Universidad Autónoma de Madrid

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Contributions relating graph coloring to topics in ergodic theory and arithmetic combinatorics

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Chapter 1

Introducción

En este capítulo, introducimos y motivamos los resultados que constituyen esta tesis doctoral. Comenzamos con algunos resultados clásicos para introducir diferentes formas de colorear grafos, y luego describimos el contenido de cada capítulo siguiente. Los capítulos 3, 4 y 5 de esta tesis son trabajos ya publicados o en proceso de revisión por pares en diferentes revistas [11–13].

1.1 Coloración de mapas, problemas de planificación y flujos de tráfico

Como se relata en [7], en 1852 el matemático Francis Guthrie observó que, utilizando sólo cuatro colores, podía colorear un mapa de los condados de Inglaterra de forma que los condados vecinos recibieran colores diferentes. Esto le llevó a conjeturar lo que ahora es el conocido *teorema de los cuatro colores*, a saber, que *todo* mapa se puede colorear, usando a lo sumo cuatro colores, de forma que cualquier par de territorios limítrofes tenga colores distintos. Francis Guthrie intentó sin éxito demostrar su conjetura. Su hermano Frederick, también matemático, compartió la conjetura con Augustus De Morgan, quien era su profesor en ese momento. De Morgan planteó el problema a William Rowan Hamilton (Cf. figura 1.1), quien aparentemente no mostró mucho interés en el problema. Décadas más tarde, en 1878, en una reunión de la Sociedad Matemática de Londres, Arthur Cayley reanimó el problema de los cuatro colores preguntando si había sido resuelto. El propio Cayley no consiguió resolverlo [16]. Finalmente, tras varias pruebas falsas (y falsas refutaciones) a lo largo del siguiente siglo desde su formulación, la conjetura fue finalmente confirmada en 1976 por Appel y Haken [2]. Aunque esta prueba fue controvertida en su momento debido a su uso de ordenadores, en la actualidad es generalmente aceptada.

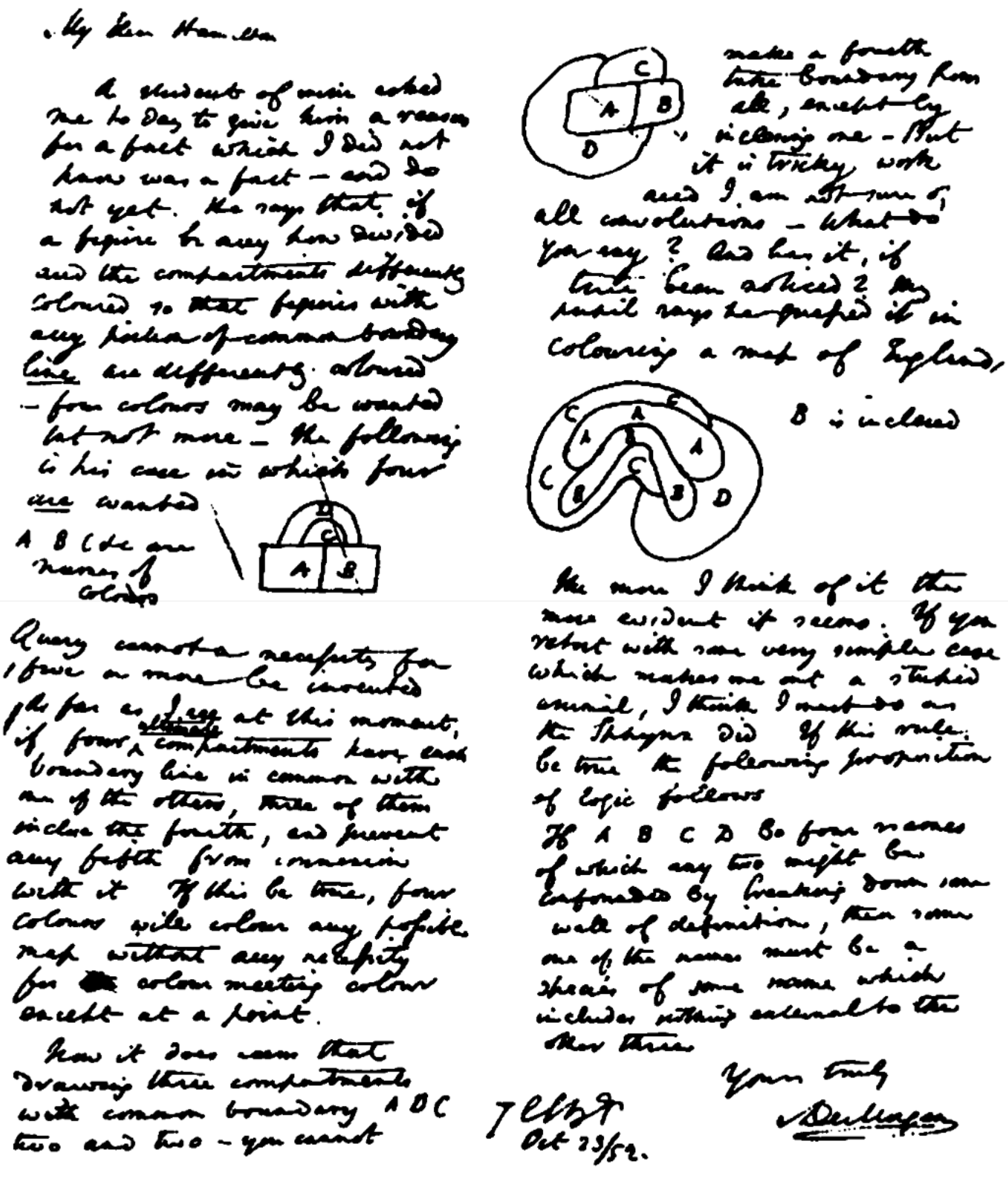
El bello y paradigmático teorema de los cuatro colores proporciona una excelente motivación para recordar los fundamentos de la teoría de grafos.

Un *grafo* G es un par (V, E), donde V es un conjunto, cuyos elementos se llaman los *vértices* de G, y E es un conjunto de pares de elementos de V, es decir, subconjuntos de V de cardinalidad 2, llamados las *aristas* de G. Podemos denotar el conjunto de todos los pares de elementos de V por $\binom{V}{2}$, y escribir por tanto $E \subset \binom{V}{2}$. De dos vértices que forman una arista, se dice que están "unidos por una arista", o que son "adyacentes", o "vecinos" en G. A lo largo de esta tesis, a menos que se indique lo contrario, asumimos que V es *finito*, que ningún vértice es adyacente a sí mismo, que no hay aristas múltiples (es decir, no más de una arista entre dos vértices), y que G es no-dirigido (es decir, las aristas no tienen orientación). Las cardinalidades $|V|$ y $|E|$ se llaman el *orden* y el *tamaño* de G respectivamente.

El teorema de los cuatro colores se formula fácilmente en términos de grafos. El conjunto de regiones de un mapa puede representarse como un grafo que tiene un vértice para cada región y una arista para cada par de regiones limítrofes (i.e. que comparten una parte de su frontera). Este grafo se dibuja en el plano, dibujando un vértice en cada región y dibujando las aristas como curvas que van desde cada vértice hasta el vértice de cada región adyacente, a través de la parte compartida de la frontera (Cf. las figuras 1.2 y 1.3). Un tal grafo, llamado *grafo plano* (o *grafo planar*), se caracteriza por esta posibilidad de dibujarlo en un plano sin que se cruce ninguna arista con otra. El teorema afirma que los vértices de todo grafo plano pueden ser coloreados con *a lo sumo* cuatro colores de manera que no haya dos vértices adyacentes que reciban el mismo color (Cf. figura 1.4).

De este modo, llegamos a la noción clásica de coloración de grafos, que formulamos de la siguiente manera.

Definición 1.1 (k-coloración de grafos)**.** Sea G un grafo, y sea k un entero positivo. Una *k-coloración* de G es una función $f : V(G) \to \{0, 1, \ldots, k-1\}$ tal que para toda arista $xy \in E(G)$ se tiene $f(x) \neq f(y)$.



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two – you cannot make a fourth take boundary from all, except by inclosing one – But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly

ADeMorgan

Oct 23/52.

Fig. 1.1: Carta de De Morgan a Hamilton. (Wikimedia Commons. Dominio público)

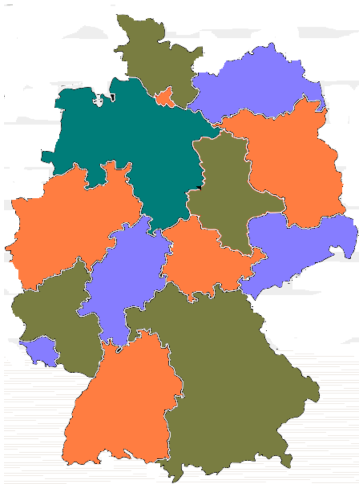


Fig. 1.2: Mapa simplificado de Alemania. (Wikimedia Commons. Dominio público)

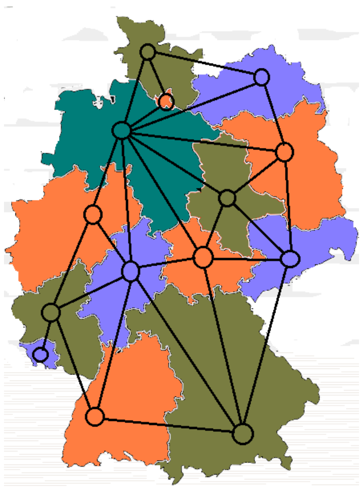


Fig. 1.3: Grafo asociado al mapa de Alemania. (Wikimedia Commons. Dominio público)

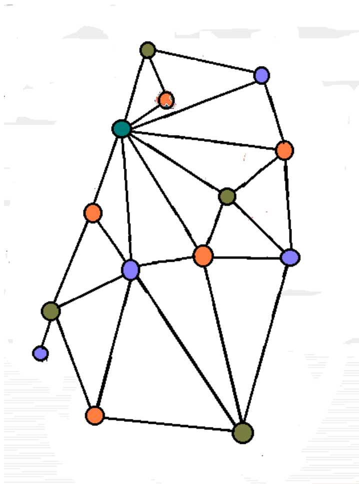


Fig. 1.4: Coloración óptima del grafo. (Wikimedia Commons. Dominio público)

Obviamente, siempre es posible colorear G con $k = \left|V\right|$ colores distintos. Una pregunta natural es entonces, dado un grafo G, cuál es el número mínimo k tal que G admite una k-coloración. Dicho número óptimo es el *número cromático* ("ordinario", o "clásico") de G.

Aquí y en el resto de la tesis, denotamos por $\mathbb{N}$ el conjunto de enteros positivos.

Definición 1.2. Sea G un grafo. Definimos el *número cromático* de G, denotado por $\chi(G)$, como sigue:

$$\chi(G) := \min \left\{k \in \mathbb{N} : G \text{ admite una } k\text{-coloración}\right\}.$$

Así pues, el teorema de los cuatro colores es el hecho que para todo grafo plano G (i.e. un grafo que puede dibujarse en un plano sin cruces de aristas) se tiene $\chi(G) \leq 4$.

El problema de determinar $\chi(G)$ para un grafo general G es altamente no-trivial, y se sabe que es difícil también desde el punto de vista computacional. Ilustramos el concepto en algunos casos sencillos, en las figuras 1.5, 1.6 y 1.7.

Otra motivación para el estudio de coloraciones de grafos, que conduce a un refinamiento del número cromático clásico, es el problema de la programación óptima de reuniones.

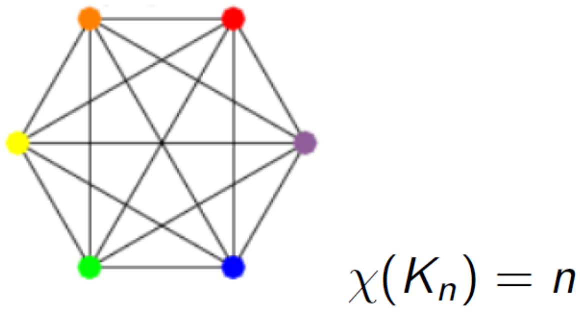


Fig. 1.5: El grafo completo K_5. En general, el grafo completo K_n ($n \in \mathbb{N}$) es el grafo de orden n con $E = \binom{V}{2}$. (Wikimedia Commons. Dominio público)

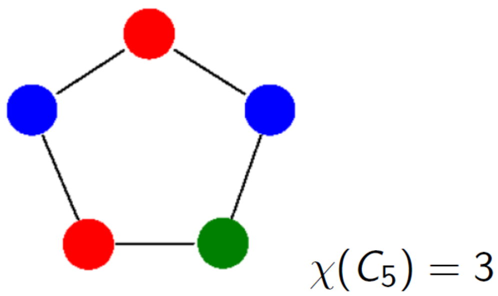


Fig. 1.6: El ciclo (o grafo cíclico) C_5. Para ciclos C_n con $n \geq 3$ entero, tenemos $\chi(C_n) = 2$ si n es par, y $\chi(C_n) = 3$ si n es impar. (Wikimedia Commons. Dominio público)

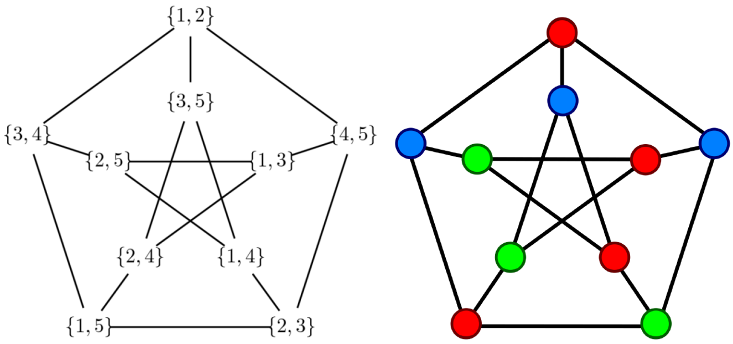


Fig. 1.7: El *grafo de Petersen* $K(5, 2)$. Es un caso particular de *grafo de Kneser* $K(n, k)$, para enteros positivos n, k con $n \geq 2k$. Los vértices de $K(n, k)$ son los k-conjuntos (i.e. subconjuntos de cardinalidad k) del conjunto $[n] := \{1, 2, \ldots, n\}$, es decir $V(K(n, k)) = \binom{[n]}{k}$. Dos tales vértices son vecinos en este grafo si los subconjuntos correspondientes son disjuntos. El número cromático del grafo de Kneser es $\chi(K(n, k)) = n - 2k + 2$, como establece un célebre teorema de Lovász. (Wikimedia Commons. Public Domain)

Siguiendo un ejemplo de [59], supongamos que hay que programar cinco reuniones de comités, cada una de 1 hora de duración. Si dos comités distintos tienen un miembro en común, no pueden reunirse al mismo tiempo. Podemos preguntar por la longitud del intervalo de tiempo más corto en el que se pueden programar todas las reuniones. Sea G el grafo cuyos vértices son los comités, y en el cual dos vértices son adyacentes si los comités correspondientes no pueden reunirse simultáneamente. Así, el grafo G representa los conflictos de programación.

La solución obvia a este problema es que la longitud del intervalo de tiempo más corto viene dada por $\chi(G)$. Supongamos que dicho grafo es el 5-ciclo, es decir $G = C_5$ (representado en la figura 1.6). Dado que $\chi(C_5) = 3$, la programación puede hacerse en 3 horas. Podemos preguntarnos si la programación se puede mejorar (i.e. acortar). Y efectivamente, se puede mejorar: la programación se puede hacer en 2,5 horas si permitimos que un comité se reúna durante media hora, y más tarde reanude su reunión durante la media hora restante, tras una interrupción. Así, es posible acortar el tiempo total de programación si permitimos que las reuniones se dividan en fracciones. El tiempo más corto necesario para programar las reuniones cuando se permiten tales divisiones no es el número cromático clásico $\chi(G)$,

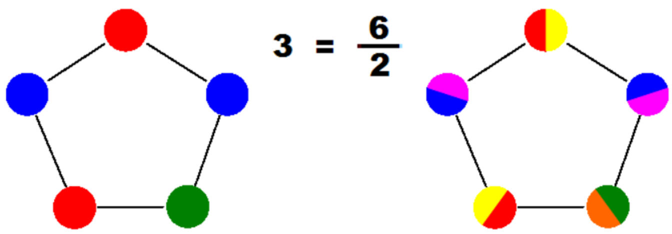


Fig. 1.8: Una $(3,1)$-coloración y una $(6,2)$-coloración de C_5.

sino el *número cromático fraccionario*, definido formalmente a continuación, y denotado por $\chi_f(G)$. En el ejemplo del 5-ciclo se puede ver, en efecto, que $\chi_f(C_5) = 2,5$ (véase la figura 1.9), lo cual muestra que $\chi_f(G)$ puede ser estrictamente menor que $\chi(G)$.

Este ejemplo ilustra el hecho que la noción de coloración de grafos puede ser refinada de maneras muy útiles. El refinamiento ilustrado en el anterior párrafo, conocido como *coloración fraccionada*, es una noción central en el desarrollo de la llamada *teoría fraccionaria de grafos*, tratada por Scheinerman y Ullman en [59] y por Berge en [5]. Según [59, §3.11], la primera publicación en la que aparece el número cromático fraccionario es [36].

Definición 1.3. (Cf. [59, §3.1]) Sea G un grafo. Una *b-coloración* de G es una asignación, a cada vértice de G, de un conjunto de b colores, de modo que los vértices adyacentes reciban conjuntos *disjuntos* de colores. Decimos que G tiene una *(d,b)-coloración* si G tiene una b-coloración en la que los colores se extraen de una paleta de $d \geq 1$ colores; es decir, si hay una función $f : V(G) \mapsto \binom{[d]}{b}$ tal que para cada arista $xy \in E(G)$ tenemos $f(x) \cap f(y) = \emptyset$.

Las figuras 1.8 y 1.9 muestran ejempos de una $(3,1)$-coloración y de una $(6,2)$-coloración, así como una $(5,2)$-coloración (realmente fraccionada).

Definición 1.4. (Número cromático fraccionario) Sea G un grafo. Definimos el número b-cromático de G por la fórmula

$$\chi_b(G) := \min \big\{ d \in \mathbb{N} : G \text{ admite una } (d,b)\text{-coloración} \big\}.$$

(Nótese que $\chi_1(G) = \chi(G)$.) Definimos entonces el *número cromático fraccionario* $\chi_f(G)$ como sigue:

$$\chi_f(G) := \lim_{b \to \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$

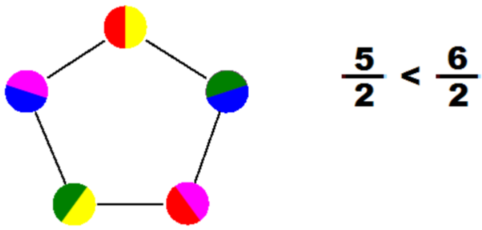


Fig. 1.9: Una $(5, 2)$-coloración de C_5. (GTBacchus, CC BY-SA 3.0, Wikimedia Commons)

La convergencia de la sucesión $\left\{\frac{\chi_n(G)}{n}\right\}_{n\geq 1}$ está garantizada por un lema básico de subaditividad (Cf. [59, Apéndice A.4]), ya que siempre tenemos $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$. De hecho, se puede ver [59] que para todo grafo G no vacío (es decir, con $E(G) \neq \emptyset$) hay un número entero positivo b tal que $\chi_f(G) = \chi_b(G)/b \geq 2$.

La determinación de $\chi_f(G)$ para un grafo general G también es de alta complejidad (como la de $\chi(G)$). No obstante, para la amplia clase de grafos G que son *vértice-transitivos*,[1] se sabe que

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)},$$

donde $\alpha(G)$ es el *número de independencia*[2] de G. En particular, los ciclos son vértice-transitivos y $\alpha(C_{2m+1}) = m$, y se deduce que $\chi_f(C_{2m+1}) = 2 + m^{-1}$. Además, los grafos de Kneser $K(n, k)$ (para n, k enteros positivos con $n \geq 2k$; Cf. figura 1.7) también son vértice-transitivos, y tenemos $\alpha(K(n, k)) = \binom{n-1}{k-1}$ y $\chi_f(K(n, k)) = n/k$.

Alternativamente, la coloración de un grafo G en el sentido clásico puede verse como un problema de *programación lineal entera*, en el que se asignan pesos 0 ó 1 a los conjuntos independientes en G, de forma que cada vértice pertenezca a conjuntos independientes cuyo peso total sea al menos 1 y la suma de los pesos de todos los conjuntos independientes esté minimizada. Desde este punto de vista, la coloración fraccionada es una *relajación lineal* de este problema de optimización: el número cromático fraccionario $\chi_f(G)$

[1] Un grafo es *vértice-transitivo* si su grupo de automorfismos actúa transitivamente sobre su conjunto de vértices.

[2] Un *conjunto independiente* en G es un conjunto $A \subset V(G)$ tal que cada par de vértices u, v en A cumple $uv \notin E(G)$. El *número de independencia* de G es la cardinalidad máxima de un conjunto independiente en G.

es el menor número real x para el cual existe una asignación de pesos no negativos a conjuntos independientes en G, tal que la suma de los pesos es x y cada vértice pertenece a conjuntos independientes cuyo peso total es al menos 1. Es bien sabido que siempre se alcanza el mínimo x en cuestión, y que $\chi_f(G)$ es siempre un número *racional* positivo [59].

Se puede demostrar (Cf. [64]) que el número cromático fraccionario es también equivalente a la siguiente definición, que utiliza la teoría de la medida.

Definición 1.5 (Número cromático fraccionario con teoría de la medida)**.** Para cualquier grafo G, tenemos $\chi_f(G) := \inf \big\{ r > 0 :$ para cada vértice v existe un conjunto medible $A_v \subseteq [0, 1)$ con medida de Lebesgue $\mu(A_v) \geq 1/r$, tal que para cada $uv \in E(G)$, $A_u \cap A_v = \emptyset \big\}$.

Esta definición, al situar la coloración fraccionada en el contexto de conjuntos medibles en un intervalo, indica un marco general que abre la puerta a nuevas nociones de coloración de grafos.

En [62], [64], Zhu propuso una tal noción nueva, llamada *coloración circular*. Esta noción utiliza *intervalos abiertos* en el grupo circular $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Podemos ver $\mathbb{T}$ como el intervalo $[0, 1)$ con la operación de suma módulo 1. Tenemos entonces la siguiente definición.

Definición 1.6 (Número cromático circular)**.** Sea G un grafo. El *número cromático circular* de G se define como sigue: $\chi_c(G) := \inf \big\{ r > 0 :$ para cada vértice x existe un intervalo abierto $A_x \subseteq \mathbb{T}$ con $\mu(A_x) \geq 1/r$ tal que para toda arista $xy \in E(G)$ se tiene $A_x \cap A_y = \emptyset \big\}$.

El número cromático circular también puede considerarse en términos de programación de reuniones. En efecto, este número es la longitud mínima t de tiempo total necesario para llevar a cabo una planificación de reuniones de comités, donde todas las reuniones deben durar 1 hora ininterrumpida, pero viendo esta hora módulo t, es decir, permitiendo que una reunión ocupe la unión de dos intervalos de tiempo $(0, a) \cup (b, t)$ (con $0 < a < b$ y $a + t - b = 1$), a saber, un intervalo al principio del periodo de reuniones y otro al final. Esta noción, diferente de la coloración clásica de grafos, es especialmente adecuada para los problemas de programación con condiciones *periódicas*. El siguiente ejemplo [64] es instructivo. Consideremos el problema de organizar un sistema de semáforos para regular de forma óptima el tráfico de vehículos en un cruce de carreteras. Un *periodo completo de tráfico* es un intervalo de tiempo durante el cual cada flujo de tráfico posible debe tener un turno de luz verde, siendo cada uno de estos turnos de igual duración, tomando esta duración de longitud 1. Este sistema se modeliza fácilmente mediante un grafo G, cada uno de cuyos vértices representa un flujo de tráfico, con cada arista representando un par de flujos de tráfico que son incompatibles, es decir, cuyos intervalos de luz verde no deben solaparse. El problema consiste en encontrar la duración mínima de un periodo completo de tráfico en este cruce vial.

Una solución que podemos dar a este problema es dividir $V(G)$ en un número mínimo k de conjuntos independientes $I_1, I_2, ..., I_k$ y asignar intervalos de tiempo unitarios sucesivos a cada conjunto independiente, obteniendo así un periodo de tráfico completo de duración total $k = \chi(G)$. A primera vista, el problema queda así resuelto. Sin embargo, si el grafo satisface la desigualdad estricta $\chi_c(G) < \chi(G)$, entonces esta solución no será óptima, y las coloraciones circulares (que utilizan la periodicidad adicional) darán una solución estrictamente mejor (Cf. Sección 4 en [28]). En particular, un resultado de Guichard [30] muestra que si un grafo G es n-crítico[3], y tiene circunferencia[4] al menos $n + 1$, entonces $\chi_c(G) < \chi(G)$.

Se puede dar una definición equivalente de $\chi_c(G)$ que es más similar a la de coloraciones clásicas (Definición 1.1), utilizando la siguiente noción [64].

Definición 1.7 (Coloración r-circular)**.** Sea G un grafo y $r \geq 1$ un número real. Una *r-coloración circular* de G es una función $f : V(G) \to [0, r)$ tal que para cada arista $xy \in E(G)$ tenemos $1 \leq \big|f(x)-f(y)\big| \leq r-1$. Podemos entonces definir el número cromático circular de G como sigue:

$$\chi_c(G) := \inf\big\{r \geq 1 : G \text{ admite una } r\text{-coloración circular}\big\}.$$

Obsérvese que si f es una k-coloración de G en el sentido clásico, entonces f es también una k-coloración circular de G, y por tanto $\chi_c(G) \leq \chi(G)$. Por otro lado, para una r-coloración circular $g : V(G) \to [0, r)$, siendo $s = \max\{g(x) : x \in V(G)\}$, podemos ver g como una $(s + 1)$-coloración clásica de G. Como $s < r$, se deduce el conocido resultado siguiente.

[3]Un grafo G es n-crítico si $\chi(G) = n$ y $\chi(G - v) = n - 1$ para cualquier vértice v de G, donde $G - v$ denota el grafo G al cual se retira el vértice v y todas las aristas que contienen a v.

[4]La circunferencia de un grafo es la longitud de su ciclo simple más corto.

Teorema 1.8. *Para cualquier grafo G tenemos*

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G), \tag{1.1}$$

en particular $\chi(G) = \lceil \chi_c(G) \rceil$.

Esto demuestra que $\chi_c(G)$ contiene más información sobre la estructura de G que $\chi(G)$, de modo que el número cromático circular puede utilizarse para cuantificar lo lejos que está G de poder colorearse con menos de $\chi(G)$ colores. En este sentido, se puede ver el parámetro $\chi_c(G)$ como un refinamiento de $\chi(G)$.

En los Capítulos 3 y 4 de esta tesis, introducimos nuevas nociones de coloración de grafos que conducen a refinamientos adicionales de $\chi_c(G)$ y $\chi_f(G)$, dando lugar a nuevos números cromáticos con propiedades interesantes. Detallamos los principales contenidos de estos capítulos en las dos secciones siguientes.

1.2 Coloración de grafos mediante traslaciones en el círculo: el número girocromático

Históricamente, la primera definición del número cromático circular fue dada por Vince en 1988 [61], con el nombre de *número cromático estelar* (en relación con grafos estelares). Esta noción se puede formular en un entorno discreto como sigue.

Definición 1.9 (Número cromático estelar)**.** Sean $1 \leq b \leq d$ enteros y sea G un grafo. Una (d, b)-$\star$-coloración de G es una función $f : V(G) \to [d]$ tal que para cada arista xy tenemos

$$b \leq |f(x) - f(y)| \leq d - b.$$

Definimos el *número cromático estelar* de G por la fórmula

$$\chi^\star(G) := \inf\left\{\frac{d}{b} : G \text{ admite una } (d, b)\text{-}\star\text{-coloración}\right\}.$$

Entre las principales propiedades básicas del número cromático estelar, destacamos dos hechos: primero, como demostró el propio Vince [61], el infimum en la definición de $\chi^\star(G)$ se alcanza para todo grafo G, y es por tanto un mínimo; segundo, el hecho (consecuencia del anterior) que $\chi^\star(G)$ es siempre un número racional (Cf. Vince [61] y Bondy–Hell [8]). [5]

Posteriormente, Zhu [62], [64] observó que su número cromático circular es efectivamente igual al número cromático estelar.

Lema 1.10 (Zhu)**.** *Para todo grafo G tenemos $\chi^\star(G) = \chi_c(G)$.*

Incluimos la breve prueba aquí.

Prueba. Supongamos que f es una (d,b)-$\star$-coloración de G. Definamos la función $g : V(G) \to [0, d/b)$ tal que $g(x) := f(x)/b$. Para cada arista xy de G tenemos $1 \leq |g(x) - g(y)| \leq \frac{d}{b} - 1$, por lo que toda (d,b)-$\star$-coloración de G corresponde a una (d/b)-coloración circular de G. Por otro lado, si g es una (d/b)-coloración circular de G, entonces $f(x) := \lfloor b \cdot g(x) \rfloor$ es una (d,b)-$\star$-coloración de G. □

A partir de las definiciones de $\chi_f(G)$ y $\chi_c(G)$ en términos de coloraciones (d,b) y (d,b)-$\star$, vemos sin dificultad las siguientes desigualdades.

Lema 1.11. *Todo grafo G satisface $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$.*

Ambas desigualdades pueden ser estrictas, y la diferencia entre $\chi_f(G)$ y $\chi_c(G)$ puede ser arbitrariamente grande. Por ejemplo, como mencionamos anteriormente, el grafo de Kneser $G = K(n,k)$ $(n \geq 2k)$ cumple $\chi(G) = n - 2k + 2$ y $\chi_f(G) = \frac{n}{k}$, y se sabe también [18] que $\chi_c(G) = \chi(G)$.

El punto de partida del Capítulo 3 fue el trabajo de Avila y Candela [3], que incluye aplicaciones combinatorias de herramientas centrales de la teoría ergódica. Estas aplicaciones condujeron a un nuevo parámetro cromático para grafos, que se encuentra entre el número cromático circular y el fraccionario. El propósito de este capítulo es estudiar este nuevo parámetro, que llamamos *número girocromático*. Este número es el recíproco de la cantidad natural definida a continuación, donde $\mathbb{T}$ denota como de costumbre el grupo circular $\mathbb{R}/\mathbb{Z}$.

Definición 1.12 (Avila-Candela, 2016)**.** Sea G un grafo. Decimos que un conjunto de Borel $A \subseteq \mathbb{T}$ es una $\mathbb{T}$-*base de coloración* para G si existe un

[5] Existen otras descripciones equivalentes del número cromático estelar, en términos de homomorfismos hacia cierta clase de grafos [8], que también implican fácilmente la racionalidad de $\chi^\star(G)$.

mapeo $f : V(G) \to \mathbb{T}$ tal que para cada arista xy de G tenemos $\big(A + f(x)\big) \cap \big(A + f(y)\big) = \emptyset$. Decimos entonces que (A, f) es una $\mathbb{T}$-coloración (o $\mathbb{T}$-*girocoloración*) de G. Definimos

$$\sigma_{\mathbb{T}}(G) := \sup\{\mu_{\mathbb{T}}(A) : A \text{ es una } \mathbb{T}\text{-base de coloración para } G\}, \tag{1.2}$$

donde $\mu_{\mathbb{T}}$ denota la medida de probabilidad de Haar sobre $\mathbb{T}$.

Definición 1.13 (Número girocromático)**.** Para cualquier grafo G definimos $\chi_g(G) := 1/\sigma_{\mathbb{T}}(G)$.

Nótese que la noción $\sigma_{\mathbb{T}}(G)$ se generaliza fácilmente sustituyendo $\mathbb{T}$ por cualquier grupo abeliano compacto Z. Para cualquier grupo de este tipo, dotado de una medida de probabilidad de Haar μ_Z, podemos definir en efecto

$$\sigma_Z(G) := \sup\{\mu_Z(A) : A \text{ es una } Z\text{-base de coloración para } G\}, \tag{1.3}$$

siendo obvia la extensión de la noción de $\mathbb{T}$-base a la de Z-base.

Una observación básica sobre el número girocromático es la siguiente.

Proposición 1.14. *Todo grafo finito G satisface $\chi_f(G) \leq \chi_g(G) \leq \chi_c(G)$.*

Prueba. Para la primera desigualdad, supongamos que (A, f) con $A \subseteq \mathbb{T}$ Borel es una $\mathbb{T}$-coloración de G. Entonces también es una coloración fraccionada de G con $A_x := A + f(x)$ para cada vértice x (como mencionamos anteriormente, solemos identificar $\mathbb{T}$ con el intervalo $[0, 1)$ dotado de la operación de suma módulo 1). Para la segunda desigualdad, si existe una coloración por intervalos abiertos (que son conjuntos de Borel) A_x, todos de la misma longitud, entonces es obviamente una $\mathbb{T}$-coloración ya que cada intervalo abierto A_x puede verse como una traslación de un único intervalo A (de modo que $A_x =: A + f(x)$ satisface $A_x \cap A_y = \emptyset$ para toda $xy \in E(G)$). $\square$

Informalmente, digamos que la coloración circular asigna arcos (intervalos en el círculo) a los vértices de G, mientras que la coloración fraccionada les asigna conjuntos de Borel en $[0, 1)$. Una $\mathbb{T}$-girocoloración es más estricta que la coloración fraccionada ya que a los vértices les debe asignar copias giradas de *un mismo* subconjunto de Borel; por otro lado es más flexible que la coloración circular, ya que el subconjunto de Borel asignado a los vértices no tiene por qué ser un arco.

Una de las motivaciones que condujeron a este nuevo parámetro de coloración de grafos es una versión de un problema paradigmático en la combinatoria aritmética. El problema general consiste en determinar el mayor tamaño que puede tener un subconjunto de un grupo abeliano si se exige que este conjunto no contenga soluciones a ecuaciones lineales prescritas. En la versión particular en cuestión, tratada en [3], se consideran números enteros $c_1, c_2, \ldots, c_d$ no nulos y *multiplicativamente independientes*[6], se fija un grafo G cualquiera con conjunto de vértices $V = [d]$, y se pregunta entonces cuál es el mayor tamaño (medida de Haar) que puede tener un subconjunto de $\mathbb{T}$ si exigimos que este conjunto evite las soluciones de las ecuaciones lineales $c_i\, x_1 = c_j\, x_2$ para toda arista ij en G. En otras palabras, preguntamos cuál es el valor de

$$d_{G,c_i} := \sup \left\{ \mu_{\mathbb{T}}(A) : A \subset \mathbb{T},\ \forall\, ij \in E(G),\ \forall\, x_1, x_2 \in A,\ c_i\, x_1 \neq c_j\, x_2 \right\}.$$

En [3], se demuestra que $d_{G,c_i} = \sigma_{\mathbb{T}}(G)$.

En el Capítulo 3, demostramos que la definición del número girocromático, al igual que los números cromáticos circulares y fraccionarios, es *robusta*, en el sentido de que puede ser reformulada de varias formas que son equivalentes. Por ejemplo, la definición a través de $\mathbb{T}$-bases de coloración tiene la siguiente variante equivalente en ámbito discreto.

Teorema 1.15. *Sea G un grafo. Entonces $\sigma_{\mathbb{Z}_N}(G) \leq \sigma_{\mathbb{T}}(G)$ para cada $N \in \mathbb{N}$, y tenemos*

$$\sigma_{\mathbb{T}}(G) = \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G).$$

Esto da lugar a la siguiente reformulación.

Corolario 1.16. *El número girocromático de un grafo G es igual al infimum de los números N/K para los cuales existe un conjunto $A \in \binom{\mathbb{Z}_N}{K}$ y una función $f : V(G) \to \mathbb{Z}_N$ tal que $(A + f(u)) \cap (A + f(v)) = \emptyset$ para cada arista $uv \in E(G)$.*

Recordemos el hecho, bien conocido, que la coloración clásica, la coloración circular y la coloración fraccionada de grafos pueden definirse utilizando homomorfismos hacia ciertas clases de grafos: grafos completos

[6] Esto significa simplemente que si $c_1^{\lambda_1} \cdot c_2^{\lambda_2} \cdots c_d^{\lambda_d} = 1$ con $\lambda_i \in \mathbb{Z}$, entonces $\lambda_i = 0$ para cada i

(cliques), cliques circulares, y grafos de Kneser, respectivamente. Demostramos que el número girocromático también sigue este patrón: en efecto, se puede definir en términos de homomorfismos a *grafos circulantes*. [7]

Teorema 1.17. *El número girocromático de un grafo G es igual a*

$$\inf_{\substack{N\in\mathbb{N}, S\subseteq\mathbb{Z}_N\\ G\to C(N,S)}} \chi_f\left(C(N,S)\right) = \inf_{\substack{N\in\mathbb{N}, S\subseteq\mathbb{Z}_N\\ G\to C(N,S)}} \frac{N}{\alpha\left(C(N,S)\right)},$$

es decir, es igual al infimum de los números cromáticos fraccionarios de los grafos circulantes que admiten un homomorfismo desde G.

Otra característica notable del número girocromático es que se puede definir de forma equivalente utilizando toros de dimensión arbitrariamente alta.

Teorema 1.18. *Para todo grafo G, para todo $d \in \mathbb{N}$ se cumple*

$$\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G).$$

Este resultado implica que la variante discreta de la definición (dada por el Teorema 1.15) es equivalente a la versión donde se consideran *todos* los grupos abelianos finitos en lugar de solamente los grupos $\mathbb{Z}_N$ (Cf. Corolarios 3.10 y 3.11). Esto muestra una cierta universalidad del número girocromático. Obsérvese que (como se detalla en el Capítulo 3) el propio número cromático fraccionario se puede considerar de una forma similar, pero abarcando *todos* los grupos finitos (no solo los abelianos). De este modo, el número girocromático puede verse como una versión del número cromático fraccionario restringida a los grupos abelianos.

Entre otros resultados sobre el número girocromático, damos la siguiente construcción de grafos que satisfacen las desigualdades estrictas $\chi_f(G) < \chi_g(G) < \chi_c(G)$.

Teorema 1.19. *Existe una sucesión de grafos $(G_k)_{k\in\mathbb{N}\setminus\{1\}}$ tal que $\chi_f(G_k) < \chi_g(G_k) < \chi_c(G_k) = k+2$, y con $\lim_{k\to\infty} \chi_g(G_k) = 2$.*

[7] Un *grafo circulante* $G = C(n,S)$ es un grafo de Cayley sobre el grupo cíclico $V(G) = \mathbb{Z}_n$, generado por un conjunto $S \subset \mathbb{Z}_n$, que suponemos *simétrico* (es decir, $S = -S$); así, dos vértices $a, b \in \mathbb{Z}_n$ son adyacentes si y sólo si $a - b \in S$ (equivalentemente $b - a \in S$).

Por último, demostramos que, de forma un tanto sorprendente, no es siempre posible alcanzar el supremum en la definición (1.2), lo cual implica también que no siempre se alcanza el infimum en la definición discreta dada por el Teorema 1.17. Esto conduce al problema abierto de si existe un grafo finito G tal que el número girocromático de G no sea racional. Proponemos este y otros problemas abiertos, entre otras observaciones finales en el capítulo.

Consideramos que estos resultados aportan pruebas convincentes de que el número girocromático es un parámetro natural y robusto, de utilidad para profundizar y afinar nuestra comprensión de la estructura de los grafos.

1.3 Números cromáticos torales de grafos

En el Capítulo 3 introducimos el número girocromático de un grafo G, y varios resultados en dicho capítulo muestran que $\chi_g(G)$ contiene más información que los números cromáticos fraccionario y circular. En particular, demostramos que se cumple $\chi_f(G) \leq \chi_g(G) \leq \chi_c(G)$ para cualquier grafo G, y que estas desigualdades pueden ser estrictas. También se demuestra que el número girocromático tiene cierta universalidad en el sentido de que el supremum en su definición original puede extenderse a conjuntos de Borel en un toro de dimensión finita arbitraria, *sin cambiar el valor* de $\chi_g(G)$. Es decir que para cualquier $r \in \mathbb{N}$ tenemos

$$\chi_g(G) = \inf\{1/\mu_{\mathbb{T}^r}(A) : A \text{ es una } \mathbb{T}^r\text{-base de coloración de } G\}. \tag{1.4}$$

Sin embargo, el número girocromático es más "escurridizo" que el número cromático circular $\chi_c(G)$, ya que también se demostró en el Capítulo 3 que el infimum en la definición original (es decir en (1.4) con $r = 1$) no siempre se alcanza, y aún no sabemos si $\chi_g(G)$ es siempre racional. Peor aún, no sabemos si siempre hay al menos una dimensión finita r tal que el infimum se alcanza en *esta* dimensión.

Esto aporta motivación para estudiar nociones de coloración intermedias, que también refinan $\chi_c(G)$ pero que son analíticamente más manejables que el número girocromático. Este es el objetivo principal del Capítulo 4.

Un candidato natural para un refinamiento más manejable de $\chi_c(G)$ consiste en colorear el grafo con traslaciones de una *caja* en el toro d-dimensional $\mathbb{T}^d$. Podemos ver este toro como $[0, 1]^d$ con operación de suma mod 1 en cada coordenada. Por una *caja abierta* en $\mathbb{T}^d$ entendemos un producto cartesiano de la forma $I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ donde I_j es un intervalo abierto (conjunto conexo abierto) en $\mathbb{T}$ para cada $j \in [d]$. El correspondiente refinamiento del número cromático circular se define entonces como sigue.

Definición 1.20. Sea G un grafo. Para cada $d \in \mathbb{N}$, definimos el *número cromático d-toral* de G, denotado por $\chi_{c^d}(G)$, por la fórmula

$$\chi_{c^d}(G) = \inf \big\{ 1/\mu_{\mathbb{T}^d}(R) : R \subset \mathbb{T}^d \text{ una caja abierta, } G \text{ es } R\text{-coloreable} \big\}, \tag{1.5}$$

donde, para un subconjunto A de un grupo abeliano Z, decimos que un grafo G es *A-coloreable* si existe un mapeo $f : V(G) \to \mathrm{Z}$ tal que $\big(A + f(x)\big) \cap \big(A + f(y)\big) = \emptyset$ para cada arista $xy \in E(G)$. (Podemos decir entonces que f es un *mapeo de coloración* de G por A.)

El número cromático 1-toral es el número cromático circular habitual. Demostramos en el Capítulo 4 que se da la siguiente sucesión de desigualdades para cualquier número entero positivo d:

$$\chi_f(G) \leq \chi_g(G) \leq \chi_{c^{d+1}}(G) \leq \chi_{c^d}(G) \leq \chi(G). \tag{1.6}$$

También se demuestra que el infimum en (1.5) siempre se alcanza y es racional, igual que ocurre con $\chi_c(G)$:

Teorema 1.21. *Sea G un grafo de orden n, y sea $d \in \mathbb{N}$. Entonces, para cada $i \in [d]$ hay enteros $r_i \leq s_i$ en $[n]$ tales que G es coloreable por la caja $R = \prod_{i \ en [d]} (0, \frac{r_i}{s_i})$ en $\mathbb{T}^d$ y se tiene $\chi_{c^d}(G) = \frac{1}{\mu_{\mathbb{T}^d}(R)} = \frac{s_1 \cdots s_d}{r_1 \cdots r_d}$.*

Notemos que para todo grafo G, se deduce de (1.6) que la sucesión decreciente $(\chi_{c^d}(G))_{d \in \mathbb{N}}$ debe converger, y podemos preguntarnos entonces cuán rápida es esta convergencia, e incluso si la sucesión siempre se vuelve constante a partir de cierto punto. Resolvemos estas preguntas con el siguiente resultado.

Proposición 1.22. *Sea G un grafo, y sea $d = \lfloor \log_2(\chi(G)) \rfloor$. Entonces, para cada $d' \geq d$ tenemos $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$.*

También podemos preguntarnos cómo varía $\chi_{c^d}(G)$ para un d fijo y G variable. En particular, podemos preguntarnos si para cada d fijo existe un grafo G tal que $\chi_{c^{d+1}}(G) < \chi_{c^d}(G)$; una respuesta positiva indicaría que cada número $\chi_{c^d}(G)$ contiene información sobre G que permite distinguir este número de los demás $\chi_{c^{d'}}(G)$, $d' > d$. En otras palabras, esto indicaría que los grafos separan los números cromáticos torales.

Nótese que, dada la Proposición 1.22, podemos definir razonablemente lo que llamaremos la *dimensión de estabilización* de un grafo G para los números cromáticos torales, a saber, el menor número entero d con la propiedad de que $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$ para todo $d' \geq d$. Denotando la dimensión de estabilización de G por $d^*(G)$, la Proposición 1.22 implica que $d^* \leq \lfloor \log_2(|G|) \rfloor$, y podemos preguntarnos cuán precisa es esta cota superior. Demostramos que esta cota es precisa módulo una constante multiplicativa, mediante el siguiente resultado, que también muestra que existen efectivamente grafos que separan los números cromáticos torales.

Teorema 1.23. *Para cada $d \in \mathbb{N}$ existe un grafo G de orden $n = 5^d$ que satisface $d^*(G) = d = \log_5(n)$.*

Por último, al final del Capítulo 4 relacionamos el número $\chi_{c^d}(G)$ con el número cromático ordinario $\chi(G)$, a través de una desigualdad que generaliza la conocida desigualdad (1.1), y que relaciona este tema con problemas conocidos de recubrimientos del toro $\mathbb{T}^d$ por cajas abiertas (Cf. la Proposición 4.21).

1.4 El problema de Motzkin en el círculo

El último capítulo de esta tesis está dedicado a un proyecto que surgió durante la investigación sobre el número girocromático. El proyecto concierne a un problema bastante conocido en la teoría combinatoria de números, que fue planteado por T. S. Motzkin en los años 1970.

El problema de Motzkin pregunta cuán grande puede ser un conjunto de números enteros si no contiene ningún par de elementos cuya diferencia se encuentre en un conjunto prescrito de "diferencias prohibidas". Más precisamente, dado un subconjunto no vacío D del conjunto de enteros positivos $\mathbb{N}$, digamos que un conjunto $A \subset \mathbb{Z}$ es *D-huidizo* si para cada $a, a' \in A$ tenemos $|a - a'| \notin D$, es decir, si el *conjunto de diferencias* $A - A = \{a - a' : a, a' \in A\}$ es disjunto de D. Sea $A(N)$ la cardinalidad $|A \cap [-N, N]|$, y sea $\bar{\delta}(A)$ la densidad superior de A, es decir $\bar{\delta}(A) = \limsup_{N \to \infty} \frac{A(N)}{2N+1}$. El problema de Motzkin plantea determinar o estimar la mayor densidad superior que puede tener un conjunto D-huidizo, es decir, la siguiente cantidad, que llamamos la *densidad de Motzkin* de D:

$$\mathrm{Md}_{\mathbb{Z}}(D) := \sup\{\bar{\delta}(A) : A \text{ subconjunto } D\text{-huidizo de } \mathbb{Z}\}. \tag{1.7}$$

Cantor y Gordon publicaron el primer artículo sobre el problema de Motzkin [14], demostrando varios resultados interesantes, en particular una solución completa para el caso de a lo sumo dos diferencias prohibidas (es decir, para $|D| \leq 2$). Desde entonces, el problema general ha motivado muchos trabajos y se han abordado varios casos especiales adicionales (como se detalla en el Capítulo 5). Aún así, el problema general sigue sin solución completa.

El problema de Motzkin constituye todo un tema dentro de la teoría combinatoria de números, que tiene interesantes conexiones con otros problemas conocidos, entre ellos el problema del número cromático fraccionario de los llamados *grafos de distancia*, o la conocida *conjetura del corredor solitario.*

El problema de Motzkin también puede verse como un caso particular de una cuestión más amplia, que puede plantearse también en cualquier grupo abeliano compacto Z: dado un conjunto no vacío $D \subset \mathrm{Z}$, denotando por μ_{Z} la medida de probabilidad de Haar sobre Z, esta cuestión pide determinar o estimar la cantidad

$$\mathrm{Md}_{\mathrm{Z}}(D) := \sup\{\mu_{\mathrm{Z}}(A) : A \subset \mathrm{Z} \text{ de Borel con } (A - A) \cap D = \emptyset\}. \tag{1.8}$$

Un caso especialmente natural de esta cuestión concierne al grupo circular $\mathrm{Z} = \mathbb{T}$. Este caso es el principal objeto de estudio del Capítulo 5. Consideramos $\mathbb{T}$ como el intervalo $[0, 1]$ con operación de suma módulo 1 (como de costumbre), y consideraremos un conjunto *finito* $D = \{t_1, \dots, t_r\}$ de diferencias no-nulas prohibidas, viendo D como un conjunto de números reales en $(0, 1)$.

El enfoque de este problema en el Capítulo 5 combina herramientas de teoría de grafos, teoría ergódica y geometría de números.

Un primer ejemplo de la aplicabilidad de la teoría ergódica se da en el caso en que el conjunto $D \cup \{1\}$ es linealmente independiente sobre $\mathbb{Q}$. Resulta entonces que podemos aplicar una versión de la importante herramienta de teoría ergódica conocida como el *lema de Rokhlin.* Este resultado se puede describir informalmente como una herramienta que permite aproximar, con precisión arbitraria, una acción no-periódica de un grupo sobre un espacio de probabilidad (e.g. las iteraciones de una rotación irracional del círculo), aproximando por estructuras, llamadas *torres*, que son casi periódicas y son mucho más sencillas de analizar (damos la formulación precisa en la sección 5.2). Mediante una aplicación de esta herramienta, para acciones libres de $\mathbb{Z}^r$ que preservan la medida, obtenemos la solución $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$ en este caso. Esto motiva la exploración de la aplicabilidad del lema de Rokhlin (y sus diversas extensiones) al caso más general del problema de Motzkin en $\mathbb{T}$, donde $D \cup \{1\}$ puede ser linealmente dependiente sobre $\mathbb{Q}$. De hecho, extensiones conocidas del lema de Rokhlin, aplicables a acciones libres de *cocientes* de $\mathbb{Z}^r$, demuestran ser efectivamente relevantes para este problema. En particular mostramos que, gracias a estas extensiones, el problema de determinar $\mathrm{Md}_{\mathbb{T}}(D)$ puede ser transferido a un problema similar en el entorno discreto del grupo abeliano finitamente generado $\mathbb{Z}^r/\Lambda$, donde Λ es el núcleo del homomorfismo $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1t_1+\cdots+n_rt_r$. En este entorno discreto, la densidad de Motzkin se puede definir utilizando sucesiones de Følner; véase la Definición 5.8. Tenemos entonces el siguiente resultado.

Teorema 1.24. *Sea $D = \{t_1, \dots, t_r\} \subset \mathbb{T}$, sea Λ el núcleo del homomorfismo $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1t_1 + \cdots + n_rt_r$, y sea E la imagen de la base estándar de $\mathbb{R}^r$ en el cociente $\mathbb{Z}^r/\Lambda$. Entonces $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$.*

De hecho, la versatilidad de dichas extensiones del lema de Rokhlin nos permite demostrar una versión de este teorema que es válida para grupos abelianos compactos más generalmente; véase el Teorema 5.9.

El Teorema 1.24 es útil como primer paso para determinar $\mathrm{Md}_{\mathbb{T}}(D)$, ya que la densidad de Motzkin correspondiente en el entorno discreto (es decir $\mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$) se puede a menudo determinar más fácilmente. En el Capítulo 5 aplicamos esta estrategia para $r \leq 2$, obteniendo las soluciones que se resumen a continuación, en particular una fórmula exacta dada en el Teorema 1.25.

Las técnicas de la teoría de grafos y de la geometría de los números entran en escena en relación con el caso del problema en el cual, en lugar de que $D \cup \{1\}$ sea linealmente independiente sobre $\mathbb{Q}$, suponemos al contrario que $D \subset \mathbb{Q}$. En efecto, este caso se reduce al problema de determinar el número de independencia de un grafo circulante que llamamos *grafo circulante asociado*. Más precisamente, suponiendo que cada elemento de D es de la forma $t_i = a_i/b_i$ con enteros positivos coprimos $a_i < b_i$, entonces el subgrupo $\langle D \rangle \leq \mathbb{T}$ es isomorfo a $\mathbb{Z}_N$ con $N = \mathrm{lcm}(b_1, \dots, b_r)$. El grafo circulante asociado es el grafo circulante conexo G con conjunto de vértices $\mathbb{Z}_N$ (visto como el conjunto de enteros $[0, N-1]$ con suma módulo N), con saltos $d_1, \dots, d_r$ donde $d_i = a_i N/b_i$. Es decir que $x, y \in \mathbb{Z}_N$ forman una arista en G si y sólo si $x - y = d_i$ o $-d_i \mod N$ para algún $i \in [r]$. La *ratio de independencia* de G es el número $\frac{\alpha(G)}{N}$, donde $\alpha(G)$ es el número de independencia de G. Como consecuencia directa del Teorema 1.24, obtenemos $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \dots, d_r\}) = \frac{\alpha(G)}{N}$. El análisis de la cantidad $\mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \dots, d_r\})$ conduce naturalmente a la geometría de números, por su relación (desarrollada en el Capítulo 5) con el retículo Λ mencionado anteriormente (que en este caso racional es un retículo de rango r).

Cabe señalar que si $d_1, \dots, d_r$ son enteros fijos, entonces, cuando $N \to \infty$, los cocientes $\frac{\alpha(G)}{N}$ convergen a $\mathrm{Md}_{\mathbb{Z}}(\{d_1, \dots, d_r\})$, es decir a la densidad de Motzkin de D *en los enteros*. En este sentido, se puede ver que el problema de Motzkin en $\mathbb{T}$ subsume el problema original en $\mathbb{Z}$.

Tras una breve solución del caso $r = 1$ (véase la Proposición 5.15), el resto del Capítulo 5 se centra en el caso $r = 2$. Distinguimos dos sub-casos. El *primer caso* es aquel en el que al menos un elemento de D es un número irracional. Aquí obtenemos la siguiente solución *exacta*:

Teorema 1.25. *Sea $D = \{t_1, t_2\} \subset (0, 1)$ con $D \subset \mathbb{Q}$. Si $D \cup \{1\}$ es linealmente independiente sobre $\mathbb{Q}$, entonces $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$. En caso contrario, siendo m_0, m_1, m_2 enteros no todos nulos tales que $m_0 = m_1 t_1 + m_2 t_2$ y $\gcd(m_0, m_1, m_2) = 1$, tenemos*

$$\mathrm{Md}_{\mathbb{T}}(D) = \frac{\lfloor k/2 \rfloor}{k}, \ \textit{donde } k = |m_1| + |m_2|. \tag{1.9}$$

Tras este resultado, nos concentramos en el *segundo caso*, en el cual ambos elementos de D son racionales. Esto equivale a determinar la ratio de independencia de los grafos circulantes con *dos* saltos. Como mencionamos anteriormente, estudiamos este problema utilizando herramientas de la geometría de números. En particular, obtenemos la siguiente estimación, asintóticamente exacta. Recordemos que la *circunferencia impar* de un grafo G es la longitud mínima de un ciclo de longitud *impar* en G.

Teorema 1.26. *Sea $D = \{t_1, t_2\} \subset \mathbb{Q} \cap (0, 1)$. Sea G el grafo circulante asociado, y sea N el orden de G. Si G es bipartito, entonces $\mathrm{Md}_{\mathbb{T}}(D) = \frac{\alpha(G)}{N} = \frac{1}{2}$. En caso contrario, denotando por k sea la circunferencia impar de G, tenemos*

$$\tfrac{k-1}{2k} \ \geq \ \mathrm{Md}_{\mathbb{T}}(D) = \tfrac{\alpha(G)}{N} \ \geq \ \tfrac{k-1}{2k} - \tfrac{3}{\sqrt{N}}. \tag{1.10}$$

La ratio de independencia de un grafo circulante G es igual al recíproco de su número cromático fraccionario $\chi_f(G)$. Por lo tanto (1.10) nos da también una estimación asintótica para el número cromático fraccionario de un grafo circulante conexo G de orden N con 2 saltos y circunferencia impar k, a saber $\frac{2k}{k-1} \leq \chi_f(G) \leq \frac{2k}{k-1} + \frac{27}{\sqrt{N}}$.

Por último, también estudiamos la cuestión de la precisión de las cotas en (1.10) para N *fijo* (no sólo cuando $N \to \infty$). En particular, proporcionamos la siguiente familia infinita de ejemplos de grafos circulantes de 2 saltos cuya ratio de independencia alcanza la cota inferior en (1.10), módulo un múltiplo constante absoluto de $1/\sqrt{N}$:

Proposición 1.27. *Sea $d \in \mathbb{N}$ impar, sea $N = 2d(d+1)$, y sea $G = \mathrm{Cay}(\mathbb{Z}_N, \{d, d+1\})$. Entonces $\alpha(G) = d^2$, y G tiene circunferencia $k = 2d+1$, con lo cual $\alpha(G) = \left\lfloor \frac{k-1}{2k} N \right\rfloor - \frac{d-1}{2}$.*

También construimos una familia infinita de ejemplos que alcanzan la cota superior en (1.10) (Cf. Proposition 5.29).

Al final del capítulo discutimos posibles direcciones futuras de estudio del problema de Motzkin en grupos abelianos compactos.

1.5 Resumen y conclusiones

Los resultados presentados en esta tesis contribuyen a la teoría de grafos y a la teoría combinatoria de números, mediante el desarrollo de conexiones (establecidas recientemente) entre temas en estas áreas y herramientas centrales de teoría ergódica y geometría de números.

Respecto de la teoría de grafos, la tesis contribuye al tema de coloración con nuevas nociones que refinan los números cromáticos fraccionarios y circulares. En particular, formalizamos y estudiamos el nuevo concepto de *girocoloración* de grafos. Nuestros resultados muestran que esta noción y su número cromático relacionado (el *número girocromático*), son conceptos naturales y robustos con propiedades interesantes. Los aspectos analíticos no triviales del número girocromático también nos condujeron a otra noción de coloración natural y más manejable, en la que los vértices de un grafo G se colorean por traslaciones de una caja en el toro de dimensión d, refinando el número cromático circular mediante el *número cromático d-toral*, y estableciendo conexiones con problemas conocidos relativos a recubrimientos de toros. Los diversos resultados que obtuvimos sobre los números girocromáticos y d-torales también conducen a problemas abiertos que motivan futuras pesquisas.

En cuanto a la teoría combinatoria de números, la tesis contribuye al tema del problema de Motzkin, una cuestión muy natural y conocida en este ámbito, que se viene estudiando desde los años 1970. Mientras que los numerosos trabajos anteriores sobre este problema se centraron en el entorno original de los números enteros, y trataron varios casos especiales del problema con técnicas muy específicas a cada caso, en esta tesis adoptamos un enfoque amplio del problema general, estudiándolo en otros grupos abelianos y relacionándolo con herramientas de la teoría ergódica (como el lema de Rokhlin) y de la geometría de los números. Ilustramos esto considerando el análogo del problema de Motzkin en el grupo circular (que subsume el problema original), obteniendo en particular una solución exacta cuando el conjunto de diferencias prohibidas tiene dos elementos, al menos uno de los cuales es irracional. En el caso en que las diferencias prohibidas sean todas racionales, el enfoque geométrico que utilizamos conduce a estimaciones asintóticamente exactas de la ratio de independencia de un grafo circulante de 2 saltos, en términos de la circunferencia impar del grafo, abriendo interesantes perspectivas de estudios similares para grafos circulantes de d saltos con $d > 2$.

Chapter 2

Introduction

In this chapter, we introduce and motivate the results that constitute this doctoral thesis. We begin with some classical results to introduce different ways of coloring graphs, then we describe the contents of each subsequent chapter. Chapters 3, 4 and 5 in this thesis are works that have already been published, or are being refereed in various journals [11–13].

2.1 Coloring maps, scheduling problems and traffic flows

As recounted in [7], in 1852 mathematician Francis Guthrie observed that, using only four colors, he could color a map of the counties of England so that neighboring counties received different colors. This led him to conjecture what is now the well-known *four color theorem*, i.e., that *every* map can be colored using at most four colors so that every pair of adjacent regions is colored differently. Francis Guthrie tried unsuccessfully to prove his conjecture. His brother Frederick, also a mathematician, shared the conjecture with Augustus De Morgan, who was his teacher at the time. De Morgan posed the problem to William Rowan Hamilton (figure 2.1), who was apparently not interested in the question. Decades later, in 1878, at a meeting of the London Mathematical Society, Arthur Cayley revived the four color problem by asking if it had been solved. Cayley himself failed to find a solution [16]. Finally, after several false proofs (and false disproofs) throughout the next century since its formulation, the conjecture was finally confirmed in 1976 by Appel and Haken [2]. Although this proof was controversial at the time due to its use of computer based algorithms, it is now generally accepted.

The beautiful and paradigmatic four color theorem provides excellent motivation for recalling the basics of graph theory.

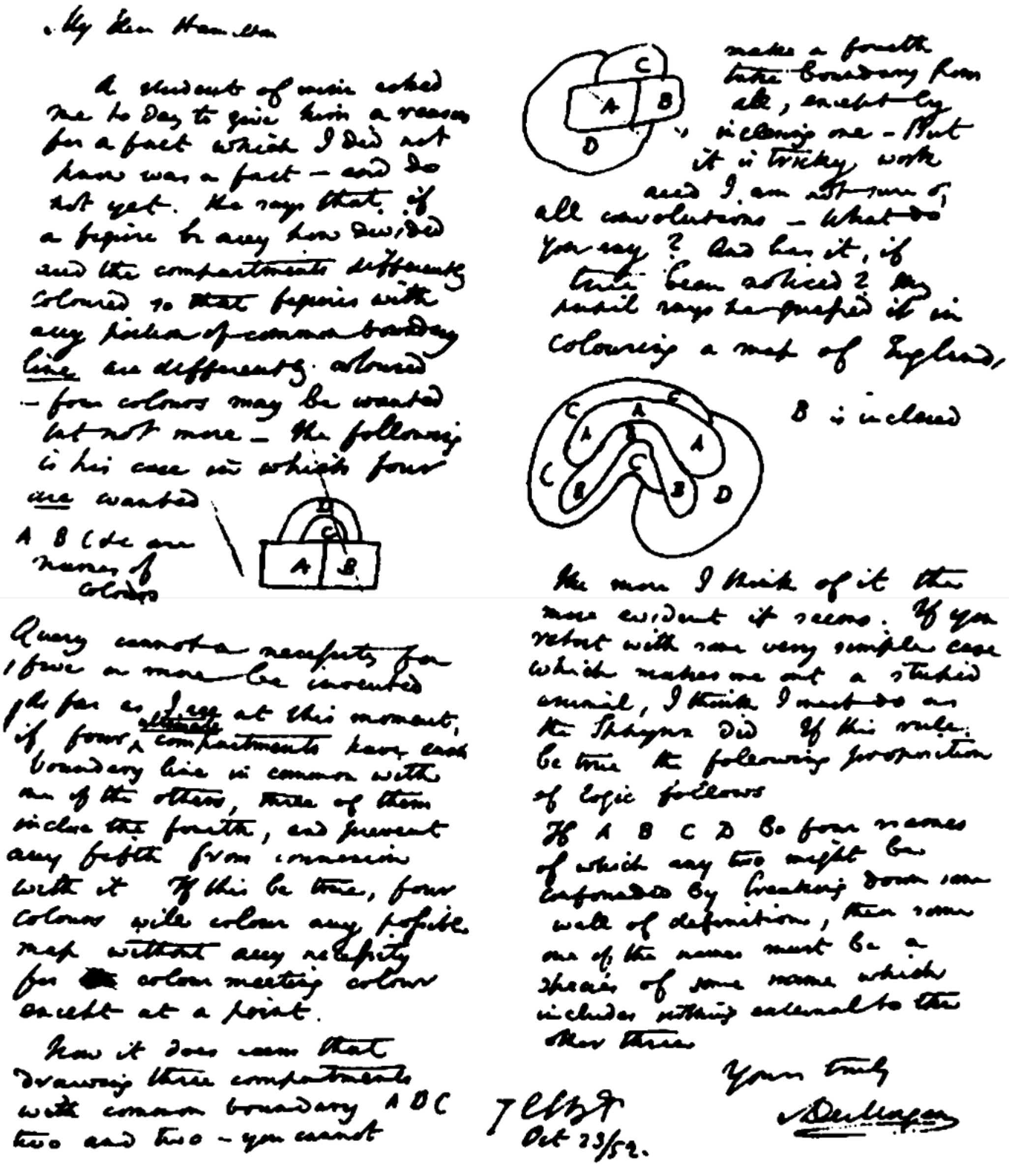
A *graph* G is a pair (V, E), where V is a set, whose elements are called the *vertices* of G, and E is a set of pairs of elements of V, i.e. subsets of V of cardinality 2, called the *edges* of G. We may denote the set of all pairs of elements of V by $\binom{V}{2}$, and thus write $E \subset \binom{V}{2}$. Two vertices in an edge are said to be "joined by an edge", or "adjacent", or "neighbors" in G. Throughout this thesis, unless stated otherwise, we assume that V is *finite*, that G is *loopless* (i.e. no vertex is adjacent to itself), that there are no multiple edges (i.e. not more than one edge between two vertices), and that G is *undirected* (i.e. there is no orientation on the edge set). The cardinalities $|V|$ and $|E|$ are called the *order* and the *size* of G respectively.

The four color theorem is easily formulated in terms of graphs. The regions of a map can be represented by a graph that has a vertex for each region and an edge for every pair of regions that share part of their boundary. This graph is drawn in the plane by drawing one vertex inside each region and drawing the edges as curves from one region's vertex (across a shared boundary part) to an adjacent region's vertex (Cf. figures 2.2 and 2.3). This kind of graph, called *planar graphs*, are characterized by this possibility of drawing them in a plane with no pair of edges crossing each other. The theorem states that the vertices of every planar graph can be colored with *at most* four colors so that no two adjacent vertices receive the same color (figure 2.4). We are thus led to the classical notion of graph coloring, which we formulate as follows.

Definition 2.1 (k-coloring of a graph)**.** Let G be a graph and let k be a positive integer. A k-coloring of G is a function $f : V(G) \mapsto \{0, 1, \ldots, k-1\}$ such that for every edge $xy \in E(G)$ we have $f(x) \neq f(y)$.

Obviously, it is always possible to color G with $k = |V|$ distinct colors. A natural question is then, given a graph G, what is the minimum number k such that G admits a k-coloring. Such an optimal number is the ("ordinary", or "classical") chromatic number of G.

Here, and in the rest of the thesis, we denote by $\mathbb{N}$ the set of positive integers.



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented

As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did — If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three

Yours truly

A De Morgan

7 C St

Oct 23/52.

Fig. 2.1: Letter of De Morgan to Hamilton. (Wikimedia Commons. Public Domain)

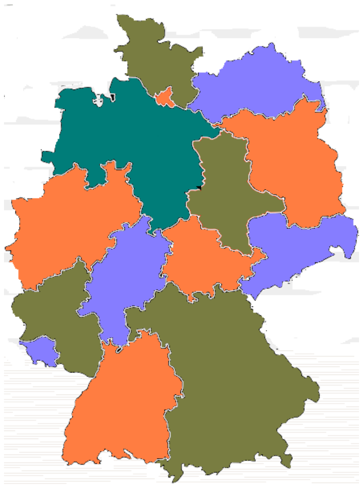


Fig. 2.2: Simplified map of Germany. (Wikimedia Commons. Public Domain)

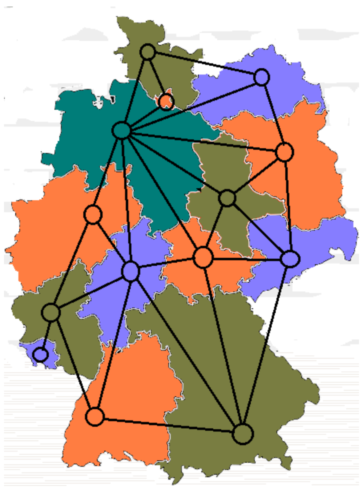


Fig. 2.3: Associated graph for Germany. (Wikimedia Commons. Public Domain)

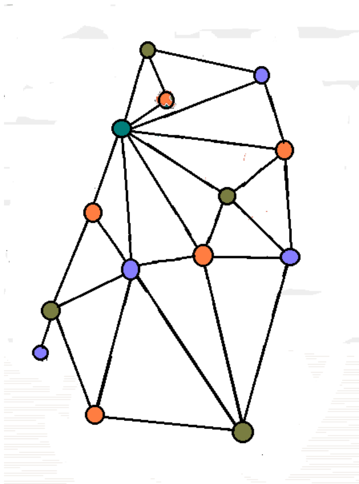


Fig. 2.4: Optimally colored graph. (Wikimedia Commons. Public Domain)

Definition 2.2. Let G be a graph. The *chromatic number* of G is denoted by $\chi(G)$ and defined by

$$\chi(G) := \min\left\{k \in \mathbb{N} : G \text{ admits a } k\text{-coloring}\right\}.$$

Thus, the four color theorem can be phrased as the fact that for every planar graph G we have $\chi(G) \leq 4$.

The problem of determining $\chi(G)$ for a general graph G is far from trivial, and it is well-known to be hard also from a computational point of view. We illustrate the concept in some simple special cases with figures 2.5, 2.6 and 2.7.

Another motivation for the study of graph colorings, which leads to a refinement of the classical chromatic number, is the problem of optimal scheduling. Following an example from [59], let us suppose that there are five committee meetings to be scheduled, each meeting being 1 hour long. If two different committees have a member in common, they cannot meet at the same time. We may ask for the length of the shortest time interval in which all the committees can be scheduled. Let G be the graph with vertex set consisting of the committees, each committee being represented by one vertex, two vertices being adjacent if their respective committees cannot meet simultaneously. Thus the graph G captures the scheduling conflicts.

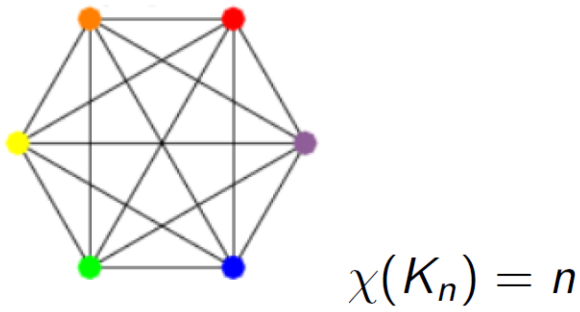


Fig. 2.5: The complete graph K_5. In general, the complete graph K_n ($n \in \mathbb{N}$) is the graph of order n with $E = \binom{V}{2}$. (Wikimedia Commons. Public Domain)

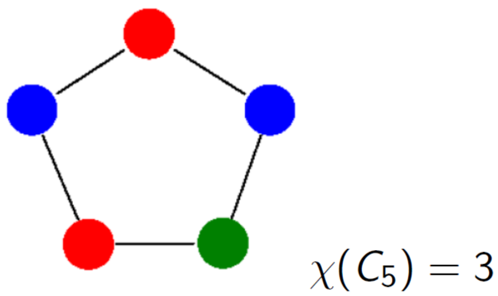


Fig. 2.6: The cycle (or cyclic graph) C_5. For cycles C_n, with $n \geq 3$ an integer, we have $\chi(C_n) = 2$ if n is even, and $\chi(C_n) = 3$ if n is odd. (Wikimedia Commons. Public Domain)

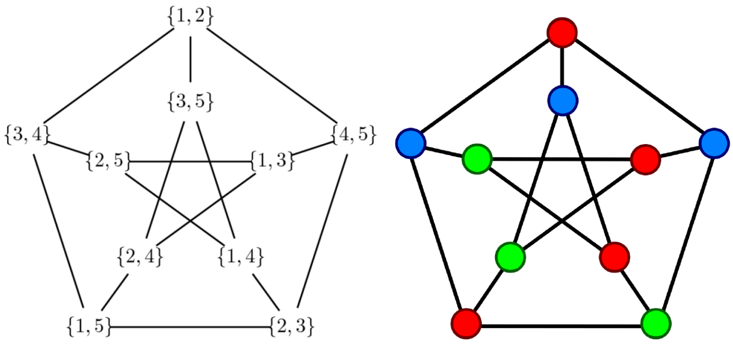


Fig. 2.7: The *Petersen* graph $K(5,2)$. This is a special case of *Kneser graphs* $K(n,k)$, for positive integers n,k satisfying $n \geq 2k$. The vertices of $K(n,k)$ are the k-subsets (subsets of cardinality k) of the set $[n] := \{1,2,\ldots,n\}$, i.e. $V(K(n,k)) = \binom{[n]}{k}$. Two such vertices are neighbors if their corresponding subsets are disjoint. The chromatic number of the Kneser graph is $\chi(K(n,k)) = n - 2k + 2$, as was famously established by Lovász. (Wikimedia Commons. Public Domain)

The obvious answer to the above scheduling problem is that the length of the shortest time interval is given by $\chi(G)$. Suppose that such a graph is the 5-cycle $G = C_5$. Since $\chi(C_5) = 3$, the scheduling can be done in 3 hours. We may wonder if this schedule can be improved. And indeed it can! The scheduling can be made in 2.5 hours if we allow a committee to meet for half an hour, and later resume its meeting for the remaining half hour, after some interruption. Thus, it becomes possible to improve (i.e. shorten) the total scheduling time if we allow the meetings to be divided into fractions. The shortest length of time needed to schedule committees when such divisions are allowed is not the classical chromatic number $\chi(G)$, but rather the *fractional chromatic number*, defined formally below, and denoted by $\chi_f(G)$. In the example of the 5-cycle it can indeed be seen that $\chi_f(C_5) = 2.5$ (see figure 2.9), which shows that $\chi_f(G)$ can be strictly less than $\chi(G)$.

This improvement is a first illustration of the fact that the notion of graph coloring can be refined in very useful ways. The refinement illustrated above, known as *fractional coloring*, is a central notion in the development of so-called *fractional graph theory*, treated by Scheinerman and Ullman [59], and by Berge in [5]. According to [59, §3.11], the first publication in which

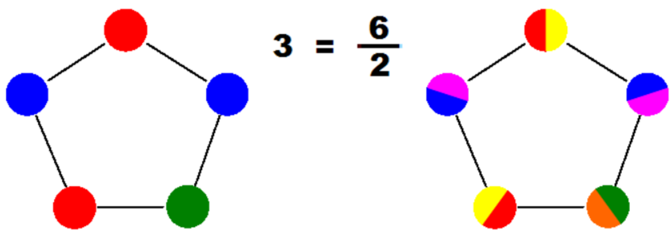


Fig. 2.8: A $(3,1)$-coloring and a $(6,2)$-coloring of C_5. (GTBacchus, CC BY-SA 3.0, Wikimedia Commons)

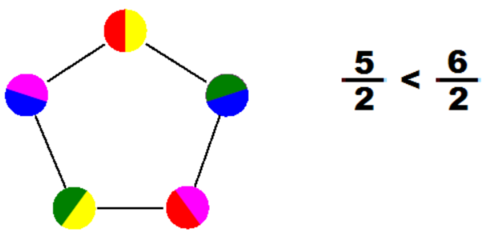


Fig. 2.9: A $(5,2)$-coloring of C_5. (GTBacchus, CC BY-SA 3.0, Wikimedia Commons)

the fractional chromatic number appears is [36].

Definition 2.3. (Cf. [59, §3.1]) Let G be a graph. A *b-fold coloring* of G is an assignment, to each vertex of G, of a set of b colors, so that adjacent vertices receive *disjoint* sets of colors. We say that G has a (d,b)-coloring if G has a b-fold coloring in which the colors are drawn from a palette of $d \geq 1$ colors; i.e., there is a function $f : V(G) \mapsto \binom{[d]}{b}$ such that for every edge $xy \in E(G)$ we have $f(x) \cap f(y) = \emptyset$.

Figures 2.8 and 2.9 show examples of a $(3,1)$-coloring, a $(6,2)$-coloring and a (properly fractional) $(5,2)$-coloring.

Definition 2.4. (Fractional chromatic number) Let G be a graph. We define the b-fold chromatic number of G by

$$\chi_b(G) := \min \big\{ d : G \text{ admits a } (d,b)\text{-coloring} \big\}.$$

(Note that $\chi_1(G) = \chi(G)$.) We then define the *fractional chromatic number* $\chi_f(G)$ as follows:

$$\chi_f(G) := \lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$

The convergence of the sequence $\left\{\frac{\chi_n(G)}{n}\right\}_{n\geq 1}$ is guaranteed by a standard subadditivity lemma (Cf. [59], Appendix A.4), since we always have $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$. In fact, it can be seen [59] that for every non-empty graph G (i.e. with $E(G) \neq \emptyset$) there is a positive integer b such that $\chi_f(G) = \chi_b(G)/b \geq 2$.

Determining $\chi_f(G)$ for a general graph G is also hard (as for $\chi(G)$), but for the wide class of *vertex-transitive*[1] graphs it is known that

$$\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$$

where $\alpha(G)$ is the *independence number*[2] of G. In particular, cycles are vertex-transitive and $\alpha(C_{2m+1}) = m$, and it follows that $\chi_f(C_{2m+1}) = 2 + m^{-1}$. Additionally, Kneser graphs $K(n,k)$ (for positive integers n, k satisfying $n \geq 2k$; Cf. figure 2.7) are also vertex-transitive and we have $\alpha(K(n,k)) = \binom{n-1}{k-1}$ and $\chi_f(K(n,k)) = n/k$.

Alternatively, coloring a graph G in the classical sense can be viewed as an *integer linear programming* problem, where independent sets in G are assigned weights 0 or 1 in such a way that every vertex belongs to independent sets whose total weight is (at least) 1 and the sum of the weights of all independent sets is minimized. From this viewpoint, fractional coloring is a *linear relaxation* of this optimization problem: the fractional chromatic number $\chi_f(G)$ is the smallest *real number* x for which there is an assignment of non-negative weights to independent sets in G such that the sum of their weights is x and each vertex belongs to independent sets whose total weight is at least 1. It is well known that the minimum is always attained and $\chi_f(G)$ is always a positive *rational* number [59].

This alternative view of the fractional chromatic number can be shown (Cf. [64]) to be equivalent to the following definition using measure theory.

Definition 2.5 (Fractional chromatic number, measure-theoretic version)**.** For any graph G, we have $\chi_f(G) := \inf\big\{r > 0 :$ for each vertex v there is a measurable set $A_v \subseteq [0,1)$ with Lebesgue measure $\mu(A_v) \geq 1/r$ so that for every $uv \in E(G)$ we have $A_u \cap A_v = \emptyset\big\}$.

[1] A graph is *vertex-transitive* if its automorphism group acts transitively on its vertices.

[2] An *independent set* A in G is a subset of $V(G)$ such that every pair of vertices $u, v \in A$ satisfies $uv \notin E(G)$. The *independence number* of G is the maximum cardinality of independent subsets in G.

By casting fractional coloring in the context of measurable subsets in an interval, this definition also suggests a general framework which opens the door to further refinements of graph coloring.

In [62], [64], Zhu proposed a different notion of graph coloring called *circular coloring*. This uses *open intervals* in the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We may view $\mathbb{T}$ as $[0, 1)$ with addition modulo 1, and we then have the following definition.

Definition 2.6 (Circular chromatic number)**.** Let G be a graph. The *circular chromatic number* of G is $\chi_c(G) := \inf \big\{ r > 0 :$ for each vertex x there is an *open interval* $A_x \subseteq \mathbb{T}$ with $\mu(A_x) \geq 1/r$ so that for every $xy \in E(G)$ we have $A_x \cap A_y = \emptyset \big\}$.

The circular chromatic number can also be viewed in terms of scheduling. Indeed, it is the minimum length t of total time needed to carry out a session of committee meetings in which all meetings must last 1 uninterrupted hour but viewed modulo t, i.e. we allow meetings also to occupy a union of two time intervals $(0, a) \cup (b, t)$ (with $0 < a < b < t$ and $a + t - b = 1$), one interval at the beginning of the sessions and the other at the end. This different notion of graph coloring is especially suitable for scheduling problems with *periodic* conditions. The following example (Zhu, [64]) is instructive. Consider the problem of organizing a system of traffic lights to regulate optimally the traffic of vehicles at a road intersection. A *complete traffic period* is a time interval during which every possible traffic flow must have a turn of green light, with every such turn being of equal duration, taken to be of unit length. This system is easily modeled by a graph G, each of whose vertices represents a traffic flow, with an edge representing a pair of traffic flows which are incompatible, i.e. whose green light intervals must not overlap. The problem consists in finding the minimum time length of a complete traffic period in this road intersection.

One solution that we can give to this traffic problem is by partitioning $V(G)$ into a minimum number k of independent sets $I_1, I_2, ..., I_k$ and assigning successive unit time intervals to each independent set, thus obtaining a complete traffic period of total duration $k = \chi(G)$. At first sight, the problem is thus solved. However, if the graph satisfies the strict inequality $\chi_c(G) < \chi(G)$ then this solution will not be optimal, and circular colorings (exploiting the additional periodicity) will yield a strictly better solution (Cf. Section 4 in [28]). In particular, a result by Guichard [30] shows that if a graph G is n-critical[3], and has girth[4] at least $n+1$, then $\chi_c(G) < \chi(G)$.

An equivalent definition of $\chi_c(G)$ can be given that is more analogous to that of the classical k-colorings (Definition 2.1), using the following notion [64].

Definition 2.7 (r-circular coloring)**.** Let G be a graph and $r \geq 1$ a real number. An *r-circular coloring* of G is a function $f : V(G) \mapsto [0, r)$ such that for every edge $xy \in E(G)$ we have $1 \leq \big|f(x) - f(y)\big| \leq r - 1$. We can then define the circular chromatic number of G as

$$\chi_c(G) := \inf \big\{r : G \text{ admits an } r\text{-circular coloring}\big\}.$$

Note that if f is a k-coloring of G in the classical sense, then f is also a k-circular coloring of G, and therefore $\chi_c(G) \leq \chi(G)$. On the other hand, for an r-circular coloring $g : V(G) \mapsto [0, r)$, letting $s = \max\{g(x) : x \in V(G)\}$, we can view g as an $(s+1)$-coloring of G. As $s < r$ we deduce the following.

Theorem 2.8. *For any finite simple graph G we have*

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G), \tag{2.1}$$

in particular $\chi(G) = \lceil \chi_c(G) \rceil$.

This shows that $\chi_c(G)$ carries more information about the structure of G than $\chi(G)$, so that the circular chromatic number can be used to quantify how far G is from being colorable with less than $\chi(G)$ colors. In this sense, $\chi_c(G)$ can be seen as a refinement of $\chi(G)$.

In Chapters 3 and 4 of this thesis, we introduce new coloring notions which lead to further refinements of $\chi_c(G)$ and $\chi_f(G)$, yielding new chromatic numbers with interesting properties. We detail the main contents of these chapters in the following two sections.

2.2 Coloring graphs by translates in the circle: the gyrochromatic number

Historically, the first definition of the circular chromatic number was given by Vince in 1988 [61], with the name *star chromatic number*. This notion

[3] A graph G is n-critical if $\chi(G) = n$ and $\chi(G - v) = n - 1$ for any vertex v of G.

[4] The *girth* of a graph is the length of a shortest cycle contained in the graph.

is formulated in a discrete setting as follows.

Definition 2.9 (Star chromatic number)**.** Let $1 \leq b \leq d$ be integers and let G be a graph. A (d,b)-$\star$-coloring of G is a function $f : V(G) \to [d]$ such that for every edge xy we have

$$b \leq |f(x) - f(y)| \leq d - b \qquad (d \geq 2b).$$

We define the *star chromatic number* of G as

$$\chi^\star(G) := \inf\left\{\frac{d}{b} : G \text{ admits a } (d,b)\text{-}\star\text{-coloring}\right\}.$$

Among the main basic properties of the star chromatic number $\chi^\star(G)$, we highlight two facts: firstly [61], the infimum in the definition of $\chi^\star(G)$ is attained for every graph G, and is therefore a minimum; secondly, the fact (consequence of the previous one) that $\chi^\star(G)$ is always a rational number (Cf. Vince [61] and Bondy & Hell [8]). [5]

It was later observed by Zhu [62], [64] that his circular chromatic number is indeed equal to the star chromatic number.

Lemma 2.10 (Zhu)**.** *For every graph G we have $\chi^\star(G) = \chi_c(G)$.*

Let us include the short proof.

Proof. Suppose f is a (d,b)-$\star$-coloring of G. Let us define the function $g : V(G) \mapsto [0, d/b)$ such that $g(x) := f(x)/b$. For every edge xy of G we have $1 \leq |g(x) - g(y)| \leq \frac{d}{b} - 1$, thus every (d,b)-$\star$-coloring of G corresponds to a (d/b)-circular coloring of G. On the other hand, if g is a (d/b)-circular coloring of G, then $f(x) := \lfloor b \cdot g(x) \rfloor$ is a (d,b)-$\star$-coloring of G. $\square$

From the definitions of $\chi_f(G)$ and $\chi_c(G)$ in terms of (d,b) and (d,b)-$\star$ colorings, we easily see the following inequalities.

Lemma 2.11. *Every graph G satisfies $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$.*

[5] There are other equivalent descriptions of this chromatic number, in terms of homomorphisms into a certain class of graphs [8], which also implies easily the rationality of $\chi^\star(G)$.

Both inequalities in this lemma can be strict, and the gap between $\chi_f(G)$ and $\chi(G)$ can be arbitrarily large. For example, as mentioned previously, the Kneser graph $G = K(n,k)$ ($n \geq 2k$) satisfies $\chi(G) = n - 2k + 2$ and $\chi_f(G) = \frac{n}{k}$, and it is also known [18] that $\chi_c(G) = \chi(G)$.

The starting point of Chapter 3 is the work by Avila and Candela [3], which included combinatorial applications of central tools from ergodic theory. These applications led to a new chromatic parameter for graphs, which lies between the circular and fractional chromatic numbers. The purpose of this chapter is to study this new parameter, which we refer to as the *gyrochromatic number* $\chi_g(G)$ of a graph G. This number is the reciprocal of the natural quantity defined as follows, where $\mathbb{T}$ denotes as usual the circle group $\mathbb{R}/\mathbb{Z}$.

Definition 2.12 (Avila-Candela, 2016)**.** Let G a finite graph. We say that a Borel set $A \subseteq \mathbb{T}$ is a $\mathbb{T}$-*coloring base* for G if there exists a map $f : V(G) \to \mathbb{T}$ such that for every edge xy of G we have $\big(A + f(x)\big) \cap \big(A + f(y)\big) = \emptyset$. We say that (A, f) is a $\mathbb{T}$-*coloring* (or $\mathbb{T}$-*gyrocoloring*) of G. We then define

$$\sigma_{\mathbb{T}}(G) := \sup\{\mu_{\mathbb{T}}(A) : A \text{ a } \mathbb{T}\text{-coloring base for } G\}, \tag{2.2}$$

where $\mu_{\mathbb{T}}$ denotes the Haar probability measure on $\mathbb{T}$.

Definition 2.13 (Gyrochromatic number)**.** For any graph G we define $\chi_g(G) := \frac{1}{\sigma_{\mathbb{T}}(G)}$.

Note that the notion $\sigma_{\mathbb{T}}(G)$ is easily generalized by replacing $\mathbb{T}$ with any compact abelian group Z. For any such group, equipped with a Haar probability measure μ_Z, we can indeed define

$$\sigma_Z(G) := \sup\{\mu_Z(A) : \ A \text{ is a Borel } Z\text{-coloring base for } G\}, \tag{2.3}$$

where the extension of the notion of $\mathbb{T}$-coloring base to a Z-base is clear.

A first basic observation on the gyrochromatic number is the following.

Proposition 2.14. *Every finite graph G satisfies $\chi_f(G) \leq \chi_g(G) \leq \chi_c(G)$.*

Proof. For the first inequality, suppose that (A, f) with Borel $A \subseteq \mathbb{T}$ is a $\mathbb{T}$-coloring of G. Then this is also a fractional coloring of G with $A_x := A + f(x)$ for each vertex x (as mentioned earlier, we usually identify $\mathbb{T}$ with the interval $[0, 1)$ equipped with addition modulo 1). For the second inequality, if there is a coloring by open intervals (these are Borel sets) A_x, all of the same length, then this is clearly a $\mathbb{T}$-coloring since every open interval A_x can be viewed as a translate of a single interval A (so that $A_x =: A + f(x)$ satisfies $A_x \cap A_y = \emptyset$ for all the edges xy). $\square$

Informally speaking, circular coloring assigns vertices of G to arcs (intervals in the circle) and fractional coloring assigns vertices of G to Borel subsets. A $\mathbb{T}$-gyrocoloring is tighter than a fractional coloring since vertices must be assigned gyrated copies of a *single* Borel subset, and it is looser than circular coloring since this Borel subset need not be an arc.

One of the motivations that led to this new graph parameter is a version of the general problem, central to arithmetic combinatorics, which consists in determining the greatest size that a subset of an abelian group can have without containing solutions to a prescribed set of linear equations. In the version in question, considered in [3], we let $c_1, c_2, \ldots, c_d$ be multiplicatively independent non-zero integers,[6] we let G be any graph with vertex set $V = [d]$, and we then ask what is the greatest size (Haar measure) that a subset of $\mathbb{T}$ can have if we require that this set avoid solutions to the linear equations $c_i\, x_1 = c_j\, x_2$ for edges ij in G. In other words, we ask what is the quantity

$$d_{G,c_i} := \sup \big\{ \mu_{\mathbb{T}}(A) : A \subset \mathbb{T},\ \forall\, ij \in E(G),\ \forall\, x_1, x_2 \in A,\ c_i\, x_1 \neq c_j\, x_2 \big\}.$$

It is proved in [3] that $d_{G,c_i} = \sigma_{\mathbb{T}}(G)$.

In Chapter 3 it is shown that the definition of the gyrochromatic number, like the circular and fractional chromatic numbers, is *robust*, in the sense that it can be reformulated in several equivalent ways. For example, the definition through coloring $\mathbb{T}$-bases yields the following equivalent variant in a discrete setting.

Theorem 2.15. *Let G be a graph. Then we have $\sigma_{\mathbb{Z}_N}(G) \leq \sigma_{\mathbb{T}}(G)$ for every $N \in \mathbb{N}$ and $\sigma_{\mathbb{T}}(G) = \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G)$.*

This yields the following reformulation.

Corollary 2.16. *The gyrochromatic number of a graph G is equal to the infimum of numbers N/K for which there exists a set $A \in \binom{\mathbb{Z}_N}{K}$ and a function $f : V(G) \to \mathbb{Z}_N$ such that $(A + f(u)) \cap (A + f(v)) = \emptyset$ for every edge $uv \in E(G)$.*

[6] This simply means that if $c_1^{\lambda_1} \cdot c_2^{\lambda_2} \cdots c_d^{\lambda_d} = 1$ with $\lambda_i \in \mathbb{Z}$, then $\lambda_i = 0$ for every i.

Recall the well-known fact that classical coloring, circular coloring and fractional coloring of graphs can be defined by using homomorphisms to certain classes of graphs: cliques, circular cliques and Kneser graphs respectively. We show that the gyrochromatic number also follows this pattern: it can be defined in terms of homomorphisms to *circulant graphs.* [7]

Theorem 2.17. *The gyrochromatic number of a graph G is equal to*

$$\inf_{\substack{N\in\mathbb{N},S\subseteq\mathbb{Z}_N\\ G\to C(N,S)}} \chi_f\left(C(N,S)\right) = \inf_{\substack{N\in\mathbb{N},S\subseteq\mathbb{Z}_N\\ G\to C(N,S)}} \frac{N}{\alpha\left(C(N,S)\right)},$$

i.e., is equal to the infimum of the fractional chromatic numbers of the circulant graphs that admit a homomorphism from G.

Another notable feature of the gyrochromatic number is that it can be defined equivalently using tori of arbitrarily high dimension.

Theorem 2.18. *For every finite simple graph G and every $d \in \mathbb{N}$, we have*

$$\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G).$$

This result implies that the discrete variant of the definition (given by Theorem 2.15) remains equivalent when *all* finite abelian groups are considered, instead of the groups $\mathbb{Z}_N$ only (Cf. Corollaries 3.10 and 3.11). This establishes a certain universality of the gyrochromatic number. Note also that (as detailed in Chapter 3) the fractional chromatic number itself can be cast in a similar way but taking into account all finite (not necessarily abelian) groups. This way, the gyrochromatic number can be viewed as a version of the fractional chromatic number restricted to abelian groups.

Among other results that we prove on the gyrochromatic number, there are constructions of graphs satisfying the strict inequalities $\chi_f(G) < \chi_g(G) < \chi_c(G)$.

Theorem 2.19. *There exists a sequence of graphs $(G_k)_{k\in\mathbb{N}\setminus\{1\}}$ such that $\chi_f(G_k) < \chi_g(G_k) < \chi_c(G_k) = k+2$ and with $\lim_{k\to\infty}\chi_g(G_k) = 2$.*

---

[7] A *circulant graph* $G = C(n, S)$ is a *Cayley graph* over the cyclic group $V(G) = \mathbb{Z}_n$, generated by a set $S \subset \mathbb{Z}_n$, which we assume to be *symmetric* (i.e. $S = -S$); thus vertices $a, b \in \mathbb{Z}_n$ are adjacent if and only if $a - b \in S$ (equivalently $b - a \in S$).

Last but not least, we show that, somewhat surprisingly, the supremum in the definition (2.2) need not be attained, which also implies that the infimum in the discrete variant given by Theorem 2.17 need not be attained either. This leads to the open problem of whether there exists a finite graph G such that the gyrochromatic number of G is not rational. Other open problems are proposed among our final remarks in the chapter.

We deem these results to be convincing evidence that the gyrochromatic number is a natural and robust parameter, which is likely to be useful to obtain a finer understanding of the structure of graphs.

2.3 On toral chromatic numbers of graphs

In Chapter 3 we introduced the gyrochromatic number of a graph G, and several results obtained in that chapter show that $\chi_g(G)$ carries more information than the fractional and circular chromatic numbers. In particular we showed that $\chi_f(G) \leq \chi_g(G) \leq \chi_c(G)$ is satisfied for any finite graph, and that these inequalities can be strict. The gyrochromatic number is also shown to have a certain universality in the sense that the infimum can be extended to Borel sets in a finite higher dimensional torus equipped with its Haar probability measure *without changing the value* of $\chi_g(G)$, that is, for every fixed $r \in \mathbb{N}$ we have

$$\chi_g(G) = \inf\{1/\mu_{\mathbb{T}^r}(A) : A \text{ is a } \mathbb{T}^r\text{-coloring base for } G\}. \tag{2.4}$$

However, the gyrochromatic number is more “elusive” than the circular chromatic number $\chi_c(G)$, since it is also proved in Chapter 3 that the infimum in the original definition (i.e. (2.4) for $r = 1$) is not always attained, and we do not yet know whether $\chi_g(G)$ is always rational. Even worse, we do not know whether there is always at least *some* finite dimension r such that the infimum is attained in *this* dimension.

This motivates the study of intermediate colorings, also refining $\chi_c(G)$ but being analytically more tractable than the gyrochromatic number. This is the main objective of Chapter 4.

A natural candidate for a more manageable refinement of $\chi_c(G)$ consists in coloring the graph with *translates of a box* in the d-dimensional torus $\mathbb{T}^d$. We may view the d-torus as $[0, 1]^d$ with addition mod 1 in each coordinate. By an *open box* in $\mathbb{T}^d$ we mean a Cartesian product of the form $I_1 \times \cdots \times$

$I_d \subset \mathbb{T}^d$ where I_j is an open interval (open connected set) in $\mathbb{T}$ for every $j \in [d]$. The corresponding refinement of the circular chromatic number is then defined as follows.

Definition 2.20 (d-toral chromatic number)**.** Let G be a graph. For each $d \in \mathbb{N}$, we define the *d-toral chromatic number* of G, denoted by $\chi_{c^d}(G)$, by the formula

$$\chi_{c^d}(G) = \inf\left\{\mu_{\mathbb{T}^d}(R)^{-1} : R \subset \mathbb{T}^d \text{ an open box such that } G \text{ is } R\text{-colorable}\right\} \tag{2.5}$$

where, for a subset A of an abelian group Z, we say that a graph G is *A-colorable* if there is a map $\varphi : V(G) \to \mathrm{Z}$ such that $\big(A+\varphi(x)\big)\cap\big(A+\varphi(y)\big) = \emptyset$ for every edge $xy \in E(G)$. (We may also say that φ is a *coloring map* of G by A).

The 1-toral chromatic number is the usual circular chromatic number. We show in Chapter 4 that the following hierarchy is satisfied for *any positive integer d*:

$$\chi_f(G) \le \chi_g(G) \le \chi_{c^{d+1}}(G) \le \chi_{c^d}(G) \le \chi(G). \tag{2.6}$$

It is also shown that the infimum in (2.5) is always attained and rational, as is the case for $\chi_c(G)$:

Theorem 2.21. *Let G be a graph of order n and let $d \in \mathbb{N}$. Then for each $i \in [d]$ there are integers $r_i \le s_i$ in $[n]$ such that G is colorable by the box $R = \prod_{i\in[d]}(0, \frac{r_i}{s_i})$ in $\mathbb{T}^d$ and $\chi_{c^d}(G) = \frac{1}{\mu_{\mathbb{T}^d}(R)} = \frac{s_1\cdots s_d}{r_1\cdots r_d}$.*

Note that for every given graph G, by (2.6) the decreasing sequence $(\chi_{c^d}(G))_{d\in\mathbb{N}}$ must converge, and we may then ask how fast it does so, and even whether it always becomes constant eventually. This is answered as follows.

Proposition 2.22. *Let G be a graph, and let $d = \lfloor \log_2(\chi(G)) \rfloor$. Then for every $d' \ge d$ we have $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$.*

One may also wonder how $\chi_{c^d}(G)$ varies for a fixed d and varying G. In particular we may ask whether for every fixed d there are graphs G for which $\chi_{c^{d+1}}(G) < \chi_{c^d}(G)$; a positive answer here would indicate that each number $\chi_{c^d}(G)$ carries certain information about G that can make it differ from other such numbers $\chi_{c^{d'}}(G)$, $d' > d$, in other words, that graphs separate the toral chromatic numbers.

Note that by Proposition 2.22 we can meaningfully define what we shall call the *stabilization dimension* of a graph G for the toral chromatic numbers, namely the least integer d with the property that $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$ for all $d' \geq d$. Denoting the stabilization dimension of G by $d^*(G)$, we have $d^* \leq \lfloor \log_2(|G|) \rfloor$ by Proposition 2.22, and we may ask how accurate this upper bound is. We prove that this bound is sharp up to a multiplicative constant, with the following result, which also shows that graphs do indeed separate the toral chromatic numbers.

Theorem 2.23. *For each $d \in \mathbb{N}$ there exists a graph G of order $n = 5^d$ satisfying $d^*(G) = d = \log_5(n)$.*

Finally, at the end of Chapter 4 we relate the number $\chi_{c^d}(G)$ to the ordinary chromatic number $\chi(G)$, via an inequality which generalizes the well-known inequality (2.1), and uses box-coverings of the d-torus (see Proposition 4.21).

2.4 On Motzkin's problem in the circle

The last chapter of this thesis is devoted to a project which arose while the research on the gyrochromatic number was in progress. The project concerns a well-known problem in combinatorial number theory which was posed by T. S. Motzkin in the 1970s.

Motzkin's problem asks how large a set of integers can be if it does not contain any pair of elements whose difference lies in a prescribed set. More precisely, given a non-empty subset D of the set of positive integers $\mathbb{N}$, let us say that a set $A \subset \mathbb{Z}$ is *D-avoiding* if for every $a, a' \in A$ we have $|a - a'| \notin D$, in other words if the *difference set* $A - A = \{a - a' : a, a' \in A\}$ is disjoint from D. Let $A(N)$ denote the cardinality $|A \cap [-N, N]|$, and let $\bar{\delta}(A)$ denote the upper density of A, namely $\bar{\delta}(A) = \limsup_{N\to\infty} \frac{A(N)}{2N+1}$. Motzkin asked to determine or estimate the greatest upper density that a D-avoiding set can have, namely the following quantity, which we call the *Motzkin density* of D:

$$\mathrm{Md}_{\mathbb{Z}}(D) := \sup\{\bar{\delta}(A) : A \text{ is a } D\text{-avoiding subset of } \mathbb{Z}\}. \tag{2.7}$$

Cantor and Gordon published the first paper on Motzkin's problem [14], proving several interesting results, including a full solution for the case of at most two forbidden differences (i.e. for $|D| \leq 2$). Since then, the general problem has motivated many works and several additional special cases have been addressed (as detailed in Chapter 5), but the problem is still open.

Motzkin's problem constitutes a topic within combinatorial number theory that has interesting connections with other well-known problems, including the fractional chromatic number of *distance graphs*, or the famous *lonely runner conjecture*.

Motzkin's problem can also be seen as a particular case of a wider question which can be asked in any compact abelian group Z: given a non-empty set $D \subset \mathrm{Z}$, letting μ_{Z} denote the Haar probability measure on Z, this question asks to determine or estimate the quantity

$$\mathrm{Md}_{\mathrm{Z}}(D) := \sup\{\mu_{\mathrm{Z}}(A) : A \subset \mathrm{Z} \text{ a Borel set with } (A-A) \cap D = \emptyset\}. \tag{2.8}$$

A particularly natural case of this question concerns the circle group $\mathrm{Z} = \mathbb{T}$. This case is the main object of study in Chapter 5. We shall view $\mathbb{T}$, as usual, as the interval $[0,1]$ with addition modulo 1, and consider a *finite* set $D = \{t_1, \dots, t_r\}$ of non-zero missing differences, viewing D as a set of real numbers in $(0,1)$.

The approach to this problem in Chapter 5 combines tools from graph theory, ergodic theory, and the geometry of numbers.

A first illustration of the applicability of ergodic theory in this chapter is given by focusing on the case where the set $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$. It then turns out that we may apply a version of the important ergodic theoretic tool known as *Rokhlin's lemma*. This result can be described informally as a tool that enables a non-periodic group-action on a probability space (e.g. an iterated irrational rotation of the circle) to be approximated, with arbitrary precision, by structures called *towers*, which are almost periodic and are much easier to analyze (we give the precise formulations in Section 5.2). By a simple application of this tool for free measure-preserving actions of $\mathbb{Z}^r$, we obtain the solution $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$ in this case. This motivates the exploration of the applicability of Rokhlin's lemma (and its various extensions) to the more general case of Motzkin's problem in $\mathbb{T}$ where $D \cup \{1\}$ can be linearly dependent over $\mathbb{Q}$. And indeed, known extensions of Rokhlin's lemma, applicable to free actions of *quotients* of $\mathbb{Z}^r$, are shown to be relevant to this problem. In particular we show that, via these extensions, the problem of determining $\mathrm{Md}_{\mathbb{T}}(D)$ can be transferred to a similar problem in the discrete setting of the finitely generated abelian group $\mathbb{Z}^r/\Lambda$, where Λ is the kernel of the homomorphism

$\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1 t_1 + \cdots + n_r t_r$. In this discrete setting, Motzkin densities can be defined using Følner sequences; see Definition 5.8. We then have the following result.

Theorem 2.24. *Let $D = \{t_1, \ldots, t_r\} \subset \mathbb{T}$, let Λ be the kernel of the homomorphism $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1 t_1 + \cdots + n_r t_r$, and let E be the image of the standard basis of $\mathbb{R}^r$ in the quotient $\mathbb{Z}^r/\Lambda$. Then $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$.*

In fact, the versatility of these extensions of Rokhlin's lemma enable us to prove a version of this theorem that is valid for compact abelian groups more generally; see Theorem 5.9.

Theorem 2.24 is useful as a first step for determining $\mathrm{Md}_{\mathbb{T}}(D)$, since the corresponding Motzkin density in the discrete setting, i.e. $\mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$, can often be simpler to determine. In Chapter 5 we pursue this approach for $r \leq 2$, obtaining the solutions summarized below, notably the exact formula in Theorem 2.25.

The techniques from graph theory and the geometry of numbers enter the picture in relation to the case of the problem in which, instead of $D \cup \{1\}$ being linearly independent over $\mathbb{Q}$, we assume on the contrary that $D \subset \mathbb{Q}$. Indeed, this case reduces to the problem of determining the independence ratio of a circulant graph which we call the *associated circulant graph*. More precisely, supposing that each element of D is of the form $t_i = a_i/b_i$ with coprime positive integers $a_i < b_i$, then the subgroup $\langle D \rangle \leq \mathbb{T}$ is isomorphic to $\mathbb{Z}_N$ with $N = \mathrm{lcm}(b_1, \ldots, b_r)$. The associated circulant graph is the connected circulant graph G with vertex set $\mathbb{Z}_N$ (viewed as the set of integers $[0, N-1]$ with addition modulo N) with jumps $d_1, \ldots, d_r$ where $d_i = a_i N/b_i$. Thus $x, y \in \mathbb{Z}_N$ form an edge in G if and only if $x - y = d_i$ or $-d_i$ mod N for some $i \in [r]$. The independence ratio of G is $\frac{\alpha(G)}{N}$, where $\alpha(G)$ is the independence number of G (i.e. the maximal cardinality of an independent set in G). As a straightforward consequence of Theorem 2.24 we have $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \ldots, d_r\}) = \frac{\alpha(G)}{N}$. The analysis of the quantities $\mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \ldots, d_r\})$ leads to the geometry of numbers through the relation (developed in Chapter 5) with the lattice Λ mentioned above (which is of full rank r in this rational case).

It is worth noting that if $d_1, \ldots, d_r$ are fixed integers then, as $N \to \infty$, the ratios $\frac{\alpha(G)}{N}$ converge to $\mathrm{Md}_{\mathbb{Z}}(\{d_1, \ldots, d_r\})$, and in this sense Motzkin's problem in $\mathbb{T}$ can be seen to subsume the original problem in $\mathbb{Z}$ (for finitely many missing differences).

After a brief solution of the case $r = 1$ (see Proposition 5.15), the rest of Chapter 5 focuses on the case $r = 2$. We distinguish two sub-cases. The *first case* is the one in which at least one element of D is an irrational number. Here we obtain the following *exact* solution:

Theorem 2.25. *Let $D = \{t_1, t_2\} \subset (0, 1)$ with $D \not\subset \mathbb{Q}$. If $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$, then $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$. Otherwise, letting m_0, m_1, m_2 be integers not all zero such that $m_0 = m_1 t_1 + m_2 t_2$ and $\gcd(m_0, m_1, m_2) = 1$, we have*

$$\mathrm{Md}_{\mathbb{T}}(D) = \frac{\lfloor k/2 \rfloor}{k}, \quad \text{where } k = |m_1| + |m_2|. \tag{2.9}$$

We then focus on the *second case*, in which both elements of D are rational. This is equivalent to determining the independence ratio of circulant graphs with two jumps. As mentioned above, we study this problem using tools from the geometry of numbers. In particular we obtain the following asymptotically sharp estimate. Recall that the *odd girth* of a graph G is the length of a shortest odd cycle in G.

Theorem 2.26. *Let $D = \{t_1, t_2\} \subset \mathbb{Q} \cap (0, 1)$. Let G be the associated circulant graph, and let N be the order of G. If G is bipartite then $\mathrm{Md}_{\mathbb{T}}(D) = \frac{\alpha(G)}{N} = \frac{1}{2}$. Otherwise, letting k be the odd girth of G, we have*

$$\tfrac{k-1}{2k} \;\geq\; \mathrm{Md}_{\mathbb{T}}(D) = \tfrac{\alpha(G)}{N} \;\geq\; \tfrac{k-1}{2k} - \tfrac{3}{\sqrt{N}}. \tag{2.10}$$

The independence ratio of a circulant graph G is equal to the reciprocal of its fractional chromatic number $\chi_f(G)$. Therefore (2.10) yields also an asymptotically sharp estimate for the fractional chromatic number of a connected circulant graph G of order N with 2 jumps and odd girth k, namely $\frac{2k}{k-1} \leq \chi_f(G) \leq \frac{2k}{k-1} + \frac{27}{\sqrt{N}}$.

Finally, we also study the question of the sharpness of the bounds in (2.10) for *fixed* N (not just as $N \to \infty$). In particular, we provide the following infinite family of examples of 2-jump circulant graphs whose independence ratios attain the lower bound in (2.10) up to the absolute constant multiple of $1/\sqrt{N}$:

Proposition 2.27. *Let $d \in \mathbb{N}$ be odd, let $N = 2d(d+1)$, and let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d, d+1\})$. Then $\alpha(G) = d^2$, and G has girth $k = 2d + 1$, so $\alpha(G) = \left\lfloor \frac{k-1}{2k} N \right\rfloor - \frac{d-1}{2}$.*

We also provide an infinite family of examples attaining the upper bound in (2.10) (see Proposition 5.29).

At the end of the chapter we discuss possible future directions of study of Motzkin's problem in compact abelian groups.

2.5 Summary and conclusions

The results presented in this thesis make contributions to graph theory and combinatorial number theory through the exploration of recently emerged connections between topics in these areas and central tools and techniques from ergodic theory and the geometry of numbers.

Concerning graph theory, the thesis contributes to the topic of graph coloring with new notions that refine the fractional and circular chromatic numbers. In particular, we formalize and study the new concept of a *gyrocoloring* of graphs. Our results show that this notion, and its related chromatic parameter (the *gyrochromatic number*), are natural and robust concepts with interesting properties. The non-trivial analytic aspects of the gyrochromatic number also led us to another natural and more tractable coloring notion, in which the vertices of a graph G are colored by translates of a box in the d-dimensional torus, refining the circular chromatic number by means of the *d-toral chromatic numbers*, and establishing connections with known problems concerning box-coverings of tori. The various results we obtained on the gyrochromatic and toral chromatic numbers also lead to open problems motivating future research.

Concerning combinatorial number theory, the thesis contributes to the topic of Motzkin's problem, a very natural question in this area that has been studied since the 1970s. While the many previous works on this problem focused on the original integer setting, and treated various special cases of the problem with very specific techniques, in this thesis we take a broad approach to the general problem by studying it in other abelian groups and relating it with tools from ergodic theory (such as Rokhlin's lemma) and from the geometry of numbers. We illustrate this by studying the analogue of Motzkin's problem in the circle group, showing that this subsumes the original problem from the integer setting, and obtaining, among other results, an exact solution when the set of forbidden differences has two elements, with at least one of them being irrational. In the case of the forbidden differences being all rational, the geometric approach that we use leads to asymptotically tight estimates of the independence ratio of a 2-jump circulant graph in terms of the graph's odd girth, opening interesting prospects for analogous studies of d-jump circulant graphs with $d > 2$.

Chapter 3

Coloring graphs by translates in the circle

The work constituting this chapter was made in collaboration with Pablo Candela, Robert Hancock, Adam Kabela, Daniel Král', Ander Lamaison and Lluís Vena. The contents of the chapter were published in in *European Journal of Combinatorics* in 2021; see [11].

3.1 Introduction

Graph coloring is one of the most studied topics in graph theory. In order to refine the basic notion of the chromatic number of a graph, various non-integral relaxations were introduced, in particular, to capture how close a graph is to being colorable with fewer colors. Among them, the two most intensively studied notions are the *circular chromatic number* and the *fractional chromatic number*. We build on the work of Avila and Candela [3] who introduced a notion of a coloring base of a graph in relation to applications of their new proof of a generalization of Rokhlin's lemma; this notion leads to a chromatic parameter of a graph which lies between the circular and fractional chromatic numbers. The purpose of the present article is to introduce this parameter, which we refer to as the *gyrochromatic number* of a graph, in the context of graph coloring.

We begin by recalling the notions of circular and fractional colorings and fixing some notation. All graphs in this paper are finite and simple. If G

is a graph, then $V(G)$ and $E(G)$ are its vertex and edge sets, and $|G|$ is the number of its vertices. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k for which there exists a mapping $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ for every edge uv of G; we use $[k]$ to denote the set of the first k positive integers. The *circular chromatic number* $\chi_c(G)$ of a graph G is the smallest real z for which there exists a mapping $f : V(G) \to [0, z)$ such that $1 \leq |f(v) - f(u)| \leq z - 1$ for every edge uv (it can be shown that the minimum is always attained). The mapping f can be viewed as a mapping of the vertices of G to unit-length arcs of a circle of circumference z such that adjacent vertices are mapped to internally disjoint arcs. We remark that there are several equivalent definitions of the circular chromatic number [64], e.g., through homomorphisms to particular graphs or through balancing edge-orientations. In relation to the chromatic number, it is not hard to show that every graph G satisfies the following:

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G),$$

in particular $\chi(G) = \lceil \chi_c(G) \rceil$. The circular chromatic number was introduced by Vince [61] in the late 1980s and has been the main subject of many papers since then; we refer to the survey by Zhu [64] for a detailed exposition.

Coloring vertices of a graph G can be viewed as an integer program such that independent sets in G are assigned weights zero and one in such a way that every vertex belongs to independent sets whose total weight is (at least) one and the sum of the weights of all independent sets is minimized. Fractional coloring is a linear relaxation of this optimization problem: the *fractional chromatic number* $\chi_f(G)$ is the smallest real z for which there is an assignment of non-negative weights to independent sets in G such that the sum of their weights is z and each vertex belongs to independent sets whose total weight is at least one (it can be shown that the minimum is always attained). Equivalently, $\chi_f(G)$ can be defined as the smallest real z such that each vertex v of G can be assigned a Borel subset of $[0, z)$ of measure one in such a way that adjacent vertices are assigned disjoint subsets. It can be shown that the circular chromatic number lies between the fractional chromatic number and chromatic number for every graph G:

$$\chi_f(G) \leq \chi_c(G) \leq \chi(G).$$

Both inequalities can be strict, and the gap between the fractional chromatic number and chromatic number (and so the circular chromatic number) can be arbitrarily large. For instance, the chromatic number of Kneser graphs can be arbitrarily large and their fractional chromatic number arbitrarily close to two. We recall that the Kneser graph $K(n,k)$ has $\binom{n}{k}$ vertices that are viewed as corresponding to k-element subsets of $[n]$; two vertices are adjacent if the corresponding subsets are disjoint. The chromatic number of $K(n,k)$ is $n-2k+2$ (if $n \geq 2k$) by the famous result of Lovász [51], and their fractional chromatic number is known to be equal to n/k. We refer to the book by Scheinerman and Ullman [59] for further results on fractional coloring and fractional graph parameters in general.

We next recall the notion of a coloring base of a graph introduced in [3, Definition 3.8], which is the starting point for the present discussion. Let G be a graph and Z be an abelian group. A set $A \subseteq Z$ is a *coloring Z-base* for G if there exists a function $f : V(G) \to A$ such that the sets $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G; we write just *coloring base* if Z is clear from the context. For a topological group Z equipped with a Haar probability measure μ, we define

$$\sigma_Z(G) = \sup\{\mu(A) : \ A \text{ is a Borel coloring } Z\text{-base for } G\}. \tag{3.1}$$

This notion is related to results in ergodic theory, and we refer the reader to [3] for the exposition of this relation. We will be particularly interested in the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the groups $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ for $N \in \mathbb{N}$. The former can be viewed as given by the unit interval $[0,1]$ with 0 and 1 identified and the usual Borel measure, and the latter is simply $\{0,\ldots,N-1\}$ with addition modulo N equipped with the uniform discrete probability measure.

The notion of a coloring base resembles equivalent definitions of circular and fractional chromatic number, which can be cast using the group $\mathbb{T}$ as follows. The circular chromatic number of a graph G is the inverse of the maximum $\mu(A)$ where A is a *connected* coloring $\mathbb{T}$-base for G (here, connected means as a subset of $\mathbb{T}$). The fractional chromatic number of a graph G is the inverse of the maximum z such that each vertex of G can be assigned a Borel subset of $\mathbb{T}$ with measure z and adjacent vertices are assigned disjoint subsets. Informally speaking, circular coloring assigns vertices of G to arcs and fractional coloring assigns vertices of G to Borel subsets. A coloring $\mathbb{T}$-base is tighter than fractional coloring since vertices must be assigned rotational copies of the same Borel subset, and it is looser than circular coloring since the Borel subset assigned to vertices need not be an arc. This leads us to the following definition: the *gyrochromatic number* $\chi_g(G)$ of a graph G is the inverse of $\sigma_{\mathbb{T}}(G)$. The equivalent definitions of the circular and fractional chromatic numbers given above yield that the gyrochromatic number of every graph G lies between its fractional and circular chromatic numbers:

$$\chi_f(G) \leq \chi_g(G) \leq \chi_c(G). \tag{3.2}$$

Similarly to the notions of the fractional and circular chromatic number, the definition of the gyrochromatic number of a graph is robust in the sense that it can be cast in several different ways. In Section 3.2, we show that the definition through coloring $\mathbb{T}$-bases is equivalent to its discrete variant using $\mathbb{Z}_N$-bases (cf. Corollary 3.3). Coloring, circular coloring and fractional coloring of a graph can be defined by using homomorphisms to special classes of graphs: cliques, circular cliques and Kneser graphs. This holds also for the gyrochromatic number of a graph, which can be defined in terms of homomorphisms to circulant graphs (cf. Theorem 3.4). More interestingly, in Section 3.4 we prove that the definition stays the same when considering higher-dimensional-torus analogues of $\mathbb{T}$ (cf. Theorem 3.9), which implies that the discrete variant of the definition is the same when all finite abelian groups are considered instead of the groups $\mathbb{Z}_N$ only (cf. Corollaries 3.10 and 3.11). We note that the fractional chromatic number can be cast in a similar way but taking into account all finite (not necessarily abelian) groups (see the discussion after Corollary 3.11), i.e., the gyrochromatic number can be viewed as an analogue of the fractional chromatic number restricted to abelian groups. We believe that these properties show that the gyrochromatic number of a graph is a natural and robust parameter, which is likely to play an important role in providing a more detailed understanding of the structure of graphs whose circular and fractional chromatic numbers differ.

In addition to presenting several equivalent definitions of the gyrochromatic number in Sections 3.2 and 3.4, we also establish some of its basic properties and in particular construct graphs with gyrochromatic number strictly between the circular and fractional chromatic numbers (cf. Theorem 3.12). Finally, in Section 3.6 we show that, somewhat surprisingly, the supremum in the definition (3.1) need not be attained, which also implies that the infimum in its discrete variant (as given in Corollary 3.3) need not be attained.

3.2 Equivalent definitions

In this section, we present alternative definitions of the gyrochromatic number, and show that these are equivalent to the original definition stated in Section 3.1. We begin by giving another definition, which is analogous to circular coloring and provides the notion of a gyrocoloring. If z is a non-negative real, a *z-gyrocoloring* of a graph G is a mapping g from $V(G)$ to Borel subsets of $[0, z)$ such that the measure of each of $g(u)$, $u \in V(G)$, is one, the sets $g(u)$ and $g(v)$ are disjoint for every edge uv, and the sets $g(u)$ and $g(v)$ are rotationally equivalent for any two vertices u and v of G, i.e., there exists $x \in [0, z)$ such that

$$g(u) = (g(v) + x) \mod z = \{(y + x) \mod z : y \in g(v)\}.$$

The gyrochromatic number of G is the infimum of all z such that G has a z-gyrocoloring. The equivalence of this definition to the one given in Section 3.1 is rather easy to see. For completeness, we include a short proof.

Proposition 3.1. *Let G be a graph. For every positive real z, the graph G has a z-gyrocoloring if and only if G has a coloring $\mathbb{T}$-base of measure z^{-1}.*

Proof. Fix a graph G and a positive real z. Suppose that g is a z-gyrocoloring. Fix a vertex v_0 of G and let $x_v \in [0, z)$ be such that $g(v) = (g(v_0) + x_v) \mod z$; in particular, $x_{v_0} = 0$. The set

$$\{x/z : x \in g(v_0)\}$$

is a coloring $\mathbb{T}$-base; this can be seen by setting the function f from the definition of a coloring base to be $f(v) = g(v)/z$.

For the other direction, let A be a coloring $\mathbb{T}$-base of measure z^{-1}, and let f be the function from the definition of a coloring base. We can now define a z-gyrocoloring of the graph G as follows:

$$g(v) = \{(x + f(v)) \cdot z \mod z : x \in A\}$$

for every vertex $v \in V(G)$. □

We next turn our attention to definitions where the equivalence is more complicated to see, and start with showing that the discrete and Borel variants of the definition of the gyrochromatic number are equivalent.

Theorem 3.2. *Let G be a graph. It holds that $\sigma_{\mathbb{Z}_N}(G) \le \sigma_{\mathbb{T}}(G)$ for every $N \in \mathbb{N}$ and*

$$\sigma_{\mathbb{T}}(G) = \sup_{N\in\mathbb{N}} \sigma_{\mathbb{Z}_N}(G) = \lim_{N\to\infty} \sigma_{\mathbb{Z}_N}(G).$$

Proof. Fix a graph G. If A is a coloring $\mathbb{Z}_N$-base for G, then the set

$$\bigcup_{x\in A} \left[\frac{x-1}{N}, \frac{x}{N}\right)$$

is a coloring $\mathbb{T}$-base. Hence, $\sigma_{\mathbb{T}}(G) \ge \sigma_{\mathbb{Z}_N}(G)$ for every $N \in \mathbb{N}$. We therefore have

$$\sigma_{\mathbb{T}}(G) \ge \sup_{N\in\mathbb{N}} \sigma_{\mathbb{Z}_N}(G) \ge \limsup_{N\to\infty} \sigma_{\mathbb{Z}_N}(G).$$

We now prove that

$$\lim_{N\to\infty} \sigma_{\mathbb{Z}_N}(G) = \sigma_{\mathbb{T}}(G),$$

which will complete the proof.

Choose any $\varepsilon > 0$ and let A be a coloring $\mathbb{T}$-base such that $\mu(A) \ge \sigma_{\mathbb{T}}(G) - \varepsilon$. We may assume without loss of generality that $\mu(A) > 0$. Let $f : V(G) \to \mathbb{T}$ be a mapping such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of the graph G.

Since the measure μ is regular, there exists a closed set $B \subseteq A$ such that $\mu(A \setminus B) < \varepsilon$. For $m \in \mathbb{N}$, let $B_m := B + \left[\frac{-1}{m}, \frac{1}{m}\right]$; note that

$$B = \bigcap_{m\in\mathbb{N}} B_m \text{ and } \lim_{m\to\infty} \mu(B_m) = \mu(B).$$

Hence, there exists an integer m such that $\mu(B_m \setminus B) < \varepsilon$, whence $\mu(B_m \triangle A) \le \mu(B_m \triangle B) + \mu(B \triangle A) < 2\varepsilon$; fix such m for the rest of the proof. Note that the measure of B_m is at least $\mu(A) - 2\varepsilon \ge \sigma_{\mathbb{T}}(G) - 3\varepsilon$.

Choose $N \in \mathbb{N}$ such that $m/N \le \varepsilon$ and define $f'(v) = \lfloor Nf(v) \rfloor / N$; here and in what follows, multiplications such as $Nf(v)$ mean that $f(v)$ is viewed as an element of $[0,1]$ and is multiplied in $\mathbb{R}$ by N. Since the sets $A + f(u)$ and $B_m + f(u)$ differ on a set of measure at most 2ε for every vertex $u \in V(G)$, the measure of the intersection $B_m + f(u)$ and $B_m + f(v)$ is at most 4ε for every edge uv of G. Since the set B_m has at most $m/2$ connected components, i.e., B_m viewed as a subset of a circle consists of at most $m/2$ closed arcs, the measure of the intersection $B_m + f'(u)$ and $B_m + f'(v)$ is at most $4\varepsilon + m/N$. Choose $z \in \mathbb{T}$ randomly according to μ

and define a set $A(z) \subseteq \mathbb{Z}_N$ to be the set containing all $i \in \{0, \dots, N-1\}$ such that $z + i/N \in B_m$ and

$$z + i/N \notin B_m + f'(v) - f'(u)$$

for every edge uv of G. The probability that a particular i satisfies that $z + i/N \in B_m$ is equal to the measure of B_m. The probability that $z + i/N$ is both in B_m and in $B_m + f'(v) - f'(u)$ for a particular edge uv is equal to the measure of the intersection of $B_m + f'(u)$ and $B_m + f'(v)$, which is at most $4\varepsilon + m/N$. Hence, the probability that i is included in the set $A(z)$ is at least

$$\mu(A) - 2\varepsilon - (4\varepsilon + m/N)|E(G)| \geq \mu(A) - 2\varepsilon - 5\varepsilon|E(G)|.$$

The expected size of $A(z)$ is therefore at least

$$N(\mu(A) - 2\varepsilon - 5\varepsilon|E(G)|).$$

Fix z such that the size of $A(z)$ is at least the expected size.

We next show that $A(z)$ is a coloring $\mathbb{Z}_N$-base for G. Consider the function $f'' : V(G) \to \mathbb{Z}_N$ defined as $f''(v) = Nf'(v)$ and observe that the sets $A(z) + f''(u)$ and $A(z) + f''(v)$ are disjoint for every edge uv. Indeed, if the intersection of $A(z) + f''(u)$ and $A(z) + f''(v)$ were non-empty, there would exist $i, j \in A(z)$ such that $i + f''(u) = j + f''(v)$, which would imply that $z + i/N \in B_m$ and $z + j/N \in B_m$; since $z + i/N = z + j/N + f'(v) - f'(u)$, it would then follow that $z + i/N \in B_m + f'(v) - f'(u)$, contradicting the definition of $A(z)$. Hence $A(z)$ is a coloring $\mathbb{Z}_N$-base for G, so $\sigma_{\mathbb{Z}_N}(G) \geq \frac{|A(z)|}{N} = \mu(A) - 2\varepsilon - 5\varepsilon|E(G)|$.

We have thus proved that for all N sufficiently large (depending on ε) we have

$$\sigma_{\mathbb{T}}(G) \geq \sigma_{\mathbb{Z}_N}(G) \geq \sigma_{\mathbb{T}}(G) - 3\varepsilon - 5\varepsilon|E(G)|.$$

Since the choice of $\varepsilon > 0$ was arbitrary, we deduce that

$$\sigma_{\mathbb{T}}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G)$$

and the result follows. □

Theorem 3.2 yields the following.

Corollary 3.3. *The gyrochromatic number of a graph G is equal to the infimum of N/K for which there exists a K-element set $A \subseteq \mathbb{Z}_N$ and a function $f : V(G) \to \mathbb{Z}_N$ such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G.*

We next turn our attention to a definition through homomorphisms to circulant graphs. Fix an integer N and a set $S \subseteq \mathbb{Z}_N$ such that $S = -S$ and $0 \notin S$; the *circulant graph* $C(N, S)$ is the graph with vertex set $\mathbb{Z}_N$ such that two vertices i and j of $C(N, S)$ are adjacent if $j - i \in S$. If G and H are two graphs, we say that G is *homomorphic* to H if there exists a mapping $h : V(G) \to V(H)$ such that $h(u)h(v)$ is an edge of H for every edge uv of G. The mapping h is a *homomorphism* from G to H. If G is homomorphic to H, we also write $G \to H$ and say that H *admits a homomorphism* from G.

Theorem 3.4. *The gyrochromatic number of a graph G is equal to*

$$\inf_{\substack{N\in\mathbb{N}, S\subseteq\mathbb{Z}_N \\ G\to C(N,S)}} \chi_f\left(C(N, S)\right) = \inf_{\substack{N\in\mathbb{N}, S\subseteq\mathbb{Z}_N \\ G\to C(N,S)}} \frac{N}{\alpha\left(C(N, S)\right)} \,,$$

i.e., is equal to the infimum of the fractional chromatic numbers of the circulant graphs that admit a homomorphism from G.

Proof. We will show that the following holds for every $N \in \mathbb{N}$:

$$\sigma_{\mathbb{Z}_N}(G) = \max_{\substack{S\subseteq\mathbb{Z}_N \\ G\to C(N,S)}} \frac{\alpha\left(C(N, S)\right)}{N}. \tag{3.3}$$

Since the graph $C(N, S)$ is vertex transitive for every choice of S, it holds that the fractional chromatic number of $C(N, S)$ is equal to $N/\alpha\left(C(N, S)\right)$. In particular, the statement of the theorem will follow from (3.3) and Theorem 3.2.

Fix $N \in \mathbb{N}$. We prove the equality in (3.3) as two inequalities, starting with showing that

$$\sigma_{\mathbb{Z}_N}(G) \le \max_{\substack{S\subseteq\mathbb{Z}_N \\ G\to C(N,S)}} \frac{\alpha\left(C(N, S)\right)}{N}.$$

Let A be a coloring $\mathbb{Z}_N$-base for G and let $f : V(G) \to \mathbb{Z}_N$ be such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. We define the set S

as the set containing $f(u)-f(v)$ and $f(v)-f(u)$ for every edge uv. We claim that the set A is independent in the circulant graph $C(N,S)$: if A was not independent, then there would exist $i,j \in A$ such that $i = j+f(v)-f(u)$ for an edge uv, which would imply that the sets $A+f(u)$ and $A+f(v)$ are not disjoint ($i+f(u)$ would be their common element). Hence, the independence number of $C(N,S)$ is at least $|A|$. Since it holds that the vertices $f(u)$ and $f(v)$ of $C(N,S)$ are adjacent for every edge uv of G, the mapping f is a homomorphism from G to $C(N,S)$ and the inequality follows.

We next prove that

$$\sigma_{\mathbb{Z}_N}(G) \geq \max_{\substack{S\subseteq \mathbb{Z}_N \\ G\to C(N,S)}} \frac{\alpha\left(C(N,S)\right)}{N}.$$

Fix a set $S \subseteq \mathbb{Z}_N$ such that there exists a homomorphism $f : V(G) \to \mathbb{Z}_N$ from G to the circulant graph $C(N,S)$, and let A be an independent set of $C(N,S)$ of size $\alpha(C(N,S))$. We claim that A is a coloring $\mathbb{Z}_N$-base for G. To establish this, it is enough to show that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Since A is an independent set in G, the sets A and $A + x$ are disjoint for every $x \in S$. Consider an edge uv of G. Since f is a homomorphism from G to $C(N,S)$, it follows that $f(v) - f(u) \in S$, in particular, the sets A and $A + f(v) - f(u)$ are disjoint. Hence, the sets $A + f(u)$ and $A + f(v)$ are disjoint. We conclude that A is a coloring $\mathbb{Z}_N$-base. □

We remark that in Section 3.4 we establish a more general statement than that of Theorem 3.4; circulant graphs are Cayley graphs of the (abelian) group $\mathbb{Z}_N$ and we will show that the gyrochromatic number of a graph G is equal to the infimum of the fractional chromatic numbers of Cayley graphs of finite abelian groups that admit a homomorphism from G (cf. Corollary 3.11). We remark that the fractional chromatic number of G is equal to the minimum of the fractional chromatic numbers of Cayley graphs of finite groups that admit a homomorphism from G (see the discussion after Corollary 3.11).

3.3 Simple properties

In this section, we establish several simple properties of the gyrochromatic number; some of them will be used within our arguments later in the paper. We start with two properties related to products of graphs. Recall that the *Cartesian product* $G\square H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ such that two vertices (v, v') and (w, w') of $G\square H$ are adjacent if either $v = w$ and $v'w'$ is an edge of H or $v' = w'$ and vw is an edge of G. Similarly to the chromatic number and the circular chromatic number, the gyrochromatic number is also preserved by the Cartesian product of a graph with itself.

Proposition 3.5. *For every graph G, it holds that $\chi_g(G) = \chi_g(G\square G)$.*

Proof. We show that $A \subseteq \mathbb{T}$ is a coloring base for G if and only if it is a coloring base for $G\square G$. If A is a coloring base for $G\square G$, then it is clearly a coloring base for G. Hence, we focus on proving the reverse implication. Let A be a coloring $\mathbb{T}$-base for G and $f : V(G) \to \mathbb{T}$ such that $A + f(v)$ and $A + f(w)$ are disjoint for any edge vw of G. We define $g : V(G)^2 \to \mathbb{T}$ by setting $g(v, v')$ to be $f(v) + f(v')$, and show that if (v, v') and (w, w') are adjacent in $G\square G$, then $A + g(v, v')$ and $A + g(w, w')$ are disjoint.

Suppose that (v, v') and (w, w') are adjacent. It holds that either $v = w$ or $v' = w'$. In the former case, v' and w' are adjacent in G and so $A + f(v')$ and $A + f(w')$ are disjoint. Consequently, $A + g(v, v') = A + f(v) + f(v')$ and $A + g(w, w') = A + f(w) + f(w') = A + f(v) + f(w')$ are disjoint. Since the case $v' = w'$ is symmetric, A is a coloring base for the graph $G\square G$. $\square$

Proposition 3.5 does not generalize to Cartesian products of distinct graphs, however the following holds.

Proposition 3.6. *For any two graphs G and H, it holds that $\chi_g(G\square H) = \chi_g(G \cup H)$, where $G \cup H$ is the disjoint union of G and H.*

Proof. Since $G\square H$ is a component of the graph $(G \cup H)\square(G \cup H)$, it holds that $\chi_g(G\square H) \le \chi_g(G\cup H)$ by Proposition 3.5. On the other hand, both G and H are subgraphs of $G\square H$, which implies that if $A \subseteq \mathbb{T}$ is a coloring base for $G\square H$, then it is also a coloring base of $G\cup H$. Indeed, if $f : V(G\square H) \to \mathbb{T}$ is a function such that $A + f(v)$ and $A + f(w)$ are disjoint for any edge vw of $G\square H$, then its restriction to the copies of G and H in $G\square H$ witness that A is a coloring base for $G \cup H$. Hence, $\chi_g(G \cup H) \le \chi_g(G\square H)$. $\square$

In the next two paragraphs, we give an example of two (small) graphs G and H such that $\chi_g(G\square H) > \max\{\chi_g(G), \chi_g(H)\}$, which is equivalent to $\chi_g(G\cup H) > \max\{\chi_g(G), \chi_g(H)\}$ by Proposition 3.6. The two graphs G and H that we identify will also satisfy that $\chi_f(G\cup H) < \chi_g(G\cup H) < \chi_c(G\cup H)$; examples of connected graphs for which both inequalities are strict are given in Theorem 3.12.

Recall that the *lexicographic product* $G[G']$ of graphs G and G' is the graph whose vertex set is $V(G)\times V(G')$ and two vertices (v,v') and (w,w') are adjacent if either vw is an edge of G or $v = w$ and $v'w'$ is an edge of G'. Consider graphs $G = K_5$ and $H = K_2[C_5]$ and note that they both are circulant graphs (the graph $K_2[C_5]$ is isomorphic to the circulant graph $C(10,S)$ for $S = \{1,3,4,5,6,7,9\}$), and therefore their gyrochromatic number is equal to their fractional chromatic number, which is five for each of them. Clearly $\chi(G) = \chi_c(G) = 5$. Moreover, since the complement of H is the union of two disjoint copies of C_5, we can see that $\chi(H) = 6$, and also (since this complement is therefore disconnected) that $\chi_c(H) = \chi(H)$ by [64, Corollary 3.1]. We next observe that $\alpha(G\square H) \leq 9$: since G is a clique of size five, the independence number of $G\square H$ is the maximum total number of vertices assembled from five pairwise disjoint independent sets of H. As $\alpha(H) = 2$ and H is not 5-colorable, the maximum number of vertices of H contained in five (disjoint) independent sets of H is at most 9. We conclude that $\alpha(G\square H) \leq 9$, which implies that $\chi_f(G\square H) \geq 50/9$.

Since $\chi_g(G\square H) \geq 50/9$, Proposition 3.6 yields that $\chi_g(G\cup H) \geq 50/9 > \max\{\chi_g(G), \chi_g(H)\} = \max\{\chi_f(K_5), \chi_f(H)\} = 5$. On the other hand, observe that $G\cup H$ admits a 40/7-gyrocoloring. The structure of such a coloring is outlined in Figure 3.1. Namely, consider the set $[0, \frac{1}{2}) \cup [\frac{15}{14}, \frac{22}{14})$ and assign its five copies shifted by 0, $\frac{1}{2}$, $\frac{29}{14}$, $\frac{36}{14}$ and $\frac{58}{14}$ to the vertices of G, its five copies shifted by $\frac{8i}{7}$ for $i = 0,1,2,3,4$ to the vertices of one copy of C_5 in H and its five copies shifted by $\frac{8i+4}{7}$ for $i = 0,\dots,4$ to the vertices of the other copy of C_5 in H. Furthermore, recalling the fractional and circular chromatic numbers of G and H, we get that $\chi_f(G\cup H) = 5$ and $\chi_c(G\cup H) = 6$ which yields that $\chi_f(G\cup H) < \chi_g(G\cup H) < \chi_c(G\cup H)$.

Considering lexicographic products, Gao and Zhu [28] showed that if $\chi_f(G) = \chi_c(G)$, then $\chi_c(G[H]) = \chi_c(G)\,\chi(H)$. In particular, $\chi_c(K_k[H]) = k\,\chi(H)$. Recall that if H is a circulant graph, then $\chi_f(K_k[H]) = \chi_g(K_k[H]) = k\,\chi_f[H]$. Hence, if H satisfies $\chi_f(H) < \chi(H)$, then $K_k[H]$ is an example of a graph such that the gyrochromatic and circular chromatic numbers differ; the graph $K_2[C_5]$ discussed above is a particular case of this more general

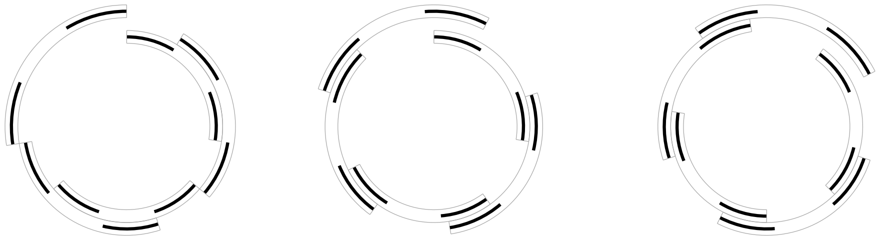


Fig. 3.1: Outline of the gyrocoloring of $K_5 \cup K_2[C_5]$. The five sets depicted on the left-hand side correspond to the vertices of K_5 (the sets are pairwise disjoint). The middle correspond to the vertices of induced C_5, and similarly the right-hand side correspond to the other C_5 (the union of the sets in the middle is disjoint from the union of the sets on the right).

argument.

We conclude this section with a lemma, which can be used to prove a lower bound on the gyrochromatic number of a graph that is larger than the fractional chromatic number. The lemma in particular yields an example of a graph such that its gyrochromatic number is strictly larger than its fractional chromatic number: consider the line graph G of the Petersen graph and note that $\omega(G) = \chi_f(G) = 3$ and $\chi(G) = 4$.

Lemma 3.7. *Let G be an n-vertex graph. If $\omega(G) < \chi(G)$, then $\chi_g(G) \geq \frac{n}{n-1}\omega(G)$.*

Proof. Let $k = \omega(H)$ and let $v_1, \ldots, v_k$ be vertices of a clique with k vertices in G. Suppose that A is a coloring $\mathbb{T}$-base for G with measure $\delta > 0$. Finally, let $f : V(G) \to \mathbb{T}$ be such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. In particular, the sets $A + f(v_1), \ldots, A + f(v_k)$ are pairwise disjoint.

For $x \in \mathbb{T}$ and $i \in [k]$, define $V_i(x) \subseteq V(G)$ as the set of vertices v such that $x + f(v_i) \in A + f(v)$. Since the element $x + f(v_i)$ is contained in the set $A + f(v)$ for every $v \in V_i(x)$, each of the sets $V_i(x)$ is an independent set in G. In addition, the sets $V_1(x), \ldots, V_k(x)$ are pairwise disjoint for every $x \in \mathbb{T}$. Indeed, if a vertex v were contained in $V_i(x)$ and $V_j(x)$, $i \neq j$, $i, j \in [k]$, then both $x + f(v_i)$ and $x + f(v_j)$ would be contained in $A + f(v)$; this would imply that the element $x - f(v) + f(v_i) + f(v_j)$ is contained both in $A + f(v_i)$ and $A + f(v_j)$, which contradicts that the sets $A + f(v_i)$ and $A + f(v_j)$ are disjoint.

Since the graph G is not k-colorable, the union $V_1(x) \cup \cdots \cup V_k(x)$ does not contain all vertices of G for any $x \in \mathbb{T}$. Hence, for every $x \in \mathbb{T}$ we have

$$|V_1(x) \cup \cdots \cup V_k(x)| = \sum_{i=1}^{k} |V_i(x)| \leq n - 1,$$

which implies that $\int_{\mathbb{T}} \sum_{i=1}^{k} |V_i(x)| \, \mathrm{d}x \leq n - 1$. On the other hand it holds that

$$\int_{\mathbb{T}} \sum_{i=1}^{k} |V_i(x)| \, \mathrm{d}x = \sum_{v \in V(G)} \sum_{i=1}^{k} \mu \left(A + f(v) - f(v_i)\right) = kn\delta.$$

It follows that δ is at most $\frac{n-1}{kn}$. We conclude that $\sigma_{\mathbb{T}}(G) \leq \frac{n-1}{kn}$, which yields that the gyrochromatic number of G is at least $\frac{n}{n-1}k$. $\square$

3.4 Universality

In this section, we show that a higher-dimensional-torus analogue of the gyrochromatic number is equal to the (one-dimensional) gyrochromatic number and use this result to prove a generalization of Theorem 3.4 to all finite abelian groups. As the first step towards this result, we observe that the proof of Theorem 3.2 readily translates to higher dimensions; we formulate the corresponding statement as a lemma.

Lemma 3.8. *Let d be any positive integer. For every graph G, it holds that*

$$\sigma_{\mathbb{T}^d}(G) = \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N^d}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N^d}(G).$$

We next state and prove the main theorem of this section.

Theorem 3.9. *Let d be any positive integer. For every graph G, it holds that*

$$\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G).$$

Proof. Fix an integer $d \geq 2$ and a graph G. Observe that if A is a coloring $\mathbb{T}$-base for G, then $A \times \mathbb{T}^{d-1}$ is a coloring $\mathbb{T}^d$-base for G; this implies that

$$\sigma_{\mathbb{T}}(G) \leq \sigma_{\mathbb{T}^d}(G).$$

The rest of the proof is devoted to establishing the reverse inequality. Choose $\varepsilon > 0$ arbitrarily and note that by Lemma 3.8, for all $N \in \mathbb{N}$ sufficiently large we have

$$\sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_{\mathbb{T}^d}(G) - \varepsilon. \tag{3.4}$$

For any such N, since the group $\mathbb{Z}_N^d$ is finite, there exists a coloring $\mathbb{Z}_N^d$-base A for G with $\sigma_{\mathbb{Z}_N^d}(G) \cdot N^d$ elements. Next choose an integer $k \in \mathbb{N}$ such that $1-\varepsilon \leq \left(\frac{k}{k+2}\right)^d$ and let N be large enough so that there are at least d distinct primes between $(k+1)N$ and $(k+2)N$ (the Prime Number Theorem implies that this holds for every N sufficiently large); let $P_1, \dots, P_d$ be such primes.

Define A' to be the subset of $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ such that

$$A' = \{(x_1 + y_1 N, \dots, x_d + y_d N) : (x_1, \dots, x_d) \in A \text{ and } y_1, \dots, y_d \in [k]\}.$$

Let $f : V(G) \to \mathbb{Z}_N^d$ be the function such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Observe that $A' + f(u)$ and $A' + f(v)$ are also disjoint for every edge uv of G when $f(u)$ and $f(v)$ are viewed as elements of $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ (here we use in particular that each of the primes $P_1, \dots, P_d$ is at least $(k+1)N$, so that no addition needs to be done modulo P_i). Let $M = P_1 P_2 \cdots P_d$ and define a mapping $g : \mathbb{Z}_M \to \mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ as

$$g(x) = (x \mod P_1, \dots, x \mod P_d).$$

Since $P_1, \dots, P_d$ are distinct primes, the mapping g is an isomorphism of the groups $\mathbb{Z}_M$ and $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$, in particular, g is a bijection.

We establish that the set $A'' = g^{-1}(A')$ is a coloring $\mathbb{Z}_M$-base for G. To do so, consider the function $f'' : V(G) \to \mathbb{Z}_M$ defined as $f''(v) = g^{-1}(f(v))$, $v \in V(G)$. Since g is an isomorphism of $\mathbb{Z}_M$ and $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$, it holds that $A'' + f''(u)$ and $A'' + f''(v)$ are disjoint for every edge uv of G. It follows that A'' is a coloring $\mathbb{Z}_M$-base for G, which implies that

$$\sigma_{\mathbb{Z}_M}(G) \geq \frac{|A''|}{M} = \frac{k^d |A|}{M} \geq \frac{k^d |A|}{(k+2)^d N^d} \geq (1-\varepsilon)\, \sigma_{\mathbb{Z}_N^d}(G). \tag{3.5}$$

The inequalities (3.4) and (3.5) yield that

$$\sigma_{\mathbb{Z}_M}(G) \geq (1-\varepsilon)\left(\sigma_{\mathbb{T}^d}(G) - \varepsilon\right) \geq \sigma_{\mathbb{T}^d}(G) - 2\varepsilon.$$

Since the choice of ε was arbitrary, it follows that $\sigma_{\mathbb{Z}_M}(G)$ is at least $\sigma_{\mathbb{T}^d}(G)$ for every sufficiently large M. Theorem 3.2 now implies that $\sigma_{\mathbb{T}}(G) \geq \sigma_{\mathbb{T}^d}(G)$, which completes the proof of the theorem. □

Theorem 3.9 yields the following corollary.

Corollary 3.10. *For every graph G, it holds that*

$$\sigma_{\mathbb{T}}(G) = \sup_{Z} \sigma_Z(G)$$

where the supremum is taken over all finite abelian groups Z.

Proof. Fix a graph G. Since every $\mathbb{Z}_N$, $N \in \mathbb{N}$, is a finite abelian group, Theorem 3.2 yields that the supremum given in the statement is at least $\sigma_{\mathbb{T}}(G)$. To establish the corollary, we need to show that $\sigma_{\mathbb{T}}(G) \geq \sigma_Z(G)$ for every finite abelian group Z.

Fix a finite abelian group Z; without loss of generality, we may assume that Z is $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$. Let A be a coloring Z-base with $\sigma_Z(G)|Z|$ elements and let $f : V(G) \to Z$ be a function such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Let N be the least common multiple of $M_1, \ldots, M_d$, and let π denote the natural homomorphism $\mathbb{Z}_N^d \to Z$, defined by $(x_1, \ldots, x_d) \mapsto (x_1 N/M_1, \ldots, x_d N/M_d)$. It is then easy to see that the preimage $A_N := \pi^{-1}(A)$ is a coloring $\mathbb{Z}_N^d$-base, with corresponding map $f' : V \to \mathbb{Z}_N^d$ where for each vertex v we let $f'(v)$ be any preimage of $f(v)$ under π. We deduce that $\sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_Z(G)$. By Lemma 3.8 and Theorem 3.9, we get that $\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G) \geq \sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_Z(G)$, and the result follows. □

We conclude this section with a generalization of Theorem 3.4 to finite abelian groups. Recall that if Z is an abelian group and S is a subset of Z such that $S = -S$ and $0 \notin S$, the *Cayley graph* $C(Z, S)$ is the graph with vertex set Z such that two vertices x and y of $C(Z, S)$ are adjacent if $y - x \in S$.

Corollary 3.11. *The gyrochromatic number of a graph G is equal to*

$$\inf_{\substack{Z, S \subseteq Z \\ G \to C(Z,S)}} \chi_f\left(C(Z, S)\right) = \inf_{\substack{Z, S \subseteq Z \\ G \to C(Z,S)}} \frac{|Z|}{\alpha\left(C(Z, S)\right)} \tag{3.6}$$

where the infimum is taken over all finite abelian groups Z.

Proof. The reasoning given in the proof of Theorem 3.4 yields that the following holds for every abelian group Z:

$$\sigma_Z(G) = \max_{\substack{S \subseteq Z \\ G \to C(Z,S)}} \frac{\alpha\left(C(Z, S)\right)}{|Z|}. \tag{3.7}$$

Since every Cayley graph is vertex-transitive, it holds for every abelian group Z and every $S \subseteq Z$:

$$\chi_f\left(C(Z,S)\right) = \max_{\substack{S \subseteq Z \\ G \to C(Z,S)}} \frac{\alpha\left(C(Z,S)\right)}{|Z|}. \tag{3.8}$$

The corollary now follows from (3.7), (3.8) and Corollary 3.10. □

We remark that if finite abelian groups Z in the infimum in (3.6) are replaced with all finite groups (with generating set S not containing the identity and satisfying $S = S^{-1}$), then the infimum is equal to the fractional chromatic number and is always attained. Indeed, it is well-known that the fractional chromatic number of a graph G is equal to the minimum fractional chromatic number of a Kneser graph that admits a homomorphism from G. Sabidussi's theorem [58] states that every vertex transitive graph (in particular, every Kneser graph) is a retract of (hence is homomorphically equivalent to) a Cayley graph, cf. [33, Theorem 3.1]. Hence, the gyrochromatic number can be viewed as a variant of the fractional chromatic number restricted to abelian groups.

3.5 Relation to circular and fractional colorings

We now identify graphs such that their gyrochromatic number is strictly between their fractional and circular chromatic numbers, and the difference with the latter number can be arbitrarily large.

Theorem 3.12. *There exists a sequence of graphs $(G_k)_{k\in\mathbb{N}\setminus\{1\}}$ such that $\chi_f(G_k) < \chi_g(G_k) \le \chi_c(G_k) = k+2$ and*

$$\lim_{k\to\infty} \chi_g(G_k) = 2.$$

Proof. We set G_k to be the Kneser graph $K(2k^3+k, k^3)$. Since the circular chromatic number and the chromatic number coincide for Kneser graphs [17, 18,49], it follows that the circular chromatic number $\chi_c(G_k)$ is equal to $k+2$. Recall that the fractional chromatic number of G_k is $\frac{2k^3+k}{k^3} = 2 + k^{-2}$, in particular, the limit of the fractional chromatic numbers of G_k is two.

We next show that the gyrochromatic numbers of the graphs G_k converge and their limit is also two. Since it holds that $\chi_f(G_k) \le \chi_g(G_k)$ for every graph G_k, the limit (assuming that it exists) must be at least two. For the upper bound, consider the Cayley graph $H_k = C(\mathbb{Z}_2^{2k^3+k}, S)$ with the generating set S consisting of those $x \in \mathbb{Z}_2^{2k^3+k}$ that have exactly $2k^3$ entries equal to one. The graph H_k admits a homomorphism from G_k: indeed, each vertex of $K(2k^3+k, k^3)$ corresponds to a k^3-element subset of $[2k^3+k]$ and we map it to the characteristic vector of this set. This mapping is indeed a homomorphism from G_k to H_k since two vertices of $G_k = K(2k^3+k, k^3)$ are adjacent if and only if their corresponding sets are disjoint, which happens if and only if the difference of their characteristic vectors (modulo two) has exactly $2k^3$ entries equal to one. Corollary 3.11 implies that

$$\limsup_{k\to\infty} \chi_g(G_k) \le \limsup_{k\to\infty} \frac{|H_k|}{\alpha(H_k)}.$$

We next show that the right limit is at most two.

Let $I_k \subseteq V(H_k)$ be the set of those elements of H_k with fewer than k^3 entries equal to one. Since I_k is an independent set in H_k, it follows that

$$\alpha(H_k) \ge |I_k| = \sum_{i=0}^{k^3-1} \binom{2k^3+k}{i} = 2^{2k^3+k} \cdot \left(\frac{1}{2} + o(1)\right),$$

which implies that

$$\limsup_{k\to\infty} \frac{|H_k|}{\alpha(H_k)} \le \lim_{k\to\infty} \frac{2^{2k^3+k}}{|I_k|} = 2.$$

We conclude that the sequence $\chi_g(G_k)$ converges and its limit is equal to two.

To complete the proof of the theorem, we need to show that $\chi_f(G_k) < \chi_g(G_k)$. To do so, it suffices to show that $\chi_f(G_k) < \chi_f(G_k \square G_k)$. Indeed, if we show this, then by Proposition 3.5 and the fact that the fractional chromatic number of every graph is at most its gyrochromatic number, we will have $\chi_f(G_k) < \chi_f(G_k \square G_k) \le \chi_g(G_k \square G_k) = \chi_g(G_k)$, as required.

Since the graphs G_k and $G_k \square G_k$ are vertex-transitive, it holds that

$$\chi_f(G_k) = \frac{|G_k|}{\alpha(G_k)} \qquad \text{and} \qquad \chi_f(G_k \square G_k) = \frac{|G_k \square G_k|}{\alpha(G_k \square G_k)} = \frac{|G_k|^2}{\alpha(G_k \square G_k)}.$$

Hence, it is enough to show that $\alpha(G_k \square G_k) < |G_k|\alpha(G_k)$. The Erdős-Ko-Rado Theorem yields that $\alpha(G_k) = \binom{2k^3+k-1}{k^3-1}$ and that for every independent set of G_k of this size, there exists $x \in [2k^3+k]$ such that the vertices of the independent set correspond to the $\binom{2k^3+k-1}{k^3-1}$ k^3-element subsets of $[2k^3+k]$ containing x. In particular, any two independent sets of the maximum cardinality in G_k have a non-empty intersection (here we use that $k \geq 2$).

Suppose that the graph $G_k \square G_k$ has an independent set of size $|G_k|\alpha(G_k)$, and let I be such an independent set. For every vertex v of G_k, the set $I_v = \{w : (v,w) \in I\}$ is an independent set in G_k (by definition of the Cartesian product). Since the set I contains $|G_k|\alpha(G_k)$ elements and $|I_v| \leq \alpha(G_k)$ for every vertex $v \in V(G_k)$, we obtain that $|I_v| = \alpha(G_k)$ for every $v \in V(G_k)$. Let v and v' be two adjacent vertices of G_k. It follows from the previous paragraph that I_v and I'_v have a common vertex w, i.e., the set I contains both (v,w) and (v',w), which contradicts that I is an independent set. Hence, the graph $G_k \square G_k$ has no independent set of size $|G_k|\alpha(G_k)$, i.e. $\alpha(G_k \square G_k) < |G_k|\alpha(G_k)$. This finishes the proof of the theorem. □

Theorem 3.12 immediately implies that the gap between the gyrochromatic number and the circular chromatic number can be arbitrarily large and there exists a graph such that its gyrochromatic number is strictly between its fractional and its circular chromatic numbers.

Corollary 3.13. *For every $k \in \mathbb{N}$, there exists a graph G such that $\chi_f(G) < \chi_g(G) \leq \chi_c(G) - k$.*

3.6 Existence of optimal gyrocoloring

In this section, we establish the existence of a graph G such that there is no coloring $\mathbb{T}$-base for G of measure $\sigma_{\mathbb{T}}(G)$, i.e., the supremum in (3.1) is not attained. In other words, there exists a graph G with no $\chi_g(G)$-gyrocoloring.

Let G_5 be the graph with vertex set $\mathbb{Z}_5^2$ and two vertices (i,j) and (i',j') adjacent if $i'-i \in \{2,3\}$ or $j'-j \in \{2,3\}$ (calculations modulo five). In other words, G_5 is the Cayley graph on $\mathbb{Z}_5^2$ with generating set $\mathbb{Z}_5^2 \setminus \{-1,0,1\}^2$. Proposition 3.15 and Theorem 3.19, which we will prove in this section, yield that G_5 has no coloring $\mathbb{T}$-base with measure $\sigma_{\mathbb{T}}(G_5)$.

We begin by analyzing the structure of independent sets of G_5.

Proposition 3.14. *The independence number of G_5 is four and the only independent sets of size four are the following 25 sets:*

$$I_v = \{v, v + (0,1), v + (1,0), v + (1,1)\}$$

where $v \in \mathbb{Z}_5^2$.

Proof. Let X be an independent set in G_5 and let $(i,j) \in \mathbb{Z}_5^2$ be any vertex of G_5 contained in X. Observe that the set X contains in addition to the vertex (i,j) at most one of the vertices $(i-1,j)$ and $(i+1,j)$, at most one of the vertices $(i,j-1)$ and $(i,j+1)$, at most one of the vertices $(i-1,j-1)$, $(i-1,j+1)$, $(i+1,j-1)$ and $(i+1,j+1)$, and no other vertex. In particular, the set X has size at most four. If the set X has size four, we can assume by symmetry that X contains the vertex $(i+1,j)$ from the first pair and the vertex $(i,j+1)$ from the second pair. If X contains the vertices (i,j), $(i+1,j)$ and $(i,j+1)$, then the only vertex from the last quadruple that it can contain is $(i+1,j+1)$. Hence, $X = I_v$ for $v = (i,j)$. □

We next compute $\sigma_{\mathbb{T}}(G_5)$. We remark that the construction of the coloring $\mathbb{Z}_5^2$-base for G_5 presented in the proof of the next proposition yields a coloring $\mathbb{T}^2$-base with measure $4/25$.

Proposition 3.15. *It holds that $\sigma_{\mathbb{T}}(G_5) = 4/25$.*

Proof. Since $\alpha(G_5) = 4$, it follows that $\chi_f(G_5) \geq 25/4$, which implies that $\sigma_{\mathbb{T}}(G_5) \leq 4/25$. On the other hand, the set $A = \{(0,0),(0,1),(1,0),(1,1)\}$ is a coloring $\mathbb{Z}_5^2$-base for G_5: indeed, the sets $A+v$ and $A+w$ are disjoint for every edge vw of G_5. Hence, $\sigma_{\mathbb{Z}_5^2}(G_5) \geq 4/25$, which yields that $\sigma_{\mathbb{T}}(G_5) \geq 4/25$ by Corollary 3.10. □

We next show that every coloring $\mathbb{T}$-base of G_5 of measure $4/25$ must have a very particular structure. In what follows, we will write $X \cong Y$ if two sets X and Y differ on a null set.

Lemma 3.16. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 and let $f : V(G_5) \to \mathbb{T}$ be such that $A + f(v)$ and $A + f(w)$ are disjoint for every edge vw; let B_v, $v \in V(G_5)$ be the set $A + f(v)$. If the measure of A is $4/25$, then there exist disjoint measurable subsets $C_{i,j}$, $(i,j) \in \mathbb{Z}_5^2$, such that*

$$B_{i,j} \cong C_{i,j} \cup C_{i+1,j} \cup C_{i,j+1} \cup C_{i+1,j+1}$$

for every $(i,j) \in \mathbb{Z}_5^2$, and the measure of each set $C_{i,j}$, $(i,j) \in \mathbb{Z}_5^2$, is $1/25$.

Proof. Let $\mathcal{I}$ be the set containing all independent sets of vertices of G_5. For each $I \in \mathcal{I}$, we define the following measurable subset of $\mathbb{T}$:

$$D_I = \{x \in \mathbb{T} : x \in A + f(v) \text{ if and only if } v \in I\}.$$

Observe that every $x \in \mathbb{T}$ belongs to D_I for some $I \in \mathcal{I}$: indeed, the set containing the vertices v such that $x \in A + f(v)$ is independent (note that $\emptyset \in \mathcal{I}$). Hence, the sets D_I, $I \in \mathcal{I}$, partition $\mathbb{T}$. We also have from the definition of D_I that

$$A + f(v) = \bigcup_{I \in \mathcal{I} : I \ni v} D_I, \tag{3.9}$$

and it follows that $\sum_{I \in \mathcal{I}} |I| \cdot \mu(D_I) = \sum_{v \in V(G_5)} \mu(A + f(v))$. On the other hand, since the measure of A is $4/25$, we have $\sum_{v \in V(G_5)} \mu(A + f(v)) = 4$. Hence, using Proposition 4.15, we have

$$4 = \sum_{I \in \mathcal{I}} |I| \cdot \mu(D_I) = 4 \sum_{I \in \mathcal{I} : |I| = 4} \mu(D_I) + \sum_{I \in \mathcal{I} : |I| < 4} |I| \cdot \mu(D_I).$$

This implies that $\mu(D_I) = 0$ for all $I \in \mathcal{I}$ with $|I| < 4$ (using $\sum_{I \in \mathcal{I}} \mu(D_I) = 1$). Setting

$$C_{i,j} = D_{\{(i-1,j-1),(i-1,j),(i,j-1),(i,j)\}},$$

we have by (4.17) that $B_{i,j} \cong C_{i,j} \cup C_{i+1,j} \cup C_{i,j+1} \cup C_{i+1,j+1}$ for every $(i, j) \in \mathbb{Z}_5^2$. To complete the proof, we need to show each set $C_{i,j}$ has measure $1/25$.

Recalling the notation I_v from Proposition 4.15, let M be the matrix with rows and columns indexed by the elements of $\mathbb{Z}_5^2$ such that $M_{(i,j),(i',j')}$ is one if $(i, j) \in I_{(i',j')}$ and zero otherwise. Further, let x be the vector with entries indexed by the elements of $\mathbb{Z}_5^2$ such that $x_{(i',j')}$ is the measure of $D_{I_{(i',j')}}$. Observe (using (4.17) and that $\mu(D_I) = 0$ for $|I| < 4$) that Mx is the vector with all entries equal to $4/25$. We next show that the matrix M is invertible. This would imply that the vector x with all entries equal to $1/25$ is the only vector such that Mx is the vector with all entries equal to $4/25$, which would complete the proof.

Assume that the matrix M is singular, i.e., there exists a non-zero vector x such that Mx is the zero vector. The entries of x can be interpreted as numbers on the toroidal 5×5 grid such that each of the 25 quadruples of entries forming a square sums to zero; a "square" stands here for a translate of $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ in the grid.

We can assume that the first row of the grid is $\alpha, \beta_1, \ldots, \beta_4$ and the first column is $\alpha, \gamma_1, \ldots, \gamma_4$, which yields that the numbers are assigned to the grid as follows.

$$\begin{array}{ccccc}
\alpha & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\gamma_1 & -\alpha-\beta_1-\gamma_1 & +\alpha-\beta_2+\gamma_1 & -\alpha-\beta_3-\gamma_1 & +\alpha-\beta_4+\gamma_1 \\
\gamma_2 & +\alpha+\beta_1-\gamma_2 & -\alpha+\beta_2+\gamma_2 & +\alpha+\beta_3-\gamma_2 & -\alpha+\beta_4+\gamma_2 \\
\gamma_3 & -\alpha-\beta_1-\gamma_3 & +\alpha-\beta_2+\gamma_3 & -\alpha-\beta_3-\gamma_3 & +\alpha-\beta_4+\gamma_3 \\
\gamma_4 & +\alpha+\beta_1-\gamma_4 & -\alpha+\beta_2+\gamma_4 & +\alpha+\beta_3-\gamma_4 & -\alpha+\beta_4+\gamma_4
\end{array}$$

Since there is a square containing β_1, β_2, $\alpha+\beta_1-\gamma_4$ and $-\alpha+\beta_2+\gamma_4$, we obtain $\beta_1 = -\beta_2$. By considering other squares wrapping around, we obtain that $\beta_1 = -\beta_2 = \beta_3 = -\beta_4$, $\gamma_1 = -\gamma_2 = \gamma_3 = -\gamma_4$ and $\beta_4 = -\gamma_4$. Hence, the table can be rewritten as follows.

$$\begin{array}{ccccc}
\alpha & \beta_1 & -\beta_1 & \beta_1 & -\beta_1 \\
-\beta_1 & -\alpha & +\alpha & -\alpha & +\alpha \\
+\beta_1 & +\alpha & -\alpha & +\alpha & -\alpha \\
-\beta_1 & -\alpha & +\alpha & -\alpha & +\alpha \\
+\beta_1 & +\alpha & -\alpha & +\alpha & -\alpha
\end{array}$$

Considering two of the squares containing the entry α in the top left corner, we obtain $2(\alpha+\beta_1) = 0$ and $2(\alpha-\beta_1) = 0$. Hence $\alpha = \beta_1 = 0$, i.e., the vector x is zero. We conclude that the matrix M is invertible, which completes the proof. $\square$

We are now ready to prove the main lemma of this section. To simplify our notation, we will understand the subscripts indexing sets $B_{i,j}$ and $C_{i,j}$ in Lemma 3.16 as pairs $(i,j) \in \mathbb{Z}_5^2$, which allows us to perform addition as with the elements of $\mathbb{Z}_5^2$, e.g., $C_{(0,1)+(1,2)}$ is the set $C_{(1,3)} = C_{1,3}$. In addition, we write $t_{v\to w}$ for $f(w) - f(v)$, when A is a coloring $\mathbb{T}$-base and $f : V(G) \to \mathbb{T}$ is a function such that $A + f(v)$ and $A + f(w)$ are disjoint for every edge vw.

Lemma 3.17. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 with measure $4/25$ and let $C_{i,j} \subseteq \mathbb{T}$, $(i,j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 3.16. For every $v \in \mathbb{Z}_5^2$ and $w \in \{(0,0),(0,1),(1,0),(1,1)\}$, it holds that*

$$C_{v+w} + t_{v\to v+(1,0)} \cong C_{v+w+(1,0)} \tag{3.10}$$
$$C_{v+w} + t_{v\to v+(0,1)} \cong C_{v+w+(0,1)}. \tag{3.11}$$

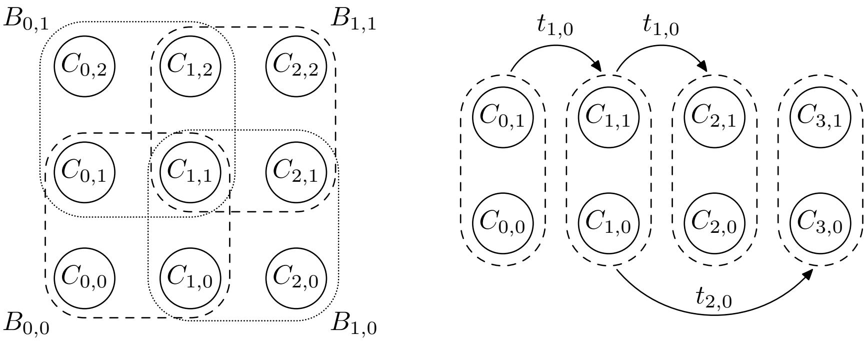


Fig. 3.2: Visualization of the notation used in the proof of Lemma 3.17 and the equalities (3.12), (3.13) and (3.14).

Proof. By symmetry, we will assume that $v = (0, 0)$ in our presentation with the exception of equality (3.15), which we formulate in the general setting. Throughout the proof, we write $B_{i,j} \subseteq \mathbb{T}$ for the sets as in Lemma 3.16; to simplify our notation, we also write $t_{i,j}$ for $t_{(0,0)\to(i,j)}$.

Our first goal is to prove the following weaker statement, which is also visualized in Figure 3.2.

$$(C_{0,0} \cup C_{0,1}) + t_{1,0} \cong C_{1,0} \cup C_{1,1} \tag{3.12}$$
$$(C_{1,0} \cup C_{1,1}) + t_{1,0} \cong C_{2,0} \cup C_{2,1} \tag{3.13}$$

Since the vertices $(0, 0)$ and $(2, 0)$ are adjacent, the sets $B_{0,0}$ and $B_{2,0} = B_{0,0} + t_{2,0}$ are disjoint, and so are these sets shifted by $t_{1,0}$, i.e., the sets $B_{0,0} + t_{1,0} = B_{1,0}$ and $B_{0,0} + t_{2,0} + t_{1,0} = B_{1,0} + t_{2,0}$. In particular, the intersection of $C_{2,0} \cup C_{2,1} \subseteq B_{1,0}$ and $(C_{1,0} \cup C_{1,1}) + t_{2,0} \subseteq B_{1,0} + t_{2,0}$ is empty. Since both $C_{2,0} \cup C_{2,1}$ and $(C_{1,0} \cup C_{1,1}) + t_{2,0}$ are subsets of $B_{2,0}$ and the measure of each of them is half of the measure of $B_{2,0}$, it follows that the sets $(C_{1,0} \cup C_{1,1}) + t_{2,0}$ and $B_{2,0} \setminus (C_{2,0} \cup C_{2,1}) = C_{3,0} \cup C_{3,1}$ are the same (up to a null set), i.e.,

$$(C_{1,0} \cup C_{1,1}) + t_{2,0} \cong C_{3,0} \cup C_{3,1}. \tag{3.14}$$

We formulate (3.14) for an arbitrary vertex v since we need the statement later:

$$(C_{v+(1,0)} \cup C_{v+(1,1)}) + t_{v\to v+(2,0)} \cong C_{v+(3,0)} \cup C_{v+(3,1)}. \tag{3.15}$$

We next apply (3.15) with $v = (1, 0)$ and $v = (3, 0)$ as follows:

$$\begin{aligned}(C_{2,0} \cup C_{2,1}) - t_{(0,0)\to(1,0)} &= (C_{2,0} \cup C_{2,1}) + t_{(1,0)\to(0,0)} \\ &= (C_{2,0} \cup C_{2,1}) + t_{(1,0)\to(3,0)} + t_{(3,0)\to(0,0)} \\ &\cong (C_{4,0} \cup C_{4,1}) + t_{(3,0)\to(0,0)} \\ &\cong C_{1,0} \cup C_{1,1},\end{aligned}$$

which proves (3.13). Since both the sets $(C_{0,0} \cup C_{0,1} \cup C_{1,0} \cup C_{1,1}) + t_{1,0}$ and $C_{1,0} \cup C_{1,1} \cup C_{2,0} \cup C_{2,1}$ are equal to $B_{1,0}$, in particular, they are the same set, and all sets $C_{i,j}$, $(i, j) \in \mathbb{Z}_5^2$, are disjoint, the equality (3.12) also follows.

An argument symmetric to the one used to prove (3.12) and (3.13) yields the following.

$$(C_{0,0} \cup C_{1,0}) + t_{0,1} \cong C_{0,1} \cup C_{1,1} \tag{3.16}$$

$$(C_{0,1} \cup C_{1,1}) + t_{0,1} \cong C_{0,2} \cup C_{1,2} \tag{3.17}$$

Next suppose for a contradiction that the intersection of $C_{0,0} + t_{1,0}$ and $C_{1,1}$ has positive measure, and let X be the set $C_{0,0} \cap (C_{1,1} - t_{1,0})$. The equality (3.17) implies that $X + t_{1,0} + t_{0,1}$ is a subset of $C_{0,2} \cup C_{1,2}$ (up to a null set). On the other hand $X + t_{0,1} \subseteq C_{0,0} + t_{0,1}$ is a subset of $C_{0,1} \cup C_{1,1}$ by (3.16), and this is a subset of $B_{0,0}$, hence $(X + t_{0,1}) + t_{1,0} \subseteq B_{0,0} + t_{1,0} = B_{1,0}$. However $B_{1,0} \cong C_{1,0} \cup C_{1,1} \cup C_{2,0} \cup C_{2,1}$ has null intersection with $C_{0,2} \cup C_{1,2}$. We have thus deduced that $X + t_{1,0} + t_{0,1}$ is included (up to a null set) in the null set $B_{1,0} \cap (C_{0,2} \cup C_{1,2})$, which contradicts the assumption that $X + t_{0,1} + t_{1,0}$ has positive measure. We conclude that the intersection of $C_{0,0} + t_{1,0}$ and $C_{1,1}$ is null. Since all the sets $C_{0,0}$, $C_{0,1}$, $C_{1,0}$ and $C_{1,1}$ have the same measure, the equality (3.12) implies that

$$C_{0,0} + t_{1,0} \cong C_{1,0} \text{ and } C_{0,1} + t_{1,0} \cong C_{1,1}. \tag{3.18}$$

A symmetric argument implies that

$$C_{0,0} + t_{0,1} \cong C_{0,1} \text{ and } C_{1,0} + t_{0,1} \cong C_{1,1}. \tag{3.19}$$

We now prove that the intersection of the sets $C_{1,1} + t_{0,1}$ and $C_{0,2}$ is null. Assume the contrary, i.e., the set $X = C_{1,1} \cap (C_{0,2} - t_{0,1})$ has positive measure. The equality (3.15) applied with $v = (0, 1)$ implies that

$$(C_{1,1} \cup C_{1,2}) + t_{(0,1)\to(2,1)} \cong C_{3,1} \cup C_{3,2}.$$

Since it holds that $B_{0,1} + t_{(0,1)\to(2,1)} = B_{2,1}$, we get that

$$(C_{0,1} \cup C_{1,1} \cup C_{0,2} \cup C_{1,2}) + t_{(0,1)\to(2,1)} \cong C_{2,1} \cup C_{3,1} \cup C_{2,2} \cup C_{3,2}.$$

Since all the sets $C_{i,j}$ are disjoint, it follows that

$$(C_{0,1} \cup C_{0,2}) + t_{(0,1)\to(2,1)} \cong C_{2,1} \cup C_{2,2}.$$

Since $X + t_{0,1}$ is a subset of $C_{0,2}$, we obtain that

$$X+t_{2,1} = X+t_{0,1}+t_{(0,1)\to(2,1)} \subseteq C_{0,2}+t_{(0,1)\to(2,1)} \subseteq C_{2,1} \cup C_{2,2} \subseteq B_{1,1}. \quad (3.20)$$

On the other hand, it holds that $X \subseteq C_{1,1} \subseteq B_{1,1}$. Hence, the intersection of the sets $B_{1,1}$ and $B_{1,1} + t_{2,1}$ contains the set $X + t_{2,1}$, in particular, it has positive measure. However, this is impossible because $B_{1,1} = B_{0,0} + t_{1,1}$ and $B_{1,1} + t_{2,1} = B_{2,1} + t_{1,1}$ and the sets $B_{0,0}$ and $B_{2,1}$ are disjoint. We conclude that the intersection of sets $C_{1,1} + t_{0,1}$ and $C_{0,2}$ is null. Using (3.17), we obtain that

$$C_{0,1} + t_{0,1} \cong C_{0,2} \text{ and } C_{1,1} + t_{0,1} \cong C_{1,2},$$

and a symmetric argument yields that

$$C_{1,0} + t_{1,0} \cong C_{2,0} \text{ and } C_{1,1} + t_{1,0} \cong C_{2,1}.$$

The proof is now complete. □

Our next step is to deduce that the elements $t_{v\to w}$ can be replaced by integer combinations of the elements $t_{(0,0)\to(1,0)}$ and $t_{(0,0)\to(0,1)}$.

Lemma 3.18. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 with measure $4/25$ and let $C_{i,j} \subseteq \mathbb{T}$, $(i,j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 3.16. It holds that*

$$C_{i \bmod 5,\ j \bmod 5} \cong C_{0,0} + it_{(0,0)\to(1,0)} + jt_{(0,0)\to(0,1)}$$

for any two non-negative integers i and j.

Proof. We proceed by induction on $i+j \in \mathbb{Z}$; all calculations with subscripts are done modulo five throughout the proof. The base of the induction is the case $i + j \in \{0, 1\}$, which is implied by Lemma 3.17. For the rest of the proof fix i and j. By symmetry, we may assume that $j > 0$. Applying

Lemma 3.17 with $v = (0,0), (1,0), \ldots, (i,0), (i,1), \ldots, (i, j-1)$, we obtain the following.

$$C_{i,j-1} \cong C_{0,0} + t_{(0,0)\to(1,0)} + \cdots + t_{(i-1,0)\to(i,0)} + t_{(i,0)\to(i,1)} + \cdots \\ \cdots + t_{(i,j-2)\to(i,j-1)} \tag{3.21}$$

$$C_{i,j} \cong C_{0,1} + t_{(0,0)\to(1,0)} + \cdots + t_{(i-1,0)\to(i,0)} + t_{(i,0)\to(i,1)} + \cdots \\ \cdots + t_{(i,j-2)\to(i,j-1)} \tag{3.22}$$

Since $C_{0,1} \cong C_{0,0} + t_{(0,0)\to(0,1)}$ by Lemma 3.17, we conclude using (3.21) and (3.22) that

$$C_{i,j} \cong C_{i,j-1} + t_{(0,0)\to(0,1)}. \tag{3.23}$$

On the other hand, the induction yields

$$C_{i,j-1} \cong C_{0,0} + it_{(0,0)\to(1,0)} + (j-1)t_{(0,0)\to(0,1)},$$

which combines with (3.23) to imply

$$C_{i,j} \cong C_{0,0} + it_{(0,0)\to(1,0)} + jt_{(0,0)\to(0,1)}.$$

□

We are now ready to prove the main theorem of this section.

Theorem 3.19. *The graph G_5 has no coloring $\mathbb{T}$-base with measure $4/25$.*

Proof. Suppose that there exists a coloring $\mathbb{T}$-base $A \subseteq \mathbb{T}$ with measure $4/25$, and let $C_{i,j} \subseteq \mathbb{T}$, $(i,j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 3.16. Further, let $\tau = t_{(0,0)\to(1,0)}$ and $\tau' = t_{(0,0)\to(0,1)}$. By Lemma 3.18 we have $C_{0,0} \cong C_{0,0} + 5\tau n$ and $C_{0,0} \cong C_{0,0} + 5\tau' n$ for any integer n. If 5τ were irrational, then the measure of $C_{0,0}$ would be either zero or one. Therefore, 5τ is rational. Similarly, $5\tau'$ is rational. It follows that both τ and τ' are also rational. Let p, q, r, p', q', r' be non-negative integers such that

$$\tau = \frac{p}{5^r q} \qquad \text{and} \qquad \tau' = \frac{p'}{5^{r'} q'},$$

p and $5^r q$ are coprime, p' and $5^{r'} q'$ are coprime, and neither q nor q' is divisible by five. By symmetry, we may assume that $r \le r'$. Let k be an integer such that $5^{r'-r}p$ and kp' are congruent modulo $5^{r'}$; note that such k exists since p' and $5^{r'}$ are coprime. Next observe that

$$q\tau = \frac{5^{r'-r}p}{5^{r'}} \qquad \text{and} \qquad kq'\tau' = \frac{kp'}{5^{r'}} = \frac{5^{r'-r}p}{5^{r'}} \mod 1. \tag{3.24}$$

By Lemma 3.18, we obtain that

$$\begin{aligned} C_{0,0} + q\tau &\cong C_{0,0} + qt_{(0,0)\to(1,0)} \cong C_{q \bmod 5,\ 0} \\ C_{0,0} + kq'\tau' &\cong C_{0,0} + kq't_{(0,0)\to(0,1)} \cong C_{0,\ kq' \bmod 5}. \end{aligned}$$

Since $q \bmod 5 \neq 0$ and the sets $C_{i,j}$ are disjoint sets of measure $1/25$ by Lemma 3.16, we obtain that the intersection of the sets $C_{0,0}+q\tau \cong C_{q \bmod 5,\ 0}$ and $C_{0,0} + kq'\tau' \cong C_{0,\ kq' \bmod 5}$ is null. However, $q\tau$ and $kq'\tau'$ is the same element of $\mathbb{T}$ by (3.24), i.e., $C_{0,0} + q\tau \cong C_{0,0} + kq'\tau'$. This contradicts our assumption on the existence of a coloring $\mathbb{T}$-base with measure $4/25$. $\square$

3.7 Conclusion

We finish with giving three open problems that we find particularly interesting and briefly mentioning a relation of the gyrochromatic number to another graph parameter, the ultimate independence ratio of a graph. The *independence ratio* $i(G)$ of a graph G is the ratio $\alpha(G)/|V(G)|$; the *ultimate independence ratio* $I(G)$, which was introduced in [35], is the limit of the independence ratios of Cartesian powers of G:

$$I(G) = \lim_{k\to\infty} \frac{\alpha(G^k)}{|V(G^k)|},$$

where G^k is the Cartesian product of k copies of G. The inverse of this quantity is the *ultimate fractional chromatic number* $\chi_F(G)$ of a graph G and the following holds [32, 63]:

$$\chi_F(G) = \frac{1}{I(G)} = \lim_{k\to\infty} \chi_f(G^k).$$

Zhu [63, p. 236] related the ultimate fractional chromatic number to coloring bases of abelian groups (though he used different terminology), via the following inequality:

$$I(G) \geq \sup\left\{ \tfrac{\alpha(H)}{|V(H)|} \,\middle|\, G \to H, H \text{ is a Cayley graph on a finite abelian group} \right\}.$$

(This inequality can be seen combining the fact that $I(G) \geq I(H)$ whenever $G \to H$ (see [32, Theorem 2.1]), together with $I(H) = \frac{\alpha(H)}{|V(H)|}$ when H is a Cayley graph on an abelian group (see [32, Section 5])). Using Corollary 3.11, we conclude that $\chi_F(G) \leq \chi_g(G)$. Hence, we obtain that the following holds for every graph G:

$$\chi_f(G) \leq \chi_F(G) \leq \chi_g(G) \leq \chi_c(G) \leq \chi(G).$$

It seems plausible that $\chi_F(G)$ and $\chi_g(G)$ differ for some graphs G, and it would be interesting to give examples of such graphs.

Question 3.20. *Construct a (connected) graph G such that $\chi_F(G) < \chi_g(G)$.*

We finish with two problems on the gyrochromatic number and its relation to fractional and circular chromatic numbers, which we believe to be of particular interest.

Question 3.21. *Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $\chi_g(G) \leq f(\chi_f(G))$ for every graph G?*

Question 3.22. *Does there exist a finite graph G such that the gyrochromatic number of G is not rational?*

Observe that a function f in Question 3.21 exists for all graphs G if and only if it exists for Kneser graphs, i.e., such a function f exists if and only if the gyrochromatic number of Kneser graph $K(m, n)$ is at most $f(m/n)$.

Also note that Theorem 3.12 implies that the circular chromatic number of a graph cannot be upper bounded by a function of its gyrochromatic number, i.e., a function f as in Question 3.21 does not exist for the gyrochromatic and the circular chromatic number.

In Section 3.6, we have constructed a graph G such that there is no coloring $\mathbb{T}$-base for G of measure $\sigma_{\mathbb{T}}(G)$, i.e. the supremum in 3.1 is not attained. However, the constructed graph G has a coloring $\mathbb{T}^2$-base with measure $\chi_g(G)^{-1}$ (see Proposition 3.15 and the remark before it), which leads to the following problem.

Question 3.23. *Does there exist for every graph G an integer d such that G has a $\mathbb{T}^d$-coloring base with measure $\chi_g(G)^{-1}$?*

These questions strongly motivate the study of intermediate coloring notions, between $\chi_g(G)$ and $\chi_c(G)$, which refine the latter number but whose analysis is more tractable than that of the former number. A natural candidate of such a refinement is introduced in the next chapter.

Chapter 4

On toral chromatic numbers of graphs

The content of this chapter is the result of joint work with Pablo Candela and Lluís Vena. The corresponding preprint [13] is currently being refereed for publication.

4.1 Introduction

The wide topic of graph coloring, central to graph theory, has been enriched by the study of various refinements of the concept of the chromatic number of a graph. One such refinement, the *star chromatic number*, was introduced by Vince in the 1980s [61], and was later shown by Zhu to be equivalent to another variant defined using the circle group [62], which led to the name *circular chromatic number* for this notion (see [64]).

Let us view the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as the interval $[0, 1]$ with addition modulo 1, and for $x \in \mathbb{T}$ let us denote by $|x|_{\mathbb{T}}$ the distance from x to the nearest integer. Then, given a graph G, which will be assumed to be finite and simple throughout this paper, the circular chromatic number of G is denoted by $\chi_c(G)$ and defined as the infimum of real numbers r such there is a map φ from the vertex set $V(G)$ to $\mathbb{T}$ satisfying $|\varphi(u) - \varphi(v)|_{\mathbb{T}} \geq 1/r$ for every edge $uv \in E(G)$.

For a subset A of an abelian group Z, we say that a graph G is A-*colorable* (or *colorable by* A) if there is a map $\varphi : V(G) \to \mathrm{Z}$ such that

$\big(A+\varphi(x)\big) \cap \big(A+\varphi(y)\big) = \emptyset$ for every edge $xy \in E(G)$; we then also say that φ is a *coloring map* of G by A, and that A is a *coloring base* in Z for G. Then $\chi_c(G)$ can be defined equivalently in terms of coloring G by open intervals $(0,\ell) \subset \mathbb{T}$:

$$\chi_c(G) = \inf\big\{1/\ell \ : \ G \text{ is colorable by } (0,\ell)\big\}. \tag{4.1}$$

Among the main basic properties of the circular chromatic number, there is the fact that the infimum in (4.1) is attained for every G, and is therefore a minimum, and also the fact that $\chi_c(G)$ is always a rational number. These properties follow immediately from results of Vince [61, Theorem 3], proved using a compactness argument in a continuous setting.

As with other central graph-coloring notions, the circular chromatic number can be described in terms of homomorphisms into a certain class of graphs. More precisely, denoting by $|G|$ the order of the graph G, it was proved by Bondy and Hell in [8] that

$$\chi_c(G) = \min\big\{s/r : r,s \in \mathbb{N}, s \leq |G|, \exists \,\text{homomorphism } G \to G_s^r\big\}, \tag{4.2}$$

for the class of *circular graphs* G_r^s, i.e. the Cayley graphs $\mathrm{Cay}(\mathbb{Z}_s, \{r, r+1, \ldots, s-r\})$, where $\mathbb{Z}_s$ denotes the group of integers with addition modulo s (see also [33, Theorem 4.20]). Formula (4.2) also implies immediately the rationality of $\chi_c(G)$, in a more elementary way than the above-mentioned analytic argument of Vince.

Another notable property of the circular chromatic number consists in the following sharp bounds describing the relation between $\chi_c(G)$ and the classical chromatic number $\chi(G)$ [61, Theorem 4], showing in particular that $\chi_c(G)$ carries more information about the structure of G than $\chi(G)$, so that $\chi_c(G)$ can be used to quantify how far G is from being colorable with less than $\chi(G)$ colors:

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G). \tag{4.3}$$

Recently, a new refinement of the circular chromatic number was introduced in [11], called the *gyrochromatic number* and denoted by $\chi_g(G)$. In this refinement, the set used as a coloring base in (4.1) is not required to be an interval, it can be any Borel set $A \subset \mathbb{T}$ of Haar probability measure $\mu_{\mathbb{T}}(A) = \ell$. More precisely,

$$\chi_g(G) := \inf\big\{1/\mu_{\mathbb{T}}(A) \ : \ G \text{ is colorable by the Borel set } A \subset \mathbb{T}\big\}. \tag{4.4}$$

Several results obtained in [11] show that $\chi_g(G)$ carries interesting information about the graph G, refining both the circular chromatic number $\chi_c(G)$ and the fractional chromatic number $\chi_f(G)$ (in particular we always have $\chi_f(G) \leq \chi_g(G) \leq \chi_c(G)$, and these inequalities can be strict; see [11, Corollary 14]). The gyrochromatic number is also shown to have a certain universality, in the sense that the infimum in (4.4) can be extended to Borel sets in a torus of arbitrary finite dimension (equipped with its Haar probability measure) without changing the value of $\chi_g(G)$. More precisely, for every positive integer r, letting $\mu_{\mathbb{T}^r}$ denote the Haar probability measure on the torus $\mathbb{T}^r$, we have (see [11, Theorem 10])

$$\chi_g(G) = \inf\{1/\mu_{\mathbb{T}^r}(A) : A \subset \mathbb{T}^r \text{ Borel such that } G \text{ is } A\text{-colorable}\}. \quad (4.5)$$

On the other hand, the number $\chi_g(G)$ is more elusive than $\chi_c(G)$, in the sense that, unlike for $\chi_c(G)$, the infimum in (4.4) is not always attained (see [11, Theorem 20]), and we do not yet know whether $\chi_g(G)$ is always rational (see [11, Problem 3], 3.22), nor do we know whether there is always at least *some* dimension $r \in \mathbb{N}$ such that the infimum in (4.5) is attained in this dimension (see [11, Problem 4], 3.23). These questions motivate the study of intermediate coloring notions, between $\chi_g(G)$ and $\chi_c(G)$, which refine the latter number but whose analysis is more tractable than that of the former number.

In this paper we study a natural candidate for such a refinement of $\chi_c(G)$, which consists in coloring the graph with translates of a box (or hyperrectangle) in the d-dimensional torus $\mathbb{T}^d$ (viewing the latter as $[0,1]^d$ with addition mod 1 in each coordinate).

By an *open box* in $\mathbb{T}^d$ we mean a Cartesian product of the form $I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ where I_j is an open interval (open connected set) in $\mathbb{T}$ for every $j \in [d]$. The corresponding refinement of the circular chromatic number is then defined as follows.

Definition 4.1 (d-toral chromatic number)**.** Let G be a graph. For each $d \in \mathbb{N}$, we define the *d-toral chromatic number* of G, denoted by $\chi_{c^d}(G)$, by the formula

$$\chi_{c^d}(G) = \inf\big\{1/\mu_{\mathbb{T}^d}(R) : R \text{ an open box in } \mathbb{T}^d,\ G \text{ is } R\text{-colorable}\big\}. \quad (4.6)$$

The 1-toral chromatic number is the circular chromatic number. Note also that by (4.5) we have $\chi_g(G) \leq \chi_{c^d}(G)$ for every d. Moreover, since for every open box $R \subset \mathbb{T}^d$ the set $R \times \mathbb{T}$ is an open box with $\mu_{\mathbb{T}^{d+1}}(R \times \mathbb{T}) = \mu_{\mathbb{T}^d}(R)$, it is readily seen that $\chi_{c^{d+1}}(G) \leq \chi_{c^d}(G)$ for every d. Hence the following inequalities hold for any positive integer d:

$$\chi_f(G) \leq \chi_g(G) \leq \chi_{c^{d+1}}(G) \leq \chi_{c^d}(G) \leq \chi(G). \tag{4.7}$$

The following questions arise. Firstly, whether the infimum in (4.6) is always attained and rational, as is the case for $\chi_c(G)$. Secondly, for every given graph G, since by (4.7) the decreasing sequence $(\chi_{c^d}(G))_{d\in\mathbb{N}}$ must converge, we may ask how fast it does so, and even whether it always becomes constant eventually. Thirdly, looking instead at how $\chi_{c^d}(G)$ varies for a fixed d and varying G, we may ask whether there are graphs G for which $\chi_{c^{d+1}}(G) < \chi_{c^d}(G)$; a positive answer here would indicate that each number $\chi_{c^d}(G)$ carries certain information about G that can make it differ from other such numbers $\chi_{c^{d'}}(G)$, $d' > d$, in other words, that graphs separate the toral chromatic numbers.

In Section 4.2 we answer positively the first question above, proving the following result.

Theorem 4.2. *Let G be a graph of order n and let d be a positive integer. Then for each $i \in [d]$ there are integers $r_i \leq s_i$ in $[n]$ such that G is colorable by the box $R = \prod_{i\in[d]}(0, \frac{r_i}{s_i})$ in $\mathbb{T}^d$ and $\chi_{c^d}(G) = \frac{1}{\mu_{\mathbb{T}^d}(R)} = \frac{s_1\cdots s_d}{r_1\cdots r_d}$.*

Our proof of this theorem extends the original analytic ideas from [61] working in the infinite setting of $\mathbb{T}^d$.

For any $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$ and any $\mathbf{r} = (r_1, \ldots, r_d) \in \prod_{i\in[d]}[s_i]$, let us define $G_{\mathbf{s}}^{\mathbf{r}}$ to be the Cayley graph $\mathrm{Cay}\big(\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_d}, (R - R)^c\big)$, where R is the box $\prod_{i\in[d]}[0, r_i - 1]$ in $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_d}$, where $R - R$ denotes the difference set $\{x - x' : x, x' \in R\}$, and where X^c denotes the complement of a set X. Thus x, y form an edge in $G_{\mathbf{s}}^{\mathbf{r}}$ if and only if the difference $x - y$ is not in $R - R$.

We note that Theorem 4.2 can be equivalently stated as the following result, which provides an expression of $\chi_{c^d}(G)$ in terms of homomorphisms from G into the class of graphs $G_{\mathbf{s}}^{\mathbf{r}}$, thus generalizing (4.2); we discuss this equivalence in Remark 4.10.

Proposition 4.3. *Let $d \in \mathbb{N}$. Then for any finite graph G of order n we* *have*

$$\chi_{c^d}(G) = \min\Big\{ \frac{s_1 \cdots s_d}{r_1 \cdots r_d} : \mathbf{s} \in [n]^d, \mathbf{r} \in \prod_{i\in[d]} [s_i], \text{ there is a homomorphism } G \to G_{\mathbf{s}}^{\mathbf{r}} \Big\}. \tag{4.8}$$

In Section 4.3 we prove the following fact, concerning the second question mentioned above.

Proposition 4.4. *Let G be a graph, and let $d = \lfloor \log_2(\chi(G)) \rfloor$. Then for every $d' \geq d$ we have $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$.*

This result justifies the definition of what we might call the *stabilization dimension* of a graph G for the toral chromatic numbers, namely the least integer d with the property that $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$ for all $d' \geq d$. Denoting the stabilization dimension of G by $d^*(G)$, we have $d^* \leq \lfloor \log_2(|G|) \rfloor$ by Proposition 4.4, and we may ask how accurate this upper bound is. In Section 4.4, we prove the following result, which tells that this bound is sharp up to a multiplicative constant. This result also answers positively the third question above, showing that graphs do indeed separate the toral chromatic numbers.

Theorem 4.5. *For each $d \in \mathbb{N}$ there exists a graph G of order $n = 5^d$ satisfying $d^*(G) = d = \log_5(n)$.*

Finally, in Section 4.5 we relate the number $\chi_{c^d}(G)$ to the classical chromatic number $\chi(G)$, via an inequality that generalizes the well-known inequality (4.3); see Proposition 4.21 and (4.39). This generalization is related to a problem of optimal coverings of the torus $\mathbb{T}^d$ by translates of a given box, an appealing combinatorial problem which has been treated previously in the special case of cubes in several works [25, 26, 54], but whose exact solution remains unknown in dimension $d \geq 3$.

4.2 Attainability and rationality of the d-toral chromatic number

Throughout the sequel we equip $[0,1]^d$ with the topology induced by the standard topology on $\mathbb{R}^d$. In this section we prove Theorem 4.2. We begin with the following topological result.

Lemma 4.6. *Let*

$$L := \big\{\ell \in [0,1]^d : \exists\, \varphi : V(G) \to \mathbb{T}^d, \\ \forall\, xy \in E(G),\, \exists\, i \in [d],\ |\varphi(x)_i - \varphi(y)_i|_{\mathbb{T}} \geq \ell_i\big\}. \tag{4.9}$$

Then L is a closed (hence compact) subset of $[0,1]^d$.

The relevance of this set L to proving Theorem 4.2 can be seen as follows: for any $\ell \in [0,1]^d$, letting $R = (0,\ell_1) \times \cdots \times (0,\ell_d)$, a map $\varphi : V(G) \to \mathbb{T}^d$ satisfies the condition in (4.9) if and only if $\varphi(x) - \varphi(y) \notin R - R = \prod_{i\in[d]}(-\ell_i,\ell_i) \subset \mathbb{T}^d$, which holds if and only if $(\varphi(x)+R)\cap(\varphi(y)+R) = \emptyset$. Hence

$$\ell \in L \quad \Longleftrightarrow \quad G \text{ is } R\text{-colorable, with } R = \prod_{i\in[d]}(0,\ell_i). \tag{4.10}$$

Proof of Lemma 4.6. We suppose that $(\ell^{(n)})_{n\in\mathbb{N}}$ is a sequence in L converging to ℓ^*, and we show that $\ell^* \in L$.

By definition of L, for each n there exists $\varphi_n : V(G) \to \mathbb{T}^d$, which we can view as a point in the compact space $(\mathbb{T}^d)^{V(G)}$, such that for every edge $xy \in E(G)$ there exists $i = i(n,xy) \in [d]$ such that $|\varphi_n(x)_i - \varphi_n(y)_i|_{\mathbb{T}} \geq \ell_i^{(n)}$. By compactness of $(\mathbb{T}^d)^{V(G)}$ there exists a subsequence $(\ell^{(m)})_{m\in I\subset\mathbb{N}}$ of $(\ell^{(n)})_{n\in\mathbb{N}}$ such that the points φ_m converge to some point $\varphi^* \in (\mathbb{T}^d)^{V(G)}$, which means that for every $x \in V(G)$ and every $i \in [d]$ we have $\varphi_m(x)_i \to \varphi^*(x)_i$ as $m \to \infty$. Moreover, by passing to further subsequences finitely many times, we can also ensure that for each $xy \in E(G)$ the map $n \mapsto i(n,xy)$ is constant. More precisely, by passing to a further subsequence for each edge of G (using each time that d is finite) we can obtain a subsequence $(\ell^{(r)})_{r\in J\subset I}$ of $(\ell^{(m)})_{m\in I\subset\mathbb{N}}$ such that the following properties hold:

1. $\ell^{(r)} \to \ell^*$ in $[0,1]^d$ as $r \to \infty$.

2. $\varphi_r \to \varphi^*$ in $(\mathbb{T}^d)^{V(G)}$ as $r \to \infty$.

3. $\forall\, xy \in E(G)$, $\exists\, i \in [d]$ such that $\forall\, r \in J$, $|\varphi_r(x)_i - \varphi_r(y)_i|_{\mathbb{T}} \geq \ell_i^{(r)}$.

Combining these properties and letting $r \to \infty$, we deduce that for every edge $xy \in E(G)$ there exists $i \in [d]$ such that $|\varphi^*(x)_i - \varphi^*(y)_i|_{\mathbb{T}} \geq \ell_i^*$, so ℓ^* is in L as required. □

We can now prove the first part of Theorem 4.2.

Theorem 4.7. *Let G be a graph and let $d \in \mathbb{N}$. Then there exists an open box $R = \prod_{i\in[d]}(0, \ell_i) \subset [0,1]^d$ such that G is R-colorable and $\chi_{c^d}(G) = \frac{1}{\mu_{\mathbb{T}^d}(R)}$.*

Proof. Consider the function $\pi : L \to \mathbb{R}$, $\ell \mapsto \prod_{i\in d} \ell_i$. By (4.10) we have

$$\frac{1}{\chi_{c^d}(G)} = \sup_{\ell\in L} \pi(\ell).$$

The function π is continuous, and by Lemma 4.6 the set L is compact, so π attains its supremum at some point $\ell = (\ell_1, \ldots, \ell_d) \in L$. This supremum $\pi(\ell)$ is the measure of the corresponding box $R = \prod_{i\in[d]}(0, \ell_i)$, and G is R-colorable since $\ell \in L$. □

We now turn to proving that $\chi_{c^d}(G)$ is a rational number. The proof is inspired by arguments in [64] and uses the following directed graphs associated with any given coloring of G by a box. We denote the elements of the standard basis of $\mathbb{R}^d$ as usual by e_i, for $i \in [d]$.

Definition 4.8 (Digraphs of a box-coloring)**.** Let G be a graph and let $\varphi : V(G) \to \mathbb{T}^d$ be a coloring of G by an open box $R \subset \mathbb{T}^d$. For each $i \in [d]$ we define a directed graph $D_i = D_i(G)$ as follows: the vertex set $V(D_i)$ is $V(G)$, and there is an arc (i.e. directed edge) $\overrightarrow{xy}$ from x to y if the following condition holds:

$$xy \in E(G) \text{ and } \forall\, \varepsilon > 0,\ \mu_{\mathbb{T}^d}[(R + \varphi(x) + \varepsilon e_i) \cap (R + \varphi(y))] > 0. \quad (4.11)$$

The measure-theoretic condition in (4.11) tells us that the box $R + \varphi(y)$ is contiguous with, and successive to, the box $R + \varphi(x)$, in the direction of e_i.

Lemma 4.9. *Let G be a graph, let $\varphi : V(G) \to \mathbb{T}^d$ be a coloring of G by an open box $R \subset \mathbb{T}^d$, and suppose that $D_i(G)$ is acyclic. Then there exists a coloring $\varphi' : V(G) \to \mathbb{T}^d$ of G by an open box $R' \subset \mathbb{T}^d$ such that $\mu_{\mathbb{T}^d}(R') > \mu_{\mathbb{T}^d}(R)$.*

Proof. For each vertex x, define the *level* $\mathcal{L}_i(x)$ to be the length of a longest directed path in $D_i(G)$ ending at x. Such a path exists since $D_i(G)$ is acyclic. Let x' be a vertex with maximum level. Then the box $R + \varphi(x')$ can be shifted in the direction of e_i by a positive distance without violating the condition that adjacent vertices receive disjoint translates of R; indeed, otherwise we could find $y \in V(G)$ such that x, y satisfy (4.11), and thus we would have a directed edge $\overrightarrow{xy}$ contradicting the maximality of $\mathcal{L}_i(x)$. After this shift of the box $R+\varphi(x')$, the vertex x' becomes an isolated vertex in the digraph $D_i(G)$. By repeating this process, we obtain a new d-toral coloring $\varphi' : V(G) \to \mathbb{T}^d$ such that the corresponding digraph D_i' has no arcs. In particular, we can replace R by a box R' obtained from R by multiplying the length in the e_i-direction by a factor $s > 1$, and still have that φ' colors G with R'. Then we have $\mu_{\mathbb{T}^d}(R') = s\mu_{\mathbb{T}^d}(R) > \mu_{\mathbb{T}^d}(R)$. □

We can now complete the proof of our first main result.

Proof of Theorem 4.2. Let $R = \prod_{i\in[d]}(0, \ell_i)$ be an open box given by Theorem 4.7 with corresponding coloring map φ. It follows from Lemma 4.9 that for every $i \in [d]$ the digraph D_i corresponding to this coloring contains a directed cycle $(x_0, x_1, ..., x_{s_i-1}, x_0)$, where clearly $s_i \leq n$. This means that the boxes $R + \varphi(x_j)$, $j \in [0, s_i - 1]$ have the property that their projections to the e_i-axis (which is isomorphic to $\mathbb{T}$) form a chain of consecutive contiguous intervals winding around this axis r_i times, for some positive integer r_i. Since the length of this e_i-axis is 1, and these intervals all have equal length ℓ_i, we deduce that $s_i\ell_i = r_i$, and the result follows. □

Remark 4.10. The equivalence of Theorem 4.2 and Proposition 4.3 is readily seen by noting that the coloring of G by the box R in $\mathbb{T}^d$ given in Theorem 4.2 can clearly be discretized to obtain a coloring by $R' := \prod_{i\in[d]}[0, r_i - 1]$ in the group $\prod_{i\in[d]} \mathbb{Z}_{s_i}$, and that such a coloring by R' yields a homomorphism $G \to G_{\mathbf{s}}^{\mathbf{r}}$; and vice versa, any such homomorphism can be viewed as a coloring by a box of adequate measure in $\mathbb{T}^d$. Let us mention that Proposition 4.3 can be proved directly, working purely in a finite setting, by extending the arguments of Bondy and Hell from [8]. In particular, the proof of [8, Proposition 2] can be extended from their setting of a single cyclic group $\mathbb{Z}_s$ to the present setting of $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_d}$. Indeed, supposing that $s > |G|$, that proof consists in a compression argument whereby a certain set S of points of $\mathbb{Z}_s$ is deleted and the remaining points are compressed into a shorter cyclic group in a way that is shown to yield a better circular coloring. This can be carried out similarly here in each component $\mathbb{Z}_{s_i}$, $i \in [d]$ that has $s_i > |G|$, by deleting not points but entire cosets of the subgroup $\mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_{i-1}} \times \{0\} \times \mathbb{Z}_{s_{i+1}} \times \cdots \times \mathbb{Z}_{s_d}$. We omit the details, as the resulting argument is in fact essentially a discrete analogue of the compression carried out in the proof of Lemma 4.9. We chose to argue in the continuous setting of $\mathbb{T}^d$ as this setting will also be more convenient for the proofs in the following sections.

4.3 On the stabilization dimension

Recall that the stabilization dimension of G for the toral chromatic numbers, which we denoted by $d^*(G)$, is the least $d \in \mathbb{N}$ such that $\chi_{c^{d'}}(G) = \chi_{c^d}(G)$ for all $d' \geq d$. The logarithmic upper bound for $d^*(G)$ mentioned in the introduction (Proposition 4.4) is a consequence of the following lemma.

Lemma 4.11. *Let G be a graph and let $d \in \mathbb{N}$ be a positive integer such that $\chi_{c^d}(G) < \chi_{c^{d-1}}(G)$. Then $\chi_{c^d}(G) \geq 2^d$.*

Proof. Let $R = (0, \ell_1) \times \cdots \times (0, \ell_d)$ be an open box in $\mathbb{T}^d$ such that G is R-colorable and $\mu_{\mathbb{T}^d}(R) = 1/\chi_{c^d}(G)$ (as guaranteed by Theorem 4.7). Our assumption $\chi_{c^d}(G) < \chi_{c^{d-1}}(G)$ implies that $\ell_i \leq 1/2$ for all $i \in [d]$. Indeed, if $\ell_i > 1/2$ for some i, then the difference set $R - R$ in $\mathbb{T}^d$ has i-th coordinate-projection covering all of $\mathbb{T}$. For every edge $xy \in E(G)$, the coloring map $\varphi : V(G) \to \mathbb{T}^d$ satisfies $R - R \not\ni \varphi(x) - \varphi(y)$ (since by definition of the coloring we have $(\varphi(x) + R) \cap (\varphi(y) + R) = \emptyset$). Then there must be $j \in [d] \setminus \{i\}$ such that $R - R \not\ni \varphi(x)_j - \varphi(y)_j$. This implies that if we define the new box R' from R by increasing ℓ_i to 1, then G is also R'-colorable with coloring map φ. Then, letting $\pi^{(i)}$ denote the projection $\mathbb{T}^d \mapsto \mathbb{T}^{d-1}$ which deletes the i-th coordinate, we see that the open box $R'' = \pi^{(i)}(R')$ and map $\varphi' = \pi^{(i)} \circ \varphi$ form an R''-coloring of G, with $\mu_{\mathbb{T}^{d-1}}(R'') = \mu_{\mathbb{T}^d}(R') \geq \mu_{\mathbb{T}^d}(R)$, so $\chi_{c^d}(G) \geq \chi_{c^{d-1}}(G)$, which contradicts our assumption. Hence $\ell_i \leq 1/2$ for all $i \in [d]$, so $\mu_{\mathbb{T}^d}(R) \leq 2^{-d}$, and the result follows. $\square$

Combining Lemma 4.11 with the inequality $\chi_{c^d}(G) \leq \chi(G)$ immediately implies Proposition 4.4.

4.4 Graphs that separate the toral chromatic numbers

In this section we produce a family of graphs $\{G_d : d \in \mathbb{N}\}$ such that $\chi_{c^d}(G_d) < \chi_{c^{d-1}}(G_d)$ for every d, thus providing examples of graphs that separate the toral chromatic numbers for different dimensions.

For each $d \in \mathbb{N}$, the graph G_d is a Cayley graph defined as follows. Let us view the cyclic group $\mathbb{Z}_5 := \mathbb{Z}/5\mathbb{Z}$ as the set of integers $\{0,1,2,3,4\}$ with addition mod 5, let Q_d be the set $\{0,1\}^d \subset \mathbb{Z}_5^d$, and let C_d be the difference set $Q_d - Q_d = \{-1,0,1\}^d \mod 5$ in $\mathbb{Z}_5^d$. The graph G_d is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_5^d, S_d)$ with generating set $S_d = \mathbb{Z}_5^d \setminus C_d$. Equivalently

$$V(G_d) := \mathbb{Z}_5^d, \quad E(G_d) = \big\{xy : x, y \in \mathbb{Z}_5^d, |x_i - y_i|_5 \geq 2 \text{ for some } i \in [d]\big\}, \tag{4.12}$$

where $|n|_5 = \min\{|n - 5m| : m \in \mathbb{Z}\}$. The r-toral chromatic number of G_d for $r \geq d$ is easily determined.

Lemma 4.12. *The Cayley graph $G_d = \mathrm{Cay}\big(\mathbb{Z}_5^d,\ \mathbb{Z}_5^d \setminus \{-1,0,1\}^d\big)$ satisfies*

$$\forall\, r \geq d, \quad \chi_{c^r}(G_d) = \chi_g(G_d) = \chi_f(G_d) = (5/2)^d. \tag{4.13}$$

Proof. Note that a subset $\mathcal{I}$ of the vertex set $\mathbb{Z}_5^d$ is independent if and only if $\mathcal{I} - \mathcal{I} \subset C_d$. Therefore $\mathcal{I}$ is independent if and only if for every $x, y \in \mathcal{I}$ we have $\max\{|x_i - y_i|_5 : i \in [d]\} \leq 1$, and it follows that $\mathcal{I}$ must be included in a translate $w + Q_d$ for some $w \in \mathbb{Z}_5^d$. Hence these translates are precisely the maximum independent sets in G_d. In particular, the independence number of G_d is

$$\alpha(G_d) = 2^d. \tag{4.14}$$

Hence, since G_d has order 5^d and is vertex-transitive (as a Cayley graph), we have $\chi_f(G_d) = 5^d/\alpha(G) = 5^d/2^d$. This immediately implies that $\chi_{c^d}(G_d) \geq 5^d/2^d$ (by (4.7)). To see that $\chi_{c^d}(G_d) = 5^d/2^d$, note that with the cube $Q' = (0, 2/5)^d \subset \mathbb{T}^d$, and the homomorphism $\varphi : \mathbb{Z}_5^d \to \mathbb{T}^d$ embedding $\mathbb{Z}_5^d$ as the subgroup $\frac{1}{5} \cdot \mathbb{Z}_5^d \subset \mathbb{T}^d$, we have a coloring of G_d by Q'. Indeed, since $C_d = Q_d - Q_d$, if xy is an edge then $(x + Q_d) \cap (y + Q_d) = \emptyset$ (otherwise

$x - y$ would be in C_d and not in S_d, contradicting that xy is an edge), so $(\varphi(x)+Q') \cap (\varphi(y)+Q') = \emptyset$. Therefore $\chi_{c^d}(G_d) \leq 5^d/2^d$. We conclude that $\chi_f(G_d) = \chi_g(G_d) = \chi_{c^d}(G_d) = 5^d/2^d$. This gives the case $r = d$ of (4.13). The case $r \geq d$ follows, since $\chi_f(G_d) \leq \chi_{c^r}(G_d) \leq \chi_{c^d}(G_d)$ for $r \geq d$. $\square$

The main result of this section is the following.

Theorem 4.13. *Fix any $d \in \mathbb{N}$. Then, for every integer $r \in [0, d-1]$ there is no Borel set $A \subset \mathbb{T}^r$ such that G_d is A-colorable and $\mu_{\mathbb{T}^r}(A) = 1/\chi_{c^d}(G_d)$.*

This theorem, combined with the fact that by Theorem 4.7 the supremum $1/\chi_{c^{d-1}}(G_d)$ is attained as the measure of some box $R \subset \mathbb{T}^{d-1}$ such that G_d is R-colorable, implies that $\chi_{c^{d-1}}(G_d) > \chi_{c^d}(G_d)$, as claimed at the beginning of this section. In particular, this together with (4.13) implies that $d^*(G_d) = d$, which implies Theorem 4.5. Note also that, since by Theorem 4.2 the numbers $\chi_{c^{d-1}}(G_d)$, $\chi_{c^d}(G_d)$ are rationals of denominator at most $|G_d|^d = 5^{d^2}$, we have in fact

$$\chi_{c^{d-1}}(G_d) \geq \chi_{c^d}(G_d) + 5^{-2d^2}. \tag{4.15}$$

Remark 4.14. Theorem 4.13 is stronger than necessary to deduce Theorem 4.5, since it tells us not just that no coloring box in any dimension $r < d$ attains the measure $1/\chi_{c^d}(G_d)$, but that in fact no Borel coloring set in any dimension $r < d$ attains this measure either. We prove this stronger result because it is also of interest concerning the gyrochromatic number, since it tells us that $\chi_g(G_d)$ is not attained on any torus of dimension less than d, i.e., the infimum in (4.5) is not attained for any $r < d$. As mentioned in the introduction, in [11, Problem 4] the question is posed of whether for each graph G there exists a finite dimension r for which the infimum in (4.5) is attained. It follows from Theorem 4.13 that such a dimension, if it exists, can be arbitrarily large depending on G. If one only aims to separate the toral chromatic numbers (i.e. just prove that $\chi_{c^d}(G_d) < \chi_{c^{d-1}}(G_d)$) then some parts of the argument that follows can be simplified; see Remark 4.20.

To prove Theorem 4.13, we need to show that no Borel set in $\mathbb{T}^r$ that is a coloring base for G_d can have measure $2^d/5^d$ (this being the value of $\chi_{c^d}(G_d)$ by Lemma 4.12). To do this, we shall generalize the main argument from [11, §6]. This task will occupy the rest of this section, and can be outlined as follows. The assumption that the coloring base has measure $2^d/5^d$ (which equals the inverse of the fractional chromatic number), and the fact that there are exactly 5^d maximal independent sets of size 2^d (see Proposition 4.15) implies that the fractional coloring is unique (i.e. the associated linear program has a unique solution when viewed in terms of Borel colorings; see Proposition 4.16). This in turn implies that such a coloring can be further decomposed into smaller 5^d coloring sets (with the property that each original coloring set is a union of these smaller sets), in such a way that the group $\mathbb{Z}_5^d$ can be viewed as acting on the family of these sets (see Lemma 4.17 and Lemma 4.18). Finally, using Lemma 4.19 to polish some possible measure-theoretic rough edges, we conclude that $\mathbb{Z}_5^d$ should be a subgroup of $\mathbb{T}^{d-1}$, leading to a contradiction. Thus, in summary, the extremal assumption that there is a coloring base of measure $2^d/5^d$ forces the coloring group (in this case $\mathbb{T}^{d-1}$) to include $\mathbb{Z}_5^d$ as a subgroup.

We begin by describing the structure of maximum independent sets of G_d. From (the proof of) Lemma 4.12, the following proposition follows clearly.

Proposition 4.15. *The only independent sets of maximal size 2^d in G_d are the following 5^d sets:*

$$I_v = v + Q_d,$$

where $v \in \mathbb{Z}_5^d$ and $Q_d := \{0,1\}^d \subset \mathbb{Z}_5^d$.

Next we show that if there existed a coloring base in $\mathbb{T}^r$ for G_d of measure $2^d/5^d$, then this base would have a very special structure, which will later be used to obtain a contradiction, implying that such a base cannot exist.

From now on we will use the notation from [11] and write $X \cong Y$ if two sets X and Y differ on a null set. Also, from now on in this section we abbreviate the notation for the Haar measure $\mu_{\mathbb{T}^r}$ to μ.

Lemma 4.16. *Let $A \subseteq \mathbb{T}^r$ be a Borel coloring base of the graph G_d and let $f : V(G_d) \to \mathbb{T}^r$ be such that $A + f(v)$ and $A + f(w)$ are disjoint for every $vw \in E(G_d)$; for each $v \in V(G_d)$ let $B_v = A + f(v) \subset \mathbb{T}^r$. If $\mu(A) = (2/5)^d$, then there exist disjoint measurable sets $C_v \subset \mathbb{T}^r$, $v \in \mathbb{Z}_5^d$, such that for every $v \in \mathbb{Z}_5^d$ we have*

$$B_v \cong \bigcup_{w \in Q_d} C_{v+w} \tag{4.16}$$

and for each $v \in \mathbb{Z}_5^d$ we have $\mu(C_v) = 1/5^d$.

Proof. Let $\mathcal{I}$ be the set containing all independent sets of vertices of G_d. For each $I \in \mathcal{I}$, we define the following measurable subset of $\mathbb{T}^r$:

$$D_I = \{x \in \mathbb{T}^r : x \in A + f(v) \text{ if and only if } v \in I\} =$$

$$= \big(\bigcup_{v\in I} A + f(v)\big) \setminus \big(\bigcup_{v\notin I} A + f(v)\big).$$

Observe that every $x \in \mathbb{T}^r$ belongs to D_I for some $I \in \mathcal{I}$. Indeed, the set containing the vertices v such that $x \in A + f(v)$ is an independent set I, by definition of A being a coloring base for G_d, and then $x \in D_I$ (note that we may have $I = \emptyset$, which we also consider to be an independent set). Since the sets D_I, $I \in \mathcal{I}$ are pairwise disjoint by definition, we conclude that these sets partition $\mathbb{T}^r$. We also have from the definition of D_I that

$$A + f(v) = \bigcup_{I\in\mathcal{I}:I\ni v} D_I, \tag{4.17}$$

and it follows that $\sum_{v\in V(G_d)} \mu(A + f(v)) = \sum_{I\in\mathcal{I}} |I| \cdot \mu(D_I)$. On the other hand, since the measure of A is $2^d/5^d$ and G_d has order 5^d, we have $\sum_{v\in V(G)} \mu(A + f(v)) = 2^d$. The last two equalities combined with the fact that $\alpha(G_d) = 2^d$ (by Proposition 4.15) imply that

$$2^d = \sum_{I\in\mathcal{I}} |I| \cdot \mu(D_I) = 2^d \sum_{I\in\mathcal{I}:|I|=2^d} \mu(D_I) + \sum_{I\in\mathcal{I}:|I|<2^d} |I| \cdot \mu(D_I).$$

This implies, using $\sum_{I\in\mathcal{I}} \mu(D_I) = 1$, that

$$\begin{aligned} 2^d &= 2^d\Big(1 - \sum_{I\in\mathcal{I}} \mu(D_I) + \sum_{I\in\mathcal{I}:|I|=2^d} \mu(D_I)\Big) + \sum_{I\in\mathcal{I}:|I|<2^d} |I| \cdot \mu(D_I) \\ &= 2^d - 2^d \sum_{I\in\mathcal{I}:|I|<2^d} \mu(D_I) + \sum_{I\in\mathcal{I}:|I|<2^d} |I| \cdot \mu(D_I) \\ &= 2^d + \sum_{I\in\mathcal{I}:|I|<2^d} (|I| - 2^d) \cdot \mu(D_I), \end{aligned}$$

which implies that $\mu(D_I) = 0$ for all $I \in \mathcal{I}$ with $|I| < 2^d$. Set

$$C_v := D_{v-Q_d}.$$

By (4.17) and the fact (given by Proposition 4.15) that the only maximal independent sets in G_d are the translates of Q_d, we have $B_v \cong \bigcup_{w\in Q_d} C_{v+w}$

for every $v \in \mathbb{Z}_5^d$. To complete the proof, we need to show each set C_v has measure $1/5^d$.

Recalling the notation I_v used in Proposition 4.15, let M be the $5^d \times 5^d$ matrix with rows and columns indexed by the elements of $\mathbb{Z}_5^d$ such that $M_{u,v} = 1$ if $u \in I_v$ and $M_{u,v} = 0$ otherwise. Let $x \in \mathbb{R}^{\mathbb{Z}_5^d}$ be the vector with entries indexed by the elements of $\mathbb{Z}_5^d$ such that $x_v = \mu(D_{I_v})$. Observe (using (4.17) and that $\mu(D_I) = 0$ for $|I| < 2^d$) that Mx is the vector with all entries equal to $\mu(A) = (2/5)^d$. Let $x' \in \mathbb{R}^{\mathbb{Z}_5^d}$ be the vector with all entries equal to $1/5^d$, and observe that Mx' also has all entries equal to $(2/5)^d$. Therefore, if we show that the matrix M is invertible, then we will have $x = x'$, which will complete the proof.

We now show that the matrix M is invertible. Let us assume that x is a vector such that $Mx = 0$, and let us prove that x must be the zero vector. The 5^d entries of x can be interpreted as an assignment of real numbers to the points of the grid $\mathbb{Z}_5^d$ such that each of the 2^d-tuples of entries corresponding to a translate of Q_d in $\mathbb{Z}_5^d$ sum to zero. It therefore suffices to prove the following claim for each $d \in \mathbb{Z}_{\geq 0}$:

$$\text{if } x \in \mathbb{R}^{\mathbb{Z}_5^d} \text{ satisfies } \sum_{w \in v+Q_d} x_w = 0 \text{ for every } v \in \mathbb{Z}_5^d, \text{ then } x = 0. \tag{4.18}$$

We prove this claim by induction on d. The case $d = 0$ is trivial. Fix $d > 0$. For $j \in [0, 4]$ let $S_j = \{v \in \mathbb{Z}_5^d : v_d = j\}$. Observe that for every j, for every $v \in S_j$ we have

$$\sum_{w \in (v+Q_d) \cap S_j} x_w = 0.$$

Indeed, by the assumption (4.18) for d, and splitting the sum $\sum_{w \in v+Q_d} x_w$ into two parts using the hyperplanes S_j and $S_{j+1 \bmod 5}$, we have

$$\sum_{w \in (v+Q_d) \cap S_j} x_w = - \sum_{w \in (v+Q_d) \cap S_{j+1 \bmod 5}} x_w.$$

Concatenating these equalities as j cycles through $\mathbb{Z}_5$, we end up deducing that

$$\sum_{v \in (w+Q_d) \cap S_j} x_v = - \sum_{v \in (w+Q_d) \cap S_j} x_v,$$

which confirms that $\sum_{w \in (v+Q_d) \cap S_j} x_w = 0$.
We have thus shown that for every j, the restriction of x to the $(d-1)$-dimensional hyperplane S_j is a vector such that its coordinates inside any

translate of $Q_d \cap S_j$ is 0, that is, this restriction satisfies (4.18) for $d-1$ (identifying S_j with $\mathbb{Z}_5^{d-1}$ the natural way). By induction, this implies that x restricted to S_j is the 0-vector. Since this holds for every j, we deduce that x is the 0 vector. This shows that the matrix M is invertible, which completes the proof. $\square$

We now prove the main lemma of this section. We will view the subscripts indexing the sets B_v and C_v in Lemma 4.16 as d-tuples $(i_1, i_2, \ldots, i_d) \in \mathbb{Z}_5^d$, which allows us to perform addition as with the elements of $\mathbb{Z}_5^d$. If A is a coloring base in $\mathbb{T}^r$ for G_d and $f : V(G_d) \to \mathbb{T}^r$ is a function such that $A + f(v)$ and $A + f(w)$ are disjoint for every edge vw, we write $t_{v\to w}$ for $f(w) - f(v)$. We also use the standard notation e_i to denote the vector with j-th coordinate equal to 1 if $j = i$ and equal to 0 otherwise.

Lemma 4.17. *With the notation and assumptions from Lemma 4.16, we have*

$$\forall\, i \in [d],\ \forall\, v \in \mathbb{Z}_5^d,\ \forall\, w \in Q_d, \quad C_{v+w} + t_{v\to v+e_i} \cong C_{v+w+e_i}. \tag{4.19}$$

Proof. We first note that Lemma 4.17 is equivalent to the following claim:

$$\forall\, S \subseteq [d],\ \forall\, \varepsilon \in \{0,1\}^S,\ \forall\, v \in \mathbb{Z}_5^d,\ \bigsqcup_{\substack{w \in Q_d: \\ w|_S = \varepsilon}} C_{v+w} + t_{v\to v+e_i} \cong \bigsqcup_{\substack{w \in Q_d: \\ w|_S = \varepsilon}} C_{v+w+e_i}. \tag{4.20}$$

Indeed, this claim implies the lemma since (4.19) is the special case of (4.20) with $S = [d]$. The opposite implication is also clear, since from (4.19) we can deduce (4.20) by taking appropriate unions.

To prove (4.20), first note that the case $S = \emptyset$ holds by definition of $t_{v\to v+e_i}$, since by (4.16) we have $\bigsqcup_{w\in Q_d} C_{v+w} \cong B_v$. Let us now show that if we prove the case $|S| = 1$ then the full claim (4.20) follows by induction on $|S|$.

Suppose that $|S| > 1$, fix any $\varepsilon \in \{0,1\}^S$, and assume by induction that (4.20) holds for every $S' \subsetneq S$ and every $\varepsilon' \in \{0,1\}^{S'}$. Note that there exist $S_1, S_2 \subsetneq S$ such that $S = S_1 \sqcup S_2$ (this requires $|S| > 1$). Then, using the inductive hypothesis, we have

$$\begin{aligned}
&\bigsqcup_{\substack{w\in Q_d:\\ w|_S=\varepsilon}} C_{v+w}+t_{v\to v+e_i}\\
&=\Bigg(\bigsqcup_{\substack{w\in Q_d:\\ w|_{S_1}=\varepsilon|_{S_1}}} C_{v+w}+t_{v\to v+e_i}\Bigg)\cap\Bigg(\bigsqcup_{\substack{w'\in Q_d:\\ w'|_{S_2}=\varepsilon|_{S_2}}} C_{v+w'}+t_{v\to v+e_i}\Bigg)\\
&\cong\Bigg(\bigsqcup_{\substack{w\in Q_d:\\ w|_{S_1}=\varepsilon|_{S_1}}} C_{v+w+e_i}\Bigg)\cap\Bigg(\bigsqcup_{\substack{w'\in Q_d:\\ w'|_{S_2}=\varepsilon|_{S_2}}} C_{v+w'+e_i}\Bigg)\\
&=\bigsqcup_{\substack{w\in Q_d:\\ w|_S=\varepsilon}} C_{v+w+e_i},
\end{aligned}$$

which proves (4.20) for S,ε. Hence it suffices to prove the case $|S|=1$ of (4.20).

Fix $i\in[d]$. We prove (4.20) for $|S|=1$ by separating the case $S=\{i\}$ from the case $S=\{j\}$, $j\neq i$.

To prove the case $S=\{i\}$, we want to show that

$$\text{for } \varepsilon\in\{0,1\},\ \forall\, v\in\mathbb{Z}_5^d,\quad \bigsqcup_{w\in Q_d:w_i=\varepsilon} C_{v+w}+t_{v\to v+e_i}\cong\bigsqcup_{w\in Q_d:w_i=\varepsilon} C_{v+w+e_i}. \tag{4.21}$$

Since v and $v-2e_i$ are neighbours in G_d, we have $(B_v+t_{v\to v-2e_i})\cap B_v=B_{v-2e_i}\cap B_v=\emptyset$. This implies that for every v' we have $(B_{v'}+t_{v\to v-2e_i})\cap B_{v'}=\emptyset$, since $B_{v'}=B_v+t_{v\to v'}$. In particular, taking $v'=v-e_i$ we have

$$(B_{v-e_i}+t_{v\to v-2e_i})\cap B_{v-e_i}=\emptyset. \tag{4.22}$$

Note that from (4.16) we clearly have both

$$\bigsqcup_{w\in Q_d:w_i=0} C_{v+w}=\bigsqcup_{w\in Q_d:w_i=1} C_{[v-e_i]+w}\ \subset\ B_{v-e_i}\quad\text{ and}$$

$$\bigsqcup_{w\in Q_d:w_i=0} C_{v-e_i+w}\ \subset\ B_{v-e_i}.$$

Combining this with (4.22), we deduce that the two sets $\big(\bigsqcup_{w\in Q_d:w_i=0} C_{v+w}\big)+t_{v\to v-2e_i}$ and $\bigsqcup_{w\in Q_d:w_i=0} C_{v-e_i+w}$ are disjoint. The latter set is one half of

B_{v-2e_i}, indeed

$$\begin{aligned}B_{v-2e_i} &= \bigsqcup_{w\in Q_d : w_i=1} C_{[v-2e_i]+w} \sqcup \bigsqcup_{w\in Q_d : w_i=0} C_{[v-2e_i]+w}\\ &= \bigsqcup_{w\in Q_d : w_i=0} C_{v-e_i+w} \sqcup \bigsqcup_{w\in Q_d : w_i=0} C_{[v-2e_i]+w}.\end{aligned}$$

Hence $\left(\bigsqcup_{w\in Q_d : w_i=0} C_{v-w}\right) + t_{v\to v-2e_i}$ must be the other half of B_{v-2e_i} (using that $B_v + t_{v\to v-2e_i} = B_{v-2e_i}$). It follows that

$$\text{for } \varepsilon\in\{0,1\}, \forall\, v\in\mathbb{Z}_5^d, \bigsqcup_{w\in Q_d : w_i=\varepsilon} C_{v+w} + t_{v\to v-2e_i} \cong \bigsqcup_{w\in Q_d : w_i=\varepsilon} C_{v-2e_i+w}. \tag{4.23}$$

We now deduce (4.21) by applying (4.23) with v, then with $v-2e_i$, and then using that $B_v + t_{v\to v+e_i} = B_v + t_{v\to v-4e_i} = B_v + t_{v\to v-2e_i} + t_{v-2e_i\to v-4e_i}$.

Now we treat the case $S=\{j\}$ with $j\neq i$, namely

$$\text{for } \varepsilon\in\{0,1\},\ \forall\, v\in\mathbb{Z}_5^d, \bigsqcup_{w\in Q_d : w_j=\varepsilon} C_{v+w} + t_{v\to v+e_i} \cong \bigsqcup_{w\in Q_d : w_j=\varepsilon} C_{v+w+e_i}. \tag{4.24}$$

It suffices to prove that

$$\forall\, v\in\mathbb{Z}_5^d,\ \forall\, w,u\in Q_d \text{ with } w_j=0 \text{ and } u_j=1; (C_{v+w}+t_{v\to v+e_i})\cap C_{v+u+e_i}\cong\emptyset. \tag{4.25}$$

Indeed this would imply the case $\varepsilon=0$ of (4.24), and the case $\varepsilon=1$ then follows easily (using the partition of B_v into two halves yielded by (4.24)).

To prove (4.25), note first that since $v+2e_j$, $v+e_i$ are neighbours in G, we have

$$(B_{v+2e_j} + t_{v+2e_j\to v+e_i}) \cap B_{v+2e_j} = \emptyset.$$

This implies that also $(B_{v+e_j} + t_{v+2e_j\to v+e_i}) \cap B_{v+e_j} = \emptyset$. Then, since C_{v+w+e_j}, C_{v+u+e_j} are both subsets of B_{v+e_j} (up to a null set), we have that $C_{v+u+e_j} + t_{v+2e_j\to v+e_i} \subset B_{v+e_j} + t_{v+2e_j\to v+e_i}$ is disjoint from $C_{v+w+e_j}\subset B_{v+e_j}$. Moreover, since C_{v+u+e_j} is also a subset of B_{v+2e_j} (since $u_j=1$), and $B_{v+2e_j} + t_{v+2e_j\to v+e_i} = B_{v+2e_j} + t_{v+2e_j\to v} + t_{v\to v+e_i}$, we have

$$C_{v+u+e_j} + t_{v+2e_j\to v+e_i} = C_{v+u+e_j} + t_{v+2e_j\to v} + t_{v\to v+e_i}.$$

Thus we have proved the following:

$$\forall\, v \in \mathbb{Z}_5^d,\ \forall\, w, u \in Q_d \text{ with } w_j = 0,\ u_j = 1,$$

$$(C_{v+u+e_j} + t_{v+2e_j \to v} + t_{v \to v+e_i}) \cap C_{v+w+e_j} = \emptyset. \tag{4.26}$$

Suppose for a contradiction that (4.25) fails, i.e.

$$\exists\, v' \in \mathbb{Z}_5^d,\ u', w' \in Q_d,\ w'_j = 0,\ u'_j = 1, \text{ with}$$

$$\mu\big((C_{v'+w'} + t_{v' \to v'+e_i}) \cap C_{v'+u'+e_i}\big) > 0. \tag{4.27}$$

We now distinguish three exhaustive cases according to possible values of u'_i, w'_i, and obtain a contradiction in each of these cases.

Case 1: $u'_i = 1$, $w'_i \in \{0, 1\}$.

In this case we shall contradict (4.26) from (4.27). To that end let us decompose the set $C_{v'+w'}$ in (4.27) using (4.23). We apply (4.23) with the index i in that formula set to be j, and with $v = v' + 2e_j$, $\varepsilon = 0$, thus obtaining

$$\bigsqcup_{w \in Q_d : w_j = 0} C_{v'+2e_j+w} + t_{v'+2e_j \to v'} = \bigsqcup_{w \in Q_d : w_j = 0} C_{v'+w}.$$

Since $C_{v'+w'}$ is among the sets in the union on the right side here (as $w'_j = 0$), and since we can write $v' + 2e_j + w$ as $v' + z + e_j$ for $z = w + e_j \in Q_d$ with $z_j = 1$ (since $w_j = 0$), we deduce that the sets $A_z := C_{v'+w'} \cap (C_{v'+z+e_j} + t_{v'+2e_j \to v'})$, $z \in Q_d$ with $z_j = 1$, form a partition of $C_{v'+w'}$ (up to a null set). Hence the sets $A_z + t_{v' \to v'+e_i}$ form a partition of $C_{v'+w'} + t_{v' \to v'+e_i}$. Therefore, by (4.27), for some $z \in Q_d$ with $z_j = 1$ we must have

$$\mu\big((A_z + t_{v' \to v'+e_i}) \cap C_{v'+u'+e_i}\big) > 0. \tag{4.28}$$

Then (4.28) contradicts (4.26), by setting u, v, w in the latter formula to be $z, v', u' - e_i - e_j$ in the former formula respectively (noting that w is then in Q with $w_j = 0$ as it should, since $u'_i = u'_j = 1$, and noting also that $u_j = z_j = 1$).

Case 2: $u'_i = 0$, $w'_i = 1$.

In this case

$$C_{v'+w'} + t_{v' \to v'+e_i} \subset \bigsqcup_{w \in Q_d : w_i = 1} C_{v'+w} + t_{v' \to v'+e_i} \overset{\text{by (4.21)}}{\cong} \bigsqcup_{w \in Q_d : w_i = 1} C_{v'+e_i+w},$$

which is disjoint from $C_{v'+u'+e_i} \subset \bigsqcup_{w\in Q_d:w_i=0} C_{v'+e_i+w}$, since the last two unions form two disjoint halves of $B_{v'+e_i}$. Therefore (4.25) holds with u, v, w equal to u', v', w' respectively, which contradicts (4.27).

Case 3 : $u'_i = 0$, $w'_i = 0$.

Recall that we are assuming from (4.27) that

$$\mu\big((C_{v'+w'} + t_{v'\to v'+e_i}) \cap C_{v'+u'+e_i}\big) > 0$$

with $u'_j = 1$, $w'_j = 0$, for a contradiction. Let $D = C_{v'+w'} \cap (C_{v'+u'+e_i} - t_{v'\to v'+e_i})$, thus by assumption $\mu(D) > 0$. By setting the index i in (4.21) to be j here (recalling that we have proved (4.21) already for every $i \in [d]$), we have

$$\forall\, \varepsilon \in \{0,1\}, \qquad \bigsqcup_{u\in Q_d:u_j=\varepsilon} C_{v'+u} + t_{v'\to v'+e_j} \cong \bigsqcup_{u\in Q_d:u_j=\varepsilon} C_{v'+u+e_j}. \tag{4.29}$$

Similarly, by applying (4.21) with index i we have

$$\bigsqcup_{u\in Q_d:u_i=0} C_{v'+u+e_i} + t_{v'\to v'+e_i} \cong \bigsqcup_{u\in Q_d:u_i=0} C_{v'+u+2e_i}. \tag{4.30}$$

Now, as $D + t_{v'\to v'+e_i} \subset C_{v'+u'+e_i}$, we have $(D + t_{v'\to v'+e_i}) + t_{v'\to v'+e_j} \subset C_{v'+u'+e_i} + t_{v'\to v'+e_j}$, and by (4.29) with $\varepsilon = 1$ (using also the assumption that $u'_j = 1$) this last set is included in $\bigsqcup_{u\in Q_d:u_j=1} C_{v'+u+e_j}$ up to a null set. Hence, up to a null set we have

$$(D + t_{v'\to v'+e_i}) + t_{v'\to v'+e_j} \subset \bigsqcup_{u\in Q_d:u_j=1} C_{v'+u+e_j}. \tag{4.31}$$

On the other hand, by (4.29) with $\varepsilon = 0$ we have

$$D + t_{v'\to v'+e_j} \subset \bigsqcup_{u\in Q_d:u_j=0} C_{v'+u+e_j},$$

and this union is included in $B_{v'}$, whence $D + t_{v'\to v'+e_j} \subset B_{v'}$ up to a null set. Thus

$$D + t_{v'\to v'+e_j} + t_{v'\to v'+e_i} \subset B_{v'} + t_{v'\to v'+e_i} = B_{v'+e_i}.$$

This implies by (4.31) that $B_{v'+e_i}$ has intersection of positive measure with $\bigsqcup_{u\in Q_d:u_j=1} C_{v'+u+e_j} \subset B_{v'+2e_j}$. But this is impossible since $B_{v'+e_i} \cap B_{v'+2e_j} = \emptyset$ (as $v'+e_i$ and $v'+2e_j$ are neighbours in G_d). This completes the proof of this last case involved in (4.25), and thus completes the proof of Lemma 4.17.

$\square$

Our next step is to deduce that the elements $t_{v\to w}$ can be replaced by integer combinations of just d such elements.

Lemma 4.18. *Let $(C_v)_{v\in\mathbb{Z}_5^d}$ be the collection of subsets of $\mathbb{T}^r$ in Lemma 4.17. For each $i \in [d]$ let $t_i = t_{0\to e_i}$. Then for every vertex $v = (v_1, \dots, v_d) \in \mathbb{Z}_5^d$, we have*

$$C_v \cong C_0 + v_1 t_1 + \cdots + v_d t_d.$$

Proof. We argue by induction on $|v| := v_1 + \cdots + v_d \in \mathbb{Z}$ (identifying $\mathbb{Z}_5^d$ with $[0,4]^d$ the natural way). All calculations with subscripts are made modulo 5 throughout the proof. The cases $|v| \in \{0,1\}$ are implied by Lemma 4.17. Let us therefore fix v with $|v| > 1$, which implies that $v_k > 0$ for some k.

There is a path $v_{(1)}, \dots, v_{(\ell)}$ in the lattice graph on $\mathbb{Z}^d$ such that $v_{(1)} = 0$, $v_{(\ell)} = v$, $|v_{(i)}| < |v|$ for $i < \ell$, $v_{(i+1)} = v_{(i)} + e_j$ for some $j = j(i) \in [d]$, for every $i \in [\ell - 1]$, and $v = v_{(\ell-1)} + e_k$. Applying (4.19) along this path, we have

$$C_{v_{(\ell-1)}} \cong C_0 + t_{v_{(1)}\to v_{(2)}} + \cdots + t_{v_{(\ell-2)}\to v_{(\ell-1)}}, \tag{4.32}$$

$$C_v \cong C_{e_k} + t_{v_{(1)}\to v_{(2)}} + \cdots + t_{v_{(\ell-2)}\to v_{(\ell-1)}}. \tag{4.33}$$

Since $C_{e_k} = C_0 + t_{0\to e_k}$ by (4.19), the last two equations above imply that

$$C_v \cong C_{v_{(\ell-1)}} + t_{0\to e_k} = C_{v-e_k} + t_{0\to e_k} = C_{v-e_k} + t_k. \tag{4.34}$$

By induction on $|v|$, we have $C_{v_{(\ell-1)}} = C_0 + v_1 t_1 + \cdots + (v_k - 1)t_k + \cdots + v_d t_d$. Combining the last two equations, the result follows. □

We need one more lemma before we can complete the proof of Theorem 4.13. For a measurable set $C \subset \mathbb{T}^r$, we say that an element $p \in \mathbb{T}^r$ is a *period* of C if $\mu_{\mathbb{T}^r}\big(C\Delta(C+p)\big) = 0$, where Δ here denotes the symmetric difference. We say that p is a *rational* element if all its coordinates are rational numbers (when p is viewed as a point in $[0,1]^r$).

Lemma 4.19. *Let C be a Borel subset of $\mathbb{T}^r$, let $p \in \mathbb{T}^r$ be a period of C, and suppose that p is not rational. Then there exists a continuous surjective homomorphism $\phi : \mathbb{T}^r \to \mathbb{T}^s$ for some $s \in [0, r-1]$, and a Borel set $C' \subset \mathbb{T}^s$ such that $\mu(C\Delta\phi^{-1}C') = 0$.*

Proof. For $k \in \mathbb{Z}^r$ and $x \in \mathbb{T}^r$, we denote by $k\cdot x$ the element $k_1x_1 + \cdots + k_rx_r$ in $\mathbb{T}$, and we denote by $\mathbb{Z}\cdot p$ the subgroup $\{np : n \in \mathbb{Z}\}$ of $\mathbb{T}^r$.

By Kronecker's theorem [9, Ch. VII, Proposition 7], the subgroup $\mathbb{Z} \cdot p$ is dense in the closed subgroup

$$V = (p^{\perp})^{\perp} := \{x \in \mathbb{T}^r : \text{ for all } k \in \mathbb{Z}^r \text{ such that } k \cdot p = 0, \text{ we have } k \cdot x = 0\} \leq \mathbb{T}^r.$$

We have that V (as a compact abelian Lie group) is isomorphic to $\mathbb{T}^{r'} \oplus \mathrm{Z}$ for some $r' \leq r$ and some finite abelian group Z, and $r' \geq 1$ (otherwise p has only rational coordinates). Let $\phi : \mathbb{T}^r \to \mathbb{T}^r/V$ be the natural quotient map, a continuous surjective homomorphism with kernel V, and let Q denote the quotient $\mathbb{T}^r/V$, a compact connected abelian Lie group, which is isomorphic to $\mathbb{T}^s$ for some s. We have $s < r$ since $r' \geq 1$.

By the quotient integral formula [23, Theorem 1.5.2], for any Borel set B and the Haar measures μ_Q, μ_V on Q and V respectively, we have

$$\mu_{\mathbb{T}^r}(B) = \int_Q \mu_t\big(B \cap \phi^{-1}(t)\big)\, \mathrm{d}\mu_Q(t),$$

where μ_t denotes the Haar measure on $\phi^{-1}(t)$ defined by $\mu_V(A-x)$ for every Borel set $A \subset \phi^{-1}(t)$ and some $x \in \phi^{-1}(t)$.

In particular, we have $\mu_{\mathbb{T}^r}(C) = \int_Q \mu_t\big(C \cap \phi^{-1}(t)\big)\, \mathrm{d}\mu_Q(t)$. Let C_1 be the Borel set obtained from C by removing all points belonging to cosets $\phi^{-1}(t)$ in which C is null, i.e. $1_{C_1}(x) = 1_C(x) 1(\mu_t\big(C \cap \phi^{-1}(\phi(x))\big) > 0)$. Using the quotient integral formula, we see that $\mu(C \Delta C_1) = 0$. In particular, the element p is also a period of C_1.

Let $B_1 = C_1 + \mathbb{Z} \cdot p$. Then by the period property and the fact that countable unions of null sets are null, we have $\mu_{\mathbb{T}^r}(C_1 \Delta B_1) = 0$. We shall now prove that

$$\mu_{\mathbb{T}^r}\big(B_1 \Delta (C_1 + V)\big) = 0. \tag{4.35}$$

This will complete the proof, since then on one hand we will have

$$0 \leq \mu_{\mathbb{T}^r}\big(C \Delta (C_1 + V)\big) \leq \mu_{\mathbb{T}^r}\big(C \Delta C_1\big) + \mu_{\mathbb{T}^r}\big(C_1 \Delta B_1\big) + \mu_{\mathbb{T}^r}\big(B_1 \Delta (C_1 + V)\big) = 0,$$

and on the other hand $C_1 + V = \phi^{-1}(C')$ for the Borel set $C' = \phi(C_1 + V) \subset Q$ (that C' is Borel follows from [41, Theorems (15.1) and (12.17)]).

To prove (4.35), since the left side is $\mu_{\mathbb{T}^r}\big((C_1 + V) \setminus B_1\big)$, by the quotient integral formula it suffices to prove that for every $\varepsilon > 0$, for all $t \in Q$ with $\mu_t\big(C_1 \cap \phi^{-1}(t)\big) > 0$ we have

$$\mu_t\big(B_1 \cap \phi^{-1}(t)\big) \geq (1 - \varepsilon)\, \mu_t\big((C_1 + V) \cap \phi^{-1}(t)\big). \tag{4.36}$$

To prove this, note first that $\mu_t\big(B_1 \cap \phi^{-1}(t)\big) > 0$ if and only if $\mu_t\big(C_1 \cap \phi^{-1}(t)\big) > 0$, whence for any such t there exists $c_t \in C_1 \cap \phi^{-1}(t)$ which is a Lebesgue density point of $C_1 \cap \phi^{-1}(t)$ (relative to the Haar measure μ_t). The fact that $c_t \in C_1 \cap \phi^{-1}(t)$ implies that $\mu_t\big((C_1+V)\cap\phi^{-1}(t)\big) = \mu_t\big(c_t+V\big) = 1$, so we have to show that $\mu_t\big(B_1 \cap \phi^{-1}(t)\big) \geq (1-\varepsilon)$.

For any (small) open ball J centered on 0 in (the identity component of) V, we have

$$\mu_t\big(B_1 \cap \phi^{-1}(t)\big) = \frac{1}{\mu_V(J)} \int_V \mu_t\big(B_1 \cap (c_t + y + J)\big)\, \mathrm{d}\mu_V(y). \tag{4.37}$$

Moreover, for every integer n, we have

$$\begin{aligned}
\mu_t\big(B_1 \cap (c_t+y+J)\big) &\geq \mu_t\big((C_1+np)\cap(c_t+y+J)\big)\\
&= \mu_t\big(C_1 \cap (c_t+y-np+J)\big)\\
&\geq \mu_t\big(C_1 \cap (c_t+J)\big) - \mu_t\big((c_t+J)\Delta(c_t+y-np+J)\big)\\
&= \mu_t\big(C_1 \cap (c_t+J)\big) - \mu_V\big(J\Delta(y-np+J)\big).
\end{aligned}$$

Since c_t is a density point, for some ball J we have $\mu_t\big(C_1 \cap (c_t+J)\big) \geq (1-\varepsilon/2)\mu_V(J)$. Then, for this J, the density of $\mathbb{Z}\cdot p$ in V implies that there is n such that np is sufficiently close to y to ensure that $\mu_V\big(J\Delta(y-np+J)\big) < \frac{\varepsilon}{2}\mu_V(J)$. It follows that

$$\mu_V\big(B_1 \cap (c_t+y+J)\big) \geq (1-\varepsilon)\,\mu_V(J).$$

Using this in (4.37), we deduce that $\mu_V\big(B_1 \cap \phi^{-1}(t)\big) \geq (1-\varepsilon)$, and (4.36) follows. □

We shall now obtain the contradiction completing the proof of the main result.

Proof of Theorem 4.13. Suppose for a contradiction that for some $r < d$ there is a Borel set $A \subset \mathbb{T}^r$ of measure $1/\chi_{c^d}(G)$ such that G is A-colorable. Assume without loss of generality that r is the minimal dimension in $[d-1]$ for which this holds.

Let t_i, $i \in [d]$ be the elements of $\mathbb{T}^r$ obtained in Lemma 4.18, and let C be the set C_0 in that lemma. Note that $t_i \neq 0$ for each i, otherwise $A \cap (A + 2t_i) = A \neq \emptyset$, contradicting that A is a coloring base (since 0 and $2e_i$ form an edge in G_d). Then the element $p_i := 5t_i$ is a non-zero period of the set C for each $i \in [d]$.

We claim that each element t_i is rational. Indeed, suppose for a contradiction that for some i the element t_i is not rational. Then p_i is not rational either. By Lemma 4.19, it follows that there is a Borel set $C' \subset \mathbb{T}^{r-1}$ such that $\mu_{\mathbb{T}^{r-1}}(C') = \mu_{\mathbb{T}^r}(C)$ and G is C'-colorable, contradicting the minimality of r.

Thus t_i is a non-zero rational element of $\mathbb{T}^r$ for each $i \in [d]$, and then we can write the finite order of t_i in the form $5^{\alpha_i} n_i$ with α_i the non-negative integer such that $\gcd(n_i, 5) = 1$. Let $t'_i = n_i t_i$. Then we have $C_0 + t'_i = C_{(0,\dots,0,n_i \bmod 5,0,\dots,0)}$. Note that $n_i \neq 0 \mod 5$. Note also that t'_i has order 5^{α_i}.

We claim that for some $j \in [d]$ there are integers λ_i, $i \in [d]\setminus\{j\}$ such that $t'_j = \sum_{i\in[d]\setminus\{j\}} \lambda_i t'_i \mod 1$. To see this, note first that by the identification of $\mathbb{T}^r$ with $[0,1]^r$ that we are using, each element t'_i is viewed as an element in $[0,1]^r$ with coordinates that are all integer multiples of $5^{-\alpha_i}$. Since d is greater than the dimension of the vector space $\mathbb{Q}^r$, these elements $t'_1, \dots, t'_d$ are linearly dependent over $\mathbb{Q}$. It follows that there are integers $c_1, \dots, c_d$ not all zero with $\gcd(c_1, \dots, c_d) = 1$ such that $c_1 t'_1 + \cdots + c_d t'_d = 0$ in $\mathbb{Q}^d$. Since the integers c_i are coprime, not all of them can be divisible by 5, so there exists $j \in [d]$ such that $\gcd(c_j, 5^{\alpha_j}) = 1$. Hence there are non-zero integers a, b such that $ac_j + b5^{\alpha_j} = 1$. Then

$$\begin{aligned}
0 \;=\; a(c_1 t'_1 + \cdots + c_d t'_d) \;&=\; ac_1 t'_1 + \cdots + ac_j t'_j + \cdots + ac_d t'_d \\
&= ac_1 t'_1 + \cdots + t'_j - b5^{\alpha_j} t'_j + \cdots + ac_d t'_d \\
&= ac_1 t'_1 + \cdots + t'_j + \cdots + ac'_d t'_d \mod 1,
\end{aligned}$$

the last equality holding because all coordinates of t'_j are integer multiples of $5^{-\alpha_j}$, so that $b5^{\alpha_j} t'_j = 0 \bmod 1$. Letting $\lambda_i = -ac'_i$, our claim follows.

We deduce the following (with calculations in the subscripts all made mod 5):

$$C_{(\lambda_1 n_1,\dots,\lambda_{j-1}n_{j-1},0,\lambda_{j+1}n_{j+1},\dots,\lambda_d n_d)} = C_0 + \sum_{i\in[d]\setminus\{j\}} \lambda_i t'_i = C_0 + t'_j = C_{(0,\dots,0,n_j,0,\dots,0)}.$$

This implies that $n_j = 0 \bmod 5$, which is impossible since $\gcd(n_j, 5) = 1$. $\square$

Remark 4.20. If we want to prove just that $\chi_{c^d}(G_d) < \chi_{c^{d-1}}(G_d)$, instead of the stronger Theorem 4.13, then there is a shorter way to finish the above proof, using the box structure of R. Indeed, note that up to Lemma 4.18 all the results work if we assume that B is an open box R. Then, let $C = C_0$ and $p_i := 5t_i$ as above, and observe that, since these elements are periods of C for each $i \in [d]$, and since the box R is the union of translates of C, it follows that the p_i are also periods of R for each i. Moreover, we can suppose that R is a proper box in $\mathbb{T}^r$ in the sense that all side-lengths of R are strictly less than 1 (otherwise we can reduce dimensions by deleting the dimensions corresponding to side-lengths equal to 1, as in the proof of Lemma 4.11). However, a proper box in $\mathbb{T}^r$ does not have periods other than 0, so $5t_i = 0 \bmod 1$ for each i. Hence $t_1, \dots, t_d$ are elements of the subgroup $\frac{1}{5} \cdot \mathbb{Z}_5^r \le \mathbb{T}^r$ and therefore cannot be independent over $\mathbb{Z}$, whence there is $v = (v_1, \dots, v_d) \in [0,4]^d$ with coordinates not all zero such that $v_1 t_1 + \cdots + v_d t_d = 0 \bmod 1$. But then $C_0 = C_v$, with $0 \neq v$, a contradiction.

4.5 Upper bound for $\chi(G)$ in terms of $\chi_{c^d}(G)$ using box coverings of $\mathbb{T}^d$

Recall that from (4.3) we have the strict upper bound

$$\chi(G) < \chi_c(G) + 1. \tag{4.38}$$

In this section we extend this to a similar upper bound for $\chi(G)$ in terms of $\chi_{c^d}(G)$, phrased in terms of optimal coverings of the torus by translates of a given box. To state the result we use the following terminology.

Given $d \in \mathbb{N}$ and $\ell = (\ell_1, \dots, \ell_d) \in (0,1]^d$, let $R = R(\ell) := \prod_{i \in [d]} (0, \ell_i)$, and let $\overline{R}$ denote the closure of R, that is $\overline{R} = \prod_{i \in [d]} [0, \ell_i]$. We shall say that a set $A \subset \mathbb{T}^d$ is an *ℓ-covering* of $\mathbb{T}^d$ if $A + \overline{R} = \mathbb{T}^d$. If R is non-empty then the compactness of $\mathbb{T}^d$ implies that there exist finite ℓ-coverings. We then define

$$M_d(\ell) := \min\{|A| \,:\, A \text{ is an } \ell\text{-covering of } \mathbb{T}^d\}.$$

We can now state the main result of this section.

Proposition 4.21. *For every $d \in \mathbb{N}$ and every finite graph G we have*

$$\chi(G) \le \min_{\ell \in \mathbb{R}^d : \chi_{c^d}(G) = (\ell_1 \cdots \ell_d)^{-1}} M_d(\ell). \tag{4.39}$$

This implies the following upper bound for $\chi(G)$ in terms of $\chi_{c^d}(G)$

$$\chi(G) \leq \chi_{c^d}(G) \cdot \min_{\ell \in \mathbb{R}^d : \chi_{c^d}(G) = (\ell_1 \cdots \ell_d)^{-1}} M_d(\ell) \ell_1 \cdots \ell_d. \tag{4.40}$$

Note that for $d = 1$ we have $M_1(\ell) = \lceil 1/\ell \rceil < 1 + 1/\ell$, so in this case (4.40) implies (4.38).

To prove Proposition 4.21 we use the following simple fact.

Lemma 4.22. *If A is an ℓ-covering of $\mathbb{T}^d$, then letting $R' = R'(\ell) := \prod_{i \in [d]} [0, \ell_i)$, we have $A + R' = \mathbb{T}^d$.*

Proof. Fix any $x \in \mathbb{T}^d$. By the covering assumption, for each $n \in \mathbb{N}$ there is a point $a^{(n)} \in A$ such that $x + \frac{1}{n}(1, \ldots, 1) \in a^{(n)} + \overline{R}$. Since we can suppose that A is finite, by passing to a subsequence if necessary we can assume that $a^{(n)}$ is the same point $a \in A$ for all $n > n_0$, and thus assume that $x + \frac{1}{n}(1, \ldots, 1) \in a + \overline{R}$ for all $n > n_0$. Then for each $i \in [d]$ we have $x_i + \frac{1}{n} \in a_i + [0, \ell_i]$ for all $n > n_0$, which implies in the limit that $x_i \in a_i + [0, \ell_i)$, whence $x \in a + R'$. □

Proof of Proposition 4.21. Fix any $\ell \in (0, 1]^d$ such that $\chi_{c^d}(G) = (\ell_1 \cdots \ell_d)^{-1}$, let $\varphi : V(G) \to \mathbb{T}^d$ be a coloring map for G by the open box $R(\ell)$, and let A be an ℓ-covering of $\mathbb{T}^d$. Let $c : V(G) \to A$ be a map that sends each vertex v to an element $a \in A$ such that $\varphi(v) \in a + R'$ with $R' = \prod_{i \in [d]} [0, \ell_i)$ (such an element a exists by Lemma 4.22). We prove that c is a coloring of G, which will imply (4.39) by taking the minimum. If $c(u) = c(v)$ then $\varphi(u)$ and $\varphi(v)$ are in the same box $a + R'$, so $|\varphi(u)_i - \varphi(v)_i|_{\mathbb{T}} < \ell_i$ for every $i \in [d]$, which by definition of the d-toral coloring φ implies that $uv \notin E(G)$. This shows that each set $c^{-1}(\{a\})$, $a \in A$ is an independent set, so c is a coloring as claimed. □

Proposition 4.21 motivates the search for good upper bounds for the quantities $M_d(\ell)$. In the case $\ell_1 = \cdots = \ell_d$, this is a known combinatorial problem of finding optimal coverings of $\mathbb{T}^d$ by translates of a cube, a problem related to information theory which was studied in particular in [26]. The problem is still open, in the sense that an exact formula for $M_d(\ell)$ in the cube case is still unknown (see a discussion on this in [26, Section 6]). As far as we know, the problem of optimal coverings of $\mathbb{T}^d$ by boxes has not been treated in the literature previously (though certain cases in dimension 2 have been used, see for instance [48]).

Question 4.23. *Give an exact formula for $M_d(\ell)$ in terms of the components ℓ_i of ℓ, $i \in [d]$.*

We do not embark on trying to solve this problem completely in this chapter. Instead we shall complete this section by proving related results.

To begin with, the simple observation that the grid

$$A = \left\{ \left(\frac{i_1}{\lceil 1/\ell_1 \rceil}, \ldots, \frac{i_d}{\lceil 1/\ell_d \rceil} \right) \ : \ (i_1, \ldots, i_d) \in \prod_{j=1}^{d} \left[0, \lceil 1/\ell_i \rceil - 1 \right] \right\}$$

is an ℓ-covering of $\mathbb{T}^d$ yields the following upper bound

$$M_d(\ell) \leq \prod_{i \in [d]} \lceil \tfrac{1}{\ell_i} \rceil. \tag{4.41}$$

Let us establish a lower bound for $M_d(\ell)$ in terms of the d projections of ℓ to $\mathbb{T}^{d-1}$.

Proposition 4.24. *For each $\ell \in (0,1)^d$ and $i \in [d]$, let $\ell^{(i)} := (\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_d) \in (0,1]^{d-1}$. Let $M_d^*(\ell) := \min\{n | \forall i, n\, \ell_i \geq M_{d-1}(\ell^{(i)})\} = \max_{i \in [d]} \left\lceil M_{d-1}(\ell^{(i)})/\ell_i \right\rceil$. Then*

$$M_d(\ell) \geq M_d^*(\ell). \tag{4.42}$$

Proof. Let A be an ℓ-covering of $\mathbb{T}^d$. By Proposition 4.22 we have $A + R' = \mathbb{T}^d$, where $R' = \prod_{i=1}^{d} [0, \ell_i)$. For each $i \in [d]$ and $t \in [0,1]$, we define the slit

$$X_{i,t} = \left\{ (x_1, \ldots, x_d) \in \mathbb{T}^d \mid x_i \in (t, t + \ell_i] \right\}. \tag{4.43}$$

For each t, since $A + R'$ covers the set $\{x : x_i = t + \ell_i\}$, there must be at least $M_{d-1}(\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_d)$ points $a \in A$ such that $a + R' \cap \{x : x_i = t + \ell_i\} \neq \emptyset$, so we must have $|A \cap X_{i,t}| \geq M_{d-1}(\ell^{(i)})$.

Assume first that ℓ_i is a rational number equal to k_i/n_i for each $i \in [d]$, and consider the k_i-fold covering of $\mathbb{T}^d$ by the n_i slits $X_{i,j/n_i}$, $j \in [n_i]$. We then have

$$\begin{aligned} k_i |A| &= \sum_{a \in A} \sum_{j \in [n_i]} 1_{X_{i,j/n_i}}(a) = \sum_{j \in [n_i]} |A \cap X_{i,t}| \\ &\geq n_i\, M_{d-1}(\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_d), \end{aligned}$$

that is $|A|\, \ell_i \geq M_{d-1}(\ell^{(i)})$. Since this holds for all i we deduce $|A| \geq M_d^*(\ell)$, and (4.42) follows in this case where $\ell \in \mathbb{Q}^d$. The result when ℓ_i is a positive real number (not necessarily rational) now follows by approximating ℓ_i arbitrarily closely by rational numbers we obtain number. $\square$

Observe that when the box is a square $[0,\varepsilon]^d$, i.e. $\ell = (\varepsilon, \dots, \varepsilon)$, from (4.42) and an induction on the dimension we recover the following lower bound from [26, Theorem 2]:

$$M_d(\ell) \geq \lceil \varepsilon^{-1} \lceil \varepsilon^{-1} \lceil \dots \lceil \varepsilon^{-1} \rceil \rceil \rceil \cdots \rceil,$$

where there are d applications of the ceiling function.

For $d = 2$, the following exact formula is obtained by generalizing [26, Theorem 3].

Proposition 4.25. *For every $\ell \in (0,1]^2$ we have*

$$M_2(\ell) = M_2^*(\ell) = \max\{ \lceil \tfrac{1}{\ell_1} \lceil \tfrac{1}{\ell_2} \rceil \rceil, \lceil \lceil \tfrac{1}{\ell_1} \rceil \tfrac{1}{\ell_2} \rceil \}.$$

Proof. Consider the line $y = \lceil \frac{1}{\ell_1} \rceil x$ in $\mathbb{T}^2$, which wraps around $\mathbb{T}$ in the second coordinate $\lceil \frac{1}{\ell_1} \rceil$-many times. Place $M_2^*(\ell)$ equally spaced centers of boxes $[0,\ell_1] \times [0,\ell_2]$ along this line. Fix any $a \in \mathbb{T}$. We show that the boxes cover the line $X_a = \{(x,y) : x = a\}$, which implies the result. Consider the slit $X_{1,a}$ (using the notation from (4.43)). The projections to the x-axis $\mathbb{T} \times \{0\}$ of the equally spaced centers give equally spaced points on this axis, so the distance between two consecutive such projections is $1/M_2^*(\ell)$. Therefore in the slit $X_{1,a}$ (of x-width ℓ_1) there are at least $\lfloor \ell_1 M_2^*(\ell) \rfloor \geq M_1(\ell_2) = \lceil 1/\ell_2 \rceil$ centers of these boxes. Observe that if two projected centers in $\mathbb{T} \times \{0\}$ are consecutive then they are also consecutive along the line $y = \lceil \frac{1}{\ell_1} \rceil x$ itself, and their projections to $\{0\} \times \mathbb{T}$ are also consecutive and distanced on this line by at most

$$\lceil 1/\ell_1 \rceil / M_2^*(\ell) \leq \lceil 1/\ell_1 \rceil / (M_1(\ell_1)/\ell_2) = \ell_2. \tag{4.44}$$

It follows that the non-empty intersections of these boxes with X_a form segments of length ℓ_2 which leave no gap between any two consecutive of them (by (4.44)). These segments thus cover X_a, and the result follows. □

We close this section with the following upper bound for $M_d(\ell)$ which applies the greedy algorithm, inspired by a similar application in [54].

Proposition 4.26. *For each $i \in [d]$, let $\ell_i = a_i/b_i \in (0,1)$, with a_i, b_i coprime positive integers. Then $M_d(\ell) \leq \frac{1}{\ell_1 \cdots \ell_d}\big(1 + \sum_{i \in [d]} \log(\ell_i b)\big)$, where $b = \operatorname{lcm}(b_1, \dots, b_d)$.*

This is easily seen to improve markedly on the simple bound (4.41), for instance by considering the diagonal case $\ell_i = \ell$ for all $i \in [d]$.

Proof. As before, let $R' = \prod_{i\in[d]}[0, \ell_i)$. Let G denote the subgroup of $\mathbb{T}^d$ isomorphic to $\mathbb{Z}_b^d$. Let $\tilde{R} = R' \cap G$. Applying [50, Theorem 4], there is a set $A \subset G$ that is an ℓ-covering and satisfies $|A| \leq \frac{1+\log|\tilde{R}|}{|\tilde{R}|} b^d = \frac{1}{\mu_{\mathbb{T}^d}(R)}(1 + \log|\tilde{R}|)$. The result now follows since $|\tilde{R}| = \frac{a_1}{b_1} b \cdots \frac{a_d}{b_d} b$. $\square$

Chapter 5

On Motzkin's problem in the circle

This chapter is the result of a collaboration with Pablo Candela, Juanjo Rué and Oriol Serra. The contents were the object of an invited publication in *Proceedings of the Steklov Institute of Mathematics*, for a special issue in 2021 commemorating the 130-th birth anniversary of Ivan Matveevich Vinogradov; see [12].

5.1 Introduction

Many interesting developments in combinatorial number theory are related to the general problem of determining how large a subset of an abelian group can be if the set avoids certain prescribed configurations. Famous examples include Szemerédi's theorem, where the configurations in question are arithmetic progressions in sets of integers. Another notable problem of this kind, posed by T. S. Motzkin, asks how large a set of integers can be if it does not contain any pair of elements whose difference lies in a prescribed set. More precisely, given a non-empty subset D of the set of positive integers $\mathbb{N}$, let us say that a set $A \subset \mathbb{Z}$ is *D-avoiding* if for every $a, a' \in A$ we have $|a - a'| \notin D$, in other words if the *difference set* $A - A = \{a - a' : a, a' \in A\}$ is disjoint from D. Let $A(N)$ denote the cardinality $|A \cap [-N, N]|$, and let $\bar{\delta}(A)$ denote the upper density of A, namely $\bar{\delta}(A) = \limsup_{N \to \infty} \frac{A(N)}{2N+1}$. Then, Motzkin's problem (posed originally for sets $A \subset \mathbb{N}$ in an unpublished problem collection; see [14]) consists in determining the following quantity, sometimes called the *Motzkin density* of D:

$$\mathrm{Md}_{\mathbb{Z}}(D) := \sup\{\bar{\delta}(A) : A \text{ is a } D\text{-avoiding subset of } \mathbb{Z}\}. \tag{5.1}$$

The first publication on Motzkin's problem is the paper [14] by Cantor and Gordon. Their results include a full solution for $|D| \leq 2$. This involves proving that the elements of D can be assumed to be coprime, then proving that $\mathrm{Md}_{\mathbb{Z}}(D) = 1/2$ for $|D| = 1$, and then proving the following formula for $D = \{d_1, d_2\}$ with $\gcd(d_1, d_2) = 1$:

$$\mathrm{Md}_{\mathbb{Z}}(D) = \frac{\lfloor \frac{d_1+d_2}{2} \rfloor}{d_1 + d_2}. \tag{5.2}$$

Motzkin's problem is still open in general. In the decades since the initial paper [14], the problem has motivated many works, and various special cases have been addressed; see for instance [31, 34, 47, 55–57]. The problem also has interesting relations with other well-known topics in combinatorics and number theory, such as the fractional chromatic number of distance graphs, or the lonely runner conjecture; see for example [46] and the references therein.

There is an analogue of Motzkin's problem for any compact abelian group Z. Namely, given a non-empty set $D \subset \mathrm{Z}$, letting μ denote the Haar probability measure on Z, the problem is to determine or estimate the quantity

$$\mathrm{Md}_{\mathrm{Z}}(D) := \sup\{\mu(A) : A \subset \mathrm{Z} \text{ a Borel set with } (A - A) \cap D = \emptyset\}. \tag{5.3}$$

In particular, a hitherto unexplored yet natural analogue of Motzkin's problem consists in taking Z to be the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which we shall view as the interval $[0, 1]$ with addition modulo 1, letting D be a set of real numbers in $(0, 1)$. In this paper we make a first treatment of this problem for D a finite set $\{t_1, \ldots, t_r\}$.

In Section 5.2 we make some observations on the problem for general $r \in \mathbb{N}$, showing in particular that it can be approached using tools from ergodic theory. We illustrate this first in the "extreme" case where $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$, applying the version of Rokhlin's lemma for free measure-preserving actions of $\mathbb{Z}^r$ to prove that in this case $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$; see Theorem 5.7. (Different applications of Rokhlin's lemma in combinatorial number theory have been given recently in [3, 27].) The general case, where $D \cup \{1\}$ may be linearly dependent over $\mathbb{Q}$, can be approached using more general versions of Rokhlin's lemma which are applicable to free actions of *quotients* of $\mathbb{Z}^r$. In particular, the problem of determining $\mathrm{Md}_{\mathbb{T}}(D)$ can thus be transferred to a similar problem in the discrete setting of the finitely generated abelian group $\mathbb{Z}^r/\Lambda$, where Λ is the kernel of the homomorphism $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1t_1 + \cdots + n_rt_r$. In this setting, a natural notion of Motzkin density can be defined using Følner sequences; see Definition 5.8. We then have the following result.

Theorem 5.1. *Let $D = \{t_1, \ldots, t_r\} \subset \mathbb{T}$, let Λ be the kernel of the homomorphism $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1t_1 + \cdots + n_rt_r$, and let E be the image of the standard basis of $\mathbb{R}^r$ in the quotient $\mathbb{Z}^r/\Lambda$. Then $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$.*

This result also holds for more general compact abelian groups; see Theorem 5.9.

Theorem 5.1 can be used as a first step in an approach towards determining $\mathrm{Md}_{\mathbb{T}}(D)$, since the corresponding Motzkin density in the discrete setting, i.e. $\mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$, can often be simpler to determine. In this paper we pursue this approach for $r \leq 2$.

Another notable special case of the problem, at the other extreme from $D \cup \{1\}$ being linearly independent over $\mathbb{Q}$, is the case in which $D \subset \mathbb{Q}$. This reduces to the problem of determining the independence ratio of a circulant graph which we call the *associated circulant graph*. More precisely, supposing that each element of D is of the form $t_i = a_i/b_i$ with coprime positive integers $a_i < b_i$, then the subgroup $\langle D \rangle \leq \mathbb{T}$ is isomorphic to $\mathbb{Z}_N$ with $N = \mathrm{lcm}(b_1, \ldots, b_r)$. The associated circulant graph is the (undirected) connected circulant graph G with vertex set $\mathbb{Z}_N$ (viewed as the set of integers $[0, N-1]$ with addition modulo N) with jumps $d_1, \ldots, d_r$ where $d_i = a_iN/b_i$. Thus $x, y \in \mathbb{Z}_N$ form an edge in G if and only if $x - y = d_i$ or $-d_i$ mod N for some $i \in [r]$. Equivalently G is the Cayley graph on $\mathbb{Z}_N$ with generating set $\{d_i, -d_i : i \in [r]\}$, which we shall denote by $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, \ldots, d_r\})$. The independence ratio of G is $\frac{\alpha(G)}{N}$, where $\alpha(G)$ is the independence number of G, i.e. the maximal cardinality of an independent (or stable) set in G. As a straightforward consequence of Theorem 5.1 we have $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \ldots, d_r\}) = \frac{\alpha(G)}{N}$; see Lemma 5.13. Let us mention also that if $d_1, \ldots, d_r$ are fixed integers then, as $N \to \infty$, the ratios $\frac{\alpha(G)}{N}$ converge to $\mathrm{Md}_{\mathbb{Z}}(\{d_1, \ldots, d_r\})$, and in this sense Motzkin's problem in $\mathbb{T}$ can be seen to subsume the original problem in $\mathbb{Z}$ for finitely many missing differences; we detail this in Remark 5.14.

Circulant graphs are extensively treated in the combinatorics and computer science literature (in the latter they are also known as *multiple-loop networks* or *chordal rings*); see for instance [6, 10, 21, 29, 38]. However, these works study mostly other parameters than the independence ratio. Works determining the independence ratio of certain circulant graphs include [28, 43].

After these remarks on the problem for general r, and a brief solution for $r = 1$ (see Proposition 5.15), we close Section 5.2 and focus on the problem for $r = 2$ for the rest of the paper. We then distinguish two cases.

In Section 5.3 we treat the case in which at least one element of D is irrational. Here we obtain the following exact solution (see Theorem 5.17).

Theorem 5.2. *Let $D = \{t_1, t_2\} \subset (0, 1)$ with $D \not\subset \mathbb{Q}$. If $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$, then $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$. Otherwise, letting m_0, m_1, m_2 be integers not all zero such that $m_0 = m_1 t_1 + m_2 t_2$ and $\gcd(m_0, m_1, m_2) = 1$, we have*

$$\mathrm{Md}_{\mathbb{T}}(D) = \frac{\lfloor k/2 \rfloor}{k}, \quad \textit{where } k = |m_1| + |m_2|. \tag{5.4}$$

In Section 5.4, we focus on the case in which both elements of D are rational. This is equivalent to determining the independence ratio of circulant graphs with two jumps. We study this problem using mainly tools from the geometry of numbers. The usefulness of such tools for the analysis of circulant graphs is well-known (see for instance [10, 22, 52]), though apparently before the present work these tools had not been used to study the independence ratio.

The independence ratio of a circulant graph G is easily seen to be $1/2$ when G is bipartite, so we can assume that G contains odd cycles. The so-called "no-homomorphism lemma" from [1] yields an upper bound for $\frac{\alpha(G)}{N}$ of the form $\frac{k-1}{2k}$, where k is the *odd girth* of G, i.e. the smallest length of an odd cycle in G (see Lemma 5.18). It is then natural to examine how accurate this upper bound is as an estimate for $\frac{\alpha(G)}{N}$. In particular, the odd girth is always one of the successive minima, relative to the ℓ^1-norm, of a 2-dimensional lattice naturally associated with G; see Lemma 5.24 (the lattice in question is just the lattice Λ from Theorem 5.1 applied in this special case). This expression of the odd girth makes the estimate $\frac{k-1}{2k}$ relatively easy to compute; see Remark 5.28, where an algorithm is outlined.

Regarding the accuracy of this estimate for $\frac{\alpha(G)}{N}$, we obtain the following result, showing that the estimate is asymptotically sharp.[1]

Theorem 5.3. *Let $D = \{t_1, t_2\} \subset \mathbb{Q} \cap (0,1)$. Let G be the associated circulant graph, and let N be the order of G. If G is bipartite then $\mathrm{Md}_{\mathbb{T}}(D) = \frac{\alpha(G)}{N} = \frac{1}{2}$. Otherwise, letting k be the odd girth of G, we have*

$$\tfrac{k-1}{2k} \quad \geq \quad \mathrm{Md}_{\mathbb{T}}(D) = \tfrac{\alpha(G)}{N} \quad \geq \quad \tfrac{k-1}{2k} - \tfrac{3}{\sqrt{N}}. \tag{5.5}$$

This is obtained as an immediate consequence of an equivalent estimate for the independence number of connected circulant graphs with two jumps, given in Theorem 5.22.

The independence ratio of a circulant graph G is equal to the reciprocal of its fractional chromatic number $\chi_f(G)$. Therefore (5.5) yields also an asymptotically sharp estimate for the fractional chromatic number of a connected circulant graph G of order N with 2 jumps and odd girth k, namely $\frac{2k}{k-1} \leq \chi_f(G) \leq \frac{2k}{k-1} + \frac{27}{\sqrt{N}}$.

We also study the question of the sharpness of the bounds in (5.5) for *fixed* N, not just as $N \to \infty$. In Proposition 5.21 we provide an infinite family of examples of 2-jump circulant graphs whose independence ratios attain the lower bound in (5.5) up to the absolute constant multiplying $1/\sqrt{N}$. In Proposition 5.29 we give an infinite family of examples attaining the upper bound in (5.5) (see also Remark 5.20).

Finally, we note that the odd-girth notion enables a unification of solutions to Motzkin's problem for two missing differences across various settings, in the non-bipartite case. For example, Theorem 5.3 can be seen to imply the formula (5.2) of Cantor and Gordon, by expressing this formula in terms of the odd girth of the corresponding distance graph $\mathrm{Cay}(\mathbb{Z}, \{d_1, d_2\})$, and viewing the corresponding Motzkin density as the limit of independence ratios of circulant graphs $\mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$. In Section 5.5 we detail such connections and discuss some questions for further research.

[1] See [19] for hardness results on estimating the independence number of general circulant graphs.

5.2 On the problem for a general finite set D

In this section we make some initial observations on the problem of determining $\mathrm{Md}_{\mathbb{T}}(D)$ for finite D, illustrating especially how tools from ergodic theory can be applied to the problem. In particular we shall use Rokhlin's lemma for free actions of finitely generated abelian groups, which we state below after recalling some terminology.

Definition 5.4. A measure-preserving action of a countable discrete group Γ on a proba- bility space $(X, \mathcal{X}, \mu)$ is a map $f : \Gamma \times X \to X$ such that for every $g \in \Gamma$ there is a measure-preserving map $f_g : X \to X$, with f_{id_Γ} being the identity map, and such that for every $g, h \in \Gamma$ and $x \in X$ we have $f_{g\,h}(x) = f_g(f_h(x))$. We say that such an action is *free* if for every $g, h \in \Gamma$ with $g \neq h$ we have $\mu(\{x \in X : f_g(x) = f_h(x)\}) = 0$.

Definition 5.5. Let f be a measure-preserving action of a countable discrete group Γ on a probability space $(X, \mathcal{X}, \mu)$, and let $K \subset \Gamma$. If $B \in \mathcal{X}$ is such that the sets $f_g(B)$, $g \in K$ are pairwise disjoint, then the union $\bigcup_{g\in K} f_g(B)$ is called a *K-tower* for f with *base* B.

A subset K of an abelian group Γ is said to *tile* Γ if there exists $C \subset \Gamma$ such that we have the partition $\Gamma = \bigsqcup_{c\in C} K + c$. The version of Rokhlin's lemma that we shall use is the following special case of [53, p. 58, Theorem 5].

Lemma 5.6. *Let Γ be a finitely generated abelian group and let f be a free measure-preserving action of Γ on a standard probability space $(X, \mathcal{X}, \mu)$. Let $K \subset \Gamma$ be a finite set that tiles Γ. Then for every $\varepsilon > 0$ there exists a K-tower for f of measure at least $1 - \varepsilon$.*

As a first simple example of the use of this lemma in this context, let us treat swiftly the case of Motzkin's problem in $\mathbb{T}$ where $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$.

5.2.1 The case of linear independence of $D \cup \{1\}$ over $\mathbb{Q}$

In this subsection we prove the following result.

Theorem 5.7. *Let D be a finite subset of $(0,1)$ such that $D \cup \{1\}$ is linearly independent over $\mathbb{Q}$. Then $\mathrm{Md}_{\mathbb{T}}(D) = \frac{1}{2}$. Moreover, no D-avoiding Borel set $A \subset \mathbb{T}$ satisfies $\mu(A) = \frac{1}{2}$.*

In the proof we use the special case of Lemma 5.6 for free actions of $\mathbb{Z}^r$, which was given in [20, Theorem 3.1] and independently in [40, Theorem 1].

Proof. Let $D = \{t_1, \ldots, t_r\}$. Clearly $\mathrm{Md}_{\mathbb{T}}(D) \le \frac{1}{2}$. Fix any $\varepsilon > 0$ and any odd $N \in \mathbb{N}$.

The translations by the elements $t_1, \ldots, t_r \in D$ generate a measure-preserving action f of $\mathbb{Z}^r$ on $\mathbb{T}$, namely $f(n, x) = x + n_1 t_1 + \cdots + n_r t_r \mod 1$. It follows from the linear independence of $D \cup \{1\}$ over $\mathbb{Q}$ that this action is free. By Lemma 5.6 there is a Borel set $B \subset \mathbb{T}$ that is the base of a $[0, N)^r$-tower for f of Haar probability at least $1 - \varepsilon$.

$$\text{Let } A = \bigsqcup_{j_1, \ldots, j_r \in [0, N-2] \colon j_1 + \cdots + j_r \text{ is even}} B + j_1 t_1 + \cdots + j_r t_r.$$

It is readily seen that $(A + t_i) \cap A = \emptyset$ for each $i \in [r]$, so A is D-avoiding. Moreover, since the translates of B in the tower have equal measure at least $(1-\varepsilon)/N^r$, and since A consists of $(N-1)^r/2$ of these sets, we have $\mu(A) \ge ((N-1)^r/2)(1-\varepsilon)/N^r \ge (1-\varepsilon)(1-1/N)^r/2$. Letting $N \to \infty$ and $\varepsilon \to 0$, we deduce that $\mathrm{Md}_{\mathbb{T}}(D) \ge \frac{1}{2}$, so $\mathrm{Md}_{\mathbb{T}}(D) = \frac{1}{2}$.

To see that the supremum $1/2$ cannot be attained, suppose for a contradiction that $A \subset \mathbb{T}$ is a measurable D-avoiding set with $\mu(A) = 1/2$. Since $A + t_1 \subset A^c := \mathbb{T} \setminus A$ and $\mu(A + t_1) = 1/2 = \mu(A^c)$, we have $\mu\big((A + t_1)\Delta A^c\big) = 0$. Hence $\mu\big((A + 2t_1)\Delta(A + t_1)^c\big) = 0$. By the triangle inequality $\mu\big((A+2t_1)\Delta A\big) \le \mu\big((A+2t_1)\Delta(A+t_1)^c\big) + \mu\big((A+t_1)^c \Delta A\big) = 0$. Hence A is an invariant set of measure $1/2$ for the map $x \mapsto x + 2t_1$, contradicting the fact that, since t_1 is irrational, this map is ergodic. $\square$

5.2.2 Transference to finitely generated abelian groups

Given a compact abelian group Z, and $D = \{t_1, \dots, t_r\} \subset \mathrm{Z}$, we consider the lattice

$$\Lambda = \{n \in \mathbb{Z}^r : n_1 t_1 + \cdots + n_r t_r = 0\}, \tag{5.6}$$

that is, the kernel of the homomorphism $\mathbb{Z}^r \to \mathrm{Z}$, $n \mapsto n_1 t_1 + \cdots + n_r t_r$. The finitely generated abelian group $\mathbb{Z}^r/\Lambda$ then has a free action f on Z, well-defined by

$$f(n + \Lambda, x) = x + (n_1 + u_1)t_1 + \cdots + (n_r + u_r)t_r, \quad \text{for any } u \in \Lambda. \tag{5.7}$$

The main idea in the proof of Theorem 5.7 is that the Rokhlin lemma enables the problem of determining $\mathrm{Md}_{\mathbb{T}}(D)$ to be transferred to a discrete setting, where it can be easier to solve. The transference part of this approach can be carried out more generally. We establish this in Theorem 5.9 below, for a general finite set D, and not just for $\mathbb{T}$ but for any compact abelian group Z such that (Z, μ) is a standard probability space, so that Lemma 5.6 can be applied with $X = \mathrm{Z}$ and $\mathcal{X}$ the Borel σ-algebra on Z. This applicability holds if Z is metrizable (as $(\mathrm{Z}, \mathcal{X})$ is then a standard Borel space [41, (4.2),(12.5)]). To avoid further technicalities, we shall assume that Z is metrizable.

Our transference result (Theorem 5.9 below) expresses the Motzkin density $\mathrm{Md}_{\mathrm{Z}}(D)$ as an analogous quantity in $\mathbb{Z}^r/\Lambda$. To detail this, we first describe a natural notion of Motzkin density in any finitely generated abelian group Γ.

For any set X we denote by $\mathcal{P}_{<\infty}(X)$ the set of all finite subsets of X. Recall that a sequence $(F_N)_{N\in\mathbb{N}}$ of sets in $\mathcal{P}_{<\infty}(\Gamma)$ is a *Følner sequence* if

$$\text{for every } g \in \Gamma \text{ we have } \lim_{N\to\infty} \frac{|F_N \Delta (g + F_N)|}{|F_N|} = 0. \tag{5.8}$$

Definition 5.8. Let Γ be a finitely generated abelian group and let $E \subset \Gamma$. Let

$$\phi_E : \mathcal{P}_{<\infty}(\Gamma) \to \mathbb{Z}_{\geq 0}, \quad S \mapsto \max\{\, |A| : A \subset S,\ (A - A) \cap E = \emptyset \,\}.$$

Then we define

$$\mathrm{Md}_\Gamma(E) := \lim_{N\to\infty} \frac{\phi_E(F_N)}{|F_N|}, \quad \text{for any Følner sequence } (F_N)_{N\in\mathbb{N}} \text{ in } \Gamma. \tag{5.9}$$

Note that the function ϕ_E is monotone relative to inclusion, subadditive relative to unions, and Γ-invariant. It follows by known results that the limit in (5.9) exists and is independent of the choice of Følner sequence (see [45, Theorem 6.1] or [24, Proposition 2.2]).

A Følner sequence $(F_N)_{N\in\mathbb{N}}$ in Γ is a *tiling Følner sequence* if F_N tiles Γ for every $N \in \mathbb{N}$. Such a sequence can be obtained using the fundamental result that there is a group isomorphism $\vartheta : \mathbb{Z}^d \times \Gamma' \to \Gamma$ for some finite group Γ' and $d \in \mathbb{Z}_{\geq 0}$. Indeed we can then take (for instance)

$$F_N = \vartheta\big([-N, N]^d \times \Gamma'\big). \tag{5.10}$$

A definition of Motzkin density in Γ can also be formulated using the notion of *upper density* relative to a *fixed* Følner sequence (see Definition 5.10 and (5.15)); the resulting definition, alternative to (5.9), is a more direct extension of the original one used in (5.1) if we use the sequence given by (5.10). Later in this section we show that for finite sets E this definition agrees with (5.9) (see Lemma 5.11). We shall use mainly the definition of Motzkin density given in (5.9), as it is more convenient for our arguments.

We can now state the transference result.

Theorem 5.9. *Let* Z *be a compact metrizable abelian group, let* $D = \{t_1, \ldots, t_r\} \subset \mathrm{Z}$*, let* Λ *be the kernel of the homomorphism* $\mathbb{Z}^r \to \mathrm{Z}$*,* $n \mapsto n_1t_1 + \cdots + n_rt_r$*, and let* E *be the image of the standard basis of* $\mathbb{R}^r$ *in the quotient* $\mathbb{Z}^r/\Lambda$*. Then* $\mathrm{Md}_{\mathrm{Z}}(D) = \mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$*.*

Proof. Let $\Gamma = \mathbb{Z}^r/\Lambda$, let $(F_N)_{N\in\mathbb{N}}$ be a tiling Følner sequence in Γ, and let us denote the elements of E by $e'_1, \ldots, e'_r$. It follows from (5.8) that

$$\forall\, \delta > 0,\ \exists\, N_0,\ \forall\, N \geq N_0,\ \forall\, i \in [r],\ \ |(F_N + e'_i) \setminus F_N| \leq \delta |F_N|. \tag{5.11}$$

We first prove that

$$\mathrm{Md}_{\mathrm{Z}}(D) \geq \mathrm{Md}_\Gamma(E). \tag{5.12}$$

Fix any $\varepsilon > 0$. By (5.9) and (5.11), we can fix N such that the following properties hold: firstly there is an E-avoiding set $A' \subset F_N$ satisfying $\frac{|A'|}{|F_N|} \geq \mathrm{Md}_\Gamma(E) - \frac{\varepsilon}{4}$, and secondly for each $i \in [r]$ we have $|(F_N + e'_i) \setminus F_N| \leq \frac{\varepsilon}{4r}|F_N|$.

Let $A'' = \{g \in A' : g + E \subset F_N\}$. We have $A' \setminus A'' \subset \{g \in F_N : g + E \not\subset F_N\} \subset \bigcup_{i\in[r]} F_N \setminus (F_N - e'_i)$. This together with the properties above implies $\frac{|A''|}{|F_N|} \geq \mathrm{Md}_\Gamma(E) - \frac{\varepsilon}{2}$.

By Lemma 5.6 applied to the action f defined in (5.7), there is a base $B \subset \mathrm{Z}$ of an F_N-tower for f of measure at least $1-\frac{\varepsilon}{2}$. Let $A = \bigsqcup_{g\in A''} f_g(B)$. For every $i \in [r]$, the set $A + t_i = \bigsqcup_{g'\in A''} f_{g'+e'_i}(B)$ is disjoint from A (otherwise, since $A'' + e'_i \subset F_N$, the tower property implies that $g' + e'_i = g$ for some $g, g' \in A''$, contradicting that A'' is E-avoiding). Hence $\mathrm{Md}_{\mathrm{Z}}(D) \geq \mu(A) = |A''|\,\mu(B) \geq |F_N|(\mathrm{Md}_\Gamma(E) - \frac{\varepsilon}{2})\,\frac{1-\frac{\varepsilon}{2}}{|F_N|} \geq \mathrm{Md}_\Gamma(E) - \varepsilon$. This yields (5.12) by letting $\varepsilon \to 0$.

We now prove that

$$\mathrm{Md}_{\mathrm{Z}}(D) \leq \mathrm{Md}_\Gamma(E). \tag{5.13}$$

Fix any D-avoiding Borel set $A \subset \mathrm{Z}$, and any $\varepsilon > 0$. By (5.9) and (5.11), we can fix N such that firstly $|(F_N + e'_i) \setminus F_N| \leq \frac{\varepsilon}{2r}|F_N|$ for every $i \in [r]$, and secondly

$$\text{every } E\text{-avoiding set } S \subset F_N \text{ satisfies } \frac{|S|}{|F_N|} \leq \mathrm{Md}_\Gamma(E) + \frac{\varepsilon}{2}. \tag{5.14}$$

By Lemma 5.6, there is a base $B \subset \mathrm{Z}$ of an F_N-tower for f of measure at least $1-\frac{\varepsilon}{2}$. We claim that there is a partition of B into non-empty measurable sets B_j, $j \in [M]$, such that there is a set $A' \subset A$ (which is therefore D-avoiding) with $\mu(A') \geq \mu(A) - \frac{\varepsilon}{2}$, and with the property that for every $j \in [M]$ there is $S_j \subset F_N$ such that A' is of the form $A' = \bigsqcup_{j\in[M]} \bigsqcup_{g\in S_j} f_g(B_j)$. Before we prove this claim, let us explain how it yields (5.13). The D-avoiding property of A' implies that each set S_j is E-avoiding. Indeed, otherwise there would be $j \in [M]$ and $i \in [r]$ such that there is $g' \in S_j$ with $g' + e'_i \in S_j$. But then the form of A' implies that $f_{g'+e'_i}(B_j) \subset A'$ (since $g := g' + e'_i \in S_j$) and $f_{g'+e'_i}(B_j) = f_{g'}(B_j) + t_i \subset A' + t_i$ (since $g' \in S_j$), so $A' \cap (A' + t_i) \supset f_{g'+e'_i}(B_j) \neq \emptyset$, contradicting that A' is D-avoiding. Hence each S_j is indeed E-avoiding. By (5.14) we then have $\frac{|S_j|}{|F_N|} \leq \mathrm{Md}_\Gamma(E) + \frac{\varepsilon}{2}$ for all $j \in [M]$. Then, using $\sum_{j\in[M]} \mu(B_j)|F_N| = \mu\big(\bigsqcup_{g\in F_N} f_g(B)\big) \leq 1$, we have $\mu(A') \leq \sum_{j\in[M]} |S_j|\mu(B_j) \leq \mathrm{Md}_\Gamma(E) + \frac{\varepsilon}{2}$, so $\mu(A) \leq \mathrm{Md}_\Gamma(E) + \varepsilon$, and (5.13) follows by letting $\varepsilon \to 0$.

We now prove the claim by finding the desired partition of B and the set A'. For every $g \in F_N$, we have the partition $B^{(g)} = \{B_{g,0}, B_{g,1}\}$ of B with atoms $B_{g,1} := B \cap f_g^{-1}(A)$ and $B_{g,0} := B \setminus B_{g,1}$. The desired partition is the common refinement (or supremum) of these partitions, i.e. the partition of B whose atoms are all the non-empty intersections of the atoms of $B^{(g)}$ as g ranges in F_N. Let $B_1, \ldots, B_M$ be the atoms in this partition. Let $A' := \bigsqcup_{g\in F_N} [A \cap f_g(B)] \subset A$. Since the F_N-tower with base B has measure at least $1-\frac{\varepsilon}{2}$, we have $\mu(A\setminus A')\leq\frac{\varepsilon}{2}$. Since $A'=\bigsqcup_{g\in F_N} f_g(B_{g,1})$, and each set $B_{g,1}$ is a union of some of the atoms B_j, it follows that A' is a union of some of the atoms of the partition $\{f_g(B_j) : j\in[M], g\in F_N\}$. Hence for every $j\in[M]$ there is a set $S_j\subset F_N$ such that $A'=\bigsqcup_{j\in[M]}\bigsqcup_{g\in S_j} f_g(B_j)$. This proves the claim and completes the proof. □

To close this subsection, let us detail the other natural definition of the Motzkin density of a finite set in a finitely generated abelian group, as announced earlier. We do this in Lemma 5.11 below. This is not used in later sections of this paper, but it can be used for instance in an alternative proof of Theorem 5.9; see Remark 5.12.

Definition 5.10. Let Γ be a finitely generated abelian group. Given any set $A\subset\Gamma$, and any Følner sequence $\mathcal{F}=(F_N)_{N\in\mathbb{N}}$ in Γ, the *upper density of A relative to $\mathcal{F}$* is defined by $\overline{\delta}_{\mathcal{F}}(A):=\limsup_{N\to\infty}\frac{|A\cap F_N|}{|F_N|}$.

Lemma 5.11. *Let Γ be a finitely generated abelian group, let $\mathcal{F}$ be a tiling Følner sequence in Γ, and let E be a finite subset of Γ. Then*

$$\mathrm{Md}_\Gamma(E)=\sup\{\overline{\delta}_{\mathcal{F}}(A) : A\subset\Gamma,\,(A-A)\cap E=\emptyset\}. \tag{5.15}$$

Proof. It is easily checked from the definitions that $\mathrm{Md}_\Gamma(E)\geq\overline{\delta}_{\mathcal{F}}(A)$ for every E-avoiding set $A\subset\Gamma$, so $\mathrm{Md}_\Gamma(E)\geq\sup\{\overline{\delta}_{\mathcal{F}}(A) : A\subset\Gamma,\,(A-A)\cap E=\emptyset\}$. We now prove that

$$\mathrm{Md}_\Gamma(E)\leq\sup\{\overline{\delta}_{\mathcal{F}}(A) : A\subset\Gamma,\,(A-A)\cap E=\emptyset\}. \tag{5.16}$$

Let $F_1,F_2,\dots$ be the sets in $\mathcal{F}$, and fix any $\varepsilon>0$. By (5.9), for all N sufficiently large we have $\frac{\phi_E(F_N)}{|F_N|}\geq\mathrm{Md}_\Gamma(E)-\frac{\varepsilon}{2}$, so there is an E-avoiding set $S\subset F_N$ with $\frac{|S|}{|F_N|}\geq\mathrm{Md}_\Gamma(E)-\frac{\varepsilon}{2}$. Since E is finite, it follows from (5.8) that for all N sufficiently large we also have $|(F_N+t)\setminus F_N|\leq\frac{\varepsilon}{2|E|}|F_N|$ for each $t\in E$. Let us now fix N with the previous two properties. The set $S':=\{g\in S : g+E\subset F_N\}$ satisfies $S\setminus S'\subset\bigcup_{t\in E}F_N\setminus(F_N-t)$, so $\frac{|S'|}{|F_N|}\geq\frac{|S|}{|F_N|}-\frac{\varepsilon}{2}\geq\mathrm{Md}_\Gamma(E)-\varepsilon$. By assumption there is a tiling $\Gamma=\bigsqcup_{c\in C}c+F_N$. It is then easily checked that $A:=C+S'$ is E-avoiding. Therefore it now suffices to prove that $\overline{\delta}_{\mathcal{F}}(A)\geq\frac{|S'|}{|F_N|}$, since then $\mathrm{Md}_\Gamma(E)\leq\overline{\delta}_{\mathcal{F}}(A)+\varepsilon$, and (5.16) follows by taking the supremum and letting $\varepsilon\to 0$. For each $N'\in\mathbb{N}$, let $C'=\{c\in C : c+F_N\subset F_{N'}\}$. Then $|F_{N'}\cap A|\geq|F_{N'}\cap(C'+S')|=\frac{|S'|}{|F_N|}|C'+F_N|=\frac{|S'|}{|F_N|}(|F_{N'}|-|F_{N'}\setminus(C'+F_N)|)$. Moreover, letting $T=F_N-F_N$, we have $F_{N'} \setminus (C'+F_N) \subset F_{N'} \setminus (T+F_{N'})$. Indeed, if $g \in F_{N'} \setminus (C'+F_N)$ then by the tiling we have $g = c+x$ for some $c \in C$, $x \in F_N$, and by the definition of C' we have $c+x' \notin F_{N'}$ for some $x' \in F_N$; so $g+x'-x \notin F_{N'}$. Thus we deduce that $\frac{|F_{N'} \cap A|}{|F_{N'}|} \geq \frac{|S'|}{|F_N|}\big(1 - \frac{|F_{N'} \setminus (T+F_{N'})|}{|F_{N'}|}\big)$. Applying now (5.8) with variable N' and every $g \in T$, we deduce that $\overline{\delta}_{\mathcal{F}}(A) = \limsup_{N' \to \infty} \frac{|F_{N'} \cap A|}{|F_{N'}|} \geq \frac{|S'|}{|F_N|}$, as required. $\square$

Remark 5.12. Using (5.15), the anonymous referee provided an alternative proof of (5.13) (the second half of the proof of Theorem 5.9) by applying the pointwise ergodic theorem for actions of finitely generated abelian groups. We gratefully include the argument here.

Let $E \subset \Gamma = \mathbb{Z}^r/\Lambda$ as defined in Theorem 5.9, let f be the action of Γ on Z, and let $\mathcal{F}$ be the tiling Følner sequence given by (5.10). Let $A \subset \mathrm{Z}$ be a Borel set with $\mu(A) > \mathrm{Md}_\Gamma(E)$. We will find a point $x \in \mathrm{Z}$ such that $A_x := \{g \in \Gamma : f_g(x) \in A\}$ satisfies $\overline{\delta}_{\mathcal{F}}(A_x) \geq \mu(A)$. Thus we will have $\overline{\delta}_{\mathcal{F}}(A_x) > \mathrm{Md}_\Gamma(E)$, implying by (5.15) that A_x is not E-avoiding, so that there are distinct elements $a, b \in A_x$ with $a - b \in E$. Then $f_a(x), f_b(x) \in A$, which implies that $A - A$ contains the element $f_a(x) - f_b(x) = f_{a-b}(0) \in D$, so A is not D-avoiding. Hence $\mathrm{Md}_{\mathrm{Z}}(D) \leq \mathrm{Md}_\Gamma(E)$.

To find the set A_x, we apply the pointwise ergodic theorem for finitely generated abelian groups with the action f (for instance as a special case of [44, Theorem 1.2], noting that $\mathcal{F}$ clearly has the required property of being tempered). Thus we deduce that the averages $x \mapsto \frac{1}{|F_N|} \sum_{g \in F_N} 1_A(f_g(x))$ converge pointwise almost everywhere to an f-invariant function $\overline{1_A} \in L^1(\mu)$. We have $\int \overline{1_A}\,\mathrm{d}\mu = \int 1_A\,\mathrm{d}\mu = \mu(A)$, and it follows that the set of points $x \in \mathrm{Z}$ with $\overline{1_A}(x) \geq \mu(A)$ has positive Haar measure. Therefore there exists $x \in \mathrm{Z}$ at which the limit of these averages is at least $\mu(A)$. Hence $\overline{\delta}_{\mathcal{F}}(A_x) \geq \mu(A)$.

5.2.3 The case $D \subset \mathbb{Q}$: the independence ratio of circulant graphs

Let us formalize the remarks, made in the introduction, about the general rational case $D \subset \mathbb{Q}$ in the circle group.

Lemma 5.13. *Let $D = \{t_1, \dots, t_r\} \subset (0,1)$, where for each $i \in [r]$ we have $t_i = a_i/b_i$ with $1 \leq a_i < b_i$ and $\gcd(a_i, b_i) = 1$. Let $N = \mathrm{lcm}(b_1, \dots, b_r)$, and* *let G be the connected circulant graph on $\mathbb{Z}_N$ with jumps $d_1, \ldots, d_r$ where $d_i = Nt_i$. Then* $\mathrm{Md}_{\mathbb{T}}(D) = \frac{\alpha(G)}{N}$.

Proof. The connectedness of G is equivalent to the elements $d_1, \ldots, d_r$ generating the full group $\mathbb{Z}_N$, which is equivalent to $\gcd(d_1, \ldots, d_r, N) = 1$, which in turn is equivalent to $\gcd(\frac{N}{b_1}, \ldots, \frac{N}{b_r}, N) = 1$ by our assumptions. This last equality can be seen to hold using the identity $\mathrm{lcm}(b_1, \ldots, b_r) = \frac{b_1 \cdots b_r}{\gcd(\pi_1, \ldots, \pi_r)}$ where $\pi_i = \prod_{j \in [r] \setminus \{i\}} b_j$ for $i \in [r]$.

Using the notation in Theorem 5.9, we have $\mathrm{Md}_{\mathbb{T}}(D) = \mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E)$. Letting ψ denote the homomorphism $\mathbb{Z}^r \to \mathbb{T}$, $n \mapsto n_1 t_1 + \cdots + n_r t_r$, by the first isomorphism theorem we have $\mathbb{Z}^r/\Lambda \cong \psi(\mathbb{Z}^r)$. Denoting by $\frac{1}{N} \cdot \mathbb{Z}_N$ the subgroup of $\mathbb{T}$ of order N, we have $\psi(\mathbb{Z}^r) = \frac{1}{N} \cdot \mathbb{Z}_N$, and $\psi(e_i') = t_i = \frac{d_i}{N}$ for $i \in [r]$. It follows that $\mathrm{Md}_{\mathbb{Z}^r/\Lambda}(E) = \mathrm{Md}_{\frac{1}{N} \cdot \mathbb{Z}_N}(D) = \mathrm{Md}_{\mathbb{Z}_N}(\{d_1, \ldots, d_r\})$. This last quantity equals $\frac{\alpha(G)}{N}$. Hence $\mathrm{Md}_{\mathbb{T}}(D) = \frac{\alpha(G)}{N}$. □

Remark 5.14. Lemma 5.13 shows that Motzkin's problem in $\mathbb{T}$ subsumes the problem of determining the independence ratio of circulant graphs. Solving the latter problem in turn yields a solution to Motzkin's original problem in $\mathbb{Z}$ for finitely many missing differences. This follows from the fact that for any finite set $D \subset \mathbb{N}$, identifying $\mathbb{Z}_N$ with the integer interval $F_N = [-\frac{N}{2}, \frac{N}{2}]$ with addition mod N, we see that $\lim_{N \to \infty} \mathrm{Md}_{\mathbb{Z}_N}(D) = \mathrm{Md}_{\mathbb{Z}}(D)$. Indeed, this can be seen from (5.9) noting that $\lim_{N \to \infty} \frac{\phi_D(F_N)}{|F_N|} - \mathrm{Md}_{\mathbb{Z}_N}(D) = 0$, and it can also be seen from previous work: by [43, Theorem 4.1] the limit $\lim_{N \to \infty} \mathrm{Md}_{\mathbb{Z}_N}(D)$ equals the reciprocal of the fractional chromatic number of the graph $\mathrm{Cay}(\mathbb{Z}, D)$; this reciprocal in turn equals $\mathrm{Md}_{\mathbb{Z}}(D)$ [47, Theorem 1].

5.2.4 The case $|D| = 1$

Proposition 5.15. *For $D = \{t\}$ with $t \in (0,1)$ we have*

$$\mathrm{Md}_{\mathbb{T}}(D) = \begin{cases} 1/2, & t \notin \mathbb{Q} \\ \lfloor N/2 \rfloor / N, & t = \frac{d}{N},\ \gcd(d, N) = 1. \end{cases}$$

Proof. The case $t \notin \mathbb{Q}$ follows from Theorem 5.7. For $t = \frac{d}{N}$ with coprime integers d, N, we have by Lemma 5.13 that $\mathrm{Md}_{\mathbb{T}}(D)$ is the independence ratio of an N-cycle. This ratio is easily seen to equal $\lfloor N/2 \rfloor / N$ by identifying the cycle's vertex set with $[0, N-1]$, where $x, y \in [0, N-1]$ form an edge if and only if $|x-y| = 1 \bmod N$, and noting that $\{0, 2, \ldots, 2(\lfloor N/2 \rfloor - 1)\}$ is a stable set of maximal cardinality in this cycle. $\square$

5.3 The case $|D| = 2$, $D \not\subset \mathbb{Q}$

In this section we suppose that $D = \{t_1, t_2\} \subset (0, 1)$ where t_1, t_2 are not both rational, and we prove Theorem 5.2. Theorem 5.7 already covers the case in which $1, t_1, t_2$ are linearly independent over $\mathbb{Q}$, in other words, the case in which the lattice Λ from (5.6) is trivial (i.e. $\Lambda = \{0\}$). The case in which Λ has full rank 2 corresponds to $D \subset \mathbb{Q}$ (treated in the next section). Therefore, here it only remains to address the case in which Λ has rank 1, that is, where Λ is a non-trivial cyclic subgroup of $\mathbb{Z}^2$. We begin by describing this subgroup more explicitly in terms of the assumption in Theorem 5.2.

Lemma 5.16. *Let $t_1, t_2 \in (0, 1)$ such that $\{t_1, t_2\} \not\subset \mathbb{Q}$ and $1, t_1, t_2$ are linearly dependent over $\mathbb{Q}$. Let Λ be the kernel of the homomorphism $\mathbb{Z}^2 \to \mathbb{T}$, $(n_1, n_2) \mapsto n_1 t_1 + n_2 t_2$, and let $m_1, m_2 \in \mathbb{Z}$. Then (m_1, m_2) generates Λ if and only if there is $m_0 \in \mathbb{Z}$ such that*

$$(m_0, m_1, m_2) \in \mathbb{Z}^3 \setminus \{0\}, \quad m_0 = m_1 t_1 + m_2 t_2 \quad \text{and} \quad \gcd(m_0, m_1, m_2) = 1. \tag{5.17}$$

Proof. If (m_1, m_2) generates Λ (i.e. $\Lambda = \mathbb{Z}(m_1, m_2)$) then in particular $(m_1, m_2) \in \Lambda$, so there is $m_0 \in \mathbb{Z}$ such that $m_0 = m_1 t_1 + m_2 t_2$ (and clearly m_1, m_2 cannot be both 0 since Λ is non-trivial); moreover $g := \gcd(m_0, m_1, m_2)$ must be 1, as otherwise $(\frac{m_1}{g}, \frac{m_2}{g})$ would be an element of $\Lambda \setminus \mathbb{Z}(m_1, m_2)$, contradicting that (m_1, m_2) generates Λ. Hence (5.17) holds.

To see the converse, note first that if (5.17) holds then $(m_1, m_2) \in \Lambda$, so $\mathbb{Z}(m_1, m_2) \subset \Lambda$ and it only remains to prove the opposite inclusion. For this, it suffices to prove that every $m' = (m'_0, m'_1, m'_2) \in \mathbb{Z}^3$ satisfying $m'_0 = m'_1 t_1 + m'_2 t_2$ is an integer multiple of $m = (m_0, m_1, m_2)$. If one of t_1, t_2 is rational, say $t_1 \in \mathbb{Q}$, then since of $t_2 \notin \mathbb{Q}$ we must have $m_2 = m'_2 = 0$, so $m'_0/m'_1 = t_1 = m_0/m_1$ and the claim is then clear. Let us therefore assume that t_1, t_2 are both irrational. We have by assumption $\begin{cases} m_0 = m_1 t_1 + m_2 t_2 \\ m'_0 = m'_1 t_1 + m'_2 t_2 \end{cases}$. Note that none of m_1, m'_1 is zero, otherwise $t_2 \in \mathbb{Q}$. Multiplying the first equation by m'_1, the second one by m_1, and subtracting, we deduce that $m_0'm_1 - m_0m_1' = (m_1m_2' - m_1'm_2)t_2$. Since $t_2 \notin \mathbb{Q}$, this implies $m_1m_2' = m_1'm_2$ and $m_0'm_1 = m_0m_1'$. The former equation implies that (m_1, m_2), (m_1', m_2') are linearly dependent over $\mathbb{Q}$. Hence there are coprime non-zero integers a, b such that $a(m_1, m_2) = b(m_1', m_2')$. Using this in the system of equations above yields $am_0 = am_1t_1 + am_2t_2 = bm_1't_1 + bm_2't_2 = bm_0'$, so $am = bm'$. This combined with $\gcd(a, b) = \gcd(m_0, m_1, m_2) = 1$ implies $|b| = 1$, so m' is indeed in $\mathbb{Z}m$. □

In view of Lemma 5.16 and Theorem 5.9, to complete the proof of Theorem 5.2 it now suffices to prove the following result.

Theorem 5.17. *Let Λ be a cyclic subgroup of $\mathbb{Z}^2$ generated by an element $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{0\}$. Let E be the image of the standard basis $\{e_1, e_2\}$ in the quotient $\mathbb{Z}^2/\Lambda$. Then*

$$\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) = \lfloor k/2 \rfloor / k, \quad \textit{where} \quad k = |m_1| + |m_2|. \tag{5.18}$$

The basic idea of the proof is that the (undirected) Cayley graph $\mathrm{Cay}(\mathbb{Z}^2/\Lambda, E)$ can be decomposed by partitioning $\mathbb{Z}^2/\Lambda$ into translates of a cycle of length k in the graph, so that $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E)$ is then easily shown to equal the independence ratio of this cycle.

Proof. We can assume without loss of generality that $m_1 \geq 0$ and $m_2 > 0$.

Let R denote the set $\mathbb{Z}(1, -1) + [0, k-1] \times \{0\} = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 + n_2 \in [0, k-1]\}$. It is easily checked that R is a fundamental domain for the action of Λ on $\mathbb{Z}^2$. For this proof we identify $\mathbb{Z}^2/\Lambda$ as a group with R equipped with addition mod Λ (i.e. addition in $\mathbb{Z}^2$ composed with reduction mod Λ into R), and we identify E with $\{e_1, e_2\} \subset R$.

Let G be the Cayley graph on R with generating set E, i.e. with uv being an edge in G if and only if $v - u \in \{e_1, -e_1, e_2, -e_2\}$ (where the operations are in R). Let $C = C_1 \cup C_2 \subset R$ where $C_1 = \{(0, i) : i \in [0, m_2 - 1]\}$ and $C_2 = \{(i, m_2 - 1) : i \in [m_1]\}$ (if $m_1 = 0$ then $C_2 = \emptyset$). Note that the subgraph of G induced by C (denoted by $G[C]$) is a k-cycle. Note also that $R = \bigsqcup_{n \in \mathbb{Z}} C + n(1, -1)$.

We now prove that $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E)$ is the independence ratio of $G[C]$, i.e. $\frac{\lfloor k/2 \rfloor}{k}$. For $N \in \mathbb{N}$ let $F_N = \bigsqcup_{n=-N}^{N} C + n(1, -1)$. It is easily seen that $(F_N)_{N \in \mathbb{N}}$ is a Følner sequence in R.

To see that $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) \geq \frac{\lfloor k/2 \rfloor}{k}$, let S be a stable subset of C of maximal size (thus $|S| = \lfloor k/2 \rfloor$) and note that S is E-avoiding. Let $A := \bigsqcup_{n \in \mathbb{Z}} S +$

$n(1,-1) = S + \mathbb{Z}(1,-1)$. We claim that A is E-avoiding. Indeed, suppose for a contradiction that there is $x \in A$ with $x + e_i \in A$ for $i = 1$ or 2. Since A is invariant under translation by elements of $\mathbb{Z}(1,-1)$, we can suppose that $x \in S$. If $x + e_1 \in A$, then we must have $x + e_1 \in S \cup (S + (1,-1))$. This implies that $(S+e_1) \cap [S \cup (S+(1,-1))] \neq \emptyset$, which implies that $S - S$ contains e_1 or e_2, which is impossible since S is E-avoiding. If $x + e_2 \in A$, then $x + e_2 \in S \cup (S - (1,-1))$, but then $(S + e_2) \cap [S \cup (S - (1,-1))] \neq \emptyset$, which similarly contradicts that S is E-avoiding. This proves our claim. Now note that $\frac{|A \cap F_N|}{|F_N|} = \frac{|S|}{|C|} = \frac{\lfloor k/2 \rfloor}{k}$ for all N. Hence by (5.9) we have $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) \geq \frac{\lfloor k/2 \rfloor}{k}$.

To see that $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) \leq \frac{\lfloor k/2 \rfloor}{k}$, note that for any $\varepsilon > 0$, by (5.9), for some $N \in \mathbb{N}$ there exists an E-avoiding set $A \subset F_N$ such that $\frac{|A|}{|F_N|} \geq \mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) - \varepsilon$. We also have $\frac{|A|}{|F_N|} = \frac{1}{(2N+1)k} \sum_{n=-N}^{N} |(C + n(1,-1)) \cap A|$. Now each set $A \cap (C + n(1,-1))$ is a stable set in (a translate of) the cycle $G[C]$, so this set has size at most $\lfloor k/2 \rfloor$. We deduce that $\frac{|A|}{|F_N|} \leq \frac{\lfloor k/2 \rfloor}{k}$, so $\mathrm{Md}_{\mathbb{Z}^2/\Lambda}(E) \leq \frac{\lfloor k/2 \rfloor}{k} + \varepsilon$ and the desired inequality follows letting $\varepsilon \to 0$. □

5.4 $|D| = 2$, $D \subset \mathbb{Q}$: the independence ratio of 2-jump circulant graphs

When both elements of D are rational, it follows from Lemma 5.13 that $\mathrm{Md}_{\mathbb{T}}(D)$ is the independence ratio of a connected circulant graph with two jumps. Thus, throughout this section we let G be a circulant graph $\mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ with $\gcd(N, d_1, d_2) = 1$. Our aim is to determine $\alpha(G)$.

It is well-known (and easily seen) that in this situation if G is bipartite then $\alpha(G) = \frac{N}{2}$. If G is not bipartite then it contains an odd cycle. Recall that the *odd girth* of G is then defined to be the smallest length of an odd cycle in G. We then have the following upper bound for $\alpha(G)$.

Lemma 5.18 (Odd-girth bound)**.** *Let G be a circulant graph of order N and odd girth k. Then*

$$\alpha(G) \leq \left\lfloor \tfrac{k-1}{2k} N \right\rfloor. \tag{5.19}$$

This is an immediate consequence of the "no homomorphism lemma" of

Albertson and Collins, which we recall here; see [1, Theorem 2] and also [33, Lemma 3.3].

Lemma 5.19. *Let G be a vertex transitive graph and let H be a subgraph of G. Then $\alpha(G)/|V(G)| \leq \alpha(H)/|V(H)|$.*

Remark 5.20. The odd-girth bound (5.19) is attained in many cases. Note first that if G is 2-regular (i.e. every vertex in G has two neighbours) then d_1 equals d_2 or $-d_2$, that is, there is just one jump of odd order N, so $\alpha(G)$ attains the odd-girth bound $\lfloor N/2 \rfloor$ in this case. If G is 3-regular (which occurs only if N is even and one of the jumps is $N/2$) then it can be seen that the odd-girth bound is attained as well, using for instance [37, Corollary 2.27]. Therefore, from now on we assume that G is 4-regular. Among 4-regular connected circulant graphs, examples attaining the odd-girth bound include those given by Gao and Zhu in [28, Theorem 7], which have jumps 1 and d_2, with d_2 sufficiently small compared to N. We establish a different family of examples in Proposition 5.29 below.

The following result provides an infinite family of examples of 2-jump circulant graphs with independence number strictly below the odd-girth bound.

Proposition 5.21. *Let $d \in \mathbb{N}$ be odd, let $N = 2d(d+1)$, and let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d, d+1\})$. Then $\alpha(G) = d^2$, and G has girth $k = 2d+1$, so $\alpha(G) = \left\lfloor \frac{k-1}{2k} N \right\rfloor - \frac{d-1}{2}$.*

Proof. We first prove that G has girth $2d+1$. Every cycle in G can be translated in $\mathbb{Z}_N$ to obtain a cycle C of same length starting from 0. To every such cycle C there correspond integers a, b such that $a(d+1) + bd = 0 \mod N$ and such that the length of the cycle is $|a| + |b|$. Supposing first that $a(d+1) + bd = cN$ for a non-zero integer c, we have $2d(d+1) = N \leq |cN| < (|a| + |b|)(d+1)$, so $|a| + |b| > 2d$ and therefore the cycle has length at least $2d+1$. Note that this length is achieved by the cycle in which first the element d is added $d+1$ times to 0, and then the element $d+1$ is added d times to reach N. The remaining possibility is that $a(d+1) + bd = 0$. Then $|a|(d+1) = |b|d \geq \mathrm{lcm}(d, d+1)$, and this least common multiple is $d(d+1)$ (since $d, d+1$ are coprime), so we deduce $|a| \geq d$ and $|b| \geq d+1$, so $|a| + |b| \geq 2d+1$.

Next, note that $\left\lfloor \frac{k-1}{2k} N \right\rfloor = \left\lfloor \frac{2d^2(d+1)}{2d+1} \right\rfloor = \left\lfloor d^2 + \frac{d^2}{2d+1} \right\rfloor = d^2 + \left\lfloor \frac{d^2}{2d+1} \right\rfloor$, and $\frac{d-1}{2}(2d+1) = d^2 - \frac{d+1}{2}$, so $\frac{d^2}{2d+1} = \frac{d-1}{2} + \frac{d+1}{2(2d+1)}$, so $\left\lfloor \frac{d^2}{2d+1} \right\rfloor = \frac{d-1}{2}$.

Finally, we prove that $\alpha(G) = d^2$. To this end, let φ denote the homomorphism $\varphi : \mathbb{Z}^2 \to \mathbb{Z}_N$, $(x, y) \mapsto x(d+1) + yd \mod N$, and let $\Lambda = \ker \varphi$. Let R denote the fundamental domain $[0, 2d-1] \times [0, d] \subset \mathbb{Z}^2$ equipped with $\mathbb{Z}^2$-addition mod Λ, and note that φ restricts to an isomorphism $R \to \mathbb{Z}_N$, so we may identify G with the Cayley graph $\mathrm{Cay}(R, \{e_1, e_2\})$. Let H denote the subgroup $[0, 2d-1] \times \{0\}$ of R. It is easily seen that for every stable set S of G, each of the $d+1$ cosets of H contains at most d elements of S. A stable set S' of size d^2 can be constructed by letting $S_0 = \{(2i, 0) : i \in [0, d-1]\} \subset H$ and then letting $S' = \varphi\big(\bigsqcup_{j\in[0,d-1]} S_0 + j(1, 1) \mod \Lambda\big)$. Hence $\alpha(G) \geq d^2$.

Now let $S \subset R$ be any maximum stable set in G. Since $|S| \geq d^2$, the average number of points of S per coset of H is greater than $d-1$, so there is a coset of H whose intersection with S has size d and is therefore a translate of the above set S_0. Hence we may assume (translating if necessary) that $H \cap S = S_0$. It follows that $S \cap (H + e_2) \subset (S_0 + e_2)^c$ and also, using that $-e_2 = (d, d) \mod \Lambda$, that $S \cap (H - e_2) = S \cap (H + (d, d)) \subset (S_0 + (d, d))^c = (S_0 + (1, d))^c$, where the last equality follows from the invariance $S_0 = S_0 + 2e_1$. Therefore $S \setminus H$ is a stable set of the subgraph of G induced by $S \setminus \big(H \cup (S_0 + e_2) \cup (S_0 + (1, d))\big)$. In this induced subgraph, every vertical line is a path of even length $d-1$, which contains at most $(d-1)/2$ elements of S. Therefore $|S| \leq d + 2d(d-1)/2 = d^2$, so $\alpha(G) = d^2$. $\square$

Proposition 5.21 shows that for a connected circulant graph $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ the independence number $\alpha(G)$ can go below the odd-girth bound by as much as a constant multiple of $\sqrt{N}$. We will show that this is the correct order of magnitude for how large the difference between these two quantities can be. Thus, in particular, the odd-girth bound is an estimate for $\alpha(G)$ that is asymptotically tight as N increases. We will establish this by proving the following result, which immediately yields Theorem 5.3.

Theorem 5.22. *Let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ with $\langle d_1, d_2\rangle = \mathbb{Z}_N$. If G is bipartite then $\alpha(G) = N/2$. Otherwise, letting k denote the odd girth of G, we have*

$$\left\lfloor \tfrac{k-1}{2k} N \right\rfloor \;\geq\; \alpha(G) \;\geq\; \left\lceil \tfrac{k-1}{2k} N - 3\sqrt{N} \right\rceil. \tag{5.20}$$

By Remark 5.20 there are arbitrarily large N for which the upper bound in (5.20) is sharp, and by Proposition 5.21 there are also arbitrarily large N for which the lower bound in (5.20) is sharp up to the absolute constant multiplying $\sqrt{N}$.

To prove Theorem 5.22 we shall use the following full-rank lattice naturally associated with G:

$$\Lambda = \{x \in \mathbb{Z}^2 : x_1 d_1 + x_2 d_2 = 0 \mod N\}, \tag{5.21}$$

that is, the lattice Λ is the kernel of the homomorphism

$$\varphi : \mathbb{Z}^2 \to \mathbb{Z}_N,\ (x_1, x_2) \mapsto x_1 d_1 + x_2 d_2 \mod N. \tag{5.22}$$

Since we suppose that $\langle d_1, d_2 \rangle = \mathbb{Z}_N$, we have that φ is surjective. Note that φ is also a graph homomorphism $\mathrm{Cay}(\mathbb{Z}^2, \{e_1, e_2\}) \to G$.

The lattice Λ is useful to analyze cycles in G. In particular, short cycles are related to the successive minima λ_1, λ_2 of Λ relative to the ℓ^1-norm, namely (see [15])

$$\lambda_1 = \min\{\rho : \dim\big(\mathrm{Span}(B_\rho \cap \Lambda)\big) \geq 1\}, \lambda_2 = \min\{\rho : \dim\big(\mathrm{Span}(B_\rho \cap \Lambda)\big) \geq 2\} \tag{5.23}$$

where B_ρ is the ball in $\mathbb{R}^2$ centered at the origin and of radius ρ relative to the ℓ^1-norm.

Remark 5.23. The lattice in (5.21) is a special case of the lattice in (5.6). These objects, as well as the role played by short cycles in the case of two missing differences, are some ideas unifying the various cases of Motzkin's problem treated in this paper. We say more about this in Section 5.5.

The following lemma shows that we can always select a convenient basis for Λ.

Lemma 5.24. *Let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ with $\langle d_1, d_2 \rangle = \mathbb{Z}_N$, and let λ_1, λ_2 be the successive minima defined in (5.23). Then there exist $u, v \in \Lambda$ with the following properties:*

1. *$\{u, v\}$ is a basis of Λ such that $\|u\|_1 = \lambda_1$, $\|v\|_1 = \lambda_2$.*
2. *If G has odd girth k, then $k \in \{\lambda_1, \lambda_2\}$.*

Proof. Property (i) is a standard result (see [15, p. 204, Lemma 1]).

To see property (ii), note first that λ_1, λ_2 are both lengths of cycles in G. Indeed, given any $w \in \mathbb{Z}^2$ let $P(w)$ denote the path in $\mathbb{Z}^2$ that starts at the origin, then adds $e_1 = (1, 0)$ if $w_1 > 0$ (resp. $-e_1$ if $w_1 < 0$) until it reaches $(w_1, 0)$ and then adds $e_2 = (0, 1)$ if $w_2 > 0$ (resp. $-e_2$ if $w_2 < 0$) until it ends at w. Note that if w is u or v, then the map φ from (5.22) is injective on $P(w) \setminus \{w\}$, so that $\varphi(P(w))$ is indeed a cycle in G of length $\|w\|_1 = \lambda_i$. Indeed, suppose for a contradiction that there exist $x, y \in P(w) \setminus \{w\}$ with $\varphi(x) = \varphi(y)$ and $\|x\|_1 < \|y\|_1$. Then $y - x \in \Lambda \setminus \{0\}$ would have $\|y - x\|_1 < \|w\|_1 \leq \lambda_2$, so $y - x$ would be in the span of u (by [15, p. 204, Lemma 1]). Hence $\|w\|_1 > \|u\|_1$, so w must be v. Then $v - (y - x) = w - (y - x)$ is an element of $\Lambda \setminus \{0\}$ of ℓ^1-norm less than $\|w\|_1 = \lambda_2$, so it is also in the span of u. This contradicts the linear independence of u, v.

If G has odd girth k, then by translating we find a k-cycle $C = (x_0 = 0, x_1, \ldots, x_k = 0)$ in G. We can then construct a walk $\tilde{C} = (\tilde{x}_0 = 0, \tilde{x}_1, \ldots, \tilde{x}_k)$ in $\mathrm{Cay}(\mathbb{Z}^2, \{e_1, e_2\})$ such that $\varphi(\tilde{C}) = C$ (in particular φ restricted to $\tilde{C} \setminus \{\tilde{x}_k\}$ is bijective onto C). Note that $\tilde{x}_k$ is in Λ and cannot be 0, since otherwise k would be even. Hence $\|\tilde{x}_k\|_1 \geq \lambda_1$, and so $k \geq \lambda_1$. If λ_1 is odd, then we must have $k = \lambda_1$, since by the previous paragraph λ_1 is the length of an odd cycle in G, and k is the minimal such length. If λ_1 is even, then λ_2 must be odd. Indeed, otherwise for every cycle $C = (x_0 = 0, x_1, \ldots, x_{n-1}, x_n = 0)$ in G, for the walk $\tilde{C} = (\tilde{x}_0 = 0, \tilde{x}_1, \ldots, \tilde{x}_n)$ in $\mathbb{Z}^2$ satisfying $\varphi(\tilde{C}) = C$, we have that $\tilde{x}_n \in \Lambda$, so $\tilde{x}_n$ is an integer combination of u, v and therefore $\|\tilde{x}_n\|_1$ would be even. This would imply that every cycle in G has even length, contradicting that G has odd cycles. Since k cannot be the even number λ_1 and is at least the ℓ^1-norm of some non-zero element in Λ, we have $k \geq \lambda_2$, whence $k = \lambda_2$ (since λ_2 is an odd-cycle length). This proves property (ii). $\square$

We shall use the basis $\{u, v\}$ to estimate $\alpha(G)$. We begin by reformulating the bipartite case in terms of the minima from (5.23).

Lemma 5.25. *We have $\alpha(G) = N/2$ if and only if λ_1 and λ_2 are both even.*

Proof. We first prove the backward implication. If λ_1 and λ_2 are both even, then, as noted in the proof of Lemma 5.24, the graph G has no odd cycles, so it is bipartite and therefore $\alpha(G) = N/2$.

For the forward implication, note that if one of λ_1, λ_2 is odd then G has odd girth $k \in \{\lambda_1, \lambda_2\}$ by Lemma 5.24, so by Lemma 5.19 we have $\alpha(G) \leq \frac{k-1}{2k} N < N/2$. $\square$

Thus, to prove Theorem 5.22 (more precisely (5.20) therein) we can assume that at least one of u, v has odd ℓ^1-norm.

Let $\mathcal{P}$ denote the parallelogram determined by u, v:

$$\mathcal{P} := [0,1)^2 \cdot (u,v) := \{\alpha u + \beta v : \alpha, \beta \in [0,1)\}.$$

By standard results, the Lebesgue measure of $\mathcal{P}$ is the absolute value of $\det\left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right)$, which is also equal to the index $|\mathbb{Z}^2/\Lambda|$, which equals N by the first isomorphism theorem and the surjectivity of φ. Moreover, since $\mathcal{P}$ is also a fundamental domain for $\mathbb{Z}^2/\Lambda$, we have that $|\mathcal{P} \cap \mathbb{Z}^2|$ is also equal to $|\mathbb{Z}^2/\Lambda|$, so

$$|\mathcal{P} \cap \mathbb{Z}^2| = N. \tag{5.24}$$

The following result tells us that for each $i \in \{1,2\}$ we can always partition a large subset of $\mathbb{Z}_N$ into useful translates of a cycle of length λ_i.

Lemma 5.26. *Let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ be 4-regular with $\langle d_1, d_2\rangle = \mathbb{Z}_N$, let Λ be the associated lattice from (5.21), and let λ_1, λ_2 be the successive minima of Λ relative to the ℓ^1-norm. Then for each $i \in \{1,2\}$ there exists a λ_i-cycle C_i in G and $\varepsilon_1, \varepsilon_2 \in \{1,-1\}$ such that we have the following union of pairwise disjoint translates of C_i in $\mathbb{Z}_N$:*

$$\bigsqcup_{t=0}^{\lfloor \frac{N}{\lambda_i} \rfloor - 2} \big(C_i + t(\varepsilon_1 d_1 + \varepsilon_2 d_2)\big). \tag{5.25}$$

The idea of the proof is that there is a lattice path in $\mathbb{Z}^2$ which represents a λ_i-cycle and has the property that, modulo Λ, one can tile a large subset of $\mathcal{P} \cap \mathbb{Z}^2$ with certain translates of this path. The images of these translates under φ then yield (5.25).

Proof. Let $\{u, v\}$ be the basis of Λ provided by Lemma 5.24. We prove (5.25) for $i = 1$; the proof for $i = 2$ is similar. Note that the operations of permuting d_1, d_2 and changing their sign all yield isomorphisms of G, and that the conclusion of the lemma is not affected by these operations. It follows that, by performing such operations if necessary, we can assume that u has both coordinates non-negative and the angle from u to v is in $(\pi, 2\pi)$ (i.e. $\det\left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right) < 0$). In the resulting more specific situation, we can prove (5.25) with $\varepsilon_1 = -\varepsilon_2 = 1$, as follows.

We first settle the case in which one of u_1, u_2 is 0. If $u_1 = 0$, then $u_2 = \lambda_1$ is the order of d_2 in $\mathbb{Z}_N$. We then set C_1 to be the cycle $\langle d_2 \rangle$. By Minkowski's second theorem [15, p. 203, (12)] we have $\lambda_1\lambda_2 \le 2N$, and each

λ_i is at least 3 (otherwise G is not 4-regular). Hence $\lambda_i < N$, so in particular C_1 is a proper subgroup of $\mathbb{Z}_N$. Since $\langle d_1, d_2\rangle = \mathbb{Z}_N$, the cosets of the form $C_1 + td_1$, $t \geq 0$ cover $\mathbb{Z}_N$. Then the smallest $t \in \mathbb{N}$ such that $td_1 \in C_1$ is $t = N/\lambda_1$ (in particular $d_1 \notin C_1$). Hence (5.25) holds with t up to $N/\lambda_1 - 1$ in this case, as a particular way to write the partition of $\mathbb{Z}_N$ into cosets of C_1. A similar argument yields (5.25) when $u_2 = 0$.

We assume from now on that $u_1, u_2 > 0$.
Let $\tilde{C}_1 = (x^{(1)} = 0, x^{(2)}, \dots, x^{(\lambda_1)} = u - e_2)$ be the lattice path in $\mathbb{Z}^2$ of length λ_1 which starts at the origin, ends at $u - e_2$, and stays as close as possible to the line $\mathbb{R}u$ while staying below this line (i.e. $x_2^{(j)} \leq \frac{u_2}{u_1} x_1^{(j)}$ for all $j \in [\lambda_1]$). We can describe $\tilde{C}_1$ inductively as follows, using the fact that the vertical distance to $\mathbb{R}u$ from a point $x^{(j)}$ below $\mathbb{R}u$ is $\frac{u_2}{u_1} x_1^{(j)} - x_2^{(j)}$:

$$x^{(1)} = 0, \quad \text{and for } j \in [\lambda_1 - 1],\ x^{(j+1)} = \begin{cases} x^{(j)} + e_1, & \frac{u_2}{u_1} x_1^{(j)} - x_2^{(j)} \in [0,1) \\ x^{(j)} + e_2, & \frac{u_2}{u_1} x_1^{(j)} - x_2^{(j)} \geq 1 \end{cases}. \tag{5.26}$$

We now estimate the greatest positive integer s such that the homomorphism φ from (5.22) is injective on $\bigcup_{t=0}^{s} \tilde{C}_1 + t(1,-1)$. First note that, for every $s \in \mathbb{N}$, there is no pair of points in this union differing by a non-zero multiple of u. Indeed, supposing that $x \in \tilde{C}_1 + i(1,-1)$ and $y \in \tilde{C}_1 + j(1,-1)$ for $j \geq i$, then y cannot be $x + u$ (let alone being $x + ru$ for any integer $r > 1$), for we have $y_2 \leq u_2 - 1 - j$, while $x_2 + u_2 \geq u_2 - i$, so $x_2 + u_2 - y_2 \geq j - i + 1 > 0$. Therefore φ is injective on $\bigcup_{t=0}^{s} \tilde{C}_1 + t(1,-1)$ if and only if no pair of points in this union differ by an element of the form $au + bv$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. A sufficient condition for this to hold is that every point in the union lies strictly above the line $v + \mathbb{R}u$. To ensure that this condition holds, it suffices to ensure that no point of $\tilde{C}_1 + (s, -s)$ lies on or below the line $v + \mathbb{R}u$. Let z denote the point in $\tilde{C}_1$ most distant from $\mathbb{R}u$ in the direction of $(1,-1)$, i.e. the point that maximizes the Euclidean length of the line segment parallel to $(1,-1)$ joining the point to the line $\mathbb{R}u$. Then the above condition holds if we set $s = \lfloor \sigma \rfloor - 1$ where σ, η are the unique real solutions to $z + \sigma(1,-1) = v + \eta u$. (We are unable to guarantee that the condition still holds with $s = \lfloor \sigma \rfloor$, because if σ happens to be an integer then $z + (\sigma, -\sigma) = v + \eta u$ is on the line $v + \mathbb{R}u$ and then we are unable to ensure that φ is injective as desired.)

Note that z is a point in $\tilde{C}_1$ maximizing the vertical distance to $\mathbb{R}u$, i.e. $\frac{u_2}{u_1} z_1 - z_2 = h := \max_{j \in [\lambda_1]} \frac{u_2}{u_1} x_1^{(j)} - x_2^{(j)}$. By (5.26), if the vertical distance from $x^{(j)}$ to $\mathbb{R}u$ is at least 1, then from $x^{(j+1)}$ the distance is smaller than that from $x^{(j)}$. Hence the maximum h occurs at $z = x^{(j+1)}$ where $x^{(j)}$ has vertical distance d to $\mathbb{R}u$ which is maximal subject to being less than 1. Thus $d \leq \max_{j\in[0,u_1-1]}\{j\frac{u_2}{u_1}\} = 1 - \frac{\gcd(u_1,u_2)}{u_1}$, so $h = d + \frac{u_2}{u_1} \leq \frac{u_1+u_2-\gcd(u_1,u_2)}{u_1}$. We therefore have $\sigma \geq \sigma'$ where $(0, -\frac{u_1+u_2-\gcd\{u_1,u_2\}}{u_1}) + \sigma'(1,-1) \in v + \mathbb{R}u$. We obtain (using $\lambda_1 = u_1+u_2$, $N = u_2v_1-u_1v_2$) that $\sigma' = \frac{N}{\lambda_1} - 1 + \frac{\gcd(u_1,u_2)}{\lambda_1} > \frac{N}{\lambda_1} - 1$. Hence, setting $s = \lfloor\sigma\rfloor - 1 \geq \lfloor\frac{N}{\lambda_1}\rfloor - 2$, we conclude that φ is injective on the set $S := \bigcup_{t=0}^{s}\big(\tilde{C}_1 + t(1,-1)\big)$. It is easily seen from (5.26) that the translates of $\tilde{C}_1$ forming S are pairwise disjoint, so by injectivity of φ the images of these translates under φ are also pairwise disjoint. Letting C_1 be the cycle $\varphi(\tilde{C}_1)$ in G, we deduce (5.25) in this case, which completes the proof. □

Using the tiling by cycles in Lemma 5.26, we can form large independent sets in G by carefully choosing a maximal independent subset in each translate of C_i in (5.25) except the last translate. This yields the following result.

Proposition 5.27. *Let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ be 4-regular with $\langle d_1, d_2\rangle = \mathbb{Z}_N$, let Λ be the associated lattice from (5.21), and let λ_1, λ_2 be the successive minima of Λ relative to the ℓ^1-norm. Then*

$$\alpha(G) \;\geq\; \max_{i\in\{1,2\}} \big(\lfloor\tfrac{N}{\lambda_i}\rfloor - 2\big)\,\lfloor\tfrac{\lambda_i}{2}\rfloor. \tag{5.27}$$

Proof. Let $\bigsqcup_{t=0}^{s} C_i + t(\varepsilon_1 d_1 + \varepsilon_2 d_2)$ be the partition in (5.25), with $s = \lfloor\frac{N}{\lambda_i}\rfloor - 2$. Let B be the independent subset of C_i of maximal size obtained by starting from 0 and picking one of every two successive elements, stopping once we have picked $\lfloor\frac{\lambda_1}{2}\rfloor$ elements. Let $A := \bigsqcup_{t=0}^{s-1} B + t(\varepsilon_1 d_1 + \varepsilon_2 d_2)$. It suffices to prove that A is stable, as then $\alpha(G) \geq |A| \geq s\lfloor\frac{\lambda_i}{2}\rfloor$. We prove this for $i = 1$; the proof for $i = 2$ is similar. By initial operations similar to those in the previous proof, we may assume that $\varepsilon_1 = -\varepsilon_2 = 1$, $u_1, u_2 \geq 0$, and $\det\left(\begin{smallmatrix} u_1 & v_1 \\ u_2 & v_2 \end{smallmatrix}\right) < 0$.

Suppose for a contradiction that vertices $x, y \in A$ form an edge in G. Since B is stable, these vertices must lie in distinct translates of C_1. Shifting and relabeling, we can suppose that $x \in B$ and $y \in B + t'(d_1 - d_2)$ for some $t' \in [1, s-1]$.

We claim that $t' = 1$. To see this let $\tilde{C}_1$ be the $\mathbb{Z}^2$-path described in (5.26), and note that since φ is bijective on $\bigsqcup_{t=0}^{s} \tilde{C}_1 + t(1,-1)$, there are unique $\tilde{x} \in \tilde{C}_1$ and $\tilde{y} \in \tilde{C}_1 + t'(1,-1)$ with $\varphi(\tilde{x}) = x$, $\varphi(\tilde{y}) = y$. The distance in G between x and y (i.e. the length of a shortest path from x to y in G) is $\|\tilde{y}-\tilde{x}\|_{\ell^1/\Lambda} := \min_{z\in\Lambda} \|\tilde{y}-\tilde{x}-z\|_1$. Since x, y are neighbours in G, this distance is 1, so there is $z \in \Lambda$ and $w \in \{\pm e_1, \pm e_2\}$ such that $\tilde{y} = \tilde{x}+w+z$. If $t' \geq 2$ then this cannot happen with z being just a multiple of u, so z must be of the form $n_1 u + n_2 v$ for $n_1 \in \mathbb{Z}$ and $n_2 \in \mathbb{N}$. But then $\tilde{y} - w = \tilde{x} + z$ lies on or below the line $v + \mathbb{R}u$, which is impossible by construction of s since $t' \leq s-1$. This proves our claim.

Since $t' = 1$, we have $\tilde{y} \in \tilde{C}_1 + (1,-1)$, and since $\tilde{y} = \tilde{x}+w+z$, we have that $\tilde{C}_1 + (1,-1)$ overlaps mod Λ with $\tilde{C}_1 + w$. By construction of $\tilde{C}_1$, this requires w to be e_1 or $-e_2$ (since $\tilde{C}_1 - e_1$ and $\tilde{C}_1 + e_2$ clearly do not overlap with $\tilde{C}_1 + (1,-1)$). We deduce that $y = \varphi(\tilde{x}+w+z)$ equals $x+d_1$ or $x-d_2$. Since $y = b + d_1 - d_2$ for some $b \in B$, we deduce that $b = x + d_2$ or $x - d_1$, so x, b are elements of B adjacent in G, contradicting that B is stable. This proves that A is stable and completes the proof. □

Since the odd girth k of G is in $\{\lambda_1, \lambda_2\}$, from (5.27) we deduce immediately that

$$\alpha(G) \geq \big(\lfloor \tfrac{N}{k} \rfloor - 2\big)\, \tfrac{k-1}{2}. \tag{5.28}$$

The lower bound here may seem to be close to the odd-girth bound, but the two bounds can in fact differ by as much as a fraction of N, when k is proportional to N. However, combining (5.27) with Minkowski's second theorem, we can now prove the main result of this section, Theorem 5.22, which ensures that the odd-girth bound itself is close to $\alpha(G)$.

Proof of Theorem 5.22. By (5.19), it suffices to prove the lower bound for $\alpha(G)$ in (5.20). As noted in Remark 5.20, if G is d-regular with $d < 4$ and has odd girth k, then we already know that $\alpha(G) = \lfloor \frac{k-1}{2k} N \rfloor$, so (5.20) holds in these cases. We therefore assume from now on that G is 4-regular.

Suppose first that $k = \lambda_1$. Then by Minkowski's second theorem we have

$$k \leq \sqrt{\lambda_1 \lambda_2} \leq \sqrt{2N}. \tag{5.29}$$

Therefore, in this case by (5.27) we have

$$\alpha(G) \geq \big(\lfloor \tfrac{N}{k} \rfloor - 2\big)\tfrac{k-1}{2} > \big(\tfrac{N}{k} - 3\big)\tfrac{k-1}{2} \geq \tfrac{k-1}{2k}N - \tfrac{3}{2}k + \tfrac{3}{2} \geq \tfrac{k-1}{2k}N - 3\sqrt{N}.$$

Supposing instead that $k = \lambda_2 > \lambda_1$, then by minimality of k and the fact that λ_1 is the length of a cycle in G, we have that λ_1 must be even, so

$\lambda_1 \geq 4$ (since G is 4-regular). By Minkowski's second theorem again we have $\lambda_1 \leq \sqrt{2N}$. Then by (5.27) we have (using that $2k \leq \lambda_1\lambda_2/2 \leq N$)

$$\alpha(G) \geq \left(\left\lfloor \tfrac{N}{\lambda_1} \right\rfloor - 2\right)\tfrac{\lambda_1}{2} > \tfrac{N}{2} - \tfrac{3}{2}\lambda_1 \geq \tfrac{k-1}{2k}N + \tfrac{N}{2k} - \tfrac{3}{2}\sqrt{2N} \geq \tfrac{k-1}{2k}N - 3\sqrt{N}. \quad \square$$

Remark 5.28. As mentioned above, from (5.20) we immediately deduce the asymptotically sharp estimate (5.5). To compute the main term $\frac{k-1}{2k}$ in this estimate, it suffices to find the vectors u, v from Lemma 5.24. This can be done using the Lagrange-Gauss reduction algorithm for the ℓ^1-norm [39], starting from any basis u', v' for Λ (for instance $u' = (\frac{d_2}{g}, -\frac{d_1}{g})$, $v' = (Na, Nb)$ where $g = \gcd(d_1, d_2)$ and $a, b \in \mathbb{Z}$ satisfy $a\frac{d_1}{g} + b\frac{d_2}{g} = 1$).

To finish this section, we consider the problem of determining for which 4-regular circulant graphs with 2 jumps the independence number matches the odd-girth bound. We do not solve this problem fully, but we provide the following family of such graphs, which is naturally described in terms of the associated lattice, and which differs significantly from the family given by Gao and Zhu in [28, Theorem 7].

Proposition 5.29. *Let $G = \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ be 4-regular with $\langle d_1, d_2\rangle = \mathbb{Z}_N$, and suppose that G has odd girth k. Let w be the basis element in $\{u, v\}$ such that $\|w\|_1 = k$. If some coordinate of w is 0, then $\alpha(G)$ equals the odd-girth bound $\frac{k-1}{2k}N$.*

Note that by permuting d_1 and d_2 if necessary, we can assume that $w_1 = 0$, so that $w = (0, k)$. In this case the subgroup $\langle d_2\rangle \leq \mathbb{Z}_N$ constitutes a k-cycle C in G (and in particular k divides N). Let $w' \in \{u, v\} \setminus \{w\}$. As noted in the proof of Lemma 5.26, we have the partition

$$\mathbb{Z}_N = \bigsqcup_{t=0}^{w_1'-1} (C + td_1),$$

with $w_1' = N/k$. In particular $w_1' d_1 \in C$, so there is $j \in \left[-\frac{k-1}{2}, \frac{k-1}{2}\right]$ which is the integer with least absolute value such that $w_1' d_1 + j d_2 = 0 \mod N$. Then, since $\|w'\|_1$ is the other smallest length of a non-trivial cycle in G, it follows that $w_2' = j$.

Proof of Proposition 5.29. We first observe that the following claim implies the conclusion of the proposition.

Claim: There is a walk $p_0, p_1, \ldots, p_{w_1'}$

$$\text{in } \mathrm{Cay}(\mathbb{Z}_k, \{1, -1\}) \text{ starting at } 0 \text{ and ending at } w_2'. \tag{5.30}$$

Indeed, if (5.30) holds then we can construct a set $A \subset \mathbb{Z}_N$ that is stable in G and has $|A| = \frac{k-1}{2k}N$, as follows. The set $A_0 = \{0, 2d_2, 4d_2, \ldots, (k-3)d_2\} \subset C$ is stable in G and has size $\frac{k-1}{2}$ (which is maximal subject to being a stable set included in C). Then, letting $p_0, p_1, \ldots, p_{w_1'}$ be a walk as described in (5.30), the following set is stable of size $\frac{N}{k}\frac{k-1}{2}$:

$$A = \bigsqcup_{t=0}^{w_1'-1} A_0 + td_1 + p_t d_2.$$

We now prove the claim (5.30), by distinguishing two cases according to the parity of $\|w'\|_1$.

Suppose that $\|w'\|_1$ is odd. Then $w_1' + |w_2'| = \|w'\|_1 \geq k$. Then $\frac{N}{k} = w_1' \geq k - |w_2'|$.

We can then see that there is a walk as claimed in (5.30), as follows. Since $w_1' - (k - |w_2'|)$ is non-negative even, we can start the walk by alternating $+1$ and -1, setting $p_0 = 0$, $p_1 = 1$, $p_2 = 0$, and so on up to $p_{w_1'-(k-|w_2'|)} = 0$. From here the walk becomes monotonic, adding only $+1$s (resp. -1s) to end at $p_{w_1'} = k - |w_2'| \equiv w_2' \mod k$ if $w_2' \in [-\frac{k-1}{2}, 0)$ (resp. $p_{w_1'} = -(k - |w_2'|) \equiv w_2' \mod k$ if $w_2' \in [0, \frac{k-1}{2}]$).

Suppose now that $\|w'\|_1$ is even. We claim that $w_1' \geq |w_2'|$. Indeed, otherwise the number $w_1' + k - |w_2'|$ is less than k, is odd (since it equals $k + \|w'\|_1 - 2|w_2'|$), and is positive (since $|w_2'| \leq \frac{k-1}{2}$). On the other hand this number is the length of an odd cycle in G. Indeed, since $w' \in \Lambda$, we have $sw_1'd_1 + |w_2'|d_2 = 0 \mod N$ for some $s \in \{1, -1\}$. Hence, since d_2 has order k, we have $-sw_1'd_1 + (k - |w_2'|)d_2 = 0 \mod N$, which indeed implies the existence of a cycle of length $w_1' + k - |w_2'|$. This proves our claim. Since $w_1' \geq |w_2'|$, we have that $w_1' - |w_2'| = \|w'\|_1 - 2|w_2'|$ is non-negative even. We can then construct a walk as claimed in (5.30) as follows. We start again by alternating $+1$ and -1, setting $p_0 = 0$, $p_1 = 1$, $p_2 = 0$, and so on up to $p_{w_1'-|w_2'|} = 0$. From here the path goes monotonically again, to end at $p_{w_1'} = w_2'$ (adding only $+1$s if $w_2' \in [0, \frac{k-1}{2}]$ and only -1s if $w_2' \in [-\frac{k-1}{2}, 0)$). $\square$

5.5 Final remarks

As mentioned in the introduction and explained in Remark 5.14, Motzkin's problem in $\mathbb{T}$ can be seen to subsume (in its rational case $D \subset \mathbb{Q}$) the original problem in $\mathbb{Z}$ for finite D. At the end of the introduction we mentioned a more specific instance of this, namely that the asymptotic solution to the case of two rational missing differences in $\mathbb{T}$ (i.e. Theorem 5.3) implies the classical solution (5.2) of Cantor and Gordon, namely $\mathrm{Md}_{\mathbb{Z}}(\{d_1, d_2\}) = \frac{\lfloor (d_1+d_2)/2 \rfloor}{d_1+d_2}$ for any coprime positive integers d_1, d_2. Let us detail this.

As explained in Remark 5.14, we have $\mathrm{Md}_{\mathbb{Z}_N}(\{d_1, d_2\}) \to \mathrm{Md}_{\mathbb{Z}}(\{d_1, d_2\})$ as $N \to \infty$. It is easily seen that if d_1, d_2 are both odd then, for N even, the circulant graph $G_N := \mathrm{Cay}(\mathbb{Z}_N, \{d_1, d_2\})$ is bipartite, while if d_1, d_2 have different parity then for large N the graph G_N has odd girth $d_1 + d_2$. Hence in all cases Theorem 5.3 indeed yields formula (5.2) in the limit. In particular, in the non-bipartite case, formula (5.2) can be written in terms of the odd girth k of $\mathrm{Cay}(\mathbb{Z}, \{d_1, d_2\})$, namely $\mathrm{Md}_{\mathbb{Z}}(\{d_1, d_2\}) = \frac{k-1}{2k}$.

Formula (5.4) from Theorem 5.2 can also be phrased in terms of the odd girth of an associated graph, namely the uncountable Cayley graph $G = \mathrm{Cay}(\mathbb{T}, \{t_1, t_2\})$. Under the assumptions of Theorem 5.2, it can be seen that if m_1, m_2 have equal parity then G is bipartite (since then every element of the associated lattice Λ is a multiple of (m_1, m_2) by Lemma 5.16 and therefore has even ℓ^1-norm, which implies that every cycle in G is even), and otherwise G has odd girth $k = |m_1| + |m_2|$, so that formula (5.4) can be written $\mathrm{Md}_{\mathbb{T}}(\{t_1, t_2\}) = \frac{k-1}{2k}$.

These connections suggest that there may be a more fundamental result, phrased in terms of the odd girth of a more general type of Cayley graph, which would imply all the above results in the case $|D| = 2$, thus shedding more light on the above connections.

It would be interesting to explore Motzkin's problem further, in at least two directions that would extend the main results of this paper.

One direction consists in considering more general compact or finite abelian groups. In this paper we have focused on the circle group, but some of our main results extend readily to more general compact abelian groups Z. For instance, the conclusion $\mathrm{Md}_{\mathbb{T}}(D) = 1/2$ in Theorem 5.7 is extended to $\mathrm{Md}_{\mathrm{Z}}(D) = 1/2$ by a similar argument using Rokhlin's lemma if the assumption of linear independence over $\mathbb{Q}$ is replaced by the triviality of the lattice Λ from (5.6). Theorem 5.3 can also be formulated more generally, whenever t_1, t_2 generate a finite subgroup Z′ of Z, as the estimate $\frac{k-1}{2k} \geq \mathrm{Md}_{\mathrm{Z}}(\{t_1, t_2\}) \geq \frac{k-1}{2k} - \frac{3}{\sqrt{N}}$ where N is the order of Z′ and k is the odd girth of $\mathrm{Cay}(\mathrm{Z}', \{t_1, t_2\})$ (assuming this graph is not bipartite). Let us mention also that there are previous results in combinatorics which can be viewed as determining Motzkin densities in other complex cases not addressed in this paper. There is for example the main result from the paper [42] by Kleitman, which can be phrased as follows (we are grateful to the anonymous referee for mentioning this).

Theorem 5.30 (Kleitman 1966)**.** *Let $k, n \in \mathbb{N}$ with $2k \leq n$. Let $G = \mathbb{Z}_2^n$, and let*

$$D = \{(x_1, \dots, x_n) \in G : \#\{j : x_j = 1\} > 2k\}.$$

Then $\mathrm{Md}_G(D) = \frac{1}{2^n} \sum_{i=0}^{k} \binom{n}{i}$.

A second natural direction of further research is to address the problem in $\mathbb{T}$ for finite sets D of cardinality at least 3. Here the transference result and tiling arguments in sections 5.2–5.4 may constitute useful elements for a general approach. A particularly appealing question in this direction is whether, and in what form, the asymptotically sharp estimate (5.5) can be refined and extended to circulant graphs with more than 2 jumps.

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