UNIVERSIDAD AUTÓNOMA DE MADRID

DOCTORAL THESIS

# Asymptotic probability techniques in monochromatic waves and fluid mechanics

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*"Logic issues in tautologies, mathematics in identities, philosophy in definitions; all trivial, but all part of the vital work of clarifying and organising our thought"*

Frank P. Ramsey

*"Probability is the most important concept in modern science, especially as nobody has the slightest notion what it means"*

Bertrand Russell

# Abstract

This thesis addresses different questions concerning probability, partial differential equations, fluid mechanics and some aspects of economic theory. We try to answer whether some events in these fields are "typical", how different probability settings modify their likelihood and how some probability techniques can give us information about expected values of important magnitudes or help us to construct deterministic realizations. The thesis is divided into two parts.

In the first part we study monochromatic random waves. First, in Chapter 2 we study monochromatic random waves on the Euclidean space defined by Gaussian variables whose variances tend to zero sufficiently fast. This has the effect that the Fourier transform of the monochromatic wave is an absolutely continuous measure on the sphere with a suitably smooth density, which connects the problem with the scattering regime of monochromatic waves. In this setting, we compute the asymptotic distribution of the nodal components of random monochromatic waves showing that the behavior changes dramatically with respect to the standard theory.

Second, in Chapter 3 we consider Gaussian random monochromatic waves $u$ on the plane depending on a real parameter $s$ that is directly related to the regularity of its Fourier transform. Specifically, the Fourier transform of $u$ is $f\, d\sigma$, where $d\sigma$ is the Hausdorff measure on the unit circle and the density $f$ is a function on the circle that, roughly speaking, has exactly $s - \frac{1}{2}$ derivatives in $L^2$ almost surely. When $s = 0$, one recovers the standard setting for random waves with a translation-invariant covariance-kernel. The main thrust of this chapter is to explore the connection between the regularity parameter $s$ and the asymptotic behavior of the number $N(\nabla u, R)$ of critical points that are contained in the disk of radius $R \gg 1$. A key step of the proof of this result is the obtention of precise asymptotic expansions for certain Neumann series of Bessel functions. When the regularity parameter is $s > 5$, we show that in fact $N(\nabla u, R)$ grows like the diameter with probability 1, albeit the ratio is not a universal constant but a random variable.

Finally, in Chapter 4 we construct deterministic solutions to the Helmholtz equation in $\mathbb{R}^m$ which behave accordingly to the Random Wave Model. We then find the number of their nodal domains, their nodal volume (Yau's conjecture) and the topologies and nesting trees of their nodal set in growing balls around the origin. The proof of the pseudo-random behavior of the functions under consideration hinges on a de-randomization technique pioneered by Bourgain and proceeds via computing their $L^p$-norms. The study of their nodal set relies on its stability properties and on the evaluation of their doubling index, in an average sense.

In the second part of this thesis we study the probability techniques applied to two different fields: fluid mechanics and economic theory. First, in Chapter 5 we show that, with probability 1, a random Beltrami field exhibits chaotic regions that coexist with invariant tori of complicated topologies. The motivation to consider this question, which arises in the study of stationary Euler flows in dimension 3, is V.I. Arnold's 1965 speculation that a typical Beltrami field exhibits the same complexity as the restriction to an energy hypersurface of a generic Hamiltonian system with two degrees of freedom. The proof hinges on the obtention of asymptotic bounds for the number of horseshoes, zeros, and knotted invariant tori and periodic trajectories that a Gaussian random Beltrami field exhibits, which we obtain through a nontrivial extension of the Nazarov–Sodin theory for Gaussian random monochromatic waves and the application of different tools from the theory of dynamical systems, including KAM theory, Melnikov analysis and hyperbolicity. Our

results hold both in the case of Beltrami fields on $\mathbb{R}^3$ and of high-frequency Beltrami fields on the 3-torus.

The second chapter in this part deals with social choice theory, a branch of theoretical economics. The Condorcet Jury Theorem or the Miracle of Aggregation are frequently invoked to ensure the competence of some aggregate decision-making processes. In Chapter 6 we explore an estimation of the prior probability of the thesis predicted by the theorem (if there are enough voters, majority rule is a competent decision procedure). We use tools from measure theory to conclude that, prima facie, it will fail almost surely. To update this prior either more evidence in favor of competence would be needed or a modification of the decision rule. Following the latter, we investigate how to obtain an almost sure competent information aggregation mechanism for almost any evidence on voter competence (including the less favorable ones). To do so, we substitute simple majority rule by weighted majority rule based on some weights correlated with epistemic rationality such that every voter is guaranteed a minimal weight equal to one.

# Resumen y conclusiones

Esta tesis aborda diferentes cuestiones relacionadas con la probabilidad, ecuaciones en derivadas parciales, mecánica de fluidos y algunos aspectos de la teoría económica. Intentamos responder si algunos eventos en estos campos son «típicos», cómo diferentes configuraciones modifican su probabilidad y cómo algunas técnicas probabilísticas pueden darnos información sobre valores esperados de magnitudes importantes o ayudarnos a construir realizaciones deterministas. La tesis está dividida en dos partes.

En la primera parte estudiamos ondas aleatorias monocromáticas. Primero, en el Capítulo 2 estudiamos ondas monocromáticas aleatorias en el espacio euclídeo definido por variables gaussianas cuyas varianzas tienden a cero lo suficientemente rápido. Esto tiene el efecto de que la transformada de Fourier de la onda es una medida absolutamente continua sobre la esfera con una densidad con la suavidad adecuada, lo que conecta el problema con el régimen de dispersión de las ondas monocromáticas. En esta configuración, calculamos la distribución asintótica de las componentes nodales de las ondas monocromáticas aleatorias mostrando que el comportamiento cambia drásticamente con respecto a la teoría estándar.

En segundo lugar, en el Capítulo 3 consideramos ondas monocromáticas aleatorias gaussianas $u$ en el plano dependiendo de un parámetro real $s$ que está directamente relacionado con la regularidad de su transformada de Fourier. En concreto, la transformada de Fourier de $u$ es $f\,d\sigma$, donde $d\sigma$ es la medida de Hausdorff en la circunferencia unidad y la densidad $f$ es una función en la circunferencia que, en términos generales, tiene exactamente $s-\frac{1}{2}$ derivadas en $L^2$ casi seguro. Cuando $s=0$, se recupera la configuración estándar para ondas aleatorias con una función de covarianza invariante ante traslaciones. El objetivo principal de este capítulo es explorar la conexión entre el parámetro de regularidad $s$ y el comportamiento asintótico del número $N(\nabla u, R)$ de puntos críticos que están contenidos en el disco de radio $R \gg 1$. Un paso clave de la demostración de este resultado es la obtención de expansiones asintóticas precisas para ciertas series de Neumann de funciones de Bessel. Cuando el parámetro de regularidad es $s > 5$, mostramos que de hecho $N(\nabla u, R)$ crece como el diámetro con probabilidad 1, aunque la relación no es una constante universal sino una variable aleatoria.

Finalmente, en el Capítulo 4 construimos soluciones deterministas a la ecuación de Helmholtz en $\mathbb{R}^m$ que se comportan de acuerdo con el «Random Wave Model». Luego encontramos el número de sus dominios nodales, su volumen nodal (conjetura de Yau) y las topologías y estructuras de árbol de su conjunto nodal en bolas crecientes alrededor del origen. La demostración del comportamiento pseudoaleatorio de las funciones bajo consideración depende de una técnica de desaleatorización iniciada por Bourgain y se realiza mediante el cálculo de sus normas $L^p$. El estudio de su conjunto nodal se basa en sus propiedades de estabilidad y en la evaluación de su «doubling index», en un sentido promedio.

En la segunda parte de esta tesis estudiamos las técnicas de probabilidad aplicadas a dos campos diferentes: la mecánica de fluidos y la teoría económica. Primero, en el Capítulo 5 mostramos que, con probabilidad 1, un campo de Beltrami aleatorio exhibe regiones caóticas que coexisten con toros invariantes de topologías complicadas. La motivación para considerar esta pregunta, que surge en el estudio de flujos estacionarios de Euler en dimensión 3, es la especulación de Arnold de 1965 de que un campo típico de Beltrami exhibe la misma complejidad que la restricción a una energía hipersuperficie de un sistema hamiltoniano genérico con dos grados de libertad. La prueba depende de la obtención de límites asintóticos para el número de herraduras, ceros y toros invariantes anudados y trayectorias periódicas que un campo de Beltrami aleatorio gaussiano posee. Esto lo obtenemos a través de una extensión no trivial de la teoría de Nazarov–Sodin para ondas monocromáticas aleatorias gaussianas y la aplicación de diferentes herramientas de la teoría de sistemas dinámicos, incluyendo la teoría KAM teoría, análisis de Melnikov e hiperbolicidad. Nuestros resultados son válidos tanto en el caso de los campos de Beltrami en $\mathbb{R}^3$ como en el de campos de Beltrami de alta frecuencia en el 3-toro.

El segundo capítulo de esta parte trata sobre la teoría de la elección social, una rama de la economía teórica. El teorema del jurado de Condorcet o el milagro de la agregación se invocan con frecuencia para garantizar la competencia de algunos procesos de toma de decisiones. En el Capítulo 6 exploramos una estimación de la probabilidad previa de la tesis predicha por el teorema (si hay suficientes votantes, la regla de la mayoría simple es un procedimiento de decisión competente). Usamos herramientas de la teoría de la medida para concluir que, *prima facie*, esto fallará casi con seguridad. Para actualizar esta probabilidad apriori (en el sentido bayesiano) se necesitarían más evidencias a favor de la competencia o una modificación de la regla de decisión. Siguiendo esto último, investigamos cómo obtener un mecanismo de agregación de información (casi seguro) competente para casi cualquier evidencia sobre la competencia de los votantes (incluidas las menos favorables). Para ello, sustituimos la regla de la mayoría simple por la regla de la mayoría ponderada basada en unos pesos correlacionados con la racionalidad epistémica de manera que a cada votante se le garantiza un peso mínimo igual a uno.

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*Esta tesis se la dedico a mis abuelos, que tanto les habría gustado ver el final de lo que ellos contribuyeron a empezar.*

[1]Siendo el exponente $\gg 1$ asumiendo trabajos $> 1$.

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# Chapter 1

# Preliminaries and main results of the thesis

## 1.1 Preliminaries

### 1.1.1 Random monochromatic waves asymptotics

On the euclidean space $\mathbb{R}^n$, we can define monochromatic waves as solutions to the Helmholtz equation on $\mathbb{R}^n$ ($n \geqslant 2$):

$$\Delta u + u = 0 \,. \tag{1.1}$$

The Helmholtz equation is an ubiquitous differential equation in theoretical physics which appears in other partial differential equations (for instance, heat, wave and Schrödinger's equation). It is known that any polynomially bounded solution to this equation is the Fourier transform of a distribution supported on the unit sphere $\mathbb{S}^{n-1}$. More specifically, let us assume that $u$ is a solution to the Helmholtz equation satisfying

$$\int_{\mathbb{R}^n} \langle x \rangle^{-N} u(x)^p dx < \infty, \tag{1.2}$$

where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$ is the Japanese bracket, for some $N > 0$ and $p \in [1, \infty)$. It is standard that then $u$ is a tempered distribution, see [Rud73, Example 7.12]. This is satisfied if, for instance, $u$ is polynomially bounded. Thus, for a test function $\varphi$,

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle = -\langle u, \Delta \widehat{\varphi} \rangle \Rightarrow \langle \hat{u}, (\|x\|^2 - 1)\varphi \rangle = 0.$$

Thus, if $\operatorname{supp} \phi \subset U \subset (\mathbb{S}^{n-1})^c$ where $U$ is an open set, then

$$\langle \hat{u}, \phi \rangle = \langle \hat{u}, (\|x\|^2 - 1)\varphi \rangle = 0$$

for the smooth test function $\varphi := \phi / (1 - \|\cdot\|^2)$. Therefore, $\operatorname{supp} \hat{u} \subset \mathbb{S}^{n-1}$. Similarly for the inverse transform $\check{u}$. By the inversion theorem,

$$u = (\check{u})\hat{} \,,$$

so $u$ is the Fourier transform of a distribution supported on the sphere $\mathbb{S}^{n-1}$.

Thus, the way one constructs monochromatic random waves is the following [CS19]. One starts with a real-valued orthonormal basis of spherical harmonics on $\mathbb{S}^{n-1}$, which we denote by $Y_{lm}$. Hence $Y_{lm}$ is an eigenfunction of the spherical Laplacian with eigenvalue $l(l + n - 2)$, the index $l$ is a nonnegative integer and $m$

ranges from 1 to the multiplicity $d_l := \frac{2l+n-2}{l+n-2}\binom{l+n-2}{l}$ of the corresponding eigenvalue.

To consider a monochromatic random wave, one now takes

$$f(\xi) := \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} i^l \, a_{lm} \, Y_{lm}(\xi) \,, \tag{1.3a}$$

where $a_{lm}$ are independent random variables of zero mean, and defines $u$ as the Fourier transform of $f\, d\sigma$, where $d\sigma$ is the area measure of the unit sphere $\mathbb{S}^{n-1}$, i.e.,

$$u = (f d\sigma)\hat{}\,.$$

This is tantamount to setting (see Proposition 2.3)

$$u(x) = (2\pi)^{\frac{n}{2}} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm} \, Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} \,. \tag{1.3b}$$

Note that $u$ is real-valued if the random variables $a_{lm}$ are. The statistical information of the field $u$ is encapsulated in the covariance kernel $K(x,y) := \mathbb{E}\,(u(x)u(y))$. It is also known that we can approximate any monochromatic wave $u$ (not necessarily satisfying (1.2)) on compact subsets in the $C^t$-topology by truncated sums of the RHS of (1.3b).

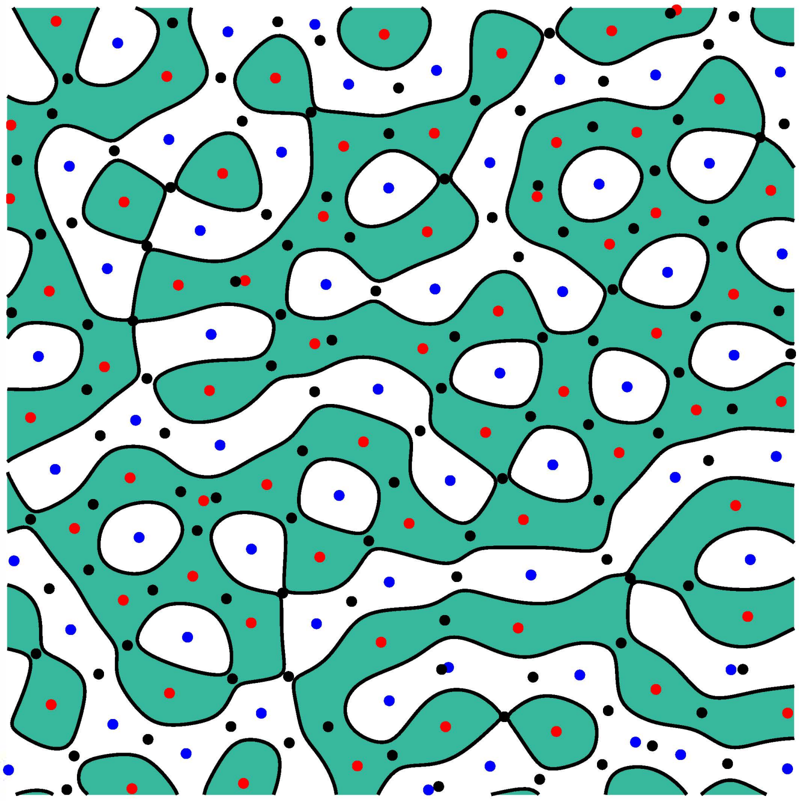


FIGURE 1.1: Nodal set (black) of a random plane wave by D. Belyaev. Nodal domains are in white and green, depending on the sign of the function. Critical points are also shown. Local extrema are painted in red and blue.

The set of points where the function $u$ vanishes is called the *nodal set*. It is like the "skeleton" of the function, it allows us to understand several aspects of the function. Thus, in this thesis we want to investigate some of its characteristics as the number of the nodal domains, the connected components of $\mathbb{R}^n \backslash u^{-1}(0)$, the "volume" of their nodal set and also its topology. The latter is a key factor in many physical properties, e.g., in Newtonian gravitation, Maxwell electromagnetic theory or quantum mechanics [EPS18]. In quantum mechanics, the nodal set of the phase function indicates the region of the space where the particle is less likely to be. See also Figure 1.3.

The breakthrough work of F. Nazarov and M. Sodin help us to understand the nodal set of monochromatic random waves. The Nazarov–Sodin theory, whose original motivation was to understand the nodal set of random spherical harmonics of large order [NS09], has been significantly extended to derive asymptotic laws for the distribution of the zero set of smooth Gaussian functions of several variables. The primary examples are the restriction to large balls of translation-invariant Gaussian functions on $\mathbb{R}^n$ and various Gaussian ensembles of large-degree polynomials on the sphere or on the torus. In this setting, one assumes that the random variables $a_{lm}$ are independent standard Gaussians (i.e., of zero mean and unit variance). Thus, one does have translational invariance, i.e., $K(x,y) = \tilde{K}(x-y)$. Indeed, a straightforward computation [CS19] shows that the covariance kernel reduces to

$$\frac{J_{\frac{n}{2}-1}(|x-y|)}{|x-y|^{\frac{n}{2}-1}}$$

up to a multiplicative constant. We can also see that our random field is isotropic (i.e., invariant under rotations) as $K(x,y) = \tilde{K}(|x-y|)$.

Let us denote by $N_u(R)$ (resp., $N_u(R;[\Sigma])$) the number of connected components of the nodal set $u^{-1}(0)$ that are contained in the ball centered at the origin of radius $R$ (resp., and diffeomorphic to $\Sigma$). Here $\Sigma$ is any smooth, closed, orientable hypersurface $\Sigma \subset \mathbb{R}^n$. It is obvious from the definition that $N_u(R;[\Sigma])$ only depends on the diffeomorphism class $[\Sigma]$ of the hypersurface. The central known results concerning the asymptotic distribution of the nodal components of monochromatic random waves in $\mathbb{R}^n$ can then be summarized as follows (see also [GW16; KW18; CS19] for related results):

**Theorem 1.1.1.** *Suppose that the random variables $a_{lm}$ in (1.3) are independent standard Gaussian variables. Then:*

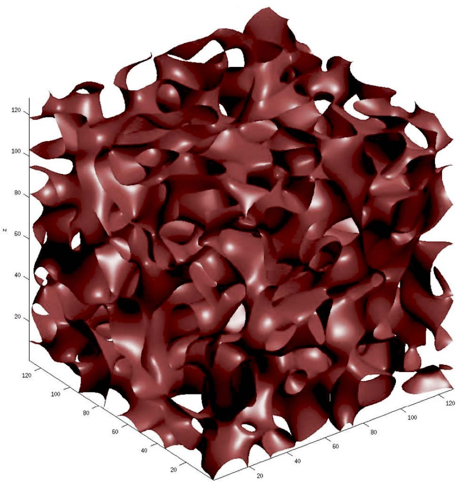
*(i) Nazarov–Sodin's estimate for the number of nodal components [NS16]: there is a constant $\nu > 0$ such that*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R)}{R^n} = \nu \right) = 1\,.$$

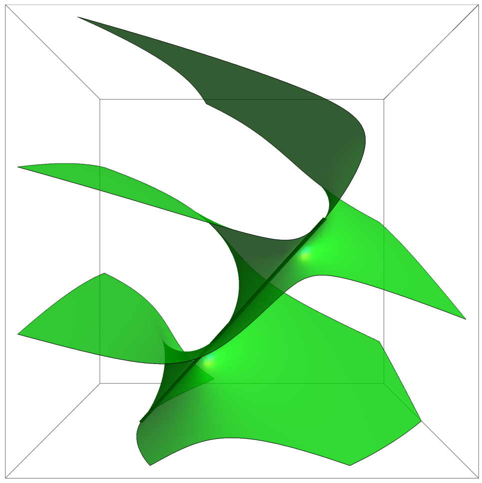
*(ii) Sarnak–Wigman's positive probability bound for the number of nodal sets of fixed topology [SW19]: for each smooth, closed, orientable hypersurface $\Sigma \subset \mathbb{R}^n$ there exists a constant $\nu([\Sigma]) > 0$, depending only on the diffeomorphism class of $\Sigma$, such that*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R;[\Sigma])}{R^n} = \nu([\Sigma]) \right) = 1\,.$$

In other words, this theorem asserts that, if $a_{lm}$ are independent standard Gaussians, the number of nodal components contained in a large ball is almost surely proportional to the volume. This volumetric growth rate holds even if one only considers nodal components of a fixed (compact) topology.



(A) Nodal set for a monochromatic wave in $\mathbb{R}^3$ by A. Barnett



(B) Nodal set for a monochromatic wave with "smooth" density, see Figure 2.1.

FIGURE 1.2: Nodal set for functions from $\mathbb{R}^3$ to $\mathbb{R}$.

In Figure 1.1 it has been represented the nodal set and critical points of a planar random wave. As we can see, the number of critical points gives an upper bound to the number of nodal domains inside a given ball. This is clear as compact nodal domains must contain at least one maximum or minimum. We can also study the nodal set in higher dimensions, see Figure 1.2. The picture on the left could be a nodal set of a "typical" random wave and on the right, the nodal set of a monochromatic wave which is the Fourier transform of a smooth enough density on the sphere. As we can see, the pattern is completely different. The reason, as we will show, lies in the regularity of the density on the sphere. In Lemma 2.4.1 and Proposition 3.2.2 with Remark 3.2.1 we will show the connection between this regularity on the sphere and the decay of the variances. In Appendix 2.A we will show the link between this regularity on the sphere and the decay at infinity of $u$.

### The Random Wave Model

Monochromatic waves can be defined on manifolds too. Given a compact Riemannian manifold $(M, g)$ without boundary of dimension $m$, let $\Delta_g$ be the Laplace-Beltrami operator. There exists an orthonormal basis for $L^2(M, g)$ consisting of eigenfunctions $\{f_{\lambda_i}\}_{i=1}^{\infty}$

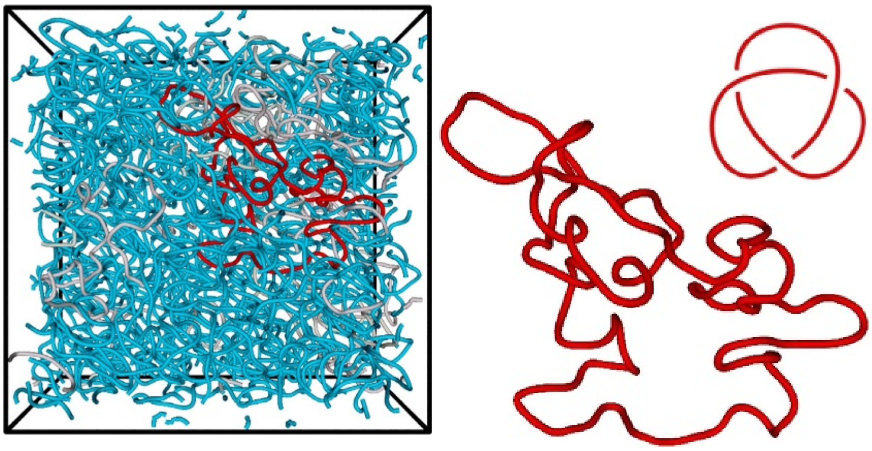
$$\Delta_g f_{\lambda_i} + \lambda_i f_{\lambda_i} = 0, \tag{1.4}$$

with $0 = \lambda_1 < \lambda_2 \leqslant ...$ listed taking into account multiplicity and $\lambda_i \to \infty$. Quantum chaos is concerned with the behaviour of $f_\lambda$ in the *high-energy* limit, i.e. $\lambda \to \infty$.

Berry [Ber77; Ber83] conjectured that "generic" Laplace eigenfunctions on negatively curved manifolds can be modelled[1] in the high-energy limit by monochromatic waves as above, that is, an isotropic Gaussian field $u$ with covariance function

$$\mathbb{E}[u(x)\overline{u(y)}] = \int_{\mathbb{S}^{m-1}} e^{2\pi i\langle x-y,\lambda\rangle} d\sigma(\lambda) = C_m \frac{J_\Lambda(|x-y|)}{|x-y|^\Lambda}, \tag{1.5}$$

where $\sigma$ is the uniform measure on the $m-1$-dimensional sphere, $J_\Lambda(\cdot)$ is the $\Lambda$-th Bessel function with $\Lambda := (m-2)/2$ and $C_m > 0$ is some constant such that $\mathbb{E}[|u(x)|^2] = 1$. This is known as the Random Wave Model (RWM) and it is supported by a large amount of numerical evidence, [HR92].



(A) Vortex lines are shown in a periodic cubic cell.

(B) The trefoil knot.

FIGURE 1.3: Tangled and knotted vortex filaments in random quantum high-energy eigenfunctions, [TD16].

Noticeably, the RWM provides a general framework to heuristically describe the zero set or *nodal set* of Laplace eigenfunctions. In particular, it provides insight into the number of their nodal domains, the connected components of $M\backslash f_\lambda^{-1}(0)$, and the volume of their nodal set and also on its topology. More precisely, let us denote by $\mathcal{N}(f_\lambda)$ the number of connected components of $M\backslash f_\lambda^{-1}(0)$ (nodal domains) and by $\mathcal{V}(f_\lambda) := \mathcal{H}^{m-1}(\{x \in M : f_\lambda(x) = 0\})$ the nodal volume of $f$, where $\mathcal{H}^{m-1}(\cdot)$ is the Hausdorff measure. Then the RWM together with the work of Nazarov and Sodin [NS16], suggests that "typically", under some conditions,

$$\mathcal{N}(f_\lambda) = c_{NS}\lambda^{m/2}(1 + o_{\lambda\to\infty}(1)), \tag{1.6}$$

where $c_{NS}$ is known as the Nazarov and Sodin constant, see Theorem 1.1.1. Similarly, the RWM together with the Kac-Rice formula suggests that "typically", under some conditions,

$$\mathcal{V}(f_\lambda) = c\lambda^{1/2}(1 + o_{\lambda\to\infty}(1)). \tag{1.7}$$

Importantly, (1.7) agrees with Yau's conjecture [Yau82], which predicts $\mathcal{V}(f_\lambda) \asymp \lambda^{1/2}$. The said conjecture is known for real-analytic manifolds thanks to the work of Donnelly-Fefferman [DF88]. In the smooth case, the lower bound was recently

[1]For a precise statement, see [Ing21; ABM18]. The idea is that, for a fixed x, $f_{\lambda_j}(\exp_x(u/\lambda_j^{1/2}))$ should behave like a random monochromatic wave as long as the geodesics on $(M,g)$ are chaotic.

proved by Logunov and Malinnikova [Log18a; Log18b; LM18] together with a polynomial upper bound.

This suggests the following questions:

- As we see in Figure 1.8, the nodal set is very different if one modifies the regularity of the density on the sphere. Can we understand, in a deterministic and probability setting, the nodal set if the density on the sphere is smooth enough? The answer is affirmative and this will be considered in Chapter 2 which is based on [EPSR22a]:
  - A. Enciso, D. Peralta-Salas, and Á . Romaniega. "Asymptotics for the nodal components of non-identically distributed monochromatic random waves". In: *International Mathematics Research Notices* 2022.1 (2022), pp. 773–799.
- As we will see in detail in Chapter 2, the behavior of random monochromatic waves changes dramatically between the Nazarov-Sodin case of Theorem 1.1.1 and the case of a smooth density of the sphere (see Theorem 1.2.1). Can we say something about the intermediate and other regions where the density is not so smooth or even less smooth than the Nazarov-Sodin case? We will treat this question in Chapter 3 for waves on the plane considering expected values. This is based on the article (submitted for publication) [EPSR21]:
  - Enciso, A., Peralta-Salas, D. and Romaniega, Á., 2021. *Critical point asymptotics for Gaussian random waves with densities of any Sobolev regularity.* In: *arXiv preprint arXiv:2107.03363.*

  A different approach for the technical (but crucial) computation of the Neumann series is given in [Rom22a]:
  - Romaniega, Á., 2022. *Integral representations and asymptotic expansions for second type Neumann series of Bessel functions of the first kind.*
- Theorem 1.1.1 says that, almost surely, monochromatic waves will have connected components of the nodal set with every topology and a given constant for the volumetric growth, but, can we find deterministic realizations of this? That is, can we find particular solutions of the Helmholtz equation satisfying these properties? The proof of the theorem is non-constructive and when we have a "good" understanding of the nodal set these monochromatic waves do not satisfy the thesis of the theorem (see Theorem 2.3.1). Hence, the answer is not trivial, but in Chapter 4 we will give an affirmative answer and connect it to Berry's and Yau's conjecture based on the article [RS22]:
  - Romaniega, Á. and Sartori, A., 2020. *Nodal set of monochromatic waves satisfying the Random Wave Model.* In: Journal of Differential Equations 333C (2022), pp. 1-54.

### 1.1.2 Fluid mechanics

Beltrami fields, that is, eigenfunctions of the curl operator satisfying[2]

$$\operatorname{curl} u \equiv \nabla \wedge u = \lambda u \tag{1.8}$$

[2] Now $u$ is a vector field, not a function.

on $\mathbb{R}^3$ or on the flat torus $\mathbb{T}^3$ for some nonzero constant $\lambda$, are a classical family of stationary solutions to the Euler equation in three dimensions. Indeed, from Euler equations for an ideal fluid

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p\,, \\ \qquad\quad \nabla \cdot u = 0\,. \end{cases}$$

If the flow is stationary and we define the Bernoulli function $B := p + \frac{1}{2}\|u\|^2$ and the vorticity $\omega := \operatorname{curl} u$, then

$$\begin{cases} u \wedge \omega = \nabla B\,, \\ \nabla \cdot u = 0\,. \end{cases}$$

because of the vector identity $(u \cdot \nabla)u = \dfrac{1}{2}\nabla \|u\|^2 - u \wedge (\nabla \wedge u)$.

Thus, Beltrami fields are solutions if we assume $B$ is constant. This assumption is motivated by the fact that if $B$ has regular level surfaces, one should expect a "laminar" behavior, see [AK21, Proposition 1.5] and below. As a critical point of $B$ at $x$, $\nabla B(x) = 0$, occurs for a nonvanishing velocity iff $\nabla \wedge u(x) = \xi(x)u(x)$ for some real number $\xi(x)$ depending on $x$, this suggests the introduction of force-free fields where $\nabla \wedge u = \xi u$ for some function $\xi$. By definition, $\xi$ is a first integral of the field $u$. Indeed,

$$\nabla\xi \cdot u = \nabla \cdot (\xi u) = 0\,,$$

where the first inequality comes from the fact that $u$ is incompressible and the second from the definition of $\xi$. Hence, every compact connected component of a regular level surface of $\xi$ is a torus provided $u$ does not have zeros . Therefore, the complex behavior is expected when $\xi = \lambda$, a constant. More specifically,

**Proposition 1.1.2** (Corollary 1.8, [AK21])**.** *If a steady analytic flow has a trajectory that is not contained in any analytic (singular) surface, then the flow is defined by a Beltrami field.*

Nevertheless, the full significance of Beltrami fields in the context of ideal fluids in equilibrium was unveiled by V.I. Arnold in his influential work on stationary Euler flows. Indeed, Arnold's structure theorem [Arn65; Arn66] ensures that, under suitable technical assumptions, a smooth stationary solution to the 3D Euler equation is either integrable or a Beltrami field. In the language of fluid mechanics, an integrable flow is usually called laminar, so complex dynamics (as expected in Lagrangian turbulence) can only appear in a fluid in equilibrium through Beltrami fields.

This connection between Lagrangian turbulence and Beltrami fields is so direct that physicists have even coined the term "Beltramization" to describe the experimentally observed phenomenon that the velocity field and its curl (i.e., the vorticity) tend to align in turbulent regions of low dissipation (see e.g. [FPS01; MRD+06]).

Motivated by Hénon's numerical studies of ABC flows [Hen66], which are the easiest examples of Beltrami fields, Arnold suggested [Arn65; Arn66] that Beltrami fields exhibit the same complexity as the restriction to an energy level of a typical mechanical system with two degrees of freedom. To put it differently, a typical Beltrami field should then exhibit chaotic regions coexisting with a positive measure set of invariant tori of complicated topology. For instance, from [Arn65]:

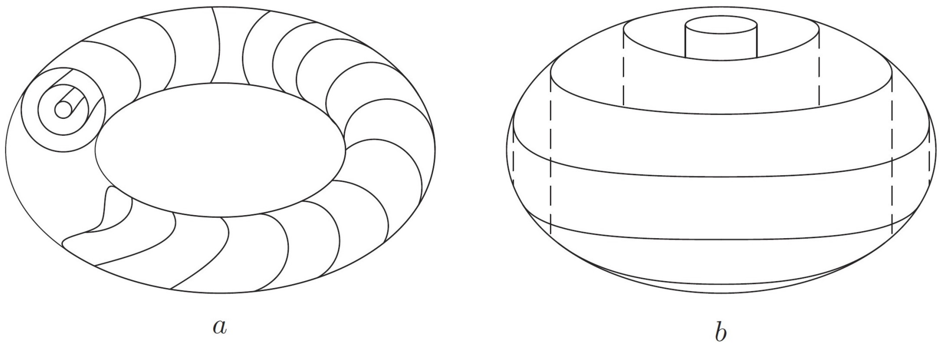



FIGURE 1.4: Arnold's structure theorem: regions fibered by invariant tori, (a) and invariant annuli (b).

> Il est probable que les écoulements tels que rot $v = \lambda v$, $\lambda = Cte$, ont des lignes de courant à la topologie compliquée. De telles complications interviennent en mécanique céleste [Arn09, Fig. 6]. La topologie des lignes de courant des écoulements stationnaires des fluides visqueux peut être semblable à celle de [Arn09, Fig. 6].

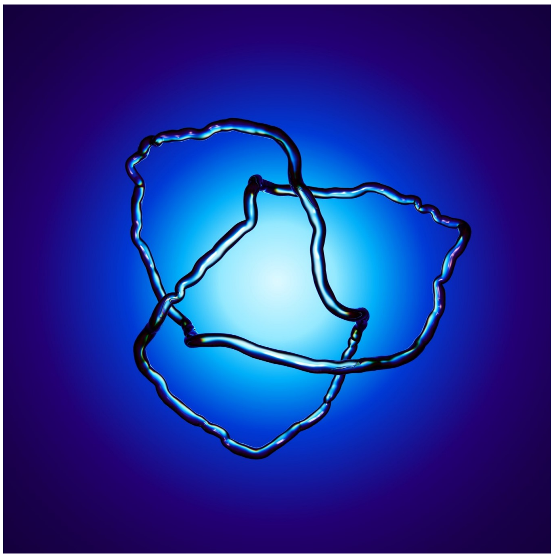
Although specific instances of chaotic ABC flows in the nearly integrable regime have been known for a long time [ZKL+93], Arnold's speculation is wide open. A major step towards the proof of this claim was developed by Alberto Enciso and Daniel Peralta-Salas. They constructed Beltrami fields on $\mathbb{R}^3$ with periodic orbits and invariant tori (possibly with homoclinic intersections [ELPS20] inside) of arbitrary knotted topology [EPS12; EPS15]. In fluid mechanics, these periodic orbits and invariant tori are usually called vortex lines and vortex tubes, respectively, and in fact the existence of vortex lines of any topology had also been conjectured by Arnold in the same papers. More precisely,

**Theorem 1.1.3** ([EPS12; EPS15]). *Let $\mathcal{S} \subset \mathbb{R}^3$ be a finite union of closed curves and tubes pairwise disjoint, but possibly knotted and linked and let $\lambda$ be any nonzero real number. Then one can deform $\mathcal{S}$ by a smooth diffeomorphism $\Phi$ of $\mathbb{R}^3$, arbitrarily close to the identity in any $C^m$ norm, such that $\Phi(\mathcal{S})$ is a set of vortex lines and tubes of a Beltrami field $u$ that satisfies the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$, and moreover, $u$ falls off at infinity as $|u(x)| < C|x|^{-1}$ (sharp decay).*

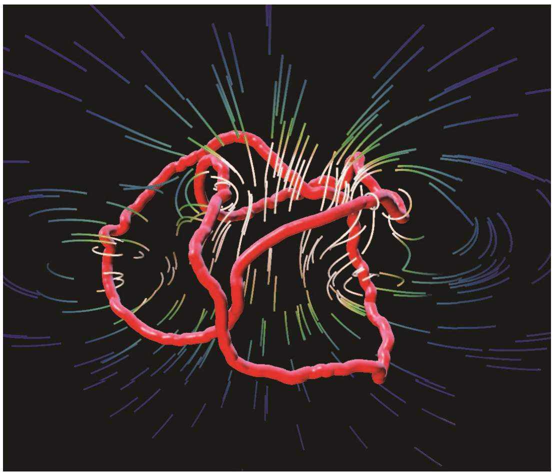
Furthermore, these vortex structures are structurally stable: if $v$, where $\operatorname{div} v \equiv \nabla \cdot v = 0$, is a $C^k$-close enough field for $k \geqslant 5$, then it will also have these set of lines and tubes up to a diffeomorphism.

These results also hold [EPSL17] in the case of Beltrami fields on $\mathbb{T}^3$, which, contrary to what happens in the case of $\mathbb{R}^3$ (by the sharp decay they cannot be square-integrable), have finite energy; this is important for applications because $\mathbb{R}^3$ and $\mathbb{T}^3$ are the two main settings in which mathematical fluid mechanics is studied. The main drawback of the approach they developed to prove these results is that, while they managed to construct structurally stable Beltrami fields exhibiting complex behavior, the method of proof provides no information whatsoever about to what extent complex behavior is typical for Beltrami fields.

This suggests the following questions:



(A) A knotted vortex tube of water obtained in the Irvine Lab at the University of Chicago. Figure courtesy of William Irvine.



(B) Reconstruction of the vortex core and flow field from raw 3D data for the trefoil knot, [KI13; KSI14].

FIGURE 1.5: Experimental realizations of a trefoil knot.

- How typical these vortex structures are? Does chaos coexist with the invariant tori? How "large" are the latter? See Figure 1.6 for a particular illustration.
- Can we prove similar results on the torus $\mathbb{T}^3$ as the frequency goes to infinity?

  The answer to both questions is affirmative, establishing Arnold's view of complexity in Beltrami fields, and it will be considered in Chapter 5 which is based on the article (submitted for publication) [EPSR20]:

  – Enciso, A., Peralta-Salas, D. and Romaniega, Á., 2020. *Beltrami fields exhibit knots and chaos almost surely.* In: *arXiv preprint arXiv:2006.15033*.

Similar techniques can also be used in the context of Hamiltonian mechanics and are explored in the article (submitted for publication) [EPSR22b]:

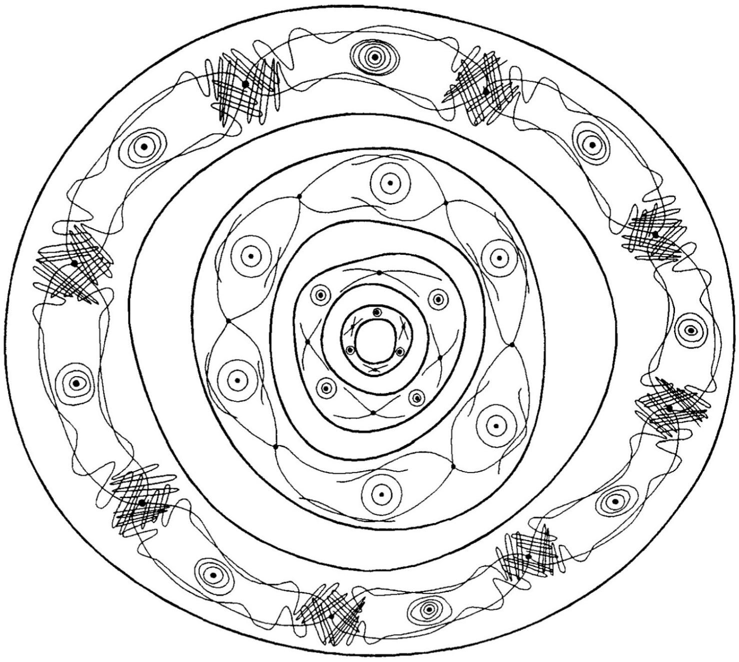


FIGURE 1.6: The Poincaré map of a perturbed Hamiltonian system in two dimensions. Some invariant tori remain undestroyed. The figure also shows stable fixed points separated by unstable fixed points and their “homoclinic tangle” ([JS98] based on Arnold and Avez, 1968).

- Enciso, A., Peralta-Salas, D. and Romaniega, Á., 2022. *Non-integrability and chaos for natural Hamiltonian systems with a random potential.* In: *arXiv preprint arXiv:2204.05964.*

### 1.1.3 Social Choice Theory

Social choice theory is a branch of theoretical economics which studies how individual preferences can be aggregated into social preferences or more directly into social decisions. Usually, this has to be done in a way compatible with the fulfillment of a variety of desirable properties. The most famous result in this area is *Arrow's impossibility theorem* which tells us that, if there are more than two options, there is no social aggregation mechanism satisfying some “natural” or desirable conditions, see [MCWG95, Chapter 21] for details.

Some ideas of this theorem are captured in Condorcet's paradox, noted by the Marquis de Condorcet in the late 18th century. Assume there are three options, $O_1, O_2, O_3$ and three voters $v_1, v_2, v_3$. If the $i$-th voter (strictly) prefers $O_j$ to $O_k$ for $j \neq k$, we write

$$O_j \succ_i O_k.$$

Can simple majority rule give us a social preference relation $\succ_{\text{social}}$? That is, if we define that $O_j$ is socially preferred to $O_k$, i.e., $O_j \succ_{\text{social}} O_k$, if more than half of the voters prefer $O_j$ than $O_k$, does $\succ_{\text{social}}$ is well-behaved? Not necessarily. For instance, if we have:

- $O_1 \succ_1 O_2 \succ_1 O_3$,

- $O_3 \succ_2 O_1 \succ_2 O_2$,
- $O_2 \succ_3 O_3 \succ_3 O_1$,

then one can easily check that $\succ_{\text{social}}$ fails to be transitive: $O_1 \succ_{\text{social}} O_2$, $O_2 \succ_{\text{social}} O_3$ and $O_3 \succ_{\text{social}} O_1$. Thus, in this simple case we do not even have transitivity, i.e., we do not have rational preferences.

**Remark 1.1.1.** "Rational preferences" is a technical term, but in a similar fashion as a Dutch book argument we can see some "irrational" implications of this. If our (strict) preferences were not transitive, as in the example above, and starting with $O_1$ (assuming it is a good now), we would sell $O_1 + \delta_{13}$ to obtain in return $O_3$ where $\delta_{13} > 0$ is some amount of some divisible good (say the numéraire). This is because $O_3$ is strictly preferred to $O_1$. Proceeding similarly with the other two relations we will end up with $O_1$ again but without $\delta_{13} + \delta_{32} + \delta_{21} > 0$. In $N$ steps (with $N$ large enough), we would have acted "locally rational", but this would be clearly seen as "globally irrational" because it would mean losing almost everything. In election processes the implications are paradoxical too. In a two-stage election, the winner will depend on the way the two stages are structured. That is, if we vote first between $O_i$ and $O_j$ and then the final option is chosen between the first winner and the remaining $O_k$ for $i \neq j \neq k \neq i$, then, choosing $i, j, k$ appropriately we can make every option win (if we want $O_l$ to win, simply set $O_k = O_l$). That is, if we want $O_1$ to win, we choose between $O_2$ and $O_3$ first.

But Condorcet also discovered an interesting result in social choice theory. Back in 1785 he published a result to show how voting could be useful to efficiently aggregate the private information of a group of agents. The result holds when we face a dichotomous choice between $A$ and $B$ which has a correct option, say $A$. For instance, the group of agents can be a jury which has to decide if the defendant is guilty in a criminal trial. Each agent is assumed to be more competent than a coin toss and their choices are assumed to be independent from each other. In this setting, Condorcet showed that if votes are aggregated using simple majority rule ($A$ wins if its number of votes is greater than the number of votes of $B$ with an odd number of voters), the probability of choosing the right option increases to one as the number of voters goes to infinity. Thus, *we can efficiently aggregate information*: if the voters are slightly competent, we can produce an (almost) perfectly competent decision procedure, i.e., the probability of being right is as close as one as we want if there are enough voters. This is the Condorcet Jury Theorem (CJT). More precisely, the $i$-th voter is a random variable $X_i$ over $\{0, 1\}$ with probability of choosing $A$ equal to $\mathbb{P}(X_i = 1) = p > 1/2$ and, obviously, the probability of choosing $B$ is $\mathbb{P}(X_i = 0) = 1 - p < 1/2$. Thus, if voters are i.i.d. random variables and the aggregation procedure is simple majority rule, i.e., $A$ is chosen if

$$\sum_{i=1}^{n} X_i > \frac{n}{2},$$

where $n$ is an odd number which represents the number of voters, then

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i > \frac{n}{2}\right) = 1. \tag{1.9}$$

If we consider a similar setting, but now most of the voters are no better than chance, i.e., $p = 1/2$, except for a group of informed voters such that $p = 1$, then (1.9) holds too. The idea is simple, most of the voters behave like noise that cancels out (because $p = 1/2$) and the informed group introduces a "bias" toward the right choice. More precisely, the result follows from the Weak Law of Large Numbers (WLLN). This second case is sometimes known in the literature as Wisdom of Crowds (WoC) or Miracle of Aggregation (MoA).

In general, we can ask whether (1.9) happens for an arbitrary distribution of voter competence, i.e., for a general sequence of probabilities $\{p_i\}_{i=1}^{\infty}$ where $p_i := \mathbb{P}(X_i = 1)$ (now voters are not necessarily identical). This is usually called the *asymptotic CJT* for independent voters. The cases considered above were:

- Condorcet: $p_i = 1/2 + \varepsilon$ where $\varepsilon \in (0, 1/2]$ $\forall i \in \mathbb{N}$.
- MoA: given $n$ voters, $p_i = 1/2$ for $(1-\varepsilon)n$ and $p_j = 1$ for $\varepsilon \cdot n$ voters where $\varepsilon \in (0, 1])$.[3]

We will denote by $\mathcal{C}_I$ the subset of $[0,1]^\infty$ or $[0,1]^\mathbb{N}$, i.e., the space of sequences with elements in $[0,1]$ such that (1.9) holds for independent voters. We will say that a sequence of probabilities satisfies the Condorcet Jury Property (CJP) if (1.9) holds, i.e., the thesis of the CJT holds. This generalizes the case considered by Condorcet of $p_i = 1/2 + \varepsilon$. $\mathcal{C}_I$ is an infinite set, i.e., there are an infinite number of sequences satisfying the CJP. This was proved in [BP98], where a complete characterization of the CJP was given.

### Weighted majority rule

Let $w := (w_i \in \mathbb{R})_{i=1}^{\infty}$ be some weights. Now, to each voter we associate the random variable

$$X_i = \begin{cases} +1 & \text{if it votes A,} \\ -1 & \text{if it votes B.} \end{cases} \tag{1.10}$$

Weighted majority rule implies that the social choice function is $\operatorname{sign}(X_n^w)$ being indifferent between the two if $X_n^w = 0$ where

$$X_n^w := \sum_{i=1}^{n} w_i X_i. \tag{1.11}$$

The larger the weight (*ceteris paribus*), the greater the influence of the voter. The previous case of simple majority rule is recovered if $w_i = w_j \ \forall\, i, j$. Notice that if we assigned $X = 0$ for the wrong option, the weights would be irrelevant in that case, so we have to consider the symmetric case $X \in \{-1, 1\}$.

Weighted majority rules have been widely explored in the literature. For instance, in [NP82] it is shown that, under some assumptions, weighted majority rule is the optimal decision rule for dichotomous choices and that the weights are given by

$$w_i = \mathcal{W}(p_i) := \log\left(\frac{p_i}{1-p_i}\right). \tag{1.12}$$

Obviously, $\mathcal{W} : [0,1] \to \mathbb{R}$ and, in particular, $\lim_{p\to 0} \mathcal{W}(p) = -\infty$ and $\lim_{p\to 1} \mathcal{W}(p) = \infty$. Also, $\mathcal{W}(p) < 0$ for $p < 1/2$. See Figure 1.7 for more details. Some intuitions of

[3] To simplify the exposition we have assumed that $\varepsilon \cdot n$ is an integer, but we should write $\varepsilon_n n := \lfloor \varepsilon \cdot n \rfloor$, i.e., take the integer part.

this result were unveiled by Nobel-prize winner Lloyd Shapely and Bernard Grofman, [SG84]. Considering the non-asymptotic CJT, suppose we have voters with competences (0.9, 0.9, 0.6, 0.6, 0.6). We have several options:

- Under expert rule, $w_i = 0$ only for $i = 2, 3, 4, 5$,

$$\mathbb{P}\left(\sum_{i=1}^{5} w_i X_i > 0\right) = 0.9\,.$$

- Under simple majority rule, $w_i = 1$ for $i = 1, 2, 3, 4, 5$,

$$\mathbb{P}\left(\sum_{i=1}^{5} w_i X_i > 0\right) \approx 0.877\,,$$

  which improves the mean competence, but it is below expert rule.

- Under weighted majority rule, $w_i = 1/3$ for $i = 1, 2$ and $w_i = 1/9$ for $i = 3, 4, 5$,

$$\mathbb{P}\left(\sum_{i=1}^{5} w_i X_i > 0\right) \approx 0.927\,,$$

  which improves the previous results.

This result might be counterintuitive, since we are assigning nonzero weights to the less competent but, nevertheless, improving the total probability with respect to the expert rule case. This result is clearer if we note that these less competent members can break the tie if the two most competent individuals disagree. The use of weights (1.12) was considered an important result by the aforementioned authors. They conclude [SG84]:

> While the results of this essay seem particularly appropriate to analysis of the problem of 'information pooling', in which the task is to weigh the advice of 'experts' or to reconcile 'expert' and 'non-expert' conflicting opinion; we believe Theorem II [this is (1.12)] to be of considerable general importance for democratic theory.

In that sense, we will also consider the CJT for a weighted majority rule and how probable it is now the CJT when weights are included. Nevertheless, we will explore a different kind of weights: they will be strictly positive, bounded from below and above and subject to some stochastic error. They will be of the form $w = w_d + \varepsilon$ where $w_d$ is a deterministic function depending on $p$ and $\varepsilon$ the error. See Figure 1.7 for a comparison between $w_d$ and $\mathcal{W}$. We use $w_d$ instead of $\mathcal{W}$ because, although mathematically optimal, they can be problematic in a real life situation. In particular, as we said, $\mathcal{W}(p) < 0$ for $p < 1/2$. This has the effect of, for $p_i < 1/2$,

$$\mathcal{W}(p_i)X_i = |\mathcal{W}(p_i)|\,(-X_i) =: |\mathcal{W}(p_i)|\tilde{X}_i = \mathcal{W}(1-p_i)\tilde{X}_i$$

and now $\mathbb{P}\left(\tilde{X}_i = 1\right) = \mathbb{P}\left(X_i = -1\right) = 1 - p_i > 1/2$. This is equivalent to reversing the outcome of the vote for these particular voters. To avoid this, we will only consider weights in an interval of the form $[1, W]$ for some $W > 1$, i.e., no voter loses, formally, its weight on the election. In the same manner, we will also assume there

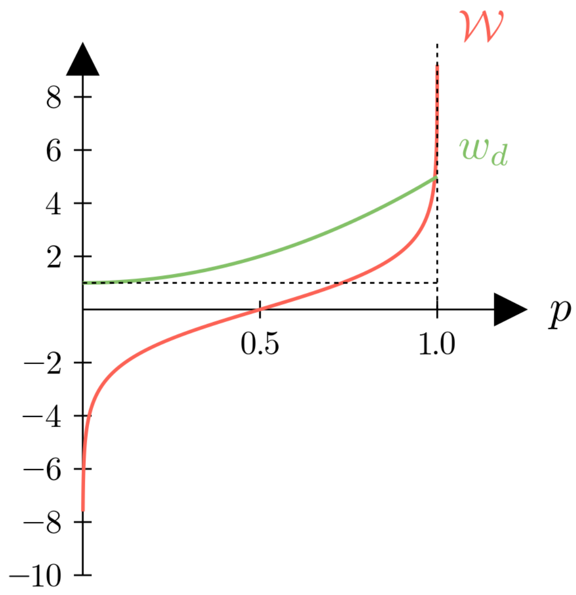



FIGURE 1.7: Comparison of optimal weights and bounded weights.

is an error of measurement, so weights are not perfectly correlated to competence. As we said, real weights, $w$, will be the sum of deterministic weights, $w_d(p)$, plus a random error, $\varepsilon$.

This motivates the following:

- Nevertheless, we can ask how "large" this set $\mathcal{C}_I$ is compared with its complement (that is, the set of sequences where the CJP does not hold). In Chapter 6 we conclude that the answer is, a priori, zero, i.e., the prior probability or measure of the CJP set is zero.

- As the prior probability or measure of this set is zero, can we modify the aggregation mechanism (following the weighted majority rule described above) so that the probability of the thesis of the CJT goes to one? The answer is also affirmative and it is given in Chapter 6, which is based on the article [Rom22b]:

  – Romaniega, Á., 2022. *On the probability of the Condorcet Jury Theorem or the Miracle of Aggregation.* In: *Mathematical Social Sciences*, 10.1016/j.mathsocsci.2022.06.002.

  Some aspects of the latter work are inspired by our previous work in theoretical physics [RRT19]:

  – Rodríguez, M. Á., Romaniega, Á. and Tempesta, P., 2019. *A new class of entropic information measures, formal group theory and information geometry.* In: *Proceedings of the Royal Society A* 475.2222 (2019), p. 20180633.

## 1.2 Main results

We now present the central results of this thesis. To ease the exposition, we analyze each chapter separately.

### 1.2.1 The results of Chapter 2

In this chapter we study monochromatic random waves on $\mathbb{R}^n$ defined by Gaussian variables whose variances tend to zero sufficiently fast. That is, we consider

$$f(\xi) := \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} i^l \, a_{lm} \, Y_{lm}(\xi) \,,$$

where $a_{lm}$ are independent random variables, and we define $u$ as the Fourier transform of $f\,dS$, where $dS$ is the area measure of the unit sphere $\mathbb{S}^{n-1}$. Instead of considering $a_{lm} \sim \mathcal{N}(0,1)$ as in the standard Nazarov-Sodin theory, we would like to consider the case where

$$a_{lm} \sim \mathcal{N}(0, \sigma_l^2)$$

are independent Gaussian variables of zero mean but distinct variances $\sigma_l^2$ and they tend to zero fast enough. This has the effect that the Fourier transform of the monochromatic wave is an absolutely continuous measure on the sphere with a suitably smooth density, which connects the problem with the scattering regime of monochromatic waves. In this setting, we compute the asymptotic distribution of the nodal components of random monochromatic waves, showing that the number of nodal components contained in a large ball $B_R$ grows asymptotically like $R/\pi$ with probability $p_n > 0$, and is bounded uniformly in $R$ with probability $1 - p_n$ (which is positive if and only if $n \geqslant 3$). In the latter case, we show the existence of a unique noncompact nodal component. More precisely, we show in Theorem 1.2.1 that:

**Theorem 1.2.1.** *Suppose that the random variables $a_{lm}$ are independent Gaussians $\mathcal{N}(0, \sigma_l^2)$, where the variances satisfy*

$$\sum_{l=0}^{\infty} (1+l)^{2s+n-2} \sigma_l^2 < \infty \tag{1.13}$$

*for some $s > \frac{n+5}{2}$. Then $f \in H^s(\mathbb{S}^{n-1})$ almost surely, so in particular $|||u||| < \infty$ (see Chapter 2 for its definition). Furthermore:*

*(i) There exists some probability $p_n$, with $p_2 = 1$ and $p_n \in (0,1)$ if $n \geqslant 3$, such that*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R)}{R} = \frac{1}{\pi} \right) = p_n \,,$$

$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R) < \infty \right) = 1 - p_n \,.$$

*(ii) If $\Sigma \subset \mathbb{R}^n$ is a smooth, compact, orientable hypersurface, then*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R;[\Sigma])}{R} = \frac{1}{\pi} \right) = p_n \qquad \text{if } [\Sigma] = [\mathbb{S}^{n-1}] \,,$$

$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R;[\Sigma]) < \infty \right) = 1 - p_n \qquad \text{if } [\Sigma] = [\mathbb{S}^{n-1}] \,,$$

$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R;[\Sigma]) < \infty \right) = 1 \qquad \text{if } [\Sigma] \neq [\mathbb{S}^{n-1}] \,.$$

The basic idea is that, with probability 1, the density $f$ is an $H^s(\mathbb{S}^{n-1})$-smooth function (and, as $s > \frac{n+5}{2}$, of class $C^3$ by the Sobolev embedding theorem) with nondegenerate zeros, and that the probability $p_n$ that $f$ does not vanish is strictly positive. When $f \in H^s(\mathbb{S}^{n-1})$ does not vanish, it is not hard to prove using asymptotic expansions that the number of nodal components contained in a large ball $B_R$ grows as the radius, and that all but a uniformly bounded number of them are diffeomorphic to a sphere. When the zero set of $f$ is regular and nonempty, one can show that the number of nodal components on $\mathbb{R}^n$ is bounded. However, the analysis is considerably subtler because it hinges on the stability of certain noncompact components of the nodal set that locally look like a helicoid. With this understanding of the deterministic case and using some technical probability results, we can arrive at Theorem 1.2.1.

It is natural to wonder which kind of asymptotic laws may arise from more general randomizations of the function $f$. As a first step in this direction, we state next a "stability result", that is, sufficient conditions for the asymptotics of Theorems 1.1.1 and 1.2.1 to hold for more general probability measures on the space of functions $f$ (or $u$).

**Theorem 1.2.2.** *Suppose that there is a nonnegative integer $l_0$ and reals $M_{lm}$ and $\sigma_{lm}$ such that the random variables $a_{lm}$ in (1.3), which we assume to be independent, follow any probability distribution on the line (absolutely continuous with respect to the Lebesgue measure) for $l < l_0$ and Gaussian distributions $\mathcal{N}(M_{lm}, \sigma_{lm}^2)$ for $l \geqslant l_0$. Then:*

*(i) The results of Theorem 1.1.1 hold, with the same constant $\nu$, if*

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{\sigma_{lm}^2 + 1} + \frac{(\sigma_{lm} - 1)^2}{\sigma_{lm}} \right] < \infty\,.$$

*(ii) The results of Theorem 1.2.1 hold if there are constants $\sigma_l$ satisfying (2.2) such that*

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{(\sigma_l^2 + \sigma_{lm}^2)} + \frac{(\sigma_l - \sigma_{lm})^2}{\sigma_l \sigma_{lm}} \right] < \infty\,.$$

These conditions are by no means obvious a priori, but the proof is based on an elementary idea: if two probability measures $\mu$ and $\widetilde{\mu}$ (on the space of functions on the sphere, which one can identify with a space of sequences $\mathbb{R}^{\mathbb{N}}$) are equivalent (i.e., mutually absolutely continuous), then these measures have the same zero-probability events. The aforementioned sufficient conditions are then derived by imposing that one of these measures correspond to the Nazarov–Sodin distribution or to the distributions considered in Theorem 1.2.1 together with a technical lemma due to Kakutani.

### 1.2.2 The results of Chapter 3

As in the previous chapter, here we also consider Gaussian random monochromatic waves $u$ but we restrict our attention to the plane. Furthermore, in this chapter we will focus on critical points. The connection is clear because the number of critical points gives an upper bound for the number of (compact) nodal components, see Figure 1.1 for an illustration.

As we saw in Section 1.1.1, when $u$ is polynomially bounded, the Helmholtz equation simply means that $u$ is the Fourier transform of a distribution supported on the unit circle, which we identify with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ via the map

$$E(\phi) := (\cos\phi, \sin\phi)\,. \tag{1.14}$$

As we know, solutions to the Helmholtz equation are necessarily analytic, but their Fourier transforms do not have any a priori regularity properties. The main thrust of this chapter is to understand the connection between the distribution of the critical points of $u$, defined as in (3.3), and the regularity of the density $f$. To this end, we consider the usual ansatz for random plane waves, (1.3), and tweak it by introducing a real parameter $s \in \mathbb{R}$ to control the regularity of $f$:

$$u(x) := \sum_{l\neq 0} a_l\, |l|^{-s} e^{il\theta}\, J_l(r)\,. \tag{1.15}$$

Here the real and imaginary part of $a_l$ are independent standard Gaussian random variables subject to the constraint $a_l = (-1)^l \overline{a_{-l}}$ (which makes $u$ real valued), $(r,\theta) \in \mathbb{R}^+ \times \mathbb{T}$ are the polar coordinates. This is equivalent to taking the Gaussian random density

$$f(\phi) := \frac{1}{2\pi} \sum_{l\neq 0} i^l a_l |l|^{-s} e^{il\phi} \tag{1.16}$$

and then defining $u$ through the formula (3.3), which must be understood in the sense of distributions. The real parameter $s$ is directly related to the regularity of its Fourier transform. Specifically, the Fourier transform of $u$ is $f\, d\sigma$, where $d\sigma$ is the Hausdorff measure on the unit circle and the density $f$ is a function on the circle that, roughly speaking, has exactly $s - \frac{1}{2}$ derivatives in $L^2$ almost surely. i.e., for any $\delta > 0$,

$$f \in \left[H^{s-\frac{1}{2}-\delta}(\mathbb{T}) \backslash H^{s-\frac{1}{2}}(\mathbb{T})\right] \cap \left[B^{s-\frac{1}{2}}_{2,\infty}(\mathbb{T}) \backslash B^{s-\frac{1}{2}+\delta}_{2,\infty}(\mathbb{T})\right]$$

with probability 1; see Proposition 3.2.2 for details.

Thus, understanding the asymptotics of critical points depending on $s$ helps us to understand what happens in between the Nazarov–Sodin theory, which ensures that the number of nodal components of $u$ contained in $B_R$ grows as

$$N(u,R) \sim \nu_0 R^2$$

almost surely for some constant $\nu_0 > 0$ (Theorem 1.1.1), and the results proven in Theorem 1.2.1

$$N(u,R) \sim \nu_\infty R$$

almost surely for $s > 4$, with $\nu_\infty := 1/\pi$. This is because when $s = 0$, one recovers the classical setting for random waves with a translation-invariant covariance-kernel and if $s$ is large enough, one recovers the setting of the previous chapter.

With that in mind, in this chapter we will prove the following theorem:

**Theorem 1.2.3.** *For any real $s$, the following statements hold:*

(i) *There exist explicit positive constants* $\kappa(s), \tilde{\kappa}_{\frac{3}{2}}, \tilde{\kappa}_{\frac{5}{2}}$ *such that the expected number of critical points of the Gaussian random function* $u$ *satisfies*

$$\mathbb{E}N(\nabla u, R) \sim \begin{cases} \kappa(s)\, R^2 & \text{if} \quad s < \frac{3}{2}, \\ \tilde{\kappa}_{\frac{3}{2}} \frac{R^2}{\sqrt{\log R}} & \text{if} \quad s = \frac{3}{2}, \\ \kappa(s)\, R^{2-(s-\frac{3}{2})} & \text{if} \quad \frac{3}{2} < s < \frac{5}{2}, \\ \tilde{\kappa}_{\frac{5}{2}} R\sqrt{\log R} & \text{if} \quad s = \frac{5}{2}, \\ \kappa(s)\, R & \text{if} \quad s > \frac{5}{2}. \end{cases}$$

(ii) *In the region where the growth of* $\mathbb{E}N(\nabla u, R)$ *is volumetric, the constant* $\kappa(s)$ *depends continuously on* $s$*. More precisely,* $\kappa(s)$ *is a* $C^\infty$ *function of* $s \in (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}]$ *but it is only Lipschitz at* $s = \frac{1}{2}$*. Furthermore, In the region* $s \in (\frac{3}{2}, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$ *the constant* $\kappa(s)$ *is also* $C^\infty$*.*

It can be seen more graphically in the following picture, Figure 1.8.

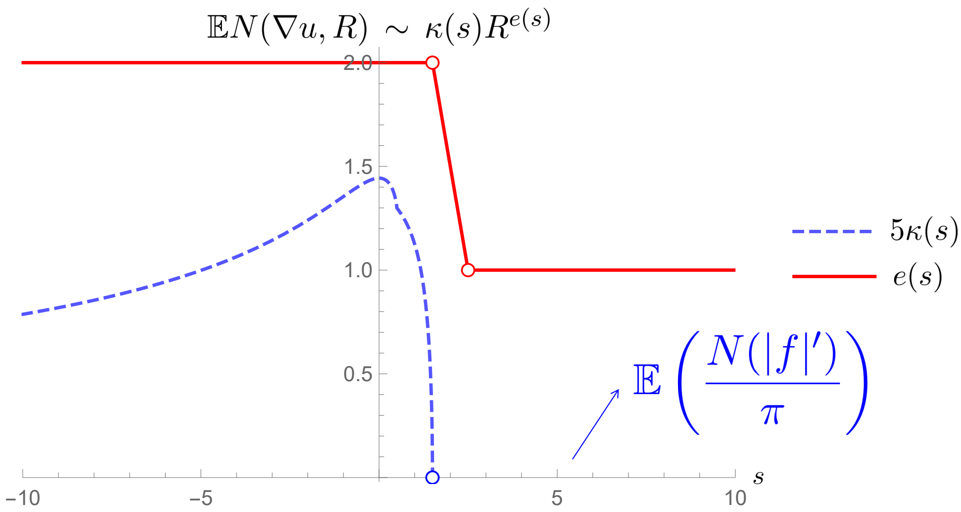



FIGURE 1.8: Consider the asymptotic behavior of $\mathbb{E}N(\nabla u, R) \sim \kappa(s)R^{e(s)}$ proved in Theorem 3.1.1. In red, we have plotted the exponent $e(s)$ as a function of $s \in \mathbb{R}\backslash\{\frac{3}{2}, \frac{5}{2}\}$. Logarithmic effects appear at the endpoints $s = 3/2$ and $s = 5/2$. In blue, we have plotted $\kappa(s)$ in the region where the asymptotic growth is volumetric, $s < \frac{1}{2}$. The maximum of $\kappa(s)$ in this region is attained at $s = 0$ and that $\kappa(s)$ is not continuously differentiable at $s = 1/2$. The reader can find a plot of $\kappa(s)$ in the range $s \in (\frac{3}{2}, \frac{5}{2})$ in Figure 3.3, cf. Section 3.4. Note that $\kappa(s) = \mathbb{E}N(|f|')/\pi$ by Theorem 3.1.3.

That is, we show that the expectation $\mathbb{E}N(\nabla u, R)$ grows like the area of the disk when the regularity is low enough ($s < \frac{3}{2}$) and like the diameter when the regularity is high enough ($s > \frac{5}{2}$), and that the corresponding exponent changes according to a linear interpolation law in the intermediate regime. The transitions occurring at the endpoint cases involve the square root of the logarithm of the radius. Interestingly, the highest asymptotic growth rate occurs only in the classical translation-invariant setting, $s = 0$.

The basic idea of the proof is to use the Kac–Rice formula to compute the expected values. The coefficients that appear in the Kac–Rice integral formula involve, via the variance matrix of $\nabla u$, weighted series of Bessel functions of the form

$$\mathcal{J}_{s,m,m'}(r) := \sum_{l=1}^{\infty} l^{-2s} J_{l+m}(r)\, J_{l+m'}(r)\,, \tag{1.17}$$

where $m$ and $m'$ are certain integers. $\mathcal{J}_{s,m,m'}$ is sometimes called in the literature a second type Neumann series. It is clear that the way each term $J_{l+m}(r)\, J_{l+m'}(r)$ contributes to the sum for $r \gg 1$ and $l \gg 1$ will depend on whether the "angular frequency" $l$ is much larger than $r$, much smaller than $r$, or roughly of the same size; moreover, the effect of each group of angular frequencies will have a different relative weight in the sum depending on the power $s$ appearing in $l^{-2s}$. More precisely, a key step of the proof is to establish the following technical result, which controls the asymptotic behavior of $\mathcal{J}_{s,m,m'}(r)$:

**Lemma 1.2.4.** *For any pair of nonnegative integers $m, m'$ and any real $s$, the large-$r$ asymptotic behavior of $\mathcal{J}_{s,m,m'}$ is*

$$\begin{aligned}
\mathcal{J}_{s,m,m'}(r) &= c^1_{s,m-m'}\, r^{-2s} + o(r^{-2s}) && \text{if} \quad s < \tfrac{1}{2}\,,\\
\mathcal{J}_{s,m,m'}(r) &= c^2_{m-m'} \frac{\log r}{r} + O(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m - m' \text{ is even}\,,\\
\mathcal{J}_{s,m,m'}(r) &= \frac{c^3_{m-m'} - c^4 \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m - m' \text{ is odd}\,,\\
\mathcal{J}_{s,m,m'}(r) &= \frac{c^5_{s,m-m'} - c^6_s \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s > \tfrac{1}{2}
\end{aligned}$$

*with some explicit constants.*

When the regularity parameter is $s > 5$, we show that in fact $N(\nabla u, R)$ grows like the diameter with probability 1, albeit the ratio is not a universal constant but a random variable. We can do this using the methods of Chapter 2, which enables us to understand the asymptotic behavior of the number of critical points (not only of its expectation value) in greater detail. Specifically, one can prove the following:

**Theorem 1.2.5.** *If $s > 5$,*

$$N(\nabla u, R) \sim \frac{N(|f|')}{\pi} R$$

*with probability 1. In particular, $N(\nabla u, R)$ grows linearly almost surely.*

Here the random variable $N(|f|') := \#\{\phi \in \mathbb{T} : |f(\phi)|' = 0\}$ (which is at least 2 almost surely) denotes the number of critical points of the (non-Gaussian) random function $|f|$. In particular, the asymptotic growth of $N(\nabla u, R)$ is linear with probability 1, albeit the ratio is not a universal constant but a random variable. In view of Theorem 1.2.3, a consequence of this asymptotic formula is an explicit formula for the expectation $\mathbb{E}N(|f|')$ when $s > 5$.

### 1.2.3 The results of Chapter 4

Sometimes we want some deterministic realizations of function satisfying, e.g., Theorem 1.1.1. The theorem ensures they exist, but it does not give a way to obtain them. Notice that this does not happen for Theorem 1.2.1, as the proof is based on deterministic results, so we understand the deterministic and the random part.

We are only aware of one instance when the RWM can be deterministically implemented to obtain information about the nodal set: Bourgain [Bou14] showed that certain eigenfunctions on the flat two dimensional torus behave accordingly to the RWM and deduced (1.6). Subsequently, Buckley and Wigman [BW16] extended Bourgain's work to "generic" toral eigenfunctions and Sartori [Sar20] proved a small scales version of (1.6).

In Chapter 4, we construct deterministic solutions to (1.4) on $\mathbb{R}^m$ which satisfy the RWM, in the sense of Bourgain [Bou14], in growing balls around the origin. We then use the RWM to study their nodal set, deduce the analogue of (1.6), (1.7) and also find the asymptotic number of nodal domains belonging to a fixed topological class. More precisely, let $m \geqslant 2$ and $\{r_n\}_{n\geqslant 1} \subset \mathbb{S}^{m-1}$ be a sequence of vectors linearly independent over $\mathbb{Q}$ such that they are not all contained in a hyperplane. We define a class of functions[4]

$$f_N \equiv f := \frac{1}{\sqrt{2N}} \sum_{|n|\leqslant N} a_n e(\langle r_n, \cdot \rangle) \tag{1.18}$$

with domain $\mathbb{R}^m$, $a_n$ are complex numbers such that $|a_n| = 1$, $e(\cdot) := e^{2\pi i \cdot}$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. Moreover, we require $\overline{a}_n = a_{-n}$ so that $f$ is real valued, as $r_{-n} := -r_n$ for $n > 0$. Differentiating:

$$\Delta f = -4\pi^2 f,$$

thus, $f$ is a solution of the Helmholtz equation in $\mathbb{R}^n$. Moreover, the *high-energy* limit of $f$ is equivalent to its behavior in $B(R) = B(R, 0)$, the ball of radius $R$ centered at the origin, as $R \to \infty$. Indeed, rescaling $f$ to $f_R := f(R\cdot)$, then

$$\Delta f_R = -4\pi^2 R^2 f_R.$$

Thus $(2\pi R)^2$ plays precisely the role of $\lambda$ of Section 1.1.1. By compactness, we assume that $\mu_r$ converges to some probability measure $\mu$ as $N \to \infty$.

Then, we prove results of the following form:

**Theorem 1.2.6.** *Let $f$ be as in (1.18), then we have*

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{N}(f_N, R)}{\operatorname{vol} B(R)} - c_{NS}(\mu) \right| = 0, \tag{1.19}$$

*where $c_{NS}(\mu)$ is the Nazarov-Sodin constant of the field $F_\mu$ and $\mu$ as in (4.1.2).*

Note that this kind of double limits gives us the deterministic realizations we are looking for. Indeed, the statement is equivalent to: given some $\varepsilon > 0$, then there exist some $N_0 = N_0(\varepsilon, m)$ such that all $N \geqslant N_0$ the following holds: there exists some $R_0 = R_0(N, \varepsilon, m)$ such that $R > R_0$, we have

$$\left| \frac{\mathcal{N}(f_N, R)}{\operatorname{vol} B(R)} - c_{NS}(\mu) \right| \leqslant \varepsilon, \tag{1.20}$$

[4]We now use $f$ for the function on the Euclidean space, not on the sphere. We have that $f = (f_0 d\sigma)\hat{}$, where $f_0$ is a linear combination of Dirac deltas at different points. This agrees with Remark 3.2.1.

that is, it satisfies the Nazarov-Sodin growth with a constant as close as we want to $c_{NS}(\mu)$. Note that $f$ is the *Fourier transform of a linear combination of Dirac deltas on* $\mathbb{S}^{m-1}$, which agrees with Remark 3.2.1: the distribution on the sphere is less regular that the ones of Chapter 2, as expected.

The question of whether we can take the limit of $N$ first will be analyzed in Section 4.7. In Chapter 4 we give analogous results for nodal sets with a given topology, nesting trees and nodal volume.

These results appear to be new for $m > 2$ (the study of the nodal volume also for $m = 2$) and they present new difficulties such as the existence of long and narrow nodal domains and the possible concentration of the nodal set in small portions of space. We overcome the far from trivial difficulties using precise bounds on the average *doubling index*, an estimate of the growth rate introduced by Donnelly-Fefferman [DF88] (see Section 4.2.3), using recent ideas of Chanillo, Logunov, Malinnikova and Mangoub, [CLM+20]. In particular, our proofs show how integrability properties of the doubling index allow to extrapolate information about the zero set of Laplace eigenfunctions from the RWM. Furthermore, our new approach (based on the weak convergence of probability measures on $C^s$ spaces, Section 4.2.2, and Thom's Isotopy Theorem 4.2.11) gives us an answer to previous questions raised by Wigman and Kulberg, see Section 4.7.2.

### 1.2.4 The results of Chapter 5

Our objective in this chapter is to establish Arnold's view of complexity in Beltrami fields. To do so, the key new tool is a theory of random Beltrami fields, which we develop there in order to estimate the probability that a Beltrami field exhibits certain complex dynamics. In particular, we will show that:

**Theorem 1.2.7.** *With probability* 1*, a Gaussian random Beltrami field on* $\mathbb{R}^3$ *exhibits infinitely many horseshoes coexisting with an infinite volume of ergodic invariant tori of each isotopy type. Moreover, the set of periodic orbits contains all knot types.*

The result we prove (see Theorem 5.6.2) is in fact considerably stronger: we do not only prescribe the topology of the periodic orbits and the invariant tori we count, but also other important dynamical quantities. Specifically, in the case of periodic orbits we have control over the periods (which we can pick in a certain interval $(T_1, T_2)$) and the maximal Lyapunov exponents (which we can also pick in an interval $(\Lambda_1, \Lambda_2)$). In the case of the ergodic invariant tori, we can control the associated arithmetic and nondegeneracy conditions. Details are provided in Section 5.6.

The blueprint for this is the Nazarov–Sodin theory for Gaussian random monochromatic waves. Heuristically, the basic idea is that a Beltrami field satisfying (1.8) can be thought of as a vector-valued monochromatic wave; however, the vector-valued nature of the solutions and the fact that we aim to control much more sophisticated geometric objects introduces essential new difficulties from the very beginning. Without getting technicalities at this stage, let us point out that this is related to analytic difficulties arising from the fact that we are dealing with quantities that are rather geometrically nontrivial. If one considers a simpler quantity such as the number of zeros of a Gaussian random Beltrami field, one can obtain an asymptotic distribution law similar to that of the nodal components of a random monochromatic wave, whose corresponding asymptotic constant can even be computed explicitly:

**Theorem 1.2.8.** *With probability* 1*, the number of zeros of a Gaussian random Beltrami field satisfies*

$$\frac{N_u^z(R)}{|B_R|} \xrightarrow[\text{a.s.}]{L^1} \nu^z$$

*as* $R \to \infty$*. The constant is explicitly given by some integral such that*

$$\nu^z := 0.00872538\ldots, \tag{1.21}$$

On the torus we have similar results. A Beltrami field on the flat 3-torus $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ (or, equivalently, on the cube of $\mathbb{R}^3$ of side length $2\pi$ with periodic boundary conditions) is a vector field on $\mathbb{T}^3$ satisfying the eigenvalue equation

$$\operatorname{curl} v = \lambda v$$

for some real number $\lambda \neq 0$. It is well-known (see e.g. [ELPS17]) that the spectrum of the curl operator on the 3-torus consists of the numbers of the form $\lambda = \pm|k|$ for some vector with integer coefficients $k \in \mathbb{Z}^3$. Restricting our attention to the case of positive eigenvalues for the sake of concreteness, one can therefore label the eigenvalue by a positive integer $L$ such that $\lambda_L = L^{1/2}$. The Beltrami fields corresponding to the eigenvalue $\lambda_L$ must obviously be of the form

$$u^L = \sum_{k \in \mathcal{Z}_L} V_k^L \, e^{ik\cdot x}\,,$$

for some vectors $V_k^L \in \mathbb{C}^3$, where $V_k^L = \overline{V_{-k}^L}$ to ensure that the Beltrami field is real-valued. Starting from this formula, in Section 5.7 we define the Gaussian ensemble of random Beltrami fields $u^L$ of frequency $\lambda_L$, which we parametrize by $L$. The natural length scale of the problem is $L^{1/2}$. We can summarize our results as follows:

**Theorem 1.2.9.** *Let us denote by* $(u^L)$ *the parametric Gaussian ensemble of random Beltrami fields on* $\mathbb{T}^3$*, where* $L$ *ranges over the set of admissible integers. Consider any contractible closed curve* $\gamma$ *and any contractible embedded torus* $\mathcal{T}$ *in* $\mathbb{T}^3$*. Then:*

*(i) With a probability tending to* 1 *as* $L \to \infty$*, the field* $u^L$ *exhibits an arbitrarily large number of approximately distributed horseshoes, zeros, periodic orbits isotopic to* $\gamma$ *and ergodic invariant tori isotopic to* $\mathcal{T}$*.*
*Furthermore, the probability that the topological entropy of the field grows at least as* $L^{1/2}$ *and that there are infinitely many ergodic invariant tori of* $u^L$ *isotopic to* $\mathcal{T}$ *also tends to* 1*.*

*(ii) The expected volume of the ergodic invariant tori of* $u^L$ *isotopic to* $\mathcal{T}$ *is uniformly bounded from below, and the expected number of horseshoes and periodic orbits isotopic to* $\gamma$ *is at least of order* $L^{3/2}$*.*
*In the case of zeros, the asymptotic expectation is explicit, with* $\nu^z$ *given by* (5.1.3)*:*

$$\lim_{L\to\infty} \frac{\mathbb{E}N_{u^L}^z}{L^{3/2}} = (2\pi)^3 \nu^z\,.$$

Note that, again, the asymptotic information that we obtain is perfectly aligned with Arnold's view of complex behavior in typical Beltrami fields. As in the case of $\mathbb{R}^3$, the result we prove in Section 5.7 is actually stronger in the sense that we have control over important dynamical quantities (which now depend strongly on $L$) describing the flow near the above invariant tori and periodic orbits.

### 1.2.5 The results of Chapter 6

In order to "measure" the set $\mathcal{C}_I$, we need to define a measure on that space. Let us start with an example on $\mathbb{R}^2$. In the square $[0,1]^2$ we "measure" a subset $A$ using the measure on $\mathbb{R}^2$ $\mu = \lambda \times \lambda = \prod_{n=1}^{2} \lambda$, that is, $\mu$ is the area. We could use this measure for some probability events as follows. For instance, if we define $X := X_1 + X_2$, $X_i \sim Bernoulli(p_i)$ and $p_i$ are unknown, then $\{\mathbb{E}[X] < 1\}$ has measure $1/2$ w.r.t. $\mu$. Indeed, $\mathbb{E}[X] = p_1 + p_2 < 1$ and by basic geometry

$$\mu\{(x,y) \in [0,1]^2 \ / \ x + y < 1\} = \frac{1}{2}.$$

In the same fashion, we can see that the measure of $\{\mathbb{E}[X] \leqslant 2\}$ is 1 as $p_1, p_2 \leqslant 1$ and, similarly, the measure of $\{\mathbb{E}[X] \geqslant 2\}$ is 0. We then say that the event $\{\mathbb{E}[X] \leqslant 2\}$ happens almost surely or $\mu-$almost surely and $\{\mathbb{E}[X] \geqslant 2\}$ does not happen $\mu-$almost surely ($\mu$-a.s.). In this setting, we can think of $\mu$ as a "*meta-probability measure*", it assigns probabilities (or measures) to some events of the parameters of the probability distributions of some random variables of our interest. Note that if we chose a different $\mu$, the associated measure of each event would probably change. In order to measure the set $\mathcal{C}_I$, we need to define a measure on $[0,1]^{\mathbb{N}}$, the space of sequences with elements in $[0,1]$, as $p_n \in [0,1]$ and the parameters of the problem are $\{p_n\}_{n=1}^{\infty}$. A natural measure to consider is

$$\mu = \prod_{n=1}^{\infty} \lambda. \tag{1.22}$$

It is well-defined by Kolmogorov's Extension Theorem. This measure has the property of being centered in the sense that the mean value (first moment) of $\lambda$ is

$$\int_{[0,1]} x \, d\lambda(x) = \frac{1}{2}. \tag{1.23}$$

However, we are going to consider more general "centered" measures than the one in (1.22), i.e., a larger class. Before the precise definition, we need to introduce the concept of distances and divergences of probability measures, say $d$. These objects tell us, in a sense to be precise in Section 6.1, how different two distinct $\mu$ and $\mu'$ assign measures to an arbitrary set $A$. If $d(\mu, \mu') = 0$, the measures are identical and if $d$ increases, so does the discrepancy for some sets. There are several ways of doing so, but two of the most important examples are the total variation distance (the statistical distance) and the Kullback–Leibler divergence (associated to the Shannon–Boltzmann entropy). In fact, we are going to consider a larger set, that will be denoted by $\mathcal{D}$ and which will be defined precisely in Remark 6.3.2. To ease the exposition here, it can be understood that $d$ below is either the total variation distance or the Kullback-Leibler divergence. We are ready to define the concept of centered measures.

**Definition 1.2.1.** *A probability measure $\mu = \prod_{n=1}^{\infty} \nu_n$ on $[0,1]^{\mathbb{N}}$ will be centered if there exists a probability measure on $[0,1]$, $\nu_0$, such that $\nu_n \ll \nu_0 \ \forall \ n \geqslant 1$ (see Section 6.1 for notation),*

$$\int_{[0,1]} x \, d\nu_0(x) = \frac{1}{2} \tag{1.24}$$

*and*

$$\sum_{n=1}^{\infty} d(\nu_n, \nu_0) < \infty. \tag{1.25}$$

*with* $d \in \mathcal{D}$.

The idea is simple, the measure $\mu$ is not too far (in the sense that the sum of distances or divergences does not go to infinity) from a product measure $\prod_{n=1}^{\infty} \nu_0$ of identical measures on $[0,1]$ and these measures have mean $1/2$. This generalizes (1.22) in two ways. First, the measures of the product are not necessarily identical. We allow the measure to be a "perturbation" of $\mu_0$., Second, the measure $\nu_0$ is not necessarily the Lebesgue measure, but a measure with mean $1/2$, i.e., we only need this measure to have the same first moment as the Lebesgue measure on $[0,1]$. For instance, we can have atomic measures, i.e., $\nu_0(\{x\}) > 0$ for some $x$. This is not allowed in the standard Lebesgue measure, as every single point has measure zero. In particular, we will denote $\epsilon_1 := \nu_0(\{1\})$, that is, there is a probability $\epsilon_1$ such that each voter is going to vote for the correct option almost surely as in the MoA. More generally, we define $\epsilon_{1-\varepsilon_0,1} := \nu_0\left([1-\varepsilon_0, 1]\right)$. In fact, the condition of the MoA is satisfied in the following sense:

**Proposition 1.2.10.** *Let* $\mu$ *a centered measure,* $0 \leqslant \varepsilon_0 < 1/2$, $0 < \varepsilon < \epsilon_{1-\varepsilon_0,1}$ *and* $\delta > 0$ *as small as we want. Then,* $\exists\, N \in \mathbb{N}$ *such that*

$$\mu_0\left(|\{1 \leqslant i \leqslant n \ / \ p_i \in [1-\varepsilon_0, 1]\}| > \varepsilon n\right) > 1 - \delta \ \ \forall\, n \geqslant N.$$

*where* $\mu_0 = \prod_{n=1}^{\infty} \nu_0$ *and*

$$\mu\left(\lim_{n\to\infty} |n^{-1}\{1 \leqslant i \leqslant n \ / \ p_i \in [1-\varepsilon_0, 1]\}| > \varepsilon\right) = 1.$$

*In particular, if* $\varepsilon_0 = 0$ *then the same holds with* $p_i = 1$ *and* $\epsilon_{1-\varepsilon_0,1} = \epsilon_1$

This means that the event that a proportion $\varepsilon > 0$ of voters is well-informed or almost well-informed will be reached if the population $n$ is greater than a (finite) $N$ with probability as close as one as we want. These voters will vote for the correct option with probability greater than $1-\varepsilon_0$ with $\varepsilon_0$ as small as we want or even zero. Despite this fact, the CJP will not hold almost surely. It is important to note that as we have a complete characterization, we are not saying that the hypothesis of the theorem (CJT) will not hold, but *that the thesis* (CJP) will not hold. The latter implies the former but the former implies the latter only if the conditions are necessary too. More precisely:

**Theorem 1.2.11.** *Almost surely independent Condorcet Jury Theorem does not hold for a centered measure* $\mu$*, that is:*

$$\mu(\mathcal{C}_I) = 0. \tag{1.26}$$

**Remark 1.2.2.** Actually, we can prove a stronger result, see Theorem 6.4.1, 6.4.3.

Therefore, Theorem 6.2.2 implies that for any measure $\mu = \prod_{n=1}^{\infty} \nu_n$ where the $\nu_n$ assign probability to both sides $\{p < 1/2\}$ and $\{p > 1/2\}$ "fairly", then $\mu$ is going to assign measure zero to the CJP, i.e., the CJP will not hold almost surely. Hence, if following a Bayesian approach we want to estimate the *prior probability* (the probability before any evidence is collected) of the CJP, we will arrive at the conclusion that the CJP fails almost surely. That is, if we try to measure the applicability of the

CJP according to a symmetrically balanced distribution (in particular, with no bias toward incompetence) without considering any evidence on voters competence, we arrive at the result that the CJT does not hold almost surely. Prior (or a priori in this case) probabilities are the baseline from which probabilities are updated when evidence is collected. So, in this setting, we would need strong evidence of voter competence to expect that the CJT can be applied.

But not everything is lost. We can try to modify the aggregation procedures to achieve a competent mechanism. The natural idea is the consideration of a weighted majority rule, i.e., we define:

$$X_n^w := \sum_{i=1}^{n} w_i X_i,$$

where now $X_i \in \{-1, 1\}$ and $w_i \in \mathbb{R}$ (in principle, they could be negative, but we will not consider that case here). The larger the weight (*ceteris paribus*), the greater the influence of the voter. Weighted majority rule implies that the social choice function is $\operatorname{sign}(X_n^w)$ being indifferent between the two if $X_n^w = 0$. The previous case of simple majority rule is recovered if $w_i = w_j \ \forall\, i, j$. The next step would be to obtain, for some positive integer $k$ and constants $\alpha, \beta > 0$,

$$w = \alpha + \beta p^k + \varepsilon, \tag{1.27}$$

i.e., competence is positively correlated with the weight we assign but the association is not perfect, there is an stochastic error $\varepsilon$. In Theorem 6.5.2 we show that if (1.27) is good enough, the CJT will hold almost surely for "almost" every measure $\mu$, even if they are strongly biased toward $p = 0$, i.e., we are not only considering centered measures but the less favorable case of measures representing voters far from competence. In other words, we are not estimating the prior probability but the probability given almost any evidence on voters competence. This gives some evidence for trying to include epistemic weights in the decision procedure if we are interested in choosing the correct option.

# Part I

# Asymptotics for monochromatic waves

# Chapter 2

# Asymptotics for the nodal components of non-identically distributed monochromatic random waves

## 2.1 Introduction

As we saw in (1.3a), (1.3b) on $\mathbb{R}^n$ monochromatic random waves are defined using a set $\{a_{lm}\}_{l,m}$ of i.i.d. random variables with distribution $\mathcal{N}(0,1)$. Our objective in this chapter is to understand the asymptotic distribution of the nodal set of $u$ when the random variables $a_{lm}$, which we will no longer assume to be identically distributed, have different distribution laws. One obvious motivation to consider this problem is that the Helmholtz equation (1.1) plays a central role in Physics, particularly in quantum mechanics and electromagnetic theory via scattering problems and in stationary solutions of the 3D Euler equation through Beltrami fields [CKK98; EPS15; RS79]. In these contexts (which are clearly different from the study of high energy eigenfunctions on a compact manifold and from problems in percolation theory), one is interested in solutions with the sharp decay at infinity, which is captured by imposing that the Agmon–Hörmander seminorm

$$|||u||| := \limsup_{R\to\infty} \left( \frac{1}{R} \int_{B_R} |u|^2 \, dx \right)^{\frac{1}{2}}$$

is finite. As we recall in Appendix 2.A, the decay properties of $u$ are closely related to the regularity of the function $f$ above; indeed, it is a classical result of Herglotz [Hör15, Theorem 7.1.28] that $|||u||| < \infty$ if and only if $u$ is the Fourier transform of a measure of the form $f\, dS$ with $\|f\|_{L^2(\mathbb{S}^{n-1})} < \infty$.

However, it is easy to see that, when $a_{lm} \sim \mathcal{N}(0,1)$ are standard Gaussians, $f$ is almost surely not in $L^2(\mathbb{S}^{n-1})$ by the law of large numbers. This means that this choice of random variables is very well suited to the study of random eigenfunctions on a compact manifold, as it is known since Nazarov and Sodin's breakthrough paper on spherical harmonics [NS09], but precisely for this reason, it cannot capture the features of random solutions to non-compact problems in the scattering regime (i.e., with finite Agmon–Hörmander seminorm). Hence one would like to consider, at least, the case where

$$a_{lm} \sim \mathcal{N}(0, \sigma_l^2)$$

are independent Gaussian variables of zero mean but distinct variances $\sigma_l^2$. The fall-off (or growth) of the covariance $\sigma_l$ as $l \to \infty$ is directly related to the expected regularity of $f$; indeed, the easiest calculation in this direction is that the expected value of the $H^s(\mathbb{S}^{n-1})$ norm of $f$ is

$$\mathbb{E}(\|f\|^2_{H^s(\mathbb{S}^{n-1})}) = \sum_{l=0}^{\infty} d_l (1+l)^{2s} \sigma_l^2 \,. \tag{2.1}$$

Since $d_l = c_n l^{n-2} + O(l^{n-3})$ for large $l$, a convenient way of stating our main result is as follows:

**Theorem 2.1.1.** *Suppose that the random variables $a_{lm}$ in (1.3) are independent Gaussians $\mathcal{N}(0, \sigma_l^2)$, where the variances satisfy*

$$\sum_{l=0}^{\infty} (1+l)^{2s+n-2} \sigma_l^2 < \infty \tag{2.2}$$

*for some $s > \frac{n+5}{2}$. Then $f \in H^s(\mathbb{S}^{n-1})$ almost surely, so in particular $|||u||| < \infty$. Furthermore:*

*(i) There exists some probability $p_n$, with $p_2 = 1$ and $p_n \in (0,1)$ if $n \geqslant 3$, such that*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R)}{R} = \frac{1}{\pi} \right) = p_n \,,$$
$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R) < \infty \right) = 1 - p_n \,.$$

*(ii) If $\Sigma \subset \mathbb{R}^n$ is a smooth, compact, orientable hypersurface, then*

$$\mathbb{P}\left( \lim_{R\to\infty} \frac{N_u(R;[\Sigma])}{R} = \frac{1}{\pi} \right) = p_n \qquad \text{if } [\Sigma] = [\mathbb{S}^{n-1}] \,,$$
$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R;[\Sigma]) < \infty \right) = 1 - p_n \qquad \text{if } [\Sigma] = [\mathbb{S}^{n-1}] \,,$$
$$\mathbb{P}\left( \lim_{R\to\infty} N_u(R;[\Sigma]) < \infty \right) = 1 \qquad \text{if } [\Sigma] \neq [\mathbb{S}^{n-1}] \,.$$

**Remark 2.1.1.** In plain words, this theorem says that, when the variances satisfy the convergence condition (2.2), the asymptotic distribution is completely different from that of the Nazarov–Sodin regime: the number of nodal components diffeomorphic to a sphere that are contained in a large ball grows like the radius with probability $p_n$ and stays uniformly bounded with probability $1 - p_n$. The number of non-spherical nodal components stays uniformly bounded almost surely. One can also study the nesting graph of the nodal structure, see [SW19; BMW19] for a definition. In the setting of Theorem 2.1.1, with probability $p_n$, the nesting graph is a tree with degree 2 internal vertices, and the number of other trees is bounded almost surely.

**Remark 2.1.2.** Arguing as in Lemma 2.4.2 using (2.12), it is easy to show that the covariance kernel of our random field is

$$\mathbb{E}\left(u(x)u(y)\right) = (2\pi)^n \sum_{l=0}^{\infty} \sigma_l^2 \frac{|d_l|}{|\mathbb{S}^{n-1}|} P_{ln}\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \frac{J_{l+\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} \frac{J_{l+\frac{n}{2}-1}(|y|)}{|y|^{\frac{n}{2}-1}} \,,$$

where $P_{ln}$ is the Legendre polynomial of degree $l$ in $n$ dimensions. Therefore, our random field is isotropic (i.e., invariant under rotations) but not translation-invariant, in general. In the case when $\sigma_l = 1$ for all $l$, one does have translational invariance; indeed, a straightforward computation [CS19] shows that the covariance kernel reduces to

$$\frac{J_{\frac{n}{2}-1}(|x-y|)}{|x-y|^{\frac{n}{2}-1}}$$

up to a multiplicative constant.

To offer some perspective into the ideas behind Theorem 2.1.1, it is convenient to start by recalling the gist of the proof of the first part of Theorem 1.1.1. Nazarov and Sodin start off with a clever (non-probabilistic) "sandwich estimate" of the form

$$\left(1-\frac{r}{R}\right)^n \fint_{B_{R-r}} \frac{N_{\tau_x u}(r)}{r^n}\,dx \leqslant \frac{N_u(R)}{R^n} \leqslant \left(1+\frac{r}{R}\right)^n \fint_{B_{R+r}} \frac{N_{\tau_x u}(r)+\mathfrak{N}_{\tau_x u}(r)}{r^n}\,dx\,,$$

where $\tau_x u(y) := u(x+y)$ is a translation of $u$ and $\mathfrak{N}_u(r)$ denotes the number of critical points of the restriction $u|_{\partial B_r}$. Now one can exploit the fact that, in the particular case when $a_{lm}$ are independent standard Gaussians, $u$ is a Gaussian random function with translation-invariant distribution, which is the setting that the Nazarov–Sodin theory applies to. Moreover, its spectral measure (which is simply $dS$, the normalized area measure on $\mathbb{S}^{n-1}$) has no atoms. Therefore, a theorem of Grenander, Fomin and Maruyama and the Kac–Rice bound respectively imply that the action of shifts on $u$ is ergodic and that the expected value of $\mathfrak{N}_u(r)$ is of order $r^{n-1}$. By taking limits $1 \ll r \ll R$, this readily implies the existence of $\lim_{R\to\infty} N_u(R)/R^n$. The fact that this limit is positive then follows from the sandwich estimate and the existence of a (non-random) solution with a structurally stable compact nodal set. Let us stress that the whole theory hinges on the fact that $a_{lm}$ are Gaussians of the same variance, as this is crucially employed both to connect the problem with the theory of Gaussian random functions and to show that one can compute limits using ergodic theory. The second item in Theorem 1.1.1 uses that, in fact, one can prescribe the topology of a robust nodal component [EPS13].

It should then come as no surprise that the proof of Theorem 2.1.1 is based on entirely different principles. The basic idea is that, with probability 1, in the setting of Theorem 2.1.1 the density $f$ is an $H^s(\mathbb{S}^{n-1})$-smooth function (and, as $s > \frac{n+5}{2}$, of class $C^3$ by the Sobolev embedding theorem) with nondegenerate zeros, and that the probability $p_n$ that $f$ does not vanish is strictly positive. When $f \in H^s(\mathbb{S}^{n-1})$ does not vanish, it is not hard to prove using asymptotic expansions that the number of nodal components contained in a large ball $B_R$ grows as the radius, and that all but a uniformly bounded number of them are diffeomorphic to a sphere. When the zero set of $f$ is regular and nonempty, one can show that the number of nodal components on $\mathbb{R}^n$ is bounded. However, the analysis is considerably subtler because it hinges on the stability of certain noncompact components of the nodal set that locally look like a helicoid. Putting these facts together, one heuristically arrives at Theorem 2.1.1.

It is worth mentioning that the regularity effect that we have striven to capture is completely different from the use of frequency-dependent weights considered by Rivera in the context of random Gaussian fields on compact manifolds [Riv19] (see also [FLL15] for frequency-dependent weights in the context of random algebraic hypersurfaces). Indeed, Rivera's central result is that the Nazarov–Sodin asymptotics still holds, with different constants, for series of the form

$$F(x) := \sum_{k=1}^{L} \lambda_k^{-s} a_k e_k(x),$$

where $L \gg 1$, $(e_k, \lambda_k)$ are the eigenfunctions and eigenvalues of the Laplacian on a compact $n$-manifold, $s \in (0, \frac{n}{2})$ and $a_l \sim \mathcal{N}(0,1)$. In contrast, we are interested in regimes with a different asymptotic behavior that correspond to a scattering situation on $\mathbb{R}^n$.

It is natural to wonder which kind of asymptotic laws may arise from more general randomizations of the function $f$. As a first step in this direction, we state next a "stability result", that is, sufficient conditions for the asymptotics of Theorems 1.1.1 and 2.1.1 to hold for more general probability measures on the space of functions $f$ (or $u$). These conditions are by no means obvious a priori, but the proof is based on an elementary idea: if two probability measures $\mu$ and $\widetilde{\mu}$ (on the space of functions on the sphere, which one can identify with a space of sequences $\mathbb{R}^{\mathbb{N}}$) are equivalent (i.e., mutually absolutely continuous), then these measures have the same zero-probability events. The aforementioned sufficient conditions are then derived by imposing that one of these measures correspond to the Nazarov–Sodin distribution or to the distributions considered in Theorem 2.1.1.

**Theorem 2.1.2.** *Suppose that there is a nonnegative integer $l_0$ and reals $M_{lm}$ and $\sigma_{lm}$ such that the random variables $a_{lm}$ in (1.3), which we assume to be independent, follow any probability distribution on the line (absolutely continuous with respect to the Lebesgue measure) for $l < l_0$ and Gaussian distributions $\mathcal{N}(M_{lm}, \sigma_{lm}^2)$ for $l \geqslant l_0$. Then:*

*(i) The results of Theorem 1.1.1 hold, with the same constant $\nu$, if*

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{\sigma_{lm}^2 + 1} + \frac{(\sigma_{lm} - 1)^2}{\sigma_{lm}} \right] < \infty.$$

*(ii) The results of Theorem 2.1.1 hold if there are constants $\sigma_l$ satisfying (2.2) such that*

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{(\sigma_l^2 + \sigma_{lm}^2)} + \frac{(\sigma_l - \sigma_{lm})^2}{\sigma_l \sigma_{lm}} \right] < \infty.$$

## 2.2 The Fourier transform of $H^s$-smooth densities on the sphere

Our goal in this section is to obtain sharp asymptotic expansions for the Fourier transform

$$u := \widehat{f\, dS}$$

of measures of the form $f\, dS$, where for the time being we can think of the integrable function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ simply as a series of spherical harmonics:

$$f(\xi) := \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} f_{lm} Y_{lm}(\xi).$$

It is well known that, for any real $s$, the $H^s(\mathbb{S}^{n-1})$ norm of $f$ can then be computed as

$$\|f\|^2_{H^s(\mathbb{S}^{n-1})} = \sum_{l=0}^{\infty}\sum_{m=1}^{d_l}(1+l)^{2s}|f_{lm}|^2 .$$

We want $u$ to be real-valued, so we impose that

$$f_{lm} := i^l a_{lm}$$

with $a_{lm} \in \mathbb{R}$. The real and imaginary parts of $f$ are then respectively given by the terms where $l$ is even and odd:

$$f_{\mathrm{R}}(\xi) := \sum_{l \text{ even}}\sum_{m=1}^{d_l}(-1)^{\frac{l}{2}} a_{lm}\, Y_{lm}(\xi) ,$$
$$f_{\mathrm{I}}(\xi) := \sum_{l \text{ odd}}\sum_{m=1}^{d_l}(-1)^{\frac{l-1}{2}} a_{lm}\, Y_{lm}(\xi) .$$

To analyze $u$, we shall start by recalling the explicit formula for the Fourier transform of a spherical harmonic, which we borrow from [CS19]. For the benefit of the reader, we include a short proof that only employs classical formulas for special functions, instead of the theory of point pair invariants and zonal spherical functions. For the ease of notation, here and in what follows we set

$$\Lambda := \frac{n}{2} - 1 .$$

Also, throughout we will often denote the radial and angular parts of $x$ by

$$r := |x| \in \mathbb{R}^+ , \qquad \theta := \frac{x}{|x|} \in \mathbb{S}^{n-1} .$$

**Proposition 2.2.1.** *The Fourier transform of the measure $Y_{lm}\, dS$ is*

$$\widehat{Y_{lm}\, dS}(x) = (2\pi)^{\frac{n}{2}}\, (-i)^l\, Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\Lambda}(|x|)}{|x|^{\Lambda}} , \tag{2.3}$$

*where $J_\alpha$ is the Bessel function of the first kind.*

*Proof.* By the Funk–Hecke formula [AH12, Theorem 2.22], we have

$$\widehat{Y_{lm}\, dS}(x) = c_l(r) Y_{lm}(\theta) , \tag{2.4}$$

where

$$c_l(r) = |\mathbb{S}^{n-2}| \int_{-1}^{1} e^{-itr}\, P_{ln}(t)(1-t^2)^{\frac{n-3}{2}}\, dt . \tag{2.5}$$

Here $P_{ln}$ is the Legendre polynomial. In turn, this last integral can be calculated using the formula [AH12, Proposition 2.26]:

$$\int_{-1}^{1} e^{-itr}\, P_{ln}(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt = \frac{(-ir)^l\, \Gamma(\frac{n-1}{2})}{2^l \Gamma(l+\frac{n-1}{2})} \int_{-1}^{1} e^{-itr} (1-t^2)^{l+\Lambda-\frac{1}{2}}\, dt .$$

The proposition then follows in view of the well-known integral representation of the Bessel function,

$$J_\alpha(z) = \frac{(\frac{z}{2})^\alpha}{\pi^{\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-1}^{1} e^{-itz}\,(1-t^2)^{\alpha-\frac{1}{2}}\,dt, \tag{2.6}$$

□

While the obtention of an asymptotic expansion for the Fourier transform of the measure $f\,dS$ hinges on the analysis of oscillatory integrals, it is convenient to employ the structure of the problem to obtain sharper results. This will be done by exploiting the expansion in spherical harmonics and then using asymptotics with uniform constants directly for Bessel functions. It is worth pointing out that, by blindly following the general approach to asymptotic expansions (e.g., [Hör15, Theorem 7.7.14]), one would need $f \in H^s(\mathbb{S}^{n-1})$ with $s > \frac{3n+1}{2}$ (without considering derivatives), while the approach we take here will lower this number to $s > \frac{n+5}{2}$.

Let us denote by

$$\partial_r u := \frac{x}{|x|}\cdot\nabla u\,, \qquad \not{\nabla} u := \nabla u - \frac{x\cdot\nabla u}{|x|^2}x$$

the radial and angular parts of the gradient. The covariant derivative on the unit sphere will be denoted by $\nabla_{\mathbb{S}}$.

**Proposition 2.2.2.** *If $f \in H^s(\mathbb{S}^{n-1})$ with $s > \frac{n+5}{2}$, then*

$$\begin{aligned}
u &= \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n-1}{2}}}\left[f_{\mathrm{R}}(\theta)\,\cos(r-r_0) + f_{\mathrm{I}}(\theta)\,\sin(r-r_0) + \mathcal{E}_1(r)\right],\\
\partial_r u &= \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n-1}{2}}}\left[-f_{\mathrm{R}}(\theta)\,\sin(r-r_0) + f_{\mathrm{I}}(\theta)\,\cos(r-r_0) + \mathcal{E}_2(r)\right],\\
\not{\nabla} u &= \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n+1}{2}}}\left[\nabla_{\mathbb{S}} f_{\mathrm{R}}(\theta)\,\cos(r-r_0) + \nabla_{\mathbb{S}} f_{\mathrm{I}}(\theta)\,\sin(r-r_0) + \mathcal{E}_3(r)\right],
\end{aligned}$$

*where $r_0 := \frac{\pi}{4}(n-1)$ and the errors are bounded as*

$$|\mathcal{E}_1(x)| + |\nabla\mathcal{E}_1(x)| + |\mathcal{E}_2(x)| + |\mathcal{E}_3(x)| \leqslant \frac{C\|f\|_{H^s(\mathbb{S}^{n-1})}}{r}\,.$$

*Proof.* By Proposition 2.2.1, $u$ is given by the series

$$\begin{aligned}
u(x) &= \sum_{l=0}^{\infty}\sum_{m=1}^{d_l} i^l a_{lm}\widehat{Y_{lm}\,dS}(x)\\
&= \frac{(2\pi)^{\frac{n}{2}}}{r^\Lambda}\sum_{l=0}^{\infty}\sum_{m=1}^{d_l} a_{lm}Y_{lm}(\theta)\,J_{l+\Lambda}(r)\,.
\end{aligned}$$

Let us now recall the following uniform bound for a Bessel function [Kra14, Theorem 4], valid for all $\alpha \geqslant 0$ and $z \geqslant 0$:

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + \left|\alpha^2 - \frac{1}{4}\right|\theta_\alpha(z)\,z^{-3/2}, \tag{2.7}$$

where $|\theta_\alpha(z)| \leqslant 1$. Setting

$$u_1(x) := \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n-1}{2}}} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm} Y_{lm}(\theta) \cos\left(r - r_0 - \frac{\pi l}{2}\right)$$
$$= \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n-1}{2}}} \left[f_{\mathrm{R}}(\theta) \cos(r-r_0) + f_{\mathrm{I}}(\theta) \sin(r-r_0)\right],$$

it then follows that

$$\mathcal{E}_1(x) := \frac{r^{\frac{n-1}{2}}}{2(2\pi)^{\frac{n-1}{2}}} \left[u(x) - u_1(x)\right]$$

can be estimated as

$$|\mathcal{E}_1(x)| \leqslant \frac{C}{r} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} (l+1)^2 |a_{lm}| |Y_{lm}(\theta)|.$$

Using the Cauchy-Schwarz inequality, we then infer

$$|\mathcal{E}_1(x)| \leqslant \frac{C}{r} \sum_{l=0}^{\infty} (1+l)^2 \left(\sum_{m=1}^{d_l} |Y_{lm}|^2\right)^{1/2} \left(\sum_{m=1}^{d_l} |a_{lm}|^2\right)^{1/2}.$$

The addition theorem [AH12, Theorem 2.9] ensures that, at any point on the sphere,

$$\sum_{m=1}^{d_l} |Y_{lm}|^2 = c_{ln} \tag{2.8}$$

with an explicit constant bounded as $c_{nl} \leqslant C(l+1)^{n-2}$. This then allows us to write

$$|\mathcal{E}_1(x)| \leqslant \frac{C}{r} \sum_{l=0}^{\infty} (1+l)^{\frac{n}{2}+1} \left(\sum_{m=1}^{d_l} |a_{lm}|^2\right)^{1/2}.$$

Applying Cauchy–Schwarz again we obtain

$$|\mathcal{E}_1(x)| \leqslant \frac{C}{r} \left(\sum_{l=0}^{\infty} (1+l)^{n-2s+2}\right)^{1/2} \left(\sum_{l=0}^{\infty} \sum_{m=1}^{d_l} (1+l)^{2s} |a_{lm}|^2\right)^{1/2} \leqslant \frac{C}{r} \|f\|_{H^s(\mathbb{S}^{n-1})},$$

as claimed.

Let us now compute the radial derivative of $u$. We start by noting that

$$\partial_r u = (2\pi)^{\frac{n}{2}} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm} \partial_r \left[Y_{lm}(\theta) \frac{J_{l+\Lambda}(r)}{r^\Lambda}\right].$$

Since

$$\partial_r \left(\frac{J_{l+\Lambda}(r)}{r^\Lambda}\right) = \frac{J_{l+\Lambda-1}(r)}{r^\Lambda} - (l+2\Lambda) \frac{J_{l+\Lambda}(r)}{r^{\Lambda+1}} \tag{2.9}$$

and this formula depends solely on Bessel functions, one can now use again the uniform estimate (2.7) to derive, with the same reasoning as above, that the error

$$\mathcal{E}_2(x) := \frac{r^{\frac{n-1}{2}}}{2(2\pi)^{\frac{n-1}{2}}} \left[\partial_r u(x) - \partial_r u_1(x)\right]$$

is bounded as

$$|\mathcal{E}_2(x)| \leqslant \frac{C}{r} \|f\|_{H^s(\mathbb{S}^{n-1})} \,.$$

The bound for the angular part of the gradient can be estimated using the same argument and the formula

$$\nabla\!\!\!\!/\, u = \frac{(2\pi)^{\frac{n}{2}}}{r^{\Lambda+1}} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm} \nabla_{\mathbb{S}} Y_{lm}(\theta)\, J_{l+\Lambda}(r)\,,$$

the only difference being that instead of the addition formula (2.8) one has to use that

$$\sum_{m=1}^{d_l} |\nabla_{\mathbb{S}} Y_{lm}|^2 = l(l+n-2)\, c_{nl} \,.$$

To prove this, it is enough to note that, by (2.8),

$$0 = \Delta_{\mathbb{S}} c_{ln} = \Delta_{\mathbb{S}} \sum_{m=1}^{d_l} Y_{lm}^2 = 2 \sum_{m=1}^{d_l} Y_{lm}\, \Delta_{\mathbb{S}} Y_{lm} + 2 \sum_{m=1}^{d_l} |\nabla_{\mathbb{S}} Y_{lm}|^2$$

and use the eigenvalue equation $\Delta_{\mathbb{S}} Y_{lm} = -l(l+n-2) Y_{lm}$. Using now that

$$\nabla \mathcal{E}_1 = \mathcal{E}_2 \, \frac{x}{r} + \mathcal{E}_3 \,,$$

the estimate for $\nabla \mathcal{E}_1$ follows from the previous bounds. The proposition is then proved. □

## 2.3 Nodal sets of non-random monochromatic waves

We recall that the nodal set of a function $F : M \to \mathbb{R}^m$, where $M$ is a manifold, is *regular* if the derivative $(DF)_x : T_x M \to \mathbb{R}^m$ has maximal rank for all $x \in F^{-1}(0)$. We say that a codimension one compact submanifold $\mathcal{S}$ of $\mathbb{R}^n$ is a *graph over the sphere* of radius $R$ centered at the origin if it can be written in spherical coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1}$ as

$$\mathcal{S} = \big\{ (r, \theta) : r = R + G(\theta)\,,\ \theta \in \mathbb{S}^{n-1} \big\}$$

for some smooth function $G : \mathbb{S}^{n-1} \to (-R, \infty)$. In particular, $\mathcal{S}$ is diffeomorphic to $\mathbb{S}^{n-1}$.

**Theorem 2.3.1.** *Let $f \in H^s(\mathbb{S}^{n-1})$ with $s > \frac{n+5}{2}$ and denote by $u$ the Fourier transform of $f\, dS$. Then:*

*(i) Suppose that $f$ does not vanish on $\mathbb{S}^{n-1}$. Then the nodal set $u^{-1}(0)$ has countably many connected components*

$$u^{-1}(0) = \bigcup_{k=1}^{\infty} \mathcal{S}_k \,,$$

*and for all large enough $k$, $\mathcal{S}_k$ is a graph over a sphere centered at the origin and is contained in the annulus $k\pi - c < |x| < k\pi + c$ for some constant $c$ depending on $f$.*

*Furthermore,*

$$\lim_{R\to\infty} \frac{\#\{k : \mathcal{S}_k \subset B_R\}}{R} = \frac{1}{\pi}\,.$$

*(ii) Suppose that the zero set $f^{-1}(0)$ is a nonempty regular subset of the sphere. Then there is a large enough $R$ such that $u^{-1}(0)\backslash B_R$ is connected.*

*Proof.* By Proposition 2.2.2, $u$ can then be written as

$$u = \frac{2(2\pi)^{\frac{n-1}{2}}}{r^{\frac{n-1}{2}}}(U + \mathcal{E}_1)\,, \tag{2.10}$$

where

$$U := f_{\mathrm{R}}(\theta)\, \cos(r - r_0) + f_{\mathrm{I}}(\theta)\, \sin(r - r_0)\,,$$

and we have the bound

$$|\mathcal{E}_1| + |\nabla \mathcal{E}_1| < C/r\,. \tag{2.11}$$

It is clear that the zero sets of $u$ and of $U + \mathcal{E}_1$ coincide, so we shall next study the latter.

Let us begin with the first case. Since $f$ does not vanish, its modulus and phase functions, defined as

$$f(\theta) =: |f(\theta)|\, e^{i\Theta(\theta)}\,,$$

are of class $H^s(\mathbb{S}^{n-1})$, and $U$ can be equivalently written as

$$U = |f(\theta)|\, \cos[r - r_0 - \Theta(\theta)]\,.$$

As $\min_{\theta\in\mathbb{S}^{n-1}} |f(\theta)| > 0$, the zero set of $U$ is given, in polar coordinates and for certain $k_0 \in \mathbb{Z}$, by

$$U^{-1}(0) = \bigcup_{k\geqslant k_0} \mathcal{U}_k\,,$$

where

$$\mathcal{U}_k := \big\{(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1} : r = \Theta(\theta) + (k + \tfrac{n+1}{4})\pi\big\}\,.$$

Obviously

$$\lim_{R\to\infty} \frac{\#\{k : \mathcal{U}_k \subset B_R\}}{R} = \frac{1}{\pi}\,.$$

For large $k$, the component $\mathcal{U}_k$ of the zero set is nondegenerate because

$$\min_{x\in\mathcal{U}_k} |\partial_r U(x)| = \min_{\theta\in\mathbb{S}^{n-1}} |f(\theta)| > 0\,.$$

In view of the bound (2.11), Thom's isotopy theorem (see e.g. [EPS13, Theorem 3.1]) then ensures that, outside a certain large compact set $K$ containing the origin, the nodal set $u^{-1}(0)$ can be written as

$$u^{-1}(0)\backslash K = \bigcup_{k\geqslant k_0} \mathcal{S}_k\,,$$

where each connected component $\mathcal{S}_k$ is of the form

$$\mathcal{S}_k = \Phi_k(\mathcal{U}_k)\,,$$

where $\Phi_k$ is a smooth diffeomorphism of $\mathbb{R}^n$ with $\|\Phi_k - \mathrm{id}\|_{C^0(\mathbb{R}^n)} < C/k$. As the number of nodal components of $u$ contained in $K$ is finite because the function $u$ is analytic, the first statement follows.

Let us now pass to the second statement. We can use again the decomposition (2.10) and study the zero set of $U$ in this case. Since

$$U^{-1}(0) = \{(r,\theta) : f_{\mathrm{R}}(\theta)\, \cos(r-r_0) = -f_{\mathrm{I}}(\theta)\, \sin(r-r_0)\}\,,$$

one has, on $U^{-1}(0)$,

$$(\partial_r U)^2 = [-f_{\mathrm{R}}(\theta)\, \sin(r-r_0) + f_{\mathrm{I}}(\theta)\, \cos(r-r_0)]^2 = f_{\mathrm{I}}(\theta)^2 + f_{\mathrm{R}}(\theta)^2\,,$$

so $\nabla U|_{U^{-1}(0)}$ can vanish at most at the points $(r,\theta) \in U^{-1}(0)$ such that $f(\theta) = 0$.

To show that $\nabla U$ is nonzero also at those points, it is enough to notice that

$$\not\nabla U = \frac{\nabla_{\mathbb{S}} f_{\mathrm{R}}(\theta)\, \cos(r-r_0) + \nabla_{\mathbb{S}} f_{\mathrm{I}}(\theta)\, \sin(r-r_0)}{r}\,,$$

so $\not\nabla U \neq 0$ at any point $(r,\theta)$ with $f(\theta) = 0$ because the set $f^{-1}(0)$ is regular (so the vectors $\nabla_{\mathbb{S}} f_{\mathrm{R}}(\theta)$ and $\nabla_{\mathbb{S}} f_{\mathrm{I}}(\theta)$ are linearly independent). Therefore, one concludes that the zero set of $U$ is regular, and in fact

$$|\nabla U|_{U^{-1}(-\epsilon,\epsilon)} \geqslant c > 0$$

for some small $\epsilon > 0$ because the function $U$ is periodic on $r$. One can now use an analog of Thom's isotopy theorem for noncompact sets [EPS13, Theorem 3.1] with the bound (2.11) to obtain that, for any large enough $R$, there exists a diffeomorphism $\Phi_R$ of $\mathbb{R}^n$, with

$$\|\Phi_R - \mathrm{id}\|_{C^0(\mathbb{R}^n)} < C/R\,,$$

such that

$$u^{-1}(0)\backslash B_R = \Phi_R[U^{-1}(0)\backslash B_R]\,.$$

Therefore, it only remains to analyze what $U^{-1}(0)$ looks like, outside a large ball. We claim that $U^{-1}(0)\backslash B_R$ is a connected set. Indeed, when the point $(r,\theta) \in U^{-1}(0)$ is such that $f(\theta) \neq 0$, it follows from the proof of the first assertion of the statement that $U^{-1}(0)$ is locally a graph over a sphere centered at the origin. When $f(\theta) = 0$, $U^{-1}(0)$ is locally a sort of helicoid. To see this, we can take advantage of the fact that $f^{-1}(0)$ is a regular set to introduce local coordinates $(y_1,\dots,y_{n-1})$ in a neighborhood of the point $\theta$ in $\mathbb{S}^{n-1}$ such that $y_1 := f_{\mathrm{R}}$ and $y_2 := f_{\mathrm{I}}$. Hence, defining functions $\rho(y_1,y_2)$ and $\phi(y_1,y_2)$ as

$$y_1 + iy_2 =: \rho\, e^{i\phi}\,,$$

we readily obtain that one can write

$$U = \rho \cos(r - r_0 - \phi)$$

locally with respect to the coordinates $(y_1,\dots,y_{n-1})$ and for all large enough $r$. In this conical sector, the zero set of $U$ then consists of the codimension 2 conical set $\rho = 0$ and the helicoidal hypersurface

$$r = r_0 + \phi + \frac{\pi}{2}\,.$$

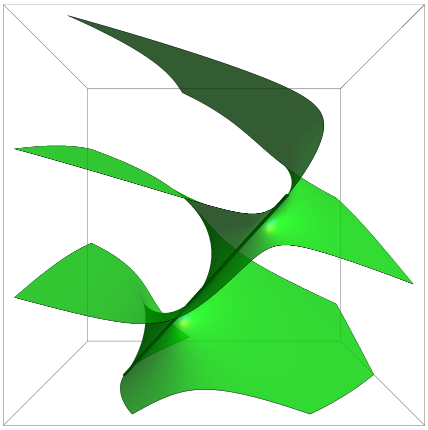


FIGURE 2.1: Local structure of the zero set $u^{-1}(0)$ when $f$ has regular zeros.

Both kinds of local description of $U^{-1}(0)$ obviously cover the whole zero set and show that it is connected. For the benefit of the reader, we have included Figure 2.1 with an illustration of what this nodal set looks like in three dimensions. □

**Remark 2.3.1.** If $s > \frac{n+5+2l}{2}$ for some integer $l \geqslant 1$, one can conclude that $\frac{r^{\frac{n-1}{2}}}{2(2\pi)^{\frac{n-1}{2}}} u$ is close to $U$ in the $C^{l+1}(\mathbb{R}^n)$ norm, so [EPS13, Theorem 3.1] then ensures that $\|\Phi_k - \mathrm{id}\|_{C^l(\mathbb{R}^n)} < C/k$. This immediately yields asymptotic formulas for the area of each nodal component $\mathcal{S}_k$.

## 2.4 Proof of Theorem 2.1.1

Let us start by introducing some notation associated with the probabilistic setting described in (1.3). We denote by $\mathbb{P}_{lm}$ the probability distribution of the random variable $a_{lm}$, which we are assuming to be a normal distribution of the form $\mathcal{N}(0, \sigma_l^2)$. By Kolmogorov's extension theorem, the associated probability measure in $\mathbb{R}^{\mathbb{N}}$ is the product measure that we will denote by $\mathbb{P}_a := \prod_{l=0}^{\infty} \prod_{m=1}^{d_l} \mathbb{P}_{lm}$. The associated measures on the space of distributions on $\mathbb{S}^{n-1}$ and on $\mathbb{R}^n$ are respectively given by the pushed forward measures $\mathbb{P}_f := f_* \mathbb{P}_a$ and $\mathbb{P} := u_* \mathbb{P}_a$, which we view as maps

$$f : \omega \in \Omega \backslash \Omega_0 \mapsto \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} i^l a_{lm}(\omega)\, Y_{lm}(\cdot) \in \mathcal{D}'(\mathbb{S}^{n-1}),$$

and

$$u : \omega \in \Omega \backslash \Omega_0 \mapsto (2\pi)^{\frac{n}{2}} \sum_{l=0}^{\infty} \sum_{m=1}^{d_l} a_{lm}(\omega)\, Y_{lm}\left(\frac{\cdot}{|\cdot|}\right) \frac{J_{l+\frac{n}{2}-1}(|\cdot|)}{|\cdot|^{\frac{n}{2}-1}} \in \mathcal{D}'(\mathbb{R}^n), \quad (2.12)$$

where $\Omega_0 \subset \Omega$ is a set of measure zero.

An important first observation is that, with the probability distribution we are considering, $f$ is an $H^s$-smooth function with probability 1:

**Lemma 2.4.1.** *The function $f$ satisfying (1.3), (2.2) is of class $H^s(\mathbb{S}^{n-1})$ almost surely.*

*Proof.* The hypothesis (2.1) implies that, for any $L > 0$, the expected value of the finite sum is bounded by a uniform constant:

$$\mathbb{E}\left(\sum_{l=0}^{L}\sum_{m=1}^{d_l} a_{lm}^2 (l+1)^{2s}\right) = \sum_{l=0}^{L} d_l \sigma_l^2 (l+1)^{2s} < C\,.$$

The monotone convergence theorem then ensures that

$$\mathbb{E}\left(\|f\|^2_{H^s(\mathbb{S}^{n-1})}\right) = \mathbb{E}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{d_l} a_{lm}^2 (l+1)^{2s}\right) < C\,,$$

which implies that $\mathbb{P}_f(H^s(\mathbb{S}^{n-1})) = 1$. □

The next result we need is that, again with probability 1, $f^{-1}(0)$ is a regular level set:

**Lemma 2.4.2.** *The zero set of $f$ is regular almost surely. Furthermore, if $n = 2$, almost surely $f$ does not vanish.*

*Proof.* Let us consider the vector field on the sphere $h(\theta,\lambda) := \nabla_{\mathbb{S}} f_{\mathrm{R}}(\theta) - \lambda \nabla_{\mathbb{S}} f_{\mathrm{I}}(\theta)$ for $\lambda \in \mathbb{R}$. If we take local coordinates $(y_1, \dots, y_{n-1})$ around a point $\theta \in \mathbb{S}^{n-1}$, the components of $h$ in this local chart are given by

$$h_j(\theta,\lambda) := \sqrt{g^{jj}}\partial_{y_j} f_{\mathrm{R}}(\theta) - \lambda\sqrt{g^{jj}}\partial_{y_j} f_{\mathrm{I}}(\theta) \tag{2.13}$$

with $1 \leqslant j \leqslant n-1$, as

$$\nabla_{\mathbb{S}} f_a = \sum_i \partial_{y_i} f_a \sqrt{g^{ii}}\mathbf{e}_i \tag{2.14}$$

where $\mathbf{e}_i$ is the unit vector in our coordinates. Recall that, by Lemma 2.4.1, $f \in C^3(\mathbb{S}^{n-1})$ almost surely.

In order to show that $(f_{\mathrm{R}}, f_{\mathrm{I}}, h)$ is a non-degenerate Gaussian vector field, we first analyze the probabilistic structure of $f$ and its derivatives, for which we need to compute the covariance matrix of $(f_{\mathrm{R}}, f_{\mathrm{I}}, \nabla_{\mathbb{S}} f_{\mathrm{R}}, \nabla_{\mathbb{S}} f_{\mathrm{I}})$. We recall that a non-degenerate Gaussian vector field means that the determinant of its covariance matrix is positive definite everywhere.

First, the covariance between $f_{\mathrm{R}}$ (or derivatives of $f_{\mathrm{R}}$) and $f_{\mathrm{I}}$ (or derivatives of $f_{\mathrm{I}}$) is zero because they depend on different independent coefficients, even and odd $l$ respectively. Second, since the Gaussian coefficients have zero mean, the expected values of $f_a$ and $\nabla_{\mathbb{S}} f_a$ are zero, where $f_a$ denotes either $f_{\mathrm{R}}$ or $f_{\mathrm{I}}$. For the covariance kernel of $f_a$, notice that if $\theta, \theta' \in \mathbb{S}^{n-1}$, we have

$$\mathbb{E}\left(f_a(\theta) f_a(\theta')\right) = \sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln} P_{ln}(\theta\cdot\theta')\,, \tag{2.15}$$

where $P_{ln}$ is the Legendre polynomial of degree $l$ in $n$ dimensions, $c_{ln}$ was defined in (2.8) and the notation parity $= a$ means that the sum is restricted to even $l$ if $a = \mathrm{R}$ and to odd $l$ if $a = \mathrm{I}$. From the kernel (2.15) we can deduce the variance of $f_a$,

$$\mathbb{E}\left(f_a(\theta) f_a(\theta)\right) = \sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln} \in (0,+\infty)\,,$$

which is independent of $\theta$ and finite by our hypothesis on $\sigma_l$.

Let us now prove that $f_a$ and its derivatives are independent. Indeed, the covariance between the function and a derivative reads as

$$\mathbb{E}\left(f_a(\theta)\partial_{y_i}f_a(\theta)\right) = \sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln} P'_{ln}(\theta\cdot\theta')\theta\cdot\partial_{y'_i}\theta'\Big|_{\theta'=\theta} = 0\,,$$

where we have used that $\theta$ is a point on the unit sphere and hence $\theta\cdot\theta = 1$. We also claim that the derivatives are independent. To prove it, we assume that the vector fields $\{\partial_{y_i}\}$ are orthogonal, i.e., $g_{ij} = 0$ if $1 \leqslant i < j \leqslant n-1$, where $g_{ij}$ is the induced metric on $\mathbb{S}^{n-1}$. This can be accomplished, for instance, by taking hyperspherical coordinates. If $g^{ij}$ denotes the inverse matrix of $g_{ij}$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sqrt{g^{ii}}\partial_{y_i}f_a(\theta)\sqrt{g^{jj}}\partial_{y_j}f_a(\theta)\right) = \\
&= \sqrt{g^{ii}}\sqrt{g^{jj}} \sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln}\left[P'_{ln}(\theta\cdot\theta')\partial_{y_i}\theta\cdot\partial_{y'_j}\theta' + P''_{ln}(\theta\cdot\theta')\left(\partial_{y_i}\theta\cdot\theta'\right)\left(\theta\cdot\partial_{y'_j}\theta'\right)\right]\Big|_{\theta=\theta'} \\
&= \sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln} P'_{ln}(1)(\partial_{y_i}\theta)^2\delta_{ij}\frac{1}{g_{ii}} = \delta_{ij}\sum_{l=0,\text{parity}=a}^{\infty} \sigma_l^2 c_{ln}P'_{ln}(1) \qquad (2.16)
\end{aligned}$$

where in the second and third equalities we have used the orthogonality condition of the coordinate system and the definition of the metric

$$g_{ij} = \partial_{y_i}\theta\cdot\partial_{y_j}\theta\,.$$

As before, this covariance matrix is strictly positive definite, independent of the point and finite. The finiteness follows from the differential equation satisfied by $P_{ln}$,

$$(1-x^2)^{\frac{3-n}{2}}\frac{d}{dx}\left[(1-x^2)^{\frac{n-1}{2}}\frac{dP_{ln}(x)}{dx}\right] + l(l+n-2)P_{ln}(x) = 0\,,$$

which allows us to compute $P'_{ln}(1) = \frac{l(l+n-2)}{n-1}$. Using (2.14) we conclude that the covariance matrix of $(f_{\mathrm{R}}, f_{\mathrm{I}}, \nabla_{\mathbb{S}}f_{\mathrm{R}}, \nabla_{\mathbb{S}}f_{\mathrm{I}})$ is diagonal and positive definite.

We are now ready to prove that $(f_{\mathrm{R}}, f_{\mathrm{I}}, h)$ is a non-degenerate Gaussian vector field. To see this, first notice that the computations above imply that $f_{\mathrm{R}}$, $f_{\mathrm{I}}$ and their derivatives are independent as they are uncorrelated (i.e., their covariance matrix vanishes), which ensures that the Gaussian vector field $(f_{\mathrm{R}}, f_{\mathrm{I}}, h)$ has zero mean (as linear combinations of independent Gaussian random variables are still Gaussian, our field is Gaussian). Also, the local expression (2.13) also ensures that

$$\mathbb{E}\left(f_{\mathrm{R}}(x)h(x)\right) = \mathbb{E}\left(f_{\mathrm{I}}(x)h(x)\right) = 0\,.$$

By (2.16),

$$\begin{aligned}
\mathbb{E}\left(h_i(\theta,\lambda)h_j(\theta,\lambda)\right) &= \delta_{ij}\left(\sum_{l=0,\text{parity}=\text{ even}}^{\infty} \sigma_l^2 c_{ln}P'_{ln}(1) + \lambda^2 \sum_{l=0,\text{parity}=\text{ odd}}^{\infty} \sigma_l^2 c_{ln}P'_{ln}(1)\right) \\
&=: \delta_{ij}(\sigma_{\mathrm{R}}^2 + \lambda^2\sigma_{\mathrm{I}}^2).
\end{aligned}$$

We can now use suitable generalizations of Bulinskaya's lemma [AW09, Proposition 6.11] to conclude that

$$\mathbb{P}_f(\{\exists\,\theta,\lambda : f(\theta)=0,\ h(\theta,\lambda)=0\})=0\,.$$

Indeed, $(f_\mathrm{R}, f_\mathrm{I}, h)$ is a non-degenerate Gaussian vector field going from an $n$-dimensional space to $\mathbb{R}^{n+1}$ and it is $C^2(\mathbb{S}^{n-1})$ almost surely. As the covariance matrix determinant, $\det\Sigma(\lambda)$, attains its minimum value at $\lambda=0$, independent of $\theta$ and strictly positive, the density of $(f_\mathrm{R}, f_\mathrm{I}, h)$ at zero is bounded for all values of $\theta, \lambda$, i.e.,

$$\rho(x,\lambda)=\frac{\exp\left(-\frac{1}{2}x^\mathrm{T}\Sigma(\lambda)^{-1}x\right)}{\sqrt{(2\pi)^{n+1}|\det\Sigma(\lambda)|}}\leqslant\rho(0,0)<\infty.$$

This shows that the zero set of $f$ is regular almost surely as $\nabla_\mathbb{S} f_\mathrm{R}$ and $\nabla_\mathbb{S} f_\mathrm{I}$ are linearly independent at $f^{-1}(0)$[1].

When $n=2$, the same argument applied to the Gaussian vector field $(f_\mathrm{R}, f_\mathrm{I})$ shows that

$$\mathbb{P}_f(\{\exists\,\theta : f(\theta)=0\})=0\,,$$

so with probability 1 the function $f$ does not vanish, and the lemma follows. □

In the next lemma we compute the probability that $f$ does not vanish and that $f$ has a nonempty regular zero set:

**Lemma 2.4.3.** *The probability that the function $f$ does not vanish on $\mathbb{S}^{n-1}$ is $p_2 := 1$ if $n=2$ and $p_n\in(0,1)$ if $n\geqslant 3$. Moreover, with probability $1-p_n$ the zero set $f^{-1}(0)$ is regular and nonempty.*

*Proof.* Given any function $f_0\in H^s(\mathbb{S}^{n-1})$, whose coefficients for the expansion in spherical harmonics we will denote by $a^0_{lm}$, and any $\epsilon>0$, we claim that

$$\mathbb{P}_f\big(\{\|f-f_0\|_{H^s(\mathbb{S}^{n-1})}<\epsilon\}\big)>0\,. \tag{2.17}$$

To prove this, we start by noting that we can take some $L$, depending on $\epsilon$, such that

$$\mathbb{P}_a\left(\sum_{l=L}^{\infty}\sum_{m=1}^{d_l}|a_{lm}-a^0_{lm}|^2(l+1)^{2s}<\frac{\epsilon}{2}\right)>0\,,$$

which is obvious because $f_0\in H^s(\mathbb{S}^{n-1})$ and $f$ is in $H^s(\mathbb{S}^{n-1})$ almost surely by Lemma 2.4.1. (2.17) then follows because

$$\begin{aligned}&\mathbb{P}_f\big(\{\|f-f_0\|_{H^s(\mathbb{S}^{n-1})}<\epsilon\}\big)\geqslant\\ &\mathbb{P}_a\left(\sum_{l=L}^{\infty}\sum_{m=1}^{d_l}|a_{lm}-a^0_{lm}|^2(l+1)^{2s}<\frac{\epsilon}{2}\right)\mathbb{P}_a\left(\sum_{l=1}^{L}\sum_{m=1}^{d_l}|a_{lm}-a^0_{lm}|^2(l+1)^{2s}<\frac{\epsilon}{2}\right)>0\,.\end{aligned}$$

For all $n\geqslant 2$, it then suffices to take $f_0:=1$ to conclude that

$$p_n:=\mathbb{P}_f(\{f>0\})>0\,;$$

[1]It remains to consider the case $\nabla_\mathbb{S} f_\mathrm{I}=0$ but $\nabla_\mathbb{S} f_\mathrm{R}\neq 0$ at some point of $f^{-1}(0)$, but we can discard this event by the same reasoning applied to the easier case $(f_\mathrm{R}, f_\mathrm{I}, \nabla_\mathbb{S} f_\mathrm{I})$.

indeed, by Lemma 2.4.2 one knows that $p_2 = 1$. Likewise, when $n \geqslant 3$, one can take any smooth function $f_0$ whose zero set is regular and nonempty to conclude, by the implicit function theorem, that

$$1 - p_n = \mathbb{P}_f(\{\min_{\mathbb{S}^{n-1}} |f| = 0\}) > 0\,.$$

Notice that this argument does not work when $n = 2$ because, as $f$ is complex-valued, the rank of $\nabla f_0$ on $f_0^{-1}(0)$ must be 2 to apply the implicit function theorem. Finally, by Lemma 2.4.2, the nodal set is regular almost surely, so the lemma follows. □

We are now ready to complete the proof of Theorem 2.1.1. Lemma 2.4.1 ensures that $f \in H^s(\mathbb{S}^{n-1})$ almost surely. Furthermore, by Lemma 2.4.3, with probability $p_n$, $f$ does not vanish, so in this case Theorem 2.3.1 ensures that the nodal set of $u$ has $R/\pi + o(R)$ components diffeomorphic to $\mathbb{S}^{n-1}$ contained in $B_R$ and only $O(1)$ components that are not diffeomorphic to $\mathbb{S}^{n-1}$. Also by Lemma 2.4.3, with probability $1 - p_n$ the zero set $f^{-1}(0)$ is regular and nonempty, so Theorem 2.3.1 ensures that $N_u(R) = O(1)$. The theorem is then proved.

## 2.5 Proof of Theorem 2.1.2

Let us denote by $\mu$ the probability measure on $\mathbb{R}^{\mathbb{N}}$ defined by the random variables $a_{lm}$, which we now assume to be absolutely continuous with respect to the Lebesgue measure for $l < l_0$ and Gaussian distributions $\mathcal{N}(M_{lm}, \sigma l m^2)$ for $l \geqslant l_0$. We denote by $\mathbb{P}_a^0$ and $\mathbb{P}_a$ the probability measures defined by random variables $a_{lm} \sim \mathcal{N}(0,1)$ and $a_{lm} \sim \mathcal{N}(0, \sigma_l^2)$ as in Theorem 2.1.1, respectively.

To prove the theorem it is enough to show that in the first (respectively, second) case, the measures $\mu$ and $\mathbb{P}_a^0$ (respectively, $\mathbb{P}_a$) are mutually absolutely continuous. Kakutani's dichotomy theorem, Proposition 2.21 in [DPZ14], ensures that, in the first case, these measures are mutually absolutely continuous if and only if the Radon–Nikodym derivative of the measures satisfies

$$\prod_{l=0}^{\infty} \prod_{m=1}^{d_l} \int_{-\infty}^{\infty} \left( \frac{d\mu_{lm}}{d\mathbb{P}_a^0} \right)^{1/2} d\mathbb{P}_a^0 > 0 \tag{2.18}$$

(being always $\leqslant 1$). Since, for $l \geqslant l_0$,

$$\frac{d\mu_{lm}}{d\mathbb{P}_a^0}(x) = \frac{1}{\sigma l m} e^{\frac{x^2}{2} - \frac{(x - M_{lm})^2}{2\sigma l m^2}}\,,$$

one has

$$\int_{-\infty}^{\infty} \left( \frac{d\mu_{lm}}{d\mathbb{P}_a^0} \right)^{1/2} d\mathbb{P}_a^0 = (2\pi\sigma l m)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{(x-M_{lm})^2}{4\sigma l m^2} - \frac{x^2}{4}}\, dx = \left( \frac{2\sigma l m}{1 + \sigma l m^2} \right)^{1/2} e^{-\frac{M_{lm}^2}{4 + 4\sigma l m^2}}\,.$$

Minus the logarithm of the product (2.18) for $l \geqslant l_0$ is then given by the series

$$C := \sum_{l=l_0}^{\infty} \sum_{m=1}^{d_l} \frac{M_{lm}^2}{4 + 4\sigma l m^2} + \frac{1}{2} \sum_{l=l_0}^{\infty} \sum_{m=1}^{d_l} \log \frac{1 + \sigma l m^2}{2\sigma l m}\,.$$

As both terms are necessarily positive and using that a sequence $a_n \geqslant 1$ satisfies

$$\sum_{n=1}^{\infty} \log a_n < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} (a_n - 1) < \infty ,$$

we then infer that that necessary and sufficient condition for $\mathcal{C} < \infty$ (or, equivalently, for the product (2.18) to be nonzero) is that

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{\sigma_{lm}^2 + 1} + \frac{(\sigma_{lm} - 1)^2}{\sigma_{lm}} \right] < \infty .$$

Likewise, $\mu$ and $\mathbb{P}_a$ are mutually absolutely continuous if (2.18) holds with $\mathbb{P}_a^0$ replaced by $\mathbb{P}_a$, which amounts to

$$\sum_{l=l_0}^{\infty} \sum_{m=0}^{d_l} \left[ \frac{M_{lm}^2}{\sigma_l^2 + \sigma_{lm}^2} + \frac{(\sigma_l - \sigma_{lm})^2}{\sigma_l \sigma_{lm}} \right] < \infty .$$

Theorem 2.1.2 then follows.

**Remark 2.5.1.** When the probability measure $\mu$ is a general Gaussian measure (not necessarily a product), the Feldman–Hajek theorem [DPZ14, Theorem 2.25] characterizes when $\mu$ and $\mathbb{P}_a$ (or $\mathbb{P}_a^0$) are mutually absolutely continuous in terms of the mean and covariance operator of $\mu$. However, the resulting condition is not very illustrative and we have opted not to include it. Nevertheless, this means that the results can be extended to coefficients which are not necessarily independent. Also, similar considerations using Kakutani's theorem can be applied to a product measure whose coefficients are not normal variables.

# Appendices

## 2.A The decay of $u$ in terms of the regularity of $f$

Standard arguments from the theory of distributions ensure that any polynomially bounded solution to the Helmholtz equation

$$\Delta u + u = 0$$

on $\mathbb{R}^n$ can be written as the Fourier transform of a distribution supported on the unit sphere. The fundamental result that connects the decay of the solution $u$ with the regularity of its Fourier transform is a classical result of Herglotz [Hör15, Theorem 7.1.28]. In order to state it, let us denote by

$$|||u|||^2 := \limsup_{R\to\infty} \frac{1}{R} \int_{B_R} u(x)^2 \, dx$$

the Agmon–Hörmander seminorm of a function $u$ on $\mathbb{R}^n$.

**Theorem 2.A.1** (Herglotz). *A solution to the Helmholtz equation satisfies the decay condition*

$$|||u||| < \infty$$

*if and only if there is a function $f \in L^2(\mathbb{S}^{n-1})$ such that*

$$u = \widehat{f\,dS}\,. \tag{2.19}$$

*Furthermore, this decay estimate is sharp in the sense that there is a universal constant such that*

$$\frac{1}{C}|||u||| \leqslant \|f\|_{L^2(\mathbb{S}^{n-1})} \leqslant C|||u|||\,.$$

An immediate consequence of this result is that the derivatives of any function of the form (2.19) with $f \in L^2(\mathbb{S}^{n-1})$ have the same decay at infinity. Indeed, for any k,

$$\left|\left|\left|\nabla^k u\right|\right|\right| \leqslant C\|\xi^k f\|_{L^2(\mathbb{S}^{n-1})} \leqslant C\|f\|_{L^2(\mathbb{S}^{n-1})}\,, \tag{2.20}$$

and in general this is obviously sharp because $\Delta u = -u$.

However, it is not hard to see that higher regularity of $f$ translates into decay rates of the *angular* derivatives of $u$. In order to state this result, let us denote by

$$\eth_\alpha u := \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_{n-1}}^{\alpha_{n-1}} u$$

for a multi-index $\alpha$ and $(\xi_j)_{j=1}^{n-1}$, for simplicity, hyper-spherical coordinates on the sphere $\mathbb{S}^{n-1}$.

**Proposition 2.A.2.** *A solution to the Helmholtz equation $u = \widehat{f\,dS}$ with $f \in L^2(\mathbb{S}^{n-1})$ satisfies*

$$|||\eth_\alpha u||| < \infty$$

*if and only if $\partial_\alpha f \in L^2(\mathbb{S}^{n-1})$.*

*Proof.* Using Proposition 2.2.1 together with the fact that $\partial_\alpha Y_{lm}$ is again an spherical harmonic with the same eigenvalue, it is straightforward to show that

$$\eth_\alpha u = \widehat{\partial_\alpha f\,dS}\,,$$

so the result follows from Herglotz's theorem. □

**Remark 2.A.1.** Roughly speaking, Herglotz's theorem asserts that a solution $u$ to the Helmholtz equation on $\mathbb{R}^n$ can decay at most as $|x|^{-\frac{n-1}{2}}$, on average, and that this sharp decay rate is attained if and only if $f$ is in $L^2(\mathbb{S}^{n-1})$. Furthermore, this proposition says that the $k^{\text{th}}$ angular derivatives of $u$ can decay at the same rate $|x|^{-\frac{n-1}{2}}$, and that this sharp rate is attained in an $L^2$-averaged sense if and only if the $k^{\text{th}}$ derivatives of $f$ are in $L^2(\mathbb{S}^{n-1})$.

The case when $f$ is of lower regularity than $L^2$, for instance $f \in H^{-k}(\mathbb{S}^{n-1})$ for a positive integer $k$, can be partly understood with a similar reasoning. In this case, one can write

$$f = \sum_{j=0}^{k} \mathcal{L}_j f_j$$

with $f_j \in L^2(\mathbb{S}^{n-1})$ and $\mathcal{L}_j$ a differential operator on $\mathbb{S}^{n-1}$ of order $j$ with smooth coefficients. Furthermore,

$$\|f\|_{H^{-k}(\mathbb{S}^{n-1})} = \sum_{j=0}^{k} \|f_j\|_{L^2(\mathbb{S}^{n-1})} .$$

Therefore, integrating by parts in the distributional formula

$$u(x) = \int_{\mathbb{S}^{n-1}} e^{ix\cdot\xi} f(\xi)\, dS(\xi) ,$$

one easily obtains that

$$\frac{1}{R} \int_{B_R} \frac{u(x)^2}{1+|x|^{2k}}\, dx \leqslant C\|f\|^2_{H^{-k}(\mathbb{S}^{n-1})} .$$

However one should note that, contrary to what happens in the previous results of this Appendix, these are not the only solutions to the Helmholtz equation with this decay rate. This is evidenced, e.g., by the solutions whose Fourier transform is

$$\widehat{u}(\xi) = \delta^{(k)}(|\xi| - 1) .$$

# Chapter 3

# Critical point asymptotics for Gaussian random waves with densities of any Sobolev regularity

## 3.1 Introduction

In this chapter we are concerned with asymptotic laws for the number of critical points (i.e., the zeros of the gradient). We consider this question in the context of Gaussian random monochromatic waves on the plane, which are solutions to the Helmholtz equation on $\mathbb{R}^2$,

$$\Delta u + u = 0\,. \tag{3.1}$$

As is well known, the study of critical points is a central topic in spectral theory [Yau82; Yau93; JN99; BLS20] (and, in general, in the geometric study of solutions to differential equations [Wal50; Ale87; AM92; EPS18]), both in the deterministic and random settings. This is partly because they are very closely related to the geometry of the nodal components.

As we saw in Chapter 1, when $u$ is polynomially bounded, the Helmholtz equation simply means that $u$ is the Fourier transform of a distribution supported on the unit circle, which we identify with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ via the map

$$E(\phi) := (\cos\phi, \sin\phi)\,. \tag{3.2}$$

Solutions to the Helmholtz equation are necessarily analytic, but their Fourier transforms do not have any a priori regularity properties. There are some connections, though, between the regularity of the Fourier transform and the decay rate of $u$ at infinity. Most important is the classical result of Herglotz ensuring that $u$ has the sharp fall-off at infinity (which is as $|x|^{-\frac{1}{2}}$ in a space-averaged sense) if and only if one can write

$$u(x) = \int_{\mathbb{T}} e^{-ix\cdot E(\phi)}\, f(\phi)\, d\phi \tag{3.3}$$

with some square-integrable density $f$, and that in this case the norm $\|f\|_{L^2(\mathbb{T})}$ quantitatively captures the decay rate of $u$. For details and generalizations, see e.g. [EPSR22a, Appendix A].

The main thrust of this chapter is to understand the connection between the distribution of the critical points of $u$, defined as in (3.3), and the regularity of the density $f$. To this end, we consider the usual ansatz for random plane waves [CS19; SW19] and tweak it by introducing a real parameter $s \in \mathbb{R}$ to control the regularity of $f$:

$$u(x) := \sum_{l\neq 0} a_l\,|l|^{-s} e^{il\theta}\,J_l(r)\,. \tag{3.4}$$

Here the real and imaginary part of $a_l$ are independent standard Gaussian random variables subject to the constraint $a_l = (-1)^l \overline{a_{-l}}$ (which makes $u$ real valued), $(r,\theta) \in \mathbb{R}^+ \times \mathbb{T}$ are the polar coordinates. This is equivalent to taking the Gaussian random density

$$f(\phi) := \frac{1}{2\pi}\sum_{l\neq 0} i^l a_l |l|^{-s} e^{il\phi} \tag{3.5}$$

and then defining $u$ through the formula (3.3), which must be understood in the sense of distributions.

Of course, the rationale behind this definition is that $\{|l|^{-s}e^{il\phi}\}_{l\neq 0}$ is an orthonormal basis of the Sobolev space $\dot{H}^s(\mathbb{T})$ of functions with zero mean and $s$ derivatives in $L^2$, which reduces to the space of square-integrable functions of zero mean when $s = 0$. The covariance kernel of $u$ is translation-invariant when $s = 0$, so the Nazarov–Sodin theory is applicable in this case (see Remark 3.4.1 for details), but this is not the case for nonzero $s$. One should note that the proofs work verbatim if one replaces the weight $|l|^{-s}$ by a more general expression such as

$$\sigma_l = \sigma_{-l} = |l|^{-s} + p_{-s-1}(l)\,, \tag{3.6}$$

where the function $p_{-s-1}(t)$ is an arbitrary classical symbol of order $-s-1$ (which does not necessarily vanish at 0). The resulting constants, however, depend on the specific sequence $\sigma_l$.

It is not hard to see that the parameter $s$ describes the regularity of the density in the sense that $f$ has exactly $s - \frac{1}{2}$ derivatives in $L^2$ almost surely, as measured using Sobolev or Besov spaces. Specifically, one can show that, for any $\delta > 0$,

$$f \in \left[H^{s-\frac{1}{2}-\delta}(\mathbb{T})\backslash H^{s-\frac{1}{2}}(\mathbb{T})\right] \cap \left[B^{s-\frac{1}{2}}_{2,\infty}(\mathbb{T})\backslash B^{s-\frac{1}{2}+\delta}_{2,\infty}(\mathbb{T})\right]$$

with probability 1; see Proposition 3.2.2 for details.

Our main result provides an asymptotic estimate for the growth of the expected number of critical points contained in a disk of large radius $R$, which we denote by

$$N(\nabla u, R) := \#\{x \in B_R : \nabla u(x) = 0\}\,,$$

as a function of the regularity parameter $s$. It is elementary that this quantity is an upper bound for the expected number of nodal components contained in $B_R$. With the usual ansatz for random plane waves, it is well known that $N(\nabla u, R)$ grows asymptotically like the area of the disk; more precisely [BCW19], when $s = 0$ one has

$$\mathbb{E}N(\nabla u, R) \sim \kappa(0)\,R^2\,,$$

where $\kappa(0) := 1/(2\sqrt{3})$ and where the notation $q(R) \sim Q(R)$ means that the quotient $q(R)/Q(R)$ tends to 1 as $R \to \infty$.

We should mention from the onset that the effect of changing the regularity parameter $s$ can be quite drastic, as one should not expect that the number of critical points grows like the area in all regularity regimes. To illustrate this, recall that, when $s = 0$, the Nazarov–Sodin theory ensures the number of nodal components of $u$ contained in $B_R$ grows as

$$N(u,R) \sim \nu_0 R^2$$

almost surely for some constant $\nu_0 > 0$. In contrast, the results proven in [EPSR22a] show that

$$N(u,R) \sim \nu_\infty R$$

almost surely for $s > 4$, with $\nu_\infty := 1/\pi$. Understanding the asymptotic behavior of the number of nodal components in other regimes is an extremely challenging open problem. Consequently, our main objective in this chapter is to analyze the intriguing transitions between distinct asymptotic regimes in the simpler case of critical points.

In the case of critical points, it is also natural to wonder about the asymptotic growth in the case of very negative regularities $s < 0$. Recall that, by the Faber–Krahn inequality, the number of nodal components of a solution to the Helmholtz equation contained in $B_R$ is at most $cR^2$, where $c$ is a universal constant. However, the number of critical points is not bounded a priori: in Appendix 3.A we show that, given any continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, there exists a solution to the Helmholtz equation on $\mathbb{R}^2$ having at least $\rho(R)$ nondegenerate critical points in $B_R$, for all $R > 1$. Thus, one could in principle expect the average number of critical points in a large ball $R$ to have a fast growth in $R$ for small enough regularities.

Our main result provides a satisfactory, and quite surprising, answer to both questions. It turns out that the growth of the expected number of critical points is like the square of the radius for $s < \frac{3}{2}$, linear for $s > \frac{5}{2}$, and the corresponding exponent changes according to a linear interpolation law in the intermediate regime $\frac{3}{2} < s < \frac{5}{2}$. The transitions occurring at the endpoint cases involve not only a power law, but also the square root of the logarithm of the radius. Furthermore, the highest asymptotic growth of the expected number of critical points is attained exactly for $s = 0$, that is, in the usual setting of random plane waves.

**Theorem 3.1.1.** *For any real $s$, the following statements hold:*

*(i) There exist explicit positive constants $\kappa(s), \tilde{\kappa}_{\frac{3}{2}}, \tilde{\kappa}_{\frac{5}{2}}$ such that the expected number of critical points of the Gaussian random function $u$ satisfies*

$$\mathbb{E}N(\nabla u, R) \sim \begin{cases} \kappa(s)\, R^2 & \text{if} \quad s < \frac{3}{2}, \\ \tilde{\kappa}_{\frac{3}{2}} \frac{R^2}{\sqrt{\log R}} & \text{if} \quad s = \frac{3}{2}, \\ \kappa(s)\, R^{2-(s-\frac{3}{2})} & \text{if} \quad \frac{3}{2} < s < \frac{5}{2}, \\ \tilde{\kappa}_{\frac{5}{2}} R\sqrt{\log R} & \text{if} \quad s = \frac{5}{2}, \\ \kappa(s)\, R & \text{if} \quad s > \frac{5}{2}. \end{cases}$$

*(ii) In the region where the growth of $\mathbb{E}N(\nabla u, R)$ is volumetric, the constant $\kappa(s)$ depends continuously on $s$. More precisely, $\kappa(s)$ is a $C^\infty$ function of $s \in (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}]$ but it is only Lipschitz at $s = \frac{1}{2}$. Furthermore, $\kappa(s)$ is strictly increasing on $(-\infty, 0)$, strictly decreasing on $(0, \frac{3}{2})$, and tends to $0$ as $s \to -\infty$ and as $s \to \frac{3}{2}^-$. In the region $s \in (\frac{3}{2}, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$ the constant $\kappa(s)$ is also $C^\infty$.*

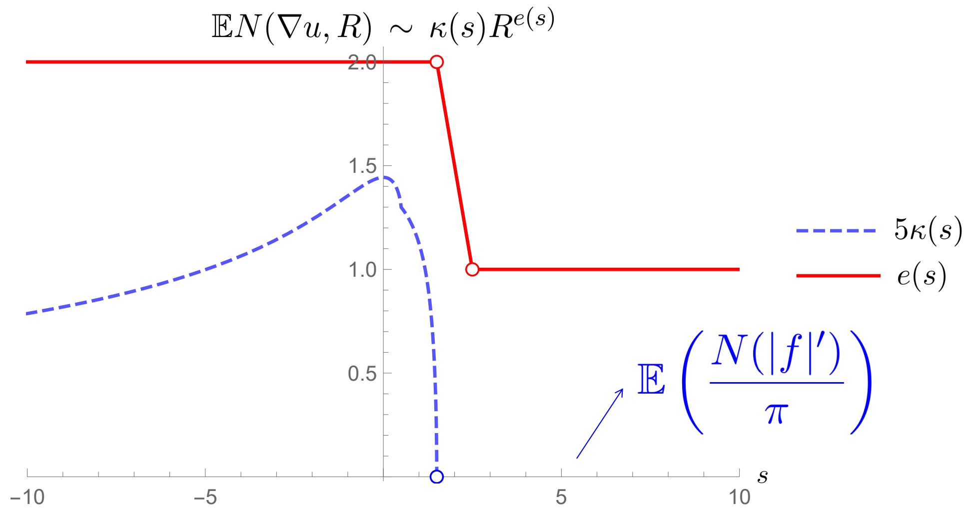



FIGURE 3.1: Consider the asymptotic behavior of $\mathbb{E}N(\nabla u, R) \sim \kappa(s)R^{e(s)}$ proved in Theorem 3.1.1. In red, we have plotted the exponent $e(s)$ as a function of $s \in \mathbb{R}\backslash\{\frac{3}{2}, \frac{5}{2}\}$. Logarithmic effects appear at the endpoints $s = 3/2$ and $s = 5/2$. In blue, we have plotted $\kappa(s)$ in the region where the asymptotic growth is volumetric, $s < \frac{1}{2}$. The maximum of $\kappa(s)$ in this region is attained at $s = 0$ and that $\kappa(s)$ is not continuously differentiable at $s = 1/2$. The reader can find a plot of $\kappa(s)$ in the range $s \in (\frac{3}{2}, \frac{5}{2})$ in Figure 3.3, cf. Section 3.4. Note that $\kappa(s) = \mathbb{E}N(|f|')/\pi$ by Theorem 3.1.3.

Figure 3.1 summarizes Theorem 3.1.1 in a more visual way. The fact that the highest asymptotic growth for the number of critical points occurs precisely in the translation-invariant case $s = 0$ is somewhat surprising. Naively one could expect that rougher density functions, which feature wilder oscillations, would exhibit more critical points. Theorem 3.1.1 shows that, strictly speaking, this is only the case for regularities $s > 0$.

Let us now discuss the proof of Theorem 3.1.1. The asymptotic analysis of $N(\nabla u, R)$ hinges on the celebrated Kac–Rice counting formula, which, under suitable technical hypotheses, expresses the expected number of zeros of a random field (in this case, the gradient $\nabla u$) has in terms of a multivariate integral. As is well known, this formula has been used profusely in the literature [EF16; NS16; BCW19; BMW19], and in particular lies at the heart of the computation of $\mathbb{E}N(\nabla u, R)$ for $s = 0$ and of the finer asymptotics bounds for the expected number of extrema and saddle points and for higher order correlations obtained in [BCW19] also in the translation-invariant case $s = 0$.

The coefficients that appear in the Kac–Rice integral formula involve, via the variance matrix of $\nabla u$, weighted series of Bessel functions of the form

$$\mathcal{J}_{s,m,m'}(r) := \sum_{l=1}^{\infty} l^{-2s} J_{l+m}(r)\, J_{l+m'}(r)\,, \tag{3.7}$$

where $m$ and $m'$ are certain integers. $\mathcal{J}_{s,m,m'}$ is sometimes called in the literature a second type Neumann series. It is clear that the way each term $J_{l+m}(r)\, J_{l+m'}(r)$ contributes to the sum for $r \gg 1$ and $l \gg 1$ will depend on whether the "angular frequency" $l$ is much larger than $r$, much smaller than $r$, or roughly of the same size;

moreover, the effect of each group of angular frequencies will have a different relative weight in the sum depending on the power $s$ appearing in $l^{-2s}$. More precisely, a key step of the proof is to establish the following technical result, which controls the asymptotic behavior of $\mathcal{J}_{s,m,m'}(r)$:

**Lemma 3.1.2.** *For any pair of nonnegative integers $m, m'$ and any real $s$, the large-$r$ asymptotic behavior of $\mathcal{J}_{s,m,m'}$ is*

$$
\begin{aligned}
\mathcal{J}_{s,m,m'}(r) &= c^1_{s,m-m'}\, r^{-2s} + o(r^{-2s}) && \text{if} \quad s < \tfrac{1}{2}\,,\\
\mathcal{J}_{s,m,m'}(r) &= c^2_{m-m'} \frac{\log r}{r} + O(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m-m' \text{ is even}\,,\\
\mathcal{J}_{s,m,m'}(r) &= \frac{c^3_{m-m'} - c^4 \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s = \tfrac{1}{2} \text{ and } m-m' \text{ is odd}\,,\\
\mathcal{J}_{s,m,m'}(r) &= \frac{c^5_{s,m-m'} - c^6_s \sin(2r - c^7_{m+m'})}{r} + o(r^{-1}) && \text{if} \quad s > \tfrac{1}{2}
\end{aligned}
$$

*with some explicit constants that will be defined later on.*

Ultimately, the different asymptotic regimes that the expectation of $N(\nabla u, R)$ can exhibit can be traced back to the asymptotic behavior of functions of the form (3.7). One should note that, in general, the highly oscillatory nature of summands in (1.17) makes the analysis of the asymptotic behavior of $\mathcal{J}_{s,m,m'}(r)$ rather subtle. An exception to this general fact is precisely the case $s = 0$, where all the associated series can be computed exactly using that the covariance kernel of $u$ is translation-invariant (or, equivalently, the addition formula for Bessel functions); this makes it much easier to analyze the corresponding asymptotic behavior of $\mathbb{E}N(\nabla u, R)$. To illustrate this fact, in the very short Appendix 3.B we carry out the analysis of the translation invariant case $s = 0$.

In the particular case of smooth enough density functions, one can use the methods of our previous chapter to understand the asymptotic behavior of the number of critical points (not only of its expectation value) in greater detail. Specifically, one can prove the following:

**Theorem 3.1.3.** *If $s > 5$,*

$$N(\nabla u, R) \sim \frac{N(|f|')}{\pi} R$$

*with probability 1. In particular, $N(\nabla u, R)$ grows linearly almost surely.*

Here the random variable $N(|f|') := \#\{\phi \in \mathbb{T} : |f(\phi)|' = 0\}$ (which is at least 2 almost surely) denotes the number of critical points of the (non-Gaussian) random function $|f|$. In particular, the asymptotic growth of $N(\nabla u, R)$ is linear with probability 1, albeit the ratio is not a universal constant but a random variable. In view of Theorem 3.1.1, a consequence of this asymptotic formula is an explicit formula for the expectation $\mathbb{E}N(|f|')$ when $s > 5$.

The chapter is organized as follows. In Section 3.2, we start by showing the relation between the parameter $s$ and the regularity of the random function $u$. Sections 3.3, 3.4 and 3.5 are respectively devoted to the proofs of Lemma 3.1.2 and Theorems 3.1.1 and 3.1.3. We have divided each of these sections into a number of subsections to emphasize the main ideas of each proof. The chapter concludes with two Appendices. In Appendix 3.A, we construct solutions to the Helmholtz equation on the plane for which the number of nondegenerate critical points contained in $B_R$ grows arbitrarily fast as $R \to \infty$. In Appendix 3.B, we revisit the translation-invariant case ($s = 0$) and explain the key simplifications that appear in this extremely important case.

## 3.2 Almost sure regularity of the random density function

Our objective in the section is to show that, with probability 1, the Gaussian random function $f$, defined in (3.5), has exactly $s - \frac{1}{2}$ derivatives in $L^2$, measured using suitable Sobolev or Besov spaces.

To prove the main result we will need the following version of the strong law of large numbers for sequences of random variables that are labeled by two integers:

**Lemma 3.2.1.** *Let $\{K_N\}_{N=1}^{\infty}$ be a sequence of positive integers such that*

$$\liminf_{M\to\infty} \frac{K_M}{\sum_{N=1}^{M} K_N} > 0\,.$$

*If $\{b_{N,k} : 1 \leqslant k \leqslant K_N,\ N \geqslant 1\}$ are i.i.d. random variables with mean $\mu$, then*

$$\lim_{N\to\infty} \frac{1}{K_N} \sum_{k=1}^{K_N} b_{N,k} = \mu$$

*almost surely.*

*Proof.* The strong law of large numbers ensures that

$$S_M := \frac{1}{Q_M} \sum_{N=1}^{M} \sum_{k=1}^{K_N} b_{N,k} - \mu \tag{3.8}$$

converges to 0 almost surely as $M \to \infty$, with $Q_M := \sum_{N=1}^{M} K_N$. Thus, from the identity

$$S_M = \frac{Q_{M-1}}{Q_M} S_{M-1} + \frac{K_M}{Q_M} \left( \frac{1}{K_M} \sum_{k=1}^{K_M} b_{M,k} - \mu \right)$$

and the fact that $Q_{M-1}/Q_M \leqslant 1$ we obtain

$$\limsup_{M\to\infty} \left| \frac{1}{K_M} \sum_{k=1}^{K_M} b_{M,k} - \mu \right| \leqslant \frac{\lim_{M\to\infty}(|S_M| + |S_{M-1}|)}{\liminf_{M\to\infty} \frac{K_M}{Q_M}} = 0$$

almost surely. Notice that we have used the assumption $\liminf_{M\to\infty} \frac{K_M}{Q_M} > 0$. The lemma then follows. □

We are now ready to prove the main result of this section. Here and in what follows, we shall use the notation $q \approx Q$ or $q \lesssim Q$ when there exists a constant $C$ (independent of the large parameter under consideration) such that $Q/C \leqslant q \leqslant CQ$ or $q \leqslant CQ$, respectively.

**Proposition 3.2.2.** *For each $\delta > 0$, the Gaussian random function* (3.5) *satisfies*

$$f \in \left[H^{s-\frac{1}{2}-\delta}(\mathbb{T})\backslash H^{s-\frac{1}{2}}(\mathbb{T})\right] \cap \left[B_{2,\infty}^{s-\frac{1}{2}}(\mathbb{T})\backslash B_{2,\infty}^{s-\frac{1}{2}+\delta}(\mathbb{T})\right]$$

*almost surely.*

*Proof.* Let us recall that the $H^\sigma(\mathbb{T})$ norm of the function $f$ defined in (3.5) is

$$\|f\|_{H^\sigma(\mathbb{T})}^2 = \sum_{l=-\infty}^{\infty} |a_l|^2 l^{2\sigma-2s}.$$

To analyze this quantity, consider the set of integers $\Lambda_N := \{l : 2^{N-1} \leqslant l < 2^N\}$ and the subsequences

$$\sum_{l=-(2^M-1)}^{2^M-1} |a_l|^2 l^{2\sigma-2s} = |a_0|^2 + 2\sum_{N=1}^{M}\sum_{l\in\Lambda_N} l^{2\sigma-2s}|a_l|^2 \approx |a_0|^2 + \sum_{N=1}^{M} 2^{N(2\sigma-2s)} \sum_{l\in\Lambda_N} |a_l|^2.$$

Since $|\Lambda_N| \approx 2^N$,

$$\frac{|\Lambda_M|}{\sum_{N=1}^{M}|\Lambda_N|} \approx \frac{2^M}{2^{M+1}} = \frac{1}{2}$$

is bounded away from zero. Hence one can apply Lemma 3.2.1 to infer that

$$\frac{1}{|\Lambda_N|}\sum_{l\in\Lambda_N}|a_l|^2 \to 1$$

almost surely as $N \to \infty$. Therefore, with probability 1,

$$\sum_{l=-(2^M-1)}^{2^M-1} |a_l|^2 l^{2\sigma-2s} \approx |a_0|^2 + \sum_{N=1}^{M} 2^{N(2\sigma-2s+1)} \frac{1}{|\Lambda_N|}\sum_{l\in\Lambda_N}|a_l|^2 \approx |a_0|^2 + \sum_{N=1}^{M} 2^{N(2\sigma-2s+1)}.$$

This shows that, with probability 1, $\|f\|_{H^\sigma(\mathbb{T})} < \infty$ if and only if $<s - \frac{1}{2}$.

The estimate for the Besov norm follows from an analogous reasoning using that

$$\|f\|_{B_{2,\infty}^\sigma(\mathbb{T})}^2 = \sup_{1\leqslant N<\infty} \sum_{l\in\Lambda_N} l^{2\sigma-2s}|a_l|^2.$$

□

**Remark 3.2.1.** The result and the proof remain valid in higher dimensions with minor modifications. Specifically, let $\{Y_{lm} : 1 \leqslant m \leqslant d_l,\ 0 \leqslant l < \infty\}$ be an orthonormal basis of spherical harmonics on the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, with $\Delta_{\mathbb{S}^{n-1}} Y_{lm} + l(l+n-2)Y_{lm} = 0$. Consider the Gaussian random function

$$f(x) := \sum_{l=1}^{\infty}\sum_{m=1}^{d_l} l^{-s} i^l a_{lm} Y_{lm}(x),$$

where $a_{lm}$ are independent standard Gaussian variables and $s \in \mathbb{R}$. Then

$$f \in \left[H^{s-\frac{n-1}{2}-\delta}(\mathbb{S}^{n-1})\backslash H^{s-\frac{n-1}{2}}(\mathbb{S}^{n-1})\right] \cap \left[B_{2,\infty}^{s-\frac{n-1}{2}}(\mathbb{S}^{n-1})\backslash B_{2,\infty}^{s-\frac{n-1}{2}+\delta}(\mathbb{S}^{n-1})\right]$$

almost surely.

To spell out the details, the proof in higher dimension starts with the formula

$$\|f\|^2_{H^\sigma(\mathbb{S}^{n-1})} := \sum_{l=1}^\infty \sum_{m=1}^{d_l} |a_{lm}|^2 l^{2\sigma-2s}\,.$$

Since $d_l = c_n l^{n-2} + O(l^{n-3})$, the set

$$\Lambda_N := \{(l,m) : 2^{N-1} \leqslant l < 2^N,\ 1 \leqslant m \leqslant d_l\}$$

satisfies $|\Lambda_N| \approx 2^{N(n-1)}$. Lemma 3.2.1 then ensures

$$\frac{1}{|\Lambda_N|} \sum_{(l,m)\in\Lambda_N} |a_{lm}|^2$$

converges to 1 almost surely as $N \to \infty$, and the result follows from the same argument as above. Obviously, the result also remains valid if one replaces the weight $l^s$ by another quantity $w_l \approx l^s$.

## 3.3 Asymptotics for weighted Bessel series

In this section we shall prove Lemma 3.1.2. In view of the well-known asymptotics

$$J_l(r) = \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} \cos\left(r - \frac{(2l+1)\pi}{4}\right) + O(r^{-1})$$

for Bessel functions, it is easy to check that the series

$$\mathcal{J}_{s,m,m'}(r) := \sum_{l=1}^\infty l^{-2s} J_{l+m'}(r)\, J_{l+m}(r)\,. \tag{3.9}$$

is locally uniformly convergent by the standard bound [OLB+10, (10.14.4)]

$$|J_l(r)| \leqslant \frac{r^l}{2^l l!}\,.$$

We are interested in the effect of the parameters $s \in \mathbb{R}$ and $m', m \in \mathbb{Z}$.

In view of the well-known integral representation formula [OLB+10, (10.9.2)] for Bessel functions of integer order,

$$J_l(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\sin x - ilx}\, dx\,,$$

one can write

$$\mathcal{J}_{s,m,m'}(r) = \frac{1}{4\pi^2} \sum_{l=1}^\infty l^{-2s} g_{\lambda_l}(r). \tag{3.10}$$

Here we have set $\lambda_l := l/r$,

$$g_\lambda(r) := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ir\varphi_\lambda(x,y) - i(m'x - my)}\, dx\, dy\,,$$

and the phase function is

$$\varphi_\lambda(x,y) := \lambda(y-x) + \sin x - \sin y \,.$$

Notice that we have used that $J_l$ is real valued, and hence $J_l = \overline{J_l}$.

A straightforward application of the stationary phase formula [Hör15, Theorem 7.7.5] gives the following asymptotic formula for $g_\lambda$. Here and in what follows, we will use the notation

$$f(\lambda) := \sqrt{1-\lambda^2} - \lambda \arccos \lambda \,, \qquad \mu := m + m' \,, \qquad \nu := m - m' \,.$$

Also, we will use the notation $O_p(r^{-k})$ to emphasize that a certain quantity of order $r^{-k}$ is not bounded uniformly with respect to the parameter $p$.

**Lemma 3.3.1.** *Suppose that $\lambda \neq 1$. For $r \gg 1$, one then has*

$$g_\lambda(r) = \frac{4\pi \left[\cos\left(\nu \arccos \lambda\right) + \sin\left(2rf(\lambda) - \mu \arccos \lambda\right)\right]}{r|1-\lambda^2|^{1/2}} + O_\lambda(r^{-2}) \,,$$

*where the error term is not bounded uniformy for large $\lambda$ or for $\lambda$ close to* 1.

*Proof.* For $\lambda \neq 1$, the phase function $\varphi_\lambda(x,y)$ has four critical points

$$\{(x_i,y_i)\}_{i=1}^4 := \{(\pm \arccos \lambda, \pm \arccos \lambda)\}$$

with the same Hessian:

$$\nabla^2 \varphi_\lambda(x_i,y_i) = \begin{pmatrix} \mp\sqrt{1-\lambda^2} & 0 \\ 0 & \pm\sqrt{1-\lambda^2} \end{pmatrix} .$$

The stationary phase method [Hör15, Theorem 7.7.5] then yields

$$\begin{aligned} g_\lambda(r) &= \frac{2\pi}{r} \sum_{i=1}^4 e^{\frac14 i\pi\sigma_i} e^{i(my_i - x_i m')} e^{ir\varphi_\lambda(x_i,y_i)} \frac{1}{|\det \nabla^2 \varphi_\lambda(x_i,y_i)|} + O_\lambda(r^{-2}) \\ &= \frac{4\pi \left[\cos\left(\nu \arccos \lambda\right) + \sin\left(2rf(\lambda) - \mu \arccos \lambda\right)\right]}{r|1-\lambda^2|^{1/2}} + O_\lambda(r^{-2}) \,, \end{aligned}$$

as claimed. In this formula, $\sigma_i$ is the signature of the matrix $\nabla^2 \varphi_\lambda(x_i,y_i)$. □

Therefore, the asymptotic analysis of $g_\lambda(r)$ becomes problematic when $\lambda$ is close to 1 (because in this case the phase function presents degenerate or "almost degenerate" critical points) and when $\lambda$ is large (because the error terms are not uniformly bounded in this case). Consequently, we will fix a small parameter $\delta > 0$ and consider smooth cutoff functions $[0,\infty) \to [0,1]$ such that

$$\chi_{\rm sm}(\lambda) := \begin{cases} 0 & \text{if } \lambda > 1-\delta \,, \\ 1 & \text{if } \lambda < 1-2\delta \,, \end{cases}$$

$$\chi_{\rm lar}(\lambda) := \begin{cases} 0 & \text{if } \lambda < 1+\delta \,, \\ 1 & \text{if } \lambda > 1+2\delta \,, \end{cases}$$

$$\chi_{\rm med}(\lambda) := 1 - \chi_{\rm sm}(\lambda) - \chi_{\rm lar}(\lambda) \,.$$

We can then split $\mathcal{J}_{s,m,m'}(r)$ as

$$\mathcal{J}_{s,m,m'}(r) = \frac{1}{4\pi^2}\,(\mathrm{I}+\mathrm{II}+\mathrm{III})$$

with

$$\mathrm{I} := \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,, \qquad \mathrm{II} := \sum_{l=1}^{\infty} \chi_{\mathrm{med}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,, \qquad \mathrm{III} := \sum_{l=1}^{\infty} \chi_{\mathrm{lar}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r)\,.$$

Note that I only involves frequencies smaller than $(1-\delta)r$, II involves frequencies close to 1 (more precisely, in the interval $(1-2\delta)r < l < (1+2\delta)r$), and III involves frequencies larger than $(1+\delta)r$.

### 3.3.1 The small frequency region

In view of the asymptotic expansion for $g_\lambda(r)$ proved in Lemma 3.3.1, it is natural to consider the closely related quantities

$$\mathrm{I}' := \frac{4\pi}{r} \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, \lambda_l^{-2s} \frac{\cos(\nu \arccos \lambda_l)}{(1-\lambda_l^2)^{1/2}}\,,$$
$$\mathrm{I}'' := \frac{4\pi}{r} \sum_{l=1}^{\infty} \chi_{\mathrm{sm}}(\lambda_l)\, \lambda_l^{-2s} \frac{\sin(2r f(\lambda_l) - \mu \arccos \lambda_l)}{(1-\lambda_l^2)^{1/2}}\,.$$

Lemma 3.3.1 obviously implies

$$\mathrm{I} = r^{-2s}(\mathrm{I}'+\mathrm{I}'') + O_\delta(r^{-2s-1})\,. \tag{3.11}$$

Let us start by analyzing the large $r$ behavior of $\mathrm{I}'$ when $s \leqslant \frac{1}{2}$:

**Lemma 3.3.2.** *For $r \gg 1$ and some $\eta > 0$ depending on $s$,*

$$\mathrm{I}' = \begin{cases} \frac{4\pi^2 2^{2s-1}\Gamma(1-2s)}{\Gamma\left(1-s-\frac{\nu}{2}\right)\Gamma\left(1-s+\frac{\nu}{2}\right)} + O(\delta^{\frac12}) + O_\delta(r^{-\eta}) & \text{if } s<\frac12,\\ 4\pi\cos\left(\frac{\pi\nu}{2}\right)\log r + O_\delta(1) & \text{if } s=\frac12 \text{ and } \nu \text{ is even},\\ 2\pi^2\sin\left(\frac{\pi}{2}|\nu|\right) + O_\delta(r^{-1}) + O(\delta^{\frac12}) & \text{if } s=\frac12 \text{ and } \nu \text{ is odd.}\end{cases}$$

*Proof.* Let us start with the case $s \leqslant 0$. The basic observation here is that, as the function

$$h(\lambda) := 4\pi\, \chi_{\mathrm{sm}}(\lambda)\, \frac{\lambda^{-2s}}{\sqrt{1-\lambda^2}} \cos(\nu \arccos \lambda)$$

is Hölder continuous,

$$\frac{1}{r}\sum_{l=1}^{(1-\delta)r} h(\lambda_l) = \int_0^{1-\delta} h(\lambda)\, d\lambda + O_\delta(r^{-1})$$

by standard results about the convergence of Riemann sums for integrands of bounded variation. If $s \leqslant 0$, the result then follows from the formula

$$\int_0^1 \frac{\lambda^{-2s}\cos(\nu\arccos\lambda)}{\sqrt{1-\lambda^2}}\, d\lambda = \frac{\pi 2^{2s-1}\Gamma(1-2s)}{\Gamma\left(1-s-\frac{\nu}{2}\right)\Gamma\left(1-s+\frac{\nu}{2}\right)} \tag{3.12}$$

and the estimate $\arcsin 1 - \arcsin(1-\delta) = O(\delta^{1/2})$.

For $s \in (0, \frac{1}{2})$, the integrand is an unbounded function in $L^1_{\mathrm{loc}}$, so the argument does not apply. Let us take a small constant $\epsilon$ such that, for simplicity of notation, $\epsilon r$ is an integer, and write

$$\mathrm{I}' = \frac{1}{r}\sum_{l=1}^{\epsilon r-1} h(\lambda_l) + \frac{1}{r}\sum_{l=\epsilon r}^{(1-\delta)r} h(\lambda_l) =: \mathrm{I}'_1 + \mathrm{I}'_2 \,.$$

Obviously, as $|h(\lambda)| \approx \lambda^{-2s}$ for small $\lambda$, and $\int_{l/r}^{(l+1)/r} \lambda^{-2s} d\lambda \approx r^{-1}\lambda_l^{-2s}$, we conclude that

$$\left| \mathrm{I}'_1 - \int_0^{\epsilon} h(\lambda)\, d\lambda \right| \lesssim \epsilon^{2-2s} + r^{-1+2s} \,.$$

To estimate $\mathrm{I}'_2$, we use that

$$\mathrm{I}'_2 - \int_{\epsilon}^{1-\delta} h(\lambda)\, d\lambda = \sum_{l=\epsilon r}^{(1-\delta)r} \int_{(l-1)/r}^{l/r} [h(\lambda_l) - h(\lambda)]\, d\lambda = \frac{1}{r}\sum_{l=\epsilon r}^{(1-\delta)r} \frac{h'(\lambda_l^*)}{r}$$

for some $\lambda_l^* \in (\frac{l-1}{r}, \frac{l}{r})$. Therefore, as $|h'(\lambda)| \lesssim \lambda^{-2s-1}$,

$$\left| \mathrm{I}'_2 - \int_{\epsilon}^{1-\delta} h(\lambda)\, d\lambda \right| \lesssim \frac{\epsilon^{-1-2s}}{r} \,,$$

where the constant in $\lesssim$ depends on $\delta$.

Putting together the estimates for $\mathrm{I}'_1$ and $\mathrm{I}'_2$ with $\epsilon \approx r^{-\frac{1}{2}}$, we obtain

$$\mathrm{I}' = \int_0^{1-\delta} h(\lambda)\, d\lambda + O_\delta(r^{s-\frac{1}{2}}) = \int_0^1 h(\lambda)\, d\lambda + O_\delta(r^{s-\frac{1}{2}}) + O(\delta^{1/2}) \,.$$

Using again the formula (3.12), this proves the lemma when $s \in (0, \frac{1}{2})$.

Let us now pass to the case $s = \frac{1}{2}$. We start by assuming that the integer $\nu$ is odd, so that $\cos\left(\frac{\pi\nu}{2}\right) = 0$. Since

$$\cos(\nu \arccos \lambda_l) = \cos\frac{\pi\nu}{2} + \lambda_l \nu \sin\frac{\pi\nu}{2} + O(\lambda_l^2) \,, \tag{3.13}$$

it turns out that the corresponding integrand is differentiable at $\lambda = 0$ in this case, so the same arguments as in the case $s < 0$ show

$$\sum_{l=1}^{(1-\delta)r} \frac{\chi_{\mathrm{sm}}(\lambda_l)}{\lambda_l r} \frac{4\pi\cos(\nu\arccos\lambda_l)}{(1-\lambda_l^2)^{1/2}} = 4\pi \int_0^{1-\delta} \frac{\chi_{\mathrm{sm}}(\lambda)\cos(\nu\arccos\lambda)}{\lambda\sqrt{1-\lambda^2}}\, d\lambda + O_\delta(r^{-1}) \,.$$

The result then follows from the formula

$$\int_0^1 \frac{\cos(\nu\arccos\lambda)}{\lambda\sqrt{1-\lambda^2}}\, d\lambda = \frac{\pi}{2}\sin\left(\frac{\pi}{2}|\nu|\right) \,.$$

To conclude, consider the case when $s = \frac{1}{2}$ and $\nu$ is even. Obviously, by (3.13),

$$\frac{4\pi}{r}\left|\sum_{l=1}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l)\left(\frac{\cos\left(\nu\arccos\lambda_l\right)}{\lambda_l(1-\lambda_l^2)^{1/2}} - \frac{\cos\frac{\pi\nu}{2}}{\lambda_l}\right)\right| \lesssim \frac{1}{r}\sum_{l=1}^{(1-\delta)r}\lambda_l \lesssim 1\,,$$

where the constant in $\lesssim$ depends on $\delta$. The leading contribution of this sum is therefore given by the harmonic series, which satisfies

$$\sum_{l=1}^{(1-\delta)r}\frac{\chi_{\mathrm{sm}}(\lambda_l)}{r\lambda_l} = \sum_{l=1}^{r/2}\frac{1}{l} + \sum_{l=\frac{r}{2}+1}^{(1-\delta)r}\frac{\chi_{\mathrm{sm}}(\lambda_l)}{l} = \log r + O(1)\,.$$

This completes the proof of the lemma. □

Now we pass to analyzing the contribution of the second term, $\mathrm{I}''$. As this term is somewhat oscillating due to the presence of the large parameter $r$ in the argument of a sine, it makes sense to expect this term should be subdominant.

**Lemma 3.3.3.** *There exists some $\eta > 0$, depending on $s$, such that*

$$\mathrm{I}'' = \begin{cases} O_\delta(r^{-\eta}) & \text{if } s < \frac{1}{2}\,,\\ -4\pi\log 2\,\sin\left(2r - \frac{\pi\mu}{2}\right) + O_\delta(r^{-\eta}) & \text{if } s = \frac{1}{2}\,. \end{cases}$$

*Proof.* We start with the case $s < \frac{1}{2}$. Let $\beta \in (0,1)$ be some constant that we will specify later and write

$$\mathrm{I}'' = \mathrm{Im}\left(\frac{1}{r}\sum_{l=1}^{\lfloor r^\beta\rfloor} h(\lambda_l)\,e^{i2rf(\lambda_l)} + \frac{1}{r}\sum_{l=\lceil r^\beta\rceil}^{(1-\delta)r} h(\lambda_l)\,e^{i2rf(\lambda_l)}\right) =: \mathrm{Im}(\mathrm{I}_1'' + \mathrm{I}_2'')\,,$$

with $h(\lambda) := 4\pi\chi_{\mathrm{sm}}(\lambda)\lambda^{-2s}e^{-i\mu\arccos\lambda}(1-\lambda^2)^{-\frac{1}{2}}$. As $s < \frac{1}{2}$, the first term can be easily estimated as

$$|\mathrm{I}_1''| \lesssim \frac{1}{r}\sum_{l=1}^{\lfloor r^\beta\rfloor}\lambda_l^{-2s} \lesssim r^{-(1-2s)(1-\beta)}\,.$$

By hypothesis, the RHS is $r^{-\eta}$ for some $\eta > 0$.

To estimate $\mathrm{I}_2''$, decompose the interval $(\lceil r^\beta\rceil, (1-\delta)r]$ as the union of $N$ disjoint intervals of the form $(l_n, l_n + \Lambda_n]$. We assume that $l_n$ are integers and that the lengths of the intervals satisfy $\Lambda_n \approx r^\gamma$ for some $\gamma \in (0,\beta)$. This implies that $N \approx r^{1-\gamma}$.

The basic idea is that, with this choice of the scales, one can expect that the function $h$ will be approximately constant in each interval but the phase of the complex exponential will oscillate rapidly. This will lead to cancellations. To make this idea precise, suppose that $\lambda - \lambda_{l_n} \in (0, \Lambda_n/r)$ and write

$$f(\lambda) =: f(\lambda_{l_n}) - (\lambda - \lambda_{l_n})\arccos(\lambda_{l_n}) + R_n(\lambda)\,, \tag{3.14}$$

where the function $R_n(\lambda)$ plays the role of an error term. Differentiating this identity with respect to $\lambda$, and noticing that $f'(\lambda) = -\arccos\lambda$, one immediately obtains that the bound $|R_n'(\lambda)| \lesssim |\lambda - \lambda_n|$ holds uniformly in $n$. As a consequence of this, setting

$L := r(\lambda - \lambda_{l_n})$, one infers that

$$\left| \frac{d}{d\lambda} \left( h(\lambda) e^{i2rR_n(\lambda)} \right) \right| \leqslant |h'(\lambda)| + |h(\lambda) 2rR_n'(\lambda)| \lesssim r^{(\beta-1)(\alpha_1-1)} + r^{(\beta-1)\alpha_0} L$$

where

$$\alpha_0 := \min\{0, -2s\}\,, \qquad \alpha_1 := \min\{1, -2s\}\,.$$

As usual, the constant in $\lesssim$ depends on $\delta$.

By the mean value theorem, observing that $R_n(\lambda_{l_n}) = 0$, one then has from Equation (3.14) that

$$\left| \sum_{l=l_n+1}^{l_n+\Lambda_n} \left( h(\lambda_l) e^{i2rf(\lambda_l)} - h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} e^{i2f'(\lambda_{l_n})L} \right) \right| \lesssim r^{(\beta-1)(\alpha_1-1)+2\gamma-1} + r^{(\beta-1)\alpha_0+3\gamma-1}\,,$$

with $\lesssim$ depending on $\delta$. As the implicit constants are uniform in $n$ and there are $N \approx r^{1-\gamma}$ intervals, this implies

$$\mathrm{I}_2'' = \frac{1}{r} \sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \sum_{L=0}^{\Lambda_n} e^{i2f'(\lambda_{l_n})L} + O_\delta(r^{(\beta-1)(\alpha_1-1)+\gamma-1} + r^{(\beta-1)\alpha_0+2\gamma-1})\,.$$

The leading contribution is therefore

$$\begin{aligned}\frac{1}{r} \sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \sum_{L=0}^{\Lambda_n} e^{i2f'(\lambda_{l_n})L} &= \frac{1}{r} \sum_{n=1}^{N} h(\lambda_{l_n}) e^{i2rf(\lambda_{l_n})} \frac{1 - e^{-2i\arccos(\lambda_{l_n})(r^\gamma+1)}}{1 + e^{-2i\arcsin(\lambda_{l_n})}} \\ &\lesssim r^{(\beta-1)\alpha_0-\gamma}\,,\end{aligned}$$

the constant in $\lesssim$ depending on $\delta$. Note that the denominator is bounded from below because $\lambda < 1-\delta$. Thus, choosing $\gamma \in (0, \frac{1}{2})$ and $\beta$ sufficiently close to 1 (depending on $\gamma$ and $s$), we conclude that

$$|\mathrm{I}_2''| \lesssim r^{-\eta'}$$

for some $\eta' > 0$.

Let us now pass to the case $s = \frac{1}{2}$. Arguing as above, one can pick some $\beta$ close to, but smaller than, 1 such that

$$\sum_{l=\lceil r^\beta \rceil}^{(1-\delta^-)r} \frac{\chi_{\mathrm{sm}}(\lambda_l) \sin\left(2rf(\lambda_l) - \mu \arccos \lambda_l\right)}{l(1-\lambda_l^2)^{1/2}} = O_\delta(r^{-\eta})$$

for some $\eta > 0$. For the sum going from $l = 1$ to $\lfloor r^\beta \rfloor$, we can disregard the $(1 - \lambda_l^2)^{1/2}$ term because

$$\left| \sum_{l=1}^{\lfloor r^\beta \rfloor} \left[ \frac{\sin\left(2rf(\lambda_l) - \mu \arccos \lambda_l\right)}{l(1-\lambda_l^2)^{1/2}} - \frac{\sin\left(2rf(\lambda_l) - \mu \arccos \lambda_l\right)}{l} \right] \right| \lesssim \sum_{l=1}^{\lfloor r^\beta \rfloor} \frac{\lambda_l}{r} \lesssim r^{-2+2\beta}\,.$$

The identity

$$\sin\left(2rf(\lambda_l) - \mu \arccos\lambda_l\right) = \sin\left(2r - \frac{\pi\mu}{2}\right)\cos\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)$$
$$+ \cos\left(2r - \frac{\pi\mu}{2}\right)\sin\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)$$

enables us to write

$$\sum_{l=1}^{\lfloor r^\beta\rfloor} \frac{\sin\left(2rf(\lambda_l) - \mu\arccos\lambda_l\right)}{l} = \sin\left(2r - \frac{\pi\mu}{2}\right)\sum_{l=1}^{\lfloor r^\beta\rfloor}\frac{\cos\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}{l}$$
$$+ \cos\left(2r - \frac{\pi\mu}{2}\right)\sum_{l=1}^{\lfloor r^\beta\rfloor}\frac{\sin\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}{l}.$$

The asymptotic expansions

$$f(\lambda) - 1 = -\frac{\pi\lambda}{2} + O(\lambda^2), \quad \frac{\pi}{2} - \arccos\lambda = \lambda + O(\lambda^2)$$

ensure that

$$2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right) = -\pi l + rO(\lambda^2).$$

The quantity $rO(\lambda^2)$ is of order $r^{2\beta'-1}$ whenever $l < r^{\beta'}$. Fixing some $\beta' \in (0, \frac{1}{2})$, we therefore have

$$\begin{aligned}\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{\cos\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}{l} &= \sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\left(\frac{\cos(\pi l)}{l} + \frac{r^2 O(\lambda_l^4)}{l}\right)\\ &= -\log 2 + O(r^{-\min\{\beta', 2-4\beta'\}}).\end{aligned}$$

Here we have used that

$$\sum_{l=1}^{L}\frac{\cos(\pi l)}{l} = -\log 2 + O(L^{-1}).$$

Similarly,

$$\sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{\sin\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}{l} = \sum_{l=1}^{\lfloor r^{\beta'}\rfloor}\frac{rO(\lambda_l^2)}{l} = O(r^{2\beta'-1}).$$

It only remains to consider the sum from $\lceil r^{\beta'}\rceil$ to $\lfloor r^\beta\rfloor$, where we can also assume that $\chi_{\mathrm{sm}}(\lambda_l) = 1$. To this end, we define the function

$$Q := \sum_{l=\lceil r^{\beta'}\rceil}^{\lfloor r^\beta\rfloor}\frac{e^{i\left(2r(f(\lambda_l)-1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_l\right)\right)}}{l} =: \sum_{l=\lceil r^{\beta'}\rceil}^{\lfloor r^\beta\rfloor}\frac{e^{-i(\pi l + \varphi(\lambda_l, r))}}{l}.$$

To show this sum goes to zero as $r \to \infty$, we are going to exploit the cancellations of consecutive terms. For this, let us define

$$\begin{aligned}\Delta_{2k} &:= \varphi(\lambda_{2k+1}, r) - \varphi(\lambda_{2k}, r) \\ &= 2r(f(\lambda_{2k}) - 1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_{2k}\right) - \left[2r(f(\lambda_{2k+1}) - 1) + \mu\left(\frac{\pi}{2} - \arccos\lambda_{2k+1}\right)\right] - \pi\,.\end{aligned}$$

More explicitly,

$$\Delta_{2k} = 2\sqrt{r^2 - 4k^2} - 2\sqrt{r^2 - (2k+1)^2} - (4k+\mu)\arccos\left(\frac{2k}{r}\right) + (4k+\mu+2)\arccos\left(\frac{2k+1}{r}\right) - \pi.$$

By the mean value theorem, there exists some $\lambda_* \in (2kr^{-1}, (2k+1)r^{-1})$ such that

$$|\Delta_{2k}| \leqslant \left|\pi r - 2\arccos\lambda_* + \frac{\mu}{r}(1-\lambda_*^2)^{-1/2}\right| r^{-1} \lesssim \frac{l}{r}$$

for $\lceil r^{\beta'} \rceil < l < \lfloor r^{\beta} \rfloor$. This enables us to estimate $Q$ as

$$\begin{aligned}|Q| &= \left| \sum_{k=\lceil r^{\beta'}\rceil/2}^{\lfloor r^{\beta}\rfloor/2} e^{i2r(f(\lambda_{2k})-1)+\mu\left(\frac{\pi}{2}-\arccos\lambda_{2k}\right)} \left(\frac{1}{2k} - \frac{e^{-i\Delta_{2k}}}{2k+1}\right)\right| \\ &\lesssim \sum_{k=\lceil r^{\beta'}\rceil/2}^{\lfloor r^{\beta}\rfloor/2} \left(\frac{1}{k^2} + \frac{1}{r}\right) \lesssim r^{-\beta'} + r^{\beta-1}\,.\end{aligned}$$

□

Let us finally consider the case $s > \frac{1}{2}$:

**Lemma 3.3.4.** *If $s > \frac{1}{2}$, there exists some $\eta > 0$ depending on $s$ such that*

$$\mathrm{I} = \frac{1}{\pi r}\zeta(2s)\left(\cos\frac{\pi\nu}{2} - \left(2^{1-2s} - 1\right)\sin\frac{\pi\mu - 4r}{2}\right) + O_\delta(r^{-1-\eta})\,.$$

*Here $\zeta$ is the Riemann's zeta function.*

*Proof.* Let us use again the integral formula for Bessel functions to write

$$J_{l+m'}(r)J_{l+m}(r) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ir(\sin x - \sin y)}\, e^{-i((l+m')x-(l+m)y)}\, dx\, dy\,.$$

Applying the stationary phase argument [Hör15, Theorem 7.7.5] with phase function $\sin x - \sin y$ and amplitude $e^{-i((l+m')x-(l+m)y)}$, one readily obtains the asymptotic expansion

$$J_{l+m'}(r)J_{l+m}(r) = \frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + R_l(r)\,,$$

where the error term satisfies the pointwise bound

$$|R_l(r)| \lesssim \frac{l^4}{r^2}\,.$$

Now, pick some $\beta \in (0, \frac{1}{4})$ and write

$$\mathrm{I} = \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} J_{l+m'}(r) J_{l+m}(r) + \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l) l^{-2s} J_{l+m'}(r) J_{l+m}(r) =: \mathrm{I}_1 + \mathrm{I}_2 \,.$$

Then

$$\begin{aligned} \mathrm{I}_1 &= \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} \left[ \frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + R_l(r) \right] \\ &=: \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} \frac{\cos\left(\frac{1}{2}\pi\nu\right) - \sin\left(\frac{1}{2}\left(2\pi l + \pi\mu - 4r\right)\right)}{\pi r} + \mathcal{R} \,, \end{aligned}$$

where the error term is bounded as

$$|\mathcal{R}| = \left| \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{-2s} R_l(r) \right| \lesssim \frac{1}{r^2} \sum_{l=1}^{\lfloor r^\beta \rfloor} l^{4-2s} \lesssim r^{-2}(1 + r^{\beta(5-2s)}) \,.$$

This decay is smaller than $r^{-1}$ if $\beta < \frac{1}{4}$. Expanding the sine, the above series can be computed in closed form in terms of the zeta function:

$$\mathrm{I}_1 = \frac{1}{\pi r} \zeta(2s) \left[ \cos\left(\frac{1}{2}\pi\nu\right) - \left(2^{1-2s} - 1\right) \sin\left(\frac{1}{2}\left(\pi\mu - 4r\right)\right) \right] + O(r^{-2} + r^{\beta(5-2s)-2}) \,.$$

To control the remaining term, we use that $s > \frac{1}{2}$ and the bound for $g_\lambda$ proved in Lemma 3.3.1 to write

$$|\mathrm{I}_2| \lesssim \left| \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} \chi_{\mathrm{sm}}(\lambda_l)\, l^{-2s} g_{\lambda_l}(r) \right| \lesssim \frac{1}{r} \sum_{l=\lceil r^\beta \rceil}^{(1-\delta)r} l^{-2s} \leqslant \frac{1}{r} \sum_{l=\lceil r^\beta \rceil}^{\infty} l^{-2s} \lesssim r^{-\beta(2s-1)-1} \,.$$

As usual, the constant in $\lesssim$ depends on $\delta$. The lemma then follows. $\square$

### 3.3.2 Intermediate frequency region

Our next goal is to derive bounds for the term

$$\mathrm{II} = \sum_{l=\lceil (1-2\delta)r \rceil}^{\lfloor (1+2\delta)r \rfloor} \chi_{\mathrm{med}}(\lambda_l)\, l^{-2s} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{il(y-x)} e^{ir(\sin x - \sin y)} e^{i(my - m'x)}\, dx\, dy \,.$$

The difficulty here is that one cannot apply the standard stationary phase method as we did above because the critical points of the phase function

$$\varphi_l(x, y) := \lambda(y - x) + \sin x - \sin y$$

are either degenerate or not uniformly non-degenerate. The main result is the following:

**Lemma 3.3.5.** *For any real $s$ and all large enough $r$ (depending on $\delta$),*

$$|\mathrm{II}| \leqslant C\delta^{\frac{1}{2}}\, r^{-2s} \,,$$

*where C is independent of $\delta$.*

*Proof.* Since

$$\varphi_l(x,y) = (1-\lambda)(x-y) - \frac{1}{6}(x^3-y^3) + O(x^5) + O(y^5)\,,$$

when $1-2\delta \leqslant \lambda \leqslant 1+2\delta$ and $\delta \ll 1$, an elementary calculation shows that

$$|\nabla\varphi_l(x,y)| \geqslant c$$

whenever $|x|+|y| > 100\,\delta^{1/2}$, where $c$ is a positive constant that depends on $\delta$. Therefore, take some $\chi(t)$ be a smooth nonnegative function that is equal to 1 for $|t| < 100\,\delta^{1/2}$ and 0 for $|t| > 200\,\delta^{1/2}$. The non-stationary phase lemma then shows that

$$\mathrm{II}' := \sum_{l=\lceil(1-2\delta)r\rceil}^{\lfloor(1+2\delta)r\rfloor} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} \int_{\mathbb{R}^2} e^{il(y-x)} e^{ir(\sin x-\sin y)} e^{i(my-m'x)}\,\chi(x)\,\chi(y)\,dx\,dy$$

coincides with II modulo an exponentially small error. More precisely,

$$\left|\mathrm{II} - r^{-2s}\mathrm{II}'\right| < C_{\delta,N}\,r^{-N}$$

for any $N$ and some constant depending on $N$ and $\delta$.

To estimate $\mathrm{II}'$, let us start by defining $z := y-x$ and writing

$$\mathrm{II}' = \sum_{l=r(1-2\delta)}^{r(1+2\delta)} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} \int_{\mathbb{R}^2} e^{ilz} e^{ir(\sin(y-z)-\sin y)} e^{i((m-m')y+m'z)} \chi(y-z)\,\chi(y)\,dy\,dz\,.$$

A first step is to consider the sum

$$S(r,z) := \frac{1}{r}\sum_{l=r(1-2\delta)}^{r(1+2\delta)} \chi_{\mathrm{med}}(\lambda_l)\,\lambda_l^{-2s} e^{ilz}$$

and to relate it to its continuous counterpart

$$F(r,z) := \int_{-\infty}^{\infty} \chi_{\mathrm{med}}(\lambda)\,\lambda^{-2s} e^{irz\lambda}\,d\lambda\,.$$

Note that it is not a priori obvious that $F(r,z)$ converges to $S(r,z)$ as $r\to\infty$ because, intuitively speaking, the sum is formally obtained by discretizing the integral with a "grid" of length $1/r$, and $r \gg 1$ is precisely the frequency at which the integrand oscillates.

We proceed as follows. Firstly, write

$$S(r,z) - F(r,z) = \sum_{l=r(1-2\delta)}^{r(1+2\delta)} \int_{\lambda_l}^{\lambda_l+\frac{1}{r}} \Big[\lambda_l^{-2s}\chi_{\mathrm{med}}(\lambda_l)\Big(\frac{e^{ir\lambda_l z}}{r} - e^{ir\lambda z}\Big) + \big(\chi_{\mathrm{med}}(\lambda_l)\lambda_l^{-2s} - \chi_{\mathrm{med}}(\lambda)\lambda^{-2s}\big)e^{ir\lambda z}\Big]\,d\lambda$$

and note that
$$\frac{e^{ilz}}{r} - \int_{\lambda_l}^{\lambda_l+\frac{1}{r}} e^{i\lambda rz}\, d\lambda = h(z)\,\frac{e^{ilz}}{r}$$
with
$$h(z) := \frac{ie^{iz}+z-i}{z}\,.$$
The function $h$ is smooth at the origin; in fact, $h(z) = O(z)$. As moreover
$$|\chi_{\mathrm{med}}(\lambda_l)\lambda_l^{-2s} - \chi_{\mathrm{med}}(\lambda)\lambda^{-2s}| \lesssim \frac{\delta^{-1}}{r} \tag{3.15}$$
if $\lambda \in [\lambda_l, \lambda_l + \frac{1}{r}]$ and $|\lambda - 1| < 2\delta$, one obtains that the error
$$R(r,z) := S(r,z) - F(r,z) - h(z)S(r,z)$$
is bounded as
$$|R(r,z)| \leqslant \frac{C}{r}\,,$$
with $C$ a constant independent of $z$ and $\delta$.

Since $z$ will eventually be small, the fact that
$$S(r,z) = \frac{F(r,z) + R(r,z)}{1-h(z)}$$
shows in which sense $S(r,z)$ and $F(r,z)$ are related. The reader can check that, had we argued as in (3.15), we would have obtained an error estimate of the form $Cz$, which is useless for our purposes.

One can thus write
$$\begin{aligned} \mathrm{II}' = r\int_{\mathbb{R}^3} \chi_{\mathrm{med}}(\lambda)\,\lambda^{-2s} e^{ir(\lambda z+\sin(y-z)-\sin y)} e^{i((m-m')y+m'z)} \frac{\chi(y-z)\chi(y)}{1-h(z)}\, d\lambda\, dz\, dy \\ + r\int_{\mathbb{R}^2} e^{ir(\sin(y-z)-\sin y)} e^{i((m-m')y+m'z)} R(r,z) \frac{\chi(y-z)\chi(y)}{1-h(z)}\, dz\, dy \\ =: \mathrm{II}_1' + \mathrm{II}_2'\,. \end{aligned}$$
The bound for $R(r,z)$ and the fact that $\chi(t)$ is supported in $|t| < 200\,\delta^{1/2}$ immediately implies
$$|\mathrm{II}_2'| \leqslant C\delta\,,$$
where the constant does not depend on $\delta$.

To analyze $\mathrm{II}_1'$, one cannot directly apply the stationary phase formula to the integral over $\mathbb{R}^3$ because the critical set of the phase has dimension 1. Instead, let us define
$$H(r,y) := r\int_{\mathbb{R}^2} e^{ir(\lambda z+\sin(y-z))} \chi_{\mathrm{med}}(\lambda)\,\lambda^{-2s} e^{im'z} \frac{\chi(y-z)}{1-h(z)} d\lambda\, dz\,.$$
Then, the phase function $\varphi_y(\lambda,z) := \lambda z + \sin(y-z)$ has a unique critical point in the support of the integrand, $(\lambda^*, z^*) := (\cos y, 0)$, and its Hessian is
$$\nabla^2\varphi_y(\lambda^*,z^*) = \begin{pmatrix} 0 & 1 \\ 1 & -\sin(y) \end{pmatrix}\,.$$

The stationary phase formula [Hör15, Theorem 7.7.6] then ensures that, if $r$ is large enough (depending on $\delta$)

$$|H(r,y)| \leqslant C$$

with a constant independent of $\delta$. Plugging this estimate into $\mathrm{II}_1'$ and using again that $\chi(t)$ is supported in $|t| < 200\,\delta^{1/2}$, one finds

$$|\mathrm{II}_1'| \leqslant \int_{-\infty}^{\infty} \chi(y)\,|H(r,y)|\,dy \leqslant C\delta^{\frac{1}{2}}$$

with a constant independent of $\delta$. Putting all the estimates together, the lemma is proven. □

### 3.3.3 Large frequency region

The last lemma of this section shows that the contribution of the large frequencies is exponentially small:

**Lemma 3.3.6.** *For any $N$, $|\mathrm{III}| \lesssim r^{-N}$ for all large enough $r$ (depending on $\delta$).*

*Proof.* Let us now use $l$ as the large parameter in the formula for $g_{\lambda_l}(r)$, which amounts to writing

$$g_{\lambda_l}(r) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{il\tilde{\varphi}_{\lambda_l}(x,y)} e^{-i(m'x-my)}\,dx\,dy$$

with

$$\tilde{\varphi}_\lambda(x,y) := y - x + \frac{\sin x - \sin y}{\lambda}\,.$$

If $\lambda > 1+\delta$, it is clear that

$$|\nabla\tilde{\varphi}_\lambda(x,y)| \geqslant c_\delta$$

for all $x, y \in [-\pi,\pi]$, where $c_\delta$ is a positive constant that only depends on $\delta$. Therefore, the non-stationary phase lemma [Hör15, Theorem 7.7.1] ensures that $g_{\lambda_l}(r)$ is an exponentially small function of $l$, meaning that for any $N'$ there exists a constant $C$ (depending on $\delta$ and $N'$) such that

$$|g_{\lambda_l}(r)| < C|l|^{-N'}\,.$$

This immediately implies that

$$|\mathrm{III}| \lesssim \sum_{l=(1+\delta)r}^{\infty} l^{-2s}|g_{\lambda_l}(r)| \lesssim r^{-N}$$

for any $N$, as claimed. □

### 3.3.4 Asymptotics for series with derivatives of Bessel functions

The results we have derived above readily yield the asymptotic bounds for weighted sums of Bessel functions that we will crucially need in the next section. Specifically, Lemma 3.1.2 follows immediately by adding the estimates derived in the previous subsections and letting $\delta \to 0^+$. The explicit constants in the lemma are:

$$c^1_{s,\nu} := \frac{2^{2s-1}\Gamma(1-2s)}{\Gamma(1-s-\frac{\nu}{2})\Gamma(1-s+\frac{\nu}{2})}, \qquad c^4 := \frac{\log 2}{\pi},$$

$$c^2_\nu := \pi^{-1}\cos\left(\frac{\pi\nu}{2}\right), \qquad c^5_{s,\nu} := \pi^{-1}\zeta(2s)\cos\left(\frac{\pi\nu}{2}\right),$$

$$c^3_\nu := 2^{-1}\sin\left(\frac{\pi|\nu|}{2}\right), \qquad c^6_{s,\nu} := \pi^{-1}\zeta(2s)(1-2^{1-2s}),$$

$$c^7_\mu := \frac{\pi\mu}{2}.$$

One should observe that, to estimate the expected number of critical points of the random monochromatic wave (3.4), we will also need asymptotic information about series with derivatives of Bessel functions. This follows easily as a byproduct of Lemma 3.1.2 using the well-known recurrence relations

$$J_l'(r) = \frac{J_{l-1}(r) - J_{l+1}(r)}{2}, \qquad J_l''(r) = \frac{J_{l+2}(r) + J_{l-2}(r) - 2J_l(r)}{4}.$$

In the following lengthly corollary of Lemma 3.1.2 we record the asymptotic formulas that we will need later on:

**Corollary 3.3.7.** *The following estimates hold:*

$$\sum_{l=1}^\infty l^{-2s} J_l(r)^2 = \begin{cases} \frac{2^{2s-1}\Gamma(1-2s)r^{-2s}}{\Gamma(1-s)^2} + o(r^{-2s}) & \text{if } s < \frac12, \\ \frac{\log r}{\pi r} + o(r^{-1}) & \text{if } s = \frac12, \\ \frac{\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r + 1\right)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12, \end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l(r)J_l'(r) = \begin{cases} o(r^{-2s}) & \text{if } s < \frac12, \\ O(r^{-1}) & \text{if } s = \frac12, \\ \frac{\left(2^{1-2s}-1\right)\cos(2r)\zeta(2s)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12, \end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l'(r)^2 = \begin{cases} \frac{\Gamma\left(\frac12-s\right)r^{-2s}}{4\sqrt{\pi}\Gamma(2-s)} + o(r^{-2s}) & \text{if } s < \frac12, \\ \frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s = \frac12, \\ \frac{\zeta(2s)\left(1-\left(2^{1-2s}-1\right)\sin 2r\right)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12, \end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l(r)J_l''(r) = \begin{cases} -\frac{\Gamma\left(\frac12-s\right)r^{-2s}}{4\sqrt{\pi}\Gamma(2-s)} + o(r^{-2s}) & \text{if } s < \frac12, \\ -\frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s = \frac12, \\ -\frac{\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r + 1\right)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12, \end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l'(r)J_l''(r) = \begin{cases} o(r^{-2s}) & \text{if } s < \frac12, \\ O(r^{-1}) & \text{if } s = \frac12, \\ -\frac{\left(2^{1-2s}-1\right)\cos(2r)\zeta(2s)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12, \end{cases}$$

$$\sum_{l=1}^\infty l^{-2s} J_l''(r)^2 = \begin{cases} \frac{3\;2^{2s-5}(2-2s)(4-2s)\Gamma(1-2s)r^{-2s}}{\Gamma(3-s)^2} + o(r^{-2s}) & \text{if } s < \frac12, \\ \frac{\log r}{\pi r} + O(r^{-1}) & \text{if } s = \frac12, \\ \frac{\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin(2r) + 1\right)}{\pi r} + o(r^{-1}) & \text{if } s > \frac12. \end{cases}$$

## 3.4 Proof of Theorem 3.1.1

We are now ready to present the proof of the main theorem, which will consist of a number of steps. Recall that we defined the random function $u$ as

$$u := \sum_l a_l \, \sigma_l \, e^{il\theta} \, J_l(r) \,, \qquad \sigma_l := \begin{cases} |l|^{-s} & \text{if } l \neq 0 \,, \\ 0 & \text{if } l = 0 \,. \end{cases} \tag{3.16}$$

It will be apparent from the proof that the argument remains valid for much more general choices of $\sigma_l$, for example of the form (3.6). Of course, the value of the constants $\kappa(s)$, $\tilde{\kappa}_{\frac{3}{2}}$, $\tilde{\kappa}_{\frac{5}{2}}$ one gets depends on the specific choice of $\sigma_l$.

### 3.4.1 A Kac–Rice formula

Our first objective is to derive an explicit, if hard to analyze, Kac–Rice type formula for the expected number of critical points of the Gaussian random function $u$.

In this subsection, we shall denote by

$$Du(r,\theta) := \begin{pmatrix} \partial_\theta u(r,\theta) \\ \partial_r u(r,\theta) \end{pmatrix} , \qquad D^2 u(r,\theta) := \begin{pmatrix} \partial_{\theta\theta} u(r,\theta) & \partial_{r\theta} u(r,\theta) \\ \partial_{r\theta} u(r,\theta) & \partial_{rr} u(r,\theta) \end{pmatrix}$$

the derivative and Hessian of $u$ in polar coordinates. To apply the Kac–Rice expectation formula, let us start by showing that $Du(r,\theta)$ has a non-degenerate distribution:

**Lemma 3.4.1.** *The variance of the Gaussian random variable $Du(r,\theta)$ is*

$$\operatorname{Var}[Du(r,\theta)] = \begin{pmatrix} 4\sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2 & 0 \\ 0 & 4\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2 \end{pmatrix} =: \begin{pmatrix} \widetilde{\Sigma}_{11}(r) & 0 \\ 0 & \widetilde{\Sigma}_{22}(r) \end{pmatrix} .$$

*Proof.* To compute the matrix

$$\operatorname{Var}[Du(r,\theta)] := \mathbb{E}[Du(r,\theta) \otimes Du(r,\theta)] \,,$$

recall the expression (3.16) for $u(r,\theta)$ and take advantage of the fact that $u(r,\theta)$ is real valued to write

$$\mathbb{E}[\partial_r u(r,\theta)^2] = \mathbb{E}[\partial_r u(r,\theta) \, \partial_r \overline{u(r,\theta)}] = \sum_{l\neq 0} \sum_{l'\neq 0} \mathbb{E}(a_l \overline{a_{l'}}) \, |l|^{-s} |l'|^{-s} e^{i(l-l')\theta} J_l'(r) \, J_{l'}'(r) \,.$$

By the definition of the random variables $a_l$,

$$\mathbb{E}(a_l \overline{a_{l'}}) = 2\delta_{l,l'} \,,$$

so one obtains

$$\mathbb{E}[\partial_r u(r,\theta)^2] = 4 \sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2$$

The same argument yields

$$\begin{aligned}
\mathbb{E}[\partial_r u(r,\theta)\,\partial_\theta u(r,\theta)] &= \mathbb{E}[\partial_\theta u(r,\theta)\,\partial_r \overline{u(r,\theta)}] \\
&= \sum_{l\neq 0}\sum_{l'\neq 0} \mathbb{E}(a_l \overline{a_{l'}})\, il|l|^{-s}|l'|^{-s} e^{i(l-l')\theta} J_l(r)\, J'_{l'}(r) \\
&= 2i \sum_{l\neq 0} l|l|^{-2s} J_l(r)\, J'_l(r) = 0
\end{aligned}$$

by parity, and

$$\mathbb{E}[\partial_\theta u(r,\theta)^2] = 4\sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2\,.$$

This easily implies that $\mathrm{Var}[Du(r,\theta)]$ is a strictly positive matrix for all $(r,\theta)$. □

**Remark 3.4.1.** The same computation as above shows that the covariance kernel of the random function (3.16) is

$$K(r,\theta;r',\theta') := \mathbb{E}[u(r,\theta)\,u(r',\theta')] = 4\sum_{l=1}^{\infty} l^{-2s} J_l(r) J_l(r') \cos[l(\theta-\theta')]\,.$$

The covariance kernel is therefore invariant under rotations but, in general, not under translation. An exception to this general fact is the case $s=0$. Indeed, it is well known that the covariance kernel of

$$\widetilde{u} := u + \sqrt{2}a_0\, J_0(r).$$

is $\widetilde{K}(x;x') = 2J_0(|x-x'|)$ by Graf's Addition Theorem. The corresponding spectral measure in this case is the Hausdorff measure on the unit circle. Observe that $\widetilde{u}$ will give the same asymptotics as $u$ for $s=0$ because, as we saw in Lemma 3.1.2, for $s=0$ the series of Bessel functions is asymptotically of order 1 but the term $J_0(r)^2$ decays like $r^{-1}$. By Lemma 3.4.2, their covariances $\Sigma_{ij}$ are then asymptotically equivalent. Note we have chosen to omit the term $l=0$ in $u$ for simplicity, especially when this term contributes to the asymptotic expansion (that is, for $s>\frac{1}{2}$ in Lemma 3.1.2).

**Lemma 3.4.2.** *The expected value of the number of critical points of the random monochromatic wave* (3.4) *is*

$$\mathbb{E}N(\nabla u,R) = \int_0^R \int_{\mathbb{R}^3} \frac{\left|z_1^2\Sigma_{13}(r) - z_2^2\Sigma_{22}(r) + z_3 z_1\sqrt{\Sigma_{11}(r)\Sigma_{33}(r)-\Sigma_{13}(r)^2}\right|}{(2\pi)^{\frac{3}{2}}\sqrt{\widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r)}}\, e^{-\frac{1}{2}|z|^2}\,dz\,dr\,,$$

*where*

$$\Sigma_{11}(r) := 4\sum_{l=1}^{\infty} l^{4-2s} J_l(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r)\right)^2}{\sum_{l=0}^{\infty} l^{-2s} J_l'(r)^2},$$

$$\Sigma_{13}(r) := 4\sum_{l=1}^{\infty} (-1) l^{2-2s} J_l(r) J_l''(r) + \frac{4\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r) \sum_{l=1}^{\infty} l^{-2s} J_l'(r) J_l''(r)}{\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2},$$

$$\Sigma_{22}(r) := 4\sum_{l=1}^{\infty} l^{2-2s} J_l'(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{2-2s} J_l(r) J_l'(r)\right)^2}{\sum_{l=1}^{\infty} l^{2-2s} J_l(r)^2},$$

$$\Sigma_{33}(r) := 4\sum_{l=1}^{\infty} l^{-2s} J_l''(r)^2 - \frac{4\left(\sum_{l=1}^{\infty} l^{-2s} J_l'(r) J_l''(r)\right)^2}{\sum_{l=1}^{\infty} l^{-2s} J_l'(r)^2}.$$

*Proof.* As $Du(r,\theta)$ is a non-degenerate Gaussian random variable by Lemma 3.4.1, the Kac–Rice integral formula in polar coordinates [AW09, Proposition 6.6] ensures that

$$\mathbb{E}\left(N(\nabla u, R)\right) = \int_{B(R)} \mathbb{E}\left\{|\det D^2 u(r,\theta)| \,\big|\, Du(r,\theta) = 0\right\} \rho_{Du(r,\theta)}(0)\, dr\, d\theta \tag{3.17}$$

where $\rho_{Du(r,\theta)} : \mathbb{R}^2 \to [0,\infty)$ denotes the probability distribution function of the $\mathbb{R}^2$-valued random variable $Du(r,\theta)$.

Next, let us reduce the computation of the conditional expectation to that of an ordinary expectation by introducing a new random variable $\zeta(r,\theta)$. Just like $D^2u(r,\theta)$, $\zeta(r,\theta)$ will take values in the space of $2\times 2$ symmetric matrices, which we shall henceforth identify with $\mathbb{R}^3$ by labeling the matrix components of a symmetric matrix as

$$\zeta =: \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix}. \tag{3.18}$$

Specifically, let us set

$$\zeta(r,\theta) := D^2u(r,\theta) - B(r,\theta)Du(r,\theta), \tag{3.19}$$

where the linear operator $B(r,\theta)$ (which we can regard as a $3\times 2$ matrix after identifying $D^2u(r,\theta)$ with a 3-component vector) is chosen so that the covariance matrix of $Du(r,\theta)$ and $\zeta(r,\theta)$ is 0:

$$B(r,\theta) := \mathbb{E}(D^2u(r,\theta)\otimes Du(r,\theta))\left[\mathbb{E}(Du(r,\theta)\otimes Du(r,\theta))\right]^{-1}$$

Indeed, one can plug (3.19) in the formula for $\mathbb{E}(\zeta(r,\theta)\otimes Du(r,\theta))$ and check that

$$\mathbb{E}(\zeta(r,\theta)\otimes Du(r,\theta)) = 0.$$

As $Du(r,\theta)$ and $\zeta(r,\theta)$ are jointly a Gaussian vector with zero mean, this condition ensures that they are independent random variables. This enables us to write the above conditional expectation as

$$\begin{aligned}\mathbb{E}\left\{|\det D^2u(r,\theta)| \,\big|\, Du(r,\theta)=0\right\} &= \mathbb{E}\left\{|\det[\zeta(r,\theta) + B(r,\theta)Du(r,\theta)]| \,\big|\, Du(r,\theta)=0\right\} \\ &= \mathbb{E}|\det\zeta(r,\theta)|.\end{aligned}$$

Let now us compute the covariance matrix of $\zeta(r,\theta)$. Since the variance matrix of $Du(r,\theta)$ is independent of $\theta$, let us simply write $\operatorname{Var} Du(r)$, and similarly with other rotation-invariant quantities. One then has

$$\operatorname{Var}\zeta(r) = \operatorname{Var} D^2u(r) - \operatorname{Cov}(D^2u, Du)(r)\cdot \operatorname{Var} Du(r)^{-1}\cdot \operatorname{Cov}(D^2u, Du)(r)^\top \quad (3.20)$$

Arguing as in Lemma 3.4.1 and using that we have identified $D^2u(r,\theta)$ with a 3-component vector, one finds that

$$\operatorname{Var} D^2u(r) := \mathbb{E}[D^2u(r,\theta)\otimes D^2u(r,\theta)]$$

is given by the $3\times 3$ matrix

$$\operatorname{Var} D^2u(r) = \begin{pmatrix} 4\sum_{l=1}^\infty l^{4-2s}J_l(r)^2 & 0 & -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l''(r) \\ 0 & 4\sum_{l=1}^\infty l^{2-2s}J_l'(r)^2 & 0 \\ -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l''(r) & 0 & 4\sum_{l=1}^\infty l^{-2s}J_l''(r)^2 \end{pmatrix}.$$

Similarly,

$$\operatorname{Cov}(D^2u, Du)(r) = \begin{pmatrix} 0 & -4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l'(r) \\ 4\sum_{l=1}^\infty l^{2-2s}J_l(r)J_l'(r) & 0 \\ 0 & 4\sum_{l=1}^\infty l^{2-2s}J_l'(r)J_l''(r) \end{pmatrix} \quad (3.21)$$

Combining these formulas, we derive that

$$\Sigma(r) := \operatorname{Var}\zeta(r,\theta) = \begin{pmatrix} \Sigma_{11}(r) & 0 & \Sigma_{13}(r) \\ 0 & \Sigma_{22}(r) & 0 \\ \Sigma_{13}(r) & 0 & \Sigma_{33}(r) \end{pmatrix}, \quad (3.22)$$

where $\Sigma_{jk}(r)$ are defined as in the statement of the lemma.

Let us now consider the Cholesky decomposition of this matrix:

$$\Sigma(r) = M(r)^\top M(r),$$

where the matrix $M(r)$ is given by

$$M(r) := \begin{pmatrix} \sqrt{\Sigma_{11}(r)} & 0 & \frac{\Sigma_{13}(r)}{\sqrt{\Sigma_{11}(r)}} \\ 0 & \sqrt{\Sigma_{22}(r)} & 0 \\ 0 & 0 & \sqrt{\Sigma_{33}(r) - \frac{\Sigma_{13}(r)^2}{\Sigma_{11}(r)}} \end{pmatrix}.$$

As the matrix $\Sigma(r)$ is positive definite and $\zeta(r,\theta)$ is a Gaussian random variable with zero mean and variance $\Sigma(r)$, one then infers that the 3-component random variable

$$Z(r,\theta) := \zeta(r,\theta)^\top M(r)^{-1}$$

is Gaussian, has zero mean and its variance matrix is the identity. It is thus straightforward that

$$\begin{aligned}\mathbb{E}|\det\zeta(r,\theta)| &= \int_{\mathbb{R}^3} |y_1y_3 - y_2^2|\,\rho_{\zeta(r,\theta)}(y)\,dy \\ &= \int_{\mathbb{R}^3} \left| z_1^2\Sigma_{13}(r) - z_2^2\Sigma_{22}(r) + z_3z_1\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2}\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{\frac{3}{2}}}\,dz,\end{aligned}$$

where

$$\rho_{\zeta(r,\theta)}(y) := \frac{\exp\Big(-\frac{1}{2}y\cdot\Sigma^{-1}y\Big)}{(2\pi)^{3/2}(\det\Sigma(r))^{1/2}}$$

is the probability density distribution of the random variable $\zeta(r,\theta)$ and we have used the change of variables

$$y_1 =: \sqrt{\Sigma_{11}(r)}z_1\,, \qquad y_2 =: \sqrt{\Sigma_{22}(r)}z_2\,, \qquad y_3 =: \frac{\Sigma_{13}(r)}{\sqrt{\Sigma_{11}(r)}}z_1 + \sqrt{\Sigma_{33}(r) - \frac{\Sigma_{13}(r)^2}{\Sigma_{11}(r)}}z_3\,.$$

and the fact that the Jacobian determinant is $\det M(r) = (\det\Sigma(r))^{\frac{1}{2}}$. The lemma follows using that the probability density function of the Gaussian random variable $Du(r,\theta)$ is

$$\rho_{Du(r,\theta)}(0) = \frac{1}{2\pi\sqrt{\widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r)}} \tag{3.23}$$

as a consequence of the formula for $\operatorname{Var} Du(r,\theta)$ computed in Lemma 3.4.1 and of the fact that the density function of an $\mathbb{R}^k$-valued Gaussian random variable $Y$ with zero mean and variance matrix $\Sigma$ is

$$\rho_Y(y) := (2\pi)^{-\frac{k}{2}}(\det\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}y\cdot\Sigma^{-1}y}\,.$$

□

### 3.4.2 Some technical lemmas

In the next subsections, we will discuss the behavior of the formula for the expected number of critical points that we have computed in Lemma 3.4.2 above. The analysis will strongly depend on the value of the parameter $s$. In the computations, we will use several technical lemmas repeatedly, often without further mention.

**Lemma 3.4.3.** *Given constants of the form $a_{jk}(r) = \tilde{a}_{jk}(r) + \epsilon_{jk}(r)$, with $1 \leqslant j,k \leqslant m$,*

$$\int_{\mathbb{R}^m}\Big|\sum_{1\leqslant j,k\leqslant m} a_{jk}(r)z_jz_k\Big|\,e^{-\frac{1}{2}|z|^2}\,dz = \int_{\mathbb{R}^m}\Big|\sum_{1\leqslant j,k\leqslant m} \tilde{a}_{jk}(r)z_jz_k\Big|\,e^{-\frac{1}{2}|z|^2}\,dz + O\Big(\max_{1\leqslant j,k\leqslant m}|\epsilon_{jk}(r)|\Big)\,.$$

*Proof.* It stems from the elementary estimate

$$\left|\Big|\sum_{1\leqslant j,k\leqslant m} a_{jk}(r)z_jz_k\Big| - \Big|\sum_{1\leqslant j,k\leqslant m} \tilde{a}_{jk}(r)z_jz_k\Big|\right| \lesssim |z|^2\max_{1\leqslant j,k\leqslant m}|\epsilon_{jk}(r)|\,.$$

□

**Lemma 3.4.4.** *Let $q : [1,\infty) \to (0,\infty)$ be a continuous function with $\int_1^\infty q(r)\,dr = \infty$. Then, for $r \gg 1$ and any fixed $r_0$,*

$$\int_{r_0}^r o(q(r'))\,dr' = o\left(\int_{r_0}^r q(r')\,dr'\right)\,.$$

*Proof.* Consider any $\epsilon > 0$ and assume, without any loss of generality, that $o(q(r')) \geqslant 0$. By definition, there is some $R_\epsilon$ such that $o(q(r)) \leqslant \epsilon q(r)$ for all $r > R_\epsilon$. Now set

$Q(r) := \int_{r_0}^{r} q(r')\,dr'$ and write

$$\frac{\int_{r_0}^{r} o(q(r'))\,dr'}{Q(r)} = \frac{\int_{r_0}^{R_\varepsilon} o(q(r'))\,dr'}{Q(r)} + \frac{\int_{R_\varepsilon}^{r} o(q(r'))\,dr'}{Q(r)}$$
$$\leqslant \frac{C_\epsilon}{Q(r)} + \frac{\epsilon \int_{R_\varepsilon}^{r} q(r')\,dr'}{Q(r)} = o(1) + \epsilon$$

as $r \to \infty$, since $Q(r) \to \infty$. Letting $\epsilon \to 0$, the result follows. □

The following lemma will be very useful in the analysis of the asymptotic behavior of the number of critical points of $u$:

**Lemma 3.4.5.** *Consider a positive smooth $\pi$-periodic function $P$ and constants $a \geqslant 0$ and $b \in \mathbb{R}$. If $a = 0$, we also assume that $b \geqslant 0$. Then, for $R \gg 1$,*

$$\int_\pi^R r^a (\log r)^b\, P(r)\,dr \sim \frac{R^{a+1}(\log R)^b}{\pi(a+1)} \int_0^\pi P(r)\,dr\,.$$

*Proof.* Let us define $J := \lfloor R/\pi \rfloor$ and write $R = J\pi + R_1$, with $0 \leqslant R_1 < \pi$. We can then write

$$\int_\pi^R r^a (\log r)^b\, P(r)\,dr = \sum_{j=1}^{J-1} \int_{\pi j}^{\pi(j+1)} r^a (\log r)^b\, P(r)\,dr + \int_{\pi J}^{\pi J + R_1} r^a (\log r)^b\, P(r)\,dr\,.$$

The second term is obviously bounded as

$$\left| \int_{\pi J}^{\pi J + R_1} r^a (\log r)^b\, P(r)\,dr \right| \lesssim R^a (\log R)^b$$

To estimate the first term, let

$$B := \int_0^\pi P(r)\,dr\,.$$

As the function $r^a(\log r)^b$ is increasing for large enough $r$, we have

$$B(\pi j)^a [\log(\pi j)]^b \leqslant \int_{\pi j}^{\pi(j+1)} r^a (\log r)^b\, P(r)\,dr \leqslant B[\pi(j+1)]^a [\log(\pi(j+1))]^b$$

if $j$ is larger that a certain integer $J_{a,b}$. With $\eta = 0, 1$, we can use the following asymptotic formula, which is an easy consequence of the Euler-Maclaurin formula,

$$\sum_{j=J_{a,b}}^{J-1} [\pi(j+\eta)]^a [\log(\pi(j+\eta))]^b \sim \frac{\pi^a (J+\eta-1)^{a+1} [\log(\pi(J+\eta-1))]^b}{a+1} \sim \frac{R^{a+1}(\log R)^b}{\pi(a+1)}$$

to derive the formula of the statement. Here we have used that $\pi J = R + O(1)$ and that the integral over $r \in [\pi, \pi J_{a,b}]$ is obviously bounded independently of $R$. □

Before discussing the behavior of $\mathbb{E}N(\nabla u, R)$ in the different regularity regimes, one should note that the integral appearing in Lemma 3.4.2 is remarkably hard to analyze. We will be able to obtain much more convenient integral representations by means of the following lemma:

**Lemma 3.4.6.** *Let $A, B, C$ be real constants. Then*

$$\int_{\mathbb{R}^3} \left|Az_1^2 + Bz_2^2 + 2Cz_1z_3\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\,dz = \frac{2}{\pi}\int_0^\infty \frac{1-a(t)\cos\frac{1}{2}\Phi(t)}{t^2}\,dt\,,$$

*where*

$$\Phi(t) := \arg\left((1-2iBt)\left(1-2iAt+4C^2t^2\right)\right),$$
$$a(t) := (1+4B^2t^2)^{-\frac{1}{4}}\left[(1+4C^2t^2)^2+4A^2t^2\right]^{-\frac{1}{4}}.$$

*Proof.* Defining the matrix

$$M := \begin{pmatrix} A & 0 & C \\ 0 & B & 0 \\ C & 0 & 0 \end{pmatrix},$$

one can write the above integral as

$$Q := \int_{\mathbb{R}^3} \left|Az_1^2 + Bz_2^2 + 2Cz_1z_3\right| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\,dz = \int_{\mathbb{R}^3} |z\cdot Mz| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\,dz\,.$$

The results about Gaussian integrals involving an absolute value function derived in [LW09, Theorem 2.1] therefore ensure that

$$Q = \frac{2}{\pi}\int_0^\infty \left[1 - \frac{\det(I-2itM)^{-\frac{1}{2}} + \det(I+2itM)^{-\frac{1}{2}}}{2}\right]\frac{dt}{t^2}\,.$$

Now a straightforward computation yields the formula in the statement. □

### 3.4.3 The case $s < \frac{1}{2}$

We are ready to compute the asymptotics for the number of critical points when $s < \frac{1}{2}$:

**Lemma 3.4.7.** *If $s < \frac{1}{2}$,*

$$\lim_{R\to\infty} \frac{\mathbb{E}N(\nabla u, R)}{R^2} = \kappa(s)$$

*with*

$$\kappa(s) := \frac{1}{2}\frac{1}{\sqrt{2-s}}\int_{\mathbb{R}^3}\left|\sqrt{\frac{1-2s}{8-4s}}\,(z_1^2-z_2^2)+z_1z_3\right|\frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\,dz\,. \tag{3.24}$$

*Proof.* Let us compute the matrix $\Sigma(r)$. From Equation (3.22) and the asymptotic formulas for sums of Bessel functions recorded in Corollary 3.3.7, it follows that

$$\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)\,,$$

where the leading contribution is

$$\Sigma^0(r) := \begin{pmatrix} \frac{2^{2s-3}\Gamma(5-2s)r^{4-2s}}{\Gamma(3-s)^2} & 0 & \frac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} \\ 0 & \frac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} & 0 \\ \frac{\Gamma\left(\frac{3}{2}-s\right)r^{2-2s}}{\sqrt{\pi}\Gamma(3-s)} & 0 & \frac{3\Gamma\left(\frac{1}{2}-s\right)r^{-2s}}{2\sqrt{\pi}\Gamma(3-s)} \end{pmatrix}$$

and the error is bounded as

$$R_{jk}(r) = o(1)\Sigma^0_{jk}(r)\,.$$

Here and in what follows, $o(1)$ denotes a quantity that tends to zero as $r \to \infty$.

Let us define

$$I(r,z) := \left| z_1^2\Sigma_{13}(r) - z_2^2\Sigma_{22}(r) + z_3 z_1\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2}\right| \tag{3.25}$$

and note that, by the formula for $\Sigma(r)$ and the asymptotics for weighted sums of Bessel functions presented in Corollary 3.3.7,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim r^{2-2s}\pi^{-1/4}2^{s-\frac{1}{2}}(2-s)\left(\frac{\Gamma\left(\frac{1}{2}-s\right)\Gamma(3-2s)}{\Gamma(3-s)^3}\right)^{1/2}.$$

Likewise, the quantity

$$\sigma(r) := \widetilde{\Sigma}_{11}(r)\widetilde{\Sigma}_{22}(r) \tag{3.26}$$

satisfies the asymptotic bound

$$\sigma(r) \sim \frac{2\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{3}{2}-s\right)}{\pi\Gamma(2-s)^2}r^{2-4s}\,.$$

Finally, the integral

$$\mathcal{I}(r) := \frac{1}{2\pi\sqrt{\sigma(r)}}\int_{\mathbb{R}^3} I(z,r)\frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}dz \tag{3.27}$$

can be then estimated, as a consequence of Lemmas 3.4.3 and 3.4.6 and of the preceding asymptotic bounds, as

$$\mathcal{I}(r) \sim \frac{\kappa(s)}{\pi}r\,,$$

where $\kappa(s)$ is defined as in the statement. Thus, the integral formula in Lemma 3.4.2 ensures that

$$\mathbb{E}N(\nabla u, R) \sim 2\int_0^R \kappa(s) r\,dr = \kappa(s)\,R^2\,.$$

□

In the next lemma, we analyze the behavior of the positive constant $\kappa(s)$ (which is written simply as $\kappa(s)$ in the statement of Theorem 3.1.1), for $s < \frac{1}{2}$. The key idea is to obtain an easier characterization of this constant as a one-dimensional integral.

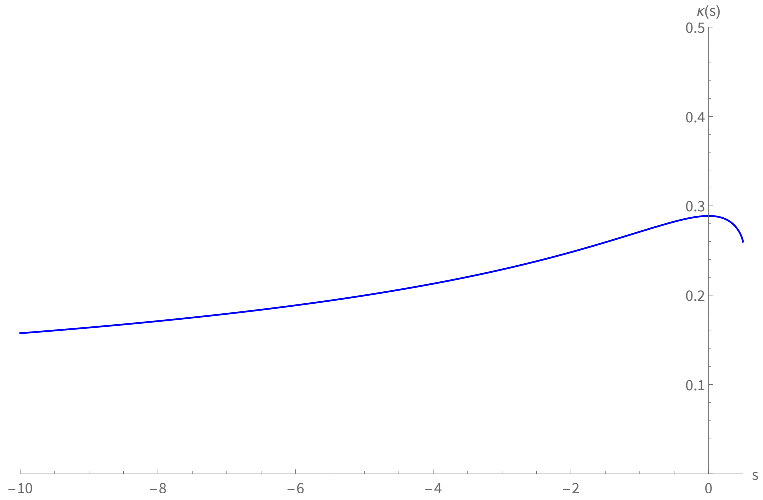



FIGURE 3.2: $\kappa(s)$ for $s < \frac{1}{2}$.

Interestingly, the global maximum of $\kappa(s)$ is attained at $s = 0$, that is, in the classical case of random waves with a translation-invariant covariance kernel. In Figure 3.2 we have plotted $\kappa(s)$ for the first region of $s < 1/2$ using the next lemma.

**Lemma 3.4.8.** *The function $\kappa(s)$ is smooth, strictly increasing on $s \in (-\infty, 0)$, and strictly decreasing on $(0, \frac{1}{2})$. Furthermore,*

$$\lim_{s \to \frac{1}{2}^-} \kappa(s) = \sqrt{\frac{2}{3}}\frac{1}{\pi}, \qquad \lim_{s \to -\infty} \kappa(s) = 0\,.$$

*Proof.* The limiting values can be computed directly from the formula for $\kappa(s)$. Indeed, the (somewhat surprising) fact that $\kappa(s) \to 0$ as $s \to -\infty$ is obvious in view of Equation (3.24), and as is the limit

$$\lim_{s \to \frac{1}{2}^-} \kappa(s) = \int_{\mathbb{R}^3} \frac{|z_1 z_3|}{\sqrt{6}} \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\, dz = \sqrt{\frac{2}{3}}\frac{1}{\pi}\,.$$

To analyze the behavior of $\kappa(s)$ for intermediate values of $s$, we use Lemma 3.4.6 to rewrite (3.24) as

$$\kappa(s) = \frac{2}{\pi} \int_0^\infty \frac{1 - a(s,t)\cos\frac{1}{2}\Phi(s,t)}{t^2}\, dt$$

with

$$a(s,t) := \frac{\sqrt{2}(4-2s)}{\left[(1-2s)t^6 + (8(2-s)^2 + 6(1-s)t^2)^2\right]^{1/4}}\,,$$

$$\Phi(s,t) := \arg\left(4 + \frac{2t^2\left(-6s + i\sqrt{1-2s}t + 6\right)}{(4-2s)^2}\right)\,.$$

Note that

$$\partial_s a(s,t) = 4s\frac{3t^2\left(16(2-s)^2+t^4+12(1-s)t^2\right)}{2\sqrt{2}\left((1-2s)t^6+\left(8(2-s)^2+6(1-s)t^2\right)^2\right)^{5/4}},$$

$$\partial_s \tan\Phi(s,t) = -4s\frac{3t^3\left(-4s+t^2+8\right)}{2\sqrt{1-2s}\left(8(2-s)^2+6(1-s)t^2\right)^2}$$

because

$$\Phi(s,t) = \arctan\left(\frac{\sqrt{1-2s}t^3}{8(2-s)^2+6(1-s)t^2}\right) = \arctan\tan\Phi(s,t).$$

Using that the polynomials appearing on the numerators are all positive for $t>0$ and $s<\frac{1}{2}$, it follows that $\kappa'(s)/s<0$ for all $s\in(-\infty,0)\cup(0,\frac{1}{2})$. The result then follows. $\square$

**Remark 3.4.2.** In the case $s=0$, where $\kappa(s)$ attains its maximum, we recover the well-known asymptotic formula (see Appendix 3.B) for the expected number of critical points:

$$\kappa(0) = \int_{\mathbb{R}^3}\frac{\left|z_1^2+2\sqrt{2}z_3z_1-z_2^2\right|}{8}\frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}dz = \frac{1}{2\sqrt{3}} = 0{,}2886\ldots$$

where we have used that for $s=0$ the integral above becomes

$$\frac{2}{\pi}\int_0^\infty \frac{1-\frac{2}{\sqrt{-\frac{it^3}{2}+3t^2+16}}-\frac{2}{\sqrt{\frac{it^3}{2}+3t^2+16}}}{t^2}dt = \frac{1}{2\sqrt{3}}.$$

### 3.4.4 The case $s=\frac{1}{2}$

We shall next show that, in spite of the appearance of logarithmic terms in the formulas, the asymptotic behavior in the case $s=\frac{1}{2}$ coincides with the limit as $s\to\frac{1}{2}^+$ of the formula derived in Lemma 3.4.7.

**Lemma 3.4.9.** *For $s=\frac{1}{2}$,*

$$\mathbb{E}N(\nabla u,R)\sim\sqrt{\frac{2}{3}}\frac{1}{\pi}R^2.$$

*Proof.* From Equation (3.22) and Corollary 3.3.7, we infer that in the case $s=\frac{1}{2}$, we can write

$$\Sigma(r) = \Sigma^0(r)+\mathcal{R}(r)$$

where

$$\Sigma^0(r) = \begin{pmatrix} \frac{8r^3}{3} & 0 & \frac{4r}{3} \\ 0 & \frac{4r}{3} & 0 \\ \frac{4r}{3} & 0 & \frac{4\log r}{r} \end{pmatrix}$$

and the error is bounded as $\mathcal{R}_{ij}(r) = \Sigma^0_{ij}(r)\,o(1)$. Therefore,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \frac{4}{3\pi} r\sqrt{6\log r}\,.$$

Likewise, the function $\sigma(r)$ defined in (3.26) satisfies

$$\sigma(r) \sim \frac{16\log r}{\pi^2}.$$

Plugging these formulas in (3.27), we obtain

$$\mathcal{I}(r) \sim \frac{r\int_{\mathbb{R}^3} |z_1 z_3| e^{-\frac{1}{2}|z|^2} dz}{\sqrt{6}\pi(2\pi)^{3/2}} = \sqrt{\frac{2}{3}}\frac{r}{\pi^2}\,.$$

□

### 3.4.5 The case $\frac{1}{2} < s < \frac{3}{2}$

We shall next show that, in the regime $\frac{1}{2} < s < \frac{3}{2}$, the expected number of critical points contained in a large disk also grows like the area. The associated proportionality constant, which we denote by $\kappa(s)$, turns out to be smooth on $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$ but only continuous at $s = \frac{1}{2}$.

**Lemma 3.4.10.** *For $\frac{1}{2} < s < \frac{3}{2}$, then $\mathbb{E}N(\nabla u, R) \sim \kappa(s) R^2$ with*

$$\kappa(s) := \frac{1}{\pi}\sqrt{\frac{3-2s}{4-2s}}\,.$$

*Proof.* By Equation (3.22) and Corollary 3.3.7, $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \begin{pmatrix} \dfrac{2^{2s-3} r^{4-2s}\Gamma(5-2s)}{\Gamma(3-s)^2} & 0 & \dfrac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} \\ 0 & \dfrac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} & 0 \\ \dfrac{r^{2-2s}\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\Gamma(3-s)} & 0 & \dfrac{4^{2-s}\left(4^s-1\right)\zeta(2s)}{\pi r\left(\left(4^s-2\right)\sin(2r)+4^s\right)} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r)\,o(1)$. Therefore, as $4-4s < 3-2s$,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \sqrt{\frac{2}{\pi}}\sqrt{\frac{\left(4^s-1\right) r^{3-2s}\zeta(2s)\Gamma(5-2s)}{\Gamma(3-s)^2\left(\left(4^s-2\right)\sin(2r)+4^s\right)}}\,.$$

Similarly, and using the same notation as in the last two subsections,

$$\sigma(r) \sim \frac{4r^{1-2s}\zeta(2s)\Gamma(3-2s)\left(\left(4^s-2\right)\sin(2r)+4^s\right)}{\pi\Gamma(2-s)^2}\,.$$

One can then plug these formulas in (3.27) to find

$$\mathcal{I}(r) \sim \frac{r}{\pi\left(1+\left(1-2^{1-2s}\right)\sin 2r\right)}\sqrt{\frac{2^{-2s}\left(1-2^{-2s}\right)(3-2s)}{(4-2s)}}\int_{\mathbb{R}^3} |z_1 z_3| \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}} dz\,.$$

As $2^{1-2s} < 1$, this immediately implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4}{\pi}\sqrt{\frac{2^{-2s}\,(1-2^{-2s})\,(3-2s)}{(4-2s)}}\int_0^R \frac{r}{1+(1-2^{1-2s})\sin 2r}\,dr\,.$$

As

$$\int_0^\pi \frac{1}{1+b\sin 2r}\,dr = \frac{\pi}{\sqrt{1-b^2}} \tag{3.28}$$

for all $|b| < 1$, the formula of the statement now follows using Lemma 3.4.5. $\square$

**Remark 3.4.3.** It follows from Lemmas 3.4.7, 3.4.9 and 3.4.10 that $\kappa(s) \in C^\infty((-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}])$, and that $\kappa(s)$ is Lipschitz at $s = \frac{1}{2}$ but not $C^1$. It also follows that

$$\lim_{s\to-\infty}\kappa(s) = \lim_{s\to\frac{3}{2}^-}\kappa(s) = 0\,.$$

### 3.4.6 The case $s = \frac{3}{2}$

Here we shall see that the expected number of critical points contained in a ball of large radius does not grow like the area of the ball any longer:

**Lemma 3.4.11.** *If $s = \frac{3}{2}$,*

$$\mathbb{E}N(\nabla u, R) \sim \frac{1}{\pi}\frac{R^2}{\sqrt{\log R}}\,.$$

*Proof.* The argument is essentially as before. Using Corollary 3.3.7 and Equation (3.22), one can write $\Sigma(r) = \sigma^0(r) + \mathcal{R}(r)$, with

$$\Sigma^0(r) := \frac{1}{\pi}\begin{pmatrix} 4r & 0 & \frac{4\log r}{r} \\ 0 & \frac{4\log r}{r} & 0 \\ \frac{4\log r}{r} & 0 & \frac{7\zeta(3)}{4r+3r\sin 2r} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r)\,o(r^0)$. Hence, keeping track of the errors using Lemmas 3.4.3-3.4.4 as before,

$$\sqrt{\Sigma_{11}(r)\Sigma_{33}(r) - \Sigma_{13}(r)^2} \sim \frac{2}{\pi}\sqrt{\frac{7\zeta(3)}{3\sin 2r+4}}\,,$$
$$\sigma(r) \sim \frac{4\zeta(3)\log r(3\sin 2r+4)}{\pi^2 r^2}\,.$$

This readily implies

$$\mathcal{I}(r) \sim \frac{r}{\sqrt{\log r}}\frac{\sqrt{7}}{\pi^2\,(3\sin 2r+4)}\,,$$

so Lemma 3.4.2 ensures that the expected number of critical points satisfies

$$\mathbb{E}N(\nabla u, R) \sim \frac{2\sqrt{7}}{\pi^2}\int_\pi^R \frac{1}{4+3\sin 2r}\frac{r}{\sqrt{\log r}}dr\,.$$

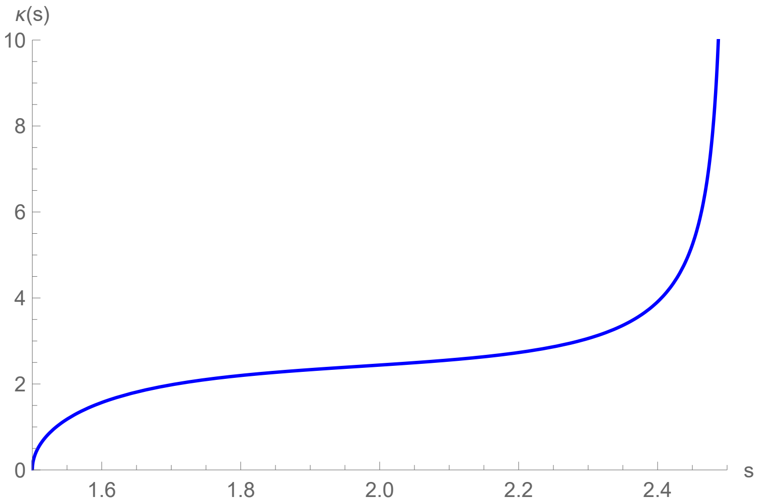



FIGURE 3.3: $\kappa(s)$ for $s \in \left(\frac{3}{2}, \frac{5}{2}\right)$.

The asymptotic behavior of this integral is

$$\int_{\pi}^{R} \frac{1}{4+3\sin 2r} \frac{r}{\sqrt{\log r}} dr \sim \frac{R^2}{2\pi\sqrt{\log R}} \int_0^{\pi} \frac{1}{4+3\sin 2r} dr = \frac{R^2}{2\sqrt{7\log R}} .$$

by Lemma 3.4.5, so the result follows. □

### 3.4.7 The case $\frac{3}{2} < s < \frac{5}{2}$

The analysis of the large $R$ asymptotics presents no new difficulties:

**Lemma 3.4.12.** *For $\frac{3}{2} < s < \frac{5}{2}$, $\mathbb{E}N(\nabla u, R) \sim \kappa(s) R^{\frac{7}{2}-s}$ with*

$$\kappa(s) := -\frac{2^{2s+\frac{1}{2}} r^{\frac{5}{2}-s} \sqrt{\frac{(4^s-1)\Gamma(5-2s)}{\zeta(2s-2)}}}{\pi^{3/2}(7-2s)\Gamma(3-s)} \int_0^{\pi} \frac{dr}{\left((4^s-2)\sin(2r)+4^s\right)\sqrt{4^s-(4^s-8)\sin(2r)}} .$$

*See Figure 3.3.*

*Proof.* Arguing as before, one finds that $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \frac{1}{\pi}\begin{pmatrix} \frac{\pi 2^{2s-3}\Gamma(5-2s) r^{4-2s}}{\Gamma(3-s)^2} & 0 & \frac{2^{3-2s}\zeta(2s-2)\left(2^{3-2s}-3\sin 2r-5\right)}{r((2^{1-2s}-1)\sin 2r-1)} \\ 0 & -\frac{2^{6-2s}\left(2^{2-2s}-1\right)\zeta(2s-2)}{(2^{3-2s}-1)r\sin 2r+r} & 0 \\ \frac{2^{3-2s}\zeta(2s-2)\left(2^{3-2s}-3\sin 2r-5\right)}{r((2^{1-2s}-1)\sin 2r-1)} & 0 & \frac{2^{4-2s}\left(2^{-2s}-1\right)\zeta(2s)}{r((2^{1-2s}-1)\sin 2r-1)} \end{pmatrix}$$

and $\mathcal{R}_{ij} = \Sigma^0_{ij}(r)o(1)$. This readily leads to the expression

$$\mathcal{I}(r) \sim \frac{\left(2^{2s-\frac{1}{2}} r^{\frac{5}{2}-s}\right)}{\sqrt{\pi}\Gamma(3-s)\left((4^s-2)\sin(2r)+4^s\right)} \sqrt{\frac{(4^s-1)\Gamma(5-2s)}{\zeta(2s-2)\left(4^s-(4^s-8)\sin(2r)\right)}} ,$$

which implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4\left(2^{2s-\frac{1}{2}}\right)}{\sqrt{\pi}\Gamma(3-s)}\sqrt{\frac{(4^s-1)\,\Gamma(5-2s)}{\zeta(2s-2)}}\times$$
$$\times\int_0^R \frac{r^{\frac{5}{2}-s}}{((4^s-2)\sin(2r)+4^s)\,(4^s-(4^s-8)\sin(2r))}dr\,.$$

Applying Lemma 3.4.5 once again, one obtains the desired formula. □

### 3.4.8 The case $s = \frac{5}{2}$

The next lemma shows that at this regularity level, there is another transition in the asymptotic behavior of the expected number of critical points of $u$:

**Lemma 3.4.13.** *If $s = \frac{5}{2}$, $\mathbb{E}N(\nabla u, R) \sim \tilde{\kappa}_{\frac{5}{2}} R\sqrt{\log R}$ with*

$$\tilde{\kappa}_{\frac{5}{2}} := \frac{4}{\pi^2}\sqrt{\frac{31}{\zeta(3)}}\int_0^\pi \frac{dr}{(16+15\sin 2r)\sqrt{4-3\sin 2r}} \approx 0.497339\,.$$

*Proof.* Arguing as before, one find that $\Sigma(r) = \Sigma^0(r) + \mathcal{R}(r)$ with

$$\Sigma^0(r) = \frac{1}{\pi}\begin{pmatrix} \frac{4\log r}{r} & 0 & \frac{\zeta(3)(12\sin 2r+19)}{r(15\sin 2r+16)} \\ 0 & \frac{7\zeta(3)}{4r-3r\sin 2r} & 0 \\ \frac{\zeta(3)(12\sin 2r+19)}{r(15\sin 2r+16)} & 0 & \frac{31\zeta(5)}{64r+60r\sin 2r} \end{pmatrix}$$

and $\mathcal{R}_{ij}(r) = \Sigma^0_{ij}(r)\,o(1)$. This eventually yields the asymptotic formula

$$\mathcal{I}(r) \sim \frac{2}{\pi^2}\sqrt{\frac{31}{\zeta(3)}}\frac{\sqrt{\log r}}{(16+15\sin 2r)\sqrt{4-3\sin 2r}}\,,$$

which implies

$$\mathbb{E}N(\nabla u, R) \sim \frac{4}{\pi}\sqrt{\frac{31}{\zeta(3)}}\int_0^R \frac{\sqrt{\log r}}{(16+15\sin 2r)\sqrt{4-3\sin 2r}}dr$$

by Lemmas 3.4.2 and 3.4.4. Lemma 3.4.5 then yields the desired asymptotic behavior. □

### 3.4.9 The case $s > \frac{5}{2}$

In this regime, the proof goes as before, showing that the expected number of critical points contained in a large ball grows asymptotically like the radius. However, the explicit formulas one obtains for the proportionality constant are extremely cumbersome.

**Lemma 3.4.14.** *For $s > \frac{5}{2}$, there exists an explicit constant $\kappa(s) > 0$ such that*

$$\mathbb{E}N(\nabla u, R) \sim \kappa(s) R\,.$$

*Proof.* As in the previous cases, let us write $\Sigma(r) = \Sigma^0(r) + \mathcal{R}$ with $\mathcal{R}_{ij} = \Sigma^0(R)\, o(1)$ and

$$\Sigma^0(r) = \frac{1}{\pi r}\begin{pmatrix} \Sigma_{11}(r) & 0 & \frac{2^{3-2s}\zeta(2s-2)\left(2^{3-2s}-3\sin 2r-5\right)}{(2^{1-2s}-1)\sin 2r-1} \\ 0 & -\frac{2^{6-2s}\left(2^{2-2s}-1\right)\zeta(2s-2)}{(2^{3-2s}-1)\sin 2r+1} & 0 \\ \frac{2^{3-2s}\zeta(2s-2)\left(2^{3-2s}-3\sin 2r-5\right)}{(2^{1-2s}-1)\sin 2r-1} & 0 & \frac{2^{4-2s}\left(2^{-2s}-1\right)\zeta(2s)}{(2^{1-2s}-1)\sin 2r-1} \end{pmatrix}.$$

Here

$$\Sigma_{11}(r) := 4\zeta(2s-4)\left(\left(2^{5-2s}-1\right)\sin 2r+1\right) + \frac{4\left(2^{3-2s}-1\right)^2\cos^2(2r)\zeta(2s-2)^2}{\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r-1\right)}\,.$$

Note that all the nonzero matrix components are exactly of order $1/r$. While this fact does not make the problem any harder from a conceptual point of view, it leads to cumbersome expressions for the various quantities appearing in the equations.

Specifically, it is not hard to show that

$$\sigma(r) \sim -\frac{16\zeta(2s-2)\zeta(2s)\left(\left(2^{1-2s}-1\right)\sin 2r-1\right)\left(\left(2^{3-2s}-1\right)\sin 2r+1\right)}{r^2}\,.$$

Plugging this formula in the expression for $I(r,z)$, one finds that

$$\mathcal{I}(r) \sim \int_{\mathbb{R}^3} |Az_1^2 + Bz_2^2 + 2Cz_1z_2|\,\frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}}\,dz\,,$$

where the constants

$$\alpha := \frac{1}{\pi}\Big[\zeta(2s-2)\zeta(2s)[1+(1-2^{1-2s})\sin 2r][1+(2^{3-2s}-1)\sin 2r]\Big]^{-\frac{1}{2}}$$

$$A := \alpha 2^{-2s}\zeta(2s-2)\frac{5-2^{3-2s}+3\sin 2r}{1-(1-2^{1-2s})\sin 2r}\,,$$

$$B := \alpha 2^{3-2s}\zeta(2s-2)\frac{2^{2-2s}-1}{1+(2^{3-2s}-1)\sin 2r}\,,$$

$$\begin{aligned} C := {}& \frac{\alpha 2^{-s-1}}{1+(1-2^{1-2s})\sin 2r}\times \\ & \times\Big[(1-2^{-2s})\zeta(2s-4)\zeta(2s)[1+(1-2^{1-2s})\sin 2r][1+(-1+2^{5-2s})\sin 2r] \\ & \quad + \zeta(2s-2)^2\big[-(1-2^{-2s})(1-2^{3-2s})^2\cos^2 2r - 2^{-2s}(2^{3-2s}-3\sin 2r-5)^5\big]^2\Big]^{\frac{1}{2}} \end{aligned}$$

are smooth functions of $\sin 2r$.

Lemma 3.4.6 then shows that

$$\mathcal{I}(r) \sim F(s, \sin 2r)$$

for some explicit smooth function of the form

$$F(s, \sin 2r) = \frac{2}{\pi} \int_0^\infty \frac{1 - a(t, s, \sin 2r) \cos \frac{1}{2}\Phi(t, s, \sin 2r)}{t^2} dt .$$

Since

$$a(t, s, \sin 2r) = \Big[(1 + 4B^2t^2)\big[(1 + 4C^2t^2)^2 + 4A^2t^2\big]\Big]^{-\frac{1}{4}} < 1$$

for all $r$ and all $t > 0$, it stems that

$$F(s, \sin 2r) > 0 .$$

Lemmas 3.4.2, 3.4.4 and 3.4.5 then ensure that

$$\mathbb{E}N(\nabla u, R) \sim \kappa(s)R$$

with

$$\kappa(s) := 2 \int_0^\pi F(s, \sin 2r)\, dr .$$

□

One can now read the asymptotic behavior of $\mathbb{E}N(\nabla u, R)$ in any regularity regime from the lemmas that we have established in this section. Theorem 3.1.1 is therefore proven.

## 3.5 Asymptotics for the number of critical points in the high regularity case

This section is devoted to the proof of Theorem 3.1.3. As all along this chapter, we shall take the definition (3.16) for the Gaussian random function $u$.

### 3.5.1 Some non-probabilistic lemmas

Before presenting the proof of this theorem, we need to prove a few auxiliary results that do not use the fact that $u$ and $f$ are random functions. Specifically, these lemmas concern solutions to the Helmholtz equation on $\mathbb{R}^2$ of the form

$$v(x) := \int_{\mathbb{T}} e^{-ix\cdot E(\phi)}\, g(\phi)\, d\phi$$

where $g \in H^m(\mathbb{T})$ for a certain real $m$ and the standard embedding $E : \mathbb{T} \to \mathbb{R}^2$ is given by (3.2).

We start by recalling the following result on the asymptotic behavior of $v$, which we proved in [EPSR22a, Proposition 2.2 and Remark 3.2]. In what follows, we will denote the real and imaginary parts of a function $g$ by $g_{\mathrm{R}}$ and $g_{\mathrm{I}}$, respectively.

**Lemma 3.5.1.** *If $m > 9/2$, for $r \gg 1$ one has*

$$\begin{aligned} v &= \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}(\theta)\, \sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}(\theta)\, \cos(r - \tfrac{\pi}{4}) + \mathcal{R}_1\right], \\ \partial_r v &= \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}(\theta)\, \cos(r - \tfrac{\pi}{4}) - g_{\mathrm{R}}(\theta)\, \sin(r - \tfrac{\pi}{4}) + \mathcal{R}_2\right], \\ \partial_\theta v &= \left(\frac{8\pi}{r}\right)^{\frac{1}{2}} \left[g_{\mathrm{I}}'(\theta)\, \sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}'(\theta)\, \cos(r - \tfrac{\pi}{4}) + \mathcal{R}_3\right], \end{aligned}$$

*where the errors are bounded as*

$$|\mathcal{R}_1| + |\nabla \mathcal{R}_1| + |\nabla^2 \mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3| \lesssim \frac{1}{r}.$$

The following theorem provides very precise asymptotic information about the critical points of $v$:

**Lemma 3.5.2.** *Assume that $m > 9/2$, that $g$ does not vanish on $\mathbb{T}$, and that all the critical points of $|g|$ are non-degenerate. If $\phi^*$ is a critical point of $|g|$, then for each large enough positive integer $n$ there exists a critical point $(r_n^*, \theta_n^*)$ of $v$ such that*

$$|\phi^* - \theta_n^*| + \left|\pi n + \tfrac{\pi}{4} + \arg g(\phi^*) - r_n^*\right| \lesssim \frac{1}{n}.$$

*Conversely, if $(r^*, \theta^*)$ is a critical point of $v$, there is some critical point $\phi^*$ of $|g|$ such that*

$$|\phi^* - \theta^*| \lesssim \frac{1}{r^*}.$$

*Proof.* Let us consider the function

$$V := \operatorname{Re}\left[g(\theta) e^{-i(r - \frac{\pi}{4})}\right] = g_{\mathrm{I}}(\theta)\, \sin(r - \tfrac{\pi}{4}) + g_{\mathrm{R}}(\theta)\, \cos(r - \tfrac{\pi}{4}),$$

whose critical points $(r^*, \theta^*)$ are the solutions to the equations

$$\operatorname{Im}\left[g(\theta^*) e^{-i(r^* - \frac{\pi}{4})}\right] = 0, \qquad \operatorname{Re}\left[g'(\theta^*) e^{-i(r^* - \frac{\pi}{4})}\right] = 0.$$

Writing $g = |g| e^{i \arg g}$, an elementary calculation shows that $(r^*, \theta^*)$ is a critical point of $V$ if and only if $r^* = \arg g(\theta^*) + \frac{\pi}{4} + \pi n$ for some integer $n$ and $\operatorname{Re}[\overline{g(\theta^*)} g'(\theta^*)] = 0$. As $g$ does not vanish on $\mathbb{T}$, the latter condition simply means that $\theta^*$ is a critical point of $|g|$. Furthermore, the Hessian of $V$ at the critical points is

$$D^2 V(r^*, \theta^*) = (-1)^n \begin{pmatrix} -|g(\theta^*)| & |g(\theta^*)| (\arg g)'(\theta^*) \\ |g(\theta^*)| (\arg g)'(\theta^*) & |g|''(\theta^*) - |g(\theta^*)| [(\arg g)'(\theta^*)]^2 \end{pmatrix}.$$

Therefore,

$$\det D^2 V(r^*, \theta^*) = -|g(\theta^*)|\, |g|''(\theta^*) \neq 0 \tag{3.29}$$

because the critical points of $|g|$ are, by hypothesis, nondegenerate.

Let us now consider the function

$$F(r, \theta) := DV(r, \theta) - \left(\frac{r}{8\pi}\right)^{\frac{1}{2}} Dv(r, \theta),$$

where $DV := (\partial_r V, \partial_\theta V)$. Lemma 3.5.1 ensures that

$$|F(r,\theta)| + |DF(r,\theta)| \lesssim \frac{1}{r}\,.$$

As the critical points of $V$ are uniformly non-degenerate by (3.29), Thom's isotopy theorem (as stated, e.g., in [EPS13]) ensures that $v$ has a critical point at a distance at most $C/n$ to each of the critical points $(r^*, \theta^*)$ of $V$ as described above, provided that $n$ is large enough. Furthermore, the asymptotic formulas for $Dv$ presented in Lemma 3.5.1 guarantee that all critical points of $v$ that are far enough from the origin must be of this form. The lemma is then proven. $\square$

### 3.5.2 Proof of Theorem 3.1.3

As $s > 5$, Proposition 3.2.2 ensures that $f \in H^{s'}(\mathbb{T})$ almost surely for some $s' > \frac{9}{2}$. Therefore, if one can prove that, with probability 1, $f$ does not vanish on $\mathbb{T}$ and all the critical points of $|f|$ are nondegenerate, Theorem 3.1.3 will follow as an easy consequence of Lemma 3.5.2.

Proving the first part of this assertion is completely standard, but the second part is quite harder. In both cases, the proof relies on Bulinskaya's lemma, which one can state as follows [AW09, Proposition 6.11]:

**Lemma 3.5.3** (Bulinskaya)**.** *Let $Y : \mathbb{T} \to \mathbb{R}^2$ be a random function that is of class $C^1(\mathbb{T})$ almost surely. Uniformly for $\phi \in \mathbb{T}$, assume that the random variable $Y(\phi)$ has a probability density $\rho_{Y(\phi)} : \mathbb{R}^2 \to [0,\infty)$ that is bounded in some fixed neighborhood of the origin. Then*

$$\mathbb{P}\{Y(\phi) = 0 \ \textit{for some}\ \phi \in \mathbb{T}\} = 0\,.$$

Armed with Bulinskaya's lemma, it is easy to show that, almost surely, $f$ does not vanish:

**Lemma 3.5.4.** *With probability 1, $f$ does not vanish on $\mathbb{T}$.*

*Proof.* By the definition of $u$, cf. Equations (3.16) and (3.5), $\tilde{Y}(\phi) := (f_{\mathrm{R}}(\phi), f_{\mathrm{I}}(\phi))$ is a Gaussian random field $\tilde{Y} : \mathbb{T} \to \mathbb{R}^2$ with zero mean. The covariance of $\tilde{Y}(\phi)$ can be computed just as in Lemma 3.4.1, obtaining the nondegenerate matrix

$$\operatorname{Var}\tilde{Y}(\phi) = \mathbb{E}[\tilde{Y}(\phi) \otimes \tilde{Y}(\phi)] = \begin{pmatrix} \pi^{-2}\sum_{l>0,\text{even}} l^{-2s} & 0 \\ 0 & \pi^{-2}\sum_{l>0,\text{odd}} l^{-2s} \end{pmatrix} =: \Sigma\,.$$

Therefore, $\tilde{Y}(\phi)$ has a bounded probability density function

$$\rho_{\tilde{Y}(\phi)}(y) := \frac{\exp\left(-\frac{1}{2}y \cdot \Sigma^{-1} y\right)}{2\pi(\det\Sigma)^{1/2}}$$

on $\mathbb{R}^2$ because $\Sigma$ is a nondegenerate matrix. Lemma 3.5.3 then ensures that $\tilde{Y}$ does not vanish with probability 1. As the zeros of $\tilde{Y}$ and $f$ obviously coincide, the lemma follows. $\square$

The crux of the proof of Theorem 3.1.3 is to show that the critical points of $|f|$ are nondegenerate. This is not direct because $|f|$ is not a Gaussian variable, and showing that it has a bounded probability density requires some work. The main ingredient of the proof is the estimate we present in the following lemma. The proof is somewhat involved, so we have relegated it to the next subsection in order to streamline the presentation of the proof of Theorem 3.1.3. To state the auxiliary result, we will write points in $\mathbb{R}^6$ as

$$z = (z', z'') \in \mathbb{R}^4 \times \mathbb{R}^2$$

with $z' := (z_1, z_2, z_3, z_4)$ and $z'' := (z_5, z_6)$.

**Lemma 3.5.5.** *Consider the nonnegative rational function on $\mathbb{R}^6$ given by*

$$Q(z) := |z'|^2 + \frac{(z_5 - z_1 z_3)^2}{z_2^2} + \frac{[(z_5 - z_1 z_3)^2 + z_2^2(z_1 z_4 + z_3^2 - z_6)]^2}{z_2^6}. \tag{3.30}$$

*For any constant $c > 0$,*

$$\sup_{|z''| < \frac{1}{2}} \int_{\mathbb{R}^4} \frac{e^{-c\,Q(z)}}{z_2^2}\, dz' < \infty .$$

Assuming for the moment that this technical lemma holds, proving that the critical points of $|f|$ are nondegenerate almost surely is straightforward:

**Lemma 3.5.6.** *With probability* $1$*, all the critical points of $|f|$ are nondegenerate.*

*Proof.* Let us start by noting that

$$|f|\,|f|' = \tfrac{1}{2}(|f|^2)' = \operatorname{Re} \overline{f}\, f' = f_{\mathrm{R}} f_{\mathrm{R}}' + f_{\mathrm{I}} f_{\mathrm{I}}' .$$

Differentiating this identity, we obtain

$$|f|\,|f|'' + (|f|')^2 = \operatorname{Re} \overline{f}\, f'' + |f'|^2 = f_{\mathrm{R}} f_{\mathrm{R}}'' + f_{\mathrm{I}} f_{\mathrm{I}}'' + (f_{\mathrm{R}}')^2 + (f_{\mathrm{I}}')^2 .$$

Therefore, all the critical points of $|f|$ are nondegenerate if and only if

$$Y := (f_{\mathrm{R}} f_{\mathrm{R}}' + f_{\mathrm{I}} f_{\mathrm{I}}', f_{\mathrm{R}} f_{\mathrm{R}}'' + f_{\mathrm{I}} f_{\mathrm{I}}'' + (f_{\mathrm{R}}')^2 + (f_{\mathrm{I}}')^2) : \mathbb{T} \to \mathbb{R}^2$$

does not vanish.

As $Y \in C^2(\mathbb{T})$ almost surely because $s > 5$, in order to apply Bulinskaya's lemma we only need to show that $Y(\phi)$ has a probability density that is bounded in a neighborhood of the origin. The random variable $Y(\phi)$ is obviously not Gaussian, so in order to compute its density we need to argue in an indirect way.

The starting point is the fact that the 2-jet of $f$,

$$Z := (f_{\mathrm{R}}, f_{\mathrm{I}}, f_{\mathrm{R}}', f_{\mathrm{I}}', f_{\mathrm{R}}'', f_{\mathrm{I}}''),$$

defines a Gaussian random variable $Z : \mathbb{T} \to \mathbb{R}^6$ with zero mean. Its variance

$$\operatorname{Var} Z(\phi) := \mathbb{E}[Z(\phi) \otimes Z(\phi)],$$

which does not depend on $\phi$, can be computed from the definition

$$f(\phi) := \frac{1}{2\pi} \sum_{l \neq 0} i^l a_l |l|^{-s} e^{il\phi}$$

by arguing just as in the proof of Lemma 3.4.1. It turns out that $\operatorname{Var} Z(\phi) = \Sigma$, where $\Sigma$ is the $6 \times 6$ matrix

$$\Sigma := \begin{pmatrix} a_0 & 0 & 0 & 0 & -b_0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & -b_1 \\ 0 & 0 & b_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & 0 & 0 \\ -b_0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & -b_1 & 0 & 0 & 0 & c_1 \end{pmatrix},$$

where

$$a_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2, \qquad b_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2 (i+2m)^2, \qquad c_i := \pi^{-2} \sum_{m=0}^{\infty} \sigma_{i+2m}^2 (i+2m)^4$$

and we have set $\sigma_l := |l|^{-s}$ for $l \neq 0$ and $\sigma_0 := 0$. We have chosen to write this formula in terms of $\sigma_l$ so that it is apparent that the result only uses the asymptotic properties of the sequence $\sigma_l$. Note that these sums are all convergent because $s > 5$.

The determinant of $\Sigma$ is

$$\det \Sigma = b_0 b_1 \left(b_0^2 - a_0 c_0\right) \left(b_1^2 - a_1 c_1\right) .$$

As $a_i c_i > b_i^2$ strictly by the Cauchy–Schwartz inequality, the matrix $\Sigma$ is invertible. Therefore, for each $\phi \in \mathbb{T}$, the probability density distribution of $Z(\phi)$ is given by the Gaussian function

$$g(z) := (2\pi)^{-3} (\det \Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2} z \cdot \Sigma^{-1} z} \in C^\infty(\mathbb{R}^6) .$$

Consider now the map $H : \mathbb{R}^6 \to \mathbb{R}^6$ given by

$$H(z) := \left(z_1, z_2, z_3, z_5, z_1 z_3 + z_2 z_4, z_1 z_5 + z_2 z_6 + z_3^2 + z_4^2\right) . \tag{3.31}$$

This map is invertible outside the hyperplane $\{z_2 = 0\}$, with inverse

$$H^{-1}(z) := \left(z_1, z_2, z_3, \frac{z_5 - z_1 z_3}{z_2}, z_4, -\frac{(z_5 - z_1 z_3)^2}{z_2^3} - \frac{z_1 z_4 + z_3^2 - z_6}{z_2}\right),$$

and its corresponding Jacobian determinant is $\det \nabla H^{-1}(z) = -z_2^{-2}$. Therefore, the probability density distribution of the random variable $H[Z(\phi)]$ is obtained by pulling back with the map $H$ the probability distribution of $Z(\phi)$:

$$\rho_{H[Z(\phi)]}(z) = |\det \nabla H^{-1}(z)|\, g[H^{-1}(z)] = (2\pi)^{-3} (\det \Sigma)^{-\frac{1}{2}} z_2^{-2} e^{-Q_H(z)} . \tag{3.32}$$

with $Q_H(z) := \frac{1}{2} H^{-1}(z) \cdot \Sigma^{-1} H^{-1}(z)$.

Now let $\widetilde{H} : \mathbb{R}^6 \to \mathbb{R}^2$ denote the last two components of the map (3.31), that is,

$$\widetilde{H}(z) := \left(z_1 z_3 + z_2 z_4, z_1 z_5 + z_2 z_6 + z_3^2 + z_4^2\right) .$$

As the random variables $Y(\phi)$ and $Z(\phi)$ are related by

$$Y(\phi) = \widetilde{H}[Z(\phi)],$$

it then follows from (3.32) that the density of $Y(\phi)$ is given by the marginal distribution

$$\rho_{Y(\phi)}(z'') = \int_{\mathbb{R}^4} \rho_{H[Z(\phi)]}(z)\, dz' \,.$$

Now notice that the function $Q(z)$ defined in (3.30) is simply

$$Q(z) = |H^{-1}(z)|^2 \,.$$

As the matrix $\Sigma$ is positive definite, therefore there is a positive constant $c > 0$ such that

$$\rho_{Y(\phi)}(z'') \lesssim \int_{\mathbb{R}^4} \frac{e^{-cQ(z)}}{z_2^2}\, dz' \,.$$

Lemma 3.5.5 then ensures that $\sup_{|z''|<\frac{1}{2}} \rho_{Y(\phi)}(z'') \lesssim 1$. Lemma 3.5.3 then guarantees that the random function $Y$ does not vanish on $\mathbb{T}$ almost surely, and the theorem follows. ☐

Theorem 3.1.3 is then proven, modulo the proof of Lemma 3.5.5, which we will address next.

### 3.5.3 Proof of the main technical lemma

Let us now present the proof of Lemma 3.5.5. To make the exposition clearer, we will divide the proof in three steps.

#### The integral $\widetilde{I}$

The first step is to rewrite the integral

$$I := \int_{\mathbb{R}^4} \frac{e^{-cQ(z)}}{z_2^2} dz'$$

in a more convenient way. For this, let us set

$$\varrho := z_1 z_3 - z_5 \,, \qquad \tau := \frac{z_1 z_3 - z_5}{z_2} \,.$$

The map $z' \mapsto (\varrho, \tau, z_3, z_4)$ is invertible outside the hyperplane $z_3 = 0$ and the set $\tau = 0$. In terms of these variables, the integral reads as

$$I = \int_{\mathbb{R}^4} \frac{e^{-c\,Q_1}}{|\varrho z_3|}\, d\varrho\, d\tau\, dz_3\, dz_4$$

with

$$\begin{aligned} Q_1 &:= Q_2 + z_4^2 \left[1 + \left(\frac{\tau(\varrho + z_5)}{\varrho z_3}\right)^2\right] + 2 z_4 (\tau^2 + z_3^2 - z_6) \frac{\tau^2(\varrho + z_5)}{\varrho^2 z_3} \,, \\ Q_2 &:= z_3^2 + \tau^2 + \frac{\varrho^2}{\tau^2} + \left(\frac{\tau(\tau^2 + z_3^2 - z_6)}{\varrho}\right)^2 + \left(\frac{\varrho + z_5}{z_3}\right)^2 . \end{aligned} \tag{3.33}$$

As $Q_1$ is a second order polynomial in $z_4$, one can explicitly integrate in this variable, obtaining

$$I(z'') = \sqrt{\frac{\pi}{c}} \int_{\mathbb{R}^3} \frac{e^{-cQ_3}}{\sqrt{\varrho^2 z_3^2 + \tau^2(\varrho + z_5)^2}} \, d\varrho \, d\tau \, dz_3 \,,$$

with

$$Q_3 := z_3^2 + \tau^2 + \frac{\varrho^2}{\tau^2} + \left( \frac{\tau z_3(\tau^2 + z_3^2 - z_6)}{(z_3^2 \varrho^2 + \tau^2(\varrho + z_5)^2)^{1/2}} \right)^2 + \left( \frac{\varrho + z_5}{z_3} \right)^2 .$$

Let us now consider polar coordinates $(\sigma, \alpha) \in \mathbb{R}^+ \times \mathbb{T}$, defined as

$$z_3 =: \sigma \cos \alpha \,, \qquad \tau =: \sigma \sin \alpha \,.$$

Still denoting by $Q_2$ the expression of (3.33) in these variables, and similarly with the other functions $Q_j$, we get

$$Q_2 = \frac{\varrho^2}{\sigma^2} \csc^2 \alpha + \left( \frac{\varrho + z_5}{\sigma} \right)^2 \sec^2 \alpha + \sigma^2 + \left( \frac{\sigma(\sigma^2 - z_6) \sin \alpha}{\varrho} \right)^2 .$$

This enables us to write

$$I = \sqrt{\frac{\pi}{c}} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2 \cos^2 \alpha + (\varrho + z_5)^2 \sin^2 \alpha}} \, d\sigma \, d\alpha \, d\varrho \,.$$

As $|z''| < \frac{1}{2}$, the denominator is nonzero for $|\varrho| > 1$, so one obviously has

$$\int_{\mathbb{R} \setminus [-1,1]} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2 \cos^2 \alpha + (\varrho + z_5)^2 \sin^2 \alpha}} \, d\sigma \, d\alpha \, d\varrho \lesssim \int_{\mathbb{R}} \int_0^{\infty} e^{-c(\sigma^2 + \frac{\varrho^2}{\sigma^2})} \, d\sigma \, d\varrho \lesssim 1 \,.$$

We can then write

$$I \lesssim 1 + \int_{-1}^{1} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-cQ_3}}{\sqrt{\varrho^2 \cos^2 \alpha + (\varrho + z_5)^2 \sin^2 \alpha}} \, d\sigma \, d\alpha \, d\varrho =: 1 + \widetilde{I} \,. \tag{3.34}$$

**The case $z_5 = 0$**

Let us start by assuming that $z_5 = 0$, so that

$$\widetilde{I} = \int_{-1}^{1} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-cQ_3}}{|\varrho|} \, d\sigma \, d\alpha \, d\varrho \leqslant 2 \int_0^1 \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c\sigma^2 - c\varrho^{-2}\sigma^2(\sigma^2 - z_6)^2 \sin^2 \alpha \cos^2 \alpha}}{\varrho} \, d\sigma \, d\alpha \, d\varrho \,.$$

The integral in $\varrho$ can be computed in terms of the incomplete Gamma function

$$\Gamma(\lambda, x) := \int_x^{\infty} t^{\lambda - 1} e^{-t} \, dt \,,$$

obtaining

$$\widetilde{I} \leqslant \int_0^{2\pi} \int_{-\infty}^{\infty} e^{-c\sigma^2} \Gamma[0, c\sigma^2(\sigma^2 - z_6)^2 \sin^2 \alpha \cos^2 \alpha] \, d\sigma \, d\alpha \,.$$

Then the bound

$$\Gamma(0,x) \lesssim \log\left(2+\frac{1}{x}\right),$$

valid for all $x>0$, immediately implies that

$$\sup_{|z_6|<\frac{1}{2}} \widetilde{I} \lesssim \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-c\sigma^2}\log\left(2+\frac{1}{c\sigma^2(\sigma^2-1/2)^2\sin^2\alpha\cos^2\alpha}\right) d\sigma\, d\alpha \lesssim 1 \tag{3.35}$$

when $z_5=0$.

**The case $z_5 \neq 0$**

In view of the estimate (3.35), from now on, we shall assume that $z_5\neq 0$. Let us now define the new variable $\tilde{\varrho} := -\varrho/z_5$, in terms of which the integral $\widetilde{I}$ reads as

$$\widetilde{I} \leqslant \int_{-1/|z_5|}^{1/|z_5|}\int_0^{2\pi}\int_0^{\infty}\frac{e^{-cQ_4}}{S(\tilde{\varrho},\alpha)}\, d\sigma\, d\alpha\, d\tilde{\varrho}\,.$$

Here we have used that

$$\sqrt{\varrho^2\cos^2\alpha+(\varrho+z_5)^2\sin^2\alpha} = |z_5|\, S(\tilde{\varrho},\alpha)$$

with

$$S(\tilde{\varrho},\alpha) := \sqrt{\tilde{\varrho}^2\cos^2\alpha+(\tilde{\varrho}-1)^2\sin^2\alpha}$$

and $Q_4$ is defined as

$$Q_4 := \sigma^2+\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2}\sin^2\alpha\cos^2\alpha.$$

Let us fix some small $\epsilon>0$ and define the sets

$$\mathcal{M}_0 := \{(\tilde{\varrho},\alpha) : |\tilde{\varrho}|<\epsilon\,,\ |\sin\alpha|<\epsilon\}\,, \qquad \mathcal{M}_1 := \{(\tilde{\varrho},\alpha) : |\tilde{\varrho}-1|<\epsilon\,,\ |\cos\alpha|<\epsilon\}\,.$$

Since $S(\tilde{\varrho},\alpha)\gtrsim 1$ for $(\tilde{\varrho},\alpha)\notin \mathcal{M}_0\cup\mathcal{M}_1$ (not uniformly in $\epsilon$), let us consider the set

$$\mathcal{M}_2 := \left(\left(-\frac{1}{|x_5|},\frac{1}{|x_5|}\right)\times\mathbb{T}\right)\backslash(\mathcal{M}_0\cup\mathcal{M}_1)$$

and split the above integral as

$$\widetilde{I} = \int_{\mathcal{M}_0}\int_0^{\infty}+\int_{\mathcal{M}_1}\int_0^{\infty}+\int_{\mathcal{M}_2}\int_0^{\infty} =: \widetilde{I}_0+\widetilde{I}_1+\widetilde{I}_2\,.$$

To estimate $\widetilde{I}_0$, observe that $\mathcal{M}_0$ consists of two connected components, which are contained in $|\tilde{\varrho}|<\epsilon$ and either $|\alpha|<C\epsilon$ or $|\alpha-\pi|<C\epsilon$, respectively. It is easy to see that both contributions to the integral are of the same size, so we will just consider the first. To analyze it, let us use the bound

$$S(\tilde{\varrho},\alpha)\gtrsim\sqrt{\tilde{\varrho}^2+\alpha^2}\,,$$

which clearly holds for $(\tilde{\varrho},\alpha) \in \mathcal{M}_0^+$, to write

$$\begin{aligned}\widetilde{I}_0 &\lesssim \int_{-\epsilon}^{\epsilon}\int_{-C\epsilon}^{C\epsilon}\int_0^\infty \frac{e^{-cQ_4}}{S(\tilde{\varrho},\alpha)}\,d\sigma\,d\alpha\,d\tilde{\varrho}\\ &\lesssim \int_{-\epsilon}^{\epsilon}\int_{-C\epsilon}^{C\epsilon}\int_0^\infty \frac{e^{-c\sigma^2}}{\sqrt{\tilde{\varrho}^2+\alpha^2}}\,d\sigma\,d\alpha\,d\tilde{\varrho}\,.\end{aligned}$$

Once can now introduce a new set of polar coordinates

$$\tilde{\varrho} =: r\cos\beta\,, \qquad \alpha =: r\sin\beta\,,$$

which yields

$$\widetilde{I}_0 \lesssim \int_0^{C\epsilon}\int_0^{2\pi}\int_0^\infty e^{-c\sigma^2}\,d\sigma\,d\beta\,dr \lesssim 1\,.$$

An analogous argument for $\mathcal{M}_1$, where $|\tilde{\varrho}-1| < \epsilon$ and either $|\alpha - \frac{\pi}{2}| < C\epsilon$ or $|\alpha - \frac{3\pi}{2}| < C\epsilon$, shows that

$$\widetilde{I}_1 \lesssim 1\,.$$

It only remains to bound $\widetilde{I}_2$. As $S(\tilde{\varrho},\alpha) \gtrsim \langle\tilde{\varrho}\rangle$ on $\mathcal{M}_2$, where $\langle x\rangle := (1+x^2)^{\frac{1}{2}}$ is the Japanese bracket, we can write

$$\begin{aligned}\widetilde{I}_2 &\lesssim \int_{-1/|z_5|}^{1/|z_5|}\int_0^{2\pi}\int_0^\infty \frac{1}{\tilde{\varrho}} e^{-c\sigma^2 - c\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2}\sin^2\alpha\cos^2\alpha}\,d\sigma\,d\alpha\,d\tilde{\varrho}\\ &= 4\int_{-1/|z_5|}^{1/|z_5|}\int_0^{\pi/2}\int_0^\infty \frac{1}{\tilde{\varrho}} e^{-c\sigma^2 - c\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2}\sin^2\alpha\cos^2\alpha}\,d\sigma\,d\alpha\,d\tilde{\varrho}\,.\end{aligned}$$

As $\cos^2\alpha\sin^2\alpha = \frac{1}{4}\sin^2(2\alpha)$ and $\sin\alpha \gtrsim \alpha$ for $|\alpha| < \frac{\pi}{2}$, the integral in $\alpha$ can be estimated as

$$\begin{aligned}&\int_0^{\pi/2} e^{-c\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 S(\tilde{\varrho},\alpha)^2}\sin^2\alpha\cos^2\alpha}\,d\alpha \leqslant \int_0^{\pi/2} e^{-C\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 \tilde{S}(\tilde{\varrho})^2}\sin^2(2\alpha)}\,d\alpha\\ &= 2\int_0^{\pi/4} e^{-C\frac{\sigma^2(\sigma^2-z_6)^2}{z_5^2 \tilde{S}(\tilde{\varrho})^2}\sin^2(2\alpha)}\,d\alpha \lesssim \left\langle \frac{\sigma(\sigma^2-z_6)}{z_5\tilde{S}(\tilde{\varrho})}\right\rangle^{-1}\,,\end{aligned}$$

where $\tilde{S}(\tilde{\varrho}) := \tilde{\varrho}^2 + (1-\tilde{\varrho})^2$. Here we have used that for $c > 0$

$$\int_0^{\pi/4} e^{-c^2x^2}dx = \frac{\sqrt{\pi}\,\mathrm{Erf}\left(\frac{\pi c}{4}\right)}{2c} \lesssim \langle c\rangle^{-1}\,,$$

where Erf is the error function. Since $|z_6| \leqslant \frac{1}{2}$, this yields

$$\begin{aligned}\widetilde{I}_2 &\lesssim \int_{-1/|z_5|}^{1/|z_5|}\int_0^\infty \frac{e^{-c\sigma^2}}{\langle\tilde{\varrho}\rangle}\left\langle\frac{\sigma(\sigma^2-z_6)}{z_5\tilde{S}(\tilde{\varrho})}\right\rangle^{-1}d\sigma\,d\tilde{\varrho}\\ &= \int_{-1}^{1}\int_0^\infty \frac{e^{-c\sigma^2}}{(z_5^2+\varrho^2)^{\frac{1}{2}}}\frac{|z_5\tilde{S}(\tilde{\varrho})|}{(\varrho^2+(\varrho+z_5)^2+\sigma^2(\sigma^2-z_6)^2)^{1/2}}\,d\sigma\,d\varrho\\ &\leqslant \int_{-1}^{1}\int_0^\infty \frac{e^{-c\sigma^2}}{(\varrho^2+(\varrho+z_5)^2+\sigma^2(\sigma^2-1/2)^2)^{1/2}}\,d\sigma\,d\varrho\,.\end{aligned}$$

where we have used that if $z_5 = a\rho$

$$\frac{\left(z_5\tilde{S}(\tilde{\varrho})\right)^2}{\rho^2+z_5^2} = \frac{\rho^2+(\rho+z_5)^2}{\rho^2+z_5^2} = \frac{a^2+2a+2}{a^2+1} < C$$

for some $C > 0$ and for all $a \in \mathbb{R}$. To integrate in $\varrho$, we need that

$$\int_{-1}^{1} \frac{1}{\sqrt{((a+\rho)^2+\rho^2)+b}}\, d\rho$$
$$= \frac{1}{\sqrt{2}} \log\left(\frac{\left(\sqrt{2}\sqrt{(a-2)a+b+2}-a+2\right)\left(\sqrt{2}\sqrt{a(a+2)+b+2}+a+2\right)}{a^2+2b}\right).$$

Using that $|z_5| < \frac{1}{2}$ we conclude

$$\widetilde{I}_2 \lesssim \int_0^\infty e^{-c\sigma^2} \log\left(\frac{\left(\sqrt{2}\sqrt{4\sigma^6-4\sigma^4+\sigma^2+13}+5\right)^2}{2\sigma^2\left(1-2\sigma^2\right)^2}\right) d\sigma$$

Thus, we obtain the bound

$$\widetilde{I}_2 \lesssim 1\,,$$

from the fact that the logarithmic singularities at $\sigma = 0$ and $\sigma = 1/\sqrt{2}$ are integrable. Lemma 3.5.5 in then proven.

# Appendices

## 3.A Monochromatic waves with many nondegenerate critical points

In this Appendix we aim to prove that there exist solutions to the Helmholtz equation

$$\Delta v + v = 0$$

on the plane with many isolated critical points. Specifically, let

$$N^*(\nabla v, R) := \left\{x \in B_R : \nabla v(x) = 0\,,\ \det \nabla^2 v(x) \neq 0\right\}$$

be the number of nondegenerate critical points of $v$ contained in the ball of radius $R$. One can then prove the following:

**Proposition 3.A.1.** *Given any continuous function $\rho : \mathbb{R}^2 :\to \mathbb{R}^+$, there exists a solution to the Helmholtz equation on $\mathbb{R}^2$ such that*

$$N^*(\nabla v, R) > \rho(R)$$

*for all $R > 1$.*

*Proof.* Without any loss of generality, let us assume that the function $\rho$ is increasing. Take a set of distinct points $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ without any accumulation points such that

$$\#\{k \in \mathbb{N} : x_k \in B_R\} > \rho(R + \tfrac{1}{2}) \tag{3.36}$$

for all $R > \frac{1}{8}$. At each point $x_k$, consider the number

$$r_k := \frac{1}{8} \min \left\{ 1, \inf_{j \in \mathbb{N} \setminus \{k\}} |x_k - x_j| \right\},$$

which is positive because the set $\{x_k\}_{k\in\mathbb{N}}$ does not have any accumulation points.

The function $v_k(x) := J_0(|x - x_k|)$ satisfies the Helmholtz equation on the plane and $x_k$ is a nondegenerate maximum of $v_k$ (in fact, $D^2 v_k(x_k) = -\frac{1}{2} I$). Therefore, the implicit function theorem ensures that there exists some $\epsilon_k > 0$ such that any function $v$ with $\|v_k - v\|_{C^2(B(x_k, 2r_k))} < \epsilon_k$ has a nondegenerate local maximum inside the ball $B(x_k, r_k)$. Notice that $B(x_k, 2r_k) \cap B(x_j, 2r_j) = \emptyset$ if $k \neq j$.

The better-than-uniform global approximation theorem for the Helmholtz equation [EPS13, Lemma 7.2] ensures that there exists a solution $v$ to the Helmholtz equation on $\mathbb{R}^2$ such that

$$\sup_{k\in\mathbb{N}} \frac{\|v_k - v\|_{C^2(B(x_k, 2r_k))}}{\epsilon_k} < 1\,.$$

One then infers that $v$ has a nondegenerate critical point in each disk $B(x_k, r_k)$. The property (3.36) then ensures that $N^*(\nabla v, R) > \rho(R)$ for all $R > 1$, as claimed. □

**Remark 3.A.1.** The result and the proof remain valid in higher dimensions. The only modification is that, on $\mathbb{R}^n$, one must define $v_k(x) := |x - x_k|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x - x_k|)$.

**Remark 3.A.2.** The function $v$ may not be polynomially bounded at infinity, so $v$ does not need to have a Fourier transform. In particular, it does not need to be the Fourier transform of a distribution supported on the unit sphere.

## 3.B The translation-invariant case

In this Appendix we shall see why the evaluation of the Kac–Rice integral that gives the asymptotic behavior of $\mathbb{E}N(\nabla u, R)$ (cf. Lemma 3.4.2) is so much easier in the translation-invariant case (that is, when $s = 0$ following Remark 3.4.1).

In the translation-invariant case, it is easy to work directly in Cartesian coordinates, instead of using polar coordinates. This is because all one needs to know about $u$ in order to apply the Kac–Rice formula are expectation values of the form $\mathbb{E}[\partial^\alpha u(x)\, \partial^\beta u(x)]$, where $\alpha, \beta$ are multiindices of order at most 2. These quantities can be computed exactly using that, as discussed in Remark 3.4.1, for $s = 0$ the covariance kernel is (up to a normalizing constant)

$$K(x, x') = J_0(|x - x'|) = \int_{\mathbb{T}} e^{i\xi\cdot(x-x')}\, d\sigma(\xi)\,. \tag{3.37}$$

Indeed, taking derivatives in this expression one finds that

$$\mathbb{E}[\partial^\alpha u(x)\, \partial^\beta u(x)] = i^{|\alpha|-|\beta|} \int_{\mathbb{T}} \xi^\alpha\, \xi^\beta\, d\sigma(\xi)\,.$$

The last integral can be computed in closed form because [Fol01]

$$\int_{\mathbb{T}} \xi^\alpha \, d\sigma(\xi) = \begin{cases} \pi^{-1}\left[\prod_{j=1}^2 \Gamma(\frac{\alpha_j+1}{2})\right]/\Gamma(\frac{|\alpha|+2}{2}) & \text{if } \alpha_1, \alpha_2 \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

These formulas readily show that $\mathbb{E}[\partial_j u \, \partial_{kl} u] = 0$, so $\nabla u$ and $\nabla^2 u$ are independent Gaussian random functions, and that the covariance matrices of the first and second derivatives of $u$ are

$$\operatorname{Var} \nabla u(x) = \frac{1}{2} I, \qquad \operatorname{Var} \nabla^2 u(x) = \frac{1}{8} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Again, we have regarded $\nabla^2 u$ as a 3-component vector. By the Kac–Rice formula, these expressions are enough to show

$$\mathbb{E} N(\nabla u, R) = \pi R^2 \int_{\mathbb{R}^3} \frac{\left|z_1^2 + 2\sqrt{2} z_1 z_2 - z_2^2\right|}{8\pi} \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{3/2}} dz = \kappa(0) R^2 \tag{3.38}$$

as in Remark 3.4.2.

In polar coordinates, one sees essentially the same simplifications. The point is that it suffices to differentiate the addition formula

$$g(r, r', \theta) := J_0\left(\sqrt{r^2 + r'^2 - 2rr' \cos\theta}\right) = \sum_{l=0}^{\infty} \epsilon_l J_l(r) J_l(r') \cos l\theta,$$

where $\epsilon_l := 2 - \delta_{l,0}$ is Neumann's factor, to compute in closed form all the sums appearing in the Kac–Rice formula (Lemma 3.4.2). Incidentally, the addition formula is equivalent to the assertion that the covariance matrix of $u$ is (3.37), written in polar coordinates. For example,

$$\begin{aligned} \sum_{l=0}^{\infty} \epsilon_l J_l(r)^2 &= g(r, r, 0) = 1, \\ \sum_{l=0}^{\infty} \epsilon_l J_l'(r)^2 &= \partial_r \partial_{r'} g(r, r, 0) = \frac{1}{2}, \\ \sum_{l=0}^{\infty} \epsilon_l l^2 J_l(r) J_l'(r) &= -\frac{1}{2} \partial_r \partial_\theta^2 g(r, r, 0) = \frac{r}{4}, \\ \sum_{l=0}^{\infty} \epsilon_l l^4 J_l(r)^2 &= \partial_\theta^4 g(r, r, 0) = \frac{r^2(4 + 3r^2)}{8}. \end{aligned}$$

These formulas are exact and easy to obtain, as one does not need to carry out the hard frequency analysis that constitutes the core of this chapter. Of course, one can plug the values of these sums in Lemma 3.4.2 to readily recover the formula (3.38) for the expected number of critical points.

# Chapter 4

# Nodal set of monochromatic waves satisfying the Random Wave model

## 4.1 Introduction

In this chapter we explore the connection between the RWM (and Yau's conjecture) and Helmholtz's, see Section 1.1.1. We are only aware of one instance when the RWM can be deterministically implemented to obtain information about the nodal set: Bourgain [Bou14] showed that certain eigenfunctions on the flat two dimensional torus behave accordingly to the RWM and deduced (1.6). Subsequently, Buckley and Wigman [BW16] extended Bourgain's work to "generic" toral eigenfunctions and A. Sartori [Sar20] proved a small scales version of (1.6).

Here, we construct deterministic solutions to (1.4) on $\mathbb{R}^m$ which satisfy the RWM, in the sense of Bourgain [Bou14], in growing balls around the origin. We then use the RWM to study their nodal set, deduce the analogue of (1.6), (1.7) and also find the asymptotic number of nodal domains belonging to a fixed topological class and with a nesting tree configuration. These results appear to be new for $m > 2$ (the study of the nodal volume also for $m = 2$) and they present new difficulties such as the existence of long and narrow nodal domains and the possible concentration of the nodal set in small portions of space. We overcome the far from trivial difficulties using precise bounds on the average *doubling index*, an estimate of the growth rate introduced by Donnelly-Fefferman [DF88] (see Section 4.2.3), using recent ideas of Chanillo, Logunov, Malinnikova and Mangoub, [CLM+20]. In particular, our proofs show how integrability properties of the doubling index allow to extrapolate information about the zero set of Laplace eigenfunctions from the RWM. Furthermore, our new approach (based on the weak convergence of probability measures on $C^s$ spaces, Section 4.2.2, and Thom's Isotopy Theorem 4.2.11) gives us an answer to previous questions raised by Wigman and Kulberg, see Section 4.7.2.

### 4.1.1 The eigenfunctions

Let $m \geqslant 2$ be a positive integer, $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ be unit sphere and $\{r_n\}_{n\geqslant 1} \subset \mathbb{S}^{m-1}$ be a sequence of vectors linearly independent over $\mathbb{Q}$ such that they are not all contained in a hyperplane[1], we will give some properties and examples of such sequences in

[1]As it will be discusses later, this is a technical, but necessary requirement for our construction to be non-degenerate.

Section 4.1.5 below. The functions we study are

$$f_N \equiv f := \frac{1}{\sqrt{2N}} \sum_{|n|\leqslant N} a_n e(\langle r_n, \cdot\rangle) \tag{4.1.1}$$

with domain $\mathbb{R}^m$, $a_n$ are complex numbers such that $|a_n| = 1$, $e(\cdot) := e^{2\pi i \cdot}$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. Moreover, we require $\overline{a}_n = a_{-n}$ so that $f$ is real valued, as $r_{-n} := -r_n$ for $n > 0$.

Differentiating term by term, we see that

$$\Delta f = -4\pi^2 f,$$

thus, $f$ is a solution of the Helmholtz equation in $\mathbb{R}^m$. Moreover, the *high-energy* limit of $f$ is equivalent to its behaviour in $B(R) = B(R, 0)$, the ball of radius $R$ centred at the origin, as $R \to \infty$. Indeed, rescaling $f$ to $f_R := f(R\cdot)$, then

$$\Delta f_R = -4\pi^2 R^2 f_R.$$

Thus $(2\pi R)^2$ plays precisely the role of $\lambda$ of Section 1.1.1.

The functions in (4.1.1) do not satisfy any boundary condition, so the spectrum is continuous; however, following Berry [Ber83], they can be adapted to satisfy either Dirichlet or Neumann boundary conditions on a straight line. It is plausible that our arguments work also in the boundary-adapted case with minor adjustments, but we do not pursue this here. Moreover, we have assumed for the sake of simplicity that $|a_n| = 1$, but more general coefficients could be considered.

Finally, it will be important to keep track of the position of the set $r := \{r_n\}_{n\geqslant 1}$ through the following probability measure supported on $\mathbb{S}^{m-1}$:

$$\mu_{r,N} = \mu_r = \frac{1}{2N} \sum_{|n|\leqslant N} \delta_{r_n}, \tag{4.1.2}$$

where $\delta_{r_n}$ is the Dirac distribution supported at $r_n$. Since the set of probability measures on $\mathbb{S}^{m-1}$ equipped with the weak$^*$ topology is compact (as a standard diagonal argument shows), up to passing to a subsequence, from now on we assume that $\mu_r$ converges to some probability measure $\mu$ as $N \to \infty$.

### 4.1.2 Statement of main results, the nodal set of $f$

Let $R > 1$ and denote by $\mathcal{N}(f, R)$ the number of nodal domains of $f$ in the ball of radius $R$ centred at 0 which do not intersect $\partial B(R)$, the boundary of $B(R)$, and let $\mathcal{V}(f, R) := \mathcal{H}^{m-1}\{x \in B(R) : f(x) = 0\}$. Moreover, given a probability measure $\mu$ on $\mathbb{S}^{m-1}$, let $c_{NS}(\mu)$ be the Nazarov-Sodin constant, see Section 4.2.4 below. Then, for the functions $f$ as in (4.1.1) we prove the following asymptotic statements:

**Theorem 4.1.1.** Let $f$ be as in (4.1.1), then we have

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{N}(f, R)}{\operatorname{vol} B(R)} - c_{NS}(\mu) \right| = 0, \tag{4.1.3}$$

where $c_{NS}(\mu)$ is the Nazarov-Sodin constant of the field $F_\mu$.

**Remark 4.1.2.** Note that this kind of double limits gives us the deterministic realizations we are looking for. Indeed, the statement is equivalent to: given some $\varepsilon > 0$, then there exist some $N_0 = N_0(\varepsilon, m)$ such that all $N \geqslant N_0$ the following holds: there exists some $R_0 = R_0(N, \varepsilon, m)$ such that $R > R_0$, we have

$$\left| \frac{\mathcal{N}(f, R)}{\operatorname{vol} B(R)} - c_{NS}(\mu) \right| \leqslant \varepsilon, \tag{4.1.4}$$

that is, it satisfies the Nazarov-Sodin growth with a constant as close as we want to $c_{NS}(\mu)$. The question of whether we can take the limit of $N$ first will be analyzed in Section 4.7.

**Theorem 4.1.3.** Let $f$ be as in (4.1.1), then we have

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{V}(f, R)}{\operatorname{vol} B(R)} - c(\mu) \right| = 0, \tag{4.1.5}$$

for some (explicit) constant $c(\mu) > 0$.

**Remark 4.1.4.** Note that, rescaling $f_R = f(R\cdot)$, then Theorem 4.1.3 gives

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{V}(f_R, 1)}{R} - c_1(\mu) \right| = 0,$$

for some constant $c_1(\mu) > 0$, in accordance with (1.7) if $(2\pi R)^2 = \lambda$.

One of the main new ingredient in the proof of Theorem 4.1.1 is, in the terminology of Nazarov and Sodin [NS16], the semi-local behaviour of the nodal domains count of $f$, that is, we have the following:

**Proposition 4.1.5.** Let $f$ be as in (4.1.1), $R \geqslant W \geqslant 1$. Then, we have

$$\frac{\mathcal{N}(f, R)}{\operatorname{vol} B(R)} = \frac{1}{\operatorname{vol} B(W)} \fint_{B(R)} \mathcal{N}(f, B(x, W))dx + O\left(W^{-1}\right) + O_{N,W}\left(R^{-\Lambda - 3/2}\right).$$

For $m = 2$, Proposition 4.1.5 follows from the bound $\mathcal{V}(f, R) \lesssim R^m$, see for example Section 4.2.3 below, which implies that most nodal domains have diameter at most $O(1)$. However, for $m > 2$, this argument does not rule out the existence of many long and narrow nodal domains. Following the recent preprint of Chanillo, Logunov, Malinnikova and Mangoubi [CLM+20], $f$ should grow fast around such nodal domains and this can be estimated in terms of the *doubling index* of $f$, see Section 4.2.3 below. The proof of Proposition 4.1.5 then relies on precise estimates on the average growth of $f$, which we obtain in Section 4.4.2. Using the aforementioned estimates, we are also able to show that there is no concentration of nodal volume of $f$ in small portion of the space. That is, we prove the following proposition which will be one of the main ingredients in the proof of Theorem 4.1.3:

**Proposition 4.1.6.** Let $F_x$ be as (4.1.6), then for some (fixed) $\alpha > 0$ there exists $R_0 = R_0(N, W, \alpha)$ such that for all $R \geqslant R_0$, we have

$$\fint_{B(R)} \mathcal{V}(f, B(x, W))^{1+\alpha} dx \lesssim W^{m(1+\alpha)} + W^{(m-1)(1+\alpha)^2} + O_{N,W,\alpha}(R^{-\Lambda - 3/2}),$$

where the constant in the $\lesssim$-notation is independent of $N$.

### 4.1.3 De-randomisation

In this section we make precise in which sense $f$ satisfies the RWM. We first need to introduce some notation: let $R \gg W > 1$ be some parameters, where $R$ is much larger than $W$, and let $F_{x,W,R,N} = F_x$ be the restriction of $f$ to $B(x, W)$, the ball of radius $W$ centred at $x \in B(R)$, that is,

$$F_x(y) := \frac{1}{\sqrt{2N}} \sum_{|n| \leqslant N} a_n e(r_n \cdot x) e(r_n \cdot y) \tag{4.1.6}$$

for $y \in B(W)$ and $x \in B(R)$. Here, we show that, as we sample $x$ uniformly in $B(R)$, the ensemble $\{F_x\}_{x \in B(R)}$ approximates, arbitrarily close, the centred stationary Gaussian field with spectral measure $\mu$. We denoted the said field by $F_\mu$ and collect the relevant background in Section 4.2.1 below.

To quantify the distance between $F_x$ and $F_\mu$, given some integer $s \geqslant 0$ and $W \geqslant 1$, we consider their *pushforward* probability measures (see Section 4.1.8 below) on the space of (probability) measures on $C^s(B(W))$, the class of $s$ continuously differentiable functions on $B(W)$. Since the space of probability measure on $C^s(B(W))$ is metrizable via the Prokhorov metric $d_P$, we define the distance between $F_x$ and $F_\mu$ as the distance between their pushforward measures. More precisely, given to random fields $F, F'$ defined on two, possibly different, probability spaces with measures $\mathbb{P}$ and $\mathbb{P}'$, we write $d_P(F, F') := d_P(F_*\mathbb{P}, F'_*\mathbb{P}')$, where $F_*\mathbb{P}$ is the pushforward probability measure. We collect the relevant background in Section 4.2.2 below.

With this notation, we prove the following:

**Theorem 4.1.7.** Let $f$ and $F_x$ be as in (4.1.1) and (4.1.6) respectively, $W > 1$, and $s \geqslant 0$. Then we have

$$\lim_{N \to \infty} \limsup_{R \to \infty} d_P(F_x, F_\mu) = 0,$$

where the convergence is with respect to the $C^s(B(W))$ topology.

One of the main ingredients in the proof of Theorem 4.1.7 will be the computation of the $L^p$-norms of $F_x$ and from these deduce its Gaussian behaviour. In particular, in Proposition 4.3.3 we will show that

$$\frac{1}{\operatorname{vol} B(R)} \int_{B(R)} |F_x(y)|^{2p} dx = \frac{(2p)!}{p! 2^p} (1 + o_{N,R \to \infty}(1)),$$

uniformly for $y \in B(W)$, that is, $F_x$ has (asymptotically) real Gaussian moments.

### 4.1.4 Topologies and nesting trees

In this section we present a strengthening of Theorem 4.1.1 in that we study nodal domains restricted to a particular topological class or nesting tree. First, we need to introduce some definitions following [SW19]. Let $\Sigma \subset \mathbb{R}^m$ be a smooth, closed, boundaryless, orientable submanifold and denote by $[\Sigma]$ its diffeomorphism class, that is, $\Sigma' \sim \Sigma$ if and only if there exists a diffeomorphism $\Phi$ such that $\Phi(\Sigma) = \Sigma'$, and let $H(m-1)$ be the set of diffeomorphism types $[\Sigma]$. Moreover, since $V(R) := f^{-1}(0) \cap B(R)$ is a smooth $m-1$-dimensional manifold (if the zero set is regular), we can decompose $V(R)$ into its connected components $V(R) = \bigcup_{c \in \mathcal{C}(f;R)} c$, where we ignore components which intersect $\partial B(R)$. Similarly, we can decompose $B(R)\backslash V(R) = \bigcup_{\alpha\in\mathcal{A}(f;R)} \alpha$ as an union of connected components. We define the tree $X(f;R)$ where the vertices are $\alpha \in \mathcal{A}(f,R)$ and there is an edge between $\alpha, \alpha' \in \mathcal{A}(f;R)$ if the share an (unique) common boundary $c \in \mathcal{C}(f;R)$. Let $\mathcal{T}$ be the set of finite rooted trees.

We define $\mathcal{N}(f,\mathcal{S},R)$ where $\mathcal{S} = \{[\Sigma_\beta]\}_{\beta\in\mathfrak{B}_\mathcal{S}} \subset H(m-1)$ as the number of nodal components of $f$ in $B(R)$, which do not intersect $\partial B(R)$ and diffeomorphic to some $\Sigma_\beta \in \mathcal{S}$. Given $T \in \mathcal{T}$, we define $\mathcal{N}(f,T,R)$ similarly. With this notation, we prove the following:

**Theorem 4.1.8.** Let $f$ be as in (4.1.1), $\mathcal{S} \subset H(m-1)$ and $T \in \mathcal{T}$, then we have

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{N}(f,\mathcal{S},R)}{\mathrm{vol}\, B(R)} - c(\mathcal{S},\mu) \right| = 0$$
$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{N}(f,T,R)}{\mathrm{vol}\, B(R)} - c(T,\mu) \right| = 0,$$

for some constants $c(\mathcal{S},\mu)$ and $c(T,\mu)$.

We observe that Theorem 4.1.1 follows from Theorem 4.1.8 choosing $\mathcal{S} = H(n-1)$. Therefore, we only need to prove Theorem 4.1.8.

### 4.1.5 Examples and properties of the $r_n$'s

In this section, we give two examples of sequences $\{r_n\} \subset \mathbb{S}^{m-1}$ being $\mathbb{Q}$-linearly independent.

**Example 4.1.9.** For $m = 2$, identifying $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z} \simeq [0,1]$, we may take a sequence of rational numbers $\{b_n\}$ in $(1,e)$ then $\log b_n = r_n$ is linearly independent over $\mathbb{Q}$ by Baker's theorem [Bak66]. For $m > 2$, we may take a vector the first co-ordinate of which is $\log b_n$.

**Example 4.1.10.** For $\mathbb{S}^{m-1}$, we can construct the sequence as follows. Let $r_1$ be a point on $\mathbb{S}^{m-1}$ and define $S_1 := \mathbb{S}^{m-1}\backslash\overline{\mathbb{Q}}r_1$, the span with algebraic coefficients of $r_1$. As we are removing a countable set from an uncountable set, $S_1$ is non-empty, in fact, uncountable, thus we may choose any $r_2 \in S_1$. For a general $n \geqslant 2$, let $S_n := S_{n-1}\backslash\overline{\mathbb{Q}}r_{n-1}$ and $r_n \in S_n$. By induction, bearing in mind that for sets $A, B, C$, $(A\backslash B)\backslash C = A\backslash(B\cup C)$, the sequence is rational independent and, by construction, we can also choose the $r_n$'s such that they uniformly distribute over $\mathbb{S}^{m-1}$. In particular, we may choose a sequence of $r_n$ such that $\mu_r$ weak* converges to the Lebesgue measure on $\mathbb{S}^{m-1}$.

In particular, Example 4.1.10 shows that if we chose the $r_n$ uniformly at random from $\mathbb{S}^{m-1}$, then the rational independence assumption would hold almost surely. This implies that our assumptions are somehow "generic". Finally, we will repeatedly use the following consequence of the $\mathbb{Q}$-linear independence of the vectors $\{r_n\}$: by a compactness argument, for any $N > 1$ and $T \geqslant 1$ there exists some $\gamma = \gamma(N,T)$ such that for any $t \leqslant T$

$$|r_{n_1} + ... + r_{n_t}| > \gamma(N,T) > 0, \tag{4.1.7}$$

for all $|n_1| \leqslant N$,...,$|n_t| \leqslant N$, unless $t$ is even and, up to permuting the indices, $r_{n_1} = -r_{n_2}, ..., r_{n_{t-1}} = -r_{n_t}$.

### 4.1.6 Plan of the proofs

*Proof of Theorem* 4.1.7, *Section* 4.3. The proof of Theorem 4.1.7 follows from an application of Bourgain's de-randomisation: roughly speaking, the linear independence of the sequence $\{r_n\}$ implies *asymptotic* independence of the waves $e(\langle r_n, x\rangle)$ under the uniform measure in $B(R)$, thus the asymptotic Gaussian behaviour of $F_x$ as in (4.1.6) is *expected* from the Central Limit Theorem, although we cannot directly apply the CLT as our waves are not independent.

To make this intuition precise, following Bourgain, we introduce an additional parameter $K \geqslant 1$ and consider an auxiliary function:

$$\phi_x(y) := \sum_{k\in\mathcal{K}} \left[ \frac{1}{(2N\mu_r(I_k))^{1/2}} \sum_{r_n\in I_k} a_n e(r_n \cdot x) \right] \mu_r(I_k)^{1/2} e\left(\zeta^k \cdot y\right) \tag{4.1.8}$$

where the $\zeta^k \in I_k \subset \mathbb{S}^{m-1}$ for $k \in \mathcal{K}$ are appropriately chosen points and the $I_k$ form a particular subset of a partition of the sphere, see (4.3.3). First, in Lemma 4.3.1, using asymptotic results for Bessel functions, we show that $\phi_x$ is, on average, a good approximation of $F_x$ as the number of $\zeta^k$ grows, that is,

$$\fint_{B(R)} \|\phi_x - F_x\|^2_{C^s(B(W))}\, dx = o(1) \quad \text{as } K, R \to \infty.$$

The advance in passing to $\phi_x$ is that we isolate the contribution of the "wave-packets"

$$b_k := \frac{1}{(2N\mu_r(I_k))^{-1/2}} \sum_{r_n\in I_k} a_n e(r_n \cdot Rx);$$

this allows us to show, see Lemma 4.3.2, that the $b_k'$s are asymptotically (as $N, R \to \infty$) i.i.d. complex standard Gaussian random variables. Thus, we can "approximate" $\phi_x$ in the $C^s(B(W))$ topology by the random field

$$\kappa_K F_{\mu_K}(y) := \sum_{k\in\mathcal{K}} \mu_r(I_k) c_k e\left(\zeta^k, y\right) \tag{4.1.9}$$

with the $c_k$ i.i.d. complex standard Gaussian random variables and $\kappa_K$ a normalizing factor, see (4.3.21). Finally, we let $K$ go to infinity so that the field $F_{\mu_K}$ will "converge" to $F_\mu$. We observe that passing to $\phi_x$ gives a stronger statement than Theorem 4.1.7 because $\phi_x$ and $F_x$ are defined on the same probability space and are $C^s$ close in $L^2$, not just with respect to the Prokhorov distance.

*Proof of Theorem* 4.1.8, *Sections* 4.4 *and* 4.5. We discuss the proof of the (simpler) Theorem 4.1.1. The starting point is Proposition 4.1.5:

$$\frac{\mathcal{N}(f,R)}{\operatorname{vol} B(R)} = \frac{1}{\operatorname{vol} B(W)} \fint_{B(R)} \mathcal{N}(F_x, W)dx + O\left(\frac{1}{W}\right) + O_{N,W}\left(\frac{1}{R^{(m+1)/2}}\right). \tag{4.1.10}$$

As mentioned in the introduction, to prove (4.1.10), we need to discard the possibility of long and narrow nodal components of $f$ which intersect many balls $B(x, W)$. Following the recent preprint of Chanillo, Logunov, Malinnikova and Mangoubi [CLM+20], $f$ has to grow very fast in balls around such nodal domains, this can be quantified using the *doubling index*[2] of $f$ in a ball $B(x, W)$:

$$\mathfrak{N}_f(B(x,W)) := \log \frac{\sup_{B(x,2W)} |f|}{\sup_{B(x,W)} |f|} + 1.$$

In Lemma 4.4.5, we show that $\mathfrak{N}_f(x, W)$ is not too big in an appropriate average sense. Therefore long and narrow nodal domains are "rare" and contribute only to the error term in (4.1.10). This will be the content of Section 4.4.

Next, we show that Theorem 4.1.7 together with the stability of the nodal set (Proposition 4.5.2) imply that

$$\mathcal{N}(F_x, W) \xrightarrow{d} \mathcal{N}(F_\mu, W) \quad \text{as } N, R \to \infty, \tag{4.1.11}$$

where the convergence is in distribution. Thanks to the Faber-Krahn inequality [Cha84, Chapter 4], see also [Man08, Theorem 1.5],

$$\sup_x \mathcal{N}(F_x, W) \lesssim W^m,$$

thus, uniform integrability or Portmanteau Theorem, together with (4.1.10) and (4.1.11) give

$$\fint_{B(R)} \mathcal{N}(F_x, W)dx = \mathbb{E}[\mathcal{N}(F_\mu, W)](1 + o(1)) \quad \text{as } N, R \to \infty. \tag{4.1.12}$$

This is proved in Proposition 4.5.1. Finally, we evaluate the right hand side of (4.1.12) using the work of Nazarov-Sodin [NS16], thus concluding the proof of Theorem 4.1.1.

*Proof of Theorem* 4.1.3, *Section* 4.6. The proof of Theorem 4.1.3 follows the same strategy as the proof of Theorem 4.1.1, with the additional difficulty that $\mathcal{V}(F_x)$ may be unbounded in the supremum norm. To circumvent this problem, and thus apply the uniform integrability theorem, we show in Proposition 4.1.6 that $\mathcal{V}(F_x, W)$ is uniformly integrable. The proof relies on the estimate on $\mathfrak{N}_f(x, W)$ which we obtained in Section 4.4.2. Once Proposition 4.1.6 is proved, the proof of Theorem 4.1.3 follows step by step the proof of Theorem 4.1.1.

Finally in Section 4.7 we collect some final comments and in the appendix some proofs for completeness.

### 4.1.7 Related work

*De-randomisation*. Ingremeau and Rivera [IR20] applied the technique on Lagrangian states, that is, functions of the form $f_h(x) = a(x)e^{i\theta(x)/h}$. The authors show that the long time evolution by the semiclassical Schrödinger operator of (a wide family of) Lagrangian states on a negatively curved compact manifold satisfy the RWM in a sense similar to Theorem 4.1.7. Thus, they provide a family of functions on negatively curved manifolds satisfying the RWM.

[2] In the literature, the doubling index is usually denote by $N$ or $\mathcal{N}$. Since this would clash with the $N$ in (4.1.1) or the $\mathcal{N}$ of nodal domains, we opted for $\mathfrak{N}(\cdot)$. We will slightly modify the definition later.

*Nodal domains*. The study of $\mathcal{N}$ for Gaussian fields started with the breakthrough work of Nazarov and Sodin [NS09; NS16]. They found the asymptotic law of the expected number for nodal domains of a stationary Gaussian field in growing balls, provided its spectral measure satisfies certain (simple) properties, importantly the spectral measure should not have atoms. That is, given a (nice) Gaussian field with spectral measure $\mu$, there exists some constant $c_{NS}(\mu) > 0$ such that

$$\lim_{R\to\infty} \frac{\mathcal{N}(F_\mu, R)}{\mathrm{vol}(B(R))} = c_{NS}(\mu), \tag{4.1.13}$$

where the convergence is a.s. and in $L^1$.

As far as deterministic results about $\mathcal{N}$ are concerned, Ghosh, Reznikov and Sarnak [GRS17; GRS13], assuming the appropriate Lindelöf hypothesis, showed that $\mathcal{N}(\cdot)$ grows at least like a power of the eigenvalue for individual Hecke-Maass eigenfunctions. Jang and Jung [JJ18] obtained unconditional results for individual Hecke-Maass eigenfunctions of arithmetic triangle groups. Jung and Zelditch [JZ16] proved, generalising the geometric argument in [GRS17; GRS13], that $\mathcal{N}(\cdot)$ tends to infinity, for most eigenfunctions on certain negatively curved manifolds, and Zelditch [Zel16] gave a logarithmic lower bound. Finally, Ingremeau [Ing18] gave examples of eigenfunctions with $\mathcal{N}(\cdot) \to \infty$ on unbounded negatively-curved manifolds.

*Topological classes*. Sarnak and Wigman [SW19] and Sarnak and Canzani [CS19] proved the analogous result of (4.1.13) for $\mathcal{N}(F_\mu, T, R)$ and $\mathcal{N}(F_\mu, H, R)$, again, for spectral measures with no atoms. For deterministic results, Enciso and Peralta-Salas [EPS13] proved the existence of functions $g$ (in the more general setting of elliptic equations and non-necessarily compact components) such that $\mathcal{N}(g, H, R) > 0$ and this property is valid even if we perturb $g$ in a $C^k$ norm. This is the key element to prove the positivity of the constants $c(H, \mu)$ of the analogous result of (4.1.13). It is also worth mentioning that Enciso and Peralta-Salas' techniques can be applied to solve another problem raised by M. Berry [Ber01] related to the existence of (complex) eigenfunctions of a quantum system whose nodal set has components with arbitrarily complicated linked and knotted structure, [EHPS18]. Furthermore, somehow related techniques for the construction of specific structurally stable examples applied to dynamical systems play a fundamental role in an extension of Nazarov-Sodin's theory to Beltrami fields. These fields are (vector-valued) eigenfunctions of the curl (instead of the Laplacian treated here) and they are a key element in fluid dynamics; turbulence can only appear in a fluid in equilibrium through Beltrami fields. This extension allows one to stablish V. I. Arnold's long standing conjecture on the complexity of Beltrami fields (i.e., a typical Beltrami field should exhibit chaotic regions coexisting with a positive measure set of invariant tori of complicated topology), see [EPSR20].

### 4.1.8 Notation

We will use the standard notation $\lesssim$ to denote $\leqslant C$, where the constant can change its value between equations, and $m \geqslant 2$ will be a positive integer which denotes the dimension of the space and $\Lambda = (m-2)/2$. Moreover, given a large parameter $R > 1$, we denote by $B(R)$ the ball of radius $R$ in $\mathbb{R}^m$ and by $\overline{B(R)}$ its closure. Given some $r > 0$ and a ball $B$, we denote by $rB$ the concentric ball with $r$-times the radius.

We write

$$\fint_{B(R)} h(x)dx := \frac{1}{\operatorname{vol} B(R)} \int_{B(R)} h(x)dx = \int_{B(R)} h(x)d\operatorname{vol}_R(x)$$

where $\operatorname{vol}_R$ for the uniform probability measure on $B(R)$. Furthermore, we denote by $(\Omega, \mathbb{P})$ an abstract probability space where every random object is defined and, given a probability measure $\mu$ on $\mathbb{S}^{m-1}$, we denote by $F_\mu$ the centred, stationary Gaussian field with spectral measure $\mu$, see Section 4.2.1 for more details.

Given two measurable spaces $(Y, \Sigma)$ and $(X, \mathcal{F})$, a measurable mapping $g\colon Y \to X$ and a measure $\mu$ on $Y$, the *pushforward* of $\mu$, denoted by $g_*\mu$, is

$$g_*\mu(B) := \mu\left(g^{-1}(B)\right)$$

for $B \in \mathcal{F}$. Note that $g_*\mu$ is well-defined as $g$ is measurable. Finally, given some function $g : \mathbb{R}^m \to \mathbb{R}$ and a set $A \subset \mathbb{R}^m$, we denote by $g|_A$ the restriction of $g$ to $A$.

## 4.2 Preliminaries

### 4.2.1 Gaussian fields background

We briefly collect some definitions about Gaussian fields (on $\mathbb{R}^m$). For us, a (real-valued) Gaussian field $F$ is a continuous map $F : \mathbb{R}^m \times \Omega \to \mathbb{R}$ for some probability space $\Omega$, such that all finite dimensional distributions $(F(x_1, \cdot), ...F(x_k, \cdot))$ are multivariate Gaussian. We say that $F$ is *centred* if $\mathbb{E}[F] \equiv 0$ and *stationary* if its law is invariant under translations $x \to x + \tau$ for $\tau \in \mathbb{R}^m$. In this script, every Gaussian field is both centred and stationary. Then, the *covariance* function of $F$ is

$$\mathbb{E}[F(x) \cdot F(y)] = \mathbb{E}[F(x-y) \cdot F(0)].$$

Since the covariance is positive definite, by Bochner's theorem, it is the Fourier transform of some measure $\mu$ on $\mathbb{R}^m$. So we have

$$\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^m} e\left(\langle x-y, \lambda\rangle\right) d\mu(\lambda).$$

The measure $\mu$ is called the *spectral measure* of $F$ and, since $F$ is real-valued, it satisfies $\mu(-I) = \mu(I)$ for any (measurable) subset $I \subset \mathbb{R}^m$, that is, $\mu$ is a symmetric measure. By Kolmogorov theorem, $\mu$ fully determines $F$, so we simply write $F = F_\mu$.

### 4.2.2 Weak convergence of probability measures in the $C^s$ space.

Let $S = C^s(V)$ be the space of $s$-times, $s \geqslant 0$ integer, continuously differentiable functions on $V$, a compact set of $\mathbb{R}^m$. In this section we review the conditions to ensure that a sequence of probability measures $\{\mu_n\}$ on $S$ converges weakly to another probability measure, $\mu$, see also [Bil13, Chapter 7] for $s = 0$.

First, since $S$ is a separable metric space, Prokhorov's Theorem [Bil13, Chapters 5 and 6] implies that $\mathcal{P}(S)$, the space of probability measures on $S$, is metrizable via the *Lévy–Prokhorov metric*. This is defined as follows: for a subset $B \subset S$, let denoted by $B_{+\varepsilon}$ the $\varepsilon$-neighbourhood of $B$, that is,

$$B_{+\varepsilon} := \{p \in S \mid \exists\, q \in B,\ \|p-q\|_{C^s} < \varepsilon\}.$$

The *Lévy–Prokhorov metric* $d_P : \mathcal{P}(S) \times P(S) \to [0, +\infty)$ is defined for two probability measures $\mu$ and $\nu$ as:

$$d_P(\mu, \nu) := \inf_{\varepsilon > 0} \{\mu(B) \leqslant \nu(B_{+\varepsilon}) + \varepsilon,\ \nu(B) \leqslant \mu(B_{+\varepsilon}) + \varepsilon\, \forall\, B \in S\}\,. \tag{4.2.1}$$

It is well-known [Pri93, Claim below Lemma 2] and [Wil86] that if the finite dimensional distributions of some sequence $X_n$ taking values on $S$ converge to some random variable $X$, that is for all $y_1, ..., y_l \in V$

$$(X_n(y_1), ...., X_n(y_l)) \xrightarrow{d} (X(y_1), ...., X(y_l)) \quad \text{as } n \to \infty \tag{4.2.2}$$

where the convergence is in distribution, and the sequence $\{(X_n)_* \mathbb{P}\}$ is *tight*, then $(X_n)_* \mathbb{P}$ converges to $(X)_* \mathbb{P}$ in $\mathcal{P}(S)$ equipped with the metric $d_P$. A set of probability measured $\Pi$ on $S$ is tight if for any $\varepsilon > 0$ there exists a compact subset $Q_\varepsilon \subset S$ such that, for all measures $\nu \in \Pi$, $\nu(Q_\varepsilon) > 1 - \varepsilon$.

A characterization of tightness in $\mathcal{P}(S)$ is given in the next lemma, which can be seen as a probabilistic version of Arzelà-Ascoli Theorem. Let us define the modulus of continuity of a function $g \in S$ as:

$$\omega_g(\delta) := \sup_{\|y-y'\| \leqslant \delta} \{|g(y) - g(y')|\}. \tag{4.2.3}$$

We then have following lemma [Pri93, Lemma 1]:

**Lemma 4.2.1.** A sequence $\{\mu_n\}$ of probability measures on $S$ is tight if and only if

i) For some $y \in V$ and $\varepsilon > 0$ there exists $M > 0$ such that, uniformly in $n$:

$$\max_{|\alpha| \leqslant s} \mu_n(g : |D^\alpha g(y)| > M) \leqslant \varepsilon \quad .$$

ii) For all multi-index $\alpha$ such that $|\alpha| = s$ and $\varepsilon > 0$, we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mu_n(g : \omega_{D^\alpha g}(\delta) \geqslant \varepsilon) = 0.$$

Finally, we will need the following result of uniform integrability [Bil13, Theorem 3.5].

**Lemma 4.2.2.** Let $X_n$ a sequence of random variables such that $X_n \xrightarrow{d} X$ (i.e., in distribution). Suppose that there exists some $\alpha > 0$ such that $\mathbb{E}[|X_n|^{1+\alpha}] \leqslant C < \infty$ for some $C > 0$, uniformly for all $n \geqslant 1$. Then,

$$\mathbb{E}X_n \to \mathbb{E}X.$$

### 4.2.3 Doubling index

Following and Donnelly-Fefferman [DF88] and Logunov and Malinnikova [Log18a; Log18b; LM18], given a function $h : \mathbb{R}^m \to \mathbb{R}$, we define the *doubling index* of $h$ in $B$ as

$$\mathfrak{N}_h(B) := \log \frac{\sup_{\varkappa_m B} |h|}{\sup_B |h|} + 1, \tag{4.2.4}$$

with $\varkappa_m := 2\sqrt{m}$. The doubling index gives a bound on the nodal volume of $f$, as in (4.1.1), thanks to the following result [DF88, Proposition 6.7] and [LM19, Lemma 2.6.1].

**Lemma 4.2.3.** Let $B \subset \mathbb{R}^m$ be the unit ball, suppose that $h : 3B \to \mathbb{R}$ is an harmonic function, that is, $\Delta h = 0$, then

$$\mathcal{V}(h, 1/2) \lesssim \mathfrak{N}_h(B).$$

Applying Lemma 4.2.4 to the lift $h(x,t) := f(x)e^{2\pi t} : \mathbb{R}^{m+1} \to \mathbb{R}$, we obtain the following:

**Lemma 4.2.4.** Let $f$ be as (4.1.1) and $r > 1$ be some parameter, then

$$\mathcal{V}(f, B(r)) \cdot r^{-m+1} \lesssim \mathfrak{N}_f(B(3r)) + r.$$

*Proof.* First, we observe that the function $h(x,t) := f(x)e^{2\pi t}$ is harmonic in a ball $B(\sqrt{2}r) \supset B(r) \times [-r, r]$ and that

$$\mathcal{H}^{m-1}\{x \in B(r) : f(x) = 0\} \times 2r \leqslant \mathcal{H}^m\{(x,t) \in B(\sqrt{2}r) : h(x,t) = 0\}$$

Therefore, rescaling $B(\sqrt{2}r)$ to a ball of radius one, the lemma follows from Lemma 4.2.3, upon noticing that

$$\mathcal{V}(f, B(r))r^{-m+1} \lesssim r + \mathfrak{N}(f, B(cr))$$

for any $c > 2\sqrt{2}$ and that the supremum norm is scale invariant. $\square$

In particular, we can control the doubling index of $f$ using the well-known Nazarov-Turan Lemma, see [Naz93] and [FM06] for the multi-dimensional version:

**Lemma 4.2.5.** Let $g(x) = \sum_{j=1}^J a_j e(\xi_j \cdot x)$ for $x \in \mathbb{R}^m$ and $\xi_1, ...\xi_J$ distinct frequencies, moreover let $B \subset \mathbb{R}^m$ be a ball and $I \subset B$ be a measurable subset. Then there exist absolute constants $c_1, c_2 > 0$ so that

$$\sup_B |g| \lesssim \left( c_1 \frac{|B|}{|I|} \right)^{c_2 J} \sup_I |g|$$

Combining Lemma 4.2.5 with Lemma 4.2.4, we obtain the following:

**Lemma 4.2.6.** Let $f$ be as (4.1.1) and $r > 0$ be some parameter, then

$$\mathcal{V}(f, B(r)) \cdot r^{-m+1} \lesssim N + r.$$

Finally, to study the nodal domains of $f$, we will to use the doubling index to control the growth of $f$ in sets which might not be balls. That is, we will need the following lemma [LM19]:

**Lemma 4.2.7** (Remez type inequality)**.** Let $B$ be the unit ball in $\mathbb{R}^m$ and suppose that $h : 2B \to \mathbb{R}$ ia an harmonic function. Then there exist constants $c_1, c_2 > 0$, independent of $h$, such that

$$\sup_B |h| \lesssim \sup_E |h| \left( c_1 \frac{|B|}{|E|} \right)^{c_2 \mathfrak{N}_h(2B)}$$

for any set $E \subset B$ of positive measure.

Using the harmonic lift $h$ of $f$ as in Lemma 4.2.4 and rescaling, we deduce the following:

**Lemma 4.2.8.** Let $B(r) \subset \mathbb{R}^m$ be a ball of radius $r > 0$ and $f$ be as in (4.1.1) then there exist constants $c_1, c_2 > 0$, such that

$$\sup_{B(r)} |f| \lesssim \sup_{E(r)} |f| \left( c_1 \frac{|B(r)|}{|E(r)|} \right)^{c_2 (\mathfrak{N}_f(B(2r)) + r)}$$

for any set $E(r) \subset B(r)$ of positive measure.

### 4.2.4 Additional Tools

In this section we extend for our purposes the work of Nazarov-Sodin [NS16] and Sarnak-Wigman [SW19] to the case of a possibly atomic symmetric spectral measure and give a sufficient condition for the positivity of the constants $c_{NS}(\cdot)$, $c(T, \cdot)$ and $c(H, \cdot)$ appearing in Theorems 4.1.7 and 4.1.8. For dimension two and for nodal domains, this was done in [KW18, Proposition 1.1], see also Section 4.7.2 below for some additional results. The proof essentially follows [NS16], we reproduce some details for completeness.

Given a probability measure $\mu$ on $\mathbb{S}^{m-1}$ and an integer $s \geqslant 1$, let $\overline{\mathcal{F} L^2_{\mathrm{H}}(\mu)}^{C^s}$, the closure in the Fréchet topology of $C^s$ compact convergence of the Fourier transform of Hermitian functions $h \colon \mathbb{R}^m \to \mathbb{C}$ with $\int |h|^2 \, d\mu < \infty$. Then, bearing in mind the notation in section 4.1.4, we have the following:

**Theorem 4.2.9.** Let $\mu$ be symmetric probability measure on $\mathbb{S}^{m-1}$. Let $\mathcal{S} \subset H(m-1)$ and $T \in \mathcal{T}$. Then, there exist constants $c(\mathcal{S}, \mu), c(T, \mu)$ such that

$$\mathbb{E}[\mathcal{N}(F_\mu, \cdot, R)] = \operatorname{vol} B(R) (c(\cdot, \mu) + o_\mu(1))$$

as $R \to \infty$. The constant $c(\mathcal{S}, \mu)$ will be positive if there is a function $F_0$ with a regular (i.e., the gradient doesn't vanish) connected component in $\mathcal{S}$ contained in $B(r)$ for some $r > 0$ and $F_0 \in \overline{\mathcal{F} L^2_{\mathrm{H}}(\mu)}^{C^s}$, similarly for $c(T, \mu)$.

The last condition means that $F_0$ can be approximated in $C^s(K)$, for $K$ any compact set, by functions in $\mathcal{F} L^2_H(\mu)$.

*Proof.* Let $r > 0$, we define:

$$\Phi_r^{a}(G) := \frac{\mathcal{N}(G,\cdot,r)}{\operatorname{vol} B(r)}, \qquad \Psi_r^{a}(G) := \frac{\mathcal{N}\mathcal{I}(G,\cdot,R)}{\operatorname{vol} B(r)},$$

where $\mathcal{N}\mathcal{I}(G,\cdot,R)$ denotes de number of nodal domains intersecting the boundary of $S(R)$. Since $F_\mu$ is translation invariant by Bochner's Theorem, Wiener's Ergodic Theorem [NS16, Section 6] implies[3] that

$$\frac{1}{\operatorname{vol} B(R)} \int_{B(R)} \Phi_r^{a}(\tau_v \cdot)\, dv \to \bar{\Phi}_r^{a} \qquad R \to \infty \tag{4.2.5}$$

a.s. and in $L^1$, where $\tau_v G := G(\cdot + v)$ for $v \in \mathbb{R}^m$. Moreover, $\bar{\Phi}_r^{a}$ is invariant under $\tau_v$, $\mathbb{E}[\bar{\Phi}_r^{a}] = \mathbb{E}[\Phi_r^{a}]$ and similarly for $\Psi_r^{a}$.

Thanks to the integral-geometric sandwich ([NS16, Lemma 1]), and following the proof of [NS16, Theorem 1], see also the proof of Proposition 4.1.5, we have that (4.2.5) implies that the limit

$$c(G,\cdot,\mu) := \lim_{R\to\infty} \frac{\mathcal{N}(G,\cdot,R)}{\operatorname{vol} B(R)}$$

exists a.s. and in $L^1$. Note that it is not a constant but a random variable, thus letting $c(\cdot,\mu) := \mathbb{E}\left(c(F_\mu,\cdot,\mu)\right)$, the first statement of the theorem follows from the $L^1$ convergence. Let us now consider the positivity of the constants. From (4.2.5) and the integral-geometric sandwich, we have

$$\mathbb{E}\left(\Phi_r^{a}\right) \leqslant c(\cdot,\mu) \leqslant \mathbb{E}\left(\Phi_r^{a}\right) + \mathbb{E}\left(\Psi_r^{a}\right).$$

Thus, in order to prove that $c(\cdot,\mu) > 0$, thanks to Chebyshev's inequality, it is enough to show that

$$\mathbb{P}\big(\big\{F_\mu \in C^s(\mathbb{R}^m) : \mathcal{N}(F_\mu,\mathcal{S},r) \geqslant 1\big\}\big) > 0. \tag{4.2.6}$$

Let $F_0$ be as in the statement of the theorem, by [NS16, Appendix A.7, A.12] for $s = 0$ and [EPSR20, Proposition 3.8] for general $s$, $F_0$ is in the support of the measure on the space of $C^s$ functions of our random field $F_\mu$, that is, for any compact set $K \subset \mathbb{R}^m$ and each $\varepsilon > 0$,

$$\mathbb{P}\big(\big\{F_\mu \in C^s(\mathbb{R}^3) : \|F_\mu - F_0\|_{C^s(K)} < \varepsilon\big\}\big) > 0\,. \tag{4.2.7}$$

Now, as the connected component of $F_0$ in $\mathcal{S}$ is regular by hypothesis, we can apply Thom's Isotopy, Theorem 4.2.11 below, to conclude that if

$$\|F - F_0\|_{C^s(K)} < \delta, \tag{4.2.8}$$

where the connected component of $F_0$ is in the interior of $K$, then $F$ also has a connected component diffeomorphic to $\mathcal{S}$. Finally (4.2.6) follows from (4.2.7), taking $\varepsilon = \delta$ in (4.2.8). We can proceed similarly for nesting trees and conclude the proof. □

[3]We can apply the Ergodic Theorem, despite our field might not be ergodic (by Fomin-Grenander-Maruyama Theorem, see, e.g., [NS16], as $\mu$ might have atoms) because we only need the translational invariance.

**Example 4.2.10.** If $\mu = \sigma_{m-1}$, the Lebesgue measure on the sphere, then it is enough to show $F_0$ is a solution to the Helmholtz equation as this set equals $\overline{\mathcal{F}L^2_{\mathbb{H}}(\sigma)}^{C^s}$ [CS19, Proposition 6]. However, in this case, the construction of the particular functions for topological classes gives $F_0$ as a (finite) sum of the form [EPS13; CS19]

$$F_0(x) = (2\pi)^{\frac{n}{2}} \sum_{l=0}^{L} \sum_{m=1}^{d_l} a_{lm} Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}},$$

so by [EPSR22a, Proposition 2.1] or by Herglotz Theorem [Hör15, Theorem 7.1.28] and the rapid decay of Bessel functions, $F_0 \in \mathcal{F}L^2_{\mathbb{H}}(\sigma)$. For instance, the example mentioned above could be the spherical Bessel functions

$$C_m \frac{J_{\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} = \int_{\mathbb{S}^{m-1}} e^{i\langle x,\omega\rangle} d\sigma_{m-1}(\omega)$$

where $C_m$ is as in (1.5). They are radial solutions to the Helmholtz equation, so the nodal sets are spheres with the radii the zeros of $J_{\frac{n}{2}-1}(|x|)$. See Figure 4.1 for the case of $n = 2$. This proves $c_{NS}(\sigma_{m-1}) := c(H(m-1), \sigma_{m-1}) > 0$ as $c(\sigma_{m-1}, \{[\mathbb{S}^{m-1}]\}) > 0$. See also [NS16, Condition ($\rho$4), Appendix C] for sufficient conditions to ensure $c_{NS}(\mu) > 0$ and [IR18] for an explicit lower bound together with some numerical estimates

The stability property of the nodal set used above is given by the following theorem.

**Theorem 4.2.11** (Compact Thom's Isotopy Theorem)**.** Let $V$ be an domain in $\mathbb{R}^m$ and let $h : V \to \mathbb{R}$ be a $C^\infty$ map. Consider a (compact) connected component $L \subset\subset V$ (i.e., which is compactly embedded in $V$) of the zero set $h^{-1}(0)$ and suppose that:

$$|\nabla h|_L| > 0.$$

Then, given any $\varepsilon > 0$ and $p \geqslant 1$, there exists some $U \subset\subset V$ neighbourhood of $L$ and $\delta > 0$ such that for any smooth function $g : U \to \mathbb{R}^m$ with

$$\|h - g\|_{C^p(U)} < \delta$$

one can transform $L$ by a diffeomorphism $\Phi$ of $\mathbb{R}^m$ so that $\Phi(L)$ is the intersection of the zero set $g^{-1}(0)$ with $U$. The diffeomorphism $\Phi$ only differs from the identity in a proper subset of $U$ (i.e., a subset $\subsetneq U$) and satisfies $\|\Phi - \mathrm{id}\|_{C^p(\mathbb{R}^m)} < \varepsilon$.

The proof follows from [EPS13, Theorem 3.1], we reproduce some details for completeness.

*Proof.* We have to construct a domain $U$ and find some $\eta > 0$ such that the component of $h^{-1}(B(0,\eta))$ connected with $L$ is contained in $U$ and $\inf_U \|\nabla h\| > 0$. For this purpose, let us define the following vector field:

$$X(x) := \frac{\nabla h(x)}{\|\nabla h(x)\|^2}$$

which is well defined if the gradient does not vanish. Denote by $\varphi^t$, the associated flow, that is, the solution to $\partial_t \varphi^t(x) = X(\varphi^t(x))$. Considering the derivative with respect to time, if $h(x) = 0$ then $h(\varphi^t(x)) = t$, and

$$\|\partial_t \varphi^t(x)\| = \|X(\varphi^t(x))\| = \frac{1}{\|\nabla h(\varphi^t(x))\|}. \tag{4.2.9}$$

By compactness and regularity of the connected component, $\|\nabla h|_L\| \in [c, C]$, with $c > 0$. Since $\varphi(t, x) := \varphi^t(x)$ is a smooth map, if we define $H : \mathbb{R} \times L$ as $H(t, x) := \|\nabla h(\varphi(t, x))\|$, then $H^{-1}(c - \delta, \infty)$ is an open set of $\mathbb{R} \times L$, for any $\delta > 0$, and it includes $\{0\} \times L$. By compactness and the product topology, there exists a finite number of $t_i > 0$, $U_i^L$ open sets of $L$ (induced topology) such that

$$\|\nabla h(\varphi(t, x))\| > c_1$$

for $t \in (-t_i, t_i)$, $x \in U_i^L$ and $c_1 := c - \delta$. If we define $\eta := \frac{1}{2}\min\{t_i, c_1 d\}$, where $d := \operatorname{dist}(L, \partial V)$, then we claim that $U := L_{+\eta/c_1}$ is the desired neighbourhood. Indeed, if $y$ is the component of $h^{-1}(B(0, \eta))$ connected with $L$, then $y = \varphi^{\eta'}(x)$ with $\eta' < \eta$, $x \in L$ so by (4.2.9) and Lagrange Theorem

$$\|y - x\| = \|\varphi^{\eta'}(x) - x\| \leqslant \frac{\eta}{c_1},$$

hence, $y \in B(x, \eta/c_1) \subset U$. Furthermore, if $y \in U$, then $\forall\, v \in \partial V$

$$\|y - v\| \geqslant \|x - v\| - \|y - x\| \geqslant d - d/2 > 0,$$

where $y \in B(x, \eta/c_1)$ and $\eta/c_1 < d/2$ by definition. $\square$

## 4.3 Bourgain's de-randomisation, proof of Theorem 4.1.7.

The content of this section follows closely the proofs in [Bou14; BW16] to extend the ideas from $\mathbb{T}^2$ to $\mathbb{R}^m$.

### 4.3.1 The function $\phi_x$

Let $m \geqslant 2$ be fixed, $R \gg W > 1$ be in section 4.1.1. Using hyperspherical coordinates, that is, writing $x \in \mathbb{S}^{m-1}$ as $x = G(\theta)$ where

$$G(\theta) := (\cos \pi\theta_1, \sin \pi\theta_2 \cos \pi\theta_2, ..., \sin \pi\theta_1 \cdots \sin \pi\theta_{m-2} \sin 2\pi\theta_{m-1})$$

such that $G|_{(0,1)^{m-1}}$ is a diffeomorphism onto $\mathbb{S}^{m-1} \backslash S'$, where $S'$ is a set of measure zero, we identify $\mathbb{S}^{m-1}$ with $[0, 1]^{m-1}$. Now, let $K > 1$ be a (large) parameter and divide $[0, 1]^{m-1}$ into $K^{m-1}$ cubes and use hyper-spherical coordinates to divide the sphere into $K^{m-1}$ regions which we call $I_k$. Let $\{\zeta^k\} \subset \mathbb{S}^{m-1}$ be the "centres "of such regions (centre is defined again picking the centre in $[0, 1]^{m-1}$ and projecting onto the sphere using hyper-spherical coordinates). Finally, pick another parameter $\delta > 0$ and let $\mathcal{K}$ to be the set of $k$'s such that $k \in \mathcal{K}$ if and only if

$$\mu_r(I_k) > \delta. \tag{4.3.1}$$

We will need the following two simple properties of this partition:

**Claim 4.3.1.** *We have the following:*

i) $\sum_{k\in\mathcal{K}} \mu_r(I_k) = 1 - \sum_{k\notin\mathcal{K}} \mu_r(I_k) = 1 + O(\delta K^{m-1})$

ii) *If* $r_n \in I_k$, *then* $\|r_n - \zeta^k\| = O(K^{-1})$.

*Proof.* i) follows from the fact that there are at most $K^{m-1}$ elements in the complement of $\mathcal{K}$. ii) follows from the fact that $G|_{[0,1]^{m-1}}$ is a smooth function so it is Lipschitz and, writing $G(\theta_k) = \zeta^k$, we have

$$\|G(\theta) - G(\theta_k)\| \leqslant C_G \|\theta - \theta_k\| .$$

□

As $r_n = -r_{-n}$, in order to count only one these points, we define $\mathcal{K}^+$ as the set of $k \in \mathcal{K}$ such that $(\zeta^k)_j > 0$ with $j := \max_{1\leqslant i\leqslant m}\{(\zeta^k)_i \neq 0\}$, where $(\zeta^k)_i$ denotes the $i$-th component of $\zeta^k$. Note that, by definition,

$$2N\delta \leqslant 2N\mu_r(I_k) = \#\{|n| \leqslant N \ / \ r_n \in I_k\} \to \infty \quad \text{as } N \to \infty. \tag{4.3.2}$$

Finally, we define the auxiliary function, as in (4.1.8)

$$\phi_x(y) := \sum_{k\in\mathcal{K}} \left[ \frac{1}{(2N\mu_r(I_k))^{1/2}} \sum_{r_n \in I_k} a_n e(r_n \cdot x) \right] \mu_r(I_k)^{1/2} e\left(\zeta^k \cdot y\right). \tag{4.3.3}$$

The next lemma shows that $\phi_x$ is, on average, a good approximation of $F_x$.

**Lemma 4.3.1.** Let $F_x$ and $\phi_x$ be as in (4.1.6) and (4.3.3) respectively, $R \gg W > 1$ and $K, \delta > 0$ be as in Section 4.3.1 with $\delta < K^{-m+1}$, $s \geqslant 0$ be some integer and $l = \lfloor \frac{m}{2} + 1 \rfloor$. Then, we have

$$\fint_{B(R)} \|F_x - \phi_x\|^2_{C^s(B(W))} \lesssim W^{2(s+l)+m} \left(\delta K^{m-1} + W^2 K^{-2}\right) \left(1 + O_N(R^{-\Lambda-3/2})\right).$$

*Proof.* Using Sobolev's Embedding Theorem, we bound the $C^s(B(W))$ norm by the $H^{s+l}(B(W))$ norm, and rescaling to a ball or radius one, we obtain

$$\fint_{B(1)} \|F_{Rx} - \phi_{Rx}\|^2_{C^s(B(W))}\, dx \lesssim \sum_{|\alpha|\leqslant s+l} \fint_{B(1)} \|D^\alpha (F_{Rx} - \phi_{Rx})\|^2_{L^2(B(W))}\, dx$$

where $D^\alpha$ is the multi-variable derivative. If $\alpha = 0$, denoting $\mathbb{S}^{m-1}_* := \mathbb{S}^{m-1} \setminus \bigcup_{k\in\mathcal{K}} I_k$ and rescaling the ball of radius $W$ to a ball of radius 1, we have

$$\begin{aligned} \fint_{B(1)} \|(F_{Rx} - \phi_{Rx})\|^2_{L^2(B(W))}\, dx &\lesssim W^m \fint_{B(1)} \|\frac{1}{2N} \sum_{r_n\in\mathbb{S}^{m-1}_*} a_n e(\langle r_n, Rx\rangle) e\left(\langle r_n, Wy\rangle\right)\|^2 dx \\ &+ W^m \fint_{B(1)} \|\sum_{k\in\mathcal{K}} (2N)^{-1/2} \sum_{r_n\in I_k} a_n e(\langle r_n, Rx\rangle) \left( e\left(\langle r_n, Wy\rangle\right) - e\left(\langle \zeta^k, Wy\rangle\right)\right) \|^2 dx. \end{aligned} \tag{4.3.4}$$

To evaluate the integrals in (4.3.4), we will need the following claim:

$$\fint_{B(1)} e(\langle r_n - r_{n'}, Rx\rangle)dx = \begin{cases} 1 & n = n' \\ C_m \dfrac{J_{\Lambda+1}(2\pi R \|r_n - r_{n'}\|)}{(R\|r_n - r_{n'}\|)^{\Lambda+1}} & n \neq n' \end{cases} = \delta_{n,n'} + O_N(R^{-\Lambda-3/2}) \tag{4.3.5}$$

Indeed, by the Fourier Transform of spherical harmonics [EPSR22a, Proposition 2.1]:

$$\int_{\mathbb{S}^{m-1}} Y_l e^{i\langle x,\cdot\rangle} d\sigma = (2\pi)^{\frac{m}{2}} (-i)^l Y_l\left(\frac{x}{|x|}\right) \frac{J_{l+\Lambda}(|x|)}{|x|^{\Lambda}} \quad \Lambda := \frac{m-2}{2}, \tag{4.3.6}$$

where $l$ is the index associated with the eigenvalue and $J_\alpha$ represents the Bessel function of first order and index $\alpha$. Setting $l = 0$ in (4.3.6) and using polar coordinates:

$$\int_{B(1)} e^{i2\pi\langle x,y\rangle} dy = (2\pi) \int_0^1 r^{m/2} \frac{J_\Lambda(r2\pi|x|)}{|x|^\Lambda} dr = \frac{J_{\Lambda+1}(2\pi|x|)}{|x|^{\Lambda+1}}. \tag{4.3.7}$$

Moreover, by the standard asymptotic expansion of Bessel functions [Wat95, Chapter 7]:

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + O_\alpha(z^{-3/2}) \tag{4.3.8}$$

Thus, for $n \neq n'$, using (4.3.7) and (4.3.8), and bearing in mind (4.1.7), (4.3.5) follows upon noticing that

$$J_{\Lambda+1}(2\pi R\|r_n - r_{n'}\|) = O\left((R\|r_n - r_{n'}\|)^{-1/2}\right) = O_N(R^{-1/2}).$$

In order to bound the first term of the RHS of (4.3.4), we expand the square, use Fubini and (4.3.5) to obtain

$$\begin{aligned} &\frac{W^m}{2N} \int_{B(1)} dy \fint_{B(1)} dx \sum_{r_n \in \mathbb{S}_*^{m-1}} \sum_{r_{n'} \in \mathbb{S}_*^{m-1}} a_n \bar{a}_{n'} e(\langle r_n - r_{n'}, Rx\rangle) e(\langle r_n - r_{n'}, Wy\rangle) = \\ &= \frac{W^m}{2N} \int_{B(1)} dy \sum_{r_n \in \mathbb{S}_*^{m-1}} |a_n|^2 + O_N\left(R^{-\Lambda-3/2}\right) \sum_{r_n \in \mathbb{S}_*^{m-1}} \sum_{r_{n'} \in \mathbb{S}_*^{m-1}} a_n \bar{a}_{n'} e(\langle r_n - r_{n'}, Wy\rangle), \end{aligned} \tag{4.3.9}$$

with $r_{n'} \neq r_n$ in the second summand. Since $|a_n| = 1$, bearing in mind (4.3.1) and using Claim 4.3.1, we can bound (4.3.9) by

$$\text{RHS}(4.3.9) \lesssim W^m \delta K^{m-1} \left(1 + O_N(R^{-\Lambda-3/2})\right). \tag{4.3.10}$$

For the second summand of the RHS of (4.3.4) we proceed similarly, taking into account Claim 4.3.1, we have

$$|e(\langle r_n, Wy\rangle) - e\left(\langle \zeta^k, Wy\rangle|\right) \leqslant \left\|r_n - \zeta^k\right\| \|Wy\| \lesssim \frac{W}{K}.$$

Thus, expanding the square and using Fubini and (4.3.5), we can bound the second term on the right hand side of (4.3.4) as

$$\begin{aligned}
&\fint_{B(1)} \| \sum_k (2N)^{-1/2} \sum_{r_n \in I_k} a_n e(\langle r_n, Rx\rangle) \left( e\left(\langle r_n, Wy\rangle\right) - e\left(\langle \zeta^k, Wy\rangle\right)\right) \|^2 dx \lesssim \\
&\lesssim \sum_k (2N)^{-1} \sum_{r_n \in I_k} |1 - e\left(\langle r_n - \zeta^k, Wy\rangle\right)|^2 + |\sum_{r_n \neq r_{n'}} C_m \frac{J_{\Lambda+1}(2\pi R\, \|r_n - r_{n'}\|)}{(R\,\|r_n - r_{n'}\|)^{\Lambda+1}} \times \\
&\times (2N)^{-1} a_n \bar{a}_m \left( e\left(\langle r_n, Wy\rangle\right) - e\left(\langle \zeta^k, Wy\rangle\right)\right)\left( e\left(\langle r_{n'}, Wy\rangle\right) - e\left(\langle \zeta^{k'}, Wy\rangle\right)\right)\Big| \lesssim \\
&\lesssim\ W^2 K^{-2}(1 + O_N(R^{-\Lambda - 3/2})).
\end{aligned} \tag{4.3.11}$$

All in all, using (4.3.10) and (4.3.11), we obtain

$$\fint_{B(1)} \|(F_{Rx} - \phi_{Rx})\|_{L^2(B(1))}\, dx \lesssim W^m \left(\delta K^{m-1} + W^2 K^{-2}\right)\left(1 + O_N(R^{-\Lambda - 3/2})\right).$$

For $\alpha \neq 0$, observe that if we differentiate with respect to $x$,

$$D^\alpha (e(y,x)) = (2\pi i)^{|\alpha|} \left( \prod_{i=1}^m y_i^{\alpha_i} \right) e(y,x).$$

Thus,

$$D^\alpha (e\left(\langle r_n, Wy\rangle\right)) = (2\pi W)^{|\alpha|} e\left(\langle r_n, Wy\rangle\right) \prod_{i=1}^m r_{n,i}^{\alpha_i}.$$

Also,

$$|D_\alpha \left( e\left(\langle r_n, Wy\rangle\right) - e\left(\langle \zeta^k, Wy\rangle\right)\right)| = (2\pi W)^{|\alpha|} \left| \prod_{i=1}^m r_{n,i}^{\alpha_i} - \prod_{i=1}^m (\zeta_i^k)^{\alpha_i} e\left(\langle \zeta^k - r_n, Wy\rangle\right)\right|.$$

Now, adding and subtracting $\prod_{i=1}^m (\zeta_i^k)^{\alpha_i}$ and using the triangle inequality, gives:

$$\begin{aligned}
&|D_\alpha \left( e\left(\langle r_n, Wy\rangle\right) - e\left(\langle \zeta^k, Wy\rangle\right)\right)| \leqslant (2\pi W)^{|\alpha|} \left( \left| \prod_{i=1}^m r_{n,i}^{\alpha_i} - \prod_{i=1}^m (\zeta_i^k)^{\alpha_i} \right| + \right. \\
&\left. + \left| \prod_{i=1}^m (\zeta_i^k)^{\alpha_i} \right| \left| 1 - e\left(\langle \zeta^k - r_n, Wy\rangle\right)\right| \right)
\end{aligned}$$

Since $|e^{ix} - e^{iy}| \leqslant |x - y|$, we bound the last expression by:

$$W^{|\alpha|} \left( \left|\zeta^k - r_n\right| + W \left|\zeta^k - r_n\right| \right) \lesssim \frac{W^{|\alpha|+1}}{K}. \tag{4.3.12}$$

Hence, following a similar argument as in the case $\alpha = 0$, combined with (4.3.12), we conclude that

$$\int_{B(1)} \|D^\alpha (F_{Rx} - \phi_{Rx})\|^2_{L^2(B(1))}\, dx \lesssim W^{2|\alpha|+m} \left(\delta K^{m-1} + W^2 K^{-2}\right)\left(1 + O_N(R^{-\Lambda - 3/2})\right)$$

finishing the proof. $\square$

### 4.3.2 Gaussian moments

Let us define:

$$b_k(x) := \frac{1}{(2N\mu_r(I_k))^{1/2}} \sum_{r_n \in I_k} a_n e(r_n \cdot Rx).$$

We are going to show that the pseudo-random vector $(b_k)_{k\in\mathcal{K}}$ approximates a Gaussian vector $(c_k)_{k\in\mathcal{K}}$, where $c_k$ are i.i.d. complex standard Gaussian random variables subject to $\bar{c}_k = c_{-k}$. More specifically, we prove the following quantitative lemma:

**Lemma 4.3.2.** Let $N \geqslant 1$, $R > 0$, $b_k$ be as above and $\mathcal{K}^+, K, \delta$ be as in Section 4.3.1. Moreover, let $D > 1$ be some large parameter and fix two sets of positive integers $\{s_k\}_{k\in\mathcal{K}^+}$ and $\{t_k\}_{k\in\mathcal{K}^+}$ such that $\sum_k s_k + t_k \leqslant D$, then we have

$$\left| \fint_{B(R)} \left[ \prod_{k\in\mathcal{K}^+} b_k^{s_k} \bar{b}_k^{t_k} \right] - \mathbb{E}\left[ \prod_{k\in\mathcal{K}^+} c_k^{s_k} \bar{c}_k^{t_k} \right] \right| = O_D(\delta^{-1}N^{-1}) + O_{N,D}(R^{-\Lambda-3/2}).$$

*Proof.* For $c_k$, first note that the independence properties of the Gaussian variables $c_k$ (which have zero mean) imply that $\mathbb{E}(c_k\overline{c_{k'}}) = 0$ if $k' \notin \{k, -k\}$. When $k' = k$ one has

$$\mathbb{E}[|c_k|^2] = \mathbb{E}[(\mathrm{Re}(c_k))^2] + \mathbb{E}[(\mathrm{Im}(c_k))^2] = 1\,,$$

and when $k' = -k$,

$$\mathbb{E}[(c_k)^2] = \mathbb{E}[(\mathrm{Re}(c_k))^2] - \mathbb{E}[(\mathrm{Im}(c_k))^2] + 2i\,\mathbb{E}[(\mathrm{Re}(c_k))(\mathrm{Im}(c_k))] = 0\,.$$

Therefore,

$$\mathbb{E}(c_k\overline{c_{k'}}) = \delta_{kk'}. \tag{4.3.13}$$

Thus by independence and a similar calculation for $\mathbb{E}\left[c_k^{t_k}\bar{c}_k^{s_k}\right]$,

$$\mathbb{E}\left[\prod_{k\in\mathcal{K}^+} c_k^{t_k}\bar{c}_k^{s_k}\right] = \prod_{k\in\mathcal{K}^+} \mathbb{E}\left[c_k^{t_k}\bar{c}_k^{s_k}\right] = \prod_{k\in\mathcal{K}^+} \delta_{t_k,s_k} s_k!. \tag{4.3.14}$$

For the moments of $b_k$, we can prove the following:

**Claim 4.3.2.** *For $k \in \mathcal{K}^+$ we have*

$$\fint_{B(R)} \left[b_k^{s_k}\bar{b}_k^{t_k}\right] = \delta_{s_k t_k}\left(s_k! + O_{s_k}\left((\delta N)^{-1}\right)\right) + O_{D,N}\left(R^{-\Lambda-3/2}\right).$$

*Proof of Claim 4.3.2.* We have that:

$$b_k^{s_k}\bar{b}_k^{t_k} = |I_k|^{-(s_k+t_k)/2} \sum_{\mathcal{C}_k} \prod_{i=1}^{s_k} \prod_{j=1}^{t_k} a_{i,k}\bar{a}_{j,k} e(\langle r_{i,k} - r_{j,k}, Rx\rangle)$$

where $r_{i,k} := r_{n_{i,k}} \in I_k$, $a_{i,k} := a_{n_{i,k}}$ and $\mathcal{C}_k$ represents the set of all possible choices of $n_{i,k}$ and $n_{j,k}$ with $i \in \{1, ..., s_k\}$ and $j \in \{1, ..., t_k\}$. Then, rescaling to a ball of radius 1, we have

$$\fint_{B(R)}\left[\prod_{k\in\mathcal{K}^+} b_k^{s_k}\overline{b}_k^{t_k}\right] = \left(\prod_{k\in\mathcal{K}^+}|I_k|^{-(s_k+t_k)/2}\right)\sum_{\mathcal{C}}\prod_{k\in\mathcal{K}^+}\prod_{i=1}^{s_k}\prod_{i=1}^{t_k} a_{i,k}\overline{a}_{j,k}\times$$
$$\times \fint_{B(1)} e\left(\sum_{k\in\mathcal{K}^+}\left(\sum_{i=1}^{s_k} r_{i,k}-\sum_{j=1}^{t_k} r_{j,k}\right)\cdot Rx\right)dx \tag{4.3.15}$$

where $\mathcal{C}$ represents the set of all possible choices of $n_{i,k}$ and $n_{j,k}$ with $i\in\{1,...,s_k\}$, $j\in\{1,...,t_k\}$ and $k\in\mathcal{K}^+$. To estimate this, let us begin by fixing $k$:

$$\fint_{B(R)}\left[b_k^{s_k}\overline{b}_k^{t_k}\right] = |I_k|^{-(s_k+t_k)/2}\sum_{\mathcal{C}_k}\prod_{i=1}^{s_k}\prod_{i=1}^{t_k} a_{i,k}\overline{a}_{j,k}\fint_{B(1)} e\left(\left(\sum_{i=1}^{s_k} r_{i,k}-\sum_{j=1}^{t_k} r_{j,n}\right)\cdot Rx\right)dx.$$

In the inner sum,

$$\sum_{i=1}^{s_k} r_{i,k}-\sum_{j=1}^{t_k} r_{j,k} = \sum_{r_n\in I_k}\alpha_n r_n - \sum_{r_n\in I_k}\beta_n r_n = \sum_{r_n\in I_k}\gamma_n r_n,$$

with $\alpha_n,\beta_n,\gamma_n$ integers. By rational independence (4.1.7), the sum vanishes if and only if $\gamma_n=0$ for every $n$. So $\mathcal{C}_k$ can be divided into the combinations where $\gamma_n=0$ for every $n$, $\mathcal{C}_{k,1}$, and the remaining terms, $\mathcal{C}_{k,2}$:

$$\fint_{B(R)}\left[b_k^{s_k}\overline{b}_k^{t_k}\right] = |I_k|^{-(s_k+t_k)/2}\left(\sum_{\mathcal{C}_{k,1}}1+\right.$$
$$\left.+\sum_{\mathcal{C}_{k,2}}\prod_{i=1}^{s_k}\prod_{i=1}^{t_k} a_{i,k}\overline{a}_{j,k}\fint_{B(1)} e\left(\left(\sum_{i=1}^{s_k} r_{i,k}-\sum_{j=1}^{t_k} r_{j,n}\right)\cdot Rx\right)dx\right). \tag{4.3.16}$$

For the first term, note that if $\gamma_n=0$, then $\alpha_n=\beta_n$, thus $s_k=\sum_{r_n\in I_k}\alpha_n=\sum_{r_n\in I_k}\beta_n=t_k$. Then, we denote by $d_k := \#\{n \mid \alpha_n\neq 0\}$ and write $\mathcal{C}_{k,1}=\bigsqcup_{d_k=1}^{s_k}\mathcal{C}_{k,1,d_k}$ where $\mathcal{C}_{k,1,d_k}$ the set of indexes of $\mathcal{C}_{k,1}$ where the number of $\alpha_n\neq 0$ equals $d_k$. Thus, the first term on the right hand side of (4.3.16) is

$$|I_k|^{-(s_k+t_k)/2}\left(\sum_{\mathcal{C}_{k,1}}1\right) = \delta_{t_k,s_k}\frac{\sum_{d_k=1}^{s_k}\#\mathcal{C}_{k,1,d_k}}{|I_k|^{s_k}} \tag{4.3.17}$$

Assume now that we fix the set of $\{\alpha_n\}$ and $d_k$, let us calculate the number of possible indexes in $\mathcal{C}_{k,1,d_k}$ the number of $\alpha_n\neq 0$ equals $d_k$. If we assume that $N$ is large enough such that $|I_k|\geqslant \max s_l$, which is possible by (4.3.2), we may write $\mathcal{C}_{k,1,d_k}=\frac{s_k!}{\prod_{n\in\mathcal{J}_k}\alpha_n!}$, where $\mathcal{J}_k$ is an index set associated with the $r_n\in I_k$. Thus,

$$\#\mathcal{C}_{k,1,d_k}=\sum\left(\frac{s_k!}{\prod_{n\in\mathcal{J}_k}\alpha_n!}\right)^2 \tag{4.3.18}$$

where the sum runs over the possible combination of $\alpha_n$ such that $d_k = \#\{n \mid \alpha_n \neq 0\}$ and $\sum_{n\in\mathcal{J}_k} \alpha_n = s_k$. If $d_k = s_k$, then $\alpha_k \leqslant 1$ and

$$\frac{\#\mathcal{C}_{k,1,d_k}}{|I_k|^{s_k}} = \frac{s_k!^2}{(2N\mu_r(I_k))^{s_k}} \frac{(2N\mu_r(I_k))!}{(2N\mu_r(I_k)-s_k)!s_k!} = s_k!(1-\varepsilon_{s_k,k}),$$

as there are $\binom{|I_k|}{s_k}$ ways of choosing the elements with $2N\mu_r(I_k) = |I_k|$. We have that $\varepsilon_{s_k,k} > 0$ will be:

$$\begin{aligned} 1-\prod_{i=0}^{s_k-1}\left(1-\frac{i}{2N\mu_r(I_k)}\right) &\leqslant 1-\left(1-\frac{s_k}{2N\mu_r(I_k)}\right)^{s_k} \\ &= \sum_{i=1}^{s_k}\binom{s_k}{i}\left(\frac{s_k}{2N\mu_r(I_k)}\right)^{i} \lesssim_{s_k} (2N\delta)^{-1}. \end{aligned}$$

Now, consider the case when $1 \leqslant d_k < s_k$, the unlabelled sum in (4.3.18) will have $\binom{2N\mu_r(I_k)}{d_k}$ elements, thus it can be bounded by

$$\begin{aligned} |\#\mathcal{C}_{k,1,d_k}| < \frac{s_k!^2}{(2N\mu_r(I_k))^{s_k}} \frac{2N\mu_r(I_k)}{(2N\mu_r(I_k)-d_k)!d_k!} &= \\ &= \frac{s_k!^2}{(2N\mu_r(I_k))^{s_k-d_k}\, d_k!} \frac{2N\mu_r(I_k)!}{(2N\mu_r(I_k)-d_k)!\,(2N\mu_r(I_k))^{d_k}}, \end{aligned}$$

with

$$\frac{(2N\mu_r(I_k))!}{((2N\mu_r(I_k))-d_k)!\,(2N\mu_r(I_k))^{d_k}} = (1-\varepsilon_{d_k,k})$$

and $\varepsilon_{d_k,k} = O_{d_k}(N\delta)^{-1}$. Therefore, the first term on the right hand side of (4.3.16) via (4.3.17) is

$$|I_k|^{(-s_k+t_k)/2}\sum_{\mathcal{C}_{k,1}} 1 = \delta_{s_k t_k}\left(s_k! + O_{s_k}\left((\delta N)^{-1}\right)\right). \tag{4.3.19}$$

For the second term, by construction, the inner sum does not vanish, so:

$$\begin{aligned} \sum_{\mathcal{C}_{k,2}}\prod_{i=1}^{s_k}\prod_{i=1}^{t_k} a_{i,k}\bar{a}_{j,k} \fint_{B(1)} e\left(\left(\sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_k} r_{j,n}\right)\cdot Rx\right)dx &= \\ &= \sum_{\mathcal{C}_{k,2}}\prod_{i=1}^{s_k}\prod_{i=1}^{t_k} a_{i,k}\bar{a}_{j,k} C_m \frac{J_{\Lambda+1}\left(2\pi R\left\|\sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_k} r_{j,n}\right\|\right)}{\left(R\left\|\sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_k} r_{j,n}\right\|\right)^{\Lambda+1}}, \end{aligned}$$

by (4.3.5). Using (4.3.5) again and (4.1.7), the second term is $O_{D,N}\left(R^{-\Lambda-3/2}\right)$. Thus, via (4.3.16) and (4.3.19), we finally obtain:

$$\mathbb{E}\left[b_k^{s_k}\bar{b}_k^{t_k}\right] = \delta_{s_k t_k}\left(s_k! + O_{s_k}\left((\delta N)^{-1}\right)\right) + O_{D,N}\left(R^{-\Lambda-3/2}\right).$$

□

Similarly, we claim that, if $k \neq k'$, then

$$\fint_{B(R)} \left[ b_k^{s_k} \overline{b}_{k'}^{t_{k'}} \right] = O_{D,N}\left( R^{-\Lambda-3/2} \right). \tag{4.3.20}$$

Indeed, as $k \neq k'$, the inner sum of (4.3.15) will be

$$\sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_{k'}} r_{j,k'} = \sum_{r_n \in I_k} \alpha_n r_n - \sum_{r_n \in I_{k'}} \beta_n r_n$$

and by rational independence (4.1.7), if the sum vanishes, then $\alpha_n = \beta_n = 0$. Thus, there is only the contribution when the inner sum doesn't vanish and, as above, this term decays as $R$ goes to infinity due to (4.3.8). Now, we can deduce the expression for the general case of (4.3.15). For the inner sum we can write:

$$\sum_{k \in \mathcal{K}^+} \left( \sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_k} r_{j,k} \right) = \sum_{k \in \mathcal{K}^+} \sum_{r_n \in I_k} \gamma_n^k r_n,$$

so the integral is, by rational independence (4.1.7) and (4.3.5),

$$\fint_{B(1)} e\left( \sum_{k \in \mathcal{K}^+} \left( \sum_{i=1}^{s_k} r_{i,k} - \sum_{j=1}^{t_k} r_{j,k} \right) \cdot Rx \right) dx =$$

$$= \begin{cases} 1 & \gamma_n^k = 0 \ \forall n, k, \\ C_m \dfrac{J_{\Lambda+1}(2\pi R \left\| \sum_{k \in \mathcal{K}^+} \sum_{r_n \in I_k} \gamma_n^k r_n \right\|)}{(R \left\| \sum_{k \in \mathcal{K}^+} \sum_{r_n \in I_k} \gamma_n^k r_n \right\|)^{\Lambda+1}} & \text{otherwise.} \end{cases}$$

This splits $\mathcal{C}$ into $\mathcal{C}_1 \subset \mathcal{C}$ of all choices of $n_{i,k}, n_{j,k}$ such that $\gamma_n^k = 0$ and $\mathcal{C}_2 := \mathcal{C} \backslash \mathcal{C}_1$, as in (4.3.16). Now,

$$\sum_{\mathcal{C}_1} \prod_{k \in \mathcal{K}^+} \prod_{i=1}^{s_k} \prod_{i=1}^{t_k} a_{i,k} \overline{a}_{j,k} = \prod_{k \in \mathcal{K}^+} \sum_{\mathcal{C}_{k,1}} \prod_{i=1}^{s_k} \prod_{i=1}^{t_k} a_{i,k} \overline{a}_{j,k} = \prod_{k \in \mathcal{K}^+} \delta_{s_k t_k} \left( s_k! + O_{s_k}\left( (\delta N)^{-1} \right) \right)$$

where we used (4.3.19) for the last equality. Finally, the sum over $\mathcal{C}_2$, arguing as above, it is going to be $O_{D,N}\left( R^{-\Lambda-3/2} \right)$. $\square$

Similarly, we can prove that the function $F_x$ has (asymptotically) real Gaussian moments for its $L^p$ norms:

**Proposition 4.3.3.** Let $p$ be a positive integer. Then,

$$\lim_{N \to \infty} \limsup_{R \to \infty} \left| \fint_{B(R)} |F_x(y)|^{2p} dx - \frac{(2p)!}{p! 2^p} \right| = 0, \ \lim_{N \to \infty} \limsup_{R \to \infty} \left| \fint_{B(R)} |F_x(y)|^{2p+1} dx \right| = 0,$$

uniformly in $y \in B(W)$.

*Proof.*

$$\fint_{B(R)} F_x^{p'}(y)dx = \frac{1}{(2N)^{p'/2}} \sum \prod_{i=1}^{p'} a_{n_i} e\left(\sum_{i=1}^{p'} r_{n_i} \cdot y\right) \fint_{B(1)} e\left(Rx \cdot \sum_{i=1}^{p'} r_{n_i}\right) dx$$

where the sum $\Sigma$ means $\sum_{\{|n_i| \leqslant N\,:\,1 \leqslant i \leqslant p'\}}$. By (4.3.7), the principal contribution will be when $\sum_{i=1}^{p'} r_{n_i} = 0$. In this case, if we define $\sum_{i=1}^{p'} r_{n_i} = \sum_{n=1}^{N} \alpha_n^+ r_n - \sum_{n=-N}^{1} \alpha_n^- r_n = \sum_{n=1}^{N} \alpha_n r_n$, $\sum_{n=1}^{N}(\alpha_n^+ + \alpha_n^-) = 2\sum_{n=1}^{N} \alpha_n^+ = p'$ must be even, so for $p' = 2p+1$ it will not be zero. For $p' = 2p$ then, fixing a vector $\alpha^+ := (\alpha_n^+)_{n=1}^N$ there are $2p!$ ways of choosing $\{|n_i| \leqslant N \,:\, 1 \leqslant i \leqslant 2p\}$ for that $\alpha^+$, so

$$\fint_{B(R)} F_x^{2p} dx = \frac{1}{(2N)^p}(2p!) \sum 1 + O_{N,p}(R^{-\Lambda - \frac{3}{2}})$$

by (4.1.7), where the sum runs over all the possible $\alpha^+$. There are $\binom{N+p-1}{p}$ of those, so the leading term is:

$$\frac{1}{(2N)^p}(2p!)\frac{N+p-1!}{p!(N-1)!} = \frac{2p!}{2^p p!}\frac{N+p-1!}{(N-1)!N^p} = \frac{2p!}{2^p p!}(1 + O_p(1/N)),$$

concluding the proof. □

### 4.3.3 From deterministic to random: passage to Gaussian fields.

The aim of this section is to prove the following technical proposition:

**Proposition 4.3.4.** Let $\phi_x$ be as in Section 4.3.1, $\varepsilon > 0$, $W > 1$ and $s \geqslant 0$ an integer. Then there exist some $K_0 = K_0(\varepsilon, W, s)$, $N_0 = N_0(K, \varepsilon, W, s)$ and $R_0 = R_0(N, \varepsilon, W, s)$ such that if $K \geqslant K_0$, $\delta \lesssim K^{-m}$, $N \geqslant N_0$ and $R \geqslant R_0$ we have

$$d_P(\phi_x, F_\mu) < \varepsilon$$

where the convergence is with respect to the $C^s(B(W))$ topology.

To ease the exposition, we divide the proof of Proposition 4.3.4 into two lemmas. In the first lemma we introduce the auxiliary field $F_{\mu_K}$ where

$$\mu_K := \mu_{K,N} := \sum_{k \in \mathcal{K}} \gamma_{K,N} \delta_{\zeta^k} \text{ where } \gamma_{K,N} := \frac{\mu_r(I_k)}{\sum_{k \in \mathcal{K}} \mu_r(I_k)} \equiv \frac{\mu_r(I_k)}{\kappa_K^2}. \tag{4.3.21}$$

We note that, by Claim 4.3.1, $\kappa_K \to 1$ as $\delta K^{m-1} \to 0$.

**Lemma 4.3.5.** Let $\varepsilon > 0$, $s \geqslant 0$ and $K \geqslant 1$, $\delta > 0$ be as in Section 4.3.1. Then there exist some $N_0 = N_0(\delta, K, \varepsilon, W, s)$ and $R_0 = R_0(N, \varepsilon, W, s)$ such that for all $N \geqslant N_0$, $R \geqslant R_0$, we have

$$d_P(\kappa_K^{-1}\phi_x, F_{\mu_K}) < \varepsilon$$

where the convergence is with respect to the $C^s(B(W))$ topology.

*Proof.* We begin by explicating the dependence of $R$ on $N$: we choose $N$ and $R$ such that the error term in Lemma 4.3.2 tends to zero for every $D$. In order to do so, we will follow a diagonal argument. Let us write explicitly the error terms in Lemma 4.3.2 as

$$|O_D(\delta^{-1}N^{-1})| \leqslant C_D\delta^{-1}N^{-1}, \quad |O_{N,D}(R^{-\Lambda-3/2})| \leqslant C_{N,D}R^{-\Lambda-3/2},$$

and notice that, up to changing the constants, we may assume that $C_{D'} \leqslant C_D$ and $C_{N,D'} \leqslant C_{N,D}$ for any $D' < D$. Now, let us define $M_D$ and $R_{M,D}$ such that

$$C_D M_D^{-1} \to 0 \quad C_{M_D,D} R_{M_D,D}^{-\Lambda-3/2} \to 0$$

as $D \to \infty$. Then, we choose $N, R$ to go to infinity as any sequence satisfying $N_j \geqslant M_j$ and $R_j \geqslant R_{N_j,j}$. Taking said sequence of $N, R$, for any fixed $D$, we have

$$O_D(\delta^{-1}N_j^{-1}) \to 0, \quad O_{N_j,D}(R_j^{-\Lambda-3/2}) \to 0 \tag{4.3.22}$$

as $j$ goes to infinity due to the fact that $C_{D'} \leqslant C_D$, $C_{N,D'} \leqslant C_{D,N}$ if $D' < D$. With this choice of $N$ and $R$ and mind, we simply say that $N, R$ tend to infinity.

Let $\{b_k\}$ and $\{c_k\}$ be defined as in Section 4.3.2. Then, since a Gaussian random variable is determined by its moments (as the moments generating functions exists) and the moments of all orders converge by (4.3.22), we can apply the method of moments, [Bil08, Theorem 30.2], to see that

$$\sum_k \alpha_k b_k \xrightarrow{d} \sum_k \alpha_k c_k \quad \text{as } R, N \to \infty \tag{4.3.23}$$

for any $\{\alpha_k\} \in \mathbb{R}^k$. Bearing in mind the definition of $\phi_x$ in (4.3.3), (4.1.9), the Cramér–Wold theorem [Bil08, Page 383], implies that, for any $y_1, ..., y_l \in B(W)$ with $l$ a positive integer and $\beta \geqslant 0$, we have

$$\begin{aligned} &\kappa_K^{-1}\left(\phi_x(y_1), ..., \phi_x(y_l)\right) \xrightarrow{d} \left(F_{\mu_K}(y_1), ..., F_{\mu_K}(y_l)\right) && R, N \to \infty \\ &\kappa_K^{-1}\left(D^\alpha\phi_x(y_1), ..., D^\alpha\phi_x(y_l)\right) \xrightarrow{d} \left(D^\alpha F_{\mu_K}(y_1), ..., D^\beta F_{\mu_K}(y_l)\right) && R, N \to \infty \end{aligned} \tag{4.3.24}$$

where we have used the multi-index notation $D^\alpha := \partial_{y_1}^{\alpha_1}...\partial_{y_n}^{\alpha_n}$. Thus, thanks to (4.3.24) and the discussion in Section 4.2.2, in order to prove the Lemma, we are left with checking the hypothesis of Proposition 4.2.1.

**Condition ii) in Proposition 4.2.1** Let $\phi_x^\alpha := \kappa_K^{-1}D_{y_i}^\alpha\phi_x$ with $\alpha$ a multi-index, the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\phi_x^\alpha(y) - \phi_x^\alpha(y')| &\lesssim (2\pi)^{|\alpha|}\left|\sum_{k\in\mathcal{K}} b_k(x)\mu_r(I_k)^{1/2}\prod_{i=1}^m (\zeta_i^k)^{\alpha_i}\left(e\left(\zeta^k\cdot y\right) - e\left(\zeta^k\cdot y'\right)\right)\right| \\ &\lesssim \|y-y'\|\sum_{k\in\mathcal{K}}|b_k(x)|. \end{aligned} \tag{4.3.25}$$

Moreover, by (4.3.23) and the Continuous Mapping Theorem [Bil13, Theorem 2.7], we have

$$\sum_{k\in\mathcal{K}}|b_k(x)| \xrightarrow{d} G \quad \text{as } R, N \to \infty \tag{4.3.26}$$

where $G$ is a random variable with finite mean, i.e. the sum of folded normal variables. By Portmanteau Theorem and Chebyshev's inequality, we deduce that

$$\limsup_{R,N\to\infty} \mathrm{vol}_R \left( \sum_{k\in\mathcal{K}} |b_k(x)| \geqslant \varepsilon' \right) \leqslant \mathbb{P}\left(G \geqslant \varepsilon'\right) \leqslant \mathbb{E}[G]\varepsilon'^{-1}, \tag{4.3.27}$$

as $[\varepsilon', \infty)$ is a closed set. Therefore, by (4.3.25) and (4.3.27), using the notation in (4.2.3), we have

$$\limsup_{R,N\to\infty} \mathrm{vol}_R \left(\omega_{\phi_x^\alpha}(\delta) \geqslant \varepsilon\right) \leqslant \limsup_{R,N\to\infty} \mathrm{vol}_R \left( \sum_{k\in\mathcal{K}} |b_k(x)| \geqslant C_{\mathcal{K}}\varepsilon\delta^{-1} \right) \leqslant C'_{\mathcal{K}}\varepsilon^{-1}\delta.$$

Hence, we can conclude that, for all $\varepsilon > 0$ and all $i$, we have

$$\lim_{\delta\to 0} \limsup_{R,N\to\infty} \mathbb{P}\left(\omega_{\phi_x^\alpha}(\delta) \geqslant \varepsilon\right) = 0.$$

This establishes (ii) in Proposition 4.2.1.

**Condition i) in Proposition 4.2.1** By (4.3.3) and (4.3.13), for any point $y \in B(W)$ we have $⨍_{B(R)} |\phi_x^0(y)|^2 dx = O(1)$ as in Proposition 4.3.3 for $p = 1$. Thus, by the Chebyshev's inequality, we have

$$\mathbb{P}_R(|\phi_x^0(y)| > M) \lesssim M^{-2}.$$

We can proceed similarly with $\phi_x^\alpha(y)$ and this establishes i). □

To prove the next result, we need the following lemma, compare the statement with [SSS16, Lemma 4] (here only a weaker version is needed).

**Lemma 4.3.6.** Let $\mu_n$ be a sequence of probability measures on $\mathbb{S}^{m-1}$ such that $\mu_n$ converges weakly to some probability measure $\mu$, then

$$d_P\left(F_{\mu_n}, F_\mu\right) \to 0 \quad \text{as } n \to \infty,$$

where the convergence is with respect to the $C^s(B(W))$ topology.

We provide a proof of Lemma 4.3.6 in Appendix 4.A. Using Lemma 4.3.6, we deduce the following:

**Lemma 4.3.7.** Let $\varepsilon > 0$ and $s \geqslant 0$. Then there exist some $K_0 = K_0(\varepsilon, W, s)$ and some $N_0 = N_0(\varepsilon, W, s)$ such that for all $K \geqslant K_0$, $\delta \lesssim K^{-m}$ and $N \geqslant N_0$, we have

$$d_P(F_{\mu_K}, F_\mu) < \varepsilon$$

where the convergence is with respect to the $C^s(B(W))$ topology.

*Proof.* Let $\mu_{K,N}$ be as in (4.3.21) and let $\delta \lesssim K^{-m}$. Then, in light of Lemma 4.3.6 it is enough to prove the following: let $\varepsilon > 0$, there exist some $K_0 = K_0(\varepsilon)$ and some $N_0 = N_0(\varepsilon)$ such that for all $K \geqslant K_0$ and $N \geqslant N_0$, we have

$$d_P(\mu_{K,N}, \mu) < \varepsilon.$$

Since $\mu_r$ weak$^\star$-converges to $\mu$ as $N \to \infty$, we may assume that $N_0 = N_0(\varepsilon)$ so that $d_P(\mu_r, \mu) < \varepsilon/2$. Therefore, using the triangle inequality, it is enough to prove that

$d_P(\mu_{K,N},\mu_r)<\varepsilon/2$ for $K$ large enough depending on $\varepsilon$ only, which bearing in mind (4.2.1), is equivalent to the following:

$$\mu_{K,N}(B)\leqslant\mu_r(B_{+\varepsilon/2})+\varepsilon/2 \qquad \mu_r(B)\leqslant\mu_{K,N}(B_{+\varepsilon/2})+\varepsilon/2 \tag{4.3.28}$$

for all Borel sets $B\subset\mathbb{S}^{m-1}$. For the sake of simplicity, from now on we write $\mu_{K,N}=\mu_K$. By Claim 4.3.1 ii) for some $C>1$ if $r_n\in I_k$, then $\|r_n-\zeta^k\|<CK^{-1}$, therefore

$$\mu_K(B)=\frac{1}{2N}\sum_{\substack{k\in\mathcal{K}\\ \zeta^k\in B}}\sum_{r_n\in I_k}1\Big/\sum_{k\in\mathcal{K}}\mu_r(I_k)\leqslant\frac{1}{2N}\sum_{r_n\in B_{+C/K}}1\Big/\sum_{k\in\mathcal{K}}\mu_r(I_k)$$

which, together with Claim 4.3.1 i) and our choice of $\delta\lesssim K^{-m}$, gives

$$\mu_K(B)\leqslant\mu_r(B_{+C/K})+C/K.$$

This proves the first part of (4.3.28), with $\varepsilon/2=C/K$. Since

$$\mu_K(B_{+C/K})=\frac{\sum_{k\in\mathcal{K}}\mu_r(I_k)\delta_{\zeta^k}(B_{+C/K})}{\sum_{k\in\mathcal{K}}\mu_r(I_k)}\geqslant\frac{\sum_{k\in\mathcal{K}}\mu_r(I_k\cap B)}{\sum_{k\in\mathcal{K}}\mu_r(I_k)}=\frac{\mu_r(B)-\mu_r(B\cap I_{\mathcal{K}}^c)}{\mu_r(I_{\mathcal{K}})},$$

where $I_{\mathcal{K}}:=\bigcup_{k\in\mathcal{K}}I_k$. Therefore, we have

$$\mu_r(B)\leqslant\mu_K(B_{+C/K})+C/K.$$

This proves the second part of (4.3.28), with $\varepsilon/2=C/K$ and hence Lemma 4.3.7. □

We are finally ready to prove Proposition 4.3.4.

*Proof of Proposition* 4.3.4. The proposition follows from Lemma 4.3.5 and Lemma 4.3.7 together with the triangle inequality for the Prokhorov distance with the following order in the choice of the parameters: $W>1$, $s\geqslant 0$ a natural number, $\varepsilon>0$ are given, then $K$ is large depending on $\varepsilon,W,s$ according to Lemma 4.3.7, $\delta\lesssim K^{-m}$; $N$ is large depending on $\delta,\varepsilon,W,s$ according to Lemma 4.3.5 and Lemma 4.3.7; finally $R$ is large depending on all the previous parameters according to Lemma 4.3.5. □

### 4.3.4 Concluding the proof of Theorem 4.1.7

We are finally ready to prove Theorem 4.1.7:

*Proof of Theorem* 4.1.7. Fix $W>1$, $s\geqslant 0$ and $\varepsilon>0$. Let $K$ large enough according to Lemma 4.3.7 applied with $\varepsilon/4$ and such that $CW^{2(s+l)+m}(\delta K^{m-1}+W^2K^{-2})<\varepsilon/4$ if $\delta\lesssim K^{-m}$ where $C,l$ were defined in Lemma 4.3.1 and also such that $|\kappa_K^{-1}-1|$ is small enough by Claim 4.3.1. Then we take an $N$ large enough according to Lemma 4.3.5 applied with $\varepsilon/4$. Finally, let $R$ large enough as in Lemma 4.3.5 applied with $\varepsilon/2$ and such that in Lemma 4.3.1, $O_N(R^{-\Lambda-3/2})<1$. With this, we have

$$\fint_{B(R)}\|F_x-\phi_x\|^2_{C^s(B(W))}<\varepsilon/2 \;\text{ and }\; d_P(\phi_x,F_\mu)<\varepsilon/2,$$

concluding the proof.

□

## 4.4 Proof of Proposition 4.1.5, semi-locality.

Let $f$ be as in (4.1.1) and denote by $\mathcal{NI}(f, x, W)$ the number of nodal domains that intersect the boundary of $B(x, W)$. In order to prove Proposition 4.1.5 we need to obtain bounds on $\mathcal{NI}(f, x, W)$. This is the content of the next section, where we follow the recent preprint of Chanillo, Logunov, Malinnikova and Mangoubi [CLM+20], see also Landis [Lan63].

### 4.4.1 A bound on $\mathcal{NI}$

We begin by introducing a piece of notation borrowed from [CLM+20]: we say that a domain $A$ is $c_0$-narrow (on scale 1) if

$$\frac{|A \cap B(x, 1)|}{|B(x, 1)|} \leqslant c_0$$

for all $x \in \overline{A}$. We will use the following bound [Section 3.2][CLM+20]:

**Lemma 4.4.1.** Let $r \in (1, R)$ and denote by $\Omega$ a nodal domain of $f$. Then, we have

$$\mathcal{NI}(f, x, r) \lesssim_m |\{\Omega : \Omega \cap B(x, r) \text{ is not } c_0\text{-narrow}\}| + \mathfrak{N}_f(B(x, r))^{m-1} + r^{m-1},$$

where $\mathfrak{N}(\cdot)$ is as in Section 4.2.3.

Since the proof of Lemma 4.4.1 follows step by step [CLM+20], we decided to present it in Appendix 4.B. We observe that, if a nodal domain is not $c_0$-narrow, then $|\Omega \cap B(x_0, 1)| > c_1$ for some constant $c_1 > 0$ and for some $x_0 \in \overline{\Omega}$. Thus, the number of *non* $c_0$-narrow nodal domains in $B(x, W)$ is $O(W^m)$. Hence, Lemma 4.2.5 together with Lemma 4.4.1 gives the following bound:

**Corollary 4.4.2.** Let $F_x$ be as in (4.1.6), then, provided that $N$ is sufficiently large with respect to $W$, we have

$$\mathcal{NI}(f, x, W) \lesssim_m W^m + N^{m-1} \lesssim N^{m-1}.$$

### 4.4.2 Small values of $f$

In this section we prove the following lemma which will also be our main tool in controlling the doubling index of $f$.

**Lemma 4.4.3.** Let $f$ be as in (4.1.1), $\beta > 0$ and $D > 1$ be two parameters, then

$$\mathrm{vol}_R(|f(x)| \leqslant \beta) \lesssim \beta + N^{-D} + O_N(R^{-\Lambda-3/2}).$$

The proof relies on the following Halàsz' anti-concentration inequality [Hal77] and [NV13, Lemma 6.2].

**Lemma 4.4.4** (Halasz' bound)**.** Let $X$ be a real-valued random variable and let $\psi(t) = \mathbb{E}[\exp(itX)]$ be its characteristic function then there exists some absolute constant $C > 0$ such that

$$\mathbb{P}(|X| \leqslant 1) \leqslant C \int_{|t|\leqslant 1} |\psi(t)| dt.$$

We are now ready to prove Lemma 4.4.3.

*Proof of Lemma 4.4.3.* Firstly we rewrite $f$ as

$$f(x) = \sum_{n=1}^{N} b_n \cos(r_n \cdot x) + \sum_{n=1}^{N} c_n \sin(r_n \cdot x) \tag{4.4.1}$$

for some $b_n, c_n$ with $b_n^2 + c_n^2 = 1/2N$. We apply Lemma 4.4.4 to obtain

$$\mathrm{vol}_R(|f(x)| \leqslant \beta) \lesssim \beta \int_{|t|\leqslant 1/\beta} |\psi_R(t)| \tag{4.4.2}$$

where $\psi_R(t) = \psi_{f,R}(t) = ⨍_{B(R)}[\exp(2\pi i t f(x))]$. From now on, we may also assume that $1/\beta \geqslant 1$, as, if $1/\beta \leqslant 1$, then we can use the trivial bound $|\psi_R(t)| \leqslant 1$ on the right hand side of (4.4.2) and conclude the proof. We need the following Jacobi–Anger expansion [AS65, page 355]:

$$e^{iz\cos\theta} = \sum_{l=-\infty}^{\infty} i^l J_l(z) e^{il\theta} \qquad\qquad e^{iz\sin\theta} = \sum_{l=-\infty}^{\infty} J_l(z) e^{il\theta}.$$

Then, by (4.4.1), we have

$$\begin{aligned}
\exp(itf(x)) &= \prod_{|n|\leqslant N} \exp(t(b_n\cos(r_n\cdot x) + c_n \sin(r_n \cdot x)) \\
&= \prod_{|n|\leqslant N} \left[\sum_{l=-\infty}^{\infty} i^l J_l(tb_n) e^{il(r_n\cdot x)}\right] \cdot \left[\sum_{l=-\infty}^{\infty} J_{l'}(tc_n) e^{il'(r_n\cdot x)}\right] \\
&= \sum_{\substack{l_1,\dots l_N \\ l'_1,\dots,l'_N}} \left(\prod_{|n|\leqslant N} i^{l_n} J_{l_n}(tb_n) J_{l'_n}(tc_n)\right) \cdot e^{(ir_1(l_1+l'_1)+\dots+r_N(l_N+l'_N))\cdot x}. \qquad (4.4.3)
\end{aligned}$$

Thanks to the rapid decay of Bessel functions as the index $\nu \to \infty$ for a fixed argument $z$, that is

$$J_\nu(z) \lesssim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,$$

and bearing in mind (4.1.7) and (4.3.5), we can integrate (4.4.3) with respect to $x$ which, using Fubini, gives

$$\psi_R(t/(2\pi)) = \sum_{l_1,\dots,l_N=-\infty}^{\infty} \prod_{|n|\leqslant N} i^{l_n} J_{l_n}(tb_n) J_{-l_n}(tc_n) + O_N\left(R^{-\Lambda-3/2}\right) \tag{4.4.4}$$

For the first term on the RHS of (4.4.4), we rewrite it as

$$\sum_{l_1,\ldots,l_N=-\infty}^{\infty} \prod_{|n|\leqslant N} i^{l_n} J_{l_n}(tb_n) J_{-l_n}(tc_n) = \prod_{|n|\leqslant N} \left( \sum_{l=-\infty}^{\infty} (-i)^l J_{l_n}(tb_n) J_{l_n}(tc_n) \right) \tag{4.4.5}$$

where we have used the fact that $J_{-l}(x) = (-1)^l J_l(x)$. By Graf's addition theorem [Wat95, page 361], we have

$$J_0(\sqrt{x^2+y^2}) = \sum_{l=-\infty}^{\infty} J_l(x) J_l(y) \cos\left(\frac{\pi}{2} l\right), \quad \sum_{l=-\infty}^{\infty} J_l(x) J_l(y) \sin\left(\frac{\pi}{2} l\right) = 0. \tag{4.4.6}$$

Writing $(-i)^l = \cos(\pi l/2) - i\sin(\pi l/2)$ and applying (4.4.6), bearing in mind that $b_n^2 + c_n^2 = 1/2N$, we obtain

$$\sum_{l=-\infty}^{\infty} (-i)^l J_{l_n}(tb_n) J_{l_n}(tc_n) = J_0\left(\frac{\sqrt{2}t}{\sqrt{N}}\right). \tag{4.4.7}$$

Finally, inserting (4.4.7) into (4.4.5), we deduce that

$$\psi_R(t/(2\pi)) = \left( J_0\left(\frac{\sqrt{2}t}{\sqrt{N}}\right)\right)^N + O_N\left(R^{-\Lambda-3/2}\right). \tag{4.4.8}$$

Let us denote the first summand by $\Psi_R^N(t)$. By the very definition of Bessel functions [AS65, Page 375], we have

$$J_0(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!\Gamma(q+1)} \left(\frac{x}{2}\right)^{2q}$$

Therefore, $J_0(x) = 1 - \Gamma(2)^{-1}x^2 + O(x^4) \leqslant e^{-cx^2}$ for some $c > 0$ and $x$ sufficiently small. Thus, bearing in mind (4.4.8), we have

$$\Psi_R^N(t) \lesssim e^{-Ct^2} \tag{4.4.9}$$

for all $t \leqslant c_1 N^{1/2}$ for some sufficiently small constant $c_1 > 0$. For $t \geqslant c_1 N^{1/2}$, we use that fact that $|J_0(x)| \leqslant \alpha < 1$ for $x > c_1$ and for some $0 < \alpha < 1$, to obtain the bound

$$\Psi_R^N(t) \lesssim \alpha^N \lesssim N^{-D}$$

for any $D \geqslant 1$. Thus,

$$\beta \int_{|t|\leqslant 1/\beta} |\Psi_R^N(t)| dt \lesssim \beta \int_{\mathbb{R}} e^{-Ct^2} dt + \beta \int_{-1/\beta}^{1/\beta} N^{-D} dt + O_N(R^{-\Lambda-3/2})$$

obtaining the desired result. □

As a consequence of Lemma 4.4.3, we deduce the following:

**Lemma 4.4.5.** Let $f$ be as in (4.1.1), $D, H > 1$ be some (arbitrary but fixed) parameters. Then, we have

$$\mathrm{vol}_R\left(\mathfrak{N}_f(B(x,H)) > Q\right) \lesssim_D \begin{cases} 1 & Q \leqslant C'H \\ \frac{1}{Q^D} + \frac{Q^{2D}}{e^Q} & Q > C'H \end{cases} + O_{N,H}\left(R^{-\Lambda-3/2}\right)$$

uniformly for all $Q \leqslant N$, $x \in B(R)$ and some absolute constant $C' > 0$.

*Proof.* Now, consider $h(\cdot,t) := f(\cdot)\cdot e^{2\pi t}$ on $\tilde{B}(H) = \tilde{B}(x,H) = B(x,H)\times[-H/2,H/2]$, then $\sup_{B(x,H)}|f| \leqslant \sup_{\tilde{B}(H)}|h|$. Write $h_H(\cdot) = h(H\cdot)$ and $f_H(\cdot) = f(H\cdot)$, using elliptic regularity [Eva98, p.332], we have

$$\sup_{B(H)}|f|e^{\pi H} = \sup_{\tilde{B}(H)}|h| = \sup_{\tilde{B}(1)}|h_H| \lesssim ||h_H||_{L^2(\tilde{B}(2))} \lesssim e^{2\pi H}||f_H||_{L^2(B(2))},$$

where the constants in the notation are independent of $H$. Thus, letting $H' = \varkappa_m H$ $B(2) = B'$, we obtain

$$\mathfrak{N}_f(B(x,H)) \leqslant \log\frac{\sup_{B(x,1)}|f_{H'}|}{|f(x)|} + 1 \leqslant C_1 + \pi H + \log\frac{||f_{H'}||_{L^2(B')}}{|f(x)|}. \tag{4.4.10}$$

Thanks to (4.4.10), we have for $B'' = B(2\varkappa_m)$

$$\mathrm{vol}_R(\mathfrak{N}_f(x,H) > Q) \leqslant \mathrm{vol}_R\left(\log\frac{||f_H||_{L^2(B'')}}{|f(x)|} > Q - cH\right),$$

for some absolute constant $c > 0$. Since $Q - cH > Q/2$ for $Q > C'H$ for $C' = 2c > 0$, we obtain that, in order to prove the lemma, it is enough to prove the following:

$$\mathrm{vol}_R\left(\log\frac{||f_H||_{L^2(B'')}}{|f(x)|} > Q/2\right) \lesssim_D 1/Q^D + \frac{Q^{2D}}{e^Q} + O_{N,H}\left(R^{-\Lambda-3/2}\right), \tag{4.4.11}$$

for $Q \geqslant C'H$. Let $I(x) := \log(||f_H||_{L^2(B'')}/|f(x)|)$ and $\beta > 0$ be some parameter to be chosen later, then

$$\mathrm{vol}_R(I > Q/2) = \mathrm{vol}_R(I > Q/2 \text{ and } |f(x)| < \beta) + \mathrm{vol}_R(I > Q/2 \text{ and } |f(x)| \geqslant \beta) \tag{4.4.12}$$

First, we bound the first term on the RHS of (4.4.12). By Lemma 4.4.3, we have

$$\mathrm{vol}_R(I \geqslant Q/2 \quad \text{and} \quad |f(x)| < \beta) \leqslant \mathrm{vol}_R(|f(x)| \leqslant \beta) \lesssim \beta + O_N\left(R^{-\Lambda-3/2}\right) \tag{4.4.13}$$

provided $\beta \geqslant N^{-D}$. For the second term on the RHS of (4.4.12), we notice that

$$\mathrm{vol}_R(I \geqslant Q/2 \text{ and } |f(x)| \geqslant \beta) \leqslant \mathrm{vol}_R\left(\|f\|_{L^2(B'')} > \beta e^{Q/2}\right). \tag{4.4.14}$$

However, bearing in mind that $B'' = B(x, 2\varkappa_m)$, using (4.1.7), (4.3.5) and Fubini, we have

$$\fint_{B(R)} ||f_H||^2_{L^2(B'')} = \mathrm{vol}\, B'' + O_N\left((HR)^{-\Lambda-3/2}\right).$$

Thus, Chebyshev's inequality gives

$$\mathrm{vol}_R\left(\|f_H\|_{L^2(B'')} > \beta e^{Q/2}\right) \lesssim \frac{1}{\beta^2 e^Q} + O_N\left((HR)^{-\Lambda-3/2}\right). \tag{4.4.15}$$

Hence, putting (4.4.12), (4.4.13), (4.4.14) and (4.4.15) together, we get

$$\mathrm{vol}_R(I > Q) \lesssim \beta + \frac{1}{\beta^2 e^Q} + O_{N,H}\left(R^{-\Lambda-3/2}\right)$$

finally, we take $\beta = 1/Q^D$, so the condition $\beta \geqslant N^{-D}$ is equivalent to $N \geqslant Q$ and conclude the proof of (4.4.11). □

### 4.4.3 Proof of Proposition 4.1.5

We are finally ready to prove Proposition 4.1.5

*Proof of Proposition 4.1.5.* For short we write $\mathcal{NI}(f,x,W) = \mathcal{NI}(F_x,W)$. By [NS16, Lemma 1] for $r = W$, we have

$$\frac{1}{\mathrm{vol}\,B(R)}\int_{B(R-W)}\frac{\mathcal{N}(F_x,W)}{\mathrm{vol}\,B(W)}dx \leqslant \frac{\mathcal{N}(f,R)}{\mathrm{vol}\,B(R)} \leqslant$$
$$\leqslant \frac{1}{\mathrm{vol}\,B(R)}\left(\int_{B(R+W)}\frac{\mathcal{N}(F_x,W)}{\mathrm{vol}\,B(W)}dx + \int_{B(R+W)}\frac{\mathcal{NI}(F_x,W)}{\mathrm{vol}\,B(W)}dx\right).$$

By Faber-Krahn inequality,

$$\int_{B(R+W)}\frac{\mathcal{N}(F_x,W)}{\mathrm{vol}\,B(W)}dx - \int_{B(R)}\frac{\mathcal{N}(F_x,W)}{\mathrm{vol}\,B(W)}dx \lesssim \mathrm{vol}\,B(R)\frac{(R+W)^m - R^m}{R^m},$$

which is $O(\mathrm{vol}\,B(R)W/R)$ by the binomial theorem, similarly for $B(R)$ and $B(R-W)$. Thus, we have

$$\frac{\mathcal{N}(f,R)}{\mathrm{vol}\,B(R)} = \frac{1}{\mathrm{vol}\,B(W)}\fint_{B(R)}\mathcal{N}(F_x,W)dx$$
$$+ O\left(\frac{1}{W^m}\fint_{B(R+W)}\mathcal{NI}(F_x,W)dx\right) + O\left(\frac{W}{R}\right). \tag{4.4.16}$$

Therefore, it is enough to prove the following:

$$\frac{1}{W^m}\fint_{B(R+W)}\mathcal{NI}(F_x,W)dx \lesssim \frac{1}{W}\left(1 + O_{N,W}\left(R^{-\Lambda-3/2}\right)\right). \tag{4.4.17}$$

First, we observe that if we cover $\partial B(x,W)$ with $O(W^{m-1})$ ($m$-dimensional) balls of radius 100 with centres $x + y_i$, then

$$\mathcal{NI}(F_x,W) = \mathcal{NI}(f,x,W) \leqslant \sum_i\left(\mathcal{NI}(f,x+y_i,100) + \mathcal{N}(f,x+y_i,100)\right)$$
$$\leqslant \sum_i \mathcal{NI}(f,x+y_i,100) + O\left(W^{m-1}\right),$$

where in the second inequality we have used the Faber-Krahn inequality. Therefore, thanks to Lemma 4.4.1 applied with $r = 100$, we have

$$\frac{1}{W^m} \fint_{B(R+W)} \mathcal{NI}(F_x, W)dx \lesssim \frac{1}{W^m} \sum_i \fint_{B(R+W)} \mathfrak{N}_f(B(x+y_i, 100))^{m-1}dx+$$
$$+\frac{1}{W^m} \sum_i \fint_{B(R+W)} |\{\Omega : \Omega \cap B(x+y_i, 100) \text{ is not } c_0\text{-narrow}\}|\, dx + O\left(\frac{1}{W}\right) \quad (4.4.18)$$

If $\Omega \cap B(x+y_i, 100)$ is not $c_0$-narrow, then $|\Omega \cap B(x+y_i, 100)| > c_1$. Thus

$$|\{\Omega : \Omega \cap B(x+y_i, 100) \text{ is not } c_0\text{-narrow}\}| = O(1),$$

bearing in mind that the sum over $i$ has $O(W^{m-1})$ terms, the second term on the right hand side of (4.4.18) is $O(W^{-1})$.

Thus, it is enough to bound the first term to the right hand side of (4.4.18). Thanks to Lemma 4.2.5, $\mathfrak{N}_f(B(x, 100)) \lesssim N$. Thus, we have

$$\fint_{B(R+W)} \mathfrak{N}_f(B(x+y_i, 100))^{m-1}dx = \int_1^{(CN)^{m-1}} \mathrm{vol}_R(\mathfrak{N}_f(B(x+y_i, 100))^{m-1} > t)dt$$
$$\lesssim 1 + O_{N,100}\left((R+W)^{-\Lambda-3/2}\right),$$

where in the second inequality we have used Lemma 4.4.5, with $D = m$ and $H = 100$. This concludes the proof of the proposition. □

## 4.5 Proof of Theorem 4.1.8

### 4.5.1 Convergence in mean

The aim of this section is to show how Theorem 4.1.7 implies convergence in mean of $\mathcal{N}(\cdot)$. That is, we prove the following proposition:

**Proposition 4.5.1.** Let $W \geqslant 1$ and $\mathcal{S} \subset H(m-1)$. Then we have

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \fint_{B(R)} \mathcal{N}(F_x, \mathcal{S}, W) - \mathbb{E}[\mathcal{N}(F_\mu, \mathcal{S}, W)] \right| = 0.$$

Moreover, the conclusion also holds for $\mathcal{N}(\cdot, T)$, as in Theorem 4.1.8.

To ease the exposition we split the proof of Proposition 4.5.1 into a series of preliminary results.

### 4.5.2 Continuity of $\mathcal{N}(\cdot)$

In this section we show that $\mathcal{N}(\cdot, W)$, $\mathcal{N}(\cdot, [\Sigma], W)$ and $\mathcal{N}(\cdot, T, W)$ are continuous functionals on a particular subspace of $C^1$ functions. This is a consequence of Thom's

Isotopy Theorem 4.2.11 and it refines the estimates of "Shell Lemma" in [NS16]. In order to state the main result of this section we need to introduce some notation:

$$\nabla\!\!\!/ g := \nabla g - \frac{x \cdot \nabla g}{|x|^2} x,$$

that is, the "spherical" part of the gradient. Also, $\Psi_g := |g| + \|\nabla g\|$, $\Psi\!\!\!/_g := |g| + \|\nabla\!\!\!/ g\|$ and for $W > 1$, let us define

$$C^1_*(W) := \left\{ g \in C^1(B(W+w)) \mid \Psi_g > 0 \text{ on } B(W+w) \text{ and } \Psi\!\!\!/_g > 0 \text{ on } \partial B(W) \right\}. \tag{4.5.1}$$

The parameter $w > 0$ could be as small as we want. For the sake of simplicity, hereafter we assume $w = 1$. We then prove the following,

**Proposition 4.5.2.** Let $W > 1$ be fixed, $\mathcal{S} \in H(m-1)$ and $T \in \mathcal{T}$ a finite tree. Then $\mathcal{N}(\cdot, \mathcal{S}, W)$ and $\mathcal{N}(\cdot, T, W)$ are continuous functionals on $C^1_*(W)$.

Before starting the proof, we observe that the condition on $\Psi\!\!\!/$ is used to rule out the possibility that the nodal set touches the boundary of the ball tangentially at one point.

*Proof of Proposition 4.5.2.* Let $V := B(W+1)$, $V_1 := B(W+1/2)$, $V_0 := B(W)$ and $h \in C^1_*(W)$, as $\Psi_h > 0$, there is a finite number of connected components in $V_0$, i.e., $h^{-1}(0) \cap V_0$ has components $\{\Sigma_i\}_{i=1}^{I} \sqcup \{\Sigma_j^*\}_{i=1}^{J}$, where $\Sigma_i \subset V_0$ for all $i \in \{1, \dots, I\}$ and $\Sigma_j^* \cap \partial V_0 \neq \emptyset$ for all $j \in \{1, \dots, J\}$. To treat the 0-level set as a boundaryless manifold, let $\chi$ be a smooth radial step function which is zero in $\overline{V_1}$ and greater than $\sup_V |h|$ on the boundary of $V$, say $T$. Moreover, we observe that the condition $\Psi\!\!\!/_h > 0$ implies that $\Sigma_j^* \cap \partial V_0$ is not a point: for $m = 2$ this follows from the definition of $\Psi\!\!\!/_h > 0$; for $m > 2$ it follows from the fact that, as the intersection is transversal, it must be a submanifold on the boundary of codimension 1.

Therefore, it is possible to define $d_j > 0$ as the maximal distance between some

$$x \in \left( \Sigma_j^* \backslash V_0 \right) \cap \overline{V_1} \tag{4.5.2}$$

and $\partial V_0$, i.e., $d_j = \max_x \operatorname{dist}\left( x \in \left( \Sigma_j^* \backslash V_0 \right) \cap \overline{V_1}, \partial V_0 \right) =: \operatorname{dist}\left( x_j, \left( \Sigma_j^* \backslash V_0 \right) \cap \overline{V_1} \right)$ for some $x_j \in \left( \Sigma_j^* \backslash V_0 \right) \cap \overline{V_1}$. Now, we are going to apply Thom's Theorem 4.2.11 to $h + \chi$ on $V$. Note that $h + \chi$ has the same nodal set as $h$ on $\overline{V_1}$ and define $\Gamma_j^*$ the connected component (boundaryless) of the nodal set of $h + \chi$ which equals $\Sigma_j^*$ on $\overline{V_1}$. Let $U_i \subset V_0$ be the open neighbourhood of $\Sigma_i$ and similarly $U_j^* \subset V$ of $\Gamma_j^*$, both given by the theorem. Let us also take $\varepsilon_i > 0$ and $d_j > \varepsilon_j^* > 0$. By Theorem 4.2.11, there is $\delta_i, \delta_j^* > 0$ such that if $g'$ satisfies

$$\|h - g'\|_{C^1(U_i)} < \delta_i, \quad \|h + \chi - g'\|_{C^1(U_j^*)} < \delta_j^*$$

then $g'$ has a nodal component in $U_i$, $U_j^*$ diffeomorphic to $\Sigma_i$, $\Sigma_j^*$ (respectively) and the diffeomorphism satisfies

$$\|\Phi_j^* - \mathrm{id}\|_{C^1(\mathbb{R}^m)} < \varepsilon_j^*.$$

If we define $\delta := \min_{i,j}\{\delta_i, \delta_j^*, (T - \sup_V |h|)/2\}$ with $\|g-h\|_{C^1(V)} < \delta$, and $g' = g + \chi$, then the connected component of $g'$ diffeomorphic to $\Gamma_j^*$ cannot lie inside $V_0$. Indeed, if $x_j$ is defined as in (4.5.2), $\left\|\Phi_j^*(x_j) - x_j\right\| < d_j$, so $\Phi_j^*(x_j)$ is outside $V_0$. Finally letting $U := \left((\cup_i U_i) \cup \left(\cup_j U_j^*\right)\right) \cap V_0$, if

$$\|g-h\|_{C^1(V)} < \min\{\delta, \min_{x\in\overline{V_0}\setminus U} \{h\} > 0\}, \tag{4.5.3}$$

then $g$ satisfies the hypotheses of Thom's Theorem 4.2.11 for all the components and it cannot vanish outside $U$. Therefore

$$\mathcal{N}(h, [\Sigma], W) = \mathcal{N}(g, [\Sigma], W) \quad \forall\, \Sigma \in H(m-1),$$

in particular, $\mathcal{N}(h, \mathcal{S}, W) = \mathcal{N}(g, \mathcal{S}, W)$. The proof of $\mathcal{N}(h, T, W)$ is similar as $\Phi_i$ of Theorem 4.2.11 is the identity outside a proper subset of $U_i$. □

**Claim 4.5.1.** *With the notation of Proposition 4.5.2, $C^1_*(W) \subset C^1(B(W+1))$ is an open set.*

*Proof.* If $h_n \to h$ in the $C^1$ topology and $|h_n|(y_n) + \|\nabla h_n\|(y_n) = 0$, we can choose $y_{n_j} \to y$ so

$$|h(y) - h_{n_j}(y_n)| \leqslant |h(y) - h(y_{n_j})| + |h(y_{n_j}) - h_{n_j}(y_{n_j})|,$$

which goes to zero as $j \to \infty$, as the convergence is uniform, and similarly for the gradient and $\nabla\!\!\!/$. Hence, the complement of $C^1_*(W)$ is closed. □

### 4.5.3 Checking the assumptions

In this section, we give a sufficient condition on $\nu$ for the Gaussian field $F_\nu$ to belong to $C^1_*(W)$ with the notation of Proposition 4.5.2. As our paths are a.s. analytic, we have the following lemma, see also [NS16, Lemma 6].

**Lemma 4.5.3** (Bulinskaya's lemma)**.** Let $F = F_\nu$, with $\nu$ an Hermitian measure supported on the sphere and $s \geqslant 1$. If $\nu$ is not supported on a hyperplane, then $F \in C^1_*(W)$ almost surely, where is $C^1_*(W)$ as in (4.5.1).

*Proof.* The proof of $\Psi > 0$ is a straightforward application of [AW09, Proposition 6.12] as the density of $(F, \nabla F)(x)$ is independent of $x \in B(W+1)$. Indeed,

$$\mathbb{E}\left(\partial_i F(x) \partial_j F(x)\right) = 4\pi^2 \int_{\mathbb{S}^{m-1}} \lambda_i \lambda_j d\nu(\lambda) =: \bar{\Sigma}_{ij}$$

for $i = 0, \ldots, m$ where $\partial_0 :=$ id and $\lambda_0 := 1$. Note that as $\nu$ is Hermitian, $\Sigma_{i0} = 0$ for $i > 0$, that is, $F(x)$ and $\nabla F(x)$ are independent. If $\Sigma = (\bar{\Sigma}_{ij})_{i,j=1}^m$, then $\det \Sigma = 0$ is equivalent to the existence of some $u \in \mathbb{R}^m$ such that

$$\int_{\mathbb{S}^{m-1}} (\lambda \cdot u)^2 d\nu(\lambda) = 0. \tag{4.5.4}$$

However, this is not possible since $\nu$ is not supported on a hyperplane, thus $\det \Sigma \neq 0$ and $\Psi > 0$.

For $\Psi > 0$, consider a local parametrization $(U, \varphi)$ of $S$ with basis $\{e_i\}_{i=1}^{m-1}$ of the tangent space $T_x S$ where $x \in S := \partial B(W)$. Then,

$$\operatorname{Var} \nabla F(x) = E(x) \cdot \Sigma \cdot E(x)^t$$

where $E^t := \begin{pmatrix} e_1, \dots, e_{m-1} \end{pmatrix}$. Thus, the variance is not positive definite if and only if there exists $v \in \mathbb{R}^{m-1}$ non-zero such that $u := \sum_{i=1}^{m-1} v_i e_i$ and $u \cdot \Sigma \cdot u^t = (4\pi^2) \int_{\mathbb{S}^{m-1}} (\lambda \cdot u)^2 d\nu(\lambda) = 0$, in contradiction (again) with the fact that $\nu$ is not supported on a hyperplane. Thus, $\det \left( \operatorname{Var} \nabla F|_{\varphi(U)} \right) > \delta > 0$, so by Bulinskaya applied to $Y := (F, \nabla F)$ we conclude $\Psi|_{\varphi(U)} > 0$, then proceed analogously with the other local parametrizations of the (finite) atlas. $\square$

As a consequence of Proposition 4.5.2 and Lemma 4.5.3, we have the following:

**Lemma 4.5.4.** Let $\varepsilon > 0$, $W > 1$ and $\mathcal{S} \subset H(m-1)$. Then there exist some $K_0 = K_0(\varepsilon, W)$, $N_0 = N_0(\varepsilon, W)$ such that for all $k \geqslant K_0$, $N \geqslant N_0$ and $\delta \lesssim K^{-m+1}$ we have

$$\left| \mathbb{E}[\mathcal{N}(F_{\mu_{K,N}}, \mathcal{S}, W)] - \mathbb{E}[\mathcal{N}(F_\mu, \mathcal{S}, W)] \right| \leqslant \varepsilon,$$

where $\mu_{K,N}$ is as in (4.3.21). Moreover, the conclusion also holds for $\mathcal{N}(\cdot, T)$, as in Theorem 4.1.8.

*Proof.* Thanks to Lemma 4.5.3, $F_{\mu_{K,N}} \in C^1_*(W)$ a.s., thus the lemma follows directly from Lemma 4.3.7 with Portmanteau Theorem. We can apply it in light of the fact that $\mathcal{N}(F_{\mu_{K,N}}, \mathcal{S}, W) \leqslant \mathcal{N}(F_{\mu_{K,N}}, W) = O(W^m)$ uniformly for all $K$ and $N$ by the Faber-Krahn inequality and Proposition 4.5.2 which ensures that $\mathcal{N}(F_{\mu_{K,N}}, \mathcal{S}, W)$ is a continuous functional on $C^1_*(W)$. $\square$

**Lemma 4.5.5.** Let $\varepsilon > 0$, $W > 1$ and $\mathcal{S} \subset H(m-1)$. Then there exist some $K_0 = K_0(\varepsilon, W)$, $N_0 = N_0(K, \varepsilon, W)$, $R_0 = R_0(N, \varepsilon, W)$ such that for all $K \geqslant K_0$, $\delta \lesssim K^{-m}$, $N \geqslant N_0$ and $R \geqslant R_0$, we have

$$\left| \fint_{B(R)} \mathcal{N}(\phi_x, \mathcal{S}, W) dx - \mathbb{E}[\mathcal{N}(F_\mu, \mathcal{S}, W)] \right| < \varepsilon,$$

Moreover, the conclusion also holds for $\mathcal{N}(\cdot, T)$, as in Theorem 4.1.8.

*Proof.* By Lemma 4.5.4, it is enough to prove that, under the assumptions of Lemma 4.5.5, we have

$$\left| \fint_{B(R)} \mathcal{N}(\phi_x, \mathcal{S}, W) dx - \mathbb{E}[\mathcal{N}(F_{\mu_K}, \mathcal{S}, W)] \right| \leqslant \varepsilon/2 \tag{4.5.5}$$

First, since $C^1_*(W)$ is open by Claim 4.5.1 and as $\mathbb{P}\left(F_{\mu_K} \in C^1_*(W)^c\right) = 0$ by Lemma 4.5.3, Portmanteau Theorem jointly with Lemma 4.3.5 gives

$$\operatorname{vol}_R \left( \phi_x \in C^1_*(W) \right) \to 1 \tag{4.5.6}$$

as $R, N$ go to infinity according to Lemma 4.3.5 depending on $K \geqslant 1$ (and thus $\delta > 0$). Thus, by the Continuous Mapping Theorem and Lemma 4.3.5 for $W+1$:

$$\mathcal{N}(\phi_x, \mathcal{S}, W) \xrightarrow{d} \mathcal{N}(F_{\mu_K}, \mathcal{S}, W).$$

Therefore Lemma 4.2.2 (using Faber-Krahn inequality) implies the desired result as long as $R, N$ go to infinity as in Lemma 4.3.5 depending on $K \geqslant 1$ (and thus $\delta > 0$). $\square$

### 4.5.4 Proof of Proposition 4.5.1

We are finally ready to prove Proposition 4.5.1.

*Proof of Proposition 4.5.1.* In light of Lemma 4.5.4 and 4.5.5 it is enough to prove that, given $\varepsilon > 0$, there exist some $K_0 = K_0(\varepsilon, W)$, $N_0 = N_0(K, \varepsilon, W)$, $R_0 = R_0(N, \varepsilon, W)$ such that for all $K \geqslant K_0$, $\delta \lesssim K^{-m}$, $N \geqslant N_0$ and $R \geqslant R_0$, we have

$$\left| \fint_{B(R)} (\mathcal{N}(F_x, \mathcal{S}, W) - \mathcal{N}(\phi_x, \mathcal{S}, W)) dx \right| < \varepsilon.$$

First, by Lemma 4.3.1, we have

$$\fint_{B(R)} \|F_x - \phi_x\|^2_{C^s(B(W'))} \, dx \leqslant CW'^{2(s+l)+m} \left(\delta K^{m-1} + W'^2 K^{-2}\right) \left(1 + O_N(R^{-\Lambda - 3/2})\right),$$

where $W' := W + 1$. Let

$$\epsilon \equiv \epsilon(K, N, W') := CW'^{2(s+l)+m} \left(\delta K^{m-1} + W'^2 K^{-2}\right) \left(1 + O_N(R^{-\Lambda - 3/2})\right). \quad (4.5.7)$$

Let us denote $\phi_x$ by $\phi_x^\epsilon$ and similarly for $F_x$, let $B(R)^*$ be the set where $\phi_x \in C^1_*(W)$. We claim that

$$\fint_{B(R)^*} (\mathcal{N}(F_x^\epsilon, \mathcal{S}, W)) - \mathcal{N}(\phi_x^\epsilon, \mathcal{S}, W))) \, dx \to 0 \text{ as } \epsilon \to 0. \quad (4.5.8)$$

For the sake of contradiction, let us suppose that there exist some $\gamma > 0$ and a sequence $\epsilon_n \to 0$, such that

$$| \fint_{B(R)^*} (\mathcal{N}(F_x^{\epsilon_n}, \mathcal{S}, W)) - \mathcal{N}(\phi_x^{\epsilon_n}, \mathcal{S}, W))) \, dx \,| \geqslant \gamma.$$

However, Lemma 4.3.1 gives $\fint_{B(R)} \|F_x^{\epsilon_n} - \phi_x^{\epsilon_n}\|^2_{C^1(B(W'))} \, dx \to 0$ as $\epsilon_n \to 0$, thus there exists a subsequence $n_j$ such that with a rescaling $B(R)$ to a ball of radius 1

$$\left\| F^{\epsilon_{n_j}}_{R_{n_j} x} - \phi^{\epsilon_{n_j}}_{R_{n_j} x} \right\|^2_{C^1(B(W'))} \to 0 \quad (4.5.9)$$

$x$-almost surely as $j \to \infty$. However, (4.5.9) together with by the continuity of $\mathcal{N}$ (Lemma 4.5.1), the Faber-Krahn inequality and the Dominated Convergence Theorem, gives

$$\left| \fint_{B(1)^*} \mathcal{N}(F^{\epsilon_{n_j}}_{R_{n_j}x}, \mathcal{S}, W)) - \mathcal{N}(\phi^{\epsilon_{n_j}}_{R_{n_j}x}, \mathcal{S}, W))dx \right| \to 0$$

as $j \to \infty$, a contradiction. Finally, bearing in mind the Faber-Krahn inequality, we have the bound

$$\begin{aligned}\fint_{B(R)} (\mathcal{N}(F^{\epsilon}_x, \mathcal{S}, W)) - \mathcal{N}(\phi^{\epsilon}_x, \mathcal{S}, W)))dx &= \fint_{B(R)^*} (\mathcal{N}(F^{\epsilon}_x, \mathcal{S}, W)) - \mathcal{N}(\phi^{\epsilon}_x, \mathcal{S}, W)))dx + \\ &+ O\left(W'^m\left(1 - \mathrm{vol}_R\left(\phi^{\epsilon} \in C^1_*(W)\right)\right)\right) \leqslant \varepsilon,\end{aligned}$$

where the inequality holds as long as $\epsilon < \epsilon_0$ given by (4.5.8) and $R, N$ large enough according to Lemma 4.3.5 to ensure (4.5.6).

Hence, we make the following choices of the parameters in order to prove the proposition (recall that the parameters $K_0, N_0, R_0$ must be chosen to satisfy i) and ii)), let $K$ large enough, with $\delta \lesssim K^{-m+1}$ as in Lemma 4.3.7, such that $CW'^{2(s+l)+m} \times \times \left(\delta K^{m-1} + W'^2 K^{-2}\right) < \epsilon_0/2$ and such that $K > K_0(\varepsilon/2, W')$ accordingly to Lemma 4.5.4. Similarly, let $N > N_0(\varepsilon/2, W')$ with $N_0$ given in Lemma 4.5.4. We also take $N$ large enough according to Lemma 4.3.5 such that

$$O\left(W'^m\left(1 - \mathrm{vol}_R\left(\phi^{\epsilon} \in C^1_*(W)\right)\right)\right) < \varepsilon/2 \tag{4.5.10}$$

provided $R$ is large enough and the same for (4.5.5). Finally, let $R$ large enough according to the two conditions mentioned below (4.5.10) and such that in the definition of $\epsilon$ in (4.5.7) we have $O_N(R^{-\Lambda-3/2}) < 1$. □

### 4.5.5 Concluding the proof of Theorem 4.1.8

We are finally ready to prove Theorem 4.1.8

*Proof of Theorem 4.1.8.* Let $\mathcal{S} \subset H(m-1)$ and $T \in \mathcal{T}$ be given, and denote by $\mathcal{N}(f, \cdot, R)$ either $\mathcal{N}(f, \mathcal{S}, R)$ or $\mathcal{N}(f, T, R)$. Thanks to Proposition 4.1.5 and the fact that the number of nodal domain with fixed topological class or tree type intersecting $B(W)$ is bounded by the total number of nodal domains intersecting $B(W)$, for $W > 1$, we have

$$\frac{\mathcal{N}(f, \cdot, R)}{\mathrm{vol}\, B(R)} = \frac{1}{\mathrm{vol}\, B(W)} \fint_{B(R)} \mathcal{N}(F_x, \cdot, R)dx + O\left(W^{-1}\right) + O_{N,W}\left(R^{-\Lambda-3/2}\right). \tag{4.5.11}$$

Now, by Theorem 4.2.9, with the same notation, we have

$$\frac{\mathbb{E}[\mathcal{N}(F_\mu, \cdot, W)]}{\mathrm{vol}\, B(W)} = c_{NS}(\cdot, \mu) + a(W),$$

where $a(W) \to 0$ as $W \to \infty$. By Proposition 4.5.1 applied with some $\varepsilon_0/2$, we have

$$\left| \fint_{B(R)} \mathcal{N}(F_x, \cdot)dx - \mathbb{E}[\mathcal{N}(F_\mu, \cdot, W)] \right| < \varepsilon_0 \tag{4.5.12}$$

if $K \geqslant K_0$ with $K_0 = K_0(\varepsilon_0/2, W)$, $N \geqslant N_0$ with $N_0 = N_0(K, \varepsilon_0/2, W)$ and $R \geqslant R_0$ with $R_0 = R_0(N, \varepsilon_0/2, W)$. Hence, putting (4.5.11), (4.5.12) together, we obtain

$$\left| \frac{\mathcal{N}(f, \cdot, R)}{\operatorname{vol} B(R)} - c_{NS}(\cdot, \mu) \right| \leqslant \frac{\varepsilon_0}{\operatorname{vol} B(W)} + a(W) + O\left(\frac{1}{W}\right) + O_{N,W}\left(R^{-\Lambda-3/2}\right).$$

Let us now pick some $\varepsilon > 0$ and choose first a $W$ large enough (and fix it) so that

$$O\left(W^{-1}\right) + a(W) < \varepsilon/3,$$

then set $K \geqslant K_0$, $N \geqslant N_0$ and $R \geqslant R_0$ with $\varepsilon_0 = \operatorname{vol} B(W)\varepsilon/3$ and $R$ large enough such that

$$O_{N,W}\left(\frac{1}{R^{\Lambda+3/2}}\right) < \varepsilon/3.$$

□

## 4.6 Proof of Theorem 4.1.3.

### 4.6.1 Uniform integrability of $\mathcal{V}(F_x, W)$

We first prove Proposition 4.1.6:

*Proof of Proposition* 4.1.6. By Lemma 4.2.4 for $r = W$, we have

$$\begin{aligned} \fint_{B(R)} \mathcal{V}(F_x, W)^{1+\alpha} dx &\lesssim (W^m)^{1+\alpha} + W^{(m-1)(1+\alpha)} \fint_{B(R)} \mathfrak{N}_f(B(x, 3W))^{1+\alpha} dx \\ &= (W^m)^{1+\alpha} + W^{(m-1)(1+\alpha)} \int_1^{CN^{1+\alpha}} \operatorname{vol}_R\left(\mathfrak{N}_f(B(x, 3W))^{1+\alpha} > t\right) dt \end{aligned}$$

Changing variables and using Lemma 4.4.5 with $D = \alpha + 2$ and $H = 3W$, we find that

$$\begin{aligned} \int_1^{CN^{1+\alpha}} \operatorname{vol}_R\left(\mathfrak{N}_f(B(x, W))^{1+\alpha} > t\right) dt &= (1+\alpha) \int_1^{CN} t^\alpha \operatorname{vol}_R\left(\mathfrak{N}_f(B(x, 3W)) > t\right) dt \\ &\lesssim W^{(1+\alpha)} + O_{N,W}(R^{-\Lambda-3/2}) \end{aligned}$$

as required. □

### 4.6.2 Continuity of $\mathcal{V}$.

In this section we prove the following proposition

**Proposition 4.6.1.** Let $W > 1$ be given and $f$ be as in (4.1.1). Then we have

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \fint_{B(R)} \mathcal{V}(F_x, W) - \mathbb{E}[\mathcal{V}(F_\mu, W)] \right| = 0$$

The proof of Proposition 4.6.1 is similar to the proof of Proposition 4.5.1. We just need the following lemma, which follows from Theorem 4.2.11.

**Lemma 4.6.2.** $\mathcal{V}(\cdot, B)$ is a continuous functional on $C^1_*(W)$.

Note that now the condition of $\Psi > 0$ is not needed here.

*Proof.* For the sake of simplicity, we set $V = B(W+1)$ and $\chi$ as in the proof of Proposition 4.5.2 so we will only consider boundaryless manifolds. Let $h$ be an arbitrary function of $C^1_*(W)$. We apply Thom's Isotopy Theorem 4.2.11 to $h' := h + \chi$, which is identical to $h$ in $B(W)$. Let $\varepsilon > 0$ and $\psi$ a local parametrization of a nodal component $L_{h'}$ of $h'$, which is a boundaryless manifold in $B(W+1)$. By Thom's Isotopy Theorem 4.2.11, and with the same notation, $\Phi \circ \psi$ is a local parametrization of a nodal component of $g'$, $L_{g'}$ provided $\|h' - g'\|_{C^1(U)} < \delta$ and $\delta > 0$ is small enough as in (4.5.3). If $J$ denotes the Jacobian matrix and the local parametrization is $\psi : A \to \mathcal{U}$, we have

$$\mathrm{vol}\,(\mathcal{U} \cap L_{g'}) = \int_A \sqrt{\det((J\psi(x))^t \cdot J\psi(x))}dx$$

and by the chain rule if $\Phi \circ \psi : A \to \mathcal{U}'$

$$\mathrm{vol}\,(\mathcal{U}' \cap L_{g'}) = \int_A |\det(J\Phi)(\psi(x))| \sqrt{\det((J\psi(x))^t \cdot J\psi(x))}dx.$$

But, by the standard series for the determinant:

$$\det(J\Phi(y)) = \det(I + J\Phi(y) - I) = 1 + \mathrm{tr}(J\Phi(y) - I) + o\left(\|J\Phi(y) - I\|\right)$$

where $I = J\mathrm{id}$ is the identity and

$$\left|\mathrm{tr}(J\Phi(y) - I) + o\left(\|J\Phi(y) - I\|\right)\right| \lesssim \|\Phi - \mathrm{id}\|_{C^1(\mathbb{R}^m)} < \varepsilon.$$

Thus, for $L_h \subset \bigsqcup_i \mathcal{U}_i \subset B(W+1)$ we have $|\,\mathrm{vol}\,(L_{h'}) - \mathrm{vol}\,(L_{g'})| \lesssim \varepsilon\,\mathrm{vol}\,(L_{h'})$. Hence, we also have

$$|\,\mathrm{vol}\,(L_{g'} \cap B(W)) - \mathrm{vol}\,(\Phi^{-1}(L_{g'} \cap B(W)))| \lesssim \varepsilon\,\mathrm{vol}\,(L_{g'} \cap B(W)) \lesssim \varepsilon(1+\varepsilon)\,\mathrm{vol}\,(L_{h'}).$$

Finally, as

$$|\,\mathrm{vol}\,(\Phi^{-1}(L_{g'} \cap B(W))) - \mathrm{vol}\,(L_{h'} \cap B(W))| \leqslant \mathrm{vol}\,(L_{h'} \cap B(W+\varepsilon) \backslash B(W))$$

the continuity of $\mathcal{V}$ at $h$ follows immediately as the are finitely many components $L_{h'}$ (because $h \in C^1_*(W)$), by the fact that $\|h - g\| = \|h' - g'\|$ for $g' = g + \chi$ and $L_{g'} \cap B(W) = L_g \cap B(W)$ (the same for $L_h$). □

We are now ready to prove Proposition 4.6.1.

*Proof of Proposition 4.6.1.* The proof is similar to the proof of Proposition 4.5.1 so we omit some details. By Claim 4.5.1 and Lemma 4.5.3 we have

$$\mathbb{P}\left(F_\mu \in C^1_*(W)\right) = 1. \tag{4.6.1}$$

Now, Lemma 4.6.2 together with (4.6.1), Theorem 4.1.7 (applied with $W+1$) and the Continuous Mapping Theorem imply that

$$\mathcal{V}(F_x, W) \xrightarrow{d} \mathcal{V}(F_\mu, W) \tag{4.6.2}$$

where the convergence is in distribution as $R, N \to \infty$ according to that theorem. Hence, Proposition 4.6.1 follows from (4.6.2), Proposition 4.1.6 and Lemma 4.2.2. $\square$

### 4.6.3 Concluding the proof of Theorem 4.1.3

Before concluding the proof of Theorem 4.1.3, we need to following direct application of the Kac-Rice formula:

**Lemma 4.6.3.** Suppose that $F_\nu$ is a centred, stationary Gaussian field defined on $\mathbb{R}^m$ such that $(F, \nabla F)$ is non-degenerate and $\nu$ is supported on $\mathbb{S}^{m-1}$ then there exists some constant $c = c(\nu)$ such that

$$\mathbb{E}[\mathcal{V}(F_\nu, W)] = c(\nu) \operatorname{vol} B(W)$$

*Proof.* For brevity let us write $F = F_\nu$. By [AT09, Theorem 6.3], since $F$ is non-degenerate (4.5.4) and almost surely analytic, we have

$$\mathbb{E}[\mathcal{V}(F_\nu, W)] = \int_{B(W)} \mathbb{E}[|\nabla F(y)||F(y)] v_{F(y)}(0) dy \tag{4.6.3}$$

where $v_{F(y)}(0)$ is the density of $F(y)$ at zero. By stationarity, the integrand in (4.6.3) is independent of $y$ and the Lemma follows. $\square$

**Remark 4.6.4.** It is possible to explicitly derive an expression for $\mathbb{E}[\mathcal{V}(F_\nu, W)] = \int_{B(W)} \mathbb{E}[|\nabla F(y)||F(y)]$ in terms of the covariance of $F_\nu$ and its derivatives, following computations similar to those of [KKW13; EPSR21]. However, since the calculations are quite long, we decided not to include them in the article.

And lastly the following lemma:

**Lemma 4.6.5.** Let $h : \mathbb{R}^m \to \mathbb{R}$ and $0 < r < R$. Then we have

$$\mathcal{V}(h, R-r) \leqslant \int_{B(R)} \frac{\mathcal{V}(h; x, r)}{\operatorname{vol} B(r)} dx \leqslant \mathcal{V}(h, R+r) \tag{4.6.4}$$

*Proof.* By the definition of $\mathcal{V}$ and Fubini, we have

$$\begin{aligned}\int_{B(R)} \mathcal{V}(h; x, r) dx &= \int_{B(R)} \int_{B(R+r)} \mathbb{1}_{B(x,r)}(y) \mathbb{1}_{h^{-1}(0)}(y) d\mathcal{H}(y) dx. \\ &= \int_{B(R+r)} \mathbb{1}_{h^{-1}(0)}(y) \operatorname{vol}\left(B(y,r) \cap B(R)\right) d\mathcal{H}(y),\end{aligned}$$

so the lemma follows from $\mathbb{1}_{B(R-r)} \leqslant \dfrac{\operatorname{vol}\left(B(\cdot, r) \cap B(R)\right)}{\operatorname{vol} B(r)} \leqslant \mathbb{1}_{B(R+r)}$. $\square$

We are finally ready to prove Theorem 4.1.3.

*Proof of Theorem* 4.1.3. The proof follows closely the proof of Theorem 4.1.1, so we omit some details. Let $\varepsilon > 0$ be given, then applying Lemma 4.6.5 with $r = W$ and dividing by $\operatorname{vol} B(R)$, we have

$$\frac{1}{\operatorname{vol} B(R)} \int_{B(R-W)} \frac{\mathcal{V}(F_x, W)}{\operatorname{vol} B(W)} dx \leqslant \frac{\mathcal{V}(f, R)}{\operatorname{vol} B(R)} \leqslant \frac{1}{\operatorname{vol} B(R)} \int_{B(R+W)} \frac{\mathcal{V}(F_x, W)}{\operatorname{vol} B(W)} dx.$$

For any $\alpha > 0$, Proposition 4.1.6 gives

$$\frac{1}{\operatorname{vol} B(R)} \int_{B(R+W) \setminus B(R)} \frac{\mathcal{V}(F_x, W)}{\operatorname{vol} B(W)} = O_{N,W,\alpha}(R^{-\gamma}),$$

for some $\gamma > 0$. Therefore, as in the proof of Proposition 4.1.5,

$$\frac{\mathcal{V}(f, R)}{\operatorname{vol} B(R)} = \fint_{B(R)} \frac{\mathcal{V}(F_x, W)}{\operatorname{vol} B(W)} dx + O_{N,W,\alpha}(R^{-\gamma}). \tag{4.6.5}$$

Finally, Proposition 4.6.1 gives, for every $\varepsilon > 0$,

$$\frac{\mathcal{V}(f, R)}{\operatorname{vol} B(R)} = \frac{1}{\operatorname{vol} B(W)} \mathbb{E}[\mathcal{V}(F_\mu, W)] + O(\varepsilon)$$

for all $N$ and $R$ sufficiently large. The theorem now follows from Lemma 4.6.3. ☐

## 4.7 Final comments.

### 4.7.1 Exact Nazarov-Sodin constant for limiting function?

As we have seen, Theorem 4.1.1 says that there are deterministic functions with the growth rate for the nodal domains count arbitrarily close, increasing $N$, to the Nazarov-Sodin constant. One may wonder whether this constant is attained if $N$ goes to infinity and the functions $\{f_N\}$ as in (4.1.1) (or a rescaling of it) converges, in some appropriate space of functions, to some function $f$. Our argument does not apply outright in the limit $N \to \infty$ and then $R \to \infty$. That is, Theorem 4.1.1 gives

$$\lim_{N\to\infty} \limsup_{R\to\infty} \left| \frac{\mathcal{N}(f_N, R)}{\operatorname{vol} B(R)} - c_{NS} \right| = 0. \tag{4.7.1}$$

However, given a sequence of functions $\tilde{f}_N := C_N f_N$ with $C_N > 0$ (so $\mathcal{N}(f_N, R) = \mathcal{N}(\tilde{f}_N, R)$) such that $\tilde{f}_N \to f$, one could hope that the following holds:

$$\lim_{R\to\infty} \lim_{N\to\infty} \left| \frac{\mathcal{N}(f_N, R)}{\operatorname{vol} B(R)} - c_{NS} \right| = \lim_{R\to\infty} \left| \frac{\mathcal{N}(f, R)}{\operatorname{vol} B(R)} - c_{NS} \right| = 0. \tag{4.7.2}$$

In this section we show that this is not true in general, that is, we give examples of sequences of functions such that $\tilde{f}_N \to f$ and (4.7.1) hold but (4.7.2) does not, in fact, the growth rate is much smaller.

Let us consider the functions

$$f_N := \frac{1}{2N^{1/2}} \sum_{|n|\leqslant N} e(\langle r_n, x\rangle),$$

assume that $\mu$ is the Lebesgue measure on the sphere and define $\tilde{f}_N = N^{-1/2} f_N$. Then, we have

$$\tilde{f}_N(x) = \frac{1}{2N} \sum_{|n|\leqslant N} e(\langle r_n, x\rangle) \xrightarrow{N\to\infty} \int_{\mathbb{S}^{n-1}} e(\langle \omega, x\rangle) d\sigma(\omega) =: f = C_m \frac{J_\Lambda(2\pi|x|)}{|x|^\Lambda}$$

where $\Lambda = (m-2)/2$ and the convergence is uniform on compact sets, see (4.A.1), with respect to $x \in \mathbb{R}^m$ and also holds after differentiating any finite number of times. Thus by Theorem 4.1.7, (4.7.1) holds, but

$$\mathcal{N}(f, R) = cR(1 + o_{R\to\infty}(1)) \tag{4.7.3}$$

for some known $c > 0$, thus (4.7.2) does not hold. We also observe that, using [EPSR22a, Theorem 3.1], it is possible to make more general choices of $a_n$, for example $a_n = \phi(r_n)$ for some sufficiently smooth function $\phi$. With this choice, either the number of nodal domains of $f$ grows as in (4.7.3) or it could even be bounded (for large enough $R$ the nodal set is a non-compact nodal component consisting on layers and an "helicoid" connecting them).

In order to illustrate this change, we show in Figure 4.1 below the nodal set for the function

$$g_N(x, y) := \frac{1}{N} \sum_{n=1}^{N} \cos(x \cos\theta_n + y \sin\theta_n)$$

for different $N$ and $\theta_n$ uniformly distributed over the sphere. As $N$ increases the connected components of the nodal sets near the origin tend to merge and they are close and diffeomorphic to the ones of $J_0$.

Finally, we mention that a similar phenomenon happens for eigenfunctions on the two dimension torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. These can be written as

$$f_{\mathbb{T}^2}(x) = \sum_{\xi\in\mathcal{E}} a_\xi e(\langle \xi, x\rangle),$$

where $\mathcal{E} = \mathcal{E}_E = \{\xi \in \mathbb{Z}^2 : |\xi|^2 = E\}$ and $a_\xi$ are complex coefficients. Under some arithmetic conditions and some constrains on the coefficients, in [Bou14; BW16] it is showed that there are deterministic realizations of the RWM, that is

$$\mathcal{N}(f) = c_{NS}(\mu) \cdot E(1 + o(1)). \tag{4.7.4}$$

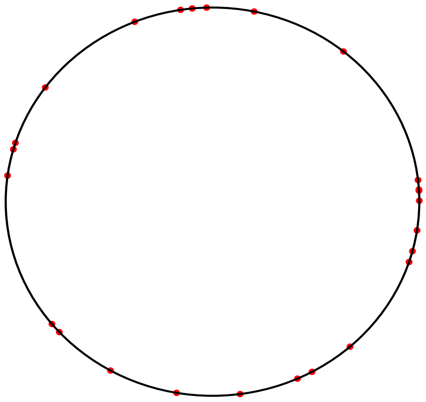
However, since the points $\xi/\sqrt{E} \in \mathbb{S}^1$ "generically" become equidistributed on $\mathbb{S}^1$ [EH99], the function

$$\tilde{f} = \frac{1}{|\mathcal{E}|} \sum_{\xi\in\mathcal{E}} e(\langle \xi, x\rangle),$$

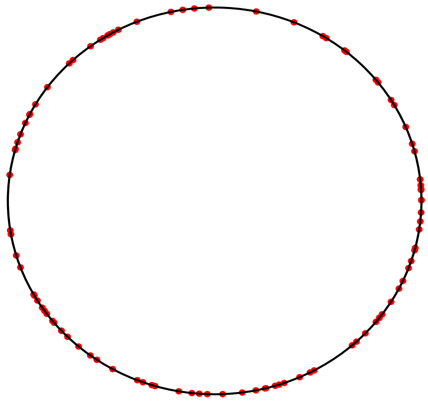
once rescaled, presents the same limiting behaviour described by (4.7.3). In particular, for $f_{1/\sqrt{E}} := f\left(E^{-1/2}\right)$, considering the periodicity, (4.7.4) and $R$ large enough,

$$\mathcal{N}(f_{1/\sqrt{E}}, R) \sim \operatorname{vol} B(R) \cdot c_{NS}(\mu)(1 + o(E^0)),$$

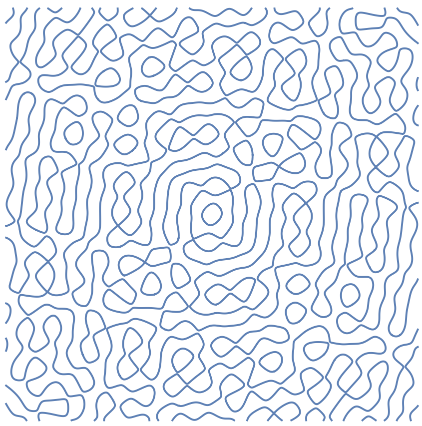
still an approximated constant.



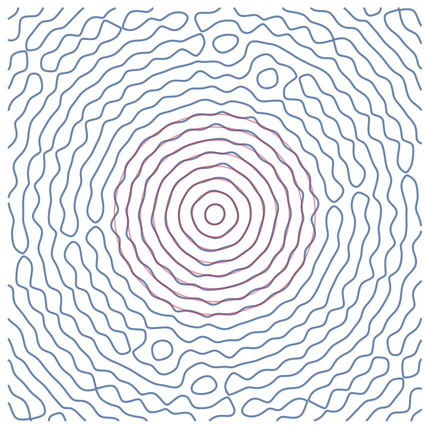
(A) 25 points.



(B) 100 points.



(C) Nodal set for the function with 25 points.



(D) Nodal set for the function with 100 points. In red for $J_0(|x|)$.

FIGURE 4.1: Nodal set for the function $g_N$ for different $N$ and the points on the sphere $\mathbb{S}^1$. We have represented in red the first eight connected components of $J_0(|\cdot|)^{-1}(0)$.

### 4.7.2 On a question of Kulberg and Wigman

The methods we have developed allow us to solve a question raised by Kulberg and Wigman, [KW18, Section 2.1], on the continuity of

$$\mu \mapsto \mathbb{E}[\mathcal{N}(F_\mu, R)]$$

on the plane. Indeed, we can strengthen Lemma 4.5.4 to give a more general result as in Proposition 4.7.1. This solves their question on any dimension, not only $m = 2$.

**Proposition 4.7.1.** Let $\mu_n$ be a sequence of measures on the sphere $\mathbb{S}^{m-1}$ not supported on a hyperplane converging weakly to $\mu$ as $n \to \infty$. Then

$$\mathbb{E}[\mathcal{N}(F_{\mu_n}, R)] \to \mathbb{E}[\mathcal{N}(F_\mu, R)]$$

for any $R > 0$.

*Proof.* Thanks to Lemma 4.5.3, $F_{\mu_n}, F_\mu \in C^1_*(B(W))$ almost surely. From Lemma 4.3.6 we can stablish $d_P(F_{\mu_n}, F_\mu) \to 0$. Thus the proposition follows directly Portmanteau Theorem, as $\mathcal{N}(F_{\mu_n}, R) = O(R^m)$ by the Faber-Krahn inequality, uniformly for all $n$, and it is continuous by Proposition 4.5.2. ☐

**Remark 4.7.2.** The same holds for topological classes and trees with an analogous, *mutatis mutandis*, proof.

With our techniques we can also extend the discrepancy functional that was introduced in [KW18, Proposition 1.2] for any dimension, topologies and nesting trees. More precisely, we have:

**Proposition 4.7.3.** The discrepancy functional exists, that is,

$$\lim_{R\to\infty} \mathbb{E}\left|\frac{\mathcal{N}(F_\mu, \cdot, R)}{\operatorname{vol} S(R)} - c(\cdot, \mu)\right| \tag{4.7.5}$$

exists and it is finite.

*Proof.* Following the proof of Theorem 4.2.9,

$$\lim_{R\to\infty} \mathbb{E}\left|\frac{\mathcal{N}(F_\mu, \cdot, R)}{\operatorname{vol} S(R)} - c(\cdot, \mu)\right| = \mathbb{E}\left|c(G, \cdot, \mu) - c(\cdot, \mu)\right|,$$

as the convergence is in $L^1$. ☐

From the proof we see that the discrepancy functional, (4.7.5), is zero if and only if $c(F_\mu, \cdot, R)$ is a.s. a constant, that is, the limit of the nodal counts is non-random. This is true, in particular, if the field is ergodic. This functional measures how far we are from the ergodic situation of $\lim_{R\to\infty} \frac{\mathcal{N}(G, \cdot, R)}{\operatorname{vol} S(R)}$ being, a.s., a constant.

## APPENDICES

## 4.A Gaussian fields lemma.

Adapting [EPSR20, Lemma 7.2] for our situation we can show:

**Lemma 4.A.1.** Let $s \geqslant 0$ and $W \geqslant 1$, be given. Moreover, suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{S}^{m-1}$ such that $\mu_n$ weak$^\star$ converges to $\mu$. Then,

$$d_P(F_{\mu_n}, F_\mu) \longrightarrow 0 \text{ as } n \to \infty$$

where the convergence is with respect to the $C^s(B(W))$ topology.

*Proof.* Since $\mu_n$ weak$^\star$-converges to $\mu$ and the exponential is a bounded continuous function, by the discussion in Section 4.2.1 and Portmanteau Theorem, we have for any $x, y \in B(W)$

$$\begin{aligned} K_n(x,y) := \mathbb{E}[F_{\mu_n}(x)F_{\mu_n}(y)] = \int e(\langle x-y, \lambda\rangle) d\mu_n(\lambda) \to \\ \to \int e(\langle x-y, \lambda\rangle) d\mu(\lambda) = \mathbb{E}[F_\mu(x)F_\mu(y)] = K(x,y) \end{aligned} \tag{4.A.1}$$

Since, $\mu_n$ and $\mu$ are compactly supported, we may differentiate under the integral sign in (4.A.1), it follows that (4.A.1) holds after taking derivatives. Thanks to the Cauchy-Schwarz inequality and the fact that $\mu_N$ is a probability measure, we have

$$|K_n(x,y) - K_n(x',y')| \leqslant C \left\|x - x' + y - y'\right\|$$

as $|e^{ix} - e^{iy}| \leqslant |x-y|$ for any $x, y \in \mathbb{R}$. Therefore $K_n$ is equicontinuous, $|K_n(x,y)| \leqslant 1$ and it converges pointwise; thus, the Arzelà-Ascoli Theorem implies that the convergence in (4.A.1) is uniform for all $x, y \in B(W)$, together with its derivatives. Now, for any integer $t \geqslant 0$, the mean of the $H^t$-norm of $F_{\mu_n}$ is uniformly bounded:

$$\begin{aligned}\mathbb{E}\|F_{\mu_n}\|^2_{H^t(B(W))} &= \sum_{|\alpha|\leqslant t} \mathbb{E}\int_{B(W)} |D^\alpha F_{\mu_n}(x)|^2\,dx = \sum_{|\alpha|\leqslant t}\int_{B(W)} D^\alpha_x D^\alpha_y K_n(x,y)\big|_{y=x}\,dx \\ &\xrightarrow[n\to\infty]{} \sum_{|\alpha|\leqslant t}\int_{B(W)} D^\alpha_x D^\alpha_y K(x,y)\big|_{y=x}\,dx < M_{t,W}\,,\end{aligned}$$

where, in the last line, we have used (4.A.1). As the constant $M_{t,W}$ is independent of $N$, Sobolev's inequality ensures that one can now take any sufficiently large $t$ to conclude that

$$\sup_n \mathbb{E}\|F_{\mu_n}\|^2_{C^{s+1}(B(W))} \leqslant C \sup_n \mathbb{E}\|F_{\mu_n}\|^2_{H^t(B(W))} < M$$

for some constant $M$ that only depends on $W, t$. Thus, for any $\varepsilon > 0$ and any sufficiently large $n$, we have

$$\nu^W_n\left(\{F_{\mu_n} \in C^{s+1}(B(W)) : \|F_{\mu_n}\|^2_{C^{s+1}(B(W))} > M/\varepsilon\}\right) < \varepsilon\,,$$

where $\nu^W_n$ is the measure on the space of $C^s(B(W))$ functions corresponding to the random field $F_{\mu_n}$. Accordingly, the sequence of probability measures $\nu^W_n$ is tight. Indeed, by Lagrange and Arzelà-Ascoli the closure of the set

$$\{F_{\mu_n} \in C^{s+1}(B(W)) : \|F_{\mu_n}\|^2_{C^{s+1}(B(W))} \leqslant M/\varepsilon\}$$

is precompact with the $C^s$ topology, so we can conclude by the very definition of tightness, see Section 4.2.2. □

## 4.B Upper bound on $\mathcal{NI}$.

In this section are going to prove Lemma 4.4.2 following [CLM+20, section 3.2] (that is, the following argument is due to F. Nazarov and we claim no originality). We will need the following (rescaled) result [CLM+20, Lemma 2.5]:

**Lemma 4.B.1.** Let $r > 1$ and $\rho > 0$, let $f$ be a Laplace eigenfunction with eigenvalue $4\pi^2 r^2$ on $B(\rho)$. Suppose that an open set $\Omega \subset B(\rho)$ is $c_0$-narrow (on scale $1/r$) and $f = 0$ on $\partial\Omega \cap B(\rho)$. Then, for every $\varepsilon > 0$, sufficiently small depending on $c_0$ and $m$, if

$$\frac{|\Omega|}{|B_\rho|} \leqslant \varepsilon^{m-1},$$

then

$$\sup_{\Omega\cap B(\rho/2)} |f| \leqslant e^{-c/\varepsilon} \sup_{\Omega\cap B(\rho)} |f|$$

We are finally ready to prove Lemma 4.4.1

*Proof of Lemma* 4.4.1. First, we rescale $f$ to $f_r$ so that $\mathcal{NI}(f,x,r) = \mathcal{NI}(f_r,x,1)$ and we may assume that every nodal domain is $c_0$-narrow. Let $Z = \mathcal{NI}(f_r,x,1)$, $\Omega_i$ be the elements of $\mathcal{NI}(f_r,x,1)$, $D = \mathfrak{N}_{f_{2r}}(B(x,1)) + r = \mathfrak{N}_f(B(x,2r)) + r$ and finally let $B(x) = B(x,1)$. Suppose that $Z > 2^{m+2}c_1^{-1} \cdot D^{m-1}$ for some constant $c_1 = c_1(m)$ to be chosen later, we are going to derive a contradiction.

Let $x_i \in \Omega_i \cap B(x,1)$ and define

$$S(p) = \left| \{\Omega_i : |\Omega_i \cap B(x_i, 2^{-j})| \leqslant c_1^{-1} D^{-m+1} |B(x_i, 2^{-j})| \text{ for } j \in \{0, ..., p\}\} \right|.$$

First we are going to show that $S(0) \neq \emptyset$. Indeed, since

$$\sum_i |\Omega_i \cap B(x_i,1)| \leqslant |B(2)|,$$

for at least $(3/4)Z$ nodal domains, we have

$$\frac{|\Omega_i \cap B(x_i,1)|}{2^m |B(1)|} \leqslant \frac{2^2}{Z},$$

thus $S(0) \geqslant (3/4)Z$.

Now, we claim the following:

**Claim 4.B.1.** *Let $p \geqslant 1$, then there are at most $Z4^{-p-2}$ nodal domains $\Omega_i \in S(p) \backslash S(p+1)$.*

*Proof.* From now on, fix some $p \geqslant 1$ and assume that $\Omega_i \in S(p)$, we wish to apply Lemma 4.B.1 with $\varepsilon = c_1^{\frac{1}{m-1}} D^{-1}$, therefore we assume $c_1$ is sufficiently small in terms of $m$ and $c_0$. Hence, we may apply Lemma 4.B.1 to $B(x_i, 2^{-j})$ for $j = 0, ..., p$, to see that

$$\frac{\sup_{\Omega_i \cap B(x_i, 2^{-p})} |f_r|}{\sup_{B(x_i,2)} |f_r|} \leqslant (2^{-p-1})^{c_2 D} \tag{4.B.1}$$

for some $c_2 = c_2(m) > 0$ and for all $i$. On the other hand, Lemma 4.2.8 applied to $f_r$, with $B(2)$ and $E = \cup_{\Omega_i \in S_p} \Omega_i \cap B(x_i, 2^{-p-1})$, in light of the fact that nodal domains are disjoint, gives

$$\left( \frac{\sum_i |\Omega_i \cap B(x_i, 2^{-p-1})|}{|B(x_i,2)|} \right)^{CD} \lesssim \frac{\sup_E |f_r|}{\sup_B |f_r|},$$

which, together with (4.B.1), implies

$$\left( \frac{\sum_i |\Omega_i \cap B(x_i, 2^{-p-1})|}{B(x_i,2)} \right)^{CD} \lesssim (2^{-p-1})^{c_2 D} \tag{4.B.2}$$

Rescaling we have $B(x_i,2) = 2^{m(p+2)} B(x_i, 2^{-p-1})$, thus (4.B.2) can be rewritten as

$$\left( \frac{\sum_i |\Omega_i \cap B(x_i, 2^{-p-1})|}{B(x_i, 2^{-p-1})} \right)^{CD} \lesssim (2^{-p-1})^{c_2 D - mCD}, \tag{4.B.3}$$

which, taking $c_2$ sufficiently large, that is, $c_1$ small depending on $m$, implies that

$$\frac{\sum_i |\Omega_i \cap B(x_i, 2^{-p-1})|}{B(x_i, 2^{-p-1})} \leqslant 4^{-p-2}.$$

Hence there are at most $Z \cdot 4^{-p-2}$ nodal domains satisfying

$$\frac{|\Omega_i \cap B(x_i, 2^{-p-1})|}{|B(x_i, 2^{-p-1})|} > c_1 D^{-m+1},$$

and the claim follows. $\square$

Using the claim and the fact that $S(p+1) \subset S(p)$, we see that, for each $p \geqslant 0$ $|S(p)| \geqslant Z - Z\sum_p 4^{-p} \geqslant Z/2$. However, since the number of nodal domains is finite and

$$\frac{|\Omega \cap B(x,\rho)|}{|B(x,\rho)|} = 1 \text{ for } \rho \text{ small enough},$$

we have that $S(p)$ is empty for $p$ sufficiently large, a contradiction. $\square$

# Part II

# Asymptotics in fluid mechanics and economics

# Chapter 5

# Almost sure existence of knots and chaos in random Beltrami fields

## 5.1 Introduction

Our objective in this chapter is to establish Arnold's view of complexity in Beltrami fields, see Chapter 1. To do so, the key new tool is a theory of random Beltrami fields, which we develop here in order to estimate the probability that a Beltrami field exhibits certain complex dynamics. The blueprint for this is the Nazarov–Sodin theory for Gaussian random monochromatic waves, which yields asymptotic laws for the number of connected nodal components of the wave. Heuristically, the basic idea is that a Beltrami field satisfying (1.8) can be thought of as a vector-valued monochromatic wave; however, the vector-valued nature of the solutions and the fact that we aim to control much more sophisticated geometric objects introduces essential new difficulties from the very beginning.

### 5.1.1 Overview of the Nazarov–Sodin theory for Gaussian random monochromatic waves

In order to be self-contained and to stress the differences with random Beltrami fields, we briefly present Nazarov-Sodin theory. See Chapter 1 for more. The Nazarov-Sodin theory [NS16], whose original motivation was to understand the nodal set of random spherical harmonics of large order [NS09], provides a very efficient tool to derive asymptotic laws for the distribution of the zero set of smooth Gaussian functions of several variables. The primary examples are various Gaussian ensembles of large-degree polynomials on the sphere or on the torus and the restriction to large balls of translation-invariant Gaussian functions on $\mathbb{R}^d$. Most useful for our purposes are their asymptotic results for Gaussian random monochromatic waves, which are random solutions to the Helmholtz equation

$$\Delta F + F = 0 \tag{5.1.1}$$

on $\mathbb{R}^d$. We will henceforth restrict ourselves to the case $d = 3$ for the sake of concreteness.

As the Fourier transform of a solution to the Helmholtz equation (5.1.1) must be supported on the sphere of radius 1, the way one constructs random monochromatic waves is the following [CS19]. One starts with a real-valued orthonormal basis of the space of square-integrable functions on the unit two-dimensional sphere $\mathbb{S}$. Although the choice of basis is immaterial, for concreteness we can think of the basis of spherical harmonics, which we denote by $Y_{lm}$. Hence $Y_{lm}$ is an eigenfunction of the spherical Laplacian with eigenvalue $l(l+1)$, the index $l$ is a non-negative integer and $m$ ranges from $-l$ to $l$. The degeneracy of the eigenvalue $l(l+1)$ is therefore $2l+1$. To consider a Gaussian random monochromatic wave, one now sets

$$\varphi(\xi) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l a_{lm} Y_{lm}(\xi) \tag{5.1.2a}$$

on the unit sphere $|\xi| = 1$, $\xi \in \mathbb{R}^3$, where $a_{lm}$ are independent standard Gaussian random variables. One then defines $F$ as the Fourier transform of the measure $\varphi\, d\sigma$, where $d\sigma$ is the area measure of the unit sphere. This is tantamount to setting

$$F(x) := (2\pi)^{\frac{3}{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{1}{2}}(|x|)}{|x|^{\frac{1}{2}}}. \tag{5.1.2b}$$

The central known result concerning the asymptotic distribution of the nodal components of Gaussian random monochromatic waves is that, almost surely, the number of connected components of the nodal set that are contained in a large ball (and even those of any fixed compact topology) grows asymptotically like the volume of the ball. More precisely, let us denote by $N_F(R)$ (respectively, $N_F(R; [\Sigma])$) the number of connected components of the nodal set $F^{-1}(0)$ that are contained in the ball centered at the origin of radius $R$ (respectively, and diffeomorphic to $\Sigma$). Here $\Sigma$ is any smooth, closed, orientable surface $\Sigma \subset \mathbb{R}^3$. It is obvious from the definition that $N_F(R; [\Sigma])$ only depends on the diffeomorphism class of the surface, $[\Sigma]$. The main result of the theory —which is due to Nazarov and Sodin [NS16] in the case of nodal sets of any topology, and to Sarnak and Wigman when the topology of the nodal sets is controlled [SW19]— can then be stated as follows. Here and in what follows, the symbol $\xrightarrow[\text{a.s.}]{L^1}$ will be used to denote that a certain sequence of random variables converges both almost surely and in mean. Morally speaking, this is a law of large numbers for the number of connected components associated with the Gaussian field $F$.

**Theorem 5.1.1.** *Let $F$ be a monochromatic random wave. Then there are positive constants $\nu$, $\nu([\Sigma])$ such that, as $R \to \infty$,*

$$\frac{N_F(R)}{|B_R|} \xrightarrow[\text{a.s.}]{L^1} \nu, \qquad \frac{N_F(R; [\Sigma])}{|B_R|} \xrightarrow[\text{a.s.}]{L^1} \nu([\Sigma]).$$

*Here $\Sigma \subset \mathbb{R}^3$ is any compact surface as above.*

### 5.1.2 Gaussian random Beltrami fields on $\mathbb{R}^3$

Our goal is then to obtain an extension of the Nazarov–Sodin theory that applies to random Beltrami fields. As we will discuss later, this is far from trivial because there are essential new difficulties that make the analysis of the problem rather involved.

The origin of many of these difficulties is strongly geometric. In contrast to the case of random monochromatic waves (or any other scalar Gaussian field), where the main geometric objects of interest are the components of its nodal set, in the study of random vector fields we aim to understand structures of a much subtler geometric nature. Among these structures, and in increasing order of complexity, one should certainly consider the following:

(i) *Zeros*, i.e., points where the vector field vanishes.

(ii) *Periodic orbits*, which can be knotted in complicated ways.

(iii) *Invariant tori*, that is, surfaces diffeomorphic to a 2-torus that are invariant under the flow of the field. They can be knotted too.

(iv) *Compact chaotic invariant sets*, which exhibit horseshoe-type dynamics and have, in particular, positive topological entropy.

Recall that a horseshoe is defined as a compact hyperbolic invariant set with a Cantor transverse section on which the time-$T$ flow of $u$ is topologically conjugate to a Bernoulli shift [GH13], for some $T$. Consequently, let us define the following quantities:

(i) $N_u^{\mathrm{z}}(R)$ denotes the number of zeros of $u$ contained in the ball $B_R$.

(ii) Given a (possibly knotted) closed curve $\gamma \subset \mathbb{R}^3$, $N_u^{\mathrm{o}}(R; [\gamma])$ denotes the number of periodic orbits of $u$ contained in $B_R$ that are isotopic to $\gamma$.

(iii) Given a (possibly knotted) torus $\mathcal{T} \subset \mathbb{R}^3$, $V_u^{\mathrm{t}}(R; [\mathcal{T}])$ is the volume (understood as the inner measure) of the set of ergodic invariant tori of $u$ that are contained in $B_R$ and are isotopic to $\mathcal{T}$. Ergodic means that we consider invariant tori on which the orbits of $u$ are dense.

(iv) $N_u^{\mathrm{h}}(R)$ denotes the number of horseshoes of $u$ contained in the ball $B_R$.

Clearly, these quantities only depend on the isotopy class of $\gamma$ and $\mathcal{T}$.

It is not hard to believe that these geometric subtleties give rise to a number of analytic difficulties. One should mention, however, that there also appear other unexpected analytic difficulties whose origin is less obvious. They are related to the fact that it is not clear how to define a random Beltrami field through an analog of (5.1.2b). This is because the characterization of a monochromatic wave as the Fourier transform of a distribution supported on a sphere is the conceptual base of the simple definition (5.1.2a), which underlies the equivalent but considerably more awkward expression (5.1.2b). Heuristically, analytic difficulties stem from the fact that there is not such a clean formula in Fourier space for a general Beltrami field. This is because the three components of the Beltrami field (which are monochromatic waves) are not independent, so the reduction to a Fourier formulation with independent variables is not trivial. We refer the reader to Section 5.3, where we explain in detail how to define Gaussian random Beltrami fields in a way that is strongly reminiscent of (5.1.2b). Later in this Introduction we shall also informally discuss the aforementioned difficulties and discuss how we manage to circumvent them using a combination of ideas from PDE, dynamical systems and probability

We can now state our main result for Gaussian random Beltrami fields on $\mathbb{R}^3$, as defined in Section 5.3. Let us emphasize that the picture that emerges from this theorem is fully consistent with Arnold's view of complexity in Beltrami fields; with probability 1, we show that a random Beltrami field is "partially integrable" in that there is a large volume of invariant tori, and simultaneously features many compact chaotic invariant sets and periodic orbits of arbitrarily complex topologies. This coexistence of chaos and order is indeed the essential feature of the restriction to an energy hypersurface of a generic Hamiltonian system with two degrees of freedom, as Arnold put it. In this direction, Corollary 5.1.3 below is quite illustrative.

**Theorem 5.1.2.** *Let $u$ be a Gaussian random Beltrami field. Then:*

*(i) The topological entropy of $u$ is positive almost surely. In fact, with probability* 1*,*

$$\liminf_{R\to\infty} \frac{N_u^{\mathrm{h}}(R)}{|B_R|} > \nu^{\mathrm{h}} .$$

*(ii) With probability* 1*, the volume of ergodic invariant tori of $u$ isotopic to a given embedded torus $\mathcal{T} \subset \mathbb{R}^3$ and the number of periodic orbits of $u$ isotopic to a given closed curve $\gamma \subset \mathbb{R}^3$ satisfy the volumetric growth estimate*

$$\liminf_{R\to\infty} \frac{V_u^{\mathrm{t}}(R;[\mathcal{T}])}{|B_R|} > \nu^{\mathrm{t}}([\mathcal{T}]) , \qquad \liminf_{R\to\infty} \frac{N_u^{\mathrm{o}}(R;[\gamma])}{|B_R|} > \nu^{\mathrm{o}}([\gamma]) .$$

*The constants $\nu^{\mathrm{h}}$, $\nu^{\mathrm{t}}([\mathcal{T}])$ and $\nu^{\mathrm{o}}([\gamma])$ above are all positive, for any choice of the curve $\gamma$ and the torus $\mathcal{T}$.*

**Corollary 5.1.3.** *With probability* 1*, a Gaussian random Beltrami field on $\mathbb{R}^3$ exhibits infinitely many horseshoes coexisting with an infinite volume of ergodic invariant tori of each isotopy type. Moreover, the set of periodic orbits contains all knot types.*

**Remark 5.1.1.** The result we prove (see Theorem 5.6.2) is in fact considerably stronger: we do not only prescribe the topology of the periodic orbits and the invariant tori we count, but also other important dynamical quantities. Specifically, in the case of periodic orbits we have control over the periods (which we can pick in a certain interval $(T_1, T_2)$) and the maximal Lyapunov exponents (which we can also pick in an interval $(\Lambda_1, \Lambda_2)$). In the case of the ergodic invariant tori, we can control the associated arithmetic and nondegeneracy conditions. Details are provided in Section 5.6.

Unlike the case of nodal set components considered in the context of the Nazarov–Sodin theory for Gaussian random monochromatic waves, we do not prove exact asymptotics for the quantities we study, but only nontrivial lower bounds that hold almost surely. Without getting technicalities at this stage, let us point out that this is related to analytic difficulties arising from the fact that we are dealing with quantities that are rather geometrically nontrivial. If one considers a simpler quantity such as the number of zeros of a Gaussian random Beltrami field, one can obtain an asymptotic distribution law similar to that of the nodal components of a random monochromatic wave, whose corresponding asymptotic constant can even be computed explicitly:

**Theorem 5.1.4.** *The number of zeros of a Gaussian random Beltrami field satisfies*

$$\frac{N_u^{\mathrm{z}}(R)}{|B_R|} \xrightarrow[\text{a.s.}]{L^1} \nu^{\mathrm{z}}$$

*as $R \to \infty$. The constant is explicitly given by*

$$\nu^{\mathrm{z}} := c^{\mathrm{z}} \int_{\mathbb{R}^5} |Q(z)|\, e^{-\widetilde{Q}(z))}\, dz = 0.00872538\ldots , \tag{5.1.3}$$

*where* $c^{\mathrm{z}} := 21^{5/2}/[143\sqrt{5}\,\pi^4]$*, and* $Q, \widetilde{Q}$ *are the following homogeneous polynomials in five variables:*

$$Q(z) := z_1 z_2^2 + z_2^3 - z_1^2 z_4 - z_1 z_2 z_4 - z_3^2 z_4 + 2 z_2 z_3 z_5 - z_1 z_5^2 \,, \tag{5.1.4}$$

$$\widetilde{Q}(z) := \frac{189}{65} z_1^2 + \frac{42}{11}(z_2^2 + z_3^2) + \frac{42}{13}(z_4^2 + z_1 z_4 + z_5^2) \,. \tag{5.1.5}$$

### 5.1.3 Random Beltrami fields on the torus

A Beltrami field on the flat 3-torus $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ (or, equivalently, on the cube of $\mathbb{R}^3$ of side length $2\pi$ with periodic boundary conditions) is a vector field on $\mathbb{T}^3$ satisfying the eigenvalue equation

$$\operatorname{curl} v = \lambda v$$

for some real number $\lambda \neq 0$. It is well-known (see e.g. [ELPS17]) that the spectrum of the curl operator on the 3-torus consists of the numbers of the form $\lambda = \pm|k|$ for some vector with integer coefficients $k \in \mathbb{Z}^3$. Restricting our attention to the case of positive eigenvalues for the sake of concreteness, one can therefore label the eigenvalue by a positive integer $L$ such that $\lambda_L = L^{1/2}$. The multiplicity of the eigenvalue is given by the cardinality of the corresponding set of spatial frequencies,

$$\mathcal{Z}_L := \{k \in \mathbb{Z}^3 : |k|^2 = L\} \,.$$

By Legendre's three-square theorem, $\mathcal{Z}_L$ is nonempty (and therefore $\lambda_L$ is an eigenvalue of the curl operator) if and only if $L$ is not of the form $4^a(8b+7)$ for nonnegative integers $a$ and $b$.

The Beltrami fields corresponding to the eigenvalue $\lambda_L$ must obviously be of the form

$$u^L = \sum_{k \in \mathcal{Z}_L} V_k^L \, e^{ik\cdot x} \,,$$

for some vectors $V_k^L \in \mathbb{C}^3$, where $V_k^L = \overline{V_{-k}^L}$ to ensure that the Beltrami field is real-valued. Starting from this formula, in Section 5.7 we define the Gaussian ensemble of random Beltrami fields $u^L$ of frequency $\lambda_L$, which we parametrize by $L$. The natural length scale of the problem is $L^{1/2}$.

Our objective is to study to what extent the appearance of the various dynamical objects described above (i.e., horseshoes, zeros, and periodic orbits and ergodic invariant tori of prescribed topology) is typical in high-frequency Beltrami fields, which corresponds to the limit $L \to \infty$. When taking this limit, we shall always assume that the integer $L$ is *admissible*, by which we mean that it is congruent with 1, 2, 3, 5 or 6 modulo 8. We will see in Section 5.7 (see also [Roz17]) that this number-theoretic condition ensures that the dimension of the space of Beltrami fields with eigenvalue $\lambda_L$ tends to infinity as $L \to \infty$.

To state our main result about high-frequency random Beltrami fields in the torus we need to introduce some notation. In parallel with the previous subsection, for any closed curve $\gamma$ and any embedded torus $\mathcal{T}$, let us respectively denote by $N^{\mathrm{z}}_{u^L}$, $N^{\mathrm{h}}_{u^L}$, $N^{\mathrm{o}}_{u^L}([\gamma])$ and $N^{\mathrm{t}}_{u^L}([\mathcal{T}])$ the number of zeros, horseshoes, periodic orbits isotopic to $\gamma$ and ergodic invariant tori isotopic to $\mathcal{T}$ of the field $u^L$, as well as the volume (i.e., inner measure) of these tori, which we denote by $V^{\mathrm{t}}_{u^L}([\mathcal{T}])$. To further control the distribution of these objects, let us define the number of approximately equidistributed ergodic invariant tori, $N^{\mathrm{t,e}}_{u^L}([\mathcal{T}])$, as the largest integer $m$ for which $u^L$ has $m$ ergodic invariant tori isotopic to $\mathcal{T}$ that are at a distance greater than $m^{-1/3}$ apart from one another. The number of approximately equidistributed horseshoes $N^{\mathrm{h,e}}_{u^L}$, periodic orbits isotopic to a curve $N^{\mathrm{o,e}}_{u^L}([\gamma])$ and zeros $N^{\mathrm{z,e}}_{u^L}$ are defined analogously. Note that, again, the asymptotic information that we obtain is perfectly aligned with Arnold's view of complex behavior in typical Beltrami fields.

**Theorem 5.1.5.** *Let us denote by $(u^L)$ the parametric Gaussian ensemble of random Beltrami fields on $\mathbb{T}^3$, where $L$ ranges over the set of admissible integers. Consider any contractible closed curve $\gamma$ and any contractible embedded torus $\mathcal{T}$ in $\mathbb{T}^3$. Then:*

*(i) With a probability tending to $1$ as $L \to \infty$, the field $u^L$ exhibits an arbitrarily large number of approximately distributed horseshoes, zeros, periodic orbits isotopic to $\gamma$ and ergodic invariant tori isotopic to $\mathcal{T}$. More precisely, for any integer $m$,*

$$\lim_{L\to\infty} \mathbb{P}\Big\{ \min\big\{N^{\mathrm{h,e}}_{u^L}, N^{\mathrm{t,e}}_{u^L}([\mathcal{T}]), N^{\mathrm{o,e}}_{u^L}([\gamma]), N^{\mathrm{z,e}}_{u^L}\big\} > m \Big\} = 1\,.$$

*Furthermore, the probability that the topological entropy of the field grows at least as $L^{1/2}$ and that there are infinitely many ergodic invariant tori of $u^L$ isotopic to $\mathcal{T}$ also tends to $1$:*

$$\lim_{L\to\infty} \mathbb{P}\{N^{\mathrm{t}}_{u^L}([\mathcal{T}]) = \infty \ \textit{and}\ \ h_{\mathrm{top}}(u^L) > \nu^{\mathrm{h}}_* L^{1/2}\} = 1\,.$$

*(ii) The expected volume of the ergodic invariant tori of $u^L$ isotopic to $\mathcal{T}$ is uniformly bounded from below, and the expected number of horseshoes and periodic orbits isotopic to $\gamma$ is at least of order $L^{3/2}$:*

$$\liminf_{L\to\infty} \min\left\{ \frac{\mathbb{E}N^{\mathrm{h}}_{u^L}}{L^{3/2}}, \frac{\mathbb{E}N^{\mathrm{o}}_{u^L}([\gamma])}{L^{3/2}}, \mathbb{E}V^{\mathrm{t}}_{u^L}([\mathcal{T}]) \right\} > \nu_*([\gamma],[\mathcal{T}])\,.$$

*In the case of zeros, the asymptotic expectation is explicit, with $\nu^{\mathrm{z}}$ given by (5.1.3):*

$$\lim_{L\to\infty} \frac{\mathbb{E}N^{\mathrm{z}}_{u^L}}{L^{3/2}} = (2\pi)^3 \nu^{\mathrm{z}}\,.$$

*Here $\nu^{\mathrm{h}}_*$ and $\nu_*([\gamma],[\mathcal{T}])$ are positive constants.*

**Remark 5.1.2.** As in the case of $\mathbb{R}^3$, the result we prove in Section 5.7 is actually stronger in the sense that we have control over important dynamical quantities (which now depend strongly on $L$) describing the flow near the above invariant tori and periodic orbits.

### 5.1.4 Some technical remarks

In a way, the cornerstone of the Nazarov–Sodin theory is their very clever (and non-probabilistic) "sandwich estimate", which relates the number $N_F(R)$ of connected components of the nodal set of the Gaussian random field $F$ that are contained in an arbitrarily large ball $B_R$ with ergodic averages of the same quantity involving the number of components contained in balls of fixed radius. Two ingredients are key to effectively apply this sandwich estimate. On the one hand, each nodal component cannot be too small by the Faber–Krahn inequality, which ensures, in dimension 3, that its volume is at least $c\lambda^{-3}$ if $\Delta F + \lambda^2 F = 0$. On the other hand, to control the connected components that intersect a large ball but are not contained in it, it suffices to employ the Kac–Rice formula to derive bounds for the number of critical points of a certain family of Gaussian random functions.

In the setting of random Beltrami fields, the need for new ideas becomes apparent the moment one realizes that there are no reasonable substitutes for these two key ingredients. That is, the frequency $\lambda$ does not provide bounds for the size of the more sophisticated geometric objects considered in this context (i.e., periodic orbits, invariant tori or horseshoes), and one cannot estimate the objects that intersect a ball but are not contained in it using a Kac–Rice formula. As a matter of fact, we have not managed to obtain any useful bounds for these quantities and, while we do use a sandwich inequality of sorts (or at least lower bounds that can be regarded as a weaker substitute thereof), even the measurability of the various objects of interest becomes a nontrivial issue due to their complicated geometric properties.

To circumvent these problems, we employ different kinds of techniques. Firstly, ideas from the theory of dynamical systems play a substantial role in our proofs. On the one hand, KAM theory and hyperbolic dynamics are important to prove that certain carefully chosen functionals are lower semicontinuous, which is key to solve measurability issues that would be very hard to deal with otherwise. Furthermore, to prove that Beltrami fields exhibit chaotic behavior almost surely, it is essential to have at least one example of a Beltrami field that features a horseshoe, and even that was not known. Indeed, the available examples of non-integrable ABC flows are known to be chaotic on $\mathbb{T}^3$ due to the non-contractibility of the domain, but not on $\mathbb{R}^3$. This technical point is fundamental, and makes them unsuitable for the study of random Beltrami fields. Therefore, an important step in our proof is to construct, using Melnikov theory, a Beltrami field on $\mathbb{R}^3$ that has a horseshoe. Techniques from Fourier analysis and from the global approximation theory for Beltrami fields are also necessary to handle the inherent difficulties that stem from the fact that the equation under consideration is more complicated than that of a monochromatic wave. As an aside, the only point of the chapter where we use the Kac–Rice formula is to compute the constant $\nu^z$ in closed form.

In the case of Beltrami fields on the torus, the results we prove concern not only the expected values of the quantities of interest, but also the probability of events. In the case of random monochromatic waves on the torus, Nazarov and Sodin [NS16] had proved results for the expectation (which apply to very general parametric scalar Gaussian ensembles), and Rozenshein [Roz17] had derived very precise exponential bounds for the probability akin to those established by Nazarov and Sodin [NS09] for random spherical harmonics. However, both results use in a crucial way that the size of nodal components can be effectively estimated in terms of the frequency: the Faber–Krahn inequality provides a lower bound for the volume and large diameter components can be ruled out using a Crofton-type formula and Bézout's theorem. No such bounds hold in the case of Beltrami fields, so the way we pass from the information that the rescaled covariant kernel of $u^L$ tends to that of $u$ to asymptotics for the distribution of invariant tori, horseshoes or periodic orbits is completely different. Specifically, we rely on a direct argument ensuring the weak convergence of sequences of probability measures, on spaces of smooth functions, provided that suitable tightness conditions are satisfied.

### 5.1.5 Outline of the chapter

In Section 5.2, we start by describing Beltrami fields in $\mathbb{R}^3$ from the point of view of Fourier analysis and provide some results about global approximation. Gaussian random Beltrami fields on $\mathbb{R}^3$ are introduced in Section 5.3, where we also establish several results about the structure of the corresponding covariance matrix and about the induced probability measure on the space of smooth vector fields. In Section 5.4 we recall, in a form that will be useful in later sections, several previous results about ergodic invariant tori and periodic orbits arising in Beltrami fields. Section 5.5 is devoted to constructing a Beltrami field on $\mathbb{R}^3$ that is stably chaotic. Finally, in Sections 5.6 and 5.7 we complete the proofs of our main results in the case of $\mathbb{R}^3$ and $\mathbb{T}^3$, respectively. The chapter concludes with an Appendix where we provide a fairly complete Fourier-theoretic characterization of Beltrami fields.

## 5.2 Fourier analysis and approximation of Beltrami fields

In what follows, we will say that a vector field $u$ on $\mathbb{R}^3$ is a Beltrami field if

$$\operatorname{curl} u = u\,.$$

Taking the curl of this equation and using that necessarily $\operatorname{div} u = 0$, it is easy to see that $u$ must also satisfy the Helmholtz equation:

$$\Delta u + u = 0\,.$$

To put it differently, the components of this vector field are monochromatic waves. An immediate consequence of this is that the Fourier transform $\widehat{u}$ of a polynomially bounded Beltrami field is a (vector-valued) distribution supported on the unit sphere

$$\mathbb{S} := \{\xi \in \mathbb{R}^3 : |\xi| = 1\}\,.$$

Since $u$ is real-valued, $\widehat{u}$ must be Hermitian, i.e., $\widehat{u}(\xi) = \overline{\widehat{u}(-\xi)}$. Furthermore, a classical result due to Herglotz [Hör15, Theorem 7.1.28] ensures that if $u$ is a Beltrami field with the sharp fall off at infinity, then there is a Hermitian vector-valued function $f \in L^2(\mathbb{S}, \mathbb{C}^3)$ such that $\widehat{u} = f\, d\sigma$; for the benefit of the reader, details on this and other related matters are summarized in Appendix 5.A. For short, we shall simply write this relation as $u = U_f$, with

$$U_f(x) := \int_{\mathbb{S}} f(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi)\,. \tag{5.2.1}$$

Obviously $U_f$ is a Beltrami field if and only if $f$ is Hermitian (which makes $U_f$ real valued) and if it satisfies the distributional equation on the sphere

$$i\xi \times f(\xi) = f(\xi)\,. \tag{5.2.2}$$

In this chapter, we are particularly interested in Beltrami fields of the form $u = U_f$, where now $f$ is a general Hermitian vector-valued distribution on the sphere. The corresponding integral, which is convergent if $f$ is integrable, must be understood in the sense of distributions for less regular $f$ (that is to say, for $f$ in the scale of Sobolev spaces $H^s(\mathbb{S}, \mathbb{C}^3)$ with $s < 0$). We recall, in particular, that for any integer

$k \geqslant 0$ the field $U_f$ is bounded as [EPSR22a, Appendix A]

$$\sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_f(x)|^2}{1+|x|^{2k}} \, dx \leqslant C \|f\|_{H^{-k}(\mathbb{S},\mathbb{C}^3)} \,. \tag{5.2.3}$$

We recall that, for any real $s$, the $H^s(\mathbb{S})$ norm of a function $f$ can be computed as

$$\|f\|^2_{H^s(\mathbb{S})} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2s} |f_{lm}|^2 \,,$$

where $f_{lm}$ are the coefficients of the spherical harmonics expansion of $f$.

With $q(t) := \frac{1}{8}(\frac{15}{\pi})^{1/2}(1+\sqrt{7} i \, t)$, let us consider the vector-valued polynomial

$$p(\xi) := q(\xi_1) \, (\xi_1^2 - 1, \xi_1\xi_2 - i\xi_3, \xi_1\xi_3 + i\xi_2) \,, \tag{5.2.4}$$

which we will regard as a Hermitian function $p : \mathbb{R}^3 \to \mathbb{C}^3$. Note that the restriction of $p$ to the sphere vanishes exactly at the poles $\xi_\pm := (\pm 1, 0, 0)$. The inessential nonvanishing normalization factor $q(\xi_1)$ has been introduced for later convenience: when we define random Beltrami fields via the function $p$ in Section 5.3, this choice of $p$ will ensure that the associated covariance matrix is the identity on the diagonal (see Corollary 5.3.3).

We next show that, away from the poles, the density $f$ of a Beltrami field $U_f$ must point in the same direction as $p$:

**Proposition 5.2.1.** *The following statements hold:*

*(i) If the vector field $U_f$ is a Beltrami field, then $p \times f = 0$ as a distribution on $\mathbb{S}$. Furthermore, if $\chi$ is a smooth real-valued function on the sphere supported in $\mathbb{S}\backslash\{\xi_+, \xi_-\}$ and $f \in H^s(\mathbb{S}, \mathbb{C}^3)$ for some real $s$, then there is a Hermitian scalar function $\varphi \in H^s(\mathbb{S})$ such that $\chi f = \varphi \, p$.*

*(ii) Conversely, for any Hermitian $\varphi \in H^s(\mathbb{S})$, the associated field $U_{\varphi p}$ is a Beltrami field.*

*Proof.* In view of Equation (5.2.2), for each vector $\xi \in \mathbb{S}$, consider the linear map $M_\xi$ on $\mathbb{C}^3$ defined as

$$M_\xi V := V - i\xi \times V \,.$$

More explicitly, $M_\xi$ is the matrix

$$M_\xi = \begin{pmatrix} -1 & -i\xi_3 & i\xi_2 \\ i\xi_3 & -1 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & -1 \end{pmatrix} .$$

The determinant of this matrix is $\det M_\xi = \xi_1^2 + \xi_2^2 + \xi_3^2 - 1$, and in fact it is easy to see that $M_\xi$ has rank 2 for any unit vector $\xi$. Since $M_\xi p(\xi) = 0$ for all $\xi \in \mathbb{S}$ and $p(\xi)$ only vanishes if $\xi = \xi_\pm$, we then obtain that the kernel of $M_\xi$ is spanned by the vector $p(\xi)$ whenever $\xi$ is not one of the poles $\xi_\pm$. In a neighborhood of the poles, the kernel of $M_\xi$ can be described as the linear span of $\tilde{p}(\xi) := q(\xi_2) \, (\xi_1\xi_2 + i\xi_3, \xi_2^2 - 1, \xi_2\xi_3 - i\xi_1)$.

Since $M_\xi f(\xi) = 0$ in the sense of distributions by (5.2.2), it stems from the above analysis that one can write

$$f(\xi) = \alpha(\xi) \, p(\xi)$$

for $\xi$ away from the poles, and

$$f(\xi) = \beta(\xi)\,\widetilde{p}(\xi)$$

in a neighborhood of the poles; here $\alpha$ and $\beta$ are complex-valued scalars. As $p(\xi) \times \widetilde{p}(\xi) = 0$ for all $\xi \in \mathbb{S}$, we immediately infer that

$$p \times f = 0\,.$$

Also, as the support of a function is a closed set, $p$ is bounded away from zero on the support of $\chi$, so we have that

$$\varphi := \chi \frac{f \cdot p}{|p|^2} \in H^s(\mathbb{S})\,.$$

As $f$ is Hermitian, this proves the first part of the proposition. The second statement follows immediately from the fact that

$$M_\xi[\varphi(\xi)p(\xi)] = \varphi(\xi)\,M_\xi p(\xi) = 0\,.$$

$\square$

**Remark 5.2.1.** A Beltrami field of the form $U_{\varphi p}$ can be written in terms of the scalar function $\psi(x) := -\int_{\mathbb{S}} e^{i\xi\cdot x} q(\xi_1)\,\varphi(\xi)\,d\sigma(\xi)$ (which satisfies the equation $\Delta\psi + \psi = 0$) as

$$U_{\varphi p} = (\operatorname{curl}\operatorname{curl} + \operatorname{curl})(\psi, 0, 0)\,.$$

When $\varphi$ is smooth, the Beltrami field has the sharp decay bound $|U_{\varphi p}(x)| \leqslant C\|\varphi\|_{L^2(\mathbb{S})}/(1 + |x|)$.

**Remark 5.2.2.** Not any Beltrami field of the form $U_f$ can be written as $U_{\varphi p}$ for some scalar function $\varphi$: an obvious counterexample is given by

$$f(\xi) := (0, 1, i)\,\delta_{\xi_+}(\xi) + (0, 1, -i)\,\delta_{\xi_-}(\xi)\,, \tag{5.2.5}$$

where $\delta_{\xi_\pm}$ is the Dirac measure supported on the pole $\xi_\pm = (\pm 1, 0, 0)$. The reason for which we cannot hope to describe all Beltrami fields using just scalar multiples of a fixed complex-valued continuous vector field $p'$ is topological. Indeed, as $u$ is divergence-free, we have that $\xi \cdot p'(\xi) = 0$, so $p'$ must be a tangent complex-valued vector field on $\mathbb{S}$. By the hairy ball theorem, the real part of $p'$ must then have at least one zero $\xi^*$. The equation $i\xi \times p'(\xi) = p'(\xi)$ implies that the imaginary part of $p'$ also vanishes at $\xi^*$, so in fact $p'(\xi^*) = 0$. This means that densities $f$ such as (5.2.5), where we can take $\xi^* := \xi_+$ without any loss of generality, cannot be written in the form $\varphi p'$.

Intuitively speaking, Proposition 5.2.1 means that any Beltrami field $U_f$ whose density $f$ is not too concentrated on $\xi_\pm$ can be approximated globally by a field of the form $U_{\varphi p}$. More precisely, one can prove the following:

**Proposition 5.2.2.** *Consider a Hermitian vector-valued distribution $f$ on $\mathbb{S}$ that satisfies the distributional equation (5.2.2), and define*

$$\epsilon_{f,k} := \inf\left\{\|\Theta f\|_{H^{-k}(\mathbb{S})} : \Theta \in C^\infty(\mathbb{S}),\ \Theta(\xi_+) = \Theta(\xi_-) = 1\right\}.$$

*If $\epsilon_{f,k}$ is finite and $\epsilon > \epsilon_{f,k}$, one can then take a Hermitian scalar distribution on the sphere $\varphi$, which is in fact a finite linear combination of spherical harmonics if $f \in H^{-k}(\mathbb{S}, \mathbb{C}^3)$,* *such that*

$$\sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_f(x) - U_{\varphi p}(x)|^2}{1+|x|^{2k}}\, dx < C\epsilon\,.$$

*Furthermore,* $\epsilon_{f,0} = 0$ *if* $f \in L^2(\mathbb{S}, \mathbb{C}^3)$.

*Proof.* The first assertion is a straightforward consequence of the first part of Proposition 5.2.1 and of the estimate (5.2.3). Indeed, since $f$ is a compactly supported distribution, then $f \in H^s(\mathbb{S}, \mathbb{C}^3)$ for some $s$. Take any $\epsilon' \in (\epsilon_{f,k}, \epsilon)$ and let us consider a function $\Theta$ as above such that $\|\Theta f\|_{H^{-k}(\mathbb{S})} < \epsilon'$. Since $\epsilon' > \epsilon_{f,k}$, it is obvious that we can assume that $\Theta = 1$ in a small neighborhood of the poles $\xi_\pm$. Applying Proposition 5.2.1 we infer that $\chi f = \varphi p$ with $\chi := 1 - \Theta$ and some Hermitian scalar function $\varphi \in H^s(\mathbb{S})$. In view of the fact that the map $f \mapsto U_f$ is linear and of the bound (5.2.3), we then have

$$\sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_f(x) - U_{\varphi p}(x)|^2}{1+|x|^{2k}}\, dx = \sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_{\Theta f}(x)|^2}{1+|x|^{2k}}\, dx \leqslant C\|\Theta f\|_{H^{-k}(\mathbb{S},\mathbb{C}^3)} < C\epsilon'\,.$$

As finite linear combinations of spherical harmonics are dense in $H^s(\mathbb{S})$, if $s = -k$ we can approximate $\varphi$ in the $H^{-k}(\mathbb{S})$ norm by a Hermitian function $\varphi'$ of this form; then

$$\begin{aligned}\sup_{R>0} \frac{1}{R} \int_{B_R} &\frac{|U_f(x) - U_{\varphi' p}(x)|^2}{1+|x|^{2k}}\, dx \\ &\leqslant \sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_f(x) - U_{\varphi p}(x)|^2}{1+|x|^{2k}}\, dx + \sup_{R>0} \frac{1}{R} \int_{B_R} \frac{|U_{(\varphi'-\varphi)p}(x)|^2}{1+|x|^{2k}}\, dx < C\epsilon\end{aligned}$$

provided that $\|\varphi - \varphi'\|_{H^{-k}(\mathbb{S})} < \epsilon - \epsilon'$.

Finally, to see that $\epsilon_{f,0} = 0$ if $f \in L^2(\mathbb{S}, \mathbb{C}^3)$, let us take a smooth function $\Theta : \mathbb{R}^3 \to [0,1]$ supported in the unit ball and such that $\Theta(0) = 1$. Setting

$$\Theta_n(\xi) := \Theta(n\xi - n\xi_+) + \Theta(n\xi - n\xi_-)\,,$$

we trivially get that $\|\Theta_n f\|_{L^2(\mathbb{S})} \leqslant \|f\|_{L^2(\mathbb{S})}$ for all $n \geqslant 2$ and that $\Theta_n f$ tends to zero almost everywhere in $\mathbb{S}$ as $n \to \infty$. The dominated convergence theorem then shows that $\|\Theta_n f\|_{L^2(\mathbb{S})} \to 0$ as $n \to \infty$, thus proving the claim. □

Another, rather different in spirit, formulation of the principle that densities of the form $\varphi p$ can approximate general Beltrami fields is presented in the following theorem. Unlike the previous corollary, the approximation is considered only locally in space, and in this direction one shows that even considering smooth functions $\varphi$ is enough to obtain a subset of Beltrami fields that is dense in the $C^k$ compact-open topology:

**Proposition 5.2.3.** *Fix any positive reals* $\epsilon$ *and* $k$ *and a compact set* $K \subset \mathbb{R}^3$ *such that* $\mathbb{R}^3 \backslash K$ *is connected. Then, given any vector field* $v$ *satisfying the equation* $\operatorname{curl} v = v$ *in an open neighborhood of* $K$*, there exists a Hermitian finite linear combination of spherical harmonics* $\varphi$ *such that the Beltrami field* $U_{\varphi p}$ *approximates* $v$ *in the set* $K$ *as*

$$\|U_{\varphi p} - v\|_{C^k(K)} < \epsilon\,.$$

*Proof.* Let us fix an open set $V \supset K$ whose closure is contained in the open neighborhood where $v$ is defined, and a large ball $B_R \supset \overline{V}$. Since $\mathbb{R}^3 \backslash K$ is connected, it is obvious that we can take $V$ so that $\mathbb{R}^3 \backslash \overline{V}$ is connected as well. By the approximation theorem with decay for Beltrami fields [EPS15, Theorem 8.3], there is a Beltrami field $w$ that approximates $v$ as

$$\|w - v\|_{C^k(V)} < \epsilon$$

and is bounded as $|w(x)| < C/|x|$. As the Fourier transform of $w$ is supported on $\mathbb{S}$, Herglotz's theorem [Hör15, Theorem 7.1.28] shows that one can write $w = U_f$ for some vector-valued Hermitian field $f \in L^2(\mathbb{S}, \mathbb{C}^3)$ that satisfies the distributional equation (5.2.2). Proposition 5.2.2 then shows that there exists some Hermitian scalar function $\varphi \in C^\infty(\mathbb{S})$ such that

$$\|U_f - U_{\varphi p}\|_{L^2(B_R)} < C\epsilon\,,$$

so that $\|v - U_{\varphi p}\|_{L^2(V)} < C\epsilon$. As the difference $v - U_{\varphi p}$ satisfies the Helmholtz equation

$$\Delta(v - U_{\varphi p}) + v - U_{\varphi p} = 0$$

in $V$, and $K \subset\subset V$, standard elliptic estimates then allow us to promote this bound to

$$\|v - U_{\varphi p}\|_{C^k(K)} < C\epsilon\,,$$

as we wished to prove. □

## 5.3 Gaussian random Beltrami fields

The Fourier-theoretical characterization of Beltrami fields presented in the previous section paves the way to the definition of random Beltrami fields.

In parallel with (5.1.2a) (see Appendix 5.A for further heuristics), let us start by setting

$$\varphi(\xi) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l\, a_{lm}\, Y_{lm}(\xi)\,,$$

where $a_{lm}$ are normally distributed independent standard Gaussian random variables and $Y_{lm}$ is an orthonormal basis of (real-valued) spherical harmonics on $\mathbb{S}$. Note that $\varphi$ is Hermitian because of the identity $Y_{lm}(-\xi) = (-1)^l Y_{lm}(\xi)$. We now define a Gaussian random Beltrami field as

$$u := U_{\varphi p}\,,$$

where we recall that $U_f$ and $p$ were respectively defined in (5.2.1) and (5.2.4).

**Remark 5.3.1.** As discussed in Proposition 5.2.1, the role of the vector field $p$ is to ensure that the density $f := \varphi p$ satisfies the Beltrami equation in Fourier space, $i\xi \times f(\xi) = f(\xi)$. Hence one could replace $p(\xi)$ by any nonvanishing multiple of it, that is, by $\widetilde{p}(\xi) := \Lambda(\xi)\, p(\xi)$ where $\Lambda : \mathbb{R}^3 \to \mathbb{C}$ is a smooth scalar Hermitian function that does not vanish on $\mathbb{S}$. All the results of the chapter about random Beltrami fields remain valid if one defines a Gaussian random Beltrami field as $u := U_{\varphi \widetilde{p}}$ with $\varphi$ as above, provided that one replaces $p$ by $\widetilde{p}$ in the formulas. Also, the results do not change if one replaces the basis of spherical harmonics by any other orthonormal basis of $L^2(\mathbb{S})$, but this choice leads to slightly more explicit formulas for certain intermediate objects that appear in the proofs.

In what follows, we will use the notation $D := -i\nabla$. An important role will be played by the vector-valued differential operator with real coefficients $p(D)$, whose expression in Fourier space is

$$\widehat{p(D)\psi}(\xi) = p(\xi)\,\widehat{\psi}(\xi)\,,$$

for any scalar function $\psi$ in $\mathbb{R}^3$. Equivalently, by Remark 5.2.1, the operator $p(D)$ reads, in physical space, as

$$p(D)\psi = -(\operatorname{curl}\operatorname{curl} + \operatorname{curl})(q(D_1)\psi, 0, 0)\,,$$

where $D_1 := -i\partial_{x_1}$.

The first result of this section shows that a Gaussian random Beltrami field is a well defined object both in Fourier and physical spaces:

**Proposition 5.3.1.** *With probability* 1*, the function $\varphi$ is in $H^{-1-\delta}(\mathbb{S})\backslash L^2(\mathbb{S})$ for any $\delta > 0$. In particular, almost surely, $u$ is a $C^\infty$ vector field and can be written as*

$$u(x) = (2\pi)^{\frac{3}{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, p(D) \left[ Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{1}{2}}(|x|)}{|x|^{1/2}} \right]. \tag{5.3.1}$$

*The series converges in $C^k$ uniformly on compact sets almost surely, for any $k$.*

*Proof.* For $l \geqslant 0$ and $-l \leqslant m \leqslant l$, $a_{lm}^2$ are independent, identically distributed random variables with expected value 1. As the number of these variables with $l \leqslant n$ is

$$\sum_{l=0}^{n} \sum_{m=-l}^{l} 1 = (n+1)^2\,,$$

the strong law of large numbers ensures that the sample average, i.e., the random variable

$$X_n := \frac{1}{(n+1)^2} \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm}^2\,,$$

converges to 1 almost surely as $n \to \infty$. Now consider the truncation

$$\varphi_n(\xi) := \sum_{l=0}^{n} \sum_{m=-l}^{l} i^l\, a_{lm}\, Y_{lm}(\xi)\,.$$

As the spherical harmonics $Y_{lm}$ are orthonormal, the $L^2$ norm of $\varphi_n$ is

$$\|\varphi_n\|_{L^2(\mathbb{S})}^2 = \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm}^2 = (n+1)^2 X_n\,,$$

and $\|\varphi_n\|_{L^2(\mathbb{S})}^2$ tends to $\|\varphi\|_{L^2(\mathbb{S})}^2$ (which may be infinite) as $n \to \infty$. Since $X_n \to 1$ almost surely, we obtain from the above formula that $(n+1)^{-2}\|\varphi_n\|_{L^2(\mathbb{S})}^2$ tends to 1 almost surely. Therefore, $\varphi$ is not in $L^2(\mathbb{S})$ with probability 1.

On the other hand, since

$$\|\varphi\|^2_{H^{-s}(\mathbb{S})} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a_{lm}^2}{(l+1)^{2s}},$$

it is straightforward to see that the expected value

$$\mathbb{E}\|\varphi\|^2_{H^{-1-\delta}(\mathbb{S})} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\mathbb{E}a_{lm}^2}{(l+1)^{2+2\delta}} = \sum_{l=0}^{\infty} \frac{2l+1}{(l+1)^{2+2\delta}}$$

is finite for all $\delta > 0$. Hence $\varphi \in H^{-1-\delta}(\mathbb{S})$ almost surely, so $u := U_{\varphi p}$ is well defined with probability 1.

To prove the representation formula for $u$ and its convergence, let us begin by noting that

$$\begin{aligned} U_{i^l Y_{lm} p}(x) &= \int_{\mathbb{S}} i^l p(\xi)\, Y_{lm}(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi) \\ &= p(D) \int_{\mathbb{S}} i^l Y_{lm}(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi)\,. \end{aligned}$$

Using either the theory of point pair invariants and zonal spherical functions [CS19, Proposition 4] or special function identities [EPSR22a, Proposition 2.1], the Fourier transform of $Y_{lm}\, d\sigma$ has been shown to be

$$\int_{\mathbb{S}} i^l Y_{lm}(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi) = (2\pi)^{\frac{3}{2}} Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{1}{2}}(|x|)}{|x|^{1/2}}\,.$$

This permits to formally write $u$ as (5.3.1). To show that this series converges in $C^k$ on compact sets, for any large $n$, any $N > n$ and any fixed positive integer $k$ consider the quantity

$$q_{n,N}(x) := \sum_{|\alpha|\leqslant k} \left| \sum_{l=n}^{N} \sum_{m=-l}^{l} a_{lm} D^\alpha p(D) \left[ Y_{lm}\left(\frac{x}{|x|}\right) \frac{J_{l+\frac{1}{2}}(|x|)}{|x|^{1/2}} \right] \right|,$$

where we are using the standard multiindex notation. Since $p(D)$ is a third-order operator, for all $|x| < R$ we obviously have

$$\begin{aligned} q_{n,N}(x) &\leqslant C_k \sum_{l=n}^{N} \sum_{m=-l}^{l} |a_{lm}| \|Y_{lm}\|_{C^{k+3}(\mathbb{S})} \left\| \frac{J_{l+\frac{1}{2}}(r)}{r^{1/2}} \right\|_{C^{k+3}((0,R))} \\ &\leqslant C_k \left( \sum_{l=n}^{N} \sum_{m=-l}^{l} \frac{a_{lm}^2}{(l+1)^{2+2\delta}} \right)^{\frac{1}{2}} \left( \sum_{l=n}^{N} \sum_{m=-l}^{l} (l+1)^{2+2\delta} \|Y_{lm}\|^2_{C^{k+3}(\mathbb{S})} \left\| \frac{J_{l+\frac{1}{2}}(r)}{r^{1/2}} \right\|^2_{C^{k+3}((0,R))} \right)^{\frac{1}{2}} \end{aligned}$$

where here $r := |x|$ and we have used the Cauchy–Schwartz inequality to pass to the second line. The Sobolev inequality immediately gives

$$\|Y_{lm}\|_{C^{k+3}(\mathbb{S})} \leqslant C\|Y_{lm}\|_{H^{k+5}(\mathbb{S})} \leqslant C(l+1)^{k+5}\,.$$

To estimate the Bessel function, recall the large-degree asymptotics

$$J_\nu(r) \sim (2\pi\nu)^{-\frac{1}{2}} \left(\frac{er}{2\nu}\right)^\nu,$$

which holds as $\nu \to \infty$ for uniformly bounded $r$. As the derivative of a Bessel function can be written in terms of Bessel functions via the recurrence relation

$$\frac{d}{dr} J_\nu(r) = -J_{\nu+1}(r) + \frac{\nu}{r} J_\nu(r)\,,$$

it follows that the $C^{k+3}$ norm of $J_{l+\frac{1}{2}}(r)/r^{1/2}$ tends to 0 exponentially as $l \to \infty$ on compact sets:

$$\left\| \frac{J_{l+\frac{1}{2}}(r)}{r^{1/2}} \right\|_{C^{k+3}((0,R))} \leqslant \left( \frac{CR}{l} \right)^{l-k-3}.$$

Since we have proven that

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a_{lm}^2}{(l+1)^{2+2\delta}} < \infty$$

almost surely, now one only has to put together the estimates above to see that, almost surely, $q_{n,N}(x)$ tends to 0 as $n \to \infty$ uniformly for all $N > n$ and for all $x$ in a compact subset of $\mathbb{R}^3$. This establishes the convergence of the series and completes the proof of the proposition. $\square$

**Remark 5.3.2.** Note that each summand $U_{i^l Y_{lm} p} = (2\pi)^{3/2} p(D)[Y_{lm}(\frac{x}{|x|})|x|^{-1/2} J_{l+\frac{1}{2}}(|x|)]$ of the series (5.3.1) is a Beltrami field.

Since $a_{lm}$ are standard Gaussian variables, it is obvious that the vector-valued Gaussian field $u$ has zero mean. Our next goal is to compute its covariance kernel, $\kappa$, which maps each pair of points $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3$ to the symmetric $3 \times 3$ matrix

$$\kappa(x,y) := \mathbb{E}[u(x) \otimes u(y)]\,. \tag{5.3.2}$$

In particular, we show that this kernel is translationally invariant, meaning that it only depends on the difference:

$$\kappa(x,y) = \varkappa(x-y)\,.$$

We recall that, by Bochner's theorem, there exists a nonnegative-definite matrix-valued measure $\rho$ such that $\varkappa$ is the Fourier transform of $\rho$: this is the spectral measure of the Gaussian random field $u$. In the statement, $p_j$ is the $j^{\text{th}}$ component of the vector field $p$.

**Proposition 5.3.2.** *The components of the covariance kernel of the Gaussian random field $u$ are*

$$\kappa_{jk}(x,y) = \varkappa_{jk}(x-y)$$

*with*

$$\varkappa_{jk}(x) := (2\pi)^{\frac{3}{2}} p_j(D) p_k(-D) \frac{J_{1/2}(|x|)}{|x|^{1/2}}\,.$$

*The spectral measure is $d\rho(\xi) = p(\xi) \otimes \overline{p(\xi)}\, d\sigma(\xi)$.*

*Proof.* As $a_{lm}$ are independent standard Gaussian variables, $\mathbb{E}(a_{lm}a_{l'm'}) = \delta_{ll'}\delta_{mm'}$, so the covariance matrix is

$$\begin{aligned}
\kappa_{jk}(x,y) &= \mathbb{E}[u_j(x)u_k(y)] = \mathbb{E}[u_j(x)\overline{u_k(y)}] \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m=-l'}^{l'} i^{l-l'}\mathbb{E}(a_{lm}a_{l'm'}) \int_{\mathbb{S}}\int_{\mathbb{S}} e^{ix\cdot\xi-iy\cdot\eta}\, p_j(\xi)\,\overline{p_k(\eta)}\, Y_{lm}(\xi)\, Y_{l'm'}(\eta)\, d\sigma(\xi)\, d\sigma(\eta) \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \int_{\mathbb{S}}\int_{\mathbb{S}} e^{ix\cdot\xi-iy\cdot\eta}\, p_j(\xi)\,\overline{p_k(\eta)}\, Y_{lm}(\xi)\, Y_{lm}(\eta)\, d\sigma(\xi)\, d\sigma(\eta)\,.
\end{aligned}$$

Here we have used that $u$ and the spherical harmonics $Y_{lm}$ are real-valued. Since $Y_{lm}$ is an orthonormal basis, one has that

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \int_{\mathbb{S}}\int_{\mathbb{S}} \psi(\xi)\,\phi(\eta)\, Y_{lm}(\xi)\, Y_{lm}(\eta)\, d\sigma(\xi)\, d\sigma(\eta) = \int_{\mathbb{S}} \psi(\xi)\,\phi(\xi)\, d\sigma(\xi)$$

for any functions $\psi, \phi \in L^2(\mathbb{S})$. Hence we can get rid of the sums in the above formula and write

$$\kappa_{jk}(x,y) = \int_{\mathbb{S}} e^{i(x-y)\cdot\xi}\, p_j(\xi)\,\overline{p_k(\xi)}\, d\sigma(\xi)\,, \tag{5.3.3}$$

which yields the formula for the spectral measure of $u$. Using now that $p$ is Hermitian (i.e., $\overline{p(\xi)} = p(-\xi)$) and a well-known representation formula for the Bessel function $J_{1/2}$, the above integral can be equivalently written as

$$\begin{aligned}
\int_{\mathbb{S}} e^{ix\cdot\xi}\, p_j(\xi)\,\overline{p_k(\xi)}\, d\sigma(\xi) &= p_j(D)\, p_k(-D) \int_{\mathbb{S}} e^{ix\cdot\xi} d\sigma(\xi) \\
&= (2\pi)^{\frac{3}{2}} p_j(D)\, p_k(-D) \frac{J_{1/2}(|x|)}{|x|^{1/2}}\,.
\end{aligned}$$

The proposition then follows. $\square$

A straightforward corollary is that the Gaussian random Beltrami field $u$ is normalized so that its covariance matrix is the identity on the diagonal:

**Corollary 5.3.3.** *For any $x \in \mathbb{R}^3$, $\kappa(x,x) = I$.*

*Proof.* The formula for the spectral measure computed in Proposition 5.3.2 implies that

$$\kappa_{jk}(x,x) = \int_{\mathbb{S}} p_j(\xi)\,\overline{p_k(\xi)}\, d\sigma(\xi)\,.$$

As $p$ is a polynomial, the computation then boils down to evaluating integrals of the form $\int_{\mathbb{S}} \xi^\alpha\, d\sigma(\xi)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex and $\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$. These integrals can be computed in closed form [Fol01]:

$$\int_{\mathbb{S}} \xi^\alpha\, d\sigma(\xi) = \begin{cases} 2\big[\prod_{j=1}^3 \Gamma(\frac{\alpha_j+1}{2})\big]/\Gamma(\frac{|\alpha|+3}{2}) & \text{if } \alpha_1, \alpha_2, \alpha_3 \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.4}$$

Here $\Gamma$ denotes the Gamma function.

Armed with this formula and taking into account the explicit expression of the polynomial $p(\xi)$ (cf. Equation (5.2.4)), a tedious but straightforward computation shows

$$\int_{\mathbb{S}} p_j(\xi)\, \overline{p_k(\xi)}\, d\sigma(\xi) = \delta_{jk}\,.$$

The result then follows. □

**Remark 5.3.3.** The probability density function of the Gaussian random vector $u(x)$ is therefore $\rho(y) := (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}|y|^2}$. That is, $\mathbb{P}\{u(x) \in \Omega\} = \int_\Omega \rho(y)\, dy$ for any $x \in \mathbb{R}^3$ and any Borel subset $\Omega \subset \mathbb{R}^3$.

Since the Gaussian field $u$ is of class $C^\infty$ with probability 1 by Proposition 5.3.1, it is standard that it defines a Gaussian probability measure, which we henceforth denote by $\mu_u$, on the space of $C^k$ vector fields on $\mathbb{R}^3$, where $k$ is any fixed positive integer. This space is endowed with its usual Borel $\sigma$-algebra $\mathfrak{S}$, which is the minimal $\sigma$-algebra containing the "squares"

$$I(x,a,b) := \{w \in C^k(\mathbb{R}^3,\mathbb{R}^3) : w(x) \in [a_1,b_1) \times [a_2,b_2) \times [a_3,b_3)\}$$

for all $x,a,b, \in \mathbb{R}^3$. To spell out the details, let us denote by $\Omega$ the sample space of the random variables $a_{lm}$ and show that the random field $u$ is a measurable map from $\Omega$ to $C^k(\mathbb{R}^3,\mathbb{R}^3)$. Since the $\sigma$-algebra of $C^k(\mathbb{R}^3,\mathbb{R}^3)$ is generated by point evaluations, it suffices to show that

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, U_{i^l Y_{lm} p}(x)$$

is a measurable function $\Omega \to \mathbb{R}^3$ for each $x \in \mathbb{R}^3$. But this is obvious because $u(x)$ is the limit of finite linear combinations (with coefficients in $\mathbb{R}^3$) of the random variables $a_{lm}$, which are of course measurable. In what follows, we will not mention the $\sigma$-algebra explicitly to keep the notation simple. Also, in view of the later applications to invariant tori, we will henceforth assume that $k \geqslant 4$. Obviously, the Gaussian probability measure $\mu_u$ is regular because the space of $C^k$ vector fields is metrizable (with the compact-open $C^k$-topology).

Following Nazarov and Sodin [NS16], the next proposition shows that from the facts that the covariance kernel $\kappa(x,y)$ only depends on $x-y$ and that the spectral measure has no atoms one can infer two useful properties of our Gaussian probability measure that will be extensively employ in the rest of the chapter. Before stating the result, let us recall that the probability measure $\mu_u$ is said to be translationally invariant if $\mu_u(\tau_y \mathcal{A}) = \mu_u(\mathcal{A})$ for all $\mathcal{A} \subset \mathfrak{S}$ and all $y \in \mathbb{R}^3$. Here $\tau_y$ denotes the translation operator on $C^k$ fields, defined as $\tau_y w(x) := w(x+y)$.

**Proposition 5.3.4.** *The probability measure $\mu_u$ is translationally invariant. Furthermore, if $\Phi$ is an $L^1$ random variable on the probability space $(C^k(\mathbb{R}^3,\mathbb{R}^3), \mathfrak{S}, \mu_u)$ , then*

$$\lim_{R\to\infty} \fint_{B_R} \Phi \circ \tau_y \, dy = \mathbb{E}\Phi$$

*both $\mu_u$-almost surely and in $L^1(C^k(\mathbb{R}^3,\mathbb{R}^3),\mu_u)$.*

*Proof.* Since the covariance kernel $\kappa(x,y)$ only depends on $x-y$, the probability measure $\mu_u$ is translationally invariant. Also, note that $(y,w) \mapsto \tau_y w$ defines a continuous map

$$\mathbb{R}^3 \times C^k(\mathbb{R}^3,\mathbb{R}^3) \to C^k(\mathbb{R}^3,\mathbb{R}^3)\,,$$

so the map $(y,w) \mapsto \Phi(\tau_y w)$ is measurable on the product space $\mathbb{R}^3 \times C^k(\mathbb{R}^3,\mathbb{R}^3)$. Wiener's ergodic theorem [NS16; Bec81] then ensures that, for $\Phi$ as in the statement, there is a random variable $\Phi^* \in L^1(C^k(\mathbb{R}^3 \times \mathbb{R}^3), \mu_u)$ such that

$$\fint_{B_R} \Phi \circ \tau_y \, dy \xrightarrow[\text{a.s.}]{L^1} \Phi^*$$

as $R \to \infty$. Furthermore, $\Phi^*$ is translationally invariant (i.e., $\Phi^* \circ \tau_y = \Phi^*$ for all $y \in \mathbb{R}^3$ almost surely) and $\mathbb{E}\Phi^* = \mathbb{E}\Phi$.

Also, as the spectral measure (computed in Proposition 5.3.2 above) has no atoms, a theorem of Grenander, Fomin and Maruyama (see e.g. [NS16, Appendix B] or [Gre50] and note that the proof carries over to the multivariate and vector-valued case) ensures that the action of the translations $\{\tau_y : y \in \mathbb{R}^3\}$ on the probability space $(C^k(\mathbb{R}^3,\mathbb{R}^3), \mathfrak{S}, \mu_u)$ is ergodic. As the measurable function $\Phi^*$ is translationally invariant, one then infers that $\Phi^*$ is constant $\mu_u$-almost surely. As $\Phi$ and $\Phi^*$ have the same expectation, then $\Phi^* = \mathbb{E}\Phi$ almost surely. The proposition then follows. $\square$

It is clear that the support of the probability measure $\mu_u$ must be contained in the space of Beltrami fields. In the last result of this section, we show that the support is in fact the whole space. This property will be key in the following sections.

**Proposition 5.3.5.** *The support of the Gaussian probability measure $\mu_u$ is the space of Beltrami fields. More precisely, $v$ is a Beltrami field iff for any compact set $K \subset \mathbb{R}^3$ and each $\epsilon > 0$,*

$$\mu_u\big(\{w \in C^k(\mathbb{R}^3,\mathbb{R}^3) : \|v - w\|_{C^k(K)} < \epsilon\}\big) > 0\,.$$

*Proof.* By Proposition 5.2.3, there exists a Hermitian finite linear combination of spherical harmonics,

$$\varphi = \sum_{l=0}^{n} \sum_{m=-l}^{l} i^l \alpha_{lm} Y_{lm}\,,$$

where $\alpha_{lm}$ are real numbers (not random variables), such that $\|v - U_{\varphi p}\|_{C^k(K)} < \epsilon/4$. Hence

$$\mu_u\big(\{w \in C^k(\mathbb{R}^3,\mathbb{R}^3) : \|w - v\|_{C^k(K)} < \epsilon\}\big) \geqslant \mathbb{P}\left(\left\{\|u - U_{\varphi p}\|_{C^k(K)} < \frac{\epsilon}{4}\right\}\right),$$

where $\mathbb{P}$ denotes the natural Gaussian probability measure on the space of sequences $(a_{lm})$.

Proposition 5.3.1 shows that the series

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} U_{i^l Y_{lm} p}$$

converges in $C^k(K)$ almost surely, so for any fixed $\delta > 0$ there exists some number $N$ (which one can assume larger than $n$) such that

$$\mathbb{P}\left(\left\{\left\|\sum_{l=N+1}^{\infty} \sum_{m=-l}^{l} a_{lm} U_{i^l Y_{lm} p}\right\|_{C^k(K)} < \frac{\epsilon}{8}\right\}\right) > 1 - \delta\,.$$

With the convention that $\alpha_{lm} := 0$ for $l > n$, note that

$$\|u - U_{\varphi p}\|_{C^k(K)} \leqslant \sum_{l=0}^{N} \sum_{m=-l}^{l} |a_{lm} - \alpha_{lm}| \|U_{i^l Y_{lm} p}\|_{C^k(K)} + \left\| \sum_{l=N+1}^{\infty} \sum_{m=-l}^{l} a_{lm} U_{i^l Y_{lm} p} \right\|_{C^k(K)} .$$

Therefore, if we set $M := 8(N+1)^2 \max_{l \leqslant N} \max_{-l \leqslant m \leqslant l} \|U_{i^l Y_{lm} p}\|_{C^k(K)}$, it follows that

$$\begin{aligned} &\mathbb{P}\left(\left\{\|u - U_{\varphi p}\|_{C^k(K)} < \frac{\epsilon}{4}\right\}\right) \\ &\geqslant \mathbb{P}\left(\left\{\left\| \sum_{l=N+1}^{\infty} \sum_{m=-l}^{l} a_{lm} U_{i^l Y_{lm} p} \right\|_{C^k(K)} < \frac{\epsilon}{8}\right\}\right) \prod_{l=0}^{N} \prod_{m=-l}^{l} \mathbb{P}\left(\left\{|a_{lm} - \alpha_{lm}| < \frac{\epsilon}{M}\right\}\right) , \end{aligned}$$

which is strictly positive. The proposition then follows. □

## 5.4 Preliminaries about hyperbolic periodic orbits and invariant tori

In this section we construct Beltrami fields that exhibit hyperbolic periodic orbits or a positive measure set of ergodic invariant tori of arbitrary topology. Our constructions are robust in the sense that these properties hold for any other divergence-free field that is $C^4$-close to the Beltrami field. Additionally, we recall some basic notions and results about periodic orbits and invariant tori that will be useful in the following sections.

### 5.4.1 Hyperbolic periodic orbits

We recall that a periodic integral curve, or periodic orbit, $\gamma$ of a vector field $u$ is hyperbolic if all the (possibly complex) eigenvalues $\lambda_j$ of the monodromy matrix of $u$ at $\gamma$ have modulus $|\lambda_j| \neq 1$. Since we are interested in divergence-free vector fields in dimension 3, in this case the eigenvalues are of the form $\lambda, \lambda^{-1}$ for some real $\lambda > 1$. The maximal Lyapunov exponent of the periodic orbit $\gamma$ is defined as $\Lambda := \frac{\log \lambda}{T} > 0$, where $T$ is the period of $\gamma$.

Given a closed curve $\gamma_0$ smoothly embedded in $\mathbb{R}^3$, we say that $\gamma$ has the knot type $[\gamma_0]$ if $\gamma$ is isotopic to $\gamma_0$. It is well known that the number of knot types is countable. Given a set of four positive numbers $\mathcal{I} = (T_1, T_2, \Lambda_1, \Lambda_2)$, with $0 < T_1 < T_2$ and $0 < \Lambda_1 < \Lambda_2$, we denote by $N_u^o(R; [\gamma], \mathcal{I})$ the number of hyperbolic periodic orbits of a vector field $u$ contained in the ball $B_R$, of knot type $[\gamma]$, whose periods and maximal Lyapunov exponents are in the intervals $(T_1, T_2)$ and $(\Lambda_1, \Lambda_2)$, respectively. Since we have fixed the intervals of the periods and Lyapunov exponents, there is a neighborhood of thickness $\eta_0$ of each periodic orbit ($\eta_0$ independent of the orbit) such that no other periodic orbit of this type intersects it. The compactness of $B_R$ then immediately implies that $N_u^o(R, [\gamma], \mathcal{I})$ is finite, although the total number of hyperbolic periodic orbits in $B_R$ may be countable.

An easy application of the hyperbolic permanence theorem [HPS06, Theorem 1.1] implies that the above periodic orbits are robust under $C^1$-small perturbations, so that

$$N_v^o(R; [\gamma], \mathcal{I}) \geqslant N_u^o(R; [\gamma], \mathcal{I})$$

for any vector field $v$ that is close enough to $u$ in the $C^1$ norm. Indeed, if $\|u - v\|_{C^1(B_R)} < \delta$, then $v$ has a periodic orbit $\gamma_\delta$ that is isotopic to, and contained in a tubular neighborhood of width $C\delta$ of, each periodic orbit $\gamma$ of $u$ that has the aforementioned properties. Moreover, the period and maximal Lyapunov exponent of $\gamma_\delta$ is also $\delta$-close to that of $\gamma$, so choosing $\delta$ small enough they still lie in the intervals $(T_1, T_2)$ and $(\Lambda_1, \Lambda_2)$, respectively. Thus we have proved the following:

**Proposition 5.4.1.** *The functional $u \mapsto N_u^o(R; [\gamma], \mathcal{I})$ is lower semicontinuous in the $C^k$ compact open topology for vector fields, for any $k \geqslant 1$. Furthermore, $N_u^o(R; [\gamma], \mathcal{I}) < \infty$ for any $C^1$ vector field $u$.*

The following result ensures that, for any fixed knot type $[\gamma]$ and any quadruple $\mathcal{I}$, there is a Beltrami field $u$ for which $N_u^o(R; [\gamma], \mathcal{I}) \geqslant 1$. This result is a consequence of [EPS12, Theorem 1.1], so we just give a short sketch of the proof.

**Proposition 5.4.2.** *Given a closed curve $\gamma_0 \subset \mathbb{R}^3$ and a set of numbers $\mathcal{I}$ as above, there exists a Hermitian finite linear combination of spherical harmonics $\varphi$ such that the Beltrami field $u_0 := U_{\varphi p}$ has a hyperbolic periodic orbit $\gamma$ isotopic to $\gamma_0$, whose period and maximal Lyapunov exponent lie in the intervals $(T_1, T_2)$ and $(\Lambda_1, \Lambda_2)$, respectively.*

*Proof.* Proceeding as in [EPS12, Section 3, Step 2], after perturbing slightly the curve $\gamma_0$ to make it real analytic (let us also call $\gamma_0$ the new curve), we construct a narrow strip $\Sigma$ that contains the curve $\gamma_0$. Using the same coordinates $(z, \theta)$ as introduced in [EPS12, Section 5], we define an analytic vector field

$$w := \frac{|\gamma_0|}{T} \nabla\theta - \Lambda\, z \nabla z\,,$$

where $|\gamma_0|$ is the length of $\gamma_0$ and $T \in (T_1, T_2)$, $\Lambda \in (\Lambda_1, \Lambda_2)$. Using the Cauchy–Kovalevskaya theorem for Beltrami fields [EPS12, Theorem 3.1], we obtain a Beltrami field $v$ on a neighborhood of $\gamma_0$ such that $v|_\Sigma = w$. A straightforward computation shows that $\gamma_0$ is a hyperbolic periodic orbit of $v$ of period $T$ and maximal Lyapunov exponent $\Lambda$. The result immediately follows by applying Proposition 5.2.3. ☐

**Corollary 5.4.3.** *There exists $R_0 > 0$ and $\delta > 0$ such that $N_w^o(R_0; [\gamma], \mathcal{I}) \geqslant 1$ for any vector field $w$ such that $\|w - u_0\|_{C^k(B_{R_0})} < \delta$, provided that $k \geqslant 1$.*

*Proof.* Taking $R_0$ large enough so that the periodic orbit $\gamma$ is contained in $B_{R_0}$, the result is a straightforward consequence of the lower semicontinuity of $N_u^o(R; [\gamma], \mathcal{I})$, cf. Proposition 5.4.1. ☐

### 5.4.2 Nondegenerate invariant tori

We recall that an invariant torus $\mathcal{T}$ of a vector field $u$ is a compact surface diffeomorphic to the 2-torus, smoothly embedded in $\mathbb{R}^3$, and such that, the field $u$ is tangent to $\mathcal{T}$ and does not vanish on $\mathcal{T}$. In other words, $\mathcal{T}$ is invariant under the flow of $u$. Given an embedded torus $\mathcal{T}_0$, we say that $\mathcal{T}$ has the knot type $[\mathcal{T}_0]$ if $\mathcal{T}$ is isotopic to $\mathcal{T}_0$. It is well known that the number of knot types of embedded tori is countable.

To study the robustness of the invariant tori of a vector field it is customary to introduce two concepts: an arithmetic condition (called Diophantine), which is related to the dynamics of $u$ on $\mathcal{T}$, and a nondegeneracy condition (called twist) that is related to the dynamics of $u$ in the normal direction to $\mathcal{T}$.

We say that the invariant torus $\mathcal{T}$ is Diophantine with Diophantine frequency $\omega$ if there exist global coordinates on the torus $(\theta_1, \theta_2) \in (\mathbb{R}/\mathbb{Z})^2$ such that the restriction of the field $u$ to $\mathcal{T}$ reads in these coordinates as

$$u|_{\mathcal{T}} = a\, e_{\theta_1} + b\, e_{\theta_2}\,, \tag{5.4.1}$$

for some nonzero real constants $a, b$, and $\omega := a/b$ modulo 1 is a Diophantine number. This means that there exist constants $c > 0$ and $\nu > 1$ such that

$$\left|\omega - \frac{p}{m}\right| \geqslant \frac{c}{m^{\nu+1}}$$

for any integers $p, m$ with $m \geqslant 1$. Here $e_{\theta_j}$ (often denoted by $\partial_{\theta_j}$) denotes the tangent vector in the direction of $\theta_j$. We recall that the set of Diophantine numbers (with all $c > 0$ and all $\nu > 1$) has full measure. It is well known that the Diophantine property (possibly changing the constant $c$) of the frequency $\omega$ is independent of the choice of coordinates.

Let us now introduce the notion of twist, which is more involved. To this end, we parameterize a neighborhood of $\mathcal{T}$ with a coordinate system $(\rho, \theta_1, \theta_2) \in (-\delta, \delta) \times (\mathbb{R}/\mathbb{Z})^2$ such that $\mathcal{T} = \{\rho = 0\}$ and $u|_{\rho=0}$ has the form (5.4.1). Let us now compute the Poincaré map $\pi$ defined by the flow of $u$ on a transverse section $\Sigma \subset \{\theta_2 = 0\}$ (which exists if $\delta$ is small enough because $b \neq 0$):

$$\pi : (-\delta', \delta') \times (\mathbb{R}/\mathbb{Z}) \to (-\delta, \delta) \times (\mathbb{R}/\mathbb{Z}) \tag{5.4.2}$$

$$(\rho, \theta_1) \mapsto (\pi_1(\rho, \theta_1), \pi_2(\rho, \theta_1))\,, \tag{5.4.3}$$

for $\delta' < \delta$. Obviously, $\pi(0, \theta_1) = (0, \theta_1 + \omega)$. Since $u$ is divergence-free, the map $\pi$ preserves an area form $\sigma$ on $\Sigma$, which one can write in these coordinates as

$$\sigma = F(\rho, \theta_1)\, d\rho \wedge d\theta_1\,, \tag{5.4.4}$$

for some positive function $F$. Notice that the area form $\sigma$ is exact because it can be written as $\sigma = dA$, where $A$ is the 1-form

$$A := h(\rho, \theta_1)\, d\theta_1\,, \qquad h(\rho, \theta_1) := \int_{-\delta}^{\rho} F(s, \theta_1)\, ds\,,$$

and the map $\pi$ is also exact in the sense that $\pi^* A - A$ is an exact 1-form. Indeed, the area preservation implies that $d(\pi^* A - A) = 0$; moreover the periodicity of $h$ in $\theta_1$ readily implies that

$$\int_0^1 (\pi^* A - A)|_{\rho=0} = \int_0^1 (h(0, \theta_1 + \omega) - h(0, \theta_1))\, d\theta_1 = 0\,,$$

so the claim follows from De Rham's theorem. The exactness of both $\sigma$ and $\pi$ is a crucial ingredient to apply the KAM theory.

**Remark 5.4.1.** It was shown in [EPS15, Proposition 7.3] that if the Euclidean volume form $dx$ reads as $H(\rho, \theta_1, \theta_2)\, d\rho \wedge d\theta_1 \wedge d\theta_2$ in coordinates $(\rho, \theta_1, \theta_2)$ for some positive function $H$, then the factor $F$ that defines the area form $\sigma$ is $F(\rho, \theta_1) = H(\rho, \theta_1, 0) u_{\theta_2}(\rho, \theta_1, 0)$, where $u_{\theta_2}$ denotes the $\theta_2$-component of the vector field $u$.

The twist of the invariant torus $\mathcal{T}$ is then defined as the number

$$\tau := \int_0^1 \frac{\partial_\rho \pi_2(0, \theta_1)}{F(0, \theta_1)} \, d\theta_1 \,. \tag{5.4.5}$$

The reason for which we consider this quantity is that it crucially appears in the KAM nondegeneracy condition of [GEL08], cf. Ref. [EPS15, Definition 7.5] for this particular case.

In the present chapter we are interested in the volume of the set of invariant tori of a divergence-free vector field $u$. More precisely, given a quadruple $\mathcal{J} := (\omega_1, \omega_2, \tau_1, \tau_2)$, where $0 < \omega_1 < \omega_2$, $0 < \tau_1 < \tau_2$, we denote by $V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})$ the inner measure of the set of Diophantine invariant tori of a vector field $u$ contained in the ball $B_R$, of knot type $[\mathcal{T}]$, whose frequencies and twists are in the intervals $(\omega_1, \omega_2)$ and $(\tau_1, \tau_2)$, respectively. One must employ the inner measure of this set (as opposed to its usual volume) because this set does not need to be measurable. When we speak of the volume of this set, it should always be understood in this sense. An efficient way of providing a lower bound for this volume is by considering, for each $V_0 > 0$, the number $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ of pairwise disjoint (closed) solid tori contained in $B_R$ whose boundaries are Diophantine invariant tori with parameters in $\mathcal{J}$ and which contain a set of Diophantine invariant tori with parameters in $\mathcal{J}$ of inner measure greater that $V_0$.

**Remark 5.4.2.** The twist defined in Equation (5.4.5) depends on several choices we made to construct the Poincaré map (i.e., the transverse section and the coordinate system). Accordingly, the functional $V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})$ has to be understood as the inner measure of the set of Diophantine invariant tori whose twists lie in the interval $(\tau_1, \tau_2)$ for some choice of (suitably bounded) coordinates and sections, and similarly with $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$. It is well known that the property of nonzero twist is independent of the aforementioned choices.

Since the Poincaré map $\pi$ that we introduced above is exact, we can apply the KAM theorem for divergence-free vector fields [KKPS14, Theorem 3.2] to show that the above invariant tori are robust for $C^4$-small perturbations, so that $V_v^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) \geqslant V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) + o(1)$ and $N_v^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0) \geqslant N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ for any divergence-free vector field $v$ that is $C^4$-close to $u$. Indeed, if $\|u - v\|_{C^4(B_R)} < \delta$, then $v$ has a set of Diophantine invariant tori of knot type $[\mathcal{T}]$ and of volume

$$V_v^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) \geqslant V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) - C\delta^{1/2} \,.$$

Here we have used that the frequency and twist of each of these invariant tori is $\delta$-close to those of $u$, so by choosing $\delta$ small enough they lie in the intervals $(\omega_1, \omega_2)$ and $(\tau_1, \tau_2)$, respectively. The argument for $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ is analogous. Summing up, we have proved the following:

**Proposition 5.4.4.** *The functionals $u \mapsto N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ and $u \mapsto V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})$ are lower semicontinuous in the $C^k$ compact open topology for divergence-free vector fields, for any $k \geqslant 4$.*

We next show that, for any knot type $[\mathcal{T}]$, one can pick a quadruple $\mathcal{J}$ and some $V_0 > 0$ for which there is a Beltrami field $u$ with $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0) \geqslant 1$. This is a straightforward consequence of [EPS15, Theorem 1.1] (see also [ELPS20, Section 3]), so we just sketch the proof.

**Proposition 5.4.5.** *Given an embedded torus $\mathcal{T} \subset \mathbb{R}^3$, there exists a set of numbers $\mathcal{J}, V_0$ as above, and a Hermitian finite linear combination of spherical harmonics $\varphi$ such that the Beltrami field $u_0 := U_{\varphi p}$ has a set of inner measure greater than $V_0 > 0$ that consists of Diophantine invariant tori of knot type $[\mathcal{T}]$ whose frequencies and twists lie in the intervals $(\omega_1, \omega_2)$ and $(\tau_1, \tau_2)$, respectively.*

*Proof.* It follows from [EPS15, Theorem 1.1] that there exists a Beltrami field $v$ that satisfies $\operatorname{curl} v = \lambda v$ in $\mathbb{R}^3$ for some small constant $\lambda > 0$, which has a positive measure set of invariant tori of knot type $[\mathcal{T}]$. These tori are Diophantine and have positive twist. It is obvious that the field $u(x) := v(x/\lambda)$ satisfies the equation $\operatorname{curl} u = u$ in $\mathbb{R}^3$, and still has a set of Diophantine invariant tori of knot type $[\mathcal{T}]$ of measure bigger than some constant $V_0$, and positive twist. The result follows taking the intervals $(\omega_1, \omega_2)$ and $(\tau_1, \tau_2)$ in the definition of $\mathcal{J}$, so that they contain the frequencies and twists of these tori of $u$, and applying Proposition 5.2.3 to approximate $u$ by a Beltrami field $U_{\varphi p}$ in a large ball containing the aforementioned set of invariant tori. $\square$

**Corollary 5.4.6.** *Take $\mathcal{J}$ and $V_0$ as in Proposition 5.4.5. There exists $R_0 > 0$ and $\delta > 0$ such that $N_w^{\mathrm{t}}(R_0; [\mathcal{T}], \mathcal{J}, V_0) \geqslant 1$ and $V_w^{\mathrm{t}}(R_0; [\mathcal{T}], \mathcal{J}) > V_0/2$ for any divergence-free vector field $w$ such that $\|w - u_0\|_{C^k(B_{R_0})} < \delta$, provided that $k \geqslant 4$.*

*Proof.* Taking $R_0$ large enough so that the aforementioned set of invariant tori of $u_0$ is contained in $B_{R_0}$, the result is a straightforward consequence of the lower semicontinuity of $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ and $V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})$, cf. Proposition 5.4.4. $\square$

## 5.5 A Beltrami field on $\mathbb{R}^3$ that is stably chaotic

Our objective in this section is to construct a Beltrami field $u$ in $\mathbb{R}^3$ that exhibits a horseshoe, that is, a compact (normally) hyperbolic invariant set with a transverse section homeomorphic to a Cantor set on which the time-$T$ flow of $u$ (or of a suitable reparametrization thereof) is topologically conjugate to a Bernoulli shift. It is standard that a horseshoe of a three-dimensional flow is a connected branched surface, and that the existence of a horseshoe is stable in the sense that any other field that is $C^1$-close to $u$ has a horseshoe too [GH13, Theorem 5.1.2]. Moreover, the existence of a horseshoe implies that the field has positive topological entropy; recall that the topological entropy of the field, which we denote as $h_{\mathrm{top}}(u)$, is defined as the entropy of its time-1 flow. Summarizing, we have the following result for the number of (pairwise disjoint) horseshoes of $u$ contained in $B_R$, $N_u^{\mathrm{h}}(R)$:

**Proposition 5.5.1.** *The functional $u \mapsto N_u^{\mathrm{h}}(R)$ is lower semicontinuous in the $C^k$ compact open topology for vector fields, for any $k \geqslant 1$. Moreover, if $u$ has a horseshoe, its topological entropy is positive.*

In short, the basic idea to construct a Beltrami field with a horseshoe, is to construct first "an integrable" Beltrami field having a heteroclinic cycle between two hyperbolic periodic orbits, which we subsequently perturb within the Beltrami class to produce a transverse heteroclinic intersection. By the Birkhoff–Smale theorem, this ensures the existence of horseshoe-type dynamics.

**Proposition 5.5.2.** *There exists a Hermitian finite linear combination of spherical harmonics $\varphi$ such that the Beltrami field $u_0 := U_{\varphi p}$ exhibits a horseshoe. In other words, $N^{\mathrm{h}}_{u_0}(R_0) \geqslant 1$ for all large enough $R_0 > 0$.*

*Proof.* Let us take cylindrical coordinates $(z, r, \theta) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{T}$, with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, defined as

$$z := x_3 , \qquad (r \cos\theta, r \sin\theta) := (x_1, x_2) .$$

We now consider the axisymmetric vector field $v$ in $\mathbb{R}^3$ given by

$$v := \frac{1}{r}\Big(\partial_r\psi\, E_z - \partial_z\psi\, E_r + \frac{\psi}{r}\, E_\theta\Big) . \tag{5.5.1}$$

Here

$$\psi := \cos z + 3rJ_1(r)$$

with $J_1$ being the Bessel function of the first kind and order 1, and the vector fields

$$E_z := (0, 0, 1) , \qquad E_r := \frac{1}{r}(x_1, x_2, 0) , \qquad E_\theta := (-x_2, x_1, 0) ,$$

which are often denoted by $\partial_z$, $\partial_r$, $\partial_\theta$ in the dynamical systems literature, have been chosen so that

$$E_z \cdot \nabla\phi = \partial_z\phi , \qquad E_r \cdot \nabla\phi = \partial_r\phi , \qquad E_\theta \cdot \nabla\phi = \partial_\theta\phi$$

for any function $\phi$. Notice that $v \cdot \nabla\psi = 0$, so the scalar function $\psi$ is a first integral of $v$. This means that the trajectories of the field $v$ are tangent to the level sets of $\psi$.

The vector field $v$ is not defined on the $z$-axis, so we shall consider the domain in Euclidean 3-space

$$\Omega := \{(z, r, \theta) : (z, r) \in \mathcal{D} , \theta \in \mathbb{T}\} ,$$

where $\mathcal{D}$ is the domain in the $(z, r)$-plane given by

$$\mathcal{D} := \left\{(z, r) : -10 < z < 10, \frac{9}{10} < r < \frac{18}{5}\right\} .$$

The reason for choosing this particular domain of $\mathbb{R}^3$ will become clear later in the proof; for the time being, let us just note that $\psi(z, r) > 0$ if $(z, r) \in \mathcal{D}$.

Also, observe that, away from the axis $r = 0$, the vector field $v$ is smooth and satisfies the Beltrami field equation $\operatorname{curl} v = v$.

We claim that, in $\Omega$, $v$ has two hyperbolic periodic orbits joined by a heteroclinic cycle. Indeed, noticing that

$$(\partial_z\psi, \partial_r\psi) = (-\sin z, 3rJ_0(r)) ,$$

where we have used the identity $\partial_r[rJ_1(r)] = rJ_0(r)$, it follows that the points $p_\pm := (\pm\pi, j_{0,1}) \in \mathcal{D}$ are critical points of $\psi$. Here $j_{0,1} = 2.4048\ldots$ is the first zero of the Bessel function $J_0$. Plugging this fact in Equation (5.5.1), this implies that, on the circles in 3-space

$$\gamma_\pm := \{(z, r, \theta) : (z, r) = p_\pm , \theta \in \mathbb{T}\} ,$$

the field $v$ takes the form

$$v(p_\pm, \theta) = \frac{c_0}{j_{0,1}^2} E_\theta$$

with $c_0 := 3j_{0,1}J_1(j_{0,1}) - 1 > 0$. Therefore, we conclude that the circles $\gamma_\pm$ are periodic orbits of $v$ contained in $\Omega$.

It is standard that the stability of these periodic orbits can be analyzed using the associated normal variational equation. Denoting by $(v_z, v_r, v_\theta)$ the components of the field $v$ in the basis $\{E_z, E_r, E_\theta\}$, this is the linear ODE

$$\dot{\eta} = A\eta\,,$$

where $\eta$ takes values in $\mathbb{R}^2$ and $A$ is the constant matrix

$$A := \left.\frac{\partial(v_z, v_r)}{\partial(z, r)}\right|_{(z,r)=p_\pm} = \begin{pmatrix} 0 & 3J_0'(j_{0,1}) \\ -1/j_{0,1} & 0 \end{pmatrix}.$$

The Lyapunov exponents of the periodic orbit $\gamma_\pm$ are the eigenvalues of the matrix $A$. Therefore, since $J_0'(j_{0,1}) < 0$, these periodic orbits have a positive and a negative Lyapunov exponent, so they are hyperbolic periodic orbits of saddle type.

Since $\psi$ is a first integral of $v$ and $\psi(p_\pm) = c_0$, the set

$$\{(z, r, \theta) : \psi(z, r) = c_0\}$$

is an invariant singular surface of the vector field $v$. This set contains two regular surfaces $\Gamma_1$ and $\Gamma_2$ diffeomorphic to a cylinder. We label them so $\Gamma_1$ is contained in the half space $\{r \leqslant j_{0,1}\}$ and $\Gamma_2$ in $\{r \geqslant j_{0,1}\}$. The boundaries of these cylinders are the periodic orbits $\gamma_\pm$. The surface $\Gamma_1$ is the stable manifold of $\gamma_+$ that coincides with an unstable manifold of $\gamma_-$, while $\Gamma_2$ is the unstable manifold of $\gamma_+$ that coincides with a stable manifold of $\gamma_-$. Hence the union $\Gamma_1 \cup \Gamma_2$ of both cylinders then form an heteroclinic cycle between the periodic orbits $\gamma_+$ and $\gamma_-$, and one can see that it is contained in $\Omega$.

Let us now perturb the Beltrami field $v$ in $\Omega$ by adding a vector field $w$ (to be fixed later) that also satisfies the Beltrami field equation $\operatorname{curl} w = w$. Our goal is to break the heteroclinic cycle $\Gamma_1 \cup \Gamma_2$ in order to produce transverse intersections of the stable and unstable manifolds of $\gamma_+^\varepsilon$ and $\gamma_-^\varepsilon$, where $\gamma_\pm^\varepsilon$ denote the hyperbolic periodic orbits of the perturbed vector field

$$X := v + \varepsilon w = \left(\frac{\partial_r \psi}{r} + \varepsilon w_z\right) E_z + \left(-\frac{\partial_z \psi}{r} + \varepsilon w_r\right) E_r + \left(\frac{\psi}{r^2} + \varepsilon w_\theta\right) E_\theta\,.$$

As before, $(w_z, w_r, w_\theta)$ denote the components of the vector field $w$ in the basis $\{E_z, E_r, E_\theta\}$, which are functions of all three cylindrical coordinates $(z, r, \theta)$. If $\varepsilon > 0$ is small enough, the $\theta$-component of $X$ is positive on the domain $\Omega$, so we can divide $X$ by the factor $X_\theta := \frac{\psi}{r^2} + \varepsilon w_\theta > 0$ to obtain another vector field $Y$ that has the same integral curves up to a reparametrization:

$$Y := \frac{X}{X_\theta} = \frac{r\partial_r \psi + \varepsilon r^2 w_z}{\psi + \varepsilon r^2 w_\theta} E_z + \frac{-r\partial_z \psi + \varepsilon r^2 w_r}{\psi + \varepsilon r^2 w_\theta} E_r + E_\theta\,. \tag{5.5.2}$$

Substituting the expression of $\psi(z, r)$ and expanding in the small parameter $\varepsilon$, the analysis of the integral curves of $Y$ reduces to that of the following non-autonomous system of ODEs in the planar domain $\mathcal{D}$:

$$\frac{dz}{dt} = \frac{3r^2 J_0(r)}{\psi(z,r)} + \varepsilon\left(\frac{r^2 w_z(z,r,t)}{\psi(z,r)} - \frac{3r^4 J_0(r) w_\theta(z,r,t)}{\psi(z,r)^2}\right) + O(\varepsilon^2), \tag{5.5.3}$$

$$\frac{dr}{dt} = \frac{r \sin z}{\psi(z,r)} + \varepsilon\left(\frac{r^2 w_r(z,r,t)}{\psi(z,r)} - \frac{r^3 \sin z\, w_\theta(z,r,t)}{\psi(z,r)^2}\right) + O(\varepsilon^2). \tag{5.5.4}$$

Notice that the dependence on $t$ is $2\pi$-periodic, and that we have replaced $\theta$ by $t$ in the function $w_z(z,r,\theta)$ (and similarly $w_r, w_\theta$) because the $\theta$-component of the vector field $Y$ is 1. When $\varepsilon = 0$, one has

$$\dot{z} = \frac{3r^2 J_0(r)}{\psi(z,r)}, \tag{5.5.5}$$

$$\dot{r} = \frac{r \sin z}{\psi(z,r)}. \tag{5.5.6}$$

Hence the unperturbed system is Hamiltonian with symplectic form $\omega := r^{-1} dz \wedge dr$ and Hamiltonian function $H(z,r) := \log \psi(z,r)$. The periodic orbits $\gamma_\pm$ of $v$ and their heteroclinic cycle $\Gamma_1 \cup \Gamma_2$ correspond to the (hyperbolic) fixed points $p_\pm$ of the unperturbed system joined by two heteroclinic connections $\widetilde{\Gamma}_k := \Gamma_k \cap \{\theta = 0\}$, $k = 1, 2$. These are precisely the two pieces of the level curve $\{H(z,r) = \log c_0\}$ that are contained in $\mathcal{D}$. Let us denote by

$$\gamma_k(t) = (Z_k(t; 0, r_k), R_k(t; 0, r_k))$$

the integral curves of the separatrices that solve Equations (5.5.5)-(5.5.6) with initial conditions $(0, r_k) \in \widetilde{\Gamma}_k$. Of course, the closure of the set $\{\gamma_k(t) : t \in \mathbb{R}\}$ is $\widetilde{\Gamma}_k$, and the stability analysis of the periodic integral curves $\gamma_\pm$ readily implies that $\lim_{t \to \pm(-1)^{k+1}\infty} \gamma_k(t) = p_\pm$.

By the implicit function theorem, the perturbed system (5.5.3)-(5.5.4) has exactly two hyperbolic fixed points $p_\pm^\varepsilon \in \mathcal{D}$ so that $p_\pm^\varepsilon \to p_\pm$ as $\varepsilon \to 0$. The technical tool to prove that the unstable (resp. stable) manifold of $p_+^\varepsilon$ and the stable (resp. unstable) manifold of $p_-^\varepsilon$ intersect transversely when $\varepsilon > 0$ is small is the Melnikov function. We define the vector fields $Y_0$, $Y_1$, respectively, as the unperturbed system and the first order in $\varepsilon$ perturbation, i.e.,

$$Y_0 := \frac{3r^2 J_0(r)}{\psi(z,r)} E_z + \frac{r \sin z}{\psi(z,r)} E_r,$$

$$Y_1 := \left(\frac{r^2 w_z}{\psi(z,r)} - \frac{3r^4 J_0(r) w_\theta}{\psi(z,r)^2}\right) E_z + \left(\frac{r^2 w_r}{\psi(z,r)} - \frac{r^3 \sin z w_\theta}{\psi(z,r)^2}\right) E_r.$$

Since the unperturbed system is Hamiltonian, we can apply Lemma 5.5.4 below (which is a variation on known results in Melnikov theory) to conclude that if the Melnikov functions

$$M_k(t_0) := \int_{-\infty}^{\infty} \omega(Y_0, Y_1)|_{\gamma_k(t-t_0)}\, dt, \tag{5.5.7}$$

have simple zeros for each $k = 1, 2$, then the aforementioned transverse intersections exist, and that actually the heteroclinic connections intersect at infinitely many points. The integrand $\omega(Y_0, Y_1)$ denotes the action of the symplectic 2-form $\omega$ on the vector fields $Y_0, Y_1$, evaluated on the integral curve $\gamma_k(t - t_0)$. It is standard that the improper integral in the definition of the Melnikov functions is absolutely convergent because of the hyperbolicity of the fixed points joined by the separatrices (see e.g. [GH13, Section 4.5]). Also notice that although [GH13, Section 4.5] concerns transverse intersections of homoclinic connections, the analysis applies verbatim to transverse intersections of heteroclinic connections.

More explicitly, the Melnikov functions are given by

$$M_k(t_0) = \frac{1}{c_0^2}\int_{-\infty}^{\infty} R_k(t)^2\big[w_z(Z_k(t), R_k(t), t)\sin Z_k(t) - 3R_k(t)J_0(R_k(t))w_r(Z_k(t), R_k(t), t)\big]\,dt\,,$$

where $R_k(t) \equiv R_k(t;0,r_k)$ and $Z_k(t) \equiv Z_k(t;0,r_k)$. It is well known that the existence of transverse intersections is independent of the choice of initial condition.

To analyze these Melnikov integrals, let us now choose the particular perturbation

$$w = J_1(r)\sin\theta\, E_z + \frac{J_1(r)}{r}\cos\theta\, E_r - \frac{J_1'(r)\sin\theta}{r}\, E_\theta\,. \tag{5.5.8}$$

It is easy to check that $\operatorname{curl} w = w$ in $\mathbb{R}^3$; in fact $w = (\operatorname{curl}\operatorname{curl} + \operatorname{curl})(J_0(r), 0, 0)$ (or, to put it differently, $w = U_{\varphi' q(\xi_1)^{-1}p}$, where the distribution $\varphi'$ on the sphere $\mathbb{S}$ is the Lebesgue measure of the equator, normalized to unit mass). With this choice, the Melnikov functions take the form

$$\begin{aligned} c_0^2 M_k(t_0) &= \int_{-\infty}^{\infty} R_k(t)^2\big[J_1(R_k(t))\sin Z_k(t)\sin(t+t_0) - 3J_0(R_k(t))J_1(R_k(t))\cos(t+t_0)\big]\,dt \\ &=: a_k \sin t_0 + b_k\cos t_0\,, \end{aligned}$$

where the constants $a_k, b_k$ are given by the integrals

$$\begin{aligned} a_k &= \int_{-\infty}^{\infty} R_k(t)^2\big[J_1(R_k(t))\sin Z_k(t)\cos t + 3J_0(R_k(t))J_1(R_k(t))\sin t\big]\,dt\,, \\ b_k &= \int_{-\infty}^{\infty} R_k(t)^2\big[J_1(R_k(t))\sin Z_k(t)\sin t - 3J_0(R_k(t))J_1(R_k(t))\cos t\big]\,dt\,. \end{aligned}$$

Since the Hamiltonian function has the symmetry $H(-z,r) = H(z,r)$, it follows that $R_k(t) = R_k(-t)$ and $Z_k(t) = -Z_k(-t)$. This immediately yields that $a_1 = a_2 = 0$. Moreover, it is not hard to compute the constants $b_1$ and $b_2$ numerically:

$$b_1 = 3.5508\ldots\,, \qquad b_2 = 0.2497\ldots$$

Therefore, the function $M_k(t_0) = b_k\cos t_0$ is a nonzero multiple of the cosine, so it obviously has exactly two zeros in the interval $[0, 2\pi)$, which are nondegenerate. It then follows from Lemma 5.5.4 below that the two heteroclinic connections joining $p_\pm^\varepsilon$ intersect transversely. In turn, this implies [WWG90, Theorem 26.1.3] that each hyperbolic fixed point $p_\pm^\varepsilon$ has transverse homoclinic intersections, so by the Birkhoff–Smale theorem [GH13, Theorem 5.3.5] the perturbed system (5.5.3)-(5.5.4) (with $w$ given by Equation (5.5.8)) has a compact hyperbolic invariant set on which the dynamics is topologically conjugate to a Bernoulli shift. This set is contained in a neighborhood of the heteroclinic cycle $\widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$, and hence in the planar domain $\mathcal{D}$ where the system is defined. This immediately implies that the vector field $Y$ defined in Equation (5.5.2), which is the suspension of the non-autonomous planar system (5.5.3), has a compact normally hyperbolic invariant set $K$ on which its time-$T$ flow is topologically conjugate to a Bernoulli shift, where $T := 2\pi N$ for some positive integer $N > 0$. The invariant set $K$ is contained in $\Omega$ because it lies in a small neighborhood of the invariant set $\Gamma_1 \cup \Gamma_2$. Since the integral curves of $X$ and $Y$ are the same, up to a reparametrization, $K$ is also a chaotic invariant set of the Beltrami field $X$ in $\Omega$.

Finally, since $\mathbb{R}^3 \backslash \overline{\Omega}$ is connected, and of course the vector field $X$ satisfies the Beltrami equation in an open neighborhood of $\overline{\Omega}$, for each $\delta > 0$, Proposition 5.2.3 shows that there is a Hermitian finite linear combination of spherical harmonics $\varphi$ such that

$$\|X - U_{\varphi p}\|_{C^1(\Omega)} < \delta\,.$$

If $\delta$ is small enough, the stability of transverse intersections implies that the Beltrami field $U_{\varphi p}$ has a compact chaotic invariant set $K_\delta$ in a small neighborhood of $K$ on which a suitable reparametrization of its time-$T$ flow is conjugate to a Bernoulli shift, so the proposition follows. □

**Corollary 5.5.3.** *There exists $R_0 > 0$ and $\delta > 0$ such that $N^{\mathrm{h}}_w(R_0) \geqslant 1$ for any vector field $w$ such that $\|w - u_0\|_{C^k(B_{R_0})} < \delta$, provided that $k \geqslant 1$.*

*Proof.* Taking $R_0$ so that the horseshoe of $u_0$ is contained in $B_{R_0}$, the result is a straightforward consequence of the lower semicontinuity of $N^{\mathrm{h}}_u(R)$, cf. Proposition 5.5.1. □

To conclude, the following lemma gives the formula for the Melnikov function that we employed in the proof of Proposition 5.5.2 above. This is an expression for the Melnikov function of perturbations of a planar system that is Hamiltonian with respect to an arbitrary symplectic form. This is a minor generalization of the well-known formulas [GH13, Theorem 4.5.3] and [Hol80, Equation (23)], which assume that the symplectic form is the standard one.

**Lemma 5.5.4.** *Let $Y_0$ be a smooth Hamiltonian vector field defined on a domain $\mathcal{D} \subset \mathbb{R}^2$ with Hamiltonian function $H$ and symplectic form $\omega$. Assume that this system has two hyperbolic fixed points $p_\pm$ joined by a heteroclinic connection $\widetilde{\Gamma}$. Take a smooth non-autonomous planar field $Y_1$, which we assume $2\pi$-periodic in time, and consider the perturbed system $Y_0 + \varepsilon Y_1 + O(\varepsilon^2)$. Then the simple zeros of the Melnikov function*

$$M(t_0) := \int_{-\infty}^{\infty} \omega(Y_0, Y_1)|_{\gamma(t-t_0;p_0)}\, dt\,,$$

*where the integrand is evaluated at the integral curve $\gamma(t - t_0; p_0)$ of $Y_0$ parametrizing the separatrix $\widetilde{\Gamma}$, give rise to a transverse heteroclinic intersection of the perturbed system, for any small enough $\epsilon$.*

*Proof.* If $\varepsilon$ is small enough, the perturbed system has two hyperbolic fixed points $p^\varepsilon_\pm$. To analyze how the heteroclinic connection is perturbed, we take a point $p_0 \in \widetilde{\Gamma}$ and we compute the so-called displacement (or distance) function $\Delta(t_0)$ on a section $\Sigma$ based at $p_0$ and transverse to $\widetilde{\Gamma}$. Recall that the function $\varepsilon\Delta(t_0)$ gives the distance of the splitting, up to order $O(\varepsilon^2)$, between the corresponding stable and unstable manifolds of the perturbed system at the section $\Sigma$.

A standard analysis, cf. [Hol80, Equation (22)] or the proof of [GH13, Theorem 4.5.3], yields the following formula for $\Delta(t_0)$:

$$\Delta(t_0) = \frac{1}{|Y_0(p_0)|} \int_{-\infty}^{\infty} Y_1(\gamma(t-t_0)) \times Y_0(\gamma(t-t_0)) e^{-\int_0^{t-t_0} \operatorname{Tr} DY_0(\gamma(s))\, ds}\, dt\,, \tag{5.5.9}$$

where we have omitted the dependence of the integral curve on the initial condition $p_0 \in \widetilde{\Gamma}$. Here we are using the notation $X \times Y := X_1 Y_2 - X_2 Y_1$ for vectors $X, Y \in \mathbb{R}^2$ and $\operatorname{Tr} DY_0$ is the trace of the Jacobian matrix of the unperturbed field $Y_0$.

Take coordinates in $\mathcal{D}$, which we will call $(z, r)$ just as in the proof of Proposition 5.5.2, and write the symplectic form as $\omega = \rho(z, r)\, dz \wedge dr$, where $\rho(z, r)$ is a smooth function that does not vanish. Let us call here $\{e_z, e_r\}$ the basis of vector fields dual to $\{dz, dr\}$ (which are usually denoted by $\partial_z$ and $\partial_r$, as they correspond to the partial derivatives with respect to the coordinates $z$ and $r$). The Hamiltonian field $Y_0$ reads in these coordinates as

$$Y_0 = \frac{1}{\rho(z,r)} \Big( \partial_r H\, e_z - \partial_z H\, e_r \Big)\,.$$

Noting that

$$Y_1(\gamma(t-t_0)) \times Y_0(\gamma(t-t_0)) = \frac{\omega(Y_0, Y_1)|_{\gamma(t-t_0)}}{\rho(\gamma(t-t_0))}$$

and

$$e^{-\int_0^{t-t_0} \operatorname{Tr} DY_0(\gamma(s))\, ds} = e^{\int_0^{t-t_0} Y_0(\gamma(s)) \cdot \nabla \log \rho(\gamma(s))\, ds} \tag{5.5.10}$$

$$= e^{\int_0^{t-t_0} \frac{d \log \rho(\gamma(s))}{ds}\, ds} = \frac{\rho(\gamma(t-t_0))}{\rho(p_0)}\,, \tag{5.5.11}$$

Equation (5.5.9) implies that

$$\Delta(t_0) = \frac{M(t_0)}{|Y_0(p_0)| \rho(p_0)}\,,$$

so the claim follows because $M(t_0)$ coincides with the displacement function up to a constant proportionality factor. □

## 5.6 Asymptotics for random Beltrami fields on $\mathbb{R}^3$

We are now ready to prove our main results about random Beltrami fields on $\mathbb{R}^3$, Theorems 5.1.2 and 5.1.4. To do this, as we saw in the two previous sections, we need to handle sets that have a rather geometrically complicated structure, which gives rise to several measurability issues. For this reason, we start this section by proving a version of the Nazarov–Sodin sandwich estimate [NS16, Lemma 1] that circumvents some of these issues and which is suitable for our purposes.

### 5.6.1 A sandwich estimate for sets of points and for arbitrary closed sets

For any subset $\Gamma \subset \mathbb{R}^3$, we denote by $N(x, r; \Gamma)$ the number of connected components of $\Gamma$ that are contained in the ball $B_r(x)$. Also, if $\mathcal{X} := \{x_j : j \in \mathcal{J}\}$, where $x_j \in \mathbb{R}^3$, is a countable set of points (which is not necessarily a closed subset of $\mathbb{R}^3$), then we define

$$\mathcal{N}(x, r; \mathcal{X}) := \#[\mathcal{X} \cap B_r(x)]$$

as the number of points of $\mathcal{X}$ contained in the open ball $B_r(x)$. For the ease of notation, we will write $N(r; \Gamma) := N(0, r; \Gamma)$ and similarly $\mathcal{N}(r; \mathcal{X})$. We remark that these numbers may be infinite.

**Lemma 5.6.1.** *Let $\Gamma$ be any subset of $\mathbb{R}^3$ whose connected components are all closed and let $\mathcal{X} := \{x_j : j \in \mathcal{J}\}$, with $x_j \in \mathbb{R}^3$, be a countable set of points of $\mathbb{R}^3$. Then the functions $\mathcal{N}(\cdot, r; \mathcal{X})$ and $N(\cdot, r; \Gamma)$ are measurable, and for any $0 < r < R$ one has*

$$\int_{B_{R-r}} \frac{\mathcal{N}(y, r; \mathcal{X})}{\operatorname{vol} B_r}\, dy \leqslant \mathcal{N}(R; \mathcal{X}) \leqslant \int_{B_{R+r}} \frac{\mathcal{N}(y, r; \mathcal{X})}{\operatorname{vol} B_r}\, dy\,,$$
$$\int_{B_{R-r}} \frac{N(y, r; \Gamma)}{\operatorname{vol} B_r}\, dy \leqslant N(R; \Gamma)\,.$$

*Proof.* Let us start by noticing that

$$\mathcal{N}(y, r; \mathcal{X}) = \#\{j \in \mathcal{J} : x_j \in B(y, r)\} = \sum_{j \in \mathcal{J}} \mathbb{1}_{B_r(x_j)}(y)\,.$$

As the ball $B_r(x)$ is an open set, it is clear that $\mathbb{1}_{B_r(x)}(\cdot)$ is a lower semicontinuous function. Recall that lower semicontinuity is preserved under sums, and that the supremum of an arbitrary set (not necessarily countable) of lower semicontinuous functions is also lower semicontinuous. Therefore, from the formula

$$\mathcal{N}(\cdot, r; \mathcal{X}) = \sup_{\mathcal{J}'} \sum_{j \in \mathcal{J}'} \mathbb{1}_{B_r(x_j)}(\cdot)\,,$$

where $\mathcal{J}'$ ranges over all finite subsets of $\mathcal{J}$, we deduce that the function $\mathcal{N}(\cdot, r; \mathcal{X})$ is lower semicontinuous, and therefore measurable.

Now let $\mathcal{J}_R := \{j \in \mathcal{J} : x_j \in B_R\}$ and note that

$$\operatorname{vol} B_r \mathcal{N}(R; \mathcal{X}) = \sum_{j \in \mathcal{J}_R} \int_{B_{R+r}} \mathbb{1}_{B_r(x_j)}(y)\, dy\,.$$

As we can interchange the sum and the integral by the monotone convergence theorem and

$$\sum_{j \in \mathcal{J}_R} \mathbb{1}_{B_r(x_j)}(y) \leqslant \sum_{j \in \mathcal{J}} \mathbb{1}_{B_r(x_j)}(y) = \mathcal{N}(y, r; \mathcal{X})\,,$$

one immediately obtains the upper bound for $\mathcal{N}(R; \mathcal{X})$. Likewise, using now that

$$\begin{aligned}
\operatorname{vol} B_r \mathcal{N}(R; \mathcal{X}) &= \sum_{j \in \mathcal{J}_R} \int_{B_{R+r}} \mathbb{1}_{B_r(x_j)}(y)\, dy \\
&\geqslant \sum_{j \in \mathcal{J}_R} \int_{B_{R-r}} \mathbb{1}_{B_r(x_j)}(y)\, dy \\
&= \sum_{j \in \mathcal{J}} \int_{B_{R-r}} \mathbb{1}_{B_r(x_j)}(y)\, dy = \int_{B_{R-r}} \mathcal{N}(y, r; \mathcal{X})\, dy\,,
\end{aligned}$$

we derive the lower bound. The sandwich estimate for $\mathcal{N}(R; \mathcal{X})$ is then proved.

Now let $\gamma$ be a connected component of $\Gamma$, which is a closed set by hypothesis. Since $\gamma \subset B_r(y)$ if and only if $y \in B_r(x)$ for all $x \in \gamma$, one has that

$$N(y, r; \Gamma) = \sum_{\gamma \subset \Gamma} \mathbb{1}_{\gamma^r}(y)\,, \tag{5.6.1}$$

where the sum is over the connected components of $\Gamma$ and the set $\gamma^r$ is defined, for each connected component $\gamma$ of $\Gamma$, as

$$\gamma^r := \bigcap_{x \in \gamma} B_r(x)\,,$$

that is, as the set of points in $\mathbb{R}^3$ whose distance to any point of $\gamma$ is less than $r$. Obviously, the set $\gamma^r$ is open, so $\mathbb{1}_{\gamma^r}$ is lower semicontinuous, and contained in the ball $B_r(x_0)$, where $x_0$ is any point of $\gamma$. Also notice that $\gamma^r$ is not the empty set provided that $2r$ is larger than the diameter of $\gamma$. Therefore, by the same argument as before, if follows from the expression (5.6.1) that the function $N(\cdot, r; \Gamma)$ is measurable. If we now define the set $\Gamma_R$ consisting of the connected components of $\Gamma$ that are contained in the ball $B_R$, the same argument as before shows that

$$\begin{aligned}
N(R; \Gamma) &\geqslant \sum_{\gamma \subset \Gamma_R} \frac{1}{|\gamma^r|} \int_{B_{R+r}} \mathbb{1}_{\gamma^r}(y)\, dy \\
&\geqslant \sum_{\gamma \subset \Gamma_R} \frac{1}{|\gamma^r|} \int_{B_{R-r}} \mathbb{1}_{\gamma^r}(y)\, dy \\
&= \sum_{\gamma \subset \Gamma} \frac{1}{|\gamma^r|} \int_{B_{R-r}} \mathbb{1}_{\gamma^r}(y)\, dy \\
&\geqslant \int_{B_{R-r}} \frac{N(y, r; \Gamma)}{\sup_{\gamma \subset \Gamma} |\gamma^r|}\, dy \\
&\geqslant \int_{B_{R-r}} \frac{N(y, r; \Gamma)}{|B_r|}\, dy\,.
\end{aligned}$$

In the first inequality we are summing over components $\gamma$ whose diameter is smaller than $2r$, and to pass to the last inequality we have used the obvious volume bound $|\gamma^r| \leqslant |B_r|$. Note that the proof of the upper bound for $\mathcal{N}(R; \mathcal{X})$ does not apply in this case, essentially because we do not have lower bounds for $|\gamma^r|$ in terms of $|B_r|$. □

### 5.6.2 Proof of Theorem 5.1.2 and Corollary 5.1.3

We are ready to prove Theorem 5.1.2. In fact, we will establish a stronger result which permits to control the parameters of the periodic orbits and the invariant tori. In what follows, we shall use the notation introduced in Sections 5.4 and 5.5 for the number of periodic orbits $N_u^{\mathrm{o}}(R; [\gamma], \mathcal{I})$, the number of Diophantine toroidal sets $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$ (and the volume of the set of invariant tori $V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})$) and the number of horseshoes $N_u^{\mathrm{h}}(R)$. This is useful in itself, since we showed in Section 5.4.1 that the quantity $N_u^{\mathrm{o}}(R; [\gamma], \mathcal{I})$ is finite but this does not need to be the case if one just counts $N_u^{\mathrm{o}}(R; [\gamma])$. Also, the choice of counting the volume of invariant tori instead of its number (which one definitely expect to be infinite) provides the trivial bound $V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) \leqslant |B_R|$. Specifically, the result we prove is the following:

**Theorem 5.6.2.** *Consider a closed curve $\gamma$ and an embedded torus $\mathcal{T}$ of $\mathbb{R}^3$. Then for any $\mathcal{I} = (T_1, T_2, \Lambda_1, \Lambda_2)$, some $\mathcal{J} = (\omega_1, \omega_2, \tau_1, \tau_2)$ and some $V_0 > 0$, where*

$$0 < T_1 < T_2\,, \quad 0 < \Lambda_1 < \Lambda_2\,, \quad 0 < \omega_1 < \omega_2\,, \quad 0 < \tau_1 < \tau_2\,,$$

*a Gaussian random Beltrami field $u$ satisfies*

$$\liminf_{R\to\infty} \frac{N_u^{\mathrm{h}}(R)}{|B_R|} \geqslant \nu^{\mathrm{h}}\,,$$
$$\liminf_{R\to\infty} \frac{N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)}{|B_R|} \geqslant \nu^{\mathrm{t}}([\mathcal{T}], \mathcal{J}, V_0)\,,$$
$$\liminf_{R\to\infty} \frac{N_u^{\mathrm{o}}(R; [\gamma], \mathcal{I})}{|B_R|} \geqslant \nu^{\mathrm{o}}([\gamma], \mathcal{I})$$

*with probability 1, with constants that are all positive. In particular, the topological entropy of $u$ is positive almost surely, and*

$$\liminf_{R\to\infty} \frac{V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J})}{|B_R|} \geqslant V_0\, \nu^{\mathrm{t}}([\mathcal{T}], \mathcal{J}, V_0)\,,$$

*with probability 1.*

*Proof.* For the ease of notation, let us denote by $\Phi_R(u)$ the quantities $N_u^{\mathrm{h}}(R)$, $N_u^{\mathrm{o}}(R; [\gamma], \mathcal{I})$ and $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$, in each case. Horseshoes are closed, and so are the set of periodic orbits isotopic to $\gamma$ with parameters in $\mathcal{I}$ and the set of closed invariant solid tori of the kind counted by $N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0)$. Therefore, the lower bound for sets $\Gamma$ whose components are closed proved in Lemma 5.6.1 ensures that, for any $0 < r < R$,

$$\frac{\Phi_R(u)}{|B_R|} \geqslant \frac{1}{|B_R|} \int_{B_{R-r}} \frac{\Phi_r(\tau_y u)}{|B_r|}\, dy \geqslant \frac{1}{|B_R|} \int_{B_{R-r}} \frac{\Phi_r^m(\tau_y u)}{|B_r|}\, dy\,,$$

where for any large $m > 1$ we have defined the truncation

$$\Phi_r^m(w) := \min\{\Phi_r(w), m\}\,.$$

We recall that the translation operator is defined as $\tau_y u(\cdot) = u(\cdot + y)$.

As the truncated random variable $\Phi_r^m$ is in $L^1(C^k(\mathbb{R}^3, \mathbb{R}^3), \mu_u)$ for any $m$, one can consider the limit $R \to \infty$ and apply Proposition 5.3.4 to conclude that

$$\liminf_{R\to\infty} \frac{\Phi_R(u)}{|B_R|} \geqslant \liminf_{R\to\infty} \frac{|B_{R-r}|}{|B_R|} \fint_{B_{R-r}} \frac{\Phi_r^m(\tau_y u)}{|B_r|}\, dy = \frac{1}{|B_r|} \mathbb{E}\Phi_r^m$$

$\mu_u$-almost surely, for any $r$ and $m$. Corollaries 5.4.3, 5.4.6 and 5.5.3 imply that (for any $\mathcal{I}$ in the case of periodic orbits, for some $\mathcal{J}$ and some $V_0 > 0$ in the case of invariant tori, and unconditionally in the case of horseshoes), there exists some $r > 0$, some $\delta > 0$ and a Beltrami field $u_0$ such that

$$\Phi_r(w) \geqslant 1$$

for any divergence-free vector field $w \in C^k(\mathbb{R}^3, \mathbb{R}^3)$ with $\|w - u_0\|_{C^4(B_r)} < \delta$. As the random variable $\Phi_r$ is nonnegative, and the measure $\mu_u$ is supported on Beltrami fields (cf. Proposition 5.3.5), which are divergence-free, it is then immediate that, when picking the parameters $\mathcal{I}$, $\mathcal{J}$ and $V_0$ as above, one has for $k \geqslant 4$

$$\mathbb{E}\Phi_r^m \geqslant \mu_u\big(\{w \in C^k(\mathbb{R}^3, \mathbb{R}^3) : \|w - u_0\|_{C^k(B_r)} < \delta\}\big) =: \mathcal{M}(u_0, \delta) .$$

This is positive again by Proposition 5.3.5. So defining the constant, in each case, as

$$\nu := \frac{\mathcal{M}(u_0, \delta)}{|B_r|} > 0$$

the first part of the theorem follows.

Finally, the topological entropy of $u$ is positive almost surely because $u$ has a horseshoe with probability 1, see Proposition 5.5.1. The estimate for the growth of the volume of Diophantine invariant tori follows from the trivial lower bound

$$V_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}) > V_0\, N_u^{\mathrm{t}}(R; [\mathcal{T}], \mathcal{J}, V_0) .$$

□

**Remark 5.6.1.** A simple variation of the proof of Theorem 5.6.2 provides an analogous result for links. We recall that a link $\mathcal{L}$ is a finite set of pairwise disjoint closed curves in $\mathbb{R}^3$, which can be knotted and linked among them. More precisely, if $N^{\mathrm{l}}(R; [\mathcal{L}], \mathcal{I})$ is the number of unions of hyperbolic periodic orbits of $u$ that are contained in $B_R$, isotopic to the link $\mathcal{L}$, and whose periods and maximal Lyapunov exponents are in the intervals prescribed by $\mathcal{I}$, then

$$\liminf_{R \to \infty} \frac{N^{\mathrm{l}}(R; [\mathcal{L}], \mathcal{I})}{|B_R|} \geqslant \nu^{\mathrm{l}}([\mathcal{L}], \mathcal{I}) > 0 .$$

To apply the lower bound obtained in Lemma 5.6.1 to estimate the number of links, it is enough to transform each link into a connected set by joining its different components by closed arcs. The proof then goes exactly as in Theorem 5.6.2 upon noticing that analogs of Proposition 5.4.2 and Corollary 5.4.3 also hold for links (the proof easily carries over to this case).

*Proof of Corollary 5.1.3.* The corollary is now an immediate consequence of the fact that the number of isotopy classes of closed curves and embedded tori is countable. Indeed, by Theorem 5.1.2, with probability 1, a Gaussian random Beltrami field has infinitely many horseshoes, an infinite volume of ergodic invariant tori isotopic to a given embedded torus $\mathcal{T}$, and infinitely many periodic orbits isotopic to a given closed curve $\gamma$. Since the countable intersection of sets of probability 1 also has probability 1, the claim follows. □

### 5.6.3 Proof of Theorem 5.1.4

We are now ready to prove the asymptotics for the number of zeros of the Gaussian random Beltrami field $u$. Let us start by noticing that, almost surely, the zeros of $u$ are nondegenerate. This is because

$$\mu_u\big(\{w \in C^k(\mathbb{R}^3, \mathbb{R}^3) : \det \nabla w(x) = 0 \text{ and } w(x) = 0 \text{ for some } x \in \mathbb{R}^3\}\big) = 0 ,$$

which is a consequence of the boundedness of the probability density function (cf. Remark 5.3.3) and that $u$ is $C^\infty$ almost surely, see [AW09, Proposition 6.5]. Hence the intersection of the zero set

$$\mathcal{X}_w := \{x \in \mathbb{R}^3 : w(x) = 0\}$$

with a ball $B_R$ is a finite set of points almost surely. The implicit function theorem then implies that these zeros are robust under $C^1$-small perturbations, so that with probability 1, $\mathcal{N}(R; \mathcal{X}_v) \geqslant \mathcal{N}(R; \mathcal{X}_w)$ for any vector field $v$ that is close enough to $w$ in the $C^1$ norm. Summarizing, we have the following:

**Proposition 5.6.3.** *Almost surely, the functional $w \mapsto \mathcal{N}(R; \mathcal{X}_w)$ is lower semicontinuous in the $C^k$ compact open topology for vector fields, for any $k \geqslant 1$. Furthermore, $\mathcal{N}(R; \mathcal{X}_w) < \infty$ with probability 1.*

Since the variance $\mathbb{E}[u(x) \otimes u(x)]$ is the identity matrix by Corollary 5.3.3, the Kac–Rice formula [AW09, Proposition 6.2] then enables us to compute the expected value of the random variable

$$\Phi_r(w) := \frac{\mathcal{N}(r; \mathcal{X}_w)}{|B_r|} \tag{5.6.2}$$

as

$$\begin{aligned} \mathbb{E}\Phi_r &= \fint_{B_r} \mathbb{E}\{|\det \nabla w(x)| : w(x) = 0\}\, \rho(0)\, dx \\ &= (2\pi)^{-\frac{3}{2}} \mathbb{E}\{|\det \nabla w(x)| : w(x) = 0\}\,. \end{aligned} \tag{5.6.3}$$

Here we have used that the above conditional expectation is independent of the point $x \in \mathbb{R}^3$ by the translational invariance of the probability measure. We recall that the probability density function $\rho(y) := (2\pi)^{-\frac{3}{2}}\, e^{-\frac{1}{2}|y|^2}$ was introduced in Remark 5.3.3.

To compute the above conditional expectation value, one can argue as follows:

**Lemma 5.6.4.** *For any $x \in \mathbb{R}^3$,*

$$\mathbb{E}\{|\det \nabla u(x)| : u(x) = 0\} = (2\pi)^{\frac{3}{2}} \nu^z\,,$$

*where the constant $\nu^z$ is given by (5.1.3).*

*Proof.* Let us first reduce the computation of the conditional expectation to that of an ordinary expectation by introducing a new random variable $\zeta$. Just like $\nabla u(x)$, this new variable takes values in the space of $3 \times 3$ matrices, which we will identify with $\mathbb{R}^9$ by labeling the matrix entries as

$$\zeta =: \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_4 & \zeta_5 & \zeta_6 \\ \zeta_7 & \zeta_8 & \zeta_9 \end{pmatrix}. \tag{5.6.4}$$

This variable is defined as

$$\zeta := \nabla u(x) - B u(x)\,, \tag{5.6.5}$$

where the linear operator $B$ (which is a $9 \times 3$ matrix if we identify $\nabla u(x)$ with a vector in $\mathbb{R}^9$) is chosen so that the covariance matrix of $u(x)$ and $\zeta$ is 0:

$$B := \mathbb{E}(\nabla u(x) \otimes u(x)) \left[\mathbb{E}(u(x) \otimes u(x))\right]^{-1} = \mathbb{E}(\nabla u(x) \otimes u(x)) .$$

Here we have used that the second matrix is in fact the identity by Corollary 5.3.3. An easy computation shows that then

$$\mathbb{E}(\zeta \otimes u(x)) = 0 ;$$

as $u(x)$ and $\zeta$ are Gaussian vectors with zero mean, this condition ensures that they are independent random variables. Therefore, we can use the identity (5.6.5) to write the conditional expectation as

$$\mathbb{E}\{|\det \nabla u(x)| : u(x) = 0\} = \mathbb{E}\{|\det[\zeta + Bu(x)]| : u(x) = 0\} = \mathbb{E}|\det \zeta| .$$

Our next goal is to compute the covariance matrix of $\zeta$ in closed form, which will enable us to find the expectation of $|\det \zeta|$. By definition,

$$\begin{aligned}\mathbb{E}(\zeta \otimes \zeta) &= \mathbb{E}[(\nabla u(x) - Bu(x)) \otimes (\nabla u(x) - Bu(x))] \\ &= \mathbb{E}[\nabla u(x) \otimes \nabla u(x)] - \mathbb{E}[\nabla u(x) \otimes u(x)]\, \mathbb{E}[u(x) \otimes \nabla u(x)] .\end{aligned}$$

The basic observation now is that, for any Hermitian polynomials in three variables $q(\xi)$ and $q'(\xi)$, the argument that we used to establish the formula (5.3.3) and Corollary 5.3.3 shows that

$$\begin{aligned}\mathbb{E}[(q(D)u_j(x))\,(q'(D)u_k(x))] &= \mathbb{E}[q(D_x)u_j(x)\,\overline{q'(D_y)u_k(y)}]|_{y=x} \\ &= \int_{\mathbb{S}} q(\xi)\, q'(-\xi)\, p_j(\xi)\, \overline{p_k(\xi)}\, e^{i\xi\cdot(x-y)}\, d\sigma(\xi)\bigg|_{y=x} \\ &= \int_{\mathbb{S}} q(\xi)\, q'(-\xi)\, p_j(\xi)\, \overline{p_k(\xi)}\, d\sigma(\xi) .\end{aligned}$$

Here we have used that $q'(D)u_k$ is real-valued because $q'$ is Hermitian. As all the matrix integrals in the calculation of $\mathbb{E}(\zeta \otimes \zeta)$ are of this form with $q(\xi) = i\xi$ or 1, the computation again boils down to evaluating integrals of the form $\int_{\mathbb{S}} \xi^\alpha\, d\sigma(\xi)$, which can be computed using the formula (5.3.4).

Tedious but straightforward computations then yield the following explicit formula for the covariance matrix of $\zeta$:

$$\Sigma := \mathbb{E}(\zeta \otimes \zeta) = \begin{pmatrix} \frac{5}{21} & 0 & 0 & 0 & -\frac{5}{42} & 0 & 0 & 0 & -\frac{5}{42} \\ 0 & \frac{11}{84} & 0 & \frac{11}{84} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{84} & 0 & 0 & 0 & \frac{11}{84} & 0 & 0 \\ 0 & \frac{11}{84} & 0 & \frac{11}{84} & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{42} & 0 & 0 & 0 & \frac{3}{14} & 0 & 0 & 0 & -\frac{2}{21} \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{84} & 0 & \frac{13}{84} & 0 \\ 0 & 0 & \frac{11}{84} & 0 & 0 & 0 & \frac{11}{84} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{84} & 0 & \frac{13}{84} & 0 \\ -\frac{5}{42} & 0 & 0 & 0 & -\frac{2}{21} & 0 & 0 & 0 & \frac{3}{14} \end{pmatrix}$$

Note that this matrix is not invertible: it has rank 5, and an orthogonal basis for the (4-dimensional) kernel is

$$\{e_1 + e_5 + e_9,\ e_2 - e_4,\ e_3 - e_7,\ e_6 - e_8\}\,,$$

where $\{e_j\}_{j=1}^9$ denotes the canonical basis of $\mathbb{R}^9$. As we are dealing with Gaussian vectors, this is equivalent to the assertion that

$$\zeta_1 + \zeta_5 + \zeta_9 = 0\,, \quad \zeta_2 = \zeta_4\,, \quad \zeta_3 = \zeta_7\,, \quad \zeta_6 = \zeta_8 \tag{5.6.6}$$

almost surely (which amounts to saying that $\zeta$ is a traceless symmetric matrix). Notice that these equations define a 5-dimensional subspace orthogonal to the kernel of $\Sigma$. The remaining random variables $\zeta' := (\zeta_1, \zeta_2, \zeta_3, \zeta_5, \zeta_6)$ are independent Gaussians with zero mean and covariance matrix

$$\Sigma' := \mathbb{E}(\zeta' \otimes \zeta') = \begin{pmatrix} \frac{5}{21} & 0 & 0 & -\frac{5}{42} & 0 \\ 0 & \frac{11}{84} & 0 & 0 & 0 \\ 0 & 0 & \frac{11}{84} & 0 & 0 \\ -\frac{5}{42} & 0 & 0 & \frac{3}{14} & 0 \\ 0 & 0 & 0 & 0 & \frac{13}{84} \end{pmatrix}$$

By construction, $\Sigma'$ is an invertible matrix, so we can immediately write down a formula for the expectation value of $|\det \zeta|$:

$$\begin{aligned} \mathbb{E}|\det \zeta| &= (2\pi)^{-\frac{5}{2}} (\det \Sigma')^{-\frac{1}{2}} \int_{\mathbb{R}^5} \left| \det \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_2 & \zeta_5 & \zeta_6 \\ \zeta_3 & \zeta_6 & -\zeta_1 - \zeta_4 \end{pmatrix} \right| e^{-\frac{1}{2}\zeta' \cdot \Sigma'^{-1} \zeta'}\, d\zeta' \\ &= (2\pi)^{-\frac{5}{2}} (\det \Sigma')^{-\frac{1}{2}} \int_{\mathbb{R}^5} |Q(\zeta')|\, e^{-\frac{1}{2}\zeta' \cdot \Sigma'^{-1} \zeta'}\, d\zeta'\,, \end{aligned}$$

with the cubic polynomial $Q$ being defined as in (5.1.4). Since $\frac{1}{2}\zeta' \cdot \Sigma'^{-1}\zeta' = \widetilde{Q}(\zeta')$, where the quadratic polynomial $\widetilde{Q}$ was defined in (5.1.5), and

$$\det \Sigma' = \frac{5 \cdot 143^2}{2^8 \cdot 21^5}\,,$$

we therefore have

$$\mathbb{E}|\det \zeta| = (2\pi)^{\frac{3}{2}} \nu^Z\,.$$

The result then follows. □

**Remark 5.6.2.** If one keeps track of the connection between $\zeta$ and $\nabla u(x)$, it is not hard to see that the first condition $\zeta_1 + \zeta_5 + \zeta_9 = 0$ in (5.6.6) is equivalent to $\operatorname{div} u(x) = 0$, while the remaining three just mean that $\operatorname{curl} u(x) = u(x)$, at the points $x \in \mathbb{R}^3$ where $u(x) = 0$.

In particular, this shows that $\Phi_R \in L^1(C^k(\mathbb{R}^3, \mathbb{R}^3), \mu_u)$. For the ease of notation, let us define the ergodic mean operator

$$\mathcal{A}_R \Phi(w) := \frac{1}{|B_R|} \int_{B_R} \Phi(\tau_y w)\, dy\,.$$

Since $\mathcal{N}(R, \mathcal{X}_w)$ is finite almost surely, cf. Proposition 5.6.3, the sandwich estimate proved in Lemma 5.6.1 implies that, almost surely,

$$\frac{1}{|B_R|}\int_{B_{R-r}} \Phi_r(\tau_y w)\, dy \leqslant \Phi_R(w) \leqslant \frac{1}{|B_R|}\int_{B_{R+r}} \Phi_r(\tau_y w)\, dy$$

for any $0 < r < R$. Therefore, and using that $|B_{R\pm r}|/|B_R| = (1 \pm r/R)^3$, one has

$$|\Phi_R - \mathcal{A}_R\Phi_r| \leqslant \left|\left(1+\frac{r}{R}\right)^3 \mathcal{A}_{R+r}\Phi_r - \mathcal{A}_R\Phi_r\right| + \left|\left(1-\frac{r}{R}\right)^3 \mathcal{A}_{R-r}\Phi_r - \mathcal{A}_R\Phi_r\right|.$$

For fixed $r$, Equation (5.6.3) and Proposition 5.3.4 ensure that

$$\mathcal{A}_R\Phi_r \xrightarrow[\text{a.s.}]{L^1} \mathbb{E}\Phi_r = \nu^z \tag{5.6.7}$$

as $R \to \infty$; also, note that the limit (which is independent of $r$) has been computed in Lemma 5.6.4 above.

Therefore, if we let $R \to \infty$ while $r$ is held fixed, the RHS of the estimate before Equation (5.6.7) tends to 0 $\mu_u$-almost surely and in $L^1(\mu_u)$, so that

$$\Phi_R - \mathcal{A}_R\Phi_r \xrightarrow[\text{a.s.}]{L^1} 0$$

as $R \to \infty$. As $\mathcal{A}_R\Phi_r \xrightarrow[\text{a.s.}]{L^1} \nu^z$ by (5.6.7), Theorem 5.1.4 is proven.

## 5.7 The Gaussian ensemble of Beltrami fields on the torus

### 5.7.1 Gaussian random Beltrami fields on the torus

As introduced in Section 5.1.3, a Beltrami field on the flat 3-torus $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ (or, equivalently, on the cube of $\mathbb{R}^3$ of side length $2\pi$ with periodic boundary conditions) is a vector field on $\mathbb{T}^3$ satisfying the equation

$$\operatorname{curl} v = \lambda v$$

for some real number $\lambda \neq 0$. To put it differently, Beltrami fields on the torus are the eigenfields of the curl operator. It is easy to see that such an eigenfield is divergence-free and has zero mean, that is, $\int_{\mathbb{T}^3} v\, dx = 0$.

Since $\Delta v + \lambda^2 v = 0$, it is well-known (see e.g. [ELPS17]) that the spectrum of the curl operator on the 3-torus consists of the numbers of the form $\lambda = \pm|k|$ for some vector with integer coefficients $k \in \mathbb{Z}^3$. For concreteness, we will henceforth assume that $\lambda > 0$; the case of negative frequencies is completely analogous. Since $k$ has integer coefficients, one can label the positive eigenvalues of curl by a positive integer $L$ such that $\lambda_L = L^{1/2}$. Let us define

$$\mathcal{Z}_L := \{k \in \mathbb{Z}^3 : |k|^2 = L\}$$

and note that the set $\mathcal{Z}_L$ is invariant under reflections (i.e., $-k \in \mathcal{Z}_L$ if $k \in \mathcal{Z}_L$).

The Beltrami fields corresponding to the eigenvalue $\lambda_L$ must be of the form

$$v = \sum_{k \in \mathcal{Z}_L} V_k \, e^{ik\cdot x} \,, \tag{5.7.1}$$

for some $V_k \in \mathbb{C}^3$. Conversely, this expression defines a Beltrami field with frequency $\lambda_L$ if and only if $V_k = \overline{V_{-k}}$ (which ensures that $v$ is real valued) and

$$\frac{ik}{L^{1/2}} \times V_k = V_k \,.$$

Since $|k| = L^{1/2}$, we infer from the proof of Proposition 5.2.1 that the vector $V_k$ must be of the form

$$V_k = \alpha_k \, p(k/L^{1/2}) \tag{5.7.2}$$

unless $k = (\pm L^{1/2}, 0, 0)$. Here $\alpha_k \in \mathbb{C}$ is an arbitrary complex number and the Hermitian vector field $p(\xi)$ was defined in (5.2.4).

The multiplicity of the eigenvalue $\lambda_L$ is given by the cardinality $d_L := \#\mathcal{Z}_L$. By Legendre's three-square theorem, $\mathcal{Z}_L$ is nonempty (and therefore $\lambda_L$ is an eigenvalue of the curl operator) if and only if $L$ is not of the form $4^a(8b+7)$ for nonnegative integers $a$ and $b$.

Based on the formulas (5.7.1)-(5.7.2), we are now ready to define a Gaussian random Beltrami field on the torus with frequency $\lambda_L$ as

$$u^L(x) := \left(\frac{2\pi}{d_L}\right)^{1/2} \sum_{k \in \mathcal{Z}_L} a_k^L \, p(k/L^{1/2}) \, e^{ik\cdot x} \,, \tag{5.7.3}$$

where the real and imaginary parts of the complex-valued random variable $a_k^L$ are standard Gaussian variables. We also assume that these random variables are independent except for the constraint $a_k^L = \overline{a_{-k}^L}$. The inessential normalization factor $(2\pi/d_L)^{1/2}$ has been introduced for later convenience.

Note that $u^L(x)$ is a smooth $\mathbb{R}^3$-valued function of the variable $x$, so it induces a Gaussian probability measure $\mu^L$ on the space of $C^k$-smooth vector fields on the torus, $C^k(\mathbb{T}^3, \mathbb{R}^3)$. As before, we will always assume that $k \geqslant 4$ to apply results from KAM theory. We will also employ the rescaled Gaussian random field

$$u^{L,z}(x) := u^L\left(z + \frac{x}{L^{1/2}}\right)$$

for any fixed point $z \in \mathbb{T}^3$.

### 5.7.2 Estimates for the rescaled covariance matrix

In what follows, we will restrict our attention to the positive integers $L$, which we will henceforth call *admissible*, that are not congruent with 0, 4 or 7 modulo 8. When $L$ is congruent with 7 modulo 8, Legendre's three-square theorem immediately implies that $\mathcal{Z}_L$ is empty. The reason to rule out numbers congruent with 0 or 4 modulo 8 is more subtle: a deep theorem of Duke [Duk88], which addresses a question raised by Linnik, ensures that the set $\mathcal{Z}_L/L^{1/2}$ becomes uniformly distributed on the unit sphere as $L \to \infty$ through integers that are congruent to 1, 2, 3, 5 or 6 modulo 8. This ensures that

$$\frac{4\pi}{d_L} \sum_{k \in \mathcal{Z}_L} \phi(k/L^{1/2}) \to \int_{\mathbb{S}} \phi(\xi)\, d\sigma(\xi) \tag{5.7.4}$$

as $L \to \infty$ through admissible values, for any continuous function $\phi$ on $\mathbb{S}$. A particular case is when $L$ goes to infinity through squares of odd values, that is, when $L = (2m+1)^2$ and $m \to \infty$.

The covariance kernel of the Gaussian random variable $u^L$ is the matrix-valued function

$$\kappa^L(x,y) := \mathbb{E}^L[u^L(x) \otimes u^L(y)]\,.$$

Following Nazarov and Sodin [NS16], we will be most interested in the covariance kernel of the rescaled field $u^{L,z}$ at a point $z \in \mathbb{T}^3$, which is given by

$$\kappa^{L,z}(x,y) = \mathbb{E}^L\left[u^L\left(z + \frac{x}{L^{1/2}}\right) \otimes u^L\left(z + \frac{y}{L^{1/2}}\right)\right].$$

The following proposition ensures that, for large admissible frequencies $L$, the rescaled covariance kernel, and suitable generalizations thereof, tend to those of a Gaussian random Beltrami field on $\mathbb{R}^3$, $\kappa(x,y)$, defined in (5.3.2):

**Proposition 5.7.1.** *For any $z \in \mathbb{T}^3$, the rescaled covariance kernel $\kappa^{L,z}(x,y)$ has the following properties:*

*(i) It is invariant under translations and independent of $z$. That is, there exists some function $\varkappa^L$ such that*

$$\kappa^{L,z}(x,y) = \varkappa^L(x-y)\,.$$

*(ii) Given any compact set $K \subset \mathbb{R}^3$, the covariance kernel satisfies*

$$\kappa^{L,z}(x,y) \to \kappa(x,y)$$

*in $C^s(K \times K)$ as $L \to \infty$ through admissible values.*

*Proof.* Let $\alpha$, $\beta$ be any multiindices, and recall the operator $D = -i\nabla$ introduced in Section 5.3. By definition, and using the fact that $u^L$ is real,

$$D_x^\alpha D_y^\beta \kappa^{L,z}(x,y) = \mathbb{E}^L\left[D_x^\alpha u^L\left(z + \frac{x}{L^{1/2}}\right) \otimes D_y^\beta u^L\left(z + \frac{y}{L^{1/2}}\right)\right]$$
$$= \mathbb{E}^L\left[D_x^\alpha u^L\left(z + \frac{x}{L^{1/2}}\right) \otimes \overline{D_y^\beta u^L\left(z + \frac{y}{L^{1/2}}\right)}\right]$$
$$= \frac{2\pi}{d_L} \sum_{k \in \mathcal{Z}_L} \sum_{k' \in \mathcal{Z}_L} \mathbb{E}^L(a_k^L \overline{a_{k'}^L})\, p\left(\frac{k}{L^{1/2}}\right) \otimes \overline{p\left(\frac{k'}{L^{1/2}}\right)} \left(\frac{k}{L^{1/2}}\right)^\alpha \left(\frac{-k'}{L^{1/2}}\right)^\beta e^{ik\cdot(z+\frac{x}{L^{1/2}}) - ik'\cdot(z+\frac{y}{L^{1/2}})}\,.$$

The independence properties of the Gaussian variables $a_k^L$ (which have zero mean) imply that $\mathbb{E}^L(a_k^L \overline{a_{k'}^L}) = 0$ if $k' \notin \{k, -k\}$. When $k' = k$ one has

$$\mathbb{E}^L[|a_k^L|^2] = \mathbb{E}^L[(\operatorname{Re} a_k^L)^2] + \mathbb{E}^L[(\operatorname{Im} a_k^L)^2] = 2\,,$$

and when $k' = -k$,

$$\mathbb{E}^L[(a_k^L)^2] = \mathbb{E}^L[(\operatorname{Re} a_k^L)^2] - \mathbb{E}^L[(\operatorname{Im} a_k^L)^2] + 2i\,\mathbb{E}^L[(\operatorname{Re} a_k^L)(\operatorname{Im} a_k^L)] = 0\,.$$

Therefore, $\mathbb{E}^L(a_k^L \overline{a_{k'}^L}) = 2\delta_{kk'}$ and we obtain

$$D_x^\alpha D_y^\beta \kappa^{L,z}(x,y) = \frac{4\pi}{d_L} \sum_{k\in\mathcal{Z}_L} p\left(\frac{k}{L^{1/2}}\right) \otimes \overline{p\left(\frac{k}{L^{1/2}}\right)} \left(\frac{k}{L^{1/2}}\right)^\alpha \left(-\frac{k}{L^{1/2}}\right)^\beta e^{ik\cdot(x-y)/L^{1/2}}.$$

In particular, this formula shows that $\kappa^{L,z}(x,y)$ is independent of $z$ and translation-invariant.

Using now the fact that $\mathcal{Z}_L$ becomes uniformly distributed on $\mathbb{S}$ as $L \to \infty$ through admissible values, we obtain via Equation (5.7.4) that

$$\begin{aligned} D_x^\alpha D_y^\beta \kappa^{L,z}(x,y) &\to \int_{\mathbb{S}} \xi^\alpha(-\xi)^\beta\, p(\xi) \otimes \overline{p(\xi)}\, e^{i\xi\cdot(x-y)}\, d\sigma(\xi) \\ &= D_x^\alpha D_y^\beta \int_{\mathbb{S}} p(\xi) \otimes \overline{p(\xi)}\, e^{i\xi\cdot(x-y)}\, d\sigma(\xi)\,. \end{aligned}$$

By Proposition 5.3.2, the RHS equals $D_x^\alpha D_y^\beta \kappa(x,y)$, so the result follows. □

### 5.7.3 A convergence result for probability measures

We shall next present a result showing that the probability measure defined by the rescaled field $u^{L,z}$ converges, as $L \to \infty$, to that defined by the Gaussian random Beltrami field on $\mathbb{R}^3$, $u$, on compact sets of $\mathbb{R}^3$:

**Lemma 5.7.2.** *Fix some $R > 0$ and denote by $\mu_R^{L,z}$ and $\mu_{u,R}$, respectively, the probability measures on $C^k(B_R, \mathbb{R}^3)$ defined by the Gaussian random fields $u^{L,z}$ and $u$. Then the measures $\mu_R^{L,z}$ converge weakly to $\mu_{u,R}$ as $L \to \infty$ through the admissible integers.*

*Proof.* Let us start by noting that all the finite dimensional distributions of the fields $u^{L,z}$ converge to those of $u$ as $L \to \infty$. Specifically, consider any finite number of points $x^1, \dots, x^n \in \mathbb{R}^3$, any indices $j^1, \dots, j^n \in \{1,2,3\}$, and any multiindices with $|\alpha^j| \leqslant k$. Then it is not hard to see that the Gaussian vectors of zero expectation

$$(\partial^{\alpha^1} u_{j^1}^{L,z}(x^1), \dots, \partial^{\alpha^n} u_{j^n}^{L,z}(x^n)) \in \mathbb{R}^n$$

converge in distribution to the Gaussian vector

$$(\partial^{\alpha^1} u_{j^1}(x^1), \dots, \partial^{\alpha^n} u_{j^n}(x^n)) \tag{5.7.5}$$

as $L \to \infty$. This follows from the fact that their probability density functions are completely determined by the $n \times n$ variance matrix

$$\Sigma^L := \left(\partial_x^{\alpha^l} \partial_y^{\alpha^m} \kappa_{j^l j^m}^{L,z}(x,y)|_{(x,y)=(x^l,x^m)}\right)_{1\leqslant l,m\leqslant n},$$

which converges to $\Sigma := (\partial_x^{\alpha^l} \partial_y^{\alpha^m} \kappa_{j^l j^m}(x,y)|_{(x,y)=(x^l,x^m)})$ as $L \to \infty$ by Proposition 5.7.1. The latter, of course, is the covariance matrix of the Gaussian vector (5.7.5).

It is well known that this convergence of arbitrary Gaussian vectors is not enough to conclude that $\mu_R^{L,z}$ converges weakly to $\mu_{u,R}$. However, notice that, for any integer $s \geqslant 0$, the mean of the $H^s$-norm of $u^{L,z}$ is uniformly bounded:

$$\begin{aligned}\mathbb{E}^{L,z}\|w\|^2_{H^s(B_R)} &= \sum_{|\alpha|\leqslant s} \mathbb{E} \int_{B_R} |D^\alpha u^{L,z}(x)|^2\, dx \\ &= \sum_{|\alpha|\leqslant s} \int_{B_R} \operatorname{Tr}\left( D_x^\alpha D_y^\alpha \kappa^{L,z}(x,y)\big|_{y=x}\right) dx \\ &\xrightarrow[L\to\infty]{} \sum_{|\alpha|\leqslant s} \int_{B_R} \operatorname{Tr}\left( D_x^\alpha D_y^\alpha \kappa(x,y)\big|_{y=x}\right) dx < M_{s,R}\,.\end{aligned}$$

To pass to the last line, we have used Proposition 5.7.1 once more. As the constant $M_{s,R}$ is independent of $L$, Sobolev's inequality ensures that

$$\sup_L \mathbb{E}^{L,z}\|w\|^2_{C^{k+1}(B_R)} \leqslant C \sup_L \mathbb{E}^{L,z}\|w\|^2_{H^{k+3}(B_R)} < M$$

for some constant $M$ that only depends on $R$. For any $\epsilon > 0$, this implies that for all admissible $L$ large enough

$$\mu_R^{L,z}\big(\{w \in C^k(B_R, \mathbb{R}^3) : \|w\|^2_{C^{k+1}(B_R)} > M/\epsilon\}\big) < \epsilon\,.$$

As the closure of the set $\{w \in C^k(B_R, \mathbb{R}^3) : \|w\|^2_{C^{k+1}(B_R)} \leqslant M/\epsilon\}$ is compact by the Arzelà–Ascoli theorem, we conclude that the sequence of probability measures $\mu_R^{L,z}$ is tight. Therefore, a straightforward extension to jet spaces of the classical results about the convergence of probability measures on the space of continuous functions [Bil13, Theorem 7.1], carried out in [Wil86], permits to conclude that $\mu_R^{L,z}$ indeed converges weakly to $\mu_{u,R}$ as $L \to \infty$. The lemma is then proven. □

### 5.7.4 Proof of Theorem 5.1.5

We are now ready to prove our asymptotic estimates for high-frequency Beltrami fields on the torus. The basic idea is that, by the definition of the rescaling,

$$\mu^L\big(\{w \in C^k(\mathbb{T}^3, \mathbb{R}^3) : N_w^{\mathrm{h}} > m\}\big) \geqslant \mu_R^{L,z}\big(\{w \in C^k(B_R, \mathbb{R}^3) : N_w^{\mathrm{h}}(r) > m\}\big)$$

provided that $r < R < L^{1/2}$: this just means that the number of horseshoes that $u^L$ has in the whole torus is certainly not less than those that are contained in a ball centered at any given point $z \in \mathbb{T}^3$ of radius $r/L^{1/2} < 1$. The same is clearly true as well when one counts invariant solid tori, periodic orbits or zeros instead.

For the ease of notation, let us denote by $\Phi_r(w)$ the quantity $N_w^{\mathrm{h}}(r)$, $N_w^{\mathrm{t}}(r; [\mathcal{T}], \mathcal{J}, V_0)$, $N_w^{\mathrm{o}}(r; [\gamma], \mathcal{I})$ or $N_w^{\mathrm{z}}(r)$ (that is, the number of nondegenerate zeros of $w$ in $B_r$), in each case. See Sections 5.4 and 5.5 for precise definitions. We recall that $N_w^{\mathrm{z}}(r) = \mathcal{N}(r; \mathcal{X}_w)$ with probability 1, cf. Section 5.6.3. Theorems 5.6.2 (for periodic orbits, invariant tori and horseshoes) and 5.1.4 (for zeros) ensure that, given any $m_1 > 0$, any $\delta_1 > 0$, any closed curve $\gamma$ and any embedded torus $\mathcal{T}$, one can find some parameters $\mathcal{I}$, $\mathcal{J}$, $V_0$ and $r > 0$ such that

$$\mu_u\big(\{w \in C^k(\mathbb{R}^3, \mathbb{R}^3) : \Phi_r(w) > m_1\}\big) > 1 - \delta_1\,.$$

Of course, here we are simply using that the volume $|B_r|$, which appears in the statements of Theorems 5.6.2 and 5.1.4 but not here, can be made arbitrarily large by taking a large $r$.

Let us now fix some $R > r$ and some point $z \in \mathbb{T}^3$. We showed in Propositions 5.4.1, 5.4.4, 5.5.1 and 5.6.3 that the functionals that we are now denoting by $\Phi_r$ are lower semicontinuous on the space $C^k(\mathbb{R}^3, \mathbb{R}^3)$ of divergence-free fields for $k \geqslant 4$. This implies that the set

$$\Omega_{r,R,m_1} := \{w \in C^k(B_R, \mathbb{R}^3) : \Phi_r(w) > m_1\}$$

is open in $C^k(B_R, \mathbb{R}^3)$. Lemma 5.7.2 ensures that the measure $\mu_R^{L,z}$ converges weakly to $\mu_{u,R}$ as $L \to \infty$ through the admissible integers. As the set $\Omega_{r,R,m_1}$ is open, this is well known to imply (see e.g. [Bil13, Theorem 2.1.iv]) that

$$\begin{aligned}\liminf_{L\to\infty} \mu_R^{L,z}(\Omega_{r,R,m_1}) &\geqslant \mu_{u,R}(\Omega_{r,R,m_1}) \\ &= \mu_u\big(\{w \in C^k(\mathbb{R}^3, \mathbb{R}^3) : \Phi_r(w) > m_1\}\big) \\ &> 1 - \delta_1 .\end{aligned}$$

We observe that $\delta_1 > 0$ can be taken arbitrarily small if $r$ is large enough (and $r/L^{1/2} < R/L^{1/2} < 1$). Now, for any $A \geqslant 1$ and $L$ large enough, we can take $A$ pairwise disjoint balls in $\mathbb{T}^3$ of radius $r/L^{1/2} < A^{-1/3}$ centered at points $\{z^a\}_{a=1}^A \subset \mathbb{T}^3$. Setting $m := Am_1$, the previous analysis, which is independent of the point $z$, readily implies that

$$\mu^L\big(\{w \in C^k(\mathbb{T}^3, \mathbb{R}^3) : N_w^{\mathrm{X},\mathrm{e}} > m\}\big) \geqslant 1 - 2A\delta_1 > 1 - \delta ,$$

where the superscript X stands for h, t, o or z, thus proving the part of the statement concerning the number of approximately equidistributed horseshoes, invariant tori isotopic to $\mathcal{T}$, periodic orbits isotopic to $\gamma$ or zeros. In fact, concerning invariant tori, we observe that obviously $N_w^{\mathrm{t}}(r; [\mathcal{T}]) = \infty$ if $N_w^{\mathrm{t}}(r; [\mathcal{T}], \mathcal{J}, V_0) \geqslant 1$. Since the previous estimate ensures that $N_w^{\mathrm{t}}(r; [\mathcal{T}], \mathcal{J}, V_0) > m_1$ with probability 1 as $L \to \infty$, we infer that the probability of having an infinite number of (Diophantine) invariant tori isotopic to $\mathcal{T}$ also tends to 1 as $L \to \infty$ through the admissible integers. However this does not provide any information about the expected volume of the invariant tori.

The result about the topological entropy follows from the following observation. If we denote by $\phi_t^L$ the time-$t$ flow of the Beltrami field $u^L(z + \cdot)$, and by $\phi_t$ the flow of the rescaled field $u^{L,z}$, it is evident that

$$\phi_t^L = \frac{1}{L^{1/2}} \phi_{L^{1/2} t} .$$

Then, the topological entropy $h_{\mathrm{top}}(u^L)$, which is defined as the entropy of its time-1 flow, satisfies

$$h_{\mathrm{top}}(u^L) = h_{\mathrm{top}}(\phi_1^L) = h_{\mathrm{top}}\Big(\frac{1}{L^{1/2}} \phi_{L^{1/2}}\Big) = h_{\mathrm{top}}(\phi_{L^{1/2}}) = L^{1/2} h_{\mathrm{top}}(\phi_1) \tag{5.7.6}$$

$$= L^{1/2} h_{\mathrm{top}}(u^{L,z}) . \tag{5.7.7}$$

In the third equality we have used that the topological entropy does not depend on the space scale (or equivalently, on the metric), and in the fourth equality we have used Abramov's well-known formula (see e.g. [GM10]). Since the rescaled field has a horseshoe in a ball of radius $r$ with probability 1 as $L \to \infty$, and a horseshoe has positive topological entropy, say larger than some constant $\nu_*^{\mathrm{h}}$ (see Proposition 5.5.1), Equation (5.7.6) implies that the topological entropy of $u^L$ is at least $\nu_*^{\mathrm{h}} L^{1/2}$.

Finally, we prove the statement about the expected values. As above, we use the functional $\Phi_r(w)$ to denote the number of different objects (horseshoes, solid tori or periodic orbits). The case of zeros will be considered later. Note that, since $\Phi_r$ is lower semicontinuous, and $\mu^{L,z}$ converges weakly to $\mu_u$ as $L \to \infty$ by Lemma 5.7.2, it is standard that [Bil13, Exercise 2.6]

$$\liminf_{L\to\infty} \mathbb{E}^{L,z} \frac{\Phi_r}{|B_r|} \geqslant \mathbb{E} \frac{\Phi_r}{|B_r|} \geqslant \eta > 0\,,$$

where we have picked some fixed, large enough $r$. Here we have used the asymptotics in $\mathbb{R}^3$, given by Theorem 5.6.2, to infer that the last expectation is positive if $r$ is large. Notice that the constant $\eta$ depends on $[\gamma], [\mathcal{T}], \mathcal{I}$ or $\mathcal{J}$ depending on the functional the we are considering, but we shall not write this dependence explicitly. Furthermore, as the distribution of the measure $\mu_R^{L,z}$ is in fact independent of $z$ by Proposition 5.7.1, this ensures that there is some $L_0$ independent of $z$ such that

$$\mathbb{E}^{L,z} \frac{\Phi_r}{|B_r|} > \frac{\eta}{2}$$

for all admissible $L > L_0$ and all $z \in \mathbb{T}^3$.

Now, given any admissible $L > L_0$, it is standard that we can cover the torus $\mathbb{T}^3$ by balls $\{B_{r_L}(z^a) : 1 \leqslant a \leqslant A_L\}$ of radius $r_L := 2r/L^{1/2}$ centered at $z^a \in \mathbb{T}^3$ such that the smaller balls $B_{r_L/2}(z^a)$ are pairwise disjoint. This implies that $A_L \geqslant c_r L^{\frac{3}{2}}$ for some dimensional constant $c_r$. The expected value of, say, the number of horseshoes of $u^L$ in $\mathbb{T}^3$ can then be controlled as follows, for any admissible $L > L_0$:

$$\begin{aligned}\frac{\mathbb{E}^L N^{\mathrm{h}}}{L^{3/2}} &\geqslant \sum_{a=1}^{A_L} \frac{|B_r|}{L^{3/2}}\, \mathbb{E}^{L,z^a} \frac{\Phi_r}{|B_r|} \\ &\geqslant \frac{c_r |B_r| \eta}{2} > \nu_*\end{aligned}$$

for some positive constant $\nu_*$ independent of $L$. An analogous estimate holds for the expected value $\mathbb{E}^L N^{\mathrm{o}}([\gamma])$.

To estimate the volume of ergodic invariant tori isotopic to $\mathcal{T}$ we can proceed as follows. For any admissible $L > L_0$ we have:

$$\begin{aligned}\mathbb{E}^L V^{\mathrm{t}}([\mathcal{T}]) &\geqslant \sum_{a=1}^{A_L} |B_{r_L/2}|\, \mathbb{E}^{L,z^a} \frac{V^{\mathrm{t}}(r; [\mathcal{T}], \mathcal{J})}{|B_r|} \\ &\geqslant \sum_{a=1}^{A_L} |B_{r_L/2}|\, V_0\, \mathbb{E}^{L,z^a} \frac{\Phi_r}{|B_r|} \\ &\geqslant \frac{V_0 \eta}{2} \sum_{a=1}^{A_L} |B_{r_L/2}| > \nu_*^{\mathrm{t}}([\mathcal{T}])\end{aligned}$$

for some positive constant $\nu_*^{\mathrm{t}}([\mathcal{T}])$ independent of $L$. Here we have used that the balls $B_{r_L/2}(z^a)$ are pairwise disjoint and the sum of their volumes is, by construction, larger than $|\mathbb{T}^3|/8$.

Lastly, in the following lemma we consider the case of zeros:

**Lemma 5.7.3.** $\mathbb{E}^L(L^{-\frac{3}{2}} N^{z}_{u^L}) \to (2\pi)^3 \nu^z$ *as* $L \to \infty$ *through admissible values.*

*Proof.* Let us use the notation

$$Q_R := (-R\pi, R\pi) \times (-R\pi, R\pi) \times (-R\pi, R\pi)$$

for the open cube of side $2\pi R$ in $\mathbb{R}^3$ and call $N^{z,*}_{u^L}$ the number of zeros of $u^L$ (or rather of its periodic lift to $\mathbb{R}^3$) that are contained in $Q_1$. By Bulinskaya's lemma [AW09, Proposition 6.11], with probability 1 the zero set of $u^L$ is nondegenerate (and hence a finite set of points) and the lift of $u^L$ does not have any zeros on the boundary $\partial Q_1$. Therefore, for any positive integer $R$,

$$N^{z}_{u^L} = N^{z,*}_{u^L}$$

almost surely. In particular, both quantities have the same expectation.

Let us now take some small positive real $r$ and denote by $N^{z}_{u^L}(y, r)$ the number of zeros of $u^L$ (or rather of its lift to $\mathbb{R}^3$) that are contained in the ball $B_r(y)$. The argument we used to prove the estimate for $\mathcal{N}(R; \mathcal{X})$ in Lemma 5.6.1 (starting now from the number of zeros in $Q_1$ instead of in $B_R$) shows that

$$\int_{Q_{1-r}} \frac{N^{z}_{u^L}(z, r)}{|B_r|}\, dz \leqslant N^{z,*}_{u^L} \leqslant \int_{Q_{1+r}} \frac{N^{z}_{u^L}(z, r)}{|B_r|}\, dz\,.$$

Note now that

$$\int_{Q_{1\pm r}} \frac{N^{z}_{u^L}(z, r)}{|B_r|}\, dz = L^{\frac{3}{2}} \int_{Q_{1\pm r}} \frac{N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|}\, dz\,.$$

The expected value of this quantity is

$$\begin{aligned} \mathbb{E}^L \int_{Q_{1\pm r}} \frac{N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|}\, dz &= \int_{Q_{1\pm r}} \frac{\mathbb{E}^{L,z} N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|}\, dz \\ &= |Q_{1\pm r}| \frac{\mathbb{E}^{L,z} N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|}\,. \end{aligned}$$

To pass to the second line we have used that the expected value inside the integral is independent of the point $z$ by Proposition 5.7.1; in particular, this value is independent of the point $z$ one considers.

We can now argue just as in the case of $\mathbb{R}^3$, discussed in detail in Subsection 5.6.3, so we will just sketch the arguments and refer to that subsection for the notation. The Kac–Rice formula ensures

$$\frac{\mathbb{E}^{L,z} N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|} = (2\pi)^{-\frac{3}{2}} \mathbb{E}^{L,z}\big(\{|\det \nabla u^{L,z}(0)| : u^{L,z}(0) = 0\}\big)\,,$$

and this conditional expectation can be transformed into an unconditional one just as in the proof of Lemma 5.6.4:

$$\begin{aligned}\frac{\mathbb{E}^{L,z} N^{z}_{u^{L,z}}(rL^{1/2})}{|B_{rL^{1/2}}|} &= (2\pi)^{-3/2}\mathbb{E}^{L,z}(|\det \zeta^{L,z}|)\\ &= \frac{(2\pi)^{-3/2}}{(2\pi)^{5/2}(\det \Sigma'^{L,z})^{1/2}} \int_{\mathbb{R}^5} Q^{L,z}(\zeta')\, e^{-\frac{1}{2}\zeta'\cdot(\Sigma'^{L,z})^{-1}\zeta'}\, d\zeta'\\ &=: \nu^{z,L,z}\,.\end{aligned}$$

The fact that the covariance matrix of $u^{L,z}$ converges to that of $u$ as $L \to \infty$ by Proposition 5.7.1 implies that

$$\lim_{L\to\infty} \nu^{z,L,z} = \nu^{z}\,.$$

Hence, writing the aforementioned sandwich estimate as

$$|Q_{1-r}|\nu^{z,L,z} \leqslant \frac{\mathbb{E}^{L} N^{z}_{u^L}}{L^{3/2}} \leqslant |Q_{1+r}|\nu^{z,L,z}$$

and letting $L \to \infty$ and then $r \to 0$, we infer that

$$\lim_{L\to\infty} \frac{\mathbb{E}^{L} N^{z}_{u^L}}{L^{3/2}} = |Q_1|\nu^{z} = (2\pi)^3\nu^{z}\,.$$

The lemma follows. □

Theorem 5.1.5 is then proven.

## APPENDICES

## 5.A Fourier-theoretic characterization of Beltrami fields

For the benefit of the reader, in this appendix we describe what polynomially bounded Beltrami fields look like in Fourier space. As Beltrami fields are a particular class of vector-valued monochromatic waves, it is convenient to start the discussion by considering polynomially bounded solutions to the Helmholtz equation

$$\Delta F + F = 0\,.$$

As before, we consider the case of monochromatic waves on $\mathbb{R}^3$, but the analysis applies essentially verbatim to any other dimension. The Fourier transform of this equation shows that

$$(1-|\xi|^2)\widehat{F}(\xi) = 0\,,$$

so the support of $\widehat{F}$ must be contained in the unit sphere, $\mathbb{S}$. In spherical coordinates $\rho := |\xi| \in \mathbb{R}^+$ and $\omega := \xi/|\xi| \in \mathbb{S}$, it is standard that this is equivalent to saying that $\widehat{F}$ is a finite sum of the form

$$\widehat{F} = \sum_{n=1}^{N} F_n(\omega)\,\delta^{(n)}(\rho-1)\,.$$

Here $\delta^{(n)}$ is the $n^{\text{th}}$ derivative of the Dirac measure and $F_n$ is a distribution on the sphere, so $F_n \in H^{s_n}(\mathbb{S})$ for some $s_n \in \mathbb{R}$ (because any compactly supported distribution is in a Sobolev space, possibly of negative order). Note that $F$ is real valued if and only if the functions $F_n$ are Hermitian. Of course, there are also monochromatic waves that are not polynomially bounded, such as $F := e^{x_1} \cos(\sqrt{2}\, x_2)$.

A classical result due to Herglotz [Hör15, Theorem 7.1.28] ensures that if $F$ is a monochromatic wave with the sharp fall off at infinity, i.e., such that

$$\limsup_{R\to\infty} \frac{1}{R} \int_{B_R} F^2\, dx < \infty\,,$$

then there is a Hermitian vector-valued function $f \in L^2(\mathbb{S})$ such that $\widehat{F} = f\,\delta(\rho - 1)$. Furthermore, the value of the above limit is in the interval $[C_1\|F\|^2_{L^2(\mathbb{S})}, C_2\|F\|^2_{L^2(\mathbb{S})}]$ for some constants $C_1, C_2$. This bound means that, on an average sense, $|F(x)|$ decays as $C/|x|$. The prime example of this behavior is given by $f = 1$, which corresponds to $F(x) = c|x|^{-1/2} J_{1/2}(|x|)$.

The expression (5.1.2) corresponds to the case $N = 0$ above, since the function $F_0$ with $\widehat{F}_0 = f(\omega)\,\delta(\rho - 1)$ is precisely

$$F_0(x) = \int_{\mathbb{S}} e^{ix\cdot\omega} f(\omega)\, d\sigma(\omega)\,.$$

Also, if $f \in H^{-k}(\mathbb{S})$ with $k \geqslant 0$ but not necessarily in $L^2(\mathbb{S})$, the function $F_0$ is bounded as [EPSR22a, Appendix A]

$$\sup_{R>0} \frac{1}{R} \int_{B_R} \frac{F_0(x)^2}{1 + |x|^{2k}}\, dx \leqslant C\|f\|^2_{H^{-k}(\mathbb{S})}\,. \tag{5.A.1}$$

Hence in this case, $F_0$ is bounded, on an average sense, by $C|x|^{k-1}$. Therefore, if $f \in H^{-1}(\mathbb{S})$, $F_0$ is uniformly bounded in average sense.

If $f$ is a Gaussian random field, as considered in the Nazarov–Sodin theory (see Equation (5.1.2a)), we showed in Proposition 5.3.1 that $f$ is almost surely in $H^{-1-\delta}(\mathbb{S})$ for all $\delta > 0$ and not in $L^2(\mathbb{S})$. This behavior morally corresponds to functions that are bounded on a average sense but do not decay at infinity, as illustrated by the function $F_0 := \cos x_1$ generated by $f := \frac{1}{2}[\delta_{\xi_+}(\xi) + \delta_{\xi_-}(\xi)]$. This is the kind of behavior one needs to describe the expected local behavior of a high energy eigenfunction on a compact manifold as one zooms in at a given point.

The monochromatic wave defined as $\widehat{F}_n := f(\omega)\,\delta^{(n)}(\rho - 1)$ reads, in physical space, as

$$F_n(x) = \int_{\mathbb{S}} \int_0^\infty e^{i\rho x\cdot\omega} f(\omega)\,\rho^2\,\delta^{(n)}(\rho - 1)\, d\rho\, d\sigma(\omega) = (-1)^n \int_{\mathbb{S}} f(\omega)\,\partial_\rho^n|_{\rho=1}(\rho^2 e^{i\rho x\cdot\omega})\, d\sigma(\omega)\,.$$

Note that the $n^{\text{th}}$ derivative term involves an $n^{\text{th}}$ power of $x$. Therefore, using the bound (5.A.1), one easily finds that $F_n$ is bounded on average as $C|x|^{n+k-1}$ if $f \in H^{-k}(\mathbb{S})$; explicit examples with this growth can be easily constructed by taking $f$ to be either a constant for $k = 0$ or the $(k-1)^{\text{th}}$ derivative of the Dirac measure for $k \geqslant 1$. Consequently, picking $f$ as in (5.1.2a), the bound (5.A.1) morally leads to thinking of $F_n$ as a function that grows as $|x|^n$ at infinity, which cannot be the localized behavior of an eigenfunction. This is the rationale for defining a random monochromatic wave as in (5.1.2a)-(5.1.2b). In this direction, let us recall that the relation between random monochromatic waves and zoomed-in high energy eigenfunctions on a various compact manifolds is an influential long-standing conjecture of Berry [Ber77]. A precise form of this relation has been recently established in the case of the round sphere and of the flat torus [NS09; NS16; Roz17], which heuristically shows that (5.1.2a)-(5.1.2b) is indeed the proper definition of random monochromatic waves for this purpose.

The reasoning leading to the definition of a random Beltrami field as (5.1.2) is completely analogous, and the fact that one can relate Gaussian random Beltrami fields on $\mathbb{R}^3$ to high-frequency Beltrami fields on the torus just as in the case of the Nazarov–Sodin theory heuristically ensures that this is indeed the appropriate definition. For completeness, let us record that, just as in the case of monochromatic random waves, the Fourier transform of a polynomially bounded Beltrami field $u$ is a finite sum of the form

$$\widehat{u} = \sum_{n=1}^{N} f_n(\omega)\,\delta^{(n)}(\rho-1)\,,$$

where now $f_n$ is a Hermitian $\mathbb{C}^3$-valued distribution on $\mathbb{S}$. For $u$ to be a Beltrami field, there is an additional constraint on $f_n$ coming from the fact that not every distribution supported on $\mathbb{S}$ satisfies the equation $i\xi \times \widehat{u}(\xi) = \widehat{u}(\xi)$. A straightforward computation shows that this constraint amounts to imposing that

$$\sum_{n=j}^{N} \binom{n}{j} \alpha_{n-j,2} f_n(\omega) = i\omega \times \sum_{n=j}^{N} \binom{n}{j} \alpha_{n-j,3} f_n(\omega)$$

on $\mathbb{S}$ for all $0 \leqslant j \leqslant N$. Here $\alpha_{k,l} := \prod_{m=0}^{k-1}(l-m)$ with the convention that $\alpha_{0,l} := 1$. To see this, it suffices to note that the action of $\widehat{u}$ and $i\xi \times \widehat{u}$ on a vector field $w \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is

$$\langle \widehat{u}, w\rangle = \sum_{n=0}^{N} (-1)^n \int_{\mathbb{S}} f_n(\omega) \cdot \partial_\rho^n|_{\rho=1} \left[\rho^2 w(\rho\omega)\right]\, d\sigma(\omega)\,,$$

$$\langle i\xi \times \widehat{u}, w\rangle = \sum_{n=0}^{N} (-1)^n \int_{\mathbb{S}} i\omega \times f_n(\omega) \cdot \partial_\rho^n|_{\rho=1} \left[\rho^3 w(\rho\omega)\right]\, d\sigma(\omega)\,,$$

expand the $n^{\text{th}}$ derivative using the binomial formula and note that $\alpha_{k,l}$ is the $k^{\text{th}}$ derivative of $\rho^l$ at $\rho = 1$.

# Chapter 6

# Unweighted Condorcet Jury Theorem and Miracle of Aggregation do not hold almost surely

The Condorcet Jury Theorem or the Miracle of Aggregation are frequently invoked to ensure the competence of some aggregate decision-making processes. Furthermore, to the best of the author's knowledge, the current literature focuses on sufficient conditions (in different circumstances) to ensure the thesis of the theorem, but less attention has been paid to the applicability of the results.

Our objective in this chapter is to set the framework for the study of the applicability of these important results. As directly checking the hypotheses of the theorem is unrealistic, we use a probabilistic approach with Bayesian grounds. Here, we study under which circumstances the thesis predicted by the theorem is likely to hold. Depending on our available evidence on voter competence, which will be measured by a bias in a second-order probability measure, the thesis of the theorem will happen almost surely or almost never. See Theorem 6.2.2 and Theorem 6.2.7 for details. As we will see in these theorems, the opposite of the CJP could occur almost surely, i.e., majority rule chooses the wrong option a.s. Therefore, this gives another reason to study the applicability in order to ensure that we are not in this situation.

Furthermore, we also apply this framework in the case of weighted majority rule with stochastic (or noisy) weights. It is concluded that these stochastic weights can fix almost any voter profile of incompetence, see Theorem 6.5.2.

The chapter is organized as follows. In Section 6.1 we introduce the notation and some definitions that will be used in the rest of the chapter. In Section 6.2 we present the first results, examples and intuitions. In Sections 6.3, 6.4 and Appendix 6.A we prove the theorems of the unweighted situation. In Section 6.5 we present the proof and statement of the theorem where weighted majority rule is used instead of simple majority rule. Section 6.6 gives an end to this chapter offering some concluding remarks.

## 6.1 Notation and some definitions

The space of sequences with elements in $[0,1]$ will be denoted by $[0,1]^{\mathbb{N}}$. The (uncentered) moments of a measure $\nu_0$ will be denoted by:

$$m^i := \int_{[0,1]} x^i \, d\nu_0(x) \leqslant 1. \tag{6.1.1}$$

In particular, $m := m^1 = b + \frac{1}{2}$ following Definition 6.2.4. We denote by $(\Omega, \mathbb{P})$ an abstract probability space where every random object is defined. Given two measures $\nu$, $\nu'$ we will say that $\nu$ is absolutely continuous with respect to $\nu'$ and write $\nu \ll \nu'$ if for every Borel set $A$, $\nu'(A) = 0$ implies $\nu(A) = 0$. We will write its Radon–Nikodym derivative as $\frac{d\nu}{d\nu'}$. If there is a $C > 0$ such that $a \leqslant Cb$ we write $a \lesssim b$.

### 6.1.1 Distances and divergences

Consider a family $M$ of probability distributions or measures.

**Definition 6.1.1** (Divergence). *Let $M$ be as above and suppose that we are given a (smooth) function $d(\cdot||\cdot) : M \times M \to \mathbb{R}$ satisfying the following properties $\forall\, p, q \in M$:*

*i)* $d(p||q) \geqslant 0$,

*ii) and* $d(p||q) = 0$ *iff* $p = q$.

*Then, $d$ is said to be a divergence.*

It is customary to add a third condition such that $d$ defines an inner product on the tangent space of $M$, see [RRT19]. Also, if only i) and ii) are satisfied, it is usually called a semidivergence. As we will not use that property here, we will use the name of divergence as in the previous definition. We also recall the standard definition of distance.

**Definition 6.1.2** (Distance). *Let $M$ be as above and suppose that we are given a function $d(\cdot, \cdot) : M \times M \to \mathbb{R}$ satisfying the following properties $\forall\, p, q, r \in M$:*

*i) (Positive definiteness)* $d(p, q) \geqslant 0$, *and* $d(p, q) = 0$ *iff* $p = q$,

*ii) (Symmetry)* $d(p, q) = d(q, p)$,

*iii) (Triangle inequality)* $d(p, r) \leqslant d(p, q) + d(q, r)$.

*Then, $d$ is said to be a distance.*

Sometimes we will also use the notation $d(\cdot, \cdot)$ for divergences too. As a divergence do not necessarily satisfies ii) and iii), it is usually called a pseudodistance. Let us explore some examples. Given two measures $\mu, \mu'$ defined on the measurable space $(X, \Sigma)$, the *total variation distance* will be denoted by:

$$\|\mu - \mu'\| := 2 \sup_{B \in \Sigma} |\mu(B) - \mu'(B)|.$$

That is, the total variation distance is twice the "maximum" difference between the measure of the same set for $\mu$ and $\mu'$. That is, we have the useful bound

$$|\mu(B) - \mu'(B)| \leqslant \frac{1}{2} \|\mu - \mu'\| \quad \forall\, B \in \Sigma, \tag{6.1.2}$$

so the smaller $\|\mu - \mu'\|$, the smaller the discrepancy between $\mu(B)$ and $\mu'(B)$ for every measurable set. In fact, it can be shown this is a norm in the space of Radon signed measures, [Fol99, Proposition 7.16]. It is a well-known identity that:

$$2 \sup_{B\in\Sigma} |\mu(A) - \mu'(A)| = \|\rho - \rho'\|_{L_1(\tau)} := \int_X |\rho(x) - \rho'(x)| d\tau(x), \tag{6.1.3}$$

where $\rho := d\mu/d\tau$ and $\rho' := d\mu'/d\tau$ for some $\tau \gg \mu, \mu'$. For instance, $\tau := \frac{1}{2}(\mu + \mu')$. Also, the relative entropy or Kullback–Leibler divergence is defined as:

$$d_{KL}(\mu\|\mu') := \int_X \log \frac{\rho(x)}{\rho'(x)} \rho(x) d\tau(x).$$

See [RRT19] for more details and for a theoretical framework relating divergences and entropies and for more (geometrical) properties of divergences.

## 6.2 On the *a priori* applicability of those results

As we saw in Chapter 1, the CJT is a powerful tool to ensure the existence of an (almost) perfectly competent decision procedure. Nevertheless, in this Chapter we investigate how likely is this result a priori and what can we do to increase its prior probability.

### 6.2.1 Preliminary example

Let $\lambda$ be the standard Lebesgue measure on $\mathbb{R}$ and $\mu = \lambda \times \lambda = \prod_{n=1}^{2} \lambda$. If we define $X := X_1 + X_2$, $X_i \sim Bernoulli(p_i)$ and $p_i$ are unknown, then $\{\mathbb{E}[X] < 1\}$ has measure $1/2$ w.r.t. $\mu$. Indeed, $\mathbb{E}[X] = p_1 + p_2 < 1$ and by basic geometry

$$\mu\{(x,y) \in [0,1]^2 \;/\; x + y < 1\} = \frac{1}{2}.$$

In the same fashion, we can see that the measure of $\{\mathbb{E}[X] \leqslant 2\}$ is 1 as $p_1, p_2 \leqslant 1$ and, similarly, the measure of $\{\mathbb{E}[X] \geqslant 2\}$ is 0. We then say that the event $\{\mathbb{E}[X] \leqslant 2\}$ happens almost surely or $\mu-$almost surely and $\{\mathbb{E}[X] \geqslant 2\}$ does not happen $\mu-$almost surely ($\mu$-a.s.). In this setting, we can think of $\mu$ as a "*meta-probability measure*" or a second-order probability measure, it assigns probabilities (or measures) to some events of the parameters of the probability distributions of some random variables of our interest. Thus, we have two different probability spaces[1]:

- *Standard probability space* $(\Omega, \mathbb{P})$: the space (with its respective probability measure) depending on some parameters (fixed) where the problem is formulated. In our previous example it was given by the random variable $X : \Omega \to \mathbb{R}$ with

[1]The $\sigma$-algebra will be the standard one in each case and thus it will be implicitly assumed.

$\Omega$ the sample space and where the distribution of $X \equiv X_{p_1,p_2}$ depends on the fixed parameters $p_1$ and $p_2$. That is, for a measurable set $A$

$$\mathbb{P}\left(X \in A\right) = P_X(A, p_1, p_2).$$

- *Meta-probability space* $(\mathfrak{P}, \mu)$: the space $\mathfrak{P}$ (with its respective probability measure $\mu$) of the parameters of the previous random variable. In our previous example it was given by $\mathfrak{P} = [0,1]^2$ and $\mu = \lambda \times \lambda$, the standard Lebesgue measure on the square $[0,1]^2$.

Now, it might be the case that we do not know the value of $p_1$ and $p_2$ but nevertheless want to know how "likely" will be that, for instance, $\mathbb{E}[X] < 1$. As we saw above, this is a problem involving the two probability spaces:

- Standard probability space, $\mathbb{E}[X_{p_1,p_2}] = \int_\Omega X_{p_1,p_2}(\omega) d\mathbb{P}(\omega) = p_1 + p_2$.
- Meta-probability space, $\mu\left(\{(p_1,p_2) \in [0,1]^2 \mid p_1 + p_2 < 1\}\right) = 1/2$.

Notice also that if we chose a different $\mu$, the associated measure of each event would probably change, i.e., we have to choose the measure on $\mathfrak{P}$. From a Bayesian point of view, if we want to consider the prior probability, it can be assumed that this measure is not "biased" in any particular direction. That is, if we have no particular evidence to assume the contrary or prior to collect any evidence, it seems reasonable to impose that, for instance,

$$\mu(\{p_1 \in [0,1/2)\}) = \mu(\{p_1 \in (1/2,1]\}).$$

### 6.2.2 The CJT and measures on $[0,1]^{\mathbb{N}}$

For the CJT, given our previous definitions, we have:

- *Standard probability space* is also denoted by $(\Omega, \mathbb{P})$ where the main random variables involved are $\frac{1}{n}\sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$ and the event of our interest is given by (1.9).
- *Meta-probability space* $(\mathfrak{P}, \mu)$ equals $\left([0,1]^{\mathbb{N}}, \mu\right)$. Here we are not interested in measures on $[0,1]^2$, but on $[0,1]^\infty$ or $[0,1]^{\mathbb{N}}$, i.e., the space of sequences with elements in $[0,1]$, as $p_n \in [0,1]$ and the parameters of the problem are $\{p_n\}_{n=1}^{\infty}$.

We now turn into the problem of finding $\mu$ (or, more precisely, a set of $\mu$). A natural measure to consider is

$$\mu = \prod_{n=1}^{\infty} \lambda, \tag{6.2.1}$$

which is the generalization of the measure on $\mathbb{R}^2$ considered above. It is well-defined by Kolmogorov's Extension Theorem. This measure has the property of being centered in the sense that the mean value (first moment) of $\lambda$ is

$$\int_{[0,1]} x \, d\lambda(x) = \frac{1}{2}. \tag{6.2.2}$$

However, we are going to consider more general "centered" measures than the one in (6.2.1), i.e., a larger class. Before the precise definition, we need to introduce the concept of distances and divergences of probability measures, say $d$. These objects tell us, in a sense to be precise in Section 4.1.8, how different two distinct $\mu$ and $\mu'$ assign measures to an arbitrary set $A$. If $d(\mu, \mu') = 0$, the measures are identical and if $d$ increases, so does the discrepancy for some sets. There are several ways of doing so, but two of the most important examples are the total variation distance (the statistical distance) and the Kullback-Leibler divergence (associated to the Shannon-Boltzmann entropy). In fact, we are going to consider a larger set, that will be denoted by $\mathcal{D}$ and which will be defined precisely in Definition 6.3.2. To ease the exposition here, it can be understood that $d$ below is either the total variation distance or the Kullback-Leibler divergence. We are ready to define the concept of centered measures.

**Definition 6.2.1.** *A probability measure $\mu = \prod_{n=1}^{\infty} \nu_n$ on $[0,1]^{\mathbb{N}}$ will be centered if there exists a probability measure on $[0,1]$, $\nu_0$, such that $\nu_n \ll \nu_0\ \forall\ n \geqslant 1$ (see Section 4.1.8 for notation),*

$$\int_{[0,1]} x \, d\nu_0(x) = \frac{1}{2} \tag{6.2.3}$$

*and*

$$\sum_{n=1}^{\infty} d(\nu_n, \nu_0) < \infty, \tag{6.2.4}$$

*with $d \in \mathcal{D}$.*

**Example 6.2.1.** The case considered in (6.2.1) corresponds to the case $\nu_0 = \lambda$ and $\nu_0 = \nu_n\ \forall\ n$ positive integers, so $d(\nu_0, \nu_n) = d(\nu_0, \nu_0) = 0$ (by definition of distance and divergence, see Definition 6.1.1 and 6.1.2) and then,

$$\sum_{n=1}^{\infty} d(\nu_n, \nu_0) = 0 < \infty.$$

The idea is simple, the measure $\mu$ is not too far (in the sense that the sum of distances or divergences does not go to infinity) from a product measure $\prod_{n=1}^{\infty} \nu_0$ of identical measures on $[0,1]$ and these measures have mean $1/2$. This generalizes (6.2.1) in two ways. First, the measures of the product are not necessarily identical. We allow the measure to be a "perturbation" of $\mu_0$. Second, the measure $\nu_0$ is not necessarily the Lebesgue measure, but a measure with mean $1/2$, i.e., we only need this measure to have the same first moment as the Lebesgue measure on $[0,1]$. For instance, we can have atomic measures, i.e., $\nu_0(\{x\}) > 0$ for some $x$. This is not allowed in the standard Lebesgue measure, as every single point has measure zero. In particular, as we said in Section 4.1.8 we will define $\epsilon_1 := \nu_0(\{1\})$, that is, there is a probability $\epsilon_1$ such that each voter is going to vote for the correct option almost surely as in the MoA. More generally, we define $\epsilon_{1-\varepsilon_0,1} := \nu_0\left([1-\varepsilon_0, 1]\right)$.

With these measures, the CJP will not hold almost surely. It is important to note that as we have a complete characterization, we are not saying that the hypothesis of the theorem (CJT) will not hold, but *that the thesis* (CJP) will not hold. The latter implies the former but the former implies the latter only if the conditions are necessary too. More precisely:

**Theorem 6.2.2.** *Almost surely independent Condorcet Jury Theorem does not hold for a centered measure $\mu$, that is:*

$$\mu(\mathcal{C}_I) = 0. \tag{6.2.5}$$

That is, no matter which measure we choose (with the reasonable condition of Definition 6.2.1), it will assign probability zero to the CJP.

**Remark 6.2.2.** We should distinguish two concepts, impossible events and probability (or measure) zero events. The first are associated with the empty set $\emptyset$ and the second with a set of probability or measure zero, i.e., a null set. For instance, take a uniform random variable $X : \Omega \to [0,1]$ over $[0,1]$ and let $x_0 := \pi/4 \in [0,1]$. $X$ will always (surely) give a number between 0 and 1. Thus, $\{X > 1\} := \{\omega \ / \ X(\omega) > 1\}$ is an impossible event. $\{X = x_0\}$ is not impossible (some number must be chosen and it also had probability zero) but will not happen almost surely (the probability is zero). This implies that if we run the variable a large number of times $n$, $\frac{\text{number of times } X=x_0}{n} \to 0$ as $n \to \infty$ (probabilistic application to frequencies).

Again, the idea is that whatever measure $\mu$ we choose with the condition that $\mu$ is centered or not "biased", the CJP will not hold almost surely. One could think that this was somehow expected as soon as we chose $m^1 = 1/2$: we are choosing $m^1 = 1/2$ but, as already Condorcet noticed, you need a probability greater than $1/2$. Thus, one does not expect the CJT to hold. Hence, the theorem is more or less trivial. Nevertheless, this intuition would be incorrect, as it would be confusing the two probability spaces. Indeed, the first and second $1/2$ belong to two different spaces

- $\mathbb{E}[X_i] = \int_\Omega X_i(\omega) d\mathbb{P}(\omega) = p_i$ and this must be greater than $1/2$ in the standard CJT (where $p_i = p \ \forall i \in \mathbb{N}$),
- $m^1 = \int_{[0,1]} x \, d\nu_0(x) = \frac{1}{2}$.

The confusion is obvious if we consider the homogeneous case $p_i = p \ \forall i \in \mathbb{N}$, the original Condorcet's theorem. Then, we would have $\mathfrak{P} = [0,1]$. Imposing that $\mu = \nu_0$ is centered around $1/2$, i.e., $\int_{[0,1]} x \, d\mu(x) = \frac{1}{2}$ would not imply that the CJT fails almost surely. In fact, it would have probability $\mu((1/2,1]) > 0$ unless $\mu = \delta_{1/2}$, the Dirac measure at $1/2$. For instance, if $\mu = \lambda$, then the CJT would have probability $1/2$.

A more subtler argument would say that, on average, probabilities are approximately (and asymptotically) $1/2$ because $m^1 = 1/2$. More precisely, $\frac{1}{n}\sum_{i=1}^n p_i \to \frac{1}{2}$ a.s. as $n \to \infty$. Thus, again, we cannot expect the CJT to hold in that situation because the probabilities, on average, should be greater than $1/2$. But, this intuition is incorrect too. In the following two examples we are going to construct an uncountable set of sequences where the CJT holds but $\frac{1}{n}\sum_{i=1}^n p_i \to \frac{1}{2}$ as $n \to \infty$, i.e., probabilities are on average $1/2$.

**Example 6.2.3.** Let

$$\mathcal{C}_1 := \left\{ (p_n)_{n=1}^\infty \in [0,1]^{\mathbb{N}} \mid p_n = \frac{1}{2} + \varepsilon_n, \ \varepsilon_n \in \left[0, \frac{1}{2}\right], \right.$$
$$\left. \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \varepsilon_i = 0, \ \lim_{n\to\infty} \frac{1}{\sqrt{n}}\sum_{i=1}^n \varepsilon_i = \infty \right\}.$$

Then, $\mathcal{C}_1 \subset \mathcal{C}_I$, i.e., the sequences in $\mathcal{C}_1$ satisfy the CJP. Indeed, let $\bar{X}_n := \frac{1}{n}\sum_{i=1}^n X_n$ as in Section 1.1.3. By definition,

$$\mathbb{E}(X_i) = p_i, \ \ \text{Var}(X_i) = p_i q_i,$$

where $q_i := 1 - p_i$, then

$$\begin{aligned} \mathbb{P}\left(\bar{X}_n \leqslant 1/2\right) &= \mathbb{P}\left(\bar{X}_n - \mathbb{E}\left(\bar{X}_n\right) \leqslant 1/2 - \mathbb{E}\left(\bar{X}_n\right)\right) \leqslant \\ &\leqslant \mathbb{P}\left(\left|\bar{X}_n - \mathbb{E}\left(\bar{X}_n\right)\right| \geqslant \mathbb{E}\left(\bar{X}_n - 1/2\right)\right) \leqslant \\ &\leqslant \frac{\operatorname{Var}(\bar{X}_n)}{\left(\mathbb{E}\left(\bar{X}_n\right) - \frac{1}{2}\right)^2} = \frac{\sum_{i=1}^n (1/4 - \varepsilon_i^2)}{\left(\sum_{i=1}^n \varepsilon_i\right)^2} = \frac{\frac{1}{n}\sum_{i=1}^n (1/4 - \varepsilon_i^2)}{\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \varepsilon_i\right)^2} \to 0 \end{aligned} \tag{6.2.6}$$

as $n \to \infty$ by Chebyshev's inequality, which can be applied because, by hypothesis, $\sum_{i=1}^n \varepsilon_i > 0$ if $n$ is large enough. We have also used that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (1/4 - \varepsilon_i^2) = \frac{1}{4} - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \varepsilon_i^2 = \frac{1}{4}$$

because $\varepsilon_i^2 \leqslant \varepsilon_i$. Thus, $\mathbb{P}\left(\bar{X}_n > 1/2\right) \to 1$ as $n \to \infty$, i.e., (1.9). But note that:

$$\frac{1}{n}\sum_{i=1}^n p_i = \frac{1}{2} + \frac{1}{n}\sum_{i=1}^n \varepsilon_i \to \frac{1}{2}.$$

Thus, $p_i$ are, on average, $1/2$ but nevertheless the CJP holds.

**Remark 6.2.3.** We can easily construct elements of this set as follows. Define $\varepsilon_i := \max\{i^\alpha, 1/2\}$. Then, by the Euler–Maclaurin formula,

$$H_n^{(-\alpha)} := \sum_{i=1}^n i^\alpha = \frac{n^{\alpha+1} - 1}{\alpha + 1} + O\left(n^\alpha\right).$$

Thus, it is enough if we take $\alpha \in (-1/2, 0)$. $H_n^{(-\alpha)}$ is the generalized harmonic number.

Now we present a second example of sets where, on average, the probabilities are $1/2$ but the CJP holds. The idea behind the construction is completely different. It will also illustrate an important fact, being an element of $\mathcal{C}_I$ *does not necessarily depend only on the tail of the sequence.*

**Example 6.2.4.** Let us fix an $m \in \mathbb{N}$ greater than 1. Consider the sequence of $(p_i)_{i=1}^\infty = (p_1, \ldots, p_m, 1, 1, 0, 1, 0, 1, \ldots)$. In what follows we assume $p_i \in \{0, 1\}$. In this setting where the probability is either 0 or 1, the CJT holds trivially iff $|\{i : 1 \leqslant i \leqslant n \text{ and } p_i = 1\}| > |\{i : 1 \leqslant i \leqslant n \text{ and } p_i = 0\}|$ for every $n$ large enough. Thus, this is equivalent to:

$$S_n := |\{i : 1 \leqslant i \leqslant n \text{ and } p_i = 1\}| = \sum_{i=1}^n p_i > \frac{n}{2} \quad \forall\, n > n_0, \tag{6.2.7}$$

both $n, n_0 \in \mathbb{O}$. If $p_i = 0\ \forall\, i \in \{1, \ldots, m\}$, then for $n = 2k + 1$:

$$\frac{S_{n+m}}{n+m} - \frac{1}{2} = \frac{1+k}{2k+1+m} - \frac{1}{2} = \frac{1-m}{2(m+2k+1)} < 0 \quad \forall\, n \in \mathbb{O}.$$

But, on the other hand, if $p_i = 1\ \forall\, i \in \{1, \ldots, m\}$, then:

$$\frac{S_{n+m}}{n+m} - \frac{1}{2} = \frac{1+k+m}{2k+1+m} - \frac{1}{2} = \frac{1+m}{2(m+2k+1)} > 0 \quad \forall\, n \in \mathbb{O}.$$

In general, if there are $m'$ $p_i = 1$ for $1 \leqslant i \leqslant m$ and $2m' + 1 > m$ then

$$\frac{S_{n+m}}{n+m} - \frac{1}{2} > 0 \quad \forall\, n \in \mathbb{O}.$$

But, note again that

$$\frac{1}{n}\sum_{i=1}^{n} p_i \to \frac{1}{2}.$$

This latter set defines $\mathcal{C}_2$.

Hence, as promised, $\mathcal{C}_1 \cup \mathcal{C}_2$ is an uncountable set of sequences where the CJT holds but $\frac{1}{n}\sum_{i=1}^{n} p_i \to \frac{1}{2}$ as $n \to \infty$, i.e., probabilities are on average $1/2$.

In the same manner, it could be argued that the hypotheses of the MoA are not satisfied. If the event that a proportion of voters is informed is quite rare for those measures, the MoA cannot be expected. Nevertheless, this condition of the MoA is satisfied in the following sense. First, recall that in Section 4.1.8 we defined $\epsilon_1 := \nu_0(\{1\})$ and $\epsilon_{1-\varepsilon_0,1} := \nu_0\left([1-\varepsilon_0, 1]\right)$. This measures the probability that an individual voter is well-informed ($\epsilon_1$) and almost well-informed ($\epsilon_{1-\varepsilon_0,1}$, the probability of choosing the correct option is greater than $1-\varepsilon_0$ for some $\varepsilon_0$ generally small). Then, we have the following result:

**Proposition 6.2.5.** *Let $\mu$ a centered measure, $0 \leqslant \varepsilon_0 < 1/2$, $0 < \varepsilon < \epsilon_{1-\varepsilon_0,1}$ and $\delta > 0$ as small as we want. Then, $\exists\, N \in \mathbb{N}$ such that*

$$\mu_0\left(\left|\{1 \leqslant i \leqslant n \;/\; p_i \in [1-\varepsilon_0, 1]\}\right| > \varepsilon n\right) > 1 - \delta \;\; \forall\, n \geqslant N.$$

*where $\mu_0 = \prod_{n=1}^{\infty} \nu_0$ and*

$$\mu\left(\lim_{n\to\infty} \left|n^{-1}\{1 \leqslant i \leqslant n \;/\; p_i \in [1-\varepsilon_0, 1]\}\right| > \varepsilon\right) = 1.$$

*In particular, if $\varepsilon_0 = 0$ then the same holds with $p_i = 1$ and $\epsilon_{1-\varepsilon_0,1} = \epsilon_1$*

This proposition means that the event that a proportion $\varepsilon > 0$ of voters is well-informed or almost well-informed will be reached if the population $n$ is greater than a (finite) $N$ with probability as close to one as we want. These voters will vote for the correct option with probability greater than $1-\varepsilon_0$ with $\varepsilon_0$ as small as we want or even zero.

These remarks warn us that the proof of Theorem 6.2.2 cannot rely on those intuitions and must use different ideas. This will be done in Appendix 6.3. The basic idea is that we can only have the CJP if the sequences satisfy something similar to (6.2.6) or (6.2.7) of the previous examples. But these conditions are too restrictive and, thus, this set will have measure zero for the measures under consideration.

### 6.2.3 On the election of $\mu$ and the prior probability

To derive $\mu(\mathcal{C}_I) = 0$, the centered condition of Definition 6.2.1 can be relaxed somehow (although this condition is essential to calculate the a priori probability as we will see below). We could define in the same fashion as in Definition 6.2.1:

**Definition 6.2.4.** *A probability measure $\mu = \prod_{n=1}^{\infty} \nu_n$ on $[0,1]^{\mathbb{N}}$ will be b-biased for $b \in [-\frac{1}{2}, \frac{1}{2}]$ if there exists a probability measure on $[0,1]$, $\nu_0$, such that $\nu_n \ll \nu_0$ $\forall\, n \geqslant 1$ (see*

*Section 4.1.8 for notation),*

$$\int_{[0,1]} x \, d\nu_0(x) = \frac{1}{2} + b \tag{6.2.8}$$

*and*

$$\sum_{n=1}^{\infty} d(\nu_n, \nu_0) < \infty. \tag{6.2.9}$$

*with* $d \in \mathcal{D}$*,*

**Example 6.2.6.** This example is a generalization of the Example 6.2.1. For instance, consider a "biased" measure

$$\mu_{b_0} = \prod_{n=1}^{\infty} \lambda_{b_0}, \tag{6.2.10}$$

where the Radon-Nikodym derivative (this is, its probability density function $\rho_{b_0}$) is given by

$$\frac{d\lambda_{b_0}}{d\lambda}(x) = \rho_{b_0}(x) = (1 - b_0/2) + b_0 x,$$

with $b_0 \in [-2, 2]$, i.e., we modify the standard Lebesgue measure ($\lambda = \lambda_0$) such that its density is affine and more concentrated on $(0, 1/2)$ if $b_0 \in [-2, 0)$ and more concentrated on $(1/2, 1)$ if $b_0 \in (0, 2]$. It is straightforward to check that in (6.2.10), $b = b_0/12$. The case of Example 6.2.1 is recovered when $b_0 = 0$.

It seems natural that the larger the positive (resp. negative) bias, the larger (resp. smaller) $\mu(\mathcal{C}_I)$ will be. This happens because we are initially assigning less (resp. more) measure to the event $\{p < 1/2\}$, i.e., to the event that the individual voter is more likely to choose wrongly. Therefore, Theorem 6.2.2, as there is no bias ($b = 0$), implies that for any measure $\mu = \prod_{n=1}^{\infty} \nu_n$ where the $\nu_n$ assign probability to both sides $\{p < 1/2\}$ and $\{p > 1/2\}$ "fairly"[2], then $\mu$ is going to assign measure zero to the CJP, i.e., the CJP will not hold almost surely. Hence, we get the same result as if $b < 0$, see Example 6.4.2. Sometimes, $b < 0$ could be justified (e.g., [Cap11]), but here we show that even if we assume $b = 0$ because, following a Bayesian approach, we want to estimate the *prior probability* (the probability before any evidence is collected) of the CJP, we will arrive at the same result: the CJP fails almost surely. That is, if we try to measure the applicability of the CJP according to a symmetrically balanced distribution (in particular, with no bias toward incompetence) without considering any evidence on voters competence, we arrive at the result that the CJT does not hold almost surely. Prior (or a priori in this case) probabilities are the baseline from which probabilities are updated when evidence is collected. So, in this setting, we would need strong evidence of voter competence to expect that the CJT can be applied.

Nevertheless, the case $b < 0$ has an important difference with respect to $b = 0$. Now we can prove that, almost surely, the anti-CJP will hold, i.e.,

$$\lim_{n \to \infty} \mathbb{P}\left( \sum_{i=1}^{n} X_i < \frac{n}{2} \right) = 1, \tag{6.2.11}$$

*the wrong option will be chosen almost surely*. Indeed, let $\Phi(x) = 1 - x$, $X' := \Phi \circ X$ and $\nu_n' := \Phi_* \nu_n$, the push-forward measure, as $\mathbb{P}(X' = 1) =: p' = \Phi(p)$ measures

[2]In the sense that $\int_{[0,1/2)} x d\nu_0(x) + \frac{1}{2}\nu_0\left(\frac{1}{2}\right) + \int_{(1/2,1]} x d\nu_0(x) = \frac{1}{2}$. If all the mass were concentrated on $[0, 1/2)$, $\nu_0[0, 1/2) = 1$, then previous sum would be $< 1/2$ and the opposite for all the mass concentrated on $(1/2, 1]$.

the probability of choosing the wrong option. Hence, for $\mu' = \prod_{n=1}^{\infty} \nu'_n$ we have

$$\int_{[0,1]} \lambda d\nu'_0(\lambda) = \int_{[0,1]} (1-\lambda) d\nu_0(\lambda) = \frac{1}{2} - b =: 1 + b' > \frac{1}{2}.$$

Thus, we can use Theorem 6.2.7 to conclude the proof.

Therefore, the CJT is a *double-edged sword*: it can either prove that majority rule is an almost perfect mechanism or an almost perfect disaster. This is partly the reason why we investigate here its applicability, to ensure that we are not in the case of a perfect disaster, but of a perfect mechanism to aggregate information.

As above, we should not confuse the Bayesian analysis in the two probability spaces. In the standard space, strategic voting and Bayesian–Nash equilibria were first analyzed by, among others, [ASB96; McL98]. Our Bayesian approach is in the meta-probability level. We want to answer whether, given a dichotomous choice and a set of voters or jurors, we can invoke the CJT to ensure they will reach the correct option as the number of members increases. More precisely, we want to know its prior probability. These two problems are completely different.

The measures considered in Definition 6.2.1 are quite general set of measures satisfying this symmetry condition of not favoring incompetence. Nevertheless, we can extend the results of Theorem 6.2.2 to a greater set of measures. This is treated in Theorem 6.4.1 and Theorem 6.4.3. As we have said, these more technical theorems extend Theorem 6.2.2 to some new measures, in particular, including the ones with $b \leqslant 0$.

### 6.2.4 The case of $b > 0$.

Can $b > 0$ be justified in some cases? As we commented in Section 1.1.3, we can achieve $p_i \geqslant 1/2$ for all $i \in \mathbb{N}$ if the original voters with $p_i < 1/2$ are assigned a weight of $-1$. In that case,

$$\mathbb{P}\left((-1)X_i = 1\right) = \mathbb{P}\left(X_i = -1\right) = 1 - p_i > 1/2,$$

where for simplicity we have assumed (see Section 1.1.3) that $X_i \in \{-1, 1\}$. As we commented in the introduction, this does not seem easy to implement because voters can reject negative weights. Nevertheless, one could think that a rational voter will self-impose this if this voter knows that $p_i < 1/2$. In other words, if the voter thinks the correct option is $A$ ($X_i = 1$), then he/she votes $B$ ($X_i = -1$) and similarly for the opposite case. Now the probability is $p'_i = 1 - p_i > 1/2$. But this strategy requires two steps:

- knowing that $p_i < 1/2$,
- be willing to reverse the outcome of one's vote.

Considering real voters (not ideal ones), it is difficult to imagine the fulfillment of these steps. First, it is an empirical fact how well people calibrate their degree of knowledge with probability estimates. The standard finding of knowledge calibration experiments is overconfidence, people tend to overestimate their probability of being right, see Chapter 8 of [SWT16] and references therein. And even if voters acknowledge that they are worse than a coin toss, it does not seem realistic to expect that, in general, they will reverse their outcome. For instance, they can rationalize their vote by introducing non-epistemic factors.

Be that as it may, our techniques can give us the result in this situation and this is the content of the following proposition:

**Theorem 6.2.7.** *Almost surely independent Condorcet Jury Theorem holds for a biased measure $\mu$ with $b > 0$, that is*

$$\mu(\mathcal{C}_I) = 1. \tag{6.2.12}$$

*Proof.* The proof of this theorem parallels the proof of Theorem 6.2.2. The main change is that the denominator of $Q_n$ is:

$$\sqrt{n} \times \frac{1}{n} \sum_{i=1}^{n} (p_i - 1/2)$$

and the second factor tends to $b > 0$ as $n \to \infty$ by the SLLN. Similarly, $m^1 - m^2 > 0$ unless $\nu_0 = \delta_1$, but in that case the theorem is trivial. Kakutani's lemma is applied in the same way.

□

### 6.2.5 Results for weighted majority rule

But not everything is lost. We can try to modify the aggregation procedures to achieve a competent mechanism. The natural idea is the consideration of a weighted majority rule, i.e., we define:

$$X_n^w := \sum_{i=1}^{n} w_i X_i,$$

where now $X_i \in \{-1, 1\}$ and $w_i \in \mathbb{R}$ (in principle, they could be negative, but we will not consider that case here). The larger the weight (*ceteris paribus*), the greater the influence of the voter. Weighted majority rule implies that the social choice function is $\text{sign}(X_n^w)$ being indifferent between the two if $X_n^w = 0$. The previous case of simple majority rule is recovered if $w_i = w_j \; \forall \; i, j$. The next step would be to obtain, for some positive integer $k$ and constants $\alpha, \beta > 0$,

$$w = \alpha + \beta p^k + \varepsilon, \tag{6.2.13}$$

i.e., competence is positively correlated with the weight we assign but the association is not perfect, there is a stochastic error $\varepsilon$. In Theorem 6.5.2 we show that if (6.2.13) is good enough, the CJT will hold almost surely for "almost" every measure $\mu$, even if they are strongly biased toward $p = 0$, i.e., we are not only considering centered measures but the less favorable case of measures representing voters far from competence. In other words, we are not estimating the prior probability but the probability given almost any evidence on voters competence. This gives some evidence for trying to include epistemic weights in the decision procedure if we are interested in choosing the correct option.

As we have said, the main ingredient is the correct assignment of weights. In Appendix 6.B we take this question seriously because there is little point in theorizing about something which cannot be practically implemented. We also consider how "fair" this situation would be.

## 6.3 Proof of Theorem 6.2.2 and Proposition 6.2.5

Let us assume first that $\mu_0 = \prod_{n=1}^{\infty} \nu_0$. By [BP98, Theorem 2], the CJP fails iff both

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} p_i - \frac{n}{2}}{\sqrt{\sum_{i=1}^{n} p_i q_i}} = \infty \tag{6.3.1}$$

where $q_i := 1 - p_i$, and $\exists\, n_0 \in \mathbb{N}$ such that

$$|\{i : 1 \leqslant i \leqslant n \text{ and } p_i = 1\}| > n/2 \quad \forall\, n > n_0 \tag{6.3.2}$$

do not hold. For the first condition, we define:

$$Q_n := \frac{\sum_{i=1}^{n} p_i - \frac{n}{2}}{\sqrt{\sum_{i=1}^{n} p_i q_i}} = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(p_i - \frac{1}{2}\right)}{\sqrt{\frac{1}{n} \sum_{i=1}^{n} p_i q_i}}$$

with $\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} \left(p_i - p_i^2\right)$. The *key* here is to realize that under the measure $\mu_0$, $p_i$ are i.i.d. random variables in $([0,1]^{\mathbb{N}}, \mu_0)$. Thus, we can apply the Strong Law of Large Numbers (SLLN), [Fol99, Theorem 10.13],

$$\frac{1}{n}\sum_{i=1}^{n} \left(p_i - p_i^2\right) \to b + \frac{1}{2} - m^2 = m^1 - m^2,$$

$\mu_0$-almost surely. Now, note that

$$m^1 - m^2 = \int_{[0,1]} (x - x^2)\, d\nu_0(x) = \int_{(0,1)} (x - x^2)\, d\nu_0(x) > 0 \tag{6.3.3}$$

as long as $\nu_0((0,1)) > 0$. This is going to be the case if $\epsilon_1 < 1/2$, as

$$\frac{1}{2} = m^1 = \epsilon_1 + \int_{(0,1)} x\, d\nu_0(x) \tag{6.3.4}$$

so $\int_{(0,1)} x\, d\nu_0(x) = 1/2 - \epsilon_1 > 0$. For the numerator, we need the other classical asymptotic result, the Central Limit Theorem (CLT), to conclude

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(p_i - \frac{1}{2}\right) \to \mathcal{N}(0, \sigma^2)$$

in distribution as $n \to \infty$. Thus, by Slutsky Theorem [Sha03, Theorem 1.11],

$$Q_n \to \mathcal{N}(0, \sigma'^2) \quad \text{where } \sigma' := \frac{\sigma}{\sqrt{m^1 - m^2}}$$

in distribution as $n \to \infty$. Let $Q \sim \mathcal{N}(0, \sigma'^2)$. So we can conclude that,

$$\mu_0\left(\lim_{n\to\infty} Q_n = \infty\right) = 0. \tag{6.3.5}$$

Indeed, let us define the events

$$\mathcal{A} := \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \;/\; Q_n((p_i)_{i=1}^{\infty}) \to \infty\}, \quad \mathcal{A}_{n,\varepsilon} := \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \;/\; Q_n((p_i)_{i=1}^{\infty}) > M_\varepsilon\}$$

where $M_\varepsilon$ satisfies $\mu_0\left(Q \leqslant M_\varepsilon\right) = 1 - \varepsilon$. By the definition of limit, for every $\varepsilon > 0$,

$$\mathcal{A} \subset \bigcup_{N\in\mathbb{N}} \bigcap_{n\geqslant N} \mathcal{A}_{n,\varepsilon}.$$

By the continuity of measures, [Fol99, Theorem 1.8.c)]

$$\mu_0\left(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geqslant N} \mathcal{A}_{n,\varepsilon}\right) = \lim_{N\to\infty} \mu_0\left(\bigcap_{n\geqslant N} \mathcal{A}_{n,\varepsilon}\right) \leqslant \lim_{N\to\infty} \mu_0\left(\mathcal{A}_{N,\varepsilon}\right) = \varepsilon,$$

where the last equality follows from the convergence in distribution. Hence, by the monotonicity of measures,

$$\mu_0\left(\mathcal{A}\right) \leqslant \varepsilon \;\; \forall\, \varepsilon > 0,$$

concluding the proof of (6.3.5). Thus, the first condition (6.3.1) will not hold almost surely. Let us see the second one (6.3.2). For that purpose, let us define for $p \in [0,1]$:

$$\tilde{p} := \begin{cases} 1 \text{ if } p = 1 \\ 0 \text{ if } p \in [0,1) \end{cases}$$

Thus, if $\mu_0(p_i \in A) = \nu_0(A)$ for $A$ a Borel set, $\tilde{p}_i \sim Bernoulli(\epsilon_1)$ where we defined $\epsilon_1$ as $\nu_0(\{1\})$. Let us also define

$$S_n := |\{i : 1 \leqslant i \leqslant n \text{ and } p_i = 1\}| = \sum_{i=1}^{n} \tilde{p}_i \tag{6.3.6}$$

and

$$\begin{aligned} \mathcal{B} &:= \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \;/\; \exists\, n_0 : S_{n_0+2k} > n_0/2 + k \;\forall\, k \geqslant 0\}, \\ \mathcal{B}_n &:= \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \;/\; S_n > n/2\}. \end{aligned} \tag{6.3.7}$$

By the SLLN,

$$\mu_0\left(\lim_{n\to\infty} \frac{S_n}{n} = \epsilon_1 < \frac{1}{2}\right) = 1.$$

But if $\lim_{n\to\infty} S_n/n = \epsilon_1 < \frac{1}{2}$, then

$$\frac{S_n}{n} < \frac{1}{2}$$

if $n$ is large enough. Thus, $\mu_0(\mathcal{B}) = 0$. Summing up,

$$\mu_0(\mathcal{C}_I) = \mu_0(\mathcal{A} \cup \mathcal{B}) = 0 \tag{6.3.8}$$

concluding the proof of the theorem for $\mu_0 = \prod_{n=1}^{\infty} \nu_0$ if $\epsilon_1 < \frac{1}{2}$.

We consider now the case of $\epsilon_1 = 1/2$ and therefore $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$. First note that condition (6.3.1) does not hold because here either $p_i = 0$ or $q_i = 0$ almost surely, so $p_i = 0$ or $q_i = 0 \;\forall\, i \in \mathbb{N}$ almost surely, i.e., $Q_n$ is not well-defined almost surely. But in this deterministic (only $p = 0$ or 1) situation, it is clear that CJP holds iff (6.3.2) holds. So let us show that the former fails. Define $\bar{p} := 1$ if $p = 1$ and $\bar{p} := -1$ otherwise. If $\bar{S}_{n_0}^{n} := \sum_{i=n_0}^{n} \bar{p}_i$, then, $\bar{S}_{n_0}^{n}$ is a symmetric random walk in $n$

starting at zero. If we denote, for $k \in \mathbb{Z}$,

$$r_k := \mu_0 \left(\{(p_i)_{i=1}^{\infty} \text{ such that } \exists\, n \;/\; \mathcal{S}_n = k\}\right)$$

where $\mathcal{S}_n$ is a symmetric random walk starting at zero. It is standard that $r_0 = 1$, in fact, the probability of returning infinitely often to 0 equals 1. Then, we have the following difference equation:

$$r_k = \frac{1}{2} r_{k-1} + \frac{1}{2} r_{k+1},$$

i.e., if the first move is up, there are $k-1$ up movements left and, similarly, if the first movement is down, we would need $k+1$ movements up. The solution is $r_k = c \in [0,1]$ because $0 \leqslant r_k \leqslant 1$. As $r_0 = 1$, we conclude $r_k = 1 \;\forall\, k \in \mathbb{Z}$.

Now, fix a $n_0 \in \mathbb{N}$. Then, if $\bar{S}_n := \bar{S}_1^n$ and (6.3.2) holds for that $n_0$, then

$$\bar{S}_{n_0} = i$$

for $i = 1, ..., n_0$ such that $\bar{S}_n > 0$ for $n \geqslant n_0$ odd. But, by the discussion above, the probability that $\bar{S}_{n_0}^n < -i < 0$ is one, so the probability that $S_n > n/2$ given a fixed $S_{n_0} \geqslant \frac{n_0+1}{2}$ is zero. Therefore, the negation of (6.3.2) holds almost surely as $\mu_0\left(\mathcal{B}\right) \leqslant \sum_{n_0} \mu_0(\cap_{k\geqslant 0} \mathcal{B}_{n_0+2k}) = 0$, where

$$\mathcal{B} = \bigcup_{n_0 \in \mathbb{O}} \bigcap_{k \geqslant 0} \mathcal{B}_{n_0+2k}.$$

**Remark 6.3.1.** One could argue that if $\mathcal{C}_I$ were a tail event, then the measure could only be 0 or 1 by Kolmogorov's zero–one law, which agrees with our results. Nevertheless, being an element of $\mathcal{C}_I$ does not necessarily depend only on the tail. For instance, consider de sequence of $(\tilde{p}_i)_{i=1}^{\infty} = (\tilde{p}_1, \ldots, \tilde{p}_m, 1, 1, 0, 1, 0, 1, \ldots)$. If $\tilde{p}_i = 0 \;\forall\, i \in \{1, \ldots, m\}$, then for $m > 1$ and $n = 2k+1$:

$$\frac{S_{n+m}}{n+m} - \frac{1}{2} = \frac{1+k}{2k+1+m} - \frac{1}{2} = \frac{1-m}{2(m+2k+1)} < 0 \quad \forall\, n \in \mathbb{O}.$$

But, on the other hand, if $\tilde{p}_i = 1 \;\forall\, i \in \{1, \ldots, m\}$, then:

$$\frac{S_{n+m}}{n+m} - \frac{1}{2} = \frac{1+k+m}{2k+1+m} - \frac{1}{2} = \frac{1+m}{2(m+2k+1)} > 0 \quad \forall\, n \in \mathbb{O}.$$

We need the following technical lemma to conclude the proof for a general centered measure.

**Lemma 6.3.1.** *$\mu = \prod_{n=1}^{\infty} \nu_n \ll \mu_0 = \prod_{n=1}^{\infty} \nu_0$ provided (6.2.9) holds.*

With this lemma the proof of Theorem 6.2.2 is concluded as $\mu(\mathcal{C}_I) = 0$ by (6.3.8).

*Proof of the technical Lemma 6.3.1.* Let

$$d_H(\nu_n, \nu_0) := \left(2\left(1 - H(\nu_n, \nu_0)\right)\right)^{1/2} = \left(\int_X \left(\sqrt{\frac{d\nu_0}{d\tau}} - \sqrt{\frac{d\nu_n}{d\tau}}\right)^2 d\tau\right)^{1/2} \tag{6.3.9}$$

where

$$H(\nu_0, \nu_n) := \int_X \sqrt{\frac{d\nu_n}{d\tau}} \sqrt{\frac{d\nu_0}{d\tau}} d\tau$$

is the Hellinger integral with $\tau$ is a measure such that $\nu_0, \nu_n$ are absolutely continuous and $X = [0,1]$ here. By Kakutani Dichotomy Theorem, [DPZ14, Proposition 2.21], if

$$\prod_{n=1}^{\infty} H(\nu_0, \nu_n) > 0, \tag{6.3.10}$$

then $\mu \ll \mu_0$. To prove (6.3.10) we need to know the following fact: for $0 \leqslant a_n < 1$, $\prod_{n=1}^{\infty}(1 - a_n)$ converges to a positive number iff $-\sum_n \log(1 - a_n)$ converges iff $\sum_n a_n$ converges, by the limit comparison test. Here $1 - a_n = H(\nu_n, \nu_0)$. Indeed, $H(\nu_n, \nu_0) \leqslant 1$ by Cauchy-Schwarz's inequality and $H(\nu_n, \nu_0) > 0$ as $\nu_n \ll \nu_0$ by hypothesis (so take $\tau = \nu_0$). Thus, it is enough if we prove that

$$1 - H(\nu_n, \nu_0) \lesssim \|\nu_n - \nu_0\|, d_{KL}(\nu_n - \nu_0) \tag{6.3.11}$$

which entails that $\sum_n (1 - H(\nu_n, \nu_0))$ converges by (6.2.9). First, using (6.1.3), (6.3.9) and

$$\left(\sqrt{\frac{d\nu}{d\tau}} - \sqrt{\frac{d\nu'}{d\tau}}\right)^2 (x) \leqslant \left(\sqrt{\frac{d\nu}{d\tau}} - \sqrt{\frac{d\nu'}{d\tau}}\right)(x) \left(\sqrt{\frac{d\nu}{d\tau}} + \sqrt{\frac{d\nu'}{d\tau}}\right)(x) = \left|\frac{d\nu}{d\tau} - \frac{d\nu'}{d\tau}\right|(x).$$

assuming w.l.o.g. that $\left(\sqrt{\frac{d\nu}{d\tau}} - \sqrt{\frac{d\nu'}{d\tau}}\right)(x) \geqslant 0$. For the second,

$$\begin{aligned}
d_{KL}(\nu\|\nu') &= \int \log \frac{\rho(x)}{\rho'(x)} \rho(x)dx = 2\int \log \frac{\sqrt{\rho(x)}}{\sqrt{\rho'(x)}} \rho(x)dx \\
&= 2\int -\log \frac{\sqrt{\rho'(x)}}{\sqrt{\rho(x)}} \rho(x)dx \geqslant 2\int \left(1 - \frac{\sqrt{\rho'(x)}}{\sqrt{\rho(x)}}\right) \rho(x)dx \\
&= \int \left(1 + 1 - 2\sqrt{\rho(x)}\sqrt{\rho'(x)}\right) dx = \int \left(\sqrt{\rho(x)} - \sqrt{\rho'(x)}\right)^2 dx = 1 - H(\nu, \nu'),
\end{aligned}$$

where we have used that $-\log(x) \geqslant 1 - x$ and defined $\rho := \frac{d\nu}{d\tau}$ and $\rho' := \frac{d\nu'}{d\tau}$. □

**Remark 6.3.2.** As we see from (6.3.11), it is enough for our purposes if the distance satisfies

$$1 - H(\nu_n, \nu_0) \lesssim d(\nu_n - \nu_0). \tag{6.3.12}$$

So this is the condition which defines $\mathcal{D}$, i.e., $d \in \mathcal{D}$ iff $d$ satisfies (6.3.12). In the last part of the proof we showed that $d_{KL}, \|\cdot\| \in \mathcal{D}$ so defined. But the set is larger than that, for instance, Bhattacharyya distance is defined as:

$$d_B(\nu, \nu') := -\log H(\nu, \nu').$$

Then, $d_B \in \mathcal{D}$ because $-\log(x) \geqslant 1 - x$ .

We finish this section with the proof of Proposition 6.2.5. We define:

$$\hat{p}^{\varepsilon_0} := \begin{cases} 1 \text{ if } p \in [1 - \varepsilon_0, 1] \\ 0 \text{ if } p \in [0, 1 - \varepsilon_0) \end{cases}$$

with $\hat{p}^0 = \tilde{p}$. Thus, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \hat{p}_i^{\varepsilon_0} = \epsilon_{1-\varepsilon_0,1} \text{ a.s.}$$

hence, by Egorov's Theorem the convergence is almost uniform. If $0 < \varepsilon < \epsilon_{1-\varepsilon_0,1}$, then $\varepsilon' := \epsilon_{1-\varepsilon_0,1} - \varepsilon > 0$ so by the definition of limit for $n > N$ large enough

$$\left|\frac{1}{n}\sum_{i=1}^{n} \tilde{p}_i - \epsilon_{1-\varepsilon_0,1}\right| < \varepsilon' \Rightarrow \frac{1}{n}\sum_{i=1}^{n} \tilde{p}_i > \varepsilon.$$

The result follows from the fact that almost uniform convergence implies that this happens (with $N$ uniform) for a set of measure no less than $1-\delta$.

## 6.4 Extending Theorem 6.2.2

We present a theorem which includes some cases not considered in Theorem 6.2.2. There is some overlapping with Theorem 6.2.2 but we opted to give a self-contained and easier proof of that theorem. This makes the exposition clearer for some readers as in the next proof we will use more technical tools, see Appendix 6.A.

**Theorem 6.4.1.** *If $\mu = \prod_{i=1}^{\infty} \nu_i$ is a measure such that:*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(m_i - \frac{1}{2}\right) < \infty, \tag{6.4.1}$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (m_i - m_i^2) > 0, \tag{6.4.2}$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \epsilon_{1i} < \frac{1}{2} \tag{6.4.3}$$

*where $\epsilon_{1i} := \nu_i(\{1\})$ and $\sigma_{T,n} := \left(\sum_{i=1}^{n} \mathbb{E}\left((p_i - m_i)^2\right)\right)^{\frac{1}{2}}$ goes to infinity, then CJP does not hold $\mu$-almost surely, i.e., $\mu(\mathcal{C}_I) = 0$.*

**Example 6.4.2.** The biased measures of Definition 6.2.4 are included in this theorem. Indeed, for $\mu_0$ with $b < 0$, $m^1 < \frac{1}{2}$ so (6.4.1) holds because $m_i = m^1\ \forall i \in \mathbb{N}$. Condition (6.4.2) holds if $\nu_0 \neq \alpha\delta_0 + (1-\alpha)\delta_1$ for[3] some $1/2 < \alpha \leqslant 1$. Indeed,

$$m^1 - m^2 = \int_{[0,1]} (x - x^2)\, d\nu_0(x) = \int_{(0,1)} (x - x^2)\, d\nu_0(x) > 0 \tag{6.4.4}$$

as long as $\nu_0((0,1)) > 0$. Now, as $m^1 < 1/2$, it must be that $\nu_0(\{1\}) = \epsilon_1 < \frac{1}{2}$ so (6.4.3) holds trivially.

We can improve the theorem as follows. First, some definitions, recall (6.3.6),

$$\mathcal{B}_{n_0,n} := \bigcap_{k=0}^{(n-n_0)/2} \mathcal{B}_{n_0+2k} = \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \ / \ S_k > k/2\ \ \forall\, k \in \{n_0, n_0+2, \ldots, n\}\},$$

$$\mathcal{B}^b_{n_0,n} := \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \ / \ S_k > k/2\ \ \forall\, k \in \{n_0, n_0+2, \ldots, n\} \text{ and } S_n = (n+1)/2\}.$$

[3]We arrive to the same conclusion in this case of $\nu_0 = \alpha\delta_0 + (1-\alpha)\delta_1$, but we should argue as in (6.3.6) and below.

The first set is given by the sequences such that the sum satisfies $S_k > k/2$ as in condition (6.3.2) for odd numbers between $n_0$ and $n$ and the second is a subset such that the last sum is in the border case $S_n = (n+1)/2$.

**Theorem 6.4.3.** *Assume that (6.4.1) and (6.4.2) hold. If the* lim inf *in (6.4.3) is* 1/2 *substitute (6.4.3) for either there is a $n \geqslant n_0$ such that $\mu\left(\mathcal{B}_{n_0,n}\backslash\mathcal{B}^b_{n_0,n}\right) = 0$ and $\mu(\{S^{n+2}_{n+1} = 0\})$ or*

$$\sum_{k=0}^{\infty} \mu(\mathcal{B}^b_{n_0,n_0+2k} \mid \mathcal{B}_{n_0,n_0+2k})(1-\epsilon_{1(n_0+2k+1)})(1-\epsilon_{1(n_0+2k+2)}) = \infty \;\; \forall n_0 \in \mathbb{O}. \quad (6.4.5)$$

*If so, we arrive at the same conclusion, the CJP does not hold $\mu$-almost surely.*

**Remark 6.4.1.** Some comments on the new hypothesis are in order. First, condition (6.4.1) is a generalization of the centered condition (6.2.3). Second, condition (6.4.2) is a generalization of $m^1 > m^2$ that we saw in (6.3.3). Third, (6.4.3) is a generalization of $\epsilon < 1/2$ in the previous theorem. Condition (6.4.5) can be used to treat the case of purely atomic measures like $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$ where there is no uncertainty as either $p = 0$ or $p = 1$, see Appendix 6.A.

We will prove these more technical theorems and give an example of application of the latter in Appendix 6.A.

## 6.5 Weighted Condorcet Jury Theorem and its applicability

But even in the case $b \leqslant 0$, not everything is lost. We can try to modify the aggregation procedures to achieve a competent mechanism. The natural idea is the consideration of a weighted majority rule described in (1.11). By hypothesis, $\mathbb{P}(X_i = 1) = p_i$ and $\{X_i\}_{i=1}^{\infty}$ are independent. With weighted majority rule we have the following version of the CJT (sufficient conditions).

**Proposition 6.5.1.** *If either*

$$\frac{\sum_{i=1}^n w_i(p_i - q_i)}{\sqrt{\sum_{i=1}^n w_i^2 p_i q_i}} \to \infty \quad (6.5.1)$$

*or, for any n large enough,*

$$\sum_{i=1}^{n} w_i \delta_{p_i 1} > \sum_{i=1}^{n} w_i (1 - \delta_{p_i 1})$$

*where $\delta_{ij}$ is the Kronecker delta, then the Condorcet Jury Property holds for the weights $w$.*

*Proof.* As we have that

$$\mathbb{E}(X_i) = 2p_i - 1 = p_i - q_i, \;\; \mathrm{Var}(X_i) = 4p_i q_i,$$

where $q_i := 1 - p_i$, then

$$\mathbb{P}(X_n^w \leqslant 0) = \mathbb{P}(X_n^w - \mathbb{E}(X_n^w) \leqslant -\mathbb{E}(X_n^w)) \leqslant \mathbb{P}(|X_n^w - \mathbb{E}(X_n^w)| \geqslant \mathbb{E}(X_n^w)) \leqslant$$
$$\leqslant \frac{\mathrm{Var}(X_n^w)}{\mathbb{E}(X_n^w)^2} = \frac{4\sum_{i=1}^n w_i^2 p_i q_i}{(\sum_{i=1}^n w_i(p_i - q_i))^2} \to 0$$

by Chebyshev's inequality as $n \to \infty$, which can be applied because $\mathbb{E}\left(X_n^w\right) > 0$ if $n$ is large enough by (6.5.1). For the second condition,

$$X_n^w \geqslant \sum_{i=1}^{n} \left(w_i \delta_{p_i 1} - w_i (1 - \delta_{p_i 1})\right) > 0 \text{ a.s.}$$

by hypothesis if $n$ is large enough. □

**Remark 6.5.1.** These conditions are the generalizations of (6.3.1) and (6.3.2). If we had that, for instance, $w_i \geqslant 1$, it can be checked that the proof given in [BP98] applies to our case and these conditions are necessary too. $w_i \geqslant 1$ corresponds to the case where no voter loses, formally, its weight on the election.

Then, we can think of a procedure such that

$$w = \alpha + \sum_{i=1}^{L} \beta_i p^i + \varepsilon,$$

i.e., the weight is correlated with the probability of "being right"(we assume the polynomial of $p$ is increasing), but there is a random error $\varepsilon$. This error can be interpreted as a measurement error, we cannot expect to obtain a perfect correlation. For simplicity we can assume

$$w = \alpha + \beta p^k + \varepsilon \tag{6.5.2}$$

for some positive $k \in \mathbb{N}$. We also assume that $w \in [1, W]$ for some $W \geqslant 1$. Thus, we choose $\alpha = 1$ and $\beta = W - 1$. That is,

$$w = w_d(p) + \varepsilon$$

where $w_d$ would be the deterministic weight for a given probability $p$ going from 1 to $W$ as a polynomial function. But there will be errors in the assignment of the weights and this is captured by $\varepsilon$. Now we are *not* going to assume that the measure $\nu$ is centered, i.e., we allow the situation $m < 1/2$ or $b < 0$. Our only requirement will be much weaker; $\nu_0\left((\frac{1}{2}, 1]\right) > 0$. Otherwise ($p > \frac{1}{2}$ does not happen almost surely), we cannot expect the CJP because in the best situation weights would reduce it to the case $\nu_0 = \delta_{1/2}$ where we know that the CJP fails.

But even though the distribution might be biased toward the wrong option, we will prove that the CJT will hold almost surely if the weights are properly chosen. This is the content of the next theorem. We define $\mathcal{C}_I^w$ as the set of sequences of probabilities $\{p_n\}_{n=1}^{\infty}$ such that for the weighted majority rule (**??**) according to (6.5.2), the CJP holds (note that the social choice function is not fixed as it depends on the weights). Also, $\tilde{\nu}_0$ is now a measure on $(p, \varepsilon)$ but $\varepsilon$ is not independent of $p$ (see (6.5.3)), i.e., it is not a product measure. Similarly, $\mu$ will be absolutely continuous (following the idea of Lemma 6.3.1) w.r.t. $\mu_0 = \prod_{n=1}^{\infty} \tilde{\nu}_0$. With this setting:

**Theorem 6.5.2.** *Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a set of random variables distributed according to:*

$$\varepsilon|_p \sim \mathcal{N}_a^b(0, \sigma_W^2) \tag{6.5.3}$$

*where $\mathcal{N}_a^b$ is the truncated Gaussian distribution restricted to the interval $(a, b)$ where $a \equiv a(p, W) := -(W-1)p$, $b \equiv b(p, W) := (W-1)(1-p)$. Let us assume that $\tilde{\nu}_0$ satisfies $\tilde{\nu}_0\left(\{p \in (\frac{1}{2}, 1]\}\right) > 0$. Then, there is a $k$ such that if $(W-1)/\sigma_W, W$ are large enough,* *then*

$$\mu(\mathcal{C}_I^w) = 1,$$

*i.e., the CJP holds almost surely with this weighted majority rule.*

The idea behind the theorem is clear: if we can find a procedure, with a suitable error, to assign weights according to competence, the (weighted) CJT will hold almost surely. Note that now we are considering the posterior probability too, as the measure is not centered any more. But, as we said, the main ingredient is the correct assignment of weights. In the next subsection we take this question seriously. We also consider how "fair" this situation would be.

*Proof.* First note that the first hypotheses ensure that $w \in [1, W]$ as if $p$ is given, then $a(p) = 1 - w_d(p)$, $b(p) = W - w_d(p)$ and $\varepsilon = w - w_d(p)$. Let us explore the first condition of Proposition 6.5.1:

$$w(p-q) = -(1+\varepsilon) + 2(1+\varepsilon)p + (1-W)p^k + 2(W-1)p^{k+1}. \tag{6.5.4}$$

We can analyze the expected values. As $p \in [0,1]$, then

$$\mathbb{E}\left(p^k\right) \geqslant \mathbb{E}\left(p^{k+1}\right).$$

Nevertheless,

$$\mathbb{E}\left(p^{k+1}\right)^{\frac{1}{k+1}} \geqslant \mathbb{E}\left(p^k\right)^{\frac{1}{k}} \tag{6.5.5}$$

by Hölder's inequality. Let us show that there is a $k$ such that:

$$2\mathbb{E}\left(p^{k+1}\right) - \mathbb{E}\left(p^k\right) > 0.$$

Indeed[4],

$$2m^{k+1} - m^k = \int_{[0,1/2)} x^k(2x-1)d\nu_0(x) + \int_{(1/2,1]} x^k(2x-1)d\nu_0(x).$$

Thus,

$$2^k\left(2m^{k+1} - m^k\right) = \int_{[0,1/2)} 2^k x^k(2x-1)d\nu_0(x) + \int_{(1/2,1]} 2^k x^k(2x-1)d\nu_0(x).$$

By the Dominated Convergence Theorem, the first term goes to zero as $k \to \infty$ and $2^k x^k(2x-1) \to \infty$ as $k \to \infty$ for the second term, so there is a $k$ such that the LHS is positive.

Now we analyze the expectations that involve the error term $\varepsilon$, in particular,

$$\mathbb{E}\left((2p-1)\varepsilon\right) = \mathbb{E}\left((2p-1)\mathbb{E}\left(\varepsilon|p\right)\right)$$

by the law of iterated expectations. It is well-known that

$$\mathbb{E}\left(\varepsilon|p\right) = \sigma\frac{\phi(\alpha)-\phi(\beta)}{\Phi(\beta)-\Phi(\alpha)}, \tag{6.5.6}$$

[4] We define $\nu_0 := \tilde{\nu}_0 \circ \pi$ where $\pi(p,\varepsilon) = p$, i.e., the push-forward measure. Thus, $\tilde{\nu}_0(\{p \in A\}) = \nu_0(A)$.

where $\alpha := \frac{a}{\sigma}$, $\beta := \frac{b}{\sigma}$, $\phi$ the p.d.f. of a standard Gaussian function and $\Phi$ its c.d.f. Then,

$$\mathbb{E}\left(\varepsilon|p\right) = (W-1)f(x,p),$$

where $x := (W-1)/\sigma$ and

$$f(x,p) := \frac{\phi((1-p)x) - \phi(-px)}{x(\Phi(-px) - \Phi((1-p)x))}.$$

It is straightforward to check that for $p \in (0,1)$, $\mathbb{E}\left(\varepsilon|p\right) \to 0$ exponentially as $x \to \infty$ because

$$\alpha = \frac{(1-W)p}{\sigma} = -px \to -\infty, \quad \beta = \frac{(W-1)(1-p)}{\sigma} = (1-p)x \to \infty$$

as $x \to \infty$. If $p = 0$, then $\beta$ still goes to infinity and if $p = 1$, $\alpha$ still goes to $-\infty$. By the Dominated Convergence Theorem (as (6.5.6) ensures the integrand is bounded by continuity on a compact set) we conclude that:

$$\lim_{x\to\infty} \mathbb{E}\left((2p-1)f(x,p)\right) = 0.$$

Therefore,

$$\frac{1}{n}\sum_{i=1}^{n} w_i(p_i - q_i) \to 2m^1 - 1 + (W-1)\mathbb{E}\left(2p^{k+1} - p^k + (2p-1)f(x,p)\right) \text{ a.s.}$$

as $n \to \infty$ by the SLLN. By the discussion above, if $x, W$ are large enough, then the limit is positive.

For the denominator of the first condition in Proposition 6.5.1 we know that

$$\mathbb{E}\left(w^2 p(1-p)\right) > 0$$

as $w \geqslant 1$, $p(1-p) \geqslant 0$ if $p \neq 0$, $p \neq 1$ $\nu_0$-almost surely. The first case is rejected because $\delta_0((1/2,1]) = 0$ and if $\nu_0 = \delta_1$, then the CJT holds for $W = 1$ trivially (in this case we do not need $W$ large). Thus, by the SLLN again,

$$\frac{\sum_{i=1}^{n} w_i(p_i - q_i)}{\sqrt{\sum_{i=1}^{n} w_i^2 p_i q_i}} = \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n} w_i(p_i - q_i)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} w_i^2 p_i q_i}} \to \infty.$$

Hence, the CJP now has $\mu_0$-measure equal to one, i.e., $\mu_0(\mathcal{C}_I^w) = 1$. As we did in the proof of Theorem 6.2.2, the same holds for a "deviation" of this measure. Indeed, it follows from $\mu \ll \mu_0$ and the fact that the complement of $\mathcal{C}_I^w$ is a $\mu_0$-null set. □

**Remark 6.5.2.** We could use weaker hypotheses, as in Theorem 6.4.1, nevertheless we opted for maintaining the simplicity. For instance, we could replace the independence of $\varepsilon$ by ergodicity and use the Ergodic Theorem instead of the SLLN, replace the Gaussianity by a nice enough distribution or make the parameters of the distribution depend on $p$.

## 6.6 Concluding remarks

We have shown that the asymptotic CJP or the CJT for independent voters (which includes the MoA and the case studied by Condorcet) are, a priori, highly unlikely (see Theorem 6.2.2, 6.4.1) unless we add some good enough epistemic weights, i.e., weights correlated with epistemic rationality. That is, if we choose an arbitrary sequence of voters, it will not satisfy the CJP almost surely. The bottom line is that applying the CJT (as it is common in some debates) might not be adequate if, using the Bayesian approach, there is no particular evidence of voter competence to update our priors (it might be the opposite case, [Cap11]) nor some weights to correct the lack of competence. If "good" epistemic weights are added, its probability goes to one by Theorem 6.5.2. Note that in this latter case we are not estimating the prior probability but the probability given almost any evidence on voters competence (including the less favorable situations). These weights must be correlated (not necessarily a perfect correlation) with epistemic rationality and they guarantee a minimal weight of one to every voter. The CJT is an important and useful result to improve the decision-making process, but we have to ensure it holds when it is supposed to hold.

Obviously, our framework is a toy model of the real world, but a good point to start and, in fact, it is the same model that is usually used when the CJT is invoked. Some complications can be added and could be the topic of future research. For instance, in some processes we do not expect the options to remain unchanged if competent voters are more influential, but this is not directly captured in a dichotomous choice. An important limitation of the framework is the independence assumption. Votes can be correlated because of a deliberation process ("contagion" in general), common sources of information or strategic voting, see [Piv17] and references therein. Some works have treated the CJT for dependent voters, see for instance [PZ12; Piv17]. In this case the known necessary and sufficient conditions involve the covariance between votes, say $\rho_{ij}$. Thus, the measure $\mu$ should include these parameters, but it cannot be a product measure as above. Indeed, if, for instance, $X_i \in \{0, 1\}$, then

$$p_{ij} := \mathbb{E}\left(X_i X_j\right) = \left|\mathbb{E}\left(X_i X_j\right)\right| \leqslant \sqrt{p_i p_j}$$

by Hölder inequality. Also, as $p_{ij} = \mathbb{P}\left(\{X_i = 1\} \cap \{X_j = 1\}\right) \leqslant p_i, p_j$ and $\rho_{ij} = p_{ij} - p_i p_j$. So, $p_{ij}$ or $\rho_{ij}$ cannot be taken independently of $p_i, p_j$ in $\mu$. A careful analysis would be needed in this situation because how to choose the measure is not trivial. Furthermore, sufficient and necessary conditions are less understood, so more analysis in that direction would be needed too. Anyway, this setting is somehow more restrictive as we not only need some competence condition, but additional requirements must be added, see [Piv17, Theorem 5.3]. For instance, the "average correlation" cannot growth too much: if votes are highly correlated there is little point in increasing the number of voters as they will vote in the same direction. So in this case, we would have to worry not only about competence but also about the correlation between votes.

# APPENDICES

## 6.A Proof of Theorem 6.4.1, 6.4.3 and an example

The proof of Theorem 6.4.1 is going to be similar (except Kakutani's Theorem) to the proof of Theorem 6.2.2, but more technical. Some steps which are already there will be omitted here (but they will be properly referenced). As we did there, we define

$$Q_n := \frac{\sum_{i=1}^n p_i - \frac{n}{2}}{\sqrt{\sum_{i=1}^n p_i q_i}} = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(p_i - \frac{1}{2}\right)}{\sqrt{\frac{1}{n} \sum_{i=1}^n p_i q_i}}.$$

As we said above, this quotient appears in the necessary and sufficient conditions of the CJT. $Q_n$ can be rewritten as:

$$Q_n = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(m_i - \frac{1}{2}\right) + \frac{\sigma_{T,n}}{\sqrt{n}} \frac{1}{\sigma_{T,n}} \sum_{i=1}^n (p_i - m_i)}{\sqrt{\frac{1}{n} \sum_{i=1}^n \left(p_i - m_i - (p_i^2 - m_i^2) + m_i - m_i^2\right)}}.$$

where $\sigma_{T,n} := \left(\sum_{i=1}^n \mathbb{E}\left((p_i - m_i)^2\right)\right)^{\frac{1}{2}}$. Now, by Kolmogorov's version of the SLLN (which we can apply because of (6.1.1)) we have almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (p_i - m_i) = 0, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (p_i^2 - m_i^2) = 0.$$

Now, by the hypotheses of the theorem we can take a subsequence such that

$$\lim_{k\to\infty} \frac{1}{\sqrt{n_k}} \left(\sum_{i=1}^{n_k} m_i - \frac{1}{2}\right) = C < \infty,$$

$$\lim_{k\to\infty} \frac{1}{n_k} \left(\sum_{i=1}^{n_k} m_i - m_i^2\right) = c > 0.$$

Finally, let us note that by (6.1.1),

$$\frac{\sigma_{T,n}}{\sqrt{n}} \leqslant 1.$$

Similarly,

$$\frac{\sum_{i=1}^n \mathbb{E}\left(p_i - m_i\right)^3}{\sigma_{T,n}^3} \leqslant \frac{\sigma_{T,n}^2}{\sigma_{T,n}^3} \to 0$$

as $\sigma_{T,n} \to \infty$. Thus, Lyapunov's condition holds, so we can apply Lindeberg's CLT. Hence, taking a subsequence (that we relabeled again) and using Slutsky Theorem as before,

$$Q_{n_k} \to \mathcal{N}(\mu_Q, \sigma_Q^2)$$

in distribution as $k \to \infty$ for some $\mu_Q, \sigma_Q \in \mathbb{R}$. Then we can apply (6.3.5) to conclude that the first condition of [BP98, Theorem 2] does not hold almost surely. Let us turn now to the second condition. For that purpose, as we did above, we define:

$$\tilde{p} := \begin{cases} 1 \text{ if } p = 1 \\ 0 \text{ if } p \in [0,1) \end{cases}$$

We also define the sums:

$$S_{n_0}^n := |\{i : 1 \leqslant n_0 \leqslant n \text{ and } p_i = 1\}| = \sum_{i=n_0}^{n} \tilde{p}_i.$$

First, if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \epsilon_{1i} < \frac{1}{2}, \tag{6.A.1}$$

by Kolmogorov's SLLN then,

$$\lim_{n\to\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{p}_i - \frac{1}{n} \sum_{i=1}^{n} \epsilon_{1i} \right) = \limsup_{n\to\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{p}_i - \frac{1}{n} \sum_{i=1}^{n} \epsilon_{1i} \right) = 0 \text{ a.s.}$$

Using that $\limsup_{n\to\infty}(x_n) + \liminf_{n\to\infty}(y_n) \leqslant \limsup_{n\to\infty}(x_n + y_n)$, then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{n=1}^{\infty} \tilde{p}_n = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \epsilon_{1i} < \frac{1}{2}$$

almost surely. Thus, for large enough $n_0$,

$$\sup_{n\geqslant n_0} \frac{S_n}{n} < \frac{1}{2} - \delta$$

for some $\delta > 0$, therefore violating the second condition for the CJT. This finishes the proof of Theorem 6.4.1.

Let us prove Theorem 6.4.3. Recall that we defined the sets, for $n_0 \leqslant n$ odd numbers:

$$\mathcal{B}_{n_0,n} := \bigcap_{k=0}^{(n-n_0)/2} \mathcal{B}_{n_0+2k} = \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \ / \ S_k > k/2 \ \ \forall\, k \in \{n_0, n_0+2, \ldots, n\}\},$$

$$\mathcal{B}_{n_0,n}^b := \{(p_i)_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \ / \ S_k > k/2 \ \ \forall\, k \in \{n_0, n_0+2, \ldots, n\} \text{ and } S_n = (n+1)/2\}.$$

If the lim inf in (6.A.1) is $1/2$, note that

$$\mathcal{B}_{n_0,n+2} = \{(p_i)_{i=1}^{\infty} \in \mathcal{B}_{n_0,n}^b \ / \ S_{n+1}^{n+2} \geqslant 1\} \bigsqcup \left( \mathcal{B}_{n_0,n} \backslash \mathcal{B}_{n_0,n}^b \right),$$

where $\sqcup$ denotes a disjoint union. The idea is that a sequence is in $\mathcal{B}_{n_0,n+2} \subset \mathcal{B}_{n_0,n}$ because it satisfies either $S_{n_0}^n = (n+1)/2$ (so we need that the next two summands are at least 1) or $S_{n_0}^n > (n+1)/2$ so the sequence is in $\mathcal{B}_{n_0,n+2}$, independently of the next two summands. Thus, applying the (product) measure we will obtain the following recurrence relation:

$$\begin{aligned} \mu(\mathcal{B}_{n_0,n+2}) &= \mu(\mathcal{B}_{n_0,n}) \left( \mu(\mathcal{B}_{n_0,n}^b \mid \mathcal{B}_{n_0,n}) \mu\left(\{S_{n+1}^{n+2} \geqslant 1\}\right) + 1 - \mu(\mathcal{B}_{n_0,n}^b \mid \mathcal{B}_{n_0,n}) \right) \\ &= \mu(\mathcal{B}_{n_0,n}) \left(1 - \alpha_{n_0,n}\beta_n\right), \end{aligned}$$

where

$$\mu(\mathcal{B}^b_{n_0,n} \mid \mathcal{B}_{n_0,n}) := \frac{\mu\left(\mathcal{B}^b_{n_0,n} \cap \mathcal{B}_{n_0,n}\right)}{\mu\left(\mathcal{B}_{n_0,n}\right)} =: \alpha_{n_0,n} \text{ and } \beta_n := \mu\left(\{S^{n+2}_{n+1} = 0\}\right).$$

Note that $\beta_n = (1 - \epsilon_{1(n+1)})(1 - \epsilon_{1(n+2)})$. Thus, for $n > n_0$ both odd:

$$\mu(\mathcal{B}_{n_0,n+2}) = \mu(\mathcal{B}_{n_0,n_0}) \prod_{k=0}^{\frac{n-n_0}{2}} (1 - \alpha_{n_0,n_0+2k}\beta_{n_0+2k}). \tag{6.A.2}$$

Now we take the limit $n \to \infty$. If there is some $k$ such that $(1 - \alpha_{n_0,n_0+2k}\beta_{n_0+2k}) = 0$, the product is zero and this corresponds to the first condition of Theorem 6.4.3. Otherwise, as we saw in the proof of Lemma 6.3.1, this infinite product will be zero iff

$$\sum_{k=0}^{\infty} \alpha_{n_0,n_0+2k}\beta_{n_0+2k} = \infty.$$

Then, similarly as we did above,

$$\mu\left(\mathcal{B}_{n_0,\infty} := \bigcap_{k=0}^{\infty} \mathcal{B}_{n_0,n_0+2k}\right) = \lim_{k\to\infty} \mu\left(\mathcal{B}_{n_0,n_0+2k}\right) = 0.$$

Therefore, $\mu\left(\mathcal{B}\right) \leqslant \sum_{n_0} \mu(\mathcal{B}_{n_0,\infty}) = 0$. This finishes the proof.

### 6.A.1 Example of application to the case $\nu_0 = \frac{1}{2}\left(\delta_0 + \delta_1\right)$: some combinatorics

Let us consider how to apply (6.4.5). For simplicity, assume $n_0 = 1$. In order to understand the set $\mathcal{B}^b_{1,n}$, we need to know how many points will satisfy $S_{1+2k} \geqslant k+1$ for $k = 0, \ldots, (n-1)/2$ and $S_n = (n+1)/2$. For $k = 0$, the only possibility is $S_1 = 1$. Thus, we need to see the number of ways in which $S^n_2 = \frac{n-1}{2}$ such that $S^{1+2k}_2 \geqslant k$ for $k = 0, \ldots, (n-1)/2 - 1$. We can see this graphically if we consider a grid where $\tilde{p}_i = 1$ is translated into moving up and $\tilde{p}_i = 0$ is translated into moving to the right. The conditions above are equivalent to the condition that for every point $(x, y)$ of the path such that $x + y = 2k$, then $x \leqslant y$ and the end point is $(m, m)$ where $n = 2m + 1$. This is illustrated in Figure 6.1. Note that the blue path satisfies these conditions while the red one does not because of the point $(3, 1)$ in black. Our conditions are not the same as lying above the diagonal (dashed line), as the blue path is below it at $(4, 3)$. Nevertheless, if we allow the path to move $(1, 1)$ at points on the diagonal, like the blue arrow in Figure 6.1 shows, we can consider that the allowed paths are always above the diagonal. So the problem reduces to counting the total number of these paths.

In order to do so, we are going to establish some bijections as it is standard in combinatorics. First, if we change the movement (1,1) to (0,1), there is a bijection with the paths starting at (0,0) and ending on $\{(x, m) \ : \ x \in \mathbb{N}\}$. Second, if we add to these paths the movement (0,1) and the complete them with $(1, 0)$ till they reach the diagonal, there is a bijection with the paths starting at $(0, 0)$ and ending at $(m + 1, m + 1)$ without going below the diagonal. It is standard[5] that the total

[5]For instance, this is the number of Dyck paths, see Problem 28, 52 and Theorem 1.5.1 in [Sta15] for more details on the bijections. Also, these numbers appear in the ballot problem: suppose $A_1$ and $A_2$

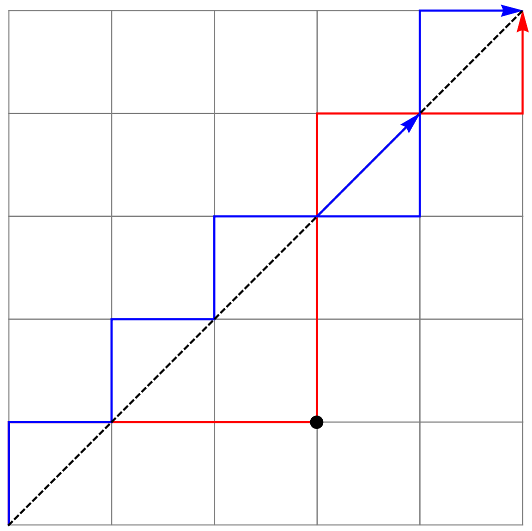


FIGURE 6.1: Illustration for the case $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$ with $n = 11$.

number of paths is $C_{m+1}$, where $C_n$ represents the $n$-th Catalan number, i.e.,

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Now, note that we can express (6.4.5) equivalently as,

$$\mu(\mathcal{B}_{n_0,n}) = \mu(\mathcal{B}_{n_0,n_0}) - \sum_{i=0}^{\frac{n-n_0}{2}-1} \beta_{n_0+2i}\gamma_{n_0,n_0+2i}$$

using the equations above (6.A.2) where

$$\gamma_{n_0,n} := \mu(\mathcal{B}_{n_0,n})\,\alpha_{n_0,n} = \mu\left(\mathcal{B}^b_{n_0,n}\right).$$

This is useful because in our case of $n_0 = 1$ and $n = 2m+1$ we can compute that sum easily,

$$\mu(\mathcal{B}_{n_0,n_0}) - \sum_{i=0}^{\frac{n-n_0}{2}-1} \beta_{n_0+2i}\gamma_{n_0,n_0+2i} = \frac{1}{2} - \frac{1}{4}\sum_{i=0}^{m-1}\frac{C_{i+1}}{2^{2i+1}} = 2^{-2(m+1)}\binom{2(m+1)}{m+1}.$$

By Stirling's approximation,

$$\mu(\mathcal{B}_{1,2m+1}) = \sqrt{\frac{1}{\pi m}} + O\left(m^{-3/2}\right) \to 0$$

as $m \to \infty$.

are candidates for some election and there are an even number of voters, say $2n$. Let us also assume that $n$ voting for $A_1$ and $n$ for $A_2$. In how many ways can the ballots be counted so that $A_1$ is always ahead of or tied with $A_2$? See the aforementioned theorem.

## 6.B Practical implementation of epistemic weights: from psychology and political philosophy

First, we must not confuse competence ($p$ near one) with other attributes that can be, under some conditions, correlated with competence, such as fluid or crystallized intelligence. Following [Kah03] we can classify cognitive processes into two broad categories: System 1 (intuition) and System 2 (reasoning). The former is autonomous (executed automatically upon encountering the triggering stimulus and independent on input from high-level control systems). Furthermore, it is fast, emotional and relies on heuristics that can lead to biases. System 2 is slow, effortful, analytic... Many processes of System 1 can operate at once in parallel, but System 2 processing is largely serial. But we can split System 2 further into two "minds", [Sta09], the algorithmic and reflective mind. The former deals with slow thinking and demanding computations (fluid intelligence, which IQ tests try to measure) and the latter is related to rational thinking dispositions and its functions are to *initiate* the override biased responses of System 1, the ones based on a "focal model" which can be biased or the simulation of alternative responses. Thus, rationality is a combination of both minds[6], not just the algorithmic one. Obviously, these systems need knowledge to work properly (and the one acquired through learning and past experiences is usually called crystallized intelligence), see Figure 3.3 of [Sta09]. Nevertheless, note that some knowledge can be useless or harmful for achieving competence ("contaminated mindware", [Sta09]) or, even if necessary, remain unused, as in the "override failure". To be more specific, [SW08]:

> ...the relevant mindware for our present discussion is not just generic procedural knowledge, nor is it the hodge-podge of declarative knowledge that is often used to assess crystallized intelligence on ability tests. Instead, it is a very special subset of knowledge related to how one views probability and chance; whether one has the tools to think scientifically and the propensity to do so; the tendency to think logically; and knowledge of some special rules of formal reasoning and good argumentation.

Thus, we must note that competence could not be achieved even if the algorithmic mind is "highly developed". There is evidence that thinking errors are relatively independent of cognitive abilities [SW08]. For instance, there is not a significant correlation between the magnitude of some classical bias popularized by Kahneman [Kah11] (e.g., anchoring effects or conjunction fallacy) and cognitive abilities. Another important example is the so-called myside bias ("people evaluate evidence, generate evidence, and test hypotheses in a manner biased toward their own prior opinions and attitudes"). The authors conjecture that fluid intelligence is only important when there is not a mindware gap (e.g., missing probability or scientific knowledge) and the need to override heuristic responses is detected. This is the case, e.g., in the rose syllogism (all flowers have petals; roses have petals; therefore, roses are flowers–which is invalid) and the belief bias, but not in the Linda problem between-subjects and conjunction fallacy. This feature of Linda problem illustrates an important fact; it is not enough to have the knowledge (here, basic probabilistic knowledge, $\mathbb{P}(A \cup B) \geqslant \mathbb{P}(A)$), but *we must have the tendency to use it when needed*,

[6] Also, the autonomous mind or System 1 can provide rational responses as it might contain normative rules that have been tightly compiled and that are automatically activated as a result of overlearning and practice.

specially when there are no cues to do so. Thinking dispositions, in contrast to cognitive abilities, are viewed as more malleable and this would predict that these skills are more teachable. As we were saying, there are some biases which are correlated with cognitive abilities as the argument evaluation test ([SW08]), but they are not naturalistic or similar to a real-life situation because subjects have been told to decouple prior beliefs from the evaluation of evidence. Then the correlation happens because "participants of differing cognitive abilities have different levels of computational power available for the override operations that make decoupling possible", [SWT13].

Furthermore, more fluid intelligence could be even worse for myside bias. Indeed, in [KPD+17] (see also references therein for more evidence) we can see why. In this experiment, subjects must draw valid causal inference from empirical data. The same empirical data is presented in two ways: in not an ideologically loaded way (skin-rash treatment) and as a partisan issue (gun-control ban). In the former (as expected), the higher the numeracy, the better the responses, but in the latter responses became polarized between liberal democrats and conservative republicans, less accurate and got worse for subjects with higher numeracy skills (algorithmic intelligence). Thus, this could be seen as a conflict between being epistemically rational (fitting one's beliefs to the real world, what is true) and instrumentally rational (optimizing goal fulfillment, what to do). This motivated reasoning can be the means to achieve our goals because sharing some political views is a symbol of membership and loyalty in political groups, expressive rationality, which can be more valuable than epistemic goals. In our day-to-day actions having true beliefs (epistemic rationality) is useful for achieving our goals (instrumental rationality). More precisely, if $\mathcal{A} := \{a_i\}_{i\in\mathcal{I}}$ are the possible actions, $\mathcal{S} := \{s_j\}_{j\in\mathcal{J}}$ the possible states of the world and $\varphi : \mathcal{A} \times \mathcal{S} \to \mathcal{S}$ maps the consequences of the action in each state of the world, then

$$U(a) = \sum_{j\in\mathcal{J}} \mathcal{P}(s_j \mid a) \cdot u\left(\varphi(s_j, a)\right),$$

where $U$ is the von Neumann–Morgenstern utility function and $\mathcal{P}$ assigns probabilities to each state of the world. In order to maximize $U$, $\max_{a\in\mathcal{A}} U(a)$ (instrumental rationality), we need to have correct beliefs about the world, $\mathcal{A}$, $\mathcal{S}$, $\mathcal{P}$ and $\varphi$, i.e., epistemic rationality. But in the political process our beliefs are dissociated from their consequences (one's beliefs on gun-control bans are unlikely to affect political decisions and their consequences), so expressive rationality makes perfect sense as epistemic and instrumental rationality are not necessarily linked and having true beliefs about the world could be less valuable than rejecting our previous beliefs or shared beliefs with our political group. As the social psychologist Jonathan Haidt puts it, we are good rationalizers but poor reasoners when thinking about politics. To achieve an epistemically rational response it could be more useful, for instance, to adopt measures that effectively shield decision-relevant science from the identity-protective motivated reasoning: behaving like a sport hooligan should not be seen an appropriate way to process information. In a recent (preregistered) replication of this study [PAK+21], the effect of motivated reasoning was found but it was less clear the motivated numeracy (motivated reasoning increases with numeracy) finding. In another study, [KS16], they corroborate the same hypothesis of expressive rationality using beliefs about human evolution.

Hence, algorithmic intelligence might not be sufficient for rational thinking and not as necessary as one could initially think, for instance, if epistemically reliable shortcuts are available instead of a direct investigation or simulation of alternative responses. For example, if there is consensus between experts, take that as the most likely option. This could reduce the need for algorithmic intelligence but it does not eliminate some minimal amount; finding a reliable shortcut is a computation demanding process. One should be cautious when assessing weights because they must be correlated with epistemic rationality and the relation between this and other typical measures of intelligence or knowledge is not trivial as we have seen. For instance, one proposal could be [Sta16; SWT16] (total or partial subsets of the CART focused on epistemic rationality) in combination with particular knowledge (mindware) or skills (algorithmic mind) needed for competence in the particular domain of the choice we face. Any other metric that is correlated with this assessment or a similar one could be used too.

Obviously the weight assignment will depend on the particular process under consideration. The assignment for a jury in a criminal trial will not be the same as the one for a democratic process (where part of the evidence presented above fits better). Nevertheless, the main idea still holds: a major part of the assignment should be based on epistemic rationality. But particular mindware should be considered in each situation. For instance, law and the particular criminal evidence for a trial and some basic knowledge of social sciences for a democratic process. Notice that some topics are more prone than others to be solved as epistemic rationality increases. For instance, discussing the means to achieve an agreed end can be easier than discussing the ends we should pursue.

Second, we could think that a more natural way to achieve (1.9) is to exclude voters with $p < 1/2$ (that is, $w = 0$), which will imply $b > 0$ and the CJP will hold almost surely (similar proof as Theorem 6.2.2 or 6.4.1). That is, as we said in the introduction we could consider:

- $w_i = 0$ if $p_i \leqslant 1/2$ (similar to expert rule) or,
- $w_i < 0$ if $p_i < 1/2$, as in (1.12).

Nevertheless, we opted to investigate the case of $w \geqslant 1$, i.e., all votes count (obviously, not in the same proportion) for several reasons. One is that it might be objected that in some circumstances not allowing some voters to participate can express disrespect, i.e., a semiotic objection based on the expressive value of the democratic process, [Bre16, Chapter 5]. To analyze it, the right of a competence decision process must be weighted against the somehow socially conceived expressive value of the restrictions. But in the setting where every potential voter is guaranteed a minimal weight, $w \geqslant 1$ for every voter, these objections are less motivated. In fact, votes have different weights in many present processes, although they are not usually weighted according to competence but other factors.

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