Facultad de Ciencias
Departamento de Matemáticas

École doctorale Galilée
Laboratoire Analyse, Géométrie et Applications, UMR 7539

Tesis

presentada y defendida publicamente por

Giuseppe NEGRO

para conseguir el título de

Doctor de la Universidad Autónoma de Madrid

Docteur de l'Université Paris 13

en Matemáticas

Sharp estimates for linear and nonlinear wave equations via the Penrose transform

Directores de tesis :
Thomas DUYCKAERTS
Keith ROGERS

Abstract. We apply the Penrose transform, which is a basic tool of relativistic physics, to the study of sharp estimates for linear and nonlinear wave equations. We disprove a conjecture of Foschi, regarding extremizers for the Strichartz inequality with data in the Sobolev space $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, for even $d \geqslant 2$. On the other hand, we provide evidence to support the conjecture in odd dimensions and refine his sharp inequality in $\mathbb{R}^{1+3}$, adding a term proportional to the distance of the initial data from the set of extremizers. Using this, we provide an asymptotic formula for the Strichartz norm of small solutions to the cubic wave equation in Minkowski space. The leading coefficient is given by Foschi's sharp constant. We calculate the constant in the second term, whose absolute value and sign changes depending on whether the equation is focusing or defocusing.

Keywords. Wave equation, Strichartz estimate, sharp inequality, Lorentz invariance.

Estimations optimales pour équations des ondes linéaire et nonlinéaire à l'aide de la transformée de Penrose.

Résumé. Nous appliquons la transformée de Penrose, qui est un outil basique de la physique relativiste, à des estimations optimales pour les équations des ondes linéaire et nonlinéaire. Nous infirmons une conjecture de Foschi concernant les points extrémaux de l'inégalité de Strichartz à données dans l'espace de Sobolev $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, où $d \geqslant 2$ est pair. En revanche, nous donnons des indications appuyant cette conjecture en dimension impaire, ainsi qu'une version raffinée de son inégalité optimale sur $\mathbb{R}^{1+3}$, en ajoutant un terme proportionnel à la distance des données initiales de l'ensemble des points extrémaux. À l'aide de ce résultat, nous obtenons une formule asymptotique pour la norme de Strichartz des solutions petites de l'équation des ondes cubique dans l'espace-temps de Minkowski. Le coefficient principal est donné par la constante optimale de Foschi. Nous calculons le terme suivant, qui change de signe et de valeur absolue selon que la non-linéarité est focalisante ou défocalisante.

Mots-clés. Équation des ondes, estimation de Strichartz, inégalité optimale, invariance de Lorentz.

Estimaciónes óptimas para ecuaciones de ondas lineales y no lineales por medio de la transformada de Penrose.

Resumen. Aplicamos la transformada de Penrose, una herramienta básica de la física relativista, a unas estimaciones óptimas para ecuaciones de ondas lineales y no lineales. Invalidamos una conjetura de Foschi, sobre extremizadores para la estimación de Strichartz con datos en el espacio de Sobolev $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, para $d \geqslant 2$ par. Por otro lado, vamos a dar indicios en favor de su conjetura en dimension impar, así como una versión refinada de su desigualdad óptima en $\mathbb{R}^{1+3}$, añadiendo un término proporcional a la distancia de los datos iniciales del conjunto de puntos extremales. Utilizando este resultado, conseguimos una fórmula asintótica para la norma de Strichartz de soluciones pequeñas de la ecuación de ondas cúbica en el espacio-tiempo de Minkowski. El coeficiente principal coincide con la constante óptima de Foschi. Calculamos explícitamente el coeficiente del otro término, cuyo módulo y signo cambian dependiendo de si estamos en el caso *focusing* o *defocusing*.

Palabras clave. Ecuación de ondas, estimación de Strichartz, desigualdad óptima, invariancia de Lorentz.

Contents

Acknowledgements

This thesis began sometime in 2013, when I walked into Keith's office asking him if he was interested in working together. He was, and after a while, he came up with this idea of cooperating with Thomas, whom I did not know at the time, and spend six months per year in France and the other six in Spain. That was not what I expected; Keith was thinking out of the box, as he always does. I admire him, both scientifically and humanely.

I accepted the challenge, and even if it's been tough at times, I am glad I did. Thomas turned out to be the perfect man for this job; he has copious creative ideas, paired with an astonishing technical prowess and a deep mathematical culture. He and Keith have been two complementary directors, with different points of view on mathematics and on scientific research, with the common trait of being two excellent mathematicians and two of the world's greatest beings. I am deeply indebted to them.

Many people helped me in the realization of this work. Sandra Lucente, Luca Fanelli and Ana Vargas encouraged me to start. I got some of the ideas through conversations with Vladimir Georgiev, Nicola Visciglia, Paolo Facchi and Jean Dolbeault, among many others. The advice of my friend Javi Ramos has been decisive for the third chapter. Jason Metcalfe, Diogo Oliveira e Silva and Lysianne Hari invited me to give talks in Chapel Hill, in Birmingham and in Besançon, which have been turning points in my research. I thank the referees, Emanuel Carneiro and Sahbi Keraani, for their careful reading of the manuscript and for their recommendations, which have improved it. I also thank Daniel Faraco, who has been my tutor in the UAM.

I received financial support from the MINECO project *Severo Ochoa*, from the ERC projects *Hade* and *Restriction*, whose main researchers are Luis Vega and Keith Rogers respectively, and from the LAGA, University of Paris XIII. I gratefully acknowledge this.

This thesis would hardly have been possible without the immense support of Anna Rita Giammetta and of Belén Martínez. I also have to thank Nieves Pérez and Gabriel Gómez, who welcomed me as a member of their family. And another very special thanks goes to my friend Jacek Jendrej.

I am thankful to Sylvie Barrier at the Institut Galilée; I owe her my command of the French language.

Finally, my warmest thanks go to my family, my mother, father, and sister. They've always been there for me, providing an ever-present helping hand without which I cannot imagine how this work could have gone through.

Introduction

The Penrose transform is a basic tool of relativistic physics. The purpose of this thesis is to show that it can be fruitfully applied to sharpen inequalities for the wave equation, both linear and nonlinear. In the linear case, such inequalities are known as Strichartz estimates, and there is a conjecture, due to Foschi, about what the optimal Strichartz estimate should be. The first chapter of this thesis deals with this question, adding some weight to support the conjecture in odd spatial dimensions, while disproving it in even dimensions.

In three spatial dimensions, Foschi proved the conjecture in the affirmative. The second chapter takes this theorem as a starting point, obtaining a refined version which improves for data away from the maximizers. This, in turn, is one of the main ingredients of the third chapter, in which a sharp estimate for solutions to the cubic wave equation is obtained.

Strichartz estimates

In 1977, Strichartz [68] proved that there is a positive constant C such that

$$\|v\|_{L^p(\mathbb{R}^{1+d})} \leqslant C\|\boldsymbol{v}(0)\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^d)}, \quad p = 2\frac{d+1}{d-1}, \tag{1}$$

where v solves the wave equation $v_{tt} = \Delta v$ on $\mathbb{R}^{1+d}$ with $d \geqslant 2$, and $\boldsymbol{v}(0) = (v(0), v_t(0))$ belongs to the Sobolev space of pairs $\boldsymbol{f} = (f_0, f_1)$ with norm defined by

$$\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^d)} = \left(\|(-\Delta)^{1/4} f_0\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{-1/4} f_1\|^2_{L^2(\mathbb{R}^d)}\right)^{1/2}.$$

This gives an integral quantification of the decay of waves, due to their dispersion. All known proofs are harmonic-analytic and a duality argument yields an estimate that restricts the Fourier transform to the cone. Tomas and Stein [72] had previously proven a similar estimate that restricted the Fourier transform to the sphere. The closely related paper of Segal [65], for the Klein-Gordon equation, should also be mentioned here.

Estimates such as (1) have been extensively studied, mainly because they have proved to be fundamental in the development of the well-posedness and scattering theory for nonlinear wave equations. The theory is far too extensive to be entirely surveyed here; a few fundamental results are [40, 41, 42, 50, 67].

Optimal constants and the Penrose transform

Foschi [37] proved the Strichartz estimate (1), for $d = 3$, with explicit constant;

$$\|v\|_{L^4(\mathbb{R}^{1+3})} \leqslant \left(\frac{3}{16\pi}\right)^{\frac{1}{4}} \|\boldsymbol{v}(0)\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)},$$

and proved that there is equality for

$$\boldsymbol{v}(0) = \left((1 + |\cdot|^2)^{-1}, 0\right), \tag{2}$$

so, in particular, the multiplicative constant is optimal, in the sense that it cannot be replaced by a smaller one. He also conjectured that, in arbitrary dimension $d \geqslant 2$, the estimate (1) should hold with constant

$$C = \frac{\|v\|_{L^p(\mathbb{R}^{1+d})}}{\|\boldsymbol{v}(0)\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^d)}}, \quad p = 2\tfrac{d+1}{d-1},$$

where

$$\boldsymbol{v}(0) = \left((1 + |\cdot|^2)^{-\frac{d-1}{2}}, 0\right), \tag{3}$$

that is, that these data should extremize the inequality in any dimension.

In the first chapter, which is dedicated to this conjecture, the Penrose transform is introduced. This is a transformation of solutions v to the wave equation on $\mathbb{R}^{1+d}$ into solutions V to the hyperbolic equation

$$\partial_T^2 V - \Delta_{\mathbb{S}^d} V + \frac{(d-1)^2}{4} V = 0, \tag{4}$$

posed on a relatively compact submanifold of $\mathbb{R} \times \mathbb{S}^d$. It involves a simple conformal mapping, first introduced by Penrose [61], and first applied to the mathematical study of wave equations by Christodoulu [25, 26].

This is relevant to the conjecture of Foschi, because the data (3) are mapped by the Penrose transform to constant initial data on $\mathbb{S}^d$;

$$\boldsymbol{V}(0) = (1/2, 0)\,.$$

The first original result presented in this thesis uses this observation, to prove that (3) is a critical point for the deficit functional of the inequality (1) if and only if the spatial dimension d is odd. In particular, the conjecture of Foschi cannot hold in even dimension.

The different behavior, according to the parity of the spatial dimension, is best explained in terms of the Penrose transform. The equation (4) is posed on a subset of $[-\pi, \pi] \times \mathbb{S}^d$ that is *not* a Cartesian product, and this, in principle, prevents the use of separation of variables. This can be overcome only if d is odd, as solutions to (4) are 2π-periodic in the conformal time variable T and satisfy an appropriate symmetry.

It is to be remarked that, for the Strichartz inequality (1), extremizing data do exist in any spatial dimension; this follows from the work of Ramos [62]. Ramos actually proved a *profile decomposition* adapted to (1), which is a by-now standard tool originating from the work of P. L. Lions [55], and introduced by Merle and Vega [56] for the Schrödinger equation, by Gérard [39] in the context of the Sobolev inequalities and by Bahouri and Gérard [9] for the wave equation.

Similar concentration-compactness techniques have been used to show the existence of maximizers in Strichartz inequalities for the Schrödinger and the wave equation in [52, 19]. However, these techniques never yield any information on the problem of uniqueness of such maximizers, up to the relevant symmetry group.

To the knowledge of the author, the use of the Penrose transform to study the Strichartz inequality is new, but the use of conformal mappings to study sharp inequalities is classical. It is especially interesting to mention the case of the Sobolev, and the closely related Hardy-Littlewood-Sobolev inequality, because constant functions on the sphere are extremal data, up to stereographic projection; see, for example, [8, 53, 69]. A conformal mapping to the hyperbolic space has been used by Tataru [71] to obtain weighted Strichartz estimates for the wave equation.

Finally, it is to be remarked that sharp space-time estimates for dispersive equations have been studied extensively; see for example [10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 36, 44, 43, 47, 48, 49, 59, 60], or the recent survey paper [35].

Sharpened inequalities

In the aforementioned paper [37], Foschi gave a complete characterization of the initial data that extremize the Strichartz inequality with $d = 3$. The full set $\mathbf{M}$ is obtained by acting a group of symmetries of the inequality on the data (3). The second chapter is mostly dedicated to the proof that (1) can be refined, by adding a term proportional to the distance from $\mathbf{M}$.

Brezis and Lieb asked if the sharp Sobolev inequality due to Aubin [8] and Talenti [69] could be sharpened in this way; see [1, question (c)]. This was solved by Bianchi and Egnell [17]; see also [23, 24, 27, 33, 34] for work in a similar spirit.

The present thesis follows the outline of Bianchi and Egnell; the key step is the proof of a local version of the sharpened inequality, meaningful in a neighborhood of $\mathbf{M}$. For this, it is necessary to establish a *transversal non-degeneracy* property of the deficit functional of (1). This means that, at all points of $\mathbf{M}$, the second derivative of the functional must be a strictly positive definite quadratic form, except on the tangent spaces of $\mathbf{M}$, on which it vanishes.

To establish this property, the Penrose transform is essential. It allows for explicit computation of these quadratic forms, using the symmetry property in odd spatial dimension mentioned in the previous section. It is remarkable that the tangent space to $\mathbf{M}$ at the maximizer (2) coincides with the sum of the first two eigenspaces of the Laplace-Beltrami operator. Analogous properties hold for the tangent spaces in the case of the Sobolev inequality; see the aforementioned paper of Bianchi and Egnell [17], and Chen, Frank and Weth [23].

A computation that is very similar in spirit is present in the work of Duyckaerts, Merle and Roudenko [31], in which a non-degeneracy property is established for the sharp Strichartz estimate for the Schrödinger equation in one and two dimensions.

The passage from the local to the global sharpened estimate is achieved by an application of the aforementioned profile decomposition of Ramos [62].

In this chapter, a five-dimensional sharpened Strichartz inequality in the energy space $\dot{H}^1 \times L^2(\mathbb{R}^5)$ is also established. This refines the sharp estimate due to Bez and Rogers [14]. The proof presents the significant additional difficulty that the relevant quadratic form is not diagonal in its expansion in spherical harmonics. This reflects the fact that such an inequality is not conformally invariant. Indeed, it is remarkable that a method based on conformal transformations works in this case.

Spacetime bounds for the cubic wave equation

The third chapter of this thesis deals with the equation

$$u_{tt} - \Delta u = \sigma u^3, \quad \text{on } \mathbb{R}^{1+3}, \tag{5}$$

where σ is the *sign* of the nonlinear term; when $\sigma > 0$, the equation is called *focusing*, and when $\sigma < 0$ it is called *defocusing*.

A standard argument using the Strichartz estimate (1) shows that, if (5) is supplied with initial data that are sufficiently small in the critical Sobolev norm $\dot{\mathcal{H}}^{1/2}$, then it admits a unique solution that belongs to the spaces $C(\mathbb{R}; \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3))$ and $L^4(\mathbb{R}^{1+3})$. In particular, such solutions are global in time and the following functional is well-defined for small $\delta > 0$;

$$I(\delta) = \sup\left\{ \|u\|^4_{L^4(\mathbb{R}^{1+3})} \ \middle|\ \lim_{t\to-\infty} \|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta \right\}. \tag{6}$$

In this chapter it is proved that the supremum is attained, and satisfies the explicit asymptotic

$$I(\delta) = \frac{3}{16\pi}\delta^4 + \sigma\delta^6 \begin{cases} \frac{29}{2^{10}\pi^3}, & \sigma > 0, \\ \frac{5}{2^{10}\pi^3}, & \sigma < 0, \end{cases} + O(\delta^8). \tag{7}$$

By the aforementioned result of Foschi,

$$\frac{3}{16\pi}\delta^4 = \max\left\{ \ \|v\|^4_{L^4(\mathbb{R}^{1+3})} \ \middle|\ v_{tt} = \Delta v,\ \|\boldsymbol{v}(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta \right\}.$$

A consequence of (7) is, therefore, that the maximal $L^4(\mathbb{R}^{1+3})$ norm is larger or smaller than the maximal norm in the linear case, for solutions to the focusing or defocusing cubic wave equation respectively. This furnishes a quantitative measure of the impact of the nonlinearity on the size of the solution.

This result is mainly inspired by the analogous one of the aforementioned Duyckaerts, Merle and Roudenko [31], for the mass-critical nonlinear Schrödinger equation in one or two spatial dimensions. Like in the Schrödinger case, an essential ingredient in establishing (7) is the sharpened Strichartz estimate. However, the nonlinear wave equation presents a significant additional difficulty. In the Schrödinger case, all symmetries to the relevant Strichartz estimate are also symmetries to the nonlinear equation. On the other hand, $\|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}}$ is not invariant under time translations, Lorentzian boosts, and phase shifts, which are all symmetries of (5).

It is to address this lack of invariance that, in (6), the limit as $t \to -\infty$ of $\|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}}$ is considered; this is manifestly invariant under time translations, and it is proved in this chapter that it is also invariant under Lorentzian boosts. This leaves out phase shifts, which is unavoidable, as these are symmetries of the linear wave equation which do not correspond to any symmetry of (5).

These invariances are necessary to explicitly compute the second-order constant in (7); this computation is carried out via the Penrose transform. These also enable the construction of a nonlinear profile decomposition, adapting the aforementioned linear decomposition of Ramos, which is then combined with a standard super-additivity argument, to prove that (6) is attained. The relation between super-additivity and maximizers is classical, and due to the aforementioned P.L. Lions [55].

The problem of uniqueness of the maximizers to (6) is also considered. Two maximizers are shown to be equal, if their metric projections on the manifold of linear maximizers coincide, up to nonlinear symmetries. The presence of the phase shifts, which leave the manifold of linear maximizers invariant, but do not correspond to any nonlinear symmetry, make this result conditional; remarkably, in the Schrödinger case this difficulty is nonexistent, and Duyckaerts, Merle and Roudenko do obtain an unconditional uniqueness. The uniqueness of maximizers is actually the most difficult part of the problem, since it cannot be resolved by concentration-compactness alone; it relies on the explicit expression, and on the geometrical structure, of the set of linear maximizers, and on the sharpened Strichartz estimate.

There is intense research going on on the dynamics of the cubic wave equation (5) in $\dot{\mathcal{H}}^{1/2}$; see [3, 2, 28, 64, 66] and the very recent [5, 4, 29]. However, to the knowledge of the author, the only paper, other than the present thesis, that deals with Lorentzian transformations is the work of Ramos [63]; see also [51] for the Klein-Gordon equation. Also related is [38], in which the Penrose transform is used.

Finally, it is remarkable that estimates of Strichartz norms for critical nonlinear problems are only known in a few cases. Duyckaerts and Merle [30] obtained a sharp bound for solutions to the focusing quintic wave equation that are close to the *threshold* solution. For the defocusing quintic wave equation in $\mathbb{R}^{1+3}$, Tao [70] gives a bound of the $L^4(\mathbb{R}; L^{12}(\mathbb{R}^3))$ norm in terms of a tower of exponentials of the $\dot{H}^1 \times L^2$ norms of initial data. This result holds for all data, not just small, but is not sharp, and it is interesting to note that a much smaller bound had previously been given in the radial case by Ginibre, Soffer and Velo [40].

Notation

Unless otherwise stated, all functions are real-valued. For $s = 1$ or $s = 1/2$,

$$\dot{\mathcal{H}}^s(\mathbb{R}^d) := \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d).$$

Boldface denotes elements of $\dot{\mathcal{H}}^s(\mathbb{R}^d)$, considered as column vectors;

$$\boldsymbol{f} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}.$$

The space $\dot{\mathcal{H}}^s(\mathbb{R}^d)$ is a real Hilbert space, with scalar product

$$\langle \boldsymbol{f} \,|\, \boldsymbol{g} \rangle_{\dot{\mathcal{H}}^s} = \int_{\mathbb{R}^d} (-\Delta)^s f_0 \cdot g_0 \, dx + \int_{\mathbb{R}^d} (-\Delta)^{s-1} f_1 \cdot g_1 \, dx.$$

The $\Box$ symbol denotes the d'Alembert operator;

$$\Box u := \partial_t^2 u - \Delta u.$$

If u is a function on the Minkowski spacetime $\mathbb{R}^{1+d}$, then its boldface denotes

$$\boldsymbol{u}(t) := \begin{bmatrix} u(t, \cdot) \\ \partial_t u(t, \cdot) \end{bmatrix}, \quad t \in \mathbb{R}.$$

The operator S denotes the propagator of the linear wave equation;

$$v = S\boldsymbol{f} \quad \Longleftrightarrow \quad \begin{cases} \Box v = 0, & \text{on } \mathbb{R}^{1+d}, \\ \boldsymbol{v}(0) = \boldsymbol{f}. \end{cases}$$

Finally, $\mathbb{S}^1$ denotes the quotient $\mathbb{R}/2\pi\mathbb{Z}$. For all $\theta_1, \theta_2 \in \mathbb{S}^1$,

$$|\theta_1 - \theta_2| := \min \left\{ \ |\theta_1' - \theta_2'| : \theta_1 \equiv \theta_1', \theta_2 \equiv \theta_2' \mod 2\pi \right\}.$$

Chapter 1

Maximizers for Strichartz estimates

In [37], Foschi conjectured that

$$\boldsymbol{f}_\star := \left(2^{\frac{d-1}{2}}(1+|\cdot|^2)^{-\frac{d-1}{2}}, 0\right)$$

is a global minimizer of the function

$$\psi(\boldsymbol{f}) := \mathcal{S}^p \|\boldsymbol{f}\|^p_{\dot{\mathcal{H}}^{1/2}} - \|S\boldsymbol{f}\|^p_{L^p(\mathbb{R}^{1+d})}, \quad p := 2\tfrac{d+1}{d-1},$$

where

$$\mathcal{S} := \frac{\|S\boldsymbol{f}_\star\|_{L^p(\mathbb{R}^{1+d})}}{\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}}.$$

In this chapter, we prove that $\boldsymbol{f}_\star$ is a critical point of ψ if and only if the spatial dimension d is odd. In particular, the conjecture cannot be true in even spatial dimension.

Theorem 1.0.1. *It holds that*

$$\tfrac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star + \varepsilon\boldsymbol{f})\big|_{\varepsilon=0} = 0, \quad \forall \boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^d),$$

if and only if d is odd.

The first step in the proof is the compactification of the Minkowski space-time by means of the Penrose transform.

1.1 The Penrose transform

We will introduce two coordinate systems on the Minkowski spacetime $\mathbb{R}^{1+d}$ and another two on the curved spacetime $\mathbb{R}\times\mathbb{S}^d$, where

$$\mathbb{S}^d = \left\{\, (X_0, X_1, \dots, X_d) \;\middle|\; X_0^2 + X_1^2 + \dots + X_d^2 = 1 \,\right\}.$$

We begin with $\mathbb{R}^{1+d}$, in which we let $t \in \mathbb{R}$ denote the time coordinate and $x \in \mathbb{R}^d$ denote the Cartesian spatial coordinates. Then, we define the *polar coordinates* by

$$r = |x|, \quad \omega = \tfrac{x}{|x|} \in \mathbb{S}^{d-1},$$

and the *light-like coordinates* as

$$x^- = t - r, \quad x^+ = t + r, \quad \text{where } x^- \leqslant x^+. \tag{1.1}$$

We now consider $\mathbb{R} \times \mathbb{S}^d$, in which we let $T \in \mathbb{R}$ denote the time coordinate and $X = (X_0, X_1, \ldots, X_d)$ denote the Cartesian coordinates on $\mathbb{S}^d$. We define the *spherical polar coordinates* via the equations

$$X_0 = \cos(R), \quad (X_1, \ldots, X_d) = \sin(R)\,\omega, \quad \omega \in \mathbb{S}^{d-1}, \; R \in [0, \pi]. \tag{1.2}$$

And finally, we define the *light-like coordinates* on $\mathbb{R} \times \mathbb{S}^d$ as

$$X^- = \tfrac{1}{2}(T - R), \quad X^+ = \tfrac{1}{2}(T + R). \tag{1.3}$$

We can now define an injective map

$$\mathcal{P} \colon \mathbb{R}^{1+d} \to \mathbb{R} \times \mathbb{S}^d, \quad (T, \cos R, \sin(R)\omega) = \mathcal{P}(t, x),$$

via the equations

$$X^- = \arctan x^-, \quad X^+ = \arctan x^+, \tag{1.4}$$

remarking that X^- and X^+ take values in the region

$$\left\{ (X^-, X^+) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^2 \;\middle|\; X^- \leqslant X^+ \right\}.$$

So, the map $\mathcal{P}$ is not surjective and its image $\mathcal{P}(\mathbb{R}^{1+d})$ is

$$\mathcal{P}(\mathbb{R}^{1+d}) = \left\{ \Big(T, (\cos R, \sin R\,\omega)\Big) \in \mathbb{R} \times \mathbb{S}^d \;\middle|\; \begin{array}{c} -\pi < T < \pi \\ 0 \leqslant R < \pi - |T| \\ \omega \in \mathbb{S}^{d-1} \end{array} \right\}; \tag{1.5}$$

see Figure 1.1.

We now discuss the conformality of $\mathcal{P}$. The metric tensor on $\mathbb{R}^{1+d}$ is $ds^2_{\mathbb{R}^{1+d}} = dt^2 - dr^2 - r^2 d\omega^2$, where $d\omega^2$ is the metric tensor on $\mathbb{S}^{d-1}$. So, using (1.1), we get the expression

$$ds^2_{\mathbb{R}^{1+d}} = \frac{1}{2}(dx^- dx^+ + dx^+ dx^-) - \frac{(x^- - x^+)^2}{4} d\omega^2. \tag{1.6}$$

The metric tensor on $\mathbb{R} \times \mathbb{S}^d$ is $ds^2_{\mathbb{R}\times\mathbb{S}^d} = dT^2 - dR^2 - (\sin R)^2 d\omega^2$, so using (1.3)

$$ds^2_{\mathbb{R}\times\mathbb{S}^d} = 2(dX^- dX^+ + dX^+ dX^-) - \sin^2(X^+ - X^-) d\omega^2.$$

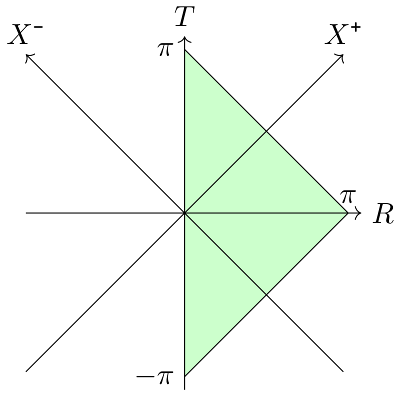

Figure 1.1: The image of the Penrose map $\mathcal{P}$.

Inserting the equations (1.4) into (1.6), and using the elementary identity

$$(\tan X^- - \tan X^+)^2 = \frac{\sin^2(X^+ - X^-)}{\cos^2 X^- \cos^2 X^+},$$

we obtain the relation

$$ds^2_{\mathbb{R}\times\mathbb{S}^d} = \Omega^2 ds^2_{\mathbb{R}^{1+d}}, \tag{1.7}$$

where Ω is the following scalar field;

$$\Omega := 2(1+(x^+)^2)^{-1/2}(1+(x^-)^2)^{-1/2} = 2\cos X^+ \cos X^-,$$

and the change of variable (1.4) is implicit. We will always omit this change of variable without further specification. The relation (1.7) expresses the fact that $\mathcal{P}$ is a conformal map.

Remark 1.1.1. The restriction of $\mathcal{P}$ to the initial time slice $\{ t = 0 \}$ is the stereographic projection from the south pole of $\mathbb{S}^d$;

$$\mathcal{P}_0 := \mathcal{P}|_{t=0} : \mathbb{R}^d \to \mathbb{S}^d \backslash \{ (-1, 0, \dots, 0) \}. \tag{1.8}$$

This is also a conformal map, whose conformal factor we denote by

$$\Omega_0 := \Omega|_{t=0} = 2(1+r^2)^{-1} = 1 + \cos R.$$

The explicit equations for $X = \mathcal{P}_0(x)$ are

$$X_0 = \Omega_0 - 1, \quad X_j = \Omega_0 x_j, \quad j = 1, \dots, d.$$

Definition 1.1.2. For all scalar field v on $\mathbb{R}^{1+d}$, we define a scalar field V on $\mathcal{P}(\mathbb{R}^{1+d})$ by

$$v = \Omega^{\frac{d-1}{2}} V,$$

The scalar field V is called the *Penrose transform* of v.

Remark 1.1.3. At $t = 0$, corresponding to $T = 0$,

$$v|_{t=0} = (\Omega^{\frac{d-1}{2}} V)\Big|_{T=0}, \quad \partial_t v|_{t=0} = (\Omega^{\frac{d+1}{2}} \partial_T V)\Big|_{T=0}, \tag{1.9}$$

where we used that $\partial_t \Omega|_{t=0} = 0$ and that $\partial_t|_{t=0} = \Omega \partial_T|_{T=0}$.

This definition is motivated by the identity

$$\Box v = \Omega^{\frac{d+3}{2}} \left(\partial_T^2 - \Delta_{\mathbb{S}^d} + \left(\frac{d-1}{2} \right)^2 \right) V, \tag{1.10}$$

which is a standard consequence of the conformality; see, for example, [46, Appendix A.4]. Here $\Delta_{\mathbb{S}^d}$ denotes the Laplace-Beltrami operator. We complement Definition 1.1.2 with the transformation laws for the initial data, modeled on (1.9);

$$f_0 = \Omega_0^{\frac{d-1}{2}} F_0, \quad f_1 = \Omega_0^{\frac{d+1}{2}} F_1, \tag{1.11}$$

where the stereographic projection (1.8) is implicit. We thus have the fundamental property

$$\begin{cases} \Box v = 0, \text{ on } \mathbb{R}^{1+d}, \\ v|_{t=0} = f_0, \\ \partial_t v|_{t=0} = f_1, \end{cases} \iff \begin{cases} \partial_T^2 V = \Delta_{\mathbb{S}^d} V - \left(\frac{d-1}{2}\right)^2 V, \quad \text{on } \mathcal{P}(\mathbb{R}^{1+d}), \\ V|_{T=0} = F_0, \\ \partial_T V|_{T=0} = F_1. \end{cases}$$

The Penrose transform is very relevant in our context, because

$$\boldsymbol{f}_\star = (\Omega_0^{\frac{d-1}{2}}, 0),$$

so, denoting $v_\star = S\boldsymbol{f}_\star$, we have the particularly simple expressions

$$F_{\star 0} = 1, \quad F_{\star 1} = 0, \quad V_\star(T, X) = \cos\left(\tfrac{d-1}{2} T\right). \tag{1.12}$$

1.1.1 Spherical harmonics

We use the notation $Y_{\ell,m}$ for normalized real-valued spherical harmonics on $\mathbb{S}^d$. Here $\ell \in \mathbb{N}_{\geqslant 0}$ denotes the degree and m the degeneracy. We have

$$-\Delta_{\mathbb{S}^d} Y_{\ell,m} = \ell(\ell + d - 1) Y_{\ell,m}, \quad m = 0, \ldots, N(\ell) := \tfrac{(2\ell+d-1)(\ell+d-2)!}{\ell!(d-1)!} - 1,$$

and

$$\int_{\mathbb{S}^d} Y_{\ell,m}(X)^2 \, dS = 1,$$

where dS is the surface measure on $\mathbb{S}^d$. We recall that $Y_{\ell,m}(X)$ is the restriction to $\mathbb{S}^d$ of a homogeneous harmonic polynomial of degree ℓ in $X = (X_0, X_1, \ldots, X_d)$; see, for example, [58]. In particular,

$$Y_{\ell,m}(-X) = (-1)^\ell Y_{\ell,m}(X). \tag{1.13}$$

For each $\ell \in \mathbb{N}_{\geqslant 0}$ there is exactly one spherical harmonic that is a function of the first coordinate X_0 only; we call it the *zonal* spherical harmonic and we denote it by $Y_{\ell,0}$.

Remark 1.1.4. The spherical harmonics of degree 0 and 1 are

$$Y_{0,0} = \frac{1}{\sqrt{|\mathbb{S}^d|}}, \quad Y_{1,m}(X) = \sqrt{\frac{d+1}{|\mathbb{S}^d|}} X_m, \quad (m = 0, 1, \dots, d).$$

We use the hat notation to denote the coefficients of expansions in spherical harmonics:

$$F(X) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{N(\ell)} \hat{F}(\ell, m) Y_{\ell,m}(X),$$

Proposition 1.1.5. *Assume that* $\Box v = 0$ *on* $\mathbb{R}^{1+d}$ *and that* $\boldsymbol{v}(0) = \boldsymbol{f}$*. Denote by* V *the Penrose transform of* v*; see Definition 1.1.2. Then*

$$\begin{aligned} V(T, X) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{N(\ell)} \cos\left(T(\ell + \frac{d-1}{2})\right) \hat{F}_0(\ell, m) Y_{\ell,m}(X) \\ + \frac{\sin\left(T\left(\ell + \frac{d-1}{2}\right)\right)}{\ell + \frac{1}{2}(d-1)} \hat{F}_1(\ell, m) Y_{\ell,m}(X). \end{aligned} \tag{1.14}$$

Proof. The equation $\partial_T^2 V = \Delta_{\mathbb{S}^d} V - \left(\frac{d-1}{2}\right)^2 V$ implies that

$$\partial_T^2 \hat{V}(T, \ell, m) = \ell(\ell + d - 1)\hat{V}(T, \ell, m) - \left(\frac{d-1}{2}\right)^2 \hat{V}(T, \ell, m),$$

with initial data $\hat{V}(0, \ell, m) = \hat{F}_0(\ell, m)$ and $\partial_T \hat{V}(0, \ell, m) = \hat{F}_1(\ell, m)$. Solving this ordinary differential equation yields (1.14). □

Remark 1.1.6. The formula (1.14) actually defines a function on $\mathbb{R} \times \mathbb{S}^d$, not just on $\mathcal{P}(\mathbb{R} \times \mathbb{S}^d)$. Thus we can consider V as defined on $\mathbb{R} \times \mathbb{S}^d$. If d is odd, V is 2π-periodic in T and it satisfies

$$V(T + \pi, -X) = (-1)^{\frac{d-1}{2}} V(T, X), \quad \forall (T, X) \in \mathbb{S}^1 \times \mathbb{S}^d, \tag{1.15}$$

because of the sign property (1.13) of $Y_{\ell,m}$. If d is even, (1.15) fails.

We conclude the section by introducing the fractional operators A_1 and A_{-1} on $\mathbb{S}^d$, defined by their action on spherical harmonics;

$$A_{\pm 1} Y_{\ell,m} := \left(-\Delta_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2\right)^{\pm\frac{1}{2}} Y_{\ell,m} = \left(\ell + \frac{d-1}{2}\right)^{\pm 1} Y_{\ell,m}. \tag{1.16}$$

These operators are the lifting to $\mathbb{S}^d$ of the euclidean fractional Laplacians $(-\Delta)^{\pm\frac{1}{2}}$ via the stereographic projection $\mathcal{P}_0$, in the sense that, for any scalar field F on $\mathbb{S}^d$:

$$(A_{\pm 1} F) \circ \mathcal{P}_0 = \Omega_0^{-\frac{1}{2}(d\pm 1)} (-\Delta)^{\pm\frac{1}{2}} \left(\Omega_0^{\frac{1}{2}(d\mp 1)} F \circ \mathcal{P}_0\right); \tag{1.17}$$

see [57, equation (2)].

1.2 Some integration formulas

We let dS denote the surface measure on $\mathbb{S}^d$. As we saw in the first section, we have the conformality properties $\Omega^2 ds^2_{\mathbb{R}^{1+d}} = ds^2_{\mathbb{R}\times\mathbb{S}^d}$ and $\Omega_0^2 ds^2_{\mathbb{R}^d} = ds^2_{\mathbb{S}^d}$, which imply the change of variable formulas

$$\begin{aligned} \int_{\mathbb{R}^d} F(\mathcal{P}_0(x))\Omega_0^d\, dx &= \int_{\mathbb{S}^d} F(X)\, dS(X), \\ \iint\limits_{\mathcal{P}(\mathbb{R}^{1+d})} V(T,X)\, dT dS(X) &= \iint\limits_{\mathbb{R}^{1+d}} V(\mathcal{P}(t,x))\Omega^{d+1}\, dt dx, \end{aligned} \tag{1.18}$$

where F and V are scalar fields on $\mathbb{S}^d$ and $\mathcal{P}(\mathbb{R}^{1+d})$ respectively. It is a consequence of the first formula and of equation (1.17) that, if $\boldsymbol{f}, \boldsymbol{g}$ are related to (F_0, F_1) and (G_0, G_1) via (1.11), then

$$\langle \boldsymbol{f} \,|\, \boldsymbol{g} \rangle_{\dot{\mathcal{H}}^{\frac12}} = \int_{\mathbb{S}^d} A_1 F_0 \cdot G_0\, dS + \int_{\mathbb{S}^d} A_{-1} F_1 \cdot G_1\, dS,$$

and so

$$\begin{aligned} \langle \boldsymbol{f} \,|\, \boldsymbol{g} \rangle_{\dot{\mathcal{H}}^{\frac12}} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{N(\ell)} &\left(\ell + \frac{d-1}{2}\right) \hat{F}_0(\ell,m)\hat{G}_0(\ell,m) \\ &+ \left(\ell + \frac{d-1}{2}\right)^{-1} \hat{F}_1(\ell,m)\hat{G}_1(\ell,m). \end{aligned} \tag{1.19}$$

In particular, from (1.12) it follows that

$$\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{\frac12}(\mathbb{R}^d)} = \frac{d-1}{2}\left|\mathbb{S}^d\right|. \tag{1.20}$$

Remark 1.2.1. The expression on the right hand side of (1.19) coincides with the scalar product of the space $H^{\frac12} \times H^{-\frac12}(\mathbb{S}^d)$; see [7, Definition 3.23].

Using the symmetry (1.15) we can considerably simplify spacetime integrals.

Lemma 1.2.2. *If V is a function on $\mathbb{S}^1 \times \mathbb{S}^d$ that satisfies*

$$V(T+\pi, -X) = V(T,X), \quad \forall (T,X) \in \mathbb{S}^1 \times \mathbb{S}^d, \tag{1.21}$$

then

$$\iint\limits_{\mathcal{P}(\mathbb{R}^{1+d})} V(T,X)\, dT dS(X) = \frac12 \iint\limits_{\mathbb{S}^1\times\mathbb{S}^d} V(T,X)\, dT dS(X). \tag{1.22}$$

Proof. We use the spherical polar coordinates (1.2), so that

$$dS = (\sin R)^{d-1}\, dR\, dS^{d-1},$$

where dS^{d-1} denotes the volume element on $\mathbb{S}^{d-1}$; see, for example, [58, §1.42]. We note that $\mathcal{P}(\mathbb{R}^{1+d})$ can be described as

$$\mathcal{P}(\mathbb{R}^{1+d}) = \left\{ \Big(T, (\cos R, \sin(R)\omega)\Big) \;\middle|\; \begin{array}{c} R \in [0,\pi) \\ -\pi + R < T < \pi - R \\ \omega \in \mathbb{S}^{d-1} \end{array} \right\};$$

see (1.5). So, setting

$$G(R) := \int_{-\pi+R}^{\pi-R} \left(\int_{\mathbb{S}^{d-1}} V(T, \cos R, \sin(R)\,\omega)\, dS^{d-1}(\omega) \right) dT,$$

the integral to evaluate can be rewritten as

$$\begin{aligned} \iint\limits_{\mathcal{P}(\mathbb{R}^{1+d})} V(T,X)\, dT dS(X) &= \int_0^\pi (\sin R)^{d-1} G(R)\, dR \\ &= \int_0^\pi (\sin R)^{d-1} \frac{1}{2} \left(G(R) + G(\pi - R) \right) dR. \end{aligned} \tag{1.23}$$

Using the changes of variable $\omega \mapsto -\omega$ and $T \mapsto T \pm \pi$,

$$\begin{aligned} G(\pi - R) = &\int_{-\pi}^{-\pi+R} \int_{\mathbb{S}^{d-1}} V(T-\pi, -\cos R, -\sin(R)\,\omega)\, dS^{d-1}(\omega)\, dT \\ &+ \int_{\pi-R}^{\pi} \int_{\mathbb{S}^{d-1}} V(T+\pi, -\cos R, -\sin(R)\,\omega)\, dS^{d-1}(\omega)\, dT. \end{aligned}$$

Now, V is 2π-periodic in T, so

$$\begin{aligned} V(T+\pi, -\cos R, -\sin(R)\omega) &= V(T-\pi, -\cos R, -\sin(R)\omega) \\ &= V(T, \cos T, \sin(R)\omega), \end{aligned}$$

by the assumption (1.21). Therefore

$$G(R) + G(\pi - R) = \int_{-\pi}^{\pi} \int_{\mathbb{S}^{d-1}} V(T, \cos R, \sin(R)\omega)\, dS^{d-1}(\omega) dT,$$

which can be inserted into (1.23) to yield the desired conclusion (1.22). □

We recall from the first section that the scalar field Ω is the conformal factor of the Penrose map $\mathcal{P}$; its explicit expression in spherical polar coordinates is

$$\Omega = \cos T + \cos R. \tag{1.24}$$

Corollary 1.2.3. *Let d be an odd integer. If $\square v = 0$ and $\square w = 0$ on $\mathbb{R}^{1+d}$, then*

$$\iint\limits_{\mathbb{R}^{1+d}} |v|^a |w|^b \, dt dx = \frac{1}{2} \iint\limits_{\mathbb{S}^1 \times \mathbb{S}^d} |\Omega|^{\frac{d-1}{2}(a+b)-(d+1)} |V|^a |W|^b \, dT dS, \tag{1.25}$$

and
$$\iint_{\mathbb{R}^{1+d}} |v|^{a-1} v\, w\, dtdx = \frac{1}{2} \iint_{\mathbb{S}^1\times\mathbb{S}^d} |\Omega|^{\frac{d-1}{2}(a+1)-(d+1)} |V|^{a-1} V\, W\, dTdS, \tag{1.26}$$

for all $a, b \in \mathbb{R}$. Here V and W denote the Penrose transforms of v and w respectively.

Proof. To prove (1.25), we need to check that

$$U(T, X) = |\Omega|^{\frac{d-1}{2}(a+b)-(d+1)} |V|^a |W|^b$$

satisfies the property (1.21), which is an immediate consequence of the symmetry property (1.15) of V and W, and of the explicit expression (1.24). We remark that these symmetry properties need not hold for even d. The proof of (1.26) is analogous. □

1.3 Proof of Theorem 1.0.1

Lemma 1.3.1. *Writing $\boldsymbol{f} = c\boldsymbol{f}_\star + \boldsymbol{f}_\perp$, with $\langle \boldsymbol{f}_\perp \mid \boldsymbol{f}_\star \rangle_{\dot{\mathcal{H}}^{1/2}} = 0$, then*

$$\frac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star + \varepsilon \boldsymbol{f})\Big|_{\varepsilon=0} = -p \iint_{\mathbb{R}^{1+d}} |S\boldsymbol{f}_\star|^{p-2} S\boldsymbol{f}_\star S\boldsymbol{f}_\perp \, dtdx. \tag{1.27}$$

Proof. This follows from the computation

$$\frac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star + \varepsilon \boldsymbol{f})\Big|_{\varepsilon=0} = p\mathcal{S}^p \langle \boldsymbol{f}_\star \mid \boldsymbol{f} \rangle_{\dot{\mathcal{H}}^{1/2}} \|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}^{p-2} - p \iint_{\mathbb{R}^{1+d}} |S\boldsymbol{f}_\star|^{p-2} S\boldsymbol{f}_\star S\boldsymbol{f} \, dtdx,$$

which holds for any $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^d)$, and then taking $\boldsymbol{f} = c\boldsymbol{f}_\star + \boldsymbol{f}_\perp$ and recalling the definition of $\mathcal{S}$. □

When d is odd, we can apply Corollary 1.2.3 to the integral on the right-hand side of (1.27);

$$\iint_{\mathbb{R}^{1+d}} |S\boldsymbol{f}_\star|^{p-2} S\boldsymbol{f}_\star S\boldsymbol{f}_\perp \, dtdx = \frac{1}{2} \iint_{\mathbb{S}^1\times\mathbb{S}^d} \left|\cos \frac{d-1}{2} T\right|^{p-2} \cos(\frac{d-1}{2} T) U_\perp \, dTdS,$$

since the Penrose transform of $S\boldsymbol{f}_\star$ is $\cos \frac{d-1}{2} T$; see (1.12). Here, $V_\perp$ denotes the Penrose transform of $v_\perp = S\boldsymbol{f}_\perp$. From the formula (1.19), we infer that the condition $\langle \boldsymbol{f}_\star \mid \boldsymbol{f}_\perp \rangle_{\dot{\mathcal{H}}^{1/2}} = 0$ is equivalent to $\hat{F}_{\perp\,0}(0, 0) = 0$. Therefore, expanding $V_\perp$ in spherical harmonics as in (1.14), we see that

$$\int_{\mathbb{S}^d} V_\perp(T, X)\, dS(X) = C \sin\left(\frac{d-1}{2} T\right) \hat{F}_1(0, 0), \quad \forall\, T \in [-\pi, \pi],$$

for some constant C. This implies that

$$\frac{1}{2}\iint\limits_{\mathbb{S}^1\times\mathbb{S}^d}\left|\cos\frac{d-1}{2}T\right|^{p-2}\cos(\frac{d-1}{2}T)V_\perp\,dTdS$$

$$=\frac{C}{2}\hat{F}_1(0,0)\int_{-\pi}^{\pi}\left|\cos\frac{d-1}{2}T\right|^{p-2}\cos(\frac{d-1}{2}T)\sin(\frac{d-1}{2}T)\,dT=0,$$

as the last integrand is odd. This completes the proof of Theorem 1.0.1 in the odd dimensional case.

The reason why this argument fails in even dimension is that Corollary 1.2.3 is not applicable in that case. In order to prove that, in fact, $\boldsymbol{f}_\star$ is not a critical point in even dimension, we need only prove that the derivative is nonzero in a single direction. A bad choice would be to take the direction $\boldsymbol{f}=(f_0,0)$, where f_0 corresponds to a spherical harmonic of degree 1 under the Penrose transform (1.11), as then we would be moving in the direction of the symmetries of the inequality. This will be proved in the forthcoming chapter; see entries 5 and 6 in Table 2.2. Instead we consider the zonal spherical harmonic of degree 2, which we denote by $Y_{2,0}$; see the previous section.

Lemma 1.3.2. *Let $d\geqslant 2$ be even and let $\boldsymbol{f}=(f_0,0)\in\dot{\mathcal{H}}^{1/2}(\mathbb{R}^d)$ be the initial data corresponding to*

$$F_0=Y_{2,0},\quad F_1=0,$$

via the Penrose transformation (1.11)*. Then*

$$\frac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star+\varepsilon\boldsymbol{f})\bigg|_{\varepsilon=0}=(-1)^{\frac{d}{2}+1}c_d,\quad \text{where}\quad c_d>0.$$

Proof. Applying the Penrose transform to (1.27) we obtain

$$\frac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star+\varepsilon\boldsymbol{f})\bigg|_{\varepsilon=0}=-p\iint\limits_{\mathcal{P}(\mathbb{R}^{1+d})}\left|\cos\frac{d-1}{2}T\right|^{p-2}\cos\left(\frac{d-1}{2}T\right)V\,dTdS,$$

where $V(T,X)=\cos\left((2+\frac{d-1}{2})T\right)Y_{2,0}(X_0)$; see (1.14). We remark that we have written the generic point $X\in\mathbb{S}^d$ as $X=(X_0,X_1,\dots,X_d)$, where $X_0\in[-1,1]$, to exploit the fact that $Y_{2,0}$ is a function of X_0 only. Taking into account the definition (1.5) of $\mathcal{P}(\mathbb{R}^{1+d})$, the right-hand side of the previous identity reads

$$-p\left|\mathbb{S}^{d-1}\right|\int_{-\pi}^{\pi}\left|\cos\frac{d-1}{2}T\right|^{p-2}\cos\left(\frac{d-1}{2}T\right)\cos\left((2+\frac{d-1}{2})T\right)dT$$
$$\times\int_0^{\pi-|T|}Y_{2,0}(\cos R)\,(\sin R)^{d-1}\,dR. \quad (1.28)$$

We have used the formula $dS = (\sin R)^{d-1} dR dS^{d-1}$ for the volume element of $\mathbb{S}^d$ in the polar coordinates (1.2). Now, the zonal spherical harmonic $Y_{2,0}$ can be expressed by the Rodrigues formula;

$$Y_{2,0}(X_0) = R_{2,d}(1 - X_0^2)^{-\frac{d-2}{2}} \frac{d^2}{dX_0^2}(1 - X_0^2)^{2+\frac{d-2}{2}},$$

see, for example, [58, Lemma 4, pg. 22]. Here $R_{2,d} > 0$ is a constant whose exact value is not important here. We compute the last integral in (1.28) using the change of variable $X_0 = \cos R$;

$$\begin{aligned} \int_0^{\pi - |T|} Y_{\ell,0}(\cos R)(\sin R)^{d-1}\, dR &= R_{2,d} \int_{-\cos T}^{1} \frac{d^2}{dX_0^2}(1 - X_0^2)^{2+\frac{d-2}{2}}\, dX_0 \\ &= C_d \cos T (\sin T)^d, \end{aligned}$$

where $C_d > 0$. Inserting this into (1.28) shows that it remains to prove the following:

$$I(d) := \frac{1}{\pi} \int_{-\pi}^{\pi} h_d(T) P_d(T)\, dT = (-1)^{\frac{d}{2}} c_d, \quad \text{for some } c_d > 0, \tag{1.29}$$

where $h_d(T) := \left|\cos \frac{d-1}{2} T\right|^{p-2}$ and

$$P_d(T) := \cos \frac{d-1}{2} T \, \cos \frac{d+3}{2} T \, \cos T \, (\sin T)^d. \tag{1.30}$$

We first consider the case $d = 2$. In this case we have that $p = 6$, so we can evaluate $I(2)$ explicitly:

$$\begin{aligned} I(2) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\cos \frac{T}{2}\right)^5 \cos \frac{5T}{2} \cos T \, (\sin T)^2 \, dT \\ &= \frac{4}{\pi} \int_0^{\pi/2} (\cos T)^5 \cos 5T \cos 2T (\sin 2T)^2 \, dT = -\frac{5}{128}. \end{aligned}$$

In the case $d \geqslant 4$ we will use the Parseval identity:

$$I(d) = \frac{\hat{h}_d(0)\hat{P}_d(0)}{2} + \sum_{k=1}^{\infty} \hat{h}_d(k)\hat{P}_d(k),$$

where $\hat{f}(k) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(T) \cos(kT)\, dT$. We remark that, with this choice of notation,

$$\text{if } f(T) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kT), \text{ then } a_k = \hat{f}(k). \tag{1.31}$$

Lemma 1.3.3. *If $k \neq m(d-1)$ where $m \in \mathbb{N}_{\geqslant 0}$ then $\hat{h}_d(k) = 0$.*

Proof of Lemma 1.3.3. Consider $u \in [-\sqrt{2}, \sqrt{2}]$. We let $|u|^{p-2} = (1+v)^{\frac{p-2}{2}}$, with $v = u^2 - 1$, and we expand it using the binomial series. This yields

$$|u|^{p-2} = \sum_{j=0}^{\infty} \binom{(p-2)/2}{j} (u^2-1)^j,$$

and the series converges uniformly by Raabe's criterion (here we use that $p > 2$). Taking $u = \cos \frac{d-1}{2} T$, we obtain

$$\left|\cos \frac{d-1}{2} T\right|^{p-2} = \sum_{j=0}^{\infty} (-1)^j \binom{(p-2)/2}{j} \left(\sin \frac{d-1}{2} T\right)^{2j}, \tag{1.32}$$

For each $j \in \mathbb{N}_{\geqslant 0}$ we can develop

$$\begin{aligned}
\left(\sin \frac{d-1}{2} T\right)^{2j} &= \frac{(-1)^j}{2^{2j}} \left(e^{i\frac{d-1}{2}T} - e^{-i\frac{d-1}{2}T}\right)^{2j} \\
&= \frac{(-1)^j}{2^{2j}} \sum_{m=0}^{2j} \binom{2j}{m} (-1)^m e^{i(j-m)(d-1)T} \\
&= \frac{1}{2^{2j}} \left(\binom{2j}{j} + 2 \sum_{m=1}^{j} \binom{2j}{j-m} (-1)^m \cos(m(d-1)T) \right).
\end{aligned}$$

This shows that each summand in (1.32) is a linear combination of the terms $\cos(m(d-1)T)$, with $m \in \mathbb{N}_{\geqslant 0}$, which, in light of (1.31), completes the proof. □

We now turn to the term P_d introduced in (1.30). Using the addition formula for the cosine, and developing $(\sin T)^d$ like we did in the previous proof, we can express P_d as a trigonometric polynomial of degree $2(d+1)$:

$$\begin{aligned}
P_d(T) =& 2^{-d-2} \left(\cos T + \cos 3T + \cos dT + \cos(d+2)T\right) \\
& \times \left(\binom{d}{d/2} + 2 \sum_{k=1}^{d/2} (-1)^k \binom{d}{d/2-k} \cos(2kT) \right);
\end{aligned} \tag{1.33}$$

so, in particular, $\hat{P}_d(k) = 0$ if $k > 2(d+1)$. Since $d \geqslant 4$, we infer from this and from Lemma 1.3.3 that $I(d)$ reduces to the sum of four terms:

$$I(d) = \frac{1}{2} \hat{h}_d(0) \hat{P}_d(0) + \sum_{m=1}^{3} \hat{h}_d(m(d-1)) \hat{P}_d(m(d-1)). \tag{1.34}$$

Actually, we have that $\hat{P}_d(3(d-1)) = 0$. This is obvious for $d \geqslant 6$, because in that case $3(d-1)$ exceeds $2(d+1)$, and can be established for $d = 4$ by inspection of the formula

$$P_4(T) = 2^{-6} \left(\cos T + \cos 3T + \cos 4T + \cos 6T\right) \left(6 - 8\cos 2T + 2\cos 4T\right),$$

again using (1.31).

To compute the remaining coefficients, we use the addition formula for the cosine to rewrite (1.33) as

$$2^{d+2}P_d(T) = P_{d,1}(T) + P_{d,3}(T) + P_{d,d}(T) + P_{d,d+2}(T),$$

where each summand is given by

$$P_{d,h}(T) = \binom{d}{d/2}\cos hT + \sum_{k=1}^{d/2}(-1)^k\binom{d}{d/2-k}(\cos(2k-h)T + \cos(2k+h)T),$$

for $h = 1, 3, d, d+2$. To compute $\hat{P}_d(0)$, we observe that the only contributing term is obtained for $2k - h = 0$, and that can only happen for $h = d$ and $k = d/2$. By (1.31) we have

$$2^{d+2}\hat{P}_d(0) = \hat{P}_{d,d}(0) = 2(-1)^{\frac{d}{2}}.$$

To compute $\hat{P}_d(d-1)$ we observe that, as $d-1$ is odd, the only contributing terms are obtained for $h = 1, 3$:

$$\begin{aligned}2^{d+2}\hat{P}_d(d-1) &= \hat{P}_{d,1}(d-1) + \hat{P}_{d,3}(d-1)\\ &= (-1)^{\frac{d}{2}} - (-1)^{\frac{d}{2}}\binom{d}{1} + (-1)^{\frac{d}{2}}\binom{d}{2}\\ &= (-1)^{\frac{d}{2}}\frac{(d-1)(d-2)}{2}.\end{aligned}$$

With analogous reasoning we obtain

$$\begin{aligned}2^{d+2}\hat{P}_d(2(d-1)) &= \hat{P}_{d,d}(2(d-1)) + \hat{P}_{d,d+2}(2(d-1))\\ &= -(-1)^{\frac{d}{2}}\binom{d}{1} + (-1)^{\frac{d}{2}}\binom{d}{2}\\ &= (-1)^{\frac{d}{2}}\left(\frac{(d-1)(d-2)}{2} - 1\right).\end{aligned}$$

Inserting the preceding computations into Parseval's identity (1.34), we obtain the formula

$$\begin{aligned}(-1)^{\frac{d}{2}}2^{d+2}I(d) &= \hat{h}_d(0) - \hat{h}_d(2(d-1))\\ &\qquad + \frac{(d-1)(d-2)}{2}\left(\hat{h}_d(d-1) + \hat{h}_d(2(d-1))\right).\end{aligned}$$

To conclude the proof of (1.29)it will suffice to prove that

$$\hat{h}_d(0) - \hat{h}_d(2(d-1)) > 0, \quad \text{and} \quad \hat{h}_d(d-1) + \hat{h}_d(2(d-1)) > 0. \tag{1.35}$$

The first inequality follows immediately from the definition (1.30) of h_d:

$$\hat{h}_d(0) - \hat{h}_d(2(d-1)) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\cos\frac{d-1}{2}T\right|^{p-2}(1-\cos 2(d-1)T)\,dT > 0.$$

To prove the second inequality we note that the change of variable $T \mapsto \frac{2}{d-1}T$ produces

$$\hat{h}_d(d-1) + \hat{h}_d(2(d-1)) = \frac{2}{\pi(d-1)} \int_{-\frac{d-1}{2}\pi}^{\frac{d-1}{2}\pi} |\cos T|^{p-2}(\cos 2T + \cos 4T)\, dT.$$

The integrand function in the right-hand side is π-periodic and even. Therefore, the integral is an integer multiple of the integral over $[0, \pi/2]$. Moreover, $\cos 2T + \cos 4T = 2\cos T \cos 3T$. We get

$$\hat{h}_d(d-1) + \hat{h}_d(2(d-1)) = \frac{4(d-2)}{\pi(d-1)} \int_0^{\pi/2} |\cos T|^{p-2} \cos T \cos 3T\, dT. \tag{1.36}$$

To conclude the proof, we notice that

$$\int_0^{\pi/6} \cos T \cos 3T\, dT = -\int_{\pi/6}^{\pi/2} \cos T \cos 3T\, dT > 0,$$

and $|\cos T|^{p-2}$ is strictly decreasing on $[0, \pi/2]$, so

$$\int_0^{\pi/6} |\cos T|^{p-2} \cos T \cos 3T\, dT > -\int_{\pi/6}^{\pi/2} |\cos T|^{p-2} \cos T \cos 3T\, dT,$$

which proves that the right-hand side in (1.36) is strictly positive. This shows that the second inequality in (1.35) holds, and the proof of Theorem 1.0.1 is complete. □

Chapter 2

Sharpened Strichartz estimates

In the aforementioned paper [37], Foschi proved the Strichartz estimate;

$$\|S\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+3})} \leqslant \mathcal{S}\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)}, \quad \text{where } \mathcal{S} := \left(\frac{3}{16\pi}\right)^{\frac{1}{4}}. \tag{2.1}$$

The constant $\mathcal{S}$ is optimal, meaning that the inequality fails if it is replaced with any strictly smaller one. Moreover, there is equality in (2.1) for $\boldsymbol{f} = \boldsymbol{f}_\star$, where

$$\boldsymbol{f}_\star = \left(\frac{2}{1+|\cdot|^2}, 0 \right).$$

In particular, the set of maximizers

$$\mathbf{M} := \left\{ \boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3) \;\middle|\; \|S\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+3})} = \mathcal{S}\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \right\}$$

is not trivial. We prove that (2.1) can be sharpened by adding a term proportional to the distance from $\mathbf{M}$, defined by

$$d(\boldsymbol{f}, \mathbf{M}) := \inf \left\{ \|\boldsymbol{f} - \boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} \;\middle|\; \boldsymbol{g} \in \mathbf{M} \right\}.$$

Theorem 2.0.1. *There is a positive constant C such that*

$$Cd(\boldsymbol{f}, \mathbf{M})^2 \leqslant \mathcal{S}^2\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+3})} \leqslant \mathcal{S}^2 d(\boldsymbol{f}, \mathbf{M})^2.$$

The upper bound is proved in a more general setting in the following section.

The sharpened version of (2.1) is the lower bound, which will follow from a local version, in which we also obtain the sharp constant. To prove the local version, one of the key ingredients is the Penrose transform, which we introduced in the previous chapter. We will also require the preliminary study of some geometrical properties of $\mathbf{M}$, which we carry out in the second section.

We conclude the proof of the theorem by an application of the profile decomposition of Ramos [62].

Finally, we dedicate the last section of this chapter to the proof of a result analogous to Theorem 2.0.1 for the sharp energy-Strichartz inequality in $\mathbb{R}^{1+5}$ of Bez and Rogers [14]. The outline of the proof is the same as in the three-dimensional case, but there is the significant additional difficulty that the quadratic term in the relevant Taylor expansion is not diagonal in its expansion in spherical harmonics.

2.1 Abstract upper bounds

Consider a bounded linear operator $S\colon \mathcal{H} \to L^p(X)$, where $\mathcal{H}$ is a real or complex Hilbert space and X is a measure space. Then, writing

$$\mathcal{S} := \sup_{f\neq 0} \frac{\|Sf\|_{L^p(X)}}{\|f\|_{\mathcal{H}}} \quad \text{and} \quad d(f,M) = \inf\{\, \|f - f_\star\|_{\mathcal{H}} : f_\star \in M \,\},$$

where $M := \{f_\star \in \mathcal{H} : \|Sf_\star\|_{L^p(X)} = \mathcal{S}\|f_\star\|_{\mathcal{H}}\}$, the following upper bound holds generally.

Proposition 2.1.1. *Let $1 < p < \infty$. Then, for all $f \in \mathcal{H}$,*

$$\mathcal{S}^2\|f\|_{\mathcal{H}}^2 - \|Sf\|_{L^p(X)}^2 \leqslant \mathcal{S}^2 d(f,M)^2. \tag{2.2}$$

Proof. For $f \in \mathcal{H}$ there exists a sequence $f_\star^n \in M$ such that

$$d(f,M)^2 = \lim_{n\to\infty} \|f - f_\star^n\|_{\mathcal{H}}^2.$$

We let $g^n = f - f_\star^n$ and we define $H_n\colon \mathbb{R} \to \mathbb{R}$ by

$$H_n(\lambda) = \|S(f_\star^n + \lambda g^n)\|_{L^p(X)}^2 - \|Sf_\star^n\|_{L^p(X)}^2.$$

The function H_n is convex and, since $p \in (1,\infty)$, it is differentiable; see [54, Theorem 2.6]. Given that $f_\star^n \in M$, the function

$$\lambda \in \mathbb{R} \mapsto \|S(f_\star^n + \lambda g^n)\|_{L^p(X)}^2 - \mathcal{S}^2\|f_\star^n + \lambda g^n\|_{\mathcal{H}}^2$$

has a global minimum and so a critical point at $\lambda = 0$, from which we infer that

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} H_n(\lambda) = \frac{d}{d\lambda}\bigg|_{\lambda=0} \mathcal{S}^2\|f_\star^n + \lambda g^n\|_{\mathcal{H}}^2 = 2\mathcal{S}^2\Re\langle f_\star^n \,|\, g^n\rangle.$$

Convexity gives $H_n(1) \geqslant H_n'(0)$; that is

$$\|S(f_\star^n + g^n)\|_{L^p(X)}^2 \geqslant \|Sf_\star^n\|_{L^p(X)}^2 + 2\mathcal{S}^2\Re\langle f_\star^n \,|\, g^n\rangle. \tag{2.3}$$

Recalling that

$$\begin{aligned}\mathcal{S}^2\|f\|_{\mathcal{H}}^2 - \|Sf\|_{L^p(X)}^2 = \mathcal{S}^2\|f_\star^n\|_{\mathcal{H}}^2 + \mathcal{S}^2\|g^n\|_{\mathcal{H}}^2 + 2\mathcal{S}^2\Re\langle f_\star^n \,|\, g^n\rangle \\ - \|S(f_\star^n + g^n)\|_{L^p(X)}^2,\end{aligned}$$

equation (2.3) yields

$$\mathcal{S}^2\|f\|_{\mathcal{H}}^2 - \|Sf\|_{L^p(X)}^2 \leqslant (\mathcal{S}^2\|f_\star^n\|_{\mathcal{H}}^2 - \|Sf_\star^n\|_{L^p(X)}^2) + \mathcal{S}^2\|g^n\|_{\mathcal{H}}^2.$$

Since $f_\star^n \in M$, the term in brackets vanishes. Letting $n \to \infty$, we find (2.2), and so the proof is complete. $\square$

Remark 2.1.2. Specializing Proposition 2.1.1 to the fractional Sobolev inequality on $\mathbb{R}^d$ gives an alternative proof of the upper bound of [23].

2.2 Geometry of the set of maximizers

Foschi proved that the set $\mathbf{M}$ is the orbit of $\boldsymbol{f}_\star$ under the action of a Lie group of symmetries, which we now describe. The following definitions and computations will be needed only for $d = 3$ or $d = 5$, but there is no added difficulty in considering the general case. We recall that

$$\boldsymbol{f}_\star = \left(2^{\frac{d-1}{2}}(1+|\cdot|^2)^{-\frac{d-1}{2}}, 0\right).$$

For $t \in \mathbb{R}$, we denote by $\vec{S}_t$ the vector-valued wave propagator;

$$\vec{S}_t\boldsymbol{f} := \begin{bmatrix} \cos(t\sqrt{-\Delta}) & \dfrac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \\ -\sin(t\sqrt{-\Delta})\sqrt{-\Delta} & \cos(t\sqrt{-\Delta}) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \tag{2.4}$$

which is characterized by the property

$$\begin{cases} \Box v = 0, \\ \boldsymbol{v}(0) = \boldsymbol{f}, \end{cases} \qquad \Longleftrightarrow \qquad \boldsymbol{v}(t) = \vec{S}_t\boldsymbol{f}.$$

For $\theta \in \mathbb{S}^1$, we denote by Ph_θ the phase shift;

$$\mathrm{Ph}_\theta\,\boldsymbol{f} := \begin{bmatrix} \cos(\theta) & \dfrac{\sin(\theta)}{\sqrt{-\Delta}} \\ -\sin(\theta)\sqrt{-\Delta} & \cos(\theta) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \tag{2.5}$$

which is characterized by

$$\vec{S}_t\,\mathrm{Ph}_\theta\,\boldsymbol{f} = \mathrm{Ph}_\theta\,\vec{S}_t\boldsymbol{f} = \begin{bmatrix} \cos(t\sqrt{-\Delta}+\theta) & \dfrac{\sin(t\sqrt{-\Delta}+\theta)}{\sqrt{-\Delta}} \\ -\sin(t\sqrt{-\Delta}+\theta)\sqrt{-\Delta} & \cos(t\sqrt{-\Delta}+\theta) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix},$$

For $\zeta_j \in \mathbb{R}$ and $j = 1, \dots, d$, we denote by $L^j_{\zeta_j}$ the Lorentzian boost along the x_j axis;

$$L^j_{\zeta_j}\boldsymbol{f} := \boldsymbol{v}_{\zeta_j}\big|_{t=0},$$

where $v(t,x) = S\boldsymbol{f}(t,x)$ and

$$\begin{aligned} v_{\zeta_1}(t,x) &= v(t\cosh\zeta_1 + x_1\sinh\zeta_1, t\sinh\zeta_1 + x_1\cosh\zeta_1, x_2, \ldots, x_d), \\ v_{\zeta_2}(t,x) &= v(t\cosh\zeta_2 + x_2\sinh\zeta_2, x_1, t\sinh\zeta_2 + x_2\cosh\zeta_2, \ldots, x_d), \\ &\vdots \\ v_{\zeta_d}(t,x) &= v(t\cosh\zeta_d + x_d\sinh\zeta_d, x_1, \ldots, x_{d-1}, t\sinh\zeta_d + x_d\cosh\zeta_d). \end{aligned}$$

We introduce the collective parameter $\boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^{2d+2}$;

$$\boldsymbol{\alpha} := (\theta, t_0, \zeta_1, \ldots, \zeta_d, \sigma, x_0), \quad \theta \in \mathbb{S}^1, t_0 \in \mathbb{R}, \zeta_j \in \mathbb{R}, \sigma \in \mathbb{R}, x_0 \in \mathbb{R}^d, \tag{2.6}$$

Then, for $\boldsymbol{f} \in \dot{\mathcal{H}}^{\frac12}$, we let $\Gamma_{\boldsymbol{\alpha}}$ denote the following element of $\dot{\mathcal{H}}^{1/2}$;

$$\vec{S}_{t_0}\,\mathrm{Ph}_\theta\, L^1_{\zeta_1}\ldots L^d_{\zeta_d}\Big(e^{\frac{d-1}{2}\sigma} f_0\left(e^\sigma(\cdot + x_0)\right), e^{\frac{d+1}{2}\sigma} f_1\left(e^\sigma(\cdot + x_0)\right)\Big). \tag{2.7}$$

We remark that $\Gamma_{\boldsymbol{0}}$ is the identity. We can now cast in this notation Foschi's characterization of $\mathbf{M}$.

Theorem 2.2.1 (Foschi [37])**.** *The set of maximizers of the three-dimensional Strichartz estimate* (2.1) *is*

$$\mathbf{M} = \left\{\, c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \;\middle|\; c \geqslant 0,\ \boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^8 \,\right\} \subset \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3).$$

Remark 2.2.2. Definition (2.7) does not contain spatial rotations; as a consequence, the operators $\Gamma_{\boldsymbol{\alpha}}$ do not form a group. Precisely, given $\Gamma_{\boldsymbol{\alpha}}, \Gamma_{\boldsymbol{\beta}}$ there is a unique $\Gamma_{\boldsymbol{\gamma}}$ such that

$$\Gamma_{\boldsymbol{\alpha}}\Gamma_{\boldsymbol{\beta}} = \Gamma_{\boldsymbol{\gamma}} R,$$

where the operator R is the representation of a rotation;

$$R\boldsymbol{f}(x) := \boldsymbol{f}(A_R x), \quad \text{for a } A_R \in SO(d).$$

This is not a nuisance, because $\boldsymbol{f}_\star$ is radially symmetric, so $\Gamma_{\boldsymbol{\alpha}}\Gamma_{\boldsymbol{\beta}}\boldsymbol{f}_\star = \Gamma_{\boldsymbol{\gamma}}\boldsymbol{f}_\star$.

Proposition 2.2.3. *The operators $\Gamma_{\boldsymbol{\alpha}}$ preserve both sides of the Strichartz inequality;*

$$\|\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \quad \textit{and} \quad \|S\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+d})} = \|S\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+d})}. \tag{2.8}$$

We will prove this proposition after introducing some more notation.

Remark 2.2.4. The full action of the symmetry group on the Strichartz inequality is the transformation $\boldsymbol{f} \mapsto c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}$. This notation has been chosen to highlight the difference between the multiplicative transformation $\boldsymbol{f} \mapsto c\boldsymbol{f}$, which is a symmetry of the inequality but does not satisfy (2.8), and the transformation $\Gamma_{\boldsymbol{\alpha}}$, which preserves both sides of the inequality. We caution that the second identity in (2.8) is specific of the space $L^4(\mathbb{R}^{1+3})$, and the operator Ph_θ does not seem to preserve the $L^p(\mathbb{R}^{1+d})$ norm unless $p = 4$.

We begin the study of the geometrical properties of $\mathbf{M}$ with the following lemma, which we state for general spatial dimension d.

Lemma 2.2.5. *The map*

$$(c, \boldsymbol{\alpha}) \in (0, \infty) \times \mathbb{S}^1 \times \mathbb{R}^{2d+2} \mapsto c\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star \tag{2.9}$$

is injective.

Proof. We need to show that $c\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star = c'\Gamma_{\boldsymbol{\alpha}'} \boldsymbol{f}_\star$ implies that $c = c'$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Now, the first identity is an immediate consequence of Proposition 2.2.3, because

$$c\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} = \|c\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} = \left\|c'\Gamma_{\boldsymbol{\alpha}'} \boldsymbol{f}_\star\right\|_{\dot{\mathcal{H}}^{1/2}} = c'\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}.$$

By the group property of Remark 2.2.2, we can also assume that $\Gamma_{\boldsymbol{\alpha}'} = \Gamma_{\mathbf{0}}$. So, letting $v_{\boldsymbol{\alpha}} := S\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star$, we are reduced to prove that

$$v_{\boldsymbol{\alpha}} = v_{\mathbf{0}} \quad \Longrightarrow \quad \boldsymbol{\alpha} = \mathbf{0}.$$

Up to a change of parameters, we can rewrite $v_{\boldsymbol{\alpha}}$ as

$$v_{\boldsymbol{\alpha}}(t, x) = e^{\frac{d-1}{2}\sigma} v_\theta(e^\sigma L^\beta(t + t_0, x + x_0)),$$

where $L^\beta \colon \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ denotes a Lorentzian boost of velocity β, where $\beta \in \mathbb{R}^d$ and $|\beta| < 1$; see the forthcoming chapter. Here $v_\theta := S\,\mathrm{Ph}_\theta\, \boldsymbol{f}_\star$. Now, we introduce the *energy* and the *momentum*, defined for an arbitrary scalar field w on $\mathbb{R}^{1+d}$ as

$$E(w) := \int_{\mathbb{R}^d} \left(|\nabla w|^2 + (\partial_t w)^2\right) dx, \quad \boldsymbol{P}(w) := \int_{\mathbb{R}^d} \partial_t w\, \nabla w\, dx.$$

These quantities are invariant with respect to all symmetries considered in this section, except for the dilations and the Lorentzian boosts; more precisely, we have the *energy-momentum relation*

$$(E(v_{\boldsymbol{\alpha}}), \boldsymbol{P}(v_{\boldsymbol{\alpha}})) = e^\sigma L^{-\beta}(E(v_{\mathbf{0}}), \boldsymbol{P}(v_{\mathbf{0}}));$$

see, for example, [51, Remark 2.5]. Since $v_{\mathbf{0}}$ is radially symmetric, $\boldsymbol{P}(v_{\mathbf{0}}) = 0$. By assumption, $(E(v_{\boldsymbol{\alpha}}), \boldsymbol{P}(v_{\boldsymbol{\alpha}}))$ must equal $(E(v_{\mathbf{0}}), \boldsymbol{P}(v_{\mathbf{0}}))$, which gives the equations

$$e^\sigma \gamma E(v_{\mathbf{0}}) = E(v_{\mathbf{0}}), \quad e^\sigma \gamma \beta E(v_{\mathbf{0}}) = 0, \quad \text{where } \gamma := (1 - |\beta|^2)^{-1/2},$$

from which we infer that $e^\sigma = 1$ and $\beta = 0$.

To conclude, we equate the spatial Fourier transforms of $v_\theta(t + t_0, \cdot + x_0)$ and $v_0(t, \cdot)$;

$$\cos((t + t_0)|\xi| + \theta) e^{ix_0\cdot\xi} \hat{f}_{\star 0}(\xi) = \cos(t|\xi|) \hat{f}_{\star 0}(\xi), \quad \forall \xi \in \mathbb{R}^d,\ t \in \mathbb{R},$$

where $f_{\star 0} = 2^{\frac{d-1}{2}}(1 + |\cdot|^2)^{-\frac{d-1}{2}}$, so $\hat{f}_{\star 0}(\xi) = Ce^{-|\xi|}/|\xi|$, for some irrelevant $C > 0$. This is non-vanishing at all ξ, so we infer that $\cos((t + t_0)|\xi| + \theta)e^{ix_0\cdot\xi}$ must be equal to $\cos(t|\xi|)$ for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^d$, which is only possible if $t_0 = 0$, $x_0 = 0$ and $\theta = 0$ modulo 2π. This completes the proof. $\square$

This lemma implies that $\mathbf{M}\backslash\{\mathbf{0}\}$ is a 10-dimensional smooth manifold, parameterized by (2.9). The tangent space at $\boldsymbol{f} \neq \mathbf{0}$ is

$$T_{\boldsymbol{f}}\mathbf{M} = \operatorname{span}\left\{\, \Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star, \partial_{\alpha_i}\Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star : i = 1, 2, \ldots, 9 \,\right\}, \quad \text{where } \boldsymbol{f} = c_0\Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star.$$

Here, ∂_{α_i} denotes the derivative with respect to the parameters (2.6). We refer to such derivatives as the *generators* of the symmetry group. In the forthcoming subsection, we will give an explicit description of the tangent space at $\boldsymbol{f}_\star$. This suffices to describe the tangent space at all points of $\mathbf{M}\backslash\{\mathbf{0}\}$, as the following proposition shows.

Proposition 2.2.6. *For all $c \neq 0$,*

$$T_{c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star}\mathbf{M} = \Gamma_{\boldsymbol{\alpha}}\left(T_{\boldsymbol{f}_\star}\mathbf{M}\right).$$

Proof. By definition,

$$T_{c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star}\mathbf{M} = \operatorname{span}\left\{\; c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star, c\,\partial_{\beta_j}(\Gamma_{\boldsymbol{\beta}}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star)\big|_{\boldsymbol{\beta}=\mathbf{0}} \;\middle|\; j = 1, \ldots, 2d+3 \right\}.$$

Now, by Remark 2.2.2, for all $\Gamma_{\boldsymbol{\beta}}$ there exists a unique $\boldsymbol{\gamma}(\boldsymbol{\beta}) \in \mathbb{S}^1 \times \mathbb{R}^{2d+2}$ such that

$$\Gamma_{\boldsymbol{\alpha}}^{-1}\Gamma_{\boldsymbol{\beta}}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star = \Gamma_{\boldsymbol{\gamma}(\boldsymbol{\beta})}\boldsymbol{f}_\star.$$

In particular, $\boldsymbol{\gamma}(\mathbf{0}) = \mathbf{0}$. We denote

$$c_{kj} := \frac{\partial\gamma_k}{\partial\beta_j}(\mathbf{0}).$$

Then, by the chain rule,

$$\partial_{\beta_j}(\Gamma_{\boldsymbol{\beta}}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star)\big|_{\boldsymbol{\beta}=\mathbf{0}} = \Gamma_{\boldsymbol{\alpha}}\big(\partial_{\beta_j}\Gamma_{\boldsymbol{\gamma}(\boldsymbol{\beta})}\boldsymbol{f}_\star\big|_{\boldsymbol{\beta}=\mathbf{0}}\big) = \Gamma_{\boldsymbol{\alpha}}\sum_{k=1}^{2d+3} c_{kj}\,\partial_{\gamma_k}\Gamma_{\boldsymbol{\gamma}}\boldsymbol{f}_\star|_{\boldsymbol{\gamma}=\mathbf{0}}\,.$$

The right-hand side is a linear combination of elements of

$$\Gamma_{\boldsymbol{\alpha}}(T_{\boldsymbol{f}_\star}\mathbf{M}) = \operatorname{span}\left\{\; \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star, \Gamma_{\boldsymbol{\alpha}}(\partial_{\gamma_k}\Gamma_{\boldsymbol{\gamma}}\boldsymbol{f}_\star|_{\boldsymbol{\gamma}=\mathbf{0}}) \;\middle|\; k = 1, \ldots, 2d+3 \right\},$$

so $T_{c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star}\mathbf{M} \subset \Gamma_{\boldsymbol{\alpha}}(T_{\boldsymbol{f}_\star}\mathbf{M})$. The reverse inclusion is proven in the same way. □

We give the explicit expression of the generators in Table 2.1. With these explicit expressions, we can prove Proposition 2.2.3.

Proof of Proposition 2.2.3. The proof of the first identity in (2.8) reduces to a check that the operators in the right column of entries 2-6 of Table 2.1 are skew-adjoint on $\dot{\mathcal{H}}^{\frac{1}{2}}(\mathbb{R}^d)$. We remark that this is true for any dimension d. The second identity in (2.8), concerning invariance of the $L^4(\mathbb{R}^{1+3})$ norm, is obvious for all symmetries except for Ph_θ (defined in (2.5)). This invariance is proved in [14, equation (2.5)]. □

	Derivative	Applied to $c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}$ at $c=1, \boldsymbol{\alpha}=\boldsymbol{0}$
1	$\frac{\partial}{\partial c}$	$\boldsymbol{f}$
2	$\frac{\partial}{\partial t_0}$	$\begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}\boldsymbol{f}$
3	$\frac{\partial}{\partial \theta}$	$\begin{bmatrix} 0 & (-\Delta)^{-\frac{1}{2}} \\ -(-\Delta)^{\frac{1}{2}} & 0 \end{bmatrix}\boldsymbol{f}$
4	$\frac{\partial}{\partial \zeta_j}$	$\begin{bmatrix} 0 & x_j \\ x_j\Delta + \frac{\partial}{\partial x_j} & 0 \end{bmatrix}\boldsymbol{f} \quad (j=1,2,\ldots,d)$
5	$\frac{\partial}{\partial \sigma}$	$\begin{bmatrix} \frac{d-1}{2} + x\cdot\nabla & 0 \\ 0 & \frac{d+1}{2} + x\cdot\nabla \end{bmatrix}\boldsymbol{f}$
6	∇_{x_0}	$\begin{bmatrix} \frac{\partial}{\partial x_j} & 0 \\ 0 & \frac{\partial}{\partial x_j} \end{bmatrix}\boldsymbol{f} \quad (j=1,2,\ldots,d).$

Table 2.1: Symmetry generators.

2.2.1 Computing the tangent spaces via the Penrose transform

We compute an explicit expression of the tangent space $T_{\boldsymbol{f}_\star}\mathbf{M}$, using the Penrose transform, which we introduced in Section 1.1 from the previous chapter. We systematically use the following identification of $x \in \mathbb{R}^d$ with

$$X = (X_0, X_1, \ldots, X_d) \in \mathbb{S}^d,$$

via the stereographic projection, whose equations we recall here;

$$\Omega_0 - 1 = X_0, \quad x_j\Omega_0 = X_j, \quad j = 1\ldots d. \tag{2.10}$$

Here $\Omega_0(x) = 2(1+|x|^2)^{-1}$ is the conformal factor of the stereographic projection; see Remark 1.1.1. In the following equations, the first computation is performed by applying (2.10), the second by applying (1.17) once, and the last by applying (1.17) twice:

$$\begin{aligned} \frac{\partial}{\partial x_j}\left(\Omega_0^{\frac{d-1}{2}}\right) &= -\frac{d-1}{2}x_j\Omega_0^{\frac{d+1}{2}} = -\frac{d-1}{2}X_j\Omega_0^{\frac{d-1}{2}}, \\ (-\Delta)^{\frac{1}{2}}\,\Omega_0^{\frac{d-1}{2}} &= \frac{d-1}{2}\Omega_0^{\frac{d+1}{2}}, \\ -\Delta\Omega_0^{\frac{d-1}{2}} &= \frac{d-1}{2}\Omega_0^{\frac{d+1}{2}}\left(\frac{d-1}{2} + \frac{d+1}{2}X_0\right). \end{aligned} \tag{2.11}$$

From (2.11) and (2.10), using $\sum_{j=1}^d X_j^2 = 1 - X_0^2$ we infer that

$$x\cdot\nabla\left(\Omega_0^{\frac{d-1}{2}}\right) = -\frac{d-1}{2}(1-X_0^2)\Omega_0^{\frac{d-3}{2}} = -\frac{d-1}{2}(1-X_0)\Omega_0^{\frac{d-1}{2}}.$$

	Generator	Applied to $\boldsymbol{f}_\star = \left(\Omega_0^{\frac{d-1}{2}}, 0\right)$
1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \Omega_0^{\frac{d-1}{2}} \\ 0 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -\frac{d-1}{2}\Omega_0^{\frac{d+1}{2}}\left(\frac{d-1}{2} + \frac{d+1}{2}X_0\right) \end{bmatrix}$
3	$\begin{bmatrix} 0 & (-\Delta)^{-\frac{1}{2}} \\ -(-\Delta)^{\frac{1}{2}} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -\frac{d-1}{2}\Omega_0^{\frac{d+1}{2}} \end{bmatrix}$
4	$\begin{bmatrix} 0 & x_j \\ x_j\Delta + \frac{\partial}{\partial x_j} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -\frac{d-1}{2}\Omega_0^{\frac{d-1}{2}}\left(\frac{(d-1)(d+1)}{4} + \frac{d+1}{2}X_0\right)X_j \end{bmatrix}$
5	$\begin{bmatrix} \frac{d-1}{2} + x\cdot\nabla & 0 \\ 0 & \frac{d+1}{2} + x\cdot\nabla \end{bmatrix}$	$\begin{bmatrix} \frac{d-1}{2}X_0\Omega_0^{\frac{d-1}{2}} \\ 0 \end{bmatrix}$
6	$\begin{bmatrix} \frac{\partial}{\partial x_j} & 0 \\ 0 & \frac{\partial}{\partial x_j} \end{bmatrix}$	$\begin{bmatrix} -\frac{d-1}{2}X_j\Omega_0^{\frac{d-1}{2}} \\ 0 \end{bmatrix} \quad (j = 1\dots d).$

Table 2.2: A basis of the tangent space at $\boldsymbol{f}_\star$ in arbitrary dimension.

We apply the generators of the symmetry group, listed in Table 2.1, to the Strichartz maximizer $\boldsymbol{f}_\star$. Using the computations (2.11), we obtain Table 2.2; we recall that we are identifying $x \in \mathbb{R}^d$ with $X \in \mathbb{S}^d$ via the stereographic projection (2.10). Since $\Omega_0 = 1 + X_0$ by (2.10), when $d = 3$ the fourth line simplifies:

$$\Omega_0^{\frac{d-1}{2}}\left(\frac{(d-1)(d+1)}{4} + \frac{d+1}{2}X_0\right)X_j = 2\Omega_0^2 X_j.$$

So, specializing the previous table to the case $d = 3$, we conclude that

$$T_{\boldsymbol{f}_\star}\mathbf{M} = \left\{ \begin{bmatrix} \Omega_0 P(X) \\ \Omega_0^2 Q(X) \end{bmatrix} : P, Q \text{ polynomials of degree} \leqslant 1 \text{ in } X \in \mathbb{S}^3 \right\}.$$

Since the restrictions of these polynomials to the sphere are spherical harmonics of degree 0 and 1, after applying the Penrose transformation (1.11) of the initial data, we see that

$$\boldsymbol{f} \in T_{\boldsymbol{f}_\star}\mathbf{M} \iff \hat{F}_0(\ell, m) = \hat{F}_1(\ell, m) = 0, \quad \ell \geqslant 2. \tag{2.12}$$

In light of the identity (1.19), expressing the $\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$ scalar product in terms of F_0, F_1, we characterize the orthogonal complement of $T_{\boldsymbol{f}_\star}\mathbf{M}$ as follows:

$$\boldsymbol{f} \perp T_{\boldsymbol{f}_\star}\mathbf{M} \iff \hat{F}_0(\ell, m) = \hat{F}_1(\ell, m) = 0, \quad \ell = 0, 1. \tag{2.13}$$

These computations immediately yield the following corollary, which we will use in the next subsection.

Corollary 2.2.7. *The matrix of scalar products*

$$M_0 := \left[\left\langle \partial_{\alpha_i}\big|_{\boldsymbol{\alpha}=\mathbf{0}}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \,\middle|\, \partial_{\alpha_j}\big|_{\boldsymbol{\alpha}=\mathbf{0}}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\right\rangle_{\dot{\mathcal{H}}^{1/2}}\right]_{i,j=1\ldots 9} \tag{2.14}$$

is nonsingular and positive definite.

2.2.2 Metric projections

We show in this subsection that every point of $\dot{\mathcal{H}}^{1/2}$ admits at least one closest point in $\mathbf{M}$. This is a crucial property for the proof of Theorem 2.0.1. We also study the uniqueness of these closest points. This will be needed in the nonlinear applications of the forthcoming chapter.

Proposition 2.2.8. *For every $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}$ there exists $P(\boldsymbol{f}) \in \mathbf{M}$ such that*

$$\|\boldsymbol{f} - P(\boldsymbol{f})\|_{\dot{\mathcal{H}}^{1/2}} = d(\boldsymbol{f}, \mathbf{M}),$$

and, if $P(\boldsymbol{f}) \neq \mathbf{0}$, then $\boldsymbol{f} - P(\boldsymbol{f}) \perp T_{P(\boldsymbol{f})}\mathbf{M}$, that is

$$\langle \boldsymbol{f} - P(\boldsymbol{f}) \,|\, \boldsymbol{g}\rangle_{\dot{\mathcal{H}}^{1/2}} = 0, \quad \forall\, \boldsymbol{g} \in T_{P(\boldsymbol{f})}\mathbf{M}, \tag{2.15}$$

Moreover, there is a constant $\rho \in (0,1)$ such that, if

$$d(\boldsymbol{f}, \mathbf{M}) < \rho \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}}, \tag{2.16}$$

then $P(\boldsymbol{f})$ is uniquely determined.

Proof. Existence. Let $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}$ be fixed. Expanding $\|\boldsymbol{f} - c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}$, we see that

$$d(\boldsymbol{f}, \mathbf{M})^2 = \inf\left\{ \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} - 2c\langle \boldsymbol{f} \,|\, \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} + c^2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} \,\middle|\, \begin{array}{c} c \geqslant 0 \\ \boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^8 \end{array} \right\}.$$

Let now $(c_n, \boldsymbol{\alpha}_n) \in [0,\infty) \times \mathbb{S}^1 \times \mathbb{R}^8$ be a minimizing sequence. Then, clearly, c_n must be bounded. Now, if $\boldsymbol{\alpha}_n$ is unbounded, then, up to a subsequence, we can assume that $|\boldsymbol{\alpha}_n| \to \infty$. This implies that

$$2c_n \langle \boldsymbol{f} \,|\, \Gamma_{\boldsymbol{\alpha}_n}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} \to 0;$$

see, for example, [62, Lemmas 3.2 and 4.1]. In this case, since $(c_n, \boldsymbol{\alpha}_n)$ is minimizing, it must be that $c_n \to 0$, and so $P(\boldsymbol{f}) = \mathbf{0}$. The only remaining possibility is that $\boldsymbol{\alpha}_n$ is also bounded, in which case, up to subsequences, $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}_0$ and $c_n \to c_0$ for some $c_0 \geqslant 0$ and $\boldsymbol{\alpha}_0 \in \mathbb{S}^1 \times \mathbb{R}^8$. Therefore, $P(\boldsymbol{f}) = c_0\Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star$.

Since $(c_0, \boldsymbol{\alpha}_0)$ is minimizing,

$$\partial_c \|\boldsymbol{f} - c\Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}\Big|_{c=c_0} = \partial_{\alpha_i}\|\boldsymbol{f} - c_0\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = 0, \quad i = 1, \ldots, 9,$$

from which the orthogonality property (2.15) follows, provided that $c_0 > 0$, which is equivalent to $P(\boldsymbol{f}) \neq \mathbf{0}$.

Uniqueness. We assume that (2.16) holds for a constant ρ to be determined, and we suppose that there exist $P(\boldsymbol{f})$ and $P'(\boldsymbol{f})$ in $\mathbf{M}\backslash\{\mathbf{0}\}$ such that

$$\boldsymbol{f} = P(\boldsymbol{f}) + \boldsymbol{f}_\perp = P'(\boldsymbol{f}) + \boldsymbol{f}'_\perp, \tag{2.17}$$

where $\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{f}'_\perp\|_{\dot{\mathcal{H}}^{1/2}} = d(\boldsymbol{f}, \mathbf{M})$. Our goal is to show that $P(\boldsymbol{f})$ and $P'(\boldsymbol{f})$ must be equal. We consider $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{S}^1 \times \mathbb{R}^8$ such that

$$P(\boldsymbol{f}) = c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \quad \text{and} \quad P'(\boldsymbol{f}) = c'\Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star,$$

and, replacing $\boldsymbol{f}$ with $\Gamma_{\boldsymbol{\alpha}}^{-1}\boldsymbol{f}$ if needed, we can assume that $\Gamma_{\boldsymbol{\alpha}} = \Gamma_{\mathbf{0}}$. The orthogonality (2.15) implies that $\langle \boldsymbol{f}_\perp \,|\, P(\boldsymbol{f})\rangle_{\dot{\mathcal{H}}^{1/2}} = \langle \boldsymbol{f}'_\perp \,|\, P'(\boldsymbol{f})\rangle_{\dot{\mathcal{H}}^{1/2}} = 0$, so using (2.17) we can expand $\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}}$, yielding

$$c = c' = \|P(\boldsymbol{f})\|_{\dot{\mathcal{H}}^{1/2}} = \left\|P'(\boldsymbol{f})\right\|_{\dot{\mathcal{H}}^{1/2}} = \sqrt{\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} - d(\boldsymbol{f}, \mathbf{M})^2}.$$

It follows from these considerations that we can rewrite (2.17) as

$$\frac{\boldsymbol{f}}{c} = \boldsymbol{f}_\star + \frac{\boldsymbol{f}_\perp}{c} = \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star + \frac{\boldsymbol{f}'_\perp}{c},$$

from which we infer the estimate

$$\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \frac{2d(\boldsymbol{f}, \mathbf{M})}{\sqrt{\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} - d(\boldsymbol{f}, \mathbf{M})^2}} \leqslant \frac{2\rho}{\sqrt{1-\rho^2}}, \tag{2.18}$$

and analogously,

$$\|\boldsymbol{f}/c - \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \frac{\rho}{\sqrt{1-\rho^2}}. \tag{2.19}$$

To finish the proof, it will suffice to show that $\boldsymbol{\alpha}' = \mathbf{0}$.

As a first step, we claim that

$$\left|\boldsymbol{\alpha}'\right| \leqslant C\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}, \tag{2.20}$$

for a $C > 0$. To prove this, we begin by squaring the left-hand side of (2.18),

$$\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} = 2 - 2\langle \boldsymbol{f}_\star \,|\, \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star\rangle_{,\dot{\mathcal{H}}^{1/2}}$$

so that

$$\langle \boldsymbol{f}_\star \,|\, \Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} \geqslant \tfrac{1-3\rho^2}{1-\rho^2}.$$

Assuming, as we may, that $\rho < 1/\sqrt{3}$, the right-hand side of this inequality is strictly positive. Now, as we have already mentioned in the proof of existence of $P(\boldsymbol{f})$, $\langle \boldsymbol{f}_\star \,|\, \Gamma_{\boldsymbol{\sigma}}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} \to 0$ as $|\boldsymbol{\sigma}| \to \infty$. Thus, there must be a $C(\rho) > 0$ such that $|\boldsymbol{\alpha}'| \leqslant C(\rho)$.

We can then assume, for a contradiction, that

$$\frac{\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}_n}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}}{|\boldsymbol{\alpha}_n|^2} \to 0, \quad \text{where } \boldsymbol{\alpha}_n \in \mathbb{S}^1 \times \mathbb{R}^8,\ |\boldsymbol{\alpha}_n| \leqslant C(\rho). \tag{2.21}$$

There exists $\boldsymbol{\alpha}_0 \in \mathbb{S}^1 \times \mathbb{R}^8$ such that $\boldsymbol{\alpha}_n \to \boldsymbol{\alpha}_0$ up to a subsequence. If $|\boldsymbol{\alpha}_0| \neq 0$, then (2.21) would imply that $\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}_0}\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} = 0$, but this is ruled out by Lemma 2.2.5. The only remaining possibility is that $|\boldsymbol{\alpha}_n| \to 0$. We record now two identities that hold for all $\boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^8$;

$$\langle \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \,|\, \partial_{\alpha_i}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} = \partial_{\alpha_i}\tfrac{1}{2}\|\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} = 0, \tag{2.22}$$

where we used that $\Gamma_{\boldsymbol{\alpha}}$ is unitary, and

$$-\left\langle \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \,\middle|\, \partial_{\alpha_i}\partial_{\alpha_j}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\right\rangle_{\dot{\mathcal{H}}^{1/2}} = \left\langle \partial_{\alpha_i}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star \,\middle|\, \partial_{\alpha_j}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\right\rangle_{\dot{\mathcal{H}}^{1/2}}, \tag{2.23}$$

which is obtained from (2.22) by differentiating. Using these we compute the expansion

$$\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} = 2\sum_{i,j=1}^{9} \alpha_i\alpha_j \left\langle \partial_{\sigma_i}\Gamma_{\boldsymbol{\sigma}}\boldsymbol{f}_\star \,\middle|\, \partial_{\sigma_j}\Gamma_{\boldsymbol{\sigma}}\boldsymbol{f}_\star\right\rangle\Big|_{\boldsymbol{\sigma}=\mathbf{0}} + O(|\boldsymbol{\alpha}|^3).$$

Since the coefficients of the quadratic term are those of the matrix M_0 defined in (2.14), the fact that $|\boldsymbol{\alpha}_n| \to 0$ implies

$$0 = \lim_{n\to\infty} \frac{\|\boldsymbol{f}_\star - \Gamma_{\boldsymbol{\alpha}_n}\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}}{|\boldsymbol{\alpha}_n|^2} \geqslant 2\lambda_0 > 0,$$

where λ_0 is the minimal eigenvalue of M_0, which is strictly positive because of Corollary 2.2.7. We have reached the desired contradiction and proved (2.20).

To conclude the proof that $\boldsymbol{\alpha}' = \mathbf{0}$, we define $\mathcal{F}\colon \mathbb{S}^1 \times \mathbb{R}^8 \times \dot{\mathcal{H}}^{1/2} \to \mathbb{R}^9$ by

$$\mathcal{F}(\boldsymbol{\alpha}, \boldsymbol{g}) := \left[\langle \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star - \boldsymbol{g} \,|\, \partial_{\alpha_i}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}}\right]_{i=1\ldots 9}.$$

By (2.15), $\Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star - \boldsymbol{f}/c = \boldsymbol{f}'_\perp/c$ is orthogonal to the tangent space at $c\Gamma_{\boldsymbol{\alpha}'}\boldsymbol{f}_\star$, which contains all the derivatives $\partial_{\alpha_i}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star$ at $\boldsymbol{\alpha}'$, so $\mathcal{F}(\boldsymbol{\alpha}', \boldsymbol{f}/c) = 0$. In the same way, we see that $\mathcal{F}(\mathbf{0}, \boldsymbol{f}/c) = 0$.

Now, obviously, $\mathcal{F}(\mathbf{0}, \boldsymbol{f}_\star) = 0$. Using the identities (2.22) and (2.23) as before, we find that the Jacobian matrix $D_{\boldsymbol{\alpha}}\mathcal{F} = \left[\partial_{\alpha_j}\mathcal{F}_i\right]_{i,j=1\ldots 9}$ at $(\mathbf{0}, \boldsymbol{f}_\star)$ is

$$D_{\boldsymbol{\alpha}}\mathcal{F}(\mathbf{0}, \boldsymbol{f}_\star) = M_0,$$

so that, in particular, it is nonsingular. We can thus rewrite the identity $\mathcal{F}(\boldsymbol{\alpha}', \boldsymbol{f}/c) = 0$ as a fixed point relation;

$$\boldsymbol{\alpha}' = P(\boldsymbol{\alpha}', \boldsymbol{f}/c), \quad \text{where } P(\boldsymbol{\alpha}, \boldsymbol{g}) := \boldsymbol{\alpha} - D_{\boldsymbol{\alpha}}\mathcal{F}(\mathbf{0}, \boldsymbol{f}_\star)^{-1}\mathcal{F}(\boldsymbol{\alpha}, \boldsymbol{g}),$$

and the function P is such that $D_{\boldsymbol{\alpha}} P(\mathbf{0}, \boldsymbol{f}_\star) = 0$. Thus, there exists an absolute constant $\varepsilon > 0$ such that

$$\|D_{\boldsymbol{\alpha}} P(\boldsymbol{\alpha}, \boldsymbol{g})\| \leqslant \tfrac{1}{2}, \quad \text{if } |\boldsymbol{\alpha}| < \varepsilon \text{ and } \|\boldsymbol{g} - \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} < \varepsilon.$$

Here, as is usual, the matrix norm is $\|M\| := \sup\left\{ |Mx|/|x| : x \in \mathbb{R}^9 \right\}$. We now require, as we may, that ρ satisfies the additional condition

$$\frac{\rho}{\sqrt{1-\rho^2}} \leqslant \frac{\varepsilon}{2C},$$

so that, combining (2.18) and (2.20), we see that $|t\boldsymbol{\alpha}'| < \varepsilon$ for all $t \in [0, 1]$, and moreover, $\|\boldsymbol{f}/c - \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} < \varepsilon$ by (2.19). Thus $\|D_{\boldsymbol{\alpha}} P(t\boldsymbol{\alpha}', \boldsymbol{f}/c)\| \leqslant \frac{1}{2}$, and from

$$\boldsymbol{\alpha}' = P(\boldsymbol{\alpha}', \boldsymbol{f}/c) = \int_0^1 \frac{d}{dt} P(t\boldsymbol{\alpha}', \boldsymbol{f}/c)\, dt = \int_0^1 D_{\boldsymbol{\alpha}} P(t\boldsymbol{\alpha}', \boldsymbol{f}/c)\boldsymbol{\alpha}'\, dt,$$

where we used that $P(\mathbf{0}, \boldsymbol{f}/c) = \mathbf{0}$, we infer that

$$\left|\boldsymbol{\alpha}'\right| \leqslant \int_0^1 \left\|D_{\boldsymbol{\alpha}} P(t\boldsymbol{\alpha}', \boldsymbol{f}/c)\right\| \left|\boldsymbol{\alpha}'\right| dt \leqslant \frac{1}{2}\left|\boldsymbol{\alpha}'\right|,$$

so that $|\boldsymbol{\alpha}'| = 0$, completing the proof. □

2.3 Proof of the lower bound in Theorem 2.0.1

In this section the spatial dimension d will be 3. We let ψ denote the deficit functional

$$\psi(\boldsymbol{f}) = \mathcal{S}^4 \|\boldsymbol{f}\|^4_{\dot{\mathcal{H}}^{1/2}} - \|S\boldsymbol{f}\|^4_{L^4(\mathbb{R}^{1+3})}, \quad \text{where } \mathcal{S} = \left(\tfrac{3}{16\pi}\right)^{\frac{1}{4}}.$$

We will use Corollary 1.2.3 from the previous chapter to compute integrals on $\mathbb{R}^{1+3}$, taking advantage of the simple expression for $v_\star = S\boldsymbol{f}_\star$ under the Penrose transform,

$$V_\star(T, X) = \cos T;$$

see (1.12). In particular, Corollary 1.2.3 yields the following representation of Foschi's constant;

$$\mathcal{S}^4 = \frac{\|S\boldsymbol{f}_\star\|^4_{L^4(\mathbb{R}^{1+3})}}{\|\boldsymbol{f}_\star\|^4_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)}} = \frac{\int_{-\pi}^{\pi} (\cos T)^4\, dT}{2|\mathbb{S}^3|}.$$

Here we have used the fact that $\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} = \left|\mathbb{S}^3\right|$; see (1.20).

2.3.1 A local version

Lemma 2.3.1. *There exists a quadratic functional* $Q\colon \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3) \to [0, \infty)$ *such that*

$$\psi(\boldsymbol{f}_\star + \boldsymbol{f}) = Q(\boldsymbol{f}) + O(\|\boldsymbol{f}\|^3_{\dot{\mathcal{H}}^{1/2}}), \tag{2.24}$$

for all $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$*. It holds that* $Q(\boldsymbol{f}) = 0$ *if and only if* $\boldsymbol{f} \in T_{\boldsymbol{f}_\star}\mathbf{M}$*, and moreover*

$$Q(\boldsymbol{f}) \geqslant \tfrac{\pi}{4}\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}}, \quad \forall \boldsymbol{f} \perp T_{\boldsymbol{f}_\star}\mathbf{M}, \tag{2.25}$$

where the constant $\frac{\pi}{4}$ *cannot be replaced by a larger one.*

Proof. We have that $\psi(\boldsymbol{f}_\star) = 0$ by definition of ψ, and we have proved in Theorem 1.0.1 that $\frac{d}{d\varepsilon}\psi(\boldsymbol{f}_\star + \varepsilon\boldsymbol{f})\big|_{\varepsilon=0} = 0$ for all $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$. So (2.24) holds with $Q(\boldsymbol{f})$ equal to $\frac{1}{2}\frac{d^2}{d\varepsilon^2}\psi(\boldsymbol{f}_\star + \varepsilon\boldsymbol{f})\Big|_{\varepsilon=0}$. Expanding we see that

$$Q(\boldsymbol{f}) = \mathcal{S}^4\left(4\langle \boldsymbol{f}_\star \,|\, \boldsymbol{f}\rangle^2_{\dot{\mathcal{H}}^{1/2}} + 2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}}\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}}\right) - 6\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^2(S\boldsymbol{f})^2\, dtdx. \tag{2.26}$$

We record that, for all $\boldsymbol{f} = (f_0, f_1) \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$, it holds that

$$Q(\boldsymbol{f}) = Q(f_0, 0) + Q(0, f_1). \tag{2.27}$$

To prove this, we start by recalling that $\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} = \langle f_0 \,|\, f_0\rangle_{\dot{H}^{1/2}} + \langle f_1 \,|\, f_1\rangle_{\dot{H}^{-1/2}}$. Moreover, since $\boldsymbol{f}_\star = (f_{\star 0}, 0)$, we have that $\langle \boldsymbol{f}_\star \,|\, \boldsymbol{f}\rangle_{\dot{\mathcal{H}}^{1/2}} = \langle f_{\star 0} \,|\, f_0\rangle_{\dot{H}^{1/2}}$, so the first summand in the right-hand side of (2.26) splits into the sum of a term depending on f_0 only and a term depending on f_1 only. The other summand splits in the same way; indeed, $S\boldsymbol{f}_\star = \cos(t\sqrt{-\Delta})f_{\star 0}$, therefore

$$\begin{aligned}\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^2(S\boldsymbol{f})^2 = \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^2(\cos t\sqrt{-\Delta} f_0)^2 + \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^2\left(\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} f_1\right)^2 \\ + 2\iint\limits_{\mathbb{R}^{1+3}} (\cos t\sqrt{-\Delta} f_{\star 0})^2 \cos t\sqrt{-\Delta} f_0 \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} f_1,\end{aligned}$$

where the last integral vanishes, as can be seen with the change of variable $t \mapsto -t$. This proves (2.27).

We now bound $Q(\boldsymbol{f})$ from below, starting with the term $Q(f_0, 0)$. We apply the Penrose transformation (1.11) to $\boldsymbol{f}$ and $\boldsymbol{f}_\star$, recalling that

$$(F_{\star 0}, F_{\star 1}) = (1, 0), \tag{2.28}$$

so, in particular, the only non-vanishing coefficient in the expansion in spherical harmonics is $\hat{F}_{\star 0}(0, 0) = |\mathbb{S}^3|^{1/2}$; see Remark 1.1.4. By the formula (1.19), that expresses the $\dot{\mathcal{H}}^{1/2}$ scalar product in terms of (F_0, F_1), we rewrite the first summand in the right-hand side of (2.26) as

$$\begin{aligned}\mathcal{S}^4\left(4\langle f_{\star 0} \,|\, f_0\rangle^2_{\dot{H}^{\frac{1}{2}}} + 2\|f_{\star 0}\|^2_{\dot{H}^{\frac{1}{2}}}\|f_0\|^2_{\dot{H}^{\frac{1}{2}}}\right) = \\ \frac{\int_{-\pi}^{\pi}(\cos T)^4\, dT}{2|\mathbb{S}^3|}\left(4|\mathbb{S}^3|\hat{F}_0(0,0)^2 + 2|\mathbb{S}^3|\sum_{\ell=0}^{\infty}\sum_{m=0}^{N(\ell)}(\ell+1)\hat{F}_0(\ell, m)^2\right).\end{aligned} \tag{2.29}$$

We compute the other summand using Corollary 1.2.3;

$$6\iint_{\mathbb{R}^{1+3}}(S\boldsymbol{f}_\star)^2(\cos(t\sqrt{-\Delta})f_0)^2 = 3\iint_{\mathbb{S}^1\times\mathbb{S}^3}\left(\cos T\sum_{\ell,m}\cos(\ell+1)T\,\hat{F}_0(\ell,m)Y_{\ell,m}\right)^2.$$

By the $L^2(\mathbb{S}^3)$-orthonormality of $Y_{\ell,m}$, the right-hand side equals

$$3\int_{-\pi}^{\pi}(\cos T)^4\,dT\hat{F}_0(0,0)^2 + 3\sum_{\ell=1}^{\infty}\sum_{m=0}^{N(\ell)}\int_{-\pi}^{\pi}(\cos T\cos(\ell+1)T)^2\,dT\hat{F}_0(\ell,m)^2.$$

For all $\ell \geqslant 1$, it holds that

$$3\int_{-\pi}^{\pi}(\cos T\cos(\ell+1)T)^2\,dT = \frac{3\pi}{2} = 2\int_{-\pi}^{\pi}(\cos T)^4\,dT,$$

so, subtracting the last equation from (2.29), the terms corresponding to $\ell = 0$ and $\ell = 1$ vanish, and we obtain that

$$Q(f_0,0) = \frac{3\pi}{4}\sum_{\ell=2}^{\infty}\sum_{m=0}^{N(\ell)}(\ell-1)\hat{F}_0(\ell,m)^2.$$

The term $Q(0,f_1)$ is computed in the same way, and the end result is:

$$Q(\boldsymbol{f}) = \frac{3\pi}{4}\sum_{\ell=2}^{\infty}\sum_{m=0}^{N(\ell)}(\ell-1)\left[\hat{F}_0(\ell,m)^2 + \frac{\hat{F}_1(\ell,m)^2}{(\ell+1)^2}\right]. \tag{2.30}$$

From this we see that $Q(\boldsymbol{f}) = 0$ if and only if $\hat{F}_0(\ell,m) = \hat{F}_1(\ell,m) = 0$ for $\ell \geqslant 2$, which is equivalent to $\boldsymbol{f} \in T_{\boldsymbol{f}_\star}\mathbf{M}$; see (2.12).

It remains to prove the sharp inequality (2.25). For $\ell \geqslant 2$, it holds that

$$3(\ell-1) \geqslant \ell+1, \quad \text{and} \quad 3\tfrac{\ell-1}{(\ell+1)^2} \geqslant \tfrac{1}{\ell+1},$$

with equality for $\ell = 2$. Therefore, (2.30) implies the sharp inequality

$$Q(\boldsymbol{f}) \geqslant \frac{\pi}{4}\sum_{\ell=2}^{\infty}\sum_{m=0}^{N(\ell)}(\ell+1)\hat{F}_0(\ell,m)^2 + (\ell+1)^{-1}\hat{F}_1(\ell,m)^2.$$

The expression on the right-hand side equals $\frac{\pi}{4}\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)}$ precisely when $\hat{F}_0(\ell,m) = \hat{F}_1(\ell,m) = 0$ for $\ell = 0,1$, which is equivalent to $\boldsymbol{f}\perp T_{\boldsymbol{f}_\star}\mathbf{M}$; see (2.13). This completes the proof. □

Remark 2.3.2. The fact that $Q(\boldsymbol{f}) = 0$ for $\boldsymbol{f} \in T_{\boldsymbol{f}_\star}\mathbf{M}$ is a consequence of the criticality of $\boldsymbol{f}_\star$ and of the invariance of ψ under the symmetries $\Gamma_{\boldsymbol{\alpha}}$ (defined in (2.7)); indeed, differentiating the identity $\psi(c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star) = 0$ twice with respect to c we get $Q(\boldsymbol{f}_\star) = 0$, and differentiating twice with respect to α_j, we get

$$Q\left(\frac{\partial}{\partial\alpha_j}\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star\bigg|_{\boldsymbol{\alpha}=\mathbf{0}}\right) = 0.$$

In Lemma 2.3.1 we proved a sharper result; namely, that $Q(\boldsymbol{f})$ vanishes *if and only if* $\boldsymbol{f} \in T_{\boldsymbol{f}_\star}\mathbf{M}$, and we gave a sharp explicit bound. In the language of the calculus of variations we can say that $\boldsymbol{f}_\star$ is a *transversally non-degenerate* local minimizer of the deficit functional ψ.

Proposition 2.3.3. *For all* $\boldsymbol{f} \in \dot{\mathcal{H}}^{\frac{1}{2}}(\mathbb{R}^3)$ *such that*

$$d(\boldsymbol{f}, \mathbf{M}) < \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{\frac{1}{2}}}, \tag{2.31}$$

it holds that

$$\frac{1}{3}\mathcal{S}^2 d(\boldsymbol{f}, \mathbf{M})^2 + O(d(\boldsymbol{f}, \mathbf{M})^3) \leqslant \mathcal{S}^2\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+3})}.$$

The result does not hold if $\frac{1}{3}\mathcal{S}^2$ *is replaced with a larger constant.*

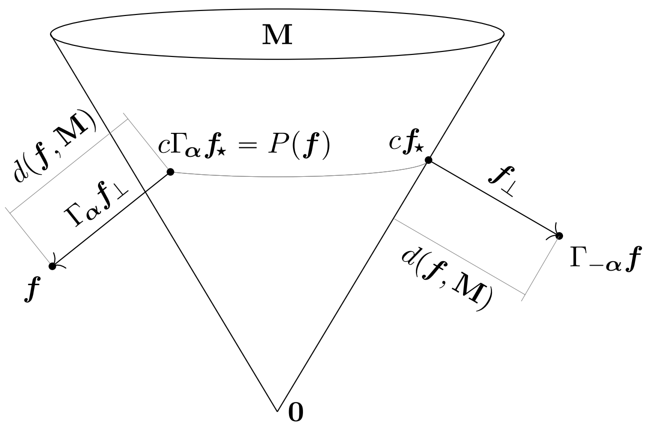

Figure 2.1: Illustration of Step 1.

Proof. Step 1: By Proposition 2.2.8, there exists $P(\boldsymbol{f}) \in \mathbf{M}$ such that

$$\|\boldsymbol{f} - P(\boldsymbol{f})\|_{\dot{\mathcal{H}}^{1/2}} = d(\boldsymbol{f}, \mathbf{M}).$$

Assuming that $P(\boldsymbol{f}) = c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star$, we define

$$\boldsymbol{f}_\perp := \Gamma_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{f} - P(\boldsymbol{f})),$$

and we claim that

$$\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^{1/2}} = d(\boldsymbol{f}, \mathbf{M}), \quad \text{and} \quad \boldsymbol{f}_\perp \perp T_{\boldsymbol{f}_\star}\mathbf{M}. \tag{2.32}$$

The first property is an immediate consequence of the fact that $\Gamma_{\boldsymbol{\alpha}}$ is a unitary operator; see Proposition 2.2.3.

To prove the second property, we begin by observing that the assumption (2.31) ensures that $P(\boldsymbol{f}) \neq \mathbf{0}$, so the tangent space $T_{P(\boldsymbol{f})}\mathbf{M}$ is well-defined, and $\boldsymbol{f} - P(\boldsymbol{f}) \perp T_{P(\boldsymbol{f})}\mathbf{M}$. By

Proposition 2.2.6, $T_{P(\boldsymbol{f})}\mathbf{M} = \Gamma_{\boldsymbol{\alpha}}(T_{\boldsymbol{f}_\star}\mathbf{M})$, and so we can conclude that

$$\left\langle \Gamma_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{f} - P(\boldsymbol{f}))\,\middle|\,\boldsymbol{g}\right\rangle_{\dot{\mathcal{H}}^{1/2}} = \langle \boldsymbol{f} - P(\boldsymbol{f})\,|\,\Gamma_{\boldsymbol{\alpha}}\boldsymbol{g}\rangle_{\dot{\mathcal{H}}^{1/2}} = 0, \quad \forall \boldsymbol{g} \in T_{\boldsymbol{f}_\star}\mathbf{M},$$

where we used that the adjoint of $\Gamma_{\boldsymbol{\alpha}}$ is $\Gamma_{\boldsymbol{\alpha}}^{-1}$, because $\Gamma_{\boldsymbol{\alpha}}$ is unitary. This proves the second identity in (2.32).

Step 2: Consider the 2-homogeneous deficit functional defined by

$$\phi(\boldsymbol{f}) := \mathcal{S}^2\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+3})}.$$

Like its 4-homogeneous counterpart ψ, the functional ϕ is $\Gamma_{\boldsymbol{\alpha}}$-invariant, so that, by Step 1,

$$\phi(\boldsymbol{f}) = \phi(c\Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\star + \Gamma_{\boldsymbol{\alpha}}\boldsymbol{f}_\perp) = \phi(c\boldsymbol{f}_\star + \boldsymbol{f}_\perp).$$

Now $\phi(c\boldsymbol{f}_\star) = 0$, and since $\langle \boldsymbol{f}_\star\,|\,\boldsymbol{f}_\perp\rangle_{\dot{\mathcal{H}}^{1/2}} = 0$, we can expand to see that

$$\frac{d}{d\varepsilon}\phi(c\boldsymbol{f}_\star + \varepsilon\boldsymbol{f}_\perp)\bigg|_{\varepsilon=0} = -\frac{2c}{\|S\boldsymbol{f}_\star\|^2_{L^4}}\iint_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3 S\boldsymbol{f}_\perp\,dtdx.$$

Combining Theorem 1.0.1 and Lemma 1.3.1 from the previous chapter, we see that the right-hand side is zero. Expanding to second order, using this fact again, we obtain

$$\begin{aligned}\phi(c\boldsymbol{f}_\star + \varepsilon\boldsymbol{f}_\perp) = \varepsilon^2\Big[\mathcal{S}^2\|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} - \frac{3}{\|S\boldsymbol{f}_\star\|^2_{L^4}}\iint_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^2(S\boldsymbol{f})^2\,dtdx\Big]\\ + O(\varepsilon^3\|\boldsymbol{f}_\perp\|^3_{\dot{\mathcal{H}}^{1/2}}).\end{aligned}$$

Evaluating at $\varepsilon = 1$, using that $\|S\boldsymbol{f}_\star\|_{L^4(\mathbb{R}^{1+3})} = \mathcal{S}\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}$, and comparing with the expression of Q given in (2.26), we obtain

$$\phi(c\boldsymbol{f}_\star + \boldsymbol{f}_\perp) = \frac{Q(\boldsymbol{f}_\perp)}{2\mathcal{S}^2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}}} + O(\|\boldsymbol{f}_\perp\|^3_{\dot{\mathcal{H}}^{\frac{1}{2}}}),$$

The proposition then follows from Lemma 2.3.1, using that $\mathcal{S}^2 = (3/16\pi)^{1/2}$ and that $\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} = |\mathbb{S}^3| = 2\pi^2$. □

2.3.2 From local to global: the profile decomposition

We now cast in our notation the profile decomposition of Ramos [62].

Theorem 2.3.4. *Let $\boldsymbol{f}_n$ be a bounded sequence in $\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$. Then there exists an at most countable set $\{\, \boldsymbol{f}^j \;:\; j = 1, 2\ldots \,\} \subset \dot{\mathcal{H}}^{1/2}$ and corresponding sequences of transformations $\Gamma_{\boldsymbol{\alpha}_n^j}$ such that, up to passing to a subsequence,*

$$\boldsymbol{f}_n = \sum_{j=1}^{J} \Gamma_{\boldsymbol{\alpha}_n^j}\boldsymbol{f}^j + \boldsymbol{r}_n^J,$$

where the remainder term $\boldsymbol{r}_n^J$ satisfies

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|S\boldsymbol{r}_n^J\right\|_{L^4(\mathbb{R}^{1+3})}=0.$$

Moreover, for each $J\geqslant 1$ the following Pythagorean expansions hold for $n\to\infty$:

$$\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{1/2}}^2=\sum_{j=1}^J\left\|\boldsymbol{f}^j\right\|_{\dot{\mathcal{H}}^{1/2}}^2+\left\|\boldsymbol{r}_n^J\right\|_{\dot{\mathcal{H}}^{1/2}}^2+o(1), \tag{2.33}$$

and

$$\|S\boldsymbol{f}_n\|_{L^4(\mathbb{R}^{1+3})}^4=\sum_{j=1}^J\left\|S\boldsymbol{f}^j\right\|_{L^4(\mathbb{R}^{1+3})}^4+\left\|S\boldsymbol{r}_n^J\right\|_{L^4(\mathbb{R}^{1+3})}^4+o(1). \tag{2.34}$$

The proof of Theorem 2.0.1 will be obtained by the combination of Proposition 2.3.3 with the following property of optimizing sequences of the Strichartz inequality. We remark that, unlike the previous proposition, in the proof of the following lemma we use the result of Foschi that $\mathcal{S}$ is the sharp constant in the Strichartz inequality.

Lemma 2.3.5. *Let $\boldsymbol{f}_n\in\dot{\mathcal{H}}^{1/2}\backslash\{\,\mathbf{0}\,\}$ be a sequence such that*

$$\lim_{n\to\infty}\frac{\|S\boldsymbol{f}_n\|_{L^4(\mathbb{R}^{1+3})}}{\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{\frac12}}}=\mathcal{S}. \tag{2.35}$$

Then, up to passing to a subsequence,

$$\lim_{n\to\infty}\frac{d(\boldsymbol{f}_n,\mathbf{M})}{\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{\frac12}}}=0.$$

Proof. By homogeneity we may assume that $\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{\frac12}}=1$. We apply the profile decomposition, Theorem 2.3.4. This produces a countable subset $\left\{\,\boldsymbol{f}^j : j\in\mathbb{N}\,\right\}$ of $\dot{\mathcal{H}}^{1/2}$. We claim that $\boldsymbol{f}^j=0$ for all but one $j\in\mathbb{N}$. To prove this we begin by showing that there is at least one $j\in\mathbb{N}$ such that $\boldsymbol{f}^j\neq 0$. Indeed, if that was not the case then from property (2.34) one would infer the contradiction $\mathcal{S}=0$. Thus we can assume that $\boldsymbol{f}^1\neq 0$.

The Pythagorean expansion (2.33) with $J=1$ reads

$$1=\left\|\boldsymbol{f}^1\right\|_{\dot{\mathcal{H}}^{\frac12}}^2+\lim_{n\to\infty}\left\|\boldsymbol{r}_n^1\right\|_{\dot{\mathcal{H}}^{\frac12}}^2.$$

On the other hand, applying the sharp Strichartz inequality to the $L^4(\mathbb{R}^{1+3})$ Pythagorean expansion (2.34) we obtain

$$\begin{aligned}\mathcal{S}^4=\lim_{n\to\infty}\|S\boldsymbol{f}_n\|_{L^4(\mathbb{R}^{1+3})}^4&=\left\|S\boldsymbol{f}^1\right\|_{L^4(\mathbb{R}^{1+3})}^4+\lim_{n\to\infty}\left\|S\boldsymbol{r}_n^1\right\|_{L^4(\mathbb{R}^{1+3})}^4\\&\leqslant\mathcal{S}^4\left(\left\|\boldsymbol{f}^1\right\|_{\dot{\mathcal{H}}^{\frac12}}^4+\lim_{n\to\infty}\left\|\boldsymbol{r}_n^1\right\|_{\dot{\mathcal{H}}^{\frac12}}^4\right).\end{aligned}$$

Now if $a, b \in \mathbb{R}$ are such that $a^2 + b^2 = 1$ and $a^4 + b^4 \geqslant 1$, then necessarily one of them must vanish. Since $\boldsymbol{f}^1 \neq \mathbf{0}$, then it must be that $\|\boldsymbol{r}_n^1\|_{\dot{\mathcal{H}}^{\frac{1}{2}}} \to 0$. We have thus shown that

$$\boldsymbol{f}_n = \Gamma_{\alpha_n^1} \boldsymbol{f}^1 + \boldsymbol{r}_n^1, \quad \|\boldsymbol{r}_n^1\|_{\dot{\mathcal{H}}^{\frac{1}{2}}} \to 0.$$

This yields, using (2.35), that $\boldsymbol{f}^1 \in \mathbf{M}$. Therefore

$$d(\boldsymbol{f}_n, \mathbf{M}) \leqslant \|\boldsymbol{r}_n^1\|_{\dot{\mathcal{H}}^{\frac{1}{2}}} \to 0,$$

and the proof is complete. □

Combining Proposition 2.3.3 and Lemma 2.3.5 we prove the lower bound in Theorem 2.0.1.

Proof of Theorem 2.0.1. Since $\mathbf{0} \in \mathbf{M}$, we have that

$$d(\boldsymbol{f}, \mathbf{M}) \leqslant \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{\frac{1}{2}}}, \quad \forall \boldsymbol{f} \in \dot{\mathcal{H}}^{\frac{1}{2}}.$$

Assume for a contradiction that the lower bound of Theorem 2.0.1 fails. This would mean that there exists a sequence $\boldsymbol{f}_n \in \dot{\mathcal{H}}^{\frac{1}{2}} \backslash \mathbf{M}$ such that

$$\lim_{n \to \infty} \frac{\mathcal{S}^2 \|\boldsymbol{f}_n\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}} - \|S\boldsymbol{f}_n\|^2_{L^4(\mathbb{R}^{1+3})}}{d(\boldsymbol{f}_n, \mathbf{M})^2} = 0. \tag{2.36}$$

By homogeneity we can assume that $\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{\frac{1}{2}}} = 1$, and so $d(\boldsymbol{f}, \mathbf{M}) \leqslant 1$. Then (2.36) implies that $\mathcal{S}^2 \|\boldsymbol{f}_n\|^2_{\dot{\mathcal{H}}^{\frac{1}{2}}} - \|S\boldsymbol{f}_n\|^2_{L^4(\mathbb{R}^{1+3})} \to 0$. By Lemma 2.3.5 we obtain that $d(\boldsymbol{f}_n, \mathbf{M}) \to 0$, and so that (2.36) would contradict our local bound, Proposition 2.3.3. □

Remark 2.3.6. The multiplicative constant $\frac{1}{3}\mathcal{S}^2$ in Proposition 2.3.3 is the optimal one for the local bound. However, the argument by contradiction just presented does not give the optimal constant for the global bound. We conjecture that the optimal constant should be $\frac{1}{3}\mathcal{S}^2$.

2.4 Sharpening the energy-Strichartz estimate

We consider the following sharp estimate, due to Bez and Rogers [14];

$$\|S\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+5})} \leqslant \mathcal{S}_5 \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}, \quad \text{where } \mathcal{S}_5 := \left(\frac{1}{8\pi}\right)^{\frac{1}{2}}. \tag{2.37}$$

There is equality in (2.37) if and only if

$$\boldsymbol{f} \in \mathbf{M}_5 := \left\{ c\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star \;\middle|\; c \geqslant 0,\ \boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^7 \right\},$$

where $\boldsymbol{f}_\star = (2^2(1+|\cdot|^2)^{-2}, 0)$, and

$$\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}(x) = \vec{S}_{t_0}\,\mathrm{Ph}_\theta\Big(e^{\frac{3}{2}\sigma} f_0\left(e^\sigma(x+x_0)\right), e^{\frac{5}{2}\sigma} f_1\left(e^\sigma(x+x_0)\right)\Big). \tag{2.38}$$

Here, the operators $\vec{S}_{t_0}$ and Ph_θ are given in (2.4) and (2.5) and

$$\boldsymbol{\alpha} = (\theta, t_0, \sigma, x_0), \quad \theta \in \mathbb{S}^1,\ t_0 \in \mathbb{R}, \sigma \in \mathbb{R},\ x_0 \in \mathbb{R}^5.$$

The only difference between these transformations and the ones in the $\dot{\mathcal{H}}^{\frac{1}{2}}$ case is that here there are no Lorentz boosts. As before, the operator $\Gamma_{\boldsymbol{\alpha}}$ defined in (2.38) preserves both sides in the Strichartz inequality (2.37);

$$\|\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} = \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}, \quad \|S\Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}\|_{L^4(\mathbb{R}^{1+5})} = \|S\boldsymbol{f}\|_{L^4(\mathbb{R}^{1+5})},$$

for all $\boldsymbol{f} \in \dot{\mathcal{H}}^1(\mathbb{R}^5)$. We consider the distance with respect to the $\dot{\mathcal{H}}^1$ norm;

$$d(\boldsymbol{f}, \mathbf{M}_5) := \inf\left\{ \|\boldsymbol{f} - c\Gamma_{\boldsymbol{\alpha}} \boldsymbol{g}\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} \;\middle|\; \begin{array}{c} c \geqslant 0 \\ \boldsymbol{\alpha} \in \mathbb{S}^1 \times \mathbb{R}^7 \end{array} \right\}.$$

We can now state the theorem which we will prove in this section.

Theorem 2.4.1. *There is a positive constant C such that*

$$Cd(\boldsymbol{f}, \mathbf{M}_5)^2 \leqslant \mathcal{S}_5^2 \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+5})} \leqslant \mathcal{S}_5^2 d(\boldsymbol{f}, \mathbf{M}_5)^2.$$

The upper bound has already been proved, as it is a special case of Proposition 2.1.1. Before proceeding with the proof of the lower bound, we need a more precise description of spherical harmonics.

2.4.1 Some more spherical harmonics

Following [58, pp. 54], we introduce the *normalized associated Legendre functions* of degree $\ell \in \mathbb{N}_{\geqslant 0}$, order $m \in \mathbb{N}_{\geqslant 0}$ with $m \leqslant \ell$, and dimension $n \in \mathbb{N}_{\geqslant 3}$ to be the functions

$$A_\ell^m(n;t) := C_{m,n}(1-t^2)^{\frac{m}{2}} P_{\ell-m}(2m+n;t), \quad t \in [-1,1], \tag{2.39}$$

where $P_\ell(n;\cdot)$ is the *Legendre polynomial* of degree ℓ in dimension n. The normalization constant

$$C_{m,n} = \sqrt{\frac{(2\ell+n-2)(\ell+n-3)!}{\ell!(n-2)!} \frac{|\mathbb{S}^{2m+n-2}|}{|\mathbb{S}^{2m+n-1}|}}$$

is chosen to ensure that

$$\int_0^\pi A_\ell^m(n;\cos R) A_{\ell'}^m(n;\cos R)(\sin R)^{n-2}\, dR = \delta_{\ell,\ell'}.$$

Now we let $X = (X_0, X_1, \ldots, X_d)$ denote the Cartesian coordinates on $\mathbb{S}^d$. If Y_m^{d-1} is a normalized spherical harmonic on $\mathbb{S}^{d-1}$ of degree $m \leqslant \ell$, then

$$Y_\ell^d(X_0, X_1, \ldots, X_d) = A_\ell^m(d+1; X_0) Y_m^{d-1}(X_1, \ldots, X_d) \tag{2.40}$$

is a normalized spherical harmonic of degree ℓ on $\mathbb{S}^d$; see [58, Section 11]. Applying (2.40) iteratively, we construct an explicit complete system of spherical harmonics on $\mathbb{S}^d$, labeled by the degree $\ell \in \mathbb{N}_{\geqslant 0}$ and by the multi-index $\boldsymbol{m} \in \mathrm{M}(\ell)$, where

$$\mathrm{M}(\ell) = \left\{ (m_1, \ldots, m_{d-1}) \in \mathbb{Z}^{d-1} \;\middle|\; \ell \geqslant m_1 \geqslant \ldots \geqslant m_{d-2} \geqslant |m_{d-1}| \right\}.$$

The spherical harmonics $Y_{\ell,\boldsymbol{0}}^d$ with $\boldsymbol{m} = \boldsymbol{0}$ are the *zonal* ones; that is, the ones that depend on X_0 only. As before, we use the hat notation to denote the coefficients of expansions in spherical harmonics;

$$F(X) = \sum_{\ell=0}^{\infty} \sum_{\boldsymbol{m} \in \mathrm{M}(\ell)} \hat{F}(\ell, \boldsymbol{m}) Y_{\ell,\boldsymbol{m}}(X).$$

Now we want to describe the $\dot{\mathcal{H}}^1$ scalar product in terms of the Penrose transform. We will need the following coefficient, related to the Clebsch-Gordan theory associated to the unitary representations of $SO(d+1)$; see [73, pp. 489-491]. Instead of applying this general theory, we obtain the formula in Lemma 2.4.3 below with a simpler direct proof, based on the recursion relation for the Legendre polynomials; see [58, Lemma 3, pg. 39].

Definition 2.4.2. For all $\ell, m_1 \in \mathbb{Z}$

$$\mathrm{C}_d(\ell, m_1) := \begin{cases} \sqrt{\frac{(\ell - m_1 + 1)(\ell + m_1 + d - 1)}{(2\ell + d + 1)(2\ell + d - 1)}}, & 0 \leqslant m_1 \leqslant \ell, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4.3. *For all $\ell \in \mathbb{N}_{\geqslant 0}$ and $\boldsymbol{m}, \boldsymbol{m}' \in \mathrm{M}(\ell)$,*

$$\int_{\mathbb{S}^d} X_0 Y_{\ell,\boldsymbol{m}}(X) Y_{\ell',\boldsymbol{m}'}(X)\, dS = \begin{cases} 0, & \boldsymbol{m} \neq \boldsymbol{m}', \\ 0, & |\ell - \ell'| \neq 1, \\ \mathrm{C}_d(\ell, m_1), & \ell' = \ell + 1, \boldsymbol{m} = \boldsymbol{m}', \\ \mathrm{C}_d(\ell', m_1), & \ell = \ell' + 1,\ \boldsymbol{m} = \boldsymbol{m}'. \end{cases} \tag{2.41}$$

Proof. We assume, without loss of generality, that $\ell' \geqslant \ell$. We consider the normalized associated Legendre functions given by (2.39), which satisfy

$$\int_{-1}^{1} A_\ell^m(n; X_0) A_{\ell'}^m(n; X_0) (1 - X_0^2)^{\frac{n-3}{2}}\, dX_0 = \delta_{\ell,\ell'}. \tag{2.42}$$

We adopt the convention that $A_\ell^m(n; X_0) = 0$ if $m > \ell$. From the aforementioned recurrence relation for the Legendre polynomials we obtain

$$\begin{aligned} 0 = {} & a(n; \ell, m_1) A_\ell^{m_1}(n; X_0) - b(n; \ell, m_1) X_0 A_{\ell-1}^{m_1}(n; X_0) \\ & + c(n; \ell, m_1) A_{\ell-2}^{m_1}(n; X_0), \end{aligned} \tag{2.43}$$

with

$$a(n;\ell,m_1)=\sqrt{\tfrac{(\ell-m_1)(\ell+m_1+n-3)}{(2\ell+n-2)(\ell+m_1+n-4)}},\quad b(n;\ell,m_1)=\sqrt{\tfrac{2\ell+n-4}{\ell+m_1+n-4}},$$
$$c(n;\ell,m_1)=\sqrt{\tfrac{\ell-m_1-1}{2\ell+n-6}}.$$

Multiplying (2.43) by $A^{m_1}_{\ell'-1}(n;X_0)(1-X_0^2)^{\frac{n-3}{2}}$ and then integrating, we infer from (2.42) that, since $\ell'\geqslant\ell$,

$$\int_{-1}^{1}A^{m_1}_{\ell-1}(n;X_0)A^{m_1}_{\ell'-1}(n;X_0)X_0(1-X_0^2)^{\frac{n-3}{2}}\,dX_0=\frac{a(n;\ell,m_1)}{b(n;\ell,m_1)}\delta_{\ell,\ell'-1}. \tag{2.44}$$

We set $n=d+1$. Letting dS and dS^{d-1} denote the volume elements of $\mathbb{S}^d$ and $\mathbb{S}^{d-1}$ respectively, we have the formula

$$dS(X_0,X_1,\ldots,X_d)=(1-X_0^2)^{\frac{d-2}{2}}dX_0\,dS^{d-1}(X_1,\ldots,X_d). \tag{2.45}$$

The integral in (2.41) is computed using the representation (2.40) and the formulas (2.45) and (2.44). □

We rewrite the scalar product as

$$\langle \boldsymbol{f}\,|\,\boldsymbol{g}\rangle_{\dot{\mathcal{H}}^1(\mathbb{R}^d)}=\int_{\mathbb{R}^d}\sqrt{-\Delta}f_0\sqrt{-\Delta}g_0\,dx+\int_{\mathbb{R}^d}f_1g_1\,dx.$$

Now we recall that the Penrose transformation (F_0,F_1) of $\boldsymbol{f}$ is

$$f_0=\Omega_0^{\frac{d-1}{2}}F_0,\quad f_1=\Omega_0^{\frac{d+1}{2}}F_1; \tag{2.46}$$

see (1.11). Here $\Omega_0(x)=2(1+|x|^2)^{-1}$ is the conformal factor of the stereographic projection, which is implicit in (2.46), and whose equations we recall here;

$$X_0=\Omega_0-1,\quad X_j=\Omega_0x_j,\quad j=1,\ldots,d,$$

see Remark 1.1.1. As a special case of formula (1.17), we have that

$$\sqrt{-\Delta}f_0=\Omega_0^{\frac{d+1}{2}}A_1F_0,$$

where A_1 is the operator on $\mathbb{S}^d$ defined by

$$A_1Y_{\ell,\boldsymbol{m}}=\left(\ell+\frac{d-1}{2}\right)Y_{\ell,\boldsymbol{m}};$$

see (1.16). Since the Jacobian determinant of the stereographic projection is Ω_0^{-d} (see (1.18)), we have that

$$\langle \boldsymbol{f}\,|\,\boldsymbol{g}\rangle_{\dot{\mathcal{H}}^1(\mathbb{R}^d)}=\int_{\mathbb{S}^d}A_1(F_0)A_1(G_0)\;\Omega_0\,dS+\int_{\mathbb{S}^d}F_1G_1\;\Omega_0\,dS.$$

Using the formula $\Omega_0 = 1 + X_0$, we can use Lemma 2.4.3 to compute

$$\int_{\mathbb{S}^d} F_1 \; G_1 \; \Omega_0 \, dS = \sum_{\ell=0}^{\infty} \sum_{\boldsymbol{m}\in \mathrm{M}(\ell)} \hat{F}_1(\ell, \boldsymbol{m}) \hat{G}_1(\ell, \boldsymbol{m}) \tag{2.47}$$
$$+ \mathrm{C}_d(\ell, m_1) \left(\hat{F}_1(\ell, \boldsymbol{m}) \hat{G}_1(\ell+1, \boldsymbol{m}) + \hat{F}_1(\ell+1, \boldsymbol{m}) \hat{G}_1(\ell, \boldsymbol{m}) \right).$$

Similarly, $\int_{\mathbb{S}^d} A_1 F_0 \; A_1 G_0 \; \Omega_0 \, dS$ is equal to

$$\sum_{\substack{\ell \geqslant 0 \\ \boldsymbol{m}\in \mathrm{M}(\ell)}} \left(\ell + \frac{d-1}{2} \right)^2 \hat{F}_0(\ell, \boldsymbol{m}) \hat{G}_0(\ell, \boldsymbol{m}) + \mathrm{C}_d(\ell, m_1) \left(\ell + \frac{d-1}{2} \right) \times \tag{2.48}$$
$$\times \left(\ell + 1 + \frac{d-1}{2} \right) \left(\hat{F}_0(\ell, \boldsymbol{m}) \hat{G}_0(\ell+1, \boldsymbol{m}) + \hat{F}_0(\ell+1, \boldsymbol{m}) \hat{G}_0(\ell, \boldsymbol{m}) \right).$$

2.4.2 The tangent spaces

By the same geometrical considerations of Section 2.2, $\mathbf{M}_5 \backslash \{\mathbf{0}\}$ is a smooth 9-dimensional manifold, and the tangent space at $\boldsymbol{f}_\star$ is

$$T_{\boldsymbol{f}_\star} \mathbf{M}_5 = \operatorname{span} \left\{ \boldsymbol{f}_\star, \, \partial_{\alpha_i} \Gamma_{\boldsymbol{\alpha}} \boldsymbol{f}_\star |_{\boldsymbol{\alpha}=\mathbf{0}} \;\middle|\; i = 1, \ldots 8 \right\}.$$

The same computations as in the three-dimensional case yield the explicit expression of $T_{\boldsymbol{f}_\star} \mathbf{M}_5$; the result is given by the entries 1, 2, 3, 5 and 6 of Table 2.2, where, due to the change in scaling, the entry number 5 is replaced by the one given below, accounting for the change in the scaling symmetry. As in the previous subsection, here we systematically identify $x \in \mathbb{R}^5$ with $X = (X_0, X_1, \ldots, X_5) \in \mathbb{S}^5$ via the stereographic projection, whose conformal factor we denote by Ω_0.

	Generator	Applied to $\boldsymbol{f}_\star = (\Omega_0^2, 0)$
5	$\begin{bmatrix} \frac{3}{2} + x \cdot \nabla & 0 \\ 0 & \frac{5}{2} + x \cdot \nabla \end{bmatrix}$	$\begin{bmatrix} \frac{3}{2} \Omega_0^2 X_0 \\ 0 \end{bmatrix}$

We thus obtain

$$T_{\boldsymbol{f}_\star} \mathbf{M}_5 = \left\{ \begin{bmatrix} \Omega_0^2 \; (\sum_{j=0}^5 a_j X_j + a_6) \\ \Omega_0^3 \; (b_0 X_0 + b_1) \end{bmatrix} : a_j, b_j \in \mathbb{R} \right\},$$

that is, applying the Penrose transformation (2.28),

$$\boldsymbol{f} \in T_{\boldsymbol{f}_\star} \mathbf{M}_5 \quad \Longleftrightarrow \quad \begin{cases} \hat{F}_0(\ell, \boldsymbol{m}) = 0, & \ell \geqslant 2, \\ \hat{F}_1(\ell, \boldsymbol{m}) = 0, & \ell \geqslant 2 \text{ or } \ell = 1, \; \boldsymbol{m} \neq \mathbf{0}, \end{cases}$$

where we used the expression of the low-degree spherical harmonics; see Remark 1.1.4.

We now specialize the formula (2.48) for the $\dot{\mathcal{H}}^1$ scalar product from the previous subsection to the case $d = 5$. We obtain

$$\int_{\mathbb{R}^5} \nabla f_0 \cdot \nabla g_0 \, dx = \sum_{\ell=0}^{\infty} \sum_{\boldsymbol{m} \in \mathrm{M}(\ell)} (\ell+2)^2 \hat{F}_0(\ell, \boldsymbol{m}) \hat{G}_0(\ell, \boldsymbol{m}) \tag{2.49}$$
$$+ \mathrm{C}_5(\ell, m_1)(\ell+2)(\ell+3)\Big(\hat{F}_0(\ell, \boldsymbol{m})\hat{G}_0(\ell+1, \boldsymbol{m}) + \hat{F}_0(\ell+1, \boldsymbol{m})\hat{G}_0(\ell, \boldsymbol{m})\Big),$$

and, similarly, we obtain from (2.47)

$$\int_{\mathbb{R}^5} f_1 g_1 \, dx = \sum_{\ell=0}^{\infty} \sum_{\boldsymbol{m} \in \mathrm{M}(\ell)} \hat{F}_1(\ell, \boldsymbol{m}) \hat{G}_1(\ell, \boldsymbol{m}) \tag{2.50}$$
$$+ \mathrm{C}_5(\ell, m_1)\left(\hat{F}_1(\ell, \boldsymbol{m})\hat{G}_1(\ell+1, \boldsymbol{m}) + \hat{F}_1(\ell+1, \boldsymbol{m})\hat{G}_1(\ell, \boldsymbol{m})\right).$$

In these formulas,

$$\mathrm{C}_5(\ell, m_1) = \frac{1}{2}\sqrt{\frac{(\ell+1-m_1)(\ell+4+m_1)}{(\ell+2)(\ell+3)}}. \tag{2.51}$$

Remark 2.4.4. These formulas show that the $\dot{\mathcal{H}}^1$ scalar product is not diagonal in the coefficients $\hat{F}_0(\ell, \boldsymbol{m}), \hat{F}_1(\ell, \boldsymbol{m})$. Therefore, the orthogonality property $\boldsymbol{f} \perp_{\dot{\mathcal{H}}^1} T_{\boldsymbol{f}_\star}\mathbf{M}_5$ cannot be characterized in terms of the coefficients $\hat{F}_0(\ell, \boldsymbol{m}), \hat{F}_1(\ell, \boldsymbol{m})$ in a simple way. We define a different orthogonality condition as follows;

$$\boldsymbol{g} \,\tilde{\perp}\, T_{\boldsymbol{f}_\star}\mathbf{M}_5 \quad \Longleftrightarrow \quad \begin{cases} \hat{G}_0(\ell, \boldsymbol{m}) = 0, \\ \hat{G}_1(\ell, \boldsymbol{0}) = 0, \end{cases} \quad \ell = 0, \ell = 1, \, \boldsymbol{m} \in \mathrm{M}(\ell). \tag{2.52}$$

We will first prove a version of Lemma 2.3.1 with respect to this notion of orthogonality, from which we will deduce a similar lemma for functions which are orthogonal with respect to $\dot{\mathcal{H}}^1(\mathbb{R}^5)$.

2.4.3 Proof of Theorem 2.4.1

Bahouri and Gérard [9] proved a profile decomposition on $\dot{\mathcal{H}}^1$ and a version of Lemma 2.3.5 follows with the same proof. Thus it remains to prove the following local version of Theorem 2.4.1.

Proposition 2.4.5. *For all* $\boldsymbol{f} \in \dot{\mathcal{H}}^1(\mathbb{R}^5)$ *such that*

$$d(\boldsymbol{f}, \mathbf{M}_5) < \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)},$$

it holds that

$$\frac{18}{85}\mathcal{S}_5^2 d(\boldsymbol{f}, \mathbf{M}_5)^2 + O(d(\boldsymbol{f}, \mathbf{M}_5)^3) \leqslant \mathcal{S}_5^2 \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+5})}. \tag{2.53}$$

Proof. Following verbatim the proof of Proposition 2.3.3, we obtain that

$$\phi_5(\boldsymbol{f}) := \mathcal{S}_5^2 \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} - \|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+5})} = \phi_5(c\boldsymbol{f}_\star + \boldsymbol{f}_\perp),$$

where $c \neq 0$ and

$$\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} = d(\boldsymbol{f}, \mathbf{M}_5), \quad \text{and} \quad \boldsymbol{f}_\perp \perp_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} T_{\boldsymbol{f}_\star}\mathbf{M}_5. \tag{2.54}$$

The same computations give the expansion

$$\phi_5(c\boldsymbol{f}_\star + \boldsymbol{f}_\perp) = \frac{Q_5(\boldsymbol{f}_\perp)}{2\mathcal{S}_5^2 \|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}} + O(\|\boldsymbol{f}_\perp\|^3_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}), \tag{2.55}$$

where the quadratic functional Q_5 is defined for $\boldsymbol{f} = (f_0, f_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^5)$ as

$$\begin{aligned} Q_5(\boldsymbol{f}) = \mathcal{S}_5^4 &\left(4\langle \boldsymbol{f}_\star \mid \boldsymbol{f}\rangle^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} + 2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}\right) \\ &- 6 \iint_{\mathbb{R}^{1+5}} (S\boldsymbol{f}_\star)^2 (S\boldsymbol{f})^2 \, dtdx. \end{aligned}$$

With the same proof as in (2.27), we see that $Q_5(\boldsymbol{f}) = Q_5(f_0, 0) + Q_5(0, f_1)$.

We will compute, in the subsequent subsection, the following expressions, where (F_0, F_1) is the Penrose transformation (2.28) of $\boldsymbol{f}$;

$$Q_5(f_0, 0) = \frac{\pi}{8}\Big[\sum_{\ell=2}^{\infty} \sum_{\boldsymbol{m} \in \mathrm{M}(\ell)} \alpha_{\ell,\boldsymbol{m}} \hat{F}_0(\ell, \boldsymbol{m})^2 + \beta_{\ell,\boldsymbol{m}} \hat{F}_0(\ell+1, \boldsymbol{m}) \hat{F}_0(\ell, \boldsymbol{m})\Big], \tag{2.56}$$

and

$$\begin{aligned} Q_5(0, f_1) = \frac{\pi}{8}\Big[&\sum_{\boldsymbol{m} \in \mathrm{M}(1), m_1 = 1} 2\alpha_{1,\boldsymbol{m}} \frac{\hat{F}_1(1, \boldsymbol{m})^2}{9} \\ &+ \sum_{\ell=2}^{\infty} \sum_{\boldsymbol{m} \in \mathrm{M}(\ell)} \alpha_{\ell,\boldsymbol{m}} \frac{\hat{F}_1(\ell, \boldsymbol{m})^2}{(\ell+2)^2} + \beta_{\ell,\boldsymbol{m}} \frac{\hat{F}_1(\ell, \boldsymbol{m}) \hat{F}_1(\ell+1, \boldsymbol{m})}{(\ell+2)(\ell+3)}\Big], \end{aligned} \tag{2.57}$$

where the coefficients are given by

$$\begin{aligned} \alpha_{\ell,\boldsymbol{m}} &= \tfrac{\ell^4 + 8\ell^3 + 11\ell^2 - 20\ell - 12 + 6m_1^2 + 18m_1}{(\ell+1)(\ell+3)}, \\ \beta_{\ell,\boldsymbol{m}} &= 2(\ell-1)(\ell+6)\, \mathrm{C}_5(\ell, m_1), \end{aligned}$$

and $\mathrm{C}_5(\ell, m_1)$ is defined in (2.51).

It remains to bound Q_5 from below. We introduce the following linear algebra criterion, which is true independently of the dimension d.

Lemma 2.4.6 (Diagonal dominance)**.** *Let $L \in \mathbb{N}_{\geqslant 0}$ and let*

$$\{\, a_{\ell,\boldsymbol{m}}, b_{\ell,\boldsymbol{m}} : \ell \in \mathbb{N}_{\geqslant L},\ \boldsymbol{m} \in \mathrm{M}(\ell) \,\}$$

be real sequences satisfying

$$\begin{cases} a_{L,\boldsymbol{m}} \geqslant \frac{1}{2}|b_{L,\boldsymbol{m}}|, \\ a_{\ell,\boldsymbol{m}} \geqslant \frac{1}{2}\left(|b_{\ell,\boldsymbol{m}}| + |b_{\ell-1,\boldsymbol{m}}|\right), \quad \ell > L. \end{cases} \tag{2.58}$$

Here, and in the rest of the paper, we use the convention that $b_{\ell-1,\boldsymbol{m}} = 0$ *if* $\ell - 1 < m_1$. *If the quadratic functional* T *is defined by*

$$T(F) = \sum_{\ell=L}^{\infty} \sum_{\boldsymbol{m}\in \mathrm{M}(\ell)} a_{\ell,\boldsymbol{m}} \hat{F}(\ell,\boldsymbol{m})^2 + b_{\ell,\boldsymbol{m}} \hat{F}(\ell,\boldsymbol{m}) \hat{F}(\ell+1,\boldsymbol{m}),$$

then

$$T(F) \geqslant 0, \quad \forall\, F \in L^2(\mathbb{S}^d).$$

Proof. With the convention that $b_{\ell,\boldsymbol{m}} = 0$ if $\ell < L$ or $\ell < m_1$, we can bound $T(F)$ from below by

$$\begin{aligned} T(F) &\geqslant \sum_{\substack{\ell\geqslant L\\ \boldsymbol{m}\in\mathrm{M}(\ell)}} \frac{|b_{\ell,\boldsymbol{m}}|}{2} \hat{F}(\ell,\boldsymbol{m})^2 + \frac{|b_{\ell-1,\boldsymbol{m}}|}{2} \hat{F}(\ell,\boldsymbol{m})^2 + b_{\ell,\boldsymbol{m}} \hat{F}(\ell,\boldsymbol{m}) \hat{F}(\ell+1,\boldsymbol{m}) \\ &\geqslant \sum_{\substack{\ell\geqslant L\\ \boldsymbol{m}\in\mathrm{M}(\ell)}} \frac{1}{2} |b_{\ell,\boldsymbol{m}}| \left(\hat{F}(\ell,\boldsymbol{m}) + \mathrm{sign}(b_{\ell,\boldsymbol{m}}) \hat{F}(\ell+1,\boldsymbol{m})\right)^2 \geqslant 0. \end{aligned}$$

□

Lemma 2.4.7. *It holds that*

$$Q_5(\boldsymbol{g}) \geqslant \tfrac{9\pi}{340} \|\boldsymbol{g}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}, \quad \forall\, \boldsymbol{g} \,\tilde{\perp}\, T_{\boldsymbol{f}_\star} \mathbf{M}_5, \tag{2.59}$$

where the relation $\tilde{\perp}$ *has been defined in* (2.52).

Proof. We consider the term $Q_5(g_0, 0)$ first. Defining the quadratic functional

$$\begin{gathered} T \colon \left\{ \hat{G}_0(\ell,\boldsymbol{m}) = 0, \text{ for } \ell = 0, \ell = 1, \ \boldsymbol{m} \in \mathrm{M}(\ell) \right\} \to \mathbb{R}, \\ T(g_0) := Q_5(g_0, 0) - \frac{9\pi}{340} \int_{\mathbb{R}^5} |\nabla g_0|^2 \, dx, \end{gathered}$$

it will suffice to show that T satisfies the conditions of Lemma 2.4.6. We perform the change of variable

$$\hat{G}_0(\ell,\boldsymbol{m}) = \frac{\hat{H}(\ell,\boldsymbol{m})}{\sqrt{(\ell+1)(\ell+3)}}, \tag{2.60}$$

so that, using (2.56) and (2.49), we have

$$T(H) = \sum_{\ell=2}^{\infty} \sum_{\boldsymbol{m}\in\mathrm{M}(\ell)} a_{\ell,\boldsymbol{m}} \hat{H}(\ell,\boldsymbol{m})^2 + b_{\ell,\boldsymbol{m}} \hat{H}(\ell,\boldsymbol{m}) \hat{H}(\ell+1,\boldsymbol{m}),$$

where

$$\begin{aligned} a_{\ell,\boldsymbol{m}} &= \tfrac{\pi}{8}\tfrac{\ell^4+8\ell^3+11\ell^2-20\ell-12+6m_1^2+18m_1}{(\ell+1)^2(\ell+3)^2} - \tfrac{9\pi}{340}\tfrac{(\ell+2)^2}{(\ell+1)(\ell+3)}, \\ b_{\ell,\boldsymbol{m}} &= \sqrt{\tfrac{(\ell+1-m_1)(\ell+4+m_1)}{(\ell+1)(\ell+4)}}\left(\tfrac{\pi}{8}\tfrac{(\ell-1)(\ell+6)}{(\ell+2)(\ell+3)} - \tfrac{9\pi}{340}\right). \end{aligned} \tag{2.61}$$

Notice that $b_{\ell,\boldsymbol{0}}$ is a rational function: the change of variable (2.60) was chosen to obtain this. Note also that $a_{\ell,\boldsymbol{m}} \geqslant a_{\ell,\boldsymbol{0}}$ and that we also have $b_{\ell,\boldsymbol{m}} \geqslant 0$ for $\ell \geqslant 2$, so that $b_{\ell,\boldsymbol{m}} \leqslant b_{\ell,\boldsymbol{0}}$. Now

$$a_{2,\boldsymbol{m}} - \frac{1}{2}b_{2,\boldsymbol{m}} \geqslant a_{2,\boldsymbol{0}} - \frac{1}{2}b_{2,\boldsymbol{0}} = \frac{3\pi}{200} - \frac{17}{30}\frac{9\pi}{340} = 0, \tag{2.62}$$

and, for $\ell > 2$, we have that

$$\begin{aligned} a_{\ell,\boldsymbol{m}} - \frac{1}{2}\left(b_{\ell,\boldsymbol{m}} + b_{\ell-1,\boldsymbol{m}}\right) &\geqslant a_{\ell,\boldsymbol{0}} - \frac{1}{2}\left(b_{\ell,\boldsymbol{0}} + b_{\ell-1,\boldsymbol{0}}\right) \\ &= \frac{\pi}{8}\frac{\ell^2+4\ell+15}{(\ell+1)^2(\ell+3)^2} - \frac{1}{(\ell+1)(\ell+3)}\frac{9\pi}{340} > 0. \end{aligned} \tag{2.63}$$

So the conditions (2.58) of Lemma 2.4.6 are satisfied and we can conclude that

$$Q_5(g_0, 0) \geqslant \frac{9\pi}{340}\int_{\mathbb{R}^5} |\nabla g_0|^2\, dx, \quad \text{if } \hat{G}_0(\ell, \boldsymbol{m}) = 0,\ \ell = 0, 1,\ \boldsymbol{m} \in \mathrm{M}(\ell).$$

To prove the analogous inequality for $Q_5(0, g_1)$ we consider the quadratic functional

$$\begin{aligned} T\colon \left\{ \hat{G}_1(\ell, \boldsymbol{0}) = 0, \text{ for } \ell = 0,\ \ell = 1 \right\} &\to \mathbb{R} \\ T(g_1) := Q_5(0, g_1) - \frac{9\pi}{340}\int_{\mathbb{R}^5} g_1^2\, dx, \end{aligned}$$

We perform the change of variable

$$\hat{G}_1(\ell, \boldsymbol{m}) = \frac{\hat{H}(\ell, \boldsymbol{m})(\ell+2)}{\sqrt{(\ell+1)(\ell+3)}},$$

so that, by (2.57) and (2.50),

$$\begin{aligned} T(H) = &\sum_{\boldsymbol{m}\in\mathrm{M}(1), m_1 = 1} a_{1,\boldsymbol{m}}\hat{H}(1,\boldsymbol{m})^2 + b_{1,\boldsymbol{m}}\hat{H}(1,\boldsymbol{m})\hat{H}(2,\boldsymbol{m}) \\ &+ \sum_{\ell=2}^{\infty}\sum_{\boldsymbol{m}\in\mathrm{M}(\ell)} a_{\ell,\boldsymbol{m}}\hat{H}(\ell,\boldsymbol{m})^2 + b_{\ell,\boldsymbol{m}}\hat{H}(\ell,\boldsymbol{m})\hat{H}(\ell+1,\boldsymbol{m}), \end{aligned}$$

where $a_{1,\boldsymbol{m}} = \frac{3\pi}{64} - \frac{9\pi}{340}\frac{9}{8}$, $b_{1,\boldsymbol{m}} = -\frac{9\pi}{340}\sqrt{\frac{3}{5}}$, and $a_{\ell,\boldsymbol{m}}$ and $b_{\ell,\boldsymbol{m}}$ equal (2.61) for $\ell \geqslant 2$. For $\ell = 1, 2$ and $m_1 = 1$ we have that

$$\begin{aligned} a_{1,\boldsymbol{m}} - \tfrac{1}{2}|b_{1,\boldsymbol{m}}| &= \tfrac{93}{5440}\,\pi - \tfrac{9}{3400}\,\pi\,\sqrt{15} > 0, \\ a_{2,\boldsymbol{m}} - \tfrac{1}{2}\left(|b_{2,\boldsymbol{m}}| + |b_{1,\boldsymbol{m}}|\right) &= \left(\tfrac{32}{1275} - \tfrac{1}{255}\sqrt{7} - \tfrac{9}{3400}\sqrt{15}\right)\pi > 0. \end{aligned}$$

For all other values of ℓ and $\boldsymbol{m}$, the assumptions of Lemma 2.4.6 have already been verified; see (2.62) for the $\ell = 2, m_1 = 2$ case (recall that, by convention, $b_{1,\boldsymbol{m}} = 0$ if $m_1 > 1$), and (2.63) for all the other cases. Since $Q_5(\boldsymbol{g}) = Q_5(g_0, 0) + Q_5(0, g_1)$, the proof of (2.59) is complete. $\square$

We want to apply Lemma 2.4.7 to $Q_5(\boldsymbol{f}_\perp)$, where $\boldsymbol{f}_\perp$ satisfies the property (2.54). To do so, we decompose $\boldsymbol{f}_\perp$ into a sum:

$$\boldsymbol{f}_\perp = \boldsymbol{g} + \boldsymbol{h}, \quad \text{where} \quad \begin{matrix} \boldsymbol{h} \in T_{\boldsymbol{f}_\star}\mathbf{M}_5 \\ \boldsymbol{g} \,\widetilde{\perp}\, T_{\boldsymbol{f}_\star}\mathbf{M}_5. \end{matrix}$$

We consider the unique bilinear functional $B_5 \colon \dot{\mathcal{H}}^1(\mathbb{R}^5) \times \dot{\mathcal{H}}^1(\mathbb{R}^5) \to \mathbb{R}$ that satisfies $Q_5(\boldsymbol{f}) = B_5(\boldsymbol{f}, \boldsymbol{f})$. By the Cauchy-Schwarz inequality we have that

$$B_5(\boldsymbol{g}, \boldsymbol{h})^2 \leqslant Q_5(\boldsymbol{g}) Q_5(\boldsymbol{h}) = 0,$$

where we used that $Q_5(\boldsymbol{h}) = 0$. Therefore

$$Q_5(\boldsymbol{f}_\perp) = Q_5(\boldsymbol{g}) + Q_5(\boldsymbol{h}) + 2B_5(\boldsymbol{g}, \boldsymbol{h}) = Q_5(\boldsymbol{g}).$$

Then by Lemma 2.4.7, combined with $\boldsymbol{g} = \boldsymbol{f}_\perp - \boldsymbol{h}$ and $\boldsymbol{f}_\perp \perp_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} \boldsymbol{h}$,

$$\begin{aligned} Q_5(\boldsymbol{f}_\perp) &\geqslant \frac{9\pi}{340} \|\boldsymbol{g}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} = \frac{9\pi}{340} \left(\|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} + \|\boldsymbol{h}\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} \right) \\ &\geqslant \frac{9\pi}{340} \|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}. \end{aligned} \tag{2.64}$$

We conclude by inserting (2.64) into (2.55), thus yielding the lower bound (2.53) with constant

$$\frac{9\pi}{340} \frac{1}{2\mathcal{S}_5^2 \|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}} = \frac{9}{340\pi} = \frac{18}{85} \mathcal{S}_5^2,$$

where we have used that $\mathcal{S}_5^2 = (8\pi)^{-1}$ and that $\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} = 4|\mathbb{S}^5| = 4\pi^3$. This last identity follows from the representation (2.49) of the norm and from the fact that $F_{\star 0} = 1 = \sqrt{|\mathbb{S}^5|} Y_{0,\mathbf{0}}$ and $F_{\star 1} = 0$; see (1.12). $\square$

2.4.4 Computation of Q_5

Here, $\boldsymbol{g} \in \dot{\mathcal{H}}^1(\mathbb{R}^5)$ and (G_0, G_1) are related through the Penrose transformation (2.28). We recall that $\mathcal{S}_5 = \left(\frac{1}{8\pi}\right)^{1/2}$. We consider the quadratic functional

$$\begin{aligned} Q_5(g_0, 0) = \mathcal{S}_5^4 &\left(4 \langle f_{\star 0} \,|\, g_0 \rangle^2_{\dot{H}^1(\mathbb{R}^5)} + 2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)} \|g_0\|^2_{\dot{H}^1(\mathbb{R}^5)} \right) \\ &- 6 \iint\limits_{\mathbb{R}^{1+5}} (S\boldsymbol{f}_\star)^2 \left(\cos t\sqrt{-\Delta} g_0 \right)^2 \, dx. \end{aligned} \tag{2.65}$$

The Penrose transform of $\boldsymbol{f}_\star$ is $F_{\star 0} = 1, F_{\star 1} = 0$; see (1.12). By the formula (2.49) for the $\dot{\mathcal{H}}^1(\mathbb{R}^5)$ scalar product, we obtain

$$\mathcal{S}_5^4\left(4\langle f_{\star 0}\,|\,g_0\rangle^2_{\dot{H}^1(\mathbb{R}^5)} + 2\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^1(\mathbb{R}^5)}\|g_0\|^2_{\dot{H}^1(\mathbb{R}^5)}\right) =$$

$$\frac{3\pi}{2}\hat{G}_0(0,\mathbf{0})^2 + \frac{3\pi\sqrt{6}}{4}\hat{G}_0(0,\mathbf{0})\hat{G}_0(1,\mathbf{0}) + \frac{3\pi}{2}\hat{G}_0(1,\mathbf{0})^2 + \frac{9\pi}{8}\sum_{\mathbf{0}\neq\boldsymbol{m}\in\mathrm{M}(1)}\hat{G}_0(1,\boldsymbol{m})^2$$

$$+\frac{\pi}{8}\sum_{\substack{\ell\geqslant 2\\ \boldsymbol{m}\in\mathrm{M}(\ell)}}(\ell+2)^2\hat{G}_0(\ell,\boldsymbol{m})^2 + 2(\ell+2)(\ell+3)\,\mathrm{C}_5(\ell,m_1)\hat{G}_0(\ell,\boldsymbol{m})\hat{G}_0(\ell+1,\boldsymbol{m}).$$

Using Corollary 1.2.3, we compute the spacetime integral;

$$6\iint_{\mathbb{R}^{1+5}}(S\boldsymbol{f}_\star)^2\,(S(g_0,0))^2\,dtdx =$$

$$3\iint_{\mathbb{S}^1\times\mathbb{S}^5}\Big[\cos(2T)(\cos T + X_0)\sum_{\substack{\ell\geqslant 0\\ \boldsymbol{m}\in\mathrm{M}(\ell)}}\cos((2+\ell)T)\hat{G}_0(\ell,\boldsymbol{m})Y_{\ell,\boldsymbol{m}}(X)\Big]^2\,dTdS.$$

Here we used that the Penrose transform of $v_\star = S\boldsymbol{f}_\star$ is $V_\star = \cos(2T)$. Now we notice that, with the convention that $Y_{\ell,\boldsymbol{m}} = 0$ if $\ell < 0$ or $\ell < m_1$, formula (2.41) implies

$$\begin{aligned}(\cos T + X_0)Y_{\ell,\boldsymbol{m}} &= \cos(T)Y_{\ell,\boldsymbol{m}} + \mathrm{C}_5(\ell-1,m_1)Y_{\ell-1,\boldsymbol{m}}\\ &\quad + \mathrm{C}_5(\ell,m_1)Y_{\ell+1,\boldsymbol{m}}.\end{aligned}$$

Combining this with the $L^2(\mathbb{S}^5)$ orthonormality of the spherical harmonics $Y_{\ell,\boldsymbol{m}}$, we obtain that the integral $6\iint(S\boldsymbol{f}_\star)^2\,(S(g_0,0))^2$ equals

$$\frac{3\pi}{2}\hat{G}_0(0,\mathbf{0})^2 + \frac{3\sqrt{6}\pi}{4}\hat{G}_0(0,\mathbf{0})\hat{G}_0(1,\mathbf{0}) + \frac{3\pi}{2}\hat{G}_0(1,\mathbf{0})^2 + \frac{9\pi}{8}\sum_{\mathbf{0}\neq\boldsymbol{m}\in\mathrm{M}(1)}\hat{G}_0(1,\boldsymbol{m})^2$$

$$+\frac{3\pi}{2}\sum_{\substack{\ell\geqslant 2\\ \boldsymbol{m}\in\mathrm{M}(\ell)}}\frac{2\ell^2+8\ell+m_1^2-3m_1-4}{2(\ell+1)(\ell+3)}\hat{G}_0(\ell,\boldsymbol{m})^2 + 2\mathrm{C}_5(\ell,m_1)\hat{G}_0(\ell,\boldsymbol{m})\hat{G}_0(\ell+1,\boldsymbol{m}).$$

Inserting these formulas into (2.65) yields formula (2.56) of the previous subsection. The proof of formula (2.57) for the functional $Q(0,g_1)$ is analogous.

Chapter 3

Maximizers for the cubic wave equation

Here we consider real-valued, global solutions u to the cubic equation

$$\Box u = \sigma u^3, \quad \text{on } \mathbb{R}^{1+3}, \tag{NLW}$$

where $\sigma \neq 0$. This equation is locally well-posed in $\dot{\mathcal{H}}^{1/2}$, and small solutions are global; see Section 3.1. We consider

$$I(\delta) = \sup \left\{ \|u\|^4_{L^4(\mathbb{R}^{1+3})} \;\middle|\; \lim_{t\to-\infty} \|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} \leqslant \delta \right\}, \tag{3.1}$$

which is manifestly invariant under translations in time, and we will prove in Section 3.2 that this is also invariant under Lorentzian boosts.

Our main concern thereafter, will be the proof of the following sharp asymptotic estimate.

Theorem 3.0.1. *Let $\delta > 0$ be sufficiently small. Then the supremum in* (3.1) *is attained and*

$$I(\delta) = \mathcal{C}_0\delta^4 + \sigma\mathcal{C}_1\delta^6 + O(\delta^8), \tag{3.2}$$

as $\delta \to 0$, where $\mathcal{C}_0 = \frac{3}{16\pi}$ and

$$\mathcal{C}_1 = \begin{cases} \frac{29}{2^{10}\pi^3}, & \sigma > 0 \quad (\textit{focusing}), \\ \frac{5}{2^{10}\pi^3}, & \sigma < 0 \quad (\textit{defocusing}). \end{cases}$$

Here, $\mathcal{C}_0$ denotes the sharp constant in the Strichartz estimate

$$\|S\boldsymbol{f}\|^4_{L^4(\mathbb{R}^{1+3})} \leqslant \mathcal{C}_0\|\boldsymbol{f}\|^4_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)}. \tag{3.3}$$

With the notation of the previous chapters, $\mathcal{C}_0 = \mathcal{S}^4$.

The outline of this chapter is the following. After having given the precise definition of *solution* in the first section, we will discuss the aforementioned Lorentzian invariance.

Then we will proceed to establish formula (3.2) via an adaptation of the argument of Duyckaerts, Merle and Roudenko [31]. We will use the Penrose transform to calculate the constant $\mathcal{C}_1$. In Section 3.5, we will prove the existence of maximizers using a standard argument based on a nonlinear profile decomposition, which will be be proved in the appendix. Finally, we give a partial result concerning the uniqueness of these maximizers.

In the appendix, we also use the Penrose transform to produce explicit solutions to focusing (NLW), one which is global and another which blows up in finite time. Finally, we prove the existence of solutions to (NLW) for which the norm $\|\boldsymbol{u}(t_0)\|_{\dot{\mathcal{H}}^{1/2}}$, at any $t_0 \in \mathbb{R}$, is neither conserved in time nor invariant under Lorentzian boosts. This explains the necessity to consider the limit as $t \to -\infty$ in the definition (3.1).

3.1 Preliminaries

We give the definition of a *solution* to (NLW). Here we will consider only global solutions which scatter to linear solutions as $t \to -\infty$. The following operator is adapted to this.

Definition 3.1.1. For $F \in L^{4/3}(\mathbb{R}^{1+3})$, we define

$$\Box^{-1} F(t, \cdot) = \int_{-\infty}^{t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (F(s, \cdot))\, ds. \tag{3.4}$$

This is well-defined because of the inhomogeneous Strichartz estimate, which follows by a standard duality argument from the Strichartz estimate previously considered; see, for example, [41, Lemma 2.3].

Proposition 3.1.2. *Let $F \in L^{4/3}(\mathbb{R}^{1+3})$ and $w = \Box^{-1} F$. Then*

$$\|w\|_{L^4(\mathbb{R}^{1+3})} + \sup_{t \in \mathbb{R}} \|\boldsymbol{w}(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant C \|F\|_{L^{4/3}(\mathbb{R}^{1+3})}. \tag{3.5}$$

Moreover, the map

$$t \in \mathbb{R} \mapsto \boldsymbol{w}(t) \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$$

is continuous.

Remark 3.1.3. Replacing F with $F\mathbf{1}_{\{t<T\}}$, we immediately see that the following estimate also holds;

$$\|w\|_{L^4((-\infty,T)\times\mathbb{R}^3)} + \sup_{t \leqslant T} \|\boldsymbol{w}(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant C \|F\|_{L^{4/3}((-\infty,T)\times\mathbb{R}^3)}, \quad \forall T \in \mathbb{R}.$$

With this we obtain existence and uniqueness of small solutions by a standard application of the fixed-point theorem.

Proposition 3.1.4. *There exists $\delta > 0$ such that, if $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} \leqslant \delta$, then there exists a unique solution u to* (NLW) *that satisfies the condition*

$$\lim_{t \to -\infty} \|\boldsymbol{u}(t) - S\boldsymbol{f}(t)\|_{\dot{\mathcal{H}}^{1/2}} = 0,$$

which we define as the fixed point of the mapping

$$w \mapsto S\boldsymbol{f} + \sigma\square^{-1}(w^3),$$

in the space $L^4(\mathbb{R}^{1+3}) \cap C(\mathbb{R}; \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3))$. *Moreover, the nonlinear operator*

$$\Phi : \boldsymbol{f} \mapsto u$$

is locally bounded on $\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$, *in the sense that*

$$\|\Phi(\boldsymbol{f})\|_{L^4(\mathbb{R}^{1+3})} + \sup_{t\in\mathbb{R}} \|\Phi(\boldsymbol{f})(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant C_\delta \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}}. \tag{3.6}$$

Thus we see that $I(\delta)$ is finite for small enough values of $\delta > 0$.

Remark 3.1.5. The nonlinear operator Φ is also differentiable for $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} < \delta$. We denote its directional derivative by

$$\Phi'(\boldsymbol{f})\boldsymbol{g} := \frac{d}{d\varepsilon}\Phi(\boldsymbol{f} + \varepsilon\boldsymbol{g})\Big|_{\varepsilon=0}, \quad \forall \boldsymbol{g} \in \dot{\mathcal{H}}^{1/2}.$$

3.2 Lorentzian invariance

For all $\alpha \in (-1, 1)$ we define a linear transformation of $\mathbb{R}^{1+3}$ as

$$L^\alpha(\tau, \xi_1, \xi_2, \xi_3) = \begin{bmatrix} \gamma & -\gamma\alpha & 0 & 0 \\ -\gamma\alpha & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix},$$

where $\gamma := (1 - \alpha^2)^{-1/2}$. Clearly, $\det L^\alpha = 1$ and $(L^\alpha)^{-1} = L^{-\alpha}$; moreover, for all $(t, x), (\tau, \xi) \in \mathbb{R}^{1+3}$,

$$L^\alpha(\tau, \xi) \cdot (t, x) = (\tau, \xi) \cdot L^\alpha(t, x).$$

Denoting $(\tilde{\tau}, \tilde{\xi}) = L^\alpha(\tau, \xi)$ we also have the fundamental property

$$\tau^2 - |\xi|^2 = \tilde{\tau}^2 - |\tilde{\xi}|^2,$$

from which it descends that, if $\tau = |\xi|$, then $\tilde{\tau} = |\tilde{\xi}|$; to see this, note that $\tilde{\tau}^2 = |\tilde{\xi}|^2$, and $\tilde{\tau} = \gamma|\xi| - \gamma\alpha\xi_1 \geqslant 0$. Analogously, if $\tau = -|\xi|$ then $\tilde{\tau} = -|\tilde{\xi}|$.

We also have the Dirac delta identity

$$2\delta(\tau^2 - |\xi|^2)\mathbf{1}_{\{\pm\tau>0\}} = \frac{\delta(\tau \mp |\xi|)}{|\xi|};$$

see, for example, [37]. By the previous considerations, the left-hand side is Lorentz-invariant, and so

$$\frac{\delta(\tau \mp |\xi|)}{|\xi|} = \delta(\tau^2 - |\xi|^2)\mathbf{1}_{\{\pm\tau>0\}} = \delta(\tilde{\tau}^2 - |\tilde{\xi}|^2)\mathbf{1}_{\{\pm\tilde{\tau}>0\}} = \frac{\delta(\tilde{\tau} \mp |\tilde{\xi}|)}{|\tilde{\xi}|},$$

which implies the integration formula

$$\int_{\mathbb{R}^3} F(L^\alpha(\pm|\xi|,\xi))G(\pm|\xi|,\xi)\frac{d\xi}{|\xi|} = \int_{\mathbb{R}^3} F(\pm|\tilde{\xi}|,\tilde{\xi})G(L^{-\alpha}(\pm|\tilde{\xi}|,\tilde{\xi}))\frac{d\tilde{\xi}}{|\tilde{\xi}|}.$$

We will now prove that $\Box^{-1}$ commutes with L^α. It is for this reason that we defined $\Box^{-1}$ as an integral over $(-\infty,t)$ rather that $(0,t)$. Ramos considered the operator as an integral over $(0,t)$, but in that case the operators do not commute precisely; see [63, Proposition 1].

Lemma 3.2.1. *Let $F \in L^{4/3}(\mathbb{R}^{1+3})$. Then, for all $\alpha \in (-1,1)$,*

$$\Box^{-1}(F \circ L^\alpha) = (\Box^{-1}F) \circ L^\alpha.$$

Proof. By the definition (3.4) and Fubini's theorem, $\Box^{-1}(F \circ L^\alpha)(t,x)$ can be written as

$$\iiint \frac{\sin((t-s)|\xi|)}{|\xi|} e^{i(x-y)\cdot\xi} F(L^\alpha(s,y))\mathbf{1}_{\{s<t\}}\, dsdy\frac{d\xi}{|\xi|},$$

modulo irrelevant factors of $(2\pi)^{-3}$. On the other hand, we divide the operator

$$\Box^{-1} = \Box_+^{-1} - \Box_-^{-1},$$

where, for an arbitrary $H \in L^{4/3}(\mathbb{R}^{1+3})$,

$$\Box_\pm^{-1}H(t,x) := \iiint \frac{e^{i(t,x)\cdot(\pm|\xi|,\xi)-i(s,y)\cdot(\pm|\xi|,\xi)}}{2i} H(s,y)\mathbf{1}_{\{s<t\}}\, dsdy\frac{d\xi}{|\xi|}.$$

We compute a convenient expression for $(\Box_\pm^{-1}F)(L^\alpha(t,x))$ using the properties of L^α that we recalled in the beginning of the section;

$$\begin{aligned}
&\iiint \frac{e^{iL^\alpha(t,x)\cdot(\pm|\xi|,\xi)-i(s,y)\cdot(\pm|\xi|,\xi)}}{2i} F(s,y)\mathbf{1}_{\{s<\gamma t-\gamma\alpha x_1\}}\, dsdy\frac{d\xi}{|\xi|}\\
=&\iiint \frac{e^{i(t,x)\cdot L^\alpha(\pm|\xi|,\xi)-i(s,y)\cdot(\pm|\xi|,\xi)}}{2i} F(s,y)\mathbf{1}_{\{s<\gamma t-\gamma\alpha x_1\}}\, dsdy\frac{d\xi}{|\xi|}\\
=&\iiint \frac{e^{i(t,x)\cdot(\pm|\xi|,\xi)-i(s,y)\cdot L^{-\alpha}(\pm|\xi|,\xi)}}{2i} F(s,y)\mathbf{1}_{\{s<\gamma t-\gamma\alpha x_1\}}\, dsdy\frac{d\xi}{|\xi|}\\
=&\iiint \frac{e^{i(t,x)\cdot(\pm|\xi|,\xi)-iL^{-\alpha}(s,y)\cdot(\pm|\xi|,\xi)}}{2i} F(s,y)\mathbf{1}_{\{s<\gamma t-\gamma\alpha x_1\}}\, dsdy\frac{d\xi}{|\xi|}\\
=&\iiint \frac{e^{i(t,x)\cdot(\pm|\xi|,\xi)-i(s,y)\cdot(\pm|\xi|,\xi)}}{2i} F(L^\alpha(s,y))\mathbf{1}_{\{\gamma s-\gamma\alpha y_1<\gamma t-\gamma\alpha x_1\}}\, dsdy\frac{d\xi}{|\xi|}.
\end{aligned}$$

We conclude that $(\Box^{-1}F)(L^\alpha(t,x))$ is equal to

$$\iiint \frac{\sin((t-s)|\xi|)}{|\xi|} e^{i(x-y)\cdot\xi} F(L^\alpha(s,y))\mathbf{1}_{\{s<t-\alpha(x_1-y_1)\}}\, dsdy\frac{d\xi}{|\xi|}.$$

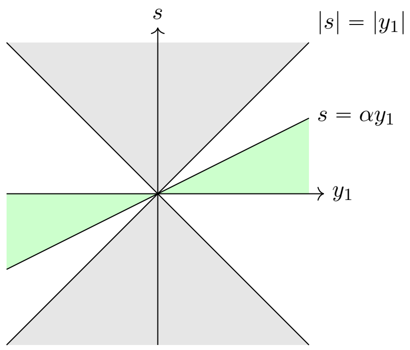

Figure 3.1: The support of $\mathbf{1}_{\{s<\alpha y_1\}} - \mathbf{1}_{\{s<0\}}$ (green) intersects the light cone (gray) only at the origin.

Using these two expressions, the difference $\square^{-1}(F \circ L^\alpha) - (\square^{-1}F) \circ L^\alpha$ can be written as

$$\iiint \frac{\sin(s|\xi|)}{|\xi|} e^{-iy\cdot\xi} G(s,y) \left(\mathbf{1}_{\{s<\alpha y_1\}} - \mathbf{1}_{\{s<0\}}\right) \, dsdyd\xi, \tag{3.7}$$

where $G(s,y) := F(L^\alpha(s+t, y+x))$. We now note that the distribution v, defined by the formal integral

$$v(s,y) := \int_{\mathbb{R}^3} \frac{\sin(s|\xi|)}{|\xi|} e^{-iy\cdot\xi} \, d\xi,$$

is a fundamental solution to the wave equation, that is,

$$\begin{cases} \square v = 0, & \text{on } \mathbb{R}^{1+3}, \\ \boldsymbol{v}(0) = (0, \delta), \end{cases}$$

where δ is the Dirac distribution. Therefore, v is supported in the cone $\{|y|^2 \leqslant s^2\}$, which intersects the support of $\mathbf{1}_{\{s<\alpha y_1\}} - \mathbf{1}_{\{s<0\}}$ only at the origin (recalling that $|\alpha| < 1$); see Figure 3.1. Thus the integral (3.7) vanishes, completing the proof. $\square$

Corollary 3.2.2. *Let $\alpha \in (-1, 1)$, let $F \in L^{4/3}(\mathbb{R}^{1+3})$, and let*

$$w_\alpha = \square^{-1} F \circ L^\alpha.$$

Then the map $t \in \mathbb{R} \mapsto \boldsymbol{w}_\alpha(t) \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$ is continuous.

The full symmetry group of solutions to (NLW) that we consider consists of Lorentzian boosts, dilations and spacetime translations. The Lorentzian boost of velocity $\beta \in \mathbb{R}^3$, with $|\beta| < 1$, is defined by

$$L^\beta(\tau, \xi) = R^{-1} \circ L^\alpha \circ R(\tau, \xi), \quad \alpha = |\beta|,$$

where $R(\tau, \xi) = (\tau, R'\xi)$, and R' is a rotation that maps $(1, 0, 0)$ to $\beta/|\beta|$. By convention we assume that $L^{(0,0,0)}$ is the identity. We denote

$$\Lambda(t, x) = L^\beta\big(\lambda(t - t_0), \lambda(x - x_0)\big),$$

where $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^3, \lambda > 0$ and $\beta \in \mathbb{R}^3$, with $|\beta| < 1$; note that Lemma 3.2.1 readily implies that, for all $F \in L^{4/3}(\mathbb{R}^{1+3})$,

$$\Box^{-1}(F \circ \Lambda) = \lambda^{-2}(\Box^{-1}F) \circ \Lambda. \tag{3.8}$$

It is well-known that these transformations act unitarily on solutions to the linear wave equation with data in $\dot{\mathcal{H}}^{1/2}$, as in the following lemma, whose proof is an immediate consequence of Proposition 2.2.3, in the previous chapter.

Lemma 3.2.3. *Let $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$. There exists a unique $\boldsymbol{f}_\Lambda \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$ such that*

$$\lambda S\boldsymbol{f}(\Lambda(t, x)) = S\boldsymbol{f}_\Lambda(t, x). \tag{3.9}$$

Moreover, $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{f}_\Lambda\|_{\dot{\mathcal{H}}^{1/2}}$.

The transformation Λ also maps smooth solutions of (NLW) to smooth solutions. Using Lemma 3.2.1, we can now describe the action of Λ on the class of solutions that we defined in Proposition 3.1.4.

Theorem 3.2.4. *Let $u \in L^4(\mathbb{R}^{1+3})$, with $\boldsymbol{u} \in C(\mathbb{R}; \dot{\mathcal{H}}^{1/2})$, satisfy the fixed point equation $u = S\boldsymbol{f} + \sigma\Box^{-1}(u^3)$. Denote*

$$u_\Lambda(t, x) = \lambda u(\Lambda(t, x)).$$

Then $u_\Lambda \in L^4(\mathbb{R}^{1+3})$, with $\|u_\Lambda\|_{L^4} = \|u\|_{L^4}$, $\boldsymbol{u}_\Lambda \in C(\mathbb{R}; \dot{\mathcal{H}}^{1/2})$ and

$$u_\Lambda = S\boldsymbol{f}_\Lambda + \sigma\Box^{-1}(u_\Lambda^3), \tag{3.10}$$

where $\boldsymbol{f}_\Lambda$ is defined in (3.9)*; in particular,*

$$\lim_{t \to -\infty} \|\boldsymbol{u}_\Lambda(t)\|_{\dot{\mathcal{H}}^{1/2}} = \lim_{t \to -\infty} \|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}}.$$

Proof. Using (3.8), we obtain from $u = S\boldsymbol{f} + \sigma\Box^{-1}(u^3)$ that

$$\begin{aligned} \lambda u \circ \Lambda &= \lambda(S\boldsymbol{f}) \circ \Lambda + \lambda\sigma\Box^{-1}(u^3) \circ \Lambda \\ &= S\boldsymbol{f}_\Lambda + \sigma\Box^{-1}(u_\Lambda^3), \end{aligned}$$

which proves (3.10). The fact that $\boldsymbol{u}_\Lambda \in C(\mathbb{R}; \dot{\mathcal{H}}^{1/2})$ follows from Corollary 3.2.2. $\Box$

3.3 The asymptotic formula

We recall the sharpened Strichartz estimate which we proved in the previous chapter.

Lemma 3.3.1. *Let $\mathcal{C}_0 = \frac{3}{16\pi}$. Then there is a constant $c > 0$ such that*

$$\|S\boldsymbol{f}\|^2_{L^4(\mathbb{R}^{1+3})} + cd(\boldsymbol{f}, \mathbf{M})^2 \leqslant \mathcal{C}_0^{1/2} \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)}, \tag{3.11}$$

where $d(\boldsymbol{f}, \mathbf{M}) = \inf\left\{\|\boldsymbol{f} - \boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} : \boldsymbol{g} \in \mathbf{M}\right\}$ *and*

$$\mathbf{M} = \left\{ \boldsymbol{g} \,:\, \|S\boldsymbol{g}\|^4_{L^4(\mathbb{R}^{1+3})} = \mathcal{C}_0 \|\boldsymbol{g}\|^4_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} \right\}. \tag{3.12}$$

Throughout this section, we consider $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$ with δ sufficiently small, so that the corresponding solution $u = \Phi(\boldsymbol{f})$ is well-defined, by Proposition 3.1.4. Recalling that

$$u = \Phi(\boldsymbol{f}) = S\boldsymbol{f} + \sigma\Box^{-1}(u^3), \tag{3.13}$$

we will require the following estimates on Picard iterations.

Lemma 3.3.2. *Let $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$. Then as $\delta \to 0$,*

$$\Phi(\boldsymbol{f}) = S\boldsymbol{f} + O(\delta^3), \tag{3.14}$$

$$\Phi(\boldsymbol{f}) = S\boldsymbol{f} + \sigma\Box^{-1}\left((S\boldsymbol{f})^3\right) + O(\delta^5), \tag{3.15}$$

where the big-O symbols refer to the norms of $L^4(\mathbb{R}^{1+3})$ and $C(\mathbb{R}; \dot{\mathcal{H}}^{1/2})$.

Proof. By the final estimate of Proposition 3.1.4, we have $u = \Phi(\boldsymbol{f}) = O(\delta)$ and so $\left\|u^3\right\|_{L^{4/3}} = O(\delta^3)$. Then, by the Strichartz estimate of Proposition 3.1.2, we obtain

$$\Box^{-1}(u^3) = O(\delta^3),$$

so the fixed point equation (3.13) yields (3.14). Now, by the Hölder inequality,

$$\left\|u^3 - (S\boldsymbol{f})^3\right\|_{L^{4/3}} \leqslant C\|u - S\boldsymbol{f}\|_{L^4}\left(\|u\|^2_{L^4} + \|S\boldsymbol{f}\|^2_{L^4}\right) \leqslant O(\delta^5),$$

where we used (3.14) to estimate $u - S\boldsymbol{f}$. We rewrite this as

$$u^3 = (S\boldsymbol{f})^3 + O(\delta^5),$$

where the big-O symbol refers to the $L^{4/3}$ norm, and inserting this into the fixed point equation yields (3.15). □

The function I, defined in the introduction to the present chapter, can be rewritten as

$$I(\delta) = \sup\left\{ \|\Phi(\boldsymbol{f})\|^4_{L^4(\mathbb{R}^{1+3})} \,\middle|\, \|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} \leqslant \delta \right\}.$$

We record some properties of the $\boldsymbol{f}$ that come close to maximize $I(\delta)$.

Lemma 3.3.3. *Let $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$ and $u = \Phi(\boldsymbol{f})$ be close to maximal in the sense that*

$$I(\delta) - \|u\|^4_{L^4(\mathbb{R}^{1+3})} = O(\delta^6). \tag{3.16}$$

Then $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \delta + O(\delta^3)$ and $d(\boldsymbol{f}, \mathbf{M}) = O(\delta^2)$. Moreover, there is a $C > 0$ such that

$$\|S\boldsymbol{f}\|^4_{L^4(\mathbb{R}^{1+3})} \leqslant \mathcal{C}_0\delta^4 - C\delta^2 d(\boldsymbol{f}, \mathbf{M})^2. \tag{3.17}$$

Proof. By squaring the sharpened Strichartz estimate (3.11), we obtain

$$\|S\boldsymbol{f}\|^4_{L^4} + 2c\|S\boldsymbol{f}\|^2_{L^4} d(\boldsymbol{f}, \mathbf{M})^2 \leqslant \mathcal{C}_0\delta^4. \tag{3.18}$$

Now, we use the first Picard estimate (3.14) for $u = \Phi(\boldsymbol{f})$ in order to find upper and lower bounds for $I(\delta)$. On the one hand, by combining it with the closeness assumption (3.16) and with (3.18), we find that

$$\begin{aligned} I(\delta) = \|u\|^4_{L^4} + O(\delta^6) &= \|S\boldsymbol{f}\|^4_{L^4} + O(\delta^6) \\ &\leqslant \mathcal{C}_0\delta^4 - 2c\|S\boldsymbol{f}\|^2_{L^4} d(\boldsymbol{f}, \mathbf{M})^2 + O(\delta^6). \end{aligned}$$

On the other hand, if $\boldsymbol{g} \in \mathbf{M}$ is such that $\|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} = 1$, then, by definition,

$$I(\delta) \geqslant \|\Phi(\delta\boldsymbol{g})\|^4_{L^4} \geqslant \mathcal{C}_0\delta^4 + O(\delta^6),$$

where the second inequality uses (3.14) and the fact that $\|S(\delta\boldsymbol{g})\|^4_{L^4} = \mathcal{C}_0\delta^4$. Combining these upper and lower bounds for $I(\delta)$ we find that

$$2c\|S\boldsymbol{f}\|^2_{L^4} d(\boldsymbol{f}, \mathbf{M})^2 \leqslant O(\delta^6), \tag{3.19}$$

and

$$\|S\boldsymbol{f}\|^4_{L^4} \geqslant \mathcal{C}_0\delta^4 + O(\delta^6). \tag{3.20}$$

Using the Strichartz inequality $\mathcal{C}_0\|\boldsymbol{f}\|^4_{\dot{\mathcal{H}}^{1/2}} \geqslant \|S\boldsymbol{f}\|^4_{L^4}$ and the assumption $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$, the bound (3.20) gives that $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \delta + O(\delta^3)$. Inserting (3.20) into (3.19) we conclude that $d(\boldsymbol{f}, \mathbf{M})^2 = O(\delta^4)$. On the other hand, reinserting (3.20) into (3.18) yields (3.17), and the proof is complete. □

For a slightly stronger version of this lemma, see Proposition 2.2.8 in the previous chapter.

Lemma 3.3.4. *For every $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$ there exists a $\boldsymbol{f}_\star \in \mathbf{M}$ such that*

$$\|\boldsymbol{f} - \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)} = d(\boldsymbol{f}, \mathbf{M}).$$

Moreover, $\langle \boldsymbol{f}_\star \mid \boldsymbol{f} - \boldsymbol{f}_\star\rangle_{\dot{\mathcal{H}}^{1/2}} = 0$ and we write $\boldsymbol{f}_\perp := \boldsymbol{f} - \boldsymbol{f}_\star$; see Figure 3.2.

Remark 3.3.5. We caution that, in the previous chapters, the symbol $\boldsymbol{f}_\star$ has been used with a different meaning.

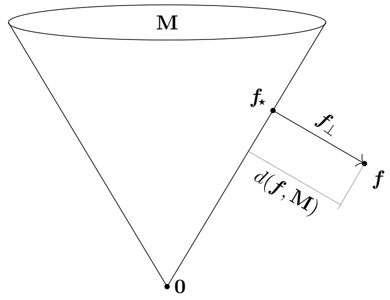

Figure 3.2: Illustration of Lemma 3.3.4

We can now obtain the asymptotic formula by combining the previous lemmas with the second Picard iteration estimate.

Proposition 3.3.6. *Let $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$ and $u = \Phi(\boldsymbol{f})$ be close to maximal in the sense that*

$$I(\delta) - \|u\|^4_{L^4(\mathbb{R}^{1+3})} = O(\delta^8).$$

Then $d(\boldsymbol{f}, \mathbf{M}) = O(\delta^3)$ and, as $\delta \to 0$,

$$I(\delta) = \mathcal{C}_0\delta^4 + \sigma\mathcal{C}_1\delta^6 + O(\delta^8),$$

where σ is the coefficient of the nonlinearity in (NLW). *The constant $\mathcal{C}_1$ satisfies*

$$\sigma\mathcal{C}_1 = \sup\left\{ \sigma \iint_{\mathbb{R}^{1+3}} (S\boldsymbol{g})^3\Box^{-1}((S\boldsymbol{g})^3)\,dtdx \;\middle|\; \begin{array}{c} \boldsymbol{g} \in \mathbf{M} \\ \|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = 1 \end{array} \right\}. \tag{3.21}$$

Proof. By Lemma 3.3.4, we can write $\boldsymbol{f} = \boldsymbol{f}_\star + \boldsymbol{f}_\perp$. Using the orthogonality, we have

$$\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} + \|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta^2,$$

from which we conclude that $\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$. This also shows that

$$\|\boldsymbol{f}_\star\|^2_{\dot{\mathcal{H}}^{1/2}} = \delta^2 + O(\delta^4), \tag{3.22}$$

because $\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} = \delta^2 + O(\delta^4)$ and $\|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} = O(\delta^4)$ by Lemma 3.3.3. Expanding, we find

$$(S\boldsymbol{f})^3 = (S\boldsymbol{f}_\star)^3 + O(\delta^2\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^{1/2}}),$$

where the big-O symbol refers to the $L^{4/3}(\mathbb{R}^{1+3})$ norm. Applying $\Box^{-1}$, we infer from the Strichartz estimates (3.5) that

$$\Box^{-1}((S\boldsymbol{f})^3) = \Box^{-1}((S\boldsymbol{f}_\star)^3) + O(\delta^2\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^{1/2}}),$$

where the big-O now refers to both the $L^4(\mathbb{R}^{1+3})$ and the $C(\mathbb{R};\dot{\mathcal{H}}^{1/2})$ norm. So, we can write

$$\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f})^3\Box^{-1}((S\boldsymbol{f})^3) = \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3\Box^{-1}((S\boldsymbol{f}_\star)^3) + O(\delta^5\|\boldsymbol{f}_\perp\|_{\dot{\mathcal{H}}^{1/2}}). \tag{3.23}$$

Now the key ingredient in this case is the second Picard estimate (3.15), from which we deduce

$$\|\Phi(\boldsymbol{h})\|_{L^4}^4 = \left\|S\boldsymbol{h} + \sigma\Box^{-1}((S\boldsymbol{h})^3)\right\|_{L^4}^4 + O(\delta^8),$$

whenever $\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$. This implies that

$$\|\Phi(\boldsymbol{h})\|_{L^4}^4 = \|S\boldsymbol{h}\|_{L^4}^4 + 4\sigma\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{h})^3\Box^{-1}((S\boldsymbol{h})^3) + O(\delta^8). \tag{3.24}$$

As $u = \Phi(\boldsymbol{f})$ with $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta$, on the one hand this yields an upper bound using our closeness hypothesis;

$$I(\delta) \leqslant \|u\|_{L^4}^4 + O(\delta^8) = \|S\boldsymbol{f}\|_{L^4}^4 + 4\sigma\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f})^3\Box^{-1}((S\boldsymbol{f})^3) + O(\delta^8).$$

Estimating the first term on the right-hand side using (3.17) of the previous lemma and the second term using (3.23), we obtain

$$\begin{aligned} I(\delta) \leqslant \mathcal{C}_0\delta^4 + 4\sigma\iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3\Box^{-1}((S\boldsymbol{f}_\star)^3) - C\delta^2 d(\boldsymbol{f},\mathbf{M})^2 \\ + O(\delta^5 d(\boldsymbol{f},\mathbf{M})) + O(\delta^8). \end{aligned} \tag{3.25}$$

For the lower bound, we let $\tilde{\boldsymbol{f}}_\star := \boldsymbol{f}_\star/\|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}$, so that $I(\delta) \geqslant \|\Phi(\delta\tilde{\boldsymbol{f}}_\star)\|_{L^4}^4$, and expanding using (3.24) we obtain

$$I(\delta) \geqslant \mathcal{C}_0\delta^4 + 4\sigma\delta^6\iint\limits_{\mathbb{R}^{1+3}} (S\tilde{\boldsymbol{f}}_\star)^3\Box^{-1}((S\tilde{\boldsymbol{f}}_\star)^3) + O(\delta^8), \tag{3.26}$$

where we used that $\|S\tilde{\boldsymbol{f}}_\star\|_{L^4}^4 = \mathcal{C}_0$. Now, using (3.22), we see that

$$\begin{aligned} \delta^6\iint\limits_{\mathbb{R}^{1+3}} (S\tilde{\boldsymbol{f}}_\star)^3\Box^{-1}((S\tilde{\boldsymbol{f}}_\star)^3) &= \|\boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}}^6\iint\limits_{\mathbb{R}^{1+3}} (S\tilde{\boldsymbol{f}}_\star)^3\Box^{-1}((S\tilde{\boldsymbol{f}}_\star)^3) + O(\delta^8) \\ &= \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3\Box^{-1}((S\boldsymbol{f}_\star)^3) + O(\delta^8), \end{aligned}$$

so combining the upper and lower bounds (3.25) and (3.26) yields

$$\delta^2 d(\boldsymbol{f},\mathbf{M})^2 \leqslant O(\delta^5 d(\boldsymbol{f},\mathbf{M}) + \delta^8).$$

Writing $X := d(\boldsymbol{f}, \mathbf{M})\delta^{-3}$, this reads $X^2 \leqslant O(1 + X)$, which implies that $X = O(1)$. Thus we find that $d(\boldsymbol{f}, \mathbf{M}) = O(\delta^3)$.

To complete the proof we observe that, since $O(\delta^5 d(\boldsymbol{f}, \mathbf{M})) = O(\delta^8)$, it follows from (3.25) and (3.26) that

$$I(\delta) = \mathcal{C}_0\delta^4 + 4\sigma \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3 \Box^{-1}((S\boldsymbol{f}_\star)^3) + O(\delta^8). \tag{3.27}$$

However, for all $\boldsymbol{g} \in \mathbf{M}$ with $\|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = \delta$, we also have

$$I(\delta) \geqslant \|\Phi(\delta\boldsymbol{g})\|_{L^4}^4 = \mathcal{C}_0\delta^4 + 4\sigma \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{g})^3 \Box^{-1}((S\boldsymbol{g})^3) + O(\delta^8),$$

and so, combining this with (3.27), we conclude that the term

$$\sigma \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{f}_\star)^3 \Box^{-1}((S\boldsymbol{f}_\star)^3)$$

must be equal to

$$\sup \left\{ \sigma \iint\limits_{\mathbb{R}^{1+3}} (S\boldsymbol{g})^3 \Box^{-1}((S\boldsymbol{g})^3) \;\middle|\; \begin{array}{c} \boldsymbol{g} \in \mathbf{M} \\ \|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = \delta \end{array} \right\} + O(\delta^8),$$

thus proving (3.21). □

It remains to evaluate this supremum, which we will do in the sequel.

3.4 Computation of the constant $\mathcal{C}_1$ via the Penrose transform

We consider the following family of elements of $\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$:

$$\boldsymbol{f}_\theta := \left(\cos\theta \frac{2}{1 + |\cdot|^2}, -\sin\theta \left(\frac{2}{1 + |\cdot|^2} \right)^2 \right),$$

and we let

$$v_\theta := S\boldsymbol{f}_\theta, \quad \boldsymbol{v}_\theta := (v_\theta, \partial_t v_\theta). \tag{3.28}$$

We caution that, in the previous chapters, we used the notation $\boldsymbol{f}_\star$ to denote what is now called $\boldsymbol{f}_0$. One can calculate that $\|\boldsymbol{f}_\theta\|_{\dot{\mathcal{H}}^{1/2}} = |\mathbb{S}^3|^{1/2}$; see (1.20).

Remark 3.4.1. For all $t \in \mathbb{R}$ it holds that $\boldsymbol{v}_\theta(t) = \mathrm{Ph}_\theta\, \boldsymbol{v}_0(t)$, where

$$\mathrm{Ph}_\theta\, \boldsymbol{f} := \begin{bmatrix} \cos(\theta) & \dfrac{\sin(\theta)}{\sqrt{-\Delta}} \\ -\sin(\theta)\sqrt{-\Delta} & \cos(\theta) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}.$$

The operator $\mathrm{Ph}_\theta \colon \dot{\mathcal{H}}^{1/2} \to \dot{\mathcal{H}}^{1/2}$ is unitary and it commutes with the linear propagator S; see the second chapter. However, Ph_θ does *not* commute with the nonlinear propagator Φ.

We recast in the notation of the present chapter the characterization of the extremizers to the Strichartz estimate (3.3); see Section 2.2, in the previous chapter, for more detail.

Proposition 3.4.2 (Foschi [37])**.** *Let* $\mathbf{M}$ *be the set of extremizing functions for the Strichartz inequality; see* (3.12)*. Then*

$$\mathbf{M} = \{\, c\, (\boldsymbol{v}_\theta \circ \Lambda)|_{t=0} \mid c, \theta, \Lambda \,\},$$

where $c \geqslant 0$, $\theta \in \mathbb{S}^1$ *and* $\Lambda(t,x) = L^\beta\big(\lambda(t-t_0), \lambda(x-x_0)\big)$.

Recalling the definition (3.21) of $\mathcal{C}_1$, we define

$$\mathcal{C}(w) := \iint\limits_{\mathbb{R}^{1+3}} w^3 \Box^{-1}(w^3), \quad \text{where } w \in L^4(\mathbb{R}^{1+3}), \tag{3.29}$$

so that $\sigma\mathcal{C}_1 = \sup\{\sigma\mathcal{C}(v)\}$, where $v = S\boldsymbol{g}$ and $\boldsymbol{g} \in \mathbf{M}$ is such that $\|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = 1$.

Proposition 3.4.3. *For all* $w \in L^4(\mathbb{R}^{1+3})$*,*

$$\mathcal{C}(w \circ \Lambda) = \lambda^2 \mathcal{C}(w). \tag{3.30}$$

In particular,

$$\sigma\mathcal{C}_1 = \max\left\{ \frac{\sigma\mathcal{C}(v_\theta)}{|\mathbb{S}^3|^3} \;\middle|\; \theta \in \mathbb{S}^1 \right\}. \tag{3.31}$$

Proof. The property (3.30) follows from the commutativity property (3.8) of $\Box^{-1}$. To conclude it suffices to note that, by Proposition 3.4.2, if $v = S\boldsymbol{g}$ with $\boldsymbol{g} \in \mathbf{M}$ and $\|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = 1$, then $v = \left|\mathbb{S}^3\right|^{-1/2} v_\theta \circ \Lambda$ for a $\theta \in \mathbb{S}^1$ and a transformation Λ with $\lambda = 1$. $\Box$

To compute the maximum in (3.31) we will use the Penrose transform, which we briefly recall here; see Section 1.1, in the first chapter, for more details. We recall that the light-like coordinates on $\mathbb{R}^{1+3}$ are defined by

$$x^- = t - r, \quad x^+ = t + r, \quad \text{where } x^- \leqslant x^+,$$

while the corresponding coordinates on the curved space-time $\mathbb{R} \times \mathbb{S}^3$ are

$$X^- = \tfrac{1}{2}(T - R), \quad X^+ = \tfrac{1}{2}(T + R), \tag{3.32}$$

where $T \in \mathbb{R}$, and R is the polar coordinate on $\mathbb{S}^3$ such that, for all $(X_0, X_1, X_2, X_3) \in \mathbb{S}^3$,

$$X_0 = \cos(R), \quad (X_1, X_2, X_3) = \sin(R)\,\omega, \quad \omega \in \mathbb{S}^2,\ R \in [0, \pi].$$

The Penrose map is the identification of $\mathbb{R}^{1+3}$ with an open subset of $\mathbb{R} \times \mathbb{S}^3$ via the equations

$$X^- = \arctan x^-, \quad X^+ = \arctan x^+, \tag{3.33}$$

so that X^- and X^+ take values in the region

$$\mathcal{T} := \left\{ (X^-, X^+) \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^2 \mid X^- \leqslant X^+ \right\}. \tag{3.34}$$

The identification (3.33) is conformal, in the sense that

$$dT^2 - dR^2 - \sin^2 R\, d\omega^2 = \Omega^2 \left(dt^2 - dr^2 - r^2 d\omega^2\right), \tag{3.35}$$

where $d\omega^2$ denotes the metric tensor of $\mathbb{S}^2$ and the conformal factor Ω is the scalar field given by

$$\Omega = 2(1 + (x^+)^2)^{-1/2}(1 + (x^-)^2)^{-1/2} = 2\cos X^+ \cos X^-.$$

In all these equations, as in the rest of the section, the change of variable (3.33) is implicit.

If v is a scalar field on $\mathbb{R}^{1+3}$, we define a scalar field V on $\mathcal{P}(\mathbb{R}^{1+3})$ by the equation

$$v = \Omega V, \tag{3.36}$$

which implies that, at $t = 0$ (corresponding to $T = 0$),

$$v|_{t=0} = (\Omega V)|_{T=0}, \quad \partial_t v|_{t=0} = (\Omega^2 \partial_T V)\big|_{T=0}.$$

The scalar field V is called the *Penrose transform* of v. We remark that v is radially symmetric if and only if V depends only on X^-, X^+, and in this case, using (3.36) and (3.33), we obtain

$$\begin{aligned} r\Box v &= (\partial_t^2 - \partial_r^2)(rv) \\ &= \Omega^2 \partial_{X^+}\partial_{X^-}(r\Omega V) \\ &= \Omega^2 \partial_{X^+}\partial_{X^-}(\sin(R)V), \end{aligned} \tag{3.37}$$

where we used the formula $r\Omega = \sin R$, which can be immediately obtained from (3.35) by comparing the factors of $d\omega^2$. We remark that there is also a more general formula, which includes the case of nonradial v; see (1.10), in the first chapter.

As already noted in the previous chapters, the Penrose transform is relevant in our context, because applying it to $\boldsymbol{v}_\theta$, as defined in (3.28), we obtain a simple expression;

$$V_\theta|_{T=0} = \cos\theta, \quad \partial_T V_\theta|_{T=0} = -\sin\theta, \quad \text{and } V_\theta = \cos(T + \theta).$$

Proposition 3.4.4. *It holds that*

$$\mathcal{C}(v_\theta) = \frac{\pi^3}{128}\left(24\cos^2\theta + 5\right). \tag{3.38}$$

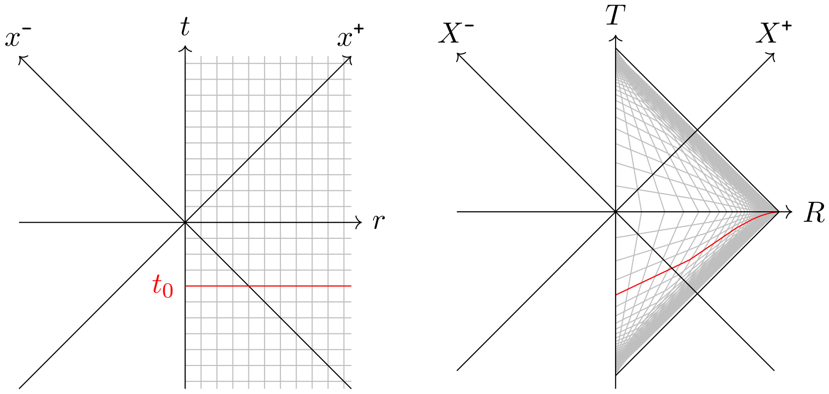

Figure 3.3: As $t_0 \to -\infty$, the image under the Penrose map $\mathcal{P}$ of the hypersurface $t = t_0$ converges uniformly to the hypersurface $X^- = -\pi/2$.

Proof. Let $w_\theta := \Box^{-1}(v_\theta^3)$. Applying the Penrose transform (3.36) to the integral (3.29) that defines $\mathcal{C}$, we obtain

$$\mathcal{C}(v_\theta) = \iint_{\mathcal{P}(\mathbb{R}^{1+3})} V_\theta^3 W_\theta \, dT dS = 4\pi \int_{-\pi}^{\pi} \int_0^{\pi-|T|} \cos^3(T+\theta) W_\theta \sin^2 R \, dT dR,$$

where $dS = \sin^2 R \, dR dS_{\mathbb{S}^2}$ denotes the volume element on $\mathbb{S}^3$. Here we used that $\Omega^4 \, dt dx = dT dS$, which follows from (3.35). Now the change of variable (3.32) yields

$$\mathcal{C}(v_\theta) = 8\pi \iint_{\mathcal{T}} \cos^3(X^+ + X^- + \theta) \sin(X^+ - X^-) \tilde{W}_\theta \, dX^- dX^+, \tag{3.39}$$

where

$$\tilde{W}_\theta := \sin(R) W_\theta,$$

and $\mathcal{T}$ is the half-square defined in (3.34). We will prove that

$$\tilde{W}_\theta(X^+, X^-) = -\tilde{W}_\theta(X^-, X^+),$$

so that the integrand of (3.39) is symmetric under permutation of the variables, allowing us to consider the integral over the full square $[-\frac{\pi}{2}, \frac{\pi}{2}]^2$.

We compute $\tilde{W}_\theta$ explicitly. From the definition of $\Box^{-1}$ it follows that

$$\begin{cases} r\Box w_\theta = r v_\theta^3, & \text{on } \mathbb{R}^{1+3}, \\ \lim_{t \to -\infty} \|\boldsymbol{w}_\theta\|_{\dot{\mathcal{H}}^{1/2}} = 0, \end{cases} \tag{3.40}$$

and using (1.11), (3.37), and the aforementioned formula $r\Omega = \sin R$, we obtain

$$r\Box w_\theta = \Omega^2 \partial_{X^+} \partial_{X^-} (\sin(R) W_\theta), \quad \text{and } r v_\theta^3 = \Omega^2 \sin(R) V_\theta^3,$$

so the factors of Ω^2 simplify and we obtain from (3.40) the differential equation

$$\partial_{X^+}\partial_{X^-}\tilde{W}_\theta = \sin(X^+ - X^-)\cos^3(X^+ + X^- + \theta).$$

The general solution $\tilde{W}_\theta$ of this can be written

$$\int_{-\frac{\pi}{2}}^{X^-}\int_{-\frac{\pi}{2}}^{X^+} \sin(Z-Y)\cos^3(Y+Z+\theta)\,dY\,dZ + F(X^+) + G(X^-), \tag{3.41}$$

where F and G are arbitrary smooth functions.

We claim that

$$F(X^+) + G(X^-) \equiv 0. \tag{3.42}$$

To prove this, we notice that for each fixed $t_0 \in \mathbb{R}$, the hypersurface of $\mathbb{R}^{1+3}$ of equation $t = t_0$ is mapped by $\mathcal{P}$ to the hypersurface of equations

$$X^- = \arctan(t_0 - r), \quad X^+ = \arctan(t_0 + r),$$

(see Figure 3.3), which, as $t_0 \to -\infty$, converges uniformly to the hypersurface $X^- = -\pi/2$. The condition $\|\boldsymbol{w}_\theta(t)\|_{\dot{\mathcal{H}}^{1/2}} \to 0$ thus implies that $\tilde{W}_\theta|_{X^-=-\frac{\pi}{2}} = 0$. We obtain another condition by observing that, since w_θ is smooth and radially symmetric, the function W_θ must be regular at $R = 0$, which implies that $\tilde{W}_\theta|_{R=0} = 0$. Now the integral of (3.41) satisfies both conditions. The first one is obvious, while the second follows from symmetry, since

$$X^-|_{R=0} = X^+|_{R=0},$$

so the domain of integration is symmetric under permutation of the variables Y, Z, while the integrand function changes sign. This proves (3.42).

Returning to (3.39), the fact that $\tilde{W}_\theta(X^+, X^-) = -\tilde{W}_\theta(X^-, X^+)$ is immediate from the explicit form of $\tilde{W}_\theta$. Thus the integral in (3.39) can be replaced by the integral over $[-\frac{\pi}{2}, \frac{\pi}{2}]^2$, with a multiplicative factor of $\frac{1}{2}$. More precisely, letting

$$F(Y, Z, \theta) := \sin(Z - Y)\cos^3(Y + Z + \theta),$$

we have the formula

$$\mathcal{C}(v_\theta) = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{X^-}\int_{-\frac{\pi}{2}}^{X^+} F(X^-, X^+, \theta)F(Y, Z, \theta)\,dX^-dX^+dY\,dZ,$$

which allows for explicit computation, yielding (3.38).

□

Combining Propositions 3.4.3 and 3.4.4 we obtain the value of the constant.

Corollary 3.4.5. *The constant $\mathcal{C}_1$ in Theorem 3.0.1 can be written*

$$\mathcal{C}_1 = \begin{cases} \dfrac{\mathcal{C}(v_0)}{|\mathbb{S}^3|^3} = \dfrac{29}{128}\Big(\dfrac{\pi}{|\mathbb{S}^3|}\Big)^3, & \sigma > 0, \\ \dfrac{\mathcal{C}(v_{\pi/2})}{|\mathbb{S}^3|^3} = \dfrac{5}{128}\Big(\dfrac{\pi}{|\mathbb{S}^3|}\Big)^3, & \sigma < 0. \end{cases}$$

3.5 Existence of maximizers

We follow the lines of [31, Section 2] to show that the supremum (3.1) is attained for small enough values of δ. We recall from Proposition 3.1.4 that $\Phi(\boldsymbol{f}) = u$ denotes the solution to the fixed point equation associated to (NLW)

$$u = S\boldsymbol{f} + \sigma\Box^{-1}(u^3),$$

provided that such a solution exists and is unique. We will require the concentration-compactness tools developed in Section 3.7 in the Appendix.

Lemma 3.5.1. *Suppose that $\delta > 0$ satisfies*

1. *Scattering: $I(\delta) < \infty$;*
2. *Superadditivity: for all $\alpha \in (0, \delta)$,*
$$I(\sqrt{\delta^2 - \alpha^2}) + I(\alpha) < I(\delta); \tag{3.43}$$
3. *Upper semicontinuity: for any sequence $\alpha_n \leqslant \delta$,*
$$\limsup_{n\to\infty} I(\alpha_n) \leqslant I(\limsup_{n\to\infty} \alpha_n). \tag{3.44}$$

Then there exists a solution u to (NLW) *such that*

$$\lim_{t\to-\infty} \|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}} = \delta \quad \textit{and} \quad \|u\|^4_{L^4(\mathbb{R}^{1+3})} = I(\delta).$$

Proof. Let u_n be a maximizing sequence of I, that is

$$u_n = \Phi(\boldsymbol{f}_n), \quad \|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{1/2}} \leqslant \delta, \quad I(\delta) = \lim_{n\to\infty} \|u_n\|^4_{L^4}.$$

We consider a profile decomposition of the sequence $\boldsymbol{f}_n$, in the sense of Theorem 3.7.3 in the Appendix, and we claim that all profiles $\{\boldsymbol{F}^j \ : \ j \in \mathbb{N}_{\geqslant 1}\}$ vanish but one.

To prove this, we denote by $\boldsymbol{g}_n$ the sequence obtained by subtracting the profile $\boldsymbol{F}^j$ from $\boldsymbol{f}_n$, that is

$$\boldsymbol{g}_n = \boldsymbol{f}_n - \lambda_n^{(j)} S\boldsymbol{F}^j \circ \Lambda_n^j \Big|_{t=0},$$

and we construct the corresponding solution $W_n = \Phi(\boldsymbol{g}_n)$. By the nonlinear profile decomposition, Corollary 3.7.5, we have that

$$u_n(t,x) = \lambda_n^{(j)} U^j(\Lambda_n^j(t,x)) + W_n(t,x) + h_n(t,x),$$

where $\|h_n\|_{L^4(\mathbb{R}^{1+3})} + \sup_{t\in\mathbb{R}} \|\boldsymbol{h}_n(t)\|_{\dot{\mathcal{H}}^{1/2}} \to 0$ as $n \to \infty$. By the Pythagorean expansion (2.33) of the $\dot{\mathcal{H}}^{1/2}$ norm,

$$\delta^2 \geqslant \|\boldsymbol{f}_n\|^2_{\dot{\mathcal{H}}^{1/2}} = \left\|\boldsymbol{F}^j\right\|^2_{\dot{\mathcal{H}}^{1/2}} + \|\boldsymbol{g}_n\|^2_{\dot{\mathcal{H}}^{1/2}} + o(1), \tag{3.45}$$

and by Remark 3.7.6,

$$\|u_n\|_{L^4}^4 = \left\|U^j\right\|_{L^4}^4 + \|W_n\|_{L^4}^4 + o(1). \tag{3.46}$$

Since u_n is a maximizing sequence, we infer from (3.45) and (3.46)

$$\begin{aligned} I(\delta) &= \left\|U^j\right\|_{L^4}^4 + \limsup_{n\to\infty} \|W_n\|_{L^4}^4 \\ &\leqslant I\big(\left\|\boldsymbol{F}^j\right\|_{\dot{\mathcal{H}}^{1/2}}\big) + I\Big(\sqrt{\delta^2 - \|\boldsymbol{F}^j\|_{\dot{\mathcal{H}}^{1/2}}^2}\Big), \end{aligned}$$

where we also used the upper semicontinuity property (3.44). Now, the superadditivity property (3.43) implies that

$$\text{either} \quad \left\|\boldsymbol{F}^j\right\|_{\dot{\mathcal{H}}^{1/2}} = 0, \quad \text{or} \quad \left\|\boldsymbol{F}^j\right\|_{\dot{\mathcal{H}}^{1/2}} = \delta.$$

It cannot be that $\boldsymbol{F}^j = \mathbf{0}$ for all $j \geqslant 1$, for otherwise the nonlinear profile decomposition (3.71) would give the contradiction $I(\delta) = 0$. On the other hand, if $\|\boldsymbol{F}^j\|_{\dot{\mathcal{H}}^{1/2}} = \delta$ then, by (3.45), $\|\boldsymbol{g}_n\|_{\dot{\mathcal{H}}^{1/2}} \to 0$ as $n \to \infty$, which means that $\boldsymbol{F}^k = \mathbf{0}$ for all $k \neq j$.

We have thus proven that there exists one and only one nonvanishing profile $\boldsymbol{F}$ for the sequence $\boldsymbol{f}_n$. Letting U denote the corresponding nonlinear profile, Corollary 3.7.5 implies that $I(\delta) = \|U\|_{L^4}^4$, and the proof is complete. ☐

We now turn to the proof that, if $\delta > 0$ is sufficiently small, then the three properties of Lemma 3.5.1 are satisfied. We already dealt with the first one in Proposition 3.1.4. The following lemma implies the third property and will also be used in the proof of the second property.

Lemma 3.5.2. *There exists $A, C_1, C_2 > 0$ such that*

$$C_1|\varepsilon|\delta^3 \leqslant |I(\delta+\varepsilon) - I(\delta)| \leqslant C_2|\varepsilon|\delta^3, \quad \forall\, \varepsilon \in (-\delta/2, \delta/2), \tag{3.47}$$

whenever $\delta \in (0, A]$. In particular, I is continuous on $(0, A/2]$.

Proof. In fact we will prove that

$$4\mathcal{C}_0\varepsilon\delta^3 + O(\varepsilon\delta^5) \leqslant I(\delta+\varepsilon) - I(\delta) \leqslant 4\mathcal{C}_0\varepsilon(\delta+\varepsilon)^3 + O(\varepsilon\delta^5), \tag{3.48}$$

from which (3.47) follows by taking $A > 0$ sufficiently small. For this we let $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \delta$ and $u = \Phi(\boldsymbol{f})$ be close to maximal in the sense that

$$I(\delta) - \|u\|_{L^4}^4 = O(\varepsilon\delta^5), \tag{3.49}$$

and we define

$$u_\varepsilon := \Phi\left((1+\tfrac{\varepsilon}{\delta})\boldsymbol{f}\right), \quad \tilde{u}_\varepsilon := (1+\tfrac{\varepsilon}{\delta})u.$$

With these definitions, since $\Box u + \sigma u^3 = 0$, we have that

$$\sigma e := \Box \tilde{u}_\varepsilon - \sigma \tilde{u}_\varepsilon^3 = -2\sigma \tfrac{\varepsilon}{\delta} u^3 + O(\tfrac{\varepsilon^2}{\delta^2} u^3),$$

where the big-O symbol refers to the $L^{4/3}(\mathbb{R}^{1+3})$ norm, and since $\|u\|_{L^4}$ is $O(\delta)$, we can conclude that

$$\|e\|_{L^{4/3}} = O(\varepsilon\delta^2).$$

Moreover, it is clear that $\|\boldsymbol{u}_\varepsilon(t) - \tilde{\boldsymbol{u}}_\varepsilon(t)\|_{\dot{\mathcal{H}}^{1/2}} \to 0$ as $t \to -\infty$, and so we can apply the forthcoming perturbation Lemma 3.7.4 to obtain

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^4} \leqslant C\varepsilon\delta^2,$$

and we infer that

$$\|u_\varepsilon\|_{L^4}^4 = \|\tilde{u}_\varepsilon\|_{L^4}^4 + O(\varepsilon\delta^5), \tag{3.50}$$

where the constant implicit in the big-O notation depends on A only.

We now insert (3.50) into the inequality $I(\delta+\varepsilon) \geqslant \|u_\varepsilon\|_{L^4}^4$, which follows from the definition of I. We obtain

$$\begin{aligned} I(\delta+\varepsilon) &\geqslant (1+\tfrac{\varepsilon}{\delta})^4 \|u\|_{L^4}^4 + O(\varepsilon\delta^5) \\ &\geqslant I(\delta) + 4\tfrac{\varepsilon}{\delta} I(\delta) + O(\varepsilon\delta^5), \end{aligned}$$

where we used the elementary inequality $(1+\frac{\varepsilon}{\delta})^4 \geqslant 1 + 4\frac{\varepsilon}{\delta}$ and the closeness condition (3.49). Now by the asymptotic Proposition 3.3.6, we know that $I(\delta) = \mathcal{C}_0\delta^4 + O(\delta^6)$ which can be inserted to complete the proof of the first inequality in (3.48).

To prove the second inequality and complete the proof of Lemma 3.5.2, we let $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \delta + \varepsilon$ and $u = \Phi(\boldsymbol{f})$ be close to maximal in the sense that

$$I(\delta+\varepsilon) - \|u\|_{L^4}^4 = O(\varepsilon(\delta+\varepsilon)^5).$$

Then we define $u_\varepsilon := \Phi((1 - \frac{\varepsilon}{\delta+\varepsilon})\boldsymbol{f})$ and $\tilde{u}_\varepsilon := (1 - \frac{\varepsilon}{\delta+\varepsilon})u$, and argue as before. □

Proposition 3.5.3. *For sufficiently small $\delta > 0$,*

$$I(\alpha) + I(\sqrt{\delta^2 - \alpha^2}) < I(\delta) \quad \forall\ \alpha \in (0, \delta).$$

Proof. This follows from the fact that I is a super-additive function of δ to main order, because $I(\delta) = \mathcal{C}_0\delta^4 + O(\delta^6)$, together with the estimates of Lemma 3.5.2, which rule out excessive fluctuations; see [31, Proposition 2.7]. □

3.6 Conditional uniqueness of maximizers

If $u = \Phi(\boldsymbol{f})$ is a maximizer to $I(\delta)$, and

$$\Lambda(t,x) = L^\beta\big(\lambda(t-t_0), \lambda(x-x_0)\big), \quad \lambda > 0, |\beta| < 1, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^3, \tag{3.51}$$

then $\lambda(u \circ \Lambda)$ is again a maximizer to $I(\delta)$; this is an immediate consequence of Theorem 3.2.4. In this section we give a partial result about the problem of uniqueness of maximizers, up to this transformation. The main tool is the local version of the sharpened Strichartz estimate of the previous chapter.

We begin by showing that each maximizer of $I(\delta)$ has a unique metric projection on the manifold $\mathbf{M}$ of linear maximizers. We refer to Section 2.2 in the previous chapter for the definition of the tangent space $T_{\boldsymbol{f}_\star}\mathbf{M}$, and we recall that, in the previous chapters, the symbol $\boldsymbol{f}_\star$ has been used with a different meaning.

Lemma 3.6.1. *Let $u = \Phi(\boldsymbol{f})$ be such that $\|u\|^4_{L^4(\mathbb{R}^{1+3})} = I(\delta)$. If $\delta > 0$ is sufficiently small, then there exists a unique $\boldsymbol{f}_\star \in \mathbf{M}\backslash\{\mathbf{0}\}$ such that*

$$\|\boldsymbol{f} - \boldsymbol{f}_\star\|_{\dot{\mathcal{H}}^{1/2}} = d(\boldsymbol{f}, \mathbf{M}).$$

Moreover, $\boldsymbol{f} - \boldsymbol{f}_\star \perp T_{\boldsymbol{f}_\star}\mathbf{M}$, where $\perp$ denotes orthogonality with respect to the $\dot{\mathcal{H}}^{1/2}$ scalar product.

Proof. This is proved in Section 2.2 in the previous chapter, the main issue being uniqueness. Lemma 3.5.1 ensures that $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \delta$, while by Proposition 3.3.6, we have $d(\boldsymbol{f}, \mathbf{M}) = O(\delta^3)$. Thus, if δ is sufficiently small, then Proposition 2.2.8 can be applied. □

The elements $\boldsymbol{f}_\star$ of $\mathbf{M}\backslash\{\mathbf{0}\}$ have the unique representation

$$\boldsymbol{f}_\star = \delta\lambda \boldsymbol{v}_\theta \circ \Lambda|_{t=0}\,, \tag{3.52}$$

where $\boldsymbol{v}_\theta = (v_\theta, \partial_t v_\theta)$ are particular solutions to the linear wave equation, as defined in (3.28); see the aforementioned Section 2.2. We let $\theta(\boldsymbol{f}_\star)$ denote the unique $\theta \in \mathbb{S}^1$. We recall that this parameter θ does not correspond to any symmetry of (NLW); see Remark 3.4.1.

We can now state the result.

Theorem 3.6.2. *Suppose that $u_{\boldsymbol{f}} = \Phi(\boldsymbol{f})$ and $u_{\boldsymbol{g}} = \Phi(\boldsymbol{g})$ satisfy*

$$\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}} = \delta, \quad \textit{and} \quad I(\delta) = \|u_{\boldsymbol{f}}\|^4_{L^4} = \|u_{\boldsymbol{g}}\|^4_{L^4},$$

with δ sufficiently small. Suppose moreover that the unique projections $\boldsymbol{f}_\star$ and $\boldsymbol{g}_\star$ satisfy

$$\theta(\boldsymbol{f}_\star) = \theta(\boldsymbol{g}_\star). \tag{3.53}$$

Then there is a transformation Λ of the form (3.51) *such that $u_{\boldsymbol{g}} = \lambda(u_{\boldsymbol{f}} \circ \Lambda)$.*

The assumption (3.53) makes this uniqueness result conditional. We conjecture that such an assumption is not necessary; that there is a single $\theta(\boldsymbol{f}_\star)$ for each maximizer $\boldsymbol{f}$ to $I(\delta)$.

We now recall the local version of the sharpened Strichartz estimate proved in the previous chapter.

Lemma 3.6.3. *Let ψ be the functional defined by*

$$\psi(\boldsymbol{f}) := \mathcal{C}_0 \|\boldsymbol{f}\|^4_{\dot{\mathcal{H}}^{1/2}} - \|S\boldsymbol{f}\|^4_{L^4(\mathbb{R}^{1+3})}.$$

Then there exists $C > 0$ such that, for all $\boldsymbol{m} \in \mathbf{M}\backslash\{\mathbf{0}\}$,

$$\frac{d^2}{d\varepsilon^2}\psi(\boldsymbol{m} + \varepsilon\boldsymbol{m}_\perp)\bigg|_{\varepsilon=0} \geqslant C\|\boldsymbol{m}\|^2_{\dot{\mathcal{H}}^{1/2}}\|\boldsymbol{m}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}}, \quad \forall\ \boldsymbol{m}_\perp \perp T_{\boldsymbol{m}}\mathbf{M}. \tag{3.54}$$

The derivative in (3.54) can be computed to be

$$\frac{1}{2}\,\frac{d^2}{d\varepsilon^2}\psi(\boldsymbol{m} + \varepsilon\boldsymbol{m}_\perp)\bigg|_{\varepsilon=0} = 2\mathcal{C}_0\|\boldsymbol{m}\|^2_{\dot{\mathcal{H}}^{1/2}}\|\boldsymbol{m}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} - 6\iint\limits_{\mathbb{R}^{1+3}}(S\boldsymbol{m})^2(S\boldsymbol{m}_\perp)^2; \tag{3.55}$$

see the proof of Lemma 2.3.1.

Proof of Theorem 3.6.2. By the unique representation (3.52), our assumption (3.53), and Lemma 3.6.1, up to changing $u_{\boldsymbol{f}}$ with $\lambda(u_{\boldsymbol{f}} \circ \Lambda)$ and $u_{\boldsymbol{g}}$ with $\lambda'(u_{\boldsymbol{g}} \circ \Lambda')$, where Λ and Λ' are transformations of the form (3.51), we can decompose

$$\boldsymbol{f} = c\delta\boldsymbol{m} + \boldsymbol{f}_\perp, \quad \boldsymbol{g} = c'\delta\boldsymbol{m} + \boldsymbol{g}_\perp, \quad \text{with } \boldsymbol{f}_\perp \perp T_{\boldsymbol{m}}\mathbf{M} \text{ and } \boldsymbol{g}_\perp \perp T_{\boldsymbol{m}}\mathbf{M},$$

where $\boldsymbol{m} = \left|\mathbb{S}^3\right|^{-1/2}\boldsymbol{f}_{\theta(\boldsymbol{f}_\star)}$, so that $\|\boldsymbol{m}\|_{\dot{\mathcal{H}}^{1/2}} = 1$. We denote

$$\boldsymbol{h} := \boldsymbol{f} - \boldsymbol{g}, \quad \text{and} \quad \boldsymbol{h}_\perp := \boldsymbol{f}_\perp - \boldsymbol{g}_\perp.$$

The proof will be complete once we show that $\boldsymbol{h} = \mathbf{0}$.

We now record the necessary estimates. First, we recall from Proposition 3.3.6 that

$$\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}} \leqslant d(\boldsymbol{f}, \mathbf{M}) + d(\boldsymbol{g}, \mathbf{M}) = O(\delta^3). \tag{3.56}$$

Now using the orthogonality, we can expand the identity $\|\boldsymbol{f}\|^2_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{g}\|^2_{\dot{\mathcal{H}}^{1/2}}$, to obtain

$$\delta^2\left|c^2 - c'^2\right| = \left|\|\boldsymbol{g}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} - \|\boldsymbol{f}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}}\right| \leqslant C\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}},$$

so that

$$(c - c')^2 = \left(\frac{c^2 - c'^2}{c + c'}\right)^2 \leqslant C\delta^2\|\boldsymbol{h}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}}. \tag{3.57}$$

In particular,

$$\|\boldsymbol{h}\|^2_{\dot{\mathcal{H}}^{1/2}} = (c - c')^2\delta^2 + \|\boldsymbol{h}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} = \|\boldsymbol{h}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}} + O(\delta^4\|\boldsymbol{h}_\perp\|^2_{\dot{\mathcal{H}}^{1/2}}). \tag{3.58}$$

We now define $w := u_{\boldsymbol{f}} - u_{\boldsymbol{g}}$; that is, $w = \Phi(\boldsymbol{f}) - \Phi(\boldsymbol{g})$. By the definition (3.13) of Φ, we have that

$$w = S\boldsymbol{h}_\perp + S\big((c - c')\delta\boldsymbol{m}\big) + \sigma\Box^{-1}\left(u^3_{\boldsymbol{f}} - u^3_{\boldsymbol{g}}\right),$$

and the Strichartz estimates (3.5) give

$$\left\|\Box^{-1}\left(u_{\boldsymbol{f}}^3 - u_{\boldsymbol{g}}^3\right)\right\|_{L^4} \leqslant C\delta^2 \|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}.$$

Thus by (3.57) and (3.58) we have

$$w = S\boldsymbol{h}_\perp + O(\delta^2\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}); \tag{3.59}$$

the big-O symbol referring to the $L^4(\mathbb{R}^{1+3})$ norm. Analogously, we see that

$$u_{\boldsymbol{g}} = S(c'\delta \boldsymbol{m}) + O(\delta^3). \tag{3.60}$$

With these estimates in hand, we may now proceed with the proof. The key step is given by the formula

$$\|u_{\boldsymbol{f}}\|_{L^4}^4 - \|u_{\boldsymbol{g}}\|_{L^4}^4 = -\frac{1}{2}\frac{d^2}{d\varepsilon^2}\psi(c'\delta\boldsymbol{m} + \varepsilon \boldsymbol{h}_\perp)\bigg|_{\varepsilon=0} + O(\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2), \tag{3.61}$$

which we will prove later. Note that the left-hand side vanishes by assumption. So, once (3.61) is proven, Lemma 3.6.3 will imply that

$$\delta^2\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2 \leqslant C\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2,$$

for an absolute constant $C > 0$, which is only possible if $\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}} = 0$, provided that $\delta < C^{-1}$. By (3.58), this would imply that $\boldsymbol{h} = \boldsymbol{0}$, concluding the proof.

In order to prove (3.61), we recall that $u_{\boldsymbol{f}} = u_{\boldsymbol{g}} + w$ and we expand

$$\begin{aligned}\iint_{\mathbb{R}^{1+3}} (u_{\boldsymbol{g}} + w)^4 - \iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^4 &= 4\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 w + 6\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^2 w^2 + O(\delta\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^3)\\ &= 4\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 w + 6\iint_{\mathbb{R}^{1+3}} (S(c'\delta\boldsymbol{m}))^2(S\boldsymbol{h}_\perp)^2 + O(\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2 + \delta\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^3),\end{aligned}$$

where we used (3.59) and (3.60). By (3.56), we know that

$$O(\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2 + \delta\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^3) = O(\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2).$$

Thus, using (3.55), to conclude the proof of (3.61) it remains to show that

$$4\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 w = -2\mathcal{C}_0 c'^2\delta^2\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2 + O(\delta^3\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}^2), \tag{3.62}$$

for which we will use the Lagrange multiplier theorem.

For $\boldsymbol{k} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$, let

$$W(\boldsymbol{k}) := \Phi(\boldsymbol{g} + \boldsymbol{k}) - \Phi(\boldsymbol{g}), \quad G(\boldsymbol{k}) := \|\boldsymbol{g} + \boldsymbol{k}\|_{\dot{\mathcal{H}}^{1/2}}^2, \tag{3.63}$$

so that $w = W(\boldsymbol{h})$, $0 = W(\boldsymbol{0})$ and $\delta^2 = G(\boldsymbol{0})$. Since $u_{\boldsymbol{g}} = \Phi(\boldsymbol{g})$ is a maximizer for $I(\delta)$, we have that

$$\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^4 = \max\left\{ \iint_{\mathbb{R}^{1+3}} (u_{\boldsymbol{g}} + W(\boldsymbol{k}))^4 \;\middle|\; G(\boldsymbol{k}) = \delta^2 \right\}; \tag{3.64}$$

that is, $\boldsymbol{k} = \boldsymbol{0}$ is a solution to the constrained optimization problem on the right-hand side of (3.64). In particular, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\mu G'(\boldsymbol{0})\boldsymbol{k} = 4 \iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 W'(\boldsymbol{0})\boldsymbol{k}, \quad \forall \boldsymbol{k} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3), \tag{3.65}$$

where the notation $F'(\boldsymbol{0})\boldsymbol{k}$ denotes the directional derivative $\frac{d}{d\varepsilon}F(\varepsilon\boldsymbol{k})\big|_{\varepsilon=0}$. We need to compute μ. First we note that, by the definition of G,

$$\mu G'(\boldsymbol{0})\boldsymbol{k} = 2\mu \langle \boldsymbol{g} \,|\, \boldsymbol{k}\rangle_{\dot{\mathcal{H}}^{1/2}}\,.$$

Now, by the definition (3.63) of W,

$$W(\boldsymbol{k}) = S\boldsymbol{k} + \sigma\Box^{-1}\left(\Phi(\boldsymbol{g}+\boldsymbol{k})^3 - \Phi(\boldsymbol{g})^3\right),$$

and the right-hand side is differentiable; see Remark 3.1.5. The directional derivative equals

$$W'(\boldsymbol{0})\boldsymbol{k} = S\boldsymbol{k} + 3\Box^{-1}(\Phi(\boldsymbol{g})^2\Phi'(\boldsymbol{g})\boldsymbol{k}) = S\boldsymbol{k} + O(\delta^2\|\boldsymbol{k}\|_{\dot{\mathcal{H}}^{1/2}}).$$

We insert this, the expansion (3.60) of $u_{\boldsymbol{g}}$ and the formula $\boldsymbol{g} = c'\delta\boldsymbol{m} + \boldsymbol{g}_\perp$, into (3.65) to obtain

$$2\mu\left\langle c'\delta\boldsymbol{m} \,\middle|\, \boldsymbol{k}\right\rangle_{\dot{\mathcal{H}}^{1/2}} + 2\mu\langle \boldsymbol{g}_\perp \,|\, \boldsymbol{k}\rangle_{\dot{\mathcal{H}}^{1/2}} = 4\iint_{\mathbb{R}^{1+3}} (S(c'\delta\boldsymbol{m}))^3 S\boldsymbol{k} + O(\delta^5\|\boldsymbol{k}\|_{\dot{\mathcal{H}}^{1/2}}).$$

We evaluate this equation at $\boldsymbol{k} = \boldsymbol{m}$, using that $\langle \boldsymbol{g}_\perp \,|\, \boldsymbol{m}\rangle_{\dot{\mathcal{H}}^{1/2}} = 0$ and that $\|S\boldsymbol{m}\|_{L^4}^4 = \mathcal{C}_0$. The result is

$$\mu = 2\mathcal{C}_0 c'^2\delta^2 + O(\delta^5).$$

We are now ready to conclude the proof of (3.62). We notice that $\|\boldsymbol{g}\|_{\dot{\mathcal{H}}^{1/2}}^2 = \|\boldsymbol{g}+\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2 = \delta^2$, so $2\langle \boldsymbol{g} \,|\, \boldsymbol{h}\rangle_{\dot{\mathcal{H}}^{1/2}} = -\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2$. Using this,

$$\begin{aligned} 4\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 w &= 4\iint_{\mathbb{R}^{1+3}} u_{\boldsymbol{g}}^3 W'(\boldsymbol{0})\boldsymbol{h} + O(\delta^3\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2), \\ &= 2\mu\langle \boldsymbol{g} \,|\, \boldsymbol{h}\rangle_{\dot{\mathcal{H}}^{1/2}} + O(\delta^3\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2) \\ &= -2\mathcal{C}_0 c'^2\delta^2\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2 + O(\delta^3\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2), \end{aligned}$$

where we used that $w = W(\boldsymbol{h}) = W'(\boldsymbol{0})\boldsymbol{h} + O(\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}^2)$. Since $\|\boldsymbol{h}\|_{\dot{\mathcal{H}}^{1/2}}$ equals $\|\boldsymbol{h}_\perp\|_{\dot{\mathcal{H}}^{1/2}}$ to main order (see (3.58)), the proof of (3.62) is complete. □

Appendix

3.7 Nonlinear profile decomposition

In this section, we adapt the linear profile decomposition of Ramos (see [62]) to sequences of solutions of (NLW). This is classical, and similar to what is done in [63], with the difference that we assign the initial data at $t = -\infty$, in the sense of Proposition 3.1.4.

We consider sequences of transformations of the form

$$\Lambda_n(t,x) = L^{\beta_n}\big(\lambda_n(t-t_n), \lambda_n(x-x_n)\big),$$

where $\lambda_n \in (0,\infty), t_n \in \mathbb{R}, x_n \in \mathbb{R}^3$ and $\beta_n \in \mathbb{R}^3$ with $|\beta_n| < 1$. Here we use the notation $a \sim b$, to mean that an absolute constant $C > 0$ exists such that $C^{-1}a \leqslant b \leqslant Ca$. The following definition is taken from [62].

Definition 3.7.1. Consider sequences $(\Lambda_n^1)_{n\in\mathbb{N}}, (\Lambda_n^2)_{n\in\mathbb{N}}$ as above and let

$$\tfrac{(\ell_n^j)^2-1}{(\ell_n^j)^2+1} = |\beta_n^j|, \quad \ell_n^j \in [1,\infty).$$

The sequences Λ_n^1 and Λ_n^2 are *orthogonal* if at least one of the following properties is satisfied:

1. Lorentz property:
$$\lim_{n\to\infty} \frac{\ell_n^1}{\ell_n^2} + \frac{\ell_n^2}{\ell_n^1} = \infty.$$

2. Rescaling property:
$$\lim_{n\to\infty} \frac{\lambda_n^{(1)}}{\lambda_n^{(2)}} + \frac{\lambda_n^{(2)}}{\lambda_n^{(1)}} = \infty.$$

3. Angular property: it holds that $\lambda_n^{(1)} \sim \lambda_n^{(2)}$, $\ell_n^1 \sim \ell_n^2$ and
$$\lim_{n\to\infty} \ell_n^1 \left| \frac{\beta_n^1}{|\beta_n^1|} - \frac{\beta_n^2}{|\beta_n^2|} \right| = \infty.$$

4. Spacetime translation property: it holds that $\lambda_n^{(1)} = \lambda_n^{(2)}, \beta_n^1 = \beta_n^2$ and

$$\lim_{n\to\infty} \left| L^{\beta_n^1}\left(\lambda_n^{(1)}(t_n^1 - t_n^2), \lambda_n^{(1)}(x_n^1 - x_n^2)\right) \right| = \infty.$$

Definition 3.7.1 is motivated by the following property.

Proposition 3.7.2. *If $w_1, w_2 \in L^4(\mathbb{R}^{1+3})$ and Λ_n^1, Λ_n^2 are orthogonal sequences of transformations, then for all $\alpha, \beta \in [0, \infty)$ such that $\alpha + \beta = 4$,*

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{1+3}} \left| \lambda_n^{(1)} w_1(\Lambda_n^1(t,x)) \right|^{\alpha} \left| \lambda_n^{(1)} w_2(\Lambda_n^2(t,x)) \right|^{\beta} dtdx = 0.$$

We can now recast, using our notation, the aforementioned linear profile decomposition of Ramos.

Theorem 3.7.3. *Let $\boldsymbol{f}_n$ be a bounded sequence in $\dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$. Then there exists an at most countable set*

$$\left\{ (\boldsymbol{F}^j, (\Lambda_n^j)_{n\in\mathbb{N}}) : j = 1, 2, 3, \ldots \right\}, \tag{3.66}$$

where $\boldsymbol{F}^j \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$ and the sequences (Λ_n^j) are pairwise orthogonal in the sense of Definition 3.7.1, such that, up to passing to a subsequence,

$$S\boldsymbol{f}_n = \sum_{j=1}^{J} \lambda_n^{(j)} (S\boldsymbol{F}^j) \circ \Lambda_n^j + S\boldsymbol{r}_n^J, \tag{3.67}$$

where the remainder term $\boldsymbol{r}_n^J$ satisfies the vanishing property

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| S\boldsymbol{r}_n^J \right\|_{L^4(\mathbb{R}^{1+3})} = 0. \tag{3.68}$$

Moreover, for each $J \geqslant 1$, we have the Pythagorean expansion, as $n \to \infty$,

$$\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{1/2}}^2 = \sum_{j=1}^{J} \left\| \boldsymbol{f}^j \right\|_{\dot{\mathcal{H}}^{1/2}}^2 + \left\| \boldsymbol{r}_n^J \right\|_{\dot{\mathcal{H}}^{1/2}}^2 + o(1). \tag{3.69}$$

To use Theorem 3.7.3 with nonlinear solutions, we will need the following lemma. We recall from Proposition 3.1.4 that a *solution* to (NLW) is a function $u \in L^4(\mathbb{R}^{1+3})$, with $\boldsymbol{u} \in C(\mathbb{R}; \dot{\mathcal{H}}^{1/2})$, that satisfies the fixed point equation

$$u = S\boldsymbol{f} + \sigma\Box^{-1}(u^3),$$

for a $\boldsymbol{f} \in \dot{\mathcal{H}}^{1/2}(\mathbb{R}^3)$. We write $u = \Phi(\boldsymbol{f})$. In particular, we are implicitly assuming that u is a *global* solution, in the sense that it is defined for all $t \in \mathbb{R}$. We will not consider non-global solutions.

Lemma 3.7.4 (Perturbation Lemma)**.** *Let $u = \Phi(\boldsymbol{f})$. For $\tilde{M} > 0$, assume that $\|\tilde{u}\|_{L^4(\mathbb{R}^{1+3})} \leqslant \tilde{M}$, where $\tilde{u}$ satisfies*

$$\lim_{t\to-\infty}\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{\dot{\mathcal{H}}^{1/2}} = 0, \quad \text{and } \|e\|_{L^{4/3}(\mathbb{R}^{1+3})} \leqslant \varepsilon, \tag{3.70}$$

where $e := \Box\tilde{u} - \sigma\tilde{u}^3$ in distributional sense. Then

$$\|u - \tilde{u}\|_{L^4(\mathbb{R}^{1+3})} + \sup_{t\in\mathbb{R}}\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{\dot{\mathcal{H}}^{1/2}} \leqslant C(\tilde{M})\,\varepsilon.$$

Proof. The assumptions (3.70) imply that $\tilde{u}$ satisfies the fixed-point equation

$$\tilde{u} = S\boldsymbol{f} + \sigma\Box^{-1}(\tilde{u}^3) + \Box^{-1}e,$$

so the difference $w := \tilde{u} - u$ satisfies $w = \sigma\Box^{-1}(\tilde{u}^3 - u^3) + \Box^{-1}e$. We now estimate w on a time interval $(-\infty, T) \subset \mathbb{R}$ via the Strichartz inequality (3.5), which holds on such time intervals because of Remark 3.1.3;

$$\begin{aligned}\|w\|_{L^4((-\infty,T)\times\mathbb{R}^3)} &\leqslant C\varepsilon + C|\sigma|\|(\tilde{u}+w)^3 - \tilde{u}^3\|_{L^{\frac{4}{3}}((-\infty,T)\times\mathbb{R}^3)} \\ &\leqslant C(\varepsilon + \|w\|^3_{L^4((-\infty,T)\times\mathbb{R}^3)}) + C\big\|\tilde{u}^2 w\big\|_{L^{\frac{4}{3}}((-\infty,T)\times\mathbb{R}^3)}.\end{aligned}$$

The Gronwall-type inequality of [32, Lemma 8.1] now implies that

$$\|w\|_{L^4((-\infty,T)\times\mathbb{R}^3)} \leqslant C_{\tilde{M}}(\varepsilon + \|w\|^3_{L^4((-\infty,T)\times\mathbb{R}^3)}).$$

Therefore, if $T \in \mathbb{R}$ is such that $\|w\|_{L^4((-\infty,T)\times\mathbb{R}^3)} \leqslant 2C_{\tilde{M}}\varepsilon$, then

$$\|w\|_{L^4((-\infty,T)\times\mathbb{R})} \leqslant C_{\tilde{M}}\varepsilon + C_{\tilde{M}}(2C_{\tilde{M}}\varepsilon)^3 \leqslant \frac{3}{2}C_{\tilde{M}}\varepsilon,$$

provided that ε is sufficiently small. By the bootstrap method, this proves the inequality $\|w\|_{L^4(\mathbb{R}^{1+3})} \leqslant \frac{3}{2}C_{\tilde{M}}\varepsilon$.

The same argument with $\sup_{t\in\mathbb{R}}\|\boldsymbol{w}(t)\|_{\dot{\mathcal{H}}^{1/2}}$ in place of $\|w\|_{L^4(\mathbb{R}^{1+3})}$ concludes the proof. ∎

Corollary 3.7.5. *Let $A > 0$ be such that, if $\|\boldsymbol{f}\|_{\dot{\mathcal{H}}^{1/2}} \leqslant A$, then there exists a unique solution $u = \Phi(\boldsymbol{f})$. If the sequence $u_n = \Phi(\boldsymbol{f}_n)$ satisfies $\|\boldsymbol{f}_n\|_{\dot{\mathcal{H}}^{1/2}} \leqslant A$, we associate to each profile $(\boldsymbol{F}^j, \Lambda^j_n)$ in* (3.66) *the* nonlinear profile

$$U^j := \Phi(\boldsymbol{F}^j).$$

Then

$$u_n(t,x) = \sum_{j=1}^{J}\lambda^{(j)}_n U^j(\Lambda^j_n(t,x)) + S\boldsymbol{r}^J_n(t,x) + h^J_n(t,x), \tag{3.71}$$

where $\boldsymbol{r}^J_n$ is the same as in (3.67)*, while h^J_n is a sequence that satisfies the vanishing condition*

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left(\big\|h^J_n\big\|_{L^4(\mathbb{R}^{1+3})} + \sup_{t\in\mathbb{R}}\big\|\boldsymbol{h}^J_n(t)\big\|_{\dot{\mathcal{H}}^{1/2}}\right) = 0. \tag{3.72}$$

Proof. To apply Lemma 3.7.4, we fix $J \in \mathbb{N}$ and we denote

$$\tilde{u}_n^J(t,x) = \sum_{j=1}^{J} \lambda_n^{(j)} U_n^j(\Lambda_n^j(t,x)) + S\boldsymbol{r}_n^J.$$

By orthogonality of the sequences Λ_n^j (see Proposition 3.7.2), and by the vanishing property (3.68) of $S\boldsymbol{r}_n^J$, we can find a sequence $\varepsilon_n^J \geqslant 0$ satisfying $\lim_J \ \limsup_n \ \varepsilon_n^J = 0$ and such that

$$\begin{aligned} \left\|\tilde{u}_n^J\right\|_{L^4(\mathbb{R}^{1+3})}^4 &= \sum_{j=1}^{J} \left\|U^j\right\|_{L^4(\mathbb{R}^{1+3})}^4 + \varepsilon_n^J \\ &\leqslant C\Big(\sum_{j=1}^{J} \left\|\boldsymbol{F}^j\right\|_{\dot{\mathcal{H}}^{1/2}}^2\Big)^2 + \varepsilon_n^J \leqslant C_A A^4, \end{aligned} \tag{3.73}$$

where we used the estimate (3.6) and the Pythagorean expansion (3.69). We remark that the estimate (3.73) is uniform in J. In order to apply the perturbation Lemma 3.7.4, we notice that, by (3.67),

$$\lim_{t\to-\infty} \left\|\boldsymbol{u}_n(t) - \tilde{\boldsymbol{u}}_n^J(t)\right\|_{\dot{\mathcal{H}}^{1/2}} = 0,$$

and, moreover,

$$\begin{aligned} e_n^J :&= \Box\, \tilde{u}_n^J - \sigma(\tilde{u}_n^J)^3 \\ &= -\sigma\left[\left(\sum_{j=1}^{J} \lambda_n^{(j)} U^j \circ \Lambda_n^j + S\boldsymbol{r}_n^J\right)^3 - \sum_{j=1}^{J}\left(\lambda_n^{(j)} U^j \circ \Lambda_n^j\right)^3\right], \end{aligned}$$

so, again by orthogonality of $\{\Lambda_n^j : j = 1 \dots J\}$ and vanishing of $S\boldsymbol{r}_n^J$,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\|e_n^J\right\|_{L^4(\mathbb{R}^{1+3})} = 0.$$

We thus obtain (3.72), concluding the proof. □

Remark 3.7.6. Proposition 3.7.2 also implies that

$$\left\|u_n\right\|_{L^4(\mathbb{R}^{1+3})}^4 = \sum_{j=1}^{J} \left\|U^j\right\|_{L^4(\mathbb{R}^{1+3})}^4 + \left\|S\boldsymbol{r}_n^J\right\|_{L^4(\mathbb{R}^{1+3})}^4 + \varepsilon_n^J,$$

where

$$\lim_{J\to\infty} \limsup_{n\to\infty} \varepsilon_n^J = 0.$$

3.8 Some explicit solutions to the cubic wave equation

The Penrose transform can be used to find the smooth solutions

$$u^a(t,x) = \frac{2}{\sqrt{1+(t-|x|)^2}\sqrt{1+(t+|x|)^2}}, \quad u^b(t,x) = \frac{2\sqrt{2}}{1+|x|^2-t^2},$$

to the focusing equation

$$\Box u = u^3, \quad \text{on } \mathbb{R}^{1+3}. \tag{3.74}$$

Note that u^b blows up at time $t = 1$. These solutions are known; for example, u^a is [6, equation (4.8)] (with $c = 1$), and u^b is [18, equation (7)] (with $a = -1, b = 1/2$), where they are computed with different methods. Our alternative method, based on the Penrose transform, involves only a very short computation.

We recall from Section 1.1, in the first chapter, that the Penrose transform associates to any function u, defined on $\mathbb{R}^{1+3}$, a function U defined on the region

$$\mathcal{P}(\mathbb{R}^{1+3}) = \left\{ \Big(T, (\cos R, \sin R\,\omega)\Big) \in \mathbb{R} \times \mathbb{S}^3 \;\middle|\; \begin{array}{c} -\pi < T < \pi \\ 0 \leqslant R < \pi - |T| \\ \omega \in \mathbb{S}^2 \end{array} \right\}.$$

They satisfy

$$u(t, r\omega) = \Omega\, U(T, \cos R, \sin R\omega), \tag{3.75}$$

where $r \geqslant 0$ and $\omega \in \mathbb{S}^2$ are the polar coordinates on $\mathbb{R}^3$, and

$$t = \arctan(T + R) + \arctan(T - R), \quad r = \arctan(T + R) - \arctan(T - R),$$

and the function Ω is

$$\Omega = 2(1 + (t + r)^2)^{-1/2}(1 + (t - r)^2)^{-1/2}.$$

As we noted in (1.7), the mapping of $\mathbb{R}^{1+3}$ onto $\mathcal{P}(\mathbb{R}^{1+3})$ is conformal; therefore

$$\Box u = \Omega^3(\partial_T^2 - \Delta_{\mathbb{S}^3} + 1)U;$$

see (1.10). We conclude that (3.74) is equivalent to the equation

$$\partial_T^2 U - \Delta_{\mathbb{S}^3} U + U = U^3, \quad \text{on } \mathcal{P}(\mathbb{R}^{1+3}). \tag{3.76}$$

Considering functions U that depend on T only, (3.76) reduces to the ordinary differential equation

$$U'' + U = U^3,$$

which has the conserved quantity

$$E = \frac{(U')^2}{2} + \frac{U^2}{2} - \frac{U^4}{4}.$$

It follows immediately from (3.75) that the stationary solution $U^a = 1$ is the Penrose transform of u^a. The blow up solution $U^b = \frac{\sqrt{2}}{\cos T}$, which is characterized by the properties $E = 0$ and $\partial_T U^b(0) = 0$, is the Penrose transform of u^b. To see this, we use the formula

$$\cos T = \frac{1}{2}(1 + |x|^2 - t^2)\Omega,$$

which can be found, for example, in [46, pag. 277].

Remark 3.8.1. The solution u^b can also be obtained by setting $x_1 = it$ in

$$Q = \frac{2\sqrt{2}}{1 + x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

Indeed, Q solves the elliptic equation $-\Delta_{\mathbb{R}^4} Q = Q^3$, which is transformed into (3.74) by the formal substitution $x_1 \mapsto it$.

3.9 The $\dot{\mathcal{H}}^{1/2}$ norm is not Lorentz-invariant

The following lemma immediately implies the existence of smooth solutions u to (NLW) such that $\|\boldsymbol{u}(t)\|_{\dot{\mathcal{H}}^{1/2}}$ is not preserved by time translations and Lorentzian transformations. We recall from Section 3.2 that, for all $\alpha \in (-1, 1)$,

$$L^\alpha(t, x) = (\gamma t - \gamma\alpha x_1, \gamma x_1 - \gamma\alpha t, x_2, x_3), \quad \text{where } \gamma = (1 - \alpha^2)^{-1/2}.$$

Lemma 3.9.1. *Let u be a smooth global solution to $\Box u = \sigma u^3$ on $\mathbb{R}^{1+3}$. Then*

$$\frac{\partial}{\partial t_0}\|\boldsymbol{u}(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}} = 2\sigma \int_{\mathbb{R}^3} (-\Delta)^{-1/2}(u_t(t_0, \cdot))u^3(t_0, x)\, dx, \tag{3.77}$$

and, letting $u_\alpha := u \circ L^\alpha$,

$$\frac{\partial}{\partial \alpha}\|\boldsymbol{u}_\alpha(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}}\Big|_{\alpha=0} = -2\sigma \int_{\mathbb{R}^3} x_1(-\Delta)^{-1/2}(u_t(t_0, \cdot))u^3(t_0, x)\, dx. \tag{3.78}$$

Proof. We recall that $\boldsymbol{u}(t_0)$ denotes the pair $(u(t_0, \cdot), u_t(t_0, \cdot))$. Using the equation, we obtain

$$\partial_{t_0}\boldsymbol{u}(t_0) = (u_t(t_0, \cdot), \Delta u(t_0, \cdot) + \sigma u^3(t_0, \cdot)). \tag{3.79}$$

Therefore

$$\begin{aligned}
&\partial_{t_0}\|\boldsymbol{u}(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}} = 2\langle \boldsymbol{u}(t_0) \,|\, \partial_{t_0}\boldsymbol{u}(t_0)\rangle_{\dot{\mathcal{H}}^{1/2}} \\
&= 2\int_{\mathbb{R}^3} (-\Delta)^{1/2}u(t_0, x)u_t(t_0, x)\, dx + 2\int_{\mathbb{R}^3} (-\Delta)^{-1/2}u_t(t_0, x)\Delta u(t_0, x)\, dx \\
&\quad + 2\sigma \int_{\mathbb{R}^3} (-\Delta)^{-1/2}u_t(t_0, x)u^3(t_0, x)\, dx.
\end{aligned}$$

Since $(-\Delta)^{-1/2}\Delta = -(-\Delta)^{1/2}$, the first two summands cancel, yielding (3.77).

To prove (3.78), we begin by observing that

$$\partial_\alpha \boldsymbol{u}_\alpha(t_0)|_{\alpha=0} = -(x_1\partial_{t_0} + t_0\partial_{x_1})\boldsymbol{u}(t_0) - (0, \partial_{x_1}u(t_0)).$$

Integration by parts immediately shows that $\langle \boldsymbol{u}(t_0) \,|\, t_0\partial_{x_1}\boldsymbol{u}(t_0)\rangle_{\dot{\mathcal{H}}^{1/2}} = 0$. So, reasoning as before and using (3.79), we obtain

$$\begin{aligned}
&-\frac{1}{2}\partial_{\alpha=0}\|\boldsymbol{u}_\alpha(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}} = \langle \boldsymbol{u}(t_0) \,|\, x_1\partial_{t_0}\boldsymbol{u}(t_0) + (0, \partial_{x_1}u(t_0))\rangle_{\dot{\mathcal{H}}^{1/2}} \\
&= \int_{\mathbb{R}^3} u_t(-\Delta)^{-\frac{1}{2}}(x_1\Delta u) + (-\Delta)^{\frac{1}{2}}u\, x_1 u_t + (-\Delta)^{-\frac{1}{2}}u_t\, \partial_{x_1}u + \sigma(-\Delta)^{-\frac{1}{2}}u_t\, x_1\, u^3.
\end{aligned}$$

Now, using the elementary commutator identity $[(-\Delta)^{-1/2}, x_1] = (-\Delta)^{-3/2}\partial_{x_1}$, we see that the first three summands cancel. This completes the proof. $\square$

It is very easy to construct solutions to (NLW) such that the derivatives in (3.77) and (3.78) do not vanish. For example, if $f_0 \neq 0$ is a smooth function with compact support, then letting $f_1 = f_0^3$, and considering a sufficiently small $\varepsilon > 0$, there exists a unique smooth solution u to

$$\begin{cases} \Box u = \sigma u^3, & \text{on } \mathbb{R}^{1+3}, \\ \boldsymbol{u}(0) = \varepsilon \boldsymbol{f}, \end{cases}$$

and by (3.77), $\partial_{t_0=0}\|\boldsymbol{u}(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}} \neq 0$. Replacing f_1 by $|x_1|^{-4/3}x_1 f_0^3$, we obtain a solution with the property that $\partial_{\alpha=0}\|\boldsymbol{u}_\alpha(0)\|^2_{\dot{\mathcal{H}}^{1/2}} \neq 0$.

Remark 3.9.2. If u is radially symmetric, then the formula (3.77) simplifies;

$$\frac{\partial}{\partial t_0}\|\boldsymbol{u}(t_0)\|^2_{\dot{\mathcal{H}}^{1/2}} = C\sigma \int_0^\infty \int_0^\infty u_t(t_0, r)u^3(t_0, s) rs \log\left|\frac{r+s}{r-s}\right| drds. \tag{3.80}$$

Indeed, rewriting the right-hand side of (3.77) as a convolution;

$$C\sigma \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_t(t_0, x)u^3(t_0, y)}{|x-y|^2}\, dxdy,$$

if u_t and u are radially symmetric, this can be further simplified as

$$C\sigma \int_0^\infty \int_0^\infty u_t(t_0, r)u^3(t_0, s) r^2 s^2 \iint\limits_{\mathbb{S}^2\times\mathbb{S}^2} \frac{dS(\omega)dS(\eta)}{|r\omega - s\eta|^2}\, drds.$$

Then (3.80) follows from the formula

$$\int_{\mathbb{S}^2} \frac{dS(\omega)}{|r\omega - s\eta|^2} = C\frac{1}{rs}\log\left|\frac{r+s}{r-s}\right|,$$

which is a standard consequence of the Funk-Hecke theorem; see for example [45, Section 3].

Using the previous remark, we can prove that

$$\left.\frac{\partial}{\partial t}\right|_{t=1} \|\boldsymbol{u}^a(t)\|^2_{\dot{\mathcal{H}}^{1/2}} < 0, \tag{3.81}$$

where $u^a(t,x) = 2(1+(t-|x|)^2)^{-1/2}(1+(t+|x|)^2)^{-1/2}$ is the explicit solution to (NLW) which we found in the previous section. To begin, we compute

$$(u^a(1,r), u^a_t(1,r)) = \left(\frac{2}{\sqrt{r^4+4}}, \frac{4(r^2-1)}{(r^4+4)^{\frac{3}{2}}}\right).$$

Applying (3.80),

$$\frac{\partial}{\partial t}\bigg|_{t=1} \|\boldsymbol{u}^a(t)\|^2_{\dot{\mathcal{H}}^{1/2}} = C \iint rs \log\left|\frac{r+s}{r-s}\right| (r^4+4)^{-3/2}(s^4+4)^{-3/2}(r^2-2)\, drds.$$

We let I denote the integral in the right-hand side. Applying the scaling $(r,s)\mapsto\sqrt{2}(r,s)$ and symmetrizing with respect to the transformation $(r,s)\mapsto(s,r)$, we have

$$2I = \iint rs \log\left|\frac{r+s}{r-s}\right| (r^4+1)^{-3/2}(s^4+1)^{-3/2}(r^2+s^2-1)\, drds.$$

We symmetrize again, this time with respect to the inversion

$$(r,s)\mapsto\left(\tfrac{r}{r^2+s^2}, \tfrac{s}{r^2+s^2}\right), \quad drds \mapsto \tfrac{drds}{(r^2+s^2)^2},$$

to obtain

$$\begin{aligned} 4I = \int_0^\infty\int_0^\infty rs \log\left|\frac{r+s}{r-s}\right| (r^2+s^2-1) \\ \left[\frac{1}{(r^4+1)^{\frac{3}{2}}(s^4+1)^{\frac{3}{2}}} - \frac{r^2+s^2}{(r^4+(r^2+s^2)^4)^{\frac{3}{2}}(s^4+(r^2+s^2)^4)^{\frac{3}{2}}}\right] drds. \end{aligned}$$

We note that the integrand function is nonpositive for all $r,s\geqslant 0$. Indeed, both the term in the round brackets and the one in the square brackets change sign only on the line $r^2+s^2=1$, so the two signs cancel each other. This proves (3.81).

Bibliography

[1] H. Brezis and E. Lieb. “Sobolev inequalities with remainder terms”. In: *J. Funct. Anal.* 62 (1985), pp. 73 –86.

[2] B. Dodson. “Global well-posedness and scattering for the radial, defocusing, cubic wave equation with initial data in a critical Besov space”. In: *Anal. PDE* 12.4 (2019), pp. 1023–1048.

[3] B. Dodson. “Global well-posedness and scattering for the radial, defocusing, cubic nonlinear wave equation with almost sharp initial data”. In: *Comm. PDE* 43.10 (2018), pp. 1413–1455.

[4] B. Dodson. “Global well-posedness for the radial, defocusing, nonlinear wave equation for $3 < p < 5$”. In: *ArXiv e-prints* (2018). arXiv: 1810.02879.

[5] B. Dodson. “Global well-posedness and scattering for the radial, defocusing, cubic nonlinear wave equation”. In: *ArXiv e-prints* (2018). arXiv: 1809.08284.

[6] S. Anco and S. Liu. “Exact solutions of semilinear radial wave equations in n dimensions”. In: *J. Math. Anal. Appl.* 297 (2004), pp. 317–342.

[7] K. Atkinson and W. Han. *Spherical harmonics and approximations on the unit sphere: an introduction.* Springer, 2012.

[8] T. Aubin. “Problèmes isopérimétriques et espaces de Sobolev”. In: *J. Differential Geometry* 11.4 (1976), pp. 573–598.

[9] H. Bahouri and P. Gérard. “High frequency approximation of solutions to critical nonlinear wave equations”. In: *Amer. J. Math.* 121.1 (1999), pp. 131–175.

[10] J. Bennett, N. Bez, and M. Iliopoulou. “Flow monotonicity and Strichartz inequalities”. In: *Int. Math. Res. Not.* 19 (2015), pp. 9415–9437.

[11] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark. “Heat-flow monotonicity of Strichartz norms”. In: *Anal. PDE* 2.2 (2009), pp. 147–158.

[12] N. Bez and C. Jeavons. “A sharp Sobolev-Strichartz estimate for the wave equation”. In: *Electron. Res. Announc. Math. Sci.* 22 (2015), pp. 46–54.

[13] N. Bez, C. Jeavons, and N. Pattakos. “Sharp Sobolev-Strichartz estimates for the free Schrödinger propagator”. In: *Current trends in analysis and its applications.* 2015, pp. 281–288.

[14] N. Bez and K. M. Rogers. "A sharp Strichartz estimate for the wave equation with data in the energy space". In: *J. Eur. Math. Soc. (JEMS)* 15.3 (2013), pp. 805–823.

[15] N. Bez, H. Saito, and M. Sugimoto. "Applications of the Funk-Hecke theorem to smoothing and trace estimates". In: *Adv. Math.* 285 (2015), pp. 1767 –1795.

[16] N. Bez and M. Sugimoto. "Optimal constants and extremisers for some smoothing estimates". In: *J. Anal. Math.* 131 (2017), pp. 159–187.

[17] G. Bianchi and H. Egnell. "A note on the Sobolev inequality". In: *J. Funct. Anal.* 100.1 (1991), pp. 18–24.

[18] P. Bizoń and A. Zenginoğlu. "Universality of global dynamics for the cubic wave equation". In: *Nonlinearity* 22.10 (2009).

[19] A. Bulut. "Maximizers for the Strichartz inequalities for the wave equation". In: *Differential Integral Equations* 23.11/12 (2010), pp. 1035–1072.

[20] E. Carneiro, D. Foschi, D. Oliveira e Silva, and C. Thiele. "A sharp trilinear inequality related to Fourier restriction on the circle". In: *Rev. Mat. Iberoam.* 33.4 (2017), pp. 1463–1486.

[21] E. Carneiro. "A sharp inequality for the Strichartz norm". In: *Int. Math. Res. Not. IMRN* 16 (2009), pp. 3127–3145.

[22] E. Carneiro and D. Oliveira e Silva. "Some sharp restriction inequalities on the sphere". In: *Int. Math. Res. Not. IMRN* 17 (2015), pp. 8233–8267.

[23] S. Chen, R. L. Frank, and T. Weth. "Remainder Terms in the Fractional Sobolev Inequality". In: *Indiana Univ. Math. J.* 62.4 (2013), pp. 1381–1397.

[24] M. Christ. "A sharpened Hausdorff-Young inequality". In: *ArXiv e-prints* (2014). arXiv: 1406.1210.

[25] D. Christodoulou. "Global solutions of nonlinear hyperbolic equations for small initial data". In: *Math. Z.* 39 (1986), pp. 267–282.

[26] D. Christodoulou. "Solutions globales des équations de champ de Yang-Mills". In: *C. R. Acad. Sci. Paris* Serie I.Tome 293 (1981), pp. 39–42.

[27] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. "The sharp Sobolev inequality in quantitative form". In: *J. Eur. Math. Soc. (JEMS)* 11 (2009), pp. 1105–1139.

[28] B. Dodson and A. Lawrie. "Scattering for the radial 3D cubic wave equation". In: *Anal. PDE* 8.2 (2015), pp. 467–497.

[29] B. Dodson, A. Lawrie, D. Mendelson, and J. Murphy. "Scattering for defocusing energy subcritical nonlinear wave equations". In: *ArXiv e-prints* (2018). arXiv: 1810.03182.

[30] T. Duyckaerts and F. Merle. "Scattering norm estimate near the threshold for energy-critical focusing semilinear wave equation". In: *Indiana Univ. Math. J.* 58.4 (2009), pp. 1971–2001.

[31] T. Duyckaerts, F. Merle, and S. Roudenko. “Maximizers for the Strichartz norm for small solutions of mass-critical NLS”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 10.2 (2011), pp. 427–476.

[32] D. Fang, J. Xie, and T. Cazenave. “Scattering for the focusing energy-subcritical nonlinear Schrödinger equation”. In: *Sci. China Math.* 54.10 (2011), pp. 2037–2062.

[33] A. Figalli, F. Maggi, and A. Pratelli. “A mass transportation approach to quantitative isoperimetric inequalities”. In: *Invent. Math.* 182.1 (2010), pp. 167–211.

[34] A. Figalli and R. Neumayer. “Gradient stability for the Sobolev inequality: the case $p \geqslant 2$”. In: *J. Eur. Math. Soc. (JEMS)* 21.2 (2019), pp. 319–354.

[35] D. Foschi and D. Oliveira e Silva. “Some recent progress on sharp Fourier restriction theory”. In: *Anal. Math.* 43.2 (2017), pp. 241–265.

[36] D. Foschi. “Global maximizers for the sphere adjoint Fourier restriction inequality”. In: *J. Funct. Anal.* 268.3 (2015), pp. 690–702.

[37] D. Foschi. “Maximizers for the Strichartz inequality”. In: *J. Eur. Math. Soc. (JEMS)* 9.4 (2007), pp. 739–774.

[38] V. Georgiev and P. Schirmer. “Global existence of low regularity solutions of non-linear wave equations”. In: *Math. Z.* 219 (1995), pp. 1–19.

[39] P. Gérard. “Description du défaut de compacité de l’injection de Sobolev”. In: *ESAIM Control Optim. Calc. Var.* 3 (1998), pp. 213–233.

[40] J. Ginibre, A. Soffer, and G. Velo. “The global Cauchy problem for the critical nonlinear wave equation”. In: *J. Funct. Anal.* 110.1 (1992), pp. 96–130.

[41] J. Ginibre and G. Velo. “Generalized Strichartz inequalities for the wave equation”. In: *J. Funct. Anal.* 133.1 (1995), pp. 50–68.

[42] J. Ginibre and G. Velo. “On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case”. In: *J. Funct. Anal.* 32.1 (1979), pp. 1–32.

[43] F. Gonçalves. “A Sharpened Strichartz Inequality For Radial Functions”. In: *ArXiv e-prints* (2017). arXiv: `1709.08100`.

[44] F. Gonçalves. “Orthogonal Polynomials and Sharp Estimates for the Schrödinger Equation”. In: *Int. Math. Res. Not.* (2017), Art. ID rnx200.

[45] W. Han, K. Atkinson, and H. Zheng. “Some integral identities for spherical harmonics in an arbitrary dimension”. In: *J. Math. Chem.* 50 (2012), pp. 1126 – 1135.

[46] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations.* Vol. 26. Springer-Verlag, Berlin, 1997, pp. viii+289.

[47] D. Hundertmark and V. Zharnitsky. “On sharp Strichartz inequalities in low dimensions”. In: *Int. Math. Res. Not.* (2006), Art. ID 34080, 18.

[48] C. Jeavons. “A sharp bilinear estimate for the Klein-Gordon equation in arbitrary space-time dimensions”. In: *Differential Integral Equations* 27.1-2 (2014), pp. 137–156.

[49] J.-C. Jiang and S. Shao. “On characterization of the sharp Strichartz inequality for the Schrödinger equation”. In: *Anal. PDE* 9.2 (2016), pp. 353–361.

[50] M. Keel and T. Tao. “Endpoint Strichartz estimates”. In: *Amer. J. Math.* 120.5 (1998), pp. 955–980.

[51] R. Killip, B. Stovall, and M. Visan. “Scattering for the cubic Klein-Gordon equation in two space dimensions”. In: *Trans. Amer. Math. Soc.* 364 (2012), pp. 1571–1631.

[52] M. Kunze. “On the existence of a maximizer for the Strichartz inequality”. In: *Comm. Math. Phys.* 243.1 (2003), pp. 137–162.

[53] E. H. Lieb. “Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities”. In: *The Annals of Mathematics* 118.2 (1983), pp. 349–374.

[54] E. H. Lieb and M. Loss. *Analysis*. Second. Vol. 14. American Mathematical Society, Providence, RI, 2001.

[55] P.-L. Lions. “The concentration-compactness principle in the calculus of variations. The limit case, Part 1”. In: *Rev. Mat. Iberoam.* 1.1 (1985), pp. 145–201.

[56] F. Merle and L. Vega. “Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2d”. In: *Int. Math. Res. Not. IMRN* 8 (1998), pp. 399–425.

[57] C. Morpurgo. “Sharp inequalities for functional integrals and traces of conformally invariant operators”. In: *Duke Math. J.* 114.3 (2002), pp. 477–553.

[58] C. Müller. *Analysis of spherical symmetries in Euclidean spaces*. Vol. 129. Springer - Verlag New York, 1998.

[59] D. Oliveira e Silva. “Extremizers for Fourier restriction inequalities: convex arcs”. In: *J. Anal. Math.* 124 (2014), pp. 337–385.

[60] T. Ozawa and K. M. Rogers. “A sharp bilinear estimate for the Klein-Gordon equation in $\mathbb{R}^{1+1}$”. In: *Int. Math. Res. Not. IMRN* 5 (2014), pp. 1367–1378.

[61] R. Penrose. “Republication of: Conformal treatment of infinity”. In: *Gen. Relativity Gravitation* 43.3 (2011), pp. 901–922.

[62] J. Ramos. “A refinement of the Strichartz inequality for the wave equation with applications”. In: *Adv. Math.* 230.2 (2012), pp. 649–698.

[63] J. Ramos. “Nonlinear profile decomposition for the $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$ energy subcritical nonlinear wave equation”. In: *J. Funct. Anal.* 275 (2018), pp. 2614–2646.

[64] C. Rodriguez. “Scattering for radial energy-subcritical wave equations in dimensions 4 and 5”. In: *Comm. PDE* 42.6 (2017).

[65] I. Segal. “Space-time decay for solutions of wave equations”. In: *Adv. Math.* 22 (1976), pp. 305–311.

[66] R. Shen. “On the energy subcritical nonlinear wave equation with radial data”. In: *Anal. PDE* 6.8 (2013), pp. 1929–1987.

[67] H. Smith and C. Sogge. “Global Strichartz estimates for nontrapping perturbations of the Laplacian”. In: *Comm. Partial Differential Equations* 25.11-12 (2000), pp. 2171–2183.

[68] R. Strichartz. “Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations”. In: *Duke Math. J.* 44.3 (1977).

[69] G. Talenti. “Best constant in Sobolev inequality”. In: *Ann. Mat. Pura Appl. (4)* 110 (1976), pp. 353–372.

[70] T. Tao. “Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions”. In: *Dynamics of PDE* 3.2 (2006), pp. 93–110.

[71] D. Tataru. “Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation”. In: *Trans. Amer. Math. Soc.* 353.2 (2000), pp. 795–807.

[72] P. A. Tomas. “A restriction theorem for the Fourier transform”. In: *Bull. Amer. Math. Soc.* 81 (1975), pp. 477–478.

[73] N. Vilenkin. *Special functions and the theory of group representations.* American Mathematical Society, 1968.

Sharp estimates for linear and nonlinear wave equations via the Penrose transform

Abstract. We apply the Penrose transform, which is a basic tool of relativistic physics, to the study of sharp estimates for linear and nonlinear wave equations. We disprove a conjecture of Foschi, regarding extremizers for the Strichartz inequality with data in the Sobolev space $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, for even $d \geqslant 2$. On the other hand, we provide evidence to support the conjecture in odd dimensions and refine his sharp inequality in $\mathbb{R}^{1+3}$, adding a term proportional to the distance of the initial data from the set of extremizers. Using this, we provide an asymptotic formula for the Strichartz norm of small solutions to the cubic wave equation in Minkowski space. The leading coefficient is given by Foschi's sharp constant. We calculate the constant in the second term, whose absolute value and sign changes depending on whether the equation is focusing or defocusing.

Keywords. Wave equation, Strichartz estimate, sharp inequality, Lorentz invariance.

Résumé. Nous appliquons la transformée de Penrose, qui est un outil basique de la physique relativiste, à des estimations optimales pour les équations des ondes linéaire et nonlinéaire. Nous infirmons une conjecture de Foschi concernant les points extrémaux de l'inégalité de Strichartz à données dans l'espace de Sobolev $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, où $d \geqslant 2$ est pair. En revanche, nous donnons des indications appuyant cette conjecture en dimension impaire, ainsi qu'une version raffinée de son inégalité optimale sur $\mathbb{R}^{1+3}$, en ajoutant un terme proportionnel à la distance des données initiales de l'ensemble des points extrémaux. À l'aide de ce résultat, nous obtenons une formule asymptotique pour la norme de Strichartz des solutions petites de l'équation des ondes cubique dans l'espace-temps de Minkowski. Le coefficient principal est donné par la constante optimale de Foschi. Nous calculons le terme suivant, qui change de signe et de valeur absolue selon que la non-linéarité est focalisante ou défocalisante.

Mots-clés. Équation des ondes, estimation de Strichartz, inégalité optimale, invariance de Lorentz.

Resumen. Aplicamos la transformada de Penrose, una herramienta básica de la física relativista, a unas estimaciones óptimas para ecuaciones de ondas lineales y no lineales. Invalidamos una conjetura de Foschi, sobre extremizadores para la estimación de Strichartz con datos en el espacio de Sobolev $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^d)$, para $d \geqslant 2$ par. Por otro lado, vamos a dar indicios en favor de su conjetura en dimension impar, así como una versión refinada de su desigualdad óptima en $\mathbb{R}^{1+3}$, añadiendo un término proporcional a la distancia de los datos iniciales del conjunto de puntos extremales. Utilizando este resultado, conseguimos una fórmula asintótica para la norma de Strichartz de soluciones pequeñas de la ecuación de ondas cúbica en el espacio-tiempo de Minkowski. El coeficiente principal coincide con la constante óptima de Foschi. Calculamos explícitamente el coeficiente del otro término, cuyo módulo y signo cambian dependiendo de si estamos en el caso *focusing* o *defocusing*.

Palabras clave. Ecuación de ondas, estimación de Strichartz, desigualdad óptima, invariancia de Lorentz.