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# Models of linear operators satisfying operator inequalities

A thesis submitted in fulfillment of the requirements for the degree of *Doctor in Mathematics* by

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*A mis padres: María Elena y Manuel*

# Resumen

Una de las principales herramientas para poder entender propiedades de un operador abstracto $T$ en un espacio de Hilbert $\mathcal{H}$ consiste en la construcción de un *modelo funcional*. Es decir, dar otro operador $\widetilde{T}$ que actúe en un espacio de Hilbert de funciones $\mathcal{K}$, de manera que $\widetilde{T}$ sea unitariamente equivalente (o bien semejante) al operador inicial $T$. De esta manera, cuestiones sobre el operador $T$ (como la existencia de subespacios invariantes, el estudio de los vectores propios y su completitud, etc) se traducen a cuestiones sobre $\widetilde{T}$, para las cuales disponemos de técnicas del análisis complejo para su estudio.

Uno de los modelos funcionales más célebres es el que dieron Nagy y Foiaş en la década de 1960 para contracciones (operadores de norma menor o igual que 1). Ese modelo ha sido la principal fuente de inspiración para esta tesis. El objetivo que nos planteamos al inicio de la misma fue la construcción de modelos funcionales en el espíritu del modelo de Nagy y Foiaş. Es inmediato ver que un operador $T$ en un espacio de Hilbert $\mathcal{H}$ es una contracción si y sólo si $I - T^*T \geq 0$ (es decir, $I - T^*T$ es un operador positivo), donde $I$ es el operador identidad en $\mathcal{H}$ y $T^*$ es el operador adjunto de $T$. Nuestro planteamiento inicial fue considerar funciones $\alpha$ representables en serie de potencias en un entorno del origen, digamos $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$. De manera formal, sin preocuparnos ahora de la convergencia, pongamos

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n.$$

La pregunta general que nos planteamos fue: bajo ciertas condiciones sobre la función $\alpha$ y el operador $T$, ¿es posible construir un modelo funcional para $T$ si $\alpha(T^*, T) \geq 0$? Notar que el modelo de Nagy y Foiaş trata el caso $\alpha(t) := 1 - t$.

El primer bloque de resultados obtenidos hace referencia al estudio de modelos unitariamente equivalentes. En trabajos previos se estudiaron los *modelos coanalíticos*, es decir, cuando el operador original $T$ es unitariamente equivalente a un operador de la forma $(B \otimes I_{\mathcal{E}}) \oplus S$, donde $B$ es un operador backward shift y $S$ una isometría. En este sentido, uno de los resultados más fuertes, obtenido recientemente por Clouâtre y Hartz, hace referencia al caso en que la función $\alpha$ es de tipo *Nevannlina-Pick* (es decir, $\alpha_0 = 1$ y $\alpha_n \leq 0$ para $n \geq 0$). En esta tesis

obtenemos nuevos casos de funciones $\alpha$ para las que los operadores $T$ satisfaciendo $\alpha(T^*, T) \geq 0$ admiten un modelo coanalítico. Además, estudiamos la unicidad de dichos modelos (que esencialmente dependerá de si $\alpha(1) = 0$ o si $\alpha(1) > 0$), y obtenemos modelos explícitos (no sólo la existencia de los mismos).

El segundo bloque de resultados hace referencia al estudio de modelos semejantes. En este caso, podemos ampliar la clase de funciones $\alpha$ que consideramos, permitiéndoles, por ejemplo, que tengan ceros dentro del disco unidad $\mathbb{D}$. La condición de que $\alpha$ no se anule en $\mathbb{D}$ es esencial en el estudio de modelos coanalíticos unitariamente equivalentes, ya que ahí juega un papel fundamental la función $k(t) = 1/\alpha(t)$ que da lugar a espacios de Hilbert con núcleo reproductivo.

Finalmente, el tercer bloque importante de resultados hace referencia a las consecuencias que se derivan de la obtención de nuestros modelos funcionales. Por un lado, obtenemos consecuencias cuando el *espacio defecto* (una generalización natural del espacio defecto de Nagy y Foiaş a nuestro contexto) tiene dimensión finita. Por otro lado, obtenemos consecuencias ergódicas para las $a$-contracciones (operadores $T$ tales que $(1-t)^a(T^*, T) \geq 0$) cuando $0 < a < 1$.

# Abstract

One of the main tools for understanding the properties of an abstract operator $T$ on a Hilbert space $\mathcal{H}$ consists on the construction of a *functional model*. That is, give another operator $\widetilde{T}$ acting on a Hilbert space of functions $\mathcal{K}$, such that $\widetilde{T}$ is unitarily equivalent (or similar) to the initial operator $T$. In that way, questions on the operator $T$ (such as the existence of invariant subspaces, the study of eigenvectors and its completeness...) can be translated into questions on $\widetilde{T}$, for which we dispose of techniques from complex analysis.

One of the most celebrated functional models is the given by Nagy and Foiaş in the 1960ies for contractions (operators of norm less than or equal to 1). That model has been the main source of inspiration for this thesis. Our initial goal was to construct models in the spirit of the Nagy-Foiaş one. It is immediate that an operator $T$ on a Hilbert space $\mathcal{H}$ is a contraction if and only if $I - T^*T \geq 0$ (that is, $I - T^*T$ is a positive operator), where $I$ is the identity operator on $\mathcal{H}$ and $T^*$ is the adjoint operator of $T$. Our initial approach was to consider functions $\alpha$ that are representable by power series on a neighborhood of the origin, say $\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n$. In a formal way, without paying attention now to convergence, put

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n.$$

The question we set was: under some conditions on the function $\alpha$ and the operator $T$, is it possible to construct a functional model for $T$ if $\alpha(T^*, T) \geq 0$? Note that the Nagy-Foiaş model treats the case $\alpha(t) := 1 - t$.

The first block of results obtained are about the study of unitarily equivalent models. In previous works, the *coanalytic models* were studied, that is, when the operator $T$ is unitarily equivalent to an operator of the form $(B \otimes I_{\mathcal{E}}) \oplus S$, where $B$ is a backward shift operator, and $S$ is an isometry. In this context, one of the strongest results, obtained recently by Clouâtre and Hartz, is about functions $\alpha$ of *Nevannlina-Pick* type (that is, $\alpha_0 = 1$ and $\alpha_n \leq 0$ for $n \geq 0$). In this thesis we obtain new cases of functions $\alpha$ for which the operators $T$ satisfying $\alpha(T^*, T) \geq 0$ admit a coanalytic model. Furthermore, we study the uniqueness of these models (which essentially depends on whether $\alpha(1) = 0$ or $\alpha(1) > 0$), and we obtain explicit models (not just their existence).

The second block of results is about the study of models up to similarity. In this case, we can enlarge the family of functions $\alpha$ considered by allowing them to have zeroes on the unit disc $\mathbb{D}$. The condition that $\alpha$ does not vanish on $\mathbb{D}$ is essential in the study of unitarily equivalent coanalytic models, since there the function $k(t) = 1/\alpha(t)$ plays a key role by giving reproducing kernel Hilbert spaces.

Finally, the third important block of results is about consequences derived from the obtainment of our models. On one hand, we get consequences when the *defect space* (a natural generalization of the defect space on Nagy and Foiaş to our context) has finite dimension. On the other hand, we obtain ergodic consequences for $a$-contractions (operators $T$ such that $(1-t)^a(T^*, T) \geq 0$) when $0 < a < 1$.

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# Contents

# Chapter 0

# Introducción (Spanish)

A lo largo de este texto, $\mathcal{H}$ será un espacio de Hilbert complejo separable. Denotamos por $\boldsymbol{B}(\mathcal{H})$ el espacio de operadores lineales y acotados en $\mathcal{H}$. Recordemos que $T \in \boldsymbol{B}(\mathcal{H})$ es una *contracción* si $\|T\| \leq 1$. Es inmediato que esto es equivalente a la desigualdad de operadores

$$I - T^*T \geq 0,$$

donde $I$ es el operador identidad en $\mathcal{H}$ y $T^*$ denota (como es usual) el operador adjunto de $T$.

Este trabajo tiene dos motivaciones principales: la teoría clásica de Nagy-Foiaş de modelos de contracciones en un espacio de Hilbert, elegantemente resumida en el libro [89] (cuya primera edición apareció en 1967), y la teoría más reciente de Agler sobre modelos de clases de operadores definidos por desigualdades hereditarias, ver el libro Agler – McCarthy [7] publicado en 2002.

Cualquier contracción $T$ se representa como una suma ortogonal de dos partes, $T = T_1 \oplus T_2$, donde $T_1$ es un operador unitario y $T_2$ es un *operador completamente no unitario* (es decir, no existe ningún subespacio reductor no nulo $\mathcal{L}$ de $T_2$ tal que $T_2|\mathcal{L}$ sea unitario). Como los operadores unitarios son normales, y para ellos tenemos el Teorema Espectral, la Teoría de Nagy-Foiaş se centra sólo en las contracciones completamente no unitarias. Dada una contracción completamente no unitaria $T$, los objetos centrales de esta teoría son los siguientes:

- El *operador defecto*, dado por

  $$D_T : \mathcal{H} \to \mathcal{H}, \quad D_T := (I - T^*T)^{1/2}.$$

  Como $T^*$ es también una contracción, de la misma manera se obtiene el operador defecto $D_{T^*}$.

- El *espacio defecto*, denotado por $\mathfrak{D}_T$, que es la clausura del rango de $D_T$. Es decir, $\mathfrak{D}_T$ es un subespacio cerrado de $\mathcal{H}$. De la misma forma se define $\mathfrak{D}_{T^*}$.

- La *función característica* $\Theta_T$, dada por

$$\Theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T] \,|\, \mathfrak{D}_T, \qquad |z| < 1.$$

  Pertenece a $\mathcal{H}^\infty(\boldsymbol{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*}))$; es decir, es una función analítica acotada que toma valores en $\boldsymbol{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ (el conjunto de operadores lineales acotados de $\mathfrak{D}_T$ en $\mathfrak{D}_{T^*}$). Resulta que $\Theta_T$ es contractiva, es decir,

$$\|\Theta_T\|_\infty := \sup_{|z|<1} \|\Theta_T(z)\| \le 1.$$

- Con la ayuda de la función característica $\Theta_T$, se definen el *espacio modelo* $K_{\Theta_T}$ y el *operador modelo* $M_{\Theta_T}$ definido en $K_{\Theta_T}$. Como las definiciones precisas no son inmediatas, quizá sea mejor tener la siguiente idea en mente: el espacio modelo $K_{\Theta_T}$ se construye usando el espacio de Hardy de funciones con valores vectoriales $H^2$ de funciones en el círculo unidad $\mathbb{T}$, mientras que el operador modelo $M_{\Theta_T}$ es, esencialmente, la multiplicación por la variable independiente $z$.

Nagy y Foiaş demostraron que si $T$ es una contracción completamente no unitaria, entonces es unitariamente equivalente a $M_{\Theta_T}$. También obtuvieron el enunciado recíproco.

Una manera natural de tratar de extender esta teoría es la siguiente. Sea $\alpha(t)$ una función representable en serie de potencias $\sum_{n=0}^\infty \alpha_n t^n$ en $\{|t| < r\}$ para algún radio $r = r(\alpha) > 0$. Ponemos

$$\alpha(T^*, T) := \sum_{n=0}^\infty \alpha_n T^{*n} T^n,$$

donde la serie se asume que converge en SOT (la topología fuerte de operadores). Cuando $\alpha$ es un polinomio, la serie de arriba es simplemente una suma finita, así que no hay ningún problema de convergencia. Una vez que sabemos que $\alpha(T^*, T) \in \boldsymbol{B}(\mathcal{H})$, la teoría de Nagy-Foiaş motiva las siguientes preguntas. Si $\alpha(T^*, T) \ge 0$,

- es posible obtener un modelo funcional para $T$?

- Si tal modelo existe, qué consecuencias espectrales se pueden derivar de él?

La teoría original de Nagy-Foias corresponde al caso $\alpha(t) = 1 - t$. De hecho, para este $\alpha$, $\alpha(T^*, T) = I - T^*T$, así que $\alpha(T^*, T) \ge 0$ si y sólo si $T$ es una contracción.

Existen muchos trabajos en este tema. Empezamos con un pequeño repaso de algunos de ellos para tener una visión de cómo encaja esta tesis en ese contexto.

## Breve repaso del tema

En su influyente trabajo [5], Agler probó que si $T$ tiene espectro $\sigma(T)$ contenido en el disco unidad $\mathbb{D} = \{|z| < 1\}$, entonces es natural modelar $T$ por partes de $B \otimes I_{\mathcal{E}}$, donde $B$ es un backward shift con peso y $I_{\mathcal{E}}$ es el operador identidad en un espacio de Hilbert auxiliar $\mathcal{E}$. (Por una *parte de un operador* entendemos su restricción a un subespacio invariante.) Más en general, cuando $\sigma(T) \subset \overline{\mathbb{D}}$, se ha visto en varios casos particulares que en vez de $B \otimes I_{\mathcal{E}}$ se deben considerar operadores de la forma $(B \otimes I_{\mathcal{E}}) \oplus S$, donde $S$ es una isometría o un operador unitario. Esta representación se llama *modelo coanalítico.* Como demostró Agler en [6], esto funciona, en particular, para *hipercontracciones de orden* $m$, es decir, operadores $T \in \boldsymbol{B}(\mathcal{H})$ tales que $(1-t)^j(T^*, T) \geq 0$ para $j = 1, \ldots, m$. El teorema de Agler se generalizó en [67] por Müller y Vasilescu a tuplas de operadores.

En [65], Müller estudió el caso donde $\alpha = p$ es un polinomio. Consideró la clase $\mathcal{C}(p)$ de operadores $T \in \boldsymbol{B}(\mathcal{H})$ tales que $p(T^*, T) \geq 0$. Demostró que cualquier contracción $T \in \mathcal{C}(p)$ tiene un modelo coanalítico siempre que $p(1) = 0, 1/p(t)$ sea analítica en $\mathbb{D}$, y $1/p(\bar{w}z)$ sea un núcleo reproductor. Esta última condición es equivalente al hecho de que todos los coeficientes de Taylor de $1/p(t)$ en el origen sean positivos. Müller también considera algunas desigualdades de operadores para $T$ con infinitos términos, con la misma propiedad de positividad. Esto le permite probar que cualquier operador $T$ es unitariamente equivalente a una parte de un backward shift con peso con el mismo radio espectral (ver [65, Corollary 2.3]).

Los primeros resultados sobre las técnicas del modelo de Agler se exponen en el libro [7] (2002) de Agler y McCarthy.

Olofsson trabajó con $m$-hipercontracciones en [71, 72] y otros artículos. En [71] obtiene, entre otras cosas, fórmulas de operadores para subespacios invariantes, relevantes en los modelos de $m$-hipercontracciones. Sus resultados fueron generalizados por Ball y Bolotnikov (ver [14]) a lo que ellos llaman *$\beta$-hipercontracciones*, que están restringidas a cierta sucesión infinita de desigualdades de operadores.

En [74], Olofsson trabaja con el caso donde $\alpha$ no es un polinomio. Sus hipótesis son que $\alpha$ sea analítica en $\mathbb{D}$, que no se anule en $\mathbb{D}$, y que $1/\alpha$ tenga los coeficientes de Taylor en el origen positivos. Bajo este contexto, estudia contracciones $T$ en $\mathcal{H}$ tales que $\alpha(rT^*, rT) \geq 0$ para todo $r \in [0, 1)$. Con más hipótesis, obtiene un modelo coanalítico para esta clase de operadores.

Hay muchos trabajos en este contexto para tuplas de operadores que conmutan, como [9, 11, 25, 26, 27, 67, 77]. En [77] Pott considera *polinomios regulares positivos.* Éstos son polinomios en varias variables complejas con coeficientes no negativos tales que el término constante es nulo y los coeficientes de los términos lineales son positivos. Dado tal polinomio $p$, Pott construye un modelo de dilatación para tuplas de operadores que conmutan $T = (T_1, \ldots, T_n)$, satisfaciendo las condiciones de positividad $(1-p)^k(T^*, T) \geq 0$ para $1 \leq k \leq m$, donde $m$ y $n$

son enteros positivos (ver [77, Theorem 3.8]). En [27], Bhattacharyya y Sarkar definieron la función característica para esta clase de tuplas y construyeron un modelo funcional en el caso puro (ver el Teorema 4.2 de ese trabajo). Ver también [25] y [26].

Modelos más generales para tuplas de operadores, sujetos a una desigualdad de operadores polinomial, se obtuvieron en el influyente artículo [9] de Ambrozie, Engliš y Müller. En [11], Arazy y Engliš extendieron estos resultados a desigualdades no polinomiales.

Recientemente, Bhattacharjee y Sarkar [24] y Eschmeier [42] han extendido los resultados de Olofsson [71] a tuplas de operadores $m$-hipercontractivas. El reciente arxiv [43] de Eschmeier y Toth contiene resultados más generales.

En [23], se discuten los modelos de contracciones "row" en el espacio de Drury-Arveson de la bola y en "espacios de Hilbert analíticos contractivos" más generales, y sus relaciones con subespacios "wandering" y funciones internas.

También hay un intenso trabajo sobre el caso de tuplas de operadores no conmutativos. Referimos al lector a los trabajos de Ball y Bolotnikov [15] y Popescu [76] para los avances hasta la fecha.

## Nuestro planteamiento

Ahora haremos más preciso el planteamiento en el que se enmarcará esta tesis. Todos los operadores que consideraremos serán lineales y acotados en espacios de Hilbert. Todas las funciones que consideraremos serán representables en series de potencias en discos centrados en el origen, digamos

$$\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n \qquad (|t| < r)$$

para cierto radio $r = r(\alpha) > 0$. Nos referiremos al coeficiente $\alpha_n$ como *el $n$-esimo coeficiente de Taylor de $\alpha$*.

Denotemos por $A_W$ el álgebra de Wiener de funciones analíticas en el disco unidad $\mathbb{D}$ cuyos coeficientes de Taylor son sumables. Es decir, $f \in A_W$ si y sólo si

$$f(t) = \sum_{n=0}^{\infty} f_n t^n \qquad (|t| \le 1)$$

donde $\sum |f_n| < \infty$.

Supongamos que $\alpha \in A_W$ y sea $T$ un operador en $\boldsymbol{B}(\mathcal{H})$ con espectro $\sigma(T) \subset \overline{\mathbb{D}}$. Si la serie

$$\sum_{n=0}^{\infty} |\alpha_n| T^{*n} T^n$$

converge en la topología fuerte en $\boldsymbol{B}(\mathcal{H})$, decimos que

$$\alpha \in \mathcal{A}_T, \qquad \text{or} \qquad T \in \mathrm{Adm}_\alpha\,.$$

Dependiendo de si queremos fijar el operador $T$ o la función $\alpha$, usaremos una notación u otra. Entonces,

$$\mathcal{A}_T \subset A_W \qquad \text{and} \qquad \mathrm{Adm}_\alpha \subset \boldsymbol{B}(\mathcal{H}).$$

Resulta que (ver Proposición 1.2) si $\alpha \in \mathcal{A}_T$, entonces la serie $\sum \alpha_n T^{*n}T^n$ (sin valores absolutos) también converge en la topología fuerte en $\boldsymbol{B}(\mathcal{H})$. Por tanto, podemos definir el operador

$$\alpha(T^*,T) := \sum_{n=0}^{\infty} \alpha_n T^{*n}T^n \ \in \boldsymbol{B}(\mathcal{H}).$$

Denotamos por $A_{W,\mathbb{R}}$ y $\mathcal{A}_{T,\mathbb{R}}$ a los subconjuntos de funciones en $A_W$ y $\mathcal{A}_T$, respectivamente, que tienen coeficientes de Taylor reales. Usaremos la notación $\mathrm{Adm}_\alpha$ sólo cuando $\alpha$ tenga coeficientes de Taylor reales. Entonces el operador $\alpha(T^*,T)$ es autoadjunto. Cuando $\alpha(T^*,T)$ es un operador positivo, decimos que $T \in \mathbf{C}_\alpha$. Es decir,

$$\mathbf{C}_\alpha := \{T \in \mathrm{Adm}_\alpha \,:\, \alpha(T^*,T) \geq 0\}\,.$$

Esta notación $\mathbf{C}_\alpha$ será muy útil para estudiar esas clases de operadores y las relaciones entre ellas. Pero normalmente, sólo escribiremos $\alpha(T^*,T) \geq 0$ en vez de $T \in \mathbf{C}_\alpha$. En este caso, ponemos

$$D := (\alpha(T^*,T))^{1/2},$$

donde se toma la raíz cuadrada no negativa. Denotamos por $\mathfrak{D}$ la clausura del rango de $D$. El operador $D$ y el espacio $\mathfrak{D}$ jugarán el mismo papel que el *operador defecto* y el *espacio defecto*, respectivamente, en la teoría de Nagy-Foiaş.

## Resultados principales por capítulos

### Resultados principales del Capítulo 1

En el primer capítulo nos concentramos en la construcción de un modelo funcional respecto de la equivalencia unitaria para un operador $T$ satisfaciendo $\alpha(T^*,T) \geq 0$. Consideramos las siguientes hipótesis.

**Hipótesis 0.1.** *Sea $\alpha$ una función en $A_{W,\mathbb{R}}$ que no se anula en $\mathbb{D}$, y pongamos*

$$k(t) = 1/\alpha(t) = \sum_{n=0}^{\infty} k_n t^n \qquad t \in \mathbb{D},$$

*con $\alpha_0 = k_0 = 1$, y $k_n > 0$ para todo $n \geq 1$.*

Bajo las Hipótesis 0.1, denotamos por $\mathcal{H}_k$ el espacio de Hilbert con pesos de series de potencias $f(t) = \sum_{n=0}^{\infty} f_n t^n$ con norma finita

$$\|f\|_{\mathcal{H}_k} := \left( \sum_{n=0}^{\infty} |f_n|^2 k_n \right)^{1/2}.$$

Sea $B_k$ el *shift hacia atrás* en $\mathcal{H}_k$, definido por

$$B_k f(t) = \frac{f(t) - f(0)}{t}. \tag{0.1}$$

**Definición 0.2.** Fijemos una función $\alpha$ que satisface las Hipótesis 0.1, y sea $T$ un operador en $\boldsymbol{B}(\mathcal{H})$. Diremos que $T$ es *$\alpha$-modelable* si $\alpha(T^*, T) \geq 0$ y $T$ es unitariamente equivalente a una parte de un operador de la forma $(B_k \otimes I_{\mathcal{E}}) \oplus S$, donde $S$ es una isometría.

Notar que $B_k \otimes I_{\mathcal{E}}$ actúa sobre el espacio de Hilbert $\mathcal{H}_k \otimes \mathcal{E}$, que se puede identificar con el espacio de Hilbert con pesos de series de potencias con valores en $\mathcal{E}$, $f(t) = \sum_{n=0}^{\infty} f_n t^n$ con norma dada por

$$\|f\|_{\mathcal{H}_k \otimes \mathcal{E}} = \left( \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{E}}^2 \, k_n \right)^{1/2}.$$

Actúa de acuerdo a la misma fórmula (0.1).

Es natural plantear la siguiente pregunta.

**Pregunta 0.3.** Dada una función $\alpha$ que satisface la Hipótesis 0.1, dar una buena condición suficiente para que un operador $T \in \boldsymbol{B}(\mathcal{H})$ sea $\alpha$-modelable.

Uno de los resultados más fuertes en esta dirección se contiene en los trabajos recientes de Bickel, Hartz y McCarthy [28] y de Clouâtre y Hartz [32]. Está formulado para tuplas simétricas esféricas. Para el caso de un sólo operador, se puede formular de la siguiente manera.

**Teorema CH** (Clouâtre y Hartz [32, Teorema 1.3])**.** *Sea $\alpha$ una función con $\alpha_0 = 1$ y $\alpha_n \leq 0$ para todo $n \geq 1$. Supongamos que $k = 1/\alpha$ tiene radio de convergencia 1, $k_n > 0$ para todo $n \geq 0$ y*

$$\lim_{n \to \infty} \frac{k_n}{k_{n+1}} = 1. \tag{0.2}$$

*Entonces $B_k$ está acotado, y para cualquier operador $T$ en un espacio de Hilbert, las siguientes afirmaciones son equivalentes.*

(i) *$T$ satisface $\alpha(T^*, T) \geq 0$.*

(ii) *$T$ es unitariamente equivalente a una parte de $(B_k \otimes I_{\mathcal{E}}) \oplus S$, donde $S$ es una isometría en $\mathcal{W}$. (Aquí $\mathcal{E}$ and $\mathcal{W}$ son espacios de Hilbert auxiliares.)*

Las funciones $\alpha(t) = (1-t)^a$, donde $a > 0$, satisfacen las Hipótesis 0.1. Si $(1-t)^a(T^*, T) \geq 0$, entonces decimos que $T$ es una *$a$-contracción*.

Notar que el Teorema CH concierne al *caso Nevannlina-Pick* (es decir, cuando $\alpha_0 = 1$ y $\alpha_n \leq 0$ para $n \geq 1$). Un ejemplo relevante es $\alpha(t) = (1-t)^a$, donde $0 < a < 1$. Clouâtre y Hartz lo mencionan como un caso importante. Por tanto, cualquier $a$-contracción, con $0 < a < 1$, es unitariamente equivalente a una parte de un operador de la forma $(B_k \otimes I_{\mathcal{E}}) \oplus S$, donde $\mathcal{E}$ es un espacio de Hilbert auxiliar y $S$ es una isometría. Podemos probar que en este caso el espacio $\mathcal{E}$ se puede escoger como $\mathfrak{D}$. Nuestro enfoque en la demostración es bastante distinto y más directo. El enunciado preciso es el siguiente.

**Teorema 0.4.** *Si $0 < a < 1$, entonces los siguientes enunciados son equivalentes.*

(i) *$T$ es una $a$-contracción.*

(ii) *Existe un espacio de Hilbert separable $\mathcal{E}$ tal que $T$ es unitariamente equivalente a una parte de un operador $(B_k \otimes I_{\mathcal{E}}) \oplus S$, donde $S$ es una isometría en un espacio de Hilbert.*

*Además, si se cumple* (ii), entonces se puede tomar como $\mathcal{E}$ es espacio $\mathfrak{D} = \overline{D\mathcal{H}}$.

Al final del Capítulo 1 discutiremos brevemente el caso $a > 1$ (ver el Teorema 1.25 y la Proposición 1.26).

El resultado principal del Capítulo 1 es el siguiente.

**Teorema 0.5.** *Supongamos las Hipótesis 0.1. Si $k \in A_{W,\mathbb{R}}$ y sus coeficientes de Taylor $\{k_n\}$ satisfacen $k_n^{1/n} \to 1, \sup k_n/k_{n+1} < \infty$ y*

$$\lim_{m\to\infty} \sup_{n\geq 2m} \sum_{m\leq j\leq n/2} \frac{k_j k_{n-j}}{k_n} = 0, \tag{0.3}$$

*entonces $B_k$ está acotado y el operador $T \in \boldsymbol{B}(\mathcal{H})$ es una parte de $B_k \otimes I_{\mathcal{E}}$ (para algún espacio de Hilbert $\mathcal{E}$) si y sólo si $\alpha$ y $k$ pertenecen a $\mathcal{A}_T$, y $\alpha(T^*, T) \geq 0$. Además, en este caso podemos tomar $\mathcal{E} = \mathfrak{D}$.*

Notar que en el teorema anterior, la parte isométrica $S$ es innecesaria (ver el Teorema 0.10 abajo para más información). El Teorema 0.5 muestra que la representación de arriba de $T$ existe en muchos casos en los que $k$ no es un núcleo de Nevanlinna-Pick, y por tanto no se puede aplicar el Teorema CH. No se conocía mucho sobre estos núcleos con anterioridad.

Dado un entero $N \geq 2$, hay ejemplos de funciones $k$ que satisfacen las hipótesis del Teorema 0.5 con cualesquiera signos prescritos de los coeficientes $\alpha_2, \ldots, \alpha_N$ (ver el Ejemplo 1.19). Notar que $\alpha_1 = -k_1$ siempre es negativo.

**Comentario 0.6.** Supongamos que la sucesión

$$\left\{\frac{k_n}{k_{n+1}}\left(1+\frac{1}{n+1}\right)^a\right\}$$

es creciente para algún $a > 1$. Entonces se cumple (0.3). Esto se parece a [87, Proposition 34]. De hecho, pongamos $k_j^* := (j+1)^{-a}$ y definamos $\rho_j := k_j/k_j^*$. Entonces nuestra condición se reduce a la condición $\rho_{n+2}/\rho_{n+1} \geq \rho_{n+1}/\rho_n$, para todo $n$, lo cual implica que $\rho_j\rho_{n-j}/\rho_n \leq C$, para $o \leq j \leq n$. Como

$$\frac{k_j k_{n-j}}{k_n} = \frac{\rho_j \rho_{n-j}}{\rho_n}\,\frac{k_j^* k_{n-j}^*}{k_n^*}$$

y $\{k_n^*\}$ satisface (0.3), se sigue que $\{k_n\}$ también satisface (0.3).

Por tanto, para sucesiones suficientemente regulares $\{k_n\}$, la condición (0.3) es equivalente a la condición $\sum k_n < \infty$. Se puede añadir que, de hecho, en el Teorema 0.5 $\{k_n\}$ no tiene por qué ser regular; más aún, los cocientes $k_n/k_{n+1}$ no tienen por qué converger (ver el Remark 1.20).

Las técnicas empleadas en la demostración son diferentes a [32]. Nosotros usamos, básicamente, una combinación de los argumentos de Müller en [65] y técnicas de álgebras de Banach.

Los teoremas de arriba abren la cuestión de describir los subespacios invariantes de $B_k \otimes I_{\mathcal{E}}$ y de construir un modelo funcional de los operadores que estudiamos, lo cual ciertamente sería interesante. No mencionaremos esa cuestión en esta tesis. En el trabajo reciente [33], Clouâtre, Hartz y Schillo establecen un teorema de tipo Beurling–Lax–Halmos para espacios de Hilbert con núcleo reproductivo en el contexto Nevannlina-Pick. Referimos al lector a [73, 37, 84, 85] para más resultados en el caso Nevannlina-Pick.

Siempre que $T$ sea $\alpha$-modelable, el operador $V_D$ dado por

$$V_D : \mathcal{H} \to \mathcal{H}_k \otimes \mathfrak{D}, \qquad V_D x(z) = D(I_{\mathcal{H}} - zT)^{-1}x, \quad x \in \mathcal{H}, \quad z \in \mathbb{D},$$

es una contracción, y podemos dar un modelo explícito para $T$ (es decir, dar explícitamente $\mathcal{E}$, $S$ y la transformación que manda el espacio inicial en el espacio modelo).

**Teorema 0.7** (Modelo explícito)**.** *Si $T$ es $\alpha$-modelable, entonces $V_D$ es una contracción, y entonces podemos definir*

$$W = (I_{\mathcal{H}} - V_D^* V_D)^{1/2}, \quad \mathcal{W} = \overline{W\mathcal{H}}.$$

*Además, $S : \mathcal{W} \to \mathcal{W}$, dado por $SWx := WTx$, es una isometría y el operador*

$$(V_D, W) : \mathcal{H} \to (\mathcal{H}_k \otimes \mathfrak{D}) \oplus \mathcal{W}, \qquad (V_D, W)h = (V_D h, Wh)$$

*da un modelo de $T$, en el sentido que $(V_D, W)$ es isométrico y*

$$((B_k \otimes I_{\mathfrak{D}}) \oplus S)(V_D, W) = (V_D, W)T.$$

Si se sabe que $T$ es $\alpha$-modelable, se puede preguntar sobre la unicidad del modelo. Para responder a esta pregunta, necesitamos las siguientes definiciones.

**Definición 0.8.** Sea $\mathcal{L}$ un subespacio invariante de $(B_k \otimes I_{\mathcal{E}}) \oplus S$, donde $S : \mathcal{W} \to \mathcal{W}$ es una isometría. Diremos que el correspondiente operador modelo

$$\big((B_k \otimes I_{\mathcal{E}}) \oplus S\big)|\mathcal{L}$$

es *minimal* si se cumplen las siguientes dos condiciones.

(i) $\mathcal{L}$ no está contenido en $(\mathcal{H}_k \otimes \mathcal{E}') \oplus \mathcal{W}$ para ningún $\mathcal{E}' \subsetneqq \mathcal{E}$.

(ii) $\mathcal{L}$ no está contenido en $(\mathcal{H}_k \otimes \mathcal{E}) \oplus \mathcal{W}'$ para ningún $\mathcal{W}' \subsetneqq \mathcal{W}$ invariante por $S$.

En el Remark 1.15 probamos que el modelo explícito obtenido en el Teorema 0.7 es de hecho minimal.

Notar que bajo las Hipótesis 0.1, $\alpha$ está definida en el círculo unidad $\mathbb{T}$, y no se anula en el intervalo $[0, 1)$. Como $\alpha(0) = \alpha_0 = 1$, obtenemos que $\alpha(1) \geq 0$.

Distinguimos los siguientes dos casos. Esta distinción ya aparece en [32, Subsection 2.3] para el caso Nevannlina-Pick.

**Definición 0.9.** Supongamos que $\alpha$ satisface las Hipótesis 0.1. Diremos que $\alpha$ es de *tipo crítico* (o, alternativamente, que tenemos el *caso crítico*) si $\alpha(1) = 0$. Si $\alpha(1) > 0$, diremos que $\alpha$ es de *tipo subcrítico* (o, alternativamente, que tenemos el *caso subcrítico*). Recordemos que $\alpha(1)$ no puede ser negativo.

**Teorema 0.10** (Unicidad del modelo minimal)**.** *Supongamos que $T$ es $\alpha$-modelable.*

(i) *En el caso crítico, el modelo minimal de $T$ es único y se obtiene tomando $V = V_D$. Más precisamente, el par de transformaciones $(V_D, W_0)$, donde $W_0 = (I - V_D^* V_D) : H \to \mathcal{W}_0$ y $\mathcal{W}_0 := \overline{\operatorname{Ran}}(I - V_D^* V_D)$ da un modelo minimal, y cualquier modelo minimal viene dado por $(V_C, W)$, donde $C = vD$, $W = wW_0 : H \to \mathcal{W}$ y $v, w$ son isomorfismos unitarios.*

(ii) *En el caso subcrítico, el modelo minimal de $T$ no es único, en general. Sin embargo, siempre existe un modelo minimal para el cual $V = V_D$ y $W$ no aparece (es decir, $\mathcal{W} = 0$).*

## Resultados principales del Capítulo 2

En el Capítulo 2 estudiamos cuándo la desigualdad $\alpha(T^*, T) \geq 0$ permite obtener un modelo funcional para $T$ bajo semejanza. Cuando tratamos con modelos semejantes en vez de unitarios, el modelo obviamente no da toda la información sobre el operador, sino la información preservada por la semejanza. Una gran ventaja de considerar sólo la semejanza es que podemos ampliar la familia de funciones $\alpha$ que consideramos. En el Capítulo 2, no usamos el núcleo $k = 1/\alpha$. De hecho, ahora permitimos que la función $\alpha \in A_{W,\mathbb{R}}$ tenga ceros en $\mathbb{D}$, excepto en el intervalo $[0, 1)$. Un ingrediente clave para nuestros argumentos es el Teorema 2.7. Con el modelo de semejanza también seremos capaces de extraer consecuencias espectrales.

Si una serie

$$\sum_{n=0}^{\infty} |\alpha_n| T^{*n} T^n$$

converge en la topología uniforme en $\boldsymbol{B}(\mathcal{H})$, diremos que

$$\alpha \in \mathcal{A}_T^0.$$

(Salvo que se afirme lo contrario, estaremos asumiendo implícitamente que $T \in \boldsymbol{B}(\mathcal{H})$ con $\sigma(T) \subset \overline{\mathbb{D}}$.) A veces necesitaremos esta convergencia más fuerte.

Obviamente, $\mathcal{A}_T^0$ está contenido en $\mathcal{A}_T$. De hecho, $\mathcal{A}_T$ es un álgebra de Banach (ver el Teorema 2.2), y $\mathcal{A}_T^0$ es su sub-álgebra separable cerrada (ver la Proposición 2.3).

Nuestro resultado principal sobre semejanza es el siguiente.

**Teorema 0.11.** *Supongamos que $\alpha$ es de la forma*

$$\alpha(t) = \eta(t)\, \widetilde{\alpha}(t),$$

*donde $\eta \in \mathcal{A}_T$ y $\widetilde{\alpha} \in \mathcal{A}_T^0$ es positivo en el intervalo $[0, 1]$. Si $\alpha(T^*, T) \geq 0$, entonces $T$ es semejante a un operador $\widehat{T}$ tal que $\eta(\widehat{T}^*, \widehat{T}) \geq 0$.*

En otras palabras, con este teorema podemos prescindir (en términos de la semejanza) del factor $\widetilde{\alpha}$. La cuestión de obtener un modelo para $T$ bajo semejanza se reduce a obtener un modelo para $\widehat{T}$ (como arriba).

Un caso importante es $\eta(t) := (1-t)^a$ para $a > 0$. Cuando $a = m$ es un entero positivo, si $(1-t)^m(T^*, T) \geq 0$ entonces $T$ se dice que es una *$m$-contracción*, y si $(1-t)^m(T^*, T) = 0$ entonces $T$ se dice que es una *$m$-isometría*. Por ejemplo, las 1-contracciones y las 1-isometrías son, precisamente, las contracciones y las isometrías (respectivamente) en $\boldsymbol{B}(\mathcal{H})$.

Notar que si $\eta(t) = 1 - t$, entonces obviamente $\eta \in \mathcal{A}_T$ (ya que es un polinomio). Luego, como un caso particular importante del teorema anterior, obtenemos el siguiente resultado.

**Corolario 0.12.**

(i) *Supongamos que $\alpha$ es de la forma*

$$\alpha(t) = (1-t)^a\, \widetilde{\alpha}(t),$$

*para algún $a > 0$, donde $(1-t)^a \in \mathcal{A}_T$ y $\widetilde{\alpha} \in A_T^0$ es positivo en el intervalo $[0, 1]$. Si $\alpha(T^*, T) \geq 0$, entonces $T$ es semejante a una $a$-contracción.*

(ii) *Supongamos que $\alpha$ es de la forma*

$$\alpha(t) = (1-t)\,\widetilde{\alpha}(t),$$

*donde $\widetilde{\alpha} \in A_T^0$ es positivo en el intervalo $[0,1]$. Si $\alpha(T^*,T) \geq 0$, entonces $T$ es semejante a una contracción.*

Hay muchos trabajos sobre el estudio de $m$-contracciones y $m$-isometrías. Aquí introducimos el caso cuando el exponente $a$ no es entero. Las definiciones de $a$-contracción y $a$-isometría son las naturales. Con la ayuda del Teorema 0.4, obtendremos el siguiente resultado ergódico.

**Teorema 0.13.** *Si $T$ es una $a$-contracción, con $0 < a < 1$, entonces $T$ es cuadráticamente $(C,b)$-acotado para todo $b > 1-a$.*

Que $T$ sea *cuadráticamente $(C,b)$-acotado* (donde la letra C hace referencia a Cesàro) significa que existe una constante $c > 0$ tal que

$$\sup_{n\geq 0} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 \leq c\|x\|^2 \qquad (\forall x \in \mathcal{H}),$$

donde los números $k^{-s}(n)$, llamados *números de Cesàro*, se definen como

$$(1-t)^s =: \sum_{n=0}^{\infty} k^{-s}(n)t^n.$$

La propiedad de ser cuadráticamente $(C,b)$-acotado se hereda por semejanza. Por tanto, podemos combinar los teoremas 0.11 y 0.13 con $\eta(t) = (1-t)^a$ y $0 < a < 1$.

Al final del Capítulo 2, estudiamos más propiedades ergódicas de las $a$-contracciones. Por ejemplo, el siguiente teorema da una conexión interesante entre el modelo funcional y una propiedad ergódica. Recordemos el modelo obtenido en el Teorema 0.4.

**Teorema 0.14.** *Sea $T$ una $a$-contracción con $0 < a < 1$ y sea $b > 1-a$. Entonces las siguientes afirmaciones son equivalentes.*

(i) *La isometría $S$ no aparece en el $(1-t)^a$-modelo de $T$.*

(ii) *Para todo $x \in \mathcal{H}$,*

$$\exists \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 = 0.$$

(iii) *Para todo $x \in \mathcal{H}$,*

$$\liminf_{n\to\infty} \|T^n x\| = 0.$$

En este capítulo también probamos que si $\alpha$ es de tipo subcrítico, $\sigma(T) \neq \overline{\mathbb{D}}$, y $\alpha(T^*, T) \geq 0$ es de rango finito, entonces las longitudes $l_\nu$ de los intervalos complementarios del conjunto $\overline{(\sigma(T) \cap \mathbb{D})} \cap \mathbb{T}$ en el círculo unidad $\mathbb{T}$ satisfacen la condición de Carleson $\sum_\nu l_\nu \log(2\pi/l_\nu) < \infty$. Ver el Teorema 2.20.

### Resultados principales del Capítulo 3

Cuando $\alpha(t) := (1-t)\widetilde{\alpha}(t)$, donde $\widetilde{\alpha} \in A_{W,\mathbb{R}}$ es positiva en el intervalo $[0,1]$, diremos que $\alpha$ es una *función admisible*. En este capítulo nos concentraremos en funciones admisibles $\alpha$. Diremos que una función $\alpha$ es *fuertemente admisible* cuando sea admisible y además $\widetilde{\alpha}$ no se anule en la circunferencia unidad $\mathbb{T} = \{|t| = 1\}$.

Dado un operador $T$ en $\mathbf{C}_\alpha$, donde $\alpha$ es fuertemente admisible, en el Teorema 3.20 construimos su modelo de tipo Nagy-Foias explícito.

Empezamos con una demostración completamente distinta del Corolario 0.12 (ii) usando *operadores defecto abstractos*. Una de las ventajas de esta prueba es que la conexión con el modelo de Nagy-Foias se manifiesta de forma más clara.

Recordar que el modelo de Nagy-Foias se construye sólo para contracciones completamente no unitarias. Uno de los pasos importantes en nuestra construcción es dar una definición de un análogo de la parte unitaria de $T$ en el contexto de operadores en $\mathbf{C}_\alpha$ (ver la Subsección 3.3.1).

Como una aplicación de este modelo, probamos que una gran parte de las consecuencias espectrales del modelo de Nagy-Foias se siguen cumpliendo cuando el defecto $D$ tiene rango finito o pertenece a las clases de Schatten-von Neumann $\mathfrak{S}_p$ $(0 < p \leq \infty)$.

Es obvio que para cualquier contracción $T$ existen los límites $\lim_{n\to\infty} \|T^n h\|$ para todo $h \in \mathcal{H}$. Ahora, si $T \in \mathbf{C}_\alpha$ (para algún $\alpha$ admisible), entonces es semejante a una contracción. Por tanto, es natural preguntarse si en este caso también existen los límites $\lim_{n\to\infty} \|T^n h\|$. Veremos que, en general, la respuesta es no (ver el Remark 3.25). Sin embargo, obtenemos una respuesta afirmativa cuando $\alpha$ es fuertemente admisible.

**Teorema 0.15.** *Si $\alpha$ es una función fuertemente admisible y $T$ pertenece a $\mathbf{C}_\alpha$, entonces existen los límites $\lim_{n\to\infty} \|T^n h\|$ para todo $h \in \mathcal{H}$.*

También discutiremos algunas propiedades elementales de las clases $\mathbf{C}_\alpha$ cuando $\alpha$ es una función admisible. En particular, en el Lemma 3.13 probamos que "casi todo" operador diagonalizable $T$ en un espacio finito dimensional $\mathcal{H}$, cuyo radio espectral no sea mayor que 1, pertenece a una clase $\mathbf{C}_\alpha$ para alguna función $\alpha$ admisible.

Al final del capítulo, probaremos un resultado de semejanza para funciones que se anulan en el origen. La idea clave es usar un teorema de Cassier y Suciu sobre *$n$-quasicontracciones*.

### Resultados principales del Capítulo 4

Dadas dos funciones $\alpha$, $\tau$ que satisfacen las Hipótesis 0.1, en este capítulo estudiaremos cuándo la clase $\mathbf{C}_\alpha$ está contenida en $\mathbf{C}_\tau$. Los resultados principales obtenidos en esta dirección son los siguientes.

En la Sección 4.1, damos algunos resultados para funciones $\alpha$ y $\tau$ generales. En particular, en el Teorema 4.3 probamos que para funciones admisibles $\alpha$, $\beta$, se tiene que $\mathbf{C}_\alpha \subset \mathbf{C}_\beta$ si y sólo si la función meromorfa $\beta/\alpha$, que es analítica en el origen, tiene coeficientes de Taylor no negativos.

En la Sección 4.2, nos concentramos en las inclusiones de las clases de $a$-contracciones. Lo primero de todo, notar que si ponemos $\alpha(t) = (1-t)^a$ y $\beta(t) = (1-t)^b$, entonces la condición de arriba se cumple para $0 < b < a$, porque en este caso los coeficientes de Taylor de la función

$$\beta(t)/\alpha(t) = (1-t)^{b-a}$$

son positivos. Sin embargo, el Teorema 4.3 no es aplicable en este caso, ya que esas fucniones $\alpha$, $\beta$ son admisibles sólo cuando $a = b = 1$.

Mostraremos que, de hecho, la clase de $a$-contracciones no está contenida en la clase de $b$-contracciones siempre que $0 < b < a$, ver la Proposición 4.20.

Cuando $\alpha(t) = (1-t)^a$, usaremos la notación $\mathrm{Adm}_a$ y $\mathbf{C}_a$ en vez de $\mathrm{Adm}_\alpha$ y $\mathbf{C}_\alpha$, respectivamente. Es decir, $\mathbf{C}_a$ es el conjunto de $a$-contracciones.

Nuestros resultados principales son los siguientes.

**Teorema 0.16.** *Sea $0 < b < a$ donde $b$ no es un entero. Si $T$ es una $a$-contracción y $T \in \mathrm{Adm}_b$, entonces $T$ es una $b$-contracción.*

**Teorema 0.17.** *Sea $a > 0$, y sea $m$ el entero tal que $m < a \leq m+1$. Entonces las siguientes afirmaciones son equivalentes.*

(i) *$T$ es una $a$-isometría.*

(ii) *$T$ es una $(m+1)$-isometría.*

(iii) *Para cada vector $h \in \mathcal{H}$, existe un polinomio $p$ de grado a lo sumo $m$ tal que $\|T^n h\|^2 = p(n)$ para todo $n \geq 0$.*

La temática de las $a$-contracciones y las $a$-isometrías está íntimamente relacionado con la temática de las diferencias finitas. Dada una sucesión de números reales $\Lambda = \{\Lambda_n\}_{n\geq 0}$, denotamos por $\nabla\Lambda$ la sucesión cuyo término $n$-ésimo, para $n \geq 0$, viene dado por $(\nabla\Lambda)_n = \Lambda_{n+1}$. En general, si $\beta(t) = \sum \beta_n t^n$ es una función analítica, denotamos por $\beta(\nabla)\Lambda$ la sucesión cuyo término $n$-ésimo viene dado por

$$\beta(\nabla)\Lambda_n = \sum_{j=0}^{\infty} \beta_j \Lambda_{j+n},$$

siempre que la serie del lado derecho converja para todo $n \geq 0$. En particular, para las funciones $(1-t)^a$ donde $a \in \mathbb{R}$, ponemos

$$(1-\nabla)^a \Lambda_n = \sum_{j=0}^{\infty} k^{-a}(j)\Lambda_{j+n}.$$

Ésta es la diferencia finita hacia adelante de orden $a$ de la sucesión $\Lambda$. Por ejemplo, para $a = 1$ obtenemos las diferencias finitas de primer orden $(1-\nabla)\Lambda_n = \Lambda_n - \Lambda_{n+1}$. Resaltamos las siguientes dos cuestiones.

(A) Determinar para qué $a, b > 0$ la desigualdad $(1-\nabla)^a \Lambda_n \geq 0$ (para todo $n \geq 0$) implica $(1-\nabla)^b \Lambda_n \geq 0$ (para todo $n \geq 0$).

(B) Dado $a > 0$, hallar el espacio de soluciones $\Lambda$ de la ecuación $(1-\nabla)^a \Lambda = 0$.

En el Teorema 4.14 damos una respuesta a (A), y como una consecuencia inmediata obtenemos el Teorema 0.16. De hecho, la idea clave es fijar un vector $x \in \mathcal{H}$ y trasladar el problema a una cuestión de diferencias finitas tomando $\Lambda_n := \|T^n x\|^2$ para $n \geq 0$.

De la misma forma, una respuesta a (B) se da en el Teorema 4.15, y como consecuencia obtenemos el Teorema 0.17.

## Artículos en los que se basa la tesis

Esta tesis se basa en los siguientes tres artículos:

1. **Operator inequalities implying similarity to a contraction**.
   G. Bello-Burguet and D. Yakubovich,
   Complex Anal. Oper. Theory, 13 (2019), 1325–1360.

2. **Operator inequalities I. Models and ergodicity**.
   L. Abadias, G. Bello-Burguet and D. Yakubovich,
   Arxiv (2019)

3. **Operator inequalities II. Models up to similarity and inclusions of operator classes**.
   L. Abadias, G. Bello-Burguet and D. Yakubovich,
   Work in progress.

En el primer artículo, consideramos funciones admisibles $\alpha$. La convergencia de $\sum \alpha_n T^{*n} T^n$ se asume en la topología uniforme en $\boldsymbol{B}(\mathcal{H})$. Por tanto, las clases de operadores estudiadas allí (que denotamos por $\mathcal{C}_\alpha$) son más pequeñas que $\mathrm{Adm}_\alpha$ y su subfamilia $\mathbf{C}_\alpha$. Los operadores estudiados allí resulta que son siempre semejantes a contracciones (ver el Teorema I en ese artículo). También damos un modelo explícito de los operadores estudiados en el espíritu del modelo de Nagy-Foias, y estudiamos algunas propiedades interesantes de las clases allí consideradas. El Capítulo 3 está basado en ese artículo.

En la tesis, trabajamos en cambio con la convergencia de la serie de arriba en la topología fuerte. Como muestra el Ejemplo 2.25, esta suposición es esencial para tener la equivalencia entre la desigualdad $\alpha(T^*, T) \geq 0$ y la existencia del modelo, como en el Teorema CH. Por tanto, este es el contexto más natural para este problema.

El segundo artículo se centra en el estudio de la equivalencia unitaria de modelos para operadores en $\mathbf{C}_\alpha$. En ese artículo la función $\alpha$ ya no se asume admisible. Como consecuencia, los operadores $T$ en $\mathbf{C}_\alpha$ ya no son semejantes a contracciones.

El tercer artículo es sobre el estudio de modelos bajo semejanza. El marco de esos dos últimos artículos es el que usamos en esta tesis. En esos dos artículos usamos las notaciones $\mathrm{Adm}_\alpha^w$ y $\mathcal{C}_\alpha^w$ en vez de $\mathrm{Adm}_\alpha$ y $\mathbf{C}_\alpha$ (como en esta tesis) para enfatizar que las condiciones de pertenencia a estas clases son más débiles que las del primer artículo.

El Capítulo 1 se basa en el segundo artículo, y el Capítulo 2 en el tercero. La primera sección del Capítulo 4 se basa en el primer artículo (pero en el contexto más general), y la segunda sección del capítulo se basa en el tercer artículo.

# Chapter 0

# Introduction

Through this text, $\mathcal{H}$ will stand for a separable complex Hilbert space. We denote by $\boldsymbol{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. Recall that $T \in \boldsymbol{B}(\mathcal{H})$ is a *contraction* if $\|T\| \leq 1$. It is immediate that this is equivalent to the operator inequality

$$I - T^*T \geq 0,$$

where $I$ is the identity operator in $\mathcal{H}$, and $T^*$ is (as usual) the adjoint operator of $T$.

This work has two main motivations: the classical Nagy-Foiaş theory of models of contractions on a Hilbert space, beautifully summarized in the book [89] (whose first edition appeared in 1967), and more recent Agler theory of models of classes of operators, defined by hereditary inequalities, see the Agler – McCarthy book [7], published in 2002.

Any contraction $T$ can be split into two parts, $T = T_1 \oplus T_2$, where $T_1$ is a unitary operator and $T_2$ is a *completely non-unitary operator* (i.e., there is no nonzero reducing subspace $\mathcal{L}$ for $T_2$ such that $T_2|\mathcal{L}$ is unitary). Since unitary operators are normal, and for them we have the Spectral Theorem, the Nagy-Foiaş Theory focuses only on completely non-unitary contractions. For a given completely non-unitary contraction $T$, the central objects of this theory are the following:

- The *defect operator*, given by

  $$D_T : \mathcal{H} \to \mathcal{H}, \quad D_T := (I - T^*T)^{1/2}.$$

  Since $T^*$ is also a contraction, in the same way, the defect operator $D_{T^*}$ is obtained.

- The *defect space*, denoted by $\mathfrak{D}_T$, which is the closure of the range of $D_T$. That is, $\mathfrak{D}_T$ is a closed subspace of $\mathcal{H}$. In the same way $\mathfrak{D}_{T^*}$ is defined.

- The *characteristic function* $\Theta_T$, given by

$$\Theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T] \,|\, \mathfrak{D}_T, \qquad |z| < 1.$$

  It belongs to $\mathcal{H}^\infty(\boldsymbol{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*}))$; i.e., it is a bounded analytic function which takes values in $\boldsymbol{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ (the set of bounded linear operators from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}$). It turns out that $\Theta_T$ is contractive-valued, i.e.,

$$\|\Theta_T\|_\infty := \sup_{|z|<1} \|\Theta_T(z)\| \le 1.$$

- With the help of the characteristic function $\Theta_T$, the *model space* $K_{\Theta_T}$ and the *model operator* $M_{\Theta_T}$ defined on $K_{\Theta_T}$ are constructed. Since the precise definitions are not immediate, maybe is better to have the following picture: the model space $K_{\Theta_T}$ is constructed using the vector-valued Hardy function spaces $H^2$ of functions on the unit circle $\mathbb{T}$, while the model operator $M_{\Theta_T}$ is, essentially, the multiplication by the independent variable $z$.

Nagy and Foiaş proved that if $T$ is a completely non-unitary contraction, then it is unitarily equivalent to $M_{\Theta_T}$. They also obtained a reciprocal statement.

A natural way to try to extend this theory is the following. Let $\alpha(t)$ be a function representable by the power series $\sum_{n=0}^\infty \alpha_n t^n$ in $\{|t| < r\}$ for some radius $r = r(\alpha) > 0$. Then we put

$$\alpha(T^*, T) := \sum_{n=0}^\infty \alpha_n T^{*n} T^n,$$

where the series is assumed to converge in SOT (the strong operator topology). When $\alpha$ is a polynomial, the series above is just a finite sum, so that there is no convergence problem. Once it is known that $\alpha(T^*, T) \in \boldsymbol{B}(\mathcal{H})$, the Nagy-Foiaş theory motivates the following questions. If $\alpha(T^*, T) \ge 0$,

- is it possible to find a functional model for $T$?

- If there is such model, what spectral consequences can be derived from it?

The original Nagy-Foias theory corresponds to the case $\alpha(t) = 1 - t$. Indeed, for this $\alpha$, $\alpha(T^*, T) = I - T^*T$, so that $\alpha(T^*, T) \ge 0$ if and only if $T$ is a contraction.

There are many works in this subject. We begin with a brief overview of some of them to get a picture of how the work of this thesis fits there.

## Brief overview of the subject

In his landmark paper [5], Agler showed that if $T$ has spectrum $\sigma(T)$ contained in the unit disc $\mathbb{D} = \{|z| < 1\}$, then it is natural to model $T$ by parts of $B \otimes I_{\mathcal{E}}$, where $B$ is a weighted backward shift and $I_{\mathcal{E}}$ is the identity operator on some auxiliary Hilbert space $\mathcal{E}$. (By *a part of an operator* we mean its restriction to an invariant subspace.) More generally, when $\sigma(T) \subset \overline{\mathbb{D}}$, it has been found in various particular cases that instead of $B \otimes I_{\mathcal{E}}$ one should consider operators of the form $(B \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an isometry or a unitary operator. This representation is called *a coanalytic model*. As Agler proved in [6], it holds, in particular, for *hypercontractions of order* $m$, i.e., operators $T \in \boldsymbol{B}(\mathcal{H})$ such that $(1-t)^j(T^*, T) \geq 0$ for $j = 1, \ldots, m$. Agler's theorem was generalized in [67] by Müller and Vasilescu to tuples of operators.

In [65], Müller studied the case where $\alpha = p$ is a polynomial. He considers the class $\mathcal{C}(p)$ of operators $T \in \boldsymbol{B}(\mathcal{H})$ such that $p(T^*, T) \geq 0$. He proves that any contraction $T \in \mathcal{C}(p)$ has a coanalytic model whenever $p(1) = 0, 1/p(t)$ is analytic in $\mathbb{D}$, and $1/p(\bar{w}z)$ is a reproducing kernel. This last condition is equivalent to the fact that all Taylor coefficients of $1/p(t)$ at the origin are positive. Müller also considers some operator inequalities for $T$ with infinitely many terms, with the same property of positivity. This permits him to show that any operator $T$ is unitarily equivalent to a part of a backward weighted shift with the same spectral radius (see [65, Corollary 2.3]).

The first results on Agler model techniques are exposed in the book [7] (2002) by Agler and McCarthy.

Olofsson worked with $m$-hypercontractions in [71, 72] and other papers. In [71], among other things, he obtained operator formulas for wandering subspaces, relevant in the models of $m$-hypercontractions. His results were generalized by Ball and Bolotnikov (see [14]) to what they call *$\beta$-hypercontractions*, which are subject to a certain infinite sequence of operator inequalities.

In [74], Olofsson deals with the case where $\alpha$ is not a polynomial. His assumptions are that $\alpha$ is analytic on $\mathbb{D}$, does not vanish on $\mathbb{D}$, and $1/\alpha$ has positive Taylor coefficients at the origin. Under this setting, he studies contractions $T$ on $\mathcal{H}$ such that $\alpha(rT^*, rT) \geq 0$ for every $r \in [0, 1)$. With more assumptions, he obtains the coanalytic model for this class of operators.

There are also many works on this context for tuples of commuting operators, such as [9, 11, 25, 26, 27, 67, 77]. In [77], Pott considered *positive regular polynomials*. These are polynomials of several complex variables with non-negative coefficients such that the constant term is 0 and the coefficients of the linear terms are positive. Given such polynomial $p$, Pott constructed a dilation model for a commuting tuple of operators $T = (T_1, \ldots, T_n)$, satisfying the positivity conditions $(1-p)^k(T^*, T) \geq 0$ for $1 \leq k \leq m$, where $m, n$ are positive integers (see [77, Theorem 3.8]). In [27], Bhattacharyya and Sarkar defined the characteristic function for this class of tuples and constructed a functional model in the pure case (see Theorem 4.2 in this work). See also [25] and [26].

General models for operator tuples, subject to a polynomial operator inequality were obtained in the influential paper [9] by Ambrozie, Engliš and Müller. This was extended by Arazy and Engliš in [11] to non-polynomial inequalities.

Recently Bhattacharjee and Sarkar [24] and Eschmeier [42] extended the results by Olofsson [71] to $m$-hypercontractive commutative tuples. The very recent arxiv [43] of Eschmeier and Toth contains more general results.

In [23], the models of row contractions in the Drury-Arveson space of the ball and in more general “contractive analytic Hilbert spaces” and their relations with wandering subspaces and inner functions are discussed.

There is also an intensive work on the case of noncommutative operator tuples. We refer to Ball and Bolotnikov [15] and Popescu [76] for up-to-day accounts.

## Our setting

Now we make more precise the setting under which the thesis will be developed. All the operators considered will be bounded and linear on some Hilbert space. All the functions considered will be representable by powers series in a disc centered at the origin, say

$$\alpha(t) = \sum_{n=0}^{\infty} \alpha_n t^n \qquad (|t| < r)$$

for some radius $r = r(\alpha) > 0$. We will refer to the coefficient $\alpha_n$ as *the $n$-th Taylor coefficient of* $\alpha$.

Denote by $A_W$ the Wiener algebra of analytic functions on the unit disc $\mathbb{D}$ whose Taylor coefficients are summable. That is, $f \in A_W$ iff

$$f(t) = \sum_{n=0}^{\infty} f_n t^n \qquad (|t| \leq 1)$$

where $\sum |f_n| < \infty$.

Suppose $\alpha \in A_W$ and let $T$ be an operator in $\boldsymbol{B}(\mathcal{H})$ with spectrum $\sigma(T) \subset \overline{\mathbb{D}}$. If the series

$$\sum_{n=0}^{\infty} |\alpha_n| T^{*n} T^n$$

converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$, then we say that

$$\alpha \in \mathcal{A}_T, \qquad \text{or} \qquad T \in \mathrm{Adm}_\alpha\,.$$

Depending on whether we want to consider fixed the operator $T$ or the function $\alpha$, we will use one notation or the other. Hence,

$$\mathcal{A}_T \subset A_W \qquad \text{and} \qquad \mathrm{Adm}_\alpha \subset \boldsymbol{B}(\mathcal{H}).$$

It turns out (see Proposition 1.2) that if $\alpha \in \mathcal{A}_T$, then the series $\sum \alpha_n T^{*n}T^n$ (without absolute values) also converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$. Therefore, we can define the operator

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n}T^n \ \in \boldsymbol{B}(\mathcal{H}).$$

We denote by $A_{W,\mathbb{R}}$ and $\mathcal{A}_{T,\mathbb{R}}$ the subsets of functions in $A_W$ and $\mathcal{A}_T$, respectively, that have real Taylor coefficients. We will use the notation $\mathrm{Adm}_\alpha$ only in case when $\alpha$ has real Taylor coefficients. Then the operator $\alpha(T^*, T)$ is selfadjoint. When $\alpha(T^*, T)$ is a positive operator, we say that $T \in \mathbf{C}_\alpha$. That is,

$$\mathbf{C}_\alpha := \{T \in \mathrm{Adm}_\alpha \, : \, \alpha(T^*, T) \geq 0\}\,.$$

This $\mathbf{C}_\alpha$ notation will be very useful for studying these classes of operators and inclusions between them. But usually, we will just write $\alpha(T^*, T) \geq 0$ instead of $T \in \mathbf{C}_\alpha$. In this case, we put

$$D := (\alpha(T^*, T))^{1/2},$$

where the non-negative square root is taken. We denote by $\mathfrak{D}$ the closure of the range of $D$. The operator $D$ and the space $\mathfrak{D}$ will play the same role as the *defect operator* and the *defect space*, respectively, in the Nagy-Foiaş theory.

# Main results by chapters

## Main results of Chapter 1

In the first chapter we focus on the construction of a functional model, respect to unitary equivalence, for an operator $T$ satisfying $\alpha(T^*, T) \geq 0$. We consider the following assumptions.

**Hypotheses 0.1.** *Let $\alpha$ be a function in $A_{W,\mathbb{R}}$ which does not vanish on $\mathbb{D}$, and put*

$$k(t) = 1/\alpha(t) = \sum_{n=0}^{\infty} k_n t^n \qquad t \in \mathbb{D},$$

*with $\alpha_0 = k_0 = 1$, and $k_n > 0$ for every $n \geq 1$.*

Under Hypotheses 0.1, we denote by $\mathcal{H}_k$ the weighted Hilbert space of power series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ with finite norm

$$\|f\|_{\mathcal{H}_k} := \left( \sum_{n=0}^{\infty} |f_n|^2 k_n \right)^{1/2}.$$

Let $B_k$ be the *backward shift* on $\mathcal{H}_k$, defined by

$$B_k f(t) = \frac{f(t) - f(0)}{t}. \tag{0.1}$$

**Definition 0.2.** Fix a function $\alpha$ satisfying Hypotheses 0.1, and let $T$ be an operator in $\boldsymbol{B}(\mathcal{H})$. We say that $T$ is $\alpha$-*modelable* if $\alpha(T^*, T) \geq 0$ and $T$ is unitarily equivalent to a part of an operator of the form $(B_k \otimes I_\mathcal{E}) \oplus S$, where $S$ is an isometry.

We remark that $B_k \otimes I_\mathcal{E}$ acts on the Hilbert space $\mathcal{H}_k \otimes \mathcal{E}$, which can be identified with the weighted Hilbert space of $\mathcal{E}$-valued power series $f(t) = \sum_{n=0}^\infty f_n t^n$ with norm, given by

$$\|f\|_{\mathcal{H}_k \otimes \mathcal{E}} = \left( \sum_{n=0}^\infty \|f_n\|_\mathcal{E}^2 \, k_n \right)^{1/2}.$$

It acts according to the same formula (0.1).

It is natural to pose the following question.

**Question 0.3.** Given a function $\alpha$ satisfying Hypotheses 0.1, give a good sufficient condition for an operator $T \in \boldsymbol{B}(\mathcal{H})$ to be $\alpha$-modelable.

One of the strongest results in this direction is contained in the recent papers by Bickel, Hartz and McCarthy [28] and by Clouâtre and Hartz [32]. It is stated for spherically symmetric tuples. For the case of a single operator, it can be formulated as follows.

**Theorem CH** (Clouâtre and Hartz [32, Theorem 1.3])**.** *Let $\alpha$ be a function with $\alpha_0 = 1$ and $\alpha_n \leq 0$ for all $n \geq 1$. Suppose that $k = 1/\alpha$ has radius of convergence 1, $k_n > 0$ for every $n \geq 0$ and*

$$\lim_{n\to\infty} \frac{k_n}{k_{n+1}} = 1. \tag{0.2}$$

*Then $B_k$ is bounded, and for any Hilbert space operator $T$ the following statements are equivalent.*

(i) *$T$ satisfies $\alpha(T^*, T) \geq 0$.*

(ii) *$T$ is unitarily equivalent to a part of $(B_k \otimes I_\mathcal{E}) \oplus S$, where $S$ is an isometry on $\mathcal{W}$. (Here $\mathcal{E}$ and $\mathcal{W}$ are auxiliary Hilbert spaces.)*

The functions $\alpha(t) = (1-t)^a$, where $a > 0$, satisfy Hypotheses 0.1. If $(1-t)^a(T^*, T) \geq 0$, then we say that $T$ is an *a-contraction.*

Note that Theorem CH concerns the *Nevannlina-Pick case* (that is, when $\alpha_0 = 1$ and $\alpha_n \leq 0$ for $n \geq 1$). One relevant example is $\alpha(t) = (1-t)^a$, where $0 < a < 1$. Clouâtre and Hartz mention it as an important one. Hence, any

$a$-contraction, with $0 < a < 1$, is unitarily equivalent to a part of an operator of the form $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $\mathcal{E}$ is an auxiliary Hilbert space and $S$ is an isometry. We can show that in this case, the space $\mathcal{E}$ can be chosen as $\mathfrak{D}$. Our approach in the proof is quite different and more direct. The precise statement of the result is the following.

**Theorem 0.4.** *If $0 < a < 1$, then the following statements are equivalent.*

(i) *$T$ is an $a$-contraction.*

(ii) *There exists a separable Hilbert space $\mathcal{E}$ such that $T$ is unitarily equivalent to a part of an operator $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is a Hilbert space isometry.*

*Moreover, if* (ii) holds, then one can take for $\mathcal{E}$ the space $\mathfrak{D} = \overline{D\mathcal{H}}$.

At the end of Chapter 1 we discuss briefly the case $a > 1$ (see Theorem 1.25 and Proposition 1.26).

The main result of Chapter 1 is as follows.

**Theorem 0.5.** *Assume Hypotheses 0.1. If $k \in A_{W,\mathbb{R}}$, and its Taylor coefficients $\{k_n\}$ satisfy $k_n^{1/n} \to 1, \sup k_n/k_{n+1} < \infty$ and*

$$\lim_{m\to\infty} \sup_{n\geq 2m} \sum_{m\leq j\leq n/2} \frac{k_j k_{n-j}}{k_n} = 0, \tag{0.3}$$

*then $B_k$ is bounded, and the operator $T \in \boldsymbol{B}(\mathcal{H})$ is a part of $B_k \otimes I_{\mathcal{E}}$ (for some Hilbert space $\mathcal{E}$) if and only if $\alpha$ and $k$ belong to $\mathcal{A}_T$, and $\alpha(T^*, T) \geq 0$. Moreover, in this case one can take $\mathcal{E} = \mathfrak{D}$.*

Notice that in the above theorem, the isometric part $S$ is unnecessary (see Theorem 0.10 below for more information). Theorem 0.5 shows that the above representation of $T$ exists in many cases when $k$ is not a Nevanlinna-Pick kernel, and so Theorem CH does not apply. Not much about these kernels has been known previously. Given an integer $N \geq 2$, there are examples of functions $k$ satisfying the hypotheses of Theorem 0.5 with whatever prescribed signs of the coefficients $\alpha_2, \dots, \alpha_N$ (see Example 1.19). Note that $\alpha_1 = -k_1$ is always negative.

**Remark 0.6.** Suppose that the sequence

$$\left\{ \frac{k_n}{k_{n+1}} \left(1 + \frac{1}{n+1}\right)^a \right\}$$

is increasing for some $a > 1$. Then (0.3) holds. This is close to [87, Proposition 34]. Indeed, put $k_j^* := (j+1)^{-a}$, and define $\rho_j := k_j/k_j^*$. Then our condition reduces to the condition $\rho_{n+2}/\rho_{n+1} \geq \rho_{n+1}/\rho_n$, for all $n$, which implies that $\rho_j\rho_{n-j}/\rho_n \leq C$, for $o \leq j \leq n$. Since

$$\frac{k_j k_{n-j}}{k_n} = \frac{\rho_j \rho_{n-j}}{\rho_n} \, \frac{k_j^* k_{n-j}^*}{k_n^*}$$

and $\{k_n^*\}$ satisfies (0.3), it follows that $\{k_n\}$ also satisfies (0.3).

Hence, for sufficiently regular sequences $\{k_n\}$, the condition (0.3) is equivalent to the condition $\sum k_n < \infty$. It can be added that, in fact, in Theorem 0.5 $\{k_n\}$ need not be regular; moreover, the quotients $k_n/k_{n+1}$ need not converge (see Remark 1.20).

The techniques employed in the proof are different from [32]. We use, basically, a combination of Müller's arguments in [65] and Banach algebras techniques.

The above theorems open the question of describing invariant subspaces of $B_k \otimes I_{\mathcal{E}}$ and of constructing a functional model of operators under the study, which certainly would be interesting. We do not address this question in this thesis. In the recent work [33], Clouâtre, Hartz and Schillo establish a Beurling–Lax–Halmos theorem for reproducing kernel Hilbert spaces in the Nevannlina-Pick context. We refer the reader to [73, 37, 84, 85] for more results in the Nevannlina-Pick case.

Whenever $T$ is $\alpha$-modelable, the operator $V_D$ given by

$$V_D : \mathcal{H} \to \mathcal{H}_k \otimes \mathfrak{D}, \qquad V_D x(z) = D(I_{\mathcal{H}} - zT)^{-1}x, \quad x \in \mathcal{H}, \quad z \in \mathbb{D},$$

is a contraction, and we can give an explicit model for $T$ (that is, give explicitly $\mathcal{E}$, $S$ and the tranform which sends the initial space into the model space).

**Theorem 0.7** (Explicit model). *If $T$ is $\alpha$-modelable, then $V_D$ is a contraction, and hence we can define*

$$W = (I_{\mathcal{H}} - V_D^* V_D)^{1/2}, \quad \mathcal{W} = \overline{W\mathcal{H}}.$$

*Moreover, $S : \mathcal{W} \to \mathcal{W}$, given by $SWx := WTx$, is an isometry and the operator*

$$(V_D, W) : \mathcal{H} \to (\mathcal{H}_k \otimes \mathfrak{D}) \oplus \mathcal{W}, \qquad (V_D, W)h = (V_D h, Wh)$$

*provides a model of $T$, in the sense that $(V_D, W)$ is isometric and*

$$((B_k \otimes I_{\mathfrak{D}}) \oplus S)(V_D, W) = (V_D, W)T.$$

If it is known that $T$ is $\alpha$-modelable, one can ask about the uniqueness of the model. For answering this question, we need the following definitions.

**Definition 0.8.** Let $\mathcal{L}$ be an invariant subspace of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S : \mathcal{W} \to \mathcal{W}$ is an isometry. We will say that the corresponding model operator

$$\big((B_k \otimes I_{\mathcal{E}}) \oplus S\big)|\mathcal{L}$$

is *minimal* if the following two conditions hold.

(i) $\mathcal{L}$ is not contained in $(\mathcal{H}_k \otimes \mathcal{E}') \oplus \mathcal{W}$ for any $\mathcal{E}' \subsetneqq \mathcal{E}$.

(ii) $\mathcal{L}$ is not contained in $(\mathcal{H}_k \otimes \mathcal{E}) \oplus \mathcal{W}'$ for any $\mathcal{W}' \subsetneqq \mathcal{W}$ invariant by $S$.

In Remark 1.15 we show that the explicit model obtained in Theorem 0.7 is indeed minimal.

Note that under Hypotheses 0.1, $\alpha$ is defined on the unit circle $\mathbb{T}$, and does not vanish on the interval $[0, 1)$. Since $\alpha(0) = \alpha_0 = 1$, we obtain that $\alpha(1) \geq 0$.

We distinguish the following two cases. This distinction appears already in [32, Subsection 2.3] for the Nevannlina-Pick case.

**Definition 0.9.** Suppose that $\alpha$ meets Hypotheses 0.1. We will say that $\alpha$ is of *critical type* (or, alternatively, that we have the *critical case*) if $\alpha(1) = 0$. If $\alpha(1) > 0$, we will say that $\alpha$ is of *subcritical type* (or, alternatively, that we have the *subcritical case*).

**Theorem 0.10** (Uniqueness of the minimal model)**.** *Suppose that $T$ is $\alpha$-modelable.*

(i) *In the critical case, the minimal model of $T$ is unique and is obtained by taking $V = V_D$. More precisely, the pair of transforms $(V_D, W_0)$, where $W_0 = (I - V_D^* V_D) : H \to \mathcal{W}_0$, and $\mathcal{W}_0 := \overline{\text{Ran}}(I - V_D^* V_D)$ gives rise to a minimal model, and any minimal model is provided by $(V_C, W)$, where $C = vD$, $W = wW_0 : H \to \mathcal{W}$ and $v, w$ are unitary isomorphisms.*

(ii) *In the subcritical case, the minimal model of $T$ is not unique, in general. However, there always exists a minimal model for which $V = V_D$ and $W$ is absent (that is, $\mathcal{W} = 0$).*

## Main results of Chapter 2

In Chapter 2 we study when the inequality $\alpha(T^*, T) \geq 0$ permits one to obtain a functional model for $T$ up to similarity. When dealing with similarity models instead of unitarily equivalent ones, the model obviously does not give all the information about the operator, but that information preserved up to similarity. One big advantage of considering just the similarity is that we are able to enlarge the family of functions $\alpha$ considered. In Chapter 2, we do not use the kernel $k = 1/\alpha$. Indeed, now we allow the function $\alpha \in A_{W,\mathbb{R}}$ to have zeroes in $\mathbb{D}$, except on the interval $[0, 1)$. A key ingredient for our arguments is Theorem 2.7. With the model up to similarity, we will also be able to extract spectral consequences from the model.

If a series

$$\sum_{n=0}^{\infty} |\alpha_n| T^{*n} T^n$$

converges in the uniform operator topology in $\boldsymbol{B}(\mathcal{H})$, then we say that

$$\alpha \in \mathcal{A}_T^0.$$

(Unless otherwise stated, we are implicitly assuming that $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subset \overline{\mathbb{D}}$.) Sometimes we will need this stronger convergence.

Obviously, $\mathcal{A}_T^0$ is contained in $\mathcal{A}_T$. In fact, $\mathcal{A}_T$ is a Banach algebra (see Theorem 2.2), and $\mathcal{A}_T^0$ is its closed separable sub-algebra (see Proposition 2.3).

Our main similarity result is the following.

**Theorem 0.11.** *Suppose that $\alpha$ has the form*

$$\alpha(t) = \eta(t)\,\widetilde{\alpha}(t),$$

*where $\eta \in \mathcal{A}_T$, and $\widetilde{\alpha} \in \mathcal{A}_T^0$ is positive on the interval $[0,1]$. If $\alpha(T^*,T) \geq 0$, then $T$ is similar to an operator $\widehat{T}$ such that $\eta(\widehat{T}^*,\widehat{T}) \geq 0$.*

In other words, with this theorem we can leave out (in terms of similarity) the factor $\widetilde{\alpha}$. The question of obtaining a model for $T$ up to similarity reduces to obtaining a model for $\widehat{T}$ (as above).

An important case is $\eta(t) := (1-t)^a$ for $a > 0$. When $a = m$ is a positive integer, if $(1-t)^m(T^*,T) \geq 0$ then $T$ is said to be an *$m$-contraction*, and if $(1-t)^m(T^*,T) = 0$ then $T$ is said to be an *$m$-isometry*. For example, the 1-contractions and 1-isometries are, precisely, the contractions and the isometries (respectively) in $\boldsymbol{B}(\mathcal{H})$.

Note that if $\eta(t) = 1-t$, then obviously $\eta \in \mathcal{A}_T$ (since it is a polynomial). Hence, as an immediate and important particular case of the above theorem we obtain the following result.

**Corollary 0.12.**

(i) *Suppose that $\alpha$ has the form*

$$\alpha(t) = (1-t)^a\,\widetilde{\alpha}(t),$$

*for some $a > 0$, where $(1-t)^a \in \mathcal{A}_T$, and $\widetilde{\alpha} \in A_T^0$ is positive on the interval $[0,1]$. If $\alpha(T^*,T) \geq 0$, then $T$ is similar to an $a$-contraction.*

(ii) *Suppose that $\alpha$ has the form*

$$\alpha(t) = (1-t)\,\widetilde{\alpha}(t),$$

*where $\widetilde{\alpha} \in A_T^0$ is positive on the interval $[0,1]$. If $\alpha(T^*,T) \geq 0$, then $T$ is similar to a contraction.*

There are many papers on the study of $m$-contractions and $m$-isometries. Here, we introduce the case when the exponent $a$ is not an integer. The definitions of $a$-contraction and $a$-isometries are the natural ones. With the help of Theorem 0.4, we will get the following ergodic result.

**Theorem 0.13.** *If $T$ be an $a$-contraction, with $0 < a < 1$, then $T$ is quadratically $(C, b)$-bounded for any $b > 1 - a$.*

That $T$ is *quadratically $(C, b)$-bounded* (where the letter C stands for Cesàro) means that there exists a constant $c > 0$ such that

$$\sup_{n\geq 0} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 \leq c\|x\|^2 \qquad (\forall x \in \mathcal{H}),$$

where the numbers $k^{-s}(n)$, called *Cesàro numbers*, are defined by

$$(1-t)^s =: \sum_{n=0}^{\infty} k^{-s}(n)t^n.$$

The property of being quadratically $(C, b)$-bounded is inherited by similarity. Hence, we can combine Theorems 0.11 and 0.13 with $\eta(t) = (1-t)^a$, and $0 < a < 1$.

At the end of Chapter 2, we study more ergodic properties of $a$-contractions. For instance, the following theorem gives an interesting connection between the functional model and an ergodic property. Recall the model obtained in Theorem 0.4.

**Theorem 0.14.** *Let $T$ be an $a$-contraction with $0 < a < 1$ and let $b > 1 - a$. Then the following statements are equivalent.*

(i) *The isometry $S$ does not appear in the $(1-t)^a$-model of $T$.*

(ii) *For every $x \in \mathcal{H}$,*

$$\exists \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 = 0.$$

(iii) *For every $x \in \mathcal{H}$,*

$$\liminf_{n\to\infty} \|T^n x\| = 0.$$

In this chapter we also prove that if $\alpha$ is of subcritical type, $\sigma(T) \neq \overline{\mathbb{D}}$, and $\alpha(T^*, T) \geq 0$ is of finite rank, then the lengths $l_\nu$ of the complementary intervals of the set $\overline{(\sigma(T) \cap \mathbb{D})} \cap \mathbb{T}$ in the unit circle $\mathbb{T}$ satisfy the Carleson condition $\sum_\nu l_\nu \log(2\pi/l_\nu) < \infty$. See Theorem 2.20.

## Main results of Chapter 3

When $\alpha(t) := (1-t)\widetilde{\alpha}(t)$, where $\widetilde{\alpha} \in A_{W,\mathbb{R}}$ is positive on the interval $[0,1]$, we say that $\alpha$ is an *admissible function*. In this chapter we focus on admissible functions $\alpha$. We say that the function $\alpha$ is *strongly admissible* when it is admissible and $\widetilde{\alpha}$ does not vanish on the unit circle $\mathbb{T} = \{|t| = 1\}$.

Given an operator $T$ in $\mathbf{C}_\alpha$, where $\alpha$ is strongly admissible, in Theorem 3.20 we construct its explicit Nagy-Foias type model.

We begin with a completely different proof of Corollary 0.12 (ii) using *abstract defect operators*. One of the advantages of this proof is that the connection with the Nagy-Foiaş model became more clear.

Remind that the Nagy-Foiaş model is constructed only for completely non-unitary contractions. One of the important steps in our construction is a definition of an analogue of the unitary part of $T$ in the context of operators in $\mathbf{C}_\alpha$ (see Subsection 3.3.1).

As an application of this model, we show that a major part of spectral consequences of the Nagy-Foias model still take place when the defect $D$ has finite rank or belongs to the Schatten-von Neumann classes $\mathfrak{S}_p$ $(0 < p \leq \infty)$.

It is obvious that for any contraction $T$ there exists the limits $\lim_{n\to\infty} \|T^n h\|$ for every $h \in \mathcal{H}$. Now, if $T \in \mathbf{C}_\alpha$ (for some admissible $\alpha$), then it is similar to a contraction; so it is natural to ask whether in this case there also exist the limits $\lim_{n\to\infty} \|T^n h\|$. We show that, in general, the answer is no (see Remark 3.25). However, we get a positive answer when $\alpha$ is strongly admissible.

**Theorem 0.15.** *If $\alpha$ is an strongly admissible function and $T$ belongs to $\mathbf{C}_\alpha$, then there exists the limit* $\lim_{n\to\infty} \|T^n h\|$ *for every $h \in \mathcal{H}$.*

We also discuss some elementary properties of the classes $\mathbf{C}_\alpha$ when $\alpha$ is an admissible function. In particular, in Lemma 3.13 we prove that "almost every" diagonalizable operator $T$ on a finite dimensional space $\mathcal{H}$, whose spectral radius is not greater than 1, belongs to a class $\mathbf{C}_\alpha$ for some admissible $\alpha$.

At the end of the chapter we prove a similarity result for functions that vanish at the origin. The key idea is the use of a theorem by Cassier and Suciu on *$n$-quasicontractions*.

## Main results of Chapter 4

Given two functions $\alpha$, $\tau$ satisfying Hypotheses 0.1, in this chapter we study when the class $\mathbf{C}_\alpha$ is contained in $\mathbf{C}_\tau$. The main results obtained in this direction are the following.

In Section 4.1, we give some results for general $\alpha$, $\tau$. In particular, in Theorem 4.3 we show that for admissible $\alpha$, $\beta$, one has $\mathbf{C}_\alpha \subset \mathbf{C}_\beta$ if and only if the meromorphic function $\beta/\alpha$, which is analytic in the origin, has nonnegative Taylor coefficients.

In Section 4.2, we concentrate on inclusions between classes of $a$-contractions. Notice first that if we put $\alpha(t) = (1-t)^a$ and $\beta(t) = (1-t)^b$, then the above condition is fulfilled for $0 < b < a$, because in this case all Taylor coefficients of the function

$$\beta(t)/\alpha(t) = (1-t)^{b-a}$$

are positive. Theorem 4.3 does not apply, however, because these functions $\alpha$, $\beta$ are admissible only in case when $a = b = 1$.

We show, in fact, that the class of $a$-contractions is not contained in the class of $b$-contractions whenever $0 < b < a$, see Proposition 4.20.

When $\alpha(t) = (1-t)^a$, we will use the notation $\mathrm{Adm}_a$ and $\mathbf{C}_a$ instead of $\mathrm{Adm}_\alpha$ and $\mathbf{C}_\alpha$, respectively. That is, $\mathbf{C}_a$ is the set of $a$-contractions.

One of our main results is as follows.

**Theorem 0.16.** *Let $0 < b < a$ where $b$ is not an integer. If $T$ is an $a$-contraction and $T \in \mathrm{Adm}_b$, then $T$ is a $b$-contraction.*

**Theorem 0.17.** *Let $a > 0$, and let the integer $m$ be defined by $m < a \leq m+1$. Then the following statements are equivalent.*

(i) *$T$ is an $a$-isometry.*

(ii) *$T$ is an $(m+1)$-isometry.*

(iii) *For each vector $h \in \mathcal{H}$, there exists a polynomial $p$ of degree at most $m$ such that $\|T^n h\|^2 = p(n)$ for every $n \geq 0$.*

The topic of $a$-contractions and $a$-isometries is closely related with the topic of finite differences. Given a sequence of real numbers $\Lambda = \{\Lambda_n\}_{n\geq 0}$, we denote by $\nabla\Lambda$ the sequence whose $n$-th term, for $n \geq 0$, is given by $(\nabla\Lambda)_n = \Lambda_{n+1}$. In general, if $\beta(t) = \sum \beta_n t^n$ is an analytic function, we denote by $\beta(\nabla)\Lambda$ the sequence whose $n$-the term is given by

$$\beta(\nabla)\Lambda_n = \sum_{j=0}^{\infty} \beta_j \Lambda_{j+n},$$

whenever the series on the right hand side converges for every $n \geq 0$. In particular, for the functions $(1-t)^a$ where $a \in \mathbb{R}$, we put

$$(1-\nabla)^a \Lambda_n = \sum_{j=0}^{\infty} k^{-a}(j) \Lambda_{j+n}.$$

This is the forward finite difference of order $a$ of the sequence $\Lambda$. For instance, for $a = 1$ we get the first order finite difference $(1-\nabla)\Lambda_n = \Lambda_n - \Lambda_{n+1}$. We address the following two questions.

(A) Determine for which $a, b > 0$ the inequality $(1-\nabla)^a \Lambda_n \geq 0$ (for every $n \geq 0$) implies $(1-\nabla)^b \Lambda_n \geq 0$ (for every $n \geq 0$).

(B) Given $a > 0$, determine the space of solutions $\Lambda$ of the equation $(1-\nabla)^a \Lambda = 0$.

In Theorem 4.14 we give an answer to (A), and as an immediate consequence Theorem 0.16 follows. Indeed, the key idea is to fix $x \in \mathcal{H}$ and translate the problem into a finite difference question taking $\Lambda_n := \|T^n x\|^2$ for $n \geq 0$.

In the same way, an answer to (B) is given in Theorem 4.15, and as a consequence we obtain Theorem 0.17.

## Papers on which the thesis is based

This thesis is based on the following three papers:

1. **Operator inequalities implying similarity to a contraction**.
   G. Bello-Burguet and D. Yakubovich,
   Complex Anal. Oper. Theory, 13 (2019), 1325–1360.

2. **Operator inequalities I. Models and ergodicity**.
   L. Abadias, G. Bello-Burguet and D. Yakubovich,
   Arxiv (2019)

3. **Operator inequalities II. Models up to similarity and inclusions of operator classes**.
   L. Abadias, G. Bello-Burguet and D. Yakubovich,
   Work in progress.

In the first paper, we consider admissible functions $\alpha$. The convergence of $\sum \alpha_n T^{*n} T^n$ is assumed to be in the uniform operator topology in $\boldsymbol{B}(\mathcal{H})$. Hence, the classes of operators studied there (which we denoted as $\mathcal{C}_\alpha$) are smaller than $\mathrm{Adm}_\alpha$ and its subfamily $\mathbf{C}_\alpha$. The operators studied there turn out to be similar to contractions (see Theorem I in that paper). We also give an explicit model for the operators studied in the spirit of the Nagy-Foias model, and studied some interesting properties of the classes considered there. Chapter 3 is based on this paper.

In the thesis, we deal instead with the convergence of the above series in the strong operator topology. As Example 2.25 shows, this assumption is essential to have the equivalence between the inequality $\alpha(T^*, T) \geq 0$ and the existence of the model, as in Theorem CH. Hence, this is the most natural setting for this problem.

The second paper focuses on the study of unitarily equivalent models for operators in $\mathbf{C}_\alpha$. In this paper, the function $\alpha$ is not considered admissible any longer. As a consequence, the operators $T$ in $\mathbf{C}_\alpha$ are no longer similar to contractions.

The third paper is on the study of models up to similarity. The framework of these last two papers is the one used for this thesis. In these two papers we used the notations $\mathrm{Adm}^w_\alpha$ and $\mathcal{C}^w_\alpha$ instead of $\mathrm{Adm}_\alpha$ and $\mathbf{C}_\alpha$ (as in this thesis) in order to emphasize that the conditions for the membership to these classes are weaker than in the first paper.

Chapter 1 is based on the second paper, and Chapter 2 on the third one. The first section of Chapter 4 is based on the first paper (but in the most general context), and the second section of the chapter is based on the third paper.

# Chapter 1

# Unitarily equivalent models

## 1.1 Preliminaries

In what follows, we assume that the functions $\alpha(t)$ and $k(t) = 1/\alpha(t)$ satisfy Hypotheses 0.1. Note that under this hypotheses, the function $k$ (analytic in $\mathbb{D}$) gives rise to a positive definite kernel $k(z, w) := k(\bar{w}z)$ defining a reproducing kernel Hilbert space $\mathcal{R}_k$ of analytic functions. That is, point evaluations in $\mathcal{R}_k$ are continuous linear functionals, and $f(w) = \langle f, k(\cdot, w)\rangle$, for any $f \in \mathcal{R}_k$. For instance, in the Nagy-Foiaş case ($\alpha(t) = 1 - t$) one obtains as $\mathcal{R}_k$ the Hardy space $H^2(\mathbb{D})$. In this chapter, we pay more attention to the weighted Hilbert space $\mathcal{H}_k$ and the backward shift $B_k$ on it defined in the Introduction. Indeed, there is a clear relation between the space $\mathcal{R}_k$ and the operator $M_t$ on it given by $g(t) \mapsto tg(t)$ on the one hand, and the space $\mathcal{H}_k$ and the backward shift $B_k$ on the other hand. It is easy to see that the operator $B_k$ can also be interpreted as $M_t^*$.

Observe that Hypotheses 0.1 do not restrict to the Nevannlina-Pick case. Let us mention briefly why these hypothesis are natural for us.

First of all, the assumption that $\alpha$ belongs to $A_{W,\mathbb{R}}$ (which, for instance, is made in the whole thesis and not only in this chapter) is natural due to Proposition 1.5, that will be seen later. To assure that $k = 1/\alpha$ is analytic in $\mathbb{D}$, we need that $\alpha$ do not vanish in $\mathbb{D}$. In order to guarantee that we can obtain a reproducing kernel Hilbert space $\mathcal{R}_k$ of analytic functions, we need to assume that $k_n > 0$ for every $n \geq 0$. The assumption $k_0 = 1$ is just a normalization of the coefficients. Finally, note that in Theorem 0.5, which is our new source of examples with respect to Theorem CH, we need that $k \in A_{W,\mathbb{R}}$. However, this assumption excludes automatically the critical case (when $\alpha(1) = 0$). Therefore, it is natural to just make the assumption that $k$ is analytic in $\mathbb{D}$, so we can still consider both cases: critical and subcritical.

The structure of this Chapter is the following. In Section 1.2 we study some interesting properties of the families $\mathrm{Adm}_\alpha$ and $\mathbf{C}_\alpha$, and characterize the membership of backward and forward weighted shifts to them. In Section 1.3, we prove Theorems 0.7 and 0.10. The proof of Theorem 0.5 is given in Section 1.4. In Section 1.5 we study the scope of the hypothesis of Theorem 0.5, and present examples satisfying the hypothesis of Theorem 0.5 where Theorem CH does not apply.

## 1.2 The classes $\mathrm{Adm}_\alpha$ and $\mathbf{C}_\alpha$

The definitions of $\mathrm{Adm}_\alpha$ and $\mathbf{C}_\alpha$ for any function $\alpha \in A_{W,\mathbb{R}}$ were given in the Introduction. In this section we give some basic properties of these families, and then we discuss the membership of the weighted shifts to them.

We begin with a well known result that will be used repeatedly.

**Lemma 1.1** (see [50, Problem 120])**.** *If an increasing sequence $\{A_n\}$ of selfadjoint Hilbert space operators satisfies $A_n \le CI$ for all $n$, where $C$ is a constant, then $\{A_n\}$ converges in the strong operator topology.*

**Proposition 1.2.** *If $T \in \mathrm{Adm}_\alpha$, then the series*

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n$$

*converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$.*

*Proof.* Let $T \in \mathrm{Adm}_\alpha$. Put

$$\alpha_n^+ := \begin{cases} \alpha_n & \text{if } \alpha_n \ge 0 \\ 0 & \text{if } \alpha_n < 0 \end{cases}, \qquad \alpha_n^- := \begin{cases} 0 & \text{if } \alpha_n \ge 0 \\ -\alpha_n & \text{if } \alpha_n < 0 \end{cases}.$$

Then, for any fixed integer $N$,

$$\sum_{n=0}^{N} \alpha_n T^{*n} T^n = \sum_{n=0}^{N} \alpha_n^+ T^{*n} T^n - \sum_{n=0}^{N} \alpha_n^- T^{*n} T^n. \tag{1.1}$$

Using Lemma 1.1, it is immediate that both summands on the right hand side of (1.1) converge in SOT as $N \to \infty$, and hence the statement follows. □

This result is generalized in Proposition 2.12.

**Notation 1.3.** If $X$ and $Y$ are two quantities (typically non-negative), then $X \lesssim Y$ (or $Y \gtrsim X$) will mean that $X \le CY$ for some absolute constant $C > 0$. If the constant $C$ depends on some parameter $p$, then we write $X \lesssim_p Y$. We put $X \asymp Y$ when both $X \lesssim Y$ and $Y \lesssim X$.

**Proposition 1.4.** *Let $\alpha$ be a function in $A_W$. Then the following statements are equivalent.*

(i) $T \in \mathrm{Adm}_\alpha$.

(ii) $\sum_{n=0}^\infty |\alpha_n| \, \|T^n x\|^2 < \infty$ *for every* $x \in \mathcal{H}$.

(iii) $\sum_{n=0}^\infty |\alpha_n| \, \|T^n x\|^2 \lesssim \|x\|^2$ *for every* $x \in \mathcal{H}$.

*Proof.* It is immediate that (i) implies (ii). Suppose that (ii) is true. Note that for every $x, y \in \mathcal{H}$ and $M > N$ we have

$$\begin{aligned}\left| \sum_{n=N+1}^{M} |\alpha_n| \langle T^n x, T^n y \rangle \right| &\le \sum_{n=N+1}^{M} |\alpha_n| \, \|T^n x\| \, \|T^n y\| \\ &\le \frac{1}{2} \left\{ \sum_{n=N+1}^{M} |\alpha_n| \, \|T^n x\|^2 + \sum_{n=N+1}^{M} |\alpha_n| \, \|T^n y\|^2 \right\} \to 0\end{aligned}$$

as $N$ and $M$ go to infinity. Therefore

$$\sum_{n=0}^{\infty} |\alpha_n| \langle T^n x, T^n y \rangle \qquad \text{converges (in } \mathbb{C}\text{)}, \tag{1.2}$$

for every $x, y \in \mathcal{H}$. Put

$$A_N := \sum_{n=0}^{N} |\alpha_n| T^{*n} T^n \in \boldsymbol{B}(\mathcal{H}) \tag{1.3}$$

for every non-negative integer $N$. Fix $x \in \mathcal{H}$. By (1.2) we know that $\langle A_N x, y \rangle$ converges for every $y \in \mathcal{H}$. This means that the sequence $\{A_N x\} \subset \mathcal{H}$ is weakly convergent. Then $\sup_N \|A_N x\| < \infty$ for any $x \in \mathcal{H}$ and therefore $\sup_N \|A_N\| < \infty$. Hence (iii) follows with absolute constant $\sup_N \|A_N\|$.

Now suppose we have (iii). This means that the operators $A_N$ given by (1.3) are uniformly bounded from above. So we can apply Lemma 1.1 to obtain (i). This completes the proof. □

**Proposition 1.5.** *Let $T \in \boldsymbol{B}(\mathcal{H})$. If a function $\beta(t) = \sum \beta_n t^n$ does not belong to $A_W$, and $\sum |\beta_n| T^{*n} T^n$ converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$, then $\sigma(T) \subset \mathbb{D}$.*

*Proof.* Suppose that $\sum |\beta_n| T^{*n} T^n$ converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$ for some $\beta$ which does not belong to $A_{W,\mathbb{R}}$. Imitating the proof of Proposition 1.4, we obtain that there exists a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} |\beta_n| \, \|T^n x\|^2 \le C \tag{1.4}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$.

Suppose that $T$ has spectral radius $\rho(T) \geq 1$. Let $\lambda$ be any point of $\sigma(T)$ such that $|\lambda| = \rho(T)$. Then $\lambda$ belongs to the boundary of the spectrum of $T$ and therefore it belongs to the approximate point spectrum. Put $R := |\lambda|^2 = \rho(T)^2 \geq 1$. Since $\sum |\beta_n| = \infty$ (because $\beta \notin A_W$), there exists an integer $N$ sufficiently large such that

$$\sum_{n=0}^{N} |\beta_n| > C + 1.$$

Now, choose a unit approximate eigenvector $h \in \mathcal{H}$ corresponding to $\lambda$ such that $\|Th - \lambda h\|$ is sufficiently small, so that

$$\left| \|T^m h\|^2 - |\lambda|^{2m} \right| < \left( \sum_{n=0}^{N} |\beta_n| \right)^{-1}, \qquad (m = 0, 1, \ldots, N).$$

Then

$$\left| \sum_{n=0}^{N} |\beta_n| R^n - \sum_{n=0}^{N} |\beta_n| \, \|T^n h\|^2 \right| \leq \sum_{n=0}^{N} |\beta_n| \left| R^n - \|T^n h\|^2 \right| < 1,$$

and therefore

$$\sum_{n=0}^{N} |\beta_n| \, \|T^n h\|^2 \geq \left( \sum_{n=0}^{N} |\beta_n| R^n \right) - 1 \geq \left( \sum_{n=0}^{N} |\beta_n| \right) - 1 > C.$$

But this contradicts (1.4). Hence $\rho(T)$ must be strictly less that 1, that is, $\sigma(T) \subset \mathbb{D}$, as we wanted to prove. □

**Remark 1.6.** We consider the classes $\mathrm{Adm}_\alpha$ (and hence the classes $\mathbf{C}_\alpha$) only for functions in $A_{W,\mathbb{R}}$ because: 1) we need that its Taylor coefficients are real in Proposition 1.2, and 2) if the Taylor coefficients were not summable, then the operators on these classes would have spectrum contained in $\mathbb{D}$, by Proposition 1.5.

Let us study now the weighted shifts. Recall some definitions from the Introduction. Given a sequence of positive numbers $\{\varkappa_n : n \geq 0\}$, we denote by $\mathcal{H}_\varkappa$ the corresponding weighted Hilbert space of power series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ with the norm

$$\|f\|_{\mathcal{H}_\varkappa} := \left( \sum_{n=0}^{\infty} |f_n|^2 \varkappa_n \right)^{1/2}.$$

Obviously, the monomials $e_n(t) := t^n$, for $n \geq 0$, form an orthogonal basis on $\mathcal{H}_\varkappa$, and

$$\|e_n\|^2_{\mathcal{H}_\varkappa} = \varkappa_n. \tag{1.5}$$

The backward and forward shifts $B_\varkappa$ and $F_\varkappa$ on $\mathcal{H}_\varkappa$ are defined by

$$B_\varkappa f(t) := \frac{f(t)-f(0)}{t} \quad \text{and} \quad F_\varkappa f(t) := tf(t) \qquad (\forall f \in \mathcal{H}_\varkappa), \tag{1.6}$$

or equivalently

$$B_\varkappa e_n := \begin{cases} e_{n-1}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases} \quad \text{and} \quad F_\varkappa e_n := e_{n+1} \qquad (\forall n \geq 0). \tag{1.7}$$

It is immediate that $\|B_\varkappa\|^2 = \sup_{n\geq 0} \varkappa_n/\varkappa_{n+1}$. Hence $B_\varkappa$ is bounded if and only if

$$\frac{\varkappa_n}{\varkappa_{n+1}} \leq C \qquad (\forall n \geq 0), \tag{1.8}$$

for a constant $C < \infty$. Analogously, $\|F_\varkappa\|^2 = \sup_{n\geq 0} \varkappa_{n+1}/\varkappa_n$, and therefore $F_\varkappa$ is bounded if and only if

$$c \leq \frac{\varkappa_n}{\varkappa_{n+1}} \qquad (\forall n \geq 0). \tag{1.9}$$

for a constant $c > 0$.

Put $\varkappa(t) = \sum_{n=0}^\infty \varkappa_n t^n$. We remark that the reproducing kernel Hilbert space $\mathcal{R}_\varkappa$, corresponding to the positive definite kernel $\varkappa(z,w) := \varkappa(\bar{w}z)$, is $\mathcal{R}_\varkappa = \mathcal{H}_{\tilde{\varkappa}}$, where $\tilde{\varkappa}_n = 1/\varkappa_n$. The pairing $\langle f, g\rangle = \sum f_n \bar{g}_n$ makes the space $\mathcal{H}_\varkappa$ naturally dual to $\mathcal{R}_\varkappa$. Notice that in this interpretation, $B_\varkappa$ is the adjoint operator to the operator $g(t) \mapsto tg(t)$, acting on $\mathcal{R}_\varkappa$.

**Notation 1.7.** Let us mention here a convenient notation that will be used in Section 2.3. When $\{\varkappa_n\}$ is precisely the sequence of Taylor coefficients of the function $(1-t)^{-s}$ for some $s > 0$, that is,

$$\varkappa_0 = 1 \quad \text{and} \quad \varkappa_n = \frac{s(s+1)\cdots(s+n-1)}{n!} \quad \text{for } n \geq 1,$$

we denote the space $\mathcal{H}_\varkappa$ by $\mathcal{H}_s$, emphasizing the exponent $s$. In the same way we use $B_s$ and $F_s$.

**Lemma 1.8.** *Let $T$ be one of the operators $B_\varkappa$ or $F_\varkappa$, for some $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$. Suppose that $T$ is bounded (i.e., assume* (1.8) *or* (1.9)*, respectively). Then:*

(i) $T \in \mathrm{Adm}_\alpha$ *if and only if*

$$\sup_{m\geq 0} \left\{ \sum_{n=0}^\infty |\alpha_n| \frac{\|T^n e_m\|^2}{\|e_m\|^2} \right\} < \infty. \tag{1.10}$$

(ii) *Suppose that $T \in \mathrm{Adm}_\alpha$. Then $T \in \mathbf{C}_\alpha$ if and only if*

$$\sum_{n=0}^{\infty} \alpha_n \|T^n e_m\|^2 \geq 0 \qquad (\forall m \geq 0). \tag{1.11}$$

*Proof.* (i) Let $T \in \mathrm{Adm}_\alpha$. By Proposition 1.4 (ii) we have

$$\sum_{n=0}^{\infty} |\alpha_n| \|T^n f\|^2 \lesssim \|f\|^2 ,$$

for every function $f \in \mathcal{H}_\varkappa$. Taking the vectors of the basis $f = e_m$ we obtain (1.10).

Conversely, let us assume now (1.10). Fix a function $f \in \mathcal{H}_\varkappa$. Then

$$T^n f = \sum_{m=0}^{\infty} f_m T^n e_m \qquad (\forall n \geq 0),$$

where the series is orthogonal. Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} |\alpha_n| \|T^n f\|^2 &= \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \|T^n e_m\|^2 \\ &= \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 \frac{\|T^n e_m\|^2}{\|e_m\|^2} \\ &= \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 \sum_{n=0}^{\infty} |\alpha_n| \frac{\|T^n e_m\|^2}{\|e_m\|^2} \\ &\lesssim \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 < \infty, \end{aligned} \tag{1.12}$$

where (1.10) allows us to justify the change of the summation indexes in the last equality. Hence $T \in \mathrm{Adm}_\alpha$.

(ii) Let $T \in \mathrm{Adm}_\alpha$. If $T \in \mathbf{C}_\alpha$, then obviously (1.11) follows. For the converse implication, note that similarly to (1.12) we get

$$\sum_{n=0}^{\infty} \alpha_n \|T^n f\|^2 = \sum_{m=0}^{\infty} |f_m|^2 \|e_m\|^2 \sum_{n=0}^{\infty} \alpha_n \frac{\|T^n e_m\|^2}{\|e_m\|^2},$$

so (1.11) implies that $T \in \mathbf{C}_\alpha$. □

Writing down this lemma for $B_\varkappa$ and $F_\varkappa$ separately, we immediately get the next two results.

**Theorem 1.9.** *Let $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$, such that the coefficients $\{\varkappa_n\}$ satisfy (1.8). Set $\beta(t) = \sum_{n\geq 0} \beta_n t^n$ with $\beta_n = |\alpha_n|$. Put $\gamma(t) = \beta(t)\varkappa(t)$. Then:*

(i) $B_\varkappa \in \mathrm{Adm}_\alpha$ *if and only if*

$$\sup_{m\geq 0} \left\{ \frac{\gamma_m}{\varkappa_m} \right\} < \infty.$$

(ii) *Suppose that $B_\varkappa \in \mathrm{Adm}_\alpha$. Then $B_\varkappa \in \mathbf{C}_\alpha$ if and only if all the Taylor coefficients of $\alpha(t)\varkappa(t)$ are non-negative.*

Recall the definition made at the end of the Introduction of the operator $\nabla$ acting on one-sided sequences $\Lambda = \{\Lambda_m\}_{m\geq 0}$, and what we understand by $\beta(\nabla)\Lambda_m$ for any function $\beta$.

**Theorem 1.10.** *Let $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$, such that the coefficients $\{\varkappa_n\}$ satisfy (1.9). Set $\beta(t) = \sum_{n\geq 0} \beta_n t^n$ with $\beta_n = |\alpha_n|$. Then:*

(i) $F_\varkappa \in \mathrm{Adm}_\alpha$ *if and only if*

$$\sup_{m\geq 0} \left\{ \frac{\beta(\nabla)\varkappa_m}{\varkappa_m} \right\} < \infty.$$

(ii) *Suppose that $F_\varkappa \in \mathrm{Adm}_\alpha$. Then $F_\varkappa \in \mathbf{C}_\alpha$ if and only if $\alpha(\nabla)\varkappa_m \geq 0$ for every $m \geq 0$.*

## 1.3 Explicit model and uniqueness of the model

In this section we prove Theorems 0.7 and 0.10.

Let us start by proving that the operator $V_D$ (given in the Introduction) is always a contraction in the Nevanlinna-Pick case. Here we do not assume that $T$ is $\alpha$-modelable.

**Theorem 1.11.** *Let $\alpha_0 = 1$ and $\alpha_n \leq 0$ for $n \geq 1$. If $T \in \boldsymbol{B}(\mathcal{H})$ satisfies $\alpha(T^*, T) \geq 0$, then the operator $V_D$ is a contraction.*

*Proof.* Recall that $D^2 = \alpha(T^*, T)$. Therefore

$$\|Dx\|^2 = \sum_{m=0}^{\infty} \alpha_m \|T^m x\|^2$$

for every $x \in \mathcal{H}$. Hence

$$\|DT^n x\|^2 = \sum_{m=0}^{\infty} \alpha_m \left\|T^{m+n} x\right\|^2$$

for every $x \in \mathcal{H}$ and every non-negative integer $n$. Fix a positive integer $N$. Then

$$\begin{aligned}\sum_{n=0}^{N} k_n \|DT^n x\|^2 &= \sum_{n=0}^{N} k_n \sum_{m=0}^{\infty} \alpha_m \left\|T^{m+n}x\right\|^2 \\ &\overset{(\star)}{=} \sum_{j=0}^{\infty} \left( \sum_{n+m=j,\, n\le N} k_n \alpha_m \right) \left\|T^j x\right\|^2 =: \sum_{j=0}^{\infty} \tau_j \left\|T^j x\right\|^2,\end{aligned}$$

where in $(\star)$ we can rearrange the series because it converges absolutely (just use Proposition 1.4 (ii)). Since $\alpha k = 1$, we get $\tau_0 = 1$ and $\tau_1 = \cdots = \tau_N = 0$. Moreover,

$$\tau_{N+i} = k_0 \alpha_{N+i} + \cdots + k_N \alpha_i \le 0$$

for every $i \ge 1$, because all the $\alpha_j$'s above are negative or zero, and the $k_j$'s are positive. Therefore

$$\sum_{n=0}^{N} k_n \|DT^n x\|^2 \le \|x\|^2$$

for every $N$ and hence the series $\sum k_n \|DT^n x\|^2$ converges for every $x \in \mathcal{H}$. This gives

$$\|V_D x\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 \le \|x\|^2,$$

as we wanted to prove. □

The following result is a simple and well-known fact.

**Proposition 1.12.** *Let $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subset \overline{\mathbb{D}}$, and let $\mathcal{E}$ be a Hilbert space. A bounded transform $V : \mathcal{H} \to \mathcal{H}_\varkappa \otimes \mathcal{E}$ satisfies*

$$VT = (B_\varkappa \otimes I_\mathcal{E})V$$

*if and only if there is a bounded linear operator $C : \mathcal{H} \to \mathcal{E}$ such that $V = V_C$, where*

$$V_C x(z) = C(I - zT)^{-1}x, \qquad x \in \mathcal{H},\, z \in \mathbb{D}.$$

*Proof.* Let us suppose first that $V = V_C$ for some bounded linear operator $C : \mathcal{H} \to \mathcal{E}$. That is,

$$V_C x(z) = \sum_{n=0}^{\infty} CT^n x z^n.$$

Recall that $B_\varkappa \otimes I_\mathcal{E}$ is the operator on $\mathcal{H}_\varkappa \otimes \mathcal{E}$ that sends

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_{n+1} z^n$$

(where $a_n \in \mathcal{E}$). Therefore

$$V_C T x(z) = \sum_{n=0}^{\infty} z^n C T^{n+1} x = (B_\varkappa \otimes I_\mathcal{E}) V_C x(z).$$

Conversely, suppose now that $VT = (B_\varkappa \otimes I_\mathcal{E})V$. Define $a_n(x)$ by

$$Vx(z) := \sum_{n=0}^{\infty} a_n(x) z^n, \qquad x \in \mathcal{H}.$$

Then

$$\sum_{n=0}^{\infty} a_n(Tx) z^n = VTx = (B_\varkappa \otimes I_\mathcal{E}) Vx = \sum_{n=0}^{\infty} a_{n+1}(x) z^n.$$

Therefore $a_{n+1}(x) = a_n(Tx)$ and the statement follows using that $a_0 : \mathcal{H} \to \mathcal{E}$ must be a bounded linear operator (and then put $C := a_0$). $\square$

**Proposition 1.13.** *Let $C : \mathcal{H} \to \mathcal{E}$ be a bounded operator and let $T \in \mathbf{C}_\alpha$. Then, there exists a bounded operator $W : \mathcal{H} \to \mathcal{W}$ such that the operator $(V_C, W)$ is isometric and transforms $T$ into a part of the operator $(B_k \otimes I_\mathcal{E}) \oplus S$, where $S \in \boldsymbol{B}(\mathcal{W})$ is an isometry, if and only if the following conditions hold.*

(i) *$V_C : \mathcal{H} \to \mathcal{H}_k \otimes \mathcal{E}$ is a contraction.*

(ii) *For every $x \in \mathcal{H}$,*

$$\|x\|^2 - \|V_C x\|^2 = \|Tx\|^2 - \|V_C T x\|^2.$$

*Proof.* Let us suppose first the existence of such operator $W$. Since $(V_C, W)$ is an isometry, (i) holds. Notice that (ii) is equivalent to proving that $\|Wx\|^2 = \|WTx\|^2$ for every $x \in \mathcal{H}$. But this is also immediate since $SWx = WTx$ and $S$ is an isometry.

Conversely, suppose now that (i) and (ii) are true. By (i), we can put $W := (I - V_C^* V_C)^{1/2}$ and $\mathcal{W} := \overline{\operatorname{Ran}} \, W$. Using (ii) we have

$$\|Wx\|^2 = \|x\|^2 - \|V_C x\|^2 = \|Tx\|^2 - \|V_C T x\|^2 = \|WTx\|^2 . \tag{1.13}$$

We define

$$S(Wx) := WTx,$$

for every $x \in \mathcal{H}$. Note that $S$ is well defined, since $\|SWx\| = \|Wx\|$ by (1.13). Since $W\mathcal{H}$ is dense in $\mathcal{W}$, $S$ can be extended to an isometry on $\mathcal{W}$. By the definition of $W$, we know that $(V_C, W)$ is an isometry and it is immediate that

$$(B_k \otimes I_\mathfrak{D}) V_C = V_C T \quad \text{ and } \quad SW = WT.$$

This completes the converse implication. $\square$

**Proposition 1.14.** *Let $T \in \mathbf{C}_\alpha$. Assume that $C : \mathcal{H} \to \mathcal{E}$ and $W : \mathcal{H} \to \mathcal{W}$ are any bounded operators, such that $(V_C, W)$ is isometric on $(\mathcal{H}_k \otimes \mathcal{E}) \oplus \mathcal{W}$ and transforms $T$ into a part of $(B_k \otimes I_\mathcal{E}) \oplus S$, where $S \in B(\mathcal{W})$ is an isometry. Then $C$ and $D$ are related by*

$$\|Dx\|^2 = \|Cx\|^2 + \alpha(1)\|Wx\|^2, \quad \forall x \in \mathcal{H}. \tag{1.14}$$

*Proof.* Since $(V_C, W)$ is isometric, we have

$$\|x\|^2 = \|V_C x\|^2 + \|Wx\|^2 = \sum_{n=0}^{\infty} k_n \|CT^n x\|^2 + \|Wx\|^2 , \tag{1.15}$$

for every $x \in \mathcal{H}$. Substituting $x$ by $T^j x$ above and multiplying by $\alpha_j$, we obtain that

$$\begin{aligned}\alpha_j \left\|T^j x\right\|^2 &= \sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{n+j} x\right\|^2 + \alpha_j \left\|WT^j x\right\|^2 \\ &= \sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{n+j} x\right\|^2 + \alpha_j \left\|Wx\right\|^2 ,\end{aligned}$$

where we have used that $\|Wx\|^2 = \|WTx\|^2$. Therefore

$$\begin{aligned}\|Dx\|^2 = \sum_{j=0}^{\infty} \alpha_j \left\|T^j x\right\|^2 &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{j+n} x\right\|^2 + \left(\sum_{j=0}^{\infty} \alpha_j\right) \|Wx\|^2 \\ &\overset{(\star)}{=} \sum_{m=0}^{\infty} \left(\sum_{j+n=m} \alpha_j k_n\right) \|CT^m x\|^2 + \alpha(1) \|Wx\|^2 .\end{aligned}$$

Since $\alpha k = 1$, the only non-vanishing summand in the last series above is for $m = 0$ and we obtain (1.14). Note that the rearrangement in $(\star)$ is correct as

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} |\alpha_j| k_n \left\|CT^{n+j} x\right\|^2 \le \sum_{j=0}^{\infty} |\alpha_j| \left\|T^j x\right\|^2 < \infty,$$

where we have used (1.15) and that $T \in \mathbf{C}_\alpha$. □

Recall the definition of *minimal model* given in the Introduction (see Definition 0.8).

**Remark 1.15.** Suppose that $T$ is $\alpha$-modelable. Then $T$ is unitarily equivalent to $\big((B_k \otimes I_\mathcal{E}) \oplus S\big)|\mathcal{L}$, where $\mathcal{L} = \overline{\operatorname{Ran}}\,(V_C, W)$. This model is minimal if and only if

(a) $\overline{\operatorname{Ran}} \, C = \mathcal{E}$; and

(b) $\overline{\operatorname{Ran}} \, W = \mathcal{W}$.

Indeed, in this case, it is easy to see that (a) is equivalent to (i), and (b) is equivalent to (ii) in Definition 0.8.

*Proof of Theorem 0.10.* (i) In the critical case (i.e., $\alpha(1) = 0$), (1.14) gives

$$\|Dx\| = \|Cx\| \qquad (\forall x \in \mathcal{H}),$$

so there exists a unitary operator $v$ such that $C = vD$. This implies the statement.

(ii) Suppose we are in the subcritical case (i.e., $\alpha(1) > 0$). First, we remark that the model is not unique in general. For instance, take $T = U$ any unitary operator. Using Proposition 1.4 and that $\alpha \in A_{W,\mathbb{R}}$, we obtain that $T \in \operatorname{Adm}_\alpha$. Since

$$\sum_{n=0}^{\infty} \alpha_n \|T^n x\|^2 = \left( \sum_{n=0}^{\infty} \alpha_n \right) \|x\|^2 \geq 0 \qquad (\forall x \in \mathcal{H}),$$

we get that $\alpha(T^*, T) = \alpha(1) I \geq 0$. Obviously, $T = U$ is a minimal model for $T$. Moreover, if $k = 1/\alpha$ fits (0.3), then Theorem 0.5 (which is proved in the next section, but its proof is completely independent) gives another model for $T$. (See Example 1.19 and Remark 1.20.)

Now suppose that $(V_C, W)$ provides a model of $T$. Let us see that there exists a minimal model of $T$ with $V = V_D$ and $W$ absent. Changing $x$ by $T^n x$ in (1.14) we obtain

$$\|DT^n x\|^2 = \|CT^n x\|^2 + \alpha(1) \|Wx\|^2 ,$$

where we have used that $\|WTx\| = \|Wx\|$. Therefore

$$\begin{aligned} \|V_D x\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 &= \sum_{n=0}^{\infty} k_n \|CT^n x\|^2 + k(1)\alpha(1) \|Wx\|^2 \\ &= \|V_C x\|^2 + \|Wx\|^2 = \|x\|^2 , \end{aligned}$$

so $V_D : \mathcal{H} \to \mathcal{H}_k \otimes \mathcal{E}$ is an isometry and (ii) follows.

This model is minimal, because $\operatorname{Ran} D$ is dense in $\mathfrak{D}$. The space $\mathcal{L}$ is just the closure of $\operatorname{Ran} V_D$ in $\mathcal{H}_k \otimes \mathfrak{D}$. (See Remark 1.15.) $\square$

*Proof of Theorem 0.7.* It is an immediate consequence of Theorem 0.10 and Proposition 1.13 (i) that $V_D$ is a contraction. Finally, for proving that $(V_D, W)$ gives a model, we just need to use the same argument employed in the reciprocal implication of Proposition 1.13. $\square$

## 1.4 Proof of Theorem 0.5

In this section we prove Theorem 0.5. For that, we need to cite some results concerning Banach algebras.

For any sequence $\omega = \{\omega_n\}_{n=0}^{\infty}$ of positive weights, define the weighted space

$$\ell^\infty(\omega) := \left\{ f(t) = \sum_{n=0}^{\infty} f_n t^n \, : \, \sup_{n\geq 0} |f_n|\omega_n < \infty \right\}.$$

In general, its elements are formal power series. We will also use the separable version of this space:

$$\ell_0^\infty(\omega) := \left\{ f(t) = \sum_{n=0}^{\infty} f_n t^n \, : \, \lim_{n\to\infty} |f_n|\omega_n = 0 \right\}.$$

**Proposition 1.16** (see [68])**.** *$\ell^\infty(\omega)$ is a Banach algebra (with respect to the formal multiplication of power series) if and only if*

$$\sup_{n\geq 0} \sum_{j=0}^{n} \frac{\omega_n}{\omega_j \omega_{n-j}} < \infty. \tag{1.16}$$

**Theorem 1.17.** *Let $\omega_n > 0$ and $\omega_n^{1/n} \to 1$. If $\sup_n \omega_{n+1}/\omega_n < \infty$ and*

$$\lim_{m\to\infty} \sup_{n\geq 2m} \sum_{m\leq j\leq n/2} \frac{\omega_n}{\omega_j \omega_{n-j}} = 0, \tag{1.17}$$

*then the following is true.*

(i) *$\ell^\infty(\omega)$ is a Banach algebra.*

(ii) *If $f \in \ell^\infty(\omega)$ does not vanish on $\overline{\mathbb{D}}$, then $1/f \in \ell^\infty(\omega)$.*

*Proof.* The hypotheses imply (1.16), so that (i) follows from Proposition 1.16. To get (ii), we apply the results of the paper [40] by El-Fallah, Nikolski and Zarrabi. We use the notation of this paper. Put $\omega'(n) = \omega(n)/(n+1)$, $A = \ell^\infty(\omega)$ and $A_0 = \ell_0^\infty(\omega)$. The hypotheses imply that $A$ (and hence $A_0$) is compactly embedded into the multiplier convolution algebra $\mathrm{mult}(\ell^\infty(\omega'))$, see [40, Lemma 3.6.3]. Hence, by [40, Theorem 3.4.1], for any $f \in A_0$, $\delta_1(A_0, \mathfrak{M}(A_0)) = 0$, see [40, Subsection 0.2.3] for the definition of this quantity. This means that for any $\delta > 0$ there is a constant $c_1(\delta) < \infty$ such that the conditions $f \in A_0$, $\|f\|_A = 1$ and $|f| > \delta$ on $\overline{\mathbb{D}}$ imply that $1/f \in A_0$ and $\|1/f\|_A \leq c_1(\delta)$. In particular, (ii) holds for $f$ in $A_0$. To get (ii) in the general case, suppose that $f \in A$ and $|f| > \delta > 0$ on $\overline{\mathbb{D}}$. Since $f(rt) \in A_0$ for all $r < 1$, we get that the norms of the functions $1/f(rt)$ in $A$ are uniformly bounded by $c_1(\delta)$ for all $r < 1$. When $r \to 1^-$, each Taylor coefficient of $1/f(rt)$ tends to the corresponding Taylor coefficient of $1/f(t)$. It follows that $1/f$ is in $A$ (and $\|1/f\|_A \leq c_1(\delta)$). $\square$

*Proof of Theorem 0.5.* Put

$$\omega_n := \frac{1}{k_n}.$$

The fist part of Theorem 0.5 (that $B_k$ is bounded) is straightforward. Also, by Theorem 1.17 (i), $\ell^\infty(\omega)$ is an algebra.

First suppose that $T$ is a part of $B_k \otimes I_{\mathcal{E}}$, and let us prove that $B_k \in \mathbf{C}_\alpha \cap \mathrm{Adm}_k$.

By Theorem 1.9 (i), we know that $B_k \in \mathrm{Adm}_k$ if and only if

$$\sum_{j=0}^{m} k_j k_{m-j} \lesssim k_m,$$

which follows from Theorem 1.17 (i) and Proposition 1.16.

Now let us see that $B_k \in \mathbf{C}_\alpha$. By Theorem 1.17 (ii), $\alpha = 1/k$ belongs to $\ell^\infty(\omega)$, and therefore $|\alpha_n| \lesssim k_n$. Then, since $B_k \in \mathrm{Adm}_k$, we obtain that $B_k \in \mathrm{Adm}_\alpha$. Finally, Theorem 1.9 (ii) gives that $B_k \in \mathbf{C}_\alpha$ (because $\alpha k = 1$ has non-negative Taylor coefficients). Hence $T$ also is in $\mathbf{C}_\alpha \cap \mathrm{Adm}_k$.

Conversely, let us assume now that $T \in \mathbf{C}_\alpha \cap \mathrm{Adm}_k$. We want to prove that $T$ is a part of $B_k \otimes I_{\mathcal{E}}$. We adapt the argument of [65, Theorem 2.2] (where the convergence of the series of operators is in the uniform operator topology).

Using that

$$V_D : \mathcal{H} \to \mathcal{H}_k \otimes \mathfrak{D}, \qquad V_D x := \{Dx, DTx, DT^2x, \dots\}.$$

it is obvious that

$$(B_k \otimes I_{\mathfrak{D}}) V_D = V_D T. \tag{1.18}$$

Moreover,

$$\begin{aligned} \|V_D x\|^2 &= \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 = \sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} \alpha_m \left\|T^{n+m}x\right\|^2 \\ &= \sum_{j=0}^{\infty} \left( \sum_{n+m=j} k_n \alpha_m \right) \left\|T^j x\right\|^2 = \|x\|^2, \end{aligned}$$

where we have used that $\sum_{n+m=j} k_n \alpha_m$ is equal to 1 if $j = 0$ and is equal to 0 if $j \geq 1$. The re-arrangement of the series is correct since, using $T \in \mathrm{Adm}_\alpha \cap \mathrm{Adm}_k$, we have

$$\sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} |\alpha_m| \left\|T^{n+m}x\right\|^2 \lesssim \sum_{n=0}^{\infty} k_n \|T^n x\|^2 \lesssim \|x\|^2,$$

and the series converges absolutely.

Hence $V_D$ is an isometry. Joined to (1.18), this proves that $T$ is unitarily equivalent to a part of $B_k \otimes I_{\mathfrak{D}}$. $\square$

**Remark 1.18.** Theorem 2.2 of Müller [65] can be formulated as follows. If $k_{n-1}k_{n+1} \geq k_n^2$ and $\sum_{j\geq 1} k_j \left\|T^j\right\|^2 \leq 1$, then $T$ is unitarily equivalent to a part of $B_k \otimes I_{\mathfrak{D}}$. The condition $\sum_{j\geq 1} k_j \left\|T^j\right\|^2 \leq 1$ can be replaced by a weaker one: $(2-k)(T^*, T) \geq 0$. In this formulation one does not have the inverse implication, because the condition $\alpha(T^*, T) \geq 0$ from Theorem 0.5 is weaker than the last condition. Müller's result only applies to Nevannlina-Pick type inequalities of special type.

## 1.5 Analysis of the scope of Theorem 0.5 and additional remarks

In this section we discuss the scope of condition (0.3) and give series of examples where Theorem CH does not apply.

Given an analytic function $f(t) = \sum f_n t^n$, we denote by $[f]_N$ its truncated polynomial of degree $N$, that is,

$$[f]_N := f_0 + f_1 t + \ldots + f_N t^N.$$

**Example 1.19.** Let $\sigma_2, \ldots, \sigma_N$ be an arbitrary sequence of signs (that is, a sequence of numbers $\pm 1$). We assert that there are functions $\alpha, k$ meeting all the hypotheses of Theorem 0.5 such that $\operatorname{sign}(\alpha_n) = \sigma_n$, for $n = 2, \ldots, N$. Indeed, take a polynomial $\widetilde{\alpha}$ of degree $N$ such that $\widetilde{\alpha}_0 = 1, \widetilde{\alpha}_1 < 0$, and for $n = 2, \ldots, N$, assume that $\widetilde{\alpha}_n < 0$ whenever $\sigma_n = -1$ and $\widetilde{\alpha}_n = 0$ whenever $\sigma_n = 1$. Put $\widetilde{k} := [1/\widetilde{\alpha}]_N$. The formula

$$\widetilde{k}_n = \sum_{\substack{s\geq 1 \\ n_1+\cdots+n_s=n}} (-1)^s \widetilde{\alpha}_{n_1} \cdots \widetilde{\alpha}_{n_s} \tag{1.19}$$

shows that all the coefficients of $\widetilde{k}$ are positive. We also require that neither $\widetilde{\alpha}$ nor the polynomial $\widetilde{k}$ vanish on $\overline{\mathbb{D}}$. It is clear that it is so if, for instance, $|\alpha_n|$ are sufficiently small for $n = 2, \ldots, N$.

Now perturb the coefficients $\widetilde{\alpha}_j$ that are equal to zero, obtaining a new polynomial $\widehat{\alpha}$ such that

$$\widehat{\alpha}_j := \begin{cases} \varepsilon & \text{if } \sigma_j = 1 \\ \widetilde{\alpha}_j & \text{otherwise} \end{cases} \qquad (2 \leq j \leq N).$$

By continuity, if $\varepsilon > 0$ is small enough, we can guarantee that the polynomial $\widehat{k} = [1/\widehat{\alpha}]_N$ also has positive Taylor coefficients, and we can also guarantee that $\widehat{k}$ (which is a slight perturbation of $\widetilde{k}$) does not vanish on $\overline{\mathbb{D}}$.

Finally, take as $k$ any function in $A_{W,\mathbb{R}}$ whose first Taylor coefficients are

$$k_0 = \widehat{k}_0 = 1, \quad k_1 = \widehat{k}_1, \quad \ldots, \quad k_N = \widehat{k}_N,$$

and such that

$$\frac{k_{n-j}}{k_n} \le C_0 \qquad (\forall n \ge 2j),$$

for some constant $C_0$. For instance, one can put $k_n = An^{-b}$ for $n > N$, with $A > 0$ (small enough) and $b > 1$. Then $k \in A_{W,\mathbb{R}}$ does not vanish on $\overline{\mathbb{D}}$.

Then obviously $k$ satisfies (1.20) and hence all the hypotheses of Theorem 0.5. The function $\alpha := 1/k$ in $A_{W,\mathbb{R}}$ has the desired pattern of signs.

Finally, it is important to note that $\alpha_1 = -k_1$ is always negative.

**Remark 1.20.** It is also easy to see that whenever $\{k_n\}$ satisfies (0.3), any other sequence $\{\tilde{k}_n\}$ with $k_0 = 1$ and $c < \tilde{k}_n/k_n < C$ for $n > 1$, where $c, C$ are positive constants, also satisfies this condition. In particular, if $\{k_n\}$ satisfies (0.3) and $\{\tilde{k}_n\}$ is as above, where $C$ is sufficiently small, then $k(t)$ is invertible in $A_{W,\mathbb{R}}$, so that all hypotheses of Theorem 0.5 are fulfilled. So there are many examples of functions $k(t)$ meeting these hypotheses, such that the quotients $k_n/k_{n+1}$ do not converge.

Let us mention now some remarks on Theorem 1.17.

**Remark 1.21.** It is immediate that the conditions

$$\frac{\omega_n}{\omega_j \omega_{n-j}} \le \tau_j \quad (\forall n \ge 2j) \qquad \text{and} \qquad \sum_{j=0}^{\infty} \tau_j < \infty, \tag{1.20}$$

imply (1.17) and (1.16) (in particular, $\sup_n \omega_{n+1}/\omega_n < \infty$). Let us give a direct proof of Theorem 1.17 for this particular case.

Statement (i) follows using Proposition 1.16.

(ii) Put $g := 1/f$. Suppose that $g \notin \ell^\infty(\omega)$. This means that

$$\sup_{n \ge 0} |g_n| \omega_n = \infty.$$

Hence, it is clear that there exists a sequence $\{\rho_n^0\}$ in $[0, 1]$ such that $\rho_n^0 \to 0$ (slowly) and

$$\sup_{n \ge 0} |g_n| \omega_n \rho_n^0 = \infty. \tag{1.21}$$

**Claim.** There exists a sequence $\{\rho_n\}$ with

$$\rho_n^0 \le \rho_n \le 1 \quad \text{ and } \quad \rho_n \to 0 \tag{1.22}$$

such that $\widetilde{\omega}_n := \rho_n \omega_n$ defines a Banach algebra $\ell^\infty(\widetilde{\omega})$.

Indeed, since $\sum \tau_j < \infty$, there exists a sequence of positive numbers $\{c_j\}$ such that $c_j \nearrow \infty$ and still $\sum c_j\tau_j < \infty$. Take any sequence $\{\rho_n\}$ that decreases, tends to zero, and satisfies $\rho_n \ge \max(\rho_n^0, 1/c_n)$. Then, for $\widetilde{\omega}_n := \rho_n\omega_n$ we have

$$\frac{\widetilde{\omega}_n}{\widetilde{\omega}_j\widetilde{\omega}_{n-j}} = \frac{\omega_n}{\omega_j\omega_{n-j}}\frac{\rho_n}{\rho_j\rho_{n-j}} \le \frac{\omega_n}{\omega_j\omega_{n-j}}\frac{1}{\rho_j} \le \tau_j c_j \qquad (\forall n \ge 2j).$$

Since $\sum \tau_j c_j < \infty$, Proposition 1.16 implies that $\ell^\infty(\widetilde{\omega})$ is a Banach algebra, and the proof of the Claim is completed.

Now fix $\{\widetilde{\omega}_n\}$ as in the Claim. We may assume that $(\rho_n^0)^{1/n} \to 1$ and therefore $(\rho_n)^{1/n} \to 1$. Since the polynomials are dense in the Banach algebra $\ell_0^\infty(\widetilde{\omega})$, any complex homomorphism $\chi$ on $\ell_0^\infty(\widetilde{\omega})$ is determined by its value on the power series $t$. So the map $\chi \mapsto \chi(t)$ is injective and continuous from the spectrum (the maximal ideal space) of $\ell_0^\infty(\widetilde{\omega})$ to $\mathbb{C}$. Since $\widetilde{\omega}_n^{1/n} \to 1$, its image contains $\mathbb{D}$ and is contained in $\overline{\mathbb{D}}$. Hence the spectrum of $\ell_0^\infty(\widetilde{\omega})$ is exactly the set $\{\chi_\lambda : \lambda \in \overline{\mathbb{D}}\}$, where $\chi_\lambda(f) = f(\lambda)$. (We borrow this argument from [40].) As

$$f_n\widetilde{\omega}_n = (f_n\omega_n)\rho_n \to 0,$$

we have $f \in \ell_0^\infty(\widetilde{\omega})$. Then, using the Gelfand theory (see, for instance, [82, Chapter 10]), we get that $g = 1/f \in \ell_0^\infty(\widetilde{\omega})$, which contradicts (1.21). Therefore, the assumption $g \notin \ell^\infty(\omega)$ is false, as we wanted to prove.

**Remark 1.22.** Notice that the above characterization of the spectrum of the algebra $\ell_0^\infty(\widetilde{\omega})$ (see the Remark 1.21) implies the following fact: the conditions (1.16) and $\omega_n^{1/n} \to 1$ imply that $\sum_n 1/\omega_n < \infty$. This also can be proved in an elementary way, without recurring to the Gelfand theory.

Indeed, by (1.16), there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \frac{\omega_n}{\omega_j\omega_{n-j}} \le C$$

for every $n \ge 1$. Fix a positive integer $L$. Then obviously, for every $n \ge L$,

$$\sum_{j=1}^{L} \frac{\omega_n}{\omega_j\omega_{n-j}} \le C. \tag{1.23}$$

Let us see that

$$\limsup_{n\to\infty} \min_{1\le j\le L} \frac{\omega_n}{\omega_{n-j}} \ge 1. \tag{1.24}$$

Indeed, if (1.24) were false, then

$$\limsup_{n\to\infty} \min_{1\le j\le L} \frac{\omega_n}{\omega_{n-j}} < r < 1$$

for some $r$. Hence, there exists a positive integer $N$ such that

$$\min_{1\le j\le L} \frac{\omega_n}{\omega_{n-j}} \le r$$

for $n \ge N$. From this, it is easy to see that

$$\omega_n \le r^{s_n} \left( \max_{0\le k\le N} \omega_k \right), \qquad s_n := \left[ \frac{n-N}{L} \right] + 1,$$

where $[a]$ denotes the integer part of $a$. Since $s_n$ behaves asymptotically as $n/L$, it follows that $\limsup_{n\to\infty} \omega_n^{1/n} \le r^{1/L} < 1$, which contradicts the hypothesis that $\omega_n^{1/n} \to 1$. Therefore, (1.24) is true.

Now, using (1.23) and (1.24), it follows that

$$C \ge \sum_{j=1}^{L} \frac{\omega_n}{\omega_j \omega_{n-j}} \ge \left( \min_{1\le j\le L} \frac{\omega_n}{\omega_{n-j}} \right) \sum_{j=1}^{L} \frac{1}{\omega_j}.$$

Taking lim sup when $n \to \infty$, and using (1.24), we get that $\sum 1/\omega_j$ converges.

The following statement shows that in the subcritical case, the hypotheses of Theorem 0.5 imply that the radius of convergence of the series for $\alpha$ is equal to one.

**Proposition 1.23.** *If $\lim k_n^{1/n} = 1$ and $\alpha$ is of subcritical type, then $\alpha$ does not continue analytically to any disc $R\mathbb{D}$, where $R > 1$.*

*Proof.* Since $k(t)$ has nonnegative Taylor coefficients, we have $|k(t)| \le k(1)$ for all $t \in \mathbb{D}$. Using that $k = 1/\alpha$, it follows that in the subcritical case, $|\alpha(t)| \ge \alpha(1) > 0$ for any $t \in \mathbb{D}$. So, $\alpha$ cannot continue analytically to any disc $R\mathbb{D}$, where $R > 1$, because in this case, the radius of convergence of the Taylor series for $k$ would be greater than 1. □

## 1.6 A direct proof of Theorem 0.4

Note that Theorem 0.4 is a particular case of Theorem CH, since it concerns the Nevannlina-Pick case. For the proof of Theorem CH, they study the reproducing kernel Hilbert spaces through the representation theory of their algebras of multipliers. Instead of this method involving $C^*$-algebras, here we give a direct proof of Theorem 0.4 using approximation in Besov spaces.

Before starting the proof of Theorem 0.4, let as mention a notation that we be used frequently in the rest of the thesis.

**Notation 1.24.** If $f(t)=\sum_{n=0}^{\infty} f_n t^n$ and $g(t)=\sum_{n=0}^{\infty} g_n t^n$, we use the notation $f \succcurlyeq g$ when $f_n \geq g_n$ for every $n \geq 0$, and the notation $f \succ g$ when $f \succcurlyeq g$ and $f_0 > g_0$.

*Proof of Theorem 0.4.* The implication (ii) $\Rightarrow$ (i) is straightforward.

So we focus on the implication (i) $\Rightarrow$ (ii).

Let $T$ be an $a$-contraction. We divide the proof into three steps:

(1) Definition of the operator $S$.

(2) Proof that $S$ is an isometry.

(3) Proof that $T$ is unitarily equivalent of a part of $(B_a \otimes I_{\mathfrak{D}}) \oplus S$.

We put $\alpha(t) := (1-t)^a$ and $k(t) := (1-t)^{-a} \succ 0$.

**Step 1.** Note that

$$\|Dx\|^2 = \sum_{j=0}^{\infty} \alpha_j \left\|T^j x\right\|^2 \qquad (\forall x \in \mathcal{H}). \tag{1.25}$$

Changing $x$ by $T^n x$ in (1.25) we obtain a more general formula:

$$\|DT^n x\|^2 = \sum_{j=0}^{\infty} \alpha_j \left\|T^{j+n} x\right\|^2 \qquad (\forall x \in \mathcal{H},\ \forall n \geq 0). \tag{1.26}$$

Multiplying (1.26) by $k_n$ and summing for $n = 0, 1, \ldots, N$ (for some fixed $N \in \mathbb{N}$) we obtain the following equation

$$\|x\|^2 = \sum_{n=0}^{N} k_n \|DT^n x\|^2 + \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} \quad (\forall x \in \mathcal{H}), \tag{1.27}$$

where

$$\rho_{N,m} = \sum_{n=N+1}^{m} k_n \alpha_{m-n} = -\sum_{j=0}^{N} k_j \alpha_{m-j} \qquad (1 \leq N+1 \leq m). \tag{1.28}$$

Since $0 < a < 1$, we have that $\alpha_n < 0$ for every $n \geq 1$ (and $\alpha_0 = 1$). Note that $\alpha_0$ does not appear in the last sum in (1.28), so since all the $k_j$'s are positive and all the $\alpha_n$'s that appear in the sum are negative, we obtain that $\rho_{N,m} > 0$. Therefore, by (1.27) we have

$$\|x\|^2 \geq \sum_{n=0}^{N} k_n \|DT^n x\|^2 .$$

Hence the series (of positive terms) $\sum k_n \|DT^n x\|^2$ converges and taking limits in (1.27) when $N \to \infty$ we obtain

$$\exists \lim_{N\to\infty} \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} = \|x\|^2 - \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 \geq 0. \tag{1.29}$$

We are going to define a new semi-inner product on our Hilbert space $\mathcal{H}$ via

$$[x, y] := \lim_{N\to\infty} \sum_{m=N+1}^{\infty} \langle T^m x, T^m y\rangle \rho_{N,m}. \tag{1.30}$$

By (1.29), $[x, x]$ is correctly defined, and $[x, x] = \langle Ax, x\rangle$, where $A$ is a self-adjoint operator with $0 \leq A \leq I$. Hence, by the polarization formula $[x, y]$ is correctly defined for any $x, y \in \mathcal{H}$, and $[x, y] = \langle Ax, y\rangle$.

Let $E := \{x \in \mathcal{H} : [x, x] = 0\}$. It is a closed subspace of $\mathcal{H}$. Put $\widehat{\mathcal{H}} := \mathcal{H}/E$. For any vector $x \in \mathcal{H}$, we denote by $\widehat{x}$ its equivalent class in $\widehat{\mathcal{H}}$. Note that $\widehat{\mathcal{H}} := \mathcal{H}/E$ is a new Hilbert space with norm $|||\cdot|||$ given by

$$|||\widehat{x}|||^2 = \lim_{N\to\infty} \sum_{m=N+1}^{\infty} \|T^m x\|^2 \rho_{N,m} = \|x\|^2 - \sum_{n=0}^{\infty} k_n \|DT^n x\|^2. \tag{1.31}$$

We define $S$ as the operator on $\widehat{\mathcal{H}}$ given by $S\widehat{x} := \widehat{Tx}$ for every $x \in \mathcal{H}$.

**Step 2.** Let us see now that the equality $|||\widehat{Tx}||| = |||\widehat{x}|||$ holds for every $x \in \mathcal{H}$. Observe that this will imply, in particular, that $S$ is well-defined.

Indeed, note that

$$\sum_{n=0}^{\infty} k_n \|DT^n x\|^2 = \lim_{r\to 1^-} \sum_{n=0}^{\infty} r^n k_n \|DT^n x\|^2 \tag{1.32}$$

since the RHS is an increasing function of $r$. Using (1.25), (1.31) and (1.32) we obtain that

$$\begin{aligned} |||\widehat{x}|||^2 &= \|x\|^2 - \lim_{r\to 1^-} \sum_{n=0}^{\infty} r^n k_n \|DT^n x\|^2 \\ &= \|x\|^2 - \lim_{r\to 1^-} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} r^n k_n \alpha_j \left\|T^{n+j} x\right\|^2 \\ &= \|x\|^2 - \lim_{r\to 1^-} \sum_{m=0}^{\infty} \|T^m x\|^2 u_r(m), \end{aligned} \tag{1.33}$$

where

$$u_r(m) := \sum_{n+j=m} r^n k_n \alpha_j \tag{1.34}$$

For each $m$, $u_r(m)$ is a continuous function of $r \in [0,1]$. We also have $u_1(m) = 0$ for $m \geq 1$ and $u_r(0) = 1$ for $r \in [0,1]$. Moreover, since $u_r(1) = a(1-r)$, we have $u_r(1) \to 0$ as $r \to 1^-$. Therefore

$$\left\|\widehat{T}x\right\|^2 - \|\widehat{x}\|^2 = \lim_{r\to 1^-} \sum_{m=2}^{\infty} \|T^m x\|^2 \left[u_r(m) - u_r(m-1)\right]. \tag{1.35}$$

We have to prove that it is zero for any $x \in \mathcal{H}$.

**Claim.** There exists a constant $C > 0$ independent of $r$ and $m$ such that

$$|u_r(m) - u_r(m-1)| \leq \frac{C}{m^{1+a}} \quad (\forall m \geq 2). \tag{1.36}$$

Indeed, it is easy to check that

$$\sum_{m=0}^{\infty} u_r(m) t^m = \frac{(1-t)^a}{(1-rt)^a} \qquad (|t| < 1 \text{ and } 0 < r < 1).$$

Multiplying by $(1-t)$, we get

$$1 + \sum_{m=1}^{\infty} [u_r(m) - u_r(m-1)] t^m = \frac{(1-t)^{a+1}}{(1-rt)^a} := f_r(t).$$

Hence (1.36) is equivalent to

$$|\widehat{f}_r(n)| \leq \frac{C}{(n+1)^{1+a}} \quad (\forall n), \tag{1.37}$$

where the constant $C$ is independent of $r$ and $n$. Equivalently, we want to prove that the Fourier coefficients of

$$\sum_{n=0}^{\infty} (1+n)^{1+a} \widehat{f}_r(n) z^n = I_{-1-a} f_r \tag{1.38}$$

are uniformly bounded, where for $\beta \in \mathbb{R}$,

$$I_\beta h(z) := \sum_{j=0}^{\infty} (1+j)^{-\beta} \widehat{h}(j) z^j,$$

as in [75, page 737]. We will prove the even stronger result

$$\left\| I_{1-a} f_r'' \right\|_{H^1} \leq C, \tag{1.39}$$

where the constant $C$ does not depend on $r$. Here, $H^p$ denotes the classical Hardy space of the unit disc. Since for any $\beta > 0$

$$I_\beta h \in H^1 \quad \Longleftrightarrow \quad \int_{\mathbb{T}} \left( \int_0^1 |h(\rho\zeta)|^2 (1-r)^{2\beta-1}\, d\rho \right)^{1/2} |d\zeta| < \infty \tag{1.40}$$

(see [75, page 737]), we obtain that (1.39) is equivalent to

$$\sup_{r\in[0,1]} \int_{\mathbb{T}} \left[ \int_0^1 |f_r''(\rho\zeta)|^2 (1-\rho)^{1-2a}\, d\rho \right]^{1/2} |d\zeta| \le C. \tag{1.41}$$

It is immediate to check that $f_r''$ can be represented as

$$f_r''(t) = \sum_{j=0}^{2} c_j(r)(1-t)^{a-1+j}(1-rt)^{-a-j},$$

where the $c_j$'s are bounded functions. Using that $|1-rt| \ge M|1-t|$ for a certain constant $M$ (for $|t|<1$ and $0<r<1$), we obtain that

$$|f_r''(t)| \le C_1 |1-t|^{-1}.$$

Therefore we just need to prove that

$$\int_{\mathbb{T}} \left[ \int_0^1 |1-\rho\zeta|^{-2} (1-\rho)^{1-2a}\, d\rho \right]^{1/2} |d\zeta| < \infty \tag{1.42}$$

because now we do not have dependence on $r$. By (1.40) this is equivalent to

$$I_{1-a} g \in H^1, \quad \text{where} \quad g(t) = (1-t)^{-1}.$$

Hence we want to prove that

$$\sum_{n=0}^{\infty} (n+1)^{a-1} t^n \in H^1.$$

For the sake of an easier notation, we will prove an equivalent statement

$$\sum_{n=1}^{\infty} n^{a-1} t^n \in H^1. \tag{1.43}$$

Recall that $k(t) = (1-t)^{-a}$. Then, by [93, Volume I, page 77 (1.18)] we have

$$k_n = \frac{1}{\Gamma(a)} n^{a-1} \left( 1 + O\left( \frac{1}{n} \right) \right).$$

Therefore

$$n^{a-1} = \Gamma(a)k_n - n^{a-1}v_n, \qquad \text{where } |v_n| \lesssim n^{-1}.$$

Since $0 < a < 1$, we know that $\sum k_n t^n = (1-t)^{-a}$ belongs to $H^1$ and the function $\sum n^{a-1} v_n t^n$ indeed belongs to $H^2$, so (1.43) follows. This finishes the proof of the Claim.

Finally, since $T$ is an $a$-contraction we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+a}} \|T^n x\|^2 \sim \sum_{n=0}^{\infty} \alpha_n \|T^n x\|^2 \quad \text{converges for every } x \in \mathcal{H}. \tag{1.44}$$

By the claim, we can estimate the series in the right hand side of (1.35) as

$$\sum_{m=2}^{\infty} \|T^m x\|^2 \, |u_r(m) - u_r(m-1)| \lesssim \sum_{m=2}^{\infty} \frac{\|T^m x\|^2}{m^{1+a}} < \infty.$$

Hence, Lebesgue's Dominated Convergence Theorem allows us to exchange the limit with the series in (1.35), and using that $u_1(j) = 0$ for every $j \geq 1$, we obtain that $|||\widehat{Tx}|||^2 = |||\widehat{x}|||^2$ for any $x \in \mathcal{H}$. Hence $S$ is well defined and it is an isometry.

**Step 3.** As usual, $\mathfrak{D}$ is the closure of $D\mathcal{H}$. Let

$$G : \mathcal{H} \to (\mathcal{H}_a \otimes \mathfrak{D}) \oplus \widehat{\mathcal{H}}, \quad Gx := (\{Dx, DTx, DT^2x, \dots\}, \boldsymbol{x}).$$

By (1.31), $G$ is an isometry. It is immediate that

$$((B_a \otimes I_{\mathfrak{D}}) \oplus S)G = GT.$$

Hence we get that $T$ is unitarily equivalent to a part of $(B_a \otimes I_{\mathfrak{D}}) \oplus S$. □

## 1.7 Remarks on the models for $a$-contractions with $a > 1$

In view of Theorem 0.4, which gives a model for $a$-contractions when $0 < a < 1$, it is natural to ask whether the statements

(a) $T$ is an $a$-contraction,

(b) There exists a separable Hilbert space $\mathcal{E}$ such that $T$ is unitarily equivalent to a part of an operator $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an $m$-isometry,

are equivalent, for $a > 1$ and $m - 1 < a \leq m$ ($m$ integer).

It turns out that one implication is true, but the other is false in general.

**Theorem 1.25.** *Let $a > 1$ and let $m$ be the positive integer such that $m - 1 < a \leq m$. Then any part of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an $m$-isometry and $\mathcal{E}$ is an auxiliary Hilbert space, is an $a$-contraction.*

We give the proof of this result right after Remark 4.16, when all the ingredients will be available. The idea is to prove that both $B_k$ and $S$ are $a$-contractions.

**Proposition 1.26.** *Let $a$ belong to the set*

$$A := \bigcup_{j \geq 1} (2j - 1, 2j] \subset \mathbb{R}.$$

*Let $m$ be the positive integer such that $m - 1 < a \leq m$, and take $s \in (m, a + 1)$ (such $s$ exists). Then the forward weighted shift $F_s$ is an $a$-contraction that cannot be modeled by a part of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an $m$-isometry.*

The proof will be given right before Proposition 4.20, when all the ingredients will be available. The idea is a comparison of operator norms.

**Remark 1.27.** It remains open the question of whether (a) implies (b) when $a > 1$ belongs to $\mathbb{R} \setminus A$.

# Chapter 2

# Models up to similarity

In this chapter we present some results related with functional models up to similarity. The main advantage of doing so is that now we can enlarge the family of functions $\alpha$ considered. For instance, now we allow the function $\alpha \in A_{W,\mathbb{R}}$ to have zeroes in $\mathbb{D}$, and $\alpha$ does not need to be of Nevannlina-Pick type. The models up to similarity still allows us to derive spectral and ergodic consequences.

## 2.1 Proof of Theorem 0.11

Recall that for a fixed operator $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subset \overline{\mathbb{D}}$, we put

$$\mathcal{A}_T := \left\{ \beta \in A_W \,:\, \sum_{n=0}^{\infty} |\beta_n| T^{*n} T^n \text{ converges in SOT} \right\}. \tag{2.1}$$

For any $\beta \in \mathcal{A}_T$, define

$$\|\beta\|_T := \sup_N \left\| \sum_{n=0}^{N} |\beta_n| T^{*n} T^n \right\|_{\boldsymbol{B}(\mathcal{H})} + \|\beta\|_{A_W}. \tag{2.2}$$

By Proposition 1.4, if $\beta \in \mathcal{A}_T$, then there exists a constant $C$ such that

$$\sum_{n=0}^{N} |\beta_n| \|T^n x\|^2 \leq C \|x\|^2 \tag{2.3}$$

for every integer $N$ and every $x \in \mathcal{H}$. Indeed, it is immediate to see that one can take $C = \|\beta\|_{\mathcal{A}_T}$ above, and hence we obtain the following result.

**Proposition 2.1.** *If $\beta$ is a function in $\mathcal{A}_T$, then for every vector $x \in \mathcal{H}$ we have*

$$\sum_{n=0}^{\infty} |\beta_n| \|T^n x\|^2 \leq \|\beta\|_{\mathcal{A}_T} \|x\|^2 .$$

**Theorem 2.2.** *For every operator $T$ in $\boldsymbol{B}(\mathcal{H})$ with spectrum contained in $\overline{\mathbb{D}}$, $\mathcal{A}_T$ is a Banach algebra with norm given by* (2.2).

*Proof.* Let us prove that $\mathcal{A}_T$ has the multiplicative property of algebras and its completeness (the rest of properties for being a Banach algebra are immediate).

Let $\beta$ and $\gamma$ belong to $\mathcal{A}_T$. We want to prove that their product $\delta$ also belongs to $\mathcal{A}_T$. Note that $\delta \in A_W$. By Proposition 1.4, we just need to prove the existence of a constant $C > 0$ such that

$$\sum_{n=0}^{N} |\delta_n| \, \|T^n x\|^2 \leq C \, \|x\|^2 \tag{2.4}$$

for every nonnegative integer $N$ and every vector $x \in \mathcal{H}$. So let us fix $N \geq 0$ and $x \in \mathcal{H}$. Put

$$|\beta|(t) := \sum_{n=0}^{\infty} |\beta_n| t^n, \quad |\gamma|(t) := \sum_{n=0}^{\infty} |\gamma_n| t^n, \quad \widetilde{\delta}(t) = |\beta|(t) \cdot |\gamma|(t).$$

Hence

$$\widetilde{\delta}_n = \sum_{j=0}^{n} |\beta_j||\gamma_{n-j}| \geq |\delta_n|.$$

Therefore (2.4) will follow if we prove the existence of a positive constant $C$ such that

$$\sum_{n=0}^{N} \widetilde{\delta}_n \, \|T^n x\|^2 \leq C \, \|x\|^2 . \tag{2.5}$$

Using that $[|\beta|]_N \cdot |\gamma| \succ [\widetilde{\delta}]_N$ (recall Notation 1.24), we have that

$$\begin{aligned}\sum_{n=0}^{N} \widetilde{\delta}_n \, \|T^n x\|^2 &= \left\langle [\widetilde{\delta}]_N (T^*, T) x, x \right\rangle \leq \langle ([|\beta|]_N \cdot |\gamma|)(T^*, T) x, x \rangle \\ &= \sum_{n=0}^{N} \langle |\gamma|(T^*, T) |\beta_n| T^n x, T^n x \rangle \leq \| |\gamma|(T^*, T) \| \sum_{n=0}^{N} |\beta_n| \, \|T^n x\|^2 \\ &\leq \| |\gamma|(T^*, T) \| \, \|\beta\|_{\mathcal{A}_T} \, \|x\|^2 .\end{aligned}$$

Note that the operator $|\gamma|(T^*, T)$ belongs to $\boldsymbol{B}(\mathcal{H})$ because $\gamma \in \mathcal{A}_T$. Now we can take $C = \| |\gamma|(T^*, T) \| \, \|\beta\|_{\mathcal{A}_T}$ (which depends neither on $N$ nor on $x$) and (2.5) follows.

Let us prove now the completeness of $\mathcal{A}_T$. Let $\{\beta^{(k)}\}_{k \geq 0}$ be a Cauchy sequence in $\mathcal{A}_T$. In other words,

$$\left\| \beta^{(k)} - \beta^{(\ell)} \right\|_T \to 0 \quad \text{when } k, \ell \to \infty. \tag{2.6}$$

We want to prove the existence of a function $\beta$ in $\mathcal{A}_T$ such that the sequence $\{\beta^{(k)}\}_{k\geq 0}$ converges to $\beta$ in the norm $\|\cdot\|_T$. Since $\|\cdot\|_T \geq \|\cdot\|_{A_W}$ and $A_W$ is complete, there exists a function $\beta$ in $A_W$ such that $\beta^{(k)}$ converge to $\beta$ in the norm of $A_W$. Now fix $\varepsilon > 0$. Then by (2.6) there exists an integer $M$ such that

$$\left\|\beta^{(k)} - \beta^{(\ell)}\right\|_T < \varepsilon$$

if $k, \ell \geq M$. In other words, for every $N$ we have

$$\left\|\sum_{n=0}^{N} |\beta_n^{(k)} - \beta_n^{(\ell)}| T^{*n} T^n\right\| + \left\|\beta^{(k)} - \beta^{(\ell)}\right\|_{A_W} < \varepsilon.$$

Now taking the limit when $\ell \to \infty$ above, we obtain

$$\left\|\sum_{n=0}^{N} |\beta_n^{(k)} - \beta_n| T^{*n} T^n\right\| + \left\|\beta^{(k)} - \beta\right\|_{A_W} \leq \varepsilon,$$

so $\left\|\beta^{(k)} - \beta\right\|_T \leq \varepsilon$ (if $k \geq M$). $\square$

Recall that in the Introduction we defined $\mathcal{A}_T^0$ as

$$\mathcal{A}_T^0 = \left\{\beta \in \mathcal{A}_T \,:\, \sum_{n=0}^{\infty} |\beta_n| T^{*n} T^n \text{ converges in norm of } \boldsymbol{B}(\mathcal{H})\right\}. \tag{2.7}$$

**Proposition 2.3.** *$\mathcal{A}_T^0$ is the closure of the polynomials in $\mathcal{A}_T$. In particular, it is a separable closed subalgebra of $\mathcal{A}_T$.*

*Proof.* Let us denote provisionally by $\mathcal{CP}$ the closure of the polynomials in $\mathcal{A}_T$. Let $\beta \in \mathcal{CP}$ and fix $\varepsilon > 0$. There exist a polynomial $p$ such that

$$\|\beta - p\|_{A_T} < \varepsilon/2.$$

Let $N$ be an integer bigger that the degree of $p$. Since $p = [p]_N$,

$$\begin{aligned}\left\|\sum_{n=N+1}^{\infty} |\beta_n| T^{*n} T^n\right\|_{\boldsymbol{B}(\mathcal{H})} &\leq \|\beta - [\beta]_N\|_{A_T} \leq \|\beta - p\|_{A_T} + \|[\beta - p]_N\|_{A_T} \\ &\leq 2\,\|\beta - p\|_{A_T} < \varepsilon.\end{aligned}$$

Hence $\sum_{n=0}^{\infty} |\beta_n| T^{*n} T^n$ converges uniformly in $\boldsymbol{B}(\mathcal{H})$. This proves the inclusion $\mathcal{CP} \subset \mathcal{A}_T^0$.

The inclusion $\mathcal{A}_T^0 \subset \mathcal{CP}$ is immediate. Indeed, any $\beta \in \mathcal{A}_T^0$ can be approximated in $\mathcal{A}_T$ by the truncations $[\beta]_N$. $\square$

**Proposition 2.4.** *Let $\beta$ be a function in $\mathcal{A}_T$.*

(i) *If $|\gamma_n| \le |\beta_n|$ for every $n$, then $\gamma$ also belongs to $\mathcal{A}_T$ and moreover $\|\gamma\|_{\mathcal{A}_T} \le \|\beta\|_{\mathcal{A}_T}$.*

(ii) *If $\gamma_n = \beta_n \tau_n$, where $\tau_n \to 0$, then $\gamma$ also belongs to $\mathcal{A}_T^0$.*

*Proof.* (i) is immediate. For the proof of (ii), put

$$C_N := \max_{n \ge N} |\tau_n|,$$

for each positive integer $N$. Then

$$\left\| \sum_{n=N}^{\infty} |\gamma_n| T^{*n} T^n \right\| \le C_N \left\| \sum_{n=N}^{\infty} |\beta_n| T^{*n} T^n \right\| \le C_N \|\beta\|_{\mathcal{A}_T} \to 0,$$

and therefore $\gamma$ belongs to $\mathcal{A}_T^0$. □

**Proposition 2.5.** *The characters of $\mathcal{A}_T^0$ are precisely the evaluation functionals at points of $\overline{\mathbb{D}}$.*

*Proof.* Let $\chi$ be a character of $\mathcal{A}_T^0$ (i.e., it is a multiplicative bounded linear functional on $\mathcal{A}_T^0$ that satisfies $\chi(1) = 1$). For the function $t$ in $\mathcal{A}_T^0$, let us put

$$\lambda := \chi(t) \in \mathbb{C}.$$

Therefore $\chi$ sends every polynomial $p(t)$ into the number $p(\lambda)$. Let us prove now that $|\lambda| \le 1$. Using the obvious fact

$$|\lambda| = (|\lambda|^n)^{1/n} = |\chi(t^n)|^{1/n}$$

we obtain

$$|\lambda| = \limsup_{n\to\infty} |\chi(t^n)|^{1/n} \le \limsup_{n\to\infty} \|t^n\|_{\mathcal{A}_T^0}^{1/n} \le 1,$$

where we have used that $\|\chi\| = 1$ (since it is a character) and that the spectral radius of $T$ is less or equal than 1.

By the continuity of $\chi$ and the density of the polynomials in $\mathcal{A}_T^0$ we obtain that $\chi$ maps every function $f(t)$ in $\mathcal{A}_T^0$ into $f(\lambda)$. □

**Corollary 2.6.** *If $\beta \in \mathcal{A}_T^0$ with $\beta(t) \ne 0$ for every $t \in \overline{\mathbb{D}}$, then $1/\beta \in \mathcal{A}_T^0$.*

Indeed, the condition on $h$ means that $\chi(h) \ne 0$ for every character $\chi$. Hence the result follows using the Gelfand Theory.

We denote by $\mathcal{A}_{T,\mathbb{R}}$ and $\mathcal{A}_{T,\mathbb{R}}^0$ the sets of elements in $\mathcal{A}_T$ and $\mathcal{A}_T^0$, respectively, whose all coefficients are real.

**Theorem 2.7.** *Let $T \in \boldsymbol{B}(\mathcal{H})$ with $\sigma(T) \subset \overline{\mathbb{D}}$ and let $f \in \mathcal{A}^0_{T,\mathbb{R}}$. If $f(t) > 0$ for every $t \in [0,1]$, then there exists a function $g \in \mathcal{A}^0_{T,\mathbb{R}}$ such that $g \succ 0$ and $fg \succ 0$.*

As an immediate consequence of this theorem, we obtain the following result.

**Corollary 2.8.** *If $f \in A_{W,\mathbb{R}}$ satisfies $f(t) > 0$ for every $t \in [0,1]$, then there exists a function $g \in A_{W,\mathbb{R}}$ such that $g \succ 0$ and $fg \succ 0$.*

*Proof.* Take as $T$ any power bounded operator. Then $\sigma(T) \subset \overline{\mathbb{D}}$ and $\mathcal{A}^0_T$ is precisely the Wiener algebra $A_W$. Then apply Theorem 2.7. □

For the proof of Theorem 2.7 we need a few preliminary facts.

**Lemma 2.9.** *If $q(t) = (t-\lambda)(t-\overline{\lambda})$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then there exists a polynomial $p$ such that $p \succ 0$ and $pq \succ 0$.*

*Proof.* Let $m$ be the smallest nonnegative integer such that $\Re(\lambda^{2^m}) \le 0$. We define

$$p(t) := \prod_{j=0}^{m-1} (t^{2^j} + \lambda^{2^j})(t^{2^j} + \bar{\lambda}^{2^j})$$

(so that $p(t) = 1$ if $m = 0$). Note that by the minimality of $m$, for each factor we have

$$(t^{2^j} + \lambda^{2^j})(t^{2^j} + \bar{\lambda}^{2^j}) = t^{2^{j+1}} + 2\Re(\lambda^{2^j})t^{2^j} + |\lambda^{2^j}| \succ 0.$$

Therefore $p \succ 0$. Moreover

$$(pq)(t) = (t^{2^m} - \lambda^{2^m})(t^{2^m} - \bar{\lambda}^{2^m}) = t^{2^{m+1}} - 2\Re(\lambda^{2^m})t^{2^m} + |\lambda^{2^m}| \succ 0. \qquad \square$$

**Corollary 2.10.** *If $q$ is a real polynomial without real roots and $q(0) > 0$, then there exists a polynomial $p$ such that $p \succ 0$ and $pq \succ 0$.*

*Proof.* Note that $q = Cq_1 \cdots q_k$, where $C$ is a positive constant and each factor $q_j$ has the form $q_j(t) = (t-\lambda_j)(t-\overline{\lambda}_j)$ for some $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$. Then for each factor $q_j$ we can construct the polynomial $p_j$ as in the previous lemma and we just take $p = p_1 \cdots p_k$. □

We denote by $\mathcal{H}(\overline{\mathbb{D}})$ the set of all analytic functions in a neighborhood of $\overline{\mathbb{D}}$.

**Lemma 2.11.** *If $q$ is a real polynomial such that $q(t) > 0$ for every $t \in [0,1]$, then there exists a rational function $u \in \mathcal{H}(\overline{\mathbb{D}})$ such that $u \succ 0$ and $uq \succ 0$.*

It is not difficult to deduce this lemma from a classic result by Bernstein [22]: if $q$ is a real polynomial such that $q(t) > 0$ for every $t \in (0,1)$, then $q$ can be written down as a linear combination of polynomials of the form $t^j(1-t)^k$ with positive coefficients. (To do it, one has to apply Bernstein's theorem to a slightly larger interval.) Bernstein's theorem can also be related with a result by Pólya (1928) on homogeneous forms, strictly positive on a tetrahedron, see [78, Section 1.3]. Here we give an independent proof of Lemma 2.11.

*Proof of Lemma 2.11.* Write $q$ as the product of polynomials $q_{nr}, q_+$ and $q_-$, where the roots of $q_{nr}$ are nonreal, the roots of $q_+$ are positive and the roots of $q_-$ are negative. Without loss of generality, $q_+(0) = 1$, $q_-(0) = 1$ and $q_{nr}(0) = q(0) > 0$. Therefore $q_- \succ 0$ and by Corollary 2.10, there exists a polynomial $p$ such that $p \succ 0$ and $pq_{nr} \succ 0$. Notice that $1/q_+ \in \mathcal{H}(\overline{\mathbb{D}})$ and $1/q_+ \succ 0$. Hence, we can take $u := p/q_+$, and the statement follows. $\square$

*Proof of Theorem 2.7.* By hypothesis, there exists a positive number $\varepsilon$ such that $f(t) > \varepsilon > 0$ for every $t \in [0, 1]$.

**Claim.** There exists a positive integer $N$ such that

$$\left\| \sum_{n=N+1}^{\infty} f_n t^n \right\|_{\mathcal{A}_T^0} < \varepsilon/2.$$

Indeed, since $f \in A_{W,\mathbb{R}}$ we deduce that there exists a positive integer $N_1$ such that for every $M \geq N_1$ we have

$$\sum_{n=M+1}^{\infty} |f_n| < \varepsilon/4.$$

Using Proposition 2.3 we obtain the existence of a positive integer $N_2$ such that for every $M \geq N_2$ we have

$$\left\| \sum_{n=M+1}^{\infty} |f_n| T^{*n} T^n \right\|_{\boldsymbol{B}(\mathcal{H})} < \frac{\varepsilon}{4}.$$

Hence, taking $N := \max\{N_1, N_2\}$ the claim follows.

In particular, we get

$$\sum_{n=N+1}^{\infty} |f_n| < \varepsilon/2.$$

Put

$$f_N(t) := \sum_{n=0}^{N} f_n t^n - \frac{\varepsilon}{2}, \qquad h(t) := \frac{\varepsilon}{2} + \sum_{n \geq N+1; f_n < 0} f_n t^n.$$

Note that $f_N$ is a polynomial and $h \in \mathcal{A}_T^0$ (by Proposition 2.4 (i)). Since

$$f_N(t) > \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 0$$

for every $t \in [0, 1]$, we can use Lemma 2.11 to obtain a function $u \in \mathcal{H}(\overline{\mathbb{D}})$ such that $u \succ 0$ and $uf_N \succ 0$. Note that it is immediate that all the functions in $\mathcal{H}(\overline{\mathbb{D}})$ also belong to the algebra $\mathcal{A}_T^0$.

For every $t \in \overline{\mathbb{D}}$ we have

$$\left| \sum_{n \geq N+1; f_n < 0} f_n t^n \right| \leq \sum_{n=N+1}^{\infty} |f_n| < \frac{\varepsilon}{2}.$$

Hence $h$ does not vanish on $\overline{\mathbb{D}}$, and therefore using Corollary 2.6 we obtain

$$v := h^{-1} \in \mathcal{A}_T^0.$$

Note that $v \succ 0$. Finally, put $g := uv$. Obviously $g \succ 0$, and since $f \succcurlyeq f_N + h$ we obtain that

$$gf \succcurlyeq g(f_N + h) = vuf_N + u \succ 0,$$

as we wanted to prove. □

Recall that, by Proposition 1.2, if $\beta \in \mathcal{A}_{T,\mathbb{R}}$, then we can define the operator

$$\beta(T^*, T) := \sum_{n=0}^{\infty} \beta_n T^{*n} T^n, \tag{2.8}$$

where the convergence of the series is in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$. The following result generalizes Proposition 1.2.

**Proposition 2.12.** *Let $f \in \mathcal{A}_{T,\mathbb{R}}$ and let $B \in \boldsymbol{B}(\mathcal{H})$ be a positive operator. Then the operator series*

$$\sum_{n=0}^{\infty} f_n T^{*n} B T^n \tag{2.9}$$

*converges in the strong operator topology in $\boldsymbol{B}(\mathcal{H})$.*

*Proof.* First we observe that the above series converges in the weak operator topology. Indeed, for every pair of vectors $x, y \in \mathcal{H}$ we have

$$\begin{aligned}\Big|\Big\langle \Big( \sum_{n=N}^{M} f_n T^{*n} B T^n \Big) x, y \Big\rangle\Big| &= \left| \sum_{n=N}^{M} f_n \langle BT^n x, T^n y \rangle \right| \leq \sum_{n=N}^{M} |f_n| \, \|B\| \, \|T^n x\| \, \|T^n y\| \\ &\leq \sum_{n=N}^{M} |f_n| \, \|B\| \, (\|T^n x\|^2 + \|T^n y\|^2) \to 0 \quad (N, M \to \infty),\end{aligned}$$

and the statement follows. Next, put

$$f_n^+ := \max\{f_n, 0\}, \quad f_n^- := \max\{-f_n, 0\}.$$

By the above, the series

$$\sum_{n=0}^{\infty} f_n^+ T^{*n} B T^n \quad \text{and} \quad \sum_{n=0}^{\infty} f_n^- T^{*n} B T^n$$

converge in the weak operator topology in $\boldsymbol{B}(\mathcal{H})$. By Lemma 1.1 these series also converges in SOT. Since $f_n = f_n^+ - f_n^-$, we also obtain the convergence in SOT of the series $\sum f_n T^{*n} B T^n$. $\square$

**Definition 2.13.** As a consequence of Proposition 2.12, for every $f \in \mathcal{A}_{T,\mathbb{R}}$ and every positive operator $B \in \boldsymbol{B}(\mathcal{H})$ we can define

$$f(T^*, T)(B) := \sum_{n=0}^{\infty} f_n T^{*n} B T^n,$$

where the convergence is in SOT. In particular, when $B$ is the identity operator in $\boldsymbol{B}(\mathcal{H})$,

$$f(T^*, T) = f(T^*, T)(I) = \sum_{n=0}^{\infty} f_n T^{*n} T^n.$$

**Remark 2.14.** Observe that, by applying Proposition 2.1 to the vector $x = B^{1/2}h$, we get

$$\|f(T^*, T)(B)\| \lesssim \|B\| \, \|f\|_{\mathcal{A}_T}. \tag{2.10}$$

**Lemma 2.15.** *Let $B \in \boldsymbol{B}(\mathcal{H})$ be a positive operator and let $f, g \in \mathcal{A}_{T,\mathbb{R}}$. Put $h := fg$. Then*

(i) *$h(T^*, T)(B) = g(T^*, T)(f(T^*, T)(B))$;*

(ii) *$h(T^*, T) = g(T^*, T)(f(T^*, T))$.*

*Proof.* Note that (ii) is just an application of (i) for $B = I$. Let us start proving (i) for the case where all the coefficients $f_n$ and $g_n$ are nonnegative. In this case, both parts of (i) are well defined by Proposition 2.12. Then

$$\begin{aligned} g(T^*, T)(f(T^*, T)(B)) &= \sum_{n=0}^{\infty} g_n T^{*n} \Big( \sum_{m=0}^{\infty} f_m T^{*m} B T^m \Big) T^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_n f_m T^{*n+m} B T^{n+m} \\ &\overset{(\star)}{=} \sum_{k=0}^{\infty} \Bigg( \sum_{n+m=k} f_m g_n \Bigg) T^{*k} B T^k = h(T^*, T)(B), \end{aligned}$$

where in $(\star)$ all the series are understood in the sense of the SOT convergence. To justify it, it suffices to pass to quadratic forms and to use that $f_m$ and $g_n$ are all nonnegative. Finally, the general case $f, g \in \mathcal{A}_T$ can be derived from the previous one by linearity and using the decompositions

$$f = f^+ - f^-, \quad g = g^+ - g^-,$$

where $f^+, f^-, g^+$ and $g^-$ have nonnegative Taylor coefficients. □

*Proof of Theorem 0.11.* By Theorem 2.7, there exists a function $\widetilde{\beta} \in \mathcal{A}_T^0$ such that $\widetilde{\beta} \succ 0$ and $f := \widetilde{\beta}\widetilde{\alpha} \succ 0$. Then $\eta(t)f(t) = \widetilde{\beta}(t)\alpha(t)$ and by Lemma 2.15 (ii) we get

$$\sum_{n=0}^{\infty} \eta_n T^{*n} f(T^*, T) T^n = \sum_{n=0}^{\infty} \widetilde{\beta}_n T^{*n} \alpha(T^*, T) T^n \geq 0. \tag{2.11}$$

Define an operator $B > 0$ by $B^2 := f(T^*, T) \geq \varepsilon I > 0$ (for some $\varepsilon > 0$). Then we have

$$\sum_{n=0}^{\infty} \eta_n \|BT^n x\|^2 \geq 0$$

for every $x \in \mathcal{H}$. Denote $y := Bx$ and put

$$\widehat{T} := BTB^{-1}.$$

We get that

$$\sum_{n=0}^{\infty} \eta_n \left\|\widehat{T}^n y\right\|^2 \geq 0$$

for every $y \in \mathcal{H}$. Therefore $\widehat{T}$ is similar to $T$ and $\eta(\widehat{T}^*, \widehat{T}) \geq 0$. □

In particular this proves Corollary 0.12. Part (ii) of this corollary can be compared with Müller's Theorem 3.10 in [65]. Notice, however, that in this theorem it is assumed that $T$ is a contraction.

## 2.2 Finite Defect

In this section, we derive some consequences of our model theorems for the case when an operator $T \in \mathbf{C}_\alpha$ is $\alpha$-modelable and the defect operator $D = (\alpha(T^*, T))^{1/2}$ is of finite rank.

We will assume that the reproducing kernel Hilbert space $\mathcal{R}_k$ is a Banach algebra with respect to the multiplication of power series. By [87, Proposition 32], it suffices to assume that

$$\sup_n \sum_{j=0}^{n} \frac{k_j^2 k_{n-j}^2}{k_n^2} < \infty;$$

compare with the condition (1.16). Put

$$m_n = \inf_j \frac{k_j}{k_{n+j}}, \quad r_1 = \lim_{n\to\infty} m_n^{1/n}$$

(this limit exists).

We will assume that

$$r_1 = \lim_{n\to\infty} k_n^{1/n} = 1. \tag{2.12}$$

We also are assuming here that the isometric part $S$ is not present in the model of $T$. Hence, $T$ is unitarily equivalent to the restriction of the backward shift $B_k \otimes I_{\mathfrak{D}}$ on the space $\mathcal{H}_k \otimes \mathfrak{D}$ to an invariant subspace $\mathcal{E}$. More generally, this applies to similarity instead of the unitary equivalence. (See Theorem 0.11.)

Here we prove the following result.

**Theorem 2.16.** *Suppose that $T$ is similar to a part of $B_k \otimes I_{\mathfrak{D}}$, acting on the space $\mathcal{H}_k \otimes \mathfrak{D}$, where $\mathcal{R}_k$ is a Banach algebra and $\mathfrak{D}$ is finite dimensional. If the spectrum $\sigma(T)$ does not cover the open disc $\mathbb{D}$, then $\sigma(T) \cap \mathbb{D}$ is contained in the zero set of a non-zero function in $\mathcal{R}_k$.*

Let us start with some preliminary remarks. Suppose that $T$ is as in the above Theorem 2.16. That is, $T$ is similar to $(B_k \otimes I_{\mathfrak{D}})|\mathcal{E}$, where $\mathcal{E} \subset \mathcal{H}_k \otimes \mathfrak{D}$ is an invariant subspace of $B_k \otimes I_{\mathfrak{D}}$. By fixing a basis in $\mathfrak{D}$, we may assume that $\mathfrak{D} = \mathbb{C}^d$, where $d = \dim \mathfrak{D}$. We will identify the space $\mathcal{H}_k \otimes \mathfrak{D}$ with $\mathcal{H}_k^d = \oplus_1^d \mathcal{H}_k$, whose elements are columns with entries in $\mathcal{H}_k$. The adjoint of $B_k$ on the space $\mathcal{H}_k^d$ is the multiplication operator $M_z$ on the space $\mathcal{R}_k^d$; this later space can be seen as a Banach module over the Banach algebra $\mathcal{R}_k$. Put

$$\mathcal{L} = \mathcal{E}^\perp \subset \mathcal{R}_k \otimes \mathfrak{D}.$$

Then $\mathcal{L}$ is $M_z$- invariant, and $T^*$ is similar to the quotient operator

$$\mathcal{M}_z : \mathcal{R}_k^d/\mathcal{L} \to \mathcal{R}_k^d/\mathcal{L}, \qquad \mathcal{M}_z[f] = [zf].$$

Here $[f] \in \mathcal{R}_k^d/\mathcal{L}$ denotes the coset of a function $f$ in $\mathcal{R}_k^d$. We adapt some ideas from Richter's work [81], which treated the case $d = 1$.

We need the following notions.

**Definition 2.17.** (see [81]). Let $\mathcal{L}$ be a subspace of $\mathcal{R}_k^d$, invariant under $M_z$.

(1) Given a point $\lambda \in \overline{\mathbb{D}}$, the space

$$\mathcal{F}(\lambda) = \mathcal{F}_{\mathcal{L}}(\lambda) := \{g(\lambda) \,:\, g \in \mathcal{L}\}$$

will be referred to as *the fiber of $\mathcal{L}$ over $\lambda$*. Note that $\mathcal{F}(\lambda)$ is a subspace of $\mathbb{C}^d$.

(2) By *the spectrum of $\mathcal{L}$* we understand the set

$$\sigma(\mathcal{L}) := \big\{\lambda \in \overline{\mathbb{D}} : \quad \mathcal{F}_{\mathcal{L}}(\lambda) \neq \mathbb{C}^d\big\}.$$

It will be shown that Theorem 2.16 is an easy consequence of the following result.

**Theorem 2.18.** *Given any subspace $\mathcal{L}$ of $\mathcal{R}_k^d$, invariant under $M_z$, one has*

$$\sigma(\mathcal{L}) \cap \mathbb{D} = \sigma(\mathcal{M}_z) \cap \mathbb{D}.$$

In the proof, we will use the following lemma

**Lemma 2.19.** *If $g \in \mathcal{R}_k$, $\lambda \in \mathbb{D}$ and $g(\lambda) = 0$, then $(z-\lambda)^{-1}g(z) \in \mathcal{R}_k$.*

*Proof.* Assume that $|\lambda| < r_1 = 1$. By [87, Proposition 13], the operator $M_z - \lambda$ is bounded from below. Notice that $(z-\lambda)^{-1}g(z) \in \mathcal{R}_k$ if and only if $g$ belongs to the closed subspace $\operatorname{Ran}(M_z - \lambda)$. This happens if and only if $g$ is orthogonal to $\ker(M_z^* - \bar{\lambda})$. This kernel is one-dimensional and is generated by the antilinear evaluation functional $g \mapsto \overline{g(\lambda)}$, which implies our assertion. $\square$

*Proof of Theorem 2.18.* First we observe that $\eta \cdot \mathcal{L} \subset \mathcal{L}$ for any $\eta$ in the algebra $\mathcal{R}_k$, which is easy to get, approximating $\eta$ by polynomials.

Assume first that $\lambda \in \mathbb{D}$, but $\lambda \notin \sigma(\mathcal{L})$. This means that $\mathcal{F}(\lambda) = \mathbb{C}^d$. Let us prove that $\lambda \notin \sigma(\mathcal{M}_z)$ (this will give the inclusion $\sigma(\mathcal{M}_z) \cap \mathbb{D} \subset \sigma(\mathcal{L}) \cap \mathbb{D}$). That is, we will see that $\mathcal{M}_z - \lambda$ is invertible in $\mathcal{R}_k^d/\mathcal{L}$.

*Claim:* If $h \in \mathcal{R}_k^d$ and $(z-\lambda)h \in \mathcal{L}$, then $h \in \mathcal{L}$.

Indeed, assume that $h$ satisfies these assumptions. Since $\mathcal{F}(\lambda) = \mathbb{C}^d$, there exist functions $\varphi_1, \ldots, \varphi_d$ in $\mathcal{L}$ such that $\varphi_j(\lambda) = e_j$ (where $\{e_j\}$ is the canonical base in $\mathbb{C}^d$). Consider the $d \times d$ matrix function

$$\Phi := (\varphi_1 | \cdots | \varphi_d) \in \mathcal{R}_k^{d\times d}, \qquad \text{and set} \qquad \varphi := \det \Phi \in \mathcal{R}_k.$$

Note that $\Phi\gamma = \gamma_1\varphi_1 + \cdots + \gamma_d\varphi_d \in \mathcal{L}$ for every $\gamma \in \mathcal{R}_k^d$. Hence, $\varphi h = \Phi\Phi^{\mathrm{ad}}h \in \mathcal{L}$. We also observe that $(\varphi - 1)h$ belongs to $\mathcal{L}$. Indeed, since $\varphi(\lambda) = 1$, we have

$$(\varphi - 1)h = \frac{\varphi(z) - \varphi(\lambda)}{z - \lambda}\,(z-\lambda)h \in \mathcal{L},$$

because $(\varphi(z) - \varphi(\lambda))/(z-\lambda) \in \mathcal{R}_k$ by Lemma 2.19 and $(z-\lambda)h \in \mathcal{L}$. Therefore,

$$h = \varphi h - (\varphi - 1)h \in \mathcal{L},$$

which proves our claim.

To check that $\mathcal{M}_z - \lambda$ is invertible in $\mathcal{R}_k^d/\mathcal{L}$, take an arbitrary element $f$ in $\mathcal{R}_k^d$, and let us study the solutions of the equation

$$(\mathcal{M}_z - \lambda)[h] = [f]$$

with respect to an unknown coclass $[h] \in \mathcal{R}_k^d/\mathcal{L}$. By the Claim, there is no more than one solution. On the other hand, if we set

$$h(z) = (z-\lambda)^{-1}\big(f(z) - \Phi(z)f(\lambda)\big),$$

then by Lemma 2.19, $h \in \mathcal{R}_k^d$, so that $[h]$ is a solution of the above equation. Note that $\Phi(\lambda) = I$. It follows that the above formula defines a bounded map $[f] \mapsto [h]$, which proves that the inverse to $\mathcal{M}_z - \lambda$ exists and is bounded on $\mathcal{R}_k^d/\mathcal{L}$. This completes the proof of the inclusion $\sigma(\mathcal{M}_z) \cap \mathbb{D} \subset \sigma(\mathcal{L}) \cap \mathbb{D}$.

To prove the opposite inclusion, take a point $\lambda$ in $\sigma(\mathcal{L}) \cap \mathbb{D}$ and let us see that $\lambda$ belongs to $\sigma(\mathcal{M}_z)$. In other words, we wish to prove that $\mathcal{M}_z - \lambda$ is not invertible in $\mathcal{R}_k^d/\mathcal{L}$.

Since $\lambda \in \sigma(\mathcal{L})$, the fiber $\mathcal{F}(\lambda)$ is not all $\mathbb{C}^d$. Hence, there exists a nonzero antilinear functional $\Psi$ on $\mathbb{C}^d$ such that $\Psi|\mathcal{F}(\lambda) \equiv 0$. It defines an antilinear functional on $\mathcal{R}_k^d$, given by

$$\widehat{\Psi}(f) = \Psi(f(\lambda)).$$

Note that $\widehat{\Psi} \neq 0$, but $\widehat{\Psi}|\mathcal{L} \equiv 0$. Hence, we obtain the antilinear functional $\widetilde{\Psi}$ on the quotient $\mathcal{R}_k^d/\mathcal{L}$, given by

$$\widetilde{\Psi} : \mathcal{R}_k^d/\mathcal{L} \to \mathbb{C}, \qquad \widetilde{\Psi}([f]) := \widehat{\Psi}(f).$$

For every $f \in \mathcal{R}_k^d$ we have

$$\Big\langle [f], (\mathcal{M}_z - \lambda)^* \widetilde{\Psi} \Big\rangle = \widetilde{\Psi}((\mathcal{M}_z - \lambda)[f]) = \widehat{\Psi}((z-\lambda)f) = 0,$$

because $(z-\lambda)f(z)$ vanishes for $z = \lambda$. Hence, $(\mathcal{M}_z - \lambda)^*\widetilde{\Psi} = 0$. Since $\widetilde{\Psi} \neq 0$, we get that $(\mathcal{M}_z - \lambda)$ is not invertible in $\mathcal{R}_k^d/\mathcal{L}$, as we wanted to prove. □

*Proof of Theorem 2.16.* We conserve the notation of the above proof. Fix any point $\lambda$ in $\mathbb{D} \setminus \sigma(T)$. Then, as above, there exist functions $\varphi_j \in \mathcal{L}$ such that $\varphi_j(\lambda) = e_j$, $j = 1, \dots d$. Define the $d \times d$ matrix function $\Phi(z)$ as above and put $\varphi(z) = \det \Phi(z)$, then $\varphi$ belongs to $\mathcal{R}_k$. Observe that $\phi \not\equiv 0$. Notice that the fiber $\mathcal{F}_{\mathcal{L}}(z)$ equals to $\mathbb{C}^d$ whenever $\Phi(z)$ is invertible, that is, whenever $\varphi(z) \neq 0$. Hence $\sigma(\mathcal{L})$ is contained in the zero set of $\varphi$. By Theorem 2.18, $\sigma(T) \cap \mathbb{D} = \sigma(\mathcal{L}) \cap \mathbb{D}$, and this implies the statement of the Theorem. □

As an important corollary to Theorem 2.16 we obtain the following result related to the Carleson condition.

**Theorem 2.20.** *Assume the hypothesis of Theorem 2.16 and also that*

$$\sum_{n=N}^{\infty} k_n \leq C N^{-\varepsilon} \qquad \forall N \geq 0, \tag{2.13}$$

*for some positive constants $C$ and $\varepsilon$ which do not depend on $N$. Suppose that the spectrum $\sigma(T)$ does not cover $\mathbb{D}$. Let*

$$E := (\overline{\sigma(T) \cap \mathbb{D}}) \cap \mathbb{T}$$

*and $\{l_\nu\}$ denotes the lengths of the finite complementary intervals of $E$ (in $\mathbb{T}$). Then the (Lebesgue) measure of $E$ is $0$, and the Carleson condition holds:*

$$\sum_\nu l_\nu \log \frac{2\pi}{l_\nu} < \infty.$$

For the proof of this theorem we need the following two lemmas. The first one is straightforward, so we omit the proof.

**Lemma 2.21.** *Suppose that* (2.13) *holds for some positive constants $C$ and $\varepsilon$ which do not depend on $N$. Then, there exists a number $s \in (0,1)$ (small enough) such that*

$$\sup_{0<t<1} t^{-s} \left( \sum_{t^{s-1}}^{\infty} k_n \right)^{1/2} < \infty. \tag{2.14}$$

**Lemma 2.22.** *Assume the hypothesis of Theorem 2.16 and also* (2.13)*. Then, there exists a positive number $s$ such that the functions in $\mathcal{R}_k$ are Hölder continuous of order $s$ on $\overline{\mathbb{D}}$.*

*Proof.* Take $s > 0$ as in Lemma 2.21, and let $f$ be a function in $\mathcal{R}_k$. To prove that $f$ is Hölder continuous of order $s$ on $\overline{\mathbb{D}}$, it is enough to show that $f$ is Hölder continuous of order $s$ in $\mathbb{T}$ (see [51]). By a rotation argument, we just need to prove that

$$\sup_{\substack{\theta \neq 0 \\ \theta \in [-\pi,\pi]}} |\theta|^{-s} |f(1) - f(e^{i\theta})| < \infty.$$

It is easy to see that this follows from the inequality

$$\sup_{\substack{\theta \neq 0 \\ \theta \in [-\pi,\pi]}} |\theta|^{-s} \left( \sum_{n=0}^{\infty} k_n |1 - e^{in\theta}|^2 \right)^{1/2} < \infty. \tag{2.15}$$

Note that $|1 - e^{in\theta}|^2 \le n^2\theta^2$, hence $|1 - e^{in\theta}|^2 \le |\theta|^{2s}$ if $n \le |\theta|^{s-1}$. Therefore

$$|\theta|^{-s} \left( \sum_{n=0}^{\infty} k_n |1 - e^{in\theta}|^2 \right)^{1/2} \le \left( \sum_{n=0}^{|\theta|^{s-1}} k_n \right)^{1/2} + 2|\theta|^{-s} \left( \sum_{|\theta|^{s-1}}^{\infty} k_n \right)^{1/2},$$

which is uniformly bounded since $\sum k_n$ converges and we are assuming (2.14). Hence (2.15) follows and the statement is proved. $\square$

*Proof of Theorem 2.20.* By Theorem 2.16, we have that $\sigma(T) \cap \mathbb{D}$ is contained in the zero set of a non-zero function $f$ in $\mathcal{R}_k$. By Lemma 2.22, $f$ is Hölder continuous of order $s$ for some $s > 0$. Note that $E$ is a set of uniqueness for $f$. Hence the statement follows using [29, Theorem 1]. $\square$

## 2.3 Ergodic properties of $a$-contractions

In this section we focus only on functions $\alpha$ of the form

$$\alpha(t) = (1-t)^a \tag{2.16}$$

for some $a > 0$. Recall that if $\alpha(T^*, T) \geq 0$ for some $T \in \boldsymbol{B}(\mathcal{H})$, then we say that $T$ is an $a$-contraction. Now

$$k(t) = (1-t)^{-a} = \sum_{n=0}^{\infty} k^a(n) t^n \qquad (|t| < 1). \tag{2.17}$$

We refer to the book [46] for a treatment of ergodic theory in the context of the theory of linear operators.

Recall that in order to emphasize the dependence on the exponent $a$ in (2.16), when $T \in \mathbf{C}_\alpha$ we will write $T \in \mathbf{C}_a$ and say that $T$ is an *$a$-contraction*, and instead of $\mathrm{Adm}_\alpha$ we will use the notation $\mathrm{Adm}_a$. For example, the 1-contractions are just the contractions in $\boldsymbol{B}(\mathcal{H})$.

Similarly, in the above case we denote the space $\mathcal{H}_k$ by $\mathcal{H}_a$, emphasizing the exponent $a$. In the same way we use the notation $B_a$ and $F_a$; these two operators act on $\mathcal{H}_a$.

Notice that $k^a(n) > 0$ for all $n \geq 0$. It follows that $\alpha(t) = (1-t)^a$ satisfies Hypotheses 0.1. Moreover, here we are in the critical case.

If $0 < a \leq 1$, then $\alpha_n \leq 0$ for $n > 0$, so that in this case we can apply Theorem 0.4 to obtain a model for $a$-contractions. This was singled out as an important particular case in [32]. With the help of this model, we will derive here some ergodic properties of $a$-contractions for these values of $a$.

Notice that

$$k^a(n) = (-1)^n \binom{a}{n} = \begin{cases} \dfrac{a(a+1)\cdots(a+n-1)}{n!} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

These numbers are called *Cesàro numbers*. See [93, Volume I, page 77].

We will need the following well-known facts about the asymptotic behavior of the Cesàro numbers.

**Proposition 2.23.** *If $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, then*

$$k^a(n) = \frac{\Gamma(n+a)}{\Gamma(a)\Gamma(n+1)} = \binom{n+a-1}{a-1} \qquad \forall n \geq 0,$$

*where $\Gamma$ is Euler's Gamma function. Therefore*

$$k^a(n) = \frac{n^{a-1}}{\Gamma(a)}(1 + O(1/n)) \qquad \text{as } n \to \infty. \tag{2.18}$$

*Moreover, if $0 < a \leq 1$, then*

$$\frac{(n+1)^{a-1}}{\Gamma(a)} \leq k^a(n) \leq \frac{n^{a-1}}{\Gamma(a)} \qquad \forall n \geq 1.$$

*Proof.* See [93, Volume I, page 77, Equation (1.18)] and [91, Equation (1)]. The last inequality follows from the Gautschi inequality (see [45, Equation (7)]). □

The *weighted space of Bergman-Dirichlet type* $\mathcal{D}_a$, where $a$ is a real parameter, consists of all the analytic functions $f$ in $\mathbb{D}$ with finite norm

$$\|f\|_{\mathcal{D}_s} := \left( \sum_{n=0}^{\infty} (n+1)^a |f_n|^2 \right)^{1/2}.$$

In fact, this is a Bergman-type space if $a < 0$, and is a Dirichlet-type space if $a > 0$. For $a = 0$, we get the Hardy space.

Theorem CH yields that for $0 < a \leq 1$, any $a$-contraction is modelable as a part of an operator $(B_a \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an isometry. Notice that the adjoint to the operator $B_a = B_k$ on $\mathcal{H}_k$ is the operator $M_t g(t) = tg(t)$ on $\mathcal{R}_k$, which is a space with the weighted norm

$$\|g\|^2_{\mathcal{R}_k} = \sum_{n=0}^{\infty} k_n^{-1} |g_n|^2.$$

Hence the characterization of invariant subspaces of $M_t$ on $\mathcal{R}_k$ becomes important. In many cases, this question is related to the description of what is called inner functions in $\mathcal{R}_k$. Since $k_n \asymp (n+1)^{a-1}$, the norm in $\mathcal{R}_k$ is equivalent to the norm in $\mathcal{D}_{-a+1}$, which is a Dirichlet-type space. One can find results in this direction in the papers [86] and [18] by Seco and coauthors; see also references therein.

Since Agler's paper [6], much work has been devoted to $m$-hypercontractions, where $m$ is a natural number. Using the above terminology, one can say that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is called $m$-hypercontraction if it is $k$-contraction for $1 \leq k \leq m$. Notice that an $m$-hypercontraction is always a contraction, whereas an $m$-contraction for $m \geq 2$ needs not to be a contraction. The papers [19, 20, 21, 47, 83] (among others) study $m$-isometries, which are defined by $(1-t)^m(T^*,T)=0$; this is a subclass of $m$-contractions. In [63] 2-isometries are studied in more detail. In [48], a more general class of $(m,p)$-isometries on Banach spaces is treated. The recent work [49] discusses $m$-isometric tuples of operators on a Hilbert space.

In [31], Chavan and Sholapurkar study another interesting class of operators: $T$ is a *joint complete hyperexpansion of order* $m$ if $(1-t)^n(T^*,T)\le 0$ for every integer $n\ge m$. That work, in fact, is devoted to tuples of commuting operators.

As a consequence of Theorem 1.9 we obtain the following result.

**Theorem 2.24.** *Let $a$ and $s$ be positive numbers. Then the following is true.*

(i) $B_s \in \mathrm{Adm}_a$.

(ii) *$B_s$ is an $a$-contraction if and only if $a\le s$.*

*Proof.* Using the notation of Theorem 1.9 we have $\varkappa(t)=(1-t)^{-s}$ and $\alpha(t)=(1-t)^a$. Hence $\beta(t)=p(t)\pm\alpha(t)$, where $p(t)$ is a polynomial, say $p(t)=p_0+p_1t+\cdots+p_Nt^N$, with all the coefficients $p_j$ positive. Then

$$\gamma(t)=(p(t)\pm\alpha(t))\varkappa(t)=p(t)\varkappa(t)\pm(1-t)^{a-s}=:\widetilde{\gamma}(t)+\widehat{\gamma}(t).$$

To prove (i), it is enough to show that

$$\sup_{m\ge 0}\frac{|\widetilde{\gamma}_m|}{\varkappa_m}<\infty,\qquad\text{and}\qquad \sup_{m\ge 0}\frac{|\widehat{\gamma}_m|}{\varkappa_m}<\infty.$$

On one hand, for $m\ge N$ we have

$$|\widetilde{\gamma}_m|\le(p_0+\cdots+p_N)\cdot\max\{\varkappa_m,\dots,\varkappa_{m-N}\}\lesssim\varkappa_m.$$

On the other hand,

$$|\widehat{\gamma}_m|\asymp\frac{1}{m^{a-s+1}}\le\frac{1}{m^{-s+1}}\asymp\varkappa_m.$$

Therefore (i) follows. Observe that (ii) is an immediate consequence of Theorem 1.9 (ii), since $\alpha(t)\varkappa(t)=(1-t)^{a-s}$. $\square$

Note that it is immediate that

$$\|B_s^m\|^2=\sup_{n\ge 0}\frac{\varkappa_n}{\varkappa_{n+m}}=\begin{cases}1 & \text{if } 1\le s\\ 1/\varkappa_m & \text{if } 0<s<1,\end{cases}\tag{2.19}$$

and

$$\|F_s^m\|^2=\sup_{n\ge 0}\frac{\varkappa_{n+m}}{\varkappa_n}=\begin{cases}\varkappa_m & \text{if } 1\le s\\ 1 & \text{if } 0<s<1,\end{cases}\tag{2.20}$$

for every $m \geq 0$. Therefore

$$\|B_s^m\|^2 \asymp (m+1)^{\max\{1-s,0\}} \quad \text{and} \quad \|F_s^m\|^2 \asymp (m+1)^{\max\{s-1,0\}}. \tag{2.21}$$

As an easy consequence we obtain the following example, which shows two relevant facts: 1) there are $a$-contractions that are not similar to contractions, and 2) the importance of considering the strong operator topology in the convergence of $\sum \alpha_n T^{*n} T^n$.

**Example 2.25.** Taking $a = s \in (0, 1/2]$ in Theorem 2.24, we get that $B_a$ is an $a$-contraction. However, it is not similar to a contraction, since it is not power bounded. Moreover,

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|B_a^n\|^2 \asymp \sum_{n=0}^{\infty} (n+1)^{-1-a} \, (n+1)^{1-a} = \sum_{n=0}^{\infty} (n+1)^{-2a} = \infty,$$

and therefore the series $\sum \alpha_n B_a^{*n} B_a^n$ does not converge in the uniform operator topology in $\boldsymbol{B}(\mathcal{H})$. Note that, obviously, the model of the $a$-contraction $B_a$ is itself.

(See also Theorem 4.23 in Chapter 4.)

Let us study now some ergodic properties of $a$-contractions.

**Definition 2.26.** Let $a \geq 0$. For any bounded linear operator $T$ on a Banach space $\mathcal{X}$, we call the operators $\{M_T^a(n)\}_{n \geq 0}$ given by

$$M_T^a(n) := \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j) T^j,$$

the *Cesàro means of order $a$ of $T$*. When this family of operators is uniformly bounded, that is,

$$\sup_{n \geq 0} \|M_T^a(n)\| < \infty,$$

we say that $T$ is *$(C, a)$-bounded*.

**Remarks 2.27.**

(i) Note that $\sum_{j=0}^{n} k^a(j) = k^{a+1}(n)$ for any complex number $a$. Also, if $a \geq 0$, then $k^a(j) \geq 0$ for every $j \geq 0$.

(ii) If $a = 0$, then $M_T^0(n) = T^n$. Hence $(C, 0)$-boundedness is just power boundedness.

(iii) If $a = 1$, then $M_T^1(n) = (n+1)^{-1} \sum_{j=0}^{n} T^j$. Hence $(C, 1)$-boundedness is just Cesàro boundedness.

(iv) It is well-known that if $0 \le a < b$, then $(C,a)$-boundedness implies $(C,b)$-boundedness. The converse is not true in general. For example, the Assani matrix

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

is $(C,1)$-bounded, but since

$$T^n = \begin{pmatrix} (-1)^n & (-1)^{n+1}2n \\ 0 & (-1)^n \end{pmatrix},$$

it is not power bounded (see [41, Section 4.7]).

**Definition 2.28.** If the sequence of operators $\{M_T^a(n)\}_{n\ge 0}$ given in Definition 2.26 converges in the strong operator topology, we say that $T$ is $(C,a)$-*mean ergodic*.

If $T$ is $(C,1)$-mean ergodic, it is conventional just to say that $T$ is *mean ergodic*.

There is a well established literature on $(C,a)$-bounded operators, which explores quite a number of properties and their interplays. Properties, characterization through functional calculus and ergodic results for $(C,a)$-bounded operators can be found in [3, 8, 36, 38, 39, 41, 61] and references therein. The connection of these operators and ergodicity dates back to the fourties of last century, see [34] and [52]. In the latter, E. Hille studies $(C,a)$-mean ergodicity in terms of Abel convergence (that is, via the resolvent operator). As application, the well known mean ergodic von Neumann's theorem for unitary groups on Hilbert spaces is extended to $(C,a)$-mean ergodicity for every $a > 0$ [52, page 255]. Also, the $(C,a)$-ergodicity on $L_1(0,1)$ of fractional (Riemann-Liouville) integrals is elucidated in [52, Theorem 11]. In particular, if $V$ is the Volterra operator then $T_V := I - V$, as operator on $L_1(0,1)$, is not power-bounded, and it is $(C,a)$-mean ergodic if and only if $a > 1/2$ [52, Theorem 11]. This result can be extended to $T_V$ acting on $L_p(0,1)$, $1 < p < \infty$, using estimates given in [64], see [2, Section 10].

In [62], Luo and Hou introduced a new notion of boundedness: a bounded linear operator $T$ on a Banach space $\mathcal{X}$ is said to be *absolutely Cesàro bounded* if

$$\sup_{n\ge 0} \frac{1}{n+1} \sum_{j=0}^{n} \left\|T^j x\right\| \lesssim \|x\|$$

for every $x \in \mathcal{X}$. This definition has been extended recently by Abadias and Bonilla in [1]: $T$ is said to be *absolutely* $(C,a)$-*Cesàro bounded* for some $a > 0$ if

$$\sup_{n\ge 0} \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j) \left\|T^j x\right\| \lesssim \|x\|$$

for every $x \in \mathcal{X}$. Note that for $a = 1$ the definition of Luo and Hou is recovered.

**Remark 2.29.** It is well-known that the following implications hold:

$$\begin{aligned}&\text{Power bounded} \;\Rightarrow\; \text{Absolutely } (C,a)\text{-bounded}\\ &\Rightarrow\; (C,a)\text{-bounded} \;\Rightarrow\; \|T^n\| = O(n^a).\end{aligned}$$

The first two implications are straightforward. For the sake of completeness, we give a proof of the last one. Suppose $T$ is $(C,a)$-bounded for some $a \geq 0$. We denote by $[a]$ the integer part of $a$. Then, for $n > [a]$, we have

$$\begin{aligned}\|T^n\| &= \left\| \sum_{j=0}^{n} k^{-a}(j) \sum_{m=0}^{n-j} k^a(n-j-m)T^m \right\| \\ &\lesssim \sum_{j=0}^{n} |k^{-a}(j)| k^{a+1}(n-j) \\ &= \sum_{j=0}^{[a]} (-1)^j k^{-a}(j) k^{a+1}(n-j) + \sum_{j=[a]+1}^{n} (-1)^{[a]+1} k^{-a}(j) k^{a+1}(n-j) \\ &= \sum_{j=0}^{[a]} \left((-1)^j + (-1)^{[a]}\right) k^{-a}(j) k^{a+1}(n-j) + (-1)^{[a]+1} \sum_{j=0}^{n} k^{-a}(j) k^{a+1}(n-j) \\ &\lesssim \sum_{j=0}^{[a]} |k^{-a}(j)| k^{a+1}(n-j) + k^1(n) \lesssim k^{a+1}(n) \asymp (n+1)^a \,.\end{aligned}$$

The following extension of the above definitions will be important for us.

**Definition 2.30.** Let $a > 0$ and $p \geq 1$. We say that a bounded linear operator $T$ on a Banach space $\mathcal{X}$ is $(C,a,p)$*-bounded* if

$$\sup_{n\geq 0} \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j) \|T^j x\|^p \lesssim \|x\|^p,$$

for all $x \in \mathcal{X}$.

Note that for $p = 1$ this definition is just the absolutely $(C,a)$-boundedness. We will use the term *quadratically* $(C,a)$*-bounded* instead of $(C,a,2)$-bounded.

Using the asymptotics $k^a(n) \asymp (n+1)^{a-1}$ given in (2.18), it is easy to see that $T$ is $(C,a,p)$-bounded if and only if

$$\sup_{n\geq 0} \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \lesssim \|x\|^p \qquad (\forall x \in \mathcal{X}). \tag{2.22}$$

The following observation will be essential for the proof of Theorem 0.13.

**Lemma 2.31.** *The following holds.*

(i) *If $T$ is $(C, a, p)$-bounded, then any part of $T$ is also $(C, a, p)$-bounded.*

(ii) *If $T_1$ and $T_2$ are $(C, a, p)$-bounded, then any direct sum $T_1 \dot{+} T_2$ is also $(C, a, p)$-bounded.*

(iii) *Let $T$ be a bounded linear operator on a Hilbert space. If $T$ is quadratically $(C, a)$-bounded, then $T \otimes I_{\mathcal{E}}$ is also quadratically $(C, a)$-bounded, where $I_{\mathcal{E}}$ is the identity operator on some Hilbert space $\mathcal{E}$.*

*Proof.* (i) and (ii) are immediate. For (iii) note that if $d = \dim \mathcal{E} \leq \infty$, then the orthogonal sum of $d$ copies of $T$ is clearly quadratically $(C, a)$-bounded (by the Pythagoras Theorem). $\square$

The following result is very useful. Its proof is simple, and we omit it.

**Lemma 2.32.** *Let $0 \leq a < b$. Then $(C, a, p)$-boundedness implies $(C, b, p)$-boundedness.*

This lemma shows an inclusion of classes of operators. By [1, Corollaries 2.2 and 2.3], if $T$ is $(C, a, 1)$-bounded then $\|T^n\| = o(n^a)$ for $0 < a \leq 1$ and $\|T^n\| = O(n)$ for $a > 1$. The following result explains why the case $a = 1$ is special.

**Theorem 2.33.** *If $a > 1$ and $p \geq 1$, then $(C, a, p)$-boundedness is equivalent to $(C, 1, p)$-boundedness.*

*Proof.* Fix $a > 1$ and $p \geq 1$. By the above Lemma, we only need to prove that any $(C, a, p)$-bounded operator $T$ is $(C, 1, p)$-bounded. Let $T$ is $(C, a, p)$-bounded. Then

$$\frac{1}{k^{a+1}(2n)} \sum_{j=0}^{2n} k^a(2n - j) \left\|T^j x\right\|^p \lesssim \|x\|^p , \tag{2.23}$$

for every $n \geq 0$, and every $x \in \mathcal{X}$. Since $a > 1$, $k^a(m)$ is an increasing function of $m$. In particular, $k^a(n) \leq k^a(2n - j)$ for $j = 0, \ldots, n$. Hence

$$k^a(n) \sum_{j=0}^{n} \left\|T^j x\right\|^p \leq \sum_{j=0}^{2n} k^a(2n - j) \left\|T^j x\right\|^p , \tag{2.24}$$

By (2.24) and (2.23),

$$\sum_{j=0}^{n} \left\|T^j x\right\|^p \lesssim \frac{k^{a+1}(2n)}{k^a(n)} \|x\|^p \lesssim (n + 1) \|x\|^p ,$$

which means that $T$ is $(C, 1, p)$-bounded. $\square$

**Theorem 2.34.** *Let $a > 0$ and $1 \leq q < p$. If $T$ is $(C, a, p)$-bounded, then it is also $(C, b, q)$-bounded for each $b > qa/p$. In particular, $(C, a, p)$-boundedness implies $(C, a, q)$-boundedness.*

*Proof.* Let us first recall that if $r > -1$, then

$$\sum_{j=1}^{m} j^r \lesssim m^{r+1} \qquad (\forall m \geq 1). \tag{2.25}$$

Let $T$ be $(C, a, p)$-bounded and let $b > qa/p$. Suppose first that $b \neq 1$, and put

$$s := \frac{p}{p-q}, \qquad s' := \frac{p}{q}, \qquad \gamma := \frac{q(a-1)}{p(b-1)}.$$

Note that $s$ and $s'$ are positive and satisfy $1/s + 1/s' = 1$. Since

$$(b-1)(1-\gamma)s = \frac{pb - qa}{p-q} - 1 > -1 \qquad \text{and} \qquad (b-1)\gamma s' = a - 1,$$

using Hölder's inequality and (2.25) it follows that

$$\begin{aligned}
&\frac{1}{(n+1)^b} \sum_{j=0}^{n} (n+1-j)^{b-1} \left\|T^j x\right\|^q \\
&\leq \frac{1}{(n+1)^b} \left( \sum_{j=0}^{n} (n+1-j)^{(b-1)(1-\gamma)s} \right)^{1/s} \left( \sum_{j=0}^{n} (n+1-j)^{(b-1)\gamma s'} \left\|T^j x\right\|^{qs'} \right)^{1/s'} \\
&\lesssim (n+1)^{-qa/p} \left( \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \right)^{q/p} \\
&= \left( \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \right)^{q/p}
\end{aligned}$$

for every $x \in \mathcal{X}$ and every non-negative integer $n$. Hence the statement follows using (2.22).

Now suppose that $b = 1$. Take any $b' \in (qa/p, 1)$. We have already proved that $T$ is $(C, b', p)$-bounded. Then, by Lemma 2.32, it follows that $T$ is $(C, 1, p)$-bounded. This completes the proof. □

**Lemma 2.35.** *Let $a > 0$ and $p \geq 1$. Then every isometry $S$ is $(C, a, p)$-bounded.*

*Proof.* This is immediate, since indeed

$$\frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j) \|S^j x\|^p = \frac{1}{k^{a+1}(n)} \left( \sum_{j=0}^{n} k^a(n-j) \right) \|x\|^p = \|x\|^p \tag{2.26}$$

for every $x \in \mathcal{X}$. □

**Lemma 2.36.** *Let $0 < s < 1$ and let $a > 0$. Then $B_s$ is quadratically $(C, a)$-bounded if and only if $1 - s < a$. Moreover, for $1 - s < a$ we have*

$$\lim_{n\to\infty} \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j)\|B_s^j x\|^2 = 0 \qquad (\forall x \in \mathcal{H}_s). \tag{2.27}$$

*Proof.* Recall the notation $e_n = t^n \in \mathcal{H}_k = \mathcal{H}_s$, where $k(t) = (1-t)^{-s}$. Suppose that $a = 1 - s$. Then

$$\begin{aligned}\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^j e_n\right\|^2 &\gtrsim \frac{1}{(n+1)^{1-s}}\sum_{j=0}^{n}(n+1-j)^{-s}(n+1-j)^{s-1}\\ &= \frac{1}{(n+1)^{1-s}}\sum_{j=1}^{n+1} j^{-1} \gtrsim \log(n+2)\left\|e_n\right\|^2\end{aligned} \tag{2.28}$$

for every $n$. Therefore $B_s$ is not quadratically $(C, 1-s)$-bounded, and by Lemma 2.32 we obtain that $B_s$ is not quadratically $(C, a)$-bounded for $a < 1 - s$.

Let us assume now that $1 - s < a \leq 1$ and fix $x \in \mathcal{H}_s$. Write $x$ in the form $x = \sum x_m e_m$, where $x_m \in \mathbb{C}$. Then

$$\left\|B_s^j x\right\|^2 = \sum_{m=j}^{\infty} k^s(m-j)|x_m|^2 \lesssim \sum_{m=j}^{\infty}(m+1-j)^{s-1}|x_m|^2,$$

for every $j \geq 0$. Hence

$$\begin{aligned}&\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^j x\right\|^2\\ &\qquad\lesssim \frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\sum_{m=j}^{\infty}(m+1-j)^{s-1}|x_m|^2\\ &\qquad= \frac{1}{(n+1)^a}\sum_{m=0}^{n}|x_m|^2\sum_{j=0}^{m}(n+1-j)^{a-1}(m+1-j)^{s-1}\\ &\qquad+ \frac{1}{(n+1)^a}\sum_{m=n+1}^{2n}|x_m|^2\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\\ &\qquad+ \frac{1}{(n+1)^a}\sum_{m=2n+1}^{\infty}|x_m|^2\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\\ &\qquad=: (I) + (II) + (III).\end{aligned}$$

In (I), note that since $1-s<a\le 1$, and $m\le n$, we have

$$\sum_{j=0}^{m}(n+1-j)^{a-1}(m+1-j)^{s-1}\le\sum_{j=0}^{m+1}(m+1-j)^{a+s-2}\lesssim(m+1)^{a+s-1}, \quad (2.29)$$

where in the last estimate we used (2.25). Therefore

$$\begin{aligned}(I)&\lesssim\frac{1}{(n+1)^a}\sum_{m=0}^{n}|x_m|^2(m+1)^{a+s-1}=\left\{\sum_{m=0}^{[\sqrt{n}]}+\sum_{m=[\sqrt{n}]+1}^{n}\right\}|x_m|^2\,\frac{(m+1)^{a+s-1}}{(n+1)^a}\\&\lesssim\frac{\|x\|^2}{\sqrt{n^a}}+\sum_{m=[\sqrt{n}]+1}^{n}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).\end{aligned}$$

In (II), using that $m>n$ and $s-1<0$, we have

$$\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\le\sum_{j=0}^{n}(n+1-j)^{a+s-2}\lesssim(n+1)^{a+s-1}.$$

Therefore

$$\begin{aligned}(II)&\lesssim\frac{1}{(n+1)^a}\sum_{m=n+1}^{2n}|x_m|^2(n+1)^{a+s-1}=(n+1)^{s-1}\sum_{m=n+1}^{2n}|x_m|^2\\&\lesssim\sum_{m=n+1}^{2n}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).\end{aligned}$$

Finally, in (III), since $m>2n$ we have that

$$\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\lesssim(m+1)^{s-1}\sum_{j=0}^{n}(n+1-j)^{a-1}\lesssim(m+1)^{s-1}(n+1)^{a}.$$

Therefore

$$(III)\lesssim\sum_{m=2n+1}^{\infty}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).$$

Hence (2.27) follows when $1-s<a\le 1$. Finally, suppose that $1<a$. Then

$$\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^j x\right\|^2\le\frac{1}{n+1}\sum_{j=0}^{n}\left\|B_s^j x\right\|^2\longrightarrow 0\quad(\text{as } n\to\infty),$$

since this is the case of $a=1$ in (2.27) (already proved). Note that (2.27) implies quadratical $(C,a)$-boundedness, so the proof is complete. $\square$

This lemma allows us to prove the following more general result.

**Theorem 2.37.** *Let $0 < s < 1$ and $1 \le q \le 2$. Then $B_s$ is $(C,b,q)$-bounded if and only if $b > q(1-s)/2$. Moreover, for $b > q(1-s)/2$ we have*

$$\lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\|B_s^j x\right\|^q = 0 \qquad (\forall x \in H). \tag{2.30}$$

*Proof.* Note that $q = 2$ is precisely Lemma 2.36. So we assume that $1 \le q < 2$. If $b = q(1-s)/2$, taking $x = e_n$, we get, as in (2.28), that

$$\frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\|B_s^j e_n\right\|^q \gtrsim \log(n+2) \left\|e_n\right\|^2$$

for every $n$. Therefore $B_s$ is not $(C, q(1-s)/2, q)$-bounded, and by Lemma 2.32 we get that $B_s$ is not $(C,b,q)$-bounded for $b < q(1-s)/2$.

Now suppose that $b > q(1-s)/2$. Then $b = qa/2$ for some $a > 1-s$. Using Hölder's inequality as in the proof of Theorem 2.34, we obtain

$$\frac{1}{(n+1)^b} \sum_{j=0}^{n} (n+1-j)^{b-1} \left\|B_s^j x\right\|^q \lesssim \left( \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|B_s^j x\right\|^2 \right)^{q/2} \xrightarrow[n\to\infty]{} 0,$$

by Lemma 2.36. Hence (2.30) follows. □

*Proof of Theorem 0.13.* Let $T \in \mathbf{C}_a$ with $0 < a < 1$ and let $b > 1-a$. By Theorem 0.4 and Theorem 0.10 (i), $T$ is unitarily equivalent to a part of $(B_a \otimes I_{\mathfrak{D}}) \oplus S$. Hence, by Lemma 2.31 (i) and (i'), it is enough to prove that $(B_a \otimes I_{\mathfrak{D}}) \oplus S$ is quadratically $(C,b)$-bounded. But this is immediate using Lemma 2.31 (ii) and (iii), and Lemmas 2.35 and 2.36. □

For the proof of Theorem 0.14 we need the following lemma, which is in the spirit of Lemma 2.31.

**Lemma 2.38.** *The following holds.*

(i) *If $T$ satisfies* (ii)*, then any part of $T$ also satisfies* (ii)*.*

(ii) *If $T_1$ and $T_2$ satisfy* (ii)*, then any direct sum $T_1 \dotplus T_2$ also satisfies* (ii)*.*

(iii) *Let $T$ be a bounded linear operator on a Hilbert space. If $T$ satisfies* (ii)*, then the operator $T \otimes I_{\mathcal{E}}$ also satisfies* (ii)*, where $I_{\mathcal{E}}$ is the identity operator on some Hilbert space $\mathcal{E}$.*

*Proof.* (i) and (ii) are immediate. For (iii) we use the same argument as in Lemma 2.31 (iii) and a simple application of Lebesgue's Dominated Convergence Theorem. □

*Proof of Theorem 0.14.* As in the proof of Theorem 0.13, we have that $T$ is unitarily equivalent to

$$(B_a \otimes I_{\mathfrak{D}}) \oplus S \,|\, \mathcal{L},$$

where $\mathcal{L}$ is a subspace of $(\mathcal{H}_a \otimes \mathfrak{D}) \oplus \mathcal{W}$ invariant by $(B_a \otimes I_{\mathfrak{D}}) \oplus S$.

Let us prove the circle of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Suppose that (i) is true. That is, $T$ is unitarily equivalent to

$$(B_a \otimes I_{\mathfrak{D}}) \,|\, \mathcal{L},$$

where $\mathcal{L}$ is a subspace of $\mathcal{H}_a \otimes \mathfrak{D}$ invariant by $B_a \otimes I_{\mathfrak{D}}$. Then (ii) follows using Lemmas 2.36 and 2.38.

Suppose now that

$$\liminf_{n\to\infty} \|T^n x\| > 0$$

for some $x \in H$. Then, obviously, $\|T^n x\| > \varepsilon > 0$ for every $n \geq 0$. Hence for this vector $x$ (ii) does not hold. Therefore we have proved that (ii) $\Rightarrow$ (iii).

Finally, suppose that the isometry $S$ appears in the minimal model. Then for some vector $\ell = (\ell_1, \ell_2) \in \mathcal{L}$, its second component $\ell_2 \in \mathcal{W}$ is not 0. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\| ((B_a \otimes I_{\mathfrak{D}}) \oplus S)^j \ell \right\|^2 \\
&= \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\| (B_a \otimes I_{\mathfrak{D}})^j \ell_1 \oplus S^j \ell_2 \right\|^2 \\
&= \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\| (B_a \otimes I_{\mathfrak{D}})^j \ell_1 \right\|^2 + \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\| S^j \ell_2 \right\|^2.
\end{aligned}$$

The second limit is $\|\ell_2\|^2 \neq 0$ because of (2.26). Hence we obtain that (iii) $\Rightarrow$ (i). □

Since $\mathbf{C}_1$ is just the set of all contractions on $H$, $T \in \mathbf{C}_1$ iff $T^* \in \mathbf{C}_1$. However, this is no longer true for $a \in (0,1)$.

**Proposition 2.39.** *If $a \in (0,1)$, then there is an operator $T \in \mathbf{C}_a$ such that $T^* \notin \mathbf{C}_a$.*

*Proof.* Note that $B_a^*$ is a forward weighted shift such that $\|B_a^{*n} f_0\| \to \infty$ as $n$ goes to $\infty$. So $B_a \in \mathbf{C}_a$, whereas its adjoint cannot belong to $\mathbf{C}_a$, because $B_a^*$ is not quadratically $(C,b)$-bounded for any $b$ (see Lemma 2.36). □

It is natural to pose the following question.

**Question 2.40.** For which functions $\alpha$, satisfying Hypotheses 0.1, is it true that $T \in \mathbf{C}_\alpha \implies T^* \in \mathbf{C}_\alpha$?

It is so for $\alpha(t) = 1 - t$ and, more generally, for $\alpha(t) = 1 - t^n$, $n \geq 1$. The author does not know other examples.

**Remarks 2.41.**

(i) If $T$ is an operator in $\mathbf{C}_a$ with $0 < a < 1$, and $0 < q < 2$, then by Theorem 0.13 and Theorem 2.34 it follows that $T$ is $(C, b, q)$-bounded for all $b > \frac{q(1-a)}{2}$.

(ii) An $m$-isometry $T$, which is not an isometry, cannot be $(C, a, p)$-bounded, because there are vectors $x$ such that the norms $\|T^n x\|$ go to infinity. The possibility for these operators to have weaker ergodic properties, such as the Cesàro boundedness and weak ergodicity, have been studied in [19].

(iii) Let $T$ be an operator in $\mathbf{C}_a$ with $0 < a < 1$. Using Theorem 0.13 (i) and Theorem 2.34 (with $p = 2$ and $q = 1$) we obtain that $T$ is $(C, b, 1)$-bounded for every $b > (1 - a)/2$.

By [1, Corollary 3.1], we get that $T$ is $(C, b)$-mean ergodic, that is, there exists
$$P_b x := \lim_{n \to \infty} M_T^b(n)x, \quad x \in H.$$
Therefore, by [2, Theorem 3.3], we have
$$H = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Ran}(I - T)}.$$
In fact,
$$\mathrm{Ker}(I - T) = \mathrm{Ran} P_b \ \text{ and } \ \overline{\mathrm{Ran}(I - T)} = \mathrm{Ker} P_b.$$
Also note that
$$M_T^b(n)x = x \text{ for } x \in \mathrm{Ker}(I-T), \text{ and } \lim_{n \to \infty} M_T^b(n)x = 0 \text{ for } x \in \overline{\mathrm{Ran}(I - T)}.$$

Let now $0 < \gamma < 1$, by [2, Proposition 4.8 and Remark 4.9] it also follows that
$$\mathrm{Ker}(I - T) = \mathrm{Ker}(I - T)^\gamma, \quad \overline{\mathrm{Ran}(I - T)} = \overline{\mathrm{Ran}(I - T)^\gamma},$$
with $\mathrm{Ran}(I - T) \subseteq \mathrm{Ran}(I - T)^\gamma$. Furthermore if $\gamma < 1 - b$, for $x \in \overline{\mathrm{Ran}(I - T)}$,
$$x \in \mathrm{Ran}(I - T)^\gamma \iff \sum_{n=1}^{\infty} \frac{1}{n^{1-\gamma}} T^n x \ \text{ converges},$$
see [2, Theorem 9.2].

(iv) By [1, Theorem 3.1], if $T$ is an operator in $\mathbf{C}_a$ with $0 < a < 1$ and $b > (1-a)/2$, then

$$\lim_{n\to\infty} \|M_T^b(n+1) - M_T^b(n)\| = 0.$$

# Chapter 3

# Similarity to contractions

In this chapter we give an alternative proof of Corollary 0.12 (ii) based on the use of abstract defect operators. In other words, here we work with

$$\alpha(t) = (1-t)\widetilde{\alpha}(t)$$

where $\widetilde{\alpha} \in A_{W,\mathbb{R}}$ is positive on the interval $[0,1]$; that is, $\alpha$ is an admissible function.

We remark that interesting abstract criteria for similarity to a contraction have been given by Holbrook, see [54] and references therein. Another criteria for similarity (which indeed looks like the condition on an operator for being a member of $\mathrm{Adm}_\alpha$ given in Proposition 1.4 (iii)) is given in [57, Proposition 1.12] by Kubrusly. In [57, Proposition 1.15] a criterion for similarity to an isometry is given.

In this chapter we also obtain an explicit model for operators in $\mathbf{C}_\alpha$ and derive some consequences from it. Finally, we prove Theorem 0.15 on the existence of the limit $\lim_{n\to\infty} \|T^n h\|$ for every $h \in \mathcal{H}$.

## 3.1 Abstract defect operators

The standard Nagy-Foias model of a contraction $S \in \boldsymbol{B}(\mathcal{H})$ makes use of its defect operator, which is defined as a nonnegative square root $D_S = (I - S^*S)^{1/2}$. This model is related with the following well-known identity

$$\|h\|^2 = \sum_{n=0}^{\infty} \|D_S S^n h\|^2 + \lim_{n\to\infty} \|S^n h\|^2, \qquad h \in \mathcal{H}, \tag{3.1}$$

valid for any contraction $S$ (see [89, Section 1.10]). This motivates the following definition.

Recall that $T \in \boldsymbol{B}(\mathcal{H})$ is a *power bounded operator* if $\sup_{n\geq 0} \|T^n\| < \infty$.

**Definition 3.1.** Let $T \in \boldsymbol{B}(\mathcal{H})$ be a power bounded operator, and let $D : \mathcal{H} \to \mathcal{F}$, where $\mathcal{H}, \mathcal{F}$ are Hilbert spaces. We will say that $D$ is *an abstract defect operator for $T$* if

$$\sum_{n=0}^{\infty} \|DT^n h\|^2 + \limsup_{n\to\infty} \|T^n h\|^2 \asymp \|h\|^2 \tag{3.2}$$

for every $h$ in $\mathcal{H}$. (See Notation 1.3.)

Let us recall the notion of *Banach limit*. If we denote by $c$ the set of all convergent complex sequences then we can define the linear functional $L : c \to \mathbb{C}$ given by $L(x) = \lim x_n$ for every $x = \{x_n\}_{n=1}^{\infty} \in c$. It is immediate that $\|L\| = 1$, $L(x') = L(x)$ if $x' = \{x_n\}_{n=2}^{\infty}$, and also $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_n \geq 0$ for every $x$). Using the Hahn-Banach Theorem, this functional can be extended to $\ell^\infty$ so that these properties still hold.

**Theorem 3.2.** *There is a linear functional $L : \ell^\infty \to \mathbb{C}$ such that*

(a) $\|L\| = 1$;

(b) $L(x) = \lim x_n$ *for every* $x \in c$;

(c) $L(x) \geq 0$ *for every* $x \in \ell^\infty$ *such that* $x \geq 0$;

(d) $L(x') = L(x)$ *if* $x \in \ell^\infty$ *and* $x' = \{x_n\}_{n=2}^{\infty}$;

(e) $\liminf x_n \leq L(x) \leq \limsup x_n$ *if* $x \in \ell^\infty$ *is a real sequence.*

*Proof.* Statements (a)-(d) are contained in [35, Theorem III.7.1], and assertion (e) is their easy consequence (and also is standard). □

A functional $L$ with the above properties is called *a Banach limit*.

**Theorem 3.3.** *Let $T$ be a power bounded operator. Then the following is true.*

(i) *$T$ is similar to a contraction if and only if it has an abstract defect operator.*

(ii) *An operator $D : \mathcal{H} \to \mathcal{F}$ is an abstract defect operator for $T$ if and only if there exists an invertible operator $W \in \boldsymbol{B}(\mathcal{H})$ such that $\widetilde{T} := WTW^{-1}$ is a contraction and $\|Dh\| = \|D_{\widetilde{T}} Wh\|$ for any $h \in \mathcal{H}$.*

*Proof.* It suffices to prove (ii). First we remark that by a lemma by Gamal [44, Lemma 2.1], for any power bounded operator $T$ one has

$$\liminf_{n\to\infty} \|T^n h\|^2 \asymp \limsup_{n\to\infty} \|T^n h\|^2 \tag{3.3}$$

for every $h \in \mathcal{H}$.

Suppose first that there exists a linear isomorphism $W \in \boldsymbol{B}(\mathcal{H})$ with the properties stated in the Theorem. Since $\widetilde{T} = WTW^{-1} \in \boldsymbol{B}(\mathcal{H})$ is a contraction, by (3.1) we have

$$\|h\|^2 = \sum_{n=0}^{\infty} \left\| D_{\widetilde{T}} \widetilde{T}^n h \right\|^2 + \lim_{n\to\infty} \left\| \widetilde{T}^n h \right\|^2$$

for $h \in \mathcal{H}$. Note that $\widetilde{T}^n = WT^nW^{-1}$, and thus

$$\|h\|^2 = \sum_{n=0}^{\infty} \left\| D_{\widetilde{T}} WT^nW^{-1}h \right\|^2 + \lim_{n\to\infty} \left\| WT^nW^{-1}h \right\|^2 .$$

Since $W$ is invertible and $\|Dh\| = \left\| D_{\widetilde{T}} Wh \right\|$, we get

$$\|h\|^2 \asymp \|Wh\|^2 = \sum_{n=0}^{\infty} \|DT^n h\|^2 + \lim_{n\to\infty} \|WT^n h\|^2 . \tag{3.4}$$

By (3.3), $\lim_{n\to\infty} \|WT^n h\|^2 \asymp \limsup_{n\to\infty} \|T^n h\|^2$. We deduce that $D$ is an abstract defect operator for $T$.

Conversely, suppose now that $D$ is an abstract defect operator for $T$. Fix a Banach limit $L$ and put

$$|||h|||^2 := \sum_{n=0}^{\infty} \|DT^n h\|^2 + L\big(\{\|T^n h\|^2\}\big). \tag{3.5}$$

Notice that (3.3) and Theorem 3.2 (e) give that

$$\limsup_{n\to\infty} \|T^n h\|^2 \asymp L\left(\{\|T^n h\|^2\}\right), \quad h \in \mathcal{H}.$$

The relation (3.2) implies that $|||h||| \asymp \|h\|$, $h \in \mathcal{H}$. It follows that $|||\cdot|||$ is an equivalent Banach space norm on $\mathcal{H}$. By applying the Cauchy-Schwarz inequality, it is easy to see that for any $x, y \in \mathcal{H}$, one can correctly define

$$[x, y] := \sum_{n=0}^{\infty} \langle DT^n x, DT^n y\rangle + L\big(\{\langle T^n x, T^n y\rangle\}\big)$$

(the sum absolutely converges). It is a semi-inner product on $\mathcal{H}$, which induces the norm $|||\cdot|||$. So, in fact, $|||\cdot|||$ is a Hilbert space norm equivalent to $\|\cdot\|$ (see [55] and [88] for similar arguments).

Therefore there exists a linear isomorphism $W : \mathcal{H} \to \mathcal{H}$ such that $\|Wh\| = |||h|||$. Observe that

$$|||Th|||^2 = |||h|||^2 - \|Dh\|^2 \le |||h|||^2.$$

Let $\widetilde{T} := WTW^{-1} \in \boldsymbol{B}(\mathcal{H})$ (similar to $T$). Take $x \in \mathcal{H}$ and put $h := W^{-1}x$. We get

$$\|WTh\| \le \|Wh\|,$$

so $\widetilde{T}$ is a contraction. Finally,

$$\left\|D_{\widetilde{T}}Wh\right\|^2 = \|Wh\|^2 - \left\|\widetilde{T}Wh\right\|^2 = |\!|\!|h|\!|\!|^2 - |\!|\!|Th|\!|\!|^2 = \|Dh\|^2$$

for every $h \in \mathcal{H}$. □

Let $\alpha$ be an admissible function and let $T \in \mathbf{C}_\alpha$. We know already that $T$ is similar to a contraction. Since $\widetilde{\alpha} \in A_{W,\mathbb{R}}$, by Corollary 2.8, there exists a function $\widetilde{\beta} \in A_{W,\mathbb{R}}$ such that $\widetilde{\beta} \succ 0$ and $f := \widetilde{\beta}\widetilde{\alpha} \succ 0$. Hence $(1-t)f(t) = \widetilde{\beta}(t)\alpha(t)$. Set

$$B := (f(T^*, T))^{1/2},$$

where the positive square root has been taken. Then $B > \varepsilon I$ for some $\varepsilon > 0$. We will assume, without loss of generality, that $\sum f_k = \|f\|_{A_{W,\mathbb{R}}} = 1$. We put

$$D := (\alpha(T^*, T))^{1/2}. \tag{3.6}$$

**Theorem 3.4.** *If $T \in \mathbf{C}_\alpha$ for some admissible function $\alpha$, then $D$ is an abstract defect operator for $T$. More specifically, if $\widetilde{\beta}, f$ and $B$ are as above, then the expression*

$$|\!|\!|h|\!|\!|^2 := \sum_{n=0}^{\infty} \|DT^n h\|^2 + \lim_{n\to\infty} \|BT^n h\|^2 \tag{3.7}$$

*defines an equivalent Hilbert space norm in $\mathcal{H}$ and $T$ is a contraction with respect to this norm. In particular, the limit in* (3.7) *exists for every $h \in \mathcal{H}$. Moreover,*

$$|\!|\!|h|\!|\!|^2 - |\!|\!|Th|\!|\!|^2 = \|Dh\|^2 \quad (\forall h \in \mathcal{H}). \tag{3.8}$$

*Proof.* Since $(1-t)f(t) = \widetilde{\beta}(t)\alpha(t)$, we have

$$B^2 - T^*B^2T = \sum_{n=0}^{\infty} \tilde{\beta}_n T^{*n} D^2 T^n.$$

Therefore, for every $h \in \mathcal{H}$ we have

$$\|Bh\|^2 - \|BTh\|^2 = \sum_{n=0}^{\infty} \tilde{\beta}_n \|DT^n h\|^2.$$

Changing $h$ by $T^j h$ we obtain

$$\left\|BT^j h\right\|^2 - \left\|BT^{j+1} h\right\|^2 = \sum_{n=0}^{\infty} \tilde{\beta}_n \left\|DT^{n+j} h\right\|^2,$$

for every $j \geq 0$. Summing these equations for $j = 0, 1, \ldots, N-1$ we obtain

$$\|Bh\|^2 - \left\|BT^N h\right\|^2 = \sum_{n=0}^{N-1} \beta_n \|DT^n h\|^2 + \sum_{n=N}^{\infty} \left( \sum_{j=n-N+1}^{n} \tilde{\beta}_j \right) \|DT^n h\|^2 , \quad (3.9)$$

where $\beta_n = \sum_{j=0}^{n} \widetilde{\beta}_j$. Since $\widetilde{\beta} \succ 0$, we have that $0 < \widetilde{\beta}_0 \leq \beta_n$, and therefore by (3.9) we get

$$\|Bh\|^2 \geq \sum_{n=0}^{N-1} \beta_n \|DT^n h\|^2 \geq \widetilde{\beta}_0 \sum_{n=0}^{N-1} \|DT^n h\|^2 .$$

Hence the series $\sum_{n=0}^{\infty} \|DT^n h\|^2$ converges. On the other hand, since

$$\sum_{j=n-N+1}^{n} \tilde{\beta}_j \leq \sum_{j=0}^{\infty} \tilde{\beta}_j = \left\|\widetilde{\beta}\right\|_{A_{W,\mathbb{R}}} < \infty,$$

we obtain that

$$\sum_{n=N}^{\infty} \left( \sum_{j=n-N+1}^{n} \tilde{\beta}_j \right) \|DT^n h\|^2 \leq \left\|\widetilde{\beta}\right\|_{A_{W,\mathbb{R}}} \sum_{n=N}^{\infty} \|DT^n h\|^2 \to 0$$

when $N \to \infty$. Therefore, taking limit in (3.9) when $N$ goes to infinity we obtain that

$$\|Bh\|^2 = \sum_{n=0}^{\infty} \beta_n \|DT^n h\|^2 + \lim_{N\to\infty} \left\|BT^N h\right\|^2 \quad (3.10)$$

(and the limit in the right hand side exists for any $h$). Recall that $B$ is invertible. Since $\widetilde{\beta}_0 \leq \beta_n \leq \|\widetilde{\beta}\|_{A_{W,\mathbb{R}}} < \infty$, it follows that $|||h|||$, given by (3.7), defines an equivalent Hilbert space norm on $\mathcal{H}$. Formula (3.8) is immediate from (3.7). $\square$

In what follows, for $T \in \mathbf{C}_\alpha$, the operator $D$, given by (3.6), will be called *the defect operator of* $T$.

**Corollary 3.5.** *If $\alpha$ is admissible and $\alpha(T^*, T) = 0$, then $T$ is similar to an isometry.*

Indeed, in this case, $D = 0$, and hence $T$ is an isometry with respect to an equivalent norm in $\mathcal{H}$, given by (3.7).

**Remark 3.6.** Notice that Theorem 0.11 (with $\eta(t) = 1 - t$), and Theorem 3.4, give two methods of finding an equivalent norm such that $T$ is a contraction in this norm. With the method of Theorem 3.4 we obtained in addition that $D$ is an abstract defect operator.

Observe that Theorem 3.4 requires $\alpha$ to be admissible, but applies to operators $T$ that are not contractions.

**Remark 3.7.** Using the notation above, note that

$$\|BT^n h\|^2 = \sum_{k=0}^{\infty} f_k \|T^{n+k}h\|^2.$$

So if we put

$$\lim{}^* x_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} f_k x_{n+k}, \tag{3.11}$$

then (3.7) can be written as

$$|||h|||^2 = \sum_{n=0}^{\infty} \|DT^n h\|^2 + \lim{}^*_{n\to\infty} \|T^n h\|^2 . \tag{3.12}$$

So, on the contrary to the general formula (3.5), the use of a non-constructive Banach limit is unnecessary in the context of an operator $T$ in the class $\mathbf{C}_\alpha$, and instead one can just use the "regularized" limit $\lim^*$. We observe that $\lim^*$ coincides with the usual limit when the sequence is convergent (here we use the above normalization assumption that $\sum f_k = 1$).

In Section 3.6, we will prove that the limit $\lim \|T^n h\|$ does exist for every $h \in \mathcal{H}$ if $\alpha$ is strongly admissible. Remark 3.25 contains an example showing that this limit does not exists in general if $\alpha$ is not strongly admissible.

## 3.2 Elementary properties of the classes $\mathbf{C}_\alpha$ with $\alpha$ admissible

In this section we will see some properties of the classes $\mathbf{C}_\alpha$ for admissible functions $\alpha$.

**Proposition 3.8.**

(a) *If $N \in \boldsymbol{B}(\mathcal{H})$ is a normal operator with $\|N\| \le 1$, then $N \in \mathbf{C}_\alpha$ for every admissible function $\alpha$. In particular, all unitary operators are in $\mathbf{C}_\alpha$.*

(b) *If $T_1, T_2 \in \mathbf{C}_\alpha$, then the orthogonal sum $T_1 \oplus T_2$ also is in $\mathbf{C}_\alpha$.*

(c) *It $T \in \mathbf{C}_\alpha$, then $\zeta T \in \mathbf{C}_\alpha$ for every $\zeta$ on the unit circle $\mathbb{T} = \{z : |z| = 1\}$.*

(d) *If $T \in \mathbf{C}_\alpha$, then $T|\mathcal{L} \in \mathbf{C}_\alpha$ for every $T$-invariant subspace $\mathcal{L} \subset \mathcal{H}$.*

(e) *If $T$ has spectral radius is less than one, then $T \in \mathbf{C}_{1-t^n}$ for $n$ sufficiently large.*

*Proof.* Statements (a)-(c) are obvious. (d) It is also immediate since the membership to the class $\mathbf{C}_\alpha$ only involves conditions on the action of the operator on each vector of the space. (See Proposition 1.4 and the definition of $\mathbf{C}_\alpha$.) (e) Note that $\sigma(T) \subset \mathbb{D}$ implies that $\|T^n\| < 1$ for $n \gg 0$. □

**Lemma 3.9.** *Let $T \in \mathrm{Adm}_\alpha$, for some admissible function $\alpha$, such that the point spectrum $\sigma_p(T)$ of $T$ is contained in the unit circle $\mathbb{T}$ and the eigenspaces* $\ker(T-\lambda)$, *$\lambda \in \mathbb{T}$, are complete. Then the following properties are equivalent.*

(i) $\alpha(T^*, T) = 0$.

(ii) *For any two eigenvalues $\lambda, \lambda'$ of $T$, either the eigenspaces* $\ker(T-\lambda)$, $\ker(T-\lambda')$ *are orthogonal or $\alpha(\lambda\bar{\lambda}') = 0$.*

*Proof.* Take any $h \in \mathcal{H}$, which can be written as a finite sum $h = \sum h_j v_j$, where $\lambda_j \in \sigma_p(T)$ and $v_j \in \ker(T - \lambda_j)$. Then

$$\begin{aligned}
\big\langle \alpha(T^*, T)h, h \big\rangle &= \sum_j \alpha_j \Big\langle T^j \sum_k h_k v_k, T^j \sum_\ell h_\ell v_\ell \Big\rangle \\
&= \sum_j \alpha_j \Big\langle \sum_k \lambda_k^j h_k v_k, \sum_\ell \lambda_\ell^j h_\ell v_\ell \Big\rangle \\
&= \sum_{k,\ell} \sum_j \alpha_j \overline{\lambda_\ell^j} \lambda_k^j \, h_k \bar{h}_\ell \langle v_k, v_\ell \rangle \\
&= \sum_{k,\ell} \alpha(\lambda_k \bar{\lambda}_\ell) \langle v_k, v_\ell \rangle \, h_k \bar{h}_\ell.
\end{aligned} \tag{3.13}$$

Notice that $\alpha(T^*, T) = 0$ if and only if the quadratic form $\langle \alpha(T^*, T)h, h\rangle$ is zero for all vectors $h$ of the above form (which are dense in $\mathcal{H}$). The latter happens if and only if all the coefficients of this quadratic form are zero, that is, $\alpha(\lambda_k \bar{\lambda}_\ell)\langle v_k, v_\ell\rangle = 0$ for all $k$, $\ell$. □

We remark that whenever $\alpha$ is admissible and $\sigma_p(T) \subset \mathbb{T}$, the condition (ii) is fulfilled automatically for the case $\lambda = \lambda'$.

**Corollary 3.10.** *Let $\alpha$ be an admissible function . Suppose that $T \in \mathrm{Adm}_{\widetilde{\alpha}}$ and also the hypotheses of the previous lemma are satisfied. If $\alpha(T^*, T) = 0$, then $T$ is similar to a diagonalizable unitary operator.*

Indeed, by Theorem 3.3, there is an invertible operator $W : \mathcal{H} \to \mathcal{H}$ such that $\widetilde{T} = WTW^{-1}$ is a contraction. The images $Wv_k$ are eigenvectors of $\widetilde{T}$, corresponding to points $\lambda_k$ on the unit circle. Hence they are pairwise orthogonal and complete. Therefore $\widetilde{T}$ is unitary.

Notice that a priori it is not supposed in Lemma 3.9 that the point spectrum $\sigma_p(T)$ is countable.

The conditions on the geometry of eigenvectors of $T$ in the above Corollary are very special, and one could ask whether more general facts hold. We refer to [80, 10, 44] for some related results.

Recall that a function $\alpha(t) = (1-t)\widetilde{\alpha}(t)$ will be called *strongly admissible* if it is admissible and $\widetilde{\alpha}$ has no roots on the unit circle $\mathbb{T}$.

In the case of a strongly admissible $\alpha$, the assertion of Corollary 3.10 can be strengthened, as follows directly from Lemma 3.9.

**Corollary 3.11.** *Suppose that the hypotheses of Corollary 3.10 hold and $\alpha$ is strongly admissible. If $\alpha(T^*, T) = 0$, then $T$ is a diagonalizable unitary operator.*

**Corollary 3.12.** *For any complex square matrix $T$ without nontrivial Jordan blocks such that $\sigma(T) \subset \mathbb{T}$, there exists an admissible polynomial $p(t)$ such that $p(T^*, T) = 0$ (and therefore $T \in \mathbf{C}_p$).*

Indeed, let $\{v_j\}$ $(1 \le j \le n)$ be a basis of eigenvectors of $T$ (here $n$ is the size of $T$), and let $\lambda_j \in \mathbb{T}$ be the corresponding eigenvalues. Put

$$p(t) = (1-t) \prod_{k<\ell,\, \lambda_k \ne \lambda_\ell} (1 - 2\Re(\lambda_k \bar{\lambda}_\ell)t + t^2);$$

this is an admissible polynomial. Since $p(\lambda_k \overline{\lambda_\ell}) = 0$ for all $k, \ell$, the assertion follows from Lemma 3.9 (applied to operators on $\mathbb{C}^n$).

**Lemma 3.13.** *Let $T$ be any complex square matrix without nontrivial Jordan blocks such that $\sigma(T) \subset \overline{\mathbb{D}}$. Then the following properties are equivalent.*

(i) *There exists an admissible function $\alpha$ such that $T \in \mathbf{C}_\alpha$ (that is, $\alpha(T^*, T) \ge 0$).*

(ii) *For any two eigenvalues $\lambda$, $\lambda'$ of $T$ such that $\lambda \in \mathbb{T}$ and $\lambda' = r\lambda$ for some $r \in [0, 1)$, the eigenspaces $\ker(T - \lambda)$ and $\ker(T - \lambda')$ are orthogonal.*

It is well-known that if $T$ satisfies $I - T^*T \ge 0$ (that is, is contractive), then the eigenspaces corresponding to eigenvalues on the unit circle are reducing ones. The above peculiar criterion shows exactly, what remains from this property if one only requires that $T$ satisfy $\alpha(T^*, T) \ge 0$ for some admissible $\alpha$, instead of being contractive.

This criterion is not invariant under Möbius transformations of the disc. On the other hand, it shows that for any diagonalizable matrix $T$ there is a Möbius automorphism $\phi$ of the disc, arbitrarily close to the identity map, such that $T' = \phi(T)$ satisfies $\alpha[T'^*, T'] \geq 0$ for some admissible $\alpha$.

We can deduce that, on the contrary to Proposition 3.8 (b), given two operators $T_1$ and $T_2$ in a class $\mathbf{C}_\alpha$, their *direct sum* might not belong to any class $\mathbf{C}_\beta$. Indeed, take $T_1$ and $T_2$ to be $1 \times 1$ matrices, whose (unique) entries are, for instance, 1 and 1/2. Then, by Lemma 3.13, $T_1 \dot{+} T_2$ does not belong to any class $\mathbf{C}_\beta$ whenever this direct sum is not orthogonal.

*Proof of Lemma 3.13.* Let $\{\lambda_1, \ldots, \lambda_N\}$ be the distinct eigenvalues of $T$.

First suppose that (i) holds, and let us see how it implies (ii). For each $\lambda_k$, choose any nonzero vector $v_k$ in the eigenspace $\ker(T - \lambda_k)$.

By (3.13), the matrix

$$A = \Big(\alpha(\lambda_k \bar{\lambda}_\ell)\langle v_k, v_\ell\rangle\Big)_{k,\ell=1}^N$$

is nonnegative. Suppose that for some $k$, $\ell$, $|\lambda_k| = 1$ and $\lambda_\ell = r\lambda_k$, where $r \in [0,1)$. Then $\alpha(|\lambda_\ell|^2) > 0$. The $2 \times 2$ submatrix of the above $N \times N$ matrix that corresponds to $k$th and $\ell$th row and column is nonnegative, and this implies that $\langle v_k, v_\ell\rangle = 0$. It follows that $\ker(T - \lambda_k) \perp \ker(T - \lambda_\ell)$.

Now let us show the converse. Suppose that (ii) holds. We have to prove that one can find an admissible $\alpha$ such that for any choice of $v_k \in \ker(T - \lambda_k)$, $A$ is nonnegative. Suppose such choice is made. Let $\widehat{A}$ be the $N \times N$ matrix whose $(k, \ell)$ entry is $\alpha(\lambda_k \bar{\lambda}_\ell)$, except when the eigenvalues satisfy the relation

$$\lambda_k \neq \lambda_\ell, \quad \max\{|\lambda_k|, |\lambda_\ell|\} = 1, \quad \lambda_k = r\lambda_\ell \text{ or } \lambda_\ell = r\lambda_k, \tag{3.14}$$

for some $r \in [0,1]$, in which case the entry is just 0. Notice that $A$ is the Schur (that is, entrywise) product of matrix $\widehat{A}$ with the Gram matrix $V$ of the vectors $\{v_j\}$. The matrix $V$ is nonnegative. Hence, by the Schur product theorem, it suffices to find an admissible $\alpha$ such that $\widehat{A}$ is nonnegative.

Let us put $S := \sigma(T) \cap \mathbb{D}$ and $S^* := \sigma(T) \cap \mathbb{T}$. Hence $\sigma(T) = S \sqcup S^*$. Notice that $S$ and $S^*$ may be empty.

Let us suppose that the elements of $S$ are ordered in increasing order of moduli:

$$\begin{aligned} |\lambda_1| = \cdots = |\lambda_{N_1}| < |\lambda_{N_1+1}| = \cdots = |\lambda_{N_1+N_2}| < \cdots \\ < |\lambda_{N_1+\cdots+N_{m-1}+1}| = \cdots = |\lambda_{N_1+\cdots+N_m}|. \end{aligned}$$

Let

$$S_1 := \{\lambda_1, \ldots, \lambda_{N_1}\}, \quad S_j := \{\lambda_{N_1+\cdots+N_{j-1}+1}, \ldots, \lambda_{N_1+\cdots+N_j}\} \quad (j = 2, \ldots, m).$$

Hence each $S_j$ has $N_j$ elements. Let $p_j := |\lambda|^2$ for any $\lambda \in S_j$ and put $P := \{p_1, \ldots, p_m\} \subset [0,1)$.

Let us fix some values of $\alpha$:

$$\alpha(\lambda_k\bar{\lambda}_\ell) = \begin{cases} 0 & \text{if } \lambda_k\bar{\lambda}_\ell \notin [0,1] \\ 1 & \text{if } \lambda_k\bar{\lambda}_\ell \in [0,1) \setminus P. \end{cases} \tag{3.15}$$

Notice that $\alpha(1)$ must be 0, because we want $\alpha$ to be admissible. So we just need to know the values that $\alpha$ takes at $P$ to be able to construct the matrix $\widehat{A}$.

The values of $\alpha(p_j) =: \mu_j$ will be determined inductively. Since $p_j \in [0,1)$ and we want $\alpha$ to be admissible, the $\mu_j$'s must be positive. The idea will be to assign them sufficiently big values in each step. We will construct inductively matrices $\widehat{A}_1, \dots, \widehat{A}_m$ with the same block structure, and then we will put $\widehat{A}_S := \widehat{A}_m$ and

$$\widehat{A} = \begin{pmatrix} \widehat{A}_S & 0 \\ 0 & 0 \end{pmatrix}$$

will have the desired property.

Fix $\mu_1 > 0$. Let $\widehat{A}_1$ be the $N_1 \times N_1$ matrix whose $(k,\ell)$ entry is $\alpha(\lambda_k\bar{\lambda}_\ell)$, where $\lambda_k, \lambda_\ell \in S_1$. The values assigned so far to $\alpha$ guarantee that $\widehat{A}_1 = \mu_1 I_{N_1} > 0$ (i.e., it is a positive defined matrix).

Now suppose $\mu_1, \mu_2, \dots, \mu_{j-1}$ have been chosen already. The $(N_1+\cdots+N_j) \times (N_1+\cdots+N_j)$ matrix $\widehat{A}_j$ whose $(k,\ell)$ entry is $\alpha(\lambda_k\bar{\lambda}_\ell)$, where $\lambda_k, \lambda_\ell \in S_1 \cup \cdots \cup S_j$, has the form

$$\widehat{A}_j = \begin{pmatrix} \widehat{A}_{j-1} & B_j \\ B_j^* & \mu_j I_{N_j} \end{pmatrix} \quad (j = 2, \dots, m), \tag{3.16}$$

where the matrices $\widehat{A}_{j-1}$ and $B_j$ do not depend on $\mu_j$ and, moreover, $\widehat{A}_{j-1} \geq \delta_{j-1} I > 0$ for some positive number $\delta_{j-1}$. We claim that, if $\mu_j$ is chosen to be sufficiently large, the matrix $\widehat{A}_j$ will also satisfy $\widehat{A}_j \geq \delta_j I > 0$.

**Claim.** *Consider a block matrix*

$$C = \begin{pmatrix} E & F \\ F^* & xI \end{pmatrix},$$

*where $E, F$ are fixed, $E$ is a square matrix and $x \geq 0$. If $E \geq \varepsilon I > 0$ for some positive number $\varepsilon$, then $C \geq (\varepsilon/2)I > 0$ for $x$ sufficiently large.*

Indeed, put $\widetilde{E} := E - (\varepsilon/2)I \geq (\varepsilon/2)I$. Then we just need to prove that

$$\widetilde{C} = \begin{pmatrix} \widetilde{E} & F \\ F^* & yI \end{pmatrix} \geq 0, \tag{3.17}$$

for $y$ sufficiently large, because then the statement follows by simply taking $x = y + \varepsilon/2$. For a vector $v = \binom{a}{b}$, we have

$$\langle \widetilde{C}v, v\rangle = a^*\widetilde{E}a + 2\mathrm{Re}(a^*Fb) + yb^*b.$$

Taking $y \geq (2/\varepsilon)\|F\|^2$ we obtain

$$\begin{aligned} a^*\widetilde{E}a + yb^*b &\geq (\varepsilon/2)\|a\|^2 + y\|b\|^2 \geq (\varepsilon/2)\|a\|^2 + (2/\varepsilon)\|F\|^2\|b\|^2 \\ &\geq 2\|a\|\|F\|\|b\| \geq -2\mathrm{Re}(a^*Fb), \end{aligned}$$

and the claim follows. □

So, by applying the claim several times and choosing $\mu_2, \ldots, \mu_m$ sufficiently large, we obtain that $\widehat{A}_S := \widehat{A}_m > 0$. Therefore, with this assignment of the values $\alpha(\lambda_k\bar{\lambda}_\ell)$ we obtain that the matrix $\widehat{A}$ has the form

$$\widehat{A} = \begin{pmatrix} \widehat{A}_S & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\widehat{A} \geq 0$, as desired.

This proves that, if $\alpha$ takes the assigned values in finitely many points of $\overline{\mathbb{D}}$, then $T \in \mathbf{C}_\alpha$. By the next lemma, there always exists an admissible function $\alpha$ that solves this interpolation problem. This finishes the proof. □

**Lemma 3.14.** *Suppose we are given an interpolation problem $\alpha(t_j) = f_j$, where $t_j \in \overline{\mathbb{D}}$ are (distinct) nodes, $t_j \neq 1$ for all $j$ and $f_j \in \mathbb{C}$. Assume the following properties of this problem:*

(i) *Symmetry: to any node $t_j$ there corresponds the symmetric node $t_k = \bar{t}_j$ and, moreover, $f_k = \bar{f}_j$.*

(ii) *Positivity: $f_j > 0$ whenever $t_j \in [0,1)$.*

*Then there exists an admissible polynomial $\alpha$, which solves this problem.*

*Proof.* Fix a small positive constant $p$. There exist symmetric data $\{g_j\}$ such that $g_j^2 + p = f_j/(1-t_j)$ and $g_j$ is real whenever $t_j \in [0,1)$. The Lagrange interpolation formula implies that there is a real polynomial $\beta$ solving the problem $\beta(t_j) = g_j$. Then the polynomial $\alpha(t) = (1-t)(\beta(t)^2 + p)$ is positive on $[0,1)$. Hence it is admissible and solves the desired interpolation problem. □

## 3.3 Explicit model

### 3.3.1 On definition of unitary part of a $\mathbf{C}_\alpha$ operator

**Definition 3.15.** Let $\mathcal{K}$ be a Hilbert space. We say that $T \in \boldsymbol{B}(\mathcal{K})$ is a *completely non-unitary operator* if there is no nonzero reducing subspace $\mathcal{L}$ for $T$ such that $T|\mathcal{L}$ is unitary.

It is well-known that every contraction $S$ can be decomposed into an orthogonal sum of a unitary operator and a completely non-unitary operator (called the *unitary part* and the *completely non-unitary part* of $S$, respectively). We recall that the standard construction of the Nagy-Foias model applies only to completely non-unitary contractions.

If one takes an operator $T$ in the class $\mathcal{C}_\alpha$ and applies to it Theorem 3.4, then one gets a *direct sum* decomposition

$$\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1 \tag{3.18}$$

such that $T|\mathcal{H}_0$ is similar to a unitary operator and $T|\mathcal{H}_1$ is similar to a completely non-unitary operator.

For a general admissible function $\alpha$, we cannot say much more. However, some extra properties hold if $\alpha$ is strongly admissible. To speak about them, we recall first the following characterization of the unitary part of a contraction given by Nagy and Foiaş.

**Theorem 3.16** (See [89, Theorem I.3.2])**.** *To every contraction $S$ on $\mathcal{H}$ there corresponds a decomposition of $\mathcal{H}$ into an orthogonal sum of two subspaces reducing $S$, say $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that $S|\mathcal{H}_0$ is unitary and $S|\mathcal{H}_1$ is completely non-unitary ($\mathcal{H}_0$ or $\mathcal{H}_1$ may equal the trivial subspace $\{0\}$). This decomposition is uniquely determined. Indeed, $\mathcal{H}_0$ consists of those elements $h$ of $\mathcal{H}$ for which*

$$\|S^n h\| = \|h\| = \|S^{*n} h\| \qquad (n = 1, 2, \ldots).$$

*$S_0 = S|\mathcal{H}_0$ and $S_1 = S|\mathcal{H}_1$ are called the unitary part and the completely non-unitary part of $S$, respectively.*

The next theorem is an analogue of this decomposition for our operators in the case of a strongly admissible function $\alpha$.

**Theorem 3.17.** *Let $T \in \mathbf{C}_\alpha$, where $\alpha$ is strongly admissible. Denote by $\mathcal{H}_0$ the elements $h \in \mathcal{H}$ for which there exists a two-sided sequence $\{h_n\}_{n\in\mathbb{Z}}$ such that $h_0 = h, Th_n = h_{n+1}$ and $\|h_n\| = \|h\|$ for every $n \in \mathbb{Z}$. Let $\widetilde{T}$ be the operator $T$ acting on $\widetilde{H} := (\mathcal{H}, |||\cdot|||)$ (the new norm, which was given in (3.7)). Then $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$ and there exists a direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1$ with the following properties:*

(i) *$T|\mathcal{H}_0$ is unitary;*

(ii) *$\mathcal{H}_0$ and $\mathcal{H}_1$ are invariant subspaces of $T$;*

(iii) *$\mathcal{H}_0$ and $\mathcal{H}_1$ are orthogonal in $\widetilde{H}$;*

(iv) *$\widetilde{T}|\mathcal{H}_0$ and $\widetilde{T}|\mathcal{H}_1$ are the unitary part and the completely non-unitary part of the contraction $\widetilde{T}$, respectively;*

(v) *For any $h \in \mathcal{H}$,* $\lim^*_{n\to\infty} \|T^n h\|^2 = \lim_{n\to\infty} |||\widetilde{T}^n h|||^2$.

**Remark 3.18.** In Corollary 3.12 we saw that if $T$ is a finite matrix without Jordan blocks and $\sigma(T) \subset \mathbb{T}$, then $T \in \mathbf{C}_p$ for some admissible polynomial $p$. In this case, $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_1 = 0$ in the decomposition (3.18), but $T|\mathcal{H}_0 = T$ is non-unitary, as a rule. As a consequence we get:

(i) Theorem 3.17 is not valid if we do require $\alpha$ to be strongly admissible;

(ii) A non-unitary finite matrix $T$ with $\sigma(T) \subset \mathbb{T}$ cannot belong to $\mathbf{C}_\alpha$ if $\alpha$ is strongly admissible.

Observe that a completely non-unitary contraction can be similar to a unitary operator. For a general operator $T$, one cannot single out the largest direct summand, which is similar to a unitary.

*Proof of Theorem 3.17.* First we notice that $\widetilde{T}$ is a contraction on $\widetilde{\mathcal{H}}$. By Theorem 3.16, $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$, where $\widetilde{\mathcal{H}}_0$ consists of those elements $h$ of $\widetilde{\mathcal{H}}$ for which

$$|||\widetilde{T}^n h||| = |||h||| = |||\widetilde{T}^{*n} h||| \qquad (n = 1, 2, \ldots),$$

we have that $\widetilde{T}_0 := \widetilde{T}|\widetilde{\mathcal{H}}_0$ is unitary and $\widetilde{T}_1 := \widetilde{T}|\widetilde{\mathcal{H}}_1$ is completely non-unitary.

The goal of the following two claims is to prove that $\mathcal{H}_0 = \widetilde{\mathcal{H}}_0$.

**Claim 1.** *Let $h \in \mathcal{H}$. Then $h \in \widetilde{\mathcal{H}}_0$ if and only in there exists a sequence $\{h_n\}_{n\in\mathbb{Z}}$ such that $h_0 = h, \widetilde{T} h_n = h_{n+1}$ and $|||h_n||| = |||h|||$ for every $n \in \mathbb{Z}$.*

(In fact, this claim is a variation of Theorem 3.16 and is valid for any contraction $\widetilde{T}$ on a Hilbert space.)

Indeed, suppose that $h \in \widetilde{\mathcal{H}}_0$. Define the sequence $\{h_n\}_{n\in\mathbb{Z}}$ by $h_0 := h, h_n := \widetilde{T}^n h$ and $h_{-n} := \widetilde{T}^{*n} h$, for $n \geq 1$. Since $\widetilde{T}|\widetilde{\mathcal{H}}_0$ is unitary we obtain that $\widetilde{T} h_{-n} = \widetilde{T}\widetilde{T}^{*n} h = \widetilde{T}^{*(n-1)} = h_{-n+1}$ for $n \geq 1$. It follows that the sequence $\{h_n\}_{n\in\mathbb{Z}}$ satisfies the conditions stated in Claim 1.

Reciprocally, suppose that a sequence $\{h_n\}_{n\in\mathbb{Z}}$ satisfies the conditions of the statement. Fix $n \geq 1$. Since $\widetilde{T}^n h_{-n} = h$, we have that $|||\widetilde{T}^n h_{-n}|||^2 = |||h|||^2 = |||h_{-n}|||^2$. Hence $\langle (I - \widetilde{T}^{*n}\widetilde{T}^n) h_{-n}, h_{-n}\rangle_{\widetilde{\mathcal{H}}} = 0$. Since $\widetilde{T}^n$ is a contraction on $\widetilde{\mathcal{H}}$, it follows that $(I - \widetilde{T}^{*n}\widetilde{T}^n) h_{-n} = 0$. Therefore, using that $\widetilde{T}^n h_{-n} = h$, we obtain that $\widetilde{T}^{*n} h = h_{-n}$. Then $h \in \widetilde{\mathcal{H}}_0$. This finishes the proof of Claim 1.

**Claim 2.** *Let $h \in \mathcal{H}$. Then $h \in \widetilde{\mathcal{H}}_0$ if and only in there exists a sequence $\{h_n\}_{n\in\mathbb{Z}}$ such that $h_0 = h, T h_n = h_{n+1}$ and $\|h_n\| = \|h\|$ for every $n \in \mathbb{Z}$.*

(Note that this claim is stated in terms of the original norm in $\mathcal{H}$.)

Indeed, we apply Claim 1. Let $\{h_n\}_{n\in\mathbb{Z}}$ be a sequence such that $h_0 = h$ and $T h_n = h_{n+1}$. We have to show that the sequence $\{|||h_n|||^2\}_{n\in\mathbb{Z}}$ is constant if and only if the sequence $\{\|h_n\|^2\}_{n\in\mathbb{Z}}$ is constant. Since the two norms are equivalent, if either of these two sequences is constant, the other is in $\ell^\infty(\mathbb{Z})$.

By (3.8) we have that

$$|||h_{n+1}|||^2 = |||Th_n|||^2 = |||h_n|||^2 - \|Dh_n\|^2 .$$

Therefore $|||h_n||| = |||h|||$ for every $n \in \mathbb{Z}$ if and only if $\|Dh_n\|^2 = 0$ for every $n \in \mathbb{Z}$. We will use the backward shift operator $\nabla$, acting on $\ell^\infty(\mathbb{Z})$ by $[\nabla a]_n = a_{n+1}$, $a \in \ell^\infty(\mathbb{Z})$. For any $f \in A_{W,\mathbb{R}}$, $f(\nabla)$ is well-defined by $\big(f(\nabla)a\big)_n := \sum_{j=0}^\infty f_j a_{n+j}$. Notice that

$$(fg)(\nabla) = f(\nabla)g(\nabla) \qquad \text{for all } f, g \in A_{W,\mathbb{R}}.$$

Denote by $\mathbf{h}$ the sequence $\{\|h_n\|^2\}_{n\in\mathbb{Z}}$. Then it is easy to obtain that $\|Dh_n\|^2 = [\alpha(\nabla)\mathbf{h}]_n$. Hence $|||h_n||| = |||h|||$ for every $n \in \mathbb{Z}$ if and only if $\alpha(\nabla)\mathbf{h} = (\dots, 0, 0, 0, \dots) = \mathbf{0}$.

So we have to show that $\mathbf{h}$ is constant if and only if $\alpha(\nabla)\mathbf{h} = \mathbf{0}$. The direct implication is obvious. For the converse, fix a factorization $\alpha(t) = (1-t)q(t)\gamma(t)$, where $q$ is a polynomial with zeros in $\mathbb{D}$ and $\gamma \in A_{W,\mathbb{R}}(\overline{\mathbb{D}})$ has no zeros in $\overline{\mathbb{D}}$ (recall that $\alpha$ is assumed to be strongly admissible). Then $1/\gamma \in A_{W,\mathbb{R}}(\overline{\mathbb{D}})$. Hence $\gamma(\nabla)[q(\nabla)(1-\nabla)\mathbf{h}] = \mathbf{0}$ implies that $q(\nabla)(1-\nabla)\mathbf{h} = \mathbf{0}$ (just multiply by $1/\gamma$). Now let $\mathbf{g} = (1-\nabla)\mathbf{h}$. We want to proof that $\mathbf{g} = \mathbf{0}$. When $q$ has just a single root, say $q(t) = 1 - at$ for some $a$ with $|a| > 1$, the result is immediate. For a general $q$ we just need to apply induction on the number of roots of $q$. This finishes the proof of Claim 2.

Now we can conclude the proof of Theorem 3.17. Claim 2 gives that $\mathcal{H}_0 = \widetilde{\mathcal{H}}_0$. Put $\mathcal{H}_1 := \widetilde{\mathcal{H}}_1$. Now (i) is obvious, since $T|\mathcal{H}_0$ is a surjective isometry. The rest of items of the theorem follow immediately using Theorem 3.16 and (3.12). □

### 3.3.2 On the Nagy-Foiaş model of a contraction

First let us briefly recall the standard Nagy-Foiaş model. We state it in the form which will be convenient for us.

Let $T$ be a completely non-unitary contraction on $\mathcal{H}$. Now

$$D_T = (I - T^*T)^{1/2} \quad \text{and} \quad D_{T^*} = (I - TT^*)^{1/2}$$

are its defect operators, and

$$\mathfrak{D}_T = \overline{D_T\mathcal{H}} \quad \text{and} \quad \mathfrak{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$$

are the corresponding defect spaces. The characteristic function of $T$ is given by

$$\Theta(z) = \Theta_T(z) := -T + zD_{T^*}(I - zT^*)^{-1}D_T\big|\mathfrak{D}_T, \quad z \in \mathbb{D}; \tag{3.19}$$

this is an operator-valued function in $\mathcal{H}^\infty(L(\mathfrak{D}_T, \mathfrak{D}_{T^*}))$ with $\|\Theta\|_\infty \le 1$. Next, we put $\Delta(\zeta) = (I - \Theta^*(\zeta)\Theta(\zeta))^{1/2}$, $\Delta_*(\zeta) = (I - \Theta(\zeta)\Theta^*(\zeta))^{1/2}$; these are measurable non-negative operator-valued functions on the unit circle $\mathbb{T} = \{|z| = 1\}$. Given a Hilbert space $\mathcal{R}$, $H^2(\mathcal{R})$ will stand for the standard Hardy space of $\mathcal{R}$-valued functions on $\overline{\mathbb{D}}$; this can also be seen as a closed subspace of the Lebesgue space $L^2(\mathcal{R}) := L^2(\mathbb{T}, \mathcal{R})$. We put $H^2_-(\mathcal{R}) = L^2(R) \ominus H^2(\mathcal{R})$; the functions in this space can be alternatively seen as $\mathcal{R}$-valued analytic functions on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ (taking value 0 at $\infty$).

As usual, we define *the model space*

$$\mathcal{K}_\Theta = \begin{pmatrix} H^2(\mathfrak{D}_{T^*}) \\ \operatorname{clos} \Delta L^2(\mathfrak{D}_T) \end{pmatrix} \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D}_T)\,.$$

It is easy to see that *the model operator*

$$M_* \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \frac{u(z)-u(0)}{z} \\ z^{-1}v(z) \end{pmatrix}. \tag{3.20}$$

is a contraction on $\mathcal{K}_\Theta$.

**Theorem 3.19** (Nagy-Foias model)**.** *For $h \in \mathcal{H}$, put*

$$\begin{aligned} \Phi_1 h(z) &= D_{T^*}(I - zT^*)^{-1}h, \quad |z| < 1, \\ \Phi_3 h(z) &= D_T(z - T)^{-1}h, \qquad |z| > 1. \end{aligned} \tag{3.21}$$

*Then $\Phi_1, \Phi_3$ are bounded operators from $\mathcal{H}$ to $H^2(\mathfrak{D}_{T^*})$, $H^2_-(\mathfrak{D}_T)$, respectively. Next, for any $h \in \mathcal{H}$, the formula*

$$\Delta\Phi_2 h = \Phi_3 h - \Theta^*\Phi_1 h \quad \textit{a.e. on } \mathbb{T} \tag{3.22}$$

*defines a unique measurable function $\Phi_2 h \in \overline{\Delta L^2(\mathfrak{D}_T)}$ on the unit circle $\mathbb{T}$. The map*

$$\Phi h = \begin{pmatrix} \Phi_1 h \\ \Phi_2 h \end{pmatrix} \tag{3.23}$$

*is an isometric isomorphism from $\mathcal{H}$ onto $\mathcal{K}_\Theta$. It has the properties*

(i) $\Phi T^* \Phi^{-1} = M_*$;

(ii) $\|\Phi_2 h(z)\|^2 = \lim_{n\to\infty} \|T^{*n}h\|^2$ *for every $h \in \mathcal{H}$.*

Since we could not find a good reference for these formulas, we derive them from the coordinate-free form by Nikolski and Vasyunin. More or less similar formulation is contained in [79].

*Proof of Theorem 3.19.* As it follows from [70], there is a unitary isomorphism $\Phi : \mathcal{H} \to \mathcal{K}_\Theta$ such that $\Phi T^* \Phi^{-1} = M_*$. The explicit formulas for $\Phi$ follow from comparing the incoming transcription of the Nagy-Foias model (which is its usual form) with its outgoing transcription, see [70]. We will use the notation of [70], with minor modifications.

The unitary dilation $U = U_T$ of operator $T$ acts on its dilation space

$$\mathcal{H}_T = H^2_-(\mathfrak{D}_{T^*}) \oplus \mathcal{H} \oplus H^2(\mathfrak{D}_T).$$

By [89] or [70, Lemma 1.1.14], $U$ is given by the formula

$$U = \begin{pmatrix} P_{H^2_-(\mathfrak{D}_{T^*})} M_{\bar{z}} & \mathbb{O} & \mathbb{O} \\ D_{T^*} j_*^* & T & \mathbb{O} \\ -jT^* j_*^* & jD_T & M_z \end{pmatrix}. \tag{3.24}$$

Here we use the notation $M_\phi$ for the multiplication operator by a function $\phi$ and $j : \mathfrak{D}_T \to H^2(\mathfrak{D}_T)$, $j_* : \mathfrak{D}_{T^*} \to H^2_-(\mathfrak{D}_T^*)$ are the isometries given by $jd(z) \equiv d$ (the image is a constant function) and $j_* d_*(z) = z^{-1} d_*$. One can observe that $G_* = H^2_-(\mathfrak{D}_{T^*})$ is an incoming space and $G = H^2(\mathfrak{D}_T)$ is an outgoing space for $U$ (in the sense that $U^* G_* \subset G_*$ and $UG \subset G$).

Introduce the space

$$\mathcal{F}_T = \begin{pmatrix} L^2(\mathfrak{D}_{T^*}) \\ \operatorname{clos} \Delta L^2(\mathfrak{D}_T) \end{pmatrix}$$

and the isometric functional embeddings

$$\pi_{*,T} : L^2(\mathfrak{D}_{T^*}) \to \mathcal{F}_T, \quad \pi_T : L^2(\mathfrak{D}_T) \to \mathcal{F}_T,$$

which are multiplication operators:

$$\pi_{*,T} = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad \pi_T = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix}. \tag{3.25}$$

Notice that

$$\mathcal{F}_T = \pi_{*,T} H^2_-(\mathfrak{D}_{T^*}) \oplus \mathcal{K}_\Theta \oplus \pi_T H^2(\mathfrak{D}_T).$$

The Nagy-Foias theory makes use of an identification $\widetilde{\Phi} : \mathcal{H}_T \to \mathcal{F}_T$. It is a unique bounded linear operator, which satisfies

$$\widetilde{\Phi} | H^2_-(\mathfrak{D}_{T^*}) = \pi_{*,T}, \quad \widetilde{\Phi} | H^2(\mathfrak{D}_T) = \pi_T, \quad \widetilde{\Phi} U \widetilde{\Phi}^{-1} = M_z. \tag{3.26}$$

The uniqueness is easy to see. Indeed, these three requirements define $\widetilde{\Phi}$ in the unique way on the set of elements of $\mathcal{H}_T$ of the form $U^k(f \oplus 0 \oplus 0) + U^\ell(0 \oplus 0 \oplus g)$, where $f \in H^2_-(\mathfrak{D}_T)$, $g \in H^2(\mathfrak{D}_{T^*})$ and $k, \ell \in \mathbb{Z}$. These elements are dense in $\mathcal{H}_T$. It is well-known that $\widetilde{\Phi} : \mathcal{H}_T \to \mathcal{F}_T$ is an isometric isomorphism.

We put $\Phi = \widetilde{\Phi} | \mathcal{H}$; it is an isometric isomorphism of $\mathcal{H}$ onto $\mathcal{K}_\Theta$. We are going to show that the formulas (3.21)–(3.23) define the same map.

The formula (3.24) implies that

$$\widetilde{\Phi}U^*r - \begin{pmatrix} z^{-1}D_{T^*}r \\ 0 \end{pmatrix} \in \begin{pmatrix} H^2(\mathfrak{D}_{T^*}) \\ \operatorname{clos}\Delta L^2(\mathfrak{D}_T) \end{pmatrix}, \qquad r \in \mathcal{H}. \tag{3.27}$$

Define $F_k : \mathcal{F}_T \to \mathfrak{D}_{T^*}$ for $k \in \mathbb{Z}$ by

$$F_k \begin{pmatrix} \sum_{n\in\mathbb{Z}} g_n z^n \\ v \end{pmatrix} = g_k.$$

Take any $h \in \mathcal{H}$, and put $g \oplus v = \Phi h \in \mathcal{K}_\Theta$. A direct calculation shows that for any $\lambda \in \mathbb{D}$,

$$F_0(I - \lambda M_*)^{-1}(g \oplus v) = g(\lambda).$$

Hence

$$\begin{aligned} g(\lambda) &= F_{-1}\bar{z}(I - \lambda M_*)^{-1}(g \oplus v) = F_{-1}\bar{z}(I - \lambda M_*)^{-1}\widetilde{\Phi}h \\ &= F_{-1}\widetilde{\Phi}U^*(I - \lambda T^*)^{-1}h = D_{T^*}(I - \lambda T^*)^{-1}h \end{aligned}$$

(the last equality follows from (3.27)). This gives the first formula in (3.21).

The formula for $\Phi_3$ follows by comparing the Nagy-Foias models of $T$ and of $T^*$. We will put the underscript $T^*$ to the objects corresponding to the model of $T^*$. First we observe that by (3.24), $U^*$ is the minimal unitary dilation of $T^*$, where now $H^2_-(\mathfrak{D}_{T^*})$ is the outgoing space and $H^2(\mathfrak{D}_T)$ is the incoming space. To formalize it, given any function $f$ on $\mathbb{T}$, we will denote by $Jf$ its "flip", $Jf(z) = z^{-1}f(z^{-1})$. Note that $J^2f = f$ for any $f$. For any Hilbert space $E$, $J$ is a unitary isomorphism sending $H^2_-(E)$ onto $H^2(E)$, and vice versa. Define the unitary isomorphism

$$W\big(f \oplus h \oplus g\big) = Jg \oplus h \oplus Jf \tag{3.28}$$

from $\mathcal{H}_T$ onto

$$\mathcal{H}_{T^*} = H^2_-(\mathfrak{D}_T) \oplus \mathcal{H} \oplus H^2(\mathfrak{D}_{T^*}).$$

Then

$$U_{T^*} = WU^*W^{-1}$$

is the unitary dilation of $T^*$, obtained from (3.24) by substituting $T^*$ in place of $T$.

Next, put $\Xi = \begin{pmatrix} \Theta^* & \Delta \\ \Delta_* & -\Theta \end{pmatrix}$. Since $\Theta\Delta = \Delta_*\Theta$, it follows that

$$M_\Xi : \mathcal{F}_T \to \mathcal{F}_{T^*} = \begin{pmatrix} L^2(\mathfrak{D}_T) \\ \operatorname{clos}\Delta_* L^2(\mathfrak{D}_{T^*}) \end{pmatrix}$$

is an isometric isomorphism. Notice also that by (3.19), $\Theta_{T^*}(z) = \Theta(\bar{z})^*$.

By inspecting (3.25), we see that the functional embeddings for $T^*$ are expressed as

$$\pi_{*,T^*} = JM_\Xi \pi_T J : H^2_-(\mathfrak{D}_T) \to \mathcal{F}_{T^*}, \quad \pi_{T^*} = JM_\Xi \pi_{*,T} J : H^2(\mathfrak{D}_{T^*}) \to \mathcal{F}_{T^*}.$$

These formulas imply that

$$\widetilde{\Phi}_{T^*} = JM_\Xi \widetilde{\Phi}_T W. \tag{3.29}$$

Indeed, one easily checks that the right hand part satisfies the requirements of (3.26), with the role of $T$ and $T^*$ interchanged and with $U_{T^*}$ in place of $U$.

To get the expression (3.22) for $\Phi_2$, take any $h \in \mathcal{H}$ and set $g \oplus v = \Phi h \in \mathcal{K}_\Theta$, $f \oplus u = \Phi_{T^*} h \in \mathcal{K}_{\Theta_{T^*}}$. By (3.29) and (3.28),

$$J \begin{pmatrix} f \\ u \end{pmatrix} = \Xi \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} \Theta^* & \Delta \\ \Delta_* & -\Theta \end{pmatrix} \begin{pmatrix} g \\ v \end{pmatrix}.$$

Since $g(z) = D_{T^*}(I - zT^*)^{-1}h$ and $f(z) = D_T(I - zT)^{-1}h$, we obtain that

$$z^{-1} D_T (I - z^{-1}T)^{-1} h = z^{-1} f(z^{-1}) = \Theta^*(z) g(z) + \Delta(z) v(z), \quad z \in \mathbb{T},$$

and (3.22) follows. The assertion (ii) of the Theorem is well-known (and follows at once by calculating the limit of $M_*^n(g \oplus v)$ as $n \to \infty$). This finishes the proof of Theorem 3.19. $\square$

## 3.4 A Nagy-Foias-type model for operators in $\mathbf{C}_\alpha$

Now let $T \in \mathbf{C}_\alpha$ for some $\alpha \in A_{W,\mathbb{R}}$. In this section we will assume that $\alpha$ is strongly admissible and $T$ is a completely non-unitary operator (see Definition 3.15).

Then, using the notation of Section 3.3.1, we have that $\widetilde{T}$ is a completely non-unitary contraction on $\widetilde{\mathcal{H}}$. We build our model for $T$ directly from the Nagy-Foias model of $\widetilde{T}^*$. Then we arrive at the following. Let $D_{\widetilde{T}}$ and $D_{\widetilde{T}^*}$ be the defect operators of $\widetilde{T}$ and let $\mathfrak{D}_{\widetilde{T}}$ and $\mathfrak{D}_{\widetilde{T}^*}$ be its defect spaces. We recall that the defect operator $D$ of $T$ has been defined by (3.6). Define the *defect space of* $T$ as $\mathfrak{D}_T = \operatorname{clos} DH$, where the closure is taken with respect to $\|\cdot\|$.

Define $V : \mathfrak{D}_{\widetilde{T}} \to \mathfrak{D}_T$ by $V(D_{\widetilde{T}} h) = Dh$, $h \in \mathcal{H}$. Then, using equation (3.8), we obtain that $V$ extends to an isometry from $\mathfrak{D}_{\widetilde{T}}$ onto $\mathfrak{D}_T$. This isometry permits us to use $\mathfrak{D}_T$ instead of $\mathfrak{D}_{\widetilde{T}}$.

Let us define the functions $\Theta_* \in \mathcal{H}^\infty(\mathfrak{D}_{\widetilde{T}^*} \to \mathfrak{D}_T)$ and $\Delta_* : \mathbb{T} \to L(\mathfrak{D}_{\widetilde{T}^*})$ by

$$\Theta_*(z)h := V(-\widetilde{T}^* + zD_{\widetilde{T}}(I - zT)^{-1} D_{\widetilde{T}^*})h, \quad h \in \mathfrak{D}_{\widetilde{T}^*},\ z \in \mathbb{D},$$

$$\Delta_*(\zeta) := (I - \Theta_*(\zeta)^* \Theta_*(\zeta))^{1/2}, \quad \zeta \in \mathbb{T}.$$

The function $\Theta_*$ will play the role of a characteristic function of $T$.

Our model space now will be

$$\mathcal{K}_{\Theta_*} := \begin{pmatrix} H^2(\mathfrak{D}_T) \\ \operatorname{clos}\Delta_* L^2(\mathfrak{D}_{\widetilde{T}^*}) \end{pmatrix} \ominus \begin{pmatrix} \Theta_* \\ \Delta_* \end{pmatrix} H^2(\mathfrak{D}_{\widetilde{T}^*}).$$

Finally, define $\Phi_{1,*} : \mathcal{H} \to H^2(\mathfrak{D}_T)$, $\Phi_{3,*} : \mathcal{H} \to H^2_-(\mathfrak{D}_{\widetilde{T}^*})$ by

$$\Phi_{1,*}h(z) = D(I - zT)^{-1}h, \quad \Phi_{3,*}h(z) = D_{\widetilde{T}^*}(z - \widetilde{T}^*)^{-1}h;$$

these are bounded operators.

**Theorem 3.20.** *Let $T \in \mathbf{C}_\alpha$ be a completely non-unitary operator, where $\alpha$ is strongly admissible. With the notation used above, introduce the model operator $M_*$, acting on the model space $\mathcal{K}_{\Theta_*}$ by* (3.20)*. Then*

(i) *For any $h \in \mathcal{H}$, the formula*

$$\Delta_*\Phi_{2,*}h = \Phi_{3,*}h - \Theta_*^*\Phi_{1,*}h \quad \textit{a.e. on } \mathbb{T}$$

*defines a unique function $\Phi_{2,*}h \in \operatorname{clos}\Delta_* L^2(\mathfrak{D}_{\widetilde{T}^*})$.*

(ii) $\Phi_* := \binom{\Phi_{1,*}}{\Phi_{2,*}} : \mathcal{H} \to \mathcal{K}_{\Theta_*}$ *is a linear isomorphism.*

(iii) $\Phi_* T = M_* \Phi_*$.

(iv) $\|\Phi_{2,*}h(z)\|^2 = \lim^*_{n\to\infty} \|T^n h\|^2$ *for every $h \in \mathcal{H}$.*

*Proof.* All these statements follows at once from Theorem 3.19 and from Theorem 3.17 (v). □

We remark that in case when $T^n \to 0$ in the SOT topology, the formula for $\Phi_*$ is given in terms of the original operator $T$, without recurring to $\widetilde{T}$.

It is a common point that this kind of model for $T$ implies that there exists a $\mathcal{H}^\infty$ functional calculus for $T$, which coincides with the natural one on polynomials. The following variant of von Neumann inequality holds for any $f \in \mathcal{H}^\infty$,

$$\|f(T)\| \le C \max_{\overline{\mathbb{D}}} |f|,$$

where $C$ is a constant depending only on $T$ (in fact, $C = \|\Phi_*\| \left\|\Phi_*^{-1}\right\|$).

Numerous relations between the properties of the characteristic function and the properties of $T$ and $T^*$, known in the Nagy-Foias theory, extend to our setting of $\mathbf{C}_\alpha$ operators. For instance, we have the following statement, which is analogous to Proposition VI.3.5 in the book by Nagy-Foias [89] (and follows at once from this Proposition and Theorem 3.20).

**Proposition 3.21.** *Let $T \in \mathbf{C}_\alpha$ be completely non-unitary, where $\alpha$ is strongly admissible. Then*

(i) $T \in C_{0\cdot}$ *(that is, $T^n h \to 0$ for all $h$),*

(ii) $T \in C_{1\cdot}$ *(that is, $T^n h \nrightarrow 0$ for all $h \neq 0$),*

*if and only if the characteristic function $\Theta_*$ is, respectively,*

(i) *inner (that is, $\Theta_*(\zeta)$ is an isometry from $\mathfrak{D}_{\widetilde{T}^*}$ to $\mathfrak{D}_T$ for almost every $\zeta \in \mathbb{T}$);*

(ii) *outer (that is, $\overline{\Theta_* H^2(\mathfrak{D}_{\widetilde{T}^*})} = H^2(\mathfrak{D}_T)$, where the closure is taken in $L^2(\mathfrak{D}_T)$).*

Similar equivalences hold in our context for classes $C_{\cdot 0}$, $C_{\cdot 1}$, which also are defined in the same way as for the case of contractions.

We should mention the works by Olofsson (see [73]), where he relates certain transfer functions associated with $n$-hypercontractions with Bergman inner functions, which are crucial in the description of invariant subspaces of Bergman spaces. His results were further generalized in the works by Ball and Bolotnikov [13, 14]. A functional model for $C_{\cdot 0}$ $m$-hypercontractions has been constructed in [24].

We remark that in [92], a scheme for constructing explicit Nagy-Foias type models of Hilbert space operators has been suggested. This was done there for rather general domains and not only for the unit disk. The setting of [92] only applies to the $C_{0\cdot}$ case. Even for the case of the disc, the term "the characteristic function" here is used in a different sense than the term "a generalized characteristic function" in [92] (see [92], §3). In the present article, only a function $\Theta$ with $\|\Theta\|_\infty \leq 1$ can be a characteristic function of an operator in $\mathbf{C}_\alpha$. If $\|\Theta\|_\infty < 1$, then by the above proposition, $T \notin C_{0\cdot}$. On the contrary, any $*$-admissible operator-valued function $\delta$ on $\mathbb{D}$ can be a generalized characteristic function, in the sense of [92], of a $C_{0\cdot}$ operator $T$, similar to a contraction. It can happen that $\|\delta\|_\infty > 1$. Moreover, in the terminology of [92], if $\delta$ is a generalized characteristic function of $T$, then $\eta_1 \delta \eta_2$ also is, whenever $\eta_1, \eta_2$ are operator-valued functions, invertible in $\mathcal{H}^\infty$.

## 3.5 Operators in $\mathbf{C}_\alpha$ whose characteristic function has a determinant

Let us see now some spectral consequences derived from the fact that if $T \in \mathbf{C}_\alpha$ for some admissible function $\alpha$ then $T$ is similar to a contraction $\widetilde{T}$.

In what follows, $\mathfrak{S}_p$ $(0 < p \leq \infty)$ will denote the Schatten-von Neumann class of operators.

**Lemma 3.22.** *Let $T \in \mathbf{C}_\alpha$ for some admissible function $\alpha$ and let $p \in [1, \infty]$.*

(i) *If* $I - T^*T \in \mathfrak{S}_p$*, then* $D^2 \in \mathfrak{S}_p$*.*

(ii) *If* $D^2 \in \mathfrak{S}_p$ *and* $\tilde{\alpha}$ *has no zeros in* $\overline{\mathbb{D}}$*, then* $I - T^*T \in \mathfrak{S}_p$*.*

*Proof.* (i) It is immediate, since $D^2 = \alpha[T^*, T] = \tilde{\alpha}(T^*, T)(I - T^*T)$.

(ii) Since $\tilde{\alpha}$ has no zeros in $\overline{\mathbb{D}}$, $1/\tilde{\alpha} \in A_{W,\mathbb{R}}$, and we obtain that

$$(1/\tilde{\alpha})(T^*, T)(D^2) = (1/\tilde{\alpha})(T^*, T)(\tilde{\alpha}(T^*, T)(I - T^*T)) = I - T^*T,$$

which proves the result. □

**Proposition 3.23.** *Let* $T \in \mathbf{C}_\alpha$ *for some admissible function* $\alpha$ *and let* $p \in [1, \infty]$*. If* $\sigma(T) \neq \overline{\mathbb{D}}$*, then* $D_{\tilde{T}} \in \mathfrak{S}_p$ *if and only if* $D_{\tilde{T}^*} \in \mathfrak{S}_p$*.*

*Proof.* In the case when $0 \notin \sigma(T)$, $D_{\tilde{T}}$ is unitarily equivalent to $D_{\tilde{T}^*}$, see [89, the proof of Theorem VIII.1.1] or [56, Lemma 9]; this implies our assertion. The general case follows from this one. Indeed, take any $\lambda \in \mathbb{D} \setminus \sigma(T)$ and consider the Möbius self-map of $\mathbb{D}$, given by $b_\lambda(z) = (z - \lambda)/(1 - \bar{\lambda}z)$. Let $\tilde{T}_\lambda$ be the contraction, defined by $\tilde{T}_\lambda = b_\lambda(\tilde{T})$. Then the formula

$$I - \tilde{T}_\lambda^* \tilde{T}_\lambda = W^*(I - \tilde{T}^*\tilde{T})W \qquad \text{with } W = (1 - |\lambda|^2)^{1/2}(I - \bar{\lambda}\tilde{T})^{-1}$$

implies that $D_{\tilde{T}} \in \mathfrak{S}_p$ if and only if $D_{\tilde{T}_\lambda} \in \mathfrak{S}_p$. Since $0 \notin \sigma(\tilde{T}_\lambda)$, the general case follows. □

We remark that the previous reasoning can be applied to a more general situation when $T$ is a power bounded operator with $\sigma(T) \neq \overline{\mathbb{D}}$ and $D$ is its abstract defect operator. Then, by Lemma 3.3, one gets a contraction $\widetilde{T}$, similar to $T$, and so for any $p$, $D \in \mathfrak{S}_p$ iff $D_{\tilde{T}} \in \mathfrak{S}_p$ iff $D_{\tilde{T}^*} \in \mathfrak{S}_p$.

We recall the well-known fact that the characteristic function of a contraction $S$ has the determinant whenever $\sigma(S) \neq \overline{\mathbb{D}}$ and $I - S^*S \in \mathfrak{S}_1$ (this is the so-called class of weak contractions). It follows that $\Theta_*$ has a determinant whenever $\sigma(T) \neq \overline{\mathbb{D}}$ and $D \in \mathfrak{S}_2$. In this case, $\det \Theta_*$ is an $\mathcal{H}^\infty$ function such that $\|\det \Theta_*\|_\infty \leq 1$. We obtain the following statement.

**Proposition 3.24.** *Suppose* $\alpha$ *is strongly admissible and* $T \in \mathbf{C}_\alpha$ *is a completely non-unitary operator. Suppose also that* $\sigma(T) \neq \overline{\mathbb{D}}$ *and* $\alpha(T^*, T) \in \mathfrak{S}_1$*. Denote by* $\sigma_p(T)$ *the point spectrum of* $T$*. Then the following assertions are equivalent.*

(i) $T$ *is complete, that is,* $\mathcal{H} = \operatorname{span}\{\ker(\lambda I - T)^k : k \geq 1, \lambda \in \sigma_p(T)\}$*;*

(ii) $T^*$ *is complete;*

(iii) $\det \Theta_*(z)$ *is a Blaschke product.*

This follows from the above observations and from an analogous fact for completely non-unitary contractions, see [70, p. 134].

In a similar way, one can extend the results by Treil [90], Nikolski - Benamara [17], Kupin [58], [59] and others to the setting of operators in $\mathbf{C}_\alpha$, where $\alpha$ is strongly admissible. We recall that a completely non-unitary contraction $T$ is said to have a scalar multiple if there is a nonzero $\mathcal{H}^\infty$ function $m$ such that $m(T) = 0$. It is known that any such operator is decomposable and admits spectral synthesis, see [69, Lecture 4].

## 3.6 Existence of the limit of $\|T^n h\|$ for operators $T$ in $\mathbf{C}_\alpha$, where $\alpha$ is admissible

In this section we prove Theorem 0.15.

Note that this result implies that whenever $T \in \mathbf{C}_\alpha$ and $\alpha$ is strongly admissible, one can write $\lim \|T^n h\|^2$ instead of $\lim^*_{n\to\infty} \|T^n h\|^2$ in the expression (3.12) for the equivalent norm $|||h|||$.

Some relationships between asymptotic properties of powers of linear operators, their spectra and their numerical ranges have been revealed in a recent paper [66] by Müller and Tomilov.

**Remark 3.25.** In general, it is not true that if $T \in \mathbf{C}_\alpha$ for some admissible function $\alpha$ then there exists the limit of $\{\|T^n h\|\}$ for every $h \in \mathcal{H}$. Indeed, take for instance

$$\alpha(t) := 1 - t^j,$$

for some $j \geq 2$. Note that in this case $\alpha$ is admissible but not strongly admissible. Then $T \in \mathbf{C}_\alpha$ if and only if $\left\|T^j\right\| \leq 1$. Let $\{e_n\}$ be an orthonormal basis on $\mathcal{H}$, and let $T$ be the weighted shift $Te_j := \omega_j e_{j+1}$, where

$$\omega_0\omega_1 \cdots \omega_{j-1} = 1, \quad \text{and} \quad \omega_m = \omega_{m-j} \quad (\forall m \geq j).$$

Then $T^j$ is an isometry (and therefore $\left\|T^j\right\| \leq 1$), but

$$\{\|T^n e_0\|^2\}_{n\geq 0} = \{1, \omega_0^2, (\omega_0\omega_1)^2, \ldots, 1, \omega_0^2, (\omega_0\omega_1)^2, \ldots\}$$

which has no limit if $\omega_i \neq 1$ for some $i$.

Recall that at the end of the Introduction we defined the operator $\nabla$ acting on sequences of real numbers $\{\Lambda_n\}_{n\geq 0}$. Now we introduce the operator $\nabla_-$.

The backward shift $\nabla$ and the shift $\nabla_-$, acting on one-sided sequences $\{a_n\}_{n=0}^\infty \in \ell^\infty$, are given by $(\nabla a)_n = a_{n+1}$ for every $n \geq 0$ and $(\nabla_- a)_0 = 0$, $(\nabla_- a)_n = a_{n-1}$ for every $n \geq 1$.

If we identify the sequence $a = \{a_n\}_{n=0}^{\infty}$ with the power series $a(z) = \sum_{n=0}^{\infty} a_n z^n$, then we can identify the operator $\nabla$ and $\nabla_-$ with the operators given by

$$(\nabla a)(z) = \frac{a(z) - a(0)}{z}, \quad (\nabla_- a)(z) = z a(z).$$

It is clear that $\nabla \nabla_- = I$ and $\nabla_- \nabla a(z) = a(z) - a(0)$.

Given a function $f \in A_{W,\mathbb{R}}$, the operators $f(\nabla)$ and $f(\nabla_-)$ are well-defined on $\ell^\infty$. Note that $c = f(\nabla_-)a$ is given by $c_n := \sum_{j=0}^{n} f_j a_{n-j}$. In terms of power series, one just has $c(z) = f(z)a(z)$. We can say, in fact, that in the power series representation, $f(\nabla_-)$ is an analytic Toeplitz operator and $f(\nabla)$ is an anti-analytic Toeplitz operator.

The following formula will be useful:

$$\nabla_-^k \nabla^j a(z) = z^{k-j}(a(z) - a_{j-1} z^{j-1} - \cdots - a_1 z - a_0) \qquad (k \geq j). \tag{3.30}$$

We need some auxiliary lemmas.

**Lemma 3.26.** *Let $f, g \in A_{W,\mathbb{R}}$ and let $a \in \ell^\infty$. Then*

$$f(\nabla)[g(\nabla)a] = (fg)(\nabla)a.$$

The proof is immediate just doing a change of summation indices.

**Lemma 3.27.** *Let $f \in A_{W,\mathbb{R}}$ and let $a \in \ell^\infty$ be a convergent sequence, say $a_n \to a_\infty$.*

(i) *If $b = f(\nabla)a$, then $b_n \to f(1)a_\infty$.*

(ii) *The same is true for $\nabla_-$ in place of $\nabla$. Namely, if $c = f(\nabla_-)a$, then also $c_n \to f(1)a_\infty$.*

*Proof.* Both statements are straightforward, and we will only check (i). Fix $\varepsilon > 0$ and let $|a_n - a_\infty| < \varepsilon / \sum_{j=0}^{\infty} |f_j|$ for every $n \geq N$. Then

$$|b_n - f(1)a_\infty| \leq \sum_{j=0}^{\infty} |f_j||a_{n+j} - a_\infty| < \varepsilon$$

for every $n \geq N$. $\square$

We can rephrase part (ii) of last lemma in terms of formal power series as follows.

**Corollary 3.28.** *Let $f \in A_{W,\mathbb{R}}$ and let $a(t) = \sum_{n=0}^{\infty} a_n t^n$ be a formal power series where the sequence $\{a_n\}_{n=0}^{\infty}$ converges to some number $a_\infty \in \mathbb{R}$. If $b(t) = f(t)a(t)$, then $b_n \to f(1)a_\infty$.*

**Lemma 3.29.** *Let $q$ be a real polynomial whose roots are in $\mathbb{D}$. Let $a \in \ell^\infty$ and put $b = q(\nabla)a$. If $b_n \to b_\infty \in \mathbb{R}$, then $a_n \to b_\infty/q(1)$.*

*Proof.* Put $q(t) = q_s t^s + \cdots + q_1 t + q_0$. Then

$$\nabla_-^s b = (q_0 \nabla_-^s + q_1 \nabla_-^s \nabla + \cdots + q_s \nabla_-^s \nabla^s)a,$$

which can be written in formal power series using (3.30) as

$$t^s b(t) = q_0 t^s a(t) + q_1 t^{s-1}\big(a(t) - a_0\big) + \cdots + q_s\big(a(t) - a_{s-1}t^{s-1} - \cdots - a_1 t - a_0\big).$$

So if we put $\tilde{q}(t) = q_0 t^s + q_1 t^{s-1} + \cdots + q_s$, then

$$t^s b(t) = \tilde{q}(t)a(t) - r(t)$$

for some polynomial $r$ of degree at most $s-1$. Note that $\tilde{q}$ has no roots in $\overline{\mathbb{D}}$, hence $1/\tilde{q} \in A_{W,\mathbb{R}}$ and therefore

$$a(t) = \frac{t^s}{\tilde{q}(t)}\, b(t) + \frac{r(t)}{\tilde{q}(t)}.$$

Since $r/\tilde{q} \in A_{W,\mathbb{R}}$, its $n$-th Taylor coefficient tends to 0. Now the statement follows using the previous corollary and that $\tilde{q}(1) = q(1)$. □

**Lemma 3.30.** *Let $q$ be a real polynomial whose roots are in $\mathbb{D}$ and put $Q(t) = (1-t)q(t)$. Let $a \in \ell^\infty$. If $b = Q(\nabla)a$ and $b_n \geq 0$, then there exists* $\lim a_n$.

*Proof.* Put $c := q(\nabla)a$. Then $b = q(\nabla)a - \nabla q(\nabla)a$, so $b_n = c_n - c_{n+1} \geq 0$. Hence $\{c_n\}_{n=0}^\infty$ is a decreasing sequence. Since $\|c\|_\infty \leq (|q_0| + \cdots + |q_s|)\, \|a\|_\infty$, the sequence $c$ is bounded. Therefore $\{c_n\}_{n=0}^\infty$ converges and by the previous lemma we obtain that $a_n \to c_\infty/q(1)$. □

*Proof of Theorem 0.15.* Let $h \in \mathcal{H}$. Since $T \in \mathbf{C}_\alpha$, we obtain that

$$\sum_{n=0}^\infty \alpha_n \|T^n h\|^2 \geq 0.$$

Changing $h$ by $T^j h$ for $j \geq 1$ we get that

$$\sum_{n=0}^\infty \alpha_n \left\|T^{n+j} h\right\|^2 \geq 0.$$

Hence, if we define the sequence $a$ by $a_n = \|T^n h\|^2$ then we have that $b := \alpha(\nabla)a$ satisfies $b_n \geq 0$ for every $n \geq 0$.

By Lemma 3.26 we have $b = \alpha(\nabla)a = (1-\nabla)\tilde{\alpha}(\nabla)a$. Since $\tilde{\alpha}$ does not vanish on $\mathbb{T}$, we can factorize it as

$$\tilde{\alpha} = q\tilde{\beta},$$

where $q$ is a polynomial with roots in $\mathbb{D}$ and $\tilde{\beta} \in A_{W,\mathbb{R}}$ does not vanish on $\overline{\mathbb{D}}$. Therefore, if we put $Q(t) = (1-t)q(t)$ and $c := \tilde{\beta}(\nabla)a$, then $b = Q(\nabla)c$. By Lemma 3.30 we know that there exists $\lim c_n$, and since $a = (1/\tilde{\beta})(\nabla)c$ and $(1/\tilde{\beta}) \in A_{W,\mathbb{R}}$, the statement follows by Corollary 3.28. □

## 3.7 On functions $\alpha(t)$ with a zero at $t = 0$

In this section we prove a similarity result for functions that vanish at the origin.

**Definition 3.31.** Let $T \in \boldsymbol{B}(\mathcal{H})$ and let $n \geq 1$ be an integer. We say that $T$ is a *$n$-quasicontraction* if it is a $T^{*n}T^n$-contraction; i.e.,

$$T^{*(n+1)}T^{n+1} \leq T^{*n}T^n.$$

**Theorem 3.32** (see [30, Theorem 4.1])**.** *An $n$-quasicontraction is similar to a contraction, for any $n \geq 1$.*

With the help of this theorem by Cassier and Suciu, we obtain the following result.

**Theorem 3.33.** *Let $\alpha(t) := t^n(1-t)\widetilde{\alpha}(t)$, where $n \geq 1$ is an integer and $\widetilde{\alpha} \in A_{W,\mathbb{R}}$ such that $\widetilde{\alpha}$ is positive on $[0,1]$. If $\sigma(T) \subset \overline{\mathbb{D}}, \widetilde{\alpha} \in \mathcal{A}_T^0$, and $\alpha(T^*,T) \geq 0$, then $T$ is similar to a contraction.*

*Proof.* By Theorem 2.7, there exists a function $\widetilde{\beta} \in \mathcal{A}_T^0$ such that $\widetilde{\beta} \succ 0$ and $f := \widetilde{\beta}\widetilde{\alpha} \succ 0$. Put $A := f(T^*,T) \geq f_0 I > 0$ and $\widetilde{T} := A^{1/2}TA^{-1/2}$ (where the positive square root is taken). Notice that $\widetilde{T}$ is similar to $T$. Now we will prove that $\widetilde{T}$ is a $n$-quasicontraction and the statement will follow from Theorem 3.32.

Using that $t^n(1-t)f = \widetilde{\beta}\alpha$ and Lemma 2.15 (ii), we have

$$T^{*n}AT^n - T^{*(n+1)}AT^{n+1} = \sum_{n=0}^{\infty} \widetilde{\beta}_n T^{*n}\alpha(T^*,T)T^n \geq 0.$$

Hence

$$A^{-1/2}T^{*(n+1)}A^{1/2}A^{1/2}T^{n+1}A^{-1/2} \leq A^{-1/2}T^{*n}A^{1/2}A^{1/2}T^nA^{-1/2},$$

which means that $\widetilde{T}$ is a $n$-quasicontraction. □

# Chapter 4

# Inclusions between operator classes $\mathbf{C}_\alpha$

We have already given in Section 3.2 some properties of the classes $\mathbf{C}_\alpha$ when $\alpha$ is an admissible function. In this chapter we focus on the inclusions of one class into another; that is, when is it possible to state that all the operators in one class $\mathbf{C}_\alpha$ belong to another class $\mathbf{C}_\beta$. In Section 4.1 we consider this question for admissible functions. It turns out that there is a very nice characterization (see Theorem 4.3). In Section 4.2 we focus on functions of the form $(1-t)^a$, for $a > 0$. That is, we study when every $a$-contraction is also a $b$-contraction. We also consider the same question for $a$-isometries.

## 4.1 Inclusions of classes $\mathbf{C}_\alpha$

Let us begin with a result which applies to functions $\alpha$ and $\tau$ that need not be admissible.

**Theorem 4.1.**

(i) *Let $T \in \boldsymbol{B}(\mathcal{H})$, and let $\alpha$ and $\gamma$ belong to $\mathcal{A}_{T,\mathbb{R}}$, where $\gamma \succ 0$. Put $\tau = \alpha \cdot \gamma$. If $\alpha(T^*, T) \geq 0$, then $\tau(T^*, T) \geq 0$.*

(ii) *Suppose that $\alpha, \tau$ belong to $A_{W,\mathbb{R}}$ with $\alpha_0 > 0$ and $\tau_0 > 0$. Consider $\gamma(t) = \tau(t)/\alpha(t) = \sum_{n\geq 0} \gamma_n t^n$ (the series converges in a neighborhood of the origin). If $\gamma \not\succ 0$, then there exists an operator $T$, similar to a contraction, such that $\alpha(T^*, T) \geq 0$, while the inequality $\tau(T^*, T) \geq 0$ does not hold.*

For the proof of this theorem we need the following lemma, which is in the spirit of Pringsheim's Theorem (see [53]).

**Lemma 4.2.** *Let $g$ be a meromorphic function in $\mathbb{D}$, analytic in the origin, such that $g \succcurlyeq 0$. If $g$ is bounded on $[0, 1)$, then $g \in A_{W,\mathbb{R}}$.*

*Proof.* Since $g \succcurlyeq 0$, we have
$$|g(z)| \le g(|z|) \tag{4.1}$$
whenever the Taylor series for $g(t)$ converges for $t = |z|$. Let $r \in (0, +\infty)$ be the radius of convergence of this series. If $r < 1$, then $g$ has poles on the circle $\{|z| = r\}$, and this contradicts the boundedness of $g$ on $[0, r)$. Hence $r \ge 1$. Since $g$ increases on $[0, 1)$ and is bounded, there exists the finite limit $\lim_{1^-} g$. It follows that $\|g\|_{A_{W,\mathbb{R}}} = \sum_n g_n = \lim_{1^-} g < \infty$. □

*Proof of Theorem 4.1.* (i) If $\alpha(T^*, T) \ge 0$, then using Lemma 2.15 (ii) we get

$$\tau(T^*, T) = (\gamma\alpha)(T^*, T) = \sum_{n=0}^{\infty} \gamma_n T^{*n} \alpha(T^*, T) T^n \ge 0.$$

(ii) Suppose that $\gamma \not\succcurlyeq 0$ and let $\ell$ be the smallest index such that $\gamma_\ell < 0$. Note that $\ell \ge 1$ because $\gamma_0 > 0$. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. By Theorem 1.10, it will suffice to find a sequence of positive numbers $\Lambda = \{\Lambda_n\}_{n=0}^{\infty}$ that is eventually constant such that $\alpha(\nabla)\Lambda \succ 0$ and $\tau(\Lambda) \not\succcurlyeq 0$, because in that case the weighted forward shift $T$ defined by $Te_n = \sqrt{\Lambda_{n+1}/\Lambda_n}\, e_{n+1}$ is similar to a contraction and satisfies $T \in \mathbf{C}_\alpha \setminus \mathbf{C}_\tau$. Let us construct that sequence.

Consider the sequence $\Gamma := (\gamma_\ell, \gamma_{\ell-1}, \dots, \gamma_0, 0, 0, \dots)$ and define the sequence $\Psi$ by
$$\Psi := (\tilde{\tau})^{-1}(\nabla)\Gamma,$$
where, as usual, $\tau(t) = (1-t)\tilde{\tau}(t)$. Note that $\Psi$ is well defined because $\Gamma$ has only finitely many nonzero terms and $\tilde{\tau}$ is invertible in a neighborhood of the origin.

Since $\Gamma_n = 0$ for $n \ge \ell + 1$ we obtain that also $\Psi_n = 0$ for $n \ge \ell + 1$. Finally, let $\Lambda$ be a the sequence that satisfies

$$\Psi = (1 - \nabla)\Lambda,$$

with $\Lambda_0$ large enough so that $\Lambda_n > 0$ for every $n$. Note that $\Lambda_n$ is constant for $n \ge \ell + 1$ and $\Lambda$ also satisfies

$$\alpha(\nabla)\Lambda = \tilde{\alpha}(\nabla)\Psi = \left(\frac{\tilde{\alpha}}{\tilde{\tau}}\right)(\nabla)\Gamma = \left(\frac{1}{\gamma}\right)(\nabla)\Gamma = (0, \dots, 0, \overset{\ell}{\smile}{1}, 0, \dots) \succ 0$$

and
$$\tau(\nabla)\Lambda = \tilde{\tau}(\nabla)\Psi = \Gamma \not\succcurlyeq 0,$$
since $\Gamma_0 = \gamma_\ell < 0$. Hence (ii) is proved. □

**Theorem 4.3.** *Let $\alpha$ and $\tau$ be admissible functions and let $\gamma := \tau/\alpha$. Then $\mathbf{C}_\alpha \subset \mathbf{C}_\tau$ if and only if $\gamma \succ 0$.*

As an immediate consequence we obtain a characterization for the equality of two classes.

**Corollary 4.4.** *Let* $\alpha$ *and* $\tau$ *be admissible functions. Then* $\mathbf{C}_\alpha = \mathbf{C}_\tau$ *if and only if* $\tau = c\alpha$ *for some positive constant* $c$.

*Proof of Theorem 4.3.* Let $\alpha$ and $\tau$ be admissible functions. By Lemma 4.2, $\gamma = \widetilde{\tau}/\widetilde{\alpha}$ is in $A_{W,\mathbb{R}}$. Now, if $\gamma \not\succ 0$, then Theorem 4.1 (ii) implies that $\mathbf{C}_\alpha$ is not contained in $\mathbf{C}_\tau$. On the other hand, if $\gamma \succ 0$, then this containment is true because of Theorem 4.1 (i). $\square$

**Remark 4.5.** Note that in general, for a rational admissible function $r$, it is not possible to find an admissible polynomial $p$ such that $\mathbf{C}_r \subset \mathbf{C}_p$. For example, consider the rational admissible function

$$r(t) = \frac{1-t}{1-t/2}.$$

If such a $p$ exists, say $p(t) = (1-t)\tilde{p}(t)$, then by Theorem 4.3 we should have $\tilde{p}(t)(1-t/2) \succ 0$. But it is immediate to check that this is impossible for any real polynomial $\tilde{p}$.

**Remark 4.6.** Let $\alpha$ be an admissible function. Then, by Theorem 4.3, we have that $\mathbf{C}_{1-t} \subset \mathbf{C}_\alpha$ if and only if $\widetilde{\alpha} \succ 0$. Notice that whenever $\mathbf{C}_{1-t} \subset \mathbf{C}_\alpha$, any operator $T$ in $\mathbf{C}_\alpha \setminus \mathbf{C}_{1-t}$ is a non-contractive operator which is similar to a contraction (see Corollary 0.12 (ii)).

**Remark 4.7.** Let $\alpha$ be any admissible function that has zeros in $\mathbb{D}$, and let $r \in (0,1)$. Then it follows from Lemma 4.2 that $\gamma(t) = \alpha(rt)/\alpha(t)$ cannot have nonnegative Taylor coefficients. So for functions $\alpha$ of this type, the corresponding kernel $k(\bar{w}, z) = 1/\alpha(\bar{w}, z)$ does not have Property 1 in the terminology of Olofsson's work [74]. He assumes this property in his main results.

**Lemma 4.8.** *Let* $\alpha$ *and* $\beta$ *be admissible functions, and consider the following conditions.*

(a) $\alpha, \beta \in \mathcal{H}(\overline{\mathbb{D}})$;

(b) $\alpha(t)/(1-t)$ *or* $\beta(t)/(1-t)$ *has no zeros on* $\overline{\mathbb{D}}$.

*If* (a) *or* (b) *holds, then there exists an admissible function* $\gamma$ *such that* $\mathbf{C}_\alpha \cup \mathbf{C}_\beta \subset \mathbf{C}_\gamma$.

*Proof.* (a) Suppose that $\alpha, \beta \in \mathcal{H}(\overline{\mathbb{D}})$. Then

$$\frac{\alpha}{\beta} = \frac{a}{b}\,\varphi$$

for some polynomials $a$ and $b$ without common roots and a function $\varphi \in \mathcal{H}(\overline{\mathbb{D}})$ which is positive on $[0,1]$. By Corollary 2.8, there exists a function $\psi \in A_{W,\mathbb{R}}$ such that $\psi \succ 0$ and $\psi\varphi \succ 0$. Put $a = a_+a_-a_{nr}$ where $a_+$ gathers the positive roots of $a$, $a_-$ gathers the negative roots of $a$, and $a_{nr}$ gathers the non-real roots of $a$. If for example $a$ does not have any positive root, then we just put $a_+ = 1$. We assume that $a$ and all its factors have positive values at 0. In the same way, put $b = b_+b_-b_{nr}$. Applying Corollary 2.10 twice, we get a polynomial $p$ without roots in $\overline{\mathbb{D}}$ such that $p \succ 0$, $pa_{nr} \succ 0$ and $pb_{nr} \succ 0$. Let

$$v := \frac{a_-a_{nr}p}{b_+}, \quad w := \frac{b_-b_{nr}p}{a_+}.$$

Note that $v \succ 0, w \succ 0$ and $a/b = v/w$. Now we simply put $\gamma := \alpha w\psi = \beta v\psi\varphi$. Since $w\psi \succ 0$ and $v\psi\varphi \succ 0$, the result follows from Theorem 4.3.

(b) Suppose that $\beta(t)/(1-t)$ has no zeros on $\overline{\mathbb{D}}$. Then $\alpha/\beta =: \varphi \in A_{W,\mathbb{R}}$ is positive on $[0,1]$. By Corollary 2.8, there exists a function $\psi \in A_{W,\mathbb{R}}$ such that $\psi \succ 0$ and $\psi\varphi \succ 0$. Now we put $\gamma := \alpha\psi = \beta\varphi\psi$. Since $\psi \succ 0$ and $\varphi\psi \succ 0$, we get the result. □

**Corollary 4.9.** *Let $T_1$ be a complex square matrix and $T_2$ be a Hilbert space operator. If $T_1$ has no Jordan blocks and $\sigma(T_1) \subset \mathbb{T}$, whereas the spectral radius of $T_2$ is less than* 1*, then there exists an admissible function $\gamma$ such that $T_1 \oplus T_2 \in \mathbf{C}_\gamma$.*

*Proof.* This follows immediately from Corollary 3.12, Proposition 3.8 (b) and (e), and Lemma 4.8 case (a). □

**Remark 4.10.** On the contrary to the above corollary and to Proposition 3.8 (b), given two operators $T_1$ and $T_2$ in a class $\mathbf{C}_\alpha$, their *direct sum* might not belong to any class $\mathbf{C}_\beta$. Indeed, take $T_1$ and $T_2$ to be $1 \times 1$ matrices whose (unique) entries are, for instance, 1 and 1/2. Then $T_1$ and $T_2$ belong to all classes $\mathbf{C}_\alpha$, whereas, by Lemma 3.13, $T_1 \dotplus T_2$ does not belong to any class $\mathbf{C}_\beta$ if this direct sum is not orthogonal.

## 4.2 Inclusions for classes of $a$-contractions and $a$-isometries

In this section we will prove Theorems 0.17 and 0.16. Let us begin with the following three results given in [60] by Kuttner.

**Theorem 4.11** (see [60, Theorem 3])**.** *Let $\sigma > -1$, $\sigma$ not an integer. Let $\Lambda = \{\Lambda_n\}_{n\geq 0}$ be a sequence of real numbers. Suppose that $(1-\nabla)^\sigma\Lambda$ is well defined. If*

$$s \geq \sigma, \quad r+s > \sigma, \quad \tau \geq \sigma - r, \quad \tau \geq 0, \tag{4.2}$$

*or if*

$$s \geq \sigma, \quad r+s = \sigma, \quad \tau > \sigma - r, \quad \tau \geq 0, \tag{4.3}$$

*then*

$$(1-\nabla)^r[(1-\nabla)^s\Lambda]_n = \sum_{j=0}^{\infty} k^{-r}(j)\left(\sum_{m=0}^{\infty} k^{-s}(m)\Lambda_{m+j+n}\right)$$

*is summable* $(C,\tau)$ *to* $(1-\nabla)^{r+s}\Lambda_n$.

**Theorem 4.12** (see [60, Theorem A]). *Let* $s > -1$ *and* $r \geq 0$. *If* $\Lambda = \{\Lambda_n\}_{n\geq 0}$ *is a sequence of real numbers, then*

$$(1-\nabla)^{r+s}\Lambda_n = (1-\nabla)^r[(1-\nabla)^s\Lambda]_n$$

*for every* $n \geq 0$, *whenever the RHS above is well defined.*

**Theorem 4.13** (see [60, Theorem B]). *Let* $s > -1, r+s > -1$ *and* $r+s$ *be non-integer. If* $\Lambda = \{\Lambda_n\}_{n\geq 0}$ *is a sequence of real numbers, then*

$$(1-\nabla)^{r+s}\Lambda_n = (1-\nabla)^r[(1-\nabla)^s\Lambda]_n$$

*for every* $n \geq 0$, *whenever both sides above are well defined.*

As a consequence of these results, we obtain the following two theorems.

**Theorem 4.14.** *Let* $\Lambda = \{\Lambda_n\}_{n\geq 0}$ *be a sequence of real numbers, and let* $0 < b < a$, *where* $b$ *is not an integer. If* $(1-\nabla)^a\Lambda_n \geq 0$ *for every* $n \geq 0$ *and* $(1-\nabla)^b\Lambda$ *is well defined, then* $(1-\nabla)^b\Lambda_n \geq 0$ *for every* $n \geq 0$.

*Proof.* Let $0 < b < a$, and let $\Lambda = \{\Lambda_n\}_{n\geq 0}$ be a sequence of real numbers such that $(1-\nabla)^a\Lambda_n \geq 0$ for every $n \geq 0$, and $(1-\nabla)^b\Lambda$ is well defined. Putting

$$\sigma = b, \quad s = a, \quad r = b-a, \quad \tau = [a]+1$$

in (4.3) (where $[a]$ denotes the biggest integer less than or equal to $a$), we obtain that the series

$$(1-\nabla)^{b-a}[(1-\nabla)^a\Lambda]_n = \sum_{j=0}^{\infty} k^{a-b}(j)\left(\sum_{m=0}^{\infty} k^{-a}(m)\Lambda_{m+j+n}\right) \tag{4.4}$$

is summable $(C,[a]+1)$ to $(1-\nabla)^b\Lambda_n$.

Since $k^{a-b}(m) \geq 0$ for every $m \geq 0$ (because $a-b > 0$) and also the series in parenthesis in (4.4) are nonnegative for every $j \geq 0$, we deduce that $(1-\nabla)^b\Lambda_n \geq 0$ for every $n \geq 0$, as we wanted to prove. $\square$

*Proof of Theorem 0.16.* Let $0 < b < a$, where $b$ is not an integer, and let $T$ be an $a$-contraction such that $T \in \mathrm{Adm}_b$. Fix $x \in \mathcal{H}$ and puts $\Lambda_n := \|T^n x\|^2$, for $n \geq 0$. If we show that

$$(1-\nabla)^b\Lambda_n = \sum_{j=0}^{\infty} k^{-b}(j)\Lambda_{j+n} = \sum_{j=0}^{\infty} k^{-b}(j)\left\|T^{j+n}x\right\|^2 \geq 0, \tag{4.5}$$

for every $n \geq 0$, since $x \in \mathcal{H}$ is arbitrary, then $T$ would be a $b$-contraction. But (4.5) follows immediately from the previous theorem, since $(1-\nabla)^a \Lambda_n \geq 0$ for every $n \geq 0$ (because $T$ is an $a$-contraction) and $(1-\nabla)^b \Lambda$ is well defined (since $T \in \mathrm{Adm}_b$). □

**Theorem 4.15.** *Let $a > 0$, and let the integer $m$ be defined by $m < a \leq m+1$. If $\Lambda = \{\Lambda_n\}_{n \geq 0}$ is a sequence of real numbers, then the following statements are equivalent.*

(i) $(1-\nabla)^a \Lambda \equiv 0$ *(i.e., all the terms of the sequence $(1-\nabla)^a \Lambda$ are 0).*

(ii) $(1-\nabla)^{m+1} \Lambda \equiv 0$.

(iii) *There exists a polynomial $p$ of degree at most $m$ such that $\Lambda_n = p(n)$ for every $n \geq 0$.*

*Proof.* The equivalence (ii) $\Leftrightarrow$ (iii) is a well known fact (see [20, Theorem 2.1]).

Suppose that (i) is true. Let us see that

$$(1-\nabla)^{m+1} \Lambda_n = (1-\nabla)^{m+1-a} [(1-\nabla)^a \Lambda]_n \tag{4.6}$$

for every $n \geq 0$. Indeed, the RHS of (4.6) is obviously 0 by assumption. Then we can apply Theorem 4.12 with $s = a > -1$ and $r = m - a \geq 0$, and (4.6) follows. Therefore we obtain that (i) $\Rightarrow$ (ii).

Suppose now that (ii) is true. Hence, we also have (iii). Obviously, if $a = m+1$ we obtain (i). Now let us prove (i) for $m < a < m+1$ (so $a$ is non-integer). We will see that

$$(1-\nabla)^a \Lambda_n = (1-\nabla)^{a-m-1} [(1-\nabla)^{m+1} \Lambda]_n \tag{4.7}$$

for every $n \geq 0$. Indeed, fix $n \geq 0$. Then, by (iii) and (2.18), we have that

$$\Lambda_{j+n} \lesssim_n (j+1)^m.$$

Therefore

$$\sum_{j=0}^{\infty} |k^{-a}(j)| \Lambda_{j+n} \lesssim_n \sum_{j=0}^{\infty} (j+1)^{-a-1} (j+1)^m = \sum_{j=0}^{\infty} (j+1)^{m-a-1} < \infty$$

since $m < a$. Hence, the series

$$\sum_{j=0}^{\infty} k^{-a}(j) \Lambda_{j+n}$$

converges for every $n \geq 0$. Thus the LHS of (4.7) is well defined. The RHS of (4.7) is obviously well defined since we are assuming (ii). Therefore, taking $s = m+1$ and $r + s = a$ in Theorem 4.12 we obtain that indeed (4.7) holds, and hence we obtain (i). □

*Proof of Theorem 0.17.* The equivalence between statements (ii) and (iii) is well-known. Suppose that (i) is true. Then, fixing $h \in \mathcal{H}$ and taking $\Lambda_n := \|T^n h\|^2$, note that (ii) follows immediately using Theorem 4.15. Suppose now that we have (ii), that is, $T$ is an $(m+1)$-isometry. Then, $\|T^n\|^2 \lesssim (n+1)^m$. Therefore

$$\sum_{n=0}^{\infty}(n+1)^{-a-1}\|T^n h\|^2 \lesssim \sum_{n=0}^{\infty} n^{m-a-1},$$

for every $h \in \mathcal{H}$. The last series above converges since $m < a$. This means that $T \in \mathrm{Adm}_a$, and then (i) follows using Theorem 4.15 again. □

**Remark 4.16.** In [12, Proposition 8], for any positive integer $m$, Athavale gives an example of an operator $T$ (indeed, a unilateral weighted shift) which is an $(m+1)$-isometry but not $n$-isometry for any positive integer $n \leq m$.

Before continuing, let us quickly prove Theorem 1.25 since now we have all the ingredients.

*Proof of Theorem 1.25.* Since $B_k$ is an $a$-contraction (see Theorem 2.24), and $S$ is also an $a$-contraction (see Theorem 0.17), it follows that $(B_k \otimes I_{\mathcal{E}}) \oplus S$ is an $a$-contraction. Obviously any part of it is also is an $a$-contraction. □

We also have the following result.

**Theorem 4.17.** *Let $0 < c < b < a$ where $c$ is not an integer. If $T$ is an $a$-contraction and $(1-t)^c$ belongs to $\mathcal{A}_T$, then $T$ is a $b$-contraction.*

*Proof.* Fix $x \in \mathcal{H}$ and let $\Lambda_n := \|T^n x\|^2$. Taking

$$\sigma = c, \quad s = a, \quad r = b - a, \quad \tau = [a - b + c] + 1$$

in (4.2), the statement follows. □

Let us study now the membership of the weighted shifts in the classes $\mathrm{Adm}_a$ and $\mathbf{C}_a$. Let $\varkappa(t) := (1-t)^{-s}$ for some $s > 0$. Recall the definition of the weighted backward and forward shifts $B_s$ and $F_s$ given in Section 1.2. (See also Notation 1.7.)

In Theorem 2.24 we obtain that for any positive numbers $a$ and $s$, it is always true that $B_s$ belongs to $\mathrm{Adm}_a$, and that $B_s$ is an $a$-contraction if and only if $a \leq s$.

In order to obtain the corresponding result for the forward shift $F_s$, we need to introduce the following sets:

$$J_{\text{even}} := \bigcup_{j \in \mathbb{Z}_{\geq 0}} (2j, 2j+1), \qquad J_{\text{odd}} := \bigcup_{j \in \mathbb{Z}_{\geq 0}} (2j+1, 2j+2). \tag{4.8}$$

Then $\{J_{\text{even}}, J_{\text{odd}}, \mathbb{N}\}$ is a partition of the interval $(0, \infty)$.

**Theorem 4.18.** *Let $a$ and $s$ be positive numbers. Then $F_s \in \mathrm{Adm}_a$ if and only if $s < a+1$ or $a$ is integer.*

*Proof.* If $a$ is a positive integer, then obviously any operator in $\boldsymbol{B}(\mathcal{H})$ belongs to $\mathrm{Adm}_a$, since in this case $(1-t)^a$ is just a polynomial. Suppose now that $a$ is not an integer. We will use the notation of Theorem 1.10. Now $\varkappa(t) = (1-t)^{-s}$ and $\alpha(t) = (1-t)^a$. Therefore

$$\beta(\nabla)\varkappa_m = \sum_{n=0}^{\infty} |k^{-a}(n)| \, \|F_s^n e_m\|^2 \asymp \sum_{n=0}^{\infty} (n+1)^{-a-1}(n+m+1)^{s-1}.$$

If $F_s \in \mathrm{Adm}_a$, then the above series must converge for every $m \geq 0$. For each $m$, this series indeed behaves as $\sum n^{s-a-2}$. Hence it implies that $s < a+1$.

Reciprocally, suppose now that $s < a+1$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)^{-a-1}(n+m+1)^{s-1} &\asymp (m+1)^{s-1}\sum_{n=0}^{m}(n+1)^{-a-1} + \sum_{n=m+1}^{\infty}(n+1)^{s-a-2} \\ &\asymp (m+1)^{s-1} + (m+1)^{s-a-1} \asymp (m+1)^{s-1} \asymp \varkappa_m.\end{aligned}$$

Therefore Theorem 1.10 (i) implies that $F_s \in \mathrm{Adm}_a$. □

**Theorem 4.19.** *Let $a$ and $s$ be positive numbers and $s < a+1$. Then:*

(i) $(1-t)^a(F_s^*, F_s) \geq 0$ *(that is, $F_s$ is an $a$-contraction) if and only if $s \in J_{\text{even}} \cup \mathbb{N}$.*

(ii) $(1-t)^a(F_s^*, F_s) \leq 0$ *if and only if $s \in J_{\text{odd}} \cup \mathbb{N}$.*

(iii) $(1-t)^a(F_s^*, F_s) = 0$ *(that is, $F_s$ is an $a$-isometry) if and only if $s \in \mathbb{N}$.*

*Proof.* We need to study the signs of

$$(1-\nabla)^a k^s(m) = \sum_{n=0}^{\infty} k^{-a}(n) k^s(n+m).$$

We assert that for any $s < a+1$,

$$(1-\nabla)^a k^s(m) = \frac{\sin(\pi s)\Gamma(1-s+a)\Gamma(s+m)}{\pi\Gamma(m+a+1)}. \tag{4.9}$$

Suppose first that $a$ is a positive integer. Then, by [4, Example 3.4 (ii)], we have that

$$(1-\nabla)^a k^s(m) = (-1)^a k^{s-a}(m+a),$$

for every non-negative integer $m$, and the statement follows easily.

Suppose that $s \in (0,1)$ and let $a$ be any real number with $a > s-1$. Using the expression for $k^s(m)$ given in Proposition 2.23 and applying the idea of the proof of [4, Lemma 1.1 ], we have

$$\begin{aligned}
(1-\nabla)^a k^s(m) &= \frac{1}{\Gamma(s)\Gamma(1-s)} \sum_{l=0}^{\infty} k^{-a}(l) \frac{\Gamma(1-s)\Gamma(s+m+l)}{\Gamma(m+l+1)}\, dx \\
&= \frac{1}{\Gamma(s)\Gamma(1-s)} \sum_{l=0}^{\infty} k^{-a}(l) \int_0^1 x^{-s}(1-x)^{s+m+l-1}\, dx \\
&= \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^1 x^{a-s}(1-x)^{s+m-1}\, dx \\
&= \frac{\Gamma(1-s+a)\Gamma(s+m)}{\Gamma(s)\Gamma(1-s)\Gamma(m+a+1)}.
\end{aligned}$$

If $s=1$, it is immediate that $(1-\nabla)^a k^s(m) = 0$. This gives (4.9) for $s \in (0,1]$.

Next, assume that $s > 0$ is arbitrary. The summation by parts formula gives

$$\begin{aligned}
&\sum_{n=0}^{N} k^{-a}(n) k^s(n+m) \\
&= k^s(m) + \sum_{n=1}^{N} (k^{-a+1}(n) - k^{-a+1}(n-1)) k^s(n+m) \\
&= k^{-a+1}(N) k^s(N+m) + \sum_{n=0}^{N-1} (k^s(n+m) - k^s(n+m+1)) k^{-a+1}(n) \\
&= k^{-a+1}(N) k^s(N+m) - \sum_{n=0}^{N-1} k^{s-1}(n+m+1) k^{-a+1}(n).
\end{aligned}$$

By passing to the limit as $N \to \infty$ and using that $a > s-1$, we obtain that

$$(1-\nabla)^a k^s(m) = -(1-\nabla)^{a-1} k^{s-1}(m+1).$$

This implies that whenever (4.9) holds for a pair $(a-1, s-1)$ (for all $m$), it also holds for the pair $(a,s)$ and for all $m$. Therefore, the case of an arbitrary pair $(a,s)$ reduces to the case of the pair $(a-n, s-n)$, where $n < s \le n+1$, for which (4.9) has been checked already. This proves this formula for the general case. The sign of $\sin(\pi s)$ depends on whether $s \in J_{\text{even}}$, $s \in J_{\text{odd}}$, or $s$ is integer, whereas for $a, s > 0$, all other values of $\Gamma$ in (4.9) are positive. This gives our statements. $\square$

Now we have all the ingredients for the proof of Proposition 1.26.

*Proof of Proposition 1.26.* Assume all the hypothesis of the statement. Let us see that the forward weighted shift $F_s$ is an $a$-contraction that cannot be modeled by a part of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S$ is an $m$-isometry.

Indeed, since $s < a+1$ and $s \in J_{\text{even}}$, Theorem 4.19 (i) gives that $F_s$ is an $a$-contraction. The second part of the claim follows by comparison of operator norms. By (2.21), $\|F_s^n\|^2 \asymp (n+1)^{s-1}$, for every $n \geq 0$. On the other hand, $B_k$ is a contraction (see (2.19)), and $\|S^n\|^2 \lesssim (n+1)^{m-1}$ since $S$ is an $m$-isometry. (This last asymptotic is well-known. For instance, it follows immediately from [20, Theorem 2.1].) Therefore

$$\|((B_k \otimes I_{\mathcal{E}}) \oplus S)^n\|^2 \lesssim (n+1)^{m-1}.$$

Since $m-1 < s-1$, we get that $F_s$ cannot be modeled by a part of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, as we wanted to prove. □

As an obvious consequence of Theorem 2.24, we obtain that it is not possible in general to pass from $b$-contractions to $a$-contractions when $0 < b < a$.

**Proposition 4.20.** *Let $0 < b < s < a$. Then $B_s$ is a $b$-contraction but not an $a$-contraction.*

We also have the following result.

**Theorem 4.21.** *Let $0 < a \leq s < 1$. Then $B_s$ is an $a$-contraction which is not similar to a contraction.*

*Proof.* Since $a \leq s$, Theorem 2.24 gives that $B_s$ is an $a$-contraction, and since $s < 1$, (2.21) gives that $B_s$ is not power bounded. □

Moreover, passing from $a$-contractions to $b$-contractions (when $0 < b < a$) is neither possible in general, as the following statement shows. Is just an immediate consequence of Theorem 4.19.

**Theorem 4.22.** *Let $1 < a \leq 2$ and $0 < b < a$. If $\max\{2, b+1\} < s < a+1$, then $F_s$ is an $a$-contraction but $F_s$ does not belong to $\mathrm{Adm}_b$ (and in particular, $F_s$ is not a $b$-contraction).*

**Theorem 4.23.** *Let $0 < s < 1$. If $0 < a \leq \min\{s, 1-s\}$ then $B_s$ is an $a$-contraction, but the series $\sum k^{-a}(n) B_s^{*n} B_s^n$ does not converge in the uniform operator topology in $\boldsymbol{B}(\mathcal{H})$.*

*Proof.* Since $a \leq s$, Theorem 2.24 gives that $B_s$ is an $a$-contraction. Moreover, using that $a \leq 1-s$ and (2.21) it is immediate that

$$\sum_{n=0}^{\infty} |k^{-a}(n)| \, \|B_s\|^2 = \infty,$$

and the statement follows. □

**Theorem 4.24.** *Let $m$ be a positive integer.*

(i) *If $T$ is a $(2m+1)$-contraction, then $T$ is a $2m$-contraction and*

$$\|T^n x\|^2 \lesssim (n+1)^{2m} \qquad (\forall x \in \mathcal{H}).$$

(ii) *If $T$ is a $(2m)$-contraction and*

$$\|T^n x\|^2 = o(n^{2m-1}) \qquad (\forall x \in \mathcal{H}),$$

*then $T$ is a $(2m-1)$-contraction.*

**Remarks 4.25.**

(a) The fact that $(2m+1)$-contractions are $2m$-contractions was already proved by Gu in [48, Theorem 2.5]. Here we give an alternative proof which also works for (ii).

(b) In general, $2m$-contractions are not $(2m-1)$-contractions. For example, the forward weighted shift $F_2$ is a 2-isometry (see Theorem 4.19 (iii)), but it is not a contraction. Indeed, it is not power bounded (see (2.21)).

*Proof of Theorem 4.24.* Fix $x \in \mathcal{H}$ and put $\Lambda_n := \|T^n x\|^2$, for every $n \geq 0$. It is easy to see (for instance, by induction on $k$) that

$$\begin{aligned}\Lambda_n &= \sum_{j=0}^{k-1}(-1)^j \binom{n}{j}(I-\nabla)^j\Lambda_0 \;+\; (-1)^k \sum_{j=0}^{n-k} \binom{n-1-j}{k-1}(I-\nabla)^k\Lambda_j \\ &=: (I) + (II),\end{aligned} \tag{4.10}$$

for $n \geq k$. (Note that this formula is a discrete analogue of the Taylor formula with the rest in the integral form.)

Let us prove (i). Suppose that $T$ is a $(2m+1)$-contraction. Taking $k = 2m+1$ (and $n$ sufficiently large) in (4.10), we obtain that $(II) \leq 0$, since $(I-\nabla)^{2m+1}\Lambda_j \geq 0$. Therefore

$$\Lambda_n \leq \Lambda_0 - \binom{n}{1}(I-\nabla)\Lambda_0 + \cdots + \binom{n}{2m}(I-\nabla)^{2m}\Lambda_0. \tag{4.11}$$

If $(I-\nabla)^{2m}\Lambda_0 < 0$, then the RHS of (4.11) is a polynomial in $n$ of degree $2m$ whose main coefficient is negative. Hence, it is negative for $n$ sufficiently large. This contradicts the fact that $\Lambda_n \geq 0$ for every $n$. Therefore, $(I-\nabla)^{2m}\Lambda_0 \geq 0$. Since the vector $x \in \mathcal{H}$, fixed at the beginning of the proof, was arbitrary, this means that $T$ is a $2m$-contraction. We have also obtained that $\Lambda_n \lesssim (n+1)^{2m}$. This completes the proof of (i).

Assume the hypotheses of (ii). Now taking $k = 2m$ (and $n$ sufficiently large) in (4.10), we obtain that $(II) \geq 0$, since $(I - \nabla)^{2m}\Lambda_j \geq 0$. Therefore

$$\Lambda_n \geq \Lambda_0 - \binom{n}{1}(I - \nabla)\Lambda_0 + \cdots - \binom{n}{2m-1}(I - \nabla)^{2m-1}\Lambda_0. \tag{4.12}$$

If $(I - \nabla)^{2m-1}\Lambda_0 < 0$, then the RHS of (4.11) is a polynomial in $n$ of degree $2m - 1$ whose main coefficient is positive. But this contradicts the hypothesis $\Lambda_n = o(n^{2m-1})$, hence it must be $(I - \nabla)^{2m-1}\Lambda_0 \geq 0$. Since the vector $x \in \mathcal{H}$, fixed at the beginning of the proof, was arbitrary, this means that $T$ is a $(2m-1)$-contraction. $\square$

# Conclusions

In this thesis we have managed to make headway in the construction of functional models for Hilbert space operators $T$ satisfying that the operator $\alpha(T^*, T)$ is positive, where $\alpha$ is a function representable by power series around the origin.

With the results of this thesis, it became clear that the appropriate topology in the definition of $\alpha(T^*, T)$ (which recall that is defined as the series $\sum \alpha_n T^{*n} T^n$) is the strong operator topology. This fact was not trivial at all. For instance, in our first work [16] we considered the uniform operator topology.

On the other hand, we have tried to consider classes of functions $\alpha$ as broadly as possible. In this way, we have obtained results going beyond the Nevannlina-Pick functions, which have been extensively studied. Also, we have seen that in the results about similarity of the models, it is possible to work even without the consideration of the kernels $k(t) = 1/\alpha(t)$. In this way we can allow, for example, that the function $\alpha$ had zeros in the unit disc $\mathbb{D}$.

One of the goals we had in mind during the thesis was, not only the obtainment of results about the existence of the unitarily equivalent model, but also consider if it was possible to give explicit models and study its uniqueness. We have been able to give explicit models where the defect space $\mathfrak{D}$ (given by the defect operator $D = \alpha(T^*, T)^{1/2}$) intervenes, and we have also characterized the uniqueness of such model (essentially, depending on whether $\alpha(1) = 0$ or $\alpha(1) > 0$).

The obtainment of explicit models was very helpful for derive conclusions from the model. For example, we give results about the spectrum of the operator $T$ when the defect space $\mathfrak{D}$ is finite-dimensional. Moreover, we have obtained an interesting connection with ergodic theory for the case of $a$-contractions with $0 < a < 1$. It is quite remarkable that this connection is in both directions, that is, we can also obtain conclusions about the model via ergodic properties of the operator $T$.

We have made an effort in the comprehension of the classes of operators $\mathbf{C}_\alpha$. In that sense, we been able to understand some elementary properties of this classes and we have also obtained some interesting results about the inclusions of the classes (in come cases, we have even characterized when $\mathbf{C}_\alpha \subset \mathbf{C}_\beta$).

The techniques we have employed in this thesis have been mixed. In advance, because of the topic of the thesis, it was clear that we will need to use machinery from operator theory, complex analysis, and harmonic analysis. What have been unexpected was that the use of Banach algebras theory and finite differences turned out to be extremely useful and natural tools for us.

To sum up, with this thesis we have made a satisfactory contribution in our initial goals, and we have also provided new techniques that for sure will help in other related problems. On the other hand, this thesis leaves some open interesting questions for future work.

## Future work

Some questions arise naturally with this thesis. Here we list some of them.

- In this thesis we have dealt with functions $\alpha$ as general as possible all the time. Despite been able to advance significantly in this sense with respect to previous works, it would be very desirable to keep enlarging the class of functions $\alpha$ for which we can obtain a model for operators $T$ satisfying that $\alpha(T^*, T) \geq 0$.

- For $0 < a \leq 1$, there exists a model for $a$-contractions. In this thesis we have seen that a quite natural conjecture about a possible model for $a$-contractions with $a > 1$ is false. Therefore, it is still pending to shed some light on this direction.

- Another interesting question is to study other types of domains apart from the unit disc $\mathbb{D}$ in this context. Thai is, instead of assume that the spectrum of $T$ is contained in $\overline{\mathbb{D}}$ as we do in this thesis, try to understand what happens when $\sigma(T)$ is contained in an annulus or any other simply connected domain.

- This thesis is focused on the study of one single Hilbert space operator $T$. However, it would be very interesting to extend the results of this thesis (at least some of them) to tuples of commuting operators $(T_1, \ldots, T_n)$.

# Conclusiones

En esta tesis hemos conseguido avanzar en la construcción de modelos funcionales para operadores $T$ en espacios de Hilbert sujetos la positividad del operador $\alpha(T^*, T)$, donde $\alpha$ es una función representable en serie de potencias alrededor del origen.

Con los resultados de esta tesis, queda claro que la topología que debemos usar para la definición de $\alpha(T^*, T)$ (que recordemos, se define como la serie $\sum \alpha_n T^{*n} T^n$) es la topología fuerte de operadores. Este hecho no era nada trivial. De hecho, en nuestro primer trabajo [16] consideramos la topología uniforme.

Por otro lado, hemos intentado considerar clases de funciones $\alpha$ lo más amplias posibles. De esta manera, hemos obtenido resultados que van más allá de las funciones de tipo Nevannlina-Pick, que han sido muy estudiadas. Además, hemos visto que en los resultados de semejanza es posible trabajar incluso sin necesidad de considerar los núcleos $k(t) = 1/\alpha(t)$. De esta forma podemos permitir, por ejemplo, que la función $\alpha$ tenga ceros en el disco unidad $\mathbb{D}$.

Uno de los objetivos que siempre se tuvo en mente durante la tesis fue, no sólo la obtención de resultados sobre la existencia de un modelo unitariamente equivalente, sino además ver si era posible dar modelos explícitos y estudiar su unicidad. Hemos conseguido dar modelos explícitos donde interviene el espacio defecto $\mathfrak{D}$ (que viene dado por el operador defecto $D = \alpha(T^*, T)^{1/2}$), y también hemos caracterizado la unicidad de dicho modelo (esencialmente, depende de si $\alpha(1) = 0$ o bien $\alpha(1) > 0$).

La obtención de modelos explícitos nos resultó de gran ayuda a la hora de obtener conclusiones derivadas del modelo. Por ejemplo, hemos obtenido resultados sobre el espectro del operador $T$ cuando el espacio defecto $\mathfrak{D}$ es finito dimensional. Además, hemos obtenido una conexión muy interesante con la teoría ergódica para el caso de $a$-contracciones con $0 < a < 1$. Resulta muy curioso que esta conexión es en ambos sentidos, es decir, también podemos obtener conclusiones sobre el modelo mediante propiedades ergódicas del operador $T$.

Hemos hecho bastantes esfuerzos por comprender las clases de operadores $\mathbf{C}_\alpha$. En este sentido, hemos conseguido entender algunas propiedades elementales de estas clases y hemos obtenido resultados interesantes para las inclusiones de clases (en algunos casos incluso hemos dado caracterizaciones sobre cuándo $\mathbf{C}_\alpha \subset \mathbf{C}_\beta$).

Las técnicas que hemos empleado en esta tesis han sido variadas. De antemano, por la temática de la tesis, estaba claro que necesitaríamos usar herramientas de la teoría de operadores, el análisis complejo y el análisis armónico. Lo interesante ha sido que hemos visto que la teoría de álgebras de Banach y las diferencias finitas han resultado ser herramientas sumamente útiles y naturales.

En resumen, con esta tesis hemos conseguido avanzar satisfactoriamente en nuestros objetivos iniciales, y además hemos aportado nuevas técnicas que seguro permitirán avanzar en problemas relacionados. Por otro lado, esta tesis deja abiertas varias cuestiones interesantes para trabajos futuros.

## Trabajo futuro

Hay varias preguntas que surgen de manera natural con esta tesis. Algunas de ellas son las siguientes.

- En esta tesis hemos tratado de considerar funciones $\alpha$ lo más generales posibles en todo momento. Aunque hemos conseguido avanzar significativamente en este sentido respecto a trabajos anteriores, sería muy deseable seguir ampliando la clase de funciones $\alpha$ para las cuales podemos obtener un modelo para operadores $T$ satisfaciendo que $\alpha(T^*, T) \geq 0$.

- Para $0 < a \leq 1$, ya hay un modelo para las $a$-contracciones. En esta tesis hemos visto que una conjetura bastante natural sobre un posible modelo para las $a$-contracciones con $a > 1$ no es cierta. Queda pendiente, por tanto, intentar arrojar algo más de luz en este sentido.

- Otra cuestión muy interesante es estudiar otro tipo de dominios aparte del disco unidad $\mathbb{D}$ en este contexto. Es decir, en vez de suponer que el espectro de $T$ se contiene en $\overline{\mathbb{D}}$ como hacemos en esta tesis, intentar entender qué pasa cuando $\sigma(T)$ está contenido, por ejemplo, en un anillo u otro dominio simplemente conexo.

- Esta tesis está centrada en el estudio de un sólo operador $T$ en un espacio de Hilbert. Sin embargo, sería muy interesante extender los resultados de esta tesis (al menos algunos) a tuplas de operadores que conmutan $(T_1, \ldots, T_n)$.

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