Departamento de Matemáticas

Análisis Armónico en dominios irregulares

Juan Cavero de Carondelet Fiscowich

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Contents

Introduction

This thesis project lies in the intersection between real harmonic analysis, partial differential equations and geometric measure theory. The research developed therein has been inspired by the recent advances in the area due to José María Martell, Steve Hofmann and Tatiana Toro, among others.

During the last years, the interest in analyzing the relation between the behavior of elliptic measure and geometric properties of the domain has increased. It is being studied how the absolute continuity of elliptic measure with respect to surface measure, in a quantitative sense, is related with some "regularity" properties of the boundary of the domain, which might be rough. Most of the new approaches are based on modern harmonic analysis tools developed in the last decades. Before going further in that direction, we will first exhibit how some of these ideas evolved along the last century. The first result appeared in 1916, working on the complex plane. F. and M. Riesz showed in [RR] that harmonic measure, the elliptic measure for the Laplace operator, was absolutely continuous with respect to arc-length measure. In order to prove that property, they assumed that the planar domain was simply connected and its boundary was a rectifiable curve. Later in 1936, Lavrentiev gave in [Lav] a quantitative version of the F. and M. Riesz's theorem. The different behavior of elliptic equations in the plane and in higher dimensions motivated further research in this topic. In 1977 the study of the Laplace operator in Lipschitz domains by Dahlberg in [Dah1] showed that in higher dimensions we still have absolute continuity of harmonic measure with respect to surface measure. Moreover the Poisson kernel, or equivalently the Radon-Nikodym derivative of harmonic measure, satisfies a reverse Hölder inequality. This is in fact a stronger version of mutual absolute continuity between harmonic and surface measures, where we have a quantitative control of the ratio between both measures in a scale invariant fashion. In the case of Lipschitz domains it is proved that the Poisson kernel belongs to the class RH_2, what implies that it has local square integrability. After that result, there was an interest in understanding whether there are more general domains for which harmonic measure satisfies a reverse Hölder inequality with a possibly smaller exponent.

In the decade of 1980s Jerison and Kenig brought up a new class of domains called "non-tangentially accesible" or NTA domains. These domains are defined in [JK] and satisfy three main properties. The first is the "Harnack chain condition" (cf. Definition 1.2), which can be seen as a quantitative version of the fact that the domain is path connected. Also we ask for quantitative versions of the openness condition, both for the domain, what is called "interior Corkscrew" (cf. Definition 1.1) and for the exterior domain, what is called "exterior Corkscrew". In [JK] it is also developed the so called Jerison-Kenig's program, that is a collection of

estimates of harmonic measure, taking into account its behavior near the boundary and its relation with the Green function. In order to be able to extend the result of Dahlberg to a larger class of domains, it came up the additional assumption that the boundary had to be "Ahlfors regular" (cf. Definition 1.3), in which case the domains are called "chord-arc" domains or CAD (cf. Definition 1.5). In 1990, both David-Jerison [DJ] and Semmes [Sem], independently proved that harmonic measure in a chord-arc domain is in some RH_p class, for $p > 1$. In terms of Muckenhoupt weights, this means that harmonic measure is always an A_∞ weight with respect to surface measure, whenever the above geometric properties are satisfied. Then, a new question arises: Are the properties defining CAD necessary to ensure that harmonic measure is in the A_∞ class? Several recent studies in this topic have been devoted to show to what extent one can drop the hypothesis of the exterior corkscrew. We say that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is a "1-sided chord-arc domain" (or 1-sided CAD) if it satisfies the interior corkscrew and Harnack chain conditions, and if its boundary is n-dimensional Ahlfors regular. It was shown in [HM3] that in a 1-sided chord-arc domain, the uniform rectifiability of $\partial\Omega$ (which is a quantitative version of rectifiability) is a sufficient condition for the harmonic measure to be an A_∞ weight with respect to the surface measure $\sigma = H^n|_{\partial\Omega}$. Actually, in the setting of 1-sided CAD both conditions are equivalent, as proven later in [HMUT]. Moreover, under the same geometric assumptions it was shown in [AHM$^+$2] that $\partial\Omega$ is uniformly rectifiable if and only if Ω satisfies an exterior corkscrew condition. Taking also into account the work of [DJ, Sem] we obtain a characterization of chord-arc domains in terms of the membership $\omega_{\mathcal{L}} \in A_\infty(\partial\Omega)$, where $\mathcal{L}$ is the Laplace operator and $\omega_{\mathcal{L}}$ the harmonic measure.

Next, we consider $Lu = -\operatorname{div}(A\nabla u)$ a variable coefficient second order divergence form real elliptic operator (cf. Definition 1.20) in a 1-sided CAD (cf. Definition 1.4). There are different strategies in order to show that ω_L, the elliptic measure associated with L, can be used to characterize the fact that the domain is actually CAD. One of them is to analyze the "smoothness" of the matrix A, what was done in [KP] introducing some additional conditions. More precisely, let us define the class $\mathbb{L}_0$ to be the collection of real elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ as above such that $A \in \operatorname{Lip}_{\operatorname{loc}}(\Omega)$, $\big\||\nabla A|\,\delta\big\|_{L^\infty(\Omega)} < \infty$, and

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} |\nabla A(X)|\, dX < \infty. \qquad (0.1)$$

With this notation, it is shown in [KP] that if Ω is CAD, then $\omega_L \in A_\infty(\partial\Omega)$ for any $L \in \mathbb{L}_0$. Recently, the authors in [HMT1] proved a free boundary result for the class $\mathbb{L}_0$. In particular, this states that for any symmetric $L \in \mathbb{L}_0$, the membership $\omega_L \in A_\infty(\partial\Omega)$ in a 1-sided CAD implies that Ω is actually CAD. These two results combine to show a new characterization of CAD using the class of symmetric operators in $\mathbb{L}_0$. Similarly, for a non-symmetric $L \in \mathbb{L}_0$ it is required both that $\omega_L \in A_\infty(\partial\Omega)$ and $\omega_{L^\top} \in A_\infty(\partial\Omega)$ in order to show that Ω is CAD, as stated in [HMT1]. Here $L^\top$ is the transpose operator of L, that is, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A. The other different strategy is to compare $Lu = -\operatorname{div}(A\nabla u)$ with some given well known operator $L_0 u = -\operatorname{div}(A_0 \nabla u)$ that satisfies $\omega_{L_0} \in A_\infty(\partial\Omega)$, or equivalently $\omega_{L_0} \in RH_p(\partial\Omega)$ for some $p > 1$. For in-

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stance, here we could think that L_0 is the Laplace operator or some $L_0 \in \mathbb{L}_0$. Over the years there has been a considerable effort to find which are the adequate conditions on the discrepancy between the matrices A and A_0 that allow us to conclude that $\omega_L \in A_\infty(\partial\Omega)$, or maybe even $\omega_L \in RH_p(\partial\Omega)$, for the same $p > 1$. This has been historically known as the problem of perturbation of elliptic operators, and it is the main topic of this thesis. Before going further, we will introduce some notation. Let us define the disagreement between A and A_0 in Ω by

$$\varrho(A, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|, \qquad X \in \Omega, \qquad (0.2)$$

where $\delta(X) := \mathrm{dist}(X, \partial\Omega)$. This disagreement induces a measure μ_{A,A_0} in Ω given by

$$\mu_{A,A_0}(U) := \iint_U \frac{\varrho(A, A_0)(X)^2}{\delta(X)} \, dX, \qquad U \subset \Omega. \qquad (0.3)$$

We say that μ_{A,A_0} is a Carleson measure with respect to σ if

$$\|\varrho(A, A_0)\| := \sup_{\substack{x \in \partial\Omega \\ 0 < r < \mathrm{diam}(\partial\Omega)}} \frac{\mu_{A,A_0}(B(x,r) \cap \Omega)}{\sigma(B(x,r) \cap \partial\Omega)} < \infty. \qquad (0.4)$$

Here, the regions $B(x, r) \cap \Omega$ where the integration takes place are usually called Carleson regions. Similarly, we say that μ_{A,A_0} is a vanishing trace Carleson measure with respect to σ if

$$\lim_{s \to 0^+} \left(\sup_{\substack{x \in \partial\Omega \\ 0 < r \le s < \mathrm{diam}(\partial\Omega)}} \frac{\mu_{A,A_0}(B(x,r) \cap \Omega)}{\sigma(B(x,r) \cap \partial\Omega)} \right) = 0. \qquad (0.5)$$

The first perturbation result in this fashion was due to Dahlberg, who in [Dah2] showed that in the unit ball, the fact that μ_{A,A_0} is a vanishing trace Carleson measure with respect to σ is sufficient to transfer the condition $RH_p(\partial\Omega)$ from ω_{L_0} to ω_L, without changing the exponent. This result has been extended to more general contexts in the work of [Esc] or [MPT2], where they treat the case of Lipschitz domains and chord-arc domains respectively. The problem of the "large constant" perturbation, that is the case when $\|\varrho(A, A_0)\| < \infty$, or equivalently when μ_{A,A_0} is a Carleson measure, was solved in 1991 by Fefferman-Kenig-Pipher [FKP]. In the setting of Lipschitz domains, they prove that if $\omega_{L_0} \in A_\infty(\partial\Omega)$ and $\|\varrho(A, A_0)\| < \infty$, then necessarily $\omega_L \in A_\infty(\partial\Omega)$. From the point of view of reverse Hölder inequalities, it is not possible to keep the same exponent from one operator to the other. Nevertheless, the $A_\infty(\partial\Omega)$ condition, which as we know can be used to characterize geometric information of the domain, is still preserved by Carleson measure type perturbations. This theorem requires a very delicate analysis and the details of its proof have been an inspiration for further results in the area. It is worth mentioning the work of [MPT1], where they extend the theorem of [FKP] to chord-arc domains, taking advantage of all the PDE machinery developed in [JK]. We note that in all of these perturbation theorems, the operators have been assumed to be symmetric.

Our first project consisted in extending the theorems in [Dah2, FKP] to the setting of 1-sided chord-arc domains. The approach is heavily inspired in the work

of [HM1, HM2], in which the upper half space is considered as a model to develop a new scheme to address Carleson perturbations. This scheme relies in the so called extrapolation of Carleson measures method, which appeared first in [LM] (see also [HL, AHLT, AHM$^+$1]) and was further developed in [HM1, HM2] (see also [HM3]). Based on the Corona construction of Carleson [Car] and Carleson-Garnett [CG], this argument is a bootstrapping that allows us to reduce the analysis to sawtooth subdomains where the perturbation is sufficiently small. Having in mind that the domains under consideration are only assumed to be 1-sided CAD, the Jerison-Kenig's program for CAD cannot be applied directly. Luckily, this program is being developed in [HMT2] for 1-sided CAD, and most of the background PDE tools are at our disposal. It is interesting to note that in the present geometric scenario, the condition $\omega_{L_0} \in RH_p(\partial\Omega)$ is equivalent to the fact that the Dirichlet problem for L_0 can be solved (in a non-tangential fashion) for boundary data in $L^{p'}(\partial\Omega)$. To be able to use the extrapolation of Carleson measures we first need to understand the case of small perturbation, that is the situation on which $\||\varrho(A, A_0)\|| < \varepsilon_1$ for a small $\varepsilon_1 > 0$ to be chosen. The study of this case led us to a new dyadic version of the Coifman-Meyer-Stein's theorem for duality of tent spaces (see [CMS]). Thanks to this property we are able to keep the same exponent $p > 1$ of the reverse Hölder inequality from one operator to the other, whenever the perturbation is small. For the "large constant" perturbation, we will follow the scheme of [HM1, HM2] and one does not expect to preserve the exponent, rather one seeks to prove a general $A_\infty(\partial\Omega)$ condition. The following theorem summarizes both the small and large constant perturbations for symmetric operators in the setting of 1-sided CAD. This corresponds to Theorem 2.1 in the text.

Theorem 1. *Let* $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, *be a 1-sided* CAD *(cf. Definition 1.4). Let* $Lu = -\operatorname{div}(A\nabla u)$ *and* $L_0 u = -\operatorname{div}(A_0\nabla u)$ *be real elliptic operators (cf. Definition 1.20) such that* A *and* A_0 *are symmetric. Suppose that there exists* p, $1 < p < \infty$, *such that the elliptic measure* $\omega_{L_0} \in RH_p(\partial\Omega)$ *(cf. Definition 1.34). The following hold:*

(a) *If* $\||\varrho(A, A_0)\|| < \infty$ *(cf. (0.4)), then there exists* $1 < q < \infty$ *such that* $\omega_L \in RH_q(\partial\Omega)$.

(b) *There exists* $\varepsilon_1 > 0$ *such that if* $\||\varrho(A, A_0)\|| \leq \varepsilon_1$, *then* $\omega_L \in RH_p(\partial\Omega)$.

Additionally, we obtain an extension of the vanishing trace Carleson perturbation result of [Dah2] to the setting of 1-sided CAD as a corollary of the case $\||\varrho(A, A_0)\|| < \varepsilon_1$. We state the result as follows, which is written more precisely in Corollary 2.12.

Corollary 2. *Suppose that* $\Omega \subset \mathbb{R}^{n+1}$ *is a **bounded** 1-sided* CAD *(cf. Definition 1.4). Let* L_0, L *be real symmetric elliptic operators (cf. Definition 1.20) and suppose that* $\omega_{L_0} \in RH_p(\partial\Omega)$ *for some* $1 < p < \infty$ *(cf. Definition 1.34). If the vanishing trace Carleson condition (0.5) holds, then we have that* $\omega_L \in RH_p(\partial\Omega)$.

We also note that a new characterization of CAD may be given with the help of Theorem 1. Indeed, similarly as considered in [KP], we introduce the class $\mathbb{L}_0'$ to be the collection of real symmetric elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ such that $A \in \operatorname{Lip}_{\operatorname{loc}}(\Omega)$, $\||\nabla A| \delta\||_{L^\infty(\Omega)} < \infty$, and (0.1) holds. We also introduce $\mathbb{L}'$, the collection

of real symmetric elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ for which there exists $L_0 = -\operatorname{div}(A_0\nabla u) \in \mathbb{L}_0'$ in such a way that $\||\varrho(A, A_0)\|| < \infty$. It is straightforward to see that all symmetric constant coefficient operators belong to the class $\mathbb{L}_0'$ and also that $\mathbb{L}_0' \subset \mathbb{L}'$. It is worth noting that the operators in $\mathbb{L}'$ may not have any regularity, however they are still appropriate in order to characterize the class CAD. The precise result will be stated in Corollary 6, where a more general case for non necessarily symmetric operators is studied. For the symmetric case we take $L \in \mathbb{L}'$ and $L_0 \in \mathbb{L}_0'$ such that $\||\varrho(A, A_0)\|| < \infty$. First, note that if Ω is CAD we have that $\omega_{L_0} \in A_\infty(\partial\Omega)$ by the main result in [KP] (see also [HMT1, Appendix A]). This combines with the large perturbation theorem of [MPT1] to show that $\omega_L \in A_\infty(\partial\Omega)$. For the converse implication, namely the fact that $\omega_L \in A_\infty(\partial\Omega)$ implies that Ω is actually CAD, we will use Theorem 1(a). In that way, we first show that $\omega_{L_0} \in A_\infty(\partial\Omega)$, which along with the fact that $L_0 \in \mathbb{L}_0'$ is sufficient to conclude that Ω is actually CAD, as seen in [HMT1].

The second project of this thesis deals with "large constant" Carleson perturbations of non-symmetric elliptic operators in the setting of 1-sided chord-arc domains. This is, we let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0\nabla u)$ be real elliptic operators, not necessarily symmetric, and we assume that $\omega_{L_0} \in A_\infty(\partial\Omega)$. Our goal is to show that under the assumption of $\||\varrho(A, A_0)\|| < \infty$ we also have that $\omega_L \in A_\infty(\partial\Omega)$. The approach used to address this problem differs from the one used in Theorem 1 (see also [HM1, HM2]), or even the one in [FKP, MPT1]. We are interested in analyzing the property that all bounded solutions of a given operator L satisfy "Carleson measure estimates" or, equivalently, CME. This means that for every bounded weak solution of $Lu = 0$ it holds

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \infty}} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |\nabla u(X)|^2 \delta(X)\, dX \leq C\|u\|_{L^\infty(\Omega)}^2. \tag{0.6}$$

This property can be found in the literature to be related with the fact that $\omega_L \in A_\infty(\partial\Omega)$. For instance, in the setting of bounded Lipschitz domains and domains above the graph of a Lipschitz function, the authors in [KKPT] show that if L satisfies "Carleson measure estimates" then we have $\omega_L \in A_\infty(\partial\Omega)$. For the converse implication we assume that $\omega_L \in A_\infty(\partial\Omega)$. The fact that every bounded weak solution of $Lu = 0$ satisfies (0.6) can be seen, by the work of [DJK], as a consequence of a more general estimate in the setting of Lipschitz and chord-arc domains (see also [HMT2] for 1-sided CAD). Indeed, assuming that $\omega_L \in A_\infty(\partial\Omega)$, it is shown that the conical square function is controlled by the non-tangential maximal function in every $L^p(\partial\Omega)$ for every $1 < p < \infty$, where both are applied to solutions of L. Applying this with $p = 2$ and with a bounded solution the desired Carleson estimate follows at once. We will present a new technique to prove this latter fact using some of the tools developed in [HMT1]. Our first problem is to show that if L is a Carleson perturbation of L_0 with $\omega_{L_0} \in A_\infty(\partial\Omega)$, then L satisfies "Carleson measure estimates". We will call this problem the $A_\infty - \text{CME}$ perturbation. First, observe that it can be used to prove that $\omega_{L_0} \in A_\infty(\partial\Omega)$ implies "Carleson measure estimates" for L_0. To see this we just take $L = L_0$, in which case the "zero perturbation" is automatically of Carleson type, hence the $A_\infty - \text{CME}$ perturbation gives the desired properties for L_0. The second problem is to find an analog of the main

theorem in [KKPT] adapted to 1-sided chord-arc domains, which combined with the $A_\infty - \mathrm{CME}$ perturbation allows us to extend Theorem 1(a) to non-symmetric elliptic operators. Here, we observe that since we are using the auxiliar condition CME, it is not possible to keep the exponent of the $RH_p(\partial\Omega)$ inequality in this setting, hence we do not obtain an analog of Theorem 1(b)). We collect these results in two different theorems. The first states the equivalence between $\omega_L \in A_\infty(\partial\Omega)$ and the CME property for L, and this corresponds to Theorem 3.1. The second is the desired generalization of Theorem 1(a), which will be called Theorem 3.2 in the text.

Theorem 3. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided* CAD *(cf. Definition 1.4) and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20). The following statements are equivalent:*

(a) *Every bounded weak solution of $Lu = 0$ satisfies the Carleson measure estimate (0.6).*

(b) $\omega_L \in A_\infty(\partial\Omega)$ *(cf. Definition 1.33).*

Theorem 4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided* CAD *(cf. Definition 1.4). Let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be real (not necessarily symmetric) elliptic operators (cf. Definition 1.20). Assume that the disagreement between A and A_0 satisfies $\|\varrho(A, A_0)\| < \infty$ (cf. (0.4)). Then, $\omega_{L_0} \in A_\infty(\partial\Omega)$ if and only if $\omega_{L_1} \in A_\infty(\partial\Omega)$ (cf. Definition 1.33).*

As noted above, the method introduced to prove Theorems 3 and 4 can be split in two different parts. For the first part, this is the $A_\infty - \mathrm{CME}$ perturbation, we take advantage of the $A_\infty(\partial\Omega)$ condition to extract a sawtooth domain with nice properties. This is done by applying a result in [HMT1], which is a stopping time argument based on the solution of the Kato square root conjecture in [HM5, HLM, AHL$^+$], that has been further developed in [HM4, HLMN]. More precisely, this sawtooth has an ample contact with the original domain, and roughly speaking, the averages of ω_{L_0} with respect to σ are essentially constant. This allows us to interchange the distance to the boundary with the Green function G_{L_0} (with a fixed pole) in (0.6), so we can integrate by parts to obtain the desired estimate in the sawtooth. Finally, using that the sawtooth has an ample contact with Ω and following again [HMT1], we can extend the Carleson measure estimate to the entire domain. The second part of this method, that is, the fact that CME for L implies $\omega_L \in A_\infty(\partial\Omega)$, is based on the ideas developed in [KKPT]. We pick a Borel set $F \subset \partial\Omega$ with small ω_L measure and try to show that $\sigma(F)$ is also small. In that way we construct a finite collection of nested sets containing F, called an ε_0-cover (cf. Definition 1.7), and a solution u associated to that collection. Once we show a lower bound for the oscillation of u, we are able to obtain a lower bound for the conical square function applied to this solution. This, along with the fact that u satisfies Carleson measure estimates help us to conclude that $\sigma(F)$ is also small, hence $\omega_L \in A_\infty(\partial\Omega)$. Note that it is not necessary to assume that every bounded weak solution satisfies CME, since we only use this property for a particular class of

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solutions. Finally, there is an interesting application of the method discussed above. In this result we would like to infer nice properties from ω_L to $\omega_{L^\top}$, where $L^\top$ is the transpose operator of L. The Carleson condition on the discrepancy between L and $L^\top$ turns out to be, after an integration by parts, a Carleson measure condition on the derivative of the antisymmetric part of A. Assuming the latter we are able to show that $\omega_L \in A_\infty(\partial\Omega)$ is equivalent to $\omega_{L^\top} \in A_\infty(\partial\Omega)$, as stated precisely in the theorem below, which stands for Theorem 3.3.

Theorem 5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided* CAD *(cf. Definition 1.4). Let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20) and let $L^\top = -\operatorname{div}(A^\top \nabla u)$ denote the transpose of L. Assume that $(A - A^\top) \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$ and let*

$$\operatorname{div}_C(A - A^\top)(X) = \left(\sum_{i=1}^{n+1} \partial_i(a_{i,j} - a_{j,i})(X) \right)_{1 \leq j \leq n+1}, \qquad X \in \Omega. \qquad (0.7)$$

Assume that the following Carleson measure estimate holds

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \left| \operatorname{div}_C(A - A^\top)(X) \right|^2 \delta(X) \, dX < \infty. \qquad (0.8)$$

Then $\omega_L \in A_\infty(\partial\Omega)$ if and only if $\omega_{L^\top} \in A_\infty(\partial\Omega)$ (cf. Definition 1.33).

Similarly as noted before, the Carleson perturbation of elliptic operators can be used to extend free boundary results. We recall that $\mathbb{L}_0$ is the collection of non symmetric elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ such that $A \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$, $\big\| |\nabla A| \delta \big\|_{L^\infty(\Omega)} < \infty$, and (0.1) holds. We note that in [HMT1] it is proved that for any non symmetric $L \in \mathbb{L}_0$, one has to assume both that $\omega_L \in A_\infty(\partial\Omega)$ and $\omega_{L^\top} \in A_\infty(\partial\Omega)$ in order to be able to ensure the existence of exterior corkscrews. In this direction we can use Theorem 5 to remove the hypothesis that $\omega_{L^\top} \in A_\infty(\partial\Omega)$. We state the new characterization of chord-arc domains in the following corollary, which corresponds to Corollary 3.4 in the text.

Corollary 6. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided* CAD *(cf. Definition 1.4). Let $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be a real elliptic operator (cf. Definition 1.20). Assume that $A_0 \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$, $|\nabla A_0| \delta \in L^\infty(\Omega)$ and that (0.1) holds for A_0. Then*

$$\omega_{L_0} \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \omega_{L_0^\top} \in A_\infty(\partial\Omega).$$

Additionally, if $Lu = -\operatorname{div}(A\nabla u)$ is a real elliptic operator (cf. Definition 1.20) such that $\|\varrho(A, A_0)\| < \infty$ (cf. (0.4)), then we have

$$\omega_L \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \Omega \text{ is a CAD (cf. Definition 1.4).} \qquad (0.9)$$

On the one hand we first observe that $|\nabla A_0| \delta \in L^\infty(\Omega)$ and the condition (0.1) for A_0 imply that

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \left| \nabla A_0(X) \right|^2 \delta(X) \, dX < \infty, \qquad (0.10)$$

hence the first part of Corollary 6 is an easy consequence of Theorem 5 with $A = A_0$. For the second part, we first analyze the backward implication. The fact that $\omega_{L_0} \in A_\infty(\partial\Omega)$ is derived from the work of [KP], and this combines with Theorem 4 to show that $\omega_L \in A_\infty(\partial\Omega)$. For the forward implication we first note that $\omega_{L_0} \in A_\infty(\partial\Omega)$ by Theorem 4. Also, by the first part of Corollary 6 we have that $\omega_{L_0^\top} \in A_\infty(\partial\Omega)$. These two conditions are sufficient to conclude that Ω is actually CAD, as stated in [HMT1].

The research in this thesis has led to the following papers:

- Juan Cavero, Steve Hofmann, and José M. Martell. Perturbations of elliptic operators in 1-sided chord-arc domains. Part I: Small and large perturbation for symmetric operators. *Transactions of the Americal Mathematical Society*, **371**(4):2797–2835, 2019. `https://doi.org/10.1090/tran/7536`.

- Juan Cavero, Steve Hofmann, José M. Martell, and Tatiana Toro. Perturbations of elliptic operators in 1-sided chord-arc domains. Part II: Non symmetric operators and Carleson measure estimates. *Work in progress*, 2019.

Introducción

Este proyecto de tesis se ubica en la intersección entre el análisis armónico real, las ecuaciones en derivadas parciales y la teoría geométrica de la medida. La investigación desarrollada se inspira en los recientes avances en el área debidos a José María Martell, Steve Hofmann y Tatiana Toro, entre otros.

Durante los últimos años ha aumentado el interés por analizar la relación entre el comportamiento de la medida elíptica y las propiedades geométricas del dominio. Se está estudiando cómo la continuidad absoluta de la medida elíptica con respecto de la medida de superficie, en términos cuantitativos, está relacionada con algunas buenas propiedades de la frontera del dominio, que podría ser irregular. Gran parte de las nuevas técnicas utilizadas se basan en herramientas de análisis armónico moderno desarrolladas en las últimas décadas. Antes de seguir en esa dirección, primero mostraremos cómo algunas de estas ideas evolucionaron a lo largo del siglo pasado. El primer resultado aparece en 1916, trabajando en el plano complejo. F. y M. Riesz prueban en [RR] que la medida armónica, la medida elíptica del Laplaciano, es absolutamente continua con respecto de la medida de "longitud de arco". Para probar esa propiedad se asumió que el dominio era simplemente conexo y que su frontera era una curva rectificable. Más tarde en 1936, Lavrentiev probó en [Lav] una versión cuantitativa del teorema de F. y M. Riesz. El distinto comportamiento de las ecuaciones elípticas en el plano y en dimensiones superiores motivó nuevas investigaciones en el tema. En 1977 el estudio del operador de Laplace en dominios Lipschitz por Dahlberg en [Dah1] mostró que en dimensiones superiores se sigue teniendo continuidad absoluta de la medida armónica con respecto de la medida de superficie. Más aún, el núcleo de Poisson, o equivalentemente la derivada de Radon-Nikodym de la medida armónica, satisface una desigualdad de tipo reverse Hölder. Esta es de hecho una versión más fuerte de la continuidad absoluta mutua entre la medida armónica y la de superficie. En el caso de dominios Lipschitz se prueba que el núcleo de Poisson pertenece a la clase RH_2, por lo que su cuadrado es localmente integrable. Tras este resultado ha habido un gran interés en entender hasta qué punto existen dominios más generales para los cuales la medida armónica satisface una desigualdad reverse Hölder, posiblemente con un menor exponente.

En la década de 1980 Jerison y Kenig introdujeron una nueva clase de dominios llamados "non-tangentially accesible" o NTA. Estos dominios se definen en [JK] y satisfacen tres propiedades principales. La primera es la "Harnack chain condition" (cf. Definición 1.2), que puede verse como una versión cuantitativa del hecho de que el dominio es conexo por caminos. También se piden versiones cuantitativas de la propiedad de ser abierto, tanto para el dominio interior, lo que se llama "interior Corkscrew" (cf. Definición 1.1) como para el dominio exterior, lo que se llama

"exterior Corkscrew". En [JK] se desarrolla el llamado programa de Jerison-Kenig, que es una colección de estimaciones de la medida armónica, teniendo en cuenta su comportamiento cerca de la frontera y su relación con la función de Green. Para ser capaces de extender el resultado de Dahlberg a una clase mayor de dominios se añadió la hipótesis de que la frontera fuera "Ahlfors regular" (cf. Definición 1.3), en cuyo caso los dominios se llaman "chord-arc" o CAD (cf. Definición 1.5). En 1990, David-Jerison [DJ] y Semmes [Sem] prueban independientemente que la medida armónica en un dominio "chord-arc" pertenece a una clase RH_p con $p > 1$. En términos de pesos de Muckenhoupt, esto significa que la medida armónica es siempre un peso A_∞ con respecto de la medida de superficie, siempre que se verifiquen las propiedades geométricas anteriores. A continuación surge una nueva pregunta: ¿Son las propiedades que definen CAD necesarias para asegurar que la medida armónica está en la clase A_∞? Muchos estudios recientes en este tema se han centrado en mostrar hasta qué punto se puede obviar la hipótesis del "exterior Corkscrew". Decimos que $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, es un dominio "1-sided chord-arc" (o 1-sided CAD) si satisface las condiciones de "Harnack Chain", de "interior Corkscrew", y si su frontera es "Ahlfors regular" n-dimensional. En [HM3] se probó que en un dominio "1-sided chord-arc", la rectificabilidad uniforme de $\partial\Omega$ (que es una versión cuantitativa de la rectificabilidad) es una condición suficiente para que la medida armónica sea un peso A_∞ con respecto de la medida de superficie $\sigma = H^n|_{\partial\Omega}$. De hecho, para un dominio "1-sided chord-arc" ambas condiciones son equivalentes, como se probó posteriormente en [HMUT]. Más aún, bajo las mismas hipótesis geométricas, en [AHM$^+$2] se demostró que $\partial\Omega$ es uniformemente rectificable si y sólo si Ω satisface la condición de "exterior Corkscrew". Teniendo en cuenta también el trabajo de [DJ, Sem], se obtiene una caracterización de los dominios "chord-arc" en función de la pertenencia $\omega_{\mathcal{L}} \in A_\infty(\partial\Omega)$, donde $\mathcal{L}$ es el operador de Laplace y $\omega_{\mathcal{L}}$ es la medida armónica.

A continuación consideramos $Lu = -\operatorname{div}(A\nabla u)$ un operador elíptico de tipo divergencia con coeficientes variables (cf. Definición 1.20) en un dominio "1-sided CAD" (cf. Definición 1.4). Existen diferentes estrategias para probar que ω_L, la medida elíptica asociada a L, puede usarse para caracterizar el hecho de que el dominio es realmente CAD. Una de ellas es analizar la "suavidad" de la matriz A, lo que se hizo en [KP] introduciendo condiciones adicionales. Más precisamente, definimos la clase $\mathbb{L}_0$ como la colección de operadores elípticos $Lu = -\operatorname{div}(A\nabla u)$ introducidos anteriormente tales que $A \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$, $\big\||\nabla A|\,\delta\big\|_{L^\infty(\Omega)} < \infty$, y

$$\sup_{\substack{x\in\partial\Omega \\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} |\nabla A(X)|\,dX < \infty. \qquad (0.11)$$

Con esta notación se demuestra en [KP] que si Ω es CAD, entonces $\omega_L \in A_\infty(\partial\Omega)$ para cualquier $L \in \mathbb{L}_0$. Recientemente, los autores de [HMT1] han probado un resultado de frontera libre para la clase $\mathbb{L}_0$. En particular se demuestra que para cualquier operador simétrico $L \in \mathbb{L}_0$, la pertenencia $\omega_L \in A_\infty(\partial\Omega)$ en un dominio 1-sided CAD implica que Ω es de hecho CAD. Estos dos resultados se combinan para dar una nueva caracterización de la clase CAD usando los operadores simétricos de $\mathbb{L}_0$. De manera similar, para un operador $L \in \mathbb{L}_0$ no necesariamente simétrico, se requiere tanto que $\omega_L \in A_\infty(\partial\Omega)$ como que $\omega_{L^\top} \in A_\infty(\partial\Omega)$ para probar que Ω es

CAD, tal y como se enuncia en [HMT1]. Aquí $L^\top$ es el operador traspuesto de L, esto es, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ con $A^\top$ la matriz traspuesta de A. La otra estrategia diferente es la de comparar $Lu = -\operatorname{div}(A\nabla u)$ con algún operador conocido $L_0 u = -\operatorname{div}(A_0 \nabla u)$ que satisface $\omega_{L_0} \in A_\infty(\partial\Omega)$, o equivalentemente $\omega_{L_0} \in RH_p(\partial\Omega)$ para algún $p > 1$. Por ejemplo, podríamos pensar que L_0 es el operador de Laplace o quizá algún $L_0 \in \mathbb{L}_0$. A lo largo de los años ha habido un esfuerzo considerable para encontrar las condiciones adecuadas en la discrepancia entre las matrices A y A_0 que nos permiten concluir que $\omega_L \in A_\infty(\partial\Omega)$, o quizá incluso $\omega_L \in RH_p(\partial\Omega)$, para el mismo $p > 1$. Esto se ha conocido históricamente como el problema de perturbación de operadores elípticos, y es el tema principal de esta tesis. Antes de seguir introduciremos algo de notación. Definimos la discrepancia entre A y A_0 en Ω como

$$\varrho(A, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|, \qquad X \in \Omega, \qquad (0.12)$$

donde $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. La discrepancia induce una medida μ_{A,A_0} en Ω dada por

$$\mu_{A,A_0}(U) := \iint_U \frac{\varrho(A, A_0)(X)^2}{\delta(X)} \, dX, \qquad U \subset \Omega. \qquad (0.13)$$

Decimos que μ_{A,A_0} es una medida de Carleson con respecto de σ si

$$\|\varrho(A, A_0)\| := \sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{\mu_{A,A_0}(B(x,r) \cap \Omega)}{\sigma(B(x,r) \cap \partial\Omega)} < \infty. \qquad (0.14)$$

Aquí, las regiones $B(x, r) \cap \Omega$ donde tiene lugar la integración se llaman regiones de Carleson. De manera similar se dice que μ_{A,A_0} es una medida de Carleson con "vanishing trace" con respecto de σ si

$$\lim_{s \to 0^+} \left(\sup_{\substack{x \in \partial\Omega \\ 0 < r \leq s < \operatorname{diam}(\partial\Omega)}} \frac{\mu_{A,A_0}(B(x,r) \cap \Omega)}{\sigma(B(x,r) \cap \partial\Omega)} \right) = 0. \qquad (0.15)$$

El primer resultado de perturbación en estos términos se debe a Dahlberg, quien en [Dah2] prueba que en la bola unidad, el hecho de que μ_{A,A_0} sea una medida de Carleson con "vanishing trace" con respecto de σ, es suficiente para transferir la condición $RH_p(\partial\Omega)$ de ω_{L_0} a ω_L, sin cambiar el exponente. Este resultado se ha extendido a contextos más generales en el trabajo de [Esc] o [MPT2], que tratan el caso de dominios Lipschitz y CAD respectivamente. El problema de la "constante grande", es decir cuando $\|\varrho(A, A_0)\| < \infty$, o equivalentemente cuando μ_{A,A_0} es una medida de Carleson, se resolvió en 1991 por Fefferman-Kenig-Pipher [FKP]. En el contexto de dominios Lipschitz, se prueba que si $\omega_{L_0} \in A_\infty(\partial\Omega)$ y $\|\varrho(A, A_0)\| < \infty$, entonces necesariamente $\omega_L \in A_\infty(\partial\Omega)$. Desde el punto de vista de desigualdades reverse Hölder, no es posible mantener el mismo exponente de un operador al otro. No obstante la condición $A_\infty(\partial\Omega)$, que como ya sabemos puede usarse para caracterizar información geométrica del dominio, se sigue preservando por perturbaciones de tipo medida de Carleson. Este teorema requiere un análisis muy delicado y los detalles de su prueba han servido como inspiración para más resultados en el área. Vale la pena mencionar el trabajo de [MPT1], en el que se extiende el teorema de [FKP] a dominios CAD, sacando partido a toda la maquinaria de EDPs desarrollada

en [JK]. Nótese que en todos los teoremas de perturbación anteriores, se asume que los operadores son simétricos.

Nuestro primer proyecto consistió en extender los teoremas de [Dah2, FKP] al contexto de dominios "1-sided CAD". El planteamiento considerado está fuertemente inspirado en el trabajo de [HM1, HM2], en el que se utiliza el semiplano superior como un caso modelo para desarrollar un nuevo esquema de prueba para teoremas de perturbación de Carleson. Este esquema se basa en el llamado método de "extrapolación de medidas de Carleson", que apareció inicialmente en [LM] (ver también [HL, AHLT, AHM$^+$1]) y fue desarrollado en [HM1, HM2] (ver también [HM3]). Basado en la construcción de Carleson [Car] y Carleson-Garnett [CG], este argumento nos permite reducir el análisis a subdominios del tipo "sawtooth" en los que la perturbación es suficientemente pequeña. Teniendo en cuenta que los dominios considerados son únicamente 1-sided CAD, el programa de Jerison-Kenig para dominios CAD no puede ser aplicado directamente. Por suerte, este programa está siendo desarrollado en [HMT2] para 1-sided CAD, y gran parte de las propiedades de EDP necesarias están a nuestra disposición. Es interesante notar que en este escenario geométrico la condición $\omega_{L_0} \in RH_p(\partial\Omega)$ es equivalente al hecho de que el problema de Dirichlet para L_0 se puede resolver (de manera no tangencial) para datos frontera en $L^{p'}(\partial\Omega)$. Para ser capaces de usar el método de extrapolación de medidas de Carleson, primero tenemos que entender el caso de la perturbación pequeña, esto es el caso en el que $\|\varrho(A, A_0)\| < \varepsilon_1$ para un $\varepsilon_1 > 0$ pequeño a escoger. El estudio de este caso trajo consigo una versión diádica del teorema de Coifman-Meyer-Stein para dualidad de "tent spaces" (ver [CMS]). Gracias a esta propiedad somos capaces de mantener el mismo exponente $p > 1$ en la desigualdad reverse Hölder de un operador al otro, siempre que la perturbación sea pequeña. Para el problema de "constante grande" seguimos el esquema de [HM1, HM2], en el que se busca probar una condición más general del tipo $A_\infty(\partial\Omega)$ en vez de $RH_p(\partial\Omega)$. El siguiente teorema contiene los casos de constante grande y pequeña para operadores simétricos en dominios 1-sided CAD. Esto se corresponde con el Teorema 2.1 en el texto.

Teorema 1. *Sea $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, un dominio 1-sided CAD (cf. Definición 1.4). Sean $Lu = -\operatorname{div}(A\nabla u)$ y $L_0 u = -\operatorname{div}(A_0\nabla u)$ operadores elípticos (cf. Definición 1.20) tales que A y A_0 son simétricas. Supongamos que existe p, $1 < p < \infty$, de manera que la medida elíptica $\omega_{L_0} \in RH_p(\partial\Omega)$ (cf. Definición 1.34). Se verifican las siguientes propiedades:*

(a) *Si $\|\varrho(A, A_0)\| < \infty$ (cf. (0.14)), entonces existe $1 < q < \infty$ tal que $\omega_L \in RH_q(\partial\Omega)$.*

(b) *Existe $\varepsilon_1 > 0$ tal que si $\|\varrho(A, A_0)\| \leq \varepsilon_1$, entonces $\omega_L \in RH_p(\partial\Omega)$.*

Adicionalmente obtenemos una extensión del teorema de perturbación "vanishing trace" de [Dah2] al contexto de 1-sided CAD como corolario del caso $\|\varrho(A, A_0)\| < \varepsilon_1$. Enunciamos el resultado como sigue, que puede verse de manera más precisa en el Corolario 2.12.

Corolario 2. *Supongamos que $\Omega \subset \mathbb{R}^{n+1}$ es un 1-sided CAD (cf. Definición 1.4) **acotado**. Sean L_0, L operadores elípticos simétricos (cf. Definición 1.20) y supong-*

amos que $\omega_{L_0} \in RH_p(\partial\Omega)$ para algún $1 < p < \infty$ (cf. Definición 1.34). Si se verifica la condición (0.15), *entonces se tiene que $\omega_L \in RH_p(\partial\Omega)$.*

Gracias al Teorema 1 obtenemos una nueva caracterización de la clase CAD. En efecto, de manera similar a como se hace en [KP], introducimos la clase $\mathbb{L}'_0$ como la colección de operadores elípticos reales y simétricos $Lu = -\mathrm{div}(A\nabla u)$ tales que $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, $\big\| |\nabla A| \, \delta \big\|_{L^\infty(\Omega)} < \infty$, y tales que se cumple (0.11). También definimos la clase $\mathbb{L}'$ como la colección de operadores elípticos reales y simétricos $Lu = -\mathrm{div}(A\nabla u)$ para los que existe $L_0 = -\mathrm{div}(A_0\nabla u) \in \mathbb{L}'_0$ tal que $\|\!|\varrho(A, A_0)|\!\| < \infty$. Es fácil ver que todos los operadores simétricos de coeficientes constantes pertenecen a $\mathbb{L}'_0$, y también que $\mathbb{L}'_0 \subset \mathbb{L}'$. Es interesante observar que los operadores de $\mathbb{L}'$ pueden ser altamente irregulares, pero siguen siendo apropiados para caracterizar la clase CAD. El resultado preciso se enunciará en el Corolario 6, donde se estudia el caso de operadores no necesariamente simétricos. Para el caso simétrico tomamos $L \in \mathbb{L}'$ y $L_0 \in \mathbb{L}'_0$ de manera que $\|\!|\varrho(A, A_0)|\!\| < \infty$. Primero, nótese que si Ω es CAD entonces $\omega_{L_0} \in A_\infty(\partial\Omega)$ por el resultado principal de [KP] (ver también [HMT1, Appendix A]). Esto se combina con el teorema de perturbación de "constante grande" de [MPT1] para mostrar que $\omega_L \in A_\infty(\partial\Omega)$. Para la implicación inversa, es decir el hecho de que si $\omega_L \in A_\infty(\partial\Omega)$ entonces Ω es realmente CAD, usaremos el Teorema 1(a). De ese modo primero se prueba que $\omega_{L_0} \in A_\infty(\partial\Omega)$, que junto con el hecho de que $L_0 \in \mathbb{L}'_0$ es suficiente para concluir que Ω es realmente CAD, como se ve en [HMT1].

El segundo proyecto de esta tesis se ocupa de perturbaciones de tipo Carleson de "constante grande", para operadores elípticos no simétricos en dominios 1-sided CAD. Consideremos $Lu = -\mathrm{div}(A\nabla u)$ y $L_0 u = -\mathrm{div}(A_0\nabla u)$ operadores elípticos reales, no necesariamente simétricos, y supongamos que $\omega_{L_0} \in A_\infty(\partial\Omega)$. Queremos probar que bajo la hipótesis de que $\|\!|\varrho(A, A_0)|\!\| < \infty$, se tiene necesariamente $\omega_L \in A_\infty(\partial\Omega)$. La manera de atacar este problema es diferente de la utilizada en el Teorema 1 (ver también [HM1, HM2]), o incluso de la de [FKP, MPT1]. Estamos interesados en analizar la propiedad de que todas las soluciones acotadas de un operador L satisfagan "Carleson measure estimates" o, equivalentemente, CME. Esto significa que para cada solución débil acotada de $Lu = 0$ se tiene que

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \infty}} \frac{1}{r^n} \iint_{B(x,r) \cap \Omega} |\nabla u(X)|^2 \delta(X) \, dX \le C \|u\|^2_{L^\infty(\Omega)}. \tag{0.16}$$

Esta propiedad puede verse relacionada con el hecho de que $\omega_L \in A_\infty(\partial\Omega)$. Por ejemplo, en el escenario de dominios Lipschitz acotados o dominios bajo la gráfica de una función Lipschitz, se prueba en [KKPT] que si L satisface "Carleson measure estimates" entonces $\omega_L \in A_\infty(\partial\Omega)$. Para la otra implicación suponemos que $\omega_L \in A_\infty(\partial\Omega)$. El hecho de que toda solución débil acotada de $Lu = 0$ satisfaga (0.16) puede verse, por el trabajo de [DJK], como consecuencia de una estimación más general en dominios Lipschitz o CAD (ver también [HMT2] para 1-sided CAD). En efecto, asumiendo que $\omega_L \in A_\infty(\partial\Omega)$ se demuestra que la función cuadrado cónica está controlada por la función maximal no tangencial en todo $L^p(\partial\Omega)$ con $1 < p < \infty$, donde ambos operadores se aplican a soluciones de L. Utilizando esta propiedad con $p = 2$ y una función acotada, se obtiene la estimación deseada.

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Presentamos una nueva técnica para probar esta última implicación, utilizando algunas de las herramientas desarrolladas en [HMT1]. El primer problema es el de mostrar que si L es una perturbación de tipo Carleson de L_0 con $\omega_{L_0} \in A_\infty(\partial\Omega)$, entonces L satisface "Carleson measure estimates". Llamaremos a este problema la perturbación $A_\infty - \text{CME}$. Primero observemos que puede usarse para probar que $\omega_{L_0} \in A_\infty(\partial\Omega)$ implica "Carleson measure estimates" para L_0. Para ver esto simplemente tomamos $L = L_0$, en cuyo caso la perturbación es evidentemente de tipo Carleson, así que la perturbación $A_\infty - \text{CME}$ nos da las propiedades deseadas para L_0. El segundo problema es el de encontrar un análogo del teorema principal de [KKPT] adaptado a dominios 1-sided CAD, que combinado con la perturbación $A_\infty - \text{CME}$ extiende el Teorema 1(a) a operadores no necesariamente simétricos. Nótese que como estamos usando la condición auxiliar CME, no es posible mantener el exponente de la desigualdad $RH_p(\partial\Omega)$, por lo que no obtenemos un análogo del Teorema 1(b). Combinamos estos resultados en dos teoremas diferentes. El primero da la equivalencia entre $\omega_L \in A_\infty(\partial\Omega)$ y la propiedad CME para L, y se corresponde con el Teorema 3.1. El segundo es la generalización deseada del Teorema 1(a), que se llamará Teorema 3.2 en el texto.

Teorema 3. *Sea $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, un dominio 1-sided CAD (cf. Definición 1.4) y sea $Lu = -\operatorname{div}(A\nabla u)$ un operador elíptico real, no necesariamente simétrico (cf. Definición 1.20). Las siguientes propiedades son equivalentes:*

(a) *Toda solución débil acotada de $Lu = 0$ satisface (0.16).*

(b) *$\omega_L \in A_\infty(\partial\Omega)$ (cf. Definición 1.33).*

Teorema 4. *Sea $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, un dominio 1-sided CAD (cf. Definición 1.4). Sean $Lu = -\operatorname{div}(A\nabla u)$ y $L_0 u = -\operatorname{div}(A_0\nabla u)$ operadores elípticos reales, no necesariamente simétricos (cf. Definición 1.20). Supongamos que la discrepancia entre A y A_0 satisface $\|\varrho(A, A_0)\| < \infty$ (cf. (0.14)). Entonces, $\omega_{L_0} \in A_\infty(\partial\Omega)$ si y sólo si $\omega_{L_1} \in A_\infty(\partial\Omega)$ (cf. Definición 1.33).*

Como hemos observado anteriormente, el método introducido para demostrar los Teoremas 3 y 4 puede ser dividido en dos partes. Para la primera, esto es la perturbación $A_\infty - \text{CME}$, aprovechamos la condición $A_\infty(\partial\Omega)$ para extraer un dominio "sawtooth" con buenas propiedades. Para ello se aplica un resultado de [HMT1], que es un argumento "stopping-time" basado en la solución de la conjetura de Kato en [HM5, HLM, AHL$^+$], y que ha sido desarrollado en [HM4, HLMN]. Más precisamente, este dominio "sawtooth" guarda un amplio contacto con el dominio original, y las medias de ω_{L_0} con respecto de σ son esencialmente constantes. Esto nos permite intercambiar la distancia a la frontera por la función de Green G_{L_0} (con un polo fijado) en (0.16), así que podemos integrar por partes para obtener la estimación deseada en el "sawtooth". Finalmente, usando que este subdominio tiene un amplio contacto con Ω y siguiendo de nuevo [HMT1], podemos extender la estimación de tipo Carleson al dominio original. La segunda parte de este método, esto es, el hecho de que CME para L implica $\omega_L \in A_\infty(\partial\Omega)$, está basado en las ideas desarrolladas en [KKPT]. Consideramos un conjunto de Borel $F \subset \partial\Omega$ con

pequeña medida ω_L e intentamos probar que $\sigma(F)$ es también pequeña. Para ello se construye una colección finita de conjuntos anidados que contienen a F, a la que llamamos un ε_0-cubrimiento (cf. Definición 1.7), y una solución u asociada a esa colección. Primero se da una cota inferior para la oscilación de u, lo que implica una cota inferior para la función cuadrado cónica aplicada esa solución. Esto, junto con el hecho de que u satisface "Carleson measure estimates" nos permite concluir que $\sigma(F)$ es también pequeña, luego $\omega_L \in A_\infty(\partial\Omega)$. Nótese que no es necesario asumir que toda solución débil y acotada satisfaga CME, ya que sólo usamos esta propiedad para una clase particular de soluciones. Finalmente, hay una aplicación interesante del método discutido anteriormente. En este resultado queremos deducir buenas propiedades de ω_L a $\omega_{L^\top}$, donde $L^\top$ es el operador traspuesto de L. La condición de Carleson en la discrepancia entre L y $L^\top$ se convierte, tras una integración por partes, en una condición de tipo Carleson sobre la derivada de la parte antisimétrica de A. Asumiendo esta condición podemos probar que $\omega_L \in A_\infty(\partial\Omega)$ es equivalente a $\omega_{L^\top} \in A_\infty(\partial\Omega)$, como se enuncia precisamente en el teorema a continuación, que se corresponde con el Teorema 3.3 en el texto.

Teorema 5. *Sea $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, un dominio 1-sided CAD (cf. Definición 1.4). Sea $Lu = -\operatorname{div}(A\nabla u)$ un operador elíptico real, no necesariamente simétrico (cf. Definición 1.20) y sea $L^\top = -\operatorname{div}(A^\top \nabla u)$ el operador traspuesto de L. Supongamos que $(A - A^\top) \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$ y que*

$$\operatorname{div}_C(A - A^\top)(X) = \left(\sum_{i=1}^{n+1} \partial_i(a_{i,j} - a_{j,i})(X) \right)_{1 \leq j \leq n+1}, \qquad X \in \Omega. \qquad (0.17)$$

Supongamos que se verifica la siguiente estimación

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \left| \operatorname{div}_C(A - A^\top)(X) \right|^2 \delta(X) \, dX < \infty. \ (0.18)$$

Entonces $\omega_L \in A_\infty(\partial\Omega)$ si y sólo si $\omega_{L^\top} \in A_\infty(\partial\Omega)$ (cf. Definición 1.33).

De manera similar a como hemos observado antes, la perturbación de Carleson de operadores elípticos puede usarse para extender resultados de frontera libre. Recordemos que $\mathbb{L}_0$ es la colección de operadores elípticos no simétricos $Lu = -\operatorname{div}(A\nabla u)$ tales que $A \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$, $\big\| |\nabla A| \delta \big\|_{L^\infty(\Omega)} < \infty$, y tales que se verifica (0.11). Nótese que en [HMT1] se prueba que para todo operador no simétrico $L \in \mathbb{L}_0$, uno tiene que asumir tanto que $\omega_L \in A_\infty(\partial\Omega)$ como que $\omega_{L^\top} \in A_\infty(\partial\Omega)$ de cara a asegurar que se verifica la propiedad de "exterior Corkscrew". En esta dirección podemos usar el Teorema 5 para quitar la hipótesis de que $\omega_{L^\top} \in A_\infty(\partial\Omega)$. Enunciamos la nueva caracterización de dominios chord-arc en el siguiente corolario, que se corresponde con el Corolario 3.4 en el texto.

Corolario 6. *Sea $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, un dominio 1-sided CAD (cf. Definitción1.4). Sea $L_0 u = -\operatorname{div}(A_0 \nabla u)$ un operador elíptico real (cf. Definición 1.20). Supongamos que $A_0 \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$, $|\nabla A_0| \delta \in L^\infty(\Omega)$ y que se verifica (0.11) para A_0. Entonces*

$$\omega_{L_0} \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \omega_{L_0^\top} \in A_\infty(\partial\Omega).$$

Adicionalmente, si $Lu = -\operatorname{div}(A\nabla u)$ es un operador elíptico real (cf. Definición 1.20) tal que $\|\varrho(A, A_0)\| < \infty$ (cf. (0.14)), entonces se tiene que

$$\omega_L \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \Omega \text{ is a CAD } (cf. \text{ Definition } 1.4). \qquad (0.19)$$

Por una parte observemos que $|\nabla A_0|\, \delta \in L^\infty(\Omega)$ junto con (0.11) para A_0 implica que

$$\sup_{\substack{x\in\partial\Omega \\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} \left|\nabla A_0(X)\right|^2 \delta(X)\, dX < \infty, \qquad (0.20)$$

luego la primera parte del Corolario 6 es una consecuencia fácil del Teorema 5 con $A = A_0$. Para la segunda parte, primero analizaremos la implicación a la izquierda. El hecho de que $\omega_{L_0} \in A_\infty(\partial\Omega)$ se deduce del trabajo de [KP], y esto se combina con el Teorema 4 para probar que $\omega_L \in A_\infty(\partial\Omega)$. Para la implicación a la derecha primero observamos que $\omega_{L_0} \in A_\infty(\partial\Omega)$ por el Teorema 4. Además, utilizando la primera parte del Corolario 6 se tiene que $\omega_{L_0^\top} \in A_\infty(\partial\Omega)$. Estas dos condiciones son suficientes para concluir que Ω es de hecho CAD, como se prueba en [HMT1].

Notation

- Our ambient space is $\mathbb{R}^{n+1}$, $n \geq 2$.

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. Moreover, if c and C depend on some given parameter η, which is somehow relevant, we write $a \lesssim_\eta b$ and $a \approx_\eta b$. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Given a domain (i.e., open and connected) $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in Ω).

- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$. A "surface ball" is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$, and unless otherwise specified it is implicitly assumed that $x \in \partial\Omega$. Also if $\partial\Omega$ is bounded, we typically assume that $0 < r \lesssim \operatorname{diam}(\partial\Omega)$, so that $\Delta = \partial\Omega$ if $\operatorname{diam}(\partial\Omega) < r \lesssim \operatorname{diam}(\partial\Omega)$.

- Given a Euclidean ball B or surface ball Δ, its radius will be denoted $r(B)$ or $r(\Delta)$ respectively.

- Given a Euclidean ball $B = B(X, r)$ or surface ball $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted by $\kappa B = B(X, \kappa r)$ or $\kappa\Delta = \Delta(x, \kappa r)$.

- For $X \in \mathbb{R}^{n+1}$, we set $\delta_{\partial\Omega}(X) := \operatorname{dist}(X, \partial\Omega)$. Sometimes, when clear from the context we will omit the subscript $\partial\Omega$ and simply write $\delta(X)$.

- We let H^n denote the n-dimensional Hausdorff measure, and let $\sigma_{\partial\Omega} := H^n|_{\partial\Omega}$ denote the "surface measure" on $\partial\Omega$. For a closed set $E \subset \mathbb{R}^{n+1}$ we will use the notation $\sigma_E := H^n|_E$. When clear from the context we will also omit the subscript and simply write σ.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\mathbf{1}_A$ denote the usual indicator function of A, i.e., $\mathbf{1}_A(x) = 1$ if $x \in A$, and $\mathbf{1}_A(x) = 0$ if $x \notin A$.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\text{int}(A)$ denote the interior of A, and $\overline{A}$ denote the closure of A. If $A \subset \partial\Omega$, $\text{int}(A)$ will denote the relative interior, i.e., the largest relatively open set in $\partial\Omega$ contained in A. Thus, for $A \subset \partial\Omega$, the boundary is then well defined by $\partial A := \overline{A} \setminus \text{int}(A)$.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we denote by $C(A)$ the space of continuous functions on A and by $C_c(A)$ the subspace of $C(A)$ with compact support in A. Note that if A is compact then $C(A) \equiv C_c(A)$.

- For an open set $\Omega \subset \mathbb{R}^{n+1}$, we denote by $C^\infty(\Omega)$ the space of infinitely differentiable functions on Ω and by $C_c^\infty(\Omega)$ the subspace of $C^\infty(\Omega)$ with compact support in Ω.

- For a Borel set $A \subset \partial\Omega$ with $0 < \sigma(A) < \infty$, we write $\fint_A f \, d\sigma := \sigma(A)^{-1} \int_A f \, d\sigma$.

- We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. We use Q to denote a dyadic "cube" on $E \subset \mathbb{R}^{n+1}$. The latter exists, given that E is AR (cf. [DS1], [Chr]), and enjoy certain properties which we enumerate in Lemma 1.6 below.

Chapter 1

Preliminaries

1.1 Some geometric aspects

In this section we state the precise definitions of the geometric properties that will be assumed throughout the text. Also, we review the constructions that allow us to define adapted Whitney and Carleson regions, which serve as basic blocks in most of the proofs. This section is entirely based in the work of [HM3], although the same geometric assumptions have been used in [HMUT, HMT1, HMT2]. We present it for the sake of completeness.

Definition 1.1 (Corkscrew condition). Following [JK], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "Corkscrew condition" if for some uniform constant $c \in (0,1)$ and for every surface ball $\Delta := \Delta(x,r) = B(x,r) \cap \partial\Omega$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x,r) \cap \Omega$. The point $X_\Delta \in \Omega$ is called a "corkscrew point" relative to Δ. Note that we may allow $r < C\operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c.

Definition 1.2 (Harnack Chain condition). Again following [JK], we say that $\Omega \subset \mathbb{R}^{n+1}$ satisfies the Harnack Chain condition if there is a uniform constant C such that for every $\rho > 0$, $\Theta \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Theta\rho$, there is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leq C(\Theta)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ and $C^{-1}\operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C\operatorname{diam}(B_k)$. The chain of balls is called a "Harnack Chain".

Definition 1.3 (Ahlfors regular). We say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional AR (or simply AR), if there is some uniform constant $C = C_{\mathrm{AR}}$ such that
$$C^{-1}r^n \leq H^n(E \cap B(x,r)) \leq Cr^n, \quad 0 < r < \operatorname{diam}(E), \quad x \in E.$$

Definition 1.4 (1-sided chord-arc domain). A connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a "1-sided chord-arc domain" (1-sided CAD for short) if it satisfies the Corkscrew and Harnack Chain conditions and if $\partial\Omega$ is AR.

Definition 1.5 (Chord-arc domain). A connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a "chord-arc domain" (CAD for short) if it is a 1-sided CAD and moreover Ω satisfies the exterior Corkscrew condition (that is, the domain $\Omega_{\mathrm{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfies the Corkscrew condition).

We give a lemma concerning the existence of a "dyadic grid":

Lemma 1.6 (**Existence and properties of the "dyadic grid"**, [DS1, DS2], [Chr]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional* AR. *Then there exist constants $a_0 > 0$, $\eta > 0$ and $C_1 < \infty$ depending only on dimension and the* AR *constant, such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \big\{ Q_j^k \subset \partial\Omega : \, j \in \mathcal{J}_k \big\},$$

where $\mathcal{J}_k$ denotes some (possibly finite) index set depending on k, satisfying:

(a) *$E = \bigcup_j Q_j^k$ for each $k \in \mathbb{Z}$.*

(b) *If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \varnothing$.*

(c) *For each $j, k \in \mathbb{Z}$ and each $m > k$, there is a unique $i \in \mathbb{Z}$ such that $Q_j^k \subset Q_i^m$.*

(d) *$\operatorname{diam}(Q_j^k) \leq C_1 \, 2^{-k}$.*

(e) *Each Q_j^k contains some "surface ball" $\Delta(x_j^k, a_0 2^{-k}) = B(x_j^k, a_0 2^{-k}) \cap E$.*

(f) *$H^n\big(\big\{ x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \tau 2^{-k} \big\}\big) \leq C_1 \tau^\eta H^n(Q_j^k)$, for all $j, k \in \mathbb{Z}$ and for all $\tau \in (0, a_0)$.*

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Chr], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0,1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the AR property, the result already appears in [DS1, DS2].

- We shall denote by $\mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D}(E) := \bigcup_k \mathbb{D}_k,$$

  where, if $\operatorname{diam}(E)$ is finite, the union runs over those $k \in \mathbb{Z}$ such that $2^{-k} \lesssim \operatorname{diam}(E)$.

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. It is clear that $\ell(Q) \approx \operatorname{diam}(Q)$. Also, for $Q \in \mathbb{D}(E)$ we will set $k(Q) = k$ if $Q \in \mathbb{D}_k$.

- Properties (d) and (e) imply that for each cube $Q \in \mathbb{D}$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$ such that $c\ell(Q) \leq r_Q \leq \ell(Q)$, for some uniform constant $c > 0$, and

$$\Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, Cr_Q) \tag{1.1}$$

for some uniform constant $C > 1$. We shall denote these balls and surface balls by

$$B_Q := B(x_Q, r_Q), \qquad \Delta_Q := \Delta(x_Q, r_Q), \tag{1.2}$$

$$\widetilde{B}_Q := B(x_Q, Cr_Q), \qquad \widetilde{\Delta}_Q := \Delta(x_Q, Cr_Q), \tag{1.3}$$

and we shall refer to the point x_Q as the "center" of Q.

- Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the Corkscrew condition and such that $\partial\Omega$ is AR. Given $Q \in \mathbb{D}(\partial\Omega)$ we define the "corkscrew point relative to Q" as $X_Q := X_{\Delta_Q}$. We note that

$$\delta(X_Q) \approx \text{dist}(X_Q, Q) \approx \text{diam}(Q).$$

Following [HM3, Section 3] we next introduce the notion of "Carleson region" and "discretized sawtooth". Given a cube $Q \in \mathbb{D}(E)$, the "discretized Carleson region" $\mathbb{D}_Q$ relative to Q is defined by

$$\mathbb{D}_Q := \big\{ Q' \in \mathbb{D}(E) : Q' \subset Q \big\}.$$

Let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}(E)$ be a family of disjoint cubes. The "global discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes $Q \in \mathbb{D}(E)$ that are not contained in any $Q_i \in \mathcal{F}$, that is,

$$\mathbb{D}_\mathcal{F} := \mathbb{D}(E) \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i}.$$

For a given $Q \in \mathbb{D}(E)$, the "local discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes in $\mathbb{D}_Q$ that are not contained in any $Q_i \in \mathcal{F}$ or, equivalently,

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_i \in \mathcal{F}} \mathbb{D}_{Q_i} = \mathbb{D}_\mathcal{F} \cap \mathbb{D}_Q.$$

We also introduce the "geometric" Carleson regions and sawtooths. In the sequel, $\Omega \subset \mathbb{R}^{n+1}$ ($n \geq 2$) will be a 1-sided CAD. Given $Q \in \mathbb{D}(\partial\Omega)$ we want to define some associated regions which inherit the good properties of Ω. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of $\Omega \subset \mathbb{R}^{n+1}$, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω, which satisfy

$$4 \, \text{diam}(I) \leq \text{dist}(4I, \partial\Omega) \leq \text{dist}(I, \partial\Omega) \leq 40 \, \text{diam}(I), \qquad \forall I \in \mathcal{W}, \qquad (1.4)$$

and

$$\text{diam}(I_1) \approx \text{diam}(I_2), \ \text{ whenever } I_1 \text{ and } I_2 \text{ touch}.$$

Let $X(I)$ denote the center of I, let $\ell(I)$ denote the sidelength of I, and write $k = k_I$ if $\ell(I) = 2^{-k}$.

Given $0 < \lambda < 1$ and $I \in \mathcal{W}$ we write $I^* = (1+\lambda)I$ for the "fattening" of I. By taking λ small enough, we can arrange matters, so that, first, $\text{dist}(I^*, J^*) \approx \text{dist}(I, J)$ for every $I, J \in \mathcal{W}$, and secondly, I^* meets J^* if and only if ∂I meets ∂J (the fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathcal{W}$ whose boundaries touch, so that the Harnack Chain property then holds locally in $I^* \cup J^*$, with constants depending upon λ). By picking λ sufficiently small, say $0 < \lambda < \lambda_0$, we may also suppose that there is $\tau \in (1/2, 1)$ such that for distinct $I, J \in \mathcal{W}$, we have that $\tau J \cap I^* = \emptyset$. In what follows we will need to work with dilations $I^{**} = (1+2\lambda)I$ or $I^{***} = (1+4\lambda)I$, and in order to ensure that the same properties hold we further assume that $0 < \lambda < \lambda_0/4$.

For every $Q \in \mathbb{D}(\partial\Omega)$ we can construct a family $\mathcal{W}_Q^* \subset \mathcal{W}(\Omega)$, and define

$$U_Q := \bigcup_{I \in \mathcal{W}_Q^*} I^*,$$

satisfying the following properties: $X_Q \in U_Q$ and there are uniform constants k^* and K_0 such that

$$k(Q) - k^* \leq k_I \leq k(Q) + k^*, \quad \forall I \in \mathcal{W}_Q^*,$$

$$X(I) \to_{U_Q} X_Q, \quad \forall I \in \mathcal{W}_Q^*,$$

$$\mathrm{dist}(I, Q) \leq K_0 2^{-k(Q)}, \quad \forall I \in \mathcal{W}_Q^*.$$

Here, $X(I) \to_{U_Q} X_Q$ means that the interior of U_Q contains all balls in a Harnack Chain (in Ω) connecting $X(I)$ to X_Q, and moreover, for any point Z contained in any ball in the Harnack Chain, we have $\mathrm{dist}(Z, \partial\Omega) \approx \mathrm{dist}(Z, \Omega \setminus U_Q)$ with uniform control of the implicit constants. The constants k^*, K_0 and the implicit constants in the condition $X(I) \to_{U_Q} X_Q$, depend on at most allowable parameters and on λ. Moreover, given $I \in \mathcal{W}(\Omega)$ we have that $I \in \mathcal{W}_{Q_I}^*$, where $Q_I \in \mathbb{D}(\partial\Omega)$ satisfies $\ell(Q_I) = \ell(I)$, and contains any fixed $\widehat{y} \in \partial\Omega$ such that $\mathrm{dist}(I, \partial\Omega) = \mathrm{dist}(I, \widehat{y})$. The reader is referred to [HM3, Section 3] for full details.

For a given $Q \in \mathbb{D}(\partial\Omega)$, the "Carleson box" relative to Q is defined by

$$T_Q := \mathrm{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \right).$$

For a given family $\mathcal{F} = \{Q_i\}$ of pairwise disjoint cubes and a given $Q \in \mathbb{D}(\partial\Omega)$, we define the "local sawtooth region" relative to $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} = \mathrm{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'} \right) = \mathrm{int}\left(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^* \right), \tag{1.5}$$

where $\mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \mathcal{W}_Q^*$. Analogously, we can slightly fatten the Whitney boxes and use I^{**} to define new fattened Whitney regions and sawtooth domains. More precisely, for every $Q \in \mathbb{D}(\partial\Omega)$,

$$T_Q^* := \mathrm{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}^* \right), \qquad \Omega_{\mathcal{F},Q}^* := \mathrm{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}^* \right), \qquad U_Q^* := \bigcup_{I \in \mathcal{W}_Q^*} I^{**}.$$

Similarly, we can define T_Q^{**}, $\Omega_{\mathcal{F},Q}^{**}$ and U_Q^{**} by using I^{***} in place of I^{**}.

To define the "Carleson box" T_Δ associated to a surface ball $\Delta = \Delta(x, r)$, let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200r \leq 2^{-k}$, and set

$$\mathbb{D}^\Delta := \{Q \in \mathbb{D}_{k(\Delta)} : Q \cap 2\Delta \neq \varnothing\}.$$

We then set

$$T_\Delta := \mathrm{int}\left(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q} \right).$$

We can also consider slight dilations of T_Δ given by

$$T_\Delta^* := \mathrm{int}\left(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q^*} \right), \qquad T_\Delta^{**} := \mathrm{int}\left(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q^{**}} \right).$$

Following [HM3, Section 3], one can easily see that there exist constants $0 < \kappa_1 < 1$ and $\kappa_0 \geq 2C$ (with C the constant in (1.3)), depending only on the allowable parameters, so that

$$\kappa_1 B_Q \cap \Omega \subset T_Q \subset T_Q^* \subset T_Q^{**} \subset \overline{T_Q^{**}} \subset \kappa_0 B_Q \cap \overline{\Omega} =: \tfrac{1}{2} B_Q^* \cap \overline{\Omega}, \qquad (1.6)$$

$$\tfrac{5}{4} B_\Delta \cap \Omega \subset T_\Delta \subset T_\Delta^* \subset T_\Delta^{**} \subset \overline{T_\Delta^{**}} \subset \kappa_0 B_\Delta \cap \overline{\Omega} =: \tfrac{1}{2} B_\Delta^* \cap \overline{\Omega}, \qquad (1.7)$$

and also

$$Q \subset \kappa_0 B_\Delta \cap \partial\Omega = \tfrac{1}{2} B_\Delta^* \cap \partial\Omega =: \tfrac{1}{2}\Delta^*, \qquad \forall Q \in \mathbb{D}^\Delta, \qquad (1.8)$$

where B_Q is defined as in (1.2), $\Delta = \Delta(x, r)$ with $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and $B_\Delta = B(x, r)$ is so that $\Delta = B_\Delta \cap \partial\Omega$.

1.2 Borel measures and weights

Throughout this section, $E \subset \mathbb{R}^{n+1}$ will be an n-dimensional AR set. We first introduce the concept of ε_0-cover associated with a given regular Borel measure. This definition is based on the work of [KKPT], with some slight modifications in order to adapt it to our geometric setting and dyadic constructions.

Definition 1.7 (A good ε_0-cover). Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional AR set. Fix $Q_0 \in \mathbb{D}(E)$ and let μ be a regular Borel measure on Q_0. Given $\varepsilon_0 \in (0, 1)$ and a Borel set $F \subset Q_0$, a good ε_0-cover of F with respect to μ, of length $k \in \mathbb{N}$, is a collection $\{\mathcal{O}_\ell\}_{\ell=1}^k$ of Borel subsets of Q_0, together with pairwise disjoint families $\mathcal{F}_\ell = \{Q_i^\ell\} \subset \mathbb{D}_{Q_0}$, such that

(a) $F \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset \cdots \subset \mathcal{O}_2 \subset \mathcal{O}_1 \subset Q_0$,

(b) $\mathcal{O}_\ell = \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} Q_i^\ell, \qquad 1 \leq \ell \leq k$,

(c) $\mu(\mathcal{O}_\ell \cap Q_i^{\ell-1}) \leq \varepsilon_0\, \mu(Q_i^{\ell-1}), \qquad \forall Q_i^{\ell-1} \in \mathcal{F}_{\ell-1}, \quad 2 \leq \ell \leq k$.

Note that the third property of the above definition can be iterated to obtain a more general condition, as stated below.

Lemma 1.8. *If $\{\mathcal{O}_\ell\}_{\ell=1}^k$ is a good ε_0-cover of F with respect to μ of length $k \in \mathbb{N}$ then*

$$\mu(\mathcal{O}_\ell \cap Q_i^m) \leq \varepsilon_0^{\ell-m} \mu(Q_i^m), \qquad \forall Q_i^m \in \mathcal{F}_m, \qquad 1 \leq m \leq \ell \leq k. \qquad (1.9)$$

Proof. Fix $1 \leq \ell \leq k$ and we proceed by induction in m. If $m = \ell$ the estimate is trivial since $\mu(\mathcal{O}_\ell \cap Q_i^\ell) = \mu(Q_i^\ell)$. If $m = \ell - 1$ (in which case necessarily $\ell \geq 2$) then (1.9) follows directly from (c) in Definition 1.7. Assume next that (1.9) holds for some fixed $2 \leq m \leq \ell$ and we prove it for $m - 1$ in place of m. We first claim that for every $Q_i^{m-1} \in \mathcal{F}_{m-1}$ there holds

$$\mathcal{O}_\ell \cap Q_i^{m-1} \subset \bigcup_{\substack{Q_j^m \in \mathcal{F}_m \\ Q_j^m \subsetneq Q_i^{m-1}}} \mathcal{O}_\ell \cap Q_j^m. \qquad (1.10)$$

To see this, take $x \in \mathcal{O}_\ell \cap Q_i^{m-1} \subset \mathcal{O}_m$. Hence, there exists a unique $Q_j^m \in \mathcal{F}_m$ such that $x \in Q_j^m$ and consequently either $Q_i^{m-1} \subset Q_j^m$ or $Q_j^m \subsetneq Q_i^{m-1}$. If $Q_i^{m-1} \subset Q_j^m$ then $\mu(Q_i^{m-1}) = \mu(\mathcal{O}_m \cap Q_i^{m-1}) \le \varepsilon_0 \mu(Q_i^{m-1})$, by (c) in Definition 1.7, and this is a contradiction since $0 < \varepsilon_0 < 1$. Thus, $Q_j^m \subsetneq Q_i^{m-1}$ and (1.10) holds and

$$\mu(\mathcal{O}_\ell \cap Q_i^{m-1}) \le \sum_{\substack{Q_j^m \in \mathcal{F}_m \\ Q_j^m \subsetneq Q_i^{m-1}}} \mu(\mathcal{O}_\ell \cap Q_j^m) \le \varepsilon_0^{\ell-m} \sum_{\substack{Q_j^m \in \mathcal{F}_m \\ Q_j^m \subsetneq Q_i^{m-1}}} \mu(Q_j^m)$$

$$\le \varepsilon_0^{\ell-m} \mu(\mathcal{O}_m \cap Q_i^{m-1}) \le \varepsilon_0^{\ell-(m-1)} \mu(Q_i^{m-1}),$$

where we have applied the induction hypothesis to the Q_j^m's and the properties of the good ε_0-cover. $\blacksquare$

Definition 1.9 (**Dyadically doubling**)**.** We say that a regular Borel measure μ on $Q_0 \in \mathbb{D}(E)$ is dyadically doubling if there exists $C_\mu \ge 1$ such that $\mu(Q^*) \le C_\mu \mu(Q)$ for every $Q \in \mathbb{D}_{Q_0} \setminus \{Q_0\}$, with $Q^* \supset Q$ and $\ell(Q^*) = 2\ell(Q)$ (i.e., Q^* is the "dyadic parent" of Q). We call C_μ the dyadically doubling constant of μ.

The next lemma is also found in a different version in [KKPT]. We note that here, the subsets used to build the ε_0-cover are defined as the level sets of a given fixed function, instead of working iteratively.

Lemma 1.10. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional* AR *set and fix $Q_0 \in \mathbb{D}(E)$. Let μ be a regular Borel measure on Q_0 and assume that it is dyadically doubling on Q_0 with constant C_μ. For every $0 < \varepsilon_0 \le e^{-1}$, if $F \subset Q_0$ with $\mu(F) \le \alpha \mu(Q_0)$ and $0 < \alpha \le \varepsilon_0^2/(2C_\mu^2)$ then F has a good ε_0-cover with respect to μ of length $k_0 = k_0(\alpha, \varepsilon_0) \in \mathbb{N}$, $k_0 \ge 2$, which satisfies $k_0 \approx \frac{\log \alpha^{-1}}{\log \varepsilon_0^{-1}}$. In particular, if $\mu(F) = 0$, then F has a good ε_0-cover of arbitrary length.*

Proof. Fix ε_0, F and α as in the statement and write $a := C_\mu/\varepsilon_0 > 1$. Note that since $0 < \alpha < \varepsilon_0^2/(2C_\mu^2) = a^{-2}/2$ there is a unique $k_0 = k_0(\alpha, \varepsilon_0) \in \mathbb{N}$, $k_0 \ge 2$, such that

$$a^{-k_0-1} < 2\alpha \le a^{-k_0},$$

and our choice of ε_0 gives that

$$\frac{1}{3(1+\log C_\mu)} \frac{\log \alpha^{-1}}{\log \varepsilon_0^{-1}} \le k_0 \le \frac{\log \alpha^{-1}}{\log \varepsilon_0^{-1}}. \tag{1.11}$$

Since $\mu(F) \le \alpha \mu(Q_0)$, by outer regularity there exists a relatively open set $U \subset E$ such that $F \subset U$ and $\mu(U \setminus F) < \alpha \mu(Q_0)$. Set $\widetilde{F} := U \cap Q_0 \subset Q_0$ and define the level sets

$$\Omega_k := \left\{ x \in Q_0 : M_{\mu,Q_0}^d(\mathbf{1}_{\widetilde{F}})(x) > a^{-k} \right\}, \qquad 1 \le k \le k_0,$$

where M_{μ,Q_0}^d is the local dyadic maximal operator with respect to μ given by

$$M_{\mu,Q_0}^d f(x) := \sup_{x \in Q \in \mathbb{D}_{Q_0}} \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y), \qquad f \in L_{\mathrm{loc}}^1(Q_0, d\mu).$$

Clearly, $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{k_0} \subset Q_0$. Moreover, $\widetilde{F} \subset \Omega_1$. To see this fix $x \in \widetilde{F}$ and use that U is relatively open to find $B_x = B(x, r_x)$ with $r_x > 0$ so that $B_x \cap E \subset U$. Take next $Q_x \in \mathbb{D}$ with $Q_x \ni x$ so that $\ell(Q_x) < \ell(Q_0)$ and $\mathrm{diam}(Q_x) < r_x$. Since $x \in \widetilde{F} \cap Q_x \subset Q_x \cap Q_0$ and $\ell(Q_x) < \ell(Q_0)$ it follows that $Q_x \in \mathbb{D}_{Q_0}$. Also since $\mathrm{diam}(Q_x) < r_x$ we easily see that $Q_x \subset B_x \cap E \subset U$ and eventually we have obtained that $Q_x \subset \widetilde{F}$ which in turn gives

$$M_{\mu, Q_0}^d (\mathbf{1}_{\widetilde{F}})(x) \geq \frac{\mu(\widetilde{F} \cap Q_x)}{\mu(Q_x)} = 1 > a^{-1}.$$

Hence, $x \in \Omega_1$ as desired.

All the previous observations show that $F \subset \widetilde{F} \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{k_0} \subset Q_0$ and in particular $\Omega_k \neq \emptyset$ for every $k \geq 1$. Moreover, by our choice of k_0, we have that for every $1 \leq k \leq k_0$

$$\mu(\widetilde{F}) \leq \mu(U) \leq \mu(U \setminus F) + \mu(F) < 2\alpha\mu(Q_0) \leq a^{-k_0}\mu(Q_0) \leq a^{-k}\mu(Q_0).$$

Subdividing Q_0 dyadically we can then select a pairwise disjoint collection of cubes $\mathcal{F}_k = \{Q_i^k\} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ which are maximal with respect to the property that

$$\mu(\widetilde{F} \cap Q_i^k) > a^{-k}\mu(Q_i^k), \tag{1.12}$$

and also $\Omega_k = \bigcup_{Q_i^k \in \mathcal{F}_k} Q_i^k$ (note that $\mathcal{F}_k \neq \emptyset$ since $\Omega_k \neq \emptyset$). By the maximality of the selected cubes we obtain that

$$\frac{\mu(\widetilde{F} \cap Q_i^k)}{\mu(Q_i^k)} \leq C_\mu \frac{\mu(\widetilde{F} \cap (Q_i^k)^*)}{\mu((Q_i^k)^*)} \leq C_\mu a^{-k}, \tag{1.13}$$

where $(Q_i^k)^*$ is the dyadic parent of Q_i^k.

Next we claim that for each $Q_j^{k+1} \in \mathcal{F}_{k+1}$ we have that $\mu(\Omega_k \cap Q_j^{k+1}) \leq \varepsilon_0 \mu(Q_j^{k+1})$. To see this we first observe that if $Q_i^k \cap Q_j^{k+1} \neq \emptyset$, then necessarily $Q_i^k \subset Q_j^{k+1}$, for otherwise $Q_j^{k+1} \subsetneq Q_i^k$ and by the maximality of the cube Q_j^{k+1} and (1.12) we would have that $a^{-k}\mu(Q_i^k) < \mu(\widetilde{F} \cap Q_i^k) \leq a^{-k-1}\mu(Q_i^k)$, which leads to a contradiction since $a > 1$. Hence, $Q_i^k \subset Q_j^{k+1}$ whenever $Q_i^k \cap Q_j^{k+1} \neq \emptyset$. Using this, (1.12), and (1.13) (for Q_j^{k+1} and $k+1$ replacing Q_i^k and k respectively), we have that

$$\mu(\Omega_k \cap Q_j^{k+1}) = \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(Q_i^k \cap Q_j^{k+1}) = \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(Q_i^k)$$

$$< a^k \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(\widetilde{F} \cap Q_i^k) \leq a^k \mu(\widetilde{F} \cap Q_j^{k+1}) \leq a^{-1} C_\mu \mu(Q_j^{k+1}) = \varepsilon_0 \mu(Q_j^{k+1}),$$

and this proves the claim.

To complete the proof of the lemma we define $\mathcal{O}_k := \Omega_{k_0 - k + 1}$ and note that the sets $\{\mathcal{O}_k\}_{k=1}^{k_0}$ form a good ε_0-cover of F, with respect to μ, of length k_0 which satisfies (1.11). Finally we observe that if $\mu(F) = 0$, then α can be taken arbitrarily small, hence k_0, the length of the good ε_0-cover of F, can be taken as large as desired by (1.11). $\blacksquare$

In what follows, we will write $\sigma = H^n|_E$ to denote the surface measure. Next, we will recall some basic facts about Muckenhoupt weights, or more precisely the A_∞ and RH_p conditions. These can be seen as quantitative scale invariant versions of the absolute continuity with respect to σ.

Definition 1.11 (A_∞ **and** $A_\infty^{\mathrm{dyadic}}$)**.** Given a surface ball $\Delta_0 = B_0 \cap E$, with $B_0 = B(x_0, r_0)$, $x_0 \in E$, $0 < r < \mathrm{diam}(E)$, a regular Borel measure ω defined on Δ_0 is said to belong to $A_\infty(\Delta_0)$ if there exist constants $0 < \alpha, \beta < 1$ such that for every surface ball $\Delta = B \cap E$ centered at E with $B \subset B_0$, and for every Borel set $F \subset \Delta$, we have that

$$\frac{\omega(F)}{\omega(\Delta)} \le \alpha \implies \frac{\sigma(F)}{\sigma(\Delta)} \le \beta.$$

Given $Q_0 \in \mathbb{D}(E)$, a regular Borel measure ω defined on Q_0 is said to belong to $A_\infty^{\mathrm{dyadic}}(Q_0)$ if there exist constants $0 < \alpha, \beta < 1$ such that for every $Q \in \mathbb{D}_{Q_0}$ and for every Borel set $F \subset Q$, we have that

$$\frac{\omega(F)}{\omega(Q)} \le \alpha \implies \frac{\sigma(F)}{\sigma(Q)} \le \beta.$$

Suppose further that ω is dyadically doubling, it is well known (see [GR], [CF], [HM3]) that since σ is a doubling measure (recall that E satisfies the AR condition), $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$ if and only if $\omega \ll \sigma$ in Q_0 and there exists $1 < p < \infty$ such that $\omega \in RH_p^{\mathrm{dyadic}}(Q_0)$, that is, there is a constant $C > 1$ such that

$$\left(\fint_Q k(x)^p \, d\sigma(x) \right)^{\frac{1}{p}} \le C \fint_Q k(x) \, d\sigma(x),$$

for every $Q \in \mathbb{D}_{Q_0}$, where $k = d\omega/d\sigma$ is the Radon-Nikodym derivative. In fact, $A_\infty^{\mathrm{dyadic}}$ defines an equivalence relationship between dyadically doubling measures. Indeed, in particular $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$ if and only if $\omega \ll \sigma$ in Q_0 and there exist constants $C_1 > 1$, $\theta_1, \theta_2 > 0$ such that

$$C_1^{-1} \left(\frac{\sigma(F)}{\sigma(Q)} \right)^{\theta_1} \le \frac{\omega(F)}{\omega(Q)} \le C_1 \left(\frac{\sigma(F)}{\sigma(Q)} \right)^{\theta_2},$$

for every $Q \in \mathbb{D}_{Q_0}$ and for every Borel set $F \subset Q$.

Definition 1.12 (**The projection operator**)**.** Fix $Q_0 \in \mathbb{D}(E)$. For each $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$, a family of pairwise disjoint dyadic cubes, and each f locally integrable, we define

$$\mathcal{P}_\mathcal{F} f(x) = f(x) \mathbf{1}_{E \setminus \left(\bigcup_i Q_i \right)}(x) + \sum_{Q_i \in \mathcal{F}} \left(\fint_{Q_i} f(y) \, d\sigma(y) \right) \mathbf{1}_{Q_i}(x).$$

If ω is a non-negative regular Borel measure on Q_0, we may naturally then define the measure $\mathcal{P}_\mathcal{F} \omega$ as $\mathcal{P}_\mathcal{F} \omega(F) = \int_E \mathcal{P}_\mathcal{F} \mathbf{1}_F \, d\omega$, that is,

$$\mathcal{P}_\mathcal{F} \omega(F) = \omega \left(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i \right) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(F \cap Q_i)}{\sigma(Q_i)} \omega(Q_i), \tag{1.14}$$

for each Borel set $F \subset Q_0$.

The next result follows easily by combining the arguments in [HM3, Lemma B.1] and [HM1, Lemma 4.1]

Lemma 1.13. *Let ω be a non-negative regular Borel measure on $Q_0 \in \mathbb{D}(E)$.*

(a) *If ω is dyadically doubling on Q_0 then $\mathcal{P}_{\mathcal{F}}\omega$ is dyadically doubling on Q_0.*

(b) *If $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$ then $\mathcal{P}_{\mathcal{F}}\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$.*

The following result shows that given an absolute continuous measure μ with respect to σ, under suitable quantitative conditions we can extract a sawtooth that has an ample contact with the original domain, such that μ and σ are comparable on the cubes above the sawtooth.

Lemma 1.14 ([HMT1, Lemma 3.5])**.** *Let μ be a non-negative regular Borel measure on $Q_0 \in \mathbb{D}(E)$. Assume that $\mu \ll \sigma$ on Q_0, and also that there exist $K_0 \geq 1$, $\theta > 0$ such that*

$$1 \leq \frac{\mu(Q_0)}{\sigma(Q_0)} \leq K_0; \qquad \frac{\mu(F)}{\sigma(Q_0)} \leq K_0 \left(\frac{\sigma(F)}{\sigma(Q_0)} \right)^\theta, \quad \forall F \subseteq Q_0. \tag{1.15}$$

Then, there exists a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subseteq \mathbb{D}_{Q_0} \setminus \{Q_0\}$ such that

$$\sigma \left(Q_0 \setminus \bigcup_{Q_j \in \mathcal{F}} Q_j \right) \geq K_1^{-1} \sigma(Q_0) \tag{1.16}$$

and

$$\frac{1}{2} \leq \frac{\mu(Q)}{\sigma(Q)} \leq K_0 K_1, \qquad \forall Q \in \mathbb{D}_{\mathcal{F}, Q_0}, \tag{1.17}$$

where $K_1 = (4K_0)^{1/\theta}$.

Note that, under the hypothesis of Lemma 1.14, the second condition in (1.15) can be seen as a consequence of the fact that $\mu \in A_\infty^{\mathrm{dyadic}}(Q_0)$. This, when combined with Lemma 1.17, will be a useful tool in order to prove that a certain measure satisfies further properties, as we will discuss in the following section.

1.3 Discrete Carleson measures

Let $\{\gamma_Q\}_{Q \in \mathbb{D}(E)}$ be a sequence of non-negative real numbers. We define the "measure" $\mathfrak{m}$ (acting on collections of dyadic cubes) by

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \gamma_Q, \qquad \mathbb{D}' \subset \mathbb{D}(E). \tag{1.18}$$

Definition 1.15 (Discrete Carleson measure). Let $Q_0 \in \mathbb{D}(E)$, we say that $\mathfrak{m}$ is a discrete "Carleson measure" on Q_0 (with respect to σ) or, equivalently, $\mathfrak{m} \in \mathcal{C}(Q_0)$ if

$$\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty. \tag{1.19}$$

In a similar way, we will say that $\mathfrak{m}$ is a discrete Carleson measure on E or, equivalently, $\mathfrak{m} \in \mathcal{C}(E)$ if

$$\|\mathfrak{m}\|_{\mathcal{C}(E)} := \sup_{Q \in \mathbb{D}(E)} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty. \qquad (1.20)$$

Usually, in order to simplify the notation, we will refer to these measures just as Carleson measures on Q_0 or E, respectively.

Definition 1.16 (**Restriction of a measure**)**.** Let $\{\gamma_Q\}_{Q \in \mathbb{D}(E)}$ be a sequence of non-negative real numbers and consider the measure $\mathfrak{m}$ defined in (1.18). Given $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$, a family of pairwise disjoint dyadic cubes, we define $\mathfrak{m}_{\mathcal{F}}$ by

$$\mathfrak{m}_{\mathcal{F}}(\mathbb{D}') = \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q \in \mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}} \gamma_Q, \qquad \mathbb{D}' \subset \mathbb{D}_{Q_0}.$$

Equivalently, the measure $\mathfrak{m}_{\mathcal{F}}$ is given by the sequence $\{\gamma_{\mathcal{F},Q}\}_{Q \in \mathbb{D}_{Q_0}}$, where

$$\gamma_{\mathcal{F},Q} = \begin{cases} \gamma_Q & \text{if } Q \in \mathbb{D}_{\mathcal{F},Q_0}, \\ 0 & \text{if } Q \in \mathbb{D}_{Q_0} \setminus \mathbb{D}_{\mathcal{F},Q_0}. \end{cases} \qquad (1.21)$$

The next result establishes that in order to show that $\mathfrak{m}$ is a Carleson measure on E, it suffices to work locally on each Q_0 and check that $\mathfrak{m}_{\mathcal{F}}$ is a Carleson measure on Q_0, whenever the sawtooth $\mathbb{D}_{\mathcal{F},Q_0}$ has an ample contact with the domain, in the sense of (1.16).

Lemma 1.17 ([HMT1, Lemma 3.12])**.** *Let $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}(E)}$ be a sequence of non-negative numbers and consider $\mathfrak{m}$ as defined above. Given $M_1 > 0$ and $K_1 \geq 1$, we assume that for every $Q_0 \in \mathbb{D}(E)$ there exists a pairwise disjoint family $\mathcal{F}_{Q_0} = \{Q_j\} \subseteq \mathbb{D}_{Q_0} \setminus \{Q_0\}$ such that*

$$\sigma\Big(Q_0 \setminus \bigcup_{Q_j \in \mathcal{F}_{Q_0}} Q_j\Big) \geq K_1^{-1}\sigma(Q_0) \qquad (1.22)$$

and

$$\mathfrak{m}(\mathbb{D}_{\mathcal{F}_{Q_0},Q_0}) \leq M_1\sigma(Q_0). \qquad (1.23)$$

Then, $\mathfrak{m}$ is a Carleson measure on E and moreover

$$\|\mathfrak{m}\|_{\mathcal{C}(E)} = \sup_{Q \in \mathbb{D}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} \leq K_1 M_1.$$

Part of the proof of Theorems 3.1, 3.2, or 3.3, will rely heavily on Lemmas 1.14 and 1.17, as we will see in Chapter 3. We have already shown that the use of the $A_\infty^{\mathrm{dyadic}}$ condition may help us to prove that certain measure is indeed a discrete Carleson measure. Next, we state a powerful result in the opposite direction, that will be essential in the proof of Theorem 2.1. This is, we use an auxiliar Carleson measure $\mathfrak{m}$ to prove that a given weight $\omega \in A_\infty^{\mathrm{dyadic}}$, reducing the work to show that for every disjoint family $\mathcal{F}$ for which the restriction $\mathfrak{m}_{\mathcal{F}}$ has small Carleson norm, the projection $\mathcal{P}_{\mathcal{F}}\omega$ satisfies an $A_\infty^{\mathrm{dyadic}}$-type condition. The precise statement is as follows.

Lemma 1.18 ([HM3, Lemma 8.5]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional* AR. *Fix $Q_0 \in \mathbb{D}(E)$, let σ, ω be a pair of non-negative dyadically doubling regular Borel measures on Q_0, and let $\mathfrak{m}$ be a discrete Carleson measure with respect to σ, with*

$$\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \leq M_0.$$

Suppose that there exists $\gamma > 0$ such that for every $Q \in \mathbb{D}_{Q_0}$ and every family of pairwise disjoint dyadic cubes $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_Q$ verifying

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} = \sup_{Q' \in \mathbb{D}_Q} \frac{\mathfrak{m}(\mathbb{D}_{\mathcal{F}, Q'})}{\sigma(Q')} \leq \gamma,$$

we have that $\mathcal{P}_{\mathcal{F}}\omega$ satisfies the following property:

$$\forall \varepsilon \in (0, 1) \quad \exists C_\varepsilon > 1 \text{ such that } \left(F \subset Q, \quad \frac{\sigma(F)}{\sigma(Q)} \geq \varepsilon \implies \frac{\mathcal{P}_{\mathcal{F}}\omega(F)}{\mathcal{P}_{\mathcal{F}}\omega(Q)} \geq \frac{1}{C_\varepsilon} \right).$$

Then, there exist $\eta_0 \in (0, 1)$ and $C_0 < \infty$ such that, for every $Q \in \mathbb{D}_{Q_0}$

$$F \subset Q, \quad \frac{\sigma(F)}{\sigma(Q)} \geq 1 - \eta_0 \implies \frac{\omega(F)}{\omega(Q)} \geq \frac{1}{C_0}.$$

In other words, $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$.

Finally, we will show a discrete localized version of [CMS, Theorem 1] adapted to our geometric setting. Fix $Q_0 \in \mathbb{D}(E)$ and consider the operators $\mathcal{A}_{Q_0}$, $\mathfrak{C}_{Q_0}$ defined by

$$\mathcal{A}_{Q_0}\alpha(x) := \left(\sum_{x \in Q \in \mathbb{D}_{Q_0}} \frac{1}{\ell(Q)^n} \alpha_Q^2 \right)^{1/2}, \quad \mathfrak{C}_{Q_0}\alpha(x) := \sup_{x \in Q \in \mathbb{D}_{Q_0}} \left(\frac{1}{\sigma(Q)} \sum_{Q' \in \mathbb{D}_Q} \alpha_{Q'}^2 \right)^{1/2}, \tag{1.24}$$

where $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q_0}}$ is a sequence of real numbers. Note that these operators are discrete analogues of the area functional and Carleson operator used in [CMS] to develop the theory of tent spaces. Sometimes, we use a truncated version of $\mathcal{A}_{Q_0}$, defined for each $k \geq 0$ by

$$\mathcal{A}_{Q_0}^k\alpha(x) := \left(\sum_{x \in Q \in \mathbb{D}_{Q_0}^k} \frac{1}{\ell(Q)^n} \alpha_Q^2 \right)^{1/2},$$

where $\mathbb{D}_{Q_0}^k$ is the collection of $Q \in \mathbb{D}_{Q_0}$ such that $\ell(Q) \leq 2^{-k}\ell(Q_0)$.

Lemma 1.19. *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional* AR, *fix $Q_0 \in \mathbb{D}(E)$, let $\mathcal{A}_{Q_0}$ and $\mathfrak{C}_{Q_0}$ be the operators defined in (1.24) respectively. There exists C, depending only on dimension and the* AR *constant, such that for every $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q_0}}$, $\beta = \{\beta_Q\}_{Q \in \mathbb{D}_{Q_0}}$ sequences of real numbers, we have that*

$$\sum_{Q \in \mathbb{D}_{Q_0}} |\alpha_Q \beta_Q| \leq C \int_{Q_0} \mathcal{A}_{Q_0}\alpha(x)\mathfrak{C}_{Q_0}\beta(x)\, d\sigma(x). \tag{1.25}$$

Proof. We first claim that it suffices to consider the case on which $\beta_Q = 0$ when $\ell(Q) \leq 2^{-N}\ell(Q_0)$ for some $N \in \mathbb{N}$, and in that scenario, we establish (1.25) with C independent of N. To obtain the general case, for every $N \geq 1$, we let $\beta^N = \{\beta_Q^N\}_{Q \in \mathbb{D}_{Q_0}}$ where $\beta_Q^N = \beta_Q$ if $2^{-N}\ell(Q_0) < \ell(Q) \leq \ell(Q_0)$ and $\beta_Q^N = 0$ when $\ell(Q) \leq 2^{-N}\ell(Q_0)$. Then by our claim, (1.25) holds for β^N with C independent of N. Observing that $\mathfrak{C}_{Q_0}\beta^N \leq \mathfrak{C}_{Q_0}\beta$ we just need to let $N \to \infty$ and the desired estimate follows at once.

Let us then show our claim. Fix β so that $\beta_Q = 0$ when $\ell(Q) \leq 2^{-N}\ell(Q_0)$ for some $N \in \mathbb{N}$. For $Q \in \mathbb{D}_{Q_0}$, let $k_Q \geq 0$ be so that $\ell(Q) = 2^{-k_Q}\ell(Q_0)$. Suppose that $Q' \in \mathbb{D}_{Q_0}$ satisfies $\ell(Q') \leq 2^{-k_Q}\ell(Q_0) = \ell(Q)$ and $Q' \cap Q \neq \emptyset$, then necessarily $Q' \in \mathbb{D}_Q$. Therefore, using the AR property we obtain

$$\int_Q \left(\mathcal{A}_{Q_0}^{k_Q}\beta(y)\right)^2 d\sigma(y) = \int_Q \sum_{Q' \in \mathbb{D}_Q} \mathbf{1}_{Q'}(y) \frac{1}{\ell(Q')^n} \beta_{Q'}^2 \, d\sigma(y) \lesssim \sum_{Q' \in \mathbb{D}_Q} \beta_{Q'}^2.$$

Dividing both sides by $\sigma(Q)$, we have proved that for every $Q \in \mathbb{D}_{Q_0}$ and every $x \in Q$ we have that

$$\eta_Q := \fint_Q \left(\mathcal{A}_{Q_0}^{k_Q}\beta(y)\right)^2 d\sigma(y) \leq C_0 \left(\mathfrak{C}_{Q_0}\beta(x)\right)^2, \tag{1.26}$$

with C_0 depending only on the AR constant. Since $\beta_Q = 0$ for $\ell(Q) \leq 2^{-N}\ell(Q_0)$, we have that $\mathcal{A}_{Q_0}\beta(x) \leq C(N) < \infty$ and hence $\eta_Q \leq C(N)^2 < \infty$. Now, we set $C_1 := 2\sqrt{C_0}$ and define

$$F_0 := \left\{x \in Q_0 : \mathcal{A}_{Q_0}^k\beta(x) > C_1\mathfrak{C}_{Q_0}\beta(x), \, \forall k \geq 0\right\}.$$

In particular, using (1.26), we have $\mathcal{A}_{Q_0}^{k_Q}\beta(x) > 2\eta_Q^{1/2}$ for each $x \in Q \cap F_0$. We claim that $4\sigma(Q \cap F_0) \leq \sigma(Q)$. Indeed, if $\eta_Q = 0$ then one can see that $\mathcal{A}_{Q_0}^{k_Q}\beta(y) = 0$ for every $y \in Q$ and hence $Q \cap F_0 = \emptyset$, which trivially gives that $4\sigma(Q \cap F_0) \leq \sigma(Q)$. On the other hand, if $\eta_Q > 0$, we have

$$4\eta_Q\sigma(Q \cap F_0) \leq \int_{Q \cap F_0} \left(\mathcal{A}_{Q_0}^{k_Q}\beta(y)\right)^2 d\sigma(y) \leq \eta_Q\sigma(Q),$$

and the desired estimate follows since $0 < \eta_Q < \infty$. Let us now consider

$$k(x) := \min\left\{k \geq 0 : \mathcal{A}_{Q_0}^k\beta(x) \leq C_1\mathfrak{C}_{Q_0}\beta(x)\right\}, \qquad x \in Q_0 \setminus F_0. \tag{1.27}$$

Setting $F_{1,Q} := \{x \in Q \setminus F_0 : k(x) > k_Q\}$ and using (1.26) we obtain

$$F_{1,Q} \subset \{x \in Q \setminus F_0 : \mathcal{A}_{Q_0}^{k_Q}\beta(x) > 2\eta_Q^{1/2}\}.$$

Applying Chebychev's inequality, it follows that

$$\sigma(F_{1,Q}) \leq \frac{1}{4\eta_Q} \int_{Q \setminus F_0} \left(\mathcal{A}_{Q_0}^{k_Q}\beta(y)\right)^2 d\sigma(y) \leq \frac{1}{4}\sigma(Q).$$

Setting $F_{2,Q} := \{x \in Q \setminus F_0 : k(x) \leq k_Q\}$, and gathering the above estimates, we have

$$\sigma(F_{2,Q}) = \sigma(Q) - \sigma(Q \cap F_0) - \sigma(F_{1,Q}) \geq \frac{1}{2}\sigma(Q).$$

Hence, the AR property, Cauchy-Schwarz's inequality and (1.27) yield

$$
\sum_{Q \in \mathbb{D}_{Q_0}} |\alpha_Q \beta_Q| \lesssim \sum_{Q \in \mathbb{D}_{Q_0}} \sigma(F_{2,Q}) \frac{|\alpha_Q \beta_Q|}{\ell(Q)^n} \leq \int_{Q_0 \setminus F_0} \sum_{Q \in \mathbb{D}_{Q_0}} \frac{|\alpha_Q \beta_Q|}{\ell(Q)^n} \mathbf{1}_{F_{2,Q}}(x) \, d\sigma(x)
$$

$$
\lesssim \int_{Q_0 \setminus F_0} \mathcal{A}_{Q_0} \alpha(x) \left(\sum_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\ell(Q)^n} \beta_Q^2 \mathbf{1}_{F_{2,Q}}(x) \right)^{1/2} d\sigma(x)
$$

$$
\lesssim \int_{Q_0 \setminus F_0} \mathcal{A}_{Q_0} \alpha(x) \mathcal{A}_{Q_0}^{k(x)} \beta(x) \, d\sigma(x)
$$

$$
\lesssim \int_{Q_0} \mathcal{A}_{Q_0} \alpha(x) \mathfrak{C}_{Q_0} \beta(x) \, d\sigma(x),
$$

where we have used that $Q \in \mathbb{D}_{Q_0}^{k(x)}$ for each $x \in F_{2,Q}$. As the implicit constant does not depend on $N \in \mathbb{N}$, this completes the proof of (1.25). ∎

1.4 PDE estimates

In this section we will begin by summarizing some of the basic facts in the theory of elliptic partial differential equations. First, we assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set, we define the elliptic operators that will be considered in the text, as well as weak solutions and interior estimates. The reader is referred to the book of [Ken] for further details concerning this topic.

Definition 1.20 (Elliptic operator). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, we say that $Lu = -\operatorname{div}(A\nabla u)$ is a variable coefficient second order divergence form elliptic operator in Ω if $A(X) = (a_{i,j}(X))_{i,j=1}^{n+1}$ is a real (not necessarily symmetric) matrix with $a_{i,j} \in L^\infty(\Omega)$ for $1 \leq i, j \leq n+1$, and A uniformly elliptic, that is, there exists $\Lambda \geq 1$ such that

$$
\Lambda^{-1}|\xi|^2 \leq A(X)\xi \cdot \xi, \qquad |A(X)\xi \cdot \zeta| \leq \Lambda|\xi||\zeta|, \tag{1.28}
$$

for all $\xi, \zeta \in \mathbb{R}^{n+1}$ and almost every $X \in \Omega$.

In what follows we will only be working with this kind of operators, we will refer to them as "elliptic operators" for the sake of simplicity. We write $L^\top$ to denote the transpose of L, or, in other words, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A.

Definition 1.21 (The spaces $W^{1,2}$, $W^{1,2}_{\mathrm{loc}}$ and $W^{1,2}_0$). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, we say that $u \in W^{1,2}(\Omega)$ if $u \in L^2(\Omega)$ is such that the weak gradient ∇u exists, and

$$
\|u\|_{W^{1,2}(\Omega)} := \left(\iint_\Omega |u(Y)|^2 \, dY \right)^{1/2} + \left(\iint_\Omega |\nabla u(Y)|^2 \, dY \right)^{1/2} < \infty.
$$

Also, we say that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ if $u \in W^{1,2}(U)$ for every relatively compact subset $U \subset \Omega$. Finally, we define the space $W^{1,2}_0(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to $W^{1,2}(\Omega)$.

Definition 1.22 (**Weak solution**). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and $Lu = -\operatorname{div}(A\nabla u)$ an elliptic operator in Ω. We say that a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak solution of $Lu = 0$ in Ω, or that $Lu = 0$ in the weak sense, if

$$\iint_\Omega A(X)\nabla u(X) \cdot \nabla\varphi(X)\, dX = 0, \qquad \forall \varphi \in C^\infty_c(\Omega).$$

Lemma 1.23 (**Interior estimates**, [Ken]). *Given an elliptic operator L, there exists a constant $C > 1$ depending only on dimension and ellipticity such that for every ball $B(X, r) \subset \mathbb{R}^{n+1}$ and every positive weak solution $u \in W^{1,2}(B(X, 2r))$ of $Lu = 0$ in $B(X, 2r)$ we have the following:*

(a) (**Caccioppoli's estimate**)

$$\iint_{B(X,r)} |\nabla u(Y)|^2\, dY \leq C r^{-2} \iint_{B\left(X, \frac{3r}{2}\right)} |u(Y)|^2\, dY. \tag{1.29}$$

(b) (**De Giorgi-Nash-Moser's estimate**)

$$\sup_{Y \in B(X,r)} u(Y) \leq C \left(\iint_{B\left(X, \frac{3r}{2}\right)} |u(Y)|^2\, dY \right)^{1/2}. \tag{1.30}$$

(c) (**Harnack's inequality**)

$$\sup_{Y \in B(X,r)} u(Y) \leq C \inf_{Y \in B(X,r)} u(Y). \tag{1.31}$$

The use of De Giorgi-Nash-Moser's estimate allows us to show that weak solutions are actually Hölder continuous, as seen in [Ken]. Associated with L and $L^\top$ one can respectively construct the elliptic measures $\{\omega_L^X\}_{X \in \Omega}$ and $\{\omega_{L^\top}^X\}_{X \in \Omega}$, and the Green functions G_L and $G_{L^\top}$ (see [HMT2] for full details). Next we will add the assumption that the boundary of Ω satisfies the AR property.

Definition 1.24 (**The space $H^{1/2}$**). Given $E \subseteq \mathbb{R}^{n+1}$ and n-dimensional AR set, let $H^{1/2}(E)$ be the set of functions $f \in L^2(E)$ such that

$$\|f\|_{H^{1/2}(E)} := \|f\|_{L^2(E)} + \left(\int_E \int_E \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}}\, d\sigma(x)\, d\sigma(y) \right)^{1/2} < \infty.$$

Lemma 1.25 (**Existence of elliptic measure**, [HMT2]). *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set such that $\partial\Omega$ satisfies the AR property, and let L be an elliptic operator. There exists a family of regular Borel measures $\omega_L = \{\omega_L^X\}_{X \in \Omega}$, called the L-elliptic measure, such that each measure has total mass at most 1 (i.e. $\omega_L^X(\partial\Omega) \leq 1$ for every $X \in \Omega$) and for every $f \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$,*

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega \tag{1.32}$$

is a weak solution in Ω of the Dirichlet problem with datum h in $\partial\Omega$. By the latter we mean that $u \in W^{1,2}_{\text{loc}}(\Omega)$ and $Lu = 0$ in the weak sense. The weak solution u satisfies the maximum principle

$$\inf_{\partial\Omega} f \le \inf_{\Omega} u \le \sup_{\Omega} u \le \sup_{\partial\Omega} f, \qquad \sup_{\Omega} |u| \le \sup_{\partial\Omega} |f|. \tag{1.33}$$

Moreover, if $f \in H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$ then $u \in W^{1,2}(\Omega)$ verifies

$$\|u\|_{W^{1,2}(\Omega)} \le C\|f\|_{H^{1/2}(\partial\Omega)}, \tag{1.34}$$

where $C > 0$ depends only on dimension, on the AR constant and on ellipticity.

Lemma 1.26 (**Bourgain's estimate**, [HMT2]). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set such that $\partial\Omega$ satisfies the AR property. Let L be an elliptic operator, there exist constants $c_1 < 1$ and $C_1 > 1$ (depending only on the AR constant and on the ellipticity of L) such that for every $x \in \partial\Omega$ and every $0 < r < \text{diam}(\partial\Omega)$, we have*

$$\omega_L^Y(\Delta(x,r)) \ge \frac{1}{C_1}, \qquad \forall Y \in B(x, c_1 r) \cap \Omega.$$

We refer the reader to [Bou, Lemma 1] for the proof in the harmonic case and to [HMT2] for general elliptic operators. See also [HKM, Theorem 6.18] and [Zha, Section 3]. A proof of the following two lemmas may be found in [HMT2]. We note that, in particular, the AR hypothesis implies that $\partial\Omega$ satisfies the Capacity Density Condition, hence $\partial\Omega$ is Wiener regular at every point (see [HLMN, Lemma 3.27]).

Lemma 1.27 (**Hölder continuity at the boundary**, [HMT2]). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set such that $\partial\Omega$ satisfies the AR property. Let L be an elliptic operator, there exist C, $0 < \gamma \le 1$ (depending only on dimension, the AR constants and the ellipticity of L), such that for every $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$, $0 < r_0 < \text{diam}(\partial\Omega)$, and $\Delta_0 = B_0 \cap \partial\Omega$, if $0 \le u \in W^{1,2}_{\text{loc}}(B_0 \cap \Omega) \cap C(\overline{B_0 \cap \Omega})$ is a weak solution of $Lu = 0$ in $B_0 \cap \Omega$ such that $u \equiv 0$ in Δ_0, then*

$$u(X) \le C \left(\frac{|X - x_0|}{r_0} \right)^\gamma \sup_{Y \in \overline{B_0 \cap \Omega}} u(Y), \qquad \forall X \in B_0 \cap \Omega.$$

Lemma 1.28 (**The Green function**, [HMT2]). *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set such that $\partial\Omega$ satisfies the AR property. Given an elliptic operator L, there exist $C > 1$ (depending only on dimension and on the ellipticity of L) and $c_\theta > 0$ (depending on the above parameters and on $\theta \in (0,1)$) such that G_L, the Green function associated with L, satisfies*

$$G_L(X, Y) \le C|X - Y|^{1-n}; \tag{1.35}$$

$$c_\theta |X - Y|^{1-n} \le G_L(X, Y), \quad \text{if } |X - Y| \le \theta\delta(X), \quad \theta \in (0,1); \tag{1.36}$$

$$G_L(\cdot, Y) \in C(\overline{\Omega} \setminus \{Y\}) \quad \text{and} \quad G_L(\cdot, Y)|_{\partial\Omega} \equiv 0 \quad \forall Y \in \Omega; \tag{1.37}$$

$$G_L(X, Y) \ge 0, \quad \forall X, Y \in \Omega, \quad X \ne Y; \tag{1.38}$$

$$G_L(X, Y) = G_{L^\top}(Y, X), \quad \forall X, Y \in \Omega, \quad X \ne Y. \tag{1.39}$$

Moreover, $G_L(\cdot, Y) \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{Y\})$ for every $Y \in \Omega$, and satisfies $LG_L(\cdot, Y) = \delta_Y$ in the weak sense in Ω, that is,

$$\int_\Omega A(X) \nabla_X G_L(X, Y) \cdot \nabla \varphi(X) \, dX = \varphi(Y), \quad \forall \, \varphi \in C_c^\infty(\Omega). \tag{1.40}$$

Remark 1.29. If we also assume that Ω is bounded, following [HMT2] we know that the Green function G_L coincides with the one constructed in [GW]. Consequently, for each $Y \in \Omega$ and $0 < r < \delta(Y)$, there holds

$$G_L(\cdot, Y) \in W^{1,2}(\Omega \setminus B(Y, r)). \tag{1.41}$$

Moreover, for every $\varphi \in C_c^\infty(\Omega)$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $B(Y, r)$ with $0 < r < \delta(Y)$, we have that

$$(1 - \varphi) G_L(\cdot, Y) \in W_0^{1,2}(\Omega). \tag{1.42}$$

The next lemma is a collection of estimates, which together with Proposition 1.35, are part of the Jerison and Kenig's program developed in [HMT2] for 1-sided CAD domains. These tools were first introduced in the setting of CAD domains in [JK] (see also [Ken]).

Lemma 1.30 ([HMT2])**.** *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided CAD. Let L and L_1 be elliptic operators, there exist $C_1 \ge 1$ (depending only on dimension, the 1-sided CAD constants and the ellipticity of L) and $C_2 \ge 1$ (depending on the above parameters and on the ellipticity of L_1), such that for every $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$, $0 < r_0 < \mathrm{diam}(\partial\Omega)$, and $\Delta_0 = B_0 \cap \partial\Omega$ we have the following properties:*

(a) (**CFMS estimate**) *If $B = B(x, r)$ with $x \in \partial\Omega$ and $\Delta = B \cap \partial\Omega$ is such that $2B \subset B_0$, then for all $X \in \Omega \setminus B_0$ we have that*

$$\frac{1}{C_1} \omega_L^X(\Delta) \le r^{n-1} G_L(X, X_\Delta) \le C_1 \omega_L^X(\Delta).$$

(b) (**Doubling**) *If $X \in \Omega \setminus 4B_0$, then*

$$\omega_L^X(2\Delta_0) \le C_1 \omega_L^X(\Delta_0).$$

(c) (**Change of pole**) *If $B = B(x, r)$ with $x \in \partial\Omega$ and $\Delta := B \cap \partial\Omega$ is such that $B \subset B_0$, then for every $X \in \Omega \setminus 2\kappa_0 B_0$ with κ_0 as in (1.7), we have that*

$$\frac{1}{C_1} \omega_L^{X_{\Delta_0}}(\Delta) \le \frac{\omega_L^X(\Delta)}{\omega_L^X(\Delta_0)} \le C_1 \omega_L^{X_{\Delta_0}}(\Delta).$$

Moreover, if we also suppose that $\omega_L \ll \sigma$, then

$$\frac{1}{C_1} k_L^{X_{\Delta_0}}(y) \le \frac{k_L^X(y)}{\omega_L^X(\Delta_0)} \le C_1 k_L^{X_{\Delta_0}}(y), \quad \text{for } \sigma\text{-a.e. } y \in \Delta_0.$$

(d) (**Comparison principle 1**) *If $B = B(x, r)$ with $x \in \Delta_0$, $0 < r < r_0/4$ and $\Delta = B \cap \partial\Omega$, then we have that*

$$\frac{1}{C_1} \omega_{L,\Omega}^{X_\Delta}(F) \leq \omega_{L,T_{\Delta_0}}^{X_\Delta}(F) \leq C_1 \omega_{L,\Omega}^{X_\Delta}(F), \quad \text{for every Borel set } F \subset \Delta.$$

This implies that $\omega_{L,\Omega} \ll \sigma$ in Δ if and only if $\omega_{L,T_{\Delta_0}} \ll \sigma$ in Δ and, in such a case,

$$\frac{1}{C_1} k_{L,\Omega}^{X_\Delta}(y) \leq k_{L,T_{\Delta_0}}^{X_\Delta}(y) \leq C_1 k_{L,\Omega}^{X_\Delta}(y), \quad \text{for } \sigma\text{-a.e. } y \in \Delta.$$

(e) (**Comparison principle 2**)) *If $L \equiv L_1$ in $B(x_0, 2\kappa_0 r_0) \cap \Omega$ with κ_0 as in (1.7), then*

$$\frac{1}{C_2} \omega_{L_1}^{X_{\Delta_0}}(F) \leq \omega_L^{X_{\Delta_0}}(F) \leq C_2 \omega_{L_1}^{X_{\Delta_0}}(F), \quad \text{for every Borel set } F \subset \Delta_0.$$

This implies that $\omega_L \ll \sigma$ in Δ_0 if and only if $\omega_{L_1} \ll \sigma$ in Δ_0 and, in such a case,

$$\frac{1}{C_2} k_{L_1}^{X_{\Delta_0}}(y) \leq k_L^{X_{\Delta_0}}(y) \leq C_2 k_{L_1}^{X_{\Delta_0}}(y), \quad \text{for } \sigma\text{-a.e. } y \in \Delta_0.$$

Remark 1.31. As a consequence of Lemma 1.30(c), one can see that if $\omega_L \ll \sigma$, there exists $C \geq 1$ (depending only on dimension, the 1-sided CAD constants and the ellipticity of L) such that for every $Q_0 \in \mathbb{D}(\partial\Omega)$ and every $Q \in \mathbb{D}_{Q_0}$ we have that

$$\frac{1}{C} k_L^{X_Q}(y) \leq \frac{k_L^{X_{Q_0}}(y)}{\omega_L^{X_{Q_0}}(Q)} \leq C k_L^{X_Q}(y), \quad \text{for } \sigma\text{-a.e. } y \in Q.$$

We also have a dyadic version of the comparison principle stated in Lemma 1.30(e), for large interior regions of $Q_0 \in \mathbb{D}(\partial\Omega)$.

Lemma 1.32. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided CAD. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$, let L and L_1 be elliptic operators such that $\omega_L \ll \sigma$, $\omega_{L_1} \ll \sigma$, and $L \equiv L_1$ in T_{Q_0}. Given $0 < \tau < 1$, there exists $C_\tau > 1$ such that*

$$\frac{1}{C_\tau} k_{L_1}^{X_{Q_0}}(y) \leq k_L^{X_{Q_0}}(y) \leq C_\tau k_{L_1}^{X_{Q_0}}(y), \quad \text{for } \sigma\text{-a.e. } y \in Q_0 \setminus \Sigma_{Q_0,\tau},$$

where $\Sigma_{Q_0,\tau}$ is the region defined by $\Sigma_{Q_0,\tau} = \{x \in Q_0 : \operatorname{dist}(x, \partial\Omega \setminus Q_0) < \tau\ell(Q_0)\}$.

Proof. Let $r = \tau\ell(Q_0)/M$ with $M > 1$ to be chosen. Using a Vitali type covering argument, we construct a maximal collection of points $\{x_j\}_{j \in \mathcal{J}} \subset Q_0 \setminus \Sigma_{Q_0,\tau}$ with respect to the property that $|x_j - x_k| > 2r/3$ for every $j, k \in \mathcal{J}$, and a disjoint family $\{\Delta_j'\}_{j \in \mathcal{J}}$ given by $\Delta_j' = \Delta(x_j, r/3)$, in such a way that $Q_0 \setminus \Sigma_{Q_0,\tau} \subset \bigcup_{j \in \mathcal{J}} 3\Delta_j'$. Note that there exists C, depending only on dimension and on the 1-sided CAD constants, such that $\Delta_j' \subset \Delta(x_{Q_0}, C\ell(Q_0))$ for every $j \in \mathcal{J}$. Hence,

$$\#\mathcal{J}\left(\frac{\tau\ell(Q_0)}{M}\right)^n \approx \sum_{j \in \mathcal{J}} \sigma(\Delta_j') = \sigma\left(\bigcup_{j \in \mathcal{J}} \Delta_j'\right) \leq \sigma(\Delta(x_{Q_0}, C\ell(Q_0))) \approx \ell(Q_0)^n.$$

We have then obtained a covering $\{\Delta_j\}_{j=1}^{N_\tau}$ of $Q_0 \setminus \Sigma_{Q_0,\tau}$ by balls $\Delta_j = \Delta(x_j, r)$ with $x_j \in Q_0 \setminus \Sigma_{Q_0,\tau}$, $r = \tau\ell(Q_0)/M$ and $N_\tau \lesssim (M/\tau)^n$. We claim that for $M \gg 1$ we have $B_j^* \cap \Omega \subset T_{Q_0}$, with $B_j^* := B_{\Delta_j}^* = B(x_j, 2\kappa_0 r)$ and κ_0 as in (1.7). Let $Y \in B_j^* \cap \Omega$ and $I \in \mathcal{W}$ be such that $Y \in I$. Take $y_j \in \partial\Omega$ such that $\mathrm{dist}(I, \partial\Omega) = \mathrm{dist}(I, y_j)$ and pick $Q_j \in \mathbb{D}(\partial\Omega)$ the unique cube such that $y_j \in Q_j$ and $\ell(Q_j) = \ell(I)$. As already observed, $I \in \mathcal{W}_{Q_j}^*$. We are going to see that $Q_j \in \mathbb{D}_{Q_0}$. First of all, note that

$$\ell(Q_j) = \ell(I) \approx \mathrm{dist}(I, \partial\Omega) \leq |x_j - Y| < 2\kappa_0\tau\ell(Q_0)/M < 2\kappa_0\ell(Q_0)/M.$$

Choosing $M \gg 1$ sufficiently large (independent of τ) we may obtain $\ell(Q_j) < \ell(Q_0)/4$ and $\mathrm{dist}(I, \partial\Omega) \leq |x_j - Y| < \tau\ell(Q_0)/4$. Also, since $x_j \in Q_0 \setminus \Sigma_{Q_0,\tau}$, we can write by (1.4)

$$\tau\ell(Q_0) \leq \mathrm{dist}(x_j, \partial\Omega \setminus Q_0) \leq |x_j - Y| + \mathrm{diam}(I) + \mathrm{dist}(I, y_j) + \mathrm{dist}(y_j, \partial\Omega \setminus Q_0)$$
$$\leq \tfrac{1}{4}\tau\ell(Q_0) + \tfrac{5}{4}\mathrm{dist}(I, \partial\Omega) + \mathrm{dist}(y_j, \partial\Omega \setminus Q_0) \leq \tfrac{9}{16}\tau\ell(Q_0) + \mathrm{dist}(y_j, \partial\Omega \setminus Q_0),$$

and hence $y_j \in \mathrm{int}(Q_0)$. Since $y_j \in Q_0 \cap Q_j$ and $\ell(Q_j) < \ell(Q_0)/4$ it follows that $Q_j \in \mathbb{D}_{Q_0}$. This and the fact that $Y \in I \in \mathcal{W}_{Q_j}^*$ allow us to conclude that $Y \in T_{Q_0}$. Consequently, we have shown that $B_j^* \cap \Omega \subset T_{Q_0}$ and thus $L \equiv L_1$ in $B_j^* \cap \Omega$ for every $j = 1, \ldots, N_\tau$.

Next, we note that $\delta(X_{Q_0}) \approx \ell(Q_0) \geq \tau\ell(Q_0)$, $\delta(X_{\Delta_j}) \approx \tau\ell(Q_0)$, and $|X_{Q_0} - X_{\Delta_j}| \lesssim \ell(Q_0)$. Hence, we can use Harnack's inequality to move from X_{Q_0} to X_{Δ_j} with constants depending on τ, and Lemma 1.30(e), we obtain

$$k_L^{X_{Q_0}}(y) \approx_\tau k_L^{X_{\Delta_j}}(y) \approx k_{L_1}^{X_{\Delta_j}}(y) \approx_\tau k_{L_1}^{X_{Q_0}}(y)$$

for σ-almost every $y \in \Delta_j = B_j \cap \partial\Omega$. Since we know that $\{\Delta_j\}_{j=1}^{N_\tau}$ covers $Q_0 \setminus \Sigma_{Q_0,\tau}$, the desired conclusion follows. ∎

In Section 1.2 we defined the A_∞ and RH_p conditions for arbitrary Borel measures. Now, we will introduce some alternative definitions in order to consider the case that we are treating with an elliptic measure, which is a family of Borel measures indexed in the points of the domain.

Definition 1.33 (A_∞ **for elliptic measures**). Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided CAD and let L be a real (non-necessarily symmetric) elliptic operator. We say that the elliptic measure $\omega_L \in A_\infty(\partial\Omega)$ if there exist constants $0 < \alpha, \beta < 1$ such that given an arbitrary surface ball $\Delta_0 = B_0 \cap \partial\Omega$, with $B_0 = B(x_0, r_0)$, $x_0 \in \partial\Omega$, $0 < r < \mathrm{diam}(\partial\Omega)$, and for every surface ball $\Delta = B \cap \partial\Omega$ centered at $\partial\Omega$ with $B \subset B_0$, and for every Borel set $F \subset \Delta$, we have that

$$\frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} \leq \alpha \implies \frac{\sigma(F)}{\sigma(\Delta)} \leq \beta. \tag{1.43}$$

With the notation introduced in Definition 1.11, we say that $\omega_L \in A_\infty(\partial\Omega)$ if $\omega_L^{X_{\Delta_0}} \in A_\infty(\Delta_0)$ with uniformly controlled constants for every Δ_0. As already noted in Section 1.2, since σ and ω_L are doubling measures (see Lemma 1.30(b)), we have

that $\omega_L \in A_\infty(\partial\Omega)$ if and only if $\omega_L \ll \sigma$ in $\partial\Omega$ and there exists $1 < q < \infty$ such that for every Δ_0 and Δ as above

$$\left(\fint_\Delta k_L^{X_{\Delta_0}}(x)^q \, d\sigma(x) \right)^{\frac{1}{q}} \leq C \fint_\Delta k_L^{X_{\Delta_0}}(x) \, d\sigma(x),$$

where $k_L^{X_{\Delta_0}} = d\omega_L^{X_{\Delta_0}}/d\sigma$ is the Radon-Nikodym derivative. This motivates the following definition.

Definition 1.34 (RH_p and RH_p^{dyadic} **for elliptic measures**). Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided CAD, let L be an elliptic operator and let $1 < p < \infty$. We say that $\omega_L \in RH_p(\partial\Omega)$ if $\omega_L \ll \sigma$ and $k_L^{X_{\Delta_0}} \in RH_p(\Delta_0)$ uniformly in Δ_0 for every surface ball $\Delta_0 \subset \partial\Omega$. That is, there exists $C \geq 1$ such that for every $B_0 := B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \mathrm{diam}(\partial\Omega)$, and for every $B = B(x, r) \subset B_0$ with $x \in \partial\Omega$, we have that

$$\left(\fint_\Delta k_L^{X_{\Delta_0}}(y)^p \, d\sigma(y) \right)^{1/p} \leq C \fint_\Delta k_L^{X_{\Delta_0}}(y) \, d\sigma(y), \qquad \Delta = B \cap \partial\Omega.$$

Analogously, we say that $\omega_L \in RH_p^{\mathrm{dyadic}}(\partial\Omega)$ if $\omega_L \ll \sigma$ and $k_L^{X_{Q_0}} \in RH_p^{\mathrm{dyadic}}(Q_0)$ uniformly in Q_0 for every $Q_0 \in \mathbb{D}(\partial\Omega)$. That is, there exists $C \geq 1$ such that for every $Q_0 \in \mathbb{D}(\partial\Omega)$ and every $Q \in \mathbb{D}_{Q_0}$, we have that

$$\left(\fint_Q k_L^{X_{Q_0}}(y)^p \, d\sigma(y) \right)^{1/p} \leq C \fint_Q k_L^{X_{Q_0}}(y) \, d\sigma(y).$$

Before going further, let us introduce the following operators (see [HMUT, Section 2.4]):

$$Su(x) := \left(\iint_{\Gamma(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2}, \qquad \widetilde{\mathcal{N}}_* u(x) := \sup_{Y \in \widetilde{\Gamma}(x)} |u(Y)|,$$

where

$$\Gamma(x) := \bigcup_{x \in Q \in \mathbb{D}(\partial\Omega)} U_Q, \qquad \widetilde{\Gamma}(x) := \bigcup_{x \in Q \in \mathbb{D}(\partial\Omega)} U_Q^*.$$

These operators are known, respectively, as the square function and non-tangential maximal operators. Also, we say that $\Gamma(x)$ is a non-tangential cone with vertex on $x \in \partial\Omega$, while $\widetilde{\Gamma}(x)$ is a slight fattening of $\Gamma(x)$, with the same vertex point. Similarly, we can define localized versions of the above operators. For a fixed $Q_0 \in \mathbb{D}(\partial\Omega)$, we define

$$S_{Q_0}u(x) := \left(\iint_{\Gamma_{Q_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2}, \qquad \widetilde{\mathcal{N}}_{Q_0,*}u(x) := \sup_{Y \in \widetilde{\Gamma}_{Q_0}(x)} |u(Y)|,$$

for each $x \in Q_0$, where

$$\Gamma_{Q_0}(x) := \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q, \qquad \widetilde{\Gamma}_{Q_0}(x) := \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q^*.$$

We summarize some of the most important equivalences to the fact that $\omega_L \in RH_p(\partial\Omega)$ in the following proposition.

Proposition 1.35 (**Solvability and** RH_p, [HMT2])**.** *Suppose that* $\Omega \subset \mathbb{R}^{n+1}$ *is a 1-sided* CAD, *let* L *be an elliptic operator and let* $1 < p < \infty$, *the following statements are equivalent:*

(a) *The Dirichlet problem is solvable in* $L^{p'}(\partial\Omega)$: *That is, there exists* $C \geq 1$ *such that*

$$\|\widetilde{\mathcal{N}}_* u\|_{L^{p'}(\partial\Omega)} \leq C\|f\|_{L^{p'}(\partial\Omega)},$$

whenever

$$u(X) = \int_{\partial\Omega} f(y)\,d\omega_L^X(y), \qquad f \in C_c(\partial\Omega). \tag{1.44}$$

(b) $\omega_L \in RH_p(\partial\Omega)$ *(cf. Definition 1.34).*

(c) $\omega_L \ll \sigma$ *and there exists* $C \geq 1$ *such that for every* $B := B(x,r)$ *with* $x \in \partial\Omega$ *and* $0 < r < \mathrm{diam}(\partial\Omega)$, *we have that*

$$\int_\Delta k_L^{X_\Delta}(y)^p\,d\sigma(y) \leq C\sigma(\Delta)^{1-p}, \quad \Delta = B \cap \partial\Omega. \tag{1.45}$$

Moreover, (a), (b) and/or (c) yield that for every $0 < q < \infty$ *there exists* C *(depending only on dimension, the 1-sided* CAD *constants, the ellipticity of* L, *the constants in (a), (b) and/or (c), and on* q*) such that for every* $Q_0 \in \mathbb{D}(\partial\Omega)$

$$\|S_{Q_0} u\|_{L^q(Q_0)} \lesssim \|\widetilde{\mathcal{N}}_{Q_0,*} u\|_{L^q(Q_0)} \tag{1.46}$$

for every u *as in* (1.44).

Remark 1.36. Note that $\omega_L \in RH_p(\partial\Omega)$, together with Lemma 1.30(b) and Harnack's inequality, imply that $\omega_L \in RH_p^{\mathrm{dyadic}}(\partial\Omega)$. This in turn gives

$$\int_Q k_L^{X_Q}(y)^p\,d\sigma(y) \leq C\sigma(Q)^{1-p}, \qquad Q \in \mathbb{D}(\partial\Omega). \tag{1.47}$$

Moreover, from (1.47) and Harnack's inequality, we can see that (1.45) holds, and hence $\omega_L \in RH_p(\partial\Omega)$. Therefore, the conditions $\omega_L \in RH_p(\partial\Omega)$, (1.45), $\omega_L \in RH_p^{\mathrm{dyadic}}(\partial\Omega)$ and (1.47) are all equivalent.

In the following lemmas we discuss some representation formulas for the difference between two elliptic measure solutions with the same given boundary value. We first begin with a result inspired in the work of [HMT2].

Lemma 1.37. *Suppose that* $\Omega \subset \mathbb{R}^{n+1}$ *is a bounded open set such that* $\partial\Omega$ *satisfies the* AR *property. Let* L_0, L_1 *be elliptic operators, and let* $u_0 \in W^{1,2}(\Omega)$ *be a weak solution of* $L_0 u_0 = 0$ *in* Ω. *Then,*

$$\iint_\Omega A_0(Y)\nabla_Y G_{L_1}(Y,X) \cdot \nabla u_0(Y)\,dY = 0, \quad \text{for a.e. } X \in \Omega. \tag{1.48}$$

Proof. Let us take a cut-off function $\varphi \in C_c([-2, 2])$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $[-1, 1]$. Fix $X_0 \in \Omega$, for each $0 < \varepsilon < \delta(X_0)/16$ we set $\varphi_\varepsilon(X) = \varphi(|X - X_0|/\varepsilon)$ and $\psi_\varepsilon = 1 - \varphi_\varepsilon$. Using (1.42) we have that $G_{L_1}(\cdot, X_0)\psi_\varepsilon \in W_0^{1,2}(\Omega)$, which together with the fact that $u_0 \in W^{1,2}(\Omega)$ is a weak solution of $L_0 u_0 = 0$ in Ω, implies

$$\iint_\Omega A_0(Y)\nabla\big(G_{L_1}(\cdot, X_0)\psi_\varepsilon\big)(Y) \cdot \nabla u_0(Y)\, dY = 0.$$

Hence, we can write

$$\iint_\Omega A_0\nabla G_{L_1}(\cdot, X_0) \cdot \nabla u_0\, dY = \iint_\Omega A_0\nabla\big(G_{L_1}(\cdot, X_0)\varphi_\varepsilon\big) \cdot \nabla u_0\, dY$$

$$= \iint_\Omega A_0\nabla G_{L_1}(\cdot, X_0) \cdot \nabla u_0\, \varphi_\varepsilon\, dY + \iint_\Omega A_0\nabla\varphi_\varepsilon \cdot \nabla u_0\, G_{L_1}(\cdot, X_0)\, dY =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon.$$

$$(1.49)$$

In order to simplify the notation we set $C_j(X_0, \varepsilon) := \{Y \in \mathbb{R}^{n+1} : 2^{-j+1}\varepsilon \leq |Y - X_0| < 2^{-j+2}\varepsilon\}$ for $j \geq 1$. For the first term, we use Cauchy-Schwarz's inequality, Caccioppoli's inequality and (1.35)

$$|\mathrm{I}_\varepsilon| \lesssim \iint_{B(X_0, 2\varepsilon)} |\nabla_Y G_{L_1}(Y, X_0)||\nabla u_0(Y)|\, dY \qquad (1.50)$$

$$\lesssim \sum_{j=1}^\infty (2^{-j}\varepsilon)^{n+1}\left(\iint_{C_j(X_0,\varepsilon)} |\nabla G_{L_1}(\cdot, X_0)|^2\, dY\right)^{1/2}\left(\iint_{B(X_0, 2^{-j+2}\varepsilon)} |\nabla u_0|^2\, dY\right)^{1/2}$$

$$\lesssim \sum_{j=1}^\infty 2^{-j}\varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0) \lesssim \varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0),$$

where $M_2 f(X) := M(|f|^2)(X)^{1/2}$, with M being the Hardy-Littlewood maximal operator on $\mathbb{R}^{n+1}$. For the second term, using again (1.35) and Jensen's inequality,

$$|\mathrm{II}_\varepsilon| \lesssim \varepsilon^{-1}\iint_{C_1(X_0,\varepsilon)} |G_{L_1}(Y, X_0)||\nabla u_0(Y)|\, dY$$

$$\lesssim \varepsilon^{-n}\iint_{B(X_0, 2\varepsilon)} |\nabla u_0(Y)|\, dY \lesssim \varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0). \quad (1.51)$$

Combining (1.50) and (1.51), we have proved that, for every $X_0 \in \Omega$ and for every $0 < \varepsilon < \delta(X_0)/16$,

$$\left|\iint_\Omega A_0(Y)\nabla_Y G_{L_1}(Y, X_0) \cdot \nabla u_0(Y)\, dY\right| \lesssim \varepsilon M_2(|\nabla u_0|\mathbf{1}_\Omega)(X_0). \qquad (1.52)$$

Recall that $M_2(|\nabla u_0|\mathbf{1}_\Omega) \in L^{1,\infty}(\Omega)$ as $|\nabla u_0| \in L^2(\Omega)$, thus $M_2(|\nabla u_0|\mathbf{1}_\Omega)(X) < \infty$ for almost every $X \in \Omega$. Taking limits as $\varepsilon \to 0$ in (1.52), we obtain as desired (1.48). ∎

Lemma 1.38. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set such that $\partial\Omega$ satisfies the* AR *property. Let L_0 and L_1 be elliptic operators, and let $g \in H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$. Consider the solutions u_0 and u_1 given by*

$$u_0(X) = \int_{\partial\Omega} g(y) \, d\omega_{L_0}^X(y), \qquad u_1(X) = \int_{\partial\Omega} g(y) \, d\omega_{L_1}^X(y), \qquad X \in \Omega.$$

Then,

$$u_1(X) - u_0(X) = \iint_\Omega (A_0 - A_1)(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla u_0(Y) \, dY, \quad \text{for a.e. } X \in \Omega. \tag{1.53}$$

Proof. Following [HMT2] we know that $u_0 = \widetilde{g} - v_0$ and $u_1 = \widetilde{g} - v_1$, where $\widetilde{g} = \mathcal{E}_{\partial\Omega} g \in W^{1,2}(\mathbb{R}^{n+1})$ is the Jonsson-Wallin extension (see [JW]), and $v_0, v_1 \in W_0^{1,2}(\Omega)$ are the Lax-Milgram solutions of $L_0 v_0 = L_0 \widetilde{g}$ and $L_1 v_1 = L_1 \widetilde{g}$ respectively. Hence, we have that $u_1 - u_0 = v_0 - v_1 \in W_0^{1,2}(\Omega)$, and following again [HMT2] we obtain

$$(u_1 - u_0)(X) = \iint_\Omega A_1(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla(u_1 - u_0)(Y) \, dY, \quad \text{for a.e. } X \in \Omega.$$

For almost every $X \in \Omega$ we then have that

$$(u_1 - u_0)(X) - \iint_\Omega (A_0 - A_1)(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla u_0(Y) \, dY =$$

$$= \iint_\Omega A_1(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla u_1(Y) \, dY - \iint_\Omega A_0(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla u_0(Y) \, dY.$$

Using Lemma 1.37 for both terms, the right side of the above equality vanishes almost everywhere, and this proves (1.53). ∎

Lemma 1.39. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set such that $\partial\Omega$ satisfies the* AR *property. Let L_0, L_1 be elliptic operators such that $K := \operatorname{supp}(A_0 - A_1) \cap \Omega$ is compact. For every $g \in H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$, let*

$$u_0(X) = \int_{\partial\Omega} g(y) \, d\omega_{L_0}^X(y), \qquad u_1(X) = \int_{\partial\Omega} g(y) \, d\omega_{L_1}^X(y), \qquad X \in \Omega.$$

Then, for almost every $X \in \Omega \setminus K$, there holds

$$u_1(X) - u_0(X) = \iint_\Omega (A_0 - A_1)(Y) \nabla_Y G_{L_1}(Y, X) \cdot \nabla u_0(Y) \, dY. \tag{1.54}$$

Proof. First, fix $x_0 \in \partial\Omega$, following [HMT2] we consider the family of bounded increasing open subsets $\{\mathcal{T}_k\}_{k \in \mathbb{Z}}$ such that $\Omega = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_k$, and $\partial\mathcal{T}_k$ satisfies the AR property, with constants possibly depending on k and $\operatorname{diam}(\partial\Omega)$ (see [HMT2]). As we can see in [JW], there exists an extension operator $\mathcal{E}_{\partial\Omega}$, which maps $H^{1/2}(\partial\Omega)$ continuously into $W^{1,2}(\mathbb{R}^{n+1})$, and a restriction operator $\mathfrak{R}_{\partial\Omega}$, which is bounded from $W^{1,2}(\mathbb{R}^{n+1})$ to $H^{1/2}(\partial\Omega)$, such that $\mathfrak{R}_{\partial\Omega} \circ \mathcal{E}_{\partial\Omega} = Id$ in $H^{1/2}(\partial\Omega)$. Moreover, we have that $\mathcal{E}_{\partial\Omega} f \in C_c(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$ for every $f \in H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$. Let $g \in H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$ and $h = \mathcal{E}_{\partial\Omega} g \in W^{1,2}(\mathbb{R}^{n+1}) \cap C_c(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$. Let $\eta \in$

$C_c^\infty([-2, 2])$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $[-1, 1]$, η monotonously decreasing in $(1, 2)$ and monotonously increasing in $(-2, -1)$. Let us consider $h_k(y) = h(y)\eta(|y - x_0|/2^k)$, as well as the solutions

$$u_0^k(X) = \int_{\partial\mathcal{T}_k} h_k(y)\, d\omega_{L_0, \mathcal{T}_k}^X(y), \quad u_1^k(X) = \int_{\partial\mathcal{T}_k} h_k(y)\, d\omega_{L_1, \mathcal{T}_k}^X(y), \qquad X \in \mathcal{T}_k.$$

We take $k_0 \gg 1$ such that $\operatorname{supp}(g), \operatorname{supp}(h) \subset B(x_0, 2^{k_0-1})$, in such a way that $h_k \equiv h$ for $k \geq k_0$. Note that by [HMT2], $B(x_0, 2^k) \cap \Omega \subset \mathcal{T}_k$, hence $h = g\mathbf{1}_{\partial\Omega}$ on $\partial\mathcal{T}_k$, and consequently $h \in H^{1/2}(\partial\mathcal{T}_k) \cap C_c(\partial\mathcal{T}_k)$ for $k \geq k_0$. Using Lemma 1.38, we have that

$$(u_1^k - u_0^k)(X) = \iint_{\mathcal{T}_k} (A_0 - A_1)(Y)\nabla_Y G_{L_1, \mathcal{T}_k}(Y, X) \cdot \nabla u_0^k(Y)\, dY, \qquad k \geq k_0, \ (1.55)$$

for almost every $X \in \mathcal{T}_k$. Let $\mathcal{G}_k$ be the set of points $X \in \mathcal{T}_k$ for which (1.55) holds, and let $\mathcal{B}_k = \mathcal{T}_k \setminus \mathcal{G}_k$. We fix $X_0 \in (\Omega \setminus K) \setminus \bigcup_{k \geq k_0} \mathcal{B}_k$ and take k_0 (possibly greater than before) such that $X_0 \in B(x_0, 2^{k_0-1}) \cap \Omega \subset \mathcal{T}_k$ and $K \subset B(x_0, 2^{k_0-1}) \cap \Omega \subset \mathcal{T}_k$. Let us consider $v_k = G_{L_1, \mathcal{T}_k}(\cdot, X_0)$, which converge to $v = G_{L_1}(\cdot, X_0)$ uniformly on compacta in $\Omega \setminus \{X_0\}$ (see [HMT2]), and hence on $W_{\text{loc}}^{1,2}(\Omega \setminus \{X_0\})$ by Caccioppoli's inequality. Also, note that for $i = 0, 1$, we have that $u_i^k \to u_i$ uniformly on compacta in Ω (see [HMT2]). In particular, Caccioppoli's inequality yields $u_0^k \to u_0$ in $W_{\text{loc}}^{1,2}(\Omega)$. Thanks to these observations, using (1.55) and Cauchy-Schwarz's inequality we obtain

$$\left| (u_1^k - u_0^k)(X_0) - \iint_\Omega (A_0 - A_1)(Y)\nabla_Y G_{L_1}(Y, X_0) \cdot \nabla u_0(Y)\, dY \right|$$

$$\leq \iint_K \left| (A_0 - A_1)(Y) \right| \left| \nabla v_k(Y) \cdot \nabla u_0^k(Y) - \nabla v(Y) \cdot \nabla u_0(Y) \right| dY$$

$$\lesssim \|\nabla v_k\|_{L^2(K)}\|\nabla u_0^k - \nabla u_0\|_{L^2(K)} + \|\nabla v_k - \nabla v\|_{L^2(K)}\|\nabla u_0\|_{L^2(K)}.$$

Taking limits as $k \to \infty$, (1.54) is then proved. ∎

Remark 1.40. Note that Lemma 1.38 ensures that there exists $\mathcal{G} \subset \Omega$ with $|\Omega \setminus \mathcal{G}| = 0$ such that (1.53) holds for all $X \in \mathcal{G}$. Let $\Delta = \Delta(x, r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ be such that $X_\Delta \notin \mathcal{G}$. Take $\widetilde{X}_\Delta \in B(X_\Delta, cr/2) \cap \mathcal{G}$ where $0 < c < 1$ is the corkscrew constant. Taking into account that $B(\widetilde{X}_\Delta, cr/2) \subset B(X_\Delta, cr)$ and slightly modifying the constants, we can use $\widetilde{X}_\Delta$ as a corkscrew point associated with Δ. Hence, we may assume that for every Δ as before, there exists a corkscrew point $X_\Delta \in \mathcal{G}$ for which (1.53) holds with $X = X_\Delta$. Similarly, we may also assume that (1.54) holds for X_Δ, as long as $X_\Delta \notin K$. In particular, for every $Q \in \mathbb{D}(\partial\Omega)$, we can choose X_Q so that (1.53) and (1.54) hold with $X = X_Q$ (the latter provided $X_Q \notin K$).

1.5 A density result

In this section we present a density result, which allows us to approximate functions in $L^q(E)$ by bounded Lipschitz functions, where $E \subseteq \mathbb{R}^{n+1}$ is an n-dimensional AR set.

Lemma 1.41. *Suppose that $E \subseteq \mathbb{R}^{n+1}$ is n-dimensional* AR *and fix a cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\mathbf{1}_{(0,1)} \leq \varphi \leq \mathbf{1}_{(0,2)}$. For $t > 0$ we define the operator $g \longmapsto P_t g$, acting over $g \in L_{\mathrm{loc}}^1(E)$, by*

$$P_t g(x) := \int_E \varphi_t(x,y) g(y)\, d\sigma(y), \qquad x \in E, \tag{1.56}$$

where

$$\varphi_t(x,y) := \frac{\varphi\big(\frac{|x-y|}{t}\big)}{\int_E \varphi\big(\frac{|x-z|}{t}\big)\, d\sigma(z)}, \qquad x, y \in E. \tag{1.57}$$

(a) *P_t is uniformly bounded on $L^q(E)$ for every $1 < q \leq \infty$.*

(b) *If $g \in L^q(E)$, $1 < q < \infty$, and $t > 0$ then $P_t g \in L^\infty(E) \cap \mathrm{Lip}(E)$.*

(c) *If $g \in L^q(E)$, $1 < q < \infty$, then $P_t g \longrightarrow g$ in $L^q(E)$ as $t \to 0^+$.*

(d) *If $g \in C_c(E)$, then $P_t g(x) \longrightarrow g(x)$ as $t \to 0^+$ for every $x \in E$.*

(e) *If $g \in L^q(E)$, $1 \leq q \leq \infty$, with $\mathrm{supp}\, g \subset \Delta(x_0, r_0)$ then $\mathrm{supp}\, P_t g \subset \Delta(x_0, r_0 + 2\,t)$.*

Proof. We first let $x \in E$, using the AR property we have that

$$t^n \approx \sigma(\Delta(x,t)) \leq \int_E \varphi\Big(\frac{|x-z|}{t}\Big)\, d\sigma(z) \leq \sigma(\Delta(x,2t)) \approx t^n,$$

hence $t^{-n} \mathbf{1}_{|x-y|<t} \lesssim \varphi_t(x,y) \lesssim t^{-n} \mathbf{1}_{|x-y|<2t}$. Also, since $\int_E \varphi_t(x,y)\, d\sigma(y) = 1$, it holds $|P_t g(x)| \leq \|g\|_{L^\infty(E)}$ for every $x \in E$, thus $P_t : L^\infty(E) \to L^\infty(E)$ is bounded. Note that

$$|P_t g(x)| \lesssim t^{-n} \int_{\Delta(x,2t)} |g(y)|\, d\sigma(y) \lesssim Mg(x), \qquad x \in E,$$

with M being the Hardy-Littlewood maximal operator, hence $P_t : L^1(E) \to L^{1,\infty}(E)$ is also bounded, with constants depending only on the AR constant. Using Marcinkiewicz's interpolation theorem we prove (a).

Suppose now that $g \in L^q(E)$, using Hölder's inequality and the AR property, we have that

$$|P_t g(x)| \lesssim t^{-n} \int_{\Delta(x,2t)} |g(y)|\, d\sigma(y) \lesssim t^{-n} t^{\frac{n}{q'}} \|g\|_{L^q(E)} = t^{-\frac{n}{q}} \|g\|_{L^q(E)}, \tag{1.58}$$

for every $x \in E$ and every $t > 0$, hence $P_t g \in L^\infty(E)$. In order to prove that $P_t g \in \mathrm{Lip}(E)$ we also take $y \in E$. First, if $|x - y| \geq 2t$ then

$$|P_t g(x) - P_t g(y)| \leq 2\|P_t g\|_{L^\infty(E)} \lesssim t^{-\frac{n}{q}} \|g\|_{L^q(E)} \lesssim t^{-\frac{n}{q}-1} \|g\|_{L^q(E)} |x - y|, \tag{1.59}$$

where we have used (1.58) in the second inequality. Suppose now that $|x - y| < 2t$, since $\Delta(x,2t) \cup \Delta(y,2t) \subseteq \Delta(x,4t)$ we have

$$|P_t g(x) - P_t g(y)| \leq \int_{\Delta(x,4t)} |\varphi_t(x,z) - \varphi_t(y,z)| |g(z)|\, d\sigma(z). \tag{1.60}$$

Given $z \in E$ we decompose $\varphi_t(x, z) - \varphi_t(y, z) = \mathrm{I} + \mathrm{II}$, where

$$
\mathrm{I} = \frac{1}{\int_E \varphi\left(\frac{|x-w|}{t}\right) d\sigma(w)} \left(\varphi\left(\frac{|x-z|}{t}\right) - \varphi\left(\frac{|y-z|}{t}\right) \right), \tag{1.61}
$$

$$
\mathrm{II} = \varphi\left(\frac{|y-z|}{t}\right) \left(\frac{1}{\int_E \varphi\left(\frac{|x-w|}{t}\right) d\sigma(w)} - \frac{1}{\int_E \varphi\left(\frac{|y-w|}{t}\right) d\sigma(w)} \right). \tag{1.62}
$$

For the first term, using the AR property, we obtain

$$
|\mathrm{I}| \lesssim t^{-n} \|\varphi'\|_{L^\infty(\mathbb{R})} \left| \frac{|x-z|}{t} - \frac{|y-z|}{t} \right| \lesssim t^{-n-1} \|\varphi'\|_{L^\infty(\mathbb{R})} |x-y|.
$$

For the second term we may write

$$
|\mathrm{II}| \lesssim t^{-2n} \int_E \left| \varphi\left(\frac{|x-w|}{t}\right) - \varphi\left(\frac{|y-w|}{t}\right) \right| d\sigma(w),
$$

where we have used again the AR property. In fact, the last integral is supported in $\Delta(x, 4t)$, thus

$$
|\mathrm{II}| \lesssim t^{-2n} \|\varphi'\|_{L^\infty(\mathbb{R})} \int_{\Delta(x,4t)} \left| \frac{|x-w|}{t} - \frac{|y-w|}{t} \right| d\sigma(w) \lesssim t^{-n-1} \|\varphi'\|_{L^\infty(\mathbb{R})} |x-y|.
$$

Since φ is fixed, for every $t > 0$ we have proved that $|\varphi_t(x, z) - \varphi_t(y, z)| \lesssim t^{-n-1} |x-y|$. Let us recall (1.60), using Hölder's inequality we have that

$$
|P_t g(x) - P_t g(y)| \lesssim t^{-n-1} |x-y| \int_{\Delta(x,4t)} |g(z)| \, d\sigma(z) \lesssim t^{-\frac{n}{q}-1} \|g\|_{L^q(E)} |x-y|.
$$

This, together with (1.59) shows that $P_t g \in \mathrm{Lip}(E)$, and (b) is proved.

As before, let $g \in L^q(E)$, and observe that $|P_t g - g| \lesssim Mg + |g| \in L^q(E)$. Given $x \in E$, since $\int_E \varphi_t(x, y) \, d\sigma(y) = 1$ we have that

$$
|P_t g(x) - g(x)| = \left| \int_E \varphi_t(x, y)(g(y) - g(x)) \, d\sigma(y) \right| \lesssim \fint_{\Delta(x,2t)} |g(y) - g(x)|. \tag{1.63}
$$

As $t \to 0^+$, the right hand side of (1.63) tends to zero σ-a.e. $x \in E$ by Lebesgue's differentiation theorem. The dominated convergence theorem proves (c). We also note that (d) is an automatic consequence of the fact that for $g \in C_c(E)$, the right hand side of (1.63) tends to zero for every $x \in E$ as $t \to 0^+$.

Finally, suppose that $\mathrm{supp}\, g \subseteq \Delta(x_0, r_0)$, and let $x \in E$ be such that $|x - x_0| \geq r_0 + 2t$. Note that the integral in (1.56) is supported in the set $\Delta(x_0, r_0) \cap \Delta(x, 2t)$, which is empty since in other case, given $y \in \Delta(x_0, r_0) \cap \Delta(x, 2t)$ we would have

$$
r_0 + 2t \leq |x - x_0| \leq |x - y| + |y - x_0| < 2t + r_0,
$$

which leads to a contradiction. Therefore $P_t g(x) = 0$ for $|x - x_0| \geq r_0 + 2t$, and that proves the last property. ∎

Chapter 2

Perturbations of symmetric operators

In this chapter we extend the Carleson perturbation theorem of [FKP] to the setting of 1-sided chord-arc domains. Implicit in the proof it is also obtained a "small perturbation" result. The vanishing trace Carleson perturbation of [Dah2] is studied in the last section. The main theorem of this chapter can be stated as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided* CAD *(cf. Definition 1.4). Let $Lu = -\operatorname{div}(A\nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be real elliptic operators (cf. Definition 1.20) such that A and A_0 are symmetric. Define the disagreement between A and A_0 in Ω by*

$$\varrho(A, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|, \qquad X \in \Omega, \qquad (2.1)$$

where $\delta(X) := \operatorname{dist}(X, \partial\Omega)$, and write

$$\||\varrho(A, A_0)\|| := \sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \frac{\varrho(A, A_0)(X)^2}{\delta(X)} \, dX. \quad (2.2)$$

Suppose that there exists p, $1 < p < \infty$, such that the elliptic measure $\omega_{L_0} \in RH_p(\partial\Omega)$ (cf. Definition 1.34). The following hold:

(a) *If $\||\varrho(A, A_0)\|| < \infty$, then there exists $1 < q < \infty$ such that $\omega_L \in RH_q(\partial\Omega)$. Here, q and the implicit constant depend only on dimension, p, the 1-sided* CAD *constants, the ellipticity of L_0 and L, $\||\varrho(A, A_0)\||$, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$.*

(b) *There exists $\varepsilon_1 > 0$ (depending only on dimension, p, the 1-sided* CAD *constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$) such that if one has $\||\varrho(A, A_0)\|| \leq \varepsilon_1$, then $\omega_L \in RH_p(\partial\Omega)$, with the implicit constant depending only on dimension, p, the 1-sided* CAD *constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$.*

2.1   Proof of Theorem 2.1(a), Carleson perturbation

We will prove Theorem 2.1(a) with the help of Lemma 1.18. In this way we consider the measure $\mathfrak{m} = \{\gamma_Q\}_{Q \in \mathbb{D}(\partial\Omega)}$, where

$$\gamma_Q := \sum_{I \in \mathcal{W}_Q^*} \frac{\sup_{I^*} |\mathcal{E}|^2}{\ell(I)} |I|, \qquad Q \in \mathbb{D}(\partial\Omega), \tag{2.3}$$

and $\mathcal{E}(Y) = A(Y) - A_0(Y)$. We are going to show that $\mathfrak{m}$ is indeed a discrete Carleson measure with respect to σ, as it is required in the hypotheses of Lemma 1.18.

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided* CAD, *let L_0 and L be elliptic operators whose disagreement in Ω is given by the function $a := \varrho(A, A_0)$ defined in (2.1), and suppose that $\|a\| < \infty$, see (3.3). Then, there exists $\kappa > 0$ (depending only on dimension and the 1-sided* CAD *constants) such that for every $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \operatorname{diam}(\partial\Omega)/\kappa_0$ (see (1.6)), the collection $\mathfrak{m} = \{\gamma_Q\}_{Q \in \mathbb{D}(\partial\Omega)}$ given by (2.3) defines a discrete Carleson measure $\mathfrak{m} \in \mathcal{C}(Q_0)$ with $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \leq \kappa\|a\|$.*

Proof. Let $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \operatorname{diam}(\partial\Omega)/\kappa_0$. First, note that for every $I \in \mathcal{W}$ and every $Y \in I$ we have that $\sup_{I^*} |\mathcal{E}| \leq a(Y)$. Indeed, since $4\operatorname{diam}(I) \leq \operatorname{dist}(I, \partial\Omega)$ (see (1.4)), we know that $I^* \subset \{X \in \Omega : |X - Y| < \delta(Y)/2\}$. Given $Q \in \mathbb{D}_{Q_0}$ we can write

$$\mathfrak{m}(\mathbb{D}_Q) = \sum_{Q' \in \mathbb{D}_Q} \gamma_{Q'} = \sum_{Q' \in \mathbb{D}_Q} \sum_{I \in \mathcal{W}_{Q'}^*} \frac{\sup_{I^*} |\mathcal{E}|^2}{\ell(I)} |I|$$

$$\lesssim \sum_{Q' \in \mathbb{D}_Q} \sum_{I \in \mathcal{W}_{Q'}^*} \iint_I \frac{a(Y)^2}{\delta(Y)} \, dY \leq \sum_{Q' \in \mathbb{D}_Q} \iint_{U_{Q'}} \frac{a(Y)^2}{\delta(Y)} \, dY \lesssim \iint_{T_Q} \frac{a(Y)^2}{\delta(Y)} \, dY. \tag{2.4}$$

where we have used that the family $\{U_{Q'}\}_{Q' \in \mathbb{D}_Q}$ has bounded overlap. Indeed, if $Y \in U_{Q'} \cap U_{Q''}$ then $\ell(Q') \approx \delta(Y) \approx \ell(Q'')$ and $\operatorname{dist}(Q', Q'') \leq \operatorname{dist}(Y, Q') + \operatorname{dist}(Y, Q'') \lesssim \ell(Q') + \ell(Q'') \approx \ell(Q')$. These readily imply that Y can be only in a bounded number of $U_{Q'}$'s.

On the other hand, by (1.6) we know that $T_Q \subset B(x_Q, \kappa_0 r_Q) \cap \Omega$. Also, $\kappa_0 r_Q \leq \kappa_0 \ell(Q) \leq \kappa_0 \ell(Q_0) < \operatorname{diam}(\partial\Omega)$. Using the AR property, from (2.4) we conclude that

$$\mathfrak{m}(\mathbb{D}_Q) \lesssim \iint_{B(x_Q, \kappa_0 r_Q) \cap \Omega} \frac{a(Y)^2}{\delta(Y)} \, dY \leq \|a\| \, \sigma(\Delta(x_Q, \kappa_0 r_Q)) \lesssim \|a\|\sigma(Q),$$

Taking the supremum over $Q \in \mathbb{D}_{Q_0}$, we obtain $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \leq \kappa\|a\|$ with κ depending on the allowable parameters. This completes the proof. ∎

Remark 2.3. We choose $M_0 > 2\kappa_0/c$, which will remain fixed during the proof of Theorem 2.1(a), where c is the corkscrew constant and κ_0 as in (1.6). Given an arbitrary $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \operatorname{diam}(\partial\Omega)/M_0$ we let $B_{Q_0} = B(x_{Q_0}, r_{Q_0})$ with $r_{Q_0} \approx \ell(Q_0)$ as in (1.1). Let $X_{M_0 \Delta_{Q_0}}$ be the corkscrew point relative to $M_0 \Delta_{Q_0}$ (note that $M_0 r_{Q_0} \leq M_0 \ell(Q_0) < \operatorname{diam}(\partial\Omega)$). By our choice of M_0, it is clear that $\delta(X_{M_0 \Delta_{Q_0}}) \geq c M_0 r_{Q_0} > 2\kappa_0 r_{Q_0}$. Hence, by (1.6),

$$X_{M_0 \Delta_{Q_0}} \in \Omega \setminus B_{Q_0}^* \subset \Omega \setminus T_{Q_0}^{**}. \tag{2.5}$$

As noted before, we will prove Theorem 2.1(a) using Lemma 1.18. To do that we need to split the proof in several steps.

2.1.1   Step 0

We first make a reduction which will allow us to use some qualitative properties of the elliptic measure. Fix $j \in \mathbb{N}$ (large enough, as we eventually let $j \to \infty$) and $\widetilde{L} = L^j$ be the operator defined by $\widetilde{L}u = -\operatorname{div}(\widetilde{A}\nabla u)$, with

$$\widetilde{A}(Y) = A^j(Y) := \begin{cases} A(Y) & \text{if } Y \in \Omega, \, \delta(Y) \geq 2^{-j}, \\ A_0(Y) & \text{if } Y \in \Omega, \, \delta(Y) < 2^{-j}. \end{cases} \tag{2.6}$$

Note that the matrix A^j is uniformly elliptic with constant $\Lambda_j = \max\{\Lambda_A, \Lambda_{A_0}\}$, where Λ_A and Λ_{A_0} are the ellipticity constants of A and A_0 respectively. Recall that $\omega_{L_0} \in RH_p(\partial\Omega)$ and that $\widetilde{L} \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$. Therefore, applying Lemma 1.30(e) we have that $\omega_{\widetilde{L}} \ll \sigma$ and there exists $k_{\widetilde{L}}^X := d\omega_{\widetilde{L}}^X/d\sigma$. The fact that $\widetilde{L}$ verifies these qualitative hypotheses will be essential in the following steps. At the end of Step 4 we will have obtained the desired conclusion for the operator $\widetilde{L} = L^j$, with constants independent of $j \in \mathbb{N}$, and in Step 5 we will prove it for L via a limiting argument. From now on, $j \in \mathbb{N}$ will be fixed and we will focus on the operator $\widetilde{L} = L^j$.

2.1.2   Step 1

Let us fix $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \operatorname{diam}(\partial\Omega)/M_0$ and M_0 as in Remark 2.3, and set $X_0 := X_{M_0\Delta_{Q_0}}$ so that (2.5) holds. Inspired by Lemma 1.18 we also fix $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ a family of disjoint dyadic subcubes such that

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)} = \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_{\mathcal{F},Q})}{\sigma(Q)} \leq \varepsilon_1, \tag{2.7}$$

with $\varepsilon_1 > 0$ sufficiently small to be chosen and where $\mathfrak{m} = \{\gamma_Q\}_{Q \in \mathbb{D}}$ with γ_Q defined in (2.3). We modify the operator L_0 inside the region $\Omega_{\mathcal{F},Q_0}$ (see (1.5)), by defining $L_1 = L_1^{\mathcal{F},Q_0}$ as $L_1 u = -\operatorname{div}(A_1 \nabla u)$, where

$$A_1(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in \Omega_{\mathcal{F},Q_0}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus \Omega_{\mathcal{F},Q_0}, \end{cases}$$

and $\widetilde{A} = A^j$ as in (2.6). By construction, it is clear that $\mathcal{E}_1 := A_1 - A_0$ verifies $|\mathcal{E}_1| \leq |\mathcal{E}|\mathbf{1}_{\Omega_{\mathcal{F},Q_0}}$ and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$. Hence, the support of $A_1 - A_0$ is contained in a compact subset contained in Ω.

Our goal in Step 1 is to prove $\|k_{L_1}^{X_{Q_0}}\|_{L^p(Q_0)} \lesssim \sigma(Q_0)^{-1/p'}$ (uniformly in j), using that $\omega_{L_0} \in RH_p(\partial\Omega)$. Note that by Harnack's inequality and Lemma 1.30(e), we have that $\omega_{L_1} \ll \sigma$ and $\|k_{L_1}^{X_{Q_0}}\|_{L^p(Q_0)} \leq C_j < \infty$ for $k_{L_1}^X := d\omega_{L_1}^X/d\sigma$. We will use this qualitatively, and the point of this step is to show that we can actually remove the dependence on j.

Take an arbitrary $0 \leq g \in L^{p'}(Q_0)$ such that $\|g\|_{L^{p'}(Q_0)} = 1$. Without loss of generality we may assume that g is defined in Ω with $g \equiv 0$ in $\Omega \setminus Q_0$. Let

$\widetilde{\Delta}_{Q_0} := \Delta(x_{Q_0}, Cr_{Q_0})$ (see (1.1)) and take $0 < t < Cr_{Q_0}/2$. Set $g_t = P_t g$ (cf. Lemma 1.41) and consider the solutions

$$u_0^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_{L_0}^X(y), \qquad u_1^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_{L_1}^X(y), \qquad X \in \Omega. \quad (2.8)$$

By Lemma 1.41, $g_t \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}(g_t) \subset 2\widetilde{\Delta}_{Q_0}$, hence $g_t \in \mathrm{Lip}_c(\partial\Omega) \subset H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$. Recall that $\mathcal{E}_1 = A_1 - A_0$ verifies $|\mathcal{E}_1| \leq |\mathcal{E}|\mathbf{1}_{\Omega_{\mathcal{F},Q_0}}$ and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$. This, (2.5), and (1.6) allow us to invoke Lemma 1.39 (see Remark 1.40), which along with Cauchy-Schwarz's inequality yields

$$F_{Q_0}^t(X_0) := |u_1^t(X_0) - u_0^t(X_0)| = \left| \iint_\Omega (A_0 - A_1)(Y)\nabla_Y G_{L_1}(Y,X_0) \cdot \nabla u_0^t(Y)\, dY \right|$$

$$(2.9)$$

$$\leq \sum_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \sum_{I \in \mathcal{W}_Q^*} \iint_{I^*} |\mathcal{E}(Y)||\nabla_Y G_{L_1}(Y,X_0)||\nabla u_0^t(Y)|\, dY,$$

$$\leq \sum_{Q \in \mathbb{D}_{\mathcal{F},Q_0}} \sum_{I \in \mathcal{W}_Q^*} \sup_{I^*} |\mathcal{E}| \left(\iint_{I^*} |\nabla_Y G_{L_1}(Y,X_0)|^2\, dY \right)^{1/2} \left(\iint_{I^*} |\nabla u_0^t(Y)|^2\, dY \right)^{1/2},$$

Note that by our choice of $X_0 = X_{M_0\Delta_{Q_0}}$, see (2.5), the function $v(Y) = G_{L_1}(Y,X_0)$ is a weak solution of $L_1 v = 0$ in I^{***} for every $I \in \mathcal{W}_Q^*$ with $Q \in \mathbb{D}_{Q_0}$. Therefore, by Caccioppoli's and Harnack's inequalities, the fact that L_1 is symmetric (hence $G_{L_1}(X_Q, X_0) = G_{L_1}(X_0, X_Q)$), and Lemma 1.30(a), we obtain

$$\iint_{I^*} |\nabla_Y G_{L_1}(Y,X_0)|^2\, dY \lesssim \ell(I)^{n-1} G_{L_1}(X_Q, X_0)^2 \approx \left(\frac{\omega_{L_1}^{X_0}(Q)}{\sigma(Q)} \right)^2 |I|. \quad (2.10)$$

Also, since $\delta(Y) \approx \ell(I) \approx \ell(Q)$ for every $Y \in I^*$ such that $I \in \mathcal{W}_Q^*$,

$$\iint_{I^*} |\nabla u_0^t(Y)|^2\, dY \approx \ell(I)^{-1}\ell(Q)^n \iint_{I^*} |\nabla u_0^t(Y)|^2 \delta(Y)^{1-n}\, dY. \quad (2.11)$$

Recalling (2.3), (1.21), we define the sequences $\alpha = \{\alpha_Q\}_{Q \in \mathbb{D}_{Q_0}}$, $\beta = \{\beta_Q\}_{Q \in \mathbb{D}_{Q_0}}$ by

$$\alpha_Q := \frac{\omega_{L_1}^{X_0}(Q)}{\sigma(Q)} \left(\ell(Q)^n \iint_{U_Q} |\nabla u_0^t(Y)|^2 \delta(Y)^{1-n}\, dY \right)^{1/2} \quad \text{and} \quad \beta_Q := \gamma_{\mathcal{F},Q}^{1/2}. \quad (2.12)$$

Using Cauchy-Schwarz's inequality and the bounded overlap of the cubes I^*, one can see that (2.9), (2.10), (2.11), and (2.12) yield

$$F_{Q_0}^t(X_0) \lesssim \sum_{Q \in \mathbb{D}_{Q_0}} \frac{\omega_{L_1}^{X_0}(Q)}{\sigma(Q)} \gamma_{\mathcal{F},Q}^{1/2} \left(\ell(Q)^n \iint_{U_Q} |\nabla u_0^t(Y)|^2 \delta(Y)^{1-n}\, dY \right)^{1/2}$$

$$= \sum_{Q \in \mathbb{D}_{Q_0}} \alpha_Q \beta_Q \lesssim \int_{Q_0} \mathcal{A}_{Q_0}\alpha(x)\mathfrak{C}_{Q_0}\beta(x)\, d\sigma(x), \quad (2.13)$$

where in the last estimate we have used Lemma 1.19, and where we recall that $\mathcal{A}_{Q_0}$, $\mathfrak{C}_{Q_0}$ were defined in (1.24). Using the bounded overlap property of U_Q with $Q \in \mathbb{D}_{Q_0}$, we have that

$$\mathcal{A}_{Q_0}\alpha(x) = \left(\sum_{x \in Q \in \mathbb{D}_{Q_0}} \left(\frac{\omega_{L_1}^{X_0}(Q)}{\sigma(Q)} \right)^2 \iint_{U_Q} |\nabla u_0^t(Y)|^2 \delta(Y)^{1-n}\, dY \right)^{1/2}$$

$$\lesssim M_{Q_0}^d k_{L_1}^{X_0}(x)\, S_{Q_0} u_0^t(x), \quad (2.14)$$

where

$$M_{Q_0}^d f(x) := \sup_{x \in Q \in \mathbb{D}_{Q_0}} \fint_Q |f(y)|\, d\sigma(y) \quad (2.15)$$

is the localized dyadic maximal Hardy-Littlewood operator.

On the other hand, (2.7) yields

$$\mathfrak{C}_{Q_0}\beta(x) = \sup_{x \in Q \in \mathbb{D}_{Q_0}} \left(\frac{1}{\sigma(Q)} \sum_{Q' \in \mathbb{D}_Q} \gamma_{\mathcal{F}, Q'} \right)^{1/2} \leq \|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q_0)}^{1/2} \leq \varepsilon_1^{1/2}. \quad (2.16)$$

Plugging (2.14), (2.16) into (2.13), using Hölder's inequality we conclude that

$$F_{Q_0}^t(X_0) \lesssim \varepsilon_1^{1/2} \|S_{Q_0} u_0^t\|_{L^{p'}(Q_0)} \|M_{Q_0}^d k_{L_1}^{X_0}\|_{L^p(Q_0)} \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(Q_0)}, \quad (2.17)$$

where we have used that $M_{Q_0}^d$ is bounded in $L^p(Q_0)$ and that

$$\|S_{Q_0} u_0^t\|_{L^{p'}(Q_0)} \lesssim \|\widetilde{\mathcal{N}}_{Q_0,*} u_0^t\|_{L^{p'}(Q_0)} \lesssim \|g_t\|_{L^{p'}(Q_0)} \lesssim \|g\|_{L^{p'}(Q_0)} = 1,$$

which follows from (1.46), Lemma 1.35(a), $\omega_{L_0} \in RH_p(\partial\Omega)$, (2.8), and Lemma 1.41.

From (2.9), (2.17), and for all $0 < t < Cr_{Q_0}/2$,

$$0 \leq u_1^t(X_0) \leq F_{Q_0}^t(X_0) + u_0^t(X_0) \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(Q_0)} + \|k_{L_0}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})},$$

where we have used Hölder's inequality, that $\|g_t\|_{L^{p'}(\partial\Omega)} \lesssim 1$ and Lemma 1.41, and the implicit constants do not depend on t. Next, using the previous estimate and Hölder's inequality we see that

$$\int_{\partial\Omega} g(y) k_{L_1}^{X_0}(y)\, d\sigma(y) = u_1^t(X_0) + \int_{\partial\Omega} (g(y) - g_t(y)) k_{L_1}^{X_0}(y)\, d\sigma(y)$$

$$\lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(Q_0)} + \|k_{L_0}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})} + \|g - g_t\|_{L^{p'}(\partial\Omega)} \|k_{L_1}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})}.$$

Note that $\|k_{L_1}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})} \leq C_j < \infty$ by Lemma 1.30(e) and Harnack's inequality ($L_0 \equiv L_1$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$). Recall that $\|g - g_t\|_{L^{p'}(\partial\Omega)} \to 0$ as $t \to 0$ (see Lemma 1.41) and hence

$$\int_{\partial\Omega} g(y) k_{L_1}^{X_0}(y)\, d\sigma(y) \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(Q_0)} + \|k_{L_0}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})}.$$

Taking the supremum over $0 \leq g \in L^{p'}(Q_0)$ with $\|g\|_{L^{p'}(Q_0)} = 1$, the latter implies

$$\|k_{L_1}^{X_0}\|_{L^p(Q_0)} \leq C\varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(Q_0)} + C\|k_{L_0}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})},$$

with C depending only on dimension, p, the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$. As mentioned above, $\|k_{L_1}^{X_0}\|_{L^p(Q_0)} \leq C_j < \infty$, thus taking $\varepsilon_1 < C^{-2}/4$ we can hide the first term in the left hand side, and consequently $\|k_{L_1}^{X_0}\|_{L^p(Q_0)} \lesssim \|k_{L_0}^{X_0}\|_{L^p(2\widetilde{\Delta}_{Q_0})}$. Recalling that $X_0 = X_{M_0\Delta_{Q_0}}$ we have that $\delta(X_{Q_0}) \approx \ell(Q_0)$, $\delta(X_0) \approx M_0\ell(Q_0) \geq \ell(Q_0)$, $\delta(X_{2\widetilde{\Delta}_{Q_0}}) \approx \ell(Q_0)$. Also, $|X_0 - X_{Q_0}| + |X_0 - X_{2\widetilde{\Delta}_{Q_0}}| \lesssim M_0\ell(Q_0)$. Hence, using Harnack's inequality (with constants depending on M_0, which has been already fixed), and the fact that $\omega_{L_0} \in RH_p(\partial\Omega)$, we conclude that

$$\int_{Q_0} k_{L_1}^{X_{Q_0}}(y)^p \, d\sigma(y) \approx \int_{Q_0} k_{L_1}^{X_0}(y)^p \, d\sigma(y) \lesssim \int_{2\widetilde{\Delta}_{Q_0}} k_{L_0}^{X_0}(y)^p \, d\sigma(y)$$

$$\approx \int_{2\widetilde{\Delta}_{Q_0}} k_{L_0}^{X_{2\widetilde{\Delta}_{Q_0}}}(y)^p \, d\sigma(y) \lesssim \sigma(2\widetilde{\Delta}_{Q_0})^{1-p} \approx \sigma(Q_0)^{1-p}. \quad (2.18)$$

2.1.3   Self-improvement of Step 1

The goal of this section is to extend (2.18) and show that it holds with the integration taking place in an arbitrary $Q \in \mathbb{D}_{Q_0}$, but with the pole of the elliptic measure being X_{Q_0}. In doing this, we will lose the exponent p, showing that a RH_q inequality holds for some fixed q.

Fix $Q \in \mathbb{D}_{Q_0}$, and let L_1^Q be the operator defined by $L_1^Q u = -\operatorname{div}(A_1^Q \nabla u)$, where

$$A_1^Q(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in \Omega_{\mathcal{F},Q}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus \Omega_{\mathcal{F},Q}, \end{cases}$$

with $\widetilde{A} = \widetilde{A}^j$ as in (2.6). Since $L_1^Q \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$, Lemma 1.30(e) implies that $\omega_{L_1^Q} \ll \sigma$, hence there exists $k_{L_1^Q}^X = d\omega_{L_1^Q}^X/d\sigma$. Our first goal is to obtain

$$\int_Q k_{L_1^Q}^{X_Q}(y)^p \, d\sigma(y) \lesssim \sigma(Q)^{1-p}. \quad (2.19)$$

We consider two cases. Suppose first that $Q \subset Q_i$ for some $Q_i \in \mathcal{F}$, then $\Omega_{\mathcal{F},Q} = \emptyset$, $L_1^Q \equiv L_0$ in Ω, and (2.19) is a consequence of the fact that $\omega_{L_0} \in RH_p(\partial\Omega)$. In other case, that is, if $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, we define $\mathcal{F}_Q = \{Q_i \in \mathcal{F} : Q_i \cap Q \neq \emptyset\} = \{Q_i \in \mathcal{F} : Q_i \subsetneq Q\}$. Note that $A_0 - A_1^Q$ is supported in $\Omega_{\mathcal{F}_Q,Q} = \Omega_{\mathcal{F},Q}$, and clearly

$$\|\mathfrak{m}_{\mathcal{F}_Q}\|_{\mathcal{C}(Q)} = \sup_{Q' \in \mathbb{D}_Q} \frac{\mathfrak{m}_{\mathcal{F}_Q}(\mathbb{D}_{Q'})}{\sigma(Q')} \leq \sup_{Q' \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}_{\mathcal{F}}(\mathbb{D}_{Q'})}{\sigma(Q')} \leq \varepsilon_1.$$

We can then repeat the argument of Step 1 for the operator L_1^Q replacing L_1, and with Q and $\mathcal{F}_Q$ in place of respectively Q_0 and $\mathcal{F}$. Hence, the estimate (2.18) becomes (2.19).

We next notice that using [HM3, Lemma 3.55], there exists $0 < \widehat{\kappa}_1 < \kappa_1$ (see (1.6)), depending only on the allowable parameters, such that $\widehat{\kappa}_1 B_Q \cap \Omega_{\mathcal{F},Q_0} =$

$\widehat{\kappa}_1 B_Q \cap \Omega_{\mathcal{F},Q}$. This easily gives that $L_1 \equiv L_1^Q$ in $\widehat{\kappa}_1 B_Q \cap \Omega$. Using now Lemma 1.30(e) and Harnack's inequality, we have

$$k_{L_1}^{X_Q}(y) \approx k_{L_1^Q}^{X_Q}(y), \quad \text{for } \sigma\text{-a.e. } y \in \eta\Delta_Q, \tag{2.20}$$

where $\eta = \widehat{\kappa}_1/(2\kappa_0)$ and κ_0 is as in (1.7), and hence $\eta\Delta_Q \subset \Delta_Q \subset Q$. Combining (2.19), (2.20), Lemma 1.26, Lemma 1.30(b) and Harnack's inequality we obtain

$$\left(\fint_{\eta\Delta_Q} k_{L_1}^{X_Q}(y)^p \, d\sigma(y) \right)^{1/p} \lesssim \left(\fint_Q k_{L_1^Q}^{X_Q}(y)^p \, d\sigma(y) \right)^{1/p} \lesssim \sigma(Q)^{-1}$$

$$\lesssim \sigma(Q)^{-1} \omega_{L_1}^{X_Q}(Q) \lesssim \fint_{\eta\Delta_Q} k_{L_1}^{X_Q}(y) \, d\sigma(y).$$

Now, using Remark 1.31 we have that

$$\left(\fint_{\eta\Delta_Q} k_{L_1}^{X_{Q_0}}(y)^p \, d\sigma(y) \right)^{1/p} \leq C_1 \fint_{\eta\Delta_Q} k_{L_1}^{X_{Q_0}}(y) \, d\sigma(y), \tag{2.21}$$

with $C_1 > 1$ depending only on dimension, p, the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$. Note that (2.21) holds then for every $Q \in \mathbb{D}_{Q_0}$. Also, by means of Lemma 1.26, Lemma 1.30(b) and Harnack's inequality, there exists $C_\eta > 1$ such that $0 < \omega_{L_1}^{X_{Q_0}}(Q) \leq C_\eta \omega_{L_1}^{X_{Q_0}}(\eta\Delta_Q)$ for every $Q \in \mathbb{D}_{Q_0}$. The following result is a generalization of [HM2, Lemma B.7] to our dyadic setting. In what follows, given $0 \leq v \in L^1_{\mathrm{loc}}(\Omega)$ and given $F \subset \partial\Omega$ we write $v(F) := \int_F v(y) d\sigma(y)$.

Lemma 2.4. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set such that $\partial\Omega$ satisfies the AR property. Fix $0 < \eta < 1$, $Q_0 \in \mathbb{D}(\partial\Omega)$ and let $v \in L^1(Q_0)$ be such that $0 < v(Q) \leq C_0 v(\eta\Delta_Q)$ for every $Q \in \mathbb{D}_{Q_0}$, for some uniform $C_0 \geq 1$. Suppose also that there exist $C_1 \geq 1$ and $1 < p < \infty$ such that*

$$\left(\fint_{\eta\Delta_Q} v(y)^p \, d\sigma(y) \right)^{1/p} \leq C_1 \fint_{\eta\Delta_Q} v(y) \, d\sigma(y), \qquad Q \in \mathbb{D}_{Q_0}, \tag{2.22}$$

then $v \in A_\infty^{\mathrm{dyadic}}(Q_0)$, with the implicit constants depending on dimension, p, C_0, C_1, η and the AR constant.

Proof. We first prove that for every $Q \in \mathbb{D}_{Q_0}$ and every Borel set $F \subset \eta\Delta_Q$, there holds

$$\frac{v(F)}{v(\eta\Delta_Q)} \leq C_1 \left(\frac{\sigma(F)}{\sigma(\eta\Delta_Q)} \right)^{1/p'}. \tag{2.23}$$

Indeed, using Hölder's inequality together with (2.22), we obtain

$$\frac{v(F)}{\sigma(\eta\Delta_Q)} = \frac{1}{\sigma(\eta\Delta_Q)} \int_F v(y) \, d\sigma(y) \leq \left(\frac{\sigma(F)}{\sigma(\eta\Delta_Q)} \right)^{1/p'} \left(\fint_{\eta\Delta_Q} v(y)^p \, d\sigma(y) \right)^{1/p}$$

$$\leq C_1 \left(\frac{\sigma(F)}{\sigma(\eta\Delta_Q)} \right)^{1/p'} \fint_{\eta\Delta_Q} v(y) \, d\sigma(y),$$

which is equivalent to (2.23).

To obtain that $v \in A_\infty^{\text{dyadic}}(Q_0)$, we observe that $\sigma(Q) \leq C\sigma(\eta\Delta_Q)$ with $C > 1$ depending only on AR and n. Fix then $0 < \alpha < (CC_1^{p'})^{-1}$ and take $E \subset Q$ such that $\sigma(E) > (1-\alpha)\sigma(Q)$. Writing $E_0 = E \cap \eta\Delta_Q$ and $F_0 = \eta\Delta_Q \setminus E$, it is clear that

$$(1-\alpha)\frac{\sigma(Q)}{\sigma(\eta\Delta_Q)} < \frac{\sigma(E)}{\sigma(\eta\Delta_Q)} \leq \frac{\sigma(E_0)}{\sigma(\eta\Delta_Q)} + \frac{\sigma(Q \setminus \eta\Delta_Q)}{\sigma(\eta\Delta_Q)} = \frac{\sigma(E_0)}{\sigma(\eta\Delta_Q)} + \frac{\sigma(Q)}{\sigma(\eta\Delta_Q)} - 1,$$

and hence

$$\frac{\sigma(F_0)}{\sigma(\eta\Delta_Q)} = 1 - \frac{\sigma(E_0)}{\sigma(\eta\Delta_Q)} < \alpha\,\frac{\sigma(Q)}{\sigma(\eta\Delta_Q)} \leq C\,\alpha. \tag{2.24}$$

Combining (2.23) and (2.24) we obtain $v(F_0)/v(\eta\Delta_Q) < C_1(C\alpha)^{1/p'}$. This and the fact that $v(Q) \leq C_0 v(\eta\Delta_Q)$ yield

$$\frac{v(E)}{v(Q)} \geq \frac{v(\eta\Delta_Q)}{v(Q)}\,\frac{v(E_0)}{v(\eta\Delta_Q)} \geq C_0^{-1}\left(1 - \frac{v(F_0)}{v(\eta\Delta_Q)}\right) > C_0^{-1}\left(1 - C_1(C\alpha)^{1/p'}\right) =: 1 - \beta,$$

with $0 < \beta < 1$ by our choice of α. This eventually proves that $v \in A_\infty^{\text{dyadic}}(Q_0)$ and the proof is complete. ∎

Using Lemma 2.4 we obtain that $\omega_{L_1}^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$. This and Lemma 1.13(b) yield $\mathcal{P}_{\mathcal{F}}\omega_{L_1}^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ and this finishes the first step.

2.1.4　Step 2

We define a new operator L_2 by changing L_1 below the region $\Omega_{\mathcal{F},Q_0}$. More precisely, set $L_2 u = -\operatorname{div}(A_2\nabla u)$ with

$$A_2(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in T_{Q_0} \setminus \Omega_{\mathcal{F},Q_0}, \\ A_1(Y) & \text{if } Y \in \Omega \setminus (T_{Q_0} \setminus \Omega_{\mathcal{F},Q_0}). \end{cases}$$

Note that by construction, $A_2 = \widetilde{A}$ in T_{Q_0} and $A_2 = A_0$ in $\Omega \setminus T_{Q_0}$. Our goal is to prove that $\mathcal{P}_{\mathcal{F}}\omega_{L_2}^{X_{Q_0}} \in A_\infty^{\text{dyadic}}(Q_0)$ by using the following lemma.

Lemma 2.5 ([HMT2])**.** *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided* CAD*. Given $Q_0 \in \mathbb{D}(\partial\Omega)$ and $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$, a family of pairwise disjoint dyadic cubes, let $\mathcal{P}_{\mathcal{F}}$ be the corresponding projection operator defined in (1.14). Given an elliptic operator L, we denote by $\omega_L = \omega_{L,\Omega}^{A_{Q_0}}$ and $\omega_{L,\star} = \omega_{L,\Omega_{\mathcal{F},Q_0}}^{A_{Q_0}}$ the elliptic measures of L with respect to Ω and $\Omega_{\mathcal{F},Q_0}$ with fixed pole at the corkscrew point $A_{Q_0} \in \Omega_{\mathcal{F},Q_0}$ (cf. [HM3, Proposition 6.4]). Let $\nu_L = \nu_L^{A_{Q_0}}$ be the measure defined by*

$$\nu_L(F) = \omega_{L,\star}\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\omega_L(F \cap Q_i)}{\omega_L(Q_i)}\omega_{L,\star}(P_i), \qquad F \subset Q_0, \tag{2.25}$$

where P_i is the cube produced by [HM3, Proposition 6.7]. Then $\mathcal{P}_{\mathcal{F}}\nu_L$ depends only on $\omega_{L,\star}$ and not on ω_L. More precisely,

$$\mathcal{P}_{\mathcal{F}}\nu_L(F) = \omega_{L,\star}\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(F \cap Q_i)}{\sigma(Q_i)}\omega_{L,\star}(P_i), \qquad F \subset Q_0. \tag{2.26}$$

Moreover, there exists θ > 0 such that for all Q ∈ $\mathbb{D}_{Q_0}$ and all F ⊂ Q, we have

$$\left(\frac{\mathcal{P}_\mathcal{F}\omega_L(F)}{\mathcal{P}_\mathcal{F}\omega_L(Q)}\right)^\theta \lesssim \frac{\mathcal{P}_\mathcal{F}\nu_L(F)}{\mathcal{P}_\mathcal{F}\nu_L(Q)} \lesssim \frac{\mathcal{P}_\mathcal{F}\omega_L(F)}{\mathcal{P}_\mathcal{F}\omega_L(Q)}. \tag{2.27}$$

For $k = 1, 2$, we write $\omega_{L_k} = \omega_{L_k,\Omega}^{A_{Q_0}}$ and $\omega_{L_k,\star} = \omega_{L_k,\Omega_{\mathcal{F},Q_0}}^{A_{Q_0}}$ for the elliptic measures of L_k with respect to the domains Ω and $\Omega_{\mathcal{F},Q_0}$, with fixed pole at A_{Q_0} (see [HM3, Proposition 6.4]). Note that since $A_1 = A_2$ in $\Omega_{\mathcal{F},Q_0}$ then $\omega_{L_1,\star} = \omega_{L_2,\star}$. Finally let $\nu_{L_k} = \nu_{L_k}^{A_{Q_0}}$ be the corresponding measures defined as in (2.25), and observe that (2.26) implies $\mathcal{P}_\mathcal{F}\nu_{L_1} = \mathcal{P}_\mathcal{F}\nu_{L_2}$.

In Step 1 we have shown that $\mathcal{P}_\mathcal{F}\omega_{L_1}^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q_0)$, thus Harnack's inequality and (2.27) give that $\mathcal{P}_\mathcal{F}\nu_{L_2} = \mathcal{P}_\mathcal{F}\nu_{L_1} \in A_\infty^{\mathrm{dyadic}}(Q_0)$. Another use of (2.27) and Harnack's inequality allows us to obtain that $\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}} \approx \mathcal{P}_\mathcal{F}\omega_{L_2} \in A_\infty^{\mathrm{dyadic}}(Q_0)$. Note that by Lemma 1.30(b), Harnack's inequality and Lemma 1.13(a) it follows that $\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}$ is dyadically doubling in Q_0.

Thus, [HM3, Lemma B.7] implies that there exist $\theta, \theta' > 0$ such that

$$\left(\frac{\sigma(E)}{\sigma(Q)}\right)^\theta \lesssim \frac{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(Q)} \lesssim \left(\frac{\sigma(E)}{\sigma(Q)}\right)^{\theta'}, \qquad Q \in \mathbb{D}_{Q_0}, \qquad E \subset Q. \tag{2.28}$$

2.1.5 Step 3

To complete the proof it remains to change the operator outside T_{Q_0}. Let us introduce $L_3 u = -\operatorname{div}(A_3\nabla u)$, where

$$A_3(Y) := \begin{cases} A_2(Y) & \text{if } Y \in T_{Q_0}, \\ \widetilde{A}(Y) & \text{if } Y \in \Omega \setminus T_{Q_0}, \end{cases}$$

and note that $L_3 \equiv \widetilde{L}$ in Ω.

We want to prove that for every $0 < \varepsilon < 1$, there exists $C_\varepsilon > 1$ such that

$$E \subset Q_0, \quad \frac{\sigma(E)}{\sigma(Q_0)} \geq \varepsilon \implies \frac{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(Q_0)} \geq \frac{1}{C_\varepsilon}. \tag{2.29}$$

Let $0 < \varepsilon < 1$ and let $E \subset Q_0$ be such that $\sigma(E) \geq \varepsilon\sigma(Q_0)$. First, we can disregard the trivial case $\mathcal{F} = \{Q_0\}$:

$$\frac{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(Q_0)} = \frac{\frac{\sigma(E)}{\sigma(Q_0)}\omega_{L_3}^{X_{Q_0}}(Q_0)}{\frac{\sigma(Q_0)}{\sigma(Q_0)}\omega_{L_3}^{X_{Q_0}}(Q_0)} = \frac{\sigma(E)}{\sigma(Q_0)} \geq \varepsilon.$$

Suppose then that $\mathcal{F} \subsetneq \mathbb{D}_{Q_0} \setminus \{Q_0\}$. For $\tau \ll 1$ we consider the sets

$$\Sigma_\tau := \left\{x \in Q_0 : \operatorname{dist}(x, \partial\Omega \setminus Q_0) < \tau\ell(Q_0)\right\}$$

and $\widetilde{Q_0} := Q_0 \setminus \bigcup_{Q' \in \mathcal{I}_\tau} Q'$, where

$$\mathcal{I}_\tau = \{Q' \in \mathbb{D}_{Q_0} : \tau\ell(Q_0) < \ell(Q') \leq 2\tau\ell(Q_0),\ Q' \cap \Sigma_\tau \neq \varnothing\}.$$

By construction, $\Sigma_\tau \subset \bigcup_{Q' \in \mathcal{I}_\tau} Q'$, and there exists $C = C(n, \mathrm{AR}) > 0$ such that every $Q' \in \mathcal{I}_\tau$ satisfies $Q' \subset \Sigma_{C\tau}$. Using Lemma 1.6(f), for $\tau = \tau(\varepsilon) > 0$ sufficiently small we have

$$\sigma(Q_0 \setminus \widetilde{Q_0}) \leq \sigma(\Sigma_{C\tau}) \leq C_1(C\tau)^\eta \sigma(Q_0) \leq \frac{\varepsilon}{2}\,\sigma(Q_0),$$

and letting $F = E \cap \widetilde{Q_0}$, it follows that

$$\varepsilon\sigma(Q_0) \leq \sigma(E) \leq \sigma(F) + \sigma(Q_0 \setminus \widetilde{Q_0}) \leq \sigma(F) + \frac{\varepsilon}{2}\sigma(Q_0).$$

Hence $\sigma(F)/\sigma(Q_0) \geq \varepsilon/2$ and by (2.28), we conclude that

$$\frac{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(Q_0)} \gtrsim \left(\frac{\sigma(F)}{\sigma(Q_0)}\right)^\theta \geq \left(\frac{\varepsilon}{2}\right)^\theta. \tag{2.30}$$

We claim that there exists $c_\varepsilon > 0$ such that $\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(F) \geq c_\varepsilon\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(F)$. Assuming this momentarily, we easily obtain (2.29):

$$\frac{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(Q_0)} \geq \mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(F) \geq c_\varepsilon\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(F) \gtrsim c_\varepsilon\frac{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(F)}{\mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(Q_0)} \geq c_\varepsilon\left(\frac{\varepsilon}{2}\right)^\theta =: \frac{1}{C_\varepsilon},$$

where we have used Lemma 1.26, (2.30), and the fact that $\mathcal{P}_\mathcal{F}\omega_{L_k}^{X_{Q_0}}(Q_0) = \omega_{L_k}^{X_{Q_0}}(Q_0)$ for $k = 2, 3$.

Let us then show our claim. First, since $L_2 \equiv L_3$ in T_{Q_0} and $\widetilde{Q_0} \subset Q_0 \setminus \Sigma_\tau$, Lemma 1.32 yields

$$k_{L_2}^{X_{Q_0}}(y) \approx_\tau k_{L_3}^{X_{Q_0}}(y), \quad \text{for } \sigma\text{-a.e. } y \in \widetilde{Q_0}. \tag{2.31}$$

This and the fact that $F \subset \widetilde{Q_0}$ give

$$\omega_{L_2}^{X_{Q_0}}\left(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\right) \approx_\tau \omega_{L_3}^{X_{Q_0}}\left(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\right),$$

which in turn yields

$$\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(F) = \omega_{L_3}^{X_{Q_0}}\left(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\right) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(F \cap Q_i)}{\sigma(Q_i)}\,\omega_{L_3}^{X_{Q_0}}(Q_i)$$

$$\geq c_\tau\omega_{L_2}^{X_{Q_0}}\left(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\right) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(F \cap Q_i)}{\sigma(Q_i)}\,\omega_{L_3}^{X_{Q_0}}(Q_i). \tag{2.32}$$

It remains to estimate the second term. Note that in the sum we can restrict ourselves to those cubes $Q_i \in \mathcal{F}$ such that $F \cap Q_i \neq \varnothing$. We consider two cases. If $Q_i \subset \widetilde{Q_0}$, using (2.31) we have that $\omega_{L_3}^{X_{Q_0}}(Q_i) \approx_\tau \omega_{L_2}^{X_{Q_0}}(Q_i)$. Otherwise, if $Q_i \setminus \widetilde{Q_0} \neq$

Ø, there exists $Q' \in \mathcal{I}_\tau$ such that $Q_i \cap Q' \neq \varnothing$. Then $Q' \subsetneq Q_i$ (if $Q_i \subset Q'$ then $Q_i \subset Q_0 \setminus \widetilde{Q_0}$, contradicting that $F \cap Q_i \neq \varnothing$ and $F \subset \widetilde{Q_0}$) and, in particular, $\ell(Q_i) > \tau\ell(Q_0)$. Let x_{Q_i} be the center of Q_i, and let $\Delta_{Q_i} = \Delta(x_{Q_i}, r_{Q_i})$ with $r_{Q_i} \approx \ell(Q_i)$ as in (1.2). Take $\widehat{Q_i} \in \mathbb{D}_{Q_i}$ with $x_{Q_i} \in \widehat{Q_i}$, $\ell(\widehat{Q_i}) = 2^{-M}\ell(Q_i)$ and $M > 1$ to be chosen. Notice that $\mathrm{diam}(\widehat{Q_i}) \approx 2^{-M}\ell(Q_i) \approx 2^{-M}r_{Q_i}$ and clearly

$$r_{Q_i} \leq \mathrm{dist}(x_{Q_i}, \partial\Omega \setminus \Delta_{Q_i}) \leq \mathrm{diam}(\widehat{Q_i}) + \mathrm{dist}(\widehat{Q_i}, \partial\Omega \setminus \Delta_{Q_i})$$
$$\approx 2^{-M}r_{Q_i} + \mathrm{dist}(\widehat{Q_i}, \partial\Omega \setminus \Delta_{Q_i}).$$

Taking $M \gg 1$ large enough (depending on the AR constant), we conclude that $c\tau\ell(Q_0) < \mathrm{dist}(\widehat{Q_i}, \partial\Omega \setminus \Delta_{Q_i}) \leq \mathrm{dist}(\widehat{Q_i}, \partial\Omega \setminus Q_0)$ and hence $\widehat{Q_i} \subset Q_0 \setminus \Sigma_{c\tau}$. Again, using Lemma 1.32 and the fact that $\omega_{L_2}^{X_{Q_0}}$ is doubling in Q_0 (which is a consequence of Lemma 1.30(b) and Harnack's inequality), we obtain

$$\omega_{L_3}^{X_{Q_0}}(Q_i) \geq \omega_{L_3}^{X_{Q_0}}(\widehat{Q_i}) \approx_\tau \omega_{L_2}^{X_{Q_0}}(\widehat{Q_i}) \gtrsim \omega_{L_2}^{X_{Q_0}}(Q_i).$$

In the two cases, since $\tau = \tau(\varepsilon)$, (2.32) turns into

$$\mathcal{P}_\mathcal{F}\omega_{L_3}^{X_{Q_0}}(F) \gtrsim_\varepsilon \omega_{L_2}^{X_{Q_0}}\Big(F \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(Q_i \cap F)}{\sigma(Q_i)}\omega_{L_2}^{X_{Q_0}}(Q_i) = \mathcal{P}_\mathcal{F}\omega_{L_2}^{X_{Q_0}}(F),$$

completing the proof of our claim.

Recalling that $\widetilde{L} \equiv L_3$, the previous argument proves the following proposition:

Proposition 2.6. *There exists $\varepsilon_1 > 0$ (depending only on dimension, p, the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$) such that the following property holds: given $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 1$ such that for every $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/M_0$ and every $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_{Q_0}$ with $\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q_0)} \leq \varepsilon_1$, there holds*

$$E \subset Q_0, \quad \frac{\sigma(E)}{\sigma(Q_0)} \geq \varepsilon \implies \frac{\mathcal{P}_\mathcal{F}\omega_{\widetilde{L}}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{\widetilde{L}}^{X_{Q_0}}(Q_0)} \geq \frac{1}{C_\varepsilon}, \tag{2.33}$$

where $\widetilde{L} = L^j$ is the operator defined in (2.6) and $j \in \mathbb{N}$ is arbitrary.

2.1.6   Step 4

What we have proved so far does not allow us to apply Lemma 1.18. We have to be able to fix the pole relative to Q_0, and show that (2.33) also holds for all $Q \in \mathbb{D}_{Q_0}$.

Proposition 2.7. *Let ε_1 be the parameter obtained in Proposition 2.6. Given $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 1$ such that for every $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/M_0$, every $Q \in \mathbb{D}_{Q_0}$, every $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_Q$ with $\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} \leq \varepsilon_1$, there holds*

$$E \subset Q, \quad \frac{\sigma(E)}{\sigma(Q)} \geq \varepsilon \implies \frac{\mathcal{P}_\mathcal{F}\omega_{\widetilde{L}}^{X_{Q_0}}(E)}{\mathcal{P}_\mathcal{F}\omega_{\widetilde{L}}^{X_{Q_0}}(Q)} \geq \frac{1}{C_\varepsilon}, \tag{2.34}$$

where $\widetilde{L} = L^j$ is the operator defined in (2.6) and $j \in \mathbb{N}$ is arbitrary. Consequently, there exists $1 < q < \infty$ such that $\omega_{\widetilde{L}}^{X_{Q_0}} \in RH_q^{\mathrm{dyadic}}(Q_0)$ uniformly in $Q_0 \in \mathbb{D}(\partial\Omega)$ provided $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/M_0$, and moreover $\omega_{\widetilde{L}} \in RH_q(\partial\Omega)$.

Proof. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/M_0$. Let $0 < \varepsilon < 1$, $Q \in \mathbb{D}_{Q_0}$. Let $\mathcal{F} = \{Q_i\} \subset \mathbb{D}_Q$ be such that $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq \varepsilon_1$ and let $E \subset Q$ satisfy $\sigma(E) \geq \varepsilon\sigma(Q)$. By Lemma 1.30(c) (see also Remark 1.31) and the fact that $\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_Q}(Q) = \omega_{\widetilde{L}}^{X_Q}(Q) \approx 1$ by Lemma 1.26, we see that

$$\frac{\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_{Q_0}}(E)}{\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_{Q_0}}(Q)} = \frac{\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_{Q_0}}(E)}{\omega_{\widetilde{L}}^{X_{Q_0}}(Q)} \approx \omega_{\widetilde{L}}^{X_Q}\Big(E \setminus \bigcup_{Q_i \in \mathcal{F}} Q_i\Big) + \sum_{Q_i \in \mathcal{F}} \frac{\sigma(E \cap Q_i)}{\sigma(Q_i)} \omega_{\widetilde{L}}^{X_Q}(Q_i)$$

$$= \mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_Q}(E) \approx \frac{\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_Q}(E)}{\mathcal{P}_{\mathcal{F}}\omega_{\widetilde{L}}^{X_Q}(Q)} \geq \frac{1}{C_\varepsilon},$$

where in the last inequality we have applied Proposition 2.6 to Q (replacing Q_0) satisfying $\ell(Q) < \mathrm{diam}(\partial\Omega)/M_0$. This shows (2.34), which together with Lemma 2.2 and our choice of M_0, allows us to invoke Lemma 1.18 and eventually conclude that $\omega_{\widetilde{L}}^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q_0)$ uniformly in Q_0, provided $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/M_0$. Thus, there exists $1 < q < \infty$, such that $\omega_{\widetilde{L}}^{X_{Q_0}} \in RH_q^{\mathrm{dyadic}}(Q_0)$ uniformly in Q_0 for the same class of cubes and, in particular,

$$\int_{Q_0} k_{\widetilde{L}}^{X_{Q_0}}(y)^q \, d\sigma(y) \lesssim \sigma(Q_0)^{1-q}, \qquad Q_0 \in \mathbb{D}(\partial\Omega), \quad \ell(Q_0) < \frac{\mathrm{diam}(\partial\Omega)}{M_0}. \quad (2.35)$$

When $\mathrm{diam}(\partial\Omega) < \infty$, we need to extend the previous estimate to all cubes with sidelength of the order of $\mathrm{diam}(\partial\Omega)$. Let us then take $Q_0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q_0) \geq \mathrm{diam}(\partial\Omega)/M_0$ and define the collection

$$\mathcal{I}_{Q_0} = \Big\{ Q \in \mathbb{D}_{Q_0} : \frac{\mathrm{diam}(\partial\Omega)}{2M_0} \leq \ell(Q) < \frac{\mathrm{diam}(\partial\Omega)}{M_0} \Big\}.$$

Note that $Q_0 = \bigcup_{Q \in \mathcal{I}_{Q_0}} Q$ is a disjoint union and using the AR property we have that

$$\#\mathcal{I}_{Q_0} \Big(\frac{\mathrm{diam}(\partial\Omega)}{2M_0}\Big)^n \leq \sum_{Q \in \mathcal{I}_{Q_0}} \ell(Q)^n \approx \sum_{Q \in \mathcal{I}_{Q_0}} \sigma(Q) = \sigma(Q_0) \approx \ell(Q_0)^n \lesssim \mathrm{diam}(\partial\Omega)^n,$$

which implies $\#\mathcal{I}_{Q_0} \lesssim M_0^n$. We can use Harnack's inequality to move the pole from X_{Q_0} to X_Q for any $Q \in \mathcal{I}_{Q_0}$ (with constants depending on M_0, which is already fixed), since $\delta(X_{Q_0}) \approx \ell(Q_0) > \ell(Q)$, $\delta(X_Q) \approx \ell(Q)$ and $|X_{Q_0} - X_Q| \lesssim M_0\ell(Q)$. Hence, we obtain

$$\int_{Q_0} k_{\widetilde{L}}^{X_{Q_0}}(y)^q \, d\sigma(y) \approx \sum_{Q \in \mathcal{I}_{Q_0}} \int_Q k_{\widetilde{L}}^{X_Q}(y)^q \, d\sigma(y) \lesssim \sum_{Q \in \mathcal{I}_{Q_0}} \sigma(Q)^{1-q}$$

$$\lesssim \#\mathcal{I}_{Q_0} \mathrm{diam}(\partial\Omega)^{(1-q)n} \lesssim \sigma(Q_0)^{1-q},$$

where we have used (2.35) for Q since $\ell(Q) < \operatorname{diam}(\partial\Omega)/M_0$, and the AR property. Therefore, we have extended (2.35) to all $Q_0 \in \mathbb{D}(\partial\Omega)$ and Remark 1.36 yields that $\omega_{\widetilde{L}} \in RH_q(\partial\Omega)$, where $\widetilde{L} = L^j$ and the implicit constants are independent of $j \in \mathbb{N}$.
∎

2.1.7   Step 5

In the previous step we have proved that $\omega_{\widetilde{L}} \in RH_q(\partial\Omega)$ where $\widetilde{L} = L^j$ and the implicit constants are all uniform in j. To complete the proof of Theorem 2.1(a) we show that $\omega_L \in RH_q(\partial\Omega)$ using the following result:

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD. Let L and L_0 be real symmetric elliptic operators with matrices A and A_0 respectively. For every $j \in \mathbb{N}$, let $L^j u = -\operatorname{div}(A^j \nabla u)$, with $A^j(Y) = A(Y)$ if $\delta(Y) \geq 2^{-j}$ and $A^j(Y) = A_0(Y)$ if $\delta(Y) < 2^{-j}$. Assume that there exists $1 < q < \infty$ such that $\omega_{L^j} = \omega_{L^j,\Omega} \in RH_q(\partial\Omega)$ uniformly in j, for every $j \geq j_0$. That is, $\omega_{L^j,\Omega} \ll \sigma$ and there exists C such that*

$$\int_\Delta k_{L^j,\Omega}^{X_\Delta}(y)^q \, d\sigma(y) \leq C\sigma(\Delta)^{1-q}, \qquad k_{L^j,\Omega}^{X_\Delta} := d\omega_{L^j,\Omega}^{X_\Delta}/d\sigma, \qquad (2.36)$$

for every $j \geq j_0$ and every $\Delta(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Then $\omega_{L,\Omega} \in RH_q(\partial\Omega)$.

Proof. Fix $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)/25$, set $\Delta_0 = B_0 \cap \partial\Omega$, and consider the subdomain $\Omega_\star := T_{20\Delta_0}$. Using [HM3, Lemma 3.61] we know that $\Omega_\star$ is a bounded 1-sided CAD, with constants depending only on those of Ω. Applying Lemma 1.30(d) it follows that $\omega_{L^j,\Omega_\star} \ll \sigma$ in $4\Delta_0$ and also

$$k_{L^j,\Omega}^{X_{4\Delta_0}}(y) \approx k_{L^j,\Omega_\star}^{X_{4\Delta_0}}(y), \quad \text{for } \sigma\text{-a.e. } y \in 4\Delta_0.$$

Recalling (1.7) we know that $25B_0 \cap \Omega \subset \Omega_\star$. In particular, $10B_0 \cap \partial\Omega = 10B_0 \cap \partial\Omega_\star$ and $\sigma_\star := H^n|_{\partial\Omega_\star}$ coincides with σ in $4\Delta_0$. Therefore, (2.36) gives

$$\int_{4\Delta_0} k_{L^j,\Omega_\star}^{X_{4\Delta_0}}(y)^q \, d\sigma_\star(y) \approx \int_{4\Delta_0} k_{L^j,\Omega}^{X_{4\Delta_0}}(y)^q \, d\sigma(y) \lesssim \sigma(\Delta_0)^{1-q} \qquad (2.37)$$

uniformly in $j \in \mathbb{N}$. Note also that $\delta_\star(X_{4\Delta_0}) = \delta(X_{4\Delta_0})$, where $\delta_\star(Y) = \operatorname{dist}(Y, \partial\Omega_\star)$:

$$\delta_\star(X_{4\Delta_0}) = \operatorname{dist}(X_{4\Delta_0}, 10B_0 \cap \partial\Omega_\star) = \operatorname{dist}(X_{4\Delta_0}, 10B_0 \cap \partial\Omega) = \delta(X_{4\Delta_0}).$$

Define, for every $g \in C_c(\partial\Omega_\star)$

$$\Phi(g) := \int_{\partial\Omega_\star} g(y) \, d\omega_{L,\Omega_\star}^{X_{4\Delta_0}}(y).$$

Let $g \in \operatorname{Lip}_c(\partial\Omega)$ be such that $\operatorname{supp}(g) \subset 4\Delta_0$ and extend g by zero to $\partial\Omega_\star \setminus 4\Delta_0$ (by a slight abuse of notation we will call the extension g) so that $g \in \operatorname{Lip}_c(\partial\Omega_\star)$ and define

$$u(X) = \int_{\partial\Omega_\star} g(y) \, d\omega_{L,\Omega_\star}^X(y), \qquad u_j(X) = \int_{\partial\Omega_\star} g(y) \, d\omega_{L^j,\Omega_\star}^X(y), \qquad X \in \Omega_\star.$$

Since $g \in \mathrm{Lip}_c(\partial\Omega_\star) \subset H^{1/2}(\partial\Omega_\star) \cap C_c(\partial\Omega_\star)$, using Lemma 1.38 with $\Omega_\star$ and slightly moving $X_{4\Delta_0}$ if needed, we can write

$$u(X_{4\Delta_0}) - u_j(X_{4\Delta_0}) = \iint_{\Omega_\star} (A^j - A)(Y) \nabla_Y G_{L,\Omega_\star}(Y, X_{4\Delta_0}) \cdot \nabla u_j(Y) \, dY.$$

Set $\Sigma_j := \{Y \in \Omega : \delta(Y) < 2^{-j}\}$, $\widehat{B}_0 := B(X_{4\Delta_0}, \delta(X_{4\Delta_0})/2)$ take $j_1 \geq j_0$ large enough so that $\widehat{B}_0 \cap \Sigma_{j_1} = \varnothing$. For every $j \geq j_1$, it is clear that $|A^j - A| \lesssim \mathbf{1}_{\Sigma_j}$, with constants depending only on the ellipticity of L_0 and L. Also we have the a priori estimate $\|\nabla u_j\|_{L^2(\Omega_\star)} \lesssim \|g\|_{H^{1/2}(\partial\Omega_\star)}$ (see [HMT2]), where the implicit constant depends on dimension, the AR constant, the ellipticity of L_0 and L, and also of $\mathrm{diam}(\partial\Omega_\star) \approx r_0$. All these and Hölder's inequality yield

$$|u(X_{4\Delta_0}) - u_j(X_{4\Delta_0})| \lesssim \iint_{\Omega_\star \cap \Sigma_j} |\nabla_Y G_{L,\Omega_\star}(Y, X_{4\Delta_0})| |\nabla u_j(Y)| \, dY \qquad (2.38)$$

$$\lesssim \|\nabla G_{L,\Omega_\star}(\cdot, X_{4\Delta_0}) \mathbf{1}_{\Sigma_j}\|_{L^2(\Omega_\star \setminus \widehat{B}_0)} \|g\|_{H^{1/2}(\partial\Omega_\star)}.$$

Since $\Omega_\star$ is bounded, our Green function coincides with the one defined in [GW], hence $\nabla G_{L,\Omega_\star}(\cdot, X_{4\Delta_0}) \in L^2(\Omega_\star \setminus \widehat{B}_0)$ (see (1.41)). Using the dominated convergence theorem, the first factor of the right hand side of (2.38) tends to zero, hence $u_j(X_{4\Delta_0}) \to u(X_{4\Delta_0})$. Recalling then (2.37) we have that Hölder's inequality gives

$$|u(X_{4\Delta_0})| = \lim_{j \to \infty} |u_j(X_{4\Delta_0})| \leq \|g\|_{L^{q'}(4\Delta_0)} \sup_{j \in \mathbb{N}} \|k_{L^j, \Omega_\star}^{X_{4\Delta_0}}\|_{L^q(4\Delta_0)}$$

$$\lesssim \|g\|_{L^{q'}(4\Delta_0)} \sigma(\Delta_0)^{-1/q'}.$$

and hence

$$|\Phi(g)| \lesssim \|g\|_{L^{q'}(4\Delta_0)} \sigma(\Delta_0)^{-1/q'}, \qquad g \in \mathrm{Lip}_c(\partial\Omega), \quad \mathrm{supp}(g) \subset 4\Delta_0. \qquad (2.39)$$

Suppose now that $g \in L^{q'}(2\Delta_0)$ is such that $\mathrm{supp}(g) \subset 2\Delta_0$, and for $0 < t < r_0$ set $g_t = P_t g$ with P_t as in Lemma 1.41. Since $g_t \in \mathrm{Lip}(\partial\Omega)$ satisfies $\mathrm{supp}(g_t) \subset 4\Delta_0$, we have by (2.39)

$$|\Phi(g_t) - \Phi(g_s)| = |\Phi(g_t - g_s)| \lesssim \|g_t - g_s\|_{L^{q'}(4\Delta_0)} \sigma(\Delta_0)^{-1/q'}$$

$$\lesssim \sigma(\Delta_0)^{-1/q'} \left(\|P_t g - g\|_{L^{q'}(\partial\Omega)} + \|P_s g - g\|_{L^{q'}(\partial\Omega)} \right)$$

for $0 < t, s < r_0$. Hence $\{\Phi(g_t)\}_{t>0}$ is a Cauchy sequence, and we can define $\widetilde{\Phi}(g) := \lim_{t \to 0} \Phi(g_t)$. Clearly, $\widetilde{\Phi}$ is a well-defined linear operator and $\widetilde{\Phi} \in L^{q'}(2\Delta_0)^*$:

$$|\widetilde{\Phi}(g)| \leq \sup_{0 < t < r_0} |\Phi(g_t)| \lesssim \sigma(\Delta_0)^{-1/q'} \sup_{0 < t < r_0} \|P_t g\|_{L^{q'}(4\Delta_0)} \lesssim \sigma(\Delta_0)^{-1/q'} \|g\|_{L^{q'}(2\Delta_0)},$$
$$(2.40)$$

where we have used (2.39) and Lemma 1.41. Consequently, there exists $h \in L^q(2\Delta_0)$ with $\|h\|_{L^q(2\Delta_0)} \lesssim \sigma(\Delta_0)^{-1/q'}$ in such a way that $\widetilde{\Phi}(g) = \int_{2\Delta_0} g(y) h(y) \, d\sigma(y)$ for every $g \in L^{q'}(2\Delta_0)$ such that $\mathrm{supp}(g) \subset 2\Delta_0$.

Let $g \in C_c(\partial\Omega)$ with $\mathrm{supp}(g) \subset 2\Delta_0$ and we extend g by zero to $\partial\Omega_\star$ so that $g \in C_c(\partial\Omega_\star)$. From Lemma 1.41 applied to $\Omega_\star$, $\|P_t g\|_{L^\infty(\partial\Omega_\star)} \leq \|g\|_{L^\infty(2\Delta_0)}$ and

$P_t g(x) \to g(x)$ as $t \to 0^+$ for every $x \in \partial\Omega_\star$. These, the definition of $\widetilde{\Phi}(g)$ and the dominated convergence theorem with respect to $\omega_{L,\Omega_\star}^{X_{4\Delta_0}}$, shows

$$\widetilde{\Phi}(g) = \lim_{t \to 0^+} \Phi(P_t g) = \lim_{t \to 0^+} \int_{\partial\Omega_\star} P_t g(y) \, d\omega_{L,\Omega_\star}^{X_{4\Delta_0}}(y) = \int_{\partial\Omega_\star} g(y) \, d\omega_{L,\Omega_\star}^{X_{4\Delta_0}}(y) = \Phi(g), \tag{2.41}$$

hence $\widetilde{\Phi}(g) = \Phi(g)$ for every $g \in C_c(\partial\Omega)$ with $\operatorname{supp}(g) \subset 2\Delta_0$.

Next, we see that $\widehat{\omega} := \omega_{L,\Omega_\star}^{X_{4\Delta_0}} \ll \sigma_\star = \sigma$ in $\frac{5}{4}\Delta_0$. Let $E \subset \frac{5}{4}\Delta_0$ and let $\varepsilon > 0$. Since $\widehat{\omega}$ and σ are both regular measures, there exist $K \subset E \subset U \subset \frac{3}{2}\Delta_0$ with K compact and U open such that $\widehat{\omega}(U \setminus K) + \sigma(U \setminus K) < \varepsilon$. Using Urysohn's lemma we construct $g \in C_c(\partial\Omega)$ such that $\mathbf{1}_K \leq g \leq \mathbf{1}_U$ and $\operatorname{supp}(g) \subset 2\Delta_0$. Thus, by (2.41) and (2.40),

$$\widehat{\omega}(E) \leq \varepsilon + \widehat{\omega}(K) \leq \varepsilon + \int_{\partial\Omega_\star} g(y) \, d\widehat{\omega}(y) = \varepsilon + \Phi(g) = \varepsilon + \widetilde{\Phi}(g)$$
$$\leq \varepsilon + \|g\|_{L^{q'}(2\Delta_0)} \|h\|_{L^q(2\Delta_0)} \lesssim \varepsilon + (\varepsilon + \sigma(E))^{1/q'} \sigma(\Delta_0)^{-1/q'}.$$

Letting $\varepsilon \to 0^+$ we conclude that $\widehat{\omega}(E) \lesssim \sigma(E)^{1/q'} \sigma(\Delta_0)^{-1/q'}$ and in particular $\widehat{\omega} \ll \sigma$ in $\frac{5}{4}\Delta_0$. Writing then $\widehat{k} = d\widehat{\omega}/d\sigma \in L^1(\frac{5}{4}\Delta_0)$ we have that

$$\int_{\frac{5}{4}\Delta_0} g(y) h(y) \, d\sigma(y) = \widetilde{\Phi}(g) = \Phi(g) = \int_{\partial\Omega_\star} g(y) \, d\widehat{\omega}(y) = \int_{\frac{5}{4}\Delta_0} g(y) \widehat{k}(y) \, d\sigma(y), \tag{2.42}$$

for every $g \in C_c(\partial\Omega)$ with $\operatorname{supp}(g) \subset \frac{5}{4}\Delta_0$. Since $(h - \widehat{k})\mathbf{1}_{\frac{5}{4}\Delta_0} \in L^1(\partial\Omega)$ by Lemma 1.41 it follows that $P_t((h - \widehat{k})\mathbf{1}_{\frac{5}{4}\Delta_0}) \longrightarrow (h - \widehat{k})\mathbf{1}_{\frac{5}{4}\Delta_0}$ in $L^1(\partial\Omega)$ as $t \to 0^+$. Moreover, for any $x \in \Delta_0$, if we let $0 < t < r_0/8$ so that $\operatorname{supp}(\varphi_t(x, \cdot)) \subset \frac{5}{4}\Delta_0$, then (2.42) applied to $g = \varphi_t(x, \cdot)$ yields that $P_t((h - \widehat{k})\mathbf{1}_{\frac{5}{4}\Delta_0})(x) = 0$. All these allow to conclude that $\widehat{k} = h$ σ-a.e. in Δ_0, hence $\|\widehat{k}\|_{L^q(\Delta_0)} \leq \|h\|_{L^q(2\Delta_0)} \lesssim \sigma(\Delta_0)^{-1/q'}$.

Note that we showed before that $\widehat{\omega} := \omega_{L,\Omega_\star}^{X_{4\Delta_0}} \ll \sigma$ in Δ_0, Lemma 1.30(d) and Harnack's inequality give $\omega_{L,\Omega}^{X_{\Delta_0}} \ll \sigma$ in Δ_0, and

$$\int_{\Delta_0} k_{L,\Omega}^{X_{\Delta_0}}(y)^q \, d\sigma(y) \approx \int_{\Delta_0} k_{L,\Omega_\star}^{X_{\Delta_0}}(y)^q \, d\sigma(y) \approx \int_{\Delta_0} \widehat{k}(y)^q \, d\sigma(y) \lesssim \sigma(\Delta_0)^{1-q},$$

Since $\Delta_0 = \Delta(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \operatorname{diam}(\partial\Omega)/25$ was arbitrary, we have proved that $\omega_L \ll \sigma$ and

$$\int_\Delta k_{L,\Omega}^{X_\Delta}(y)^q \, d\sigma(y) \leq C\sigma(\Delta)^{1-q}, \qquad \Delta = \Delta(x, r), \qquad 0 < r < \frac{\operatorname{diam}(\partial\Omega)}{25}, \tag{2.43}$$

for $C > 1$ depending only on dimension, p, the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$. By a standard covering argument and Harnack's inequality, (2.43) extends to all $0 < r < \operatorname{diam}(\partial\Omega)$. Using Lemma 1.35, we have shown that $\omega_L = \omega_{L,\Omega} \in RH_q(\partial\Omega)$ completing the proof of Proposition 2.8. ∎

2.2   Proof of Theorem 2.1(b), small perturbation

We first note that by Theorem 2.1(a), the fact that $\|\varrho(A, A_0)\| \leq \varepsilon_1$ gives that $\omega_L \in RH_q(\partial\Omega)$ for some $1 < q < \infty$, and in particular $\omega_L \ll \sigma$. The goal of Theorem 2.1(b) is to see that if $\varepsilon_1 > 0$ is taken sufficiently small, then we indeed have that $\omega_L \in RH_p(\partial\Omega)$, that is, L_0 and L are in the same reverse Hölder class. To this aim, we split the proof in several steps.

Remark 2.9. We choose $M_0 > 400\,\kappa_0/c$, which will remain fixed during the proof of Theorem 2.1(b), where c is the corkscrew constant and κ_0 as in (1.6). Given an arbitrary ball $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $0 < r_0 < \mathrm{diam}(\partial\Omega)/M_0$, let $\Delta_0 = B_0 \cap \partial\Omega$ and take $X_{M_0 \Delta_{Q_0}}$ the corkscrew point relative to $M_0 \Delta_{Q_0}$ (note that $M_0 r_0 < \mathrm{diam}(\partial\Omega)$). If $Q_0 \in \mathbb{D}^{\Delta_0}$ then $\ell(Q_0) < 400\,r_0 < \mathrm{diam}(\partial\Omega)/\kappa_0$. Also $\delta(X_{M_0\Delta_0}) \geq cM_0 r_0 > 2\kappa_0 r_0$, and by (1.6),

$$X_{M_0\Delta_0} \in \Omega \setminus 2\kappa_0 B_0 \subset \Omega \setminus T^{**}_{\Delta_0}. \tag{2.44}$$

2.2.1   Step 0

As done in Step 0 of the proof of Theorem 2.1(a), we let work with $\widetilde{L} = L^j$, associated with the matrix $\widetilde{A} = A^j$ defined in (2.6). As there we have that $\omega_{\widetilde{L}} \ll \sigma$, hence we let $k_{\widetilde{L}}^X := d\omega_{\widetilde{L}}^X/d\sigma$. This qualitative property will be essential in the first two steps. At the end of Step 2 we will have obtained the desired conclusion for the operator $\widetilde{L} = L^j$, with constants independent of $j \in \mathbb{N}$, and in Step 3 we will transfer it to L via a limiting argument. From now on, $j \in \mathbb{N}$ will be fixed and we will focus on the operator $\widetilde{L} = L^j$.

2.2.2   Step 1

We start by fixing $B_0 = B(x_0, r_0)$ with $x_0 \in \partial\Omega$, $0 < r_0 < \mathrm{diam}(\partial\Omega)/M_0$ and M_0 as in Remark 2.9. Set $\Delta_0 = B_0 \cap \partial\Omega$ and $X_0 := X_{M_0\Delta_{Q_0}}$ so that (2.44) holds. We define the operator $L_1 u = L_1^{\Delta_0} u = -\mathrm{div}(A_1 \nabla u)$ where

$$A_1(Y) := \begin{cases} \widetilde{A}(Y) & \text{if } Y \in T_{\Delta_0}, \\ A_0(Y) & \text{if } Y \in \Omega \setminus T_{\Delta_0}, \end{cases}$$

and $\widetilde{A} = A^j$ as in (2.6). By construction, it is clear that $\mathcal{E}_1 := A_1 - A_0$ verifies $|\mathcal{E}_1| \leq |\mathcal{E}|\mathbf{1}_{T_{\Delta_0}}$, and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$. Hence, the support of $A_1 - A_0$ is contained in a compact subset of Ω.

In order to simplify the notation, we set $\widehat{\Delta}_0 := \frac{1}{2}\Delta_0^* = \Delta(x_0, \kappa_0 r_0)$ and let $0 \leq g \in L^{p'}(\widehat{\Delta}_0)$ be such that $\|g\|_{L^{p'}(\widehat{\Delta}_0)} = 1$. Without loss of generality, we may assume that g is defined in $\partial\Omega$ with $g \equiv 0$ in $\Omega \setminus \widehat{\Delta}_0$. For $0 < t < \kappa_0 r_0/2$, we consider $g_t = P_t g \geq 0$ with $P_t g$ defined as in (1.56), together with the solutions

$$u_0^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_{L_0}^X(y), \qquad u_1^t(X) = \int_{\partial\Omega} g_t(y)\, d\omega_{L_1}^X(y), \qquad X \in \Omega.$$

By Lemma 1.41, $g_t \in \mathrm{Lip}(\partial\Omega)$ verifies $\mathrm{supp}(g_t) \subset \Delta_0^*$ and hence $g_t \in \mathrm{Lip}_c(\partial\Omega) \subset H^{1/2}(\partial\Omega) \cap C_c(\partial\Omega)$. Since $\mathcal{E}_1 = A_1 - A_0$ verifies $|\mathcal{E}_1| \leq |\mathcal{E}|\mathbf{1}_{T_{\Delta_0}}$ and also $\mathcal{E}_1(Y) = 0$ if $\delta(Y) < 2^{-j}$, (2.44) and (1.6) allow us to invoke Lemma 1.39 (see Remark 1.40) which along with Cauchy-Schwarz's inequality yields

$$F^t(X_0) := |u_1^t(X_0) - u_0^t(X_0)| = \left| \iint_\Omega (A_0 - A_1)(Y)\nabla_Y G_{L_1}(Y, X_0) \cdot \nabla u_0^t(Y)\, dY \right|$$

$$\leq \sum_{Q_0 \in \mathbb{D}^{\Delta_0}} \sum_{Q \in \mathbb{D}_{Q_0}} \sum_{I \in \mathcal{W}_Q^*} \iint_{I^*} |\mathcal{E}(Y)||\nabla_Y G_{L_1}(Y, X_0)||\nabla u_0^t(Y)|\, dY$$

$$\leq \sum_{Q_0 \in \mathbb{D}^{\Delta_0}} \sum_{Q \in \mathbb{D}_{Q_0}} \sum_{I \in \mathcal{W}_Q^*} \sup_{I^*} |\mathcal{E}| \left(\iint_{I^*} |\nabla_Y G_{L_1}(Y, X_0)|^2\, dY \right)^{1/2}$$

$$\times \left(\iint_{I^*} |\nabla u_0^t(Y)|^2\, dY \right)^{1/2}.$$

Note that for every $Q_0 \in \mathbb{D}^{\Delta_0}$ and our choice of M_0, we have that $\ell(Q_0) < \mathrm{diam}(\partial\Omega)/\kappa_0$. Thus by Lemma 2.2 the estimate $\|a\| \leq \varepsilon_1$ implies that $\mathfrak{m} = \{\gamma_Q\}_{Q \in \mathbb{D}(\partial\Omega)} \in \mathcal{C}(Q_0)$ (see (2.3)) and $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \leq \kappa\varepsilon_1$, where $\kappa > 0$ depends only on dimension and on the 1-sided CAD constants. Also, note that L_1 is a symmetric operator. At this point we just need to repeat the arguments in (2.9)–(2.17) in every $Q_0 \in \mathbb{D}^{\Delta_0}$ with $\mathcal{F} = \varnothing$ and hence $\mathbb{D}_{\mathcal{F},Q_0} = \mathbb{D}_{Q_0}$. This ultimately gives

$$F^t(X_0) \lesssim \varepsilon_1^{1/2} \sum_{Q_0 \in \mathbb{D}^{\Delta_0}} \|k_{L_1}^{X_0}\|_{L^p(Q_0)} \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)},$$

where the last inequality is justified by the bounded cardinality of $\mathbb{D}^{\Delta_0}$. Therefore,

$$0 \leq u_1^t(X_0) \leq F^t(X_0) + u_0^t(X_0) \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} + \|k_{L_0}^{X_0}\|_{L^p(\Delta_0^*)},$$

where we have used Hölder's inequality, and the facts that $\|g_t\|_{L^{p'}(\partial\Omega)} \lesssim 1$ and $\mathrm{supp}(g_t) \subset \Delta_0^*$ by Lemma 1.41, and where the implicit constants do not depend on t. Next, we write

$$\int_{\partial\Omega} g(y) k_{L_1}^{X_0}(y)\, d\sigma(y) = u_1^t(X_0) + \int_{\partial\Omega} (g(y) - g_t(y)) k_{L_1}^{X_0}(y)\, d\sigma(y)$$

$$\lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} + \|k_{L_0}^{X_0}\|_{L^p(\Delta_0^*)} + \|g - g_t\|_{L^{p'}(\partial\Omega)} \|k_{L_1}^{X_0}\|_{L^p(\Delta_0^*)}.$$

Notice that $g_t \to g$ in $L^{p'}(\partial\Omega)$ by Lemma 1.41, which along with the fact that $\|k_{L_1}^{X_0}\|_{L^p(\Delta_0^*)} \leq C_j < +\infty$, by Lemma 1.30(e) and Harnack's inequality, implies

$$\int_{\partial\Omega} g(y) k_{L_1}^{X_0}(y)\, d\sigma(y) \lesssim \varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} + \|k_{L_0}^{X_0}\|_{L^p(\Delta_0^*)}.$$

Taking the supremum over all $0 \leq g \in L^{p'}(\widehat{\Delta}_0)$ with $\|g\|_{L^{p'}(\widehat{\Delta}_0)} = 1$ we obtain

$$\|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} \leq C\varepsilon_1^{1/2} \|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} + C\|k_{L_0}^{X_0}\|_{L^p(\Delta_0^*)},$$

where C depends on the allowable parameters. Since $\|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} \leq C_j < \infty$, taking $\varepsilon_1 < C^{-2}/4$, we can hide the first term in the left hand side to obtain $\|k_{L_1}^{X_0}\|_{L^p(\widehat{\Delta}_0)} \lesssim \|k_{L_0}^{X_0}\|_{L^p(\Delta_0^*)}$. Using then that $\omega_{L_0} \in RH_p(\partial\Omega)$ and Harnack's inequality to change the pole from $X_0 = X_{M_0\Delta_0}$ to $X_{\Delta_0^*}$ (with constants depending on M_0, which is already fixed), we conclude that

$$\int_{\Delta_0} k_{L_1}^{X_{\Delta_0}}(y)^p \, d\sigma(y) \lesssim \int_{\widehat{\Delta}_0} k_{L_1}^{X_0}(y)^p \, d\sigma(y) \lesssim \int_{\Delta_0^*} k_{L_0}^{X_0}(y)^p \, d\sigma(y)$$
$$\approx \int_{\Delta_0^*} k_{L_0}^{X_{\Delta_0^*}}(y)^p \, d\sigma(y) \lesssim \sigma(\Delta_0^*)^{1-p} \approx \sigma(\Delta_0)^{1-p}. \quad (2.45)$$

2.2.3   Step 2

We introduce the operator $L_2 := -\operatorname{div}(A_2\nabla u)$, where

$$A_2(Y) := \begin{cases} A_1(Y) & \text{if } Y \in T_{\Delta_0}, \\ \widetilde{A}(Y) & \text{if } Y \in \Omega \setminus T_{\Delta_0}, \end{cases}$$

and hence $A_2 = \widetilde{A}$ in Ω. As seen in Step 0, since $\widetilde{L} \equiv L_0$ in $\{Y \in \Omega : \delta(Y) < 2^{-j}\}$, we have that $\omega_{L_2} = \omega_{\widetilde{L}} \ll \sigma$, and there exists $k_{L_2} = d\omega_{L_2}/d\sigma$. Set $B_0' := B(x_0, r_0/(2\kappa_0))$ and $\Delta_0' = B_0' \cap \partial\Omega$. By (1.7), $2\kappa_0 B_0' \cap \Omega \subset \frac{5}{4}B_0 \cap \Omega \subset T_{\Delta_0}$ and since $L_2 \equiv L_1$ in T_{Δ_0}, Lemma 1.30(e) implies

$$k_{\widetilde{L}}^{X_{\Delta_0'}}(y) = k_{L_2}^{X_{\Delta_0'}}(y) \approx k_{L_1}^{X_{\Delta_0'}}(y), \quad \text{for } \sigma\text{-a.e. } y \in \Delta_0'.$$

Consequently, using (2.45) and Harnack's inequality (with constants depending on M_0, which is already fixed), we obtain

$$\int_{\Delta_0'} k_{\widetilde{L}}^{X_{\Delta_0'}}(y)^p \, d\sigma(y) \approx \int_{\Delta_0'} k_{L_1}^{X_{\Delta_0'}}(y)^p \, d\sigma(y)$$
$$\lesssim \int_{\Delta_0} k_{L_1}^{X_{\Delta_0}}(y)^p \, d\sigma(y) \lesssim \sigma(\Delta_0)^{1-p} \approx \sigma(\Delta_0')^{1-p}.$$

Since the surface ball $\Delta_0 = \Delta(x_0, r_0)$ with $x_0 \in \partial\Omega$ and $r_0 < \operatorname{diam}(\partial\Omega)/M_0$ was arbitrary, we have proved that

$$\int_\Delta k_{\widetilde{L}}^{X_\Delta}(y)^p \, d\sigma(y) \lesssim \sigma(\Delta)^{1-p}, \qquad \Delta = \Delta(x,r), \quad 0 < r < \frac{\operatorname{diam}(\partial\Omega)}{2M_0\kappa_0}. \quad (2.46)$$

By a standard covering argument and Harnack's inequality, (2.46) extends to all $\Delta = \Delta(x,r)$ with $0 < r < \operatorname{diam}(\partial\Omega)$. This and Lemma 1.35 show that $\omega_{\widetilde{L}} \in RH_p(\partial\Omega)$ where we recall that $\widetilde{L} = L^j$ is the operator defined in (2.6), $j \in \mathbb{N}$ is arbitrary, and the implicit constant is independent of $j \in \mathbb{N}$.

2.2.4 Step 3

Using the previous step and Proposition 2.8 with $q = p$ we conclude as desired that $\omega_L \in RH_p(\partial\Omega)$ and the proof of Theorem 2.1(b) is complete.

Remark 2.10. One can easily see from the previous proof that $\|\|a\|\| \leq \varepsilon_1$ could be slightly weakened by simply assuming that $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)}$ is small enough, with $\mathfrak{m} = \{\gamma_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ and γ_Q defined in (2.3). Further details are left to the interested reader.

2.3 Vanishing trace perturbation

In this section we will present an extension of the main theorem in [Dah2] to the setting of 1-sided CAD domains. With the help of Lemma 1.30(e), it will appear as an easy corollary of Theorem 2.1(b). Given L_0, L elliptic operators with matrices A_0, A respectively, we say that their disagreement defined in (2.1) verifies a *vanishing trace* Carleson condition if

$$\lim_{s\to 0^+} \left(\sup_{\substack{x\in\partial\Omega \\ 0<r\leq s<\mathrm{diam}(\partial\Omega)}} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r)\cap\Omega} \frac{\varrho(A,A_0)(X)^2}{\delta(X)}\, dX \right) = 0. \qquad (2.47)$$

Note that since this condition is not scale invariant, we do not expect that a vanishing trace perturbation could transfer a scale invariant condition like $RH_p(\partial\Omega)$ from one operator to the other. That is only achieved in the case of bounded domains. Next, we state the precise results.

Corollary 2.11. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided CAD. Let L_0, L be real symmetric elliptic operators whose disagreement in Ω is given by the function $\varrho(A,A_0)$ defined in (2.1). If $\omega_{L_0} \in RH_p(\partial\Omega)$ for some $1 < p < \infty$ and the vanishing trace Carleson condition (2.47) holds, then $\omega_L \ll \sigma$ and there exist $C_0 > 0$ (depending only on dimension, p, the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$), and $0 < r_0 < \mathrm{diam}(\partial\Omega)$ (depending on the above parameters and the condition (2.47)), such that*

$$\int_\Delta k_L^{X_\Delta}(y)^p\, d\sigma(y) \leq C_0\, \sigma(\Delta)^{1-p}, \quad \Delta = \Delta(x,r), \quad x \in \partial\Omega, \quad 0 < r \leq r_0. \quad (2.48)$$

Proof. Take $\varepsilon_1 > 0$ from Theorem 2.1(b) and let $M > 1$ to be chosen. Thanks to (2.47), there exists $s_0 = s_0(\varepsilon_1, M) < \mathrm{diam}(\partial\Omega)$ such that for every $\Delta = \Delta(x,r)$ with $x \in \partial\Omega$ and $0 < r \leq s_0$, we have that

$$\frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r)\cap\Omega} \frac{a(X)^2}{\delta(X)}\, dX \leq \frac{\varepsilon_1}{M}, \qquad (2.49)$$

where $a := \varrho(A,A_0)$. Given $s > 0$, set $\Sigma_s := \{Y \in \Omega : \delta(Y) < s\}$ and consider the operator $\widetilde{L}u = -\mathrm{div}(\widetilde{A}\nabla u)$ with

$$\widetilde{A}(Y) := \begin{cases} A_0(Y) & \text{if } Y \in \Omega \setminus \Sigma_{s_0/4}, \\ A(Y) & \text{if } Y \in \Sigma_{s_0/4}. \end{cases}$$

Note that $\widetilde{A}$ is uniformly elliptic with constant $\widetilde{\Lambda} = \max\{\Lambda_A, \Lambda_{A_0}\}$, where Λ_A and Λ_{A_0} are the ellipticity constants of A and A_0 respectively. Setting $\widetilde{\mathcal{E}} := \widetilde{A}(Y) - A_0(Y)$ and $\widetilde{a}(X) := \sup_{|X-Y|<\delta(X)/2} |\widetilde{\mathcal{E}}(Y)|$, it is clear that $\widetilde{\mathcal{E}}(Y) = \mathcal{E}(Y)\mathbf{1}_{\Sigma_{s_0/4}}(Y)$. Therefore, since $B(X, \delta(X)/2) \subset \Omega \setminus \Sigma_{s_0/4}$ for each $X \in \Omega \setminus \Sigma_{s_0/2}$, we have that

$$\widetilde{a}(X) \leq a(X)\mathbf{1}_{\Sigma_{s_0/2}}(X), \qquad X \in \Omega. \tag{2.50}$$

Now, we claim that

$$\|\widetilde{a}\| = \sup_{\substack{x \in \partial\Omega \\ 0 < r \leq s < \text{diam}(\partial\Omega)}} \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r) \cap \Omega} \frac{\widetilde{a}(X)^2}{\delta(X)} \, dX \leq \varepsilon_1, \tag{2.51}$$

provided M is chosen large enough depending only on dimension and the AR constant. To prove the claim we take $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$. Suppose first that $0 < r \leq s_0$, using (2.49) and (2.50), we obtain

$$\frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r) \cap \Omega} \frac{\widetilde{a}(X)^2}{\delta(X)} \, dX \leq \frac{1}{\sigma(\Delta(x,r))} \iint_{B(x,r) \cap \Omega} \frac{a(X)^2}{\delta(X)} \, dX \leq \frac{\varepsilon_1}{M} \leq \varepsilon_1.$$

On the other hand, if $r > s_0$, using (2.50) we have that

$$\iint_{B(x,r) \cap \Omega} \frac{\widetilde{a}(X)^2}{\delta(X)} \, dX \leq \iint_{B(x,r) \cap \Sigma_{s_0/2}} \frac{a(X)^2}{\delta(X)} \, dX.$$

By a standard Vitali type covering argument, there exists a family $\{\Delta_j\}_j$ of disjoint surface balls $\Delta_j = \Delta(x_j, s_0/2)$ with $x_j \in \Delta(x, 2r)$, satisfying $\Delta(x, 2r) \subset \bigcup_j 3\Delta_j$ and $\Delta_j \subset \Delta(x, 3r)$. Note that by construction, $B(x,r) \cap \Sigma_{s_0/2} \subset \bigcup_j B(x_j, s_0)$, hence by (2.49), we have that

$$\iint_{B(x,r) \cap \Sigma_{s_0/2}} \frac{a(X)^2}{\delta(X)} \, dX \leq \sum_j \iint_{B(x_j, s_0) \cap \Omega} \frac{a(X)^2}{\delta(X)} \, dX \leq \frac{\varepsilon_1}{M} \sum_j \sigma(\Delta(x_j, s_0))$$

$$\approx \frac{\varepsilon_1}{M} \sum_j \sigma(\Delta_j) \leq \frac{\varepsilon_1}{M} \sigma(\Delta(x, 3r)) \approx \frac{\varepsilon_1}{M} \sigma(\Delta(x,r)) \leq \varepsilon_1 \sigma(\Delta(x,r)),$$

for M sufficiently large, depending only on dimension and on the AR constant. Gathering the above estimates, we have proved as desired (2.51).

Next we apply Theorem 2.1(b) to L_0 and $\widetilde{L}$, to conclude that $\omega_{\widetilde{L}} \in RH_p(\partial\Omega)$ and, in particular,

$$\fint_\Delta k_{\widetilde{L}}^{X_\Delta}(y)^p \, d\sigma(y) \lesssim \sigma(\Delta)^{1-p}, \qquad \Delta = \Delta(x,r), \quad x \in \partial\Omega, \quad 0 < r < \text{diam}(\partial\Omega). \tag{2.52}$$

Set $r_0 := s_0/(8\kappa_0)$ and let $\Delta = \Delta(x,r)$ with $x \in \partial\Omega$ and $0 < r \leq r_0$. Note that $B(x, 2\kappa_0 r) \cap \Omega \subset B(x, s_0/4) \cap \Omega \subset \Sigma_{s_0/4}$, hence $\widetilde{L} \equiv L$ in $B(x, 2\kappa_0 r) \cap \Omega$. Using Lemma 1.30(e) we have that $\omega_L \ll \sigma$ in Δ and

$$k_L^{X_\Delta}(y) \approx k_{\widetilde{L}}^{X_\Delta}(y), \quad \text{for } \sigma\text{-a.e. } y \in \Delta.$$

This and (2.52) proves (2.48) and the proof is complete. ∎

Corollary 2.12. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a **bounded** 1-sided CAD. Let L_0, L be real symmetric elliptic operators whose disagreement in Ω is given by the function $a(X)$ defined in (2.1), and suppose that $\omega_{L_0} \in RH_p(\partial\Omega)$ for some $1 < p < \infty$. If the vanishing trace Carleson condition (2.47) holds, then we have that $\omega_L \in RH_p(\partial\Omega)$, with constants depending on $\mathrm{diam}(\partial\Omega)$, dimension, p, the condition (2.47), the 1-sided CAD constants, the ellipticity of L_0 and L, and the constant in $\omega_{L_0} \in RH_p(\partial\Omega)$.*

Chapter 3

Perturbations of non-symmetric operators

This chapter is devoted to generalize Theorem 2.1(a) in order to allow non symmetric operators. First of all, we show the equivalence between the A_∞ condition for elliptic measures and the fact that bounded weak solutions satisfy Carleson measure estimates (see also [KKPT]). Then, we prove a Carleson perturbation result for non symmetric operators in 1-sided chord-arc domains and a slight analog when the perturbation is considered between an operator and its transpose. We state below the precise theorems.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided CAD and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20). The following statements are equivalent:*

(a) *Every bounded weak solution of $Lu = 0$ satisfies a Carleson measure estimate, that is, there exists C such that every $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ with $Lu = 0$ in the weak sense in Ω satisfies the Carleson measure condition*

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \infty}} \frac{1}{r^n} \iint_{B(x,r) \cap \Omega} |\nabla u(X)|^2 \delta(X) \, dX \leq C \|u\|^2_{L^\infty(\Omega)}. \qquad (3.1)$$

(b) *$\omega_L \in A_\infty(\partial\Omega)$ (cf. Definition 1.33).*

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD (cf. Definition 1.4). Let $L_1 u = -\operatorname{div}(A_1 \nabla u)$ and $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be real (not necessarily symmetric) elliptic operators (cf. Definition 1.20). Define the disagreement between A_1 and A_0 in Ω by*

$$\varrho(A_1, A_0)(X) := \sup_{Y \in B(X, \delta(X)/2)} |A_1(Y) - A_0(Y)|, \qquad X \in \Omega, \qquad (3.2)$$

where $\delta(X) := \operatorname{dist}(X, \partial\Omega)$, and assume that it satisfies the Carleson measure condition

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \frac{\varrho(A_1, A_0)(X)^2}{\delta(X)} \, dX < \infty. \qquad (3.3)$$

Then, $\omega_{L_0} \in A_\infty(\partial\Omega)$ if and only if $\omega_{L_1} \in A_\infty(\partial\Omega)$ (cf. Definition 1.33).

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD (cf. Definition 1.4). Let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20) and let $L^\top$ denote the transpose of L (i.e, $L^\top u = -\operatorname{div}(A^\top \nabla u)$ with $A^\top$ being the transpose matrix of A). Assume that $(A - A^\top) \in \operatorname{Lip}_{\operatorname{loc}}(\Omega)$ and let*

$$\operatorname{div}_C(A - A^\top)(X) = \left(\sum_{i=1}^{n+1} \partial_i(a_{i,j} - a_{j,i})(X) \right)_{1 \leq j \leq n+1}, \qquad X \in \Omega. \qquad (3.4)$$

Assume that the following Carleson measure estimate holds

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \left| \operatorname{div}_C(A - A^\top)(X) \right|^2 \delta(X)\, dX < \infty. \quad (3.5)$$

Then $\omega_L \in A_\infty(\partial\Omega)$ if and only if $\omega_{L^\top} \in A_\infty(\partial\Omega)$ (cf. Definition 1.33).

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD (cf. Definition 1.4). Let $L_0 u = -\operatorname{div}(A_0 \nabla u)$ be a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20). Assume that $A_0 \in \operatorname{Lip}_{\operatorname{loc}}(\Omega)$, $|\nabla A_0|\delta \in L^\infty(\Omega)$ and that (0.1) holds for A_0. Then*

$$\omega_{L_0} \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \omega_{L_0^\top} \in A_\infty(\partial\Omega).$$

Additionally, if $Lu = -\operatorname{div}(A\nabla u)$ is a real (not necessarily symmetric) elliptic operator (cf. Definition 1.20) such that $\||\varrho(A, A_0)\|| < \infty$, then we have

$$\omega_L \in A_\infty(\partial\Omega) \qquad \Longleftrightarrow \qquad \Omega \text{ is a CAD (cf. Definition 1.4).} \qquad (3.6)$$

3.1   Proof of Theorem 3.1

In this section we will prove Theorem 3.1 in two steps. We will assume that $\Omega \subset \mathbb{R}^{n+1}$ is a 1-sided CAD and $Lu = -\operatorname{div}(A\nabla u)$ a real (not necessarily simmetric) elliptic operator (cf. Definition 1.20).

3.1.1   Proof of CME $\Longrightarrow A_\infty$

Given $Q_0 \in \mathbb{D}(\partial\Omega)$ and for every $\eta \in (0,1)$ we define the modified non-tangential cone

$$\Gamma_{Q_0}^\eta(x) := \bigcup_{\substack{Q \in \mathbb{D}_{Q_0} \\ Q \ni x}} U_{Q,\eta^3}, \qquad U_{Q,\eta^3} = \bigcup_{\substack{Q' \in \mathbb{D}_Q \\ \ell(Q') > \eta^3 \ell(Q)}} U_{Q'}. \qquad (3.7)$$

As already noted in Section 2, the sets $\{U_{Q,\eta^3}\}_{Q \in \mathbb{D}_{Q_0}}$ have bounded overlap with constant depending on η.

Lemma 3.5. *There exist $0 < \eta \ll 1$, depending only on dimension, the 1-sided CAD constants and the ellipticity of L, and $\alpha_0 \in (0,1)$, $C_\eta \geq 1$ both depending on the same parameters and additionally on η, such that for every $Q_0 \in \mathbb{D}$, for every $0 < \alpha < \alpha_0$, and for every Borel set $F \subset Q_0$ satisfying $\omega_L^{X_{Q_0}}(F) \leq \alpha \omega_L^{X_{Q_0}}(Q_0)$, there exists a Borel set $S \subset Q_0$ such that the bounded weak solution $u(X) = \omega_L^X(S)$ satisfies*

$$
S_{Q_0}^\eta u(x) := \left(\iint_{\Gamma_{Q_0}^\eta(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} \geq C_\eta^{-1} \left(\log \alpha^{-1} \right)^{\frac{1}{2}}, \qquad \forall \, x \in F,
$$
(3.8)

Assuming this result momentarily, we can now prove Theorem 3.1.

Proof of Theorem 3.1: (a) $\Longrightarrow$ (b). Our first goal is to see that given $\beta \in (0,1)$ there exists $\alpha \in (0,1)$ so that for every $Q_0 \in \mathbb{D}$ and every Borel set $F \subset Q_0$, we have that

$$
\frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)} \leq \alpha \qquad \Longrightarrow \qquad \frac{\sigma(F)}{\sigma(Q_0)} \leq \beta.
$$
(3.9)

Fix then $\beta \in (0,1)$ and $Q_0 \in \mathbb{D}$, and take a Borel set $F \subset Q_0$ so that $\omega_L^{X_{Q_0}}(F) \leq \alpha \omega_L^{X_{Q_0}}(Q_0)$ where $\alpha \in (0,1)$ is to be chosen. Applying Lemma 3.5, if we assume that $0 < \alpha < \alpha_0$, then $u(X) = \omega_L^X(S)$ satisfies (3.8) and therefore

$$
\begin{aligned}
C_\eta^{-2} \log \alpha^{-1} \sigma(F) &\leq \int_F S_{Q_0}^\eta u(x)^2 \, d\sigma(x) \\
&\leq \int_{Q_0} \left(\iint_{\Gamma_{Q_0}^\eta(x)} |\nabla u(Y)|^2 \delta(Y)^{1-n} \, dY \right) d\sigma(x) \\
&= \iint_{B_{Q_0}^* \cap \Omega} |\nabla u(Y)|^2 \delta(Y)^{1-n} \left(\int_{Q_0} \mathbf{1}_{\Gamma_{Q_0}^\eta(x)}(Y) \, d\sigma(x) \right) dY \quad (3.10)
\end{aligned}
$$

where we have used that $\Gamma_{Q_0}^\eta(x) \subset T_{Q_0} \subset B_{Q_0}^* \cap \Omega$ (see (1.6)), and we have used Fubini's theorem. To estimate the inner integral we fix $Y \in B_{Q_0}^* \cap \Omega$ and $\widehat{y} \in \mathbb{D}(\partial\Omega)$ such that $|Y - \widehat{y}| = \delta(Y)$. We claim that

$$
\left\{ x \in Q_0 : Y \in \Gamma_{Q_0}^\eta(x) \right\} \subset \Delta(\widehat{y}, C\eta^{-3}\delta(Y)).
$$
(3.11)

To show this let $x \in Q_0$ be such that $Y \in \Gamma_{Q_0}^\eta(x)$. Then there exists $Q \in \mathbb{D}_{Q_0}$ such that $x \in Q$ and $Y \in U_{Q,\eta^3}$. Hence, there is $Q' \in \mathbb{D}_Q$ with $\ell(Q') > \eta^3 \ell(Q)$ such that $Y \in U_{Q'}$ and consequently $\delta(Y) \approx \mathrm{dist}(Y, Q') \approx \ell(Q')$. Then,

$$
|x - \widehat{y}| \leq \mathrm{diam}(Q) + \mathrm{dist}(Y, Q') + \delta(Y) \lesssim \ell(Q) + \delta(Y) \leq C\eta^{-3}\delta(Y),
$$

thus $x \in \Delta(\widehat{y}, C\eta^{-3}\delta(Y))$ as desired. If we now use (3.11) and the AR property we conclude that for every $Y \in B_{Q_0}^* \cap \Omega$

$$
\int_{Q_0} \mathbf{1}_{\Gamma_{Q_0}^\eta(x)}(Y) \, d\sigma(x) \leq \sigma(\Delta(\widehat{y}, C\eta^{-3}\delta(Y))) \lesssim \eta^{-3n}\delta(Y)^n.
$$

Plugging this into (3.10) and using (3.1), since $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^{\infty}(\Omega)$ with $Lu = 0$ in the weak sense in Ω, we obtain

$$C_\eta^{-2} \log \alpha^{-1} \sigma(F) \lesssim \eta^{-3n} \iint_{B^*_{Q_0} \cap \Omega} |\nabla u(Y)|^2 \delta(Y) \, dY \lesssim \eta^{-3n} \sigma(\Delta^*_{Q_0}) \leq C\eta^{-3n} \sigma(Q_0),$$

where we have used that $\Delta^*_{Q_0} = B^*_{Q_0} \cap \partial\Omega$, that $0 \leq u(X) \leq \omega^X(\partial\Omega) \leq 1$ and that $\partial\Omega$ is AR. Rearranging the terms we see that $\sigma(F)/\sigma(Q_0) \leq \beta$ provided $0 < \alpha < \min\{\alpha_0, e^{-CC_\eta^2 \eta^{-3n} \beta^{-1}}\}$ and (3.9) follows.

Next we see that (3.9) implies that $\omega_L \in A_\infty(\partial\Omega)$. To see this we first obtain a dyadic-A_∞ condition. Fix $Q^0, Q_0 \in \mathbb{D}$ with $Q_0 \subset Q^0$. Lemma 1.30 parts (b) and (c), Harnack's inequality and Lemma 1.26 gives for every $F \subset Q_0$

$$\frac{1}{C_1} \frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)} \leq \frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q_0)} \leq C_1 \frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)}. \tag{3.12}$$

With all these in hand we fix $\beta \in (0,1)$ and take the corresponding $\alpha \in (0,1)$ so that (3.9) holds. Let $M \geq 1$ be large enough to be chosen and we are going to see that

$$\frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q)} \leq \frac{\alpha}{C_1} \implies \frac{\sigma(F)}{\sigma(Q_0)} \leq \beta. \tag{3.13}$$

Assuming that the first estimate holds we see that (3.12) yields $\frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q)} \leq \alpha$. Thus we can apply (3.9) to obtain that $\frac{\sigma(F)}{\sigma(Q_0)} \leq \beta$ as desired. To complete the proof we need to see that (3.13) gives (1.43). The argument is standard and is left to the the interested reader. This completes the proof of Theorem 3.1 modulo the proof of Lemma 3.5. ∎

Before proving Lemma 3.5 we need some notation and some estimates. Let $\eta = 2^{-k_*} < 1$. Given $Q \in \mathbb{D}(\partial\Omega)$ we define $\widetilde{Q} \in \mathbb{D}_Q$ to be the unique cube such that $x_Q \in \widetilde{Q}$, and $\ell(\widetilde{Q}) = \eta\ell(Q)$. Using this notation we have the following estimates which will be used later:

$$\omega_L^{X_{\widetilde{Q}}}(\partial\Omega \setminus Q) = \omega_L^{X_{\widetilde{Q}}}(\partial\Omega) - \omega_L^{X_{\widetilde{Q}}}(Q) \leq 1 - \omega_L^{X_{\widetilde{Q}}}(Q) \leq C\eta^\gamma \tag{3.14}$$

where C depends on dimension, the 1-sided CAD constants and the ellipticity of L and γ is the parameter in Lemma 1.27. To see this, keeping in mind the notation introduced in (1.1), let $\varphi(X) = \varphi_0((X - x_Q)/r_Q)$ where $\varphi_0 \in C_c(\mathbb{R}^{n+1})$ with $\mathbf{1}_{B(0,1)} \leq \varphi_0 \leq \mathbf{1}_{B(0,2)}$. Note that $\varphi \in C_c(\mathbb{R}^{n+1})$ with $0 \leq \varphi \leq 1$, $\text{supp}(\varphi) \subset 2B_Q$, and $\varphi \equiv 1$ in B_Q. In particular, $\varphi|_{\partial\Omega} \leq \mathbf{1}_{2\Delta_Q} \leq \mathbf{1}_Q$ and hence

$$v(X) := \int_{\partial\Omega} \varphi(y) d\omega_L^{X_{\widetilde{Q}}}(y) \leq \omega_L^{X_{\widetilde{Q}}}(Q) \tag{3.15}$$

Note that $v \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a weak solution with $0 \leq v \leq 1$ and $v|_{\partial\Omega} = \varphi|_{\partial\Omega} \equiv 1$ in B_Q. Thus, $\widetilde{v} = 1 - v \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a weak solution with $0 \leq \widetilde{v} \leq 1$ and $\widetilde{v}|_{\partial\Omega} = 1 - \varphi|_{\partial\Omega} \equiv 0$ in B_Q. Thus we can use (3.15) and Lemma 1.27 to see that

$$1 - \omega_L^{X_{\widetilde{Q}}}(Q) \leq 1 - v(X) = \widetilde{v}(X) \lesssim \left(\frac{|X_{\widetilde{Q}} - x_Q|}{r_Q}\right)^\gamma \|\widetilde{v}\|_{L^\infty(\Omega)} \leq C\eta^\gamma, \tag{3.16}$$

where the last estimate follows from

$$|X_{\widetilde{Q}} - x_Q| \le |X_{\widetilde{Q}} - x_{\widetilde{Q}}| + |x_{\widetilde{Q}} - x_Q| \lesssim \ell(\widetilde{Q}) = \eta\ell(Q),$$

since $x_Q \in \widetilde{Q}$ and $X_{\widetilde{Q}}$ is a corkscrew point relative to $\widetilde{Q}$.

We also claim that there exists $c_0 \in (0,1)$ depending only on the AR constant and on the ellipticity of L so that if η is small enough (depending only on the AR constant) then

$$c_0 \le \omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}) \le 1 - c_0. \tag{3.17}$$

The first inequality follows at once from Lemma 1.26 and Harnack's inequality. For the second one we claim that if η is small enough we can find $\widetilde{Q}' \in \mathbb{D}$ with $\ell(\widetilde{Q}') = \ell(\widetilde{Q})$, $\widetilde{Q}' \cap \widetilde{Q} = \varnothing$ and $\operatorname{dist}(\widetilde{Q}, \widetilde{Q}') \lesssim \ell(\widetilde{Q})$. Indeed, if we write $\widetilde{Q}^j$ for the j-th ancestor of $\widetilde{Q}$ (that is, the unique cube satisfying $\ell(\widetilde{Q}^j) = 2^j\ell(\widetilde{Q})$ and $\widetilde{Q} \subset \widetilde{Q}^j$) then $\sigma(\widetilde{Q}^j) \gtrsim \ell(\widetilde{Q}^j)^n = 2^{jn}\ell(\widetilde{Q})^n > \sigma(\widetilde{Q})$ for j large enough depending on the AR constant. Note that in the previous estimates we are implicitly using that $\ell(\widetilde{Q}) \lesssim \operatorname{diam}(\partial\Omega)$, fact that follows by choosing η small enough depending on the AR constant. Once j has been chosen we must have $\widetilde{Q} \subsetneq \widetilde{Q}^j$, and we can easily pick $\widetilde{Q}' \in \mathbb{D}_{\widetilde{Q}^j}$ with all the desired properties. In turn by Harnack's inequality and Lemma 1.26 one can see that $\omega^{X_{\widetilde{Q}}}(\widetilde{Q}') \gtrsim \omega^{X_{\widetilde{Q}'}}(\widetilde{Q}') \ge C^{-1}$ with $C > 1$ and consequently

$$\omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}) = \omega_L^{X_{\widetilde{Q}}}(\partial\Omega) - \omega_L^{X_{\widetilde{Q}}}(\partial\Omega \setminus \widetilde{Q}) \le 1 - \omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}') \le 1 - C^{-1},$$

which is the desired estimate.

Proof of Lemma 3.5. Let $\eta = 2^{-k_*} < 1$ be a small dyadic number to be chosen (in particular (3.14) and (3.17) hold). Fix $Q_0 \in \mathbb{D}$ and note that $\omega := \omega_L^{X_{Q_0}}$ is a regular Borel measure on $\partial\Omega$ which is dyadically doubling with constants C_0 (depending only on dimension, the 1-sided CAD constants and the ellipticity of L) by Lemma 1.30(b) and Harnack's inequality. Let $0 < \varepsilon_0 < e^{-1}$ and $0 < \alpha < \varepsilon_0^2/(2C_0^2)$, sufficiently small to be chosen later, and let $F \subset Q_0$ be a Borel set such that $\omega(F) \le \alpha\omega(Q_0)$. By Lemma 1.10 applied to $\mu = \omega$, it follows that F has a good ε_0-cover of length $k \approx \frac{\log\alpha^{-1}}{\log\varepsilon_0^{-1}}$, with $k \ge 2$. Let $\{\mathcal{O}_\ell\}_{\ell=1}^k$ be the corresponding collection of Borel sets so that $F \subset \mathcal{O}_k \subset \cdots \subset \mathcal{O}_1 \subset Q_0$ and $\mathcal{O}_\ell = \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} Q_i^\ell$, with disjoint families $\mathcal{F}_\ell = \{Q_i^\ell\} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$. Now, using the notation above we define $\widetilde{\mathcal{O}}_\ell := \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} \widetilde{Q}_i^\ell$ and consider the Borel set $S := \bigcup_{j=2}^k (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)$. Note that the union of sets comprising S is disjoint, hence

$$\mathbf{1}_S(y) = \sum_{j=2}^k \mathbf{1}_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}(y), \qquad y \in \partial\Omega. \tag{3.18}$$

Now we introduce some notation. For each $y \in F$ and $1 \le \ell \le k$, there exists a unique $Q_i^\ell(y) \in \mathcal{F}_\ell$ such that $y \in Q_i^\ell(y)$. We also let $P_i^\ell(y) \in \mathbb{D}_{Q_i^\ell(y)}$ be the unique cube verifying $y \in P_i^\ell(y)$ and $\ell(P_i^\ell(y)) = \eta\ell(Q_i^\ell(y))$. Associated with $P_i^\ell(y)$ we can construct $\widetilde{P}_i^\ell(y)$ as above, that is, $\widetilde{P}_i^\ell(y) \in \mathbb{D}_{P_i^\ell(y)}$ satisfies $\ell(\widetilde{P}_i^\ell(y)) = \eta\ell(P_i^\ell(y))$ and

$x_{P_i^\ell(y)} \in \widetilde{P}_i^\ell(y)$, where $x_{P_i^\ell(y)}$ is the center of $P_i^\ell(y)$. As usual we write $X_{\widetilde{Q}_i^\ell(y)}$ and $X_{\widetilde{P}_i^\ell(y)}$ to denote, respectively, the corkscrew points associated to $\widetilde{Q}_i^\ell(y)$ and $\widetilde{P}_i^\ell(y)$.

Let $u(X) := \omega_L^X(S)$ be so that

$$u(X) = \int_{\partial\Omega} \mathbf{1}_S(y)\, d\omega_L^X(y) = \sum_{j=2}^{k} \omega_L^X(\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j). \tag{3.19}$$

In the following lemma we obtain a lower bound for the oscillation of u.

Lemma 3.6. *If η and ε_0 are taken sufficiently small (depending only on dimension, the 1-sided CAD constants and the ellipticity of L), then for each $y \in F$, and each $1 \le \ell \le k-1$, we have that*

$$\left| u(X_{\widetilde{Q}_i^\ell(y)}) - u(X_{\widetilde{P}_i^\ell(y)}) \right| \ge \frac{c_0}{2}, \tag{3.20}$$

where c_0 is the constant in (3.17)

Assume this result momentarily and fix the corresponding η and ε_0. Fix also $y \in F$, $1 \le \ell \le k-1$, and write $Q_i^\ell := Q_i^\ell(y) \in \mathbb{D}_{Q_0}$, and $P_i^\ell := P_i^\ell(y) \in \mathbb{D}_{Q_i^\ell}$. By construction $X_{\widetilde{Q}_i^\ell} \in U_{\widetilde{Q}_i^\ell}$ and $X_{\widetilde{P}_i^\ell} \in U_{\widetilde{P}_i^\ell}$, hence we can find Whitney cubes $I_{\widetilde{Q}_i^\ell} \in \mathcal{W}_{\widetilde{Q}_i^\ell}^*$ and $I_{\widetilde{P}_i^\ell} \in \mathcal{W}_{\widetilde{P}_i^\ell}^*$ so that $X_{\widetilde{Q}_i^\ell} \in I_{\widetilde{Q}_i^\ell}$ and $X_{\widetilde{P}_i^\ell} \in I_{\widetilde{P}_i^\ell}$.

Also, note that $\ell(\widetilde{Q}_i^\ell) = \eta\ell(Q_i^\ell)$ and $\ell(\widetilde{P}_i^\ell) = \eta^2\ell(Q_i^\ell)$ which imply $\ell(\widetilde{Q}_i^\ell) > \ell(\widetilde{P}_i^\ell) > \eta^3\ell(Q_i^\ell)$ since $\eta < 1$. On the other hand, $\widetilde{Q}_i^\ell \subset Q_i^\ell$ and $\widetilde{P}_i^\ell \subset P_i^\ell \subset Q_i^\ell$, which in turn yield that $I_{\widetilde{Q}_i^\ell}^*$ and $I_{\widetilde{P}_i^\ell}^*$ are both contained in $U_{Q_i^\ell, \eta^3}$. Using (3.20), the notation $[u]_{U_{Q_i^\ell, \eta^3}} := \fint\!\!\!\fint_{U_{Q_i^\ell, \eta^3}} u\, dX$, De Giorgi-Nash-Moser's estimate and the previous observations we can obtain

$$\frac{c_0}{2} \le \left| u(X_{\widetilde{Q}_i^\ell}) - [u]_{U_{Q_i^\ell, \eta^3}} \right| + \left| [u]_{U_{Q_i^\ell, \eta^3}} - u(X_{\widetilde{P}_i^\ell}) \right|$$

$$\lesssim \left(\fint\!\!\!\fint_{I_{\widetilde{Q}_i^\ell}^*} \left| u(Y) - [u]_{U_{Q_i^\ell, \eta^3}} \right|^2 dY \right)^{1/2} + \left(\fint\!\!\!\fint_{I_{\widetilde{P}_i^\ell}^*} \left| u(Y) - [u]_{U_{Q_i^\ell, \eta^3}} \right|^2 dY \right)^{1/2}$$

$$\le C_\eta \left(\ell(Q_i^\ell)^{-n-1} \iint_{U_{Q_i^\ell, \eta^3}} \left| u(Y) - [u]_{U_{Q_i^\ell, \eta^3}} \right|^2 dY \right)^{1/2}$$

$$\le C_\eta \left(\iint_{U_{Q_i^\ell, \eta^3}} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY \right)^{1/2},$$

where the last estimate follows from the Poincaré's inequality in [HMT1, Lemma 3.1]), and the fact that $\delta(Y) \approx_\eta \ell(Q_i^\ell)$ for every $Y \in U_{Q_i^\ell, \eta^3}$. Summing up the above estimate, taking into account that the sets $\{U_{Q, \eta^3}\}_{Q \in \mathbb{D}_{Q_0}}$ have bounded overlap with constant depending on η, and using Lemma 1.10, we obtain if α is small enough

$$\frac{c_0^2}{4} \frac{\log \alpha^{-1}}{\log \varepsilon_0^{-1}} \approx \frac{c_0^2}{4}(k-1) \le C_\eta \sum_{\ell=1}^{k-1} \iint_{U_{Q_i^\ell, \eta^3}} |\nabla u(Y)|^2 \delta(Y)^{1-n}\, dY \le C_\eta \left(S_{Q_0}^\eta(u)(y) \right)^2.$$

This completes the proof of Lemma 3.5. ∎

Proof of Lemma 3.6. Fix $y \in F$ and write $Q_i^\ell := Q_i^\ell(y)$, $P_i^\ell := P_i^\ell(y)$. Our first goal is to estimate $u(X_{\widetilde{Q}_i^\ell})$. For starters, by (3.14)

$$u(X_{\widetilde{Q}_i^\ell}) = \omega_L^{X_{\widetilde{Q}_i^\ell}}(S) \leq \omega_L^{X_{\widetilde{Q}_i^\ell}}(\partial\Omega \setminus Q_i^\ell) + \omega_L^{X_{\widetilde{Q}_i^\ell}}(S \cap Q_i^\ell)$$
$$\leq C\eta^\gamma + \omega_L^{X_{\widetilde{Q}_i^\ell}}(S \cap Q_i^\ell) =: C\eta^\gamma + \mathrm{I}. \quad (3.21)$$

For $1 \leq \ell \leq k-1$ we have that $Q_i^\ell \subset \mathcal{O}_\ell \subset \mathcal{O}_j$ for each $2 \leq j \leq \ell$ and hence

$$\mathrm{I} = \sum_{j=2}^{k} \omega_L^{X_{\widetilde{Q}_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\big) = \sum_{j=\ell+1}^{k} \omega_L^{X_{\widetilde{Q}_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\big)$$
$$= \sum_{j=\ell+2}^{k} \omega_L^{X_{\widetilde{Q}_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\big) + \omega_L^{X_{\widetilde{Q}_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1})\big) =: \mathrm{I}_1 + \mathrm{I}_2, \quad (3.22)$$

with the understanding that if $\ell = k-1$ then $\mathrm{I}_1 = 0$.

Next, we claim that $\mathrm{I}_1 \leq C_\eta \varepsilon_0$. This is clear if $\ell = k-1$ and for $1 \leq \ell \leq k-2$, using Harnack's inequality to move from $X_{\widetilde{Q}_i^\ell}$ to $X_{Q_i^\ell}$ (with constants depending on η), Lemma 1.30 parts (b) and (c) (recall that $\omega = \omega_L^{X_{Q_0}}$), we have that

$$\mathrm{I}_1 \leq C_\eta \sum_{j=\ell+2}^{k} \omega_L^{X_{Q_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\big) \leq \frac{C_\eta}{\omega(Q_i^\ell)} \sum_{j=\ell+2}^{k} \omega\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\big)$$
$$\leq \frac{C_\eta}{\omega(Q_i^\ell)} \sum_{j=\ell+2}^{k} \omega(Q_i^\ell \cap \mathcal{O}_{j-1}) \leq C_\eta \sum_{j=\ell+2}^{k} \varepsilon_0^{j-1-\ell} \leq C_\eta \varepsilon_0, \quad (3.23)$$

where the next-to-last estimate follows from Lemma 1.8 with $\mu = \omega$, and the last one uses that $\varepsilon_0 < e^{-1}$. Let us now focus on I_2. Note that $Q_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell$, hence (3.17) yields

$$\mathrm{I}_2 = \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \setminus \mathcal{O}_{\ell+1}) \leq \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell) \leq 1 - c_0.$$

Collecting this with (3.21), (3.22), (3.23), we conclude that

$$u(X_{\widetilde{Q}_i^\ell}) \leq C\eta^\gamma + C_\eta \varepsilon_0 + 1 - c_0 \leq 1 - \frac{3}{4}c_0, \quad (3.24)$$

by choosing first η small enough so that $C\eta^\gamma < c_0/8$ and then ε_0 small enough so that $C_\eta \varepsilon_0 < c_0/8$.

To get a lower bound for $u(X_{\widetilde{Q}_i^\ell})$ we use that $Q_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell$ and (3.17):

$$u(X_{\widetilde{Q}_i^\ell}) = \omega_L^{X_{\widetilde{Q}_i^\ell}}(S) \geq \omega_L^{X_{\widetilde{Q}_i^\ell}}\big(Q_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1})\big)$$
$$= \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \setminus \mathcal{O}_{\ell+1}) = \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell) - \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \geq c_0 - \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}).$$

Using Harnack's inequality to move from $X_{\widetilde{Q}_i^\ell}$ to $X_{Q_i^\ell}$ (with constants depending on η), Lemma 1.30 parts (b) and (c) (recall that $\omega = \omega_L^{X_{Q_0}}$), we have that

$$\omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \omega_L^{X_{Q_i^\ell}}(Q_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \frac{\omega(Q_i^\ell \cap \mathcal{O}_{\ell+1})}{\omega(Q_i^\ell)} \leq C_\eta \varepsilon_0, \quad (3.25)$$

where the last estimate follows from Lemma 1.8 with $\mu = \omega$ and since $1 \le \ell \le k-1$. Assuming further that $C_\eta \varepsilon_0 < c_0/4$ we arrive at

$$u(X_{\widetilde{Q}_i^\ell}) \ge c_0 - C_\eta \varepsilon_0 \ge \frac{3}{4}c_0. \tag{3.26}$$

Let us now focus on estimating $u(X_{\widetilde{P}_i^\ell})$ and we consider two cases:

Case 1: $P_i^\ell \cap \widetilde{Q}_i^\ell = \varnothing$. Much as before by (3.14)

$$u(X_{\widetilde{P}_i^\ell}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(S) \le \omega_L^{X_{\widetilde{P}_i^\ell}}(\partial\Omega \setminus P_i^\ell) + \omega_L^{X_{\widetilde{P}_i^\ell}}(S \cap P_i^\ell)$$

$$\le C\eta^\gamma + \omega_L^{X_{\widetilde{P}_i^\ell}}(S \cap P_i^\ell) =: C\eta^\gamma + \widehat{\mathrm{I}}. \tag{3.27}$$

For $1 \le \ell \le k-1$ we have that $P_i^\ell \subset Q_i^\ell \subset \mathcal{O}_\ell \subset \mathcal{O}_j$ for each $2 \le j \le \ell$ and hence

$$\widehat{\mathrm{I}} = \sum_{j=2}^{k} \omega_L^{X_{\widetilde{P}_i^\ell}}\left(P_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\right) = \sum_{j=\ell+1}^{k} \omega_L^{X_{\widetilde{P}_i^\ell}}\left(P_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\right)$$

$$= \sum_{j=\ell+2}^{k} \omega_L^{X_{\widetilde{P}_i^\ell}}\left(P_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\right) + \omega_L^{X_{\widetilde{P}_i^\ell}}\left(P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1})\right) =: \widehat{\mathrm{I}}_1 + \widehat{\mathrm{I}}_2, \tag{3.28}$$

with the understanding that if $\ell = k-1$ then $\widehat{\mathrm{I}}_1 = 0$. The estimate for $\widehat{\mathrm{I}}_1$ (when $\ell \le k-2$) follows from that of I_1 since using Harnack's inequality to move from $X_{\widetilde{P}_i^\ell}$ to $X_{\widetilde{Q}_i^\ell}$ and the fact that $P_i^\ell \subset Q_i^\ell$ we easily obtain from (3.23)

$$\widehat{\mathrm{I}}_1 \le C_\eta \sum_{j=\ell+2}^{k} \omega_L^{X_{\widetilde{Q}_i^\ell}}\left(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j)\right) = C_\eta \mathrm{I}_1 \le C_\eta \varepsilon_0. \tag{3.29}$$

On the other hand, note that $P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1}) = (P_i^\ell \cap \widetilde{Q}_i^\ell) \setminus \mathcal{O}_{\ell+1} = \varnothing$ and hence $\widehat{\mathrm{I}}_2 = 0$. Thus (3.27), (3.28), and (3.29) yield

$$u(X_{\widetilde{P}_i^\ell}) \le C\eta^\gamma + C_\eta \varepsilon_0 \le \frac{1}{4}c_0, \tag{3.30}$$

by choosing first η small enough so that $C\eta^\gamma < c_0/8$ and then ε_0 small enough so that $C_\eta \varepsilon_0 < c_0/8$. This estimate along with (3.26) give at once

$$|u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell})| = u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell}) \ge \frac{3}{4}c_0 - \frac{1}{4}c_0 = \frac{1}{2}c_0,$$

which is the desired estimate.

Case 2: $P_i^\ell \cap \widetilde{Q}_i^\ell \ne \varnothing$. Notice that since both cubes have the same sidelength it follows that $P_i^\ell = \widetilde{Q}_i^\ell$. Our goal is to get a lower bound for $u(X_{\widetilde{P}_i^\ell})$. We use that $P_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell = P_i^\ell$ and (3.14):

$$u(X_{\widetilde{P}_i^\ell}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(S) \ge \omega_L^{X_{\widetilde{P}_i^\ell}}\left(P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1})\right) = \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \setminus \mathcal{O}_{\ell+1})$$

$$= \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell) - \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}) \ge 1 - C\eta^\gamma - \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}).$$

Moreover, using Harnack's inequality to move from $X_{\widetilde{P}_i^\ell}$ to $X_{\widetilde{Q}_i^\ell}$ (with constants depending on η) and (3.25) we observe that

$$\omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \varepsilon_0.$$

Collecting the obtained estimates we conclude that

$$u(X_{\widetilde{Q}_i^\ell}) \geq 1 - C\eta^\gamma - C_\eta \varepsilon_0 \geq 1 - \frac{1}{4}c_0, \tag{3.31}$$

if we choose first η small enough so that $C\eta^\gamma < c_0/8$ and then ε_0 small enough so that $C_\eta \varepsilon_0 < c_0/8$. If we now gather (3.24) and (3.31) we eventually obtain the desired estimate

$$\left| u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell}) \right| = u(X_{\widetilde{P}_i^\ell}) - u(X_{\widetilde{Q}_i^\ell}) \geq \left(1 - \frac{1}{4}c_0\right) - \left(1 - \frac{3}{4}c_0\right) = \frac{1}{2}c_0.$$

This completes the proof. ∎

3.1.2 Proof of $A_\infty \implies$ CME

We begin with a preliminary result showing that the desired Carleson measure estimate (3.1) can be obtained as a consequence of the fact that a certain sequence of coefficients indexed on $Q \in \mathbb{D}(\partial\Omega)$ generates a discrete Carleson measure. This is inspired in the work of [HMM] and is stated precisely in the lemma below.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided CAD and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) elliptic operator. Let $u \in W_{\mathrm{loc}}^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfy $Lu = 0$ in the weak sense in Ω and define*

$$\alpha := \{\alpha_Q\}_{Q \in \mathbb{D}} := \left\{ \iint_{U_Q} |\nabla u(X)|^2 \delta(X)\, dX \right\}_{Q \in \mathbb{D}}. \tag{3.32}$$

Suppose that there exist $C_0, M_0 \geq 1$ such that $\|\mathfrak{m}_\alpha\|_{\mathcal{C}(Q)} \leq C_0 \|u\|_{L^\infty(\Omega)}^2$ for every $Q \in \mathbb{D}(\partial\Omega)$ verifying $\ell(Q) < \operatorname{diam}(\partial\Omega)/M_0$. Then,

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \infty}} \frac{1}{r^n} \iint_{B(x,r) \cap \Omega} |\nabla u(X)|^2 \delta(X)\, dX \leq C(1 + C_0 + M_0)\|u\|_{L^\infty(\Omega)}^2, \tag{3.33}$$

where C depends only on dimension, the 1-sided CAD constants, and the ellipticity of L.

Proof. By homogeneity we may assume that $\|u\|_{L^\infty(\Omega)} = 1$. First, we claim that

$$\sup_{Q \in \mathbb{D}(\partial\Omega)} \frac{1}{\sigma(Q)} \iint_{T_Q} |\nabla u(X)|^2 \delta(X)\, dX \lesssim C_0 + M_0. \tag{3.34}$$

Given $Q_0 \in \mathbb{D}(\partial\Omega)$ such that $\ell(Q_0) < \operatorname{diam}(\partial\Omega)/M_0$, we have that

$$\iint_{T_{Q_0}} |\nabla u(X)|^2 \delta(X)\, dX \leq \sum_{Q \in \mathbb{D}_{Q_0}} \alpha_Q = \mathfrak{m}_\alpha(\mathbb{D}_{Q_0}) \leq \|\mathfrak{m}_\alpha\|_{\mathcal{C}(Q_0)} \sigma(Q_0) \leq C_0 \sigma(Q_0).$$

Otherwise, if $\ell(Q_0) \geq \operatorname{diam}(\partial\Omega)/M_0$ (this happens only if $\operatorname{diam}(\partial\Omega) < \infty$), there exists a unique $k_0 \geq 1$ so that

$$2^{k_0-1}\frac{\operatorname{diam}(\partial\Omega)}{M_0} \leq \ell(Q_0) < 2^{k_0}\frac{\operatorname{diam}(\partial\Omega)}{M_0}.$$

As observed before if $\operatorname{diam}(\partial\Omega) < \infty$ then $\ell(Q_0) \lesssim \operatorname{diam}(\partial\Omega)$ hence $2^{k_0} \lesssim M_0$. Define the disjoint collection $\mathcal{D}_0 := \{Q' \in \mathbb{D}_{Q_0} : \ell(Q') = 2^{-k_0}\ell(Q_0)\}$ and let

$$\mathbb{D}_{Q_0}^{\text{small}} := \{Q \in \mathbb{D}_{Q_0} : \ell(Q) < 2^{-k_0}\ell(Q_0)\}, \quad \mathbb{D}_{Q_0}^{\text{big}} := \{Q \in \mathbb{D}_{Q_0} : \ell(Q) \geq 2^{-k_0}\ell(Q_0)\}.$$

Note that

$$\iint_{T_{Q_0}} |\nabla u(X)|^2 \delta(X)\,dX \leq \sum_{Q \in \mathbb{D}_{Q_0}^{\text{small}}} \alpha_Q + \sum_{Q \in \mathbb{D}_{Q_0}^{\text{big}}} \alpha_Q =: \mathrm{I}_{Q_0} + \mathrm{II}_{Q_0}.$$

Note that if $Q \in \mathbb{D}_{Q_0}^{\text{small}}$, there exists a unique $Q' \in \mathcal{D}_0$ such that $Q \in \mathbb{D}_{Q'}$, hence

$$\mathrm{I}_{Q_0} = \sum_{Q' \in \mathcal{D}_0} \sum_{Q \in \mathbb{D}_{Q'}} \alpha_Q = \sum_{Q' \in \mathcal{D}_0} \mathfrak{m}_\alpha(\mathbb{D}_{Q'}) \leq \sum_{Q' \in \mathcal{D}_0} \|\mathfrak{m}_\alpha\|_{\mathcal{C}(Q')}\sigma(Q') \leq C_0\sigma(Q_0).$$

where we have used our hypothesis since $\ell(Q') = 2^{-k_0}\ell(Q_0) < \operatorname{diam}(\partial\Omega)/M_0$. For the second term, since $\delta(X) \approx \ell(Q)$ for $X \in U_Q$, we write

$$\mathrm{II}_{Q_0} \lesssim \sum_{Q \in \mathbb{D}_{Q_0}^{\text{big}}} \ell(Q) \iint_{U_Q} |\nabla u(X)|^2\,dX \lesssim \sum_{Q \in \mathbb{D}_{Q_0}^{\text{big}}} \ell(Q)^{-1} \iint_{U_Q^*} |u(X)|^2\,dX$$

$$\lesssim 2^{k_0}\ell(Q_0)^{-1}|T_{Q_0}^*| \lesssim M_0\sigma(Q_0),$$

where we have used Caccioppoli's inequality, the fact that the family $\{U_Q^*\}_{Q \in \mathbb{D}}$ has bounded overlap, the normalization $\|u\|_{L^\infty(\Omega)} = 1$, (1.6), the AR property, and that $2^{k_0} \lesssim M_0$. Gathering the above we have proved that (3.34) holds.

Our next goal is to see that (3.34) yields (3.33). Fix then $x \in \partial\Omega$ and $0 < r < \infty$. Set

$$\mathcal{I} = \{I \in \mathcal{W} : I \cap B(x,r) \neq \emptyset\}.$$

Given $I \in \mathcal{I}$, let $Z_I \in I \cap B(x,r)$ and note that by (1.4)

$$\operatorname{diam}(I) \leq \operatorname{dist}(I,\partial\Omega) \leq |Z_I - x| < r. \tag{3.35}$$

Set

$$\mathcal{I}^{\text{small}} = \{I \in \mathcal{I} : \ell(I) < \operatorname{diam}(\partial\Omega)/4\}, \qquad \mathcal{I}^{\text{big}} = \{I \in \mathcal{I} : \ell(I) \geq \operatorname{diam}(\partial\Omega)/4\},$$

with the understanding that $\mathcal{I}^{\text{big}} = \emptyset$ if $\operatorname{diam}(\partial\Omega) = \infty$. Then,

$$\iint_{B(x,r)\cap\Omega} |\nabla u|^2 \delta\,dX \leq \sum_{I \in \mathcal{I}^{\text{small}}} \iint_I |\nabla u|^2\delta\,dX + \sum_{I \in \mathcal{I}^{\text{big}}} \iint_I |\nabla u|^2\delta\,dX = \mathrm{I} + \mathrm{II},$$

here we understand that $\mathrm{II} = 0$ if $\mathcal{I}^{\text{big}} = \emptyset$.

To estimate I we set $r_0 = \min\{r, \operatorname{diam}(\partial\Omega)/4\}$ and pick $k_2 \in \mathbb{Z}$ so that $2^{k_2-1} \leq r_0 < 2^{k_2}$. Set

$$\mathcal{D}_1 = \{Q \in \mathbb{D} : \ell(Q) = 2^{k_2}, \, Q \cap \Delta(x, 3r) \neq \varnothing\}.$$

Given $I \in \mathcal{I}^{\text{small}}$ we pick $y \in \partial\Omega$ so that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, y)$. Hence there exists a unique $Q_I \in \mathbb{D}$ so that $y \in Q_I$ and $\ell(Q_I) = \ell(I) < r_0 \leq \operatorname{diam}(\partial\Omega)/4$ by (3.35). This as mentioned above implies that $I \in \mathcal{W}_{Q_I}^*$. On the other hand by (3.35)

$$|y - x| \leq \operatorname{dist}(y, I) + \operatorname{diam}(I) + |Z_I - x| < 3r,$$

hence there exists a unique $Q \in \mathcal{D}_1$ so that $y \in Q$. Since $\ell(Q_I) < r_0 < 2^{k_2} = \ell(Q)$ we conclude that $Q_I \subset Q$ and consequently $I \subset \operatorname{int}(U_{Q_I}) \subset T_Q$. In short we have shown that if $I \in \mathcal{I}^{\text{small}}$ then there exists $Q \in \mathcal{D}_1$ so that $I \subset T_Q$. Thus,

$$\text{I} \leq \sum_{Q \in \mathcal{D}_1} \iint_{T_Q} |\nabla u|^2 \delta \, dX \lesssim (C_0 + M_0) \sum_{Q \in \mathcal{D}_1} \sigma(Q) = (C_0 + M_0)\sigma\Big(\bigcup_{Q \in \mathcal{D}_1} Q\Big)$$
$$\leq (C_0 + M_0)\sigma(\Delta(x, Cr)) \lesssim (C_0 + M_0)r^n,$$

where we have used that the Whitney boxes have non-overlapping interiors, (3.34), the fact that $\mathcal{D}_1$ is a pairwise disjoint family, that $\bigcup_{Q \in \mathcal{D}_1} Q \subset \Delta(x, Cr)$ (C depends on dimension and AR), and that $\partial\Omega$ is AR.

We now estimate II using (1.4), Caccioppoli's inequality and our assumption $\|u\|_{L^\infty(\Omega)} = 1$:

$$\text{II} \lesssim \sum_{I \in \mathcal{I}^{\text{big}}} \ell(I) \iint_I |\nabla u|^2 \, dX \lesssim \sum_{I \in \mathcal{I}^{\text{big}}} \ell(I)^{-1} \iint_{I^*} |u|^2 \, dX$$
$$\lesssim \sum_{I \in \mathcal{I}^{\text{big}}} \ell(I)^n \leq \sum_{\frac{\operatorname{diam}(\partial\Omega)}{4} \leq 2^k < r} 2^{kn} \#\{I \in \mathcal{I}^{\text{big}} : \ell(I) = 2^k\}.$$

To estimate the last term we observe that if $Y \in I \in \mathcal{I}^{\text{big}}$ we have by (1.4)

$$|Y - x| \leq \operatorname{diam}(I) + \operatorname{dist}(I, \partial\Omega) + \operatorname{diam}(\partial\Omega) \lesssim \ell(I).$$

This and the fact that Whitney boxes have non-overlapping interiors imply

$$\#\{I \in \mathcal{I}^{\text{big}} : \ell(I) = 2^k\} = 2^{-k(n+1)} \sum_{I \in \mathcal{I}^{\text{big}}:\ell(I)=2^k} |I|$$
$$= 2^{-k(n+1)} \Big| \bigcup_{I \in \mathcal{I}^{\text{big}}:\ell(I)=2^k} I \Big| \leq 2^{-k(n+1)} |B(x, C2^k)| \lesssim 1.$$

Therefore,

$$\text{II} \lesssim \sum_{\frac{\operatorname{diam}(\partial\Omega)}{4} \leq 2^k < r} 2^{kn} \lesssim r^n.$$

Collecting the estimates for I and II we obtain the desired estimate. ∎

Proof of Theorem 3.1: $(b) \implies (a)$. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^{\infty}(\Omega)$ be so that $Lu = 0$ in the weak sense in Ω and our goal is to prove that (3.1) holds. By homogeneity we may assume, without loss of generality, that $\|u\|_{L^{\infty}(\Omega)} = 1$. On the other hand, by Lemma 3.7 we can reduce matters to establish that $\|\mathfrak{m}_{\alpha}\|_{\mathcal{C}(Q)} \leq C_0$, for every $Q \in \mathbb{D}(\partial\Omega)$ such that $\ell(Q) < \mathrm{diam}(\partial\Omega)/M_0$ and where α is given in (3.32). To show this we fix $M_0 > 2\kappa_0/c$, where c is the corkscrew constant and κ_0 as in (1.6). We also fix a cube $Q^0 \in \mathbb{D}(\partial\Omega)$ with $\ell(Q^0) < \mathrm{diam}(\partial\Omega)/M_0$. Applying Lemma 1.17 it suffices to show that for every $Q_0 \in \mathbb{D}_{Q^0}$ we can find some pairwise disjoint family $\mathcal{F}_{Q_0} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ satisfying

$$\sigma\Big(Q_0 \setminus \bigcup_{Q_j \in \mathcal{F}_{Q_0}} Q_j\Big) \geq K_1^{-1}\sigma(Q_0), \tag{3.36}$$

and prove that

$$\mathfrak{m}_{\alpha}(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) \leq M_1\sigma(Q_0). \tag{3.37}$$

With all the previous reductions our main goal is to find $\mathcal{F}_{Q_0}$ so that (3.36) holds and establish (3.37). Having these in mind we let $B_{Q_0} := B(x_{Q_0}, r_{Q_0})$ with $r_{Q_0} \approx \ell(Q_0)$ as in (1.1). Let $X_0 := X_{M_0\Delta_{Q_0}}$ be the corkscrew point relative to $M_0\Delta_{Q_0}$ (note that $M_0 r_{Q_0} \leq M_0\ell(Q_0) < \mathrm{diam}(\partial\Omega)$). By our choice of M_0, it is clear that $Q_0 \subset M_0\Delta_{Q_0}$ and also that $\delta(X_0) \geq cM_0 r_{Q_0} > 2\kappa_0 r_{Q_0}$. Hence, by (1.6),

$$X_0 \in \Omega \setminus B_{Q_0}^*. \tag{3.38}$$

On the other hand, $\delta(X_{Q_0}) \approx \ell(Q_0)$, $\delta(X_0) \approx M_0\ell(Q_0) \geq \ell(Q_0)$, and $|X_0 - X_{Q_0}| \lesssim M_0\ell(Q_0)$. Using Lemma 1.26 and Harnack's inequality, there exists $C_0 \geq 1$ depending on the 1-sided CAD constants, the ellipticity of L, and on M_0 (which is already fixed), such that $\omega_L^{X_0}(Q_0) \geq C_0^{-1}$.

Next, we define the normalized elliptic measure and Green function as

$$\omega_0 := C_0\,\sigma(Q_0)\omega_L^{X_0}, \qquad \text{and} \qquad \mathcal{G}_0(\cdot) := C_0\,\sigma(Q_0)G_L(X_0, \cdot). \tag{3.39}$$

Note the fact that $\omega_L^{X_0}(\partial\Omega) \leq 1$ implies

$$1 \leq \frac{\omega_0(Q_0)}{\sigma(Q_0)} \leq C_0.$$

Recall that we have assumed that $\omega_L \in A_{\infty}(\partial\Omega)$ and, as observed above, this means after passing to the previous renormalization that $\omega_0 \ll \sigma$ and we write $k_0 = d\omega_0/d\sigma$ for the Radon-Nikodym derivative. Since $Q_0 \subset M_0\Delta_{Q_0}$, using (1.45) we have that

$$\left(\fint_{Q_0} k_0(y)^q \, d\sigma(y)\right)^{1/q} \leq C_2.$$

In particular, for any Borel set $F \subset Q_0$, using Hölder's inequality we obtain

$$\frac{\omega_0(F)}{\sigma(Q_0)} \leq \left(\fint_{Q_0} \mathbf{1}_F(y)^{q'} \, d\sigma(y)\right)^{1/q'}\left(\fint_{Q_0} k_0(y)^q \, d\sigma(y)\right)^{1/q} \leq C_2\Big(\frac{\sigma(F)}{\sigma(Q_0)}\Big)^{1/q'}.$$

Hence we can apply Lemma 1.14 to $\mu = \omega_0$, and extract a pairwise disjoint family $\mathcal{F}_{Q_0} = \{Q_j\} \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ verifying (3.36), as well as

$$\frac{1}{2} \leq \frac{\omega_0(Q)}{\sigma(Q)} \leq K_0 K_1, \qquad \forall Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}, \tag{3.40}$$

with $K_1 = (4K_0)^{1/\theta}$, $K_0 = \max\{C_0, C_2\}$, and $\theta = 1/q'$.

We next observe that if $I \in \mathcal{W}_Q^*$ with $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$ then $2B_Q \subset B_{Q_0}^*$ (see (1.6)). Hence, using Harnack's inequality, parts (a) and (b) of Lemma 1.30, (3.40) and the AR property we have

$$\frac{\mathcal{G}_0(X_I)}{\ell(I)} \approx \frac{\mathcal{G}_0(X_I)}{\delta(X_I)} \approx \frac{\omega_0(\Delta_Q)}{\sigma(Q)} \approx 1, \tag{3.41}$$

where X_I is the center of I.

At this point, we are looking for M_1 independent of Q_0 and Q^0 such that (3.37) holds. Recalling (3.32) we note that

$$\mathfrak{m}_\alpha(\mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}) = \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}} \iint_{U_Q} |\nabla u(X)|^2 \delta(X) \, dX$$

$$\approx \sum_{Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}} \iint_{U_Q} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX \lesssim \iint_{\Omega_{\mathcal{F}_{Q_0}, Q_0}} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX, \tag{3.42}$$

where we have used Harnack's inequality, (3.41), and the bounded overlap of the family $\{U_Q\}_{Q \in \mathbb{D}}$.

As in Section 1.1 for every $N \geq 1$ we can consider the pairwise disjoint collection $\mathcal{F}_N := \mathcal{F}_{Q_0}\big(2^{-N}\ell(Q_0)\big)$ which is the family of maximal cubes of the collection $\mathcal{F}_{Q_0}$ augmented by adding all of the cubes $Q \in \mathbb{D}_{Q_0}$ such that $\ell(Q) \leq 2^{-N}\ell(Q_0)$. In particular, $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F}_{Q_0}, Q_0}$ and $\ell(Q) > 2^{-N}\ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N, Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'}, Q_0}$ if $N \leq N'$, and therefore $\Omega_{\mathcal{F}_N, Q_0} \subset \Omega_{\mathcal{F}_{N'}, Q_0} \subset \Omega_{\mathcal{F}_{Q_0}, Q_0}$. This and the monotone convergence theorem give that

$$\iint_{\Omega_{\mathcal{F}_{Q_0}, Q_0}} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX = \lim_{N \to \infty} \iint_{\Omega_{\mathcal{F}_N, Q_0}} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX. \tag{3.43}$$

We now formulate an auxiliary result that will lead us to the desired estimate.

Proposition 3.8. *Given $C_1 \geq 1$, one can find C such that if $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$, $N \in \mathbb{N}$, is a family of pairwise disjoint dyadic cubes satisfying*

$$C_1^{-1} \leq \frac{\omega_0(Q)}{\sigma(Q)} \leq C_1 \qquad and \qquad \ell(Q) > 2^{-N}\ell(Q_0), \qquad \forall Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}, \tag{3.44}$$

then

$$\iint_{\Omega_{\mathcal{F}_N, Q_0}} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX \leq C\sigma(Q_0). \tag{3.45}$$

Here, C depends only on dimension, the 1-sided CAD *constants, and the ellipticity of L.*

Assuming this result momentarily, (3.40) and the construction of $\mathcal{F}_N$ give (3.44). Next, we combine (3.42), (3.43) and (3.45) to conclude (3.37). This completes the proof of $(b) \implies (a)$ Theorem 3.1, modulo obtaining the just stated proposition. ∎

Proof of Proposition 3.8. We introduce an adapted cut-off function which can be obtained from a straightforward modification of [HMT1, Lemma 4.44] by simply replacing λ by 2λ (recall that λ appearing in Section 1.1 can be chosen arbitrarily small).

Lemma 3.9. *There exists $\Psi_N \in C_c^\infty(\mathbb{R}^{n+1})$ such that*

(a) $\mathbf{1}_{\Omega_{\mathcal{F}_N,Q_0}} \lesssim \Psi_N \leq \mathbf{1}_{\Omega^*_{\mathcal{F}_N,Q_0}}$.

(b) $\sup_{X \in \Omega} |\nabla \Psi_N(X)| \delta(X) \lesssim 1$.

(c) Set

$$\mathcal{W}_N := \bigcup_{Q \in \mathbb{D}_{\mathcal{F}_N,Q_0}} \mathcal{W}_Q^*, \qquad \mathcal{W}_N^\Sigma := \left\{ I \in \mathcal{W}_N : \exists J \in \mathcal{W} \setminus \mathcal{W}_N, \quad \partial I \cap \partial J \neq \varnothing \right\}.$$

Then

$$\nabla \Psi_N \equiv 0 \quad in \quad \bigcup_{I \in \mathcal{W}_N \setminus \mathcal{W}_N^\Sigma} I^{**} \qquad and \qquad \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \lesssim \sigma(Q_0),$$

with implicit constants depending only on the allowable parameters but uniform in N.

Taking then Ψ_N as above, Leibniz's rule leads us to

$$A\nabla u \cdot \nabla u\, \mathcal{G}_0\, \Psi_N^2 = A\nabla u \cdot \nabla(u\, \mathcal{G}_0\, \Psi_N^2) - \tfrac{1}{2} A\nabla(u^2\, \Psi_N^2) \cdot \nabla \mathcal{G}_0$$
$$+ \tfrac{1}{2} A\nabla(\Psi_N^2) \cdot \nabla \mathcal{G}_0\, u^2 - \tfrac{1}{2} A\nabla(u^2) \cdot \nabla(\Psi_N^2)\, \mathcal{G}_0. \quad (3.46)$$

Note that $u\, \mathcal{G}_0\, \Psi_N^2 \in W_0^{1,2}(\Omega^{**}_{\mathcal{F}_N,Q_0})$ since $\overline{\Omega^{**}_{\mathcal{F}_N,Q_0}}$ is a compact subset of Ω (indeed by construction $\text{dist}(\overline{\Omega^{**}_{\mathcal{F}_N,Q_0}}, \partial\Omega) \gtrsim 2^{-N}\ell(Q_0))$, $u \in W_{\text{loc}}^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\mathcal{G}_0 \in W_{\text{loc}}^{1,2}(\Omega \setminus \{X_0\})$, $\overline{\Omega^{**}_{\mathcal{F}_N,Q_0}} \subset \overline{T_{Q_0}^{**}} \subset \frac{1}{2}B_{Q_0}^*$ (cf. (1.6)), and (3.38). Moreover, since $u \in W_{\text{loc}}^{1,2}(\Omega)$ it follows that $u \in W_{\text{loc}}^{1,2}(\Omega) \subset W^{1,2}(\Omega^{**}_{\mathcal{F}_N,Q_0})$. All these and the fact that $Lu = 0$ in the weak sense in Ω easily give

$$\iint_\Omega A\nabla u \cdot \nabla(u\, \mathcal{G}_0 \Psi_N^2)\, dX = \iint_{\Omega^{**}_{\mathcal{F}_N,Q_0}} A\nabla u \cdot \nabla(u\, \mathcal{G}_0 \Psi_N^2)\, dX = 0. \quad (3.47)$$

On the other hand, much as before $u^2\, \Psi_N^2 \in W_0^{1,2}(\Omega^{**}_{\mathcal{F}_N,Q_0})$. Also, Lemma 1.28 (see in particular (1.40)) gives at once that $\mathcal{G}_0 \in W^{1,2}(\Omega^{**}_{\mathcal{F}_N,Q_0})$ and $L^\top \mathcal{G}_0 = 0$ in the weak sense in $\Omega \setminus \{X_0\}$. Thus, we easily obtain

$$\iint_\Omega A\nabla(u^2\, \Psi_N^2) \cdot \nabla \mathcal{G}_0\, dX = \iint_{\Omega^{**}_{\mathcal{F}_N,Q_0}} A^\top \nabla \mathcal{G}_0 \cdot \nabla(u^2\, \Psi_N^2)\, dX = 0. \quad (3.48)$$

Using ellipticity, (3.46), (3.47), (3.48), the fact that $\|u\|_{L^\infty(\Omega)} = 1$, and Lemma 3.9, we have

$$\iint_\Omega |\nabla u|^2\, \mathcal{G}_0\, \Psi_N^2\, dX \lesssim \iint_\Omega A\nabla u \cdot \nabla u\, \mathcal{G}_0\, \Psi_N^2\, dX$$
$$\lesssim \iint_\Omega \left(|\nabla \mathcal{G}_0| + |\nabla u|\, \mathcal{G}_0 \right) |\nabla \Psi_N|\, dX =: \text{I}. \quad (3.49)$$

To estimate I we use Lemma 3.9, Caccioppoli's and Harnack's inequalities, and the fact that $\|u\|_{L^\infty(\Omega)} = 1$:

$$\text{I} \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^{-1} \bigg(\iint_{I^{**}} |\nabla \mathcal{G}_0| \, dX + \iint_{I^{**}} |\nabla u| \, \mathcal{G}_0 \, dX \bigg) \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^{n-1} \mathcal{G}_0(X_I),$$
$$(3.50)$$

where X_I is the center of I. Note that for every $I \in \mathcal{W}_N^\Sigma$ there is $Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}$ such that $I \in \mathcal{W}_Q^*$. Hence we can use (3.41) to obtain

$$\text{I} \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^{n-1} \mathcal{G}_0(X_I) \lesssim \sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^n \lesssim \sigma(Q_0). \qquad (3.51)$$

Plugging this into (3.49) we get the desired estimate and the proof is complete. ∎

3.2 Proof of Theorems 3.2 and 3.3

We will prove Theorems 3.2 and 3.3 by showing that all bounded weak solutions satisfy the Carleson measure estimate (3.1), in which case Theorem 3.1 will give the A_∞ properties. Before that we need some integration by parts equality.

Lemma 3.10. *Let $D = (d_{i,j})_{i,j=1}^{n+1} \in L^\infty(\Omega) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be an antisymmetric real matrix and set*

$$\mathrm{div}_C D(X) := \big(\mathrm{div} \, (d_{\cdot,j}(X)) \big)_{1 \le j \le n+1} = \bigg(\sum_{i=1}^{n+1} \partial_i d_{i,j}(X) \bigg)_{1 \le j \le n+1}, \quad X \in \Omega,$$
$$(3.52)$$

which is the vector formed by taking the divergence operator acting on the columns of D. Then,

$$\iint_\Omega D(X) \nabla u(X) \cdot \nabla v(X) \, dX = - \iint_\Omega \mathrm{div}_C D(X) \cdot \nabla u(X) \, v(X) \, dX, \qquad (3.53)$$

for every $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and every $v \in W^{1,2}(\Omega)$ such that $K = \mathrm{supp}(v) \subset \Omega$ is compact.

Proof. We first consider the case on which $u, v \in C_c^\infty(\Omega)$. Using Leibniz's rule and the fact that D is antisymmetric we have that

$$\mathrm{div}(D \nabla u) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \partial_i d_{i,j} \partial_j u + \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} d_{i,j} \partial_i \partial_j u = \mathrm{div}_C D \cdot \nabla u.$$

Using this we integrate by parts to obtain

$$\iint_\Omega D \nabla u \cdot \nabla v \, dX = - \iint_\Omega \mathrm{div}(D \nabla u) \, v \, dX = - \iint_\Omega \mathrm{div}_C D \cdot \nabla u \, v \, dX.$$

To obtain the general case let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $v \in W^{1,2}(\Omega)$ such that $K = \mathrm{supp}(v) \subset \Omega$ is compact. It is standard to see, using for instance the Whitney

covering, that we can find $\Phi_K \in C_c^\infty(\Omega)$ so that $\Phi_K \equiv 1$ in K. Write $K^* = \text{supp}(\Phi_K)$ which is a compact subset of Ω and define

$$U := \{X \in \Omega : \text{dist}(X, K^*) < \text{dist}(K^*, \partial\Omega)/2\}$$

which satisfies $\text{dist}(\overline{U}, \partial\Omega) \geq \text{dist}(K^*, \partial\Omega)/2 > 0$, hence $\overline{U}$ it is also a compact subset of Ω. Since $u \in W_{\text{loc}}^{1,2}(\Omega)$ we clearly have that $u\Phi_K \in W_0^{1,2}(U)$ and hence we can find $\{u_j\}_j \subset C_c^\infty(U)$ so that $u_j \to u\Phi_K$ in $W^{1,2}(U)$. Also, since $v \in W^{1,2}(\Omega)$ verifies $K = \text{supp}(v) \subset \Omega$ it is also easy to see that $v \in W_0^{1,2}(U)$ and hence we can find $\{v_j\}_j \subset C_c^\infty(U)$ so that $v_j \to v$ in $W^{1,2}(U)$. Notice that extending the u_j's and v_j's as 0 outside of U one sees that $\{u_j\}_j, \{v_j\}_j \subset C_c^\infty(\Omega)$. Thus, we can use (3.53) and for every j

$$\iint_\Omega D\nabla u_j \cdot \nabla v_j \, dX = -\iint_\Omega \text{div}_C D \cdot \nabla u_j \, v_j \, dX. \tag{3.54}$$

Note that using that $\text{supp}(v_j), \text{supp}(v) \subset U$ and that $\Phi_K \equiv 1$ in $K \subset U$ we have

$$\left| \iint_\Omega D\nabla u \cdot \nabla v \, dX - \iint_\Omega D\nabla u_j \cdot \nabla v_j \, dX \right|$$

$$= \left| \iint_\Omega D\nabla(u\Phi_K) \cdot \nabla v \, dX - \iint_\Omega D\nabla u_j \cdot \nabla v_j \, dX \right|$$

$$\leq \|D\|_{L^\infty(\Omega)} \big(\|\nabla(u\Phi_K)\|_{L^2(U)} \|\nabla v - \nabla v_j\|_{L^2(U)}$$

$$+ \|\nabla(u\Phi_K) - \nabla u_j\|_{L^2(U)} \|\nabla v_j\|_{L^2(U)} \big),$$

and the last term converges to 0 as $j \to \infty$ since $D \in L^\infty(\Omega)$. Analogously,

$$\left| \iint_\Omega \text{div}_C D \cdot \nabla u \, v \, dX - \iint_\Omega \text{div}_C D \cdot \nabla u_j \, v_j \, dX \right|$$

$$= \left| \iint_\Omega \text{div}_C D \cdot \nabla(u\Phi_K)) \, v \, dX - \iint_\Omega \text{div}_C D \cdot \nabla u_j \, v_j \, dX \right|$$

$$\leq \|\nabla D\|_{L^\infty(U)} \big(\|\nabla(u\Phi_K)\|_{L^2(U)} \|v - v_j\|_{L^2(U)}$$

$$+ \|\nabla(u\Phi_K) - \nabla u_j\|_{L^2(U)} \|v_j\|_{L^2(U)} \big),$$

which also converges to $j \to \infty$ since $D \in \text{Lip}_{\text{loc}}(\Omega)$. All these and (3.54) readily gives (3.53). ∎

We are going to show that Theorems 3.2 and 3.3 follow easily from the following more general result which is interesting on its own right:

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided CAD (cf. Definition 1.4). Let $L_1 u = -\text{div}(A_1 \nabla u)$ and $L_0 u = -\text{div}(A_0 \nabla u)$ be real (not necessarily symmetric) elliptic operators (cf. Definition 1.20). Suppose that $A_0 - A_1 = A + D$ where $A, D \in L^\infty(\Omega)$ are real matrices satisfying the following conditions:*

(i) Define

$$a(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y)|, \qquad X \in \Omega, \tag{3.55}$$

where $\delta(X) := \text{dist}(X, \partial\Omega)$, and assume that it satisfies the Carleson measure condition

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \text{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} \frac{a(X)^2}{\delta(X)} \, dX < \infty. \tag{3.56}$$

(ii) $D \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ *is antisymmetric and suppose that* $\mathrm{div}_C D$ *defined in* (3.52) *satisfies the Carleson measure condition*

$$\sup_{\substack{x \in \partial\Omega \\ 0 < r < \mathrm{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r) \cap \partial\Omega)} \iint_{B(x,r) \cap \Omega} |\mathrm{div}_C D(X)|^2 \delta(X)\, dX < \infty. \quad (3.57)$$

Then, $\omega_{L_0} \in A_\infty(\partial\Omega)$ *if and only if* $\omega_{L_1} \in A_\infty(\partial\Omega)$ *(cf. Definition 1.33).*

Assuming this result momentarily we can easily prove Theorems 3.2 and 3.3:

Proof of Theorem 3.2. For L_0 and L_1 as in the statement of Theorem 3.2 we set $A = A_0 - A_1$ and $D = 0$. Thus, it suffices to check that A and D satisfy the required conditions in Theorem 3.11. For (i) notice that $a = \varrho(A_1, A_0)$ (cf. (3.55) and (2.1)), hence (3.3) gives immediately (3.56). On the other hand since $D = 0$ we clearly have all the conditions in (ii). With all these in hand, Theorem 3.11 gives at once the desired conclusion. ∎

Proof of Theorem 3.3. Set $A_0 = A$, $A_1 = A^\top$, $\widetilde{A} = 0$ and $D = A - A^\top$ so that $A_0 - A_1 = \widetilde{A} + D$. As before we can easily see that $\widetilde{A}$ and D satisfy the required conditions in Theorem 3.11. This time (i) is trivial. For (ii) notice that by assumption $D = A - A^\top \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and also that (3.5) yields (3.57) since (3.4) agrees with (3.52). As a result, we can invoke Theorem 3.11 obtaining the desired conclusion. ∎

Besides the previous results one can easily get other interesting perturbation results from Theorem 3.11. For instance suppose that $L_0 u = -\mathrm{div}(A_0 \nabla u)$ has an associated elliptic measure satisfying $\omega_{L_0} \in A_\infty(\partial\Omega)$. Let D be a real antisymmetric matrix with locally Lipschitz coefficients and assume that $\|D\|_{L^\infty(\Omega)} < \lambda_0$ where $\lambda_0 > 0$ is so that $A(X)\xi \cdot \xi \geq \lambda_0 |\xi|^2$ for all $\xi \in \mathbb{R}^{n+1}$ and a.e. $X \in \Omega$. The latter ensures that $A_1 = A_0 + D$ is uniformly elliptic and hence if we assume that $\mathrm{div}_C D$ satisfies (3.57) then Theorem 3.11 gives immediately that $\omega_{L_1} \in A_\infty(\partial\Omega)$ where $L_1 u = -\mathrm{div}(A_1 \nabla u)$. In particular, the A_∞ property is preserved under perturbations by antisymmetric "sufficiently small" matrices D with locally Lipschitz coefficients so that $|\nabla D|^2 \delta$ satisfies a Carleson measure condition.

Proof of Theorem 3.11. By symmetry it suffices to assume that $\omega_{L_0} \in A_\infty(\partial\Omega)$ and our goal is to see that $\omega_{L_1} \in A_\infty(\partial\Omega)$. By Theorem 3.1 we only need to show that given $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ with $L_1 u = 0$ in the weak sense in Ω then (3.1) holds. As before, by homogeneity we may assume without loss of generality that $\|u\|_{L^\infty(\Omega)} = 1$. We can now follow closely the proof of (b) $\Longrightarrow$ (a) in Theorem 3.1 with the following changes. Here we are assuming that $\omega_{L_0} \in A_\infty(\partial\Omega)$ and hence (3.39) needs to be replaced by

$$\omega_0 := C_0\, \sigma(Q_0)\omega_{L_0}^{X_0}, \qquad \text{and} \qquad \mathcal{G}_0(\cdot) := C_0\, \sigma(Q_0)G_{L_0}(X_0, \cdot), \qquad (3.58)$$

where $X_0 := X_{M_0 \Delta_{Q_0}}$ is chosen as before so that (3.38) holds.

Notice that in the present situation u satisfies $L_1 u = 0$ (as opposed to what happened above where both u and $\mathcal{G}_0$ where associated with the same operator). Other than that, and keeping in mind (3.58), all estimates (3.40)–(3.43) hold. Thus it is straightforward to see that everything reduces to obtain the following analog of Proposition 3.8:

Proposition 3.12. *Given $C_1 \geq 1$, one can find C such that if $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$, $N \in \mathbb{N}$, is a family of pairwise disjoint dyadic cubes satisfying*

$$C_1^{-1} \leq \frac{\omega_0(Q)}{\sigma(Q)} \leq C_1 \qquad and \qquad \ell(Q) > 2^{-N}\ell(Q_0), \qquad \forall Q \in \mathbb{D}_{\mathcal{F}_N, Q_0}, \qquad (3.59)$$

then

$$\iint_{\Omega_{\mathcal{F}_N, Q_0}} |\nabla u(X)|^2 \mathcal{G}_0(X) \, dX \leq C\sigma(Q_0). \qquad (3.60)$$

Here, C depends only on dimension, the 1-sided CAD *constants, the ellipticity of L_0 and L_1, and on the quantity* (3.3) *in the scenario of Theorem 3.2 or* (3.5) *in the scenario of Theorem 3.3.*

Much as before, assuming this result momentarily, the proof of Theorem 3.11 is complete modulo obtaining the just stated proposition. ∎

Proof of Proposition 3.12. Take Ψ_N from Lemma 3.9 and write $\mathcal{E}(X) := A_1(X) - A_0(X)$. Then Leibniz's rule leads us to

$$A_1 \nabla u \cdot \nabla u \, \mathcal{G}_0 \, \Psi_N^2 = A_1 \nabla u \cdot \nabla(u \, \mathcal{G}_0 \, \Psi_N^2) - \tfrac{1}{2} A_0 \nabla(u^2 \, \Psi_N^2) \cdot \nabla \mathcal{G}_0$$
$$+ \tfrac{1}{2} A_0 \nabla(\Psi_N^2) \cdot \nabla \mathcal{G}_0 \, u^2 - \tfrac{1}{2} A_0 \nabla(u^2) \cdot \nabla(\Psi_N^2) \, \mathcal{G}_0 - \tfrac{1}{2} \mathcal{E} \, \nabla(u^2) \cdot \nabla(\mathcal{G}_0 \, \Psi_N^2). \quad (3.61)$$

Note that $u \, \mathcal{G}_0 \, \Psi_N^2 \in W_0^{1,2}(\Omega_{\mathcal{F}_N, Q_0}^{**})$ since $\overline{\Omega_{\mathcal{F}_N, Q_0}^{**}}$ is a compact subset of Ω (indeed by construction $\text{dist}(\overline{\Omega_{\mathcal{F}_N, Q_0}^{**}}, \partial\Omega) \gtrsim 2^{-N}\ell(Q_0)$), $u \in W_{\text{loc}}^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\mathcal{G}_0 \in W_{\text{loc}}^{1,2}(\Omega \setminus \{X_0\})$, $\overline{\Omega_{\mathcal{F}_N, Q_0}^{**}} \subset \overline{T_{Q_0}^{**}} \subset \tfrac{1}{2} B_{Q_0}^*$ (cf. (1.6)), and (3.38). Moreover, since $u \in W_{\text{loc}}^{1,2}(\Omega)$ it follows that $u \in W_{\text{loc}}^{1,2}(\Omega) \subset W^{1,2}(\Omega_{\mathcal{F}_N, Q_0}^{**})$. All these and the fact that $L_1 u = 0$ in the weak sense in Ω easily give

$$\iint_\Omega A_1 \nabla u \cdot \nabla(u \, \mathcal{G}_0 \Psi_N^2) \, dX = \iint_{\Omega_{\mathcal{F}_N, Q_0}^{**}} A_1 \nabla u \cdot \nabla(u \, \mathcal{G}_0 \Psi_N^2) \, dX = 0. \qquad (3.62)$$

On the other hand, much as before $u^2 \, \Psi_N^2 \in W_0^{1,2}(\Omega_{\mathcal{F}_N, Q_0}^{**})$. Also, Lemma 1.28 (see in particular (1.40)) gives at once that $\mathcal{G}_0 \in W^{1,2}(\Omega_{\mathcal{F}_N, Q_0}^{**})$ and $L_0^\top \mathcal{G}_0 = 0$ in the weak sense in $\Omega \setminus \{X_0\}$. Thus, we easily obtain

$$\iint_\Omega A_0 \nabla(u^2 \, \Psi_N^2) \cdot \nabla \mathcal{G}_0 \, dX = \iint_{\Omega_{\mathcal{F}_N, Q_0}^{**}} A_0^\top \nabla \mathcal{G}_0 \cdot \nabla(u^2 \, \Psi_N^2) \, dX = 0. \qquad (3.63)$$

Using ellipticity, (3.61), (3.62), (3.63), the fact that $\|u\|_{L^\infty(\Omega)} = 1$, and Lemma 3.9, we have

$$\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \lesssim \iint_\Omega A_1 \nabla u \cdot \nabla u \, \mathcal{G}_0 \, \Psi_N^2 \, dX$$
$$\lesssim \iint_\Omega \left(|\nabla \mathcal{G}_0| + |\nabla u| \, \mathcal{G}_0 \right) |\nabla \Psi_N| \, dX + \left| \iint_\Omega \mathcal{E} \nabla(u^2) \cdot \nabla(\mathcal{G}_0 \, \Psi_N^2) \, dX \right| =: \text{I} + \text{II}.$$
$$(3.64)$$

Much as in (3.50) and (3.51) we can show that I $\lesssim \sigma(Q_0)$. To estimate II note that since $\mathcal{E} = A_1 - A_0 = -(A + D)$ it follows that

$$
\text{II} \leq \left| \iint_\Omega A \nabla(u^2) \cdot \nabla(\mathcal{G}_0 \, \Psi_N^2) \, dX \right| + \left| \iint_\Omega D \nabla(u^2) \cdot \nabla(\mathcal{G}_0 \, \Psi_N^2) \, dX \right| = \text{II}_1 + \text{II}_2.
$$

(3.65)

For the term II$_1$ we use that $A \in L^\infty(\Omega)$ and the fact that $\|u\|_{L^\infty(\Omega)} = 1$ to obtain

$$
\text{II}_1 \lesssim \iint_\Omega |A| \, |\nabla u| \, |\nabla \mathcal{G}_0| \, \Psi_N^2 \, dX + \iint_\Omega |\nabla(u^2)| \, |\nabla(\Psi_N^2)| \, \mathcal{G}_0 \, dX =: \text{III}_1 + \text{III}_2. \quad (3.66)
$$

For III$_1$ we note that $\sup_{I^{**}} |A| \leq \inf_{I^*} a$ for every $I \in \mathcal{W}$, since $I^{**} \subset \{Y \in \Omega : |Y - X| < \delta(X)/2\}$ for every $X \in I^*$ (see (1.4)). Hence, Lemma 3.9, Caccioppoli's and Harnack's inequalities, (3.41), the fact that the family $\{I^{**}\}_{I \in \mathcal{W}}$ has bounded overlap, and (1.6) yield

$$
\text{III}_1 \lesssim \sum_{I \in \mathcal{W}_N} \sup_{I^{**}} |A| \left(\iint_{I^{**}} |\nabla u|^2 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\iint_{I^{**}} |\nabla \mathcal{G}_0|^2 \, dX \right)^{\frac{1}{2}} \quad (3.67)
$$

$$
\lesssim \sum_{I \in \mathcal{W}_N} \left(\iint_{I^{**}} |\nabla u|^2 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\sup_{I^{**}} |A|^2 \, \mathcal{G}_0(X_I)^2 \, \ell(I)^{n-1} \right)^{\frac{1}{2}}
$$

$$
\lesssim \sum_{I \in \mathcal{W}_N} \left(\iint_{I^{**}} |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\iint_{I^*} \frac{a^2}{\delta} \, dX \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\iint_{B_{Q_0}^*} \frac{a^2}{\delta} \, dX \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \sigma(Q_0)^{\frac{1}{2}},
$$

where in the last estimate we have used (3.56) and AR along with the fact that $r(B_{Q_0}^*) = 2\kappa_0 r_{Q_0} \leq 2\kappa_0 \ell(Q_0) \leq 2\kappa_0 \, \text{diam}(\partial\Omega)/M_0 < \text{diam}(\partial\Omega)$ by our choice of M_0. On the other hand, we observe that

$$
\text{III}_2 \lesssim \iint_\Omega |\nabla u| \, |\nabla \Psi_N| \, \mathcal{G}_0 \, \Psi_N \, dX \quad (3.68)
$$

$$
\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\iint_\Omega |\nabla \Psi_N|^2 \, \mathcal{G}_0 \, dX \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\sum_{I \in \mathcal{W}_N^\Sigma} \ell(I)^{n-1} \mathcal{G}_0(X_I) \right)^{\frac{1}{2}}
$$

$$
\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \sigma(Q_0)^{\frac{1}{2}},
$$

where we have used Lemma 3.9, the normalization $\|u\|_{L^\infty(\Omega)} = 1$, Harnack's inequality and the last estimate follows as in (3.51).

Let us now turn our attention to estimating II_2. Note that $u^2 \in W^{1,2}_{\mathrm{loc}}(\Omega)$ since $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$; $\mathrm{supp}(\mathcal{G}_0 \, \Psi_N^2) \subset \overline{\Omega^*_{\mathcal{F}_N, Q_0}}$ which is a compact subset of Ω since by construction $\mathrm{dist}(\overline{\Omega^*_{\mathcal{F}_N, Q_0}}, \partial\Omega) \gtrsim 2^{-N}\ell(Q_0)$; and finally $\mathcal{G}_0 \, \Psi_N^2 \in W^{1,2}(\Omega)$ since $\mathcal{G}_0 \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{X_0\})$, $\overline{\Omega^*_{\mathcal{F}_N, Q_0}} \subset \overline{T^*_{Q_0}} \subset \frac{1}{2}B^*_{Q_0}$ (cf. (1.6)), and (3.38). Thus we can invoke Lemma 3.10 to see that

$$\mathrm{II}_2 = \left| \iint_\Omega \mathrm{div}_C \, D \cdot \nabla(u^2) \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right| \tag{3.69}$$

$$\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \left(\iint_\Omega |\mathrm{div}_C \, D|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}}.$$

$$\lesssim \left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \sigma(Q_0)^{\frac{1}{2}},$$

where we have used $\|u\|_{L^\infty(\Omega)} = 1$ and the last estimate is obtained as follows:

$$\iint_\Omega |\mathrm{div}_C \, D|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \lesssim \sum_{I \in \mathcal{W}_N} \mathcal{G}_0(X_I) \iint_{I^{**}} |\mathrm{div}_C \, D|^2 \, dX$$

$$\lesssim \sum_{I \in \mathcal{W}_N} \ell(I) \iint_{I^{**}} |\mathrm{div}_C \, D|^2 \, dX \lesssim \iint_{B^*_{Q_0} \cap \Omega} |\mathrm{div}_C \, D(X)|^2 \delta(X) \, dX \ \leq C\sigma(Q_0),$$

where we have used Harnack's inequality, (3.41), the fact that the family $\{I^{**}\}_{I \in \mathcal{W}}$ has bounded overlap, (1.6), and the last estimate follows from (3.57), the fact that $r(B^*_{Q_0}) = 2\kappa_0 r_{Q_0} \leq 2\kappa_0 \ell(Q_0) \leq 2\kappa_0 \, \mathrm{diam}(\partial\Omega)/M_0 < \mathrm{diam}(\partial\Omega)$ by our choice of M_0, and AR.

At this point we can collect (3.64)–(3.69) and use Young's inequality to conclude that

$$\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \leq C\sigma(Q_0) + C\left(\iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \right)^{\frac{1}{2}} \sigma(Q_0)^{\frac{1}{2}}$$

$$\leq \frac{C(2+C)}{2}\sigma(Q_0) + \frac{1}{2} \iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX.$$

The last term is finite since $\mathrm{supp}(\Psi_N) \subset \overline{\Omega^*_{\mathcal{F}_N, Q_0}}$ which is a compact subset of Ω, $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $\mathcal{G}_0 \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{X_0\})$, (3.38), and (1.6). Hence we can hide it and use Lemma 3.9 to conclude as desired that

$$\iint_{\Omega_{\mathcal{F}_N, Q_0}} |\nabla u|^2 \, \mathcal{G}_0 \, dX \lesssim \iint_\Omega |\nabla u|^2 \, \mathcal{G}_0 \, \Psi_N^2 \, dX \lesssim \sigma(Q_0).$$

This completes the proof.   ∎

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