ choices for c , and $n - l - 2$ for b .

Accounting for (1.1), (1.2) and (1.3), for a given vertex $a \in V$ we have

$$\sum_{\substack{c, b, d \in (\text{II.2.2}) \\ d=a}} E_{abcd}(z) = [l(n - 2l + q) + (n - l - 1)(n - l - 2)]z^3 E_a(z)$$

- (2) $d \in \text{Lk}(a)$: Here we have l choices for d . We divide now the analysis into the following disjoint subcases:

- (2.1) $c \in \{a, d\}$: The case $c = d$ is impossible. In the case $c = a$, we have one choice for c and b can be any vertex of $V \setminus (\text{St}(a) \cup \text{St}(d))$. As $|\text{St}(a) \cap \text{St}(d)| = q + 2$, we have $n - 2l + q$ possibilities for b .
- (2.2) $c \in \text{Lk}(\{a, d\})$: which gives us q choices for c . We have a triangle $\{a, c, d\}$ in Γ and by hypothesis of case (II.2.2), $b \notin \text{St}(a) \cup \text{St}(c) \cup \text{St}(d) = \text{Lk}(a) \cup \text{Lk}(c) \cup \text{Lk}(d)$. We have that $\text{Lk}(x) \cap \text{Lk}(y)$ has q elements, for $x \neq y$, $x, y \in \{a, c, d\}$. As Γ has no tetrahedra, $\text{Lk}(\{a, b, c\}) = \text{Lk}(a) \cap \text{Lk}(c) \cap \text{Lk}(d)$ is empty. Using the inclusion-exclusion principle, we have $n - 3l + 3q$ choices for b .
- (2.3) $c \notin \text{St}(\{a, d\})$: we subdivide this case into the following disjoint subcases:
- (2.3.1) $c = a$: This is impossible since $c \notin \text{St}(\{a, d\})$.
- (2.3.2) $c \in \text{Lk}(a)$: In this case, necessarily, $c \notin \text{St}(d)$. Here we get $l - 1 - q$ choices for c . Also $b \notin \text{Lk}(a) \cup \text{Lk}(c)$ and we have $n - 2l + q$ choices for b .
- (2.3.3) $c \notin \text{St}(a)$: We do now again, three subcases:
- (2.3.3.1) $c = d$: This is impossible since $c \notin \text{St}(\{a, d\})$.
- (2.3.3.2) $c \in \text{Lk}(d)$: In this case $c \in \text{Lk}(d) \setminus \text{St}(a)$ here we get $l - 1 - q$ choices for c . Also $b \notin \text{Lk}(a) \cup \text{Lk}(d)$ and we have $n - 2l + q$ choices for b .
- (2.3.3.3) $c \notin \text{St}(d)$: Here one has $|V \setminus (\text{St}(a) \cup \text{St}(d))| = n - 2l + q$ choices for c . Note that since a, d span an edge and b is not star of a in (II.2.2) we have that b can not be equal to d neither to a . We subdivide this case into the following disjoint subcases:
- (2.3.3.3.1) $b = d$: impossible.
- (2.3.3.3.2) $b \in \text{Lk}(d)$: then $b \in \text{Lk}(d) \setminus \text{St}(a)$ and we have $l - 1 - q$ choices for b .
- (2.3.3.3.3) $b \notin \text{St}(d)$: here $b \neq c$ to get a geodesic, and $b \notin \text{St}(a) \cup \text{St}(d)$. We have $n - 2l + q - 1$ choices for b .

Accounting for (2.1), (2.2) and (2.3), for a given vertex $a \in V$ we have

$$\sum_{\substack{c, b, d \in (\text{II.2.2}) \\ d \in \text{Lk}(a)}} E_{abcd}(z) = [lq(n - 3l + 3q) + l(n - 2l + q)(n + l - 2q - 3)]z^3 E_a(z).$$

(3) $d \notin \text{St}(a)$:

- (3.1) $b = d$: here we have $n - 1 - l$ choices for d , 1 choice for b . As $abcb$ is a geodesic, c does not belong to $\text{St}(b)$, and we have $n - l - 1$ choices for c .
- (3.2) $b \neq d$: here we split into these following disjoint cases:
- (3.2.1) $c = a$: here we get 1 choice for c . Since $\{a, b\} \not\subseteq \text{St}(c)$ and $c = a$ we get $b \notin \text{St}(a)$. One has that b, d are any pair of different vertices of $V \setminus \text{St}(a)$, and they are distinct, thus we have $(n - l - 1)(n - l - 2)$ possibilities for b and d .

(3.2.2) $c \in \text{Lk}(a)$: here we get l choices for c . The hypothesis of case II.2, $\{a, b\} \not\subseteq \text{St}(c)$ implies that $b \notin \text{St}(c)$. We consider the following disjoint subcases:

(3.2.2.1) $d = c$: which is impossible since $abcd$ is geodesic.

(3.2.2.2) $d \in \text{Lk}(c)$: here get $|\text{Lk}(c) \setminus \text{St}(a)| = l - 1 - q$ choices for d . We have $|V \setminus (\text{St}(a) \cup \text{St}(c))| = n - 2l + q$ for b .

(3.2.2.3) $d \notin \text{St}(c)$: we get $|V \setminus (\text{St}(a) \cup \text{St}(c))| = n - 2l + q$ choices for d and $|V \setminus (\text{St}(a) \cup \text{St}(c) \cup \{d\})| = n - 2l + q - 1$ choices for b .

(3.2.3) $c \notin \text{St}(a)$: here for b, c, d we get $(n - l - 1)(n - l - 2)(n - l - 3)$ choices as b, c, d can be any vertex outside of $\text{St}(a)$, $b \neq c$, $c \neq d$ because $abcd$ is geodesic, and $b \neq d$ by hypothesis.

Accounting for (3.1) and (3.2), we get

$$\sum_{\substack{c, b, d \in (\text{II.2.2}) \\ d \notin \text{St}(a)}} E_{abcd}(z) = [(n-1-l)^2 + (n-l-1)(n-l-2)^2 + l(n-2l+q)(n-l-2)]z^3 E_a(z)$$

Ultimately, in case (II.2.2), we get

$$\sum_{(\text{II.2.2})} E_{abcd}(z) = [(n-l-1)^3 + 2l(n-2l+q)(n-q-2) + lq(n-3l+3q)]z^3 \mathcal{E}_v(z).$$

We finally collect all these cases together, and we conclude that

$$\sum_{(a,b,c,d) \in (\text{VI})^4} E_{abcd}(z)$$

is equal to:

$$\begin{aligned} & (q-1)z\mathcal{E}_\Delta(z) \\ & + (l-2)z\mathcal{E}_\Delta(z) \\ & + [(l-1-q)^2 + q(l-2q)]z^2\mathcal{E}_e(z) \\ & + (n-3)z\mathcal{E}_\Delta(z) \\ & + [(l-1-q)(n-q-2) + q(l-2q)]z^2\mathcal{E}_e(z) \\ & + [2(n-l-1) + q(n-q-l) + (n-q-2)(n-q-3)]z^2\mathcal{E}_e(z) \\ & + [(n-l-1)^3 + 2l(n-2l+q)(n-q-2) + lq(n-3l+3q)]z^3\mathcal{E}_v(z) \end{aligned}$$

After grouping similar expressions we can express $\sum_{(a,b,c,d) \in (\text{VI})^4} E_{abcd}(z)$ as:

$$\begin{aligned} & (n+q+l-6)z \sum_{\Delta \in \Delta\Gamma} E_\Delta(z) \\ & + (l^2 + ln + n^2 - 2lq - 2nq - 2q^2 - 6l - 4n + 10q + 7)z^2\mathcal{E}_e(z) \\ & + [(n-l-1)^3 + 2l(n-2l+q)(n-q-2) + lq(n-3l+3q)]z^3\mathcal{E}_v(z) \end{aligned}$$

Now, summing everything up, we get Equation 2.5 of the main theorem, and ultimately the proof of the theorem.

Chapter 3

Twisted RAAGs

The goal of this chapter is to describe a normal form for elements in a twisted right-angled Artin group. Using the normal form, we will conclude that a tRAAG has the same spherical and geodesic growth as the corresponding RAAG based on the underlying naïve graph, and in the case of spherical growth one gets the same formulas as in [5]. Moreover we will discuss some algebraic properties that point out differences and similarities with RAAGs.

3.1 Introduction

The topic of this chapter are twisted right-angled Artin groups (tRAAGs shortly). One defines them using a mixed graph (see Definition 1.2.6), which is a simplicial graph as in the case of RAAGs, however some edges can be directed edges, as presented in the following example (see also Example 3.1.3 for other mixed graphs).

Example 3.1.1. Consider the graph consisting of only one directed edge, as below:

$$\Gamma = \begin{array}{c} a \\ \bullet \longrightarrow \bullet \\ b \end{array}$$

To this graph we associate a group G_Γ by defining its presentation as:

$$G_\Gamma = \langle a, b | abab^{-1} = 1 \rangle,$$

i.e. its generators are the vertices of the graph, while the relations can be read from the edges. In this case we have only one edge, so we have only one corresponding relation $abab^{-1} = 1$. In this case the group G_Γ is the fundamental group K of the Klein bottle, and for this reason we will call the relation $abab^{-1} = 1$, the Klein relation.

Let $\Gamma = (V, E)$ be a mixed graph as in Definition 1.2.6 of Chapter 1. We give a presentation for the corresponding tRAAG, denoted by G_Γ . The set of vertices V is going to serve as the generating set for G_Γ . Now we define the relations using the edges. Recall that:

$$E = \overrightarrow{E\Gamma} \sqcup \overrightarrow{E\Gamma},$$

where $\overrightarrow{E\Gamma}$ is the set of undirected edges, and $\overrightarrow{E\Gamma}$ is the set of directed edges.

If $e = \{a, b\}$ is an edge in E which belongs to $\overrightarrow{E\Gamma}$ we denote it by $[a, b]$, and if it belongs to $\overrightarrow{E\Gamma}$ we denote it by $[a, b\rangle$ where a the origin and b the terminus of e . These two types of edges are represented in the figure below by cases (i) and (ii) respectively.



We want graph (i) to represent the commutation of a, b , and graph (ii) to represent the Klein relation. Therefore, we fix the following notation:

$$(i) \quad [a, b] = aba^{-1}b^{-1}, \text{ and}$$

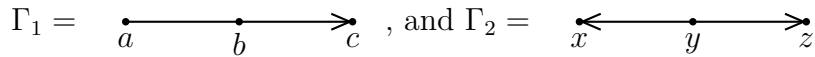
$$(ii) \quad [a, b\rangle = abab^{-1}.$$

Definition 3.1.2. Let $\Gamma = (V, E)$ be a mixed graph. Define a group G_Γ , corresponding to Γ as

$$G_\Gamma = \langle v \in V \mid [a, b] = 1 \text{ if } [a, b] \in \overrightarrow{E\Gamma}, [a, b\rangle = 1 \text{ if } [a, b\rangle \in \overrightarrow{E\Gamma} \rangle.$$

We call G_Γ the *twisted right-angled Artin group* based on Γ . And we call Γ , the *defining graph* for G_Γ .

Example 3.1.3 (see [8]). Consider the graphs Γ_1 and Γ_2 as drawn below:



Associated to each Γ_i we get a tRAAG, denoted by $G_i = G_{\Gamma_i}$, and presented as:

$$\begin{aligned} G_1 &= \langle a, b, c \mid ab = ba, bc = cb^{-1} \rangle, \\ G_2 &= \langle x, y, z \mid yx = xy^{-1}, yz = zy^{-1} \rangle. \end{aligned}$$

Consider the map $f: G_1 \rightarrow G_2$ defined by $f(a) = xz^{-1}, f(b) = y, f(c) = z$. Then f is a well-defined morphism of groups because

$$\begin{aligned} f(aba^{-1}b^{-1}) &= (xz^{-1})y(xz^{-1})^{-1}y^{-1} = xz^{-1}yzx^{-1}y^{-1} \\ &= x(z^{-1}yz)x^{-1}y^{-1} && \text{use } yz = zy^{-1} \\ &= xy^{-1}x^{-1}y^{-1} && \text{use } yx = xy^{-1} \\ &= 1 \end{aligned}$$

and similarly $f(bcbc^{-1}) = yzyz^{-1} = 1$. In a similar fashion we conclude that the map $g: G_2 \rightarrow G_1$ defined by $g(x) = ac, f(y) = b, f(z) = c$ is a well-defined morphism of groups. Moreover $g \circ f = 1_{G_1}$, and $f \circ g = 1_{G_2}$ which means that $G_1 \simeq G_2$.

Remark 3.1.4. The example provided above shows that we can have isomorphic tRAAGs based on non-isomorphic twisted graphs.

Definition 3.1.5. Let Γ be a mixed graph and G_Γ the tRAAG over Γ . Denote by $\bar{\Gamma}$ the underlying naïve graph of Γ , and by $G\bar{\Gamma}$ the RAAG over $\bar{\Gamma}$. We call $G\bar{\Gamma}$, *the underlying RAAG corresponding to G_Γ* .

Remark 3.1.6. The correspondence provided above seems quite natural, as a lot of geometric properties of G_Γ , agree with the corresponding properties of $G\bar{\Gamma}$.

Notation 3.1.7. When we talk about languages, words, and growth in tRAAGs, we will use $S = V \sqcup V^{-1}$ as our alphabet (or preferred monoid generating set in the context of groups), where V will always denote the vertex set of the defining graph Γ .

3.2 Abelianization on tRAAGs

In this part we will characterize the abelianization of tRAAGs. In some cases this provides a way to distinguish two tRAAGs up to isomorphism.

Definition 3.2.1. Let G be a group.

- (i) The commutator subgroup $[G, G]$ is the subgroup of G generated by all commutators, which are the elements of the form $[x, y] = xyx^{-1}y^{-1}$, for x, y in G .
- (ii) The abelianization of G is the quotient of the group G by its commutator subgroup and we denote it by G^{ab} . Shortly, $G^{ab} = G/[G, G]$.

In the case of RAAGs the abelianization depends only on the number of vertices on the graph. Indeed, let Γ be a graph with $|\Gamma| = n$, and A the RAAG based on Γ . Then $A^{ab} = \mathbb{Z}^n$.

In the case of tRAAGs the situation is a bit different, and it depends on the direction of the edges as well. To get a better grasp at the situation let us discuss first the notion of indicability.

Definition 3.2.2. A group G is called *indicable* if G is trivial or if there is a surjective homomorphism $\varphi: G \rightarrow \mathbb{Z}$.

RAAGs are indicable. An example of a surjective homomorphism $\varphi: G \rightarrow \mathbb{Z}$ would be defining $\varphi(v) = 1$ for any generator $v \in V\Gamma$. On the other hand, tRAAGs are not always indicable. And the following result gives a characterization.

Theorem 3.2.3. A non-trivial tRAAG is indicable if and only if there is a vertex v which is not the origin of any directed edge.

Proof. If such v exists, we can define a map from G to \mathbb{Z} by sending v to 1 and all the other vertices to 0. This map respects the relations on all the generators, and therefore induces a surjective homomorphism. So the group is indicable.

On the other hand, by contradiction, assume that any vertex of our graph is the origin of at least one directed edge. Assume that there is a morphism $\varphi : G \rightarrow \mathbb{Z}$. Pick any vertex a in Γ , then by assumption there is an oriented edge $e = [a, b]$ for some b . The relation coming from edge e is: $aba = b$, and this implies $2\varphi(a) = 0$. This means that φ is trivial, as all generators are mapped to 0. In this case the group is not indicable. \square

Now we go back to the abelianization of tRAAGs. Let Γ be a directed graph, and G the tRAAG based on Γ . Then G^{ab} can be computed by adding the relations $xy = yx$ to the presentation of G for all $x, y \in V\Gamma$. If $[a, b]$ is a directed edge, we have the relation $abab^{-1} = 1$. Since the relation $aba^{-1}b^{-1} = 1$ is added to the presentation as well, we get $a^2 = 1$. This holds for any vertex a when it is the origin of a directed edge. If we denote by V_o the set of vertices that are the origin of a directed edge, then

$$G^{ab} = \mathbb{Z}^{|V\Gamma - V_o|} \times \mathbb{Z}_2^{|V_o|}.$$

Since isomorphic groups have isomorphic abelianizations we have the following corollary.

Corollary 3.2.4. Let Γ_1 and Γ_2 be two mixed graphs. If

- $|V\Gamma_1| \neq |V\Gamma_2|$, or
- Γ_1 and Γ_2 have different numbers of vertices which are the origin of a directed edge,

then the tRAAGs G_{Γ_1} and G_{Γ_2} are not isomorphic.

3.3 Overview of graph products

Here we give an overview of graph products, and a normal form for their elements as appearing in [22]. This will serve as a motivation for the corresponding normal form in tRAAGs.

Definition 3.3.1. Let $\Gamma = (V, E)$ be a simplicial graph, and let G_v be groups, indexed by $v \in V$. Define $G\Gamma$ by the presentation:

$$G\Gamma := \langle G_v, v \in V \mid [G_u, G_v] = 1, \forall \{u, v\} \in E \rangle.$$

Call $G\Gamma$ the *graph product* of groups $(G_v)_{v \in V}$ based on the graph Γ . One refers to the groups G_v (for $v \in V$) as *generating groups* of $G\Gamma$, and to the graph Γ as the *underlying graph*.

Notation 3.3.2. When representing geometrically the underlying graph Γ from Definition 3.3.1, we make the following conventions:

1. we label its vertex v , by the group G_v , for all $v \in V$,
2. if $G_v = \mathbb{Z} = \langle g \rangle$ for some $v \in V$, we can label the vertex v by the corresponding generator g .

Example 3.3.3. Consider the graph $\Gamma = (V, E)$ with

$$V = \{1, 2, 3, 4\}, \quad E = \{\{1, 2\}, \{2, 3\}\}.$$

and let G_1, G_2, G_3, G_4 be groups. Geometrically we draw Γ as in case (i) of Figure 3.1, and we obtain the graph product

$$G\Gamma = \langle G_1, G_2, G_3, G_4 \mid [G_1, G_2] = 1, [G_2, G_3] = 1 \rangle \simeq ((G_1 * G_3) \times G_2) * G_4.$$

Notice that if $G_i = \mathbb{Z} = \langle g_i \rangle$, for all $1 \leq i \leq 4$, using Notation 3.3.2 we present Γ as in case (ii) of Figure 3.1, which defines the graph product

$$G\Gamma = \langle g_1, g_2, g_3, g_4 \mid g_1 g_2 = g_2 g_1, g_2 g_3 = g_3 g_1 \rangle,$$

and this is the right-angled Artin group G_Γ based on Γ .

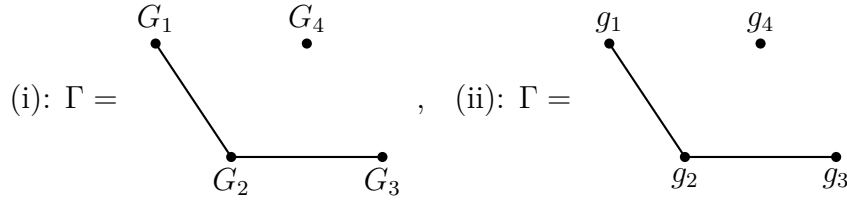


Figure 3.1: Geometric presentations of underlying graphs

Note 3.3.4. In fact, the situation occurring in Example 3.3.3 is a general one, because taking $G_v = \mathbb{Z}$ for all $v \in V$ in Definition 3.3.1, then $G\Gamma$ is the RAAG based on Γ , and denoted by G_Γ . So RAAGs are in particular, graph products.

Remark 3.3.5. With the notation of Definition 3.3.1 we notice that:

- (i) if Γ is complete we get a direct product: $G\Gamma = \times_{v \in V} G_v$, and
- (ii) if Γ is totally disconnected we get a free product: $G\Gamma = *_{v \in V} G_v$.

Definition 3.3.6. Let $G\Gamma$ be the graph product of groups G_1, G_2, \dots, G_n , and w a word of $G\Gamma$, written as $w = w_1 w_2 \dots w_r$ where each w_i is a word on generators of only one of the generating groups, no w_i is the empty word, and w_i, w_{i+1} are not in the same generating group for all $1 \leq i \leq r - 1$. Then:

- the words w_i (for $1 \leq i \leq r$) are called the *syllables* of w ,
- (w_1, w_2, \dots, w_r) is called a sequence of syllables, representing $g = w_1 \dots w_r$ as an element in $G\Gamma$,

- the syllable length $\lambda(w)$ of w is equal to r , and
- the syllable length $\lambda(g)$ of an element $g \in G\Gamma$ is the minimal syllable length of words representing g , i.e. $\lambda(g) = \min\{\lambda(w) \mid w = g \text{ as elements in } G\Gamma\}$.

Note 3.3.7. The syllable length has the following basic properties:

- $\lambda(1) = 0$ for the identity $1 \in G\Gamma$,
- $\lambda(g) = 1$ when $g \neq 1$ belongs to one of the generating groups, and
- $\lambda(g) \geq 2$ when g does not belong to one of the generating groups.

Definition 3.3.8. Let (w_1, w_2, \dots, w_r) be a sequence of syllables representing an element g . For $1 \leq i < j \leq r$ we will say that the syllables w_i and w_j can be *joined together* if w_i and w_j belong to the same generating group, and for all $k \in \{i+1, \dots, j-1\}$ one has $w_i w_k = w_k w_i$.

In this case one can group together w_i and w_j , so we can present g with less syllables. Also notice that one has $w_j w_k = w_k w_j$, as both w_i, w_j belong to the same generating group.

To develop a normal form for elements in a graph product, we introduce the notion of reduced sequences of syllables.

Definition 3.3.9. The sequence of syllables (w_1, \dots, w_r) is called a *reduced sequence* in $G\Gamma$ if it is either empty, or if $w_i \neq 1$ for all $1 \leq i \leq r$ and no two syllables of the sequence can be joined.

Note 3.3.10. The identity element $1 \in G$ is represented by the empty sequence. We will adopt the convention to denote the empty sequence by \emptyset .

Definition 3.3.11. Introduce an equivalence relation for reduced sequences, denoted by \cong , generated by:

$$(w_1, \dots, w_i, w_{i+1}, \dots, w_r) \cong (w_1, \dots, w_{i+1}, w_i, \dots, w_r) \iff [w_i, w_{i+1}] = 1 \quad (3.1)$$

i.e. when w_i, w_{i+1} belong to generating groups whose corresponding vertices are joined by an edge in Γ .

Remark 3.3.12. We refer to the relation on Equation (3.1) as *shuffling of syllables* or *syllable shuffling*.

Now we can state the normal form theorem for graph products.

Theorem 3.3.13 (Theorem 3.9 in [22]). Let $G\Gamma$ be a graph product of the groups G_1, \dots, G_n . Each element $g \in G\Gamma$ can be uniquely expressed, up to syllable shuffling (i.e. up to the equivalence \cong), as a product:

$$g = g_1 g_2 \cdots g_r$$

where (g_1, g_2, \dots, g_r) is a reduced sequence of syllables.

3.4 Normal Form theorem for tRAAGs

Now we describe a normal form for elements of tRAAGs using a similar approach as for the case of graph products of groups in [22]. See also Section 3.3 of this thesis for an overview.

With reference to Definition 3.3.1, in the following notation, we adopt a point of view to consider tRAAGs in an analogous fashion of graph products.

Notation 3.4.1. Let $\Gamma = (V, E)$ be a twisted graph, and let G_Γ be the tRAAG based on Γ . Treat any vertex $v \in V$ as a generator of an infinite cyclic group G_v ; in other words $G_v = \langle v \rangle \simeq \mathbb{Z}$. One refers to the groups G_v (for $v \in V$) as *generating groups* of G_Γ , and to the graph Γ , as the *underlying twisted graph*.

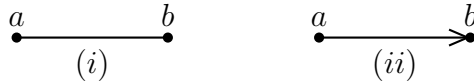
By convention, refer to G_Γ as a *twisted graph product* of groups $(G_v)_{v \in V}$, based on the twisted graph Γ .

Definition 3.4.2. By a *sequence of syllables* in G_Γ we mean a finite sequence (g_1, \dots, g_r) of elements of G_Γ , where each g_i belongs to a generating group, i.e. each g_i is of the form $v_i^{s_i}$ for some $v_i \in V\Gamma$ and $s_i \in \mathbb{Z}$.

Any element $g \in G_\Gamma$ can be represented as a word $w \equiv g_1 \dots g_r$ where (g_1, \dots, g_r) is a sequence of syllables in G . Here $g = g_1 \dots g_r$ and we call each g_i a syllable of w and also a syllable of the sequence (g_1, \dots, g_r) .

Example 3.4.3. Let $G = \langle a, b | abab^{-1} = 1 \rangle$. The element $g = a^3ba^{-1}a^2b^{-1}$ can be represented by the word $w = a^3bab^{-1}$, and (a^3, b, a, b^{-1}) is a sequence of syllables.

Remark 3.4.4. Let $a, b \in V\Gamma$ be adjacent vertices as in the figure below. The edge that they belong to, can be either undirected, as in case (i), or directed as in case (ii).



Let m, n any two integers. Then:

- (i) $ab = ba$ implies $a^m b^n = b^n a^m$, and
- (ii) $ab = ba^{-1}$ implies $a^m b^n = b^n a^{(-1)^n m}$.

Notation 3.4.5. If $g_1 = a^m$, and $g_2 = b^n$ we are going to write $g_1 g_2 = g'_2 g'_1$, where

- (i) $g'_1 = a^m = g_1$, and $g'_2 = b^n = g_2$ if $ab = ba$, or
- (ii) $g'_1 = a^{(-1)^n m} \in \{g_1, g_1^{-1}\}$, and $g'_2 = b^n = g_2$ if $ab = ba^{-1}$.

We refer to $g_1 g_2 = g'_2 g'_1$ as *syllable shuffling*, as one can shuffle g_1 and g_2 (up to 'signs of powers' in the case of the Klein relation). We also write $(g_1, g_2) \longleftrightarrow (g'_2, g'_1)$.

Definition 3.4.6. Let (g_1, \dots, g_r) be a sequence of syllables with $g_t = v_t^{s_t}$ with $v_t \in V\Gamma$ for all $1 \leq t \leq r$.

1. For $1 \leq i < j \leq r$ we will say that the syllables g_i and g_j can be joined together if $v_i = v_j = v \in V\Gamma$ and for any $k \in \{i+1, \dots, j-1\}$ one has $v_k \in \text{St}(v)$. In this case one can group together powers of v appearing in g_i and g_j , and get the same element with less syllables.
2. If in the sequence (g_1, \dots, g_r) , the syllables g_i, g_{i+1} are not joined we say that the syllable length of the word $w = g_1 \dots g_r$ is equal to r , and denote $\lambda(w) = r$.
3. For the group element g , its syllable length $\lambda(g)$ is defined as the minimal syllable length of the words representing it.

Note 3.4.7. The identity element $1 \in G_\Gamma$ is represented by the empty sequence. We will adopt the convention to denote the empty sequence by \emptyset .

Example 3.4.8. The syllable length has the following properties:

- $\lambda(1) = 0$,
- $\lambda(g) = 1$ if and only if $g \neq 1$, and g belongs to a generating group i.e. $g = v^m$ where $v \in V\Gamma$ and $m \in \mathbb{Z} \setminus \{0\}$,
- in all the other cases $\lambda(g) \geq 2$.

Using Notation [3.4.1](#) and the shuffling in Notation [3.4.5](#), we have a way to use techniques about graph products appearing in [\[22\]](#), also in the case of tRAAGs. The following definition is the most important one in this chapter.

Definition 3.4.9. We say that the sequence of syllables (g_1, \dots, g_r) is a *reduced sequence* in $G\Gamma$ if either

- (g_1, \dots, g_r) is the empty sequence \emptyset , or
- if $g_i \neq 1$ for all i , and no two syllables of the sequence (g_1, \dots, g_r) can be joined.

Definition 3.4.10. Introduce an equivalence relation for reduced sequences, denoted by \cong , and generated by:

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_r) \cong (g_1, \dots, g_{i-1}, g'_{i+1}, g'_i, g_{i+2}, \dots, g_r)$$

if and only if g_i, g_{i+1} belong to generating groups whose corresponding vertices are joined by an edge in Γ , and $g_i g_{i+1} = g'_{i+1} g'_i$ is the syllable shuffling from Notation [3.4.5](#). The sequences that are in the same class of equivalence are called *equivalent*.

Notation 3.4.11. In terms of shufflings of positions we denote the equivalence

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_r) \cong (g_1, \dots, g_{i-1}, g'_{i+1}, g'_i, g_{i+2}, \dots, g_r)$$

by the notation:

$$(1, \dots, i-1, \mathbf{i}, \mathbf{i}+1, i+2, \dots, r) \longleftrightarrow (1, \dots, i-1, \mathbf{i}+1, \mathbf{i}, i+2, \dots, r),$$

which corresponds to the transposition $(i, i+1) \longleftrightarrow (i+1, i)$ regarding the positions.

Now we will provide a notation to express the process of shuffling of a syllable through a reduced sequence.

Assume that $\{v_1, \dots, v_r\} \subseteq \text{Lk}(v)$ in Γ , and let (g, g_1, \dots, g_r) be a reduced sequence, with $g = v^s$, $g_i = v_i^{s_i}$ for $1 \leq i \leq r$, and s, s_1, \dots, s_r are non-zero integers.

We can shuffle the first term of the sequence (g, g_1, \dots, g_r) to the end. In the process of shuffling, the powers of generators might change their sign, and we get a sequence of the form (g'_1, \dots, g'_r, g') ; here $g' = v^{s'}$ where $s' \in \{s, -s\}$, and for any $1 \leq i \leq r$ we have $g'_i = v_i^{s'_i}$ with $s'_i \in \{s_i, -s_i\}$.

By Definition [3.4.10](#) we obtain:

$$(g, g_1, \dots, g_r) \cong (g'_1, \dots, g'_r, g') \quad (3.2)$$

Using Notation [3.4.11](#) one can think of the right hand side of Equation [\(3.2\)](#) as the effect of a chain of shuffling by transpositions $(i, i+1) \longleftrightarrow (i+1, i)$ for $0 \leq i \leq n-1$, where position 0 corresponds to g .

Notation 3.4.12. Another way to express the result of shuffling appearing on the right hand side of Equation [\(3.2\)](#), using only the sequence appearing on the left hand side of that equation, is:

$$(g, g_1, \dots, g_r) \cong ((g_1, \dots, g_r)^g, (g_1, \dots, g_r)_g) \quad (3.3)$$

where $(g_1, \dots, g_r)^g$, and $(g_1, \dots, g_r)_g$ are the sub-sequences of (g'_1, \dots, g'_r, g') , defined as:

$$\begin{aligned} (g_1, \dots, g_r)^g &:= (g'_1, \dots, g'_r) \\ (g_1, \dots, g_r)_g &:= (g'). \end{aligned} \quad (3.4)$$

Now we introduce the main theorem of the current chapter.

Theorem 3.4.13 (Normal form theorem for tRAAGs). Let G_Γ be a tRAAG. Each element $g \in G$ can be expressed uniquely (up to shuffling) as a product $g = g_1 \cdots g_n$, where (g_1, \dots, g_n) is a reduced sequence of syllables in G_Γ .

Proof. Here we present a strategy on the proof of the normal form theorem. The individual steps appearing in the strategy are proved in the upcoming sections.

The proof has the following two main parts:

- **existence** of such an expression, and
- **uniqueness** up to shuffling of syllables.

Start by writing g as a word w in G_Γ . Moves, which define the equivalence of words in G_Γ , consist of:

- (id) inserting or deleting the identity $1 \in G_\Gamma$,

- (j/s) joining together two syllables g_i, g_{i+1} belonging to the same generating group; or splitting a syllable into two, and
- (sh) shuffling two syllables g_i, g_{i+1} belonging to adjacent generating groups.

The following list presents the main steps of the proof:

- (1) First we define a map ρ from sequences of elements in G_Γ to reduced sequences. The map ρ can also be considered as a map from words in G_Γ to reduced sequences.
- (2) If $\rho(w) = (g_1, \dots, g_r)$ then $w = g_1 \cdots g_r$ as elements in G_Γ .
- (3) If (g_1, \dots, g_r) is a reduced sequence of syllables, then $\rho(g_1, \dots, g_r) = (g_1, \dots, g_r)$.
- (4) For any word $w \in G_\Gamma$ the equivalence class of $\rho(w)$ is preserved under the operations (id), (j/s), and (sh).
- (5) Use the previous steps to show existence, and uniqueness up to shuffling of syllables.
 - The existence is straightforward, as for $g \in G_\Gamma$, we take a word w presenting g in G_Γ . According to point (1), $\rho(w) = (g_1, \dots, g_r)$ is a reduced sequence, and by point (2) the element g is represented by $\rho(w)$. So there exists a reduced sequence of syllables, namely (g_1, \dots, g_r) given by $\rho(w)$, such that $g = g_1 \cdots g_r$.
 - To show uniqueness suppose (g_1, \dots, g_r) and (h_1, \dots, h_s) are two reduced sequences such that the words $U = g_1 \cdots g_r$ and $V = h_1 \cdots h_s$ define the same element $g \in G_\Gamma$, i.e. $g_1 \cdots g_r = g = h_1 \cdots h_s$.

The **goal** of uniqueness is to show the equivalence:

$$(g_1, \dots, g_r) \cong (h_1, \dots, h_s). \quad (*)$$

Since U, V present the same element in G_Γ , there is a sequence of words

$$U_0, U_1, \dots, U_i, U_{i+1}, \dots, U_{n-1}, U_n$$

going from $U = U_0$ to $V = U_n$, where one goes from a member U_i of the sequence, to the next member U_{i+1} by performing one of the moves (id), (j/s), or (sh).

According to (4), one has $\rho(U_i) \cong \rho(U_{i+1})$. Now, applying (4) several times for the sequence $U_0, \dots, U_i, U_{i+1}, \dots, U_n$ we get $\rho(U) \cong \rho(V)$. From (3), we get the equalities $\rho(U) = (g_1, \dots, g_r)$, and $\rho(V) = (h_1, \dots, h_s)$, which means that the equation (*) is satisfied. This completes point (5), and hence the proof. □

Remark 3.4.14. Step (1) of the strategy above is discussed in Section 3.5 where we define the map ρ . Also see Lemma 3.5.7 for steps (1), (2), and (3). For step (4) see Lemma 3.5.12, Remark 3.5.13, and see Section 3.7 for the actual proof.

3.5 The reduction procedure

We introduce a map ρ to reduce any sequence of elements (or any word) to a reduced sequence of syllables.

Remark 3.5.1. In the following definition, the apostrophe is used to note the effect of shuffling on every step. This effect can be read from the graph and from the parity of the powers of generators. After the definition there is a note providing an alternative view of the inductive steps.

Definition 3.5.2. Define inductively a map:

$$\rho: \{\text{sequences of elements in } G_\Gamma\} \longrightarrow \{\text{reduced sequences of syllables in } G_\Gamma\}$$

by defining first the base cases: as

$$\rho(\emptyset) = \emptyset, \quad \rho(1) = \emptyset, \quad \text{and} \quad \rho(g) = (g) \text{ where } g = v^s \text{ for some } v \in V \text{ and } s \neq 0 \text{ in } \mathbb{Z}.$$

Suppose by the induction hypothesis that we have computed:

$$\rho(g_1, \dots, g_n) = (h_1, \dots, h_m).$$

To complete the inductive process, suppose that $g = v^s$ for some $v \in V$, and $s \in \mathbb{Z}$. We define $\rho(g_1, \dots, g_n, g)$, by the following procedure:

(I) If $g = 1$, define:

$$(i) \quad \rho(g_1, \dots, g_n, g) = (h_1, \dots, h_m).$$

(II) If $g \neq 1$, consider the following cases:

(II.1) There is h_j (unique), which shuffles to the end as h'_j , and belongs to the same generating group as g . Consider two cases:

$$(ii) \quad \text{If } h'_j \cdot g = 1, \text{ define } \rho(g_1, \dots, g_n, g) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_m)$$

$$(iii) \quad \text{If } h'_j \cdot g \neq 1, \text{ define } \rho(g_1, \dots, g_n, g) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_m, (h'_j g))$$

(II.2) Any h_j that shuffles to the end, does not belong to the same generating group as g . Define:

$$(iv) \quad \rho(g_1, \dots, g_n, g) = (h_1, \dots, h_m, g)$$

Remark 3.5.3. The element h_j appearing in case (II.1) of Definition [3.5.2](#) is unique because (h_1, \dots, h_m) is reduced. If there were h_j and h_k shuffling to the end, and belonging to the same generating group as g , then they could already be joined in the sequence (h_1, \dots, h_m) , which would mean that the sequence was not reduced.

Moreover the map ρ is well-defined in the sense that its outputs are reduced sequences of syllables. This is straightforward from the definition of reduced sequences of syllables and the definition of ρ (see part (b) of Lemma [3.5.7](#)).

Note 3.5.4. Using Notation [3.4.12](#) we can express conditions (ii), and (iii) of the inductive step of Definition [3.5.2](#) as:

- (ii) If h_j can be shuffled to the end, and $(h_{j+1}, \dots, h_m)_{h_j} \cdot g = 1$, define

$$\rho(g_1, \dots, g_n, g) = (h_1, \dots, h_{j-1}, (h_{j+1}, \dots, h_m)^{h_j})$$

- (iii) If h_j belongs to the same generating group as g , it shuffles to the end, but $(h_{j+1}, \dots, h_m)_{h_j} \cdot g \neq 1$, define

$$\rho(g_1, \dots, g_n, g) = (h_1, \dots, h_{j-1}, (h_{j+1}, \dots, h_r)^{h_j}, (h_{j+1}, \dots, h_m)_{h_j} \cdot g)$$

Naturally, the map ρ , can also be seen as a map from words in G_Γ , to reduced sequences of syllables, as in the following definition.

Definition 3.5.5. If w is a word in G_Γ with syllables w_1, \dots, w_n and g_i are the elements in G_i representing w_i then we define $\rho(w) = \rho(w_1 \dots w_n) = \rho(g_1, \dots, g_n)$.

Example 3.5.6. Let $G_\Gamma = \langle a, b | ab = ba^{-1} \rangle$, and let $w = aa^{-3}bb^{-2}a^2$. The syllables of w are: $w_1 = aa^{-3}, w_2 = bb^{-2}, w_3 = a^2$. The elements representing these syllables in their respective generating groups are $g_1 = a^{-2}, g_2 = b^{-1}$, and $g_3 = a^2$. Hence we have $\rho(w) = \rho(g_1, g_2, g_3)$, and by definition of ρ we get $\rho(g_1) = (g_1)$. Since g_1, g_2 belong to different generating groups, use case (iv), to obtain $\rho(g_1, g_2) = (g_1, g_2)$. Finally, to compute $\rho(g_1, g_2, g_3)$ use case (II.1) of definition of ρ , as g_1 belongs to the same generating group as g_3 and g_1 shuffles to the end. Indeed,

$$g_1 g_2 = a^{-2} b^{-1} = b^{-1} a^2 = g_2 g_1^{-1} = g_2' g_1'.$$

Since $g_1' g_3 = a^2 a^2 = a^4 \neq 1$ we use case (iii) in (II.1) to obtain $\rho(g_1, g_2, g_3) = (g_2', (g_1' g_3))$, which gives $\rho(w) = \rho(g_1, g_2, g_3) = (b^{-1}, a^4)$. Moreover $w = b^{-1} a^4$ in G_Γ .

From Definition [3.5.2](#), we deduce the following properties of ρ , given by the lemma below.

Lemma 3.5.7. The map ρ satisfies the following properties:

- (a) the map ρ is a retraction, i.e. if the sequence (g_1, \dots, g_n) is reduced then:

$$\rho(g_1, \dots, g_n) = (g_1, \dots, g_n).$$

- (b) The sequence $\rho(g_1, \dots, g_n)$ is reduced of syllable length at most n .

- (c) If $\rho(g_1, \dots, g_n) = (h_1, \dots, h_m)$, then $g_1 \cdots g_n = h_1 \cdots h_m$ as group elements in G_Γ .

Proof. We use induction on n , and application of the inductive Definition [3.5.2](#). The base cases, i.e. $n = 0$ and $n = 1$ follow from the definition of the base cases.

- (a) Assume that (g_1, \dots, g_n, g) is a reduced sequence of syllables. Then (g_1, \dots, g_n) is reduced in particular, so $\rho(g_1, \dots, g_n) = (g_1, \dots, g_n)$. To compute $\rho(g_1, \dots, g_n, g)$ one has to go in case (iv) of Definition 3.5.2, as no g_i can be joined with g . Hence $\rho(g_1, \dots, g_n, g) = (g_1, \dots, g_n, g)$, as desired.
- (b) Let λ denote the syllable length. By the base case definition of ρ for $n = 0$ and $n = 1$, we see that the sequences $\rho(\emptyset) = \emptyset$ and $\rho(g_1)$ are reduced and moreover $\lambda(\emptyset) = 0$, $\lambda(\rho(g_1)) \leq 1$. Now assume that $\rho(g_1, \dots, g_n) = (h_1, \dots, h_m)$ is reduced with $\lambda(\rho(g_1, \dots, g_n)) \leq n$.

For $n + 1$, by Definition 3.5.2, the output of $\rho(g_1, \dots, g_n, g)$ can be:

- (i) (h_1, \dots, h_m) , if $g = 1$
- (ii) $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_m)$
- (iii) $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_m, (h'_j g))$
- (iv) (h_1, \dots, h_m, g) if any h_j that shuffles to the end, does not belong to the same generating group as g ,

where in cases (ii), and (iii) there is a unique h_j which belongs to the same generating group as g and it shuffles to the end; moreover $(h'_j g) = 1$ in (ii), and $(h'_j g) \neq 1$ in (iii).

By the inductive definition of ρ the outputs for $\rho(g_1, \dots, g_n, g)$ are still reduced, and

$$\lambda(\rho(g_1, \dots, g_n, g)) \leq \lambda(\rho(g_1, \dots, g_n)) + 1 \leq n + 1,$$

where it actually grows only in case (iv). So $\rho(g_1, \dots, g_n, g)$ is reduced of length at most $n + 1$, hence (b) holds.

- (c) This property holds in the base cases of definition of ρ . Assume that for

$$\rho(g_1, \dots, g_n) = (h_1, \dots, h_m)$$

we have $g_1 \cdots g_n = h_1 \cdots h_m$. Now consider $\rho(g_1, \dots, g_n, g)$ using the inductive definition of ρ .

For $g = 1$, i.e. case (i), the result is immediate, as $\rho(g_1, \dots, g_n, 1) = (h_1, \dots, h_m)$, and $g_1 \cdots g_n \cdot 1 = h_1 \cdots h_m$. In case (iii) we want to show that

$$g_1 \cdots g_n \cdot g = h_1 \cdots h_{j-1} \cdot h'_{j+1} \cdots h'_m \cdot (h'_j g),$$

and since $g_1 \cdots g_n = h_1 \cdots h_m$ the result to show becomes:

$$h_1 \cdots h_m \cdot g = h_1 \cdots h_{j-1} \cdot h'_{j+1} \cdots h'_m \cdot (h'_j g),$$

which is obvious by the setting of case (iii). Similarly we obtain case (ii). For case (iv) we have $\rho(g_1, \dots, g_n, g) = (h_1, \dots, h_m, g)$ and here:

$$g_1 \cdots g_n \cdot g = (g_1 \cdots g_n)g = (h_1 \cdots h_m) \cdot g = h_1 \cdots h_m \cdot g.$$

As all cases of the definition yield the required result, by induction (c) holds as well.

□

Remark 3.5.8. The definition of ρ and the properties presented in Lemma 3.5.7, present the first 3 steps of the strategy to prove the normal form theorem.

Now we give another property of ρ . For a word $w = w_1 \dots w_n$ and a sequence of syllables (g_1, \dots, g_n) , we consider $wg_1 \dots g_n$ as a word, namely $w_1 \dots w_n g_1 \dots g_n$, so we can apply the map ρ on it.

Corollary 3.5.9. Let w be a word, and (g_1, \dots, g_n) be a sequence of syllables. Then:

$$\rho(w, g_1, \dots, g_n) = \rho(\rho(w), g_1, \dots, g_n)$$

Proof. We proceed by induction on n . Denote $\rho(w) = (h_1, \dots, h_r)$.

For $n = 0$, $\rho(\rho(w)) = \rho(w)$ as ρ is a retraction, and $\rho(w)$ is reduced.

Now consider $n = 1$. We have $\rho(h_1, \dots, h_r) = (h_1, \dots, h_r)$ because (h_1, \dots, h_r) is reduced, which we use on the definition of ρ to compute $\rho(w, g_1)$.

(I) If $g_1 = 1$, then:

$$(i) \quad \rho(w, g_1) = \rho(w, 1) = (h_1, \dots, h_r) = \rho(h_1, \dots, h_r, g_1)$$

(II) If $g_1 \neq 1$, consider the following cases:

(II.1) There is h_j (unique), which shuffles to the end as h'_j , and belongs to the same generating group as g_1 . Consider two cases:

(ii) If $h'_j \cdot g_1 = 1$, we have

$$\rho(w, g_1) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r) = \rho(h_1, \dots, h_r, g_1)$$

(iii) If $h'_j \cdot g_1 \neq 1$, we have

$$\rho(w, g_1) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_1)) = \rho(h_1, \dots, h_r, g_1)$$

(II.2) Any h_j that shuffles to the end, does not belong to the same generating group as g_1 . We obtain

$$(iv) \quad \rho(w, g_1) = (h_1, \dots, h_r, g_1) = \rho(h_1, \dots, h_r, g_1).$$

As we notice, in all cases we have $\rho(w, g_1) = \rho(h_1, \dots, h_r, g_1)$. Since $(h_1, \dots, h_r) = \rho(w)$ we obtain

$$\rho(w, g_1) = \rho(\rho(w), g_1), \tag{3.5}$$

which proves the case $n = 1$. For $n > 1$ we obtain:

$$\begin{aligned} \rho(w, g_1, \dots, g_n) &= \rho(\rho(w, g_1, \dots, g_{n-1}), g_n) && \text{by Equation (3.5)} \\ &= \rho(\rho(\rho(w), g_1, \dots, g_{n-1}), g_n) && \text{by induction hypothesis} \\ &= \rho(\rho(\rho(w), g_1, \dots, g_{n-1}, g_n)) && \text{by Equation (3.5)} \\ &= \rho(\rho(w), g_1, \dots, g_{n-1}, g_n) && \text{as } \rho \text{ is a retraction} \end{aligned}$$

□

The next lemma provides a property of the equivalences of sequences of syllables.

Lemma 3.5.10. Let (g_1, \dots, g_n, g) , and (h_1, \dots, h_n, g') be equivalent reduced sequences of syllables, with g and g' in the same generating group. Then $g = g'$.

Proof. One has

$$(g_1, \dots, g_n, g) \cong (h_1, \dots, h_n, g')$$

so there is a sequence of shufflings going from the first sequence to the second. Sequences are reduced, so any syllable g_i or h_i belonging to the same generating group as g cannot shuffle to the end (otherwise one could join syllables). Let $g = v^s$ for some $v \in V$ and some $s \in \mathbb{Z}$. As g shuffles, its power s can change the sign (see Notation 3.4.5), so $g' = g$, or $g' = g^{-1}$. Anytime g changes the sign by shuffling with some g_i (or g_i^{-1}), while going towards the front, it will change the sign by reshuffling with one of g_i, g_i^{-1} while going towards the back. So at the end the sign will be the same, and hence $g = g'$. Note that even some g_j and g_k might shuffle, the parity of their powers does not change, and if the sign of the power changes, it depends only on the parity of the powers. \square

By concatenating two words $u = u_1 \dots u_n$ and $w = w_1 \dots w_n$ we obtain a new word, expressed as: $uw = u_1 \dots u_n w_1 \dots w_n$.

Lemma 3.5.11. If w_1, w_2 are words in G_Γ that satisfy $\rho(w_1) \cong \rho(w_2)$, then for any word w in G_Γ we have

$$\rho(w_1 w) \cong \rho(w_2 w)$$

Proof. To prove the result, by Corollary 3.5.9, one can equivalently show:

$$\rho(\rho(w_1), w) \cong \rho(\rho(w_2), w).$$

Let $w = (g_1, \dots, g_n)$ be expressed by a sequence of syllables.

The reduced sequences $\rho(w_1)$, and $\rho(w_2)$ are equivalent, so they have the same length. Let $\rho(w_1) = (h_1, \dots, h_r)$ and $\rho(w_2) = (l_1, \dots, l_r)$.

We will proceed by induction on n and r . More precisely, for a fixed r we show that the case $n = 1$ is sufficient to get the result for an arbitrary n .

The result is immediate if $n = 0$. Also it is enough to show the result when w is a syllable, i.e. $n = 1$, because for $n > 1$ we have

$$\begin{aligned} \rho(\rho(w_1), g_1, \dots, g_n) &= \rho(\rho(w_1, g_1, \dots, g_{n-1}), g_n) \\ &= \rho(\rho(\rho(w_1), g_1, \dots, g_{n-1}), g_n) \end{aligned}$$

and similarly

$$\rho(\rho(w_2), g_1, \dots, g_n) = \rho(\rho(\rho(w_2), g_1, \dots, g_{n-1}), g_n)$$

By the induction hypothesis we have

$$\rho(\rho(w_1), g_1, \dots, g_{n-1}) \cong \rho(\rho(w_2), g_1, \dots, g_{n-1})$$

so applying the case of $n = 1$ we have the result.

Therefore to show the result, it amounts to showing

$$\rho(\rho(w_1), g) \cong \rho(\rho(w_2), g)$$

where g is a non-trivial syllable.

The case when $r = 0$ is trivial. For $r = 1$ we must have $h_1 = l_1$ as they represent the same element, and in this case the result is also trivial.

Now assume that $r > 1$.

We have the equations:

$$\rho(\rho(w_1), g) = \rho(h_1, \dots, h_r, g) \quad \text{and} \quad \rho(\rho(w_2), g) = \rho(l_1, \dots, l_r, g).$$

Now we follow the definition of ρ for $g \neq 1$. If (h_1, \dots, h_r, g) is reduced (i.e. case (iv) of Definition 3.5.2), then also (l_1, \dots, l_r, g) is reduced, because $(h_1, \dots, h_r) \cong (l_1, \dots, l_r)$. In this case it follows that $\rho(\rho(w_1), g) \cong \rho(\rho(w_2), g)$.

Now consider case (II.1.). There is a unique h_i which shuffles to the end as h'_i and belongs to the same generating group as g . By equivalence of sequences, it follows that there is a unique l_j which shuffles to the end as l'_j and belongs to the same generating group as g . We must have $h'_i = l'_j$ as they both are brought to the last position (see Lemma 3.5.10).

If $h'_i g = 1$ then $l'_j g = 1$, so we obtain:

$$\rho(\rho(w_1), g) = (h_1, \dots, h_{i-1}, h'_{i+1}, \dots, h'_r) \quad \text{and} \quad \rho(\rho(w_2), g) = (l_1, \dots, l_{j-1}, l'_{j+1}, \dots, h'_r)$$

which are equivalent.

If $h'_i g \neq 1$ then also $l'_j g \neq 1$, and in fact $h'_i g = l'_j g$ so we obtain:

$$\rho(\rho(w_1), g) = (h_1, \dots, h_{i-1}, h'_{i+1}, \dots, h'_r, h'_i g),$$

and

$$\rho(\rho(w_2), g) = (l_1, \dots, l_{j-1}, l'_{j+1}, \dots, h'_r, l'_j g),$$

which are again equivalent. □

Lemma 3.5.12. The map ρ satisfies also the following properties:

- (d) $\rho(g_1, \dots, g_k, 1, g_{k+1}, \dots, g_n) = \rho(g_1, \dots, g_k, g_{k+1}, \dots, g_n)$
- (E) If g_i, g_{i+1} belong to the same generating group, then:
 $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) = \rho(g_1, \dots, g_{i-1}, g_i g_{i+1}).$

- (F) If g_i, g_{i+1} shuffle and $g_i g_{i+1} = g'_{i+1} g'_i$, then:
 $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) \cong \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i)$.

Proof. We present the proof of (E) and (F) later in Section [3.7](#).

- (d) We have the equality:

$$\rho(g_1, \dots, g_k, 1) = \rho(g_1, \dots, g_k)$$

by definition of ρ . Now from Corollary [3.5.9](#) we obtain:

$$\begin{aligned} \rho(g_1, \dots, g_k, 1, g_{k+1}, \dots, g_n) &= \rho(\rho(g_1, \dots, g_k, 1), g_{k+1}, \dots, g_n) \\ &= \rho(\rho(g_1, \dots, g_k), g_{k+1}, \dots, g_n) \\ &= \rho(g_1, \dots, g_n) \end{aligned}$$

□

Remark 3.5.13. Assuming that (E), and (F) are proved, Corollary [3.5.9](#) implies

- (e) If g_i, g_{i+1} belong to the same generating group, then:
 $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) = \rho(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n)$.

Using also Lemma [3.5.11](#) we obtain:

- (f) If g_i, g_{i+1} shuffle and $g_i g_{i+1} = g'_{i+1} g'_i$, then:
 $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \cong \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i, g_{i+2}, \dots, g_n)$.

Remark 3.5.14. Properties (d), (e), (f) imply that for $w \in G_\Gamma$, the equivalence class of $\rho(w)$ is preserved under the operations (id), (j/s), and (sh). This shows step (4) of the strategy for proving the normal form theorem.

3.6 Applications of the normal form theorem

Let $G = G_\Gamma$ be the tRAAG based on a mixed graph $\Gamma = (V, E)$. For applications of the normal form we choose S a monoid generating set of G as

$$S = V \sqcup V^{-1} = \{v, v^{-1} \mid v \in V\}.$$

Notation 3.6.1. We will denote by $l(w)$ the length of a word w with respect to S ; and by $|g|$ the length of the element $g \in G$ with respect to S .

3.6.1 The word problem

In a group G with a monoid generating set S the *word problem* consists of determining whether or not a given word w over S represents the identity $1 \in G$.

Lemma 3.6.2. Properties (a) - (f) of the map ρ imply:

$$w_1 = w_2 \text{ in } G_\Gamma \iff \rho(w_1) \cong \rho(w_2).$$

Proof. The equivalence class of $\rho(w)$ is preserved under the operations (id), (j/s), (sh). This means that if $w_1 = w_2$ in G_Γ , then $\rho(w_1) \cong \rho(w_2)$.

For the other direction, write $\rho(w_1) = (g_1, \dots, g_r)$, and $\rho(w_2) = (h_1, \dots, h_s)$, and assume that $\rho(w_1) \cong \rho(w_2)$. Then $r = s$ and one can perform only shufflings to go from one sequence to the other, hence $g_1 \cdots g_r = h_1 \cdots h_r$. Property (c) implies $w_1 = g_1 \cdots g_r$, and $w_2 = h_1 \cdots h_r$ as group elements, so $w_1 = w_2$ in G_Γ . \square

As a corollary of the normal form theorem for tRAAGs we get the following:

Corollary 3.6.3. In the class of tRAAGs the word problem is solvable.

Proof. Let w be a word in G_Γ . Then $w = 1$ in $G_\Gamma \iff \rho(w) = \emptyset$. \square

3.6.2 Growth in tRAAGs

One of the aforementioned applications of the normal form theorem is to compare the spherical and the geodesic growth of tRAAGs with the respective growth of RAAGs.

As a first result, we show that reduced sequences of syllables represent geodesics. Recall that $S = V \sqcup V^{-1}$ is our monoid generating set.

Lemma 3.6.4. Let (g_1, \dots, g_r) be a reduced sequence of syllables, with $g_i = v_i^{s_i}$, where for all $1 \leq i \leq r$ we have $v_i \in V$, and $s_i \in \mathbb{Z} \setminus \{0\}$. Put $g = g_1 \cdots g_r$. Then

$$|g| = \sum_{i=1}^r |s_i|.$$

Moreover, the word $g_1 \dots g_r$ is a geodesic.

Proof. By considering all cases of Definition 3.5.2 we conclude that $l(w) \geq l(\rho(w))$, for any word w in G_Γ .

Let w be a word of minimal length, representing g , and consider $\rho(w) = (h_1, \dots, h_s)$. Now we have two reduced sequences representing the same element g , so we obtain

$$(g_1, \dots, g_r) \cong (h_1, \dots, h_s)$$

By the properties of ρ , one gets $r = s$ and $g_1 \cdots g_r = h_1 \cdots h_r$ as elements. Moreover as words, both (g_1, \dots, g_r) , and (h_1, \dots, h_r) have equal length $\sum_{i=1}^r |s_i|$, because one goes from one to the other using shufflings (which preserve the length). Now,

$$|g| = l(w) \geq l(\rho(w)) = l(h_1 \dots h_r) = \sum_{i=1}^r |s_i|.$$

The word $g_1 \dots g_r$ represents g , so obviously, we also have:

$$|g| \leq \sum_{i=1}^r |s_i|,$$

as required.

Since the length of the word $w = g_1 \dots g_r$ is equal to $\sum_{i=1}^r |s_i|$ which is equal to the length of the element $g = g_1 \dots g_r$ we conclude that the length of the word w was already minimal, and hence it is a geodesic. \square

Remark 3.6.5. The lemma above can also be expressed as

$$|g| = \sum_{i=1}^r |g_i|.$$

Now we state one of the main applications of the normal form theorem on this chapter.

Theorem 3.6.6. The spherical and the geodesic growth of a tRAAG over Γ agrees with the corresponding growth of the RAAG based on the underlying naïve graph $\bar{\Gamma}$.

Proof. Let G_Γ be the tRAAG based on Γ , and $G\bar{\Gamma}$ the corresponding RAAG based on the underlying naïve graph $\bar{\Gamma}$. Both Γ and $\bar{\Gamma}$ have the same set of vertices V . Put an order on V by labeling the vertices with $\{1, 2, \dots, n\}$. By the normal forms in both groups, each element has a geodesic representative with a reduced sequence of syllables, which is unique up to shuffling. If for any g in G_Γ we choose a representative with the minimal lexicographic order we notice that g viewed as an element of $G\bar{\Gamma}$ has the same representative (which is of minimal order as well, because any shuffling that can be done in $G\bar{\Gamma}$, has a corresponding shuffling in G_Γ). So there is a bijection (not a morphism in general)

$$i: G\bar{\Gamma} \longrightarrow G_\Gamma$$

which preserves the syllable length. This implies the same spherical growth.

Let w be a geodesic word representing the element g in $G\bar{\Gamma}$, with the minimal lexicographic order. So we can express the elements with the same generators $S = V \sqcup V^{-1}$.

For the geodesic growth, assume w_1 and w_2 are geodesics in G_Γ representing the same element g . Then $\rho(w_1) \cong \rho(w_2)$. As w_1 is a geodesic, the reduction procedure to arrive at $\rho(w_1)$ (which also represents g) does not shorten the length, so we can go only through cases (iii) and (iv) of Definition 3.5.2 (moreover in case (iii) the signs of the powers of the last syllable have to agree). So to go from w_1 to $\rho(w_1)$ in G we use only shufflings. Similarly for going from w_2 to $\rho(w_2)$. Since $\rho(w_1) \cong \rho(w_2)$ we use shufflings to go from $\rho(w_1)$ to $\rho(w_2)$. Ultimately, one can go from w_1 to w_2 only by using shufflings. The same is true for any two geodesics in $G\bar{\Gamma}$ representing the same element. The set of geodesics representing an element is finite, and the shufflings that we can use are determined by Γ in both groups, therefore the sets of geodesics for any given element are in bijection, and hence we get the same geodesic growths. \square

3.6.3 Torsion

Twisted RAAGs have a lot of similarities with RAAGs, especially when we consider their geometric and combinatorial nature. However, when we work on their algebraic properties, we notice many remarkable differences. One was pointed out by indicability, and another one is the presence of torsion, as illustrated by the following example.

Example 3.6.7. Consider the graph $\Delta = (V, E)$, in Figure 3.2, representing a triangle with $V = \{a, b, c\}$, and $E = \{[a, b], [b, c], [c, a]\}$.

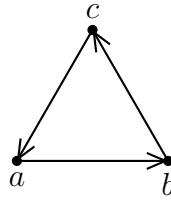


Figure 3.2: Triangle graph Δ

The corresponding tRAAG to the graph Δ has the presentation:

$$G_{\Delta} = \langle a, b, c \mid aba = b, bcb = c, cac = a \rangle.$$

By the normal form theorem, any element $g \in G_{\Delta}$ can be represented as $g = a^m b^n c^p$ for some m, n, p in \mathbb{Z} .

Suppose we have two elements $g_i = a^{m_i} b^{n_i} c^{p_i}$ for some m_i, n_i, p_i in \mathbb{Z} for $i \in \{1, 2\}$. The way to express $g_1 g_2$ in the normal form would be:

$$g_3 = g_1 g_2 = a^{m_3} b^{n_3} c^{p_3},$$

with $m_3 = m_1 + (-1)^{n_1} m_2$, $n_3 = n_1 + (-1)^{p_1} n_2$, $p_3 = p_2 + (-1)^{m_2} p_1$.

If we choose $(m_1, n_1, p_1) = (m_2, n_2, p_2)$, and all of them odd then we get $g^2 = 1$. This means that there is torsion in these groups.

Remark 3.6.8. Let $g \in G$. If g has torsion then the vertices in the support of g (those vertices that are used to express g in the normal form) form a clique. Indeed, by the normal form theorem, any two vertices in g have to be connected (otherwise we cannot bring together the corresponding syllables).

Theorem 3.6.9. G has torsion if and only if there is a clique C in Γ whose vertices form a closed cycle.

Proof. If there is such a clique C then we have torsion. One can pick $g \in G$ to be the product of elements in this cycle in an order given by the cycle (in the case of the triangle above we have the product $g = abc$, with $g^2 = 1$).

For the converse, assume that any clique in Γ does not form a closed cycle, and there is $g \in G$ which has torsion. By Remark 3.6.8, we know that the support of g lies in a clique C with $|C| \geq 3$. Assume that g has the minimal number of elements in the support among the elements which have torsion. By assumption the clique C does not form a closed cycle, nor does any of its sub-cliques. So, there is a vertex v in C which is not the origin of any directed edges. By Theorem 3.2.3 one has a map $\varphi : C \rightarrow \mathbb{Z}$ with $\varphi(v) = 1$, and $\varphi(v') = 0$ for $v' \in C \setminus \{v\}$, so $\varphi(g) \neq 0$, and hence g cannot have torsion. \square

3.7 Shuffling property of ρ

Before proving cases (E) and (F) of Lemma 3.5.12, we discuss the following 2 lemmas.

Lemma 3.7.1. Let v_1, v_2, v_3 be vertices of a triangle, and $g_1 = v_1^m, g_2 = v_2^n, g_3 = v_3^p$ for $m, n, p \in \mathbb{Z}$. In the product $g_1 g_2 g_3$ consider the following two ways of shuffling, which bring g_3 (or g_3^{-1}) to the front:

- (I) Shuffle first (g_2, g_3) to get (g'_3, g'_2) and then shuffle (g_1, g'_3) to bring the power of v_3 in front.
- (II) Shuffle first (g_1, g_2) to get (g'_2, g'_1) and then perform the shuffling of case (I) to the sequence (g'_2, g'_1, g_3) .

In terms of shuffling of positions one can express the cases above as:

$$\begin{aligned} (I) \quad (1, 2, 3) &\rightarrow (1, 3, 2) \rightarrow (3, 1, 2) \\ (II) \quad (1, 2, 3) &\rightarrow (2, 1, 3) \rightarrow (2, 3, 1) \rightarrow (3, 2, 1) \end{aligned}$$

In both cases the power of v_3 in the front is the same.

Proof. Shuffling might affect the sign of the power, which depends on the parity of the power of the other shuffling element. In both cases (I) and (II) the third syllable shuffles twice to the left among powers of v_1 and v_2 , whose parity is the same in both cases, so the effect in the power of v_3 is the same. \square

Now we provide a generalization of the lemma above.

Lemma 3.7.2. Suppose that the sequence (h_1, \dots, h_r) is reduced. Assume that there are indices $1 \leq j < k \leq r$ such that h_j, h_k shuffle to the end. Shuffling first with h_j to the end of the sequence and then with h_k as given in diagram (I) below in terms of positions:

$$\begin{aligned} (I) \quad & (1, \dots, j-1, j, j+1, \dots, k-1, k, k+1, \dots, r) \\ & \rightarrow (1, \dots, j-1, j+1, \dots, k-1, k, k+1, \dots, r, j) \\ & \rightarrow (1, \dots, j-1, j+1, \dots, k-1, k+1, \dots, r, j, k) \end{aligned}$$

of first shuffling to the end with h_k and then with h_j as in:

$$\begin{aligned}
 (II) \quad & (1, \dots, j-1, j, j+1, \dots, k-1, k, k+1, \dots, r) \\
 \longrightarrow & (1, \dots, j-1, j, j+1, \dots, k-1, k+1, \dots, r, k) \\
 \longrightarrow & (1, \dots, j-1, j+1, \dots, k-1, k+1, \dots, r, j, k)
 \end{aligned}$$

results on identical reduced sequences.

Proof. Let h_1, \dots, h_r be powers of vertices v_1, \dots, v_r respectively. Since the sequence (h_1, \dots, h_r) is reduced, and both v_j and v_k shuffle to the end we have:

$$\{v_{j+1}, \dots, v_r\} \subseteq \text{Lk}(v_j) \text{ and } \{v_{k+1}, \dots, v_r\} \subseteq \text{Lk}(v_k).$$

Now we can draw an approximate naïve underlying graph for the vertices $\{v_j, \dots, v_r\}$. We say approximate, because some vertices could be equal, and there can be edges between some vertices; the only edges that we have for sure are the ones between v_j , v_k to vertices in $\text{Lk}(v_j)$ and $\text{Lk}(v_k)$ respectively, as drawn in Figure 3.3 below.

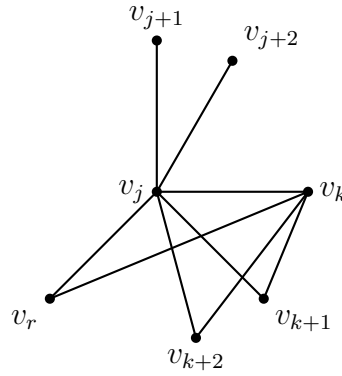


Figure 3.3: Partial naïve graph

Shuffling first h_j to the end, and then h_k as described by (I) we obtain:

$$\begin{aligned}
 (h_1, \dots, h_r) &\cong (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j) \\
 &\cong (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_{k-1}, h''_{k+1}, \dots, h''_r, h''_j, h''_k)
 \end{aligned} \tag{3.6}$$

Instead, shuffling first h_k to the end, and then h_j as described by (II) we obtain:

$$\begin{aligned}
 (h_1, \dots, h_r) &\cong (h_1, \dots, h_{k-1}, h'_{k+1}, \dots, h'_r, h'_k) \\
 &\cong (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_{k-1}, h''_{k+1}, \dots, h''_r, h''_j, h''_k)
 \end{aligned} \tag{3.7}$$

Now we look closely at the sequences in equations (3.6) and (3.7). Notice see that the elements with indices $1, \dots, j-1$ in both equations are not effected by shuffling, so they remain the same.

The elements with indices $j+1, \dots, k-1$ are affected in both equations only by shuffling with h_j (or h_j^{-1}), so this part is equal to $(h'_{j+1}, \dots, h'_{k-1})$ for both expressions. Here after the syllable h_j shuffles with $(h_{j+1}, \dots, h_{k-1})$ it might change the sign of its power, and we denote it by h'_j .

The elements with indices $k+1, \dots, r$ are affected twice by shuffling, and the effect is not on the same order, so let us discuss by induction the equality of terms of both sequences in Equations (3.6) and (3.7) with indices $k+1, \dots, r$.

First look at h''_{k+1} in both equations which is obtained by bringing the syllable with index $k+1$ in the front, from the expression $h'_i h_k h_{k+1}$.

Since v_j, v_k, v_{k+1} form a triangle, we are left with the same syllable in front, no matter which one of two orders of shuffling we perform (by Lemma 3.7.1). The same procedure applied to other h''_i yields the corresponding equality.

Since both sequences represent the same element we also have that the last two elements agree. As desired. \square

To prove cases (E) and (F) of Lemma 3.5.12 it is enough to show that

$$\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) \cong \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i) \quad (3.8)$$

for the following two cases:

- (I) g_i, g_{i+1} are powers of the same generator.
- (II) g_i, g_{i+1} are powers of adjacent generators.

Remark 3.7.3. Case (I) here will imply property (E), i.e. in case (I) we will show that if g_i, g_{i+1} are powers of the same generator, then:

$$\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) = \rho(g_1, \dots, g_{i-1}, g_i g_{i+1})$$

Consider case (I). Here the relation $g_i g_{i+1} = g'_{i+1} g'_i$ and the fact that g_i, g_{i+1} are powers of the same generator implies $g'_i = g_i$ and $g'_{i+1} = g_i$. When one of g_i, g_{i+1} is 1 then the result follows by applying property (d) of Lemma 3.5.12. Also the case when g_i, g_{i+1} are equal is trivial, so assume that none of them are the identity and they are different.

Assume $\rho(g_1, \dots, g_{i-1}) = (h_1, \dots, h_r)$, and distinguish the following subcases:

- (I.1) There is a unique h_j which shuffles as h'_j to the end of (h_1, \dots, h_r) and is in the same generating group as g_i .
- (I.2) Any h_j which shuffles to the end of (h_1, \dots, h_r) is on a different generating group of g_i .

Note that in case (I.1) both h_j, g_i are in the same vertex group, so $h'_j g_i$ is a power of a generator. Now let us consider two subcases.

(I.1.1) If $h'_j g_i = 1$ then $\rho(g_1, \dots, g_{i-1}, g_i) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r)$. Using this we get:

$$\begin{aligned} \rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}) \end{aligned}$$

as no other h_i which can shuffle to the end can be in the same vertex group as g_{i+1} , because g_i and g_{i+1} belong to the same generating group, and the sequence h_1, \dots, h_r is reduced. We must have $h'_j g_{i+1} \neq 1$, because $g_i \neq g_{i+1}$, and also $h'_j g_{i+1} g_i = g_{i+1} \neq 1$, which implies:

$$\begin{aligned} \rho(g_1, \dots, g_{i-1}, g_{i+1}, g_i) &= \rho(h_1, \dots, h_r, g_{i+1}, g_i) \\ &= \rho(\rho(h_1, \dots, h_r, g_{i+1}), g_i) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_{i+1}), g_i) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_{i+1} g_i)) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_i g_{i+1})) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}). \end{aligned}$$

So, we got $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) = \rho(g_1, \dots, g_{i-1}, g_{i+1}, g_i)$

(I.1.2) $h'_j g_i \neq 1$ but h'_j , and g_i are in the same vertex group. In this case we have

$$\rho(g_1, \dots, g_{i-1}, g_i) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i). \quad (3.9)$$

Now the right hand side of Equation (3.9) is reduced. By applying Corollary 3.5.9 several times, we obtain:

$$\begin{aligned} \rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) &= \rho(\rho(g_1, \dots, g_{i-1}, g_i), g_{i+1}) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i, g_{i+1}) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_i g_{i+1})) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_{i+1} g_i)) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_{i+1}), g_i) \\ &= \rho(\rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, (h'_j g_{i+1})), g_i) \\ &= \rho(h_1, \dots, h_r, g_{i+1}, g_i) \end{aligned}$$

because g_i and g_{i+1} commute as they both are powers of the same generator.

Now $\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) = \rho(h_1, \dots, h_r, g_{i+1}, g_i) = \rho(g_1, \dots, g_{i-1}, g_{i+1}, g_i)$, as desired.

In case (I.2) g_i is in a different generating group from any h_j which can be shuffled to the end, so we have:

$$\rho(g_1, \dots, g_{i-1}, g_i) = (h_1, \dots, h_r, g_i). \quad (3.10)$$

The right hand side of Equation (3.10) is reduced, so if $g_{i+1} = g_i^{-1}$ then by the definition of ρ (case (ii)) we obtain:

$$\begin{aligned}\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) &= \rho(h_1, \dots, h_r, g_i, g_{i+1}) \\ &= (h_1, \dots, h_r) \\ &= \rho(h_1, \dots, h_r, g_{i+1}, g_i) \\ &= \rho(g_1, \dots, g_{i-1}, g_{i+1}, g_i),\end{aligned}$$

because also $(h_1, \dots, h_r, g_{i+1})$ is reduced.

So assume that $g_{i+1} \neq g_i^{-1}$. Then again by the definition of ρ (case (iii)) we have:

$$\begin{aligned}\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) &= (h_1, \dots, h_r, g_i g_{i+1}) \\ &= (h_1, \dots, h_r, g_{i+1} g_i) \\ &= \rho(h_1, \dots, h_r, g_{i+1}, g_i) \\ &= \rho(g_1, \dots, g_{i-1}, g_{i+1}, g_i)\end{aligned}$$

because g_i, g_{i+1} commute.

Remark 3.7.4. Note that in all subcases of (I) we also have:

$$\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) = \rho(g_1, \dots, g_{i-1}, g_i g_{i+1}),$$

which ultimately implies property (E) of Lemma 3.5.12.

Now consider case (II). Here g_i and g_{i+1} belong to adjacent generating groups, hence they shuffle. Let $\rho(g_1, \dots, g_{i-1}) = (h_1, \dots, h_r)$. Recall that we want to show:

$$\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) \cong \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i),$$

where g_i, g_{i+1} shuffle and $g_i g_{i+1} = g'_{i+1} g'_i$ (property (F) of Lemma 3.5.12).

Remark 3.7.5. Consider the sequence (h_1, \dots, h_r) above, and suppose that there are two distinct indices j, k (for example $j < k$), such that both h_j, h_k shuffle to the end of the sequence (h_1, \dots, h_r) , and h_j, h_k belong to the same generating groups as g_i, g_{i+1} respectively. By Lemma 3.7.2, no matter which one of two ways of shuffling we choose, we obtain the following equivalence of sequences:

$$(h_1, \dots, h_r) \cong (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_{k-1}, h''_{k+1}, \dots, h''_r, h''_j, h''_k).$$

By Corollary 3.5.9 and Lemma 3.5.11, to prove that

$$\rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) \cong \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i)$$

it is enough to prove the case when $j = r - 1$, and $k = r$, i.e. we can equivalently (up to renaming the variables) show the equivalence:

$$\rho(h_1, \dots, h_{r-2}, h_{r-1}, h_r, g_i, g_{i+1}) \cong \rho(h_1, \dots, h_{r-2}, h_{r-1}, h_r, g'_{i+1}, g'_i)$$

where h_{r-1} and h_r are in the same generating groups as g_i and g_{i+1} respectively. In fact the part $\rho(h_1, \dots, h_{r-2}) = (h_1, \dots, h_{r-2})$ does not interfere with the calculations, so it is enough to show that

$$\rho(h_1, h_2, g_1, g_2) \cong \rho(h_1, h_2, g'_1, g'_1)$$

where h_1 and h_2 are in the same generating groups as g_1 , and g_2 respectively.

As before, put:

$$\rho(g_1, \dots, g_{i-1}) = (h_1, \dots, h_r),$$

and distinguish the following two cases for $\rho(g_1, \dots, g_{i-1}, g_i)$:

- (II.1) There is a unique h_j which shuffles as h'_j to the end of (h_1, \dots, h_r) and is in the same generating group as g_i .
- (II.2) Any h_j which shuffles to the end of (h_1, \dots, h_r) is on a different generating group of g_i .

In case (II.1) h'_j and g_i are in the same generating group, so $h'_j g_i$ is a power of a generator. Now consider two subcases for (II.1) as follows:

- (II.1.1) $h'_j g_i = 1$. By definition of ρ we obtain:

$$\rho(g_1, \dots, g_{i-1}, g_i) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r).$$

Now we have:

$$\begin{aligned} \rho(g_1, \dots, g_i, g_{i+1}) &= \rho(\rho(g_1, \dots, g_{i-1}, g_i), g_{i+1}) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}) \end{aligned}$$

We know that $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r)$ is reduced. Now we distinguish two cases for computing $\rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1})$:

- (II.1.1.1) A unique element from the sequence $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r)$ shuffles to the end, and is in the same generating group as g_{i+1} . Here the element that shuffles to the end, can have an index smaller than j , or greater than j . It cannot be equal to j because h_j and g_i were on the same generating group, but g_i and g_{i+1} are on different generating groups.

Suppose that the element shuffling to the end is of the form h'_k with $j < k \leq r$, because the other case $k < j$ becomes analogous by trying to show:

$$\rho(g_1, \dots, g'_{i+1}, g'_i) \cong \rho(g_1, \dots, g_i, g_{i+1})$$

This is the case when we can use Remark [3.7.5](#). And it is enough to prove that

$$\rho(h_1, h_2, g_1, g_2) \cong \rho(h_1, h_2, g'_2, g'_1)$$

where h_1, h_2 stand for h_j, h_k respectively, and g_1, g_2 stand for g_i, g_{i+1} respectively.

So far we have $h'_1 g_1 = 1$. Now we ultimately distinguish 2 cases:

(a) $h'_2 g_2 = 1$. In this case:

$$\begin{aligned}\rho(h_1, h_2, g_1, g_2) &= \rho(\rho(h_1, h_2, g_1), g_2) \\ &= \rho(h'_2, g_2) \\ &= \emptyset\end{aligned}$$

Now let us compute $\rho(h_1, h_2, g'_2, g'_1)$.

$$\begin{aligned}\rho(h_1, h_2, g'_2, g'_1) &= \rho(\rho(h_1, h_2, g'_2), g'_1) \\ &= \rho(\rho(h_1, h_2 g'_2), g'_1) \\ &= \rho(\rho(h_1, 1), g'_1) \\ &= \rho(h_1, g'_1) \\ &= \emptyset\end{aligned}$$

Here we had $h'_1 g_1 = 1$, and $h'_2 g_2 = 1$, where $h_1 h_2 = h'_2 h'_1$. So $h'_1 = g_1^{-1}$, and $h'_2 = g_2^{-1}$. Now $g'_2 g'_1 = g_1 g_2 = h_1^{-1} h_2^{-1} = h_2^{-1} h_1^{-1}$, so $h_2 g'_2 = 1$, and $h_1 g'_1 = 1$, as we have used them.

(b) $h'_2 g_2 \neq 1$. In this case:

$$\begin{aligned}\rho(h_1, h_2, g_1, g_2) &= \rho(\rho(h_1, h_2, g_1), g_2) \\ &= \rho(h'_2, g_2) \\ &= (h'_2 g_2)\end{aligned}$$

and the right hand side is reduced. Now let us compute $\rho(h_1, h_2, g'_2, g'_1)$.

$$\begin{aligned}\rho(h_1, h_2, g'_2, g'_1) &= \rho(\rho(h_1, h_2, g'_2), g'_1) \\ &= \rho(\rho(h_1, h_2 g'_2), g'_1) \\ &= \rho(h_1, h_2 g'_2, g'_1) \\ &= (h'_2 g_2)\end{aligned}$$

Here we had $h'_1 g_1 = 1$, and $h'_2 g_2 \neq 1$, where $h_1 h_2 = h'_2 h'_1$. So $h'_1 = g_1^{-1}$. Now $g'_2 g'_1 = g_1 g_2 = h_1^{-1} g_2$ which implies $h'_1 g'_2 g'_1 = g_2$, so shuffling h_1 past $h_2 g'_2$ we obtain:

$$h_1(h_2 g'_2) = h'_2(h'_1 g'_2) = h'_2(g_2 g'_1)^{-1}$$

as we have used them.

(II.1.1.2) Any of the syllables of the sequence $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r)$ which shuffles to the end, is in a different generating group from g_{i+1} . Here we get:

$$\begin{aligned}\rho(g_1, \dots, g_i, g_{i+1}) &= \rho(\rho(g_1, \dots, g_{i-1}, g_i), g_{i+1}) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1}).\end{aligned}$$

Now we compute $\rho(g_1, \dots, g'_{i+1}, g'_i)$:

$$\begin{aligned}\rho(g_1, \dots, g'_{i+1}, g'_i) &= \rho(\rho(g_1, \dots, g_{i-1}, g'_{i+1}), g'_i) \\ &= \rho(h_1, \dots, h_r, g'_{i+1}) \\ &= (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, g_{i+1})\end{aligned}$$

because h_j shuffles to the end, and as it arrives in front of g_{i+1} is denoted by h'_j and it is equal to g_i^{-1} . So $h'_j g'_{i+1} = g_i^{-1} g'_{i+1} = g_{i+1} g_i'^{-1}$, and $g_i'^{-1} g'_i = 1$.

(II.1.2) $h'_j g_i \neq 1$ but h'_j , and g_i are in the same vertex group. Now we have

$$\rho(g_1, \dots, g_i) = (h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i).$$

Using the equation above we have:

$$\begin{aligned}\rho(g_1, \dots, g_i, g_{i+1}) &= \rho(\rho(g_1, \dots, g_{i-1}, g_i), g_{i+1}) \\ &= \rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i, g_{i+1})\end{aligned}$$

We know that $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i)$ is reduced. Now we distinguish two cases for computing $\rho(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r, h'_j g_i, g_{i+1})$:

(II.1.2.1) A unique element from the sequence $(h_1, \dots, h_{j-1}, h'_{j+1}, \dots, h'_r)$ shuffles to the end, and is in the same generating group as g_{i+1} . Here the element that shuffles to the end, can have an index smaller than j , or greater than j . It cannot be equal to j because h_j and g_i were on the same generating group, but g_i and g_{i+1} are on different generating groups.

Suppose that the element shuffling to the end is of the form h'_k with $j < k \leq r$, because the other case $k < j$ becomes analogous by trying to show:

$$\rho(g_1, \dots, g'_{i+1}, g'_i) \cong \rho(g_1, \dots, g_i, g_{i+1})$$

This is the case when we can use Remark [3.7.5](#). And it is enough to prove that

$$\rho(h_1, h_2, g_1, g_2) \cong \rho(h_1, h_2, g'_2, g'_1)$$

where h_1, h_2 stand for h_j, h_k respectively, and g_1, g_2 stand for g_i, g_{i+1} respectively. So far we have $h'_1 g_1 \neq 1$, and therefore:

$$\begin{aligned}\rho(h_1, h_2, g_1, g_2) &= \rho(\rho(h_1, h_2, g_1), g_2) \\ &= \rho(h'_2, h'_1 g_1, g_2)\end{aligned}$$

Now we ultimately distinguish 2 cases:

- (a) $h_2''g_2 = 1$, where h_2'' is obtained by shuffling h_2' past $h_1'g_1$, where the part in front of h_2'' is now $(h_1'g_1)'$. In this case:

$$\rho(h_1, h_2, g_1, g_2) = \rho(h_2', h_1'g_1, g_2) = (h_1'g_1)'$$

Now we compute

$$\rho(h_1, h_2, g_2', g_1') = h_1g_1'$$

Here we have these calculations: $h_1h_2 = h_2'h_1'$. Then $h_2'(h_1'g_1) = h_1(h_2g_1)$. Since $h_2 = g_2$ or $h_2 = g_2^{-1}$ we have $h_2g_1 = g_1'h_2''$, with $h_2'' = g_2^{-1}$ by assumption. Hence $(h_1'g_1)' = h_1g_1'$. Also, from $h_2g_1 = g_1'h_2''$ we get $h_2g_2'g_1' = (h_2g_1)g_2 = g_1'(h_2''g_2) = g_1'$, hence $h_2g_2' = 1$. So both expressions above are equal.

- (b) $h_2''g_2 \neq 1$, where h_2'' is obtained by shuffling h_2' past $h_1'g_1$. In this case:

$$\rho(h_1, h_2, g_1, g_2) = \rho(h_2', h_1'g_1, g_2) = (h_1'g_1)', h_2''g_2$$

Now we also compute

$$\rho(h_1, h_2, g_2', g_1') = \rho(h_1, h_2g_2', g_1') = (h_2g_2')', h_1'g_1'$$

Here we have these calculations: $h_1h_2 = h_2'h_1'$, $g_1g_2 = g_2'g_1'$, $h_2'(h_1'g_1) = (h_1'g_1)'h_2''$, and $h_1(h_2g_2') = (h_2g_2')'h_1''$. Multiplying the third and the fourth by g_2 , and g_1' respectively, and using the first two equalities we get that both expressions are congruent.

- (II.1.2.2) Any of the elements from the sequence $(h_1, \dots, h_{j-1}, h_{j+1}', \dots, h_r')$ which can shuffle to the end, is in a different generating group from g_{i+1} . Here we get:

$$\begin{aligned} \rho(g_1, \dots, g_i, g_{i+1}) &= \rho(\rho(g_1, \dots, g_{i-1}, g_i), g_{i+1}) \\ &= \rho(h_1, \dots, h_{j-1}, h_{j+1}', \dots, h_r', h_j'g_i, g_{i+1}) \\ &= (h_1, \dots, h_{j-1}, h_{j+1}', \dots, h_r', h_j'g_i, g_{i+1}). \end{aligned}$$

Also

$$\begin{aligned} \rho(g_1, \dots, g_{i+1}', g_i') &= \rho(\rho(g_1, \dots, g_{i-1}, g_{i+1}'), g_i') \\ &= \rho(h_1, \dots, h_r, g_{i+1}', g_i') \\ &= (h_1, \dots, h_{j-1}, h_{j+1}', \dots, h_r', (g_{i+1}')', h_j''g_i') \end{aligned}$$

where $h_j'g_{i+1}' = (g_{i+1}')'h_j''$.

As $h_j'g_{i+1}' = h_j'g_{i+1}'g_i' = (g_{i+1}')'h_j''g_i'$ we conclude that both expressions are congruent.

In case (II.2) we have the following equation:

$$\rho(g_1, \dots, g_{i-1}, g_i) = (h_1, \dots, h_r, g_i). \quad (3.11)$$

We also assume that there is no h_k which shuffles to the end and is on the same generating group as g_{i+1} . Otherwise by analogy we could consider it as a subcase of (II.1) by trying to show the equivalence:

$$\rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i) \cong \rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}).$$

Under the assumptions above, we obtain:

$$\begin{aligned} \rho(g_1, \dots, g_{i-1}, g_i, g_{i+1}) &= (h_1, \dots, h_r, g_i, g_{i+1}) \\ &\cong (h_1, \dots, h_r, g'_{i+1}, g'_i) \\ &= \rho(g_1, \dots, g_{i-1}, g'_{i+1}, g'_i). \end{aligned}$$

Chapter 4

Parabolic subgroups in even Artin groups

In this chapter we discuss properties of parabolic subgroups in even Artin groups. We will especially focus on intersections of parabolic subgroups, and whether these intersections are again parabolic subgroups. Using Bass-Serre theory we show that the set of parabolic subgroups in a certain subclass \mathcal{C} (defined in Section 4.3) is stable under intersections.

4.1 Introduction

Let $G = G_\Gamma$ be an Artin group based on a labeled graph $\Gamma = (V, E)$. Let $S \subset V$ and consider the induced graph $\Gamma' = (S, E')$, where E' is the set of labeled edges in E whose endpoints lie in S . Let G_S be the subgroup of G generated by the vertices of S . Then by a theorem of Van der Lek in [27] the subgroup G_S is isomorphic to the Artin group $G_{\Gamma'}$.

Definition 4.1.1. The subgroups of G of the form G_S for $S \subset V$ are called *standard parabolic subgroups*. Their conjugates gG_Sg^{-1} , for some $g \in G$, are called *parabolic subgroups* of G . A parabolic subgroup $P = gG_Sg^{-1}$ is called *of type S* as it is a conjugate of the standard parabolic G_S , and sometimes we will also say that P is a *parabolic subgroup over S* .

One core question about parabolic subgroups in Artin Groups is the following:

Question 1. Given an Artin group G , is the set of parabolic subgroups in G stable under intersections?

If we limit ourselves to standard parabolic subgroups, Question 1 has a positive answer (see [27]), and for any $S, T \subset V$ one has the equality:

$$G_S \cap G_T = G_{S \cap T}.$$

However, Question 1 is still open for Artin groups. The same problem has a positive answer in the class of Coxeter groups (see [33] and the references within).

In the introduction of Chapter 1 we expressed the importance of parabolic subgroups and we mentioned some subclasses of Artin groups where Question 1 has a positive answer. Among those subclasses is the class of RAAGs.

In this chapter we study parabolic subgroups of even Artin groups (motivated by [2] for graph products and the case of RAAGs in particular). We also provide a subclass \mathcal{C} of even Artin groups, where Question 1 has a positive answer.

Remark 4.1.2. Throughout the chapter we use the letter V to mean the vertex set of the defining graph Γ .

4.2 Retractions and parabolic subgroups

In this section we present retractions, and we show that any parabolic subgroup of an even Artin group is a retract.

Definition 4.2.1. Let G be a group and H a subgroup in G . Call H a *retract* of G if there is a group morphism $\rho: G \rightarrow H$ with $\rho(h) = h$ for any $h \in H$. The morphism ρ is called a *retraction*.

Let $\Gamma = (E, V)$ be a simplicial graph labeled with even natural numbers, or ∞ , and let $G = G_\Gamma$ be the corresponding Artin group based on Γ . We call G an even Artin group, and from now on we work with these groups.

Definition 4.2.2. In even Artin groups one has a *retraction* $\rho_S : G_\Gamma \rightarrow G_S$ for any $S \subseteq V$, defined on the generators of G_Γ as:

$$\rho(s) = s \text{ for } s \in S, \text{ and } \rho(v) = 1 \text{ for } v \in V \setminus S.$$

Any parabolic subgroup $K = fG_Sf^{-1}$, with $f \in G$ is a retract of G , with the *retraction*:

$$\rho_K = \rho_S^f : G \rightarrow K = fG_Sf^{-1} \text{ defined by } \rho_S^f(g) := f\rho_S(f^{-1}gf)f^{-1} \text{ for all } g \in G.$$

Retractions of the type above make sense only on even Artin groups, as explained in the following example.

Example 4.2.3. Consider the odd Artin group $G_\Gamma = \langle a, b \mid aba = bab \rangle$. For $S = \{a\}$, we cannot have a retraction of the form $\rho_S : G_\Gamma \rightarrow G_S$ defined on the generators a, b as $\rho(a) = a, \rho(b) = 1$. This would not induce a morphism of groups as $aba = bab$ in G_Γ , but $\rho_S(aba) = a^2$, and $\rho_S(bab) = a$. Nevertheless, there are retractions $\rho : G_\Gamma \rightarrow G_S$, for example, defining $\rho(a) = \rho(b) = a$ induces a well defined retraction.

Remark 4.2.4. Consider the graph Γ given below:

The corresponding Artin group is presented as:

Lemma 4.2.6. Let $G = G_\Gamma$ be an even Artin group with $A, B \subseteq V$ and $g, h \in G$. Then $gG_Ag^{-1} \subsetneq hG_Bh^{-1}$ implies $A \subsetneq B$.

Proof. One can write the proper inclusion $gG_Ag^{-1} \subsetneq hG_Bg^{-1}h^{-1}$ in the equivalent form $fG_Af^{-1} \subsetneq G_B$, for $f = h^{-1}g$.

Applying ρ_B we obtain:

$$fG_Af^{-1} = \rho_B(fG_Af^{-1}) = \rho_B(f)G_{A \cap B}\rho_B(f)^{-1} \subsetneq G_B.$$

So, the proper inclusion $fG_Af^{-1} \subsetneq G_B$ is equivalent to the proper inclusion

$$\rho_B(f)G_{A \cap B}\rho_B(f)^{-1} \subsetneq G_B,$$

which after conjugating by $\rho_B(f)^{-1}$ becomes equivalent to $G_{A \cap B} \subsetneq G_B$, and this implies that $A \cap B \subsetneq B$.

Instead, applying ρ_A to $fG_Af^{-1} \subsetneq G_B$ we obtain

$$G_A = \rho_A(f)G_A\rho_A(f)^{-1} = \rho_A(fG_Af^{-1}) \subseteq \rho_A(G_B) = G_{A \cap B}.$$

The inclusion $G_A \subseteq G_{A \cap B}$ implies $A \subseteq A \cap B$.

Ultimately $A \subseteq A \cap B \subsetneq B$, which means that $A \subsetneq B$, as required. \square

Now we characterize the intersection of two parabolic subgroups in even Artin groups. Once again, we will make use of retractions.

Lemma 4.2.7. Let $G = G_\Gamma$ be an even Artin group. The intersection of any two parabolic subgroups H, K can be expressed as an intersection of two parabolic subgroups over the same set, i.e.

$$H \cap K = g_1G_Sg_1^{-1} \cap g_2G_Sg_2^{-1},$$

for some $g_1, g_2 \in G$, and some $S \subseteq V$.

Proof. Denote $H = fG_Af^{-1}$ and $K = gG_Bg^{-1}$ for some $A, B \subseteq V$, and some $f, g \in G$. As one has the equality

$$fG_Af^{-1} \cap gG_Bg^{-1} = f[G_A \cap (f^{-1}g)G_B(f^{-1}g)^{-1}]f^{-1},$$

and since $f(X \cap Y)f^{-1} = (fXf^{-1}) \cap (fYf^{-1})$ for any sets $X, Y \subseteq G$, we can assume that $f = 1$, i.e. it is sufficient to show the result for $H = G_A$ and $K = gG_Bg^{-1}$.

Now the goal is to express $P = G_A \cap gG_Bg^{-1}$ as an intersection of parabolic subgroups over the same set.

Using $P \subseteq G_A$, and $G_A \cap G_B = G_{A \cap B}$ (see Lemma 4.2.5) we obtain:

$$\begin{aligned} P &= \rho_A(P) = \rho_A(G_A \cap gG_Bg^{-1}) \subseteq \rho_A(G_A) \cap \rho_A(gG_Bg^{-1}) \\ &= G_A \cap \rho_A(g)\rho_A(G_B)\rho_A(g^{-1}) \\ &= \rho_A(g)G_{A \cap B}\rho_A(g)^{-1}. \end{aligned}$$

Setting $g_1 = \rho_A(g)$ and $A \cap B = B'$ we can write the inclusion above as $P \subseteq g_1 G_{B'} g_1^{-1}$, and we notice that $g_1 G_{B'} g_1^{-1} \subseteq G_A$. Also, $P = G_A \cap g G_B g^{-1}$, so we have

$$\begin{aligned} P &= (G_A \cap g G_B g^{-1}) \cap g_1 G_{B'} g_1^{-1} = g G_B g^{-1} \cap (G_A \cap g_1 G_{B'} g_1^{-1}) \\ &= g G_B g^{-1} \cap g_1 G_{B'} g_1^{-1}, \end{aligned}$$

where $B' \subseteq B$. Now, to show the result, it is sufficient to show that $P' = g^{-1} P g$ is an intersection of parabolic subgroups over the same set. This new subgroup is given as:

$$P' = G_B \cap h G_{B'} h^{-1}$$

for $h = g^{-1} g_1$.

Applying the same procedure as for P above, we obtain:

$$\begin{aligned} P' &= \rho_B(P') = \rho_B(G_B \cap h G_{B'} h^{-1}) \\ &\subseteq \rho_B(G_B) \cap \rho_B(h G_{B'} h^{-1}) \\ &= \rho_B(h) G_{B \cap B'} \rho_B(h)^{-1} \\ &= \rho_B(h) G_{B'} \rho_B(h)^{-1}. \end{aligned}$$

Setting $h_1 = \rho_B(h) \in G_B$ we express the inclusion above as $P' \subseteq h_1 G_{B'} h_1^{-1} \subseteq G_B$. Putting together $P' = G_B \cap h G_{B'} h^{-1}$ and $P' \subseteq h_1 G_{B'} h_1^{-1}$ we have:

$$P' = (G_B \cap h G_{B'} h^{-1}) \cap h_1 G_{B'} h_1^{-1} = h G_{B'} h^{-1} \cap (G_B \cap h_1 G_{B'} h_1^{-1}),$$

which ultimately yields:

$$P' = h G_{B'} h^{-1} \cap h_1 G_{B'} h_1^{-1},$$

which expresses P' as an intersection of two parabolic subgroups over $B' = A \cap B$. \square

Remark 4.2.8. Note that, following the proof above we have:

$$f G_A f^{-1} \cap g G_B g^{-1} = f' G_{A \cap B} f'^{-1} \cap g' G_{A \cap B} g'^{-1}$$

for some $f', g' \in G$.

From now on we can substitute the intersection of two parabolic subgroups with the intersection of two parabolic subgroups over the same set.

As an immediate corollary of Lemma 4.2.7 we get the following:

Corollary 4.2.9. Let $G = G_\Gamma$ be an even Artin group. If the intersection $G_A \cap g G_A g^{-1}$ is a parabolic subgroup for any $g \in G$, and any $A \subseteq V$, then the intersection of any two parabolic subgroups is a parabolic subgroup.

In the next proposition we use the notion of the link on a graph (which appears in Section 2.2). For our even labeled graph $\Gamma = (V, E)$ and for $x \in V$, we denote the link of x in Γ by $\text{Lk}(x)$ and define it as:

$$\text{Lk}(x) = \{v \in V \mid \{x, v\} \in E\}.$$

Proposition 4.2.10. Let $G = G_\Gamma$ be an even Artin group based on $\Gamma = (V, E)$, and let $A \subset V$ and $g \in G$. Suppose that $G_A \cup gG_Ag^{-1}$ is not contained in a proper parabolic subgroup of G and that $G_A \cap gG_Ag^{-1}$ is not contained in a parabolic subgroup over a proper subset of A . Then for all $x \in V \setminus A$, one has $\text{Lk}(x) \supseteq A$.

Proof. Consider $P = G_A \cap gG_Ag^{-1}$. Assume by contradiction that there is an $x \in V \setminus A$ with the property $\text{Lk}(x) \not\supseteq A$.

If $G_{V \setminus \{x\}} = gG_{V \setminus \{x\}}$, then $g \in G_{V \setminus \{x\}}$. This means that both G_A and gG_Ag^{-1} are parabolic subgroups in $G_{V \setminus \{x\}}$, and hence $G_A \cup gG_Ag^{-1}$ is contained in the proper parabolic subgroup $G_{V \setminus \{x\}}$ of G . This contradicts the assumptions of the proposition, so suppose that $G_{V \setminus \{x\}} \neq gG_{V \setminus \{x\}}$. Using Bass-Serre theory (see Appendix [A.1](#)), one has a splitting of G as:

$$G = G_{\text{St}(x)} *_{G_{\text{Lk}(x)}} G_{V \setminus \{x\}}. \quad (4.2)$$

Consider the Bass-Serre tree T corresponding to this splitting. There are two types of vertices in T : left cosets of $G_{\text{St}(x)}$, and left cosets of $G_{V \setminus \{x\}}$ in G . Only vertices of different type can be adjacent in T . The group G acts naturally on T , without edge inversions. Moreover, the vertex stabilizers correspond to conjugates of $G_{\text{St}(x)}$ and conjugates of $G_{V \setminus \{x\}}$ for the respective type of vertices, while the edge stabilizers are conjugates of $G_{\text{Lk}(x)}$.

In the tree T , both $G_{V \setminus \{x\}}$ and $gG_{V \setminus \{x\}}$, are distinct vertices of the same type. Their stabilizers are $G_{V \setminus \{x\}}$, and $gG_{V \setminus \{x\}}g^{-1}$ respectively. As we are on a tree, there is a unique geodesic p in T connecting $G_{V \setminus \{x\}}$ and $gG_{V \setminus \{x\}}$.

Now for our parabolic subgroups G_A and gG_Ag^{-1} we have:

$$G_A \subseteq G_{V \setminus \{x\}}, \quad gG_Ag^{-1} \subseteq gG_{V \setminus \{x\}}g^{-1},$$

which means that they stabilize the vertices labeled by $G_{V \setminus \{x\}}$ and $gG_{V \setminus \{x\}}$ respectively. The intersection $G_A \cap gG_Ag^{-1}$ stabilizes the geodesic p connecting those vertices, and hence it stabilizes any edge belonging to p . Since stabilizers of edges in T are conjugates of $G_{\text{Lk}(x)}$, we have:

$$P = G_A \cap gG_Ag^{-1} \subseteq hG_{\text{Lk}(x)}h^{-1}$$

for some $h \in G$. Now one can write P as:

$$P = G_A \cap gG_Ag^{-1} \cap hG_{\text{Lk}(x)}h^{-1} = (G_A \cap hG_{\text{Lk}(x)}h^{-1}) \cap (gG_Ag^{-1} \cap hG_{\text{Lk}(x)}h^{-1}) \quad (4.3)$$

By Lemma [4.2.7](#), one can express $G_A \cap hG_{\text{Lk}(x)}h^{-1}$ as an intersection of two parabolic subgroups over $\text{Lk}(x) \cap A \subsetneq A$ (because $\text{Lk}(x) \not\supseteq A$). This means that $P = G_A \cap gG_Ag^{-1}$ is contained in a parabolic subgroup over a proper subset of A , again contradicting the assumption of the proposition.

As in both cases our assumption brings us to a contradiction with our hypothesis, we get that $\text{Lk}(x) \supseteq A$ for all $x \in V \setminus A$. \square

Remark 4.2.11. In Equation [4.3](#) one can express P as an intersection of 4 parabolic subgroups over $\text{Lk}(x) \cap A$. Indeed, by Theorem [4.2.7](#), both $(G_A \cap hG_{\text{Lk}(x)}h^{-1})$ and $(gG_Ag^{-1} \cap hG_{\text{Lk}(x)}h^{-1})$ can be expressed as intersections of two parabolic subgroups over $\text{Lk}(x) \cap A$.

In the following sections we proceed to find some subclasses of even Artin groups where arbitrary intersections of parabolic subgroups are parabolic subgroups.

4.3 Main Theorem

In this section we work on a certain subclass \mathcal{C} and we prove that the set of parabolic subgroups in \mathcal{C} is closed under intersections.

Definition 4.3.1. Let \mathcal{C} be the class of those even Artin groups G_Γ where Γ is finite, and for any $v \in V$ that belongs in a triangle in Γ , all the edges having an endpoint in v are labeled by 2's.

Notice that RAAGs belong in \mathcal{C} . Another family that belongs in \mathcal{C} is the class of 2-dimensional even Artin group (i.e. even Artin groups that do not contain triangles). In particular, any triangle in Γ , with G_Γ in \mathcal{C} , has all of its edges labeled by 2's.

Definition 4.3.2. An Artin group G_Γ is of *FC type* if for any subset $S \subseteq V$ which spans a complete subgraph, the parabolic subgroup G_S is of spherical type.

Remark 4.3.3. The class \mathcal{C} is a subclass of even Artin groups of FC type, as all the edges of any triangle in \mathcal{C} are labeled by 2's, which means that complete subgraphs generate free abelian groups, hence of spherical type (see also Lemma 3.1 in [7](#), which states that in even Artin groups of FC type, triangles have at least two edges labeled by 2's).

Before stating the main theorem, we give the following result, which is a special case of the main theorem, and will be proved in Section [4.4](#).

Result (Lemma [4.4.1](#)). Let $G = G_\Gamma$ be an even Artin group based on $\Gamma = (V, E)$. Let $A \subset V$ with G_A a free subgroup, $V \setminus A = \{x\}$, and $\text{Lk}(x) = A$. Then for any $g \in G$ the intersection $P = G_A \cap gG_Ag^{-1}$ is a parabolic subgroup.

Theorem 4.3.4. Let $G = G_\Gamma$ be an even Artin group belonging to the class \mathcal{C} . Then the intersections $P = G_A \cap gG_Ag^{-1}$ are parabolic subgroups for any $g \in G$, and any $A \subseteq V$.

Proof. Recall that Γ is finite for G_Γ in \mathcal{C} . Therefore, we can use induction on $n = |V|$, and $m = |A|$. If $G_A = G_\Gamma$ then for any $g \in G$ we obtain $P = G_A$, which is a parabolic subgroup. Hence, we can assume that G_A is a proper subgroup.

If $n = 1$ then $G_\Gamma \simeq \mathbb{Z}$, and the result is obvious. The result also holds for any Γ when $m = 1$, because intersections of parabolic subgroups of spherical type in FC

type Artin groups are again parabolic subgroups (see [31]); moreover, using the same reference, we have the result when $n = 2$ as the proper parabolic subgroups are of spherical type.

Now assume that $n > 2$, and $m \geq 2$.

The case when there is $x \in V \setminus A$ with the property $\text{Lk}(x) \not\supseteq A$ gives two subcases:

1. If $gG_{V \setminus \{x\}} = G_{V \setminus \{x\}}$ then both G_A, gG_Ag^{-1} are parabolic subgroups in $G_{V \setminus \{x\}}$. By induction on n the intersection $P = G_A \cap gG_Ag^{-1}$ is a parabolic subgroup in $G_{V \setminus \{x\}}$, and hence it is a parabolic subgroup in G_Γ as well.
2. If $gG_{V \setminus \{x\}} \neq G_{V \setminus \{x\}}$ by the discussion after the splitting in Equation (4.2), we can express P as an intersection of 4 parabolic subgroups over $\text{Lk}(x) \cap A \subsetneq A$ (i.e. proper subsets of A). By induction on m , we get that P is a parabolic subgroup.

Now consider the case where any $x \in V \setminus A$ satisfies $\text{Lk}(x) \supseteq A$. Suppose that $V \setminus A$ has k disjoint connected components.

Here we proceed by induction on k . Suppose that $k = 1$, and let C_1 be the only connected component.

If $|C_1| = 1$ then $V \setminus A = \{x\}$ and $\text{Lk}(x) = A$. The case when G_A is free is the result stated above (Lemma 4.4.1). Instead, if G_A is not free, there is an edge $\{a, b\}$ in Γ for a pair of vertices a, b in A . As (a, b, x) form a triangle, by definition of our class \mathcal{C} , we have that x commutes with A (all the edges having x as an endpoint are labeled by 2's). Hence $G_A \cap gG_Ag^{-1} = G_A$.

Now consider $|C_1| > 1$. Since any $x \in V \setminus A$ satisfies $\text{Lk}(x) \supseteq A$ we have that any vertex $a \in A$ forms an edge with any vertex $c \in C_1$. Moreover, C_1 is connected, so every vertex $c \in C_1$ belongs to an edge in C_1 . Hence, any vertex $a \in A$ belongs to a triangle, the other two vertices of which lie in $C_1 = V \setminus A$. Since all triangles in \mathcal{C} are labeled by 2's, we have:

$$G = G_A \times G_{V \setminus A}.$$

The direct product above implies that $gG_A = G_Ag$, and the result follows as:

$$G_A \cap gG_Ag^{-1} = G_A \cap G_A = G_A.$$

Now assume $k > 1$.

If at least one of the connected components C_1, \dots, C_k is not a singleton, we follow the same procedure as when $k = 1$ and $|C_1| > 1$, to deduce that:

$$G = G_A \times G_{\Gamma-A},$$

and the result follows as earlier. We get the same splitting if G_A is not free as any other vertex in $V \setminus A$ would belong to a triangle.

Finally assume that all the connected components are singletons, i.e. $C_i = \{x_i\}$ for some $x_i \in V \setminus A$ for all $1 \leq i \leq k$, and G_A free. Choose one $x \in V \setminus A$. Then we have a splitting:

$$G = G_{A \cup \{x\}} *_{G_A} G_{V \setminus \{x\}}$$

and consider the Bass-Serre tree T corresponding to this splitting.

Consider the edges G_A and gG_A on T . If $G_A = gG_A$, then $G_A \cap gG_Ag^{-1} = G_A$, so assume that they are different. When $G_A \neq gG_A$, there is a unique geodesic p connecting them. Let $G_A, g_1G_A, \dots, g_nG_A, gG_A$ be the labels of the geodesic p .

By the construction of T , one has either $g_i^{-1}g_{i+1} \in G_{A \cup \{x\}}$ or $g_i^{-1}g_{i+1} \in G_{V \setminus \{x\}}$, for any $i = 0, \dots, n$ where $g_0 = 1$ and $g_{n+1} = g$.

The intersection $G_A \cap gG_Ag^{-1}$ stabilizes the endpoints of the geodesic p , hence it stabilizes the whole p . As the stabilizer of a geodesic is the intersection of stabilizers of its edges, we have the equality

$$G_A \cap gG_Ag^{-1} = G_A \cap g_1G_Ag_1^{-1} \cap \dots \cap g_nG_Ag_n^{-1} \cap gG_Ag^{-1}.$$

The intersections $g_iG_Ag_i^{-1} \cap g_{i+1}G_Ag_{i+1}^{-1}$ can be expressed as:

$$g_iG_Ag_i^{-1} \cap g_{i+1}G_Ag_{i+1}^{-1} = g_i[G_A \cap g'_iG_Ag_i'^{-1}]g_i^{-1},$$

where $g'_i = g_i^{-1}g_{i+1}$ is either in $G_{A \cup \{x\}}$, or in $G_{V \setminus \{x\}}$.

If g'_i is in $G_{A \cup \{x\}}$ then by the case we discuss in Section 4.4 (Lemma 4.4.1) we obtain:

$$G_A \cap g'_iG_Ag_i'^{-1} = G_{S_i}$$

for some $S_i \subseteq A$.

On the other hand, if g'_i is in $G_{V \setminus \{x\}}$ then by induction on n we know that the intersections $G_A \cap g'_iG_Ag_i'^{-1}$ are parabolic subgroups in $G_{V \setminus \{x\}}$, and hence parabolic subgroups in G . Moreover one can express these intersections (by applying ρ_A) as

$$G_A \cap g'_iG_Ag_i'^{-1} = a_iG_{S_i}a_i^{-1}$$

for $S_i \subseteq A$, and $a_i \in A$.

If for at least one of the intersecting pairs we get $S_i \subsetneq A$, then $G_A \cap gG_Ag^{-1}$ could be expressed as an intersection of parabolic subgroups over a proper subset of A , and hence by induction on m this case is proved.

Otherwise, we get $S_i = A$ for all i , and hence:

$$G_A \cap gG_Ag^{-1} = G_A \cap g_1G_Ag_1^{-1} \cap \dots \cap g_nG_Ag_n^{-1} = G_A.$$

In all the cases the intersection is again a parabolic subgroup. \square

Using induction and Theorem 4.3.4, we obtain the following corollary.

Corollary 4.3.5. Let $n \in N$, and P_1, \dots, P_n be parabolic subgroups in \mathcal{C} . Then the intersection

$$P_1 \cap \dots \cap P_n,$$

is a parabolic subgroup in \mathcal{C} .

So any finite intersection of parabolic subgroups in \mathcal{C} is again a parabolic subgroup in \mathcal{C} . One can extend this result to arbitrary intersections. We follow the strategy of Section 10 in [15] for the proof of this result, stated in the following corollary.

Corollary 4.3.6. Let $G = G_\Gamma$ be an even Artin group belonging to the class \mathcal{C} . Then any arbitrary intersection of parabolic subgroups in G is a parabolic subgroup.

Proof. Let \mathcal{P} be the set of parabolic subgroups in G . The set \mathcal{P} is countable, as any parabolic subgroup can be defined as gG_Sg^{-1} for $S \subseteq V$ and $g \in G$. We have finitely many such subsets S (as Γ is finite for $G_\Gamma \in \mathcal{C}$), and countably many elements $g \in G$ (as G is finitely presented), so there are countably many parabolic subgroups in G .

For an arbitrary indexing set I , we want to show that:

$$Q = \bigcap_{i \in I, P_i \in \mathcal{P}} P_i$$

is a parabolic subgroup in \mathcal{C} .

If I is finite, the claim follows from Corollary 4.3.5. So, we can assume that the indexing set I is countable, and we can index its elements by natural numbers. Write:

$$\bigcap_{i \in I} P_i = \bigcap_{n \in \mathbb{N}} \left(\bigcap_{i \leq n} P_i \right),$$

and set $Q_n = \bigcap_{i \leq n} P_i$. We know that Q_n is a parabolic subgroup for any n . Moreover we have a chain of parabolic subgroups:

$$Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \cdots$$

where the intersection of all members Q_i of the chain above is equal to Q . We cannot have an infinite chain of nested distinct parabolic subgroups. Indeed, using Lemma 4.2.6 we have that $gG_Ag^{-1} \subsetneq hG_Ah^{-1}$ implies $A \subsetneq B$. Hence there are at most $|V| + 1$ distinct parabolic subgroups in the chain above.

Ultimately, Q is an intersection of at most $|V| + 1$ parabolic subgroups and hence it is a parabolic subgroup. \square

4.4 Particular free subgroups

Now we want to see if in our class \mathcal{C} , the intersection $P = G_A \cap gG_Ag^{-1}$ is parabolic, when $V \setminus A = \{x\}$, $\text{Lk}(x) = A$ and G_A is free. This is the only missing piece in the proof of Theorem 4.3.4

Lemma 4.4.1. Let $G = G_\Gamma$ be an even Artin group based on $\Gamma = (V, E)$. Let $A \subset V$ with G_A free and $V \setminus A = \{x\}$. Then for any $g \in G$ the intersection $P = G_A \cap gG_Ag^{-1}$ is a parabolic subgroup.

Before giving a proof for the lemma above, we compute the kernel of the retraction map $\rho_{\{x\}}$ without assuming that G_A is free.

Denote the retraction map $\rho_{\{x\}}$ by ρ and let $K = \ker(\rho)$. We use the Reidemeister-Schreier procedure (see Appendix [A.3](#), and Example [1.2.12](#)) to obtain a presentation of K . Our setting is:

$$G = G_\Gamma = \langle V, R \rangle \text{ where } V = A \sqcup \{x\}, \text{ and } R = \text{even Artin relations in } G.$$

So we have a map:

$$\rho: G_\Gamma \rightarrow \mathbb{Z}, \quad \forall a \in A : a \mapsto 0, \quad x \mapsto 1,$$

with $K = \ker(\rho)$.

The set $T = \{x^i \mid i \in \mathbb{Z}\}$ gives a Schreier transversal for K in G .

The set of generators for K is $Y = \{tv(\overline{tv})^{-1} \mid t \in T, v \in V, tv \notin T\}$ where \overline{w} is the representative of w in T .

Any $t \in T$ can be written as x^i for $i \in \mathbb{Z}$, so:

$$tv(\overline{tv})^{-1} = x^i v(\overline{x^i v})^{-1} = \begin{cases} x^i x(\overline{x^i x})^{-1} & \text{if } v = x \\ x^i a(\overline{x^i a})^{-1} & \text{if } v = a \in A \end{cases}.$$

As $x^i x \in T$ for all $i \in \mathbb{Z}$ the first line of the cases above does not produce any generators.

On the other hand $x^i a \notin T$ for all $a \in A$ and all $i \in \mathbb{Z}$. Moreover $\overline{x^i a} = x^i$ in T . Hence the second lines above, for any $a \in A$, gives $s_{i,a} = x^i a x^{-i}$ as a generator of K .

Ultimately, the set

$$Y = \{s_{i,a} = x^i a x^{-i} \mid a \in A, i \in \mathbb{Z}\},$$

gives a set of generators for K .

To get relations for K , rewrite each trt^{-1} for $t \in T$ and $r \in R$, using generators in Y .

Write any $t \in T$ as x^i for some $i \in \mathbb{Z}$. We divide the relations in R in two types:

- (i) relations involving only elements of A . Suppose $a, b \in A$ satisfy $r = (ab)^m (ba)^{-m}$,
- (ii) relations involving x . Suppose $a \in A$ and x satisfy $r = (ax)^n (xa)^{-n}$.

In case (i) we have $trt^{-1} = x^i ((ab)^m (ba)^{-m}) x^{-i}$. Introducing $x^i x^{-i}$ between letters, and recalling that $s_{i,a} = x^i a x^{-i}$ we obtain:

$$trt^{-1} = (s_{i,a} s_{i,b})^m (s_{i,b} s_{i,a})^{-m} \tag{4.4}$$

which is an even Artin relation, for the pair $s_{i,a}, s_{i,b}$ for all $i \in \mathbb{Z}$, with the same label as the Artin relation for the pair a, b .

In case (ii) we have $trt^{-1} = x^i ((ax)^n (xa)^{-n}) x^{-i}$. Again we put $x^i x^{-i}$ between letters, and use $s_{i,a} = x^i a x^{-i}$ to obtain:

$$trt^{-1} = s_{i,a} s_{i+1,a} \cdots s_{i+n-1,a} (s_{i+1,a} s_{i+2,a} \cdots s_{i+n,a})^{-1}. \tag{4.5}$$

This looks messier, so let us illustrate the commutative case ($ax = xa$) for inspiration. We have $r = ax(xa)^{-1}$ (i.e. $n = 1$ above), now for some $i \in \mathbb{Z}$ and $t = x^i$ we obtain:

$$\begin{aligned} trt^{-1} &= x^i(axa^{-1}x^{-1})x^{-i} = (x^i ax^{-i})(x^i xx^{-i})(x^i a^{-1}x^{-i})(x^i x^{-1}x^{-i}) \\ &= (x^i ax^{-i})x(x^i a^{-1}x^{-i})x^{-1} \\ &= (x^i ax^{-i})(x^{i+1}a^{-1}x^{-i-1}) \\ &= (x^i ax^{-i})(x^{i+1}ax^{-(i+1)})^{-1} \\ &= s_{i,a}s_{i+1,a}^{-1}. \end{aligned}$$

To recap, the presentation for K is given as:

$$K = \langle Y \mid S \rangle,$$

where $Y = \{s_{i,a} = x^i ax^{-i} \mid a \in A, i \in \mathbb{Z}\}$, and the relations are described as below:

- (i) if $a, b \in A$ satisfy $(ab)^m = (ba)^m$, then for all $i \in \mathbb{Z}$: $(s_{i,a}s_{i,b})^m = (s_{i,b}s_{i,a})^m$
- (ii) if $a \in A$ and x satisfy $(ax)^n = (xa)^n$ then for all $i \in \mathbb{Z}$:

$$s_{i,a}s_{i+1,a} \cdots s_{i+n-1,a} = s_{i+1,a}s_{i+2,a} \cdots s_{i+n,a}.$$

So far, the presentation for K has infinitely many generators, and some of the relations do not seem like something we know. However there is a silver lining, expressed by the following lemma.

Lemma 4.4.2. If G is even Artin group of FC type, then so is the kernel K .

Proof. Look at type (ii) relations: if $a \in A$ and x satisfy $(ax)^n = (xa)^n$ then $\forall i \in \mathbb{Z}$ we have $s_{i,a}s_{i+1,a} \cdots s_{i+n-1,a} = s_{i+1,a}s_{i+2,a} \cdots s_{i+n,a}$. It is enough to use the generators:

$$s_{0,a}, s_{1,a}, \dots, s_{n-1,a},$$

as the other generators $s_{i,a}$ are obtained recursively by them. Indeed, setting:

$$\sigma_a = s_{0,a}s_{1,a} \cdots s_{n-1,a},$$

we obtain:

$$s_{l,a} = \sigma_a^{-q} s_{r,a} \sigma_a^q, \tag{4.6}$$

where $l = n \cdot q + r$ with $0 \leq r < n$.

Since x is linked to any $a \in A$ by a label $2k_a$, we see that it is enough to use only the generating set:

$$Y_1 = \{s_{j,a} = x^j ax^{-j} \mid a \in A, 0 \leq j \leq k_a - 1 \text{ in } \mathbb{Z}\},$$

which is finite. Now, with the new generating set, the relations in case (ii) are vanished.

Now we only consider the relations in case (i) assuming that G is even Artin group of FC type. The condition of being FC type implies that in any triangle in Γ , there are

at least two edges labeled by 2's. Let us build a simplicial graph Γ_1 (to represent K as an Artin group) where the vertex set is Y_1 and now we define the edges using relations of case (i).

Consider these two cases:

- (1) if $a, b \in A$ are linked by an edge labeled by m with $m \geq 4$, put an edge with label m for the pair $s_{0,a}, s_{0,b}$ in Γ_1 as well. This is because (a, b, x) is a triangle so $2k_a = 2k_b = 2$, which means that the generators in Y_1 corresponding to a, b are $s_{0,a}, s_{0,b}$ respectively.
- (2) If $a, b \in A$ commute, since (x, a, b) form a triangle, only one of the labels $2k_a, 2k_b$ can be bigger than 2. Assume $2k_a \geq 4$. The generators in Y_1 corresponding to a are $s_{i,a}$ with $0 \leq i \leq k_a - 1$, and there is only one (namely $s_{0,b}$) corresponding to b . Now put edges with labels equal to 2 for the pairs $s_{i,a}, s_{0,b}$ for all $0 \leq i \leq k_a - 1$. This comes from $s_{0,b} = s_{i,b}, \forall i \in \mathbb{Z}$ prior to vanishing the non-necessary generators.

We notice that all the edges in Γ_1 are labeled by even numbers, and whenever we put a triangle, at least 2 edges were labeled by 2's. Hence K is represented by Γ_1 and it is an even Artin group of FC type as well, and its presentation $K = \langle Y_1 \mid E \rangle$ can be read from the graph Γ_1 . Here E denotes the relations from (1) and (2), or equivalently, as seen from the labeled edges in Γ_1 . \square

Remark 4.4.3. Both parabolic subgroups G_A , and gG_Ag^{-1} lie in K . We want to use this to calculate $G_A \cap gG_Ag^{-1}$ in K and then see how the intersection looks like in G_Γ . One obstacle appearing here is that g is not necessarily in K , so we cannot interpret $G_A \cap gG_Ag^{-1}$ as an intersection of parabolic subgroups in K .

Since the relations in K come from the relations between elements of A , we obtain immediately the following corollary.

Corollary 4.4.4. If G_A is free, then the kernel K is free as well, on $\sum_{a \in A} k_a$ generators, where $2k_a$ is the label of the edge in Γ for the pair x, a with $a \in A$.

Remark 4.4.5. The case when K is free is more helpful for computing intersections (usage of Stallings foldings, see [35], and [25]). So will assume that G_A is free in the rest of the section.

Proof of Lemma 4.4.1. Let $\rho(g) = x^j$ for some $j \in \mathbb{Z}$, and write $g = (gx^{-j})x^j = hx^j$, where $h = gx^{-j} \in K$.

Now we can write gG_Ag^{-1} as:

$$hx^jG_Ax^{-j}h^{-1} = h\langle s_{j,a} \mid a \in A \rangle h^{-1}, \quad (4.7)$$

where $s_{j,a}$ is a generator of K (coming from the Reidemeister-Schreier procedure), and for any $a \in A$, we have $s_{0,a} = a$.

Define $m \in \mathbb{Z}_+$ to be the least common multiple of all k_a for $a \in A$, where $2k_a$ is the label of the edge between x and a in Γ . Distinguish these two cases:

1. The number j appearing in Equation (4.7) satisfies $j \notin m\mathbb{Z}$. This means that there is an $a \in A$, such that $k_a \nmid j$, which implies that $s_{j,a}$ is not a conjugate of $s_{0,a}$ (coming from Equation (4.6)). Consider $Q = \rho_A(gG_Ag^{-1})$. One notices that

$$Q = \rho_A(h)\langle s_{0,a} \mid a \in A, k_a \mid j \rangle \rho_A(h)^{-1}.$$

Since there is at least on k_a with $k_a \nmid j$, we conclude that the set

$$\{s_{0,a} \mid a \in A, k_a \mid j\},$$

is a proper subset of A . Moreover $P = G_A \cap gG_Ag^{-1}$ satisfies $P \subset Q$, so $P = P \cap Q$, which is going to be an intersection of parabolic subgroups where the set of generators of parabolic subgroups is a proper subset of A , so again by induction we conclude that P is parabolic.

2. The number j satisfies $j \in m\mathbb{Z}$, which implies that $s_{j,a}$ is a conjugate of $s_{0,a}$. Here one can express $P = G_A \cap gG_Ag^{-1}$ as:

$$P = G_A \cap h\langle s_{j,a} \mid a \in A \rangle h^{-1} = G_A \cap h\langle \sigma_a^{-q_a} s_{0,a} \sigma_a^{q_a} \mid a \in A \rangle h^{-1},$$

where $j = m \cdot q_a$.

If $j = 0$ then $g \in K$ and hence $G_A \cap gG_Ag^{-1}$ is an intersection of parabolic subgroups in K which is a free group, so $G_A \cap gG_Ag^{-1}$ would be a parabolic subgroup. From now on assume that $j \neq 0$.

Express the set A as a disjoint union $A = B \sqcup C$ such that any $b \in B$ is linked to x by a label greater than 2, and any $c \in C$ is linked to x by a label equal to 2.

To consider the intersection above, for combinatorial simplicity, we take an isomorphism $\varphi: K \rightarrow K$, which fixes every generator in K , except $s_{1,b}$ for $b \in B$, and we ask that $\varphi(\sigma_b) = s_{1,b}$ where $\sigma_b = s_{0,b}s_{1,b} \cdots s_{k_b-1,b}$.

Now computing the intersection $P = G_A \cap h\langle \sigma_a^{-q_a} s_{0,a} \sigma_a^{q_a} \mid a \in A \rangle h^{-1}$ is equivalent to computing

$$P = G_{A_0} \cap h\langle s_{1,b}^{-q_b} s_{0,b} s_{1,b}^{q_b}, s_{0,c} \mid b \in B, c \in C \rangle h^{-1},$$

where $A_0 = \{s_{0,a} \mid a \in A\} = A$, because $s_{0,a} = a$. Note that h might change when we apply φ but we use the same letter. Similarly we identify B with B_0 and C with C_0 .

Finally, we arrive at the place where we compute intersections of subgroups in free groups. See Appendix A.2 for an overview (see [35], and [25] for further details).

To each of the subgroups G_{A_0} and $h\langle s_{1,b}^{-q_b} s_{0,b} s_{1,b}^{q_b}, s_{0,c} \mid b \in B, c \in C \rangle h^{-1}$ as finitely generated subgroups of a free group K we associate an Y_1 -digraph whose language recognizes the respective subgroup. Here Y_1 is the generating set of K (as described in Lemma 4.4.2). We use foldings and then take the product of

digraphs to obtain an Y_1 -digraph which recognizes the intersection of subgroups. See Appendix [A.2](#) for the procedure of the construction.

Assume h is a reduced word in K . If $h \in G_{A_0}$ then the intersection is G_{C_0} ; so assume that $h \notin G_{A_0}$. Let h_0 be the maximal suffix of h belonging to C_0 ; note that, h_0 can be empty. Write h as $h = h_1 h_0$ with reduced h_1, h_0 . The part h_0 can be folded in the digraph among the elements of C_0 , so we can assume that $h = h_1$.

Now assume that h_1 ends with a letter not belonging to $B_1 = \{s_{1,b} \mid b \in B\}$. In this case there are no more possible foldings. Since we assumed that $h \notin A_0$ the intersection graph for G_{A_0} and $h \langle s_{1,b}^{-q_b} s_{0,b} s_{1,b}^{q_b}, s_{0,c} \mid b \in B, c \in C \rangle h^{-1}$ gives a trivial intersection.

Lastly assume that h_1 ends with a letter belonging to B_1 . Here we can do foldings (if the signs disagree) only in a single branch, and if we cannot do folding there, we get a trivial intersection. Let us say that h_1 ends in b_1^t , with $t \cdot q_b < 0$. Now if $|t| < |q_b|$ or $|t| > |q_b|$ we get a trivial intersection. In the case $|t| = |q_b|$ we can get nontrivial intersection if and only if $h_1 b_1^{-t}$ belongs to A_0 , and this gives a parabolic, actually G_{b_0} .

In all the cases we treated we obtained a parabolic subgroup. This implies the result of the Lemma.

□

Appendix A

Appendix - Specific preliminaries

Here we discuss about Bass-Serre theory, Stallings graphs, and RS procedure.

A.1 Bass-Serre Theory

Bass-Serre theory is a standard tool in geometric group theory which analyzes the algebraic structure of groups acting by automorphisms on simplicial trees.

Graphs of groups, are the basic objects of Bass-Serre theory, and the notion of the fundamental group of a graph of groups is used to relate group actions on trees with decomposing groups as iterated applications of the operations of free product with amalgamation and HNN extension.

Here we will state the fundamental theorem of Bass-Serre theory, and we refer to [34] for the notation and proofs. Notice that the notion of a graph here is different.

Definition A.1.1 (Graphs in Serre's formalism). A graph $A = (V, E, o, t, \bar{\cdot})$ consists of a vertex set V , an edge set E , an edge reversal map $\bar{\cdot}: E \rightarrow E$, $e \mapsto \bar{e}$ with $e \neq \bar{e}$ and $\bar{\bar{e}} = e$ for any $e \in E$, and an extremities map $(o, t): E \rightarrow V \times V$, $e \mapsto (o(e), t(e))$ with $o(\bar{e}) = t(e)$.

One calls the vertex $o(e)$ the origin of e , the vertex $t(e)$ is called the terminus of e , and \bar{e} is called the formal inverse of edge e .

Here we allow multiple edges and loops (i.e. edges e with $o(e) = t(e)$)

An orientation on the A is a partition of E into two disjoint subsets E^+ , and E^- so that for every edge e one has

$$|E^+ \cap \{e, \bar{e}\}| = 1 = |E^- \cap \{e, \bar{e}\}|.$$

Definition A.1.2 (Graph of groups). A graph of groups $\mathcal{A} = (A, \mathcal{G})$ consists of a connected graph $A = (V, E, o, t, \bar{\cdot})$, and two families of groups $\mathcal{G} = (\mathcal{G}_V, \mathcal{G}_E)$ with an

assignment of a vertex group G_v from the family \mathcal{G}_V to every vertex $v \in V$, and an edge group G_e from the family \mathcal{G}_E to every edge $e \in E$, such that:

- For every edge $e \in E$ we have $G_e = G_{\bar{e}}$,
- For all $e \in E$, there are boundary monomorphisms $\alpha_e : G_e \rightarrow G_{o(e)}$.

Let T be a spanning tree of $A = (V, E, o, t, \bar{\cdot})$, i.e. a maximal sub-tree in A . Now we define one of the key concepts in Bass-Serre theory.

Definition A.1.3. The *fundamental group of a graph of groups* $\mathcal{A} = (A, \mathcal{G})$ with respect to T , is the group $\pi = \pi(\mathcal{A}, \mathcal{G}, T)$ given by the presentation:

$$\pi := \left\langle \begin{array}{ll} G_v, & v \in V \\ e, & e \in E \end{array} \left| \begin{array}{ll} \bar{e}\alpha_e(g)e = \alpha_{\bar{e}}(g), & \forall e \in E, \forall g \in G_e \\ \bar{e}e = 1, & \forall e \in E \\ e = 1, & \forall e \in T \end{array} \right. \right\rangle.$$

This definition does not depend on the choice of the spanning tree, as one gets isomorphic groups.

Remark A.1.4 (A piece of the fundamental theorem). If G is the fundamental group of a graph of groups, then G acts without inversion of edges on a tree T (called Bass-Serre tree). We will see that free products with amalgamation can be seen as fundamental groups of graph of groups, and we will explain how the tree T looks like.

Consider a graph of groups \mathcal{A} with a single edge e , which is not a loop, together with its formal inverse e^{-1} with two distinct endpoints $u = o(e)$ and $v = t(e)$. Take $H = G_u$ and $K = G_v$ as vertex groups, and $C = G_e$ as an edge group, as in the figure below:

$$A = \begin{array}{c} H \quad C \quad K \\ \bullet \text{---} \text{---} \bullet \end{array}$$

Let also $\alpha = \alpha_e : C \rightarrow H$ and $\omega = \omega_e : C \rightarrow K$ be the boundary monomorphisms. Then $T = A$ is a spanning tree in A and the fundamental group $\pi = \pi(\mathcal{A}, \mathcal{G}, T)$ is isomorphic to the amalgamated free product:

$$G = H *_C K = (H * K) / \langle \alpha_e(c) = \omega_e(c) \mid c \in C \rangle.$$

In this case the Bass-Serre tree T can be described as:

$$VT = \{gH \mid g \in G\} \sqcup \{gK \mid g \in G\}.$$

Two vertices fH and gK are adjacent in T if there is $k \in K$ such that $fH = gkH$, or equivalently if there is $h \in H$ such that $gK = fhK$.

The G -stabilizer of the vertex fH is equal to fHf^{-1} , and the G -stabilizer of the vertex gK is equal to gKg^{-1} .

For an edge $[fH, fhK]$ of T its G -stabilizer is equal to $fh\alpha(C)h^{-1}f^{-1}$

This is important for our splittings of Artin groups which will appear in Chapter [4](#)

Remark A.1.5. If (G, S) is an Artin groups with $A, B \subseteq S$ such that $S = A \cup B$ then

$$G = G_A *_{G_{A \cap B}} G_B$$

where G_D represents the parabolic subgroup with generating set $D \subseteq S$.

A.2 Stallings' foldings

Here we present an algorithm for computing the intersection of two subgroups of a free group. It is an application of the folded graphs technique (we follow Section 9 in [25]).

Definition A.2.1. Let X be a finite alphabet. By an X -*digraph* we mean a combinatorial graph Γ where every edge e has an arrow (or direction) and is labeled by a letter from X , denoted $\mu(e)$.

Here we allow loops as well. For an edge e denote by $o(e), t(e)$ its origin and terminus respectively. If $o(e) = t(e)$ then e is a loop.

Given an X -digraph Γ , one can turn it into an oriented graph with labels in $X \sqcup X^{-1}$, by adding formal edges e^{-1} for any edge $e \in E\Gamma$. The label of e^{-1} is $\mu(e)^{-1}$, and the endpoints are given as $o(e^{-1}) = t(e)$, and $t(e^{-1}) = o(e)$. Denote the new graph by $\hat{\Gamma}$.

Using $\hat{\Gamma}$, define a path p in Γ as a sequence e_1, \dots, e_n of edges in $\hat{\Gamma}$ with $t(e_i) = o(e_{i+1})$ for all $1 \leq i \leq n-1$. Define $o(p) = o(e_1)$, and $t(p) = t(e_n)$. Moreover the path p has a label $\mu(p) = \mu(e_1) \cdots \mu(e_n)$ and we can regard it as a word on the alphabet $X \sqcup X^{-1}$. For a vertex v in Γ , the single term sequence $p = v$ is a path with $o(p) = t(p) = v$ and $\mu(p) = 1$ (the empty word). A path p in an X -digraph Γ is called *reduced* if p does not contain subpaths of the form e, e^{-1} or e^{-1}, e for $e \in E\hat{\Gamma}$.

Definition A.2.2. Let Γ be an X -digraph, and $v \in V\Gamma$. Define the *language of Γ with respect to v* to be:

$$L(\Gamma, v) = \{\mu(p) \mid p \text{ is a reduced path in } \Gamma \text{ with } o(p) = t(p)\}.$$

Definition A.2.3. The X -digraph Γ is called *folded*, if for each vertex $v \in V\Gamma$, and each letter $x \in X$ there is at most one edge in $E\Gamma$ with origin v and label x and at most one edge with terminus v and label x .

There is a natural procedure to fold any X -digraph, one just looks at edges with the same label and the same origin (or the same terminus) and glues them together. For a folded X -digraph Γ and a vertex $v \in V\Gamma$, all the words of $L(\Gamma, v)$ are freely reduced.

One can see a connection with finite state automata (see Section 1.2.4). Indeed, we regard the pair $(\hat{\Gamma}, v)$ as an automaton M over the alphabet $X \sqcup X^{-1}$. Vertices represent states and edges represent transitions. The initial state is unique, given by v , which also defines the unique accepting state. If Γ is folded, then M is deterministic.

Remark A.2.4. If H is a finitely generated subgroup of $F(X)$, then there is a finite folded connected X -digraph Γ and a vertex v such that $L(\Gamma, v) = H$.

Indeed, assume $H = \langle h_1, \dots, h_m \rangle$ with each h_i freely reduced over $X \sqcup X^{-1}$. Define an X -digraph Γ_1 as a wedge of m circles at a vertex v_1 , where the i -th circle is subdivided into $|h_i|$ edges which are directed and labeled by $X \sqcup X^{-1}$ such that the label of the path h_i , read from v_1 to v_1 is the word h_i . Now fold the graph Γ_1 to obtain a graph Γ with a distinguished vertex v (corresponding to v_1) along the folding.

Definition A.2.5 (Product-graph). Let Δ_1 and Δ_2 be X -digraphs. The product-graph $\Delta = \Delta_1 \times \Delta_2$ is an X -digraph, with vertex set $V\Delta = V\Delta_1 \times V\Delta_2$. For a pair of vertices $u = (u_1, u_2), v = (v_1, v_2)$ in $V\Delta$ and a letter $x \in X$ introduce an edge e in $E\Delta$ labeled by x , with origin in u and terminus in v if and only if there are edges $e_i \in E\Delta_i$ labeled by x with origin in u_i and terminus in v_i for $i \in \{1, 2\}$.

Lemma A.2.6. Let Δ_1 and Δ_2 be folded X -digraphs. Let $H_i = L(\Delta_i, v_i)$ for some vertices $v_i \in V\Delta_i$ and $i \in \{1, 2\}$. Let $v = (v_1, v_2) \in V(\Delta_1 \times \Delta_2)$. Then $\Delta = \Delta_1 \times \Delta_2$ is a folded X -digraph, and $L(\Delta, v) = H_1 \cap H_2$.

A.3 Reidemeister-Schreier procedure

There are examples in the thesis where we need to find a presentation for a subgroup of a group, and usually the situation for us will be the following:

- we have a finitely presented group G ,
- there is a surjective morphism $\rho : G \rightarrow C$ where C is a cyclic group,

and we want to find a presentation for $K = \ker(\rho)$.

There is an algorithm, called the *Reidemeister-Schreier procedure* that, given a presentation of a group G and enough suitable information about a subgroup H of G , yields a presentation of the subgroup H . For this topic we give [29] as the main reference.

Definition A.3.1. Let $F = F(X)$ be a free group with basis X and H a subgroup of F . A *Schreier transversal* for H in F , is a subset T of F which is a *right transversal* i.e.

- the union $\bigcup_{t \in T} Ht$ is equal to F , and
- $Ht_1 \neq Ht_2$ as for $t_1 \neq t_2$ in T ,

and furthermore any prefix of any element of T belongs to T as well.

In [29], the procedure is as follows:

- the group G is given by a presentation $\langle X \mid R \rangle$.
- $\pi : F(X) \rightarrow G$ is the canonical epimorphism from the free group $F = F(X)$ onto G .
- T is a Schreier transversal for $\pi^{-1}(H)$ in $F(X)$.

- The map $\bar{\cdot}: F \rightarrow T$ maps $w \in F$ to its coset representative $\bar{w} \in T$.

then $Y = \{tx(\bar{tx})^{-1} \mid t \in T, x \in X, tx \notin T\}$ gives a set of generators for H .

Furthermore, there is a map $\tau: F(X) \rightarrow F(Y)$ (see [29]); the map τ rewrites each word $w \in \pi^{-1}(H)$ in terms of generators of Y . Define

$$S = \{\tau(trt^{-1}) \mid t \in T, r \in R\}.$$

Then $\langle Y, S \rangle$ is a presentation for our subgroup H .

Remark A.3.2. In our examples we refer to T as Schreier transversal for H in G .

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