Instituto de Ciencias Matemáticas
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# Minimal Surfaces and Splitting Results on Riemannian Manifolds. Duality and Approximation in Variable Lebesgue Spaces

A thesis submitted in fulfillment of the requirements
for the degree of *Doctor in Mathematics* by

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*Puede haber dos cosas infinitas:*
*los cardinales y mi amor hacia ti,*
*y de la primera no estoy tan seguro.*

# Acknowledgments

*"When you have eliminated the impossible, whatever remains, however improbable, must be the truth".* Esta frase del famosísimo detective Sherlock Holmes me ha acompañado a lo largo de mis años de doctorado. No ha sido un camino sencillo, y las dificultades que me he encontrado me han llevado a pensar en cosas imposibles. Y espero que lo que ha quedado de todo esto en la tesis sea verdad.

No podría haber hecho estas deducciones sin mi director de tesis, Antonio Córdoba. Gracias a él he conocido en mayor profundidad la belleza de las matemáticas que me enamoran más cada día. Nuestro resultado sobre superficies mínimas y su relación con funciones armónicas generalizadas es el fruto de nuestra relación investigadora.

Quisiera agradecer a todos los docentes que me han guiado hasta aquí desde nivel universitario y anterior. En especial a Pepe Martínez que me dedicaba la tarde de los viernes a proponerme problemas de congruencias, triángulos e inducción, cuando aún estaba en el instituto. A los profesores de la Universidad de Murcia, Universidad de Granada y la Universidad Autónoma de Madrid que me han acogido con los brazos abiertos y de los que tanto he aprendido sobre Análisis Funcional, Geometría Convexa, Geometría Diferencial y Ecuaciones en Derivadas Parciales.

No quiero olvidarme de agradecer a los revisores de este manuscrito Fernando Chamizo, Alberto Farina y David Cruz-Uribe. Gracias a sus consejos y sugerencias ha mejorado sustancialmente la exposición de los resultados. Y a su vez también quiero acordarme de los coautores de los resultados presentes en esta tesis. Ha sido muy gratificante colaborar con ellos.

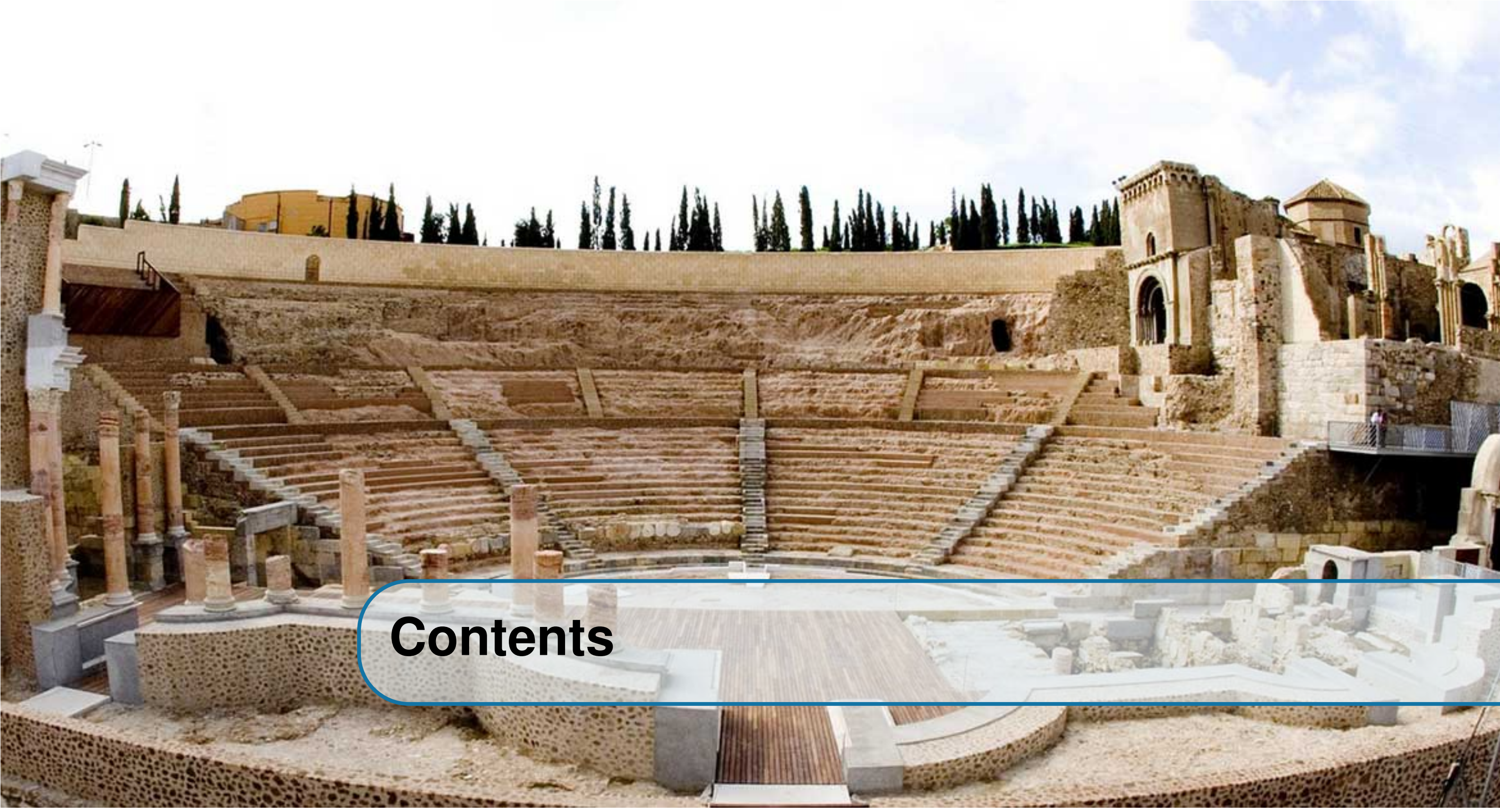
También han sido importantes los compañeros y amigos que me he ido encontrando por el camino, desde mis mosqueteros murcianos Alex y Juanma hasta la maravillosa gente con la que he compartido experiencias en el doctorado como Glenier, el número 1 del ajedrez, y Samuel, con nuestras sobremesas del ICMAT. No quiero enumerar una larga lista de gente y dejarme alguien fuera, pero quiero que sepáis que agradezco todo el apoyo y buenos ratos con la gente que he convivido en el ICMAT, UAM y diversos congresos y escuelas (como las de Bilbao, Chicago y Princeton).

I would like to specially thank the hosts of my various research stays. Thanks to them my knowledge has notably broadened. I really appreciate the huge help I have received from Alberto. He invited me to spend three months with him in an early stage of my doctorate, where nothing seemed to go too well. He made the impossible come true and we obtained a very nice paper with splitting results of which I am very proud. I also thank Wen for inviting me to Academia Sinica for two months. It helped me very much to learn from a new type of culture and mathematic tools like neural networks, which motivated me to study the problem that led to the last chapter of this manuscript. Muchas gracias Ángel, me invitaste a Princeton y la experiencia fue una de las mejores de mi vida. Finally, I am grateful for the opportunity that Nam has give men to work with him in Munich for some months.

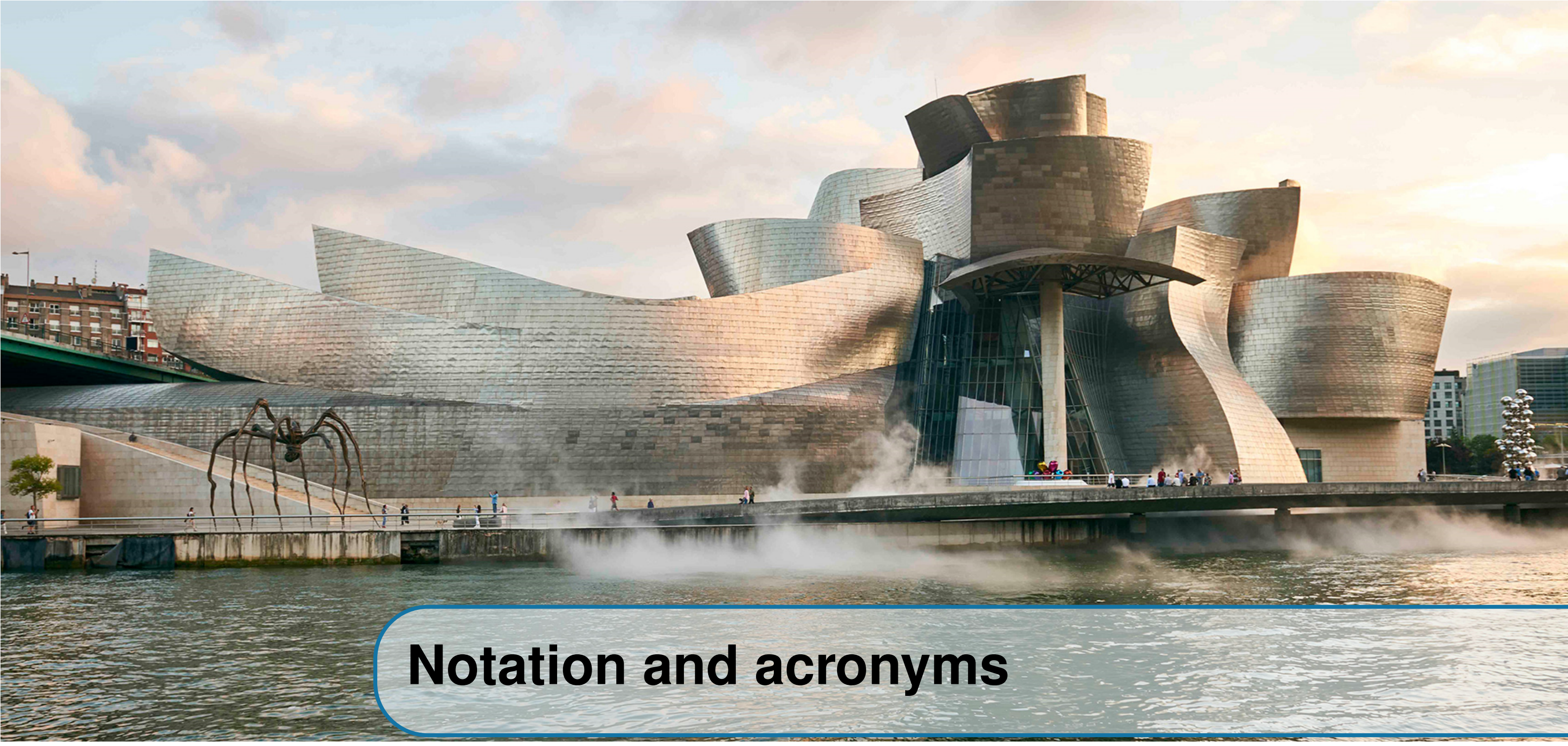
Y si tengo que hablar de gente que ha estado siempre allí en esta travesía, es mi familia. A mis padres, Tomás y Pilar, mi hermano Patxi, y toda la familia política que ya la considero como mía. Y especialmente a Ángela, ella me ha guiado con sus pasos en los buenos y malos momentos de estos últimos años.

Y por supuesto, muchas gracias a usted lector por dedicar parte de su preciado tiempo en leer las palabras de este loco prototipo de doctor. Ya sin más dilación:

*The game is afoot*!



# Contents



# Notation and acronyms

### Acronyms and math symbols

| | |
|---|---|
| TFAE | The following are equivalent. |
| WLOG | Without loss of generality. |
| $\int$ | The integral in the Lebesgue sense. |
| $\fint$ | The mean value. |
| $\Delta_g$ | The Laplace-Beltrami operator. |
| $\cong$ | Isometric isomorphic. |
| $\mathbf{1}_A$ | The characteristic function on $A$. |

### Geometric spaces

| | |
|---|---|
| $\Omega$ | An open subset of $\mathbb{R}^n$. |
| $(\mathbb{R}^n, \lvert\cdot\rvert)$ | The euclidean space of dimension $n$ with the euclidean norm. |
| $B(x,r)$ | The open ball centred at $x$ with radius $r$, $\{y \in \mathbb{R}^n : \lvert y-x\rvert < r\}$. |
| $S(x,r)$ | The sphere centred at $x$ with radius $r$, $\{y \in \mathbb{R}^n : \lvert y-x\rvert = r\}$. |
| $(\mathscr{M}, g)$ | A complete, connected, smooth, boundaryless Riemannian manifold with nonnegative Ricci curvature. |

### Space of functions

| | |
|---|---|
| $C(\Omega)$ | The space of continuous functions over $\Omega$. |
| $C^k(\Omega)$ | The space of $k$-times differentiable functions over $\Omega$ with continuous derivative. |
| $C^\infty(\Omega)$ | The space of infinitely-times differentiable functions over $\Omega$. |
| $C_c^\infty(\Omega)$ | The space of infinitely-times differentiable functions over $\Omega$ whose support is compactly contained in $\Omega$. |
| $L^1(\Omega)$ | The space of Lebesgue-integrable functions over $\Omega$. |
| $L^1_{loc}(\Omega)$ | The space of locally Lebesgue-integrable functions over $\Omega$. |
| $L^\infty_{loc}(\Omega)$ | The space of locally bounded functions over $\Omega$. |
| $L^{p(\cdot)}$ | A variable Lebesgue space. |
| $L_b^{p(\cdot)}$ | The closure of the subspace of a variable Lebesgue space consisting of functions where the exponent function is bounded at their support. |
| $L_Q^{p(\cdot)}$ | The quotient space $L^{p(\cdot)}/L_b^{p(\cdot)}$. |
| $\ell^{p(\cdot)}$ | A variable sequence space. |
| $H_\sigma$ | The subspace of functions that a neural network can represent. |

# LIST OF FIGURES

# Summary


Mathematics is a branch of science which focuses on solving problems in a rigorous, deductive and analytic way through infallible proofs. This text displays solutions to four problems of different disciplines of mathematics which I have studied during my doctorate in Madrid.

**1.** A generalized harmonic function extends harmonic functions. This generalization is obtained through the study of the local or infinitesimal mean value property over balls. Recall that harmonic functions verify the mean value property over balls.

It is well-known that if a generalized harmonic function is continuous, then it has to be smooth and actually harmonic. Nevertheless, there are examples of discontinuous functions that are generalized harmonic (and obviously not harmonic).

Based on several results, it seems that the set of discontinuities of a generalized harmonic function, if existing, has codimension one. It is natural to study the geometry of such sets, at least, in the easiest case: smooth hypersurfaces.

**Problem:** What are the sufficient and necessary hypotheses so that a hypersurface is the set of discontinuities of a generalized harmonic function?

The answer to the previous question led to the following characterization of minimal surfaces.

**Solution:** If the hypersurface is differentiable with bounded gradient, the sufficient and necessary condition is the minimality (the mean curvature vanishes everywhere).

**2.** Bounded solutions of the Poisson equation with nonnegative potential in the euclidean space verify an upper bound for the gradient known as Modica's estimate. Several generalizations of it have been obtained in the last years.

It is evident that unidimensional bounded solutions of the equation verify the equality everywhere in Modica's estimate or its generalizations. It is highly interesting to know if the reciprocal holds in the most general ambient of Riemannian manifolds.

**Problem:** What are the geometric implications of achieving equality at Modica's estimate or its generalizations in Riemannian manifolds?

The answer gives results about the splitting of the Riemannian manifold.


The image above shows the Institute for Advanced Study in Princeton. I was invited to attend *Summer School in Geometric Analysis* at Princeton University. In these places you could breath profound mathematics at every corner.

**Solution:** Equality at Modica's estimate and its generalizations provide several splitting results of the manifold. One of them is global, whereas the other is local, but it is necessarily achieved at a regular point.

The splitting of the Riemannian manifold means that a part of the totality of it is isometric to a product space of a submanifold of codimension one and an interval. Furthermore, the solution is unidimensional there.

**3.** Variable Lebesgue spaces are a generalization of classical Lebesgue spaces where the exponent is a function instead of a constant. To provide a structure it is required to use generalized Musielak-Orlicz spaces.

It is well-known that the dual of a Lebesgue space with finite exponent is another Lebesgue space whose exponent is its Hölder conjugate. It would be natural to think that the dual of a variable Lebesgue space whose exponent function is finite everywhere must be another variable Lebesgue space whose exponent function is pointwise its Hölder conjugate. Unfortunately, the situation is much more complex and it depends on the boundedness of the exponent function. If it is bounded, the dual is known and it is another variable Lebesgue space. Then,

**Problem:** What is the dual of a variable Lebesgue space whose exponent function is unbounded?

If the exponent function is unbounded, the dual has many components.

**Solution:** The dual of a variable Lebesgue space whose exponent function is unbounded consists of two pieces: the first one is a variable Lebesgue space whose exponent function is its Hölder conjugate and the second one is an *abstract* **L**-*space*.

An abstract **L**-space is intuitively like the space $L^1$. Even so, it is difficult to characterize these spaces in full generality. Therefore, we provide explicit examples of this space for variable sequence spaces. Depending on the growth of the exponent function we can analyze the complexity of it.

**4.** One of the most important properties of neural networks is the universal approximation, i.e., a neural network with adequate activation function can represent a continuous function defined on a compact set with any desired precision.

This property has been studied for other function spaces with more derivatives or integration conditions. Nevertheless, it had not been previously studied for variable Lebesgue spaces. Consequently, it was natural to consider the following question:

**Problem:** Does the property of universal approximation of neural networks hold to approximate functions belonging to variable Lebesgue spaces?

Since approximating a function has strong connections to the dual of the space, the results related to the previous problem were key to give a solution.

**Solution:** If the variable Lebesgue space is bounded, then it has the universal approximation property. If it is unbounded, this property is false and, in some cases, we can characterize the subspace of functions which can actually be approximated.

# Resumen


Las matemáticas son la rama de la ciencia centrada en resolver problemas de manera analítica, deductiva y rigurosa a través de demostraciones infalibles. En este escrito recogemos soluciones a cuatro problemas de diferentes disciplinas de las matemáticas que he abordado durante mi doctorado en Madrid.

**1.** Una función armónica generalizada extiende a las funciones armónicas. Esta generalización se obtiene estudiando de manera local o infinitesimal la propiedad de la media en bolas. Recordemos que las funciones armónicas verifican la propiedad de la media en bolas.

Es conocido que si una función armónica generalizada es continua, entonces tiene que ser suave y de hecho armónica. No obstante, hay ejemplos de funciones discontinuas que son armónicas generalizadas (y obviamente no armónicas).

Basado en ciertos resultados, parece que el conjunto de discontinuidades de una función armónica generalizada, si existe, tiene codimensión uno. Es natural plantearse el estudio de la geometría de dichos conjuntos, al menos, en el caso más sencillo: hipersuperficies suaves.

**Problema:** ¿Cuáles son las hipótesis necesarias o suficientes para que una hipersuperficie sea el conjunto de puntos de discontinuidad de una función armónica generalizada?

La respuesta a la anterior pregunta dio lugar a una caracterización de superficies mínimas.

**Solución:** Si la hipersuperficie es diferenciable con gradiente localmente acotado, la condición necesaria y suficiente es que sea mínima (su curvatura media se anule en todo punto).

**2.** Las soluciones acotadas de la ecuación de Poisson con potencial nonegativo en el espacio euclídeo verifican una acotación superior del gradiente conocida como estimación de Modica. A lo largo de los últimos años, se han estudiado diversas generalizaciones.

Es evidente que las soluciones unidimensionales acotadas de la ecuación verifican la igualdad en la estimación de Modica o en la de su generalización en todo punto. Es interesante saber si el reciproco es cierto en el ambiente más general posible de variedades Riemanianas.

**Problema:** ¿Qué implicaciones geométricas tiene alcanzar la igualdad en la estimación de Modica y su generalización en variedades Riemanianas?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Esta foto muestra la Alhambra de Granada. En esta ciudad empecé a estudiar superficies mínimas con una Beca Séneca durante la carrera y durante el doctorado pude regresar para asistir al congreso *Geometry and PDE in front of the Alhambra* que se celebró en un sitio con excelentes vistas de este monumento.

La solución propició los resultados de separación en variedades Riemanianas.

**Solución:** La igualdad en la estimación de Modica y su generalización proporciona sendos resultados de separación de la variedad. El primero de ellos global y el segundo de manera local, pero es necesario que sea alcanzada en un punto regular.

La separación de una variedad Riemaniana viene a través de que una parte o la totalidad de la misma sea isómetrica a un espacio producto de una variedad de una dimensión menor junto a un intervalo. Además, la solución que alcanza la igualdad en dicha estimación tiene un comportamiento unidimensional en dicho entorno.

**3.** Los espacios de Lebesgue variables son una generalización de los espacios de Lebesgue clásicos donde el exponente en vez de ser una constante pasa a ser una función. Para dar una estructura a estos espacios es necesario usar los espacios generalizados de Musielak-Orlicz.

Es altamente conocido que el dual de un espacio de Lebesgue con exponente finito es otro espacio de Lebesgue cuyo exponente es su conjugado de Hölder. Sería entonces natural pensar que el dual de un espacio de Lebesgue variable cuyo exponente es siempre finito tiene que ser otro espacio de Lebesgue variable cuyo función exponente es puntualmente la conjugada de Hölder de la original. Desafortunadamente, la situación es más compleja y depende de la acotación de la función exponente. Si está acotada, el dual se conoce y es otro espacio de Lebesgue variable. Entonces,

**Problema:** ¿Cuál es el dual de un espacio de Lebesgue variable con exponente no acotado?

Si la función exponente no está acotada, da lugar a un dual más rico.

**Solución:** El dual de un espacio de Lebesgue variable con función exponente no acotada está formado por dos partes: la primera de ellas es un espacio de Lebesgue variable cuya función exponente es la conjugada de Hölder y la segunda es un *abstract* **L**-*space*.

Un abstract **L**-space es intuitivamente como el espacio $L^1$. Aún así, sigue siendo difícil de caracterizar en general estos espacios. Por tanto, damos ejemplos explícitos de este espacio para los espacios de sucesiones variable. Dependiendo del crecimiento de la función exponente podemos analizar la complejidad del mismo.

**4.** Una de las propiedades más importantes de las redes neuronales es la de aproximación universal, es decir, una red neuronal con función de activación adecuada puede representar una función continua definida en un compacto con una precisión tan pequeña como se quiera.

Esta propiedad se ha estudiado para otros espacios de funciones relacionados con más derivadas o condiciones de integrabilidad. No obstante, no se había tratado para los espacios de Lebesgue variables. Por tanto, fue natural considerar la siguiente cuestión:

**Problema:** ¿Es cierta la propiedad de aproximación universal de las redes neuronales artificiales para aproximar funciones pertenecientes a espacios de Lebesgue variables?

Como poder aproximar una función tiene alta relación con el dual del espacio, los resultados del problema anterior fueron claves para dar la solución.

**Solución:** Si el espacio de Lebesgue variable es acotado, entonces tiene la propiedad universal de aproximación. Si no está acotado, no la tiene, y en algunos casos podemos describir el subespacio de funciones que sí se pueden aproximar.

# Introduction

This manuscript covers results of multidisciplinary and interdisciplinary nature in the field of mathematics, all of them related to mathematical analysis in some way or another.

We have obtained several results mainly employing two different type of techniques: one of them concerning the analysis of partial differential equations and the other, functional analysis. Therefore, this text is divided into two parts, clearly differentiated by the tools used.

Both parts are again split into two chapters. Each chapter describes one problem and its solution. These problems are related to the respected titles:

1. Minimal surfaces,
2. Splitting Results on Riemannian Manifolds,
3. Duality in Variable Lebesgue Spaces,
4. Approximation in Variable Lebesgue Spaces.

Next, we introduce the problems and answers of each chapter with a brief motivation.

## Minimal Surfaces

Minimal surfaces and their study are a fundamental and key part of differential geometry. Nevertheless, the motivation of the problem solved in this chapter comes from a more analytic nature. More specifically, jointly with my PhD advisor, Antonio Córdoba, we were interested in the study of generalized harmonic functions and regularizing theorems about them in the euclidean space.

A generalized harmonic function is a way to extend the harmonic functions. It is well-known that harmonic functions verify the mean value property over balls. Therefore, to generalize this type of functions we define various ways to verify locally or infinitesimally the mean value property.

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Pisa is photographed above. There I attended my first international conference titled *A mathematical tribute to Ennio de Giorgi*. He was a relevant mathematician in mathematical analysis whose contribution is highly important up to the present.

**Definition — $\mathrm{MVP}_{\mathrm{loc}}$: Local mean value property.**
A function $u$ verifies $\mathrm{MVP}_{\mathrm{loc}}$ (the local mean value property) if $\forall x \in \Omega$ there is $r_x > 0$ such that $\forall 0 < r < r_x$, $B(x,r) \subset \Omega$ and

$$u(x) = ⨍_{B(x,r)} u(y)\mathrm{d}y.$$

**Definition — Generalized harmonic.**
A function $u \in L^1_{loc}(\Omega)$ is *generalized harmonic* (GH abbreviated) if $\forall x \in \Omega$,

$$\lim_{r \to 0^+} \frac{1}{r^2} ⨍_{B(x,r)} (u(y) - u(x))\mathrm{d}y = 0.$$

There is an important regularizing theorem for these types of generalization. If a generalized harmonic function is continuous, then it has to be smooth and, actually, harmonic.

Nevertheless, there are examples of discontinuous functions that are generalized harmonic (and obviously not harmonic). To know up to what function space (such as Sobolev spaces) we have gaining of regularity, it is necessary to comprehend the size and geometry of the possible set of discontinuities of a generalized harmonic function.

Based on several results, it seems that the set of discontinuities has codimension 1. Consequently, it is natural to study the geometry of this set, at least, in the easiest case: smooth hypersurfaces.

**Problem:** What are the sufficient and necessary hypotheses so that a hypersurface is the set of discontinuities of a generalized harmonic function?

The theorem which solves this question is the one that inspires the title of the chapter.

**Teorema — [CO20b] Minimal surfaces and GH.**
Let $S$ be a hypersurface $C^1$ that splits the domain $\Omega$ into two nonempty open components: $\Omega = \Omega^+ \cup \Omega^- \cup S$, such that $S = \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega$.
Consider the associated function to $S$, $f_S$, which takes the value 1 on $\Omega^+$, $-1$ on $\Omega^-$ and 0 on $S$.
The function $f_S$ is GH if, and only if, $S$ is minimal.

The key of the proof to this result lies in its validity if the hypersurface is smooth enough through some estimations. Later on, we can relax the regularity hypothesis about the hypersurface using resularity results of viscosity solutions since the parametrization of the hypersurface is a viscosity solution of the minimal surface equation.

## Splitting Results on Riemannian Manifolds

Riemannian manifolds are another key part of differential geometry. Nevertheless, the splitting results of this chapter came again from a more analytical matter. In a research stay at *Université de Picardie Jules Verne* with Alberto Farina, we were interested in understanding better some unidimensional results for some partial differential equations. One of the main open problems concerning this topic is De Giorgi's conjecture, based on Allen-Cahn equation. This conjecture states that bounded solutions of Allen-Cahn in the euclidean space which are monotone in one direction must be unidimensionals (i.e. the level sets must be hyperplanes) at least up to certain dimension. This conjecture is partially solved and it is extremely interesting for the techniques used.

We study solutions of

$$-\Delta_g u + f(u) = 0, \tag{1}$$

where $f = F'$ is the first derivative of a function $F \in C^2(\mathbb{R})$ and $\Delta_g$ is the Laplace-Beltrami operator on the manifold.

Bounded solutions of this equation verify an upper bound of the gradient known as Modica's estimate, since he was the first in proving them in [Mod85] for the euclidean space. When $F \geq 0$,

$$\frac{1}{2}|\nabla u|^2(x) \leq F(u(x)) \qquad \forall x \in \mathbb{R}^m. \tag{2}$$

Later, several generalizations of this estimate have been obtained, where Alberto Farina has unarguably contributed extending the result to Riemannian manifolds and dropping the hypothesis $F \geq 0$, leading to the generalized Modica's estimate:

$$\frac{1}{2}|\nabla_g u|^2(x) \leq F(u(x)) - c_u \qquad \forall x \in \mathscr{M}, \tag{3}$$

where the constant $c_u$ is efficiently computed using a proposition in the main text.

It is obvious that a unidimensional solution verifies equality everywhere in Modica's estimate or its generalization. The matter we wanted to comprehend is whether the reciprocal was true in the most general possible ambient, Riemannian manifolds. More specifically, let $\mathscr{M}$ be a complete, connected, smooth ($C^\infty$), boundaryless Riemannian manifold of dimension $m \geq 2$ with nonnegative Ricci curvature.

**Problem:** What are the geometric implications of achieving equality at Modica's estimate or its generalization in Riemannian manifolds?

Achievement of Modica's estimate and its generalization give several splitting results on the manifold. One of them is global, whereas the other is local, but it is necessarily achieved at a regular point.

**Theorem — [FO20] Global Splitting via Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded nonconstant solution of (1) with $F \geq 0$. Suppose that there is a point $x_0$ that achieves the equality in (2). Then, $\mathscr{M}$ splits as the Riemannian product $\mathscr{N} \times \mathbb{R}$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$. Furthermore, $u : \mathscr{N} \times \mathbb{R} \to \mathbb{R}$ is such that $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

**Theorem — [FO20] Local splitting via generalized Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded solution of (1). Suppose that equality is achieved in (3) at a regular point $x_0$, i.e. $\nabla_g u(x_0) \neq 0$. Denote by $\mathscr{U}$ the open connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ that contains $x_0$. Then,

- equality in (3) holds in $\mathscr{U}$,
- $Ric_g(\nabla_g u, \nabla_g u)$ vanishes at $\mathscr{U}$,
- $\mathscr{U}$ splits as the Riemannian product $\mathscr{N} \times I$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$ and $I \subseteq \mathbb{R}$ is an interval,
- the solution $u$ restricted to the neighborhood $\mathscr{U}$, $u : \mathscr{N} \times I \to \mathbb{R}$, is equal to $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

The splitting of the Riemannian manifold must be understood as the fact that a part or the totality of it is isometric to a product space consisted of a submanifold of codimension one and an interval. Furthermore, the solution that achieves the equality in that estimate has a unidimensional behavior in that neighborhood.

The main part of the proofs of these results is the construction of a harmonic function whose gradient has constant length. This type of function provides the splitting of the manifold into the product space commented above.

## Duality in Variable Lebesgue Spaces

My interest in variable Lebesgue spaces arises from a conference held at ICMAT, where the following problem was discussed and I decided to understand it better.

**Problem:** What is the dual of a variable Lebesgue space whose exponent function is unbounded?

Variable Lebesgue spaces are a generalization of the classical Lebesgue spaces where the exponent is a function instead of a constant. To give a structure to these spaces it is necessary to use generalized Musielak-Orlicz spaces.

**Definition — Variable Lebesgue spaces.**
Let $(\Omega, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space. We denote by $\mathscr{P}(\Omega)$ the collection of all measurable functions $p\colon \Omega \to [1, +\infty]$; we will refer to the elements of $\mathscr{P}(\Omega)$ as exponent functions. For each $p \in \mathscr{P}(\Omega)$, define the modular

$$\rho_{p(\cdot)}(f) := \left( \int_{\Omega \setminus \Omega_\infty} \frac{2}{p(x)} |f(x)|^{p(x)} \, \mathrm{d}\mu(x) + \|f\|_{L^\infty(\Omega_\infty)} \right),$$

where $\Omega_\infty := \{x \in \Omega : p(x) = +\infty\}$. Given a measurable function $f$, we say that $f \in L^{p(\cdot)}(\Omega)$ if there exists $\lambda > 0$ such that $\rho_{p(\cdot)}(f/\lambda) < +\infty$. This set becomes a Banach function space when equipped with the Luxemburg norm (introduced in [Lux55])

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

It is well-known that the dual of a nonvariable Lebesgue space with finite exponent, $p$, is another Lebesgue space whose exponent is its Hölder conjugate $(1/p + 1/q = 1)$. It would be logical to assume that the dual of a variable Lebesgue space whose exponent function is everywhere finite must be another variable Lebesgue space whose exponent function is pointwise the Hölder conjugate of the original. But fortunately for me, the situation is much more complex than it seems.

We need the exponent function to be bounded so that the dual is just its Hölder conjugate. This condition is vital, since it differentiates crucially the properties that variable Lebesgue spaces have. If the exponent function is bounded, it is like a Lebesgue space with finite exponent. However, if the exponent function is unbounded (even though it is everywhere finite), some properties of $L^\infty$ appear.

Jointly with Alex Amenta, José Conde and David Cruz-Uribe, we arrived at the conclusion that the dual of an unbounded variable Lebesgue space can be split into two parts: one that coincides with the variable Lebesgue space whose exponent function is the Hölder conjugate and a second part which was extremely difficult to give a structure to and to understand.

This second part is highly related to the dual of a certain quotient space. Therefore, we achieved to prove an important characterization of the quotient norm, which contributed to prove that the quotient space is an abstract **M**-space (a generalization of $L^\infty$ spaces). Then, its dual is an abstract **L**-space, which it is intuitively like a $L^1$ space.

**Definition — $L_b^{p(\cdot)}$ y $L_Q^{p(\cdot)}$.**
Let $p:\Omega\to[1,+\infty)$ be an unbounded exponent function. We consider

$$L_b^{p(\cdot)} := \overline{\left\{f\in L^{p(\cdot)} : p \text{ is bounded at } \operatorname{supp}(f)\right\}},$$

where the closure must be understood with the $L^{p(\cdot)}$ norm. The behavior of this subspace is very similar to bounded Lebesgue spaces. and we define the quotient space

$$L_Q^{p(\cdot)} := L^{p(\cdot)}/L_b^{p(\cdot)}.$$

The characterization of the quotient norm that provides structure to this space is the following.

**Theorem — [Ame+19] Characterization of the quotient norm.**
Let $p:\Omega\to[1,+\infty)$ be an unbounded exponent function. Then,

$$\|[f]\|_Q = \inf\left\{\lambda>0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu(x) < +\infty\right\}.$$

Consequently, $L_Q^{p(\cdot)}$ es un abstract **M**-space.

We recall the exact definition of this type of spaces.

**Definition — Abstract M-space.**
A Banach lattice $X$ is an *abstract* ***M**-space* whenever given $x_1, x_2\in X$ such that $x_1\wedge x_2 = 0$, then

$$\|x_1\vee x_2\| = \max\{\|x_1\|,\|x_2\|\}. \tag{4}$$

The second part of the dual is given by the following splitting result.

**Theorem — [Ame+19] Splitting of the dual.**
Let $p:\Omega\to[1,+\infty)$ be an unbounded exponent function, then

$$\left(L^{p(\cdot)}\right)^* \cong \left(L^{q(\cdot)}, \|\cdot\|_{q(\cdot)}\right) \oplus \left(L_Q^{p(\cdot)}, \|\cdot\|_Q\right)^*,$$

where $\cong$ denotes an isometric isomorphism. The second part is an abstract **L**-space.

Even so, it is still difficult in general to characterize this second part. Therefore I tried to describe it explicitly in a particular case, variable sequence spaces (the domain is discrete: $\mathbb{N}$). This case is easier, since variable sequence spaces are always contained in $\ell^\infty$. However, it is well-known that there are unbounded functions belonging to variable Lebesgue spaces.

The description of the dual for this case is showed in the following result. Note that the growth of the exponent function is really important.

**Theorem — [Ame+19] Dual of variable sequence spaces.**
Given $p:\Omega\to[1,+\infty)$ an exponent function, the dual of the variable sequence space $\ell^{p(\cdot)}$ is:

- $\ell^{q(\cdot)}$ if, and only if, $p$ is bounded.
- $\ell^{q(\cdot)}\oplus pba(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N})=1$.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N})<\infty$.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{A})$ if, and only if, $S_{p(\cdot)}$ is dense.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{A})\oplus pba(\mathscr{Z}(\mathscr{A},c_0))$ in general.

The second part of the dual of a variable sequence space coincides with the second part of the dual of $\ell^\infty$ (or an isomorphic generalization) if the exponent function diverges quickly enough. And it is way more complex if it diverges in a slow way. In the following, we show the notions needed to comprehend this second part of the dual.

**Definition — [Ame+19] $\omega$ and $\mathscr{A}$.**
Let $A \subseteq \Omega$ be measurable and $p : \Omega \to [1, +\infty)$ is an unbounded exponent function, we define

$$\omega(A) := \omega^{p(\cdot)}(A) := \|[\mathbf{1}_A]\|_Q.$$

And $\mathscr{A}$ or $\mathscr{A}_{p(\cdot)}$ is the collection of subsets $A$ such that $\omega(A) < \infty$.

$\omega$ is in spirit a weight that measures how important is a set in the quotient space.

Finally, $pba_\omega(\mathscr{A})$ and $pba(\mathscr{Z}(\mathscr{A}, c_0))$ are two spaces of purely finite additive measures. The first one is easier to describe since it generalizes the classical space *pba*. Nevertheless, the second space is more complex to introduce and we recommend to read Subsection 3.6.2 to understand better the dense subspace where it is defined.

**Definition — [Ame+19] $pba_\omega(\mathscr{A})$.**
Given $p \in \mathscr{P}(\mathbb{N})$, define $pba_\omega(\mathscr{A})$ to be the vector space of set functions $\mu$ defined on $\mathscr{A}$ satisfying the following properties:

1. $\mu(A \cup B) = \mu(A) + \mu(B)$ for any pair of disjoint sets $A, B \in \mathscr{A}$.
2. There exists $C > 0$ such that given any collection $\{A_i\}_{i=1}^n$ of pairwise disjoint sets in $\mathscr{A}$,

$$\sum_{i=1}^n \frac{|\mu(A_i)|}{\omega(A_i)} \leq C.$$

Define a norm on $pba_\omega(\mathscr{A})$ by

$$\|\mu\|_{pba_\omega} := \inf\{C > 0 : \text{ condition (2) holds}\}.$$

The tools used in this piece of the text to obtain characterization of norms, description of dense subspaces and more properties concerning the dual are of analytic nature.

## Approximation in Variable Lebesgue Spaces

My interest in artificial neural networks was born during a research stay with Wen-Liang Hwang at *Academia Sinica*. One of the main properties of neural networks that caught my eye was universal approximation, i.e., a neural network with adequate activation function can represent any continuous function defined on a compact set with any arbitrary and desired precision.

This property has been studied for another function spaces with more derivatives or integrability conditions. Nevertheless, variable Lebesgue spaces had not been considered in the past. Therefore, it was straightforward to consider the next matter:

**Problem:** Does the property of universal approximation for neural networks hold to approximate functions belonging to variable Lebesgue spaces?

Let us introduce first some preliminary notions related to artificial neural networks and the functions that they can represent.

**Definition — Feedforward shallow artificial neural network.**
A *feedforward shallow artificial neural network* (ANN) can be described by a finite linear combination of the form

$$g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j),$$

where $x \in \mathbb{R}^d$ represents the *input* to the neural network, $g(x) \in \mathbb{R}$ the *output*, $w_j \in \mathbb{R}^d$ and $\alpha_j \in \mathbb{R}$ are the *weights* between first and second layer, and second and third layer, respectively, $b_j \in \mathbb{R}$ are the *biases*, $\sigma : \mathbb{R} \to \mathbb{R}$ is the *activation function* and $M$ the *height*. The subspace of functions represented by an ANN is denoted by

$$H_\sigma := \left\{ g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) \right\}.$$

Since approximating a function has strong connections to the dual of the space, the results of the previous chapter were fundamental to give an answer to this problem. Recently a joint work with Ángela Capel was written on this topic.

**Theorem — [CO20a] Universal approximation for bounded variable Lebesgue spaces.**
Let $\Omega \subseteq \mathbb{R}^d$, consider $p : \Omega \to [1, +\infty)$ a bounded exponent function and $\sigma : \mathbb{R} \to \mathbb{R}$ discriminatory for $L^{p(\cdot)}(K)$ for every compact $K \subset \Omega$. Then, truncated finite sums of the form

$$g(x) = \begin{cases} \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) & x \in K, \\ 0 & x \in \Omega \setminus K, \end{cases}$$

with $K$ compact are dense in $L^{p(\cdot)}(\Omega)$.

The condition required for $\sigma$ means essentially that if a functional of $L^{p(\cdot)}(K)$ vanishes on $H_\sigma$, then the functional must be zero everywhere. Most of the activation functions considered in the literature verify this property.

When the exponent function is unbounded, the situation is more subtle. In fact, due to separability not every function can be approximated. Consequently, it is interesting to find the subspace of functions which actually can be approximated.

**Theorem — [CO20a] Approximation for unbounded variable Lebesgue spaces.**
Let $\Omega = [1, +\infty)$ and $p : \Omega \to [1, +\infty)$ an unbounded exponent function such that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ and it is bounded in every compact subset of $\Omega$. Let $\sigma \in L^\infty(\mathbb{R})$ be a non-constant, sigmoidal activation function. Then, the following conditions are equivalent for $f \in L^{p(\cdot)}(\Omega)$:

1. For every $\varepsilon > 0$, there is a $g_\varepsilon \in H_\sigma$ such that $\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$.
2. There is a scalar $\beta \in \mathbb{R}$ such that

$$\|[f - \beta \mathbf{1}_\Omega]\|_Q = 0,$$

where $\|\cdot\|_Q$ is the quotient norm.

It is intuitive that universal approximation holds when the exponent function is bounded since these spaces behave like classical Lebesgue spaces with finite exponent. However, when the function exponent is unbounded, since the space is like $L^\infty$, only functions with limit in some sense when the exponent diverges can be approximated.

## Publications

The results contained in this thesis are covered in the following scientific publications.

(CO20b) A. Córdoba and J. Ocáriz, ***A note on generalized Laplacians and minimal surfaces***, *Bull. London Math. Soc* 52.1 (2020), 153-157,
(Chapter 1).

(FO20) A. Farina and J. Ocáriz, ***Splittings theorems on complete Riemannian manifolds with nonnegative Ricci curvature***, *Discrete Contin. Dyn. Syst.*, (to appear)
(Chapter 2).

(Ame+19) A. Amenta, J. Conde-Alonso, D. Cruz-Uribe and J. Ocáriz, ***On the dual of variable Lebesgue spaces with unbounded exponent***, *preprint* (2019), arXiv: 1909.05987,
(Chapter 3).

(CO20a) Á. Capel and J. Ocáriz, ***Approximation with neural networks in variable Lebesgue spaces***, *preprint* (2020), arXiv: 2007.04166,
(Chapter 4).

# Introducción

Este manuscrito recoge resultados de carácter multidisciplinar e interdisciplinar en el área de las matemáticas, todos ellos relacionados con el análisis matemático de una forma u otra.

Principalmente hemos obtenido resultados utilizando dos tipos diferentes de técnicas: uno de ellos relacionado con análisis de las ecuaciones en derivadas parciales y el otro con análisis funcional. Por tanto, este texto está dividido en dos partes claramente diferenciadas por las herramientas utilizadas.

Ambas partes están a su vez divididas en dos capítulos. Cada capítulo describe un problema y su solución. Estos problemas están relacionados con el título:

1. Superficies Mínimas,
2. Resultados de Separación en Variedades Riemanianas,
3. Dualidad en Espacios de Lebesgue Variables,
4. Aproximación en Espacios de Lebesgue Variables.

A continuación vamos a introducir los problemas y resoluciones de cada capítulo junto con una breve motivación.

## Superficies Mínimas

Las superficies mínimas y su estudio son una parte fundamental y clave de la geometría diferencial. No obstante, la motivación del problema que aparece en este capítulo nace de una raíz más analítica. Más en concreto, junto con mi director de tesis, Antonio Córdoba, nos interesaba el estudio de las funciones armónicas generalizadas y teoremas regularizantes al respecto en el espacio euclídeo.

Una función armónica generalizada, como bien describe su nombre, es una forma de generalizar las funciones armónicas. Es claramente conocido que las funciones armónicas verifican la propiedad de la media en bolas. Por tanto, para poder generalizar este tipo de funciones definimos varias maneras de verificar localmente la propiedad de la media (una localmente y otra de manera infinitesimal).

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Esta foto muestra una panorámica de Barcelona. Ciudad que he visitado para asistir a congresos y escuelas de ecuaciones en derivadas parciales durante el doctorado.

**Definición — MVP$_{\text{loc}}$: Propiedad local de la media.**
Una función $u$ verifica MVP$_{\text{loc}}$ (*propiedad local de la media*) si $\forall x \in \Omega$ existe $r_x > 0$ tal que $\forall 0 < r < r_x$, $B(x,r) \subset \Omega$ y

$$u(x) = \fint_{B(x,r)} u(y)\mathrm{d}y.$$

**Definición — Función armónica generalizada.**
Una función $u \in L^1_{loc}(\Omega)$ es *armónica generalizada* (GH abreviada) si $\forall x \in \Omega$,

$$\lim_{r \to 0^+} \frac{1}{r^2} \fint_{B(x,r)} (u(y) - u(x))\mathrm{d}y = 0.$$

Existe un teorema de regularidad importante para estos tipos de generalizaciones. Si la función armónica generalizada es continua, entonces tiene que ser suave y de hecho armónica.

No obstante, hay ejemplos de funciones discontinuas que son armónicas generalizadas (y obviamente no armónicas). Para saber hasta qué espacio de funciones (como los espacios de Sobolev) podemos tener resultados de ganancia de regularidad es necesario comprender el tamaño y geometría del posible conjunto de puntos de discontinuidad de una función armónica generalizada.

Avalado por ciertos resultados, parece que el conjunto de discontinuidades tiene codimensión 1. Por tanto, es natural poder estudiar la geometría de dicho conjunto, al menos, en el caso más sencillo: hipersuperficies suaves.

**Problema:** ¿Cuáles son las hipótesis necesarias o suficientes para que una hipersuperficie sea el conjunto de puntos de discontinuidad de una función armónica generalizada?

El teorema que resuelve la pregunta y que motiva el título del capítulo es el siguiente.

**Teorema — [CO20b] Superficies mínimas y GH.**
Sea $S$ una hipersuperficie $C^1$ que separa el dominio $\Omega$ en dos componentes abiertas no vacías: $\Omega = \Omega^+ \cup \Omega^- \cup S$, de tal manera que $S = \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega$.
Considera la función asociada a $S$, $f_S$, que toma el valor 1 en $\Omega^+$, $-1$ en $\Omega^-$ y 0 en $S$.
La función $f_S$ es GH si, y sólo si, $S$ es minimal.

La clave de la prueba de este resultado está en la validez del resultado si la hipersuperficie es suficientemente derivable a través de unas estimaciones. Posteriormente, se puede relajar la regularidad de la hipersuperficie a través de resultados de regularidad de soluciones viscosas, ya que se puede ver que la parametrización de la hipersuperficie es una solución viscosa de la ecuación de las superficies mínimas.

## Resultados de Separación en Variedades Riemanianas

Las variedades Riemanianas son otra parte clave de la geometría diferencial. Sin embargo, los resultados de separación de este capítulo vinieron, otra vez, de una cuestión más analítica. En una estancia en la *Université de Picardie Jules Verne* con Alberto Farina nos interesaba comprender mejor los resultados de unidimensionalidad de ciertas ecuaciones en derivadas parciales. Uno de los problemas abiertos más importantes en esta rama es la conjetura de De Giorgi, basada en la ecuación de Allen-Cahn. Esta conjetura afirma que las soluciones acotadas de Allen-Cahn en el espacio euclideo que son crecientes en una dirección tienen que ser unidimensionales (es decir los conjuntos de nivel son hiperplanos) al menos hasta cierta dimensión. Esta conjetura está parcialmente resuelta y es muy interesante por las técnicas empleadas.

Nos interesa estudiar soluciones de

$$-\Delta_g u + f(u) = 0, \tag{5}$$

donde $f = F'$ es la primera derivada de una función $F \in C^2(\mathbb{R})$ y $\Delta_g$ es el operador de Laplace-Beltrami sobre la variedad.

Las soluciones acotadas de esta ecuación verifican una acotación superior del gradiente conocida como estimación de Modica, por ser el primero en haberlas estudiado en [Mod85] en el espacio euclídeo. Cuando $F \geq 0$,

$$\frac{1}{2}|\nabla u|^2(x) \leq F(u(x)) \qquad \forall x \in \mathbb{R}^m. \tag{6}$$

Posteriormente, se han estudiado diversas generalizaciones de esta estimación en las que Alberto Farina ha contribuido enormemente para variedades Riemanianas y eliminando la hipótesis de que $F \geq 0$, dando lugar a la estimación de Modica generalizada:

$$\frac{1}{2}|\nabla_g u|^2(x) \leq F(u(x)) - c_u \qquad \forall x \in \mathscr{M}, \tag{7}$$

donde la constante $c_u$ se calcula eficientemente usando la Proposición 2.4.

Es evidente que una solución unidimensional verifica la igualdad en la estimación de Modica o su generalización en todo punto. La cuestión que quisimos entonces estudiar es si el recíproco era cierto y lo contextualizamos en el ambiente más general posible de variedades Riemanianas. Más en concreto, $\mathscr{M}$, una variedad Riemaniana de dimensión $m \geq 2$, completa, conexa, suave ($C^\infty$), sin frontera y con curvatura de Ricci nonegativa.

**Problema:** ¿Qué implicaciones geométricas tiene alcanzar la igualdad en la estimación de Modica y su generalización en variedades Riemanianas?

La igualdad en la estimación de Modica y su generalización proporciona sendos resultados de separación de la variedad. El primero de ellos global y el segundo de manera local, pero es necesario que sea alcanzada en un punto regular.

**Teorema — [FO20] Separación global usando la estimación de Modica.**
Sea $u \in C^3(\mathscr{M})$ una solución acotada no constante de (5) con $F \geq 0$. Supongamos que hay un punto $x_0$ que alcance la igualdad en (6). Entonces, $\mathscr{M}$ es isométrico al espacio producto $\mathscr{N} \times \mathbb{R}$ donde $\mathscr{N} \subset \mathscr{M}$ es una hipersuperficie totalmente geodésica e isoparamétrica con $\mathrm{Ric}(\mathscr{N}) \geq 0$. Además, $u : \mathscr{N} \times \mathbb{R} \to \mathbb{R}$ es tal que $u(p,s) = \varphi(s)$ donde $\varphi$ es una solución acotada y estrictamente monótona de la EDO, $\varphi'' = f(\varphi)$.

**Teorema — [FO20] Separación local usando la estimación generalizada de Modica.**
Sea $u \in C^3(\mathscr{M})$ una solución acotada de (5). Supongamos que se alcanza la igualdad en (7) en un punto regular $x_0$, i.e. $\nabla_g u(x_0) \neq 0$. Denotemos por $\mathscr{U}$ la componente conexa y abierta de $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ que contiene a $x_0$. Entonces,

- la igualdad en (7) es cierta en $\mathscr{U}$,
- $Ric_g(\nabla_g u, \nabla_g u)$ se anula en $\mathscr{U}$,
- $\mathscr{U}$ es isométrico al producto Riemaniano $\mathscr{N} \times I$ donde $\mathscr{N} \subset \mathscr{M}$ es una hipersuperficie totalmente geodésica e isoparamétrica con $\mathrm{Ric}(\mathscr{N}) \geq 0$ e $I \subseteq \mathbb{R}$ es un intervalo,
- la solución $u$ restringida al entorno $\mathscr{U}$, $u : \mathscr{N} \times I \to \mathbb{R}$, coincide con $u(p,s) = \varphi(s)$ donde $\varphi$ es una solución acotada y estrictamente monótona de la EDO, $\varphi'' = f(\varphi)$.

La separación de una variedad Riemaniana viene a través de que una parte o la totalidad de la misma es isómetrica a un espacio producto de una variedad de una dimensión menor junto a un intervalo. Además, la solución que alcanza la igualdad en dicha estimación tiene un comportamiento unidimensional en dicho entorno.

La parte principal de las pruebas de estos resultados es la construcción de una función armónica cuyo gradiente tenga longitud constante. Este tipo de función proporciona la separación de la variedad en la variedad producto descrita anteriormente.

## Dualidad en Espacios de Lebesgue Variables

Mi interés por los espacios de Lebesgue de exponente variable radica en un congreso celebrado en el ICMAT en el que se propuso el siguiente problema que me fascinó al momento y me propuse entender mejor.

**Problema:** ¿Cuál es el dual de un espacio de Lebesgue variable con exponente no acotado?

Los espacios de Lebesgue variables son una generalización de los espacios de Lebesgue clásicos donde el exponente en vez de ser una constante pasa a ser una función. Para dar una estructura a estos espacios es necesario usar los espacios generalizados de Musielak-Orlicz.

**Definición — Espacios de Lebesgue variables.**
Sea $(\Omega,\mathscr{A},\mu)$ un espacio de medida $\sigma$-finito. Denotamos por $\mathscr{P}(\Omega)$ la colección de todas las funciones medibles $p\colon \Omega\to[1,+\infty]$; nos referimos a los elementos $\mathscr{P}(\Omega)$ como funciones exponentes. Para cada $p\in\mathscr{P}(\Omega)$, definimos la forma modular

$$\rho_{p(\cdot)}(f) := \left(\int_{\Omega\setminus\Omega_\infty} \frac{2}{p(x)}\,|f(x)|^{p(x)}\ \mathrm{d}\mu(x) + \|f\|_{L^\infty(\Omega_\infty)}\right),$$

donde $\Omega_\infty := \{x\in\Omega : p(x)=+\infty\}$. Dada una función medible $f$, decimos que $f\in L^{p(\cdot)}(\Omega)$ si existe $\lambda>0$ tal que $\rho_{p(\cdot)}(f/\lambda)<+\infty$. Este conjunto se convierte en un espacio de Banach cuando se equipa con la norma de Luxemburg norm [Lux55].

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda>0:\ \rho_{p(\cdot)}(f/\lambda)\ \le\ 1\right\}.$$

Es altamente conocido que el dual de un espacio de Lebesgue con exponente finito es otro espacio de Lebesgue cuyo exponente es su conjugado de Hölder ($1/p+1/q=1$). Sería entonces natural pensar que el dual de un espacio de Lebesgue variable cuyo exponente es siempre finito tiene que ser otro espacio de Lebesgue variable cuyo función exponente es puntualmente la conjugada de Hölder de la original. Pero afortunadamente para mí, la situación es más compleja de lo que parece.

Para que el dual sea su conjugado de Hölder es necesario que la función exponente esté acotada superiormente. Esta condición es vital, ya que diferencia crucialmente las propiedades que tienen los espacios de Lebesgue variables. Si la función exponente está acotada, se comporta como un espacio de Lebesgue con exponente finito. En cambio, si la función exponente no está acotada (aunque sea finita en todo punto) se parece más a $L^\infty$.

Junto a Alex Amenta, José Conde y David Cruz-Uribe, llegamos a la conclusión de que el dual de un espacio de Lebesgue variable no acotado se podía dividir en dos partes: una coincide con el espacio de Lebesgue variable cuya función exponente es la conjugada de Hölder y una segunda parte a la que era difícil de dar estructura y entender.

Esta segunda parte tiene mucho que ver con el dual de un espacio cociente. Por tanto, conseguimos probar una caracterización de su norma que permitió probar que este cociente es un abstract **M**-space (que extiende los espacio $L^\infty$). Entonces, su dual es un abstract **L**-space, que es intuitivamente como el espacio $L^1$.

**Definición** — $L_b^{p(\cdot)}$ y $L_Q^{p(\cdot)}$.
Sea $p:\Omega\to[1,+\infty)$ una función exponente no acotada. Consideramos entonces

$$L_b^{p(\cdot)} := \overline{\{f\in L^{p(\cdot)} : p \text{ está acotado en } \operatorname{supp}(f)\}},$$

donde la clausura se toma con la norma de $L^{p(\cdot)}$. Este subespacio se comporta de manera parecida a los espacios de Lebesgue variables acotados. Y definimos el espacio cociente

$$L_Q^{p(\cdot)} := L^{p(\cdot)}/L_b^{p(\cdot)}.$$

La caracterización de la norma cociente que da estructura a este espacio es la siguiente.

**Teorema** — **[Ame+19] Caracterización de la norma cociente.**
Sea $p:\Omega\to[1,+\infty)$ una función exponente no acotado. Entonces,

$$\|[f]\|_Q = \inf\left\{\lambda>0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu(x) < +\infty\right\}.$$

Consecuentemente, $L_Q^{p(\cdot)}$ es un abstract **M**-space.

Recordemos la definición exacta.

**Definición** — **Abstract M-space.**
Una retícula de Banach $X$ es un *abstract **M**-space* cuando dados $x_1,x_2\in X$ tales que $x_1\wedge x_2=0$, entonces

$$\|x_1\vee x_2\| = \max\{\|x_1\|,\|x_2\|\}. \tag{8}$$

La segunda parte del dual la tenemos descrita por el siguiente resultado de separación.

**Teorema** — **[Ame+19] Separación del dual.**
Sea $p:\Omega\to[1,+\infty)$ una función exponente no acotada, entonces

$$\left(L^{p(\cdot)}\right)^* \cong \left(L^{q(\cdot)},\|\cdot\|_{q(\cdot)}\right)\oplus\left(L_Q^{p(\cdot)},\|\cdot\|_Q\right)^*,$$

donde $\cong$ denota un isomorfismo isométrico. La segunda parte es un abstract **L**-space.

Aún así, sigue siendo difícil de caracterizar en general esta segunda parte. Así que me propuse también describir explícitamente un caso particular, el discreto, es decir, los espacios de sucesiones variables (el dominio es $\mathbb{N}$). Este caso es más sencillo, ya que los espacios de sucesiones variables siempre están contenidos en $\ell^\infty$. Sin embargo, es conocido que hay funciones no acotadas que pertenecen a los espacios de Lebesgue variables.

La descripción del dual para este caso viene recopilada en el siguiente resultado.

**Teorema** — **[Ame+19] Dual de los espacios de sucesiones variable.**
Sea $p:\Omega\to[1,+\infty)$ una función exponente. El dual del espacio de sucesiones $\ell^{p(\cdot)}$ es:

- $\ell^{q(\cdot)}$ si, y solamente si, $p$ está acotada.
- $\ell^{q(\cdot)}\oplus pba(\mathscr{B}(\mathbb{N}))$ si $\omega(\mathbb{N})=1$.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{B}(\mathbb{N}))$ si $\omega(\mathbb{N})<\infty$.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{A})$ si, y solamente si, $S_{p(\cdot)}$ es denso.
- $\ell^{q(\cdot)}\oplus pba_\omega(\mathscr{A})\oplus pba(\mathscr{Z}(\mathscr{A},c_0))$ en general.

La segunda parte del dual de un espacio de sucesiones variable coincide con la segunda parte del dual de $\ell^\infty$ (o una generalización suya isomorfa) si la función exponente diverge lo suficientemente rápido. Y es mucho más complejo si diverge de manera lenta. A continuación presentamos las nociones para entender está segunda parte del dual.

**Definición — [Ame+19] $\omega$ y $\mathscr{A}$.**
Sea $A \subseteq \Omega$ medible y $p : \Omega \to [1, +\infty)$ una función exponente no acotada, definimos

$$\omega(A) := \omega^{p(\cdot)}(A) := \|[\mathbf{1}_A]\|_Q.$$

Y $\mathscr{A}$ o $\mathscr{A}_{p(\cdot)}$ es la colección de subconjuntos de $A$ tales que $\omega(A) < \infty$.

$\omega$ es una especie de peso que mide como de importante es un conjunto en cierto espacio cociente.

Finalmente, $pba_\omega(\mathscr{A})$ y $pba(\mathscr{X}(\mathscr{A}, c_0))$ son dos espacios de medidas puramente finitamente aditivas. El primero es más fácil de describir ya que se trata de una extensión del clásico espacio *pba*. Sin embargo, el segundo espacio es más complejo de introducir y recomendamos leer la Subsección 3.6.2 para entender los subespacios densos donde está definido.

**Definición — [Ame+19] $pba_\omega(\mathscr{A})$.**
Sea $p \in \mathscr{P}(\mathbb{N})$, definimos $pba_\omega(\mathscr{A})$ como el espacio vectorial de medidas con signo $\mu$ definidas en $\mathscr{A}$ que satisfacen las siguientes propiedades:

1. $\mu(A \cup B) = \mu(A) + \mu(B)$ para cada par de conjuntos disjuntos $A, B \in \mathscr{A}$.
2. Existe $C > 0$ tal que dada cualquier colección $\{A_i\}_{i=1}^n$ disjuntos dos a dos de conjuntos que pertenecen a $\mathscr{A}$,
$$\sum_{i=1}^{n} \frac{|\mu(A_i)|}{\omega(A_i)} \leq C.$$

Definimos una norma en $pba_\omega(\mathscr{A})$ de la siguiente manera

$$\|\mu\|_{pba_\omega} := \inf\{C > 0 : \text{ condición (2) es cierta}\}.$$

Las herramientas utilizadas en esta parte son de carácter analítico para obtener caracterizaciones de normas, descripciones de subespacios densos y más propiedades para describir el dual.

## Aproximación en Espacios de Lebesgue Variables

Mi interés por las redes neuronales artificiales comenzó estando de estancia con Wen-Liang Hwang en la *Academia Sinica*. Una de las propiedades de las redes neuronales que me fascinó es la de aproximación universal, es decir, una red neuronal con función de activación adecuada puede representar una función continua definida en un compacto con una precisión tan pequeña como se quiera.

Esta propiedad se ha estudiado para otros espacios de funciones relacionados con más derivadas o condiciones de integrabilidad. No obstante, en el pasado no se había tratado el caso de los espacios de Lebesgue variables. Por tanto, fue natural considerar la siguiente cuestión:

**Problema:** ¿Es cierta la propiedad de aproximación universal de las redes neuronales artificiales para aproximar funciones pertenecientes a espacios de Lebesgue variables?

Introduzcamos primero las funciones descritas por una red neuronal artificial.

**Definición — Red neuronal artificial.**
Una red neuronal artificial (ANN) puede estar descrita por una combinación linear finita de la forma

$$g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j)\,,$$

donde $x \in \mathbb{R}^d$ representa el *input*, $g(x) \in \mathbb{R}$ el *output*, $w_j \in \mathbb{R}^d$ y $\alpha_j \in \mathbb{R}$ son los *pesos* entre la primera y segunda capa, y segunda y tercera, respectivamente, $b_j \in \mathbb{R}$ son los *sesgos*, $\sigma : \mathbb{R} \to \mathbb{R}$ es la *función de activatión* y $M$ la *altura*. El subespacio de funciones que se describen usando ANN lo denotamos por

$$H_\sigma := \left\{ g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) \right\}.$$

Como poder aproximar una función tiene alta relación con el dual del espacio, los resultados del capítulo anterior fueron claves para dar la solución al problema anterior. Recientemente junto a Ángela Capel escribimos la siguiente solución.

**Teorema — [CO20a] Aproximación para espacios de Lebesgue variables acotados.**
Sea $\Omega \subseteq \mathbb{R}^d$, considera $p : \Omega \to [1, +\infty)$ una función exponente acotada y $\sigma : \mathbb{R} \to \mathbb{R}$ discriminatoria para $L^{p(\cdot)}(K)$ para cada compacto $K \subset \Omega$. Entonces, las sumas finitas truncadas

$$g(x) = \begin{cases} \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) & x \in K\,, \\ 0 & x \in \Omega \setminus K\,, \end{cases}$$

con $K$ compacto son densas en $L^{p(\cdot)}(\Omega)$.

La condición impuesta en $\sigma$ esencialmente significa que si un funcional de $L^{p(\cdot)}(K)$ se anula en $H_\sigma$, entonces el funcional tiene que ser nulo. La mayoría de las funciones de activaciones consideradas en la literatura verifican esta propiedad.

Cuando la función exponente es no acotada, la situación es mucho más compleja. De hecho por tema de separabilidad ya no se puede aproximar toda función. Por tanto, nos interesa estudiar el subespacio que sí se puede aproximar.

**Teorema — [CO20a] Aproximación para espacios de Lebesgue variables no acotados.**
Sea $\Omega = [1, +\infty)$ y $p : \Omega \to [1, +\infty)$ una función exponente no acotada tal que $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ y es acotada en cada compacto contenido en $\Omega$. Sea $\sigma \in L^\infty(\mathbb{R})$ una función de activación no constante y sigmoidal (que tiene límite en $-\infty$ y $+\infty$). Entonces las siguientes condiciones son equivalentes para $f \in L^{p(\cdot)}(\Omega)$:

1. $f \in \overline{H_\sigma}$.
2. Existe $\beta \in \mathbb{R}$ tal que
$$\|[f - \beta \mathbf{1}_\Omega]\|_Q = 0\,,$$
donde $\|\cdot\|_Q$ es la norma del espacio cociente.

Es natural que cuando la función exponente sea acotada, se tenga la propiedad universal de aproximación ya que el comportamiento es similar a los espacios de Lebesgue de exponente finito. No obstante, cuando no está acotada la función exponente, en cierta manera el espacio tiene semejanzas con $L^\infty$. Consecuentemente, solamente se pueden aproximar las funciones que tienen límite en cierto sentido cuando la función exponente diverge.

## Publicaciones

Los resultados abarcados en esta tesis están cubiertos en las siguientes publicaciones.

(CO20b) A. Córdoba and J. Ocáriz, ***A note on generalized Laplacians and minimal surfaces***, *Bull. London Math. Soc* 52.1 (2020), 153-157,
(Capítulo 1).

(FO20) A. Farina and J. Ocáriz, ***Splittings theorems on complete Riemannian manifolds with nonnegative Ricci curvature***, *Discrete Contin. Dyn. Syst.*, (to appear)
(Capítulo 2).

(Ame+19) A. Amenta, J. Conde-Alonso, D. Cruz-Uribe and J. Ocáriz, ***On the dual of variable Lebesgue spaces with unbounded exponent***, *preprint*(2019), arXiv: 1909.05987,
(Capítulo 3).

(CO20a) Á. Capel and J. Ocáriz, ***Approximation with neural networks in variable Lebesgue spaces***, *preprint* (2020), arXiv: 2007.04166,
(Capítulo 4).

# Conclusions and open questions

We have solved four different problems along this text. Even though the solutions provide satisfactory answer to these problems, more questions arose from them.

On the one hand, we have shown that a necessary and sufficient condition so that a hypersurface gives a generalized harmonic function is the minimality of the hypersurface. A regularity hypothesis about the manifold must be assumed so that the proof holds. Could we relax this hypothesis to be just Lipschitz (or even just continuous)?

Another interesting matter concerning this would be to study those minimal surfaces whose associated function verifies the local mean value property. We know that the plane and helicoid verify this, whereas the catenoid does not. It would be nice for future work to know if the rest of minimal surfaces verify this property. In case all the others fail, it would be an interesting problem to obtain a characterization result with the spirit of Catalan's theorem, the one that states that the unique ruled minimal surfaces are the plane and the helicoid.

In the first part of the thesis we also give some splitting results on Riemannian manifolds. These results are complete and difficult to improve since there are counterexamples if we remove any hypothesis.

On the other hand, in the second part we have studied variable Lebesgue spaces. We have characterized the dual of unbounded variable Lebesgue spaces. One of the components of this dual is difficult to describe in detail and thus it would be nicer to obtain a better description of it, even though it seems impossible for the general case.

For the variable sequence space, an interesting question is to know if the density of simple sequences is equivalent to the hypothesis $\omega(\mathbb{N}) < \infty$ about the divergence of the exponent function. One of the implications is already proven in the manuscript and the other is partially done by using a counterexample. To prove this equivalence it seems that the key tool appears already in the counterexample. With a slight modification of it, keeping just an infinite and sparse number of indices, we could get the desired growth of the exponent function and the simple sequences would not be dense.

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This is the IST in Lisbon. I gave a talk on the dual of variable Lebesgue spaces in this city during the conference *IWOTA*.

Finally, we have provided some approximation results for variable Lebesgue spaces using artificial neural networks with one hidden layer. In reality, it seems that neural networks with more hidden layers, known as deep neural networks, are more useful. A future project is thus to study approximation results for this type of neural networks.

# Conclusiones y preguntas abiertas

En este texto hemos resuelto cuatro problemas de diversa índole. Aunque las soluciones a los mismos han dado respuestas satisfactorias, más preguntas han ido surgiendo.

Por un lado, hemos visto que una condición suficiente y necesaria para que una hipersuperficie dé lugar a una función armónica generalizada es la minimalidad de la hipersuperficie. Para que la prueba de este resultado sea cierto es necesario una mínima regularidad de la hipersuperficie. ¿Podríamos poder relajar esta condición de regularidad a simplemente Lipschitz (o incluso continuidad)?

Otra cuestión interesante al respecto podría ser estudiar aquellas superficies mínimas cuya función asociada verifica la propiedad local de la media. Hemos visto que el plano y el helicoide lo verifican pero el catenoide no. Podría ser viable como trabajo en el futuro estudiar esta propiedad en las otras superficies mínimas, y en caso de fallar en el resto, poder obtener un resultado de caracterización con el espíritu del resultado de Catalan de que las únicas superficies mínimas regladas son el plano y el helicoide.

En la primera parte también presentamos resultados de separación de variedades Riemanianas. Estos resultados son completos y difíciles de mejorar, ya que al quitar alguna de las hipótesis podemos encontrar contraejemplos.

Por otro lado, en la segunda parte hemos estudiado los espacios de Lebesgue de exponente variable. Hemos podido caracterizar su dual en el caso de que el exponente no esté acotado. Una de las componentes de este dual es difícil de explicar con detalle y por tanto nos gustaría poder obtener una mejor descripción de la misma, aunque parece una tarea imposible para el caso general.

En el caso del espacio de sucesiones variables una pregunta interesante es saber si la densidad de las sucesiones simples es equivalente a la hipótesis $\omega(\mathbb{N}) < \infty$ sobre la divergencia de la función exponente. Una de las implicaciones ya está probada en el texto y la otra parcialmente con un contraejemplo. Para poder probar esta equivalencia parece que la clave se basa en el contraejemplo. Modificando ligeramente el contraejemplo, quedándote con una cantidad infinita y separada de índices, conseguimos que la función exponente verifique la condición de crecimiento requerida y las sucesiones simples no serían densas.

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Esta foto muestra el centro de Castellón. Fue en esta ciudad donde presenté nuestro resultado sobre superficies mínimas durante el *Congreso de Jóvenes Investigadores de la RSME*.

Por último, conseguimos probar ciertas aproximaciones en espacios de Lebesgue variable usando redes neuronales artificiales con una capa oculta. En la práctica, parece que son más útiles las redes neuronales con más capas ocultas, las conocidas como redes neuronales profundas. Para el futuro sería interesante estudiar propiedades de aproximación para este tipo de redes neuronales.

# Part I

# Minimal Surfaces and Splitting Results on Riemannian Manifolds

# 1. Minimal Surfaces via Generalized Harmonic Functions

In this chapter, we link two different areas such as Mathematical Analysis and Differential Geometry through an interdisciplinary and interesting result. More specifically, we present in the following pages a characterization of minimal hypersurfaces in euclidean space through a generalization of the Laplace operator. This is related to the publication [CO20b].

The structure of the chapter is the following:

- First, we introduce some historical results such as the characterization of harmonicity using the mean value property and some generalizations of the Laplace operator concerning the Blaschke and Privaloff operators. This leads to the definition of generalized harmonic function.
- Later, since generalized harmonic functions coincides with harmonic functions except from a singular set, we study this singular set. We see that it includes the closure of points of discontinuity of the function and it must have zero Lebesgue measure.
- Motivated by the fact that the singular set of a generalized harmonic function is small, we consider generalized harmonic functions in the euclidean space of dimension $n$ whose singular set is a $n-1$ hypersurface. Then, we can state the main result of this chapter, i.e, the conditions of minimality of the hypersurface and the generalized harmonicity of the function are equivalent.
- Finally, we provide the proof of this characterization result.

## 1.1 Introduction

In this chapter, the geometric ambient is the following: an open subset, $\Omega$, of the euclidean space of dimension $n$, $\mathbb{R}^n$. Since we are going to deal with interior and local results, we will not impose any further condition about the border of $\Omega$.

### Harmonic functions

We recall that a harmonic function is a function $u \in C^2(\Omega)$ which belongs to the kernel of the Laplacian operator, that is, $\Delta u = 0$ where $\Delta$ is the trace of the Hessian $\left(\sum_{i=1}^{n} \partial_{ii}\right)$.

This landscape of Murcia is the image of this first chapter. Murcia is the region where my PhD advisor and I were born. Since the results of our joint work are explained throughout the following pages, this image is totally adequate. Furthermore, in Murcia was where I started my degree in Math.

Harmonic functions are fundamental in the theory of elliptic PDEs and they have very nice properties. Some of them are more related to analysis, like their gain of regularity (they are, actually, infinitely differentiable, $C^\infty$), whereas others are of different nature. We give now different characterizations of harmonic functions, which we subsequently generalize. Therefore, we start with the following geometric property.

**Definition 1.1 — MVP: Mean Value Property.**
A function $u$ verifies MVP (the mean value property) in $\Omega$ if for every ball $B(x,r) \subset \Omega$ the next identity holds

$$u(x) = \fint_{B(x,r)} u(y)\mathrm{d}y,$$

where $B(x,r)$ is the open ball centred at $x$ with radius $r$.

Note that verifying MVP is quite strong because we have a restriction for every ball contained in $\Omega$, of which there are infinitely many. In the following figure we show a particular example.

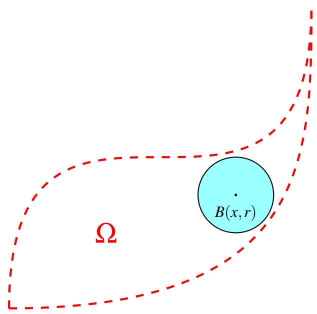


Figure 1.1: Example of a ball contained in $\Omega$.

The relation of this property with harmonic functions follows from the following well-known result.

**Proposition 1.2 — Geometric characterization of harmonic functions.**
The following conditions are equivalent:

1. $u \in C^2(\Omega)$ and $u$ is harmonic.
2. $u \in C(\Omega)$ and $u$ verifies MVP.

*Proof.* This is a classical result whose proof can be read, for example in [GT15] (Theorem 2.1 and Theorem 2.7). ■

However, it has been widely studied what happens when we relax this condition to hold only for some specific radii. The following two examples appear in the literature.

■ **Example 1.3 — One Circle problem [Huc54].** Suppose that we have a continuous function over $\Omega$, $u \in C(\Omega)$, with the following property:

* $\forall x \in \Omega\ \exists r_x > 0$ such that $u(x) = \fint_{B(x,r_x)} u(y)\mathrm{d}y$.

Should the function be harmonic?

- Yes, if the function is continuous in the closure of $\Omega$ (see [Kel34]).
- Not in general, as shown in an example provided by J.E. Littlewood for the unit ball $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. This counterexample is based on the following construction. Consider the function defined piecewise in the following way: for $n \in \mathbb{N} \cup \{0\}$ there are two constants

$a_n$ and $b_n$ to be determined such that $u(x) = a_n \log(|x|) + b_n$ for $2^{-(n+1)} < 1 - |x| < 2^{-n}$. We can take $a_n$ and $b_n$ such that the function is continuous in $\Omega$ (in particular $a_0 = 0$) and verifying the mean value property for one radius at points on the circumferences of radius $1 - 2^{-n}$. Away from these circumferences, the function verifies the mean value property because the function is harmonic in the annuli.

■

■ **Example 1.4 — Two Circle problem [Zal80].** Suppose that we have $u \in C(\mathbb{R}^n)$ verifying the following property:

** $\exists r_1, r_2 > 0$ different such that $u(x) = \fint_{B(x,r_1)} u(y)\mathrm{d}y = \fint_{B(x,r_2)} u(y)\mathrm{d}y$ holds $\forall x \in \Omega$.

Is the function harmonic?

- The function is harmonic if, and only if, the quotient of the radii is not a zero of a determined Bessel function (see [Del61]). This result is known as the two-radius theorem.

Some generalizations of the previous theorem can be seen in [BG86], [Vol96] among others. We highlight an extension of this result for $u \in L^1_{loc}(\mathbb{R}^n)$ (the space of locally Lebesgue integrable functions) where equality everywhere is replaced for equality almost everywhere. ■

**Remark 1.5** As a direct application of the local version of the two-radius theorem, we obtain that MVP property characterizes harmonic functions for functions in $u \in L^1_{loc}(\Omega)$.

Another characterization of harmonicity needs to introduce the following concept related to distribution.

**Definition 1.6 — Weakly harmonic.**
A function $u \in L^1_{loc}(\Omega)$ is *weakly harmonic* in $\Omega$ if for every $\varphi \in C^\infty_c(\Omega)$ we have that

$$\int_\Omega u \cdot \Delta\varphi = 0.$$

The term weakly harmonic is justified because they are weak solutions of the Laplace equation.

**Remark 1.7** It is straightforward that harmonic functions are weakly harmonic using the technique of integration by parts.

Conversely, we have the other implication thanks to the following classical result.

**Lemma 1.8 — Weyl's Lemma.**
Let $u \in L^1_{loc}(\Omega)$ be weakly harmonic. Then, $u \in C^2(\Omega)$ and $u$ is harmonic up to a redefinition of the function over a set of measure zero.

*Proof.* It was first proved in [Wey40], see Lemma 2 in page 415. ■

Therefore, the concepts harmonic, weakly harmonic and the mean value property are equivalent.

### Generalization of harmonic functions

After introducing harmonic functions and showing some of its characterization, we show how to generalize the concept of harmonicity for functions in several ways. One way is to extend the Laplace operator to a bigger function space and another is to study a local version of one of the properties that characterizes harmonicity, since the property that $\Delta u = 0$ is local. More precisely, we introduce the local version of the mean value property in the following definition. This concept is related to the One Circle problem exposed in Example 1.3.

**Definition 1.9 — $\text{MVP}_{\text{loc}}$: Local Mean Value Property.**
A function $u$ verifies $\text{MVP}_{\text{loc}}$ (the local mean value property) if $\forall x \in \Omega$ there is $r_x > 0$ such that $\forall 0 < r < r_x$, $B(x,r) \subset \Omega$ and

$$u(x) = \fint_{B(x,r)} u(y)\mathrm{d}y.$$

**Remark 1.10** Note that MVP implies $\text{MVP}_{\text{loc}}$ because for any $x \in \Omega$ we can take any sufficiently small $r_x$ guaranteeing that balls with smaller radii are contained in $\Omega$.

As far as we are concerned, this definition is not standard in the literature because, among other reasons, both concepts coincide in the class of continuous functions.

**Proposition 1.11 — MVP = $\text{MVP}_{\text{loc}}$ in $C(\Omega)$.**
Let $u \in C(\Omega)$. TFAE:

1. $u$ verifies MVP.
2. $u$ verifies $\text{MVP}_{\text{loc}}$.

*Proof.* 1. implies 2. follows from Remark 1.10, meanwhile the other implication can be read in the literature for example in Theorem 1.17 of [HK76]. ■

However, we can find easily discontinuous functions which verify $\text{MVP}_{\text{loc}}$ and do not verify MVP . See the following example:

■ **Example 1.12 — Sign function.** Let $\Omega = \mathbb{R}$ and consider the following representation of the sign function:

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

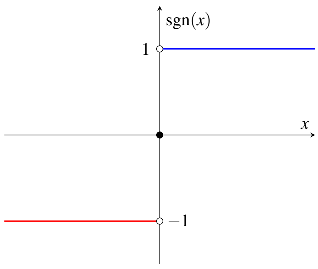


Figure 1.2: The sign function

It clearly verifies $\text{MVP}_{\text{loc}}$. For example, we can take $r_x = |x|$ for $x \neq 0$ and $r_0$ any positive number. However, by taking a ball centred at a nonzero number $x$ it fails MVP for radii greater than $|x|$. ■

From all the equivalent conditions of harmonicity, the easiest to compute is with the usual definition determined by the Laplace operator, since the others rely to check a condition for many elements such as the balls contained in the domain or all the possible test functions (smooth functions compactly supported). Therefore, next we give a generalization of harmonicity linked with an extension of the Laplace operator. More specifically, we show two extensions of the Laplace operator for a larger class of functions which includes, at least, the space of smooth functions. This strategy was used by Blaschke in [Bla16] and Privaloff in [Pri25] and these operators were named after them, Blaschke and Privaloff operator respectively (see [Sak41] and [Pot48]). These are the precise definitions of these operators.

**Definition 1.13 — Blaschke Operator.**
Let $\Omega \subseteq \mathbb{R}^n$. We define *Blaschke Operator* for functions $u : \Omega \to \mathbb{R}$, whenever the following limit exists pointwise,

$$\Delta_B u(x) := \lim_{r \to 0^+} \frac{C_B(n)}{r^2} \fint_{S(x,r)} (u(y) - u(x))\mathrm{d}y,$$

where $S(x,r)$ is the sphere centred at $x$ with radius $r$ and $C_B(n)$ is a constant which only depends on the dimension $n$ ($C_B(n) = 2n$).

Blaschke operator is intuitively computing in an infinitesimal way the difference between the function and its averages over spheres. Privaloff operator acts in a similar way but using balls instead of spheres.

**Definition 1.14 — Privaloff Operator.**
Let $\Omega \subseteq \mathbb{R}^n$. We define *Privaloff Operator* for functions $u : \Omega \to \mathbb{R}$, whenever the following limit exists pointwise,

$$\Delta_P u(x) := \lim_{r \to 0^+} \frac{C_P(n)}{r^2} \fint_{B(x,r)} (u(y) - u(x))\mathrm{d}y,$$

where $C_P(n)$ is again a constant which only depends on the dimension $n$ ($C_P(n) = 2(n+2)$).

**Remark 1.15** The domains of Blaschke and Privaloff operators include smooth functions because, using Taylor's theorem, the previous limits always exist pointwise and they coincide, up to a multiplicative constant, with the Laplacian of the function. The constants $C_B(n)$ and $C_P(n)$ are chosen so that these operators extend the Laplacian, i.e., $C_B(n) = 2n$ and $C_P(n) = 2(n+2)$.

These two operators are connected in the following way.

**Lemma 1.16 — Relation between Blaschke and Privaloff Operators.**
If $\Delta_B$ exists, then $\Delta_P$ exists and both coincide.

*Proof.* It follows from a simple argument with integrals. For a full proof, see Theorem 3 in [Pot48]. ■

From the previous lemma we deduce that the Privaloff operator is more general, and thus we prefer to consider Privaloff operator for the following definition.

**Definition 1.17 — Generalized harmonic.**
A function $u \in L^1_{loc}(\Omega)$ is *generalized harmonic* (GH in short) if $\Delta_P u$ exists and vanishes everywhere in $\Omega$. In other words, the function belongs to the kernel of the Privaloff operator.

**Remark 1.18** Note that $\mathrm{MVP}_{\mathrm{loc}}$ implies GH, because for any $x \in \Omega$ and for any sufficiently small $r_x$ the integral is equal to 0. Therefore, the limit exists and it is equal to 0. However, the converse is false and we will give a counterexample in Example 1.26. We can also construct more counterexamples in a geometric way using the main result of [CO20b] and Example 1.35.

Intuitively, we could say that a GH function verifies the mean value property in an infinitesimal way. We can see how these new definitions are related for functions $u \in L^1_{loc}(\Omega)$ in the following figure where all the inclusions are strict, which reminds of a traffic light.

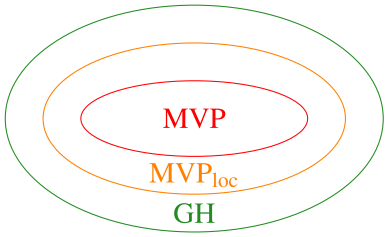


Figure 1.3: In the center, we have harmonic functions in red (because they verify the mean value property). Then, we have the ones verifying locally the mean value property in orange. And finally, in green the generalized harmonic functions.

However, there is a curious phenomena concerning these notions. All the previous definitions coincide for the space of continuous functions by virtue of the following result.

**Proposition 1.19 — Harmonic = GH in $C(\Omega)$ [Rie26].**
Let $u \in C(\Omega)$. TFAE:

1. $u$ is GH.
2. $u$ is harmonic.

*Proof.* The fact that harmonic implies GH is straightforward, as it was mentioned in Remark 1.18. The proof of the other implication follows from the maximum principle: let $x_0 \in \Omega$ and let $v$ be the solution of the Dirichlet problem (ordinary harmonic in $B(x_0,r)$ with boundary values given by $u$) and $u_\varepsilon^+(x) = u(x) + \varepsilon(r^2 - |x-x_0|^2)$, $u_\varepsilon^-(x) = u(x) - \varepsilon(r^2 - |x-x_0|^2)$, for $\varepsilon > 0$. Then $\Delta u_\varepsilon^+ = -2n\varepsilon$, $\Delta u_\varepsilon^- = +2n\varepsilon$, implying that $u_\varepsilon^+$ is superharmonic and $u_\varepsilon^-$ is subharmonic having boundary values $u$. Therefore $u_\varepsilon^- \le v \le u_\varepsilon^+$ and the conclusion follows taking $\varepsilon \to 0^+$. ■

The previous result states that a generalized harmonic function is harmonic in open subsets where the function is continuous. Therefore, the difference between a generalized harmonic function relies on the discontinuity points of the functions. Next, we give some results that show how small is the set of discontinuities of a generalized harmonic function.

**Proposition 1.20 — Regularity I of GH.**
Let $u \in L^\infty_{loc}(\Omega)$ be generalized harmonic and $K \subset \Omega$ compact with nonempty interior. Then, there is a nonempty open $U \subset K$ such that $u$ is harmonic in $U$.

*Proof.* The key tool that we use in this proof is Baire category theorem. Let $k \in \mathbb{N}$ and define for large enough $k$ the following subsets

$$F_k := \left\{ x \in K : \frac{1}{r^2} \left| \fint_{B(x,r)} (u(y) - u(x))\mathrm{d}y \right| \le 1, \quad \forall 0 < r \le \frac{1}{k} \right\}.$$

This is well defined for $k \ge k_0$ for some $k_0$ because $K$ is compactly contained in $\Omega$. We check that these sets verify the following properties:

- $F_k$ are closed.
- $F_k \subseteq F_{k+1}$.
- $\bigcup_{k\geq k_0} F_k = K$.

The sets $F_k$ are closed because for every sequence contained in $F_k$ we can find a subsequence that converges to a point in $F_k$. Let $\{x_k\}$ be a sequence contained in $F_k$. Since $K$ is compact and $u \in L^\infty_{loc}$, there is a subsequence, WLOG we rename it by $\{x_k\}$, such that $x_k$ converges to some $x \in K$ and $u(x_k)$ converges to some value $L$. From the hypothesis, it holds for $r \in (0, 1/k)$

$$\frac{1}{r^2}\left|\fint_{B(x_k,r)} (u(y) - u(x_k))\mathrm{d}y\right| \leq 1.$$

Taking limits we obtain for $r \in (0, 1/k)$

$$\frac{1}{r^2}\left|\fint_{B(x,r)} (u(y) - L)\mathrm{d}y\right| \leq 1.$$

Since $u$ is GH, it is clear that $u(x) = L$. Therefore, $x \in F_k$. The other two properties are straightforward.

Since we are in a locally compact Hausdorff space (and therefore a Baire space), we know that there is some $k$ such that $F_k$ has nonempty interior. Let $U$ be the interior of this $F_k$. Then we can show that $u$ is weakly harmonic in $U$. Consider $\varphi \in C^\infty_c(U)$, i.e.

$$\begin{aligned}\int_U u \cdot \Delta\varphi &= \int_U u(x) \lim_{r\to 0^+} \frac{C_P}{r^{n+2}} \int_{B(x,r)} (\varphi(y) - \varphi(x))\,\mathrm{d}y\mathrm{d}x \\ &= \lim_{r\to 0^+} \frac{C_P}{r^{n+2}} \int_U \int_{B(x,r)} (u(x)\varphi(y) - u(x)\varphi(x))\,\mathrm{d}y\mathrm{d}x\end{aligned}$$

The first equality holds because the Privaloff operator coincides with the Laplacian for smooth functions. Later, we have applied the dominated convergence theorem to interchange the limit with the integral in the second line. Adding and substracting the same quantity we obtain the following, where the last integral vanishes due to a symmetry argument:

$$\begin{aligned}&\lim_{r\to 0^+} \frac{C_P}{r^{n+2}} \left(\int_U \int_{B(x,r)} (u(y)\varphi(x) - u(x)\varphi(x))\,\mathrm{d}y\mathrm{d}x + \int_U \int_{B(x,r)} (u(x)\varphi(y) - u(y)\varphi(x))\,\mathrm{d}y\mathrm{d}x\right) \\ &= \int_U \varphi(x) \left(\lim_{r\to 0^+} \frac{C_P}{r^{n+2}} \int_{B(x,r)} (u(y) - u(x))\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_U \varphi \cdot \Delta_P u = 0.\end{aligned}$$

Finally, we have appealed again to the dominated convergence theorem and used the hypothesis that $u$ is GH to prove that $u$ is weakly harmonic in $U$.

To conclude, using Lemma 1.8 we get that the function is harmonic in $U$ and there is no need to redefine the function in a set of measure zero, since the function is indeed generalized harmonic. ■

**Corollary 1.21 — Regularity II of GH.**
Let $u \in L^\infty_{loc}(\Omega)$ be generalized harmonic. Then, for almost every point in $\Omega$, we can find a sufficiently small ball where the function is indeed harmonic.

*Proof.* The proof follows by an easy contradiction argument using Proposition 1.20. ■

Therefore, locally bounded generalized harmonic function must be almost everywhere smooth (in particular, also continuous). We study in the next section the behavior of the function away from the regular points, that is the singular set which will be introduced there.

## 1.2 Singular set of generalized harmonic function

The result of the previous section motivates the following definition.

**Definition 1.22 — Singular set of GH function.**
Given a generalized harmonic function $u$ in $\Omega$, we say that a point $x$ belongs to the *singular set*, denoted by $S_u(\Omega)$, if there is no ball centred at that point where the function is harmonic. When the domain is not relevant we simply write $S_u$ instead.

The singular set of a generalized harmonic function might not be empty as it was shown in Example 1.12 with the sign function. However, under the natural assumption of being locally bounded, this set must be small, since it was proved in Corollary 1.21 that it necessarily has zero Lebesgue measure. Furthermore, it should be closed, because its complementary is clearly open. In fact, the singular set coincides with the closure of the points of discontinuity of the function.

**Lemma 1.23** Let $u$ be GH. Then, $S_u$ is the closure of the points of discontinuity of $u$.

*Proof.* It is direct from the definition of singular set (Definition 1.22) and Proposition 1.19. ■

With Example 1.12 with the sign function, we have already showed that points with finite jump of discontinuity might appear in the singular set of a generalized harmonic function. The following examples point out other types of points that might belong to the singular set.

■ **Example 1.24 — Essential discontinuity point in the singular set.** Let $\{x_k\}$ be any decreasing sequence converging to 0. We define the function piecewise in the following way:

$$u(x) = \begin{cases} 1 & \text{if } x > x_1 \\ 0 & \text{if } x = x_k \\ (-1)^k & \text{if } x \in (x_{k+1}, x_k) \\ 0 & \text{if } x = 0 \\ -u(-x) & \text{if } x < 0 \end{cases}$$

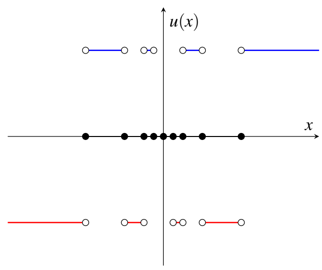


Figure 1.4: The function is equal to 0 in the sequence $x_k$ and constant with an alternating value of $-1$ or 1 in the intervals $(x_{k+1}, x_k)$. We take the odd extension. This function is generalized harmonic. Note that the condition over 0 comes given from the fact that the function is odd. When we compute the singular set is clearly $\{x_k\} \cup \{-x_k\} \cup \{0\}$. And 0 is an essential discontinuity point because the lateral limits do not exist.

■

■ **Example 1.25 — Continuity point in the singular set.** Let $\{x_k\}$ be any decreasing sequence converging to 0 and $\{c_k\}$ any sequence converging to 0. We define the function piecewise in the following way:

$$u(x) = \begin{cases} c_1 & \text{if } x > x_1 \\ \frac{1}{2}(c_k + c_{k+1}) & \text{if } x = x_k \\ c_{k+1} & \text{if } x \in (x_{k+1}, x_k) \\ 0 & \text{if } x = 0 \\ -u(-x) & \text{if } x < 0 \end{cases}$$

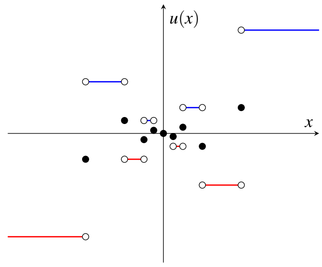


Figure 1.5: The function is equal to the average of the extremes of the jump for the sequence $\{x_k\}$ and a constant $c_k$ in the intervals $(x_{k+1}, x_k)$. We take the odd extension. This functions is generalized harmonic. Note again that the condition over 0 follows from the fact that the function is odd. The singular set of this function is the same as the previous one, that is, $\{x_k\} \cup \{-x_k\} \cup \{0\}$. And 0 is a continuity point this time because $c_k$ converges to 0. The previous plot is an example with $c_k = (-1)^k \frac{1}{2^k}$.

■

Note that the functions given in the previous Examples 1.24 and 1.25 also verify $\mathrm{MVP}_{\mathrm{loc}}$. We provide one easy counterexample to show that GH does not imply $\mathrm{MVP}_{\mathrm{loc}}$.

■ **Example 1.26 — GH $\not\Rightarrow$ $\mathrm{MVP}_{\mathrm{loc}}$.** Let $x_k = 2^{-k}$ and take $\{c_k\}$ be any decreasing sequence such that $4^k c_k$ converges to 0. For example, $c_k = 5^{-k}$. We define the function piecewise in the following way:

$$u(x) = \begin{cases} \frac{1}{2}(c_k + c_{k+1}) & \text{if } x = x_k \\ c_k & \text{if } x \in (x_{k+1}, x_k) \\ 0 & \text{if } x = 0 \\ u(-x) & \text{if } x < 0 \end{cases}$$

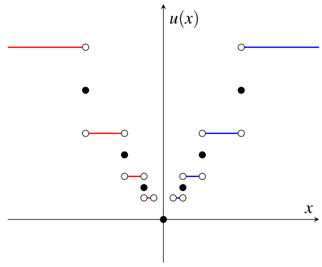


Figure 1.6: The function is equal to the mean value of the extremes of the jump for the sequence $x_k$ and constant to some value $c_k$ in the intervals $(x_{k+1}, x_k)$. We take the even extension. This function is generalized harmonic because away from 0, it is $\mathrm{MVP}_{\mathrm{loc}}$. However, in 0 it is clearly not $\mathrm{MVP}_{\mathrm{loc}}$. In 0, we have to estimate the ratio of the mean values in the following way:

$$\left| \frac{1}{r^2} \int_0^r u(y)\mathrm{d}y \right| \leq (2^{k+1})^2 c_k = 4 \cdot 4^k c_k$$

where $r \in [x_{k+1}, x_k)$. Since this quantity tends to 0, the function is generalized harmonic.

■

The function is regular in the complement of the singular set. Therefore, it is smooth in an open set. In the one dimensional case we can characterize

**Lemma 1.27** Let $U \subseteq \mathbb{R}$ open and nonempty. Then, there is a finite or countable subset $J$ and pairwise disjoint intervals $I_j$ such that $U = \bigcup_{j \in J} I_j$.

*Proof.* We say that two points $x$ and $y$ are equivalent in $U$ if, and only if, $[\min\{x,y\}, \max\{x,y\}] \subseteq U$. This defines an equivalence relation on $U$ whose equivalence classes are pairwise disjoint open intervals and their union is equal to $U$. Since for every interval we can choose one rational contained in the interval, there is at most a countable number of equivalence classes. ■

It is clear then that the singular part coincides with the extreme of the pairwise disjoint intervals that decompose the open set where the function is smooth. Therefore, the following proposition shows that points of finite jump discontinuity appears when two of these intervals have one same endpoint.

**Proposition 1.28** Let $u$ be GH defined over a nonempty open interval $I$ of $\mathbb{R}$. Suppose that there is $x \in \overline{I_{j_1}} \cap \overline{I_{j_2}}$ where $j_1 \neq j_2$ and $\bigcup_{j\in J} I_j$ coincides with the smooth part of $u$. Then,

- $u(x) = \dfrac{1}{2}\left(\lim_{y\to x^+} u(y) + \lim_{y\to x^-} u(y)\right)$.
- $\lim_{y\to x^+} u'(y) = \lim_{y\to x^-} u'(y)$.

*Proof.* It is straightforward from the definition of generalized harmonic functions and the fact that harmonic functions defined over intervals are affine functions. ■

Furthermore, when this type of jump discontinuity appears the slope of the function must be the same near the point. Thus to fulfil these restrictions we can restrict to the case of functions that are constants over at most a countable number of pairwise disjoint intervals.

It is clear that if the quantity of intervals is finite, any arbitrary choice of the constants leads to a generalized harmonic function whose singular set is finite and consisting of jump discontinuities.

However, when the number of intervals is countable the situation is more subtle. We can have essential discontinuity points and continuity points in the singular set as shown in Example 1.24 and 1.25 respectively. Furthermore, not any arbitrary choice of constants lead to a generalized harmonic function as displayed in the following example with the Cantor function.

■ **Example 1.29 — Cantor function.** Let $C$ be the Cantor set in $(0,1)$. Note that every element $x \in C$ has an unique representation of the form $x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}$ where $a_k \in \{0,1\}$. The terms $a_k$ are identified with the representation of the number $x$ in base 3. The Cantor function is defined by:

$$C(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{a_k}{2^k} & \text{if } x = \displaystyle\sum_{k=1}^{\infty} \frac{2a_k}{3^k} \in C \\ \displaystyle\sup_{\substack{y\le x,\\ y\in C}} C(y) & \text{if } x \in (0,1)\setminus C \end{cases}$$

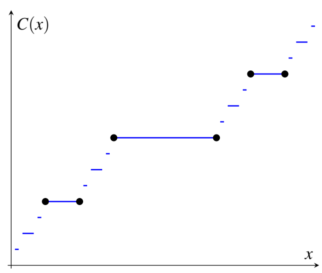


Figure 1.7: The Cantor function has the shape of a stair. It is a continuous function and the image above just displays a finite number of the steps. Though it can be constructed as the pointwise limit of GH functions with jump discontinuities, it is not GH because it is continuous but it is not harmonic (recall Proposition 1.19). Therefore, the Cantor function is an example that the limit of GH function might not be GH and that not every choice of constants over a countable number of intervals can build a GH function.

■

Note that even if the regular part of a generalized harmonic function is the union of a countable number of intervals, its singular set might be uncountable. For example, with the same choice of intervals chosen in Example 1.29 we might construct, a priori, a generalized harmonic function whose singular set is the Cantor set. Furthermore, in the following proposition, we show that uncountable singular set must have a homeomorphic image of the Cantor set.

**Proposition 1.30 — Singular set of GH in 1D.**
Consider $I \subset \mathbb{R}$ a bounded interval and $u \in L^\infty(I)$ generalized harmonic. If $S_u$ is uncountable, then except for a finite or countable set, it is homeomorphic to the Cantor set.

*Proof.* The singular set is closed and bounded. Using Cantor theorem (or its generalization to Polish spaces known as Cantor-Bendixson theorem) the closed set $S_u$ can be uniquely written as the disjoint union of a countable set and a perfect set (we recall that a perfect set is closed and has no isolated points, or equivalently, it coincides with its derivative set). We focus on the perfect part of the singular set, which is also totally disconnected (it cannot contain open sets due to Corollary 1.21).

Therefore, this part is a compact perfect totally disconnected metric space and thus it is homeomorphic to the Cantor set (see for example Theorem 7.8 in [Kec12]). ■

Motivated by above, we try to construct a generalized harmonic function whose singular set is the Cantor set. In fact, we prove that this type of construction is impossible leading to the conjecture that the singular set must be countable.

First, we deal with the case that the function is constant over the intervals and with all the symmetries possible from the self-similarity of the Cantor set.

■ **Example 1.31 — Singular set and Cantor set.** Let $C$ be again the Cantor set in $(0,1)$. Let $x \in (0,1)$. Then, we can write $x$ in base 3 as

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where $a_k \in \{0,1,2\}$. This representation is unique up to the following cases:

Case 1 There is $k_0 \in \mathbb{N}$ such that $a_{k_0} = 0$ and $a_k = 2\ \forall k > k_0$. This coincides with the representation $a_{k_0} = 1$ and $a_k = 0\ \forall k > k_0$.

Case 2 There is $k_0 \in \mathbb{N}$ such that $a_{k_0} = 1$ and $a_k = 2\ \forall k > k_0$. This coincides with the representation $a_{k_0} = 2$ and $a_k = 0\ \forall k > k_0$.

We split the Cantor into three pieces $C_1$, $C_2$ and $C_3$. We say that $x \in C$ is in $C_1$ (respectively in $C_2$) if it has a representation in the case 1 (respect. case 2). And it belongs to $C_3$ when it does not belong to $C_1$ or $C_2$. Note that we can characterize the elements in the Cantor set as the only ones that have a representation where $a_k \neq 1\ \forall k$.

Let $j(x) := \min\{k \in \mathbb{N} : a_k = 1\}$. Note that this function is well defined for $x \in (0,1) \setminus C$. We can now describe the function using the previous auxiliary function and a sequence $\{s_k\}$:

It is a direct to check that if the function is GH then the sequence converges in mean to some value that for simplicity we have taken to be 0.

It is clear that the function it is generalized harmonic in the open set $(0,1) \setminus C$ because it is harmonic in every connected component (the function is constant there). Therefore, we just need to show if the condition of GH also holds in the Cantor set.

Note that the function is symmetric with respect to $1/2$, that is, $u(x) = u(1-x)$. Therefore, it is enough to study the points in $C_2$ verifying the generalized harmonic condition to understand the case $C_1$.

$$u(x) = \begin{cases} s_{j(x)} & \text{if } x \in (0,1) \setminus C \\ \frac{1}{2} \lim_k c(x+3^{-k}) & \text{if } x \in C_1 \\ \frac{1}{2} \lim_k c(x-3^{-k}) & \text{if } x \in C_2 \\ 0 & \text{if } x \in C_3 \end{cases}$$

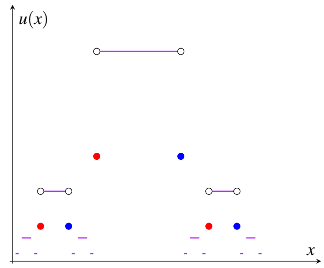


Figure 1.8: The construction of this function via steps is analogous to the Cantor function. However, instead of looking for continuity everywhere, we include some jump discontinuities in the points in red and blue. If the sequence the sequence $\{s_k\}$ converges to 0, the function is elsewhere continuous (also in some points of the Cantor set, i.e, in $C_3$).

Fix $x \in C_2$. Then, $\lim\limits_{y \to x^-} u(y) = s(k)$ for some $k$ and $\lim\limits_{y \to x^+} u(y) = 0$. At $x$, we have a jump discontinuity. And $c(x)$ is the average of the jump. Since coming from the left the function is eventually constant, to prove that it is generalized harmonic we just need to check that

$$\lim_{r \to 0^+} \frac{1}{r^3} \int_0^r u(x+s) \mathrm{d}s = 0.$$

To simplify this computation, we construct a function $v$ from $u$. We are going to split every interval where the function $u$ is constant to $\alpha$ in two equal pieces. In the left part of the interval is going to be equal to $|\alpha|$, in the right part to $-|\alpha|$ and in the middle point is going to be equal to zero. This produces many cancellations and therefore if $u$ is GH then the function $v$ is also GH.

Note that we can take $\{|s_k|\}$ to be nonincreasing without loss of generality. Now we can check for $x \in C_2$ the condition

$$\lim_{r \to 0^+} \frac{1}{r^3} \int_0^r v(x+s) \mathrm{d}s = 0.$$

Since $x \in C_2$ there is $k_0$ such that the digits of the representation of $x$ in base 3 are eventually 0. For $k > k_0$ it is clear that there are radii that touches points where the function takes the value $|s_k|$ $\left(\text{more precisely, for } 3^{-k} \leq r < \frac{3}{2} \cdot 3^{-k}\right)$. Since $|s_k|$ is nonincreasing, the maximum of $\frac{1}{r^3} \int_0^r u(x+s)\mathrm{d}s$ for $r \in \left[3^{-k}, 3 \cdot 3^{-k}\right]$ is achieved in the interval $\left[3^{-k}, \frac{3}{2} \cdot 3^{-k}\right]$. A simple analysis lead to that the maximum is achieved at $\frac{3}{2} \cdot 3^{-k}$ and therefore the condition of generalized harmonic at $x$ can be rewritten as

$$\lim_{k \to \infty} \frac{4}{27} 9^k |s_k| = 0.$$

If we assume that $9^k |s_k|$ converges to 0, then $v$ verifies the GH condition for points in $C_1$ and $C_2$. This implies that there is some $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ $|s_{k+1}| \leq \frac{1}{6}|s_k|$.

We are going to see that unless $|s_k|$ is eventually zero then it is impossible for $v$ to verify GH condition on $C_3$. More precisely, suppose that we have $x = x_2 + \varepsilon$ with $x_2 \in C_2$ of the form $\left(x_2 = \sum_{j=1}^k a_j \cdot 3^{-k} \text{ with } k \geq k_0\right)$ and $0 < \varepsilon < 3^{-k-2}$ . Then for $r \in \left[\varepsilon, \frac{3}{2}\varepsilon\right]$, the maximum of

$$\frac{1}{r^3} \left| \int_{-r}^r v(x+s) \mathrm{d}s \right|$$

is lower bounded by

$$\left(\frac{2}{3\varepsilon}\right)^3 \left(\frac{1}{2}|s_k|\varepsilon - \frac{3}{2}\varepsilon|s_{k+1}|\right) \geq \frac{2}{27} \frac{|s_k|}{\varepsilon^2}$$

Note that there is $x \in C_3$ such that

$$\limsup_{r\to 0^+} \frac{1}{r^3}\left|\int_{-r}^{r} v(x+s)\mathrm{d}s\right| \geq \frac{2}{27}$$

just taking $x = \sum_{j=1}^{k} a_j \cdot 3^{-k} + \varepsilon_k$ for infinitely many $k$ and $0 < \varepsilon_k^2 \leq |s_k|$. This is possible to do unless $|s_k|$ is eventually zero.

**Remark 1.32** An analogous argument works when we consider affine functions instead of constants over the intervals. Nevertheless, we omit the details because the idea is the same and the computations are similar.

■

From the previous example, we can infer that, at least from the unidimensional case, that the singular set of generalized harmonic functions cannot have dimension greater than zero. In higher dimension $n$, it is natural to conjecture that its dimension cannot exceed $n-1$. In the next section, we study further the geometric properties of the singular set having this in mind. Before that we construct one example of a function $u$ verifying the following weaker version of GH

$$\lim_{r\to 0^+} \frac{1}{r^{2-\delta}} \int_{-r}^{r} f(x+s)\mathrm{d}s = 0$$

with $\delta > 0$ for every $x$. The idea is analogous to the previous example.

■ **Example 1.33 — Weaker GH and Cantor set.** Let $C$ be again the Cantor set in $(0,1)$. We split the Cantor into three pieces $C_1$, $C_2$ and $C_3$ defined in Example 1.31.

For $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ not belonging to the Cantor set, let $j(x) := \min\{k \in \mathbb{N} : a_k = 1\}$, $A(x) := \sum_{k=1}^{j(x)} a_k + 3^{-k}$ and $B(x) := A(x) + 3^{-j(x)}$. Note that $A(x) \in C_1$ and $B(x) \in C_2$. Given a positive bounded sequence $\{m_k\}$ we define the function

$$u(x) = \begin{cases} m_k(x - A(x)) & \text{if } x < \frac{A(x)+B(x)}{2} \\ m_k(x - B(x)) & \text{if } x > \frac{A(x)+B(x)}{2} \\ 0 & \text{if } x = \frac{A(x)+B(x)}{2} \\ 0 & \text{if } x \in C \end{cases}$$

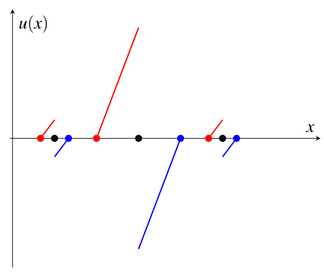


Figure 1.9: This function is constructed via affine functions whose slopes are going to tend to zero. We split the interval into two pieces so cancellation appears.

It is GH clearly outside the Cantor set and for points belonging in the Cantor set the estimation follows from the following:

$$\left|\frac{1}{r^{2-\delta}} \int_{0}^{r} f(x+s)\mathrm{d}s\right| = \frac{|m_k| s^2}{2(\varepsilon+s)^{2-\delta}} \leq \frac{|m_k| s^{\delta}}{2}$$

where $\varepsilon \geq 0$ is the distance from the point $x$ to a point in $C_1$. ■

## 1.3 Geometric motivation of the main result

We have shown that there are discontinuous functions which are harmonic in the generalized sense in Example 1.12 with the sign function. It is straightforward to generalize this to $\mathbb{R}^n$ using hyperplanes in the following way: let $a \in \mathbb{R}^n$, and $b, \alpha_+, \alpha_- \in \mathbb{R}$ such that $\alpha_- \neq \alpha_+$. Consider

$$f(x) = \begin{cases} \alpha_+ & \text{if } a \cdot x > b \\ \frac{1}{2}(\alpha_+ + \alpha_-), & \text{if } a \cdot x = b \\ \alpha_- & \text{if } a \cdot x < b \end{cases}$$

which is clearly discontinuous and GH (actually, $\mathrm{MVP}_{\mathrm{loc}}$). Note that any hyperplane splits any domain into at most two pieces. And given an orientation, the function takes the value $\alpha_+$ above the hyperplane and the value $\alpha_-$ below the hyperplane. WLOG we can assume that $\alpha_+ > 0$ and $\alpha_- < 0$; furthermore, we can take these constants to have the same absolute value (i.e. $\alpha_- = -\alpha_+$) and therefore the function takes the value 0 at the set of discontinuities.

The set of zeros of a harmonic function has been widely studied. Therefore, it is interesting to study the same question for this type of generalization. Hence, we aim to understand the hypersurfaces in $\mathbb{R}^n$ that can appear as the set of discontinuities of a GH function (or even a $\mathrm{MVP}_{\mathrm{loc}}$ function).

**Definition 1.34 — $f_S$ associated function.**
Given a $n$ dimensional manifold embedded $S$ in $\mathbb{R}^{n+1}$ which splits a neighbourhood of it in two connected components, we define its associated function, $f_S$, to be equal to 0, $+1$ and $-1$ for the manifold and each one of the two connected components respectively.

For example, if the manifold is orientable and boundaryless, the normal splits a neighbourhood of the manifold in two different connected components.

**Problem:** What are the hypothesis we need to impose to the hypersurface $S$ to ensure that its associated function is GH (or even $\mathrm{MVP}_{\mathrm{loc}}$)?

Note that proving $\mathrm{MVP}_{\mathrm{loc}}$ is generally more difficult to GH and therefore we deal first with the problem related to generalized harmonic. Observe that since GH is intuitively a condition related to verifying the mean value property over balls in an infinitesimal way (more precisely, linked to the square of the radius of the ball), GH might be associated to a condition of second order in the manifold, that is, something related to the second fundamental form of the hypersurface like the following conditions:

- The second fundamental form vanishes everywhere. This is equivalently to state that the manifold is a piece of a hyperplanes.
- The trace of the second fundamental form is zero everywhere. This implies, by definition, that the mean curvature is zero and therefore the hypersurface is minimal.

The last condition is similar to the condition of harmonicity of a function, since we recall that a function is harmonic whenever the trace of its hessian is zero everywhere. Furthermore, another equivalent definition for minimal hypersurface with a more analytic spirit is the following: the hypersurface, with an appropriate set of coordinates, is given locally by the graph of the function $x_n = \varphi(x_1, \ldots, x_{n-1})$, thus minimality is equivalent to $\varphi$ being a solution of the minimal surface equation, that is, $\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+(\nabla\varphi)^2}}\right) = 0$.

The answer to the problem above for finding a geometric property equivalent to the generalized harmonicity of the associated function is the main result of publication [CO20b] for smooth, orientable and boundaryless hypersurfaces.

**Answer:** Let $S$ be $C^1$, orientable and boundaryless. Then, $f_S$ is GH if, and only if, $S$ is minimal.

Obtaining more smooth examples seems rather difficult. In the following example, we show that the function associated with the catenoid does not verify $\mathrm{MVP}_{\mathrm{loc}}$.

■ **Example 1.35 — Catenoid and $\mathrm{MVP}_{\mathrm{loc}}$.** It is well-known that the there are only two types of minimal surfaces of revolution. One is the hyperplane and the other is the catenoid. Being of revolution simplifies the computations of the averages of balls to check if the function verifies $\mathrm{MVP}_{\mathrm{loc}}$ property. We recall that a parametrization of the catenoid is the following

$$\begin{cases} x = \cosh u \cdot \cos\theta \\ y = \cosh u \cdot \sin\theta \\ z = u \end{cases}$$

where $u \in \mathbb{R}$ and $0 \leq \theta < 2\pi$. If we take a foliation by planes perpendicular to the axis $z$, the intersection of the catenoid with the plane $z = z_0$ is the circumference of radius $\cosh(z_0)$. Therefore, the function associated to the catenoid candidate to be $\mathrm{MVP}_{\mathrm{loc}}$ takes the value 1 inside that circle and $-1$ outside that circle. To check if this function verifies $\mathrm{MVP}_{\mathrm{loc}}$ property, it is enough to see that, at some point of the catenoid, the volume of the points of the ball centred there where the function takes the value 1 is half the volume of the entire ball. To compute the volume of the positive part, we appeal to Fubini's theorem to compute the integral as an iterated integral along the axis $z$. The area of these slices are the intersection between two circles of radii $R$, $r$ whose centers are at distance $d$. These intersections are lens whose area can be computed by the following formula:

$$\begin{aligned} A(R,r,d) := {} & r^2 \arccos\left(\frac{d^2+r^2-R^2}{2dr}\right) + R^2 \arccos\left(\frac{d^2-r^2+R^2}{2dR}\right) \\ & - \frac{1}{2}\sqrt{(d+r+R)(-d+r+R)(d-r+R)(d+r-R)} \end{aligned}$$

Therefore, given $u_0 \in \mathbb{R}$ and $\theta_0 \in [0, 2\pi)$, to satisfy $\mathrm{MVP}_{\mathrm{loc}}$ we need to find $r_0 > 0$ such that $\forall r \in (0, r_0)$ the following identity follows

$$\int_{-r}^{r} A\left(\cosh(u_0+z), \sqrt{r^2-z^2}, \cosh(u_0)\right) \mathrm{d}z = \frac{2\pi}{3} r^3.$$

Since we cannot find this $r_0$, this function does not verify $\mathrm{MVP}_{\mathrm{loc}}$. ■

However, we can show that the helicoid verifies $\mathrm{MVP}_{\mathrm{loc}}$. Consequently, it would be interesting to characterize the minimal surfaces verifying $\mathrm{MVP}_{\mathrm{loc}}$. If they are only the plane and the helicoid like the characterization of ruled minimal surfaces due to Catalan ([Cat42]) or there are even more.

■ **Example 1.36 — Helicoid and $\mathrm{MVP}_{\mathrm{loc}}$.** Consider the following parametrization of the helicoid

$$\begin{cases} x = v \cdot \cos u \\ y = v \cdot \sin u \\ z = u \end{cases}$$

where $u, v \in \mathbb{R}$. It is easy show that given a point $(x_0, y_0, z_0)$ of the helicoid we can take a radius $r$ small enough so that the function $f(s)$ that computes the average of the function on the ball centered at that point with that radius intersected with the plane $\{z = z_0 + s\}$ is clearly odd. Consequently, this gives that the helicoid verify $\mathrm{MVP}_{\mathrm{loc}}$. ■

### Main result

After the previous motivation, we are ready to state the main result of this chapter and of the publication [CO20b]. Let $S$ be a $C^1$-hypersurface separating the domain $\Omega$ into two non empty components: $\Omega = \Omega^+ \cup \Omega^- \cup S$, that is,

$$S = \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega.$$

The associated function to the hypersurface $S$ is the function $f_S$, which is equal to $\alpha_+$ inside $\Omega^+$, $\alpha_-$ inside $\Omega^-$ and $\frac{1}{2}(\alpha_+ + \alpha_-)$ in $S$.

The main result is stated in the following theorem.

**Theorem 1.37 — Minimal and GH.**
The function $f_S$ (with $\alpha_+ \neq \alpha_-$) is GH if and only if $S$ is minimal.

Roughly speaking, Theorem 1.37 is an infinitesimal generalization of the fact that $S$ is a hyperplane if, and only if, every sufficiently small ball centred on $S$ is divided into two regions of exactly equal volume.

The smoothness of the hypersurface is important for the proof. However, this result might be generalized for Lipschitz hypersurfaces or even just continuous hypersurfaces. If we weaken further our hypothesis more geometric objects appear as it can be seen in the following examples.

**■ Example 1.38 — Simons Cone and GH.**
If we consider $S$ to be the Simons cone, that is,

$$S = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\} \subset \mathbb{R}^8$$

is a non-smooth hypersurface in $\mathbb{R}^8$ whose associated function is GH. ■

**■ Example 1.39 — Union of hyperplanes and $\mathrm{MVP}_{\mathrm{loc}}$.**
If $S$ is the union of some determined collection of hyperplanes (we need some condition about their intersection), its associated function is $\mathrm{MVP}_{\mathrm{loc}}$ as we can see in the following figure:

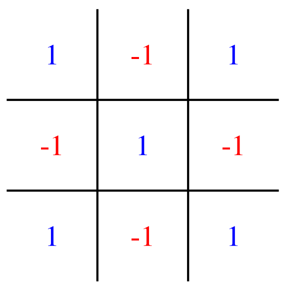


Figure 1.10: Example of $\mathrm{MVP}_{\mathrm{loc}}$ function with the union of four lines in $\mathbb{R}^2$.

■

## 1.4 Proof of the main result

The idea of the proof of the main result is the following: first, we prove the result for infinitely differentiable hypersurfaces with a basic calculus lemma. Later, from a $C^1$ hypersurface we apply some gain of regularity results to show that it is actually $C^\infty$. For example, it is well-known that $C^1$ minimal hypersurfaces are smooth. However, to gain regularity with the hypothesis that its associated function is GH is trickier and we need to use the notion of viscosity solution of uniformly elliptic equations. Namely, we show that if $\Delta_p f_S = 0$, then, locally, $S$ will be given as the graph of a viscosity solution of the minimal surface equation and, therefore, it has to be smooth. Once this regularity is established, the proof is completed again via the basic calculus lemma. We shall use also several well-known properties of minimal surfaces and elliptic equations for which [CC93], [Giu84] and [CC95] are appropriate references.

Note that we can apply our result even to Lipschitz surfaces at almost every point. On the one hand, if $S$ is minimal, we can use the basic calculus lemma due to the gain of regularity of $S$. On the other hand, if $f_S$ is generalised harmonic, almost every point of the surface has a tangent hyperplane and at those points we can analogously prove that they are a viscosity solution of the minimal surfaces (the Lipschitz hypothesis guarantees the uniformly ellipticity). However, we cannot proceed with this argument at those points without tangent hyperplane. It is an interesting problem to get Theorem 1.37 with the weaker hypothesis $S$ Lipschitz or even $S$ continuous.

### A basic calculus lemma

Let $S$ be a smooth ($C^4$) hypersurface in $\mathbb{R}^n$ with unit normal vector field $\nu(x)$. Given $x \in S$ and $r > 0$, small enough, the ball $B_r^{(n)}(x)$ is separated by $S$ in two connected components, $B_r^{n,+}(x)$, $B_r^{n,-}(x)$, where $B_r^{n,+}(x)$ (respectively $B_r^{n,-}(x)$) consists of the points inside $B_r^{(n)}(x)$ which are placed above (respect. below) $S$ in the given normal direction.

Then we have:

**Lemma 1.40 — Basic calculus lemma.**
Let $S \in C^4$. As $r \to 0^+$,

$$\operatorname{vol}(B_r^{n,+}(x)) - \operatorname{vol}(B_r^{n,-}(x)) = -c_n H_S(x) \cdot r^{n+1} + O(r^{n+3}).$$

Here $H_S(x)$ denotes the mean curvature of $S$ at the point $x$ and $c_n > 0$ is a universal constant which only depends upon the dimension $n$.

*Proof.* Without loss of generality we can assume that $x = 0$ is the origin of a coordinate system such that the tangent space of $S$ at $x$ is horizontal, i.e., the normal vector is $\nu(0) = (0, \cdots, 0, 1)$. Hence, near $x = 0$, $S$ is the graph of a smooth ($C^4$)) function $x_n = \varphi(x_1, \cdots, x_{n-1})$ satisfying:

1. $\varphi(0) = 0$.
2. $\nabla\varphi(0) = 0$.

Then, inside the cylinder $B_r^{(n-1)}(0) \times \mathbb{R}$, $r$ small enough, we have the inclusion

$$S \cap \left(B_r^{(n-1)}(0) \times \mathbb{R}\right) \subset \{x \in \mathbb{R}^n : |x_n| \leq c_1 r^2\}$$

for a positive constant $c_1$ depending upon the size of the second derivatives of $\varphi$.

An elementary computation shows that the vertical projection of $S \cap B_r^{(n)}(0)$ onto $B_r^{(n-1)}(0)$ must contain the ball

$$B_{r-c_2 r^3}^{(n-1)}(0)$$

for a fixed constant $c_2$.

Let

$$\begin{aligned} D_r^+ &= B_r^{n,+}(0) \cap \{|x_n| \le c_1 r^2\}, \\ D_r^- &= B_r^{n,-}(0) \cap \{|x_n| \le c_1 r^2\}. \end{aligned}$$

Then, since $S$ is contained in the strip, by symmetry we get the following equality

$$\begin{aligned} \mathrm{vol}(B_r^{n,+}(0)) - \mathrm{vol}(B_r^{n,-}(0)) &= \mathrm{vol}(D_r^+) - \mathrm{vol}(D_r^-) \\ &= \int_{B^{(n-1)}_{r-c_2r^3}(0)} \big(c_1 r^2 - \varphi(x)\big)\,\mathrm{d}x - \int_{B^{(n-1)}_{r-c_2r^3}(0)} \big(c_1 r^2 + \varphi(x)\big)\,\mathrm{d}x + I \end{aligned}$$

where $I$ is the correction of restricting the domain of integration.

A direct computation of the term $I$ shows that

$$|I| \lesssim r^2 \cdot (r^{n-1} - (r - c_2 r^3)^{n-1}) \lesssim r^{n+3} = O(r^{n+3}).$$

Hence

$$\mathrm{vol}(B_r^{n,+}(0)) - \mathrm{vol}(B_r^{n,-}(0)) = -2 \int_{B^{(n-1)}_{r-c_2r^3}(0)} \varphi(x)\mathrm{d}x + O(r^{n+3}).$$

Another elementary computation of the volume of the cylinders yields

$$\left| \int_{B^{(n-1)}_{r}(0)} \varphi(x)\mathrm{d}x - \int_{B^{(n-1)}_{r-c_2r^3}(0)} \varphi(x)\mathrm{d}x \right| \lesssim r^{n+3} = O(r^{n+3}).$$

Then using Taylor's expansion we obtain

$$⨍_{B^{(n-1)}_{r}(0)} \varphi(y)\mathrm{d}y = \frac{1}{2(n+1)} \Delta\varphi(0) \cdot r^2 + O(r^4).$$

This allows us to finish the proof of the lemma: because we know that

$$H_S(x', \varphi(x')) = \frac{1}{n-1} \mathrm{div} \left( \frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) (x'),$$

and since $\nabla \varphi(0) = 0$, we have

$$H_S(0) = \frac{1}{n-1} \Delta \varphi(0).$$

∎

First, without loss of generality, one can assume that $\alpha_+ = +1$ and $\alpha_- = -1$. Next, let us observe that one of the two implications of the theorem follows immediately: namely if $S$ is minimal and $C^1$ then, by the classical theorem of de Giorgi-Nash, $S$ has to be smooth and we can apply Lemma 1.40 to observe that at any point $x \in S$ we have:

$$\begin{aligned}
\Delta_P f(x) &= \lim_{r\to 0^+} \frac{2(n+2)}{r^2} \fint_{B_r^{(n)}(x)} (f(y)-0)\mathrm{d}y \\
&= \lim_{r\to 0^+} \frac{2(n+2)}{|B_r^{(n)}(x)|r^2} \left[\mathrm{vol}(B_r^{n,+}(x)) - \mathrm{vol}(B_r^{n,-}(x))\right] \\
&= 0,
\end{aligned}$$

because of the minimality condition $H_S(x) = 0$.

Note, that a similar argument with Lemma 1.40 also works to prove that $f_S$ being generalized harmonic implies that $S$ is minimal. Therefore, to finish the proof we just need to prove the regularity ($C^4$) of $S$.

To continue the proof let us recall now that a real continuous function $F$ defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times M^n$, where $M^n$ denotes the vector space of $n$ x $n$ symmetric matrices, yields an elliptic equation $F(x,u,Du,D^2u) = 0$ if

$$F(x,u,\eta,\delta) \leq F(x,u,\eta,\delta+\sigma)$$

for all $(x,u,\eta,\delta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times M^n$ and $\sigma \in M^n$ non-negative.

The elliptic equation is called uniformly elliptic if there exist positive constant $\lambda, \Lambda$ satisfying the estimate:

$$0 < \lambda\|\sigma\| < F(x,u,\eta,\delta+\sigma) - F(x,u,\eta,\delta) \leq \Lambda\|\sigma\|$$

where $\|\sigma\|$ denotes the $(L^2,L^2)$-norm (i.e. $\|\sigma\| = \sup_{\|x\|=1}\|\sigma x\|$ = maximum of the eigenvalues of $\sigma$).

**Definition 1.41 — Viscosity solution.**
A continuous function $u$ is called a viscosity subsolution (respectively supersolution) of $F(x,u,Du,D^2u) = 0$ if, for any quadratic polynomial (if it exists) $\Psi \in C^2(\Omega)$ such that we have a local maximum (respectively local minimum) of $u - \Psi$ at $x_0$ then

$$F(x_0,u(x_0),D\Psi(x_0),D^2\Psi(x_0)) \geq 0 \qquad \text{(respectively } \leq 0).$$

Finally, $u$ is a viscosity solution if it is both a viscosity subsolution and supersolution.

Reference [CC95] contains the result about regularity (Corollary 5.7) and uniqueness (Corollary 5.4) of viscosity solutions of uniformly elliptic equations, which we shall invoke to conclude the proof. Namely, we have: if $u$ is a viscosity solution of $F(D^2u) = 0$ in $B_1(x)$ then $u \in C^{1,\alpha}(\overline{B}_{\frac{1}{2}}(x_0))$ and $||u||_{C^{1,\alpha}(\overline{B}_{\frac{1}{2}}(x_0))} \leq C\left(||u||_{L^\infty(B_1(x))} + F(x)\right)$. Then we can apply some Schauder's estimates to conclude the smoothness ($C^\infty$) of $u$ (see Chapter 8 in [CC95]).

That is, under the hypothesis that $\Delta_P f \equiv 0$, let $P$ be a paraboloid tangent from below to our hypersurface $S = \{x_n = \varphi(x_1,\cdots,x_{n-1})$ at a point $x = (x_1,\cdots,x_{n-1},x_n) \in S \cap P$.

Let $f_P$ be its corresponding function defined in the introduction, we have the inequality

$$\fint_{B_r^{(n)}(x)} f_P \geq \fint_{B_r^{(n)}(x)} f_S$$

On the other hand, Lemma 1.40 applied to the hypersurface $P$ yields

$$\int_{B_r^{(n)}(x)} f_P = -c_n H_P(x) \cdot r^{n+1} + O(r^{n+3})$$

which together with the hypothesis

$$\lim_{r \to 0^+} \frac{1}{r^2} \fint_{B_r^{(n)}(x)} f_S = 0$$

implies that $H_P(x) \leq 0$.

Similarly if $P$ is now a paraboloid tangent to $S$ from above at the point $x$, then we must have $H_P(x) \geq 0$. Therefore $\varphi$ is a viscosity solution of the equation

$$\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) = 0$$

whose uniform ellipticity is ensured by the hypothesis that $S$ is of class $C^1$.

Then, the regularity of such solutions ([CC95]) allows us to conclude the smoothness of $\varphi$ (and $S$) and, therefore, its minimality.

# 2. Splitting Results on Riemannian Manifolds

This chapter displays the results of the accepted paper [FO20]. The theorems contained here are the result of a fruitful research stay at *Université Picardie Jules Verne* with Alberto Farina. More specifically, we obtained some splitting results concerning Riemannian manifolds, relating them to the existence of solutions of any (non)linear Poisson equation which achieve somewhere a gradient estimate, known as Modica's estimate, or its generalization.

The splitting is highly related to the one dimensionality of the solutions of some elliptic PDEs. A relevant and highly interesting problem related to this property is De Giorgi's conjecture, which states that bounded and monotone in one direction solutions of the Allen-Cahn equation in the euclidean space must be 1D, at least for low dimensions. For the state of the art of this interesting problem see, for instance, [FV09] and [Sav09].

## 2.1 Introduction

Throughout this chapter we shall denote by $(\mathscr{M},g)$, or simply by $\mathscr{M}$, a complete, connected, smooth ($C^\infty$) boundaryless Riemannian manifold of dimension $m \geq 2$ with nonnegative Ricci curvature. The neccesity of the hypotheses about the manifold are discussed in Remark 2.3.

We shall consider the, a priori, nonlinear Poisson equation over $\mathscr{M}$,

$$-\Delta_g u + f(u) = 0, \tag{2.1}$$

where $f = F'$ is the first derivative of a function $F \in C^2(\mathbb{R})$ and $\Delta_g$ is the Laplace-Beltrami operator on $\mathscr{M}$, that is, the extension of the Laplacian for Riemannian manifolds. Equation 2.1 can be interpreted as the Euler-Lagrange equation of a mechanical system whose potential comes from the function $F$. Therefore, it is natural to consider the particular case: $F \geq 0$.

An important property of bounded solutions of 2.1 is that the potential controls the gradient of the solution. More precisely, in the seminal paper [Mod85], Modica proved the following pointwise gradient estimate for bounded solutions $u \in C^3(\mathbb{R}^m)$ of (2.1) with $F \geq 0$ in the euclidean case:

$$\frac{1}{2}|\nabla u|^2(x) \leq F(u(x)) \qquad \forall x \in \mathbb{R}^m. \tag{2.2}$$

The picture of this chapter reflects the beauty of France (and Amiens in particular). It was really nice to have pondered along the amazing riversides and later have discussed these thoughts in a nice French cafeteria leading to the results displayed in this chapter.

This important inequality, known as Modica's estimate, has been widely studied. It has been studied in a more general geometric setting and arbitrary potential function. More specifically, when there is no assumption about the nonnegativeness of the function $F$ and additive constant can be subtracted in the right side of the estimate to make it sharp.

The inequality (2.2) is no longer true if we drop the assumption: $F(t) \geq 0$ for every $t \in \mathbb{R}$, as one can easily see by considering the function $u(x_1, \cdots, x_m) = \cos(x_1)$, which solves $\Delta u = F'(u)$ in $\mathbb{R}^m$ with $F(t) = -\frac{t^2}{2}$. This problem is overcome in the next result where a refined and sharp form of Modica's estimate is obtained for any possible nonlinear function $F \in C^2(\mathbb{R})$ and the geometric setting introduced at the beginning.

**Proposition 2.1 — Generalized Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded solution of (2.1). Then,

$$\frac{1}{2}|\nabla_g u|^2(x) \leq F(u(x)) - c_u \qquad \forall x \in \mathscr{M}, \tag{2.3}$$

where

$$c_u := \inf_{y \in \mathscr{M}} F(u(y)). \tag{2.4}$$

The above proposition recovers and improves the results in [FSV08], [FV10] established in the euclidean setting as well as those proved in [FV11] and [RR95] for some compact and noncompact Riemannian manifolds respectively.

**Remark 2.2** Note that the bound 2.3 is always sharp (by definition of $c_u$) and that it is achieved everywhere by any entire 1D solution of a semilinear Poisson equation over $\mathbb{R}^m$ endowed with the flat metric (just multiply the ODE $u'' = F'(u)$ by $u'$ and then integrate the resulting identity). On the other hand, the bound (2.2) is not necessarily sharp as shown by any nonconstant 1D periodic solution of the Allen-Cahn equation $\Delta u = u^3 - u$ on $\mathbb{R}^m$ $\left(\text{here } F(t) = \frac{(t^2-1)^2}{4}\right)$.

**Remark 2.3** If $F \equiv 0$, we obtain, in particular, the classical result that the only bounded harmonic functions are the constant functions. It is well-known that the topological properties of connectivity, completeness and the absence of boundary are necessary so that Proposition 2.1 is true. With respect to the curvature condition, on the one hand, if $\mathrm{Ric}(\mathscr{M}) \geq 0$ with the previous properties implies the statement (Corollary 1 in [Yau75]) and, on the other hand, if $\mathrm{Ric}(\mathscr{M}) \leq -C$ (with $C > 0$), there may exist nonconstant bounded harmonic functions, see [And83].

The following proposition explains how the constant $c_u$ of the generalized Modica's estimate can be computed effectively using the extrema of the solution $u$.

**Proposition 2.4 — About the constant in the generalized Modica estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded nonconstant solution of (2.1). Then, (2.3) holds with constant

$$c_u = \min\left\{F\left(\inf_{x \in \mathscr{M}} u(x)\right), F\left(\sup_{x \in \mathscr{M}} u(x)\right)\right\}$$

and

$$c_u < F(t) \qquad \forall\, t \in \left(\inf u, \sup u\right)$$

## 2.2 Main results

The main results of this chapter are splitting results, that is, some criteria to determine if the manifold or a piece of it is isometric to a product space. While the classical splitting result in [CG71] has a geometric flavor (the condition there is that the manifold has a straight line), these results possess a more analytic spirit since the criteria is the existence of a bounded solution of any Poisson equation which achieves at a regular point Modica's estimate or its generalization. The achievement of Modica's estimate leads to the splitting of the whole manifold whereas the achievement of the generalized Modica's estimate at a regular point just provides the splitting of a piece of the manifold.

**Theorem — Global Splitting via Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded nonconstant solution of (2.1) with $F \geq 0$. Suppose that there is a point $x_0$ such that

$$\frac{1}{2}|\nabla_g u|^2(x_0) = F(u(x_0)).$$

Then, $\mathscr{M}$ splits as the Riemannian product $\mathscr{N} \times \mathbb{R}$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$. Furthermore, $u : \mathscr{N} \times \mathbb{R} \to \mathbb{R}$ is such that $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

**Remark 2.5** An analogous conclusion about the one dimensionality of the solutions also appears in [CGS94] for the euclidean case, i.e. $\mathbb{R}^m$ endowed with its standard flat metric. Nevertheless, the splitting result is totally new in the general geometric setting that we are considering here.

Studying what happens when equality holds, at a regular point $x_0 \in \mathscr{M}$, in the generalization of Modica's estimate (2.3) leads to a local splitting of the manifold in a neighborhood of $x_0$ and to a precise description of the solution in this neighborhood.

**Theorem — Local splitting via generalized Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded solution of (2.1). Suppose that equality is achieved in (2.3) at a regular point $x_0$, i.e. $\nabla_g u(x_0) \neq 0$. Denote by $\mathscr{U}$ the open connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ that contains $x_0$. Then,

- equality in (2.3) holds in $\mathscr{U}$,
- $Ric_g(\nabla_g u, \nabla_g u)$ vanishes at $\mathscr{U}$,
- $\mathscr{U}$ splits as the Riemannian product $\mathscr{N} \times I$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$ and $I \subseteq \mathbb{R}$ is an interval,
- the solution $u$ restricted to the neighborhood $\mathscr{U}$, $u : \mathscr{N} \times I \to \mathbb{R}$, is equal to $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

**Remark 2.6** We underline that the result of local splitting obtained in the aforementioned Theorem 2.13 is sharp. Indeed, a global splitting result, like the one demonstrated in Theorem 2.14, is no longer true when $F$ is not nonnegative on $\mathbb{R}$. An example showing this phenomenon can be built in the following way: consider any Riemannian manifold $\mathscr{N}$ of dimension $m-1$, with nonnegative Ricci tensor and which does not contain any line (e.g. a round sphere) and set $\mathscr{M} = \mathscr{N} \times \mathbb{S}^1$. The function $u(n,s) = \sin(s)$ is a solution of $\Delta_g u = -u$ for which the refined Modica's estimate (2.3) is achieved everywhere on $\mathscr{M}$ and in particular in many regular points of $u$. Nevertheless $\mathscr{M}$ does not split any euclidean factor.

**Remark 2.7** If a critical point achieves the equality in (2.3), we cannot assure that there are more points which saturate the inequality, as shown by the following example in the euclidean space. For $m \geq 3$ consider the function $u(x) = \left(\frac{\sqrt{m(m-2)}}{1+|x|^2}\right)^{\frac{m-2}{2}}$, which is a radial smooth, positive, bounded solution of $-\Delta u = u^{\frac{m+2}{m-2}}$ on the euclidean space $\mathbb{R}^m$. In this case $F(t) = -\frac{|t|^{p+1}}{p+1}$, with $p = \frac{m+2}{m-2}$, and $c_u = -\frac{(u(0))^{p+1}}{p+1}$ so, by integration we immediately get that $\frac{1}{2}|\nabla u|^2(x) = \frac{1}{2}|u'|^2(|x|) = F(u(x)) + \frac{(u(0))^{p+1}}{p+1} - \int_0^{|x|} \frac{m-1}{r}(u')^2(r)\mathrm{d}r$ for every $x \in \mathbb{R}^m$. Therefore, the equality in (2.3) is achieved at the origin, which is a critical point, while elsewhere inequality (2.3) is strict.

Some other global splitting theorems were obtained in [FMV13] and [FSV12] for the special subclass of stable solutions to (2.1). The novelty of our aforementioned results is that they hold true for any bounded solution of equation (2.1). Since in the present work we do not assume any stability assumption on the considered solutions, the method that we develop here is completely different from those used in [FMV13] and [FSV12]. In our approach, the main hypothesis required is the achievement of the Modica's estimate or generalized Modica's estimate at a regular point to obtain the global and local splitting respectively stated in Theorems 2.14 and 2.13. For analogous questions in RCD metric measure spaces see the very recent paper [ABS19].

The rest of the chapter is devoted to prove these results. First, in Section 2.3, we prove some geometric results that shows that the existence of a local harmonic function whose gradient has constant length implies splitting. Later, in Section 2.4, we provide the proofs of the generalized Modica's estimate and the computation of the constant. Finally, in Section 2.5, we show that, as long as the gradient of the solution does not vanish, we can construct a harmonic function whose gradient has constant length, which implies the splitting. We recall that a similar approach was previously used in [CG71] where an harmonic function whose gradient has constant length is established from the hypothesis that the manifold has a straight line.

## 2.3 Some geometric results concerning the splitting

To prove our splitting results we recall, for the sake of completeness, some results of Riemannian geometry (see for instance [Pet06]).

**Lemma 2.8** Let $E$ and $F$ be two distributions on $(\mathscr{M}, g)$, which are orthogonal complements of each other in $T\mathscr{M}$, and suppose that the distributions are parallel (i.e. if two vector fields $X$ and $Y$ are tangent to, say, $E$, then $\nabla_X Y$ is also tangent to $E$). Then,

- The distributions are integrable.
- $\mathscr{M}$ is locally a product metric, i.e., there is a product neighborhood $U = V_E \times V_F$ such that $(U, g) = (V_E \times V_F, g_{|E} + g_{|F})$, where $g_{|E}$ and $g_{|F}$ are the restrictions of $g$ to the two distributions.

*Proof.* To prove that the distributions are integrable we just need to show they are involutive because these concepts are equivalents thanks to Frobenius Theorem. Thus, we want to show that having two vector fields $X$ and $Y$ tangent to, say, $E$, then the Lie bracket $[X,Y]$ is also tangent to $E$. Then,

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

is tangent to $E$ because $E$ is parallel. Analogously with $F$.

Since $E$ and $F$ are integrable distributions, for any $p \in \mathscr{M}$ there is a neighborhood of $p$

where we can take local coordinates $x_i$ and $y_\alpha$ where $1 \le i \le n$ and $n+1 \le \alpha \le m$ such that

$$E = \mathrm{span}\left\{\frac{\partial}{\partial x_i} : 1 \le i \le n\right\},$$
$$F = \mathrm{span}\left\{\frac{\partial}{\partial y_\alpha} : n+1 \le \alpha \le m\right\}.$$

We just need to show that $g_{|E}$ (respect. $g_{|F}$) is independent of $y_\alpha$ (respect. $x_i$), i.e. $\partial_\alpha g_{ij} = 0$ (respect. $\partial_i g_{\alpha\beta} = 0$)

$$\partial_\alpha g_{ij} = \partial_\alpha g_{ij} + \partial_i g_{j\alpha} - \partial_j g_{i\alpha} = 2\Gamma_{ij\alpha} = 2g\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right) = 0,$$

where the previous equalities holds because $E$ and $F$ are orthogonal and $E$ is parallel. The equality $\partial_i g_{\alpha\beta} = 0$ is proven analogously. ■

**Lemma 2.9** Let $X$ be a parallel vector field on $(\mathscr{M}, g)$. Then,

- $X$ has constant length.
- $X$ generates parallel distributions, one that contains $X$ and the other that is the orthogonal complement to $X$.
- Locally the metric is a product with an interval, $(U, g) = (V \times I, g_{|TV} + \mathrm{d}t^2)$.

*Proof.* Since $X$ is parallel, we have by definition that $\nabla X = 0$. From the Fundamental Theorem of Riemanian Geometry we get directly that $X$ has constant length:

$$\nabla g(X,X) = g(\nabla X, X) + g(X, \nabla X) = 0.$$

We can consider the distribution $E$ and $F$ to be respectively the ones that for every $p \in \mathscr{M}$, $E_p = \mathrm{span}\{X(p)\}$ and $F_p = (\mathrm{span}\{X(p)\})^\perp$. These are parallel because if we have two vector fields $Y_1$ and $Y_2$ tangent to $E$, then clearly $\nabla_{Y_1} Y_2$ is tangent to $E$.

Using the previous lemma we get directly that the metric is locally a product with an interval. ■

At last we recall the following result.

**Lemma 2.10** Let $u$ be a harmonic function on a manifold $\mathscr{M}$ with $\mathrm{Ric}(\mathscr{M}) \ge 0$ such that $g(\nabla u, \nabla u) = 1$. Then, $X = \nabla u$ is a parallel vector field.

*Proof.* Let $p \in \mathscr{M}$ and $\sigma$ be the integral curve of $\nabla u$ through $p$. Set $X := \nabla u$ and consider $\{e_1, \cdots, e_{m-1}, X\}$ an orthonormal frame in a neighborhood of $p$ which is parallel along $\sigma$. Then, $\nabla_X X = 0$ because from $g(X,X) = 1$ we get

$$0 = \nabla_X g(X,X) = 2g(X, \nabla_X X),$$

and for $e_i$ we get from the orthogonality and the condition that $e_i$ are parallel along $\sigma$,

$$g(e_i, \nabla_X X) = \nabla_X g(e_i, X) - g(\nabla_X e_i, X) = \nabla_X 0 - g(0, X) = 0.$$

At $p$, we have that

$$\begin{aligned}
\mathrm{Ric}(X) &= \sum_{i=1}^{m-1} g(R(e_i,X)X,e_i) + g(R(X,X)X,X) && \big(R(X,X)X=0 \text{ from } \nabla_X X=0\big)\\
&= \sum_{i=1}^{m-1} g\left(\nabla_{e_i}\nabla_X X - \nabla_X\nabla_{e_i}X - \nabla_{[e_i,X]}X, e_i\right) && \big(\text{using } \nabla_X X = 0 \wedge \nabla_X e_i = 0\big)\\
&= \sum_{i=1}^{m-1} \left[-g\left(\nabla_X\nabla_{e_i}X,e_i\right) - g\left(\nabla_{\nabla_{e_i}X}X,e_i\right)\right] && \Big(\text{note } \nabla_{e_i}X = \sum_{j=1}^{m-1} g(\nabla_{e_i}X,e_j)e_j\Big)\\
&= \sum_{i=1}^{m-1} \nabla_X\left[-g\left(\nabla_{e_i}X,e_i\right)\right] - \sum_{1\le i,j\le m-1} g(\nabla_{e_i}X,e_j)g(\nabla_{e_j}X,e_i) && \big(\text{using orthogonality}\big)\\
&= \nabla_X\left[-\mathrm{div}(X)\right] - \sum_{1\le k\le m-1} g(\nabla_{e_k}X,e_k)^2 && \big(\mathrm{div}(X)=0 \text{ from harmonicity}\big)\\
&= -\|\nabla X\|^2
\end{aligned}$$

From the hypothesis that Ric$\geq 0$, we get that $\nabla X = 0$, proving that $X$ is parallel. ■

The previous geometric results are applied in the next section to prove Theorem 2.14 and Theorem 2.13. For that reason, a harmonic function, whose gradient has constant length, is constructed in a neighborhood of the point where the generalized Modica's estimate is achieved.

## 2.4 Generalization of Modica's estimate. Proofs of Propositions 2.1 and 2.4

In this section we incorporate the proofs of the generalized Modica's estimate and the computation of the constant that appears there.

First, we are going to provide, using the literature, the reason why the generalized Modica's estimate stated in Proposition 2.1 is valid.

*Proof.* The result is already known for $\mathscr{M} = \mathbb{R}^m$ (see [FSV08], [FV10]) and when $\mathscr{M}$ is a compact manifold (see [FV11]).

In the case of a complete, noncompact manifold we note that the gradient of the solution $u$ is bounded (see for instance Remark 48 in Appendix 1 of [FMV13] or Proposition 1 of [RR95]) and it satisfies $\inf_{\mathscr{M}} |\nabla u| = 0$ (see for instance Appendix 1 of [FMV13]). The desired inequality (2.3) then follows from Theorem B of [RR95] applied with $Q(u) = F(u) - c_u$. This concludes the proof of Proposition 2.1. ■

The key to prove Proposition 2.4 is throught the following interesting result. It claims that if $c_u$ is achieved at an interior point of the interval $[\inf u, \sup u]$, then $u$ must be constant. To prove it we follow [FV10] and [Mod85].

**Proposition 2.11** Let $u \in C^3(\mathscr{M})$ be a bounded solution of (2.1) such that $c_u$ is achieved in an interior point of the interval $[\inf u, \sup u]$. Then, $u$ is constant.

*Proof.* By Proposition 2.1, we know that the generalization of Modica's estimate holds. Suppose that $c_u$ is achieved at a interior point $u(x_0) = \alpha$, then $\alpha$ is a local minimum of $F$ and therefore verifies the following properties: $F(\alpha) = 0$, $F'(\alpha) = 0$ and $F''(\alpha) \geq 0$. Hence there exists $k \geq 0$ and $\delta > 0$ such that

$$F(s) - c_u \leq k(s-\alpha)^2$$

for every $s$ such that $|s-\alpha| < \delta$.

Since $\mathscr{M}$ is connected, to obtain the desired conclusion it is enough to prove that the nonempty and closed set $A := \{x \in \mathscr{M} : u(x) = \alpha\}$ is also open in $\mathscr{M}$. Pick $a \in A$. There is a normal neighborhood of $a$ of the form $B(a,r)$ for some $r = r(a) > 0$ such that $|u(x) - \alpha| = |u(x) - u(a)| < \delta$ for every $x \in B(a,r)$ using the continuity of the function $u$.

For any $x$ in $B(a,r)$ consider the unit speed minimizing geodesic $\gamma\colon [0,t_x] \to \mathscr{M}$ such that $\gamma(0) = a$ and $\gamma(t_x) = x$, where $t_x := d(a,x) < r$. For all $t \in [0,t_x]$ set

$$\varphi(t) := u(\gamma(t)) - u(a)$$

and observe that

$$\frac{1}{2}|\varphi'(t)|^2 \leq \frac{1}{2}|\nabla_g u(\gamma(t))|^2 \leq F(u(\gamma(t)) - c_u \leq k(u(\gamma(t)) - \alpha)^2 = k\varphi(t)^2$$

since $\gamma(t) \in B(a,r)$ for any $t \in [0,t_x]$ (recall that $\gamma$ is minimizing).

Therefore $|\varphi'(t)| \leq C|\varphi(t)|$ for some $C > 0$ and for all $t \in [0,t_x]$ and so $\varphi \equiv 0$ on $[0,t_x]$, since $\varphi(0) = 0$. This means that $u(x) = u(a) = \alpha$ and so $B(a,r) \subset A$, which concludes the proof. ■

**Remark 2.12** Note that in the previous proof we have proved, in particular, the classical Liouville result. More specifically, if $c_u$ is a local minimum of $F$ and $u^{-1}(c_u) \neq \emptyset$, then $u$ must be constant. In particular, if a solution achieves the Modica's estimate 2.2 somewhere, it is either constant or it doesn't have any critical points. Observe that all the hypothesis are necessary because otherwise there are easy counterexamples.

Finally, we show the proof of Proposition 2.4.

*Proof.* It is clear that the above Proposition 2.11 and Proposition 2.4 and since we have already proved the first one, the latest one should also be true. ■

## 2.5 Proofs of splitting results

Finally, we show the proofs of the contributed results that appear in [FO20], that is, the splitting results stated in Theorem 2.13 and 2.14.

We start recalling the local splitting result.

**Theorem 2.13 — Local splitting via generalized Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded solution of (2.1). Suppose that equality is achieved in (2.3) at a regular point $x_0$, i.e. $\nabla_g u(x_0) \neq 0$. Denote by $\mathscr{U}$ the open connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ that contains $x_0$. Then,

- equality in (2.3) holds in $\mathscr{U}$,
- $Ric_g(\nabla_g u, \nabla_g u)$ vanishes at $\mathscr{U}$,
- $\mathscr{U}$ splits as the Riemannian product $\mathscr{N} \times I$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$ and $I \subseteq \mathbb{R}$ is an interval,
- the solution $u$ restricted to the neighborhood $\mathscr{U}$, $u : \mathscr{N} \times I \to \mathbb{R}$, is equal to $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

*Proof.* Let us consider the function

$$P := P(u,x) = \frac{1}{2}|\nabla_g u(x)|^2 - F(u(x)) + c_u, \tag{2.5}$$

then by proceeding as in the proof of Theorem 1 of [FV11] we get

$$|\nabla_g u|^2 \Delta_g P - 2f(u)\langle \nabla_g u, \nabla_g P\rangle - |\nabla_g P|^2 \geq |\nabla_g u|^2 Ric_g(\nabla_g u, \nabla_g u) \geq 0 \quad \text{on} \quad \mathscr{M} \tag{2.6}$$

Recall that $|\nabla_g u(x_0)| > 0$ and $P(u,x_0) = 0$ by assumption. Also $P \leq 0$ on $\mathscr{M}$ by (2.3) and therefore, in the light of (2.6), the strong maximum principle gives that $P(u,x) = P(u,x_0) = 0$ in the connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ that contains $x_0$. The latter and (2.6) then imply $Ric_g(\nabla_g u, \nabla_g u) = 0$ in the connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ that contains $x_0$. Therefore we have proved the first two claims of Theorem 2.13.

Using the first statements of Theorem 2.13 we know that equality in 2.3 holds in the connected component of $\mathscr{M} \cap \{\nabla_g u \neq 0\}$ where $x_0$ belongs, $\mathscr{U}$.

Now, from $u$ we are going to construct a harmonic function $v$ on $\mathscr{U}$ with constant gradient using a change of variables. We set $v := H(u)$ where

$$H(u) := \int_{u_0}^{u} (2F(s) - 2c_u)^{-\frac{1}{2}}\, ds$$

for some $u_0 \in u(\mathscr{U})$. Then, $v$ has constant gradient in $\mathscr{U}$

$$|\nabla_g v|^2 = H'(u)^2 |\nabla_g u|^2 = \frac{|\nabla_g u|^2}{2F(u) - 2c_u} = 1,$$

where we have used $\frac{1}{2}|\nabla_g u|^2 = F(u) - c_u$ in the last equality. Let us check that $v$ is harmonic in $\mathscr{U}$

$$\begin{aligned}\Delta_g v &= H''(u)|\nabla_g u|^2 + H'(u)\Delta_g u \\ &= -\frac{f(u)}{(2F(u)-2c_u)^{3/2}}|\nabla_g u|^2 + \frac{f(u)}{(2F(u)-2c_u)^{1/2}} = 0\end{aligned}$$

using again that $\frac{1}{2}|\nabla_g u|^2 = F(u) - c_u$ and $u$ being a solution of (2.1).

Hence, since $v$ is a harmonic function whose gradient has length 1, we know that this generates a parallel vector field and we obtain a local splitting using the results from Section 3.

Finally, the solution restricted to this neighborhood is 1D and monotone is straightforward. And this part will be explained in more detail in the next proof.

This concludes the proof of Theorem 2.13. ■

Now we recall the global splitting result. Note that this is a particular case of Theorem 2.13 where the gradient of the solution does not vanish. Therefore, we show more carefully the one dimensionality of the solution.

**Theorem 2.14 — Global Splitting via Modica's estimate.**
Let $u \in C^3(\mathscr{M})$ be a bounded nonconstant solution of (2.1) with $F \geq 0$. Suppose that there is a point $x_0$ such that

$$\frac{1}{2}|\nabla_g u|^2(x_0) = F(u(x_0)).$$

Then, $\mathscr{M}$ splits as the Riemannian product $\mathscr{N} \times \mathbb{R}$ where $\mathscr{N} \subset \mathscr{M}$ is a totally geodesic and isoparametric hypersurface with $\mathrm{Ric}(\mathscr{N}) \geq 0$. Furthermore, $u : \mathscr{N} \times \mathbb{R} \to \mathbb{R}$ is such that $u(p,s) = \varphi(s)$ where $\varphi$ is a bounded and strictly monotone solution of the ODE, $\varphi'' = f(\varphi)$.

*Proof.* If $u$ is not constant, then by proceeding as in the first part of the proof of the previous result we see that $|\nabla v| = 1$ on $\mathscr{M}$. Then, the Bochner formula implies that the Hessian of $v$ is everywhere zero on $\mathscr{M}$ and so, the level sets of $v$ are totally geodesic (and isoparametric) smooth hypersurfaces (see Proposition 18 of [FMV13]). Moreover, from lemma 2.10 we also know that $\nabla v$ is a parallel vector field on $\mathscr{M}$. Therefore, if we denote by $\mathscr{N}$ the level hypesurface $\{v=0\}$ and by $\phi_t$ the flow of $\nabla v$, it is well-known that the map $\Phi : \mathscr{N} \times \mathbb{R} \to \mathscr{M}$ defined by

$$\Phi(x,t) := \phi_t(x)$$

is a Riemannian isometry with respect to the product metric on $\mathscr{N} \times \mathbb{R}$ (see e.g. pp. 219-220 of [Sak96] or pp. 206-207 of [PRS08]. See also Remark 2 on p. 145 of [EH84]).

Finally, in a local Darboux frame $\{e_j, \nu = \nabla v\}$ for the level surface $\mathscr{N}$,

$$\begin{aligned} 0 &= |II|^2 \Longrightarrow \nabla \mathrm{d}v(e_i, e_j) = 0, \\ 0 &= \langle \nabla |\nabla v|, e_j \rangle = \nabla \mathrm{d}v(\nu, e_j), \end{aligned} \tag{2.7}$$

so the unique nonzero component of $\nabla \mathrm{d}v$ is that corresponding to the pair $(\nu, \nu)$. Let $\gamma$ be any integral curve of $\nu$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(v \circ \gamma) = \langle \nabla v, \nu \rangle = |\nabla v| \circ \gamma > 0$$

and so

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}(v \circ \gamma) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(|\nabla v| \circ \gamma) = \langle \nabla |\nabla v|, \nu \rangle(\gamma) = \nabla \mathrm{d}v(\nu, \nu)(\gamma) \\ &= \Delta v(\gamma) = F'(v \circ \gamma). \end{aligned}$$

Therefore $y = v \circ \gamma$ is a solution of the ODE $y'' = F'(y)$ and $y' > 0$.

■

# Part II

# Duality and Approximation in Variable Lebesgue Spaces


# 3. Dual Space of Variable Lebesgue Spaces

The purpose of this chapter is to fully characterize the dual space of variable Lebesgue spaces. Variable Lebesgue spaces are a generalization of the classical Lebesgue spaces where the exponent is a function instead of a constant. Furthermore, they are a very important particular case of generalized Musielak-Orlicz spaces.

The characterization of the dual space for variable Lebesgue spaces was stated explicitly as an open problem in [CF13, Problem A.3], but prior to this the question was part of the folklore in the study of variable Lebesgue spaces. Actually, I discovered this topic in the conference Harmonic Analysis in Winter held at ICMAT (see the image above).

We prove that the dual of a variable Lebesgue space can be split into two pieces. One was already known and it is related to its Hölder conjugate associated space. And the other, we show that it is an abstract **L**-space. Furthermore, in the particular case of variable sequence spaces, we provide results characterizing this abstract **L**-space. The results displayed at this chapter appear in [Ame+19].

## 3.1 Introduction

Variable Lebesgue spaces are a generalization of the classical Lebesgue spaces $L^p$, in which the exponent $p \in [1,+\infty]$ is replaced by a function. The motivation for the study of variable Lebesgue spaces is the following: Consider the real function $g(x) := \frac{1}{\sqrt{|x|}}$, which presents a singularity at $x = 0$. This natural function does not belong to any $L^p(\mathbb{R})$ for $1 \leq p \leq +\infty$, because it either grows too fast at the origin or decays too slow at infinity. An idea to solve this matter and find a Lebesgue space to which it could belong would be to split the domain. With this on mind, we can prove $g \in L^1([-1,1])$ and $g \in L^4(\mathbb{R} \setminus [-1,1])$, for instance. The disadvantage of this approach is that for more complex examples, the splitting of the domain becomes increasingly difficult. Then, letting the exponent vary pointwise, we could control in a better way the singularities that the function might present at each point. This leads to the origin of *variable Lebesgue spaces*.

Variable Lebesgue spaces were first introduced by Orlicz [Orl31]. They have been widely studied for the past thirty years ([GM15], [EGK18] among others), both for their intrinsic

ICMAT is the image of this chapter. This building represents a great deal of my PhD years. Here, I've met very fascinating people (including all the coauthors of the paper explained in this chapter) and I've plenty of nice conversations about mathematics.

interest as function spaces and for their applications to PDEs and the calculus of variations with nonstandard growth conditions. For further details of this history, including extensive references, we refer the reader to the monographs [CF13; Die+11]. The variable Lebesgue spaces are an important special case of the more general Musielak–Orlicz spaces [HH19; Mus06].

Let $(\Omega, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space. We denote by $\mathscr{P}(\Omega)$ the collection of all measurable functions $p\colon \Omega \to [1, +\infty]$; we will refer to the elements of $\mathscr{P}(\Omega)$ as exponent functions. For each $p \in \mathscr{P}(\Omega)$, define the modular

$$\rho_{p(\cdot)}(f) := \left( \int_{\Omega \setminus \Omega_\infty} \frac{2}{p(x)} |f(x)|^{p(x)} \, \mathrm{d}\mu(x) + \|f\|_{L^\infty(\Omega_\infty)} \right),$$

where $\Omega_\infty := \{x \in \Omega : p(x) = +\infty\}$. Given a measurable function $f$, we say that $f \in L^{p(\cdot)}(\Omega)$ if there exists $\lambda > 0$ such that $\rho_{p(\cdot)}(f/\lambda) < +\infty$. This set becomes a Banach function space when equipped with the Luxemburg norm (introduced in [Lux55])

**Definition 3.1 — Luxemburg norm.**
Given a measurable function $f$ and an exponent function $p$, the *Luxemburg norm* is defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \ \rho_{p(\cdot)}(f/\lambda) \ \leq \ 1\right\}.$$

We can now introduce the formal definition of variable Lebesgue spaces as follows.

**Definition 3.2 — Variable Lebesgue space.**
Let $p : \Omega \to [1, +\infty)$ be an exponent function and consider the space of functions given by:

$$L^{p(\cdot)}(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ measurable and } \|f\|_{p(\cdot)} < +\infty\}.$$

$\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a Banach space, which we call *variable Lebesgue space*.

In the rest of the chapter, we usually omit the dependence on $\Omega$, writing $L^{p(\cdot)}$ instead of $L^{p(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$, $\mathscr{P}$ instead of $\mathscr{P}(\Omega)$, and so on.

When $\Omega = \mathbb{N}$, $\mathscr{A} = \mathscr{B}(\mathbb{N})$, and $\mu$ is the discrete counting measure, we will denote the space by $\ell^{p(\cdot)}$. This is a variable sequence space and we analyze them in detail in Section 3.5.

There exist in the literature different ways to define a norm for these spaces. We have chosen the Luxemburg norm with that particular choice of modular (see Section 3.7 for the explanation of this choice), but it is also possible to use a different modular such as $\rho(f(x)) = \int |f(x)|^{p(x)}$ or even use a different definition of the norm like the Amemiya norm.

If the exponent function is constantly equal to a value $p$, we recover the classical Banach space $(L^p, \|\cdot\|_p)$ with $1 \leq p \leq \infty$, showing thus that the previous definition is a generalized version of the so-called Lebesgue spaces.

Note that we can split the measurable space into two pieces $\Omega_\infty$ and $\Omega \setminus \Omega_\infty$ and therefore our $L^{p(\cdot)}$ space is the internal direct sum of functions with support in these two pieces, that is, [1]

$$L^{p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega \setminus \Omega_\infty) \oplus L^\infty(\Omega_\infty).$$

Consequently, the dual is the internal direct sum of the dual of these two pieces.

[1] This means that the whole space is the direct sum of these two closed subspaces in an algebraic way. Note that we are not stating that the norm decomposes in these two subspaces in any direct way.

**Remark 3.3** Since the dual of $L^\infty(\Omega_\infty)$ is well known, see for instance [Con10]), it is enough to comprehend the dual of $L^{p(\cdot)}(\Omega\setminus\Omega_\infty)$ to understand the dual of $L^{p(\cdot)}(\Omega)$. Therefore, from now on we suppose that $\mu(\Omega_\infty)=0$, i.e. that $p$ is almost everywhere finite.

Next, we present an essential definition which allows to split the behavior of variable Lebesgue spaces into two different cases.

**Definition 3.4 — Bounded exponent function.**
We say that an exponent function $p:\Omega\to[1,+\infty)$ is *bounded* when the essential supremum of $p$ verifies:

$$p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < +\infty\,,$$

i.e. it is uniformly bounded except possibly in a set of zero Lebesgue measure. If $p$ is not bounded, we say that it is *unbounded*.

The importance of the boundedness of the exponent function resides in the fact that variable Lebesgue spaces with bounded exponent function behave similarly to classical Lebesgue spaces. Moreover, they verify certain important properties, like the ones presented below, whose proofs can be found in [CF13].

**Proposition 3.5 — Separable variable Lebesgue spaces.**
The variable Lebesgue space $\left(L^{p(\cdot)},\|\cdot\|_{p(\cdot)}\right)$ is separable if, and only if, the exponent function is bounded.

Therefore, whenever the exponent $p$ is bounded, the space $L^{p(\cdot)}$ is separable. For the next property, we need to consider the subspaces of $L^{p(\cdot)}$ of compactly supported functions and smooth compactly supported functions, denoted by $L_c^{p(\cdot)}$ and $C_c^\infty$, respectively. Now, one can prove the following characterization of bounded exponent for variable Lebesgue spaces, concerning their density in $L^{p(\cdot)}$.

**Proposition 3.6 — On the denseness of some subspaces.**
$\left(L_c^{p(\cdot)},\|\cdot\|_{p(\cdot)}\right)$ is dense in $L^{p(\cdot)}$ if, and only if, the exponent function is bounded.
Moreover, $\left(C_c^\infty,\|\cdot\|_{p(\cdot)}\right)$ is dense in $L^{p(\cdot)}$ if, and only if, the exponent function is bounded.

Next, we present an essential result concerning the duality of this class of spaces. Before stating it, we recall that for any exponent function $p:\Omega\to[1,+\infty)$, its *Hölder conjugate* is the exponent function $q$ defined pointwise by $\frac{1}{q(x)}+\frac{1}{p(x)}=1$, for almost every $x\in\Omega$. We take by, convention, that if $p(x)=1$, then $q(x)=+\infty$.

**Proposition 3.7 — On the dual of bounded variable Lebesgue spaces.**
If $p\in\mathscr{P}$ is bounded,

$$(L^{p(\cdot)})^*\cong L^{q(\cdot)},$$

where $q$ is pointwise the Hölder conjugate of $p$ and the symbol $\cong$ denotes an isometric isomorphism. Furthermore, there is an equivalence between functionals, $T$, and functions in $L^{q(\cdot)}$, $g$, through the following identity

$$T(f)=\int_\Omega f(x)g(x)\mathrm{d}\mu(x)\,,\qquad \text{for every } f\in L^{p(\cdot)}\,.$$

Moreover, when the domain is compact, there are some inclusions between variable Lebesgue spaces similar to the case of classical Lebesgue spaces.

**Proposition 3.8 — Inclusion variable Lebesgue spaces when domain is compact.**
Let $K$ be compact. Whenever we have two exponent functions $p_1$ and $p_2$ verifying that $p_1 \leq p_2$ pointwise in $K$, then
$$L^{p_2(\cdot)}(K) \subseteq L^{p_1(\cdot)}(K)\,.$$

On the other hand, if the exponent function is unbounded, the situation is much more subtle. Variable Lebesgue spaces with unbounded exponent functions are much more complex in general. They are non-separable, we cannot approximate their elements by compactly supported functions and the dual is more complicated. In the following section we present our results needed to deal with unbounded exponent functions and discussed in [Ame+19].

Our contribution is presented with the following structure. In Section 3.3 we decompose the dual as the sum of $L^{q(\cdot)}$ and the dual of a quotient space. Motivated by above, we study this quotient space obtaining a characterization of its norm, which is used to show that the quotient space is an abstract **M**-space in Section 3.4. This implies that the second part of the dual is an abstract **L**-space that might be very complex. To understand better this part of the dual we restrict to the case of variable sequence spaces in Section 3.5. A very interesting phenomena happens in this case: depending on the growth of the exponent function, this second part is exactly the space of purely finitely additive measures, a generalization of this type of measures or even something strictly bigger. When something strictly bigger appears it is difficult to provide a precise description. Therefore, in Section 3.6 we show the existence of very nice dense subsets which help to describe it a bit better. Finally, in the Section 3.7 we explain the choice of the modular for the definition instead of the more canonical one.

## 3.2 Main results

Before stating precisely the main results, we introduce some preliminary ideas.

**Definition — $L_b^{p(\cdot)}$.**
Let $p : \Omega \to [1, +\infty)$ be an unbounded exponent function and consider the subspace
$$L_b^{p(\cdot)} := \overline{\left\{ f \in L^{p(\cdot)} : p \text{ is bounded at } \operatorname{supp}(f) \right\}}\,.$$
If $p$ is continuous and everywhere finite, this space coincides with $L_c^{p(\cdot)}$ the space of functions with support compactly contained in the domain.

The behavior of this subspace is very similar to the one of bounded variable Lebesgue spaces.

**Definition — Quotient space.**
Let $p : \Omega \to [1, +\infty)$ be an unbounded exponent function and consider the quotient space
$$L_Q^{p(\cdot)} := L^{p(\cdot)} / L_b^{p(\cdot)}\,,$$
where we have taken the closure in the $L^{p(\cdot)}$ norm of the subspace $L_b^{p(\cdot)}$.
This quotient space is equipped with the quotient norm $\|\cdot\|_Q$ defined by
$$\|[f]\|_Q := \inf_{g \in L_b^{p(\cdot)}} \|f - g\|_{p(\cdot)}\,.$$
Note that since $p$ is unbounded, this is nontrivial (Proposition 3.6).

This way of computing the quotient norm is impractical. To overcome that issue, our first result is the following useful characterization.

**Proposition — Characterization of the quotient norm.**
Let $p : \Omega \to [1, +\infty)$ be an unbounded exponent function. Then,

$$\|[f]\|_Q = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu(x) < +\infty\right\}.$$

Note that this expression has the same flavor than the norm of $L^{p(\cdot)}$. Actually, it is equivalent to

$$\inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) < \infty\}.$$

Furthermore, note that $\|[f]\|_Q \leq \|f\|_{p(\cdot)}$, because of the inclusion of the subsets of concerning the infimums.

This characterization is vital to prove easily that the quotient space is an abstract **M**-space. A good reference for these properties related to classical Banach spaces is [LT13].

**Definition — Abstract M-space.**
A Banach lattice $X$ is an *abstract **M**-space* whenever given $x_1, x_2 \in X$ such that $x_1 \wedge x_2 = 0$, then

$$\|x_1 \vee x_2\| = \max\{\|x_1\|, \|x_2\|\}. \tag{3.1}$$

Intuitively, these spaces are similar to $L^\infty$ and the condition above can be understood in this case that whenever we have two functions with disjoint support, the norm of their sum is the maximum of their norms.

**Proposition — $L_Q^{p(\cdot)}$ is an abstract M-space.**
The quotient space $L_Q^{p(\cdot)}$ is an abstract **M**-space.

The motive why we have studied of this quotient space is the following splitting result of the dual.

**Theorem — Splitting of the dual.**
Given $p : \Omega \to [1, +\infty)$ an unbounded exponent function, then

$$\left(L^{p(\cdot)}\right)^* \cong \left(L^{q(\cdot)}, \|\cdot\|_{q(\cdot)}\right) \oplus \left(L_Q^{p(\cdot)}, \|\cdot\|_Q\right)^*,$$

where $\cong$ denotes an isometric isomorphism.

In [Kak41] it is proved that if $X$ is an abstract **M**-space then $X^*$ is an abstract **L**-space. In a sense, abstract **L**-spaces behave like $L^1$ spaces. Indeed, abstract **L**-spaces are order isometric to $L^1(\nu)$ for some measure $\nu$ (see for example [LT13]). In [Kak41], they are also characterized to be isometric and lattice isomorphic to a closed linear subspace of the space of all completely additive regular real-valued set-functions $\nu(E)$ for all Borel sets $E$ of some compact Hausdorff space.

Finally, we have provided a precise description of the dual of variable Lebesgue spaces. The first part of the dual is very easy to comprehend whereas the second part is very abstract. Consequently, we restrict to the special case of variable sequence spaces to give a better insight of this second component.

**Theorem — Dual of variable sequence spaces.**
Given $p : \Omega \to [1, +\infty)$ an exponent function, the dual of the variable sequence space $\ell^{p(\cdot)}$ is:

- $\ell^{q(\cdot)}$ if, and only if, $p$ is bounded.
- $\ell^{q(\cdot)} \oplus pba(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N}) = 1$.
- $\ell^{q(\cdot)} \oplus pba_\omega(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N}) < \infty$.
- $\ell^{q(\cdot)} \oplus pba_\omega(\mathscr{A})$ if, and only if, $S_{p(\cdot)}$ is dense.
- $\ell^{q(\cdot)} \oplus pba_\omega(\mathscr{A}) \oplus pba(\mathscr{Z}(\mathscr{A}, c_0))$ in general.

Some notions used in the previous theorem are the following:

**Definition — $\omega$ and $\mathscr{A}$.**
Given $A \subseteq \Omega$ measurable and $p : \Omega \to [1, +\infty)$ an unbounded exponent function, we define

$$\omega(A) := \omega^{p(\cdot)}(A) := \|[\mathbf{1}_A]\|_Q.$$

And $\mathscr{A}$ or $\mathscr{A}_{p(\cdot)}$ is the collection of subsets $A$ such that $\omega(A) < \infty$.

$\omega$ is a kind of weight that shows how relevant a set is in the quotient space. Furthermore, there is a huge relation to contain all the bounded functions.

**Theorem — Characterization of the containment of $L^\infty$.**
Let $p : \Omega \to [1, +\infty)$ be an unbounded exponent function. Then,
$L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ if, and only if, $\omega(\Omega) < +\infty$.

Finally, $pba_\omega(\mathscr{A})$ and $pba(\mathscr{Z}(\mathscr{A}, c_0))$ are some spaces of purely finitely additive measures. The first one is easy to describe and it is a direct generalization of the classical space $pba$. The formal definition of this first space is introduced below. Nevertheless, since the second space is trickier to describe, we recommend to read first Subsection 3.6.2 to understand the dense subspace where the measures are defined.

**Definition — $pba_\omega(\mathscr{A})$.**
Given $p \in \mathscr{P}(\mathbb{N})$, define $pba_\omega(\mathscr{A})$ to be the vector space of set functions $\delta$ defined on $\mathscr{A}$ satisfying the following properties:

1. $\delta(A \cup B) = \delta(A) + \delta(B)$ for any pair of disjoint sets $A, B \in \mathscr{A}$.
2. There exists $C > 0$ such that given any collection $\{A_i\}_{i=1}^n$ of pairwise disjoint sets in $\mathscr{A}$,

$$\sum_{i=1}^{n} \frac{|\delta(A_i)|}{\omega(A_i)} \leq C.$$

Define a norm on $pba_\omega(\mathscr{A})$ by

$$\|\delta\|_{pba_\omega} := \inf\{C > 0 : \text{ condition (2) holds}\}.$$

## 3.3 Splitting of the dual

The goal of this section is to show that the dual of $L^{p(\cdot)}$ can be split in two pieces. One of them is a familiar one, because it is what one gets when $p$ is bounded. The second one is the more interesting one, and we shall study it in greater detail in the following sections. We start with the easier part. One way to build a functional $\phi$ over $L^{p(\cdot)}$ is to use a function $g \in L^{q(\cdot)}$ in the following way: for each $f \in L^{p(\cdot)}$,

$$\phi_g(f) := \int_\Omega f \cdot g \, \mathrm{d}\mu.$$

It is clear that $\|\phi_g\|_{(L^{p(\cdot)})^*} = \|g\|_{q(\cdot)}$. As in the constant exponent case, it turns out that every functional $\phi$ over $L^{p(\cdot)}$ determines a function $g_\phi \in L^{q(\cdot)}$ such that for every $f \in X$

$$\phi(f) = \int_\Omega f \cdot g_\phi \mathrm{d}\mu$$

for some Banach space that we shall determine next. To that end, recall that

$$p_+(E) := \operatorname*{ess\,sup}_{x \in E} p(x).$$

**Definition 3.9 — $L_b^{p(\cdot)}$ and $p(\cdot)$-bounded sets.**
We say that at measurable set $E \subset \Omega$ is *$p(\cdot)$-bounded* if $p_+(E) < +\infty$. We set

$$L_b^{p(\cdot)} := \overline{\left\{ f \in L^{p(\cdot)} : \ \operatorname{supp}(f) \text{ is } p(\cdot)\text{-bounded} \right\}},$$

where the closure is taken in the $\|\cdot\|_{p(\cdot)}$ norm. We equip this subspace of $L^{p(\cdot)}$with the induced norm.

**Remark 3.10** Observe that $L_b^{p(\cdot)}$ is not the whole $L^{p(\cdot)}$ whenever $p_+(\Omega) = \infty$, see for example [CF13, Theorem 2.77].

The next result establishes the relation between $(L^{p(\cdot)})^*$ and $L^{q(\cdot)}$.

**Proposition 3.11** Given $\phi \in (L^{p(\cdot)})^*$, there exists a unique function $g_\phi \in L^{q(\cdot)}$ such that for all $f \in L_b^{p(\cdot)}$,

$$\phi(f) = \int_\Omega f(x) g_\phi(x) \, \mathrm{d}\mu(x), \tag{3.2}$$

and

$$\|g_\phi\|_{q(\cdot)} \le \|\phi\|_{(L^{p(\cdot)})^*}.$$

*Proof.* Fix $\phi \in (L^{p(\cdot)})^*$. First note that if $f \in L^{p(\cdot)}$ and $E \subset \Omega$, then $f|_E \in L^{p(\cdot)}(E)$ with $\|f|_E\|_{p(\cdot)} = \|\mathbf{1}_E f\|_{p(\cdot)}$. Conversely, if $h \in L^{p(\cdot)}(E)$, then $\tilde{h}$, the extension by zero of $h$ to $\Omega$, is in $L^{p(\cdot)}$ with $\|\tilde{h}\|_{p(\cdot)} = \|h\|_{p(\cdot)}$. Given a $p(\cdot)$-bounded set $E \subset \Omega$, for all $h \in L^{p(\cdot)}(E)$ define

$$\phi_E(h) := \phi(\tilde{h}).$$

Then we have

$$|\phi_E(h)| = |\phi(\tilde{h})| \le \|\phi\|_{(L^{p(\cdot)})^*} \|\tilde{h}\|_{p(\cdot)} = \|\phi\|_{(L^{p(\cdot)})^*} \|h\|_{p(\cdot)},$$

so $\phi_E \in (L^{p(\cdot)}(E))^*$ with $\|\phi_E\|_{(L^{p(\cdot)}(E))^*} \le \|\phi\|_{(L^{p(\cdot)})^*}$. Since $p_+(E) < \infty$, there exists a unique function $g_\phi^E \in L^{q(\cdot)}(E)$ such that for all $f \in L^{p(\cdot)}(E)$,

$$\phi_E(f) = \int_E f(x) g_\phi^E(x) \, \mathrm{d}\mu(x),$$

and with $\|g_\phi^E\|_{L^{q(\cdot)}(E)} = \|\phi_E\|_{(L^{p(\cdot)}(E))^*} \le \|\phi\|_{(L^{p(\cdot)})^*}$.

Now, notice that since $(\Omega, \mathscr{A}, \mu)$ is $\sigma$-finite we can write

$$\Omega = \bigcup_{k=1}^{\infty} \{x : \ p(x) \le k\} =: \bigcup_{k=1}^{\infty} F_k.$$

Therefore, by a standard patching argument there exists a unique measurable function $g_\phi$ on $\Omega$ such that for all $k$, $g_\phi|_{F_k} = g_\phi^{F_k}$ with $\|\mathbf{1}_{F_k} g_\phi\|_{q(\cdot)} \le \|\phi\|_{(L^{p(\cdot)})^*}$. Furthermore, for every $h \in L_b^{p(\cdot)}$, we have

$$\int_\Omega h(x) g_\phi(x)\, \mathrm{d}\mu(x) = \phi(g). \tag{3.3}$$

It remains to show that $g_\phi$ is in $L^{q(\cdot)}$ and that (3.3) holds for all $h \in L_b^{p(\cdot)}$. The functions $\mathbf{1}_{F_k}|g_\phi|$ increase pointwise a.e. to $|g_\phi|$, so by the monotone convergence theorem for $L^{q(\cdot)}$ [CF13, Theorem 2.59] we have that $g_\phi \in L^{q(\cdot)}$ and

$$\|g_\phi\|_{q(\cdot)} = \lim_{k\to\infty} \|\mathbf{1}_{F_k} g_\phi\|_{q(\cdot)} \le \|\phi\|_{(L^{p(\cdot)})^*}.$$

Finally, since

$$\int_\Omega |h(x) g_\phi(x)|\, \mathrm{d}\mu(x) \le \|h\|_{q(\cdot)} \|g_\phi\|_{p(\cdot)} < +\infty$$

for all $h \in L_b^{p(\cdot)}$, by the dominated convergence theorem the representation (3.3) holds. ■

**Remark 3.12** Proposition 3.11 shows in particular that the dual of $(L_b^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is isometrically isomorphic to $(L^{q(\cdot)}, \|\cdot\|_{q(\cdot)})$.

The main result of this section is the decomposition of the dual into two pieces, one of which is $L^{q(\cdot)}$.

**Theorem 3.13 — Splitting of the dual.**
Given an exponent function $p : \Omega \to [1, +\infty)$,

$$(L^{p(\cdot)})^* \cong (L^{q(\cdot)}, \|\cdot\|_{q(\cdot)}) \oplus \left(L^{p(\cdot)}/L_b^{p(\cdot)}, \|\cdot\|_Q\right)^*,$$

where $\|\cdot\|_Q$ is the quotient norm and $\cong$ means isometric isomorphic.

*Proof.* We define $P : (L^{p(\cdot)})^* \to (L^{p(\cdot)})^*$ by

$$P(\phi)(f) := \int_\Omega f(x) g_\phi(x)\, \mathrm{d}\mu(x),$$

where $g_\phi$ is the function in $L^{q(\cdot)}$ obtained by Proposition 3.11. Note that $P$ is a well-defined, bounded linear projection. Therefore, using Theorem 5.16 in [Rud91] we get that

$$(L^{p(\cdot)})^* \cong R(P) \oplus K(P),$$

where $R(P)$ is the range of $P$. $R(P)$ is isometrically isomorphic to $L^{q(\cdot)}$ and $K(P)$ is the kernel of $P$, which by Theorem 10.2 in [Con10] is isometrically isomorphic to the dual of the quotient space.

■

**Remark 3.14** The previous result's significance is that every element of $(L^{p(\cdot)})^*$ can be expressed uniquely as a sum of two functionals, one belonging to each addend in the direct sum. We are not saying anything about the norm of this sum beyond the trivial bounds

$$\max\{\|a\|, \|b\|\} \leq \|a \oplus b\| \leq \|a\| + \|b\|$$

for each $a \oplus b \in (L^{p(\cdot)})^*$.

## 3.4 The quotient $L_Q^{p(\cdot)}$

We next wish to study the quotient space that appears in Theorem 3.13. We recall its definition.

**Definition 3.15 — Quotient space.**
Let $p : \Omega \to [1, +\infty)$ be an unbounded exponent function and consider the quotient space

$$L_Q^{p(\cdot)} := L^{p(\cdot)} / L_b^{p(\cdot)},$$

where we have taken the closure in the $L^{p(\cdot)}$ norm of the subspace $L_b^{p(\cdot)}$.
This quotient space is equipped with the quotient norm $\|\cdot\|_Q$ defined by

$$\|[f]\|_Q := \inf_{g \in L_c^{p(\cdot)}} \|f - g\|_{p(\cdot)}.$$

Note that since $p$ is unbounded, this is nontrivial (Proposition 3.6).

We first find different characterizations of its norm that we will later use to explain its structure, and so that of $(L^{p(\cdot)})^*$.

### 3.4.1 Characterization of the norm

The following results will be applied to the modular $\rho_{p(\cdot)}$. However, they hold for a more general class of modulars $\rho$, and so we will prove the characterization of the norm in that level of generality. For the remainder of this subsection, we assume that $\rho$ is a convex modular defined over $L^{p(\cdot)}$ satisfying the following properties:

- If $f_k \to 0$ a.e., then $\rho(f_k) \to 0$.
- Fix $f \in L^{p(\cdot)}$ such that $\rho(f) < +\infty$ and a measurable set $E$. Then

$$\inf_{\operatorname{supp}(g) \subset E} \rho(f - g) = \rho(f \mathbf{1}_{E^c}).$$

- If $f, g \in L^{p(\cdot)}$ are such that $f \wedge g = 0$, then $\rho(f \vee g) = \rho(f) + \rho(g)$.

For $k \geq 1$, let $E_k = F_k^c$, where the sets $F_k$ are the ones defined in the proof of Proposition 3.11, so that

$$E_k = \{x : \ p(x) > k\}.$$

**Proposition 3.16 — Characterization of the quotient norm.**
Given $f \in L^{p(\cdot)}$, the following quantities coincide:

1. $\|[f]\|_Q := \inf\left\{\|f - g\| : g \in L_b^{p(\cdot)}\right\}$,
2. $\inf\{\lambda > 0 : \ \rho(f/\lambda) < +\infty\}$,

3. $\lim_{k\to+\infty} \|f\cdot \mathbf{1}_{E_k}\|$.

*Proof.* We show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Take $\lambda > 0$ such that $\rho(f/\lambda) < +\infty$. $f_k := f\cdot \mathbf{1}_{E_k} \to 0$ pointwise, so $\rho(f_k/\lambda) \to 0$. Therefore, for $k$ large enough $\rho(f_k/\lambda) \leq 1$ and so $\|f_k\| \leq \lambda$. Therefore,

$$\inf\left\{\|f-g\| : g \in L_b^{p(\cdot)}\right\} \leq \|f_k\| \leq \lambda.$$

$(2) \Rightarrow (3)$: Take $k \in \mathbb{N}$ and $\lambda > 0$ such that $\rho(f\cdot \mathbf{1}_{E_k}/\lambda) \leq 1$. Then

$$\rho(f/\lambda) = \rho(f\cdot \mathbf{1}_{E_k}/\lambda) + \rho(f\cdot \mathbf{1}_{F_k}/\lambda) \leq 1 + \frac{(\|f\|+\varepsilon)}{\lambda}\rho(f\cdot \mathbf{1}_{F_k}/(\|f\|+\varepsilon)) < +\infty,$$

where $\varepsilon > 0$ and using the convexity of $\rho$ on the second summand. This proves that

$$\inf\{\lambda > 0 : \rho(f/\lambda) < +\infty\} \leq \|f\cdot \mathbf{1}_{E_k}\|.$$

$(3) \Rightarrow (1)$: Fix $\varepsilon > 0$. Then, there exists $g_0 \in L_b^{p(\cdot)}$ such that:

- The support of $g_0$ is $p(\cdot)$-bounded;
- $\inf\left\{\|f-g\| : g \in L_b^{p(\cdot)}\right\} + \varepsilon \geq \|f-g_0\|$.

Pick $k$ large enough so that $\mathrm{supp}(g_0) \cap E_k = \emptyset$. Then,

$$\inf\left\{\|f-g\| : g \in L_b^{p(\cdot)}\right\} + \varepsilon \geq \|f-g_0\| \geq \|f\cdot \mathbf{1}_{E_k}\|.$$

This finishes the proof because $\varepsilon$ can be taken arbitrarily small. ■

### 3.4.2 Structure of the quotient space

We recall the definition of abstract **M**-spaces. A good reference for these classical Banach spaces is [LT13].

**Definition 3.17 — Abstract M-space.**
A Banach lattice $X$ is an *abstract **M**-space* whenever given $x_1, x_2 \in X$ such that $x_1 \wedge x_2 = 0$, then

$$\|x_1 \vee x_2\| = \max\{\|x_1\|, \|x_2\|\}. \tag{3.4}$$

We are going to show that the quotient space is an abstract **M**-space, that is, it is a vector lattice which verifies that whenever we have $[f]\wedge[g] = 0$,

$$\|[f]\vee[g]\| = \max\{\|[f]\|, \|[g]\|\}.$$

**Proposition 3.18** $(L_Q^{p(\cdot)}, \|\cdot\|_Q)$ is an abstract **M**-space.

*Proof.* It is clear that $L^{p(\cdot)}$ is a vector lattice. Since the subspace $L_b^{p(\cdot)}$ is an ideal (i.e. a subspace that inherits the lattice structure), $L_Q^{p(\cdot)}$ is again a vector lattice. Therefore, we just need to check (3.4). Take $[f],[g] \in L_Q^{p(\cdot)}$ such that $[f]\wedge[g] = 0$. Then, by Proposition 3.16,

$$\begin{aligned}\|[f]\vee[g]\|_Q = \|[f\vee g]\|_Q &= \inf\{\lambda > 0 :\ \rho((f\vee g)/\lambda) < +\infty\}\\ &= \inf\{\lambda > 0 :\ \rho(f/\lambda) + \rho(g/\lambda) < +\infty\}\\ &= \max\{\|[f]\|_Q, \|[g]\|_Q\}.\end{aligned}$$

■

Abstract **M**-spaces are a generalization of $L^\infty$ spaces. In [Kak41], abstract **M**-spaces are characterized: an **M**-space $X$ is isometrically isomorphic to a subspace of $C(K)$, the family of bounded continuous real-valued functions defined on a compact Hausdorff space $K$. $K$ is uniquely determined up to homeomorphism. Furthermore, in the same article the dual of abstract **M**-spaces have been identified leading to the next definition.

**Definition 3.19 — Abstract L-space.**
A Banach lattice $X$ is a an abstract **L**-space whenever we have $x_1, x_2 \in X$ such that $x_1, x_2 \geq 0$

$$\|x_1 \vee x_2\| = \|x_1\| + \|x_2\|.$$

The result in [Kak41] says that if $X$ is an abstract M-space then $X^*$ is an abstract **L**-space. In a sense, abstract **L**-spaces behave like $L^1$ spaces. Indeed, abstract **L**-spaces are order isometric to $L^1(\nu)$ for some measure $\nu$ (see for example [LT13]). In [Kak41], they are also characterized to be isometric and lattice isomorphic to a closed linear subspace of the space of all completely additive regular real-valued set-functions $\nu(E)$ for all Borel sets $E$ of some compact Hausdorff space.

With the discussion above in mind, Theorem 3.13 can be naturally viewed as follows: when $p$ is unbounded, the dual of $L^{p(\cdot)}$ is the direct sum of the dual of the bounded exponent part, which is $L^{q(\cdot)}$, and the dual of an $L^\infty$ part, which is a generalized $L^1$ space in the sense explained above. This can be summarized in the following corollary:

**Corollary 3.20**

$$(L^{p(\cdot)})^* \cong L^{q(\cdot)} \oplus Y,$$

where $Y$ is the abstract **L**-space dual of the abstract **M**-space $L_Q^{p(\cdot)}$. We recall that the symbol $\cong$ means that both spaces are isometrically isomorphic.

## 3.5 Variable sequence spaces

In this section we explore in detail the structure of $L_Q^{p(\cdot)}$ and its dual for the case of variable sequence spaces, $\ell^{p(\cdot)}$, that is, when the space is the natural numbers, $\Omega = \mathbb{N}$, the measurable subsets are the subsets of the natural numbers, $\mathscr{A} = \mathscr{B}(\mathbb{N})$, and the measure is the counting measure. In this case the fact that $\Omega$ is countable and $p$ can be unbounded only at infinity partly simplifies the problem. We recall that we are assuming that the exponent function must be finite. But even in this special case we will see that the dual might be still very complicated unless we impose additional restrictions on the growth of the exponent function. The sequence spaces have been much less studied than their continuous counterparts; for the known results, primarily for bounded exponents, see [EN02; Nek02; Nek07].

We fix some notation specific to this setting. Given a sequence $x = \{x(k)\}_{k=1}^\infty$, the modular $\rho_{p(\cdot)}(\cdot)$ and norm $\|\cdot\|_{\ell^{p(\cdot)}}$ are given by

$$\rho_{p(\cdot)}(x) = \sum_{k\in\mathbb{N}} \frac{2}{p(k)} |x(k)|^{p(k)}, \qquad \|x\|_{\ell^{p(\cdot)}} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(x/\lambda) \leq 1\right\}.$$

We consider several cases, which intuitively correspond to how "far" $\ell^{p(\cdot)}$ is from $\ell^\infty$.

### 3.5.1 $\ell^{p(\cdot)}$ is isomorphic to $\ell^\infty$

We prove first that $\ell^{p(\cdot)} \subseteq \ell^\infty$, similar to the case of the classical sequence spaces $\ell^p$.

**Lemma 3.21** Given any $p \in \mathscr{P}(\mathbb{N})$, there is $C = C_{p(\cdot)} > 0$ such that for all $x \in \ell^{p(\cdot)}$,

$$\|x\|_{\ell^\infty} \le C\|x\|_{\ell^{p(\cdot)}}.$$

*Proof.* Fix $x \in \ell^{p(\cdot)}$ and $\lambda > \|x\|_{\ell^{p(\cdot)}}$. Then

$$\sum_{k\in\mathbb{N}} \frac{2}{p(k)} \left|\frac{x(k)}{\lambda}\right|^{p(k)} \le 1,$$

and so for every $k \in \mathbb{N}$, $|x(k)|^{p(k)} \le \frac{p(k)}{2}\lambda^{p(k)}$. Therefore,

$$\|x\|_{\ell^\infty} \le \sup_k \{(p(k)/2)^{1/p(k)}\}\lambda,$$

and taking the infimum over all such $\lambda$ we get the desired inequality. ■

Lemma 3.21 gives the claimed inclusion $\ell^{p(\cdot)} \subseteq \ell^\infty$. If we had the reverse one, we could use the classical description of the dual of $\ell^\infty$ to characterize $(\ell^{p(\cdot)})^*$. We can determine exactly when this happens.

**Lemma 3.22** Given $p \in \mathscr{P}(\mathbb{N})$, the following are equivalent:

1. $\ell^\infty \subseteq \ell^{p(\cdot)}$ and there is $C = C_{p(\cdot)} > 0$ such that for every $x \in \ell^\infty$, $\|x\|_{\ell^{p(\cdot)}} \le C\|x\|_{\ell^\infty}$.
2. There exists $B > 1$ such that
$$\sum_{k\in\mathbb{N}} B^{-p(k)} < \infty. \tag{3.5}$$
3. $\mathbf{1}_{\mathbb{N}} \in \ell^{p(\cdot)}$.

*Proof.* To prove that $(1)$ implies $(2)$, note that

$$\|\mathbf{1}_{\mathbb{N}}\|_{\ell^{p(\cdot)}} \le C\|\mathbf{1}_{\mathbb{N}}\|_{\ell^\infty} = C,$$

and so by the definition of the norm, taking $A > C$

$$\sum_{k\in\mathbb{N}} \frac{2}{p(k)A^{p(k)}} \le 1 < \infty.$$

We just need to take $B = \max_{p\ge 1}(p/2)^{1/p} \cdot A$. On the other hand, if $(2)$ holds then by the dominated convergence theorem, if we take $B$ sufficiently large we have that

$$\sum_{k\in\mathbb{N}} B^{-p(k)} \le 1.$$

Therefore, by the definition of the norm in $\ell^{p(\cdot)}$, $\|\mathbf{1}_{\mathbb{N}}\|_{\ell^{p(\cdot)}} \le B/\max_{p\ge 1}(p/2)^{1/p}$. Thus, $(3)$ holds.

Finally, if $(3)$ is true, let $C_{p(\cdot)} = \|\mathbf{1}_{\mathbb{N}}\|_{\ell^{p(\cdot)}}$. Then for any $x \in \ell^\infty$,

$$\|x\|_{\ell^{p(\cdot)}} = \||x|\|_{\ell^{p(\cdot)}} \le \|x\|_{\ell^\infty}\|\mathbf{1}_{\mathbb{N}}\|_{\ell^{p(\cdot)}} = C_{p(\cdot)}\|x\|_{\ell^\infty},$$

and so $(1)$ holds. ■

**Remark 3.23** When $r<\infty$, Nekvinda [Nek02] characterized the exponents $p$ such that $\ell^{p(\cdot)}$ is isomorphic to $\ell^r$. Lemma 3.22 extends his result to the case $r=\infty$.

**Corollary 3.24** Given $p\in\mathscr{P}(\mathbb{N})$, $\ell^{p(\cdot)}$ is isomorphic to $\ell^\infty$ if and only if for some $B>1$, (3.5) holds. Furthermore, in this case, $\ell^{q(\cdot)}$ is isomorphic to $\ell^1$.

*Proof.* The isomorphism of $\ell^{p(\cdot)}$ and $\ell^\infty$ is immediate. The second follows from the associate space characterization of the norm. (For $\ell^p$ this is classical; for $\ell^{p(\cdot)}$ see [Die+11, Corollary 3.2.14].) For all $f\in\ell^{q(\cdot)}$ we have

$$\|f\|_{\ell^{q(\cdot)}}=\sup_{\substack{g\in\ell^{p(\cdot)}\\ \|g\|_{\ell^{p(\cdot)}}\neq 0}}\left|\sum_{n\in\mathbb{N}}f(n)g(n)\right|\|g\|_{\ell^{p(\cdot)}}^{-1}\simeq\sup_{\substack{g\in\ell^{\infty}\\ \|g\|_{\ell^{\infty}}\neq 0}}\left|\sum_{n\in\mathbb{N}}f(n)g(n)\right|\|g\|_{\ell^{\infty}}^{-1}=\|f\|_{\ell^1}.$$

■

We now characterize the dual of $\ell^{p(\cdot)}$ in this case using that of $\ell^\infty$ and the isomorphism seen above: $(\ell^\infty)^*$ is isomorphic to a space of finitely additive measures [YH52]. More precisely,

$$(\ell^\infty)^*\cong\mathrm{ba}(\mathscr{B}(\mathbb{N})),$$

where $\mathscr{B}(\mathbb{N})$ is the collection of subsets of natural numbers and $\mathrm{ba}(\mathscr{B}(\mathbb{N}))$ is the set of finitely additive signed measures $\mu$ on $\mathscr{B}(\mathbb{N})$ with $|\mu|(\mathbb{N})<\infty$. The dual pairing can be identified with an integral: there exists an isomorphism $\psi\colon(\ell^\infty)^*\to\mathrm{ba}(\mathscr{B}(\mathbb{N}))$ such that for all $\phi\in(\ell^\infty)^*$ and $x\in\ell^\infty$,

$$\phi(x)=\int_{\mathbb{N}}x\,\mathrm{d}\psi(\phi).$$

Moreover, $\mathrm{ba}(\mathscr{B}(\mathbb{N}))$ is isomorphic to the direct sum $\ell^1\oplus pba(\mathscr{B}(\mathbb{N}))$, where sequences in $\ell^1$ are identified with countably additive measures on $\mathscr{B}(\mathbb{N})$ and $pba(\mathscr{B}(\mathbb{N}))$ is the space of purely finitely additive measures on $\mathscr{B}(\mathbb{N})$, that is, the measures $\mu$ such that $\mu(E)=0$ for all finite subsets $E\subset\mathbb{N}$. Combining these observations with Theorem 3.13 and Proposition 3.24, we get the following characterization of the dual of $\ell^{p(\cdot)}$.

**Theorem 3.25** Given $p\in\mathscr{P}(\mathbb{N})$, suppose that for some $B>1$, (3.5) holds. Then $(\ell^{p(\cdot)})^*$ is isometrically isomorphic to the external direct sum

$$\ell^{q(\cdot)}\oplus Y,$$

where the abstract $\mathbf{L}$-space $Y$ is isomorphic to $pba(\mathscr{B}(\mathbb{N}))$.

**Remark 3.26** The restriction that (3.5) holds is a very strong one. It is essentially a growth condition on $p$, and requires $p(k)\to\infty$ quickly. To see this, one can consider exponent functions of the form $p_a(k)=\log(k)^a$, $k\geq 2$ and $a>0$. It is easy to see that (3.5) holds for $p_a$ if and only if $a\geq 1$.

### 3.5.2 The subspace of $p(\cdot)$-simple sequences $S_{p(\cdot)}$

Next, we study further the abstract $\mathbf{L}$-space $Y$ in theorem 3.25 trying to find a better version up to isometric isomorphism. Therefore, we introduce the following notion.

**Definition 3.27 — Finite $p(\cdot)$-content.**
Given a set $A \subseteq \mathbb{N}$, we say that it has *finite $p(\cdot)$-content* if there exists a constant $B > 1$ such that

$$\sum_{k\in A} B^{-p(k)} < \infty.$$

And denote by $\mathscr{A}$ the collection of all subsets of natural numbers with finite $p(\cdot)$-content.

If $p_+ < \infty$ then only finite subsets have finite $p(\cdot)$-content. If $\ell^\infty \subset \ell^{p(\cdot)}$, then by Lemma 3.22, $\mathbb{N}$ has finite $p(\cdot)$-content. However, given any unbounded exponent $p$, there are always infinite subsets which have finite $p(\cdot)$-content: no matter how slowly $p$ grows, we can choose a set $A$ that is sufficiently sparse that it will have finite $p(\cdot)$-content.

**Remark 3.28** The set $\mathscr{A}$ is closed under intersections and finite unions, but it is not closed under complements unless $\mathbb{N}$ has finite $p(\cdot)$-content. If $\mathbb{N}$ does not have finite $p(\cdot)$-content, then the complement of any set which has finite $p(\cdot)$-content will not. Thus, $\mathscr{A}$ is not in general a $\Sigma$-algebra. It is however, a distributive lattice; it is unbounded when $\mathbb{N}$ does not have finite $p(\cdot)$-content because finite subsets of $\mathbb{N}$ always have finite $p(\cdot)$-content and it is impossible to find a maximal element of $\mathscr{A}$ with respect to inclusion.

We now define a set function on $\mathscr{A}$, which, in a sense, measures how much a given set is affected by the singularity of $p$. Given an exponent $p \in \mathscr{P}(\mathbb{N})$, and a set $A \in \mathscr{A}$, we define the set function $w^{p(\cdot)}$ by

$$\omega(A) := \omega^{p(\cdot)}(A) := \|[\mathbf{1}_A]\|_Q.$$

**Lemma 3.29** If $A, B \in \mathscr{A}$ have $p(\cdot)$-bounded intersection, then

$$\omega(A\cup B) = \max\{\omega(A), \omega(B)\} \le \omega(A) + \omega(B).$$

*Proof.* By Proposition 3.16,

$$\begin{aligned}
\omega(A\cup B) &= \|[\mathbf{1}_{A\cup B}]\|_Q \\
&= \inf\left\{\lambda > 0 : \sum_{k\in A\cup B} \lambda^{-p(k)} < +\infty\right\} \\
&= \max\left(\inf\left\{\lambda > 0 : \sum_{k\in A} \lambda^{-p(k)} < +\infty\right\}, \inf\left\{\lambda > 0 : \sum_{k\in B} \lambda^{-p(k)} < +\infty\right\}\right) \\
&= \max\{\omega(A), \omega(B)\}.
\end{aligned}$$

■

We can use the set function $\omega$ to compute the norms of certain sequences in the quotient space $\ell^{p(\cdot)}/\ell_b^{p(\cdot)}$, which we denote by $\ell_Q^{p(\cdot)}$.

**Lemma 3.30** Suppose $x \in \ell^{p(\cdot)}$ is supported on an infinite set $A \subset \mathbb{N}$, and furthermore that $x$ converges to $\alpha$ along all divergent sequences in $A$. Then

$$\|[x]\|_Q = |\alpha|\omega(A).$$

*Proof.* Fix $\varepsilon > 0$; then there exists $N \in A$ such that $|x(k) - \alpha| < \varepsilon$, $\forall k \in A_N := \{k \in A : k \ge N\}$.

We can trap the norm in the following way

$$|\alpha-\varepsilon|\omega(A_N) \;=\; \|[(\alpha-\varepsilon)\mathbf{1}_{A_N}]\|_Q \;\leq\; \|[x\mathbf{1}_{A_N}]\|_Q \;\leq\; \|[(\alpha+\varepsilon)\mathbf{1}_{A_N}]\|_Q \;=\; |\alpha+\varepsilon|\omega(A_N).$$

Since finite sets are $p(\cdot)$-bounded, $\omega(A_N)=\omega(A)$ and $\|[x\mathbf{1}_{A_N}]\|_Q=\|[x]\|_Q$. Therefore, since the above inequality holds for all $\varepsilon>0$, we get the desired equality. ■

We can extend Lemma 3.30 to more general sequences. Given $x\in\ell^{p(\cdot)}$, let $\operatorname{acc}(x)$ denote the set of limit points of $x$. We first assume that $\operatorname{acc}(x)$ is finite.

**Proposition 3.31** Given $x\in\ell^{p(\cdot)}$, suppose $\operatorname{acc}(x)=\{\alpha_i\}_{i=1}^n$. Fix $\delta>0$ such that $|\alpha_i-\alpha_j|\geq\delta$, $i\neq j$, and define $A_i=\{k\in\mathbb{N}:|x(k)-\alpha_i|<\delta/2\}$. Then

$$\|[x]\|_Q=\max_{1\leq i\leq n}|\alpha_i|\omega(A_i). \tag{3.6}$$

*Proof.* This is a clear consequence of Lemma 3.30 and Proposition 3.18. ■

Motivated by the above result, we consider the following subspace of $\ell^{p(\cdot)}$: given $p\in\mathscr{P}(\mathbb{N})$, define the set of finite $p(\cdot)$-content simple functions to be

$$S_{p(\cdot)}:=\left\{x=\sum_{k=1}^{N}\alpha_k\mathbf{1}_{A_k}:A_k\in\mathscr{A}\text{ disjoint},\alpha_k\in\mathbb{R}\right\}.$$

$S_{p(\cdot)}$ generates a set which is larger than $\ell_b^{p(\cdot)}$, which we recall that

$$\ell_b^{p(\cdot)}:=\overline{\left\{x\in\ell^{p(\cdot)}:\ p\text{ is bounded at }\operatorname{supp}(x)\right\}},$$

Note that when $p$ is unbounded, $\ell_b^{p(\cdot)}$ coincides with the closure of the space $c_{00}$, i.e., the space consisting of sequences taking a nonzero value at most at a finite set.

**Lemma 3.32** Given $p\in\mathscr{P}(\mathbb{N})$, $\ell_b^{p(\cdot)}\subseteq\overline{S_{p(\cdot)}}$.

*Proof.* It is clear that $c_{00}\subseteq S_{p(\cdot)}$. And therefore, $\ell_b^{p(\cdot)}\subseteq\overline{S_{p(\cdot)}}$. ■

If $S_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$, we can characterize the dual of $\ell^{p(\cdot)}$, generalizing Theorem 3.25. This is a better approach because $\ell_b^{p(\cdot)}$ is dense if, and only if, $p$ is bounded. And we are going to give sufficient conditions to show when $S_{p(\cdot)}$ is dense when $p$ is unbounded.

To state our result, we need the following definition, which generalizes the concept of a finitely additive measure to the set $\mathscr{A}$.

**Definition 3.33 — $pba_\omega(\mathscr{A})$.**
Given $p\in\mathscr{P}(\mathbb{N})$, define $pba_\omega(\mathscr{A})$ to be the vector space of set functions $\delta$ defined on $\mathscr{A}$ satisfying the following properties:

1. $\delta(A\cup B)=\delta(A)+\delta(B)$ for any pair of disjoint sets $A,B\in\mathscr{A}$.
2. There exists $C>0$ such that given any collection $\{A_i\}_{i=1}^n$ of pairwise disjoint sets in $\mathscr{A}$,

$$\sum_{i=1}^{n}\frac{|\delta(A_i)|}{\omega(A_i)}\leq C.$$

Define a norm on $pba_\omega(\mathscr{A})$ by

$$\|\delta\|_{pba_\omega} := \inf\{C > 0 : \text{ condition (2) holds}\}.$$

**Remark 3.34** The connection of $pba_\omega(\mathscr{A})$ and finitely additive measures can be made explicit: if we assume that $\omega(A) = 1$ when $|A| = \infty$ and 0 otherwise, then definition 3.33 collapses to that of $pba(\mathscr{B}(\mathbb{N}))$. In that case, condition (2) implies that $\delta(A) = 0$ for finite sets $A$ and that $\delta$ has finite variation. Note that this happens when $\omega(\mathbb{N}) = 1$.

The following is the main result of this subsection.

**Theorem 3.35** Given $p \in \mathscr{P}(\mathbb{N})$, suppose $S_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$. Then $(\ell^{p(\cdot)})^*$ is isometrically isomorphic to the external direct sum

$$\ell^{q(\cdot)} \oplus pba_\omega(\mathscr{A}).$$

Furthermore, if $\omega(\mathbb{N}) = 1$, $pba_\omega(\mathscr{A})$ is the classical space of purely finitely additive measures over $B(\mathbb{N})$.

*Proof.* By Theorem 3.13, it suffices to prove that $Y$ is isometrically isomorphic to $pba_\omega(\mathscr{A})$. In fact, we will show that there exists an isometric isomorphism.

First, given $\delta \in pba_\omega(\mathscr{A})$ we will construct a linear functional $\phi_\delta$. Given $x = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{A_k}$ in $S_{p(\cdot)}$, we define

$$\phi_\delta(x) := \sum_{k=1}^{N} \alpha_k \delta(A_k).$$

It is immediate that $\phi_\delta$ is linear. Furthermore, by Propositon 3.31 and the definition of $\|\delta\|_{pba_\omega(\mathscr{A})}$ we have that

$$|\phi_\delta(x)| = \left| \sum_{k=1}^{N} \alpha_k \omega(A_k) \frac{\delta(A_k)}{\omega(A_k)} \right| \leq \left( \max_{1\leq k\leq N} |\alpha_k| \omega(A_k) \right) \sum_{k=1}^{N} \frac{\delta(A_k)}{\omega(A_k)} = \|[x]\|_Q \|\delta\|_{pba_\omega}.$$

Consequently, $\phi_\delta$ may be extended to the whole space by density of $S_{p(\cdot)}$. We then have $\phi_\delta \in (\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*$, with

$$\|\phi_\delta\|_{(\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*} \leq \|\delta\|_{pba_\omega}.$$

Conversely, fix $\phi \in (\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*$. Define a set function $\delta_\phi : \mathscr{A} \to \mathbb{R}$ by setting

$$\delta_\phi(A) := \phi([\mathbf{1}_A]);$$

since $A \in \mathscr{A}$, $\mathbf{1}_A$ is in $\ell^{p(\cdot)}$, and so $\delta_\phi$ is well-defined. Property (1) in the definition of $pba_\omega$ follows immediately from the definition. To prove property (2), suppose that $\{A_i\}_{i=1}^n$ is a finite collection of pairwise disjoint sets in $\mathscr{A}$. Then, again by Proposition 3.31,

$$\begin{aligned} \left| \sum_{i=1}^{n} \frac{\delta_\phi(A_i)}{\omega(A_i)} \right| &= \left| \phi\left( \sum_{i=1}^{n} \frac{1}{\omega(A_i)} \mathbf{1}_{A_i} \right) \right| \\ &\leq \|\phi\|_{(\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*} \left\| \sum_{i=1}^{n} \frac{1}{\omega(A_i)} \mathbf{1}_{A_i} \right\|_Q = \|\phi\|_{(\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*} \max_{1\leq i\leq n} \omega(A_i)^{-1} \omega(A_i) = \|\phi\|_{(\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*}. \end{aligned}$$

Therefore,

$$\|\delta_\phi\|_{pba_\omega} \leq \|\phi\|_{(\ell^{p(\cdot)}/\ell_b^{p(\cdot)})^*}.$$

Finally, it is clear from the definitions that

$$\delta_{\phi_\delta} = \delta \qquad \text{and} \qquad \phi_{\delta_\phi} = \phi.$$

Hence, the mapping $\delta \mapsto \phi_\delta$ is an isometric isomorphism and our proof is complete. ■

The following remarks are in order:

**Remark 3.36**

1. The functional $\phi_\delta$ defined above can be thought of as a generalized integral with respect to $\delta \in pba_\omega(\mathscr{A})$; we first define the integral on the dense set $S_{p(\cdot)}$ and then extend it in the usual way. When $\mathscr{A}$ is not a $\Sigma$-algebra, $\phi_\delta$ is not a classical integral.
2. Since $\mathscr{A}$ is not a $\Sigma$-algebra in general, the elements of $pba_\omega(\mathscr{A})$ are not always finitely additive measures. However, we can construct elements of $pba_\omega(\mathscr{A})$ from finitely additive measures defined on a fixed $D \in \mathscr{A}$: given $\delta \in pba(\mathscr{B}(D))$ and $A \in \mathscr{A}$, define $\delta_D(A) = \delta(A \cap D)$. Then $\delta_D \in pba_\omega(\mathscr{A})$. Indeed, property (1) in definition 3.33 follows immediately, while property (2) follows from Theorem 3.25. Let $\phi$ be the bounded linear functional associated to $\delta$. Then we have that

$$\sum_{i=1}^{n} \frac{|\delta_D(A_i)|}{\omega(A_i)} \leq \sum_{i=1}^{n} \frac{|\delta(A_i \cap D)|}{\omega(A_i \cap D)} = \phi\left(\sum_{i=1}^{n} \frac{\operatorname{sgn}(\delta(A_i \cap D))}{\omega(A_i \cap D)} \mathbf{1}_{A_i \cap D}\right) \leq \|\phi\|;$$

the last inequality follows from the fact that by Proposition 3.31,

$$\left\| \sum_{i=1}^{n} \frac{\operatorname{sgn}(\delta(A_i \cap D))}{\omega(A_i \cap D)} \mathbf{1}_{A_i \cap D} \right\|_Q = 1.$$

3. If $\omega(\mathbb{N}) < \infty$, then $\mathscr{A}$ is a $\Sigma$-algebra.

So far, we have characterized the dual when the $p(\cdot)$-simple sequences are dense. In this case the dual is isomorphic to the dual of $\ell^\infty$. We can decompose the dual into two pieces: the first one is the Hölder conjugate variable sequence space which is isomorphic to $\ell^1$, and the second one is isomorphic to the purely finitely additive measures space. We can further provide a precise description of this second part using the generalization of the pba space. Note that when $\omega(\mathbb{N}) = 1$, $\ell^{p(\cdot)}/\ell_b^{p(\cdot)}$ is isometrically isomorphic to the space of continuous functions over the nonprincipal ultrafilters over the natural numbers, $C(\beta\mathbb{N} \setminus \mathbb{N})$ and its dual is the classical pba space.

### 3.5.3 On the density of $S_{p(\cdot)}$

In light of Theorem 3.35 we would like to characterize when $S_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$. One could think this could always happen. However, the following example shows that it is not the case.

■ **Example 3.37** There exists $p \in \mathscr{P}(\mathbb{N})$ such that $S_{p(\cdot)}$ is not dense in $\ell^{p(\cdot)}$. ■

*Proof.* Partition $\mathbb{N}$ as

$$\mathbb{N} = \bigcup_{s=1}^{\infty} A_s,$$

where the sets $A_s$ are infinite and disjoint. On each set $A_s$, define $p$ to be an increasing exponent such that for $k \in A_s$, $p(k)$ is equal to $n$ exactly $s^n$ times for each $n \geq s$, and such that $p$ takes no other values on $A_s$. Define

$$x = \sum_{s=1}^{\infty} s^{-1} \mathbf{1}_{A_s}$$

(the sum is to be interpreted pointwise, using that the sets $A_s$ are disjoint). Then we have that $x \in \ell^{p(\cdot)}$, with $\|[x]\|_Q = 1$. To see this via Proposition 3.16, fix any $\lambda > 1$; then

$$\sum_{k=1}^{\infty} \left(\frac{|x(k)|}{\lambda}\right)^{p(k)} = \sum_{s=1}^{\infty} \sum_{k \in A_s} \left(\frac{|x(k)|}{\lambda}\right)^{p(k)} = \sum_{s=1}^{\infty} \sum_{n=s}^{\infty} s^n \left(\frac{1/s}{\lambda}\right)^n = \frac{1}{(1-\lambda^{-1})^2 \lambda} < \infty,$$

and similarly the sum is infinite for any $\lambda \leq 1$. We note also that

$$\sum_{k \in A_s} \lambda^{p(k)} = \sum_{n \geq s} s^n \lambda^{-n}$$

is finite if and only if $\lambda > s$, so $\omega(A_s) = s$.

Now suppose $y \in S_{p(\cdot)}$. By definition of $S_{p(\cdot)}$, there exists $K \in \mathbb{N}$ such that $K > \omega(\operatorname{supp}(y))$. We then have that the norm of the difference is controlled by what happens outside $\operatorname{supp}(y)$. Let us denote by $B_s := A_s \setminus (A_s \cap \operatorname{supp}(y))$.

$$\|[x] - [y]\|_Q \geq \left\| \sum_{s \geq K} [x \mathbf{1}_{B_s}] \right\|_Q,$$

where the convergence of the series is again pointwise. Since the sets $B_s$ are disjoint, we can use that the quotient space is an abstract **M**-space (Proposition 3.18) to control this norm by

$$\begin{aligned} \left\| \sum_{s \geq K} [x \mathbf{1}_{B_s}] \right\|_Q &\geq \sup_{s \geq K} \|[x \mathbf{1}_{B_s}]\|_Q \\ &= \sup_{s \geq K} s^{-1} \omega(B_s) \end{aligned}$$

since $x = s^{-1}$ on $B_s$. Finally, using Lemma 3.29 and $K > \omega(\operatorname{supp}(y))$ we show that $\omega(B_s) = s$ for $s \geq K$.

$$s = \omega(A_s) = \omega(B_s \cup (A_s \cap \operatorname{supp}(y))) = \max\{\omega(B_s), \omega(A_s \cap \operatorname{supp}(y))\} \Longrightarrow \omega(B_s) = s.$$

Therefore, we have shown that

$$\|[x] - [y]\|_Q \geq \sup_{s \geq K} s^{-1} \omega(B_s) = 1.$$

Thus $x \notin \overline{S_{p(\cdot)}}$, and the proof is complete. ■

Understanding which sequences lie in the closure of $S_{p(\cdot)}$ in $\ell^{p(\cdot)}$ is important for the last characterization of the dual that we give below. We next give a straightforward sufficient condition for the denseness of $p(\cdot)$-simple sequences.

**Proposition 3.38** Let $x \in \ell^{p(\cdot)}$ and suppose that $\operatorname{supp}(x) \in \mathscr{A}$. Then $x \in \overline{S_{p(\cdot)}}$; furthermore,

$$\|x\|_{\ell^{p(\cdot)}} \leq C \|x\|_{\ell^\infty},$$

where the constant $C$ depends only on $\omega(\operatorname{supp}(x))$. In particular, if $\omega(\mathbb{N}) < \infty$, then $S_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$.

*Proof.* Let $A := \operatorname{supp}(x)$. Since $x$ is supported on $A$, we have

$$\|x\|_{\ell^{p(\cdot)}} = \|x\|_{\ell^{p(\cdot)|_A}(A)},$$

where $p(\cdot)|_A$ is the restriction of $p(\cdot)$ to $A$. Since $A$ has finite $p(\cdot)|_A$-content, by Corollary 3.24 we have that $\ell^{p(\cdot)|_A}(A)$ is isomorphic to $\ell^\infty(A)$. Since simple functions are dense in $\ell^\infty(A)$ (see [DS58, page IV.13.69]), $x$ can be approximated by simple functions in $\ell^\infty(A)$, which are $p(\cdot)$-simple functions in $\ell^{p(\cdot)}(\mathbb{N})$. ∎

The converse to Proposition 3.38 fails as shown in the next example.

**Proposition 3.39** Given $p \in \mathscr{P}(\mathbb{N})$, suppose $\omega(\mathbb{N}) = \infty$ and

$$\lim_{k\to 0} \frac{p(k)}{k} = 0.$$

Then there exists $x \in \ell^{p(\cdot)}$ such that $\operatorname{supp}(x) = \mathbb{N}$ (and so $\operatorname{supp}(x) \notin \mathscr{A}$) but $x \in \overline{S_{p(\cdot)}}$.

*Proof.* Define $x$ by $x(k) = 2^{-k/p(k)}$. Then for any $\lambda > 0$,

$$\lim_{k\to\infty} \left(\frac{x(k)}{\lambda}\right)^{p(k)/k} = \frac{1}{2},$$

and so by the root test,

$$\sum_{k=1}^{\infty} \left(\frac{x(k)}{\lambda}\right)^{p(k)} < \infty.$$

Hence, by Proposition 3.16, $\|[x]\|_Q = 0$. Thus, $x \in \ell_b^{p(\cdot)}$ and so by Lemma 3.32, $x \in \overline{S_{p(\cdot)}}$. ∎

Finally, if $S_{p(\cdot)}$ is not dense in $\ell^{p(\cdot)}$ we can further decompose the **L**-abstract piece of the dual of $(\ell^{p(\cdot)})^*$.

**Theorem 3.40**

$$\left(\ell^{p(\cdot)}\right)^* \cong \ell^{q(\cdot)} \oplus pba_\omega(\mathscr{A}) \oplus \left(\ell^{p(\cdot)}/\overline{S_{p(\cdot)}}\right)^*.$$

*Proof.* Applying Theorem 3.13 yields

$$(\ell^{p(\cdot)})^* \cong \ell^{q(\cdot)} \oplus \left(\ell^{p(\cdot)}/\ell_b^{p(\cdot)}\right)^*.$$

Therefore, we just need to decompose the second piece of the dual. We can argue as in the proof of Theorem 3.13 and define a projection associated to some $\delta pba_\omega(\mathscr{A})$. Alternatively, we can observe that $\overline{S_{p(\cdot)}}/\ell_b^{p(\cdot)}$ is a closed sublattice of the abstract **M**-space $\ell^{p(\cdot)}/\ell_b^{p(\cdot)}$, so we can apply [Theorem 2, pag 141] in [Lac12] to obtain a positive simultaneous linear extension. This yields

$$\left(\ell^{p(\cdot)}/\ell_b^{p(\cdot)}\right)^* \cong pba_\omega(\mathscr{A}) \oplus \left(\ell^{p(\cdot)}/\overline{S_{p(\cdot)}}\right)^*,$$

which completes the proof. ∎

In the next section we are providing the existence of some dense subspaces which help to give a description of $\left(\ell^{p(\cdot)}/\overline{S_{p(\cdot)}}\right)^*$.

## 3.6 Dense subspaces of $L^{p(\cdot)}$ and $\ell^{p(\cdot)}$

As the proofs of our main results in Sections 3.3 and 3.5 show, there is a close connection between the problem of characterizing $(L^{p(\cdot)})^*$ and finding practical dense subsets of $L^{p(\cdot)}$ when $p$ is unbounded. This problem is of independent interest and was raised as an open question in [CF13, Problem A.2], though the connection with dual spaces was not noted there. In this section we give two answers to this problem, one for general $L^{p(\cdot)}$ spaces, and one specific to $\ell^{p(\cdot)}$. Neither yields a satisfactory characterization of the dual space but we believe that they are interesting in their own right and may provide a foundation for further work.

### 3.6.1 A dense subset of $L^{p(\cdot)}$

Define the set of $p(\cdot)$-*countable functions* in $L^{p(\cdot)}$ by

$$\mathscr{C}_{p(\cdot)} := \{f \in L^{p(\cdot)} : f \text{ restricted to any } p(\cdot)\text{-bounded subset is a simple function}\}.$$

It is clear that any $p(\cdot)$-countable function takes at most a countable number of values. If $p_+ < \infty$, then a function is $p(\cdot)$-countable if and only if it is simple.

**Proposition 3.41** $\mathscr{C}_{p(\cdot)}$ is dense in $L^{p(\cdot)}$.

*Proof.* For each $k \in \mathbb{N}$ define $\Omega_k := p^{-1}([k, k+1))$; then the $\Omega_k$ are measurable, $p(\cdot)$-bounded, pairwise disjoint, and their union is all of $\Omega$. Fix $\varepsilon > 0$. The simple functions are dense in $L^{p(\cdot)}(\Omega_k)$, so for every $k \in \mathbb{N}$ there exists a simple function $g_k$, $\operatorname{supp}(g_k) \subset \Omega_k$, such that $\|f\mathbf{1}_{\Omega_k} - g_k\|_{p(\cdot)} < \varepsilon 2^{-k}$.

Define the function $g$ to coincide with $g_k$ in $\Omega_k$ for every $k \in \mathbb{N}$; then $g$ is $p(\cdot)$-countable. Moreover, by Minkowski's inequality,

$$\|f - g\|_{L^{p(\cdot)}} = \left\| \sum_{k=1}^{\infty} (f\chi_{\Omega_k} - g_k) \right\|_{L^{p(\cdot)}} \leq \sum_{k=1}^{\infty} \|f\chi_{\Omega_k} - g_k\|_{L^{p(\cdot)}} < \varepsilon.$$

■

### 3.6.2 A dense subset of $\ell^{p(\cdot)}$

In the discrete setting, $\mathscr{C}_{p(\cdot)} = \ell^{p(\cdot)}$. So we need to find another subspace. We have already showed in Example 3.37 that $S_{p(\cdot)}$ may not be dense in $\ell^{p(\cdot)}$, so we have to look for a larger nontrivial subspace. Fix $p \in \mathscr{P}(\mathbb{N})$. For each $m \in \mathbb{Z}$, define

$$S^m_{p(\cdot)} = \left\{x \in S_{p(\cdot)} : |x(k)| \in (2^{m-1}, 2^m] \cup \{0\}\right\}$$

Then, define $\mathscr{Z}_{p(\cdot)}$ to be the set of all $x \in \ell^{p(\cdot)}$ such that for some $M \in \mathbb{Z}$, the following conditions are satisfied:

1. There exists $\{y_m\}_{m=-\infty}^{M}$ such that $y_m \in S^m_{p(\cdot)}$.
2. $\operatorname{supp}(y_m) \cap \operatorname{supp}(y_n) = \emptyset$ if $n \neq m$.
3. $x = \sum_{m=-\infty}^{M} y_m$.

The sum above always makes sense pointwise, since for each $k$ there is at most one $y_m$ which contains a non-zero element in the $k$-th coordinate. However, $\sum_{m=-\infty}^{M} y_m$ may not converge in norm.

**Proposition 3.42** $\mathscr{Z}_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$.

*Proof.* Since $x \in \ell^\infty$ by Lemma 3.21, there exists $M \in \mathbb{Z}$ such that $0 \leq x(k) \leq 2^M$. For each $m \leq M$, define

$$A_m := \{k \in \mathbb{N} : |x(k)| \in (2^{m-1}, 2^m]\}.$$

Then, for some $\lambda > 0$ we have

$$\sum_{k \in A_m} \lambda^{-p(k)} \leq \sum_{k \in A_m} \left(\frac{x(k)}{2^m \lambda}\right)^{p(k)} \leq \sum_{k=1}^{\infty} \left(\frac{x(k)}{2^m \lambda}\right)^{p(k)} < \infty,$$

and so $\omega(A_m) < \infty$. Therefore, by Proposition 3.38,

$$x_m := x\mathbf{1}_{A_m} \in \overline{S_{p(\cdot)}}.$$

Moreover, since $x_m(k) \in \{0\} \cup (2^{m-1}, 2^m]$, $x_m \in \overline{S^m_{p(\cdot)}}$. Hence, fix $\varepsilon > 0$ and $y_m \in S^m_{p(\cdot)}$ such that $\|x_m - y_m\|_{\ell^{p(\cdot)}} < 2^{m-M-1}\varepsilon$. Define

$$y = \sum_{m=-\infty}^{M} y_m.$$

By the triangle inequality,

$$\|y\|_{\ell^{p(\cdot)}} \leq \|x\|_{\ell^{p(\cdot)}} + \|x-y\|_{\ell^{p(\cdot)}},$$

and since the sets $A_m$ are disjoint,

$$\|x-y\|_{\ell^{p(\cdot)}} \leq \sum_{m=-\infty}^{M} \|x_m - y_m\|_{\ell^{p(\cdot)}} < \varepsilon.$$

Therefore, we have that $y \in \mathscr{Z}_{p(\cdot)}$ and, since $\varepsilon > 0$ is arbitrary, it follows that $\mathscr{Z}_{p(\cdot)}$ is dense in $\ell^{p(\cdot)}$. ■

We conclude this section by using the set $\mathscr{Z}_{p(\cdot)}$ to sketch a characterization of the third term of the dual space $(\ell^{p(\cdot)})^*$ given in Theorem 3.40, which we consider a starting point for future work since it is not too concrete.

To describe this characterization, note first that each element of $\mathscr{Z}_{p(\cdot)}$ can be identified with a sequence of ordered pairs $\{(A_k, \alpha_k)\}_{k=1}^{\infty}$, where the $A_k$ are pairwise disjoint sets in $\mathscr{A}$, and the $\alpha_k$ are real numbers such that $\alpha_k \to 0$ as $k \to \infty$. For ease of notation, we denote the associated element of $\mathscr{Z}_{p(\cdot)}$ by $x(A_k, \alpha_k)$ which means that the sequence takes the value $\alpha_k$ exactly in the subset $A_k$.

Note that not every such sequence induces an element of $\mathscr{Z}_{p(\cdot)}$: if $p(k) \to \infty$ slowly, then we can choose the $A_k$ so that $\|\mathbf{1}_{A_k}\|_{\ell^{p(\cdot)}} \to \infty$ while we choose the $\alpha_k \to 0$ so slowly that $x(A_k, \alpha_k)$ is not in $\ell^{p(\cdot)}$. We will denote the collection of all such sequences where the associated sequence is in $\ell^{p(\cdot)}$ by $\mathscr{Z}_{p(\cdot)}(\mathscr{A}^{\mathbb{N}}, c_0)$.

The idea is the following:

- The first part of the dual is isometrically isomorphic to $\ell^{q(\cdot)}$, (which actually is isomorphic to $\ell^1$). It is a measure of sets of cardinal one, $\{k\}$ with $k \in \mathbb{N}$.
- The second part of the dual is isometrically isomorphic $pba_\omega(\mathscr{A})$ (which is isomorphic to $pba(\mathscr{B}(\mathbb{N}))$ the space of purely finitely additive measures). It is a measure over infinite subsets of natural numbers which vanishes on finite subsets (an infinite set can be regarded as an infinite collection of sets of cardinal one).

Therefore, the next step would be a measure over infinite number of disjoint infinite subsets. However, this infinite sequence of infinite subsets is weighted by a sequence converging to 0. Furthermore, it vanishes if the sequence is eventually zero.

We can think that the last part of the dual is related to measures defined over $\mathscr{Z}_{p(\cdot)}(\mathscr{A}^{\mathbb{N}}, c_0)$, denoted by $pba(\mathscr{Z}_{p(\cdot)}(\mathscr{A}^{\mathbb{N}}, c_0))$. The linearity condition of the functional is related to the next conditions for such a measure $\nu$: given $\{(A_k, \alpha_k)\}_{k=1}^{\infty}$, $\{(B_k, \beta_k)\}_{k=1}^{\infty}$ in $\mathscr{Z}(\mathscr{A}, c_0)$,

1. if $A_k \cap B_k = \emptyset$ for all $k \in \mathbb{N}$, then
$$\nu(\{(A_k \cup B_k, \alpha_k)\}_{k=1}^{\infty}) = \nu(\{(A_k, \alpha_k)\}_{k=1}^{\infty}) + \nu(\{(B_k, \alpha_k)\}_{k=1}^{\infty});$$
2. $\nu(\{(A_k, \alpha_k + \beta_k)\}_{k=1}^{\infty}) = \nu(\{(A_k, \alpha_k)\}_{k=1}^{\infty}) + \nu(\{(A_k, \beta_k)\}_{k=1}^{\infty})$.

And the boundedness condition on the functional over $\ell^{p(\cdot)}/\overline{S_{p(\cdot)}}$ is determined by the analogous condition over the measure $\nu$: there is $C > 0$ such that if $x(A_k, \alpha_k) \in \ell^{p(\cdot)}$, then
$$|\nu(\{(A_k, \alpha_k)\}_{k=1}^{\infty})| \leq C \|[x(A_k, \alpha_k)]\|_{\ell^{p(\cdot)}/\overline{S_{p(\cdot)}}}.$$

The norm of a measure $\nu$, $\|\nu\|$, is given by the infimum of the positive constants verifying above.

Given these properties, using the denseness of this subspace we can mimic the proof of Theorem 3.35 to show that $(\ell^{p(\cdot)})^*$ is isometrically isomorphic to the external direct sum
$$\ell^{q(\cdot)} \oplus pba_{\omega}(\mathscr{A}) \oplus pba(\mathscr{Z}(\mathscr{A}, c_0)).$$

Details are left to the interested reader.

Finally, we are going to summarize all the previous results provided for duals of variable sequence spaces in the following.

**Theorem 3.43 — Dual of variable sequence spaces.**
The dual of the variable sequence space $\ell^{p(\cdot)}$ is:
- $\ell^{q(\cdot)}$ if, and only if, $p$ is bounded.
- $\ell^{q(\cdot)} \oplus pba(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N}) = 1$.
- $\ell^{q(\cdot)} \oplus pba_{\omega}(\mathscr{B}(\mathbb{N}))$ if $\omega(\mathbb{N}) < \infty$.
- $\ell^{q(\cdot)} \oplus pba_{\omega}(\mathscr{A})$ if, and only if, $S_{p(\cdot)}$ is dense.
- $\ell^{q(\cdot)} \oplus pba_{\omega}(\mathscr{A}) \oplus pba(\mathscr{Z}(\mathscr{A}, c_0))$ in general.

We believe that the conditions $\omega(\mathbb{N}) < \infty$ and $S_{p(\cdot)}$ is dense are equivalent. However, we have only proved one implication in our work.

## 3.7 On the choice of modular

It would be desirable to choose a modular that through the Luxemburg norm we could recover the classical $L^p$ spaces when the exponent function $p$ is constant. Observe that this does not happen with the modular considered through all this chapter, that is,
$$\rho_{p(\cdot)}(f) := \int_{\Omega \setminus \Omega_\infty} \frac{2}{p(x)} |f(x)|^{p(x)} \, \mathrm{d}\mu(x) + \|f\|_{L^\infty(\Omega_\infty)}.$$

Note that the term $\frac{2}{p(x)}$ is the one responsible of this phenomena. Nevertheless, it is key to get an isometric isomorphism via Holder's inequality. Furthermore, when the exponent is constant, this factor only contribuites to a dilatation of the norm. The following picture shows the unit ball in $\mathbb{R}^2$ for some constant $p$ and its asymptotic behavior when this constant tends to $+\infty$.

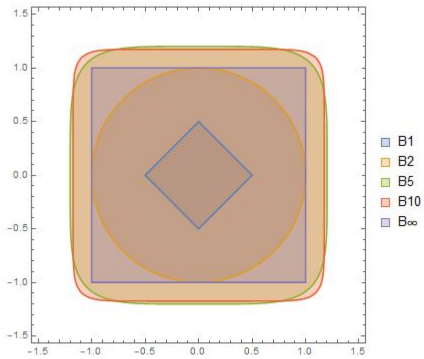


Figure 3.1: For $1 \leq p < 2$, the unit ball is obtained by a dilation by a factor less than 1, for $p = 2$ we get the same euclidean ball, for $p > 2$ we get a dilated ball by a factor greater than 1. However, this factor tends to 1 when the constant tends to $+\infty$.

The standard (or canonical) modular used is the following

$$\rho_S(f) := \left( \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} \, d\mu(x) + \|f\|_{L^\infty(\Omega_\infty)} \right)$$

since it recovers classical Lebesgue spaces when the exponent function is constant. However, the sharp constant in Hölder's inequality with this canonical modular is greater than 1 unless $p$ is constant (to read a proof of Holder's inequality with this modular see [CF13, Theorem 2.26]).

To get isometric isomorphism results having a sharp constant equal to 1 in Hölder's inequality is crucial. Some of the results we are going to show have already appeared in [Sam98] without proof.

**Proposition 3.44 — Hölder's inequality for variable Lebesgue spaces.**
Given a $\sigma$-finite measure space $(\Omega, \mathscr{A}, \mu)$ and $p \in \mathscr{P}$ such that $\mu(\Omega_\infty) = 0$. With our choice of modular, if $f \in L^{p(\cdot)}$ and $g \in L^{q(\cdot)}$,

$$\int_\Omega |f(x)g(x)| \, d\mu(x) \leq \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}. \tag{3.7}$$

The proof follows from using Young's inequality pointwise and the definition of the norm. This version of Hölder's inequality let us characterize the dual of bounded variable Lebesgue spaces.

*Proof.* By homogeneity of (3.7), it is enough to prove it assuming $\|f\|_{p(\cdot)} = \|g\|_{q(\cdot)} = 1$. Take $\lambda_1, \lambda_2 > 1$. Using Young's inequality pointwise with $a = \left|\frac{f(x)}{\lambda_1}\right|$ and $b = \left|\frac{g(x)}{\lambda_2}\right|$ we get

$$\begin{aligned}
\lambda_1\lambda_2 \int_\Omega \left|\frac{f(x)}{\lambda_1}\right| \cdot \left|\frac{g(x)}{\lambda_2}\right| d\mu(x) &\leq \lambda_1\lambda_2 \left( \int_\Omega \frac{1}{p(x)} \left|\frac{f(x)}{\lambda_1}\right|^{p(x)} d\mu(x) + \int_\Omega \frac{1}{q(x)} \left|\frac{g(x)}{\lambda_2}\right|^{q(x)} d\mu(x) \right) \\
&\leq \lambda_1\lambda_2 \left( \frac{1}{2} + \frac{1}{2} \right) = \lambda_1\lambda_2.
\end{aligned}$$

We finish letting $\lambda_1, \lambda_2 \to 1$. ■

**Corollary 3.45 — Dual of bounded variable Lebesgue spaces.**
If $p \in \mathscr{P}$ is bounded,

$$(L^{p(\cdot)})^* \cong L^{q(\cdot)},$$

where the symbol $\cong$ denotes an isometric isomorphism.

*Proof.* The proof is straightforward using this version of Hölder's inequality proved before and the denseness of the subspace of bounded and compact support functions in $L^{p(\cdot)}$. ■

Corollary 3.45 justifies our choice of modular to study duality because the canonical choice of modular just provides an isomorphism.

### 3.7.1 Equivalence of norms

Let us denote by $\|\cdot\|_S$ and $\|\cdot\|_{p(\cdot)}$ the Luxemburg norm obtained through the standard modular $\rho_S$ and our choice of modular $\rho_{p(\cdot)}$ respectively. In [Sam98], it is already stated that these two norms are equivalent. For sake of completeness, we are providing a proof of this fact.

First, note that any scalar multiple of any of the modular $\rho$ provides an equivalent norm. It is enough to show it for scalars bigger than 1.

**Lemma 3.46** For any $M > 1$ and any $f \in L^{p(\cdot)}$, then

- $\inf\{\lambda > 0 : \rho(f/\lambda) \leq 1\} \leq \inf\{\lambda > 0 : M\rho(f/\lambda) \leq 1\}$.
- $\inf\{\lambda > 0 : M\rho(f/\lambda) \leq 1\} \leq M \inf\{\lambda > 0 : \rho(f/\lambda) \leq 1\}$.

*Proof.* The first inequality follows from subset inclusion. For the second inequality, we use the convexity of the modular $M\rho(f) \leq \rho(Mf)$ and the norm takes scalars outside, completing the proof. ■

**Proposition 3.47** $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_S$ are two equivalent norms.

*Proof.* On the one hand, using Lemma 3.46 and

$$\int_\Omega \frac{1}{p(x)} \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leq \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x)$$

since $p(x) \geq 1$, we prove that $\|f\| \leq C\|f\|_S$ because the standard modular controls the other modular.

On the other hand, denote by $C = \max_{p\geq 1}\{\left(\frac{p}{2}\right)^{1/p}\}$ (the maximum exists because the limit of this quantity when $p$ tends to $+\infty$ is 1). Using again Lemma 3.46 and

$$\int_\Omega \left|\frac{f(x)}{C\lambda}\right|^{p(x)} \mathrm{d}\mu(x) \leq \int_\Omega \left|\frac{2^{1/p(x)} f(x)}{p(x)^{1/p(x)}\lambda}\right|^{p(x)} \mathrm{d}\mu(x) \leq 1,$$

we obtain the $\|f\|_S \leq C\|f\|$. Now our choice of modular controls the standard one. ■

Since both norms are equivalent, these Banach spaces have the same dense subspaces and very similar properties. Furthermore, we prove a new result about the fact that the second part of the dual is independent of the choice of the modular between the one we have chosen and the canonical one, since their quotient spaces are isometrically isomorphic.

**Proposition 3.48** The quotient spaces $L^{p(\cdot)}/L_b^{p(\cdot)}$ when we take the modulars $\rho_{p(\cdot)}$ and $\rho_S$ are isometrically isomorphic.

*Proof.* The key of the proof is that we can control one modular by the other and we have a nice characterization of the quotient norm through the modulars (see Proposition 3.16). ■

# 4. Approximation in Variable Lebesgue Spaces

In this final chapter we show the results of the recent preprint [CO20a]. The idea behind these results came from an exotic research stay at *Academia Sinica* with Wen-Liang Hwang. More specifically, in Taipei I discovered some really interesting things about neural networks.

Combined with my interest in variable Lebesgue spaces I decided to study if neural networks can approximate functions in these spaces. Consequently, we could prove that bounded variable Lebesgue spaces can be well approximated by shallow neural networks. And in the more fascinating and complex case of unbounded variable Lebesgue spaces we could give a precise description of the subspace that neural networks can approximate.

## 4.1 Introduction

Artificial neural networks are a model created with the purpose of imitating the behavior of biological neural networks using digital computing. Their origins are tied back to [MP43] and [Ros58], and since then numerous applications have been found in a wide range of fields, varying from machine learning to computer vision, speech recognition or mathematical finance, among many others. A major problem in the theory of neural networks is that of approximating within a desired accuracy a generic class of functions using neural networks, initially motivated by the behavior for neural networks observed in the Representation Theorem due to Arnold [Arn57] and Kolmogorov [Kol57] and the aim of providing a theoretical justification for it.

The starting point of approximation theory for neural networks was the *Universal Approximation Theorem* of [Cyb89] and [HSW89], which shows that every continuous function on a compact set can be uniformly approximated by shallow neural networks with a continuous, non-polynomial activation function. Subsequent extensions of this result addressed the analogous problem for Lebesgue spaces with a finite exponent [Hor91] and locally integrable spaces [PS91], meanwhile some others also considered the derivatives of the neural networks to show that shallow neural networks with a sufficiently smooth activation function and unrestricted width are dense in the space of sufficiently differentiable functions [Pin99]. Soon after this wave of results for shallow neural networks, various surveys on the topic appeared in the literature, such as [Pin97], [ST98], [TKG+03] and [San08].

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Taipei 101 is the vivid image of this part of the thesis. I spent there two months in a fruitful research stay where I learned a new culture, my mathematical knowledge widened and my interest with neural networks began.

In the last years, many directions have been explored in the approximation theory for both shallow and deep neural networks. For neural networks with ReLU activation functions, there are a number of recent papers concerning various topics: approximation for Besov spaces [Suz19], regression [Sch17] and optimization [GS09] problems, restriction to encodable weights [PV18], negative results of approximation [Alm+18], estimates for the errors obtained in the approximation [Pet99], [Yar17], or, in general, deep neural networks [KL19], [SCC18], among many others. A simpler and inspiring new proof for the universality theory in deep neural networks can be found in [HHH20]. Moreover, in [HH20] the authors develop a new technique named "un-rectifying" which transfers piecewise continuous non-linear activation functions into piecewise continuous linear functions and then use it to show that ReLU networks and MaxLU networks are indeed deep trees.

Furthermore, for more regular activation functions there are also numerous recent articles, namely [Bar94], [Böl+19], [Lin19], [LTY20], [Mha96], [OK19], [TLY19], most of which focus on deep neural networks. Some other directions are currently being studied for the problem of approximation too, such as the topological approach presented in [Kra19], the application of these results to finding solutions of partial differential equations [GR20] or the comparison to approximation with tensor networks [AN20].

In this chapter, we take a step forward in the theory of approximation with neural networks and address the **problem of approximating any function in a variable Lebesgue space** with enough precision using shallow neural networks with various activation functions. Variable Lebesgue spaces are in particular locally integrable spaces. Even though there exist some previous results of universal approximation in locally integrable spaces, they depend on a metric defined in a way that the behavior of a function away from a compact set is always negligible. Moreover, since that distance is not constructed from a norm, the approximations are not stable by dilations. In this manuscript, we overcome this situation and prove our approximation results employing the distance determined by the usual norm of the space.

Variable Lebesgue spaces are a generalization of Lebesgue spaces that might contain functions which do not belong to any Lebesgue space [CF13]. In particular, a variable Lebesgue space might include all the bounded functions even if the domain is not compact. Therefore, a natural problem that arises in this setting is that of approximating functions in noncompact domains. For example, continuous functions defined on an unbounded interval with a suitable asymptotic limit. This type of functions may appear as the representation of a quantity that follows a diffusion process with time (like the temperature at a certain point in a closed system).

More specifically, in this chapter we show approximation results with neural networks for variable Lebesgue spaces depending on the boundedness of their exponent function. If the exponent of the space is bounded, we prove universal approximation, yielding thus an analogous behavior to that of usual Lebesgue spaces. On the other hand, if the exponent is unbounded, the situation is much more subtle, but we can characterize in some cases the subspace of functions which can be approximated with neural networks relying on some results of the previous chapter. We first address the simpler case of variable sequence spaces to subsequently lift our results to a more general domain. Our results hold for most of the activation functions present in the literature, namely any sigmoidal function (logistic sigmoid, hyperbolic tangent, Heaviside function, etc) or the rectifier function.

The outline of this chapter is the following: First, we give some basic notions about ANN in Section 4.2; in Section 4.3, we present a formal exposition of the main results of the present article. In Section 4.4, we collect some of the previous results on universal approximation in certain function spaces and provide several improvements for them. Finally, in Section 4.5, we present our results on approximation with neural networks in variable Lebesgue spaces.

## 4.2 Preliminaries: Some basics of artificial neural networkks

Before showing the main results, we introduce the concepts and properties associated to neural networks that we will need for the rest of the chapter. Some appropiate references for the mathematical formulation of neural networks in this context are [Hay98], [Bis06], [KS93], [Roj96], among many others.

In this chapter, we focus on *feedforward artificial neural networks* (denoted ANN hereafter), the simplest model for neural networks. It is known that ANN with no hidden layer are not capable of approximating generic, non-linear, continuous functions [Wid90]. On the opposite side, ANN with two or more hidden layers, known as *deep neural networks*, have given rise to a broad field of research in the past years. However, for simplicity we focus specifically in the case of one hidden layer, commonly known as *shallow neural networks*, for which typical results on universal approximation have been studied in the past.

An *artificial neural network* (**ANN**) is a model with the purpose of imitating the behavior of biological neural networks. The original goal of ANN was to solve problems in a similar way that a human brain would. Nevertheless, ANNs have more other applications, including computer vision, speech recognition, machine translation, social network filtering, playing board and video games, medical diagnosis, etc.

The key model of a biological neural network is the human brain. It presents very important abilities, which usually outplay the computational power of our computers. To illustrate this, something that one can find frequently in some webpages is a captcha, used to show that those entering the site are humans and not computers. It is essentially a task where we have to write the word of an image or select all the images where something appears. These tasks are quite natural for the human brain, whereas rather difficult for computers. Actually, there is a complete field in science whose goal is to implement pattern recognition and computer vision efficiently in computers.


Figure 4.1: Example of a capcha where the task is to type the two words. Source: https://commons.wikimedia.org/wiki/File:Captchacat.png

Understanding how the human brain works is an incredibly difficult task. A good reference to the interested reader in a mathematical aproach for neuroscience is the book [ET10]. Using this text, along with [Wol20], in the following lines we explain a very simplified model of the functioning of the brain with all the necessary concepts to understand it, so that we can subsequently introduce the definitions of their analogous artificial notions and relate them.

### 4.2.1 Biological Neural Networks

A human brain contains around 86 billion neurons. The *neuron* is the basic unit of transmision of information and it usually consists of a *cell body*, *dendrites* that receive incoming signals from other neurons, and an *axon*, from which the signals are transmitted to other neurons. Typically, there are many dendrites in each neuron, originating from the cell body towards the neighborhood of the neuron to serve as a connection to other neurons, meanwhile there is an only axon, which may present a branching at a distance from the cell body.

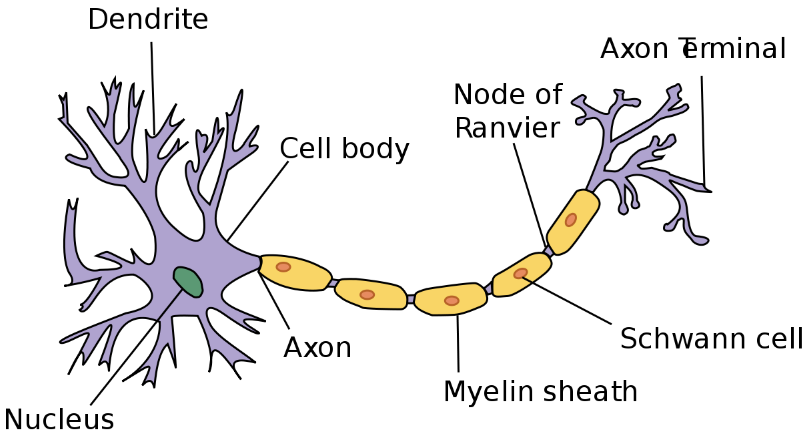


Figure 4.2: Schematic picture of a neuron. Source: https://de.wiktionary.org/wiki/neuron/media/Datei:Neuron.svg

A neuron is an electrically excitable cell that shares information with other neurons via specialized connections called *synapses*, which connect the axon of one neuron with the dendrite of another one. The signaling process is chemical and electrical. The network of neurons is very complex and leads to the existence of different types of neurons.

Indeed, the signals between neurons create a change of the electrical potential of about $100mV$. This pulse is called the *action potential*, and it takes about $1ms$ to travel down the axon, where it reaches the synapses at the branches of the axon. As mentioned above, the signal transmission within most synapses is mainly chemical, as the electrical pulse that arrives induces a chemical process inside the synapse and this results in a change of electrical potential in the postsynaptic neuron. The whole time for this process to take place is about $1ms$.

The reponse of the postsynaptic neuron to this signal in its dendritic part depends on the potential. Neurons usually verify the all-or-none law, first established in 1871 by the American physiologist H. Pickering Bowditch. Basically, this principle says that a neuron is either active or not, and its condition depends on the synapses with other neurons. More specifically, there is a threshold, below which the neuron is not active and above which it is.

As we have already mentioned, in the case of a neuron the stimulus strength is computed, among other things, to the voltage changes of the synapses. The effect of an incoming pulse on the postsynaptical neuron can vary depending on several factors, such as the duration and the strength. Moreover, a neuron can emit signals several hundred times per second, although there is a limiting factor for high rates, namely the duration of each pulse and a corresponding refractory period (usually about $1ms$) during which no stimilus can lead to signal emision.

### 4.2.2 Perceptron

The first and easiest example to translate what biological neurons do to an artificial way is the *perceptron*. In this subsection, we explain in a basic way the functioning of a perceptron, which essentially consists of:

- Some **inputs** whose effect is conditioned by variable **weights**.
- A single **output**.

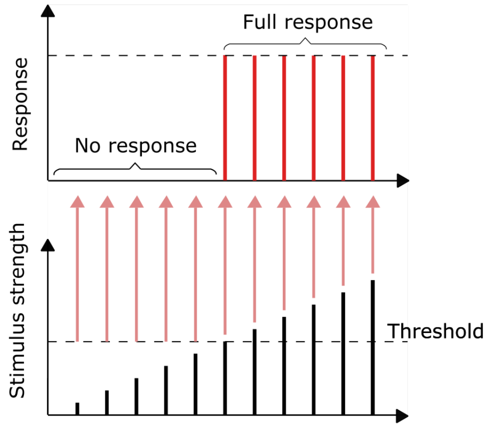


Figure 4.3: This image represents the All-or-none law. Here, we are intensifying the stimulus strength with time. During the first five steps the threshold is not achieved and thus there is no response. Later, the stimulus is higher than the threshold and, then, there is full response. Source: https://en.wikipedia.org/wiki/File:All-or-none_law_en.svg

- Integration of input signals.
- An all-or-nothing process.

Indeed, a biological neuron receives some information from the other neurons linked to it. This information corresponds to some *input* data $x_1,\ldots,x_n$, which are just real numbers. Then, we can represent this by a vector $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$.

Later, it has to process this information and decide whether to be active or not. Therefore, we need to find the analogous of the potential. One way to obtain this is by means of the following *propagation rule*:

$$V(x_1,\ldots,x_n)=\sum_{i=1}^{n} w_i\cdot x_i+b>0\,,$$

where $w_i$ are real parameters usually called *weights* (or *synaptic weights* from the inspiration from the biological model) and $b$ is another parameter called *bias*, which plays the role of a threshold value and which will also frequently written as $w_0$ to simplify notation when assuming that it is the weight of a constant $x_0=1$.

Then, the *activation function* $\sigma:\mathbb{R}\to\mathbb{R}$ determines whether the potential associated to the input is positive or not, namely:

$$\sigma(x_1,\cdots,x_n)=\begin{cases}1 & \text{if } V(x_1,\cdots,x_n)>0\\ 0 & \text{if } V(x_1,\cdots,x_n)\le 0\end{cases}$$

Note that this activation function is a step function and therefore it is not continuous. This is the functional originally associated to the perceptron, but nowadays many generalizations of this model with different activation functions are considered. This is addressed in the next subsection.

### 4.2.3 Artificial neurons and activation functions

There is a way to generalize perceptrons to a more general context. For that, we take the same potential $V$ but change the activation function from the Heaviside step function. The main

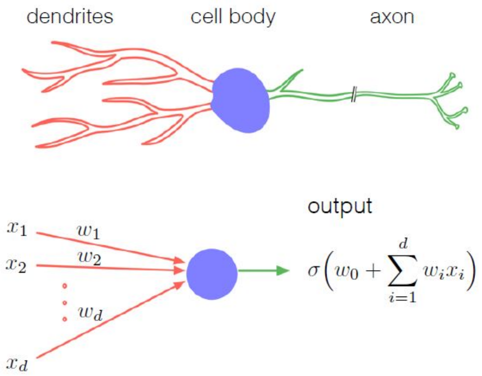


Figure 4.4: Schematic version of a perceptron in comparison with the different parts of a biological neuron. Source: Page 71 of [Wol20].

reason to do that is that these different activation functions enable gradient descent techniques for learning algorithms. Below, we present some of these frequently used activation functions, starting with the aforementioned Heaviside step function.

**■ Example 4.1** • Heaviside step function.The heaviside step function is a piecewise defined function over the real numbers, $\mathbb{R}$, in the following way:

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

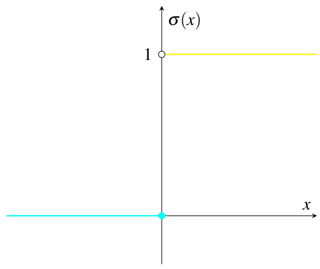


Figure 4.5: Heaviside step function. Drawn using tikzpicture.

• Sign function. The sign function is a piecewise defined function given by:

$$\sigma(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

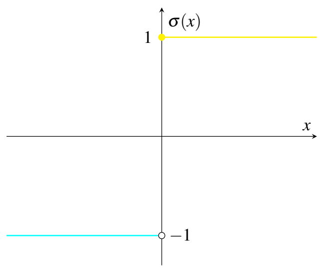


Figure 4.6: Heaviside step function. Drawn using tikzpicture.

- Rectifier. The rectifier function is a piecewise defined function over $\mathbb{R}$ in the following way:

$$\sigma(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

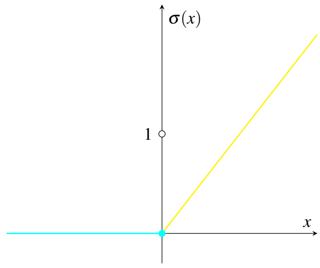


Figure 4.7: Rectifier activation function. Drawn using tikzpicture.

A unit employing the rectifier is also called a rectified linear unit (ReLU). The rectifier is one of the most popular activation functions for deep neural networks.

- Logistic sigmoid. The logistic sigmoid is frequently used in this context. It is given by:

$$\sigma(x) = \frac{1}{1+\mathrm{e}^{-x}}$$

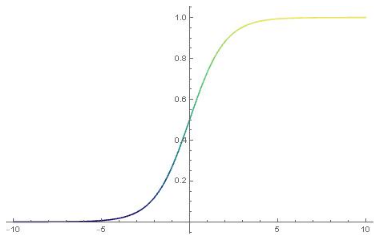


Figure 4.8: Logistic sigmoid. Drawn using Mathematica.

- Hyperbolic tangent. The hyperbolic tangent is an affinely transformed logistic sigmoid. It is given by:

$$\sigma(x) = \mathrm{Tanh}(x)$$

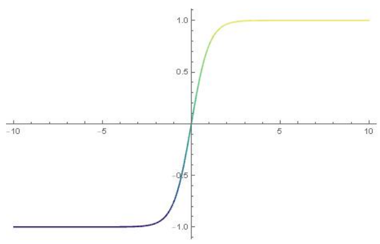


Figure 4.9: Hyperbolic tangent. Drawn using Mathematica.

■

Note that both the logistic sigmoid and the hyperbolic tangent functions are smooth versions of the Heaviside step function and the sign function, respectively. They were used in practice for many years, although they are frequently replaced by the rectified functions nowadays. For further examples of activation functions, see [ST98] for instance.

When we denote the bias by $w_0$, define the constant $x_0 = 1$, and write the vectors of data:

$$\mathbf{x} := (x_0, \ldots, x_n) \in \mathbb{R}^{n+1},$$
$$\mathbf{w} := (w_0, \ldots, w_n) \in \mathbb{R}^{n+1},$$

the previous process can be seen as a function of $\mathbf{x} \in \mathbb{R}^{n+1}$ of the form:

$$\mathbf{x} \mapsto \sigma(\mathbf{x} \cdot \mathbf{w})$$

with $\sigma : \mathbb{R} \to \mathbb{R}$ and where we are denoting by $\mathbf{x} \cdot \mathbf{w}$ the Euclidean scalar product of the two vectors. Such function is often called in the context of approximation theory a *ridge function*, and these functions present the feature that they are always constant on hyperplanes characterized by

$$\mathbf{x} \cdot \mathbf{w} = c\,,$$

for $c \in \mathbb{R}$. In the particular case of the perceptron, it is clear that the subset of $\mathbb{R}^{n+1}$ that is mapped to 1 forms a closed half space in a bijective way. This bijection provides a useful geometric interpretation, but it is often too simple for a model to express enough. This yields the necessity to obtain more complex and richer structures based on this one, what results on the appearance of the notion of neural networks.

### 4.2.4 Neural networks

The strategy followed to construct a *neural network* consists of composing several artificial neurons of the kinds introduced above. In a nutshell, an artificial neural network consists of a collection of simple processing units which communicate among themselves by sending signals to each other over a large number of weighted connections. For a deep introduction to neural networks, we refer the reader to [KS93] or [Bis06], for instance.

The main elements of a neural network, from which we will construct them below, can be summarized as follows:

- A set of processing units, which are usually called **neurons** or **cells**.
- A collection of data $\{x_1, \ldots, x_D\} \in \mathbb{R}^D$, one for each unit, which can be interpreted as the **input** of the units.
- A collection of data $\{y_1, \ldots, y_K\} \in \mathbb{R}^K$, one for each unit, which can be interpreted as the **output** of the units.
- Some connections between the units. Generally, each connection is given by a **weight** $w_{ij} \in \mathbb{R}$ that determines the effect of the signal from unit $i$ on unit $j$.
- A **propagation rule**, which determines the effective input $s_i$ of a unit from its external inputs (this was denoted in the case of the perceptron above by $V$).
- An **activation function** $\sigma : \mathbb{R} \to \mathbb{R}$ that determines the new level of activation based on the effective input $s_i$ and the updated activation $y_i$.
- A **bias** $b_k$ for each unit, which we usually write as $w_{k0}$ too.
- A method for **learning**.
- An **environment** within which the system must operate, providing input signals and error signals if necessary.

Let us now write explicitly the construction of a neural network. First, note that, in general, neural networks use basis functions $\phi_j(\mathbf{x})$ such that

$$y(\mathbf{x}, \mathbf{w}) = f\left(\sum_{j=1}^{M} w_j \phi_j(\mathbf{x})\right),$$

(for $f$ a non-linear activation function) so that each basis function is a nonlinear function of a linear combination of the inputs, where the coefficients in the linear combination, i.e. the weights, are adaptive parameters that will be adjusted during training. Here we only focus on the simplest case, that of a feedforward neural network.

The feedforward neural network was the first model of artificial neural network, as well as the simplest one. It contains several neurons, which we will denote by nodes, arranged in layers serving differently depending on their position. Moreover, nodes from adjacent layers have connections between them, with certain weights associated to them.

A feedforward neural network consists of three kinds of nodes:

- **Input nodes:** These nodes provide information from the environment to the network and altogether constitute the *Input layer*. There is no performed computation in any of them, since their purpose is just to pass on the information to the next layers, i.e. the hidden nodes.
- **Hidden nodes:** These nodes lack direct connection with the environment, and hence the label *hidden*. They serve both for performing computations and transferring information from the input nodes to the output ones. The collection of these nodes constitutes the *Hidden layer*.
- **Output nodes:** These nodes are responsible for computations and for the transmision of information from the network to the environment. They are collectively referred to as *Output layer*.

It is important to remark that a feedforward neural network only has a single input layer and a single output layer. However, it can have zero or multiple hidden layers. In a feedforward neural network, the information only moves in one direction, forward, i.e. from the input nodes to the output nodes through the hidden nodes, in case there are any. Moreover, there are no cycles or loops in the network.

The simplest example of a feedforward neural network is the single layer perceptron, which does not contain any hidden layer and was explained in detail in Subsection 4.2.2.

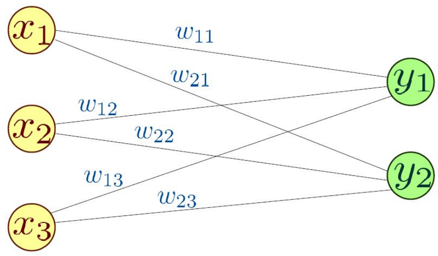


Figure 4.10: A single layer neural network.

In that subsection, we mentioned that the biases could be rewritten as weights. This is graphically shown in the next figure.

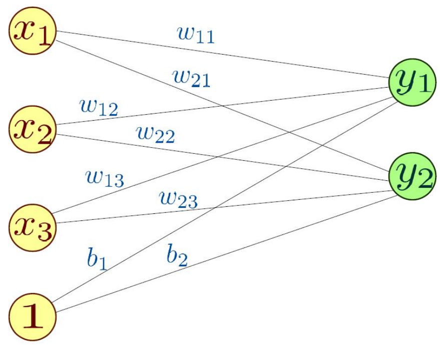


Figure 4.11: Implementation for the bias.

Another interesting and slightly less simple example is the multi layer perceptron, which presents one or more hidden layers. They are in general more useful than the single perceptron for applications and we explain them in more detail below.

In the next figure, we show an example of a multi layer feedforward neural network.

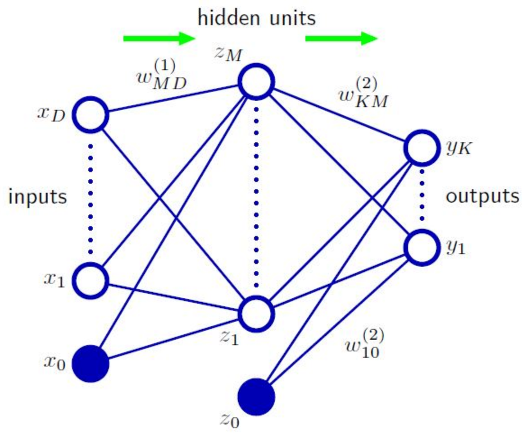


Figure 4.12: Example of a feedforward neural network extracted from [Bis06].

However, we are going to restrict to the ones with only one hidden layer.

**Definition 4.2 — Feedforward shallow artificial neural network.**
A *feedforward shallow artificial neural network* (ANN) can be described by a finite linear combination of the form

$$g(x) = \sum_{j=1}^{M} \alpha_j \, \sigma(w_j \cdot x + b_j)\,,$$

where $x \in \mathbb{R}^d$ represents the *input* to the neural network, $g(x) \in \mathbb{R}$ the *output*, $w_j \in \mathbb{R}^d$ and $\alpha_j \in \mathbb{R}$ are the *weights* between first and second layer, and second and third layer, respectively, $b_j \in \mathbb{R}$ are the *biases*, $\sigma : \mathbb{R} \to \mathbb{R}$ is the *activation function* and $M$ the *height*.

The subspace of all functions that can be obtained using an ANN with activation function $\sigma$ will be denoted hereafter by

$$H_\sigma := \left\{ g(x) = \sum_{j=1}^{M} \alpha_j \, \sigma(w_j \cdot x + b_j) \right\},$$

where $w_j, x \in \mathbb{R}^d$, $\alpha_j, b_j \in \mathbb{R}$ and $M \in \mathbb{N}$. Throughout the text, we will impose different conditions on $\sigma$ to obtain diverse results, since the role of the activation function is essential in the results of approximation of generic function spaces. Here we summarize some of the most typical examples for activation functions.

Recall the activation functions showed in Example 4.1. Now, we introduce a property concerning the activation function which will appear often throughout the rest of the text.

**Definition 4.3** Given an activation function $\sigma : \mathbb{R} \to \mathbb{R}$, we say that $\sigma$ is *sigmoidal* if

$$\sigma(t) = \begin{cases} c_{+\infty} & \text{as } t \to +\infty\,, \\ c_{-\infty} & \text{as } t \to -\infty\,, \end{cases}$$

where the constants $c_{+\infty}$ and $c_{-\infty}$ are finite.

Usually, $c_{+\infty}$ and $c_{-\infty}$ are taken to be 1 for $c_{+\infty}$ and $-1$ or 0 for $c_{-\infty}$.

**Remark 4.4** Note that all the activation functions presented in Example 4.1 satisfy this condition. Indeed, even though it goes against the intuition for the ReLU, we can obtain another activation function which is continuous and bounded by combining properly two ReLU.

## 4.3 Main results

Given an activation function $\sigma$ and a function normed space $(X, \|\cdot\|)$, the *Universal Approximation (UA)* property for shallow neural networks can be formally stated as follows:

$$\boxed{\text{For every } f \in X \text{ and } \forall \varepsilon > 0, \text{ there is a function } g \in H_\sigma \text{ such that } \|f-g\| < \varepsilon.}$$

The main results of this article concern the UA property for variable Lebesgue spaces, i.e. spaces of the form

$$L^{p(\cdot)}(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ measurable and } \|f\|_{p(\cdot)} < +\infty\},$$

for an open $\Omega \subseteq \mathbb{R}^d$, where $p : \Omega \to [1, +\infty)$ is an exponent function and the norm is given by

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}(x) \leq 1\right\}.$$

The first of these results (which appears in the main text as Theorem 4.18) shows that universal approximation holds whenever the exponent function of the space is bounded.

**Theorem — UA for bounded variable Lebesgue spaces.**
Let $\Omega \subseteq \mathbb{R}^d$, consider $p : \Omega \to [1, +\infty)$ a bounded exponent function and $\sigma : \mathbb{R} \to \mathbb{R}$ discriminatory for $L^{p(\cdot)}(K)$ for every compact $K \subset \Omega$. Then, truncated finite sums of the form

$$g(x) = \begin{cases} \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) & x \in K, \\ 0 & x \in \Omega \setminus K, \end{cases}$$

with $K$ compact are dense in $L^{p(\cdot)}(\Omega)$.

The condition imposed on $\sigma$ essentially means that if a functional over $L^{p(\cdot)}(K)$ vanishes on $H_\sigma$, then the functional is identically null. Most of the activation functions considered in the literature verify this property, such as any continuous function different from an algebraic polynomial, for example.

Whenever the exponent function of a variable Lebesgue space is unbounded, the situation is much more complex. Indeed, as a consequence of the separability of the space and the previous theorem, we provide a characterization of universal approximation in terms of the boundedness of the exponent function, which appears in the main text as Corollary 4.20.

**Proposition — UA and the boundedness of the exponent .**
Let $\Omega \subseteq \mathbb{R}^d$, $p : \Omega \to [1, +\infty)$ a exponent function and $\sigma : \mathbb{R} \to \mathbb{R}$ continuous and with finite limit at $+\infty$ and $-\infty$. Then, UA holds for $L^{p(\cdot)}(\Omega)$ if, and only if, $p$ is bounded.

Subsequently, since for unbounded exponent functions the UA fails, we study conditions to describe the subspace of functions which can be approximated with neural networks. In the following result, the condition of characterization is a generalization of the concept of having an asymptotic limit.

**Theorem — Approximation for unbounded variable Lebesgue spaces.**
Let $\Omega = [1,+\infty)$ and $p : \Omega \to [1,+\infty)$ an unbounded exponent function such that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ and it is bounded in every compact subset of $\Omega$. Let $\sigma \in L^\infty(\mathbb{R})$ be a non-constant, sigmoidal activation function. Then, the following conditions are equivalent for $f \in L^{p(\cdot)}(\Omega)$:

1. For every $\varepsilon > 0$, there is a $g_\varepsilon \in H_\sigma$ such that $\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$.
2. There is a scalar $\beta \in \mathbb{R}$ such that

$$\|[f - \beta \mathbf{1}_\Omega]\|_Q = 0\,,$$

where $\|\cdot\|_Q$ is the quotient norm given in Definition 3.15.

This theorem appears as Theorem 4.25 in the main text. In a nutshell, the quotient norm encodes the behavior of the function at $\infty$. Therefore, the functions that can be approximated are those which converge, in some sense, at $\infty$.

Finally, prior to these results we recall and adapt some classical results of universal approximation for some function spaces. More specifically, we address the space of continuous functions over compact sets and locally integrable spaces. For the former, we prove an extension of the original result of universal approximation from continuous functions on a compact set to the space of functions which vanish at infinity, in the unidimensional case, and a negative result of universal approximation, whereas for the latter we extend a result for radial activation functions to non-radial ones. Moreover, even though variable Lebesgue spaces are locally integrable spaces, our contribution for those spaces is a major improvement, because our concept of distance comes from a norm and, therefore, approximations in this context are more interesting, since for example we can control the error of dilation.

## 4.4 Universal approximation in function spaces

The aim of this section is to present some of the known results about the property of *Universal Approximation* (UA) for classical function spaces, as well as some improvements to some of them. A nice survey of these results in soft computing techniques is [TKG+03]. We are particularly interested in the ones related to the artificial neural networks (ANN) introduced in the previous section.

Originally, in the mathematical theory of artificial neural networks, the Universal Approximation Theorem stated that a feedforward neural network with a single hidden layer containing a finite number of neurons can approximate arbitrarily well real-valued continuous functions on compact subsets of $\mathbb{R}^d$. This result was later extended to several other spaces (see [PS91], [Hor91], [HSW89] [Cyb89]).

In this section, we focus on results of universal approximation in various function spaces, namely those of continuous functions and spaces of locally integrable functions. For each of these classes, we review some of the previous literature and present some new results, with improvements concerning either the domain of the functions, the domain of the activation functions or the techniques involved to prove them.

### 4.4.1 Space of continuous functions

In 1989, in both [Cyb89] and [HSW89], the authors proved simultaneously for many different activation functions that for any continuous function on a compact set $K$ of $\mathbb{R}^d$ and any $\varepsilon > 0$, there exists a feedforward neural network with one hidden layer which uniformly approximates the function with precision $\varepsilon$. Before stating formally the result and proving it, we introduce a concept concerning activation functions that is necessary to understand the statement of the theorem.

**Definition 4.5** Given an activation function $\sigma : \mathbb{R} \to \mathbb{R}$ and a compact $K \subset \mathbb{R}^d$, we say that $\sigma$ is *discriminatory for* $C(K)$ if for $\mu$ a finite, signed regular Borel measure on $K$, the fact that the following holds

$$\int_K \sigma(w \cdot x + b)\, \mathrm{d}\mu(x) = 0$$

for all $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ implies that $\mu \equiv 0$.

Now, we can state the first result on universal approximation using ANN, namely the universal approximation theorem for continuous functions on a compact set, whose proof can be found in [Cyb89, Theorem 1] or [HSW89, Theorem 2.1].

**Theorem 4.6** Let $\sigma$ be any continuous discriminatory for $C(K)$ activation function. Then, the finite sums of the form

$$\sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j)$$

are dense in $C(K)$, the space of continuous functions over a compact $K$.

Note from this result that a natural question is whether an activation function is discriminatory for $C(K)$. We can provide the following sufficient conditions to check when a continuous function $\sigma$ is discriminatory for $C(K)$, whose proof also appears in [Cyb89, Lemma 1] and [Hor91, Theorem 5].

**Proposition 4.7** Any bounded, measurable non-constant function, $\sigma$, is discriminatory for $C(K)$ for every compact $K \subset \mathbb{R}^d$. In particular, any continuous sigmoidal function is discriminatory.

By virtue of this result, we can check that all the examples of activation functions provided in Example 4.1 are discriminatory for $C(K)$. However, to get a continuous approximation the activation function needs to be continuous; thus, to apply Theorem 4.6 we can only use the examples that are continuous, namely the rectifier, the logistic sigmoid and the hyperbolic tangent.

As a generalization of the previous result, shortly after it the following characterization for discriminatory was provided in [LPS93].

**Proposition 4.8** Let $\sigma$ be locally bounded and continuous almost everywhere. Then, it is discriminatory for $C(K)$ for every compact $K \subset \mathbb{R}^d$ if, and only if, it is different from an algebraic polynomial almost everywhere.

Building on these results, we present now some improvements to some of them. The following is a slight improvement of Theorem 4.6, in the sense that it extends that result from a compact set to the space of functions which vanish at infinity. As a limitation, it only works in the unidimensional case.

**Proposition 4.9** Let $\sigma \in C(\mathbb{R})$ be sigmoidal and non-constant, and let $f \in C(\mathbb{R})$ with compact support. Then, for every $\varepsilon > 0$ there is $g_\varepsilon \in H_\sigma$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - g_\varepsilon(x)| < \varepsilon\,.$$

*Proof.* First, we know that $\sigma$ is discriminatory for $C(K)$ with $K$ compact because it is bounded and non-constant (Proposition 4.7).

Next, we recall that $C_0(\mathbb{R})$ is the subspace of continuous functions that vanishes at infinity. This space coincides with the closure in the supremum norm of the union of $C([-\ell,\ell])$ with $\ell \in \mathbb{N}$, viewed as subspaces of $L^\infty(\mathbb{R})$. Thus, $\sigma$ is also discriminatory for $C_0(\mathbb{R})$, since there are continuous functions vanishing at infinity in the subspace $H_\sigma$, due to the fact that $\sigma$ is sigmoidal.

Therefore, it is dense and we can find $\forall \varepsilon > 0$ a function $g_\varepsilon \in H_\sigma$ such that

$$\sup_{x\in\mathbb{R}} |f(x) - g_\varepsilon(x)| < \varepsilon\,. \tag{4.1}$$

■

The following result takes a further step in the approximation of continuous functions, since the domain is now the whole $\mathbb{R}^d$ instead of just a compact $K$. It has the same spirit as [PS91, Theorem 2], although the latter result concerns radial activation functions and we have considered a different structure slicing with hyperplanes. Here we check that this result also holds in our setting.

**Theorem 4.10** Let $\sigma$ be continuous and discriminatory for $C(K)$ for every $K \subset \mathbb{R}^d$ compact. Given $\varepsilon > 0$ and $f \in C(\mathbb{R}^d)$, there is a function $g \in H_\sigma$ such that $d(f,g) < \varepsilon$, where the distance is defined in the following way:

$$d(f,g) := \sum_{k=1}^{\infty} 2^{-k} \frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_\infty}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_\infty}\,,$$

where $[-k,k]^d$ is the centred $d$-dimensional cube of side $2k$.

We omit the proof of this result, as it is completely analogous to that of Theorem 4.14 for locally integrable functions. However, we remark a couple of facts concerning this result.

$\mathscr{R}$ **Remark 4.11** The previous theorem also holds for $\sigma$ the rectifier. Moreover, as we can see in the proof of Theorem 4.14, we only use the assumption of continuity for $f$ to justify that it is bounded over compact sets. Then, we could have stated the theorem above for $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$.

### 4.4.2 Negative results

Next, we explore the opposite direction, i.e. negative results of universal approximation. First, we show that the following is a necessary condition for a space to satisfy the UA.

**Lemma 4.12** If a metric space $(X,d)$ of functions can be universally approximated by finite sums of the form

$$\sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j),$$

where $\sigma$ is continuous and sigmoidal, then $(X,d)$ is separable.

*Proof.* The proof easily follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Indeed, due to the uniform continuity of $\sigma$, the countable subspace consisting of the finite sums

$$\sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j),$$

where the variables $\alpha_j, b_j$ and the components of the vector $w_j$ are taken to be rational numbers is a dense subspace. ■

Since every sigmoidal continuous activation function is uniformly continuous, Lemma 4.12 states that the size of the function space is an important fact to consider in the problem of UA. Furthermore, it can be used to show negative results of UA, such as the following example:

■ **Example 4.13** $(L^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ is a non-separable space. Therefore, UA fails for continuous sigmoidal activation functions. Moreover, it even fails for continuous activation functions.

Indeed, if $\sigma$ is continuous, every finite sum of the form

$$g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j)$$

is continuous. The Heaviside step function belongs to $L^\infty(\mathbb{R})$ and has a jump discontinuity at $x=0$ of size 1. Therefore, at $x=0$ the approximated continuous function $g$ must be simultaneously close to the value 0 and 1. Thus, using the continuity of $g$ we can see easily that

$$\|f-g\|_\infty := \sup_{x\in\mathbb{R}} \left| h(x) - \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) \right| \geq \frac{1}{3},$$

concluding thus the proof that for $(L^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ UA fails for continuous activation functions.

■

### 4.4.3 Locally integrable functions

The space of locally integrable functions, $L^1_{loc}(\mathbb{R}^d)$, is the largest space that we deal with in this manuscript. It includes as special cases the previous function spaces. However, this space lacks an associated norm. One can solve this problem by constructing a distance in a similar way that we have already done for $C(\mathbb{R}^d)$ in Theorem 4.10.

**Theorem 4.14** Given $\sigma$ discriminatory for $C(K)$ for every $K \subset \mathbb{R}^d$ compact, $\varepsilon > 0$ and $f \in L^1_{loc}(\mathbb{R}^d)$, there is a function $g$ of the form:

$$g(x) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j),$$

for every $x \in \mathbb{R}^d$, such that $d(f,g) < \varepsilon$, where the distance is defined in the following way:

$$d(f,g) := \sum_{k=1}^{\infty} 2^{-k} \frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_1},$$

where $[-k,k]^d$ is the centred $n$-dimensional cube of side $2k$.

*Proof.* The key idea of the proof relies on the fact that, due to the construction of the distance, the behavior away from compact sets of the form $[-k,k]^d$ has little impact on the computation of the distance, because of the multiplicative factor $2^{-k}$. Therefore, given $\varepsilon > 0$, there is a large enough $k$ so that we can restrict to approximating the function $f$ on $[-k,k]^d$.

Indeed, fix $f : \mathbb{R}^d \to \mathbb{R}$ continuous and $\varepsilon > 0$. Take $m \in \mathbb{N}$ such that

$$\frac{1}{2^m} < \frac{\varepsilon}{2}.$$

Hence, this clearly implies that

$$\sum_{k=m+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}.$$

Now, by virtue of Theorem 4.6, there exists a function $g$ of the form $g(x) = \sum_{j=1}^{\ell} \alpha_j \sigma(w_j \cdot x + b_j)$ for every $x \in \mathbb{R}^d$ such that

$$\left\|(f-g)\mathbf{1}_{[-m,m]^d}\right\|_1 \le \left\|(f-g)\mathbf{1}_{[-m,m]^d}\right\|_\infty (2m)^d < \frac{\varepsilon}{2}.$$

Therefore, it is clear that

$$\begin{aligned}
d(f,g) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_1} \\
&= \sum_{k=1}^{m} 2^{-k} \frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_1} + \sum_{k=m+1}^{\infty} 2^{-k} \frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_1} \\
&\le \|(f-g)\mathbf{1}_{[-m,m]^d}\|_1 + \sum_{k=m+1}^{\infty} 2^{-k} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where we are using in the second line the fact that $f-g$ is locally integrable, so that $\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1$ is finite for every $k \in \mathbb{N}$ and thus

$$\frac{\|(f-g)\mathbf{1}_{[-k,k]^d}\|_1}{1 + \|(f-g)\mathbf{1}_{[-k,k]^d}\|_1} \le 1 \quad \text{for every } k \in \mathbb{N}.$$

■

**Remark 4.15** As in the case of Theorem 4.10, a result in a similar spirit appeared in [PS91], although the latter result concerned radial activation functions.

## 4.5 Approximation in variable Lebesgue spaces

In the previous section, we have collected some results concerning the universal approximation property with neural networks in certain function spaces. However, to the best of our knowledge, there is no result about universal approximation for variable Lebesgue spaces.

In this section, we present the main results of this chapter. We have split them into three different cases, depending on the exponent function of the space and its domain. First, we show some results of universal approximation whenever the exponent function is bounded, subsequently proving that, under certain conditions on the activation function, this is the only case for which a universal approximation is possible. Later, we shift towards the unbounded exponent function setting, starting with the case in which the domain is discrete and subsequently lifting these results to a general domain case.

### 4.5.1 Case I: Bounded exponent function

In this section, we discuss the results of approximation for variable Lebesgue spaces obtained when the exponent function is bounded, i.e. whenever $p : \Omega \to [1,+\infty)$ for $\Omega \subseteq \mathbb{R}^d$ verifies

$$\operatorname*{ess\,sup}_{x\in\Omega} p(x) < +\infty .$$

In the case that the exponent function is bounded, the variable Lebesgue space $L^{p(\cdot)}$ is separable. Furthermore, smooth functions with compact support are dense. Thus, we can present a first version of universal approximation result for bounded variable Lebesgue spaces relying on the previous fact and Proposition 4.8.

**Theorem 4.16** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be locally bounded, continuous almost everywhere and different from an algebraic polynomial. Then, truncated finite sums of the form

$$g(x) = \begin{cases} \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot x + b_j) & x \in K, \\ 0 & x \in \mathbb{R}^d \setminus K, \end{cases}$$

with $K$ compact are dense in $L^{p(\cdot)}(\mathbb{R}^d)$ with bounded exponent $p$.

*Proof.* Fix $\varepsilon > 0$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^{p(\cdot)}(\mathbb{R}^d)$ by Proposition 3.6, we can find $f_1 \in C(\mathbb{R}^d)$ with compact support $K$, such that $\|f - f_1\|_{L^{p(\cdot)}(\mathbb{R}^d)} < \varepsilon/2$.

Using Theorem 4.6, we can find $g_\varepsilon \in H_\sigma$ such that it is truncated to be zero outside $K$ and

$$\sup_{x\in K} |f_1(x) - g_\varepsilon(x)| < \frac{\varepsilon}{2|K|} .$$

Therefore,

$$\|f - g_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^d)} \le \|f - f_1\|_{L^{p(\cdot)}(\mathbb{R}^d)} + \|f_1 - g_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^d)} < \frac{\varepsilon}{2} + |K| \frac{\varepsilon}{2|K|} = \varepsilon.$$

∎

Furthermore, when the exponent function is bounded, the dual of variable Lebesgue spaces is fully characterized (see Proposition 3.7) and it coincides with $L^{q(\cdot)}$, where $q$ is pointwise the Hölder conjugate of $p$, i.e. $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ for every $x\in\Omega$. By virtue of this result, and analogously to the previous notions of discriminatory activation functions, we can give the following definition:

**Definition 4.17** Let $\sigma:\mathbb{R}\to\mathbb{R}$ be an activation function and $K\subset\mathbb{R}^d$ compact. For a bounded exponent function $p:\Omega\to[1,+\infty)$, for $\Omega\subseteq\mathbb{R}^d$, we say that $\sigma$ is *discriminatory for* $L^{p(\cdot)}(K)$ if for any $h\in L^{q(\cdot)}(K)$, where $q$ is pointwise the Hölder conjugate of $p$ (i.e. $1/p(x)+1/q(x)=1$ for every $x\in\Omega$), whenever

$$\int_K \sigma(w\cdot x+b)\,h(x)\,dx=0$$

for all $w\in\mathbb{R}^d$ and $b\in\mathbb{R}$ implies that $h=0$ almost everywhere.

As mentioned above, this is indeed a reasonable definition due to the fact that $L^{q(\cdot)}$ can be identified with the dual of $L^{p(\cdot)}$. Now, considering this definition, we can state and prove the following result in this setting.

**Theorem 4.18** Let $\Omega\subseteq\mathbb{R}^d$, $p:\Omega\to[1,+\infty)$ a bounded exponent function and $\sigma:\mathbb{R}\to\mathbb{R}$ discriminatory for $L^{p(\cdot)}(K)$ for every compact $K\subset\Omega$. Then, truncated finite sums of the form

$$g(x)=\begin{cases}\sum_{j=1}^{M}\alpha_j\,\sigma(w_j\cdot x+b_j) & x\in K\,,\\ 0 & x\in\Omega\setminus K\,,\end{cases}$$

with $K$ compact are dense in $L^{p(\cdot)}(\Omega)$.

*Proof.* Given $f\in L^{p(\cdot)}(\Omega)$ and $\varepsilon>0$, we apply Proposition 3.6 to find a compact $K$ where

$$\|f-f\mathbf{1}_K\|_{L^{p(\cdot)}(\Omega)}<\frac{\varepsilon}{2}\,.$$

Then, we can show that there is a $g$ of the form $\sum_{j=1}^{M}\alpha_j\,\sigma(w_j\cdot x+b_j)$ such that

$$\left\|f-\sum_{j=1}^{M}\alpha_j\,\sigma(w_j\cdot x+b_j)\right\|_{L^{p(\cdot)}(K)}<\frac{\varepsilon}{2}\,,$$

finishing thus the proof. Indeed, if the above estimate were false, we would get a contradiction as a consequence of Hahn-Banach theorem. By virtue of its version as a hyperplane separation theorem (see [Rud91, Theorem 3.5]), we can find a nontrivial functional, $T$, which vanishes over the all the finite sums of the form

$$\sum_{j=1}^{M}\alpha_j\,\sigma(w_j\cdot x+b_j)\,.$$

From Proposition 3.7, we know that the dual of $L^{p(\cdot)}(K)$ is $L^{q(\cdot)}(K)$ where $q$ is pointwise the Hölder conjugate of $p$ and

$$T(f)=\int_K f(x)h(x)\mathrm{d}x\,.$$

Therefore, we have obtained a relation between functionals, $T$, and elements of $L^{q(\cdot)}(K)$, $h$. Since $\sigma$ is discriminatory, and $h$ can in particular be taken to be $\sigma$ above (as $\sigma$ belongs to $L^{q(\cdot)}(K)$), we conclude $T = 0$, getting a contradiction with the nontriviality of $T$. ∎

A natural question that yields the previous result is when the activation function is discriminatory for variable Lebesgue spaces. We can hence prove the following relation between the properties of being discriminatory, which in particular yields the fact that Theorem 4.18 is more general than Theorem 4.16.

**Lemma 4.19** Given $K \subset \mathbb{R}^d$ compact and two bounded exponent functions $p_1$ and $p_2$ satisfying $p_1(x) \le p_2(x)$ for all $x \in K$,

$$\{\sigma \text{ disc. for } C(K)\} \subseteq \{\sigma \text{ disc. for } L^{p_2(\cdot)}(K)\} \subseteq \{\sigma \text{ disc. for } L^{p_1(\cdot)}(K)\}.$$

In particular, non-constant, bounded activation functions $\sigma$ and the rectifier are discriminatory for $L^{p(\cdot)}(K)$ (with $p$ bounded).

*Proof.* We have the following inclusion of variable Lebesgue space when the domain is compact: Whenever $p_1 \le p_2$, it holds that $L^{p_2(\cdot)}(K) \subseteq L^{p_1(\cdot)}(K)$ (Proposition 3.8). We recall that the dual of bounded variable Lebesgue space $L^{p(\cdot)}$ is the variable Lebesgue space $L^{q(\cdot)}$ where $q$ is pointwise the Hölder conjugate of $p$ (Proposition 3.7). Therefore,

$$L^{q_1(\cdot)}(K) \subseteq L^{q_2(\cdot)}(K),$$

where $q_1$ and $q_2$ are the Hölder conjugates of $p_1$ and $p_2$, respectively. This proves that a function $\sigma$ that is discriminatory for $L^{p_2(\cdot)}(K)$ is also discriminatory for $L^{p_1(\cdot)}(K)$.

Moreover, since for every $h \in L^1(K)$, $h(x)\mathrm{d}x$ is a Radon measure, we have if $\sigma$ is discriminatory for $C(K)$, it is also discriminatory for $L^{p(\cdot)}(K)$ with $p$ bounded.

Finally, using Proposition 4.7 and Remark 4.4, non-constant, bounded activation functions $\sigma$ and the rectifier are discriminatory for $L^{p(\cdot)}(K)$ (with $p$ bounded). ∎

From Lemma 4.19 and Proposition 4.7, it easily follows that all the examples of activation function discussed are discriminatory for $L^{p(\cdot)}(K)$. Therefore, we can use Theorem 4.18 with any of the activation functions from Example 4.1.

We conclude this subsection showing the connection between the boundedness of the exponent function and the universal approximation property for $L^{p(\cdot)}$ spaces.

**Corollary 4.20** Let $\Omega \subseteq \mathbb{R}^d$, $p : \Omega \to [1, +\infty)$ a exponent function and $\sigma : \mathbb{R} \to \mathbb{R}$ continuous and sigmoidal. Then, UA holds for $L^{p(\cdot)}(\Omega)$ if, and only if, $p$ is bounded.

*Proof.* On the one hand, from Theorem 4.18, Proposition 4.7 and Lemma 4.19 we deduce that, when $p$ is bounded, we have UA for $L^{p(\cdot)}(\Omega)$. On the other hand, when $p$ is unbounded the space $L^{p(\cdot)}(\Omega)$ lacks UA, due to Proposition 3.5 and Lemma 4.12. ∎

### 4.5.2 Case II: Unbounded exponent function, discrete case

After having dealt with the bounded case in the last section, we now shift towards the unbounded case. For that, it is better to show first the results that can be obtained in the discrete case, i.e. in variable sequence spaces. Even though variable sequence spaces are easier to describe than variable Lebesgue spaces, they are complex enough to show that it is impossible to achieve a universal approximation property.

More precisely, we consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ denotes all the subsets of the natural numbers and $\mu$ is the counting measure (it associates to every subset of the natural numbers its cardinality). As in the continuum case, other measures could be considered and the results would follow with minor modifications.

Let us recall the definition of a variable sequence space: Given an exponent function $p : \mathbb{N} \to [1, +\infty)$, consider the modular

$$\rho_{p(\cdot)}(\{x(k)\}) := \sum_{j=1}^{\infty} |x(j)|^{p(j)}.$$

Then, the norm is given by

$$\|\{x(k)\}\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\{x(k)\}/\lambda) \leq 1\right\}.$$

Let us recall that we are denoting the subspace of all sequences that can be obtained using an ANN with 1 hidden layer and activation function $\sigma$ by

$$H_\sigma := \left\{\{y(k)\} : y(k) = \sum_{j=1}^{M} \alpha_j \sigma(w_j \cdot k + b_j)\right\},$$

where $\alpha_j, w_j, b_j \in \mathbb{R}$ and $M \in \mathbb{N}$. Hereafter, we will assume that the activation function satisfies $\sigma \in L^\infty(\mathbb{R})$ non-constant and sigmoidal (existence of the limits at $+\infty$ and $-\infty$). Therefore, $H_\sigma \subseteq \ell^\infty$. More specifically, since $\sigma$ is sigmoidal, $H_\sigma$ is a subspace of the space of convergent sequences, i.e. $H_\sigma \subseteq c$.

Before proceeding to the main results of this section, here we collect some results concerning variable sequence spaces and its relation with $\ell^\infty$ addressed in [Ame+19].

**Proposition 4.21** Let $p : \mathbb{N} \to [1, +\infty)$ be an exponent function. Then,

- $\ell^{p(\cdot)} \subseteq \ell^\infty$.
- $\ell^{p(\cdot)} = \ell^\infty$ as vector spaces if, and only if, $\|\mathbf{1}_{\mathbb{N}}\|_{p(\cdot)} < +\infty$.

The condition $\|\mathbf{1}_{\mathbb{N}}\|_{p(\cdot)} < +\infty$ is related to the divergence of the exponent function $p$, that is, we need that $p(k) \to +\infty$ $(k \to +\infty)$ fast enough so that the series

$$\sum_{j=1}^{\infty} s^{p(j)}$$

converges for some $s > 0$. For example, $p(k) = 1 + \log k$ or any other exponent function which diverges faster verifies the previous condition. However, there are other unbounded exponent functions which do not verify the previous condition, such as $p(k) = 1 + \log(1 + \log k)$ or any other exponent function which diverges slower than this.

**Proposition 4.22 — Approximation in unbounded variable sequence spaces.** Let $p : \mathbb{N} \to [1, +\infty)$ be an exponent function such that $\|\mathbf{1}_{\mathbb{N}}\|_{p(\cdot)} < +\infty$ and $\sigma \in L^\infty(\mathbb{R})$ sigmoidal and non-constant. Then, $\overline{H_\sigma} = c$, where the closure is taken in the $\|\cdot\|_{p(\cdot)}$ norm.

*Proof.* Note that using the hypothesis of the exponent function, $(\ell^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ and $(\ell^\infty, \|\cdot\|_\infty)$ is the same space with two equivalent norms.

On the one hand, since $c$ is closed in $(\ell^\infty, \|\cdot\|_\infty)$, from the obvious inclusion $H_\sigma \subseteq c$ we get that $\overline{H_\sigma} \subseteq c$.

On the other hand, to prove the reverse inclusion it is enough to show that the space of sequences with finite number of nonvanishing terms $c_{00}$ is contained in $\overline{H_\sigma}$, since the closure in $(\ell^\infty, \|\cdot\|_\infty)$ of $c_{00}$ is $c_0$ (the space of sequences which converges to 0) and we have good approximations of constants in $H_\sigma$ because $\sigma$ is sigmoidal.

Finally, $c_{00}$ can be approximated by a an analogous discrete argument as Proposition 4.9. ■

**Remark 4.23** The statement of Proposition 4.22 holds for $\sigma$ the rectifier. More precisely, we can prove that $\overline{H_\sigma \cap \ell^{p(\cdot)}} = c$, since using two ReLU we can obtain a non-constant, sigmoidal, bounded activation function and appeal then to Proposition 4.22.

### 4.5.3 Case III: Unbounded exponent function, general case

Now we analyze the general situation in which the exponent function is unbounded. In this case, the variable Lebesgue space is nonseparable (see Proposition 3.5). Using Lemma 4.12, we deduce that the separability of the function space is necessary for a Universal Approximation result to hold for the sigmoidal, continuous activation function. Therefore, in the current context, the difficulty to obtain approximation results is much higher. For example, note that we cannot even approximate the function by its restriction to compact domains (Proposition 3.6).

Moreover, there is an additional problem associated to finding a precise characterization of the dual, since many subtleties appear in this setting. For more information about the dual in this case, we refer the interested reader to some previous work of one of the authors [Ame+19].

For the reasons aforementioned, here we just focus in the unidimensional case, having in mind the toy model $\Omega = [1, +\infty)$ and an exponent function $p : \Omega \to \Omega$ given by $p(x) = x$ or $p(x) = [x]$, where $[x]$ denotes the integer part of $x$. In this case, we show a characterization of the subspace of functions which actually can be approximated using an artificial neural network.

First, since we are focusing on the unidimensional case, we start by showing that we can approximate bounded functions with limit at $\infty$.

**Proposition 4.24** Let $\Omega = [1, +\infty)$ and $p : \Omega \to [1, +\infty)$ an unbounded exponent function such that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ and it is bounded in every compact subset of $\Omega$. Let $\sigma \in L^\infty(\mathbb{R})$ be a non-constant, sigmoidal activation function. Then, given $f \in L^\infty(\mathbb{R})$ with limit at $\infty$, and $\varepsilon > 0$, there is a function of the form

$$g_\varepsilon(x) := \sum_{j=1}^{n} \alpha_j \sigma(w_j \cdot x + b_j)$$

such that $\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$.

*Proof.* Let $\beta := \lim_{x \to +\infty} f(x)$. For simplicity, we take $\beta = 0$. Indeed, since $\sigma$ is sigmoidal, there is function $h(x) = \alpha \sigma(w \cdot x + b)$ such that $f - h$ is a bounded function with limit 0 at $+\infty$, for suitable $\alpha$ and $w$.

Fix $\varepsilon > 0$. Given $\delta > 0$ there is $L > 0$ such that $|f(x)| < \delta$ if $x > L$. Since $L^\infty(\Omega) \subset L^{p(\cdot)}$, we know that $\omega(\Omega) < +\infty$. Hence, we can take

$$\delta = \frac{\varepsilon}{4\|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)}}.$$

Now, since $p$ is bounded in $[1, M]$ for some $M > L$, we can use Theorem 4.18 and Proposition

4.9 to find $g_\varepsilon$ of the form

$$g_\varepsilon(x) := \sum_{j=1}^{n} \alpha_j \sigma(w_j \cdot x + b_j),$$

such that the following holds

$$\|f - g_\varepsilon\|_{L^{p(\cdot)}([1,M])} < \frac{\varepsilon}{2},$$

and $|g_\varepsilon(x)| < \delta$ for $x > M$. Then,

$$\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} \leq \|f - g_\varepsilon\|_{L^{p(\cdot)}([1,M])} + \|f - g_\varepsilon\|_{L^{p(\cdot)}([M,+\infty))} < \frac{\varepsilon}{2} + 2\delta \|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)} = \varepsilon$$

■

Next, we can proceed to the main result of this chapter. In the theorem below, we provide a characterization of the set of functions of a variable Lebesgue space with an unbounded exponent that can be approximated using neural networks.

**Theorem 4.25 — Approximation in unbounded variable Lebesgue spaces.** Let $\Omega = [1,+\infty)$ and $p : \Omega \to [1,+\infty)$ be an unbounded exponent function such that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ and it is bounded in every compact subset of $\Omega$. Let $\sigma \in L^\infty(\mathbb{R})$ be a non-constant, sigmoidal activation function. Then, the following conditions are equivalent for $f \in L^{p(\cdot)}(\Omega)$:

1. For every $\varepsilon > 0$, there is a function of the form

$$g_\varepsilon(x) := \sum_{j=1}^{n} \alpha_j \sigma(w_j \cdot x + b_j),$$

   such that $\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$.
2. There is a scalar $\beta \in \mathbb{R}$ such that

$$\|[f - \beta \mathbf{1}_\Omega]\|_Q = 0,$$

   where $\|\cdot\|_Q$ is the quotient norm given in Definition 3.15.

**Remark 4.26** The rectifier in Example 4.1 is also a valid activation function for Theorem 4.25, since we can obtain a continuous, bounded activation function as a combination of two ReLU.

*Proof of Theorem 4.25.* First we show that $\underline{1 \Rightarrow 2}$. We recall that, since $\sigma$ is sigmoidal (see Definition 4.3), then

$$\sigma(t) = \begin{cases} c_{+\infty} & \text{as } t \to +\infty, \\ c_{-\infty} & \text{as } t \to -\infty. \end{cases}$$

Therefore, every function of the form

$$g(x) = \sum_{j=1}^{n} \alpha_j \sigma(w_j \cdot x + b_j)$$

converges to

$$\sum_{\{j:w_j>0\}} \alpha_j c_{+\infty} + \sum_{\{j:w_j<0\}} \alpha_j c_{-\infty} + \sum_{\{j:w_j=0\}} \alpha_j \sigma(b_j)$$

when $x$ tends to $+\infty$.

From hypothesis 1, we know that $\forall \varepsilon > 0$, there is a function $g_\varepsilon$ with the previous form such that $\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$. Let us denote by $\beta_\varepsilon$ the limit at $+\infty$ of $g_\varepsilon$. First, we can show that $g_\varepsilon$ and $\beta_\varepsilon \mathbf{1}_\Omega$ belong to the same class in the quotient space.

Indeed, since $\beta_\varepsilon$ is the limit of $g_\varepsilon(x)$ at $+\infty$, for every $\delta > 0$ we can find $M > 0$ such that for every $x > M$, it holds that $|g_\varepsilon(x) - \beta_\varepsilon| < \delta$. Then,

$$\|[g_\varepsilon - \beta_\varepsilon \mathbf{1}_\Omega]\|_Q = \|[(g_\varepsilon - \beta_\varepsilon)\mathbf{1}_{(M,+\infty)}]\|_Q \le \|[\delta \mathbf{1}_{(M,+\infty)}]\|_Q = \delta\omega(\Omega),$$

where the first equality follows from the fact that the quotient space is taken over the closure of the compactly supported functions in $L^{p(\cdot)}$. Note that $\omega(\Omega)$ is finite. Then, as $\delta > 0$ can be arbitrary small, we conclude $\|[g_\varepsilon - \beta_\varepsilon \mathbf{1}_\Omega]\|_Q = 0$.

Since these two functions are equivalent in the quotient space, we can deduce the following:

$$\|[f - \beta_\varepsilon \mathbf{1}_\Omega]\|_Q = \|[f - g_\varepsilon]\|_Q \le \|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} < \varepsilon\,.$$

Next, we claim that the sequence $\{\beta_\varepsilon\}$ is bounded. Indeed, this holds due to the fact that:

$$|\beta_{\varepsilon_1} - \beta_{\varepsilon_2}|\,\omega(\Omega) = \|[(\beta_{\varepsilon_1} - \beta_{\varepsilon_2})\mathbf{1}_\Omega]\|_Q \le \|[\beta_{\varepsilon_1}\mathbf{1}_\Omega - f]\|_Q + \|[f - \beta_{\varepsilon_2}\mathbf{1}_\Omega]\|_Q < \varepsilon_1 + \varepsilon_2\,,$$

and thus,

$$|\beta_{\varepsilon_1} - \beta_{\varepsilon_2}| \le \frac{\varepsilon_1 + \varepsilon_2}{\omega(\Omega)}\,.$$

Since $\{\beta_\varepsilon\}$ is bounded, there is subsequence $\{\beta_{\varepsilon_n}\}$, with $\varepsilon_n$ tending to 0 and $\beta_{\varepsilon_n}$ converging to some scalar $\beta$ when $n$ tends to $+\infty$. To conclude, we now have to show that

$$\|[f - \beta \mathbf{1}_\Omega]\|_Q = 0\,,$$

finishing thus the proof of 2. Indeed:

$$\|[f - \beta \mathbf{1}_\Omega]\|_Q \le \|[f - \beta_{\varepsilon_n}\mathbf{1}_\Omega]\|_Q + \|[(\beta_{\varepsilon_n} - \beta)\mathbf{1}_\Omega]\|_Q \le \varepsilon_n + |\beta_{\varepsilon_n} - \beta|\omega(\Omega)\,.$$

which clearly vanishes when $n$ tends to $+\infty$.

Now we prove $\underline{2 \Rightarrow 1}$. Fix $\varepsilon > 0$. From hypothesis 2, we can find $f_c \in L_c^{p(\cdot)}$ such that

$$\|(f - \beta \mathbf{1}_\Omega) - f_c\|_{L^{p(\cdot)}(\Omega)} < \frac{\varepsilon}{3}\,.$$

The support of $f_c$ is contained in a compact set of the form $[1,L]$ for some $L$. Since $p$ is bounded on $[1,L]$ by hypothesis, using Theorem 4.18 we can find a function $g_1$ of the form

$$g_1 = \sum_{j=1}^{n} \alpha_j \sigma(w_j \cdot x + b_j)\,,$$

such that

$$\|f_c - g_1\|_{L^{p(\cdot)}([1,L])} < \frac{\varepsilon}{3}\,.$$

Since $\beta \mathbf{1}_\Omega - g_1 \mathbf{1}_{(l,+\infty)}$ is clearly a bounded function whose limit exits (and is finite) at $+\infty$, we can appeal to Proposition 4.24 to find a function function $g_2$ of the form

$$g_2 = \sum_{j=n+1}^{n+m} \alpha_j \sigma(w_j \cdot x + b_j)\,,$$

such that

$$\|(\beta\mathbf{1}_\Omega - g_1\mathbf{1}_{(l,+\infty)}) - g_2\|_{L^{p(\cdot)}(\Omega)} < \frac{\varepsilon}{3}\,.$$

Then, we can take $g_\varepsilon$ in the statement of the theorem to be:

$$g_\varepsilon(x) := g_1(x) + g_2(x) = \sum_{j=1}^{n+m} \alpha_j \sigma(w_j \cdot x + b_j)\,,$$

since

$$\begin{aligned}
\|f - g_\varepsilon\|_{L^{p(\cdot)}(\Omega)} &\le \|(f - \beta\mathbf{1}_\Omega) - f_c\|_{L^{p(\cdot)}(\Omega)} + \|f_c - g_1\|_{L^{p(\cdot)}([1,L])} \\
&\quad + \|(\beta\mathbf{1}_\Omega - g_1\mathbf{1}_{(L,+\infty)}) - g_2\|_{L^{p(\cdot)}(\Omega)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.
\end{aligned}$$

■

# Bibliography

## Books


[Bis06] C. Bishop. *Pattern Recognition and Machine Learning*. Springer New York, 2006 (cited on pages 63, 68, 70).

[CC95] L. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*. Volume 43. American Mathematical Soc., 1995 (cited on pages 19, 21, 22).

[Cas+13] B. Cascales et al. *Análisis Funcional*. E-lectolibris, 2013.

[Con85] J. B. Conway. *A Course in Functional Analysis*. Volume 96. Springer-Verlag New York, 1985.

[Con10] J. B. Conway. *A course in functional analysis*. Volume 96. Springer, 2010 (cited on pages 37, 42).

[CF13] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces*. Birkhäuser/Springer, Heidelberg, 2013 (cited on pages 35–37, 41, 42, 54, 57, 62).

[Del61] J. Delsarte. *Lectures on topics in mean periodic functions and the two-radius theorem*. 22-23. Tata Institute of Fundamental Research Bombay, 1961 (cited on page 5).

[Die+11] L. Diening et al. *Lebesgue and Sobolev Spaces with Variable Exponents*. Volume 2017. Springer, 2011 (cited on pages 36, 47).

[DS58] N. Dunford and J. T. Schwartz. *Linear operators part I: general theory*. Volume 243. Interscience publishers New York, 1958 (cited on page 53).

[ET10] G. B. Ermentrout and D. H. Terman. *Mathematical foundations of neuroscience*. Volume 35. Springer Science & Business Media, 2010 (cited on page 63).

[GT15] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. New York: Springer, 2015 (cited on page 4).

[Giu84] E. Giusti. *Minimal surfaces and functions of bounded variation*. Volume 80. Springer, 1984 (cited on page 19).


This picture corresponds to a beautiful sight of the city Málaga where I gave a talk about the dual of Variable Lebesgue spaces in EARCO conference. Furthermore, the city is very similar to Cartagena, the place where I was born.

[HH19] P. Harjulehto and P. Hästö. *Orlicz spaces and generalized Orlicz spaces*. Volume 2236. Lecture Notes in Mathematics. Springer, 2019 (cited on page 36).

[Hay98] S. Haykin. *Neural Networks: A Comprehensive Foundation*. Prentice Hall, 1998 (cited on page 63).

[HK76] W.K. Hayman and P.B. Kennedy. *Subharmonic functions*. Volume 1. London Mathematical Soc., 1976 (cited on page 6).

[Kec12] A. Kechris. *Classical descriptive set theory*. Volume 156. Springer Science & Business Media, 2012 (cited on page 13).

[Lac12] H. E. Lacey. *The isometric theory of classical Banach spaces*. Volume 208. Springer Science & Business Media, 2012 (cited on page 53).

[LT13] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces II: function spaces*. Volume 97. Springer Science & Business Media, 2013 (cited on pages 39, 44, 45).

[Lux55] W. A. J. Luxemburg. *Banach function spaces*. Thesis, Technische Hogeschool te Delft, 1955 (cited on pages xii, xx, 36).

[Mus83] J. Musielak. *Orlicz Spaces and Modular Spaces*. Volume 1034. Springer-Verlag Berlin Heidelberg, 1983.

[Mus06] Julian Musielak. *Orlicz spaces and modular spaces*. Volume 1034. Springer, 2006 (cited on page 36).

[Pet06] P. Petersen. *Riemannian geometry*. Volume 171. Springer, 2006 (cited on page 26).

[PRS08] S. Pigola, M. Rigoli, and A. G. Setti. *Vanishing and finiteness results in geometric analysis*. Volume A generalization of the Bochner technique. Progress in Mathematics, 266. Birkhäuser Verlag, Basel, 2008 (cited on page 31).

[Roj96] R. Rojas. *Neural Networks - A Systematic Introduction*. Springer-Verlag New York, 1996 (cited on page 63).

[Rud91] W. Rudin. *Functional analysis*. McGraw-Hill, 1991 (cited on pages 42, 78).

[Sak96] T. Sakai. *Riemannian geometry*. American Mathematical Society, Providence, RI, 1996 (cited on page 31).

[Wol20] M. M. Wolf. *Mathematical Foundations of Supervised Learning*. lecture notes, 2020 (cited on pages 63, 66).

## ARTICLES

[AN20] M. Ali and A. Nouy. "Approximation with tensor networks. Part I: Approximation spaces". In: *preprint, arXiv:2007.00118* (2020) (cited on page 62).

[Alm+18] J. Almira et al. "Some negative results for single layer and multilayer feedforward neural networks". In: *preprint, arXiv:1810.10032* (2018) (cited on page 62).

[ABS19] L. Ambrosio, E. Brué, and D. Semola. "Rigidity of the 1-Bakry–Émery inequality and sets of finite perimeter in RCD spaces". In: *Geom. Funct. Anal.* 29.4 (2019), pages 949–1001 (cited on page 26).

[Ame+19] A. Amenta et al. "On the dual of variable Lebesgue spaces with unbounded exponent". In: *arXiv preprint arXiv:1909.05987* (2019) (cited on pages xiii, xiv, xxi, xxii, 35, 38, 80, 81).

[And83] M. T. Anderson. "The Dirichlet problem at infinity for manifolds of negative curvature". In: *J. Differential Geom.* 18.4 (1983), pages 701–721 (cited on page 24).

[Arn57] V. I. Arnold. “On functions of three variables”. In: *Doklady Akademii Nauk USSR* 114 (1957), pages 679–681 (cited on page 61).

[Bar94] A. Barron. “Approximation and Estimation Bounds for Artificial Neural Networks”. In: *Mach. Learn.* 14.1 (1994), pages 115–133 (cited on page 62).

[BG86] C.A. Berenstein and R. Gay. “A local version of the two-circles theorem”. In: *Israel J. Math.* 55.3 (1986), pages 267–288 (cited on page 5).

[Bla16] W. Blaschke. “Ein Mittelwertsatz und eine kennzeichnende Eigenschaft des logarithmischen Potentials”. In: *Leipz. Ber* 68 (1916), pages 3–7 (cited on page 6).

[Böl+19] H. Bölcskei et al. “Optimal approximation with sparsely connected deep neural networks”. In: *SIAM J. Math. Data Sci.* 1.1 (2019), pages 8–45 (cited on page 62).

[CC93] L. Caffarelli and A. Córdoba. “An elementary regularity theory of minimal surfaces”. In: *Differential Integral Equations* 6.1 (1993), pages 1–13 (cited on page 19).

[CGS94] L. Caffarelli, N. Garofalo, and F. Segala. “A gradient bound for entire solutions of quasi-linear equations and its consequences”. In: *Comm. Pure Appl. Math.* 47.11 (1994), pages 1457–1473 (cited on page 25).

[CO20a] A. Capel and J. Ocáriz. “Approximation with neural networks in variable Lebesgue spaces”. In: *arXiv preprint arXiv:2007.04166* (2020) (cited on pages xv, xxiii, 61).

[Cat42] E. Catalan. “Sur les surfaces réglées dont l’aire est un minimum”. In: *J. Math. Pures Appl.* 7 (1842), pages 203–211 (cited on page 17).

[CG71] J. Cheeger and D. Gromoll. “The splitting theorem for manifolds of nonnegative Ricci curvature”. In: *J. Differential Geom.* 6.1 (1971), pages 119–128 (cited on pages 25, 26).

[CO20b] A. Córdoba and J. Ocáriz. “A note on generalized Laplacians and minimal surfaces”. In: *Bull. London Math. Soc.* 52.1 (2020), pages 153–157 (cited on pages x, xviii, 3, 8, 16, 18).

[Cyb89] G. Cybenko. “Approximation by superpositions of a sigmoidal function”. In: *Mathematics of control, signals and systems* 2.4 (1989), pages 303–314 (cited on pages 61, 72, 73).

[EN02] D. E. Edmunds and A. Nekvinda. “Averaging operators on $l^{\{p_n\}}$ and $L^{p(x)}$”. In: *Math. Inequal. Appl.* 5 (2002), pages 235–246 (cited on page 45).

[EGK18] D. Edmunds, A. Gogatishvili, and T. Kopaliani. “Construction of function spaces close to $L^\infty$ with associate space close to $L^1$”. In: *J. Fourier Anal. Appl.* 24.6 (2018), pages 1539–1553 (cited on page 35).

[EH84] J. Eschenburg and E. Heintze. “An elementary proof of the Cheeger-Gromoll splitting theorem”. In: *Ann. Global Anal. Geom.* 2.2 (1984), pages 141–151 (cited on page 31).

[FMV13] A. Farina, L. Mari, and E. Valdinoci. “Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds”. In: *Comm. Partial Differential Equations* 38.10 (2013), pages 1818–1862 (cited on pages 26, 28, 31).

[FO20] A. Farina and J. Ocáriz. “Splitting theorems on complete Riemannian manifolds with nonnegative Ricci curvature”. In: *Discrete Contin. Dyn. Syst. (to appear)* (2020) (cited on pages xi, xix, 23, 29).

[FSV08] A. Farina, B. Sciunzi, and E. Valdinoci. “Bernstein and De Giorgi type problems: new results via a geometric approach”. In: *Ann. Sc. Norm. Supe. Pisa Cl. Sci. (4)* 7 (2008), pages 741–791 (cited on pages 24, 28).

[FSV12] A. Farina, Y. Sire, and E. Valdinoci. “Stable solutions of elliptic equations on Riemannian manifolds with Euclidean coverings”. In: *Proc. Amer. Math. Soc.* 140.3 (2012), pages 927–930 (cited on page 26).

[FV09] A. Farina and E. Valdinoci. “The state of the art for a conjecture of De Giorgi and related problems”. In: *Recent progress on reaction-diffusion systems and viscosity solutions* 7496 (2009) (cited on page 23).

[FV10] A. Farina and E. Valdinoci. “A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature”. In: *Adv. Math.* 225.5 (2010), pages 2808–2827 (cited on pages 24, 28).

[FV11] A. Farina and E. Valdinoci. “A pointwise gradient bound for elliptic equations on compact manifolds with nonnegative Ricci curvature”. In: *Discrete Contin. Dyn. Syst* 30.4 (2011), pages 1139–1144 (cited on pages 24, 28, 30).

[GS09] S. Giulini and M. Sanguineti. “Approximation Schemes for Functional Optimization Problems”. In: *J. Optim. Theory Appl.* 140 (2009), pages 33–54 (cited on page 62).

[GM15] P. Górka and A. Macios. “Almost everything you need to know about relatively compact sets in variable Lebesgue spaces”. In: *J. Funct. Anal.* 269.7 (2015), pages 1925–1949 (cited on page 35).

[GR20] I. Gühring and M. Raslan. “Approximation Rates for Neural Networks with Encodable Weights in Smoothness Spaces”. In: *preprint, arXiv:2006.16822* (2020) (cited on page 62).

[HHH20] A. Heinecke, J. Ho, and W. Hwang. “Refinement and universal approximation via sparsely connected ReLU convolution nets”. In: *IEEE Signal Processing Letters* (2020) (cited on page 62).

[Hor91] K. Hornik. “Approximation capabilities of multilayer feedforward networks”. In: *Neural networks* 4.2 (1991), pages 251–257 (cited on pages 61, 72, 73).

[HSW89] K. Hornik, M. Stinchcombe, and H. White. “Multilayer feedforward networks are universal approximators”. In: *Neural networks* 2.5 (1989), pages 359–366 (cited on pages 61, 72, 73).

[Huc54] F. Huckemann. “On the “One-Circle” Problem for Harmonic Functions”. In: *J. Lond. Math. Soc.* 1.4 (1954), pages 491–497 (cited on page 4).

[HH20] W. Hwang and A. Heinecke. “Un-Rectifying Non-Linear Networks for Signal Representation”. In: *IEEE Transactions on Signal Processing* 68 (2020), pages 196–210 (cited on page 62).

[Kak41] S. Kakutani. “Concrete representation of abstract **M**-spaces (A characterization of the space of continuous functions)”. In: *Ann. of Math.* (1941), pages 994–1024 (cited on pages 39, 45).

[Kel34] O.D. Kellogg. “Converses of Gauss’ theorem on the arithmetic mean”. In: *Trans. Amer. Math. Soc.* 36.2 (1934), pages 227–242 (cited on page 4).

[KL19] P. Kidger and T. Lyons. “Universal Approximation with Deep Narrow Networks”. In: *preprint, arXiv:1905.08539* (2019) (cited on page 62).

[Kol57] A. N. Kolmogorov. “On the representation of continuous functions of many variables by superpositions of continuous functions of one variable and addition”. In: *Dokl. Akad. SSSRl* 114 (1957), pages 953–956 (cited on page 61).

[Kra19] A. Kratsios. “The Universal Approximation Property: Characterizations, Existence, and a Canonical Topology for Deep-Learning”. In: *preprint, arXiv:1910.03344* (2019) (cited on page 62).

[KS93] B. Kröse and P. van der Smagt. “An introduction to neural networks”. In: *J. Comput. Sci.* 48 (Jan. 1993) (cited on pages 63, 68).

[LPS93] V. Y. Leshno M.and Lin, A. Pinkus, and S. Schocken. “Multilayer feedforward networks with a nonpolynomial activation function can approximate any function”. In: *Neural Netw.* 6.6 (1993), pages 861–867 (cited on page 73).

[LTY20] B. Li, S. Tang, and H. Yu. “Better Approximations of High Dimensional Smooth Functions by Deep Neural Networks with Recti
ed Power Units”. In: *Commun. in Comp. Phys.* 27 (2020), pages 379–411 (cited on page 62).

[Lin19] S.-B. Lin. “Generalization and Expressivity for Deep Nets”. In: *IEEE T. Neur. Net. Lear.* 30.5 (2019), pages 1392–1406 (cited on page 62).

[MP43] W. S. McCulloch and W. Pitts. “A logical calculus of the ideas immanent in nervous activity”. In: *The bulletin of mathematical biophysics* 5.4 (1943), pages 115–133 (cited on page 61).

[Mha96] M. Mhaskar. “Neural networks for optimal approximation of smooth and analytic functions”. In: *Neural Comput.* 8.1 (1996), pages 164–177 (cited on page 62).

[Mod85] L. Modica. “A gradient bound and a Liouville theorem for nonlinear Poisson equations”. In: *Comm. Pure Appl. Math.* 38.5 (1985), pages 679–684 (cited on pages xi, xix, 23, 28).

[Nek02] A. Nekvinda. “Equivalence of $l^{\{p_n\}}$ norms and shift operators”. In: *Math. Inequal. Appl* 5.4 (2002), pages 711–723 (cited on pages 45, 47).

[Nek07] A. Nekvinda. “A note on maximal operator on $l^{\{p_n\}}$ and$L^{p(x)}(\mathbb{R})$”. In: *J. Funct. Spaces* 5.1 (2007), pages 49–88 (cited on page 45).

[OK19] I. Ohn and Y. Kim. “Smooth function approximation by deep neural networks with general activation functions”. In: *Entropy* 21.7 (2019), page 627 (cited on page 62).

[Orl31] W. Orlicz. “Über konjugierte exponentenfolgen”. In: *Studia Math.* 3.1 (1931), pages 200–211 (cited on page 35).

[PS91] J. Park and I. W. Sandberg. “Universal approximation using radial-basis-function networks”. In: *Neural computation* 3.2 (1991), pages 246–257 (cited on pages 61, 72, 74, 77).

[PV18] P. Petersen and F. Voigtländer. “Optimal approximation of piecewise smooth functions using deep ReLU neural networks”. In: *Neural Netw.* 108 (2018), pages 296–330 (cited on page 62).

[Pet99] P. P. Petrushev. “Approximation by ridge function and neural networks”. In: *SIAM Journal on Mathematical Analysis* 30 (1999), pages 155–189 (cited on page 62).

[Pin97] A. Pinkus. “Approximating by ridge functions”. In: *Surface fitting and multiresolution methods* (1997), pages 279–292 (cited on page 61).

[Pin99] A. Pinkus. “Approximation theory of the MLP model in neural networks”. In: *Acta Numer.* 8 (1999), pages 143–195 (cited on page 61).

[Pot48] D.H. Potts. “A note on the operators of Blaschke and Privaloff”. In: *Bull. Amer. Math. Soc.* 54.8 (1948), pages 782–787 (cited on page 7).

[Pri25] J. Privaloff. “Sur les fonctions harmoniques”. In: *Mat. Sb.* 32 (1925), pages 464–469 (cited on page 6).

[RR95] A. Ratto and M. Rigoli. “Gradient bounds and Liouville’s type theorems for the Poisson equation on complete Riemannian manifolds”. In: *Tohoku Math. J. (2)* 47.4 (1995), pages 509–519 (cited on pages 24, 28).

[Rie26] F. Riesz. “Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel”. In: *Acta Math.* 48.3-4 (1926), pages 329–343 (cited on page 8).

[Ros58] F. Rosenblatt. “The perceptron: a probabilistic model for information storage and organization in the brain”. In: *Psychological review* 65.6 (1958), page 386 (cited on page 61).

[Sak41] S. Saks. “On the operators of Blaschke and Privaloff for subharmonic functions”. In: *Mat. Sb.* 9.2 (1941), pages 451–456 (cited on page 7).

[Sam98] S. G. Samko. “Convolution type operators in $L^{p(x)}$”. In: *Integral Transforms Spec. Funct.* 7.1-2 (1998), pages 123–144 (cited on pages 57, 58).

[San08] M. Sanguineti. “Universal Approximation by Ridge Computational Models and Neural Networks: A Survey”. In: *The Open Applied Mathematics Journal* 2.1 (2008), pages 31–58 (cited on page 61).

[Sav09] O. Savin. “Phase transitions, minimal surfaces and a conjecture of De Giorgi”. In: *Curr. Dev. Math. Sci.* (2009), pages 59–113 (cited on page 23).

[ST98] F. Scarselli and A. C. Tsoi. “Universal Approximation Using Feedforward Neural Networks: A Survey of Some Existing Methods, and Some New Results”. In: *Neural Networks* 11.1 (1998), pages 15–37 (cited on pages 61, 67).

[Sch17] J. Schmidt-Hieber. “Nonparametric regression using deep neural networks with ReLU activation function”. In: *preprint, arXiv:1708.06633* (2017) (cited on page 62).

[SCC18] U. Shaman, A. Cloninger, and R. Coifman. “Provable approximation properties for deep neural networks”. In: *Appl. Comput. Harmon. Anal.* 44.3 (2018), pages 537–557 (cited on page 62).

[Suz19] T. Suzuki. “Adaptivity of deep ReLU network for learning in Besov and mixed smooth Besov spaces: optimal rate and curse of dimensionality”. In: *7th International Conference on Learning Representations, ICLR 2019, New Orleans, LA, USA, May 6-9, 2019* (2019) (cited on page 62).

[TLY19] S. Tang, B. Li, and H. Yu. “Efficient and Stable Constructions of Deep Neural Networks with Rectified Power Units using Chebyshev Approximations”. In: *preprint, arXiv:1911.05467* (2019) (cited on page 62).

[TKG+03] D. Tikk, L. T Kóczy, T. D. Gedeon, et al. “A survey on universal approximation and its limits in soft computing techniques”. In: *International Journal of Approximate Reasoning* 33.2 (2003), pages 185–202 (cited on pages 61, 72).

[Vol96] V.V. Volchkov. “The final version of the mean value theorem for harmonic functions”. In: *Math. Notes* 59.3 (1996), pages 247–252 (cited on page 5).

[Wey40] H. Weyl. “The method of orthogonal projection in potential theory”. In: *Duke Math. J.* 7.1 (1940), pages 411–444 (cited on page 5).

[Wid90] B. Widrow. “30 years of adaptive neural networks: Perceptron, Madaline, and Backpropagation”. In: *IEEE Transactions on Neural Networks* 78.9 (1990), pages 1415–1442 (cited on page 63).

[Yar17] D. Yarotsky. “Error bounds for approximations with deep ReLU networks”. In: *Neural Netw.* 94 (2017), pages 103–114 (cited on page 62).

[Yau75] S.-T. Yau. “Harmonic functions on complete Riemannian manifolds”. In: *Comm. Pure and Appl. Math.* 28.2 (1975), pages 201–228 (cited on page 24).

[YH52] K. Yosida and E. Hewitt. “Finitely additive measures”. In: *Trans. Amer. Math. Soc.* 72 (1952), pages 46–66 (cited on page 47).

[Zal80] L. Zalcman. “Offbeat integral geometry”. In: *Amer. Math. Monthly* 87.3 (1980), pages 161–175 (cited on page 5).