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Stability Near Hydrostatic Equilibrium in Fluid Mechanics

Tesis presentada por Daniel Lear Claveras
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A Lisardo Claveras y Abundia Pilar Lafuente,
también a mi abuelo y mi abuela,
pero sobre todo, a mis yayos.

Resumen y conclusiones

Un reto fundamental de la física–matemática es entender la dinámica de los sistemas físicos a medida que evolucionan para tiempos suficientemente largos. Este problema es particularmente interesante cuando abordamos el estudio de sistemas sin disipación ni fuerzas externas. En concreto, en esta memoria nos centraremos en la estabilidad del *equilibrio hidrostático*, en dos problemas dentro del campo de la mecánica de los fluidos.

En mecánica de fluidos, se dice que un fluido está en *equilibrio hidrostático* cuando este está en reposo. Si el fluido está en reposo, entonces las fuerzas que actúan sobre él deben compensarse. Por lo tanto, surge de manera natural la siguiente pregunta:

¿Qué sucede si partimos de un estado proximo a la solución de equilibrio hidrostático?

El campo de la *estabilidad hidrodinámica* tiene una larga historia que comienza en el siglo XIX. Uno de los problemas más antiguos tratados es el de la estabilidad e inestabilidad de los flujos cortantes, que se remonta a la época de Lord Rayleigh y Lord Kelvin.

Para nosotros, el problema a tratar es considerar una pequeña perturbación del *equilibrio hidrostático*, en cuyo caso el fluido debe comenzar a moverse, y estudiar el comportamiento a largo plazo de la solución. En particular, nos centraremos en *equilibrios laminares*, en los cuales el fluido se mueve en capas bien ordenadas. Sin embargo, incluso para estas configuraciones tan simples, poco se sabe acerca de la dinámica de la solución.

En esta memoria consideramos dos problemas diferentes dentro del campo de la mecánica de fluidos. En el segundo capítulo estudiaremos las ecuaciones que rigen la dinámica de un fluido incompresible en un medio poroso. Y en el tercer capítulo presentamos la dinámica de un fluido bajo la aproximación de *Boussinesq*, que consiste en eliminar la dependencia de la densidad en todos los términos, excepto el que involucra a la fuerza de la gravedad.

La ecuación (IPM) incompresible de los medios porosos: El movimiento de los fluidos a través de un medio poroso es de gran intereses, ya que aparece en una amplia gama de problemas reales que vienen de muchas áreas de las ciencias aplicadas y de la ingeniería. El efecto del medio tiene importantes consecuencias y las ecuaciones habituales para la conservación del momento, es decir, las ecuaciones de Euler o Navier–Stokes, deben ser reemplazadas por la experimental *Ley de Darcy*.

Este principio físico, observado por primera vez por Henry Darcy en 1856, proporciona una descripción macroscópica de un flujo donde la velocidad del fluido es proporcional al gradiente de la presión y a las fuerzas externas. Desde un punto de vista matemático, la ecuación IPM pertenece a una clase más general de ecuaciones a las que a menudo se denomina como *escalares activos*, que consisten en resolver el problema de Cauchy para una ecuación de transporte donde el campo de velocidades está relacionado con el escalar que es transportado por el flujo mediante un operador. Quizás el mejor ejemplo de uno de estos escalares activos es la ecuación quasi-geostrófica superficial (SQG). En geofísica, la evolución de fluidos atmosféricos y oceánicos se modelan considerando la importancia de la fuerza de Coriolis en la dinámica. Concretamente, SQG proporciona soluciones particulares de la evolución de la temperatura de un sistema quasi-geostrófico general para números pequeños de Rossby y Ekman.

Un aspecto de gran importancia de la ecuación SQG, desde el punto de vista matemático, fue señalado por Constantin, Majda y Tabak en un trabajo en la que la propusieron como un modelo escalar y dosdimensional de la ecuación de Euler tridimensional. Desde entonces, esta ecuación ha sido una fuente de inspiración para la ecuación de Euler, de tal manera que los resultados principales para SQG pueden extenderse a la ecuación de Euler.

A pesar del hecho de que existen grandes similitudes entre las ecuaciónes de IPM y SQG, también hay importantes diferencias. Cabe destacar que tanto para IPM como para SQG, el operador que relaciona el campo de velocidades con el escalar activo, es un *operador integral singular* (SIO) de orden cero. Para datos iniciales regulares, se han probado resultados similares tanto para IPM como SQG, mientras que para soluciones débiles se pueden encontrar resultados diferentes debido a la paridad/imparidad del operador. Para soluciones débiles de tipo *patch*, los dos sistemas presentan comportamientos completamente diferentes. El problema de existencia global para el problema de Cauchy con datos iniciales regulares cualesquiera sigue siendo un problema abierto particularmente desafiante tanto para la ecuación de IPM como para la de SQG.

La idea de considerar una ecuación no-lineal donde la existencia global no se conoce y probarla para una perturbación cercana a una solución estacionaria de la ecuación es natural.

Es bien sabido que las funciones con simetria radial son soluciones estacionarias de SQG debido a la estructura del término no-lineal. Los primeros ejemplos de soluciones globales suaves no-triviales que conocemos fueron obtenidas recientemente por Castro, Córdoba y Gómez-Serrano. Sus soluciones son una perturbación suave en una adecuada dirección de una determinada función radial. La prueba se basa en la desingularización y la bifurcación desde el problema del *vortex-patch*. Señalamos que el perfil de la vorticidad es constante fuera de una región muy fina donde ocurre la transición, y que el grosor de esta región sirve como parámetro de bifurcación.

Para la ecuación de IPM, el primer resultado de una solucion global no-trivial se debe a Elgindi. La idea principal que esta detrás de este resultado es que la estratificación puede ser una fuerza estabilizadora. Uno puede imaginar que un fluido cuya densidad es proporcional a la profundidad es, en cierto modo *estable*. El mecanismo detrás de la estabilidad es que la ecuación de IPM linealizada alrededor del estado estratificado exhibe ciertas propiedades de amortiguamiento. Esta convergencia de regreso al equilibrio, a pesar de la falta de mecanismos disipativos, se conoce como *amortiguamiento no viscoso* y es un pariente cercano de la amortiguación de Landau en la física de plasmas. En un gran avance, Mouhot y Villani demostraron que la amortiguación de Landau proporciona una estabilidad similar para las ecuaciones de Vlasov–Poisson.

Como Elgindi trabaja en todo el plano, sus soluciones tienen energía finita pero densidad no acotada. Nosotros podemos evitar este inconveniente trabajando en un escenario físico confinado con condiciones de frontera antideslizantes. El trabajo del capítulo 2 parece ser el primero en encontrar un escenario para probar la existencia global de soluciones suaves con densidad acotada y energía finita para la ecuación de IPM inviscida. Este resultado ha sido publicado en [10].

Las ecuaciones de Boussinesq: En los fenómenos de *convección natural*, en qué el movimiento del fluido no es generado por ninguna fuente externa sino por gradientes de temperatura, las variaciones de densidad son insignificantes en términos de inercia. Esto da lugar a la llamada *aproximación de Boussinesq*, que consiste en eliminar la dependencia de la densidad en todos los términos, salvo el que involucra la gravedad.

El sistema Boussinesq se usa extensamente como una aproximación precisa de las ecuaciones de fluidos dependientes de la densidad para modelar fenómenos dominados por la convección natural. Desde un punto de vista físico, el sistema de Boussinesq se utiliza para modelar la dinámica del océano o la atmósfera. Desde el punto de vista matemático, el principal interés radica en la conexión entre el sistema dosdimensional de Boussinesq y las ecuaciones tridimensionales de Navier–Stokes y Euler. Al contrario que sucede para la ecuación dosdimensional de Navier–Stokes, donde la ecuación de la vorticidad no tiene término cuadrático, la ecuacion dosdimensional de Boussinesq aún captura el fenómeno de *vortex stretching*. Al igual que sucede para las ecuacion tridimensional de Euler y Navier–Stokes, el problema de existencia global para el sistema dosdimensional de Boussinesq inviscido y no difusivo sigue siendo un destacado problema abierto. De hecho, las ecuaciones de Boussinesq dosdimensional pueden identificarse formalmente con las ecuaciones de Euler tresdimensional en el caso axisimétrico con rotación, lejos del eje.

La existencia global de las soluciones se conoce cuando la disipación está presente en al menos una de las ecuaciones, o bajo una variedad más general de condiciones sobre la disipación. En contraste, el problema de regularidad global para las ecuaciones de Boussinesq dosdimensionales inviscidas y no difusivas parece estar fuera de alcance a pesar del progreso en los resultado de existencia local y los criterios de regularidad.

El capítulo 3 se centra en comprender el problema de la existencia global mediante el estudio de cómo la amortiguación afecta a la regularidad de las soluciones en las ecuaciones de Boussinesq dosdimensionales inviscidas y no difusivas. En una frase, vamos a estudiar el caso opuesto de la *inestabilidad de Rayleigh-Bénard*.

El fenómeno conocido como convección de Rayleigh-Bénard es un tipo de convección natural, que ha sido estudiado por numerosos autores durante muchos años. La idea es simple: tomar un recipiente lleno de agua que está en reposo y comenzar a calentar la parte inferior y enfríar la parte superior del recipiente. Se ha observado experimentalmente y matemáticamente que si la diferencia de temperatura entre la parte superior y la inferior va más allá de cierto valor crítico, el agua comenzará a moverse y los rollos convectivos comenzarán a formarse. Este efecto se llama inestabilidad de Rayleigh-Bénard.

Ahora, en el caso opuesto, cuando uno enfría la parte inferior y calienta la parte superior, se espera que el sistema permanezca estable. Para ello supondremos que la temperatura y la densidad están relacionadas proporcionalmente, de modo que el fluido más frío es más denso. Por lo tanto, en este caso se espera que la fuerza gravitacional estabilice dicha distribución de densidad. En presencia de viscosidad no es difícil probar este hecho. Sin embargo, sin los efectos de la viscosidad (o la difusion de la temperatura), es plausible que tal configuración sea inestable.

Resumiendo, en el capítulo 3 vamos a estudiar el caso opuesto a la inestabilidad de Rayleigh-Bénard. Más concretamente, tratamos de entender el problema de la existencia global al examinar cómo un término de amortiguación de la velocidad afecta a la regularidad de las soluciones de las ecuaciones de Boussinesq dosdimensionales inviscidas y no difusivas. Este resultado se puede encontrar en [9].

Abstract and conclusions

A fundamental challenge in mathematical physics is to understand the dynamics of physical systems as they evolve over long-times. This is particularly interesting when it comes to the study of the long-time behavior of such systems without dissipation and external forces. In particular, my thesis research has been centered on the stability near hydrostatic equilibriumn in two problems inside the field of fuid mechanics.

In fluid mechanics, a fluid is said to be in *hydrostatic equilibrium* when it is at rest. If the fluid is at rest, then the forces acting on it must balance it. A natural question therefore arises:

What happens if our initial data is close to an hydrostatic equilibrium solution?

The field of *hydrodinamic stability* has a long history starting in the 19th century. One of the oldest problems considered is the stability and instability of shear flows, dating back to Lord Rayleigh and Lord Kelvin.

For us, the basic problem is to consider a perturbation of the *hydrostatic equilibrium*, in which case the fluid must start to move, and to study the long-time behavior of the solution. In particular, we focus on *laminar equilibria*, simple equilibria in which the fluid is moving in well ordered layers. However, even for these simple configurations, surprisingly little is understood about the near equilibrium dynamics.

In this dissertation we consider two different problems inside the field of fluid mechanics. In the first chapter we treat the inviscid *incompressible porous media* (IPM) equation, which describes the dynamic of an incompressible fluid, flowing through a porous medium. In the second chapter we present the dynamics of a fluid under the *Boussinesq approximation*, which consists in neglecting the density dependence in all the terms but the one involving the gravity.

The Incompressible Porous Media (IPM) equation: Fluids in porous media are of particular interest as they arise in a wide array of real problems coming from many areas of applied science and engineering. The effect of the medium has important consequences and the usual equations for the conservation of momentum, i.e. the Euler or Navier-Stokes equations, must be replaced with the empirical *Darcy's Law*.

This physical principle, first noted by Henry Darcy in 1856, provides a macroscopic description of a flow where the velocity of the fluid is proportional to the pressure gradient and the external forces. From a mathematical point of view, the IPM belongs to a general class of equations is often referred to as *active scalars*. It consists of solving Cauchy's problem for a transport equation where the velocity field is related to the scalar that is transported by the flow through some operator. Maybe the best example of one of this active scalar is the Surface Quasi-Geostrophic (SQG) equation. This equation is a model of geophysical origin and is obtained as an aproximation of the general Quasi-Geostrophic system which considers the dynamics of atmospheric fluids taking into account the Coriolis force. Specifically, SQG measures the evolution of the temperature of the fluid when both the Rossby and Ekman numbers are small.

An important aspect of the SQG equation, from a mathematical point of view, was pointed out by Contantin, Majda and Tabak, in a paper where they proposed it as 2D scalar model of the 3D Euler equation. We can understand the relation between both equations by observing that the equation for the perpendicular gradient of the temperature in the SQG equation has the same structure that the equation for the vorticity in the 3D Euler equation. Since then, this equation has been a source of inspiration for the Euler equation, in such a way that the main results for SQG can be extended to the Euler equation.

Despite the fact that there are great similarities between the IPM and SQG equation, there are also important differences. It is important to note that, both IPM and SQG, the operator relating the velocity and the active scalar is a *singular integral operator* (SIO) of zero order. For regular initial data, similar results have been proved for IPM and SQG, while for weak solutions one can find different outcomes due to

evenness/oddness of the operator, and for patch-type weak solutions the two systems present completely different behaviors. The global existence problem for the Cauchy problem with a general smooth initial data remains as a particularly challenging open problem for both the IPM and SQG equation.

The idea of taking a non-linear equation where global well-posedness is unknown and to prove it for a perturbation *close* to a stationary solution of the equation is natural.

It is well known that radially symmetric functions are stationary solutions of SQG due to the structure of the nonlinear term. The first examples of non-trivial global smooth solutions we are aware of were recently provided by Castro, Córdoba and Gómez-Serrano. Their solutions are a smooth perturbation in a suitable direction of a specific radial function. The proof relies on the desingularization and bifurcation from the vortex patch problem. We point out that the profile of the vorticity is constant outside a very thin region where the transition occurs, and the thickness of this region serves as a bifurcation parameter.

For the IPM equation, the first construction of a non-trivial global smooth solution is due Elgindi, where the main idea is that stratification can be a stabilizing force. One can imagine that a fluid with density that is proportional to depth is in some sense *stable*. The mechanism behind the stability is that the linearized IPM equation around the stratified state exhibit certain damping properties. This convergence back to equilibrium, despite the lack of dissipative mechanisms, is known as *inviscid damping* and is a close relative of Landau damping in plasma physics. It was proved that Landau damping provides a similar stability for Vlasov–Poisson in Mouhot and Villani's breakthrough work.

As Elgindi works in the whole space, their solutions have finite energy but unbounded density. We can bypass this disadvantage considering a confined physical scenario with non-slip boundary conditions. The work of chapter 2 appears to be the first to find an scenario to prove global existence of smooth solutions with bounded density and finite energy for the inviscid IPM equation. This result has been published in [10].

The Boussinesq system: In *natural convection* phenomena, this is when the fluid motion is induced by temperature gradients without external sources, density variations are usually negligible in inertia terms. This leads to the so-called *Boussinesq approximation*, which consists in neglecting the density dependence in all the terms but the one involving the gravity.

The Boussinesq system are widely used as an accurate approximation of the full density dependent fluid equations to model phenomena dominated by natural convection. From a physical point of view, Boussinesq systems are widely used to model the dynamics of the ocean or the atmosphere. From the mathematical point of view, the main interest lies on the connection between the 2D Boussinesq system and the 3D Navier-Stokes and Euler equations. In contrast with Navier-Stokes on the plane, where the vorticity equation does not have a quadratic term, 2D Boussinesq still captures the phenomenon of *vortex stretching*. As in 3D Euler and Navier-Stokes equations, global well-posedness of the 2D inviscid and non-diffusive Boussinesq system remains an outstanding open problem.

Indeed, in this setting, the 2D Boussinesq equations are identical to the 3D Euler equations under the hypothesis of axial symmetry with swirl. The behavior of solutions to the 2D Boussinesq system and the axi-symmetric 3D Euler equations away from the symmetry axis should be "identical".

Global regularity of solutions is known when classical dissipation is present in at least one of the equations, or under a variety of more general conditions on dissipation. In contrast, the global regularity problem on the inviscid and non-diffusive 2D Boussinesq equations appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria.

The work of chapter 3 is partially aimed to understand the global existence problem by examining how damping affects the regularity of the solutions to the 2D inviscid and non-diffusive Boussinesq equations. In one sentence, we are going to study the opposite of the *Rayleigh-Bénard instability*.

The phenomenon known as Rayleigh-Bénard convection is a type of natural convection, which has been studied by a number of authors for many years. The idea is simple: take a container filled with water which is at rest. Now heat the bottom of the container and cool the top of the container. It has been observed experimentally and mathematcally that if the temperature difference between the top and the bottom goes beyond a certain critical value, the water will begin to move and convective rolls will begin to form. This effect is called Rayleigh-Bénard instability.

Now, in the inverse case, when one cools the bottom and heats the top, it is expected that the system remains stable. Here the temperature and density are assumed to be proportionally related, so that the cooler fluid is more dense. The gravitational force is thus expected to stabilize such a density (or temperature) distribution. In the presence of viscosity it is not difficult to prove this fact. However, without the effects of viscosity (or temperature dissipation), it is conceivable for such a configuration to be unstable.

In one sentence, in chapter 3 we are going to study the opposite of the Rayleigh-Bénard instability. More specifically, we try to understand the global existence problem by examining how a velocity damping term affects the regularity of the solutions to the inviscid and non-diffusive 2D Boussinesq equations. This result can be found in [9]

Conozco la utilidad de la inutilidad.
Y tengo la riqueza de no querer ser rico.

Joan Brossa

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En este pequeño espacio me gustaria dejar por escrito mi más sincero agradecimiento a todos aquellos que de una forma u otra me han brindado su apoyo, ya sea en lo profesional o en lo personal.

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Daniel Lear Claveras,
Madrid, 1 de Diciembre de 2018.

CONTENTS

CHAPTER 1

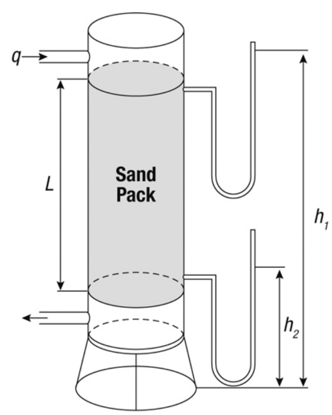
PRELIMINARIES

In this chapter we present a brief introduction of the equations which give rise to the problems studied in this dissertation. All of them came from the field of fluid mechanics. The general characteristics and some known results are presented.

1.1 Dynamics of fluids in porous media

The process of flow through porous media is of interest to a wide range of engineers, scientists, and mathematicians (see for instance [1], [48] and [59]). The effect of the *porous medium* has important consequences and the usual equations for the conservation of momentum, i.e. the Euler or Navier–Stokes equations do not provide a satisfactory model. The work of Henry Darcy (1803-1858), a french engineer who studied this phenomenon while studying the fountains of the city of Dijon [20], provides a satisfactory answer to our needs.

Darcy's Experiments: In 1855, Henry Darcy, oversaw a series of experiments aimed to understand the rates of water flow through sand layers, and their relationship to pressure loss along the flow paths. Darcy's experiments consisted of a vertical steel column of section A and length L filled with a porous medium (sand) through which water is passed. The water pressure was controlled at the inlet and outlet ends of the column using reservoirs with constant water levels (denoted h_1 and h_2)



Specifically, Darcy's experiments revealed proportionalities between the flux of water Q (volume per time) and different characteristics of the experimental system.

1. Q was directly proportional to the difference in water levels from inlet to outlet: $Q \propto h_1 - h_2$.

2. Q was directly proportional to the cross sectional area of the tube: $Q \propto A$.

3. Q was inversely proportional to the length of the column: $Q \propto L^{-1}$.

Combining these proportionalities leads to Darcy's Law, the empirical law that describes groundwater flow:

$$\mu Q = -\kappa \frac{A(h_1 - h_2)}{L}$$

where μ is the dynamic viscosity of the fluid, κ is the permeability of the porous medium, which measures the ability of the medium to transmit a fluid (see [1] Table 1.1 to find permeabilities of several isotropic porous media).

Darcy's Law: In modern notation, *Darcy's law* is given by the momentum equation

$$\frac{\mu}{\kappa}\mathbf{u} = -\nabla p - g(0, \rho),$$

where ρ is the density of the fluid, $\mathbf{u}$ is the velocity and p is the pressure. The symbols μ is the dynamic viscosity of the fluid, κ is the permeability of the porous medium and g is the acceleration due to gravity.

In the momentum equation, the velocity, instead of the acceleration, is proportional to the gradient of the pressure and external forces. This law, first determined by Darcy based on physical experiments, can also be deduced from Stokes equations using homogenization [35], [57]. The basic idea is that the porosity of the medium restrains the fluid motion, so that the inertia terms become negligible and the viscosity force acts as a restoring force linearly with the velocity, the permeability being the proportionality constant.

With this new law our model for the dynamics of incompressible flows through a porous medium are governed by the following equations

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= 0, \\ \dfrac{\mu}{\kappa}\mathbf{u} &= -\nabla p - g(0, \rho), \\ \nabla \cdot \mathbf{u} &= 0. \end{cases}$$

1.2 Dynamics under the Boussinesq approximation

In natural convection phenomena fluid flow generates due to the effect of buoyancy forces. Temperature gradients induce density variations from an equilibrium state, which gravity tends to restore. These flows are usually characterized by small deviations of the density with respect to a stratified reference state in hydrostatic balance. Potential energy is thus the main agent of movement, compared to inertia. Oberbeck was the first to notice by linearization that the buoyancy effect was proportional to temperature deviations [49], and later Boussinesq [4] completed the model based on physical assumptions. It has been since then one of the main ingredients in geophysical models, from ocean and atmosphere dynamics to mantle and solar inner convection, as well as a basic tool in building environmental engineering.

In dimensionless variables, the Boussinesq equations in the plane are given by the following expression

$$\begin{cases} \nabla \cdot \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \frac{1}{\mathrm{Re}}\Delta \mathbf{u} - \nabla p + g(0, \theta), \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= \frac{1}{\mathrm{Pe}}\Delta \theta, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ denotes the velocity, p represents the pressure deviation from the hydrostatic one and θ symbolizes the temperature variations. The *Reynolds number* Re, indicates the ratio of fluid inertial and viscous forces while the *Péclet number* Pe, compares the rates of advective and diffusive heat transport. They are thus inversely proportional to the viscosity and thermal diffusivity constants, respectively.

Roughly speaking (see [56] for a rigorous justification), to obtain the system (1.1) from the equations for density-dependent fluids

$$\left\{\begin{aligned} \nabla\cdot\mathbf{u} &= 0,\\ \rho\left(\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= \tfrac{1}{\mathrm{Re}}\Delta\mathbf{u}-\nabla p-g(0,\rho),\\ \partial_t\rho+\mathbf{u}\cdot\nabla\rho &= \tfrac{1}{\mathrm{Pe}}\Delta\rho, \end{aligned}\right. \tag{1.2}$$

one first replaces the exact density by a constant representative value in many terms of the equations of motion. We may divide the exact density into a constant part ρ_0, and a residual one $\rho'(x,y,t)$:

$$\rho(x,y,t)=\rho_0+\rho'(x,y,t).$$

In the *Boussinesq approximation*, we assume that the density variations $\rho'(x,y,t)$ are small compared to the background state ρ_0. This is, we assume that $|\rho'|\ll\rho_0$. The residual part ρ' represents density variations primarily caused by temperature variation inside the fluid. Since the two have comparable importance, he density is assumed to be linear with respect to the temperature

$$\rho'=-\beta\rho_0(\Theta-\Theta_0)$$

where β is the thermal expansion coefficien and the real temperature Θ is related to θ by the *Richardson number* Ri, as follows

$$\theta=-\mathrm{Ri}\frac{\rho'}{\rho_0}=\mathrm{Ri}\beta(\Theta-\Theta_0).$$

The Richardson number, which coincides with the inverse of the Froude number squared and measures the ratio of potential over kinetic energy, is assumed to be large enough so that density variations are not negligible in the gravity term.

With these assumptions, we can write the density-dependent fluid equations (1.2) as the Boussinesq system (1.1). For inviscid and non diffusive fluids, i.e., $\frac{1}{\mathrm{Re}}=\frac{1}{\mathrm{Pe}}=0$, the equations can be formally identified with the 3D Euler equations in vorticity form for axisymmetric swirling flows away from the axis [46]. It is well-known that the global regularity of these equations is still an outstanding open problem.

In this dissertation, we will consider the two-dimensional inviscid and non-diffusive Boussinesq system with a damping velocity term, so our equations will read as follows

$$\left\{\begin{aligned} \nabla\cdot\mathbf{u} &= 0,\\ \partial_t\theta+\mathbf{u}\cdot\nabla\theta &= 0, \qquad\qquad\qquad \text{DAMPING TERM}\\ \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p+g(0,\theta)-\boxed{\mathbf{u}}. \end{aligned}\right.$$

Our work is partially aimed to understand the global existence problem by examining how damping affects the regularity of the solutions to the inviscid and non-diffusive 2D Boussinesq equations.

CHAPTER 2

THE CONFINED IPM EQUATION

ABSTRACT: We consider a confined physical scenario to prove global existence of smooth solutions with bounded density and finite energy for the inviscid incompressible porous media (IPM) equation. The result is proved using the stability of stratified solutions, combined with an additional structure of our initial perturbation, which allows us to get rid of the boundary terms in the energy estimates.

2.1 Introduction

In this chapter we study the global in time existence of smooth solutions with bounded density and finite energy of the (2D) Incompressible Porous Media equation in a strip domain Ω. That is, we consider the following active scalar equation:

$$\partial_t \varrho + \mathbf{u} \cdot \nabla \varrho = 0,$$

with a velocity field $\mathbf{u}$ satisfying the momentum equation given by Darcy's law:

$$\frac{\mu}{\kappa}\mathbf{u} = -\nabla p - g(0, \varrho), \tag{2.1}$$

where $(\mathbf{x}, t) \in \Omega \times \mathbb{R}^+$, $\mathbf{u} = (u_1, u_2)$ is the incompressible velocity (that is, $\nabla \cdot \mathbf{u} = 0$), p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, g is the acceleration due to gravity and ϱ corresponds to the density transported without diffusion by the fluid.

Due to the direction of gravity, the horizontal and the vertical coordinates play different roles. Here we assume spatial periodicity in the horizontal space variable, says $\varrho\,(x + 2\pi k, y, t) = \varrho\,(x, y, t)$ and similarly $p(x + 2\pi k, y, t) = p(x, y, t)$. Finally, as these equations are studied on a bounded domain, we assume that our physical domain is impermeable, which is exactly satisfied if $\mathbf{u}$ satisfies the no-slip boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where $\mathbf{n}$ denotes the exterior normal vector.

In this work we will focus on the case in which the evolution problem is posed on a porous strip with width $2l$. That is, the domain is the two-dimensional flat strip $\Omega := \mathbb{T} \times [-l, l]$ with $0 < l < \infty$.

This problem is known as the *confined* IPM equation. Without loss of generality we will assume from now on that $\mu = \kappa = g = l = 1$. To summarize, we have the following system of equations in Ω:

$$\begin{cases} \partial_t \varrho + \mathbf{u} \cdot \nabla \varrho &= 0, \\ \mathbf{u} &= -\nabla p - (0, \varrho), \\ \nabla \cdot \mathbf{u} &= 0, \end{cases} \tag{2.3}$$

with the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega \equiv \{y = \pm 1\}$. In our case, this implies that $u_2|_{\partial\Omega} = 0$. In our physical system where there is gravity and stratification ($\mathbf{u} \equiv 0$ and $\varrho \equiv \varrho\,(y)$ is a stationary solution), vertical movement may be penalized while horizontal movement is not. This opens up the possibility of treating the corresponding initial value problem from a perturbative point of view. As in [27], this chapter studies the solutions of (2.3) in the perturbative regime near the stratified state $\Theta(y) := -y$ for a specific type of perturbations:

$$\varrho\,(x,y,t) = \Theta(y) + \rho(x,y,t) \qquad (\mathbf{x},t) \in \Omega \times \mathbb{R}^+. \tag{2.4}$$

The main result of the chapter is that small perturbations ρ in a suitable Sobolev space $X^k(\Omega)$, which we define below in (2.5), converge to a shear and nearby stationary flow in the sense that $\varrho\,(x,y,t) \equiv \Theta(y) + \rho(x,y,t) \to \Theta(y) + \rho_\infty(y)$ and $\mathbf{u}(x,y,t) \to 0$ as $t \to \infty$. The main mechanism of decay can be seen from the linearized equation

$$\partial_t \rho(x,y,t) = -\Theta'(y)\, u_2(x,y,t)$$

which, after solving the velocity $\mathbf{u} = (u_1, u_2)$ in terms of ρ yields

$$\partial_t \rho(x,y,t) = \Theta'(y)\left(\rho(x,y,t) + (-\Delta_\Omega)^{-1}\partial_y^2 \rho(x,y,t)\right).$$

Setting $\Theta(y) := -y$, the previous equation clearly shows the frequency dependent exponential decay over time of ρ, except the zero mode in x. The goal of the chapter is to show how to control the nonlinearity, so that it does not destroy the decay provided by the linearized equation.

To do this, controlling the boundary terms is the new additional difficulty. This can be done by working with perturbations in the appropriate Sobolev space $X^k(\Omega)$. Using standard techniques, we will prove the local in time existence of solutions for the perturbated problem in the space $X^k(\Omega)$. For the sake of completeness we include the proof, where the cornerstone will be the properties of an orthonormal basis adapted to $X^k(\Omega)$. The reason for working with initial perturbations with that additional structure will be seen in the apriori energy estimates. There, all the boundary terms that appear in the computations vanish thanks to periodicity in the horizontal variable and by the additional structure of our initial perturbations, which is preserved in time by the local existence result, as long as the solution exists.

Namely, we will prove the following result:

Theorem *The stratified state Θ of the confined IPM equation is asymptotically stable in $X^\kappa(\Omega)$ for $\kappa \geqslant 10$. In other words, there exists $\varepsilon_0 > 0$ such that if we solve (2.3) with initial data $\varrho\,(0) = \Theta + \rho(0)$ and $\rho(0) \in X^\kappa(\Omega)$ with $||\rho||_{H^\kappa(\Omega)}(0) \leqslant \epsilon_0$ then, the solution exists globally in time and satisfies:*

i) $||\mathbf{u}||_{H^3(\Omega)}(t) \lesssim \varepsilon_0\,(1+t)^{-\frac{5}{4}}$,

ii) $||\bar{\varrho}||_{H^3(\Omega)}(t) \lesssim \varepsilon_0\,(1+t)^{-\frac{5}{4}}$,

iii) $||\tilde{\varrho} - \Theta||_{H^\kappa(\Omega)}(t) \leqslant 2\,\varepsilon_0$,

where $\varrho\,(x,y,t) := \bar{\varrho}(x,y,t) + \tilde{\varrho}(y,t)$ such that $\bar{\varrho} \perp \tilde{\varrho}$ and $\bar{\varrho}$ is given by the projection operator onto the subspace of functions with zero average in the horizontal variable.

Remark: If we perturb the stratified state by a function of y only then there should be no decay. For this reason, the orthogonal decomposition $\varrho = \bar{\varrho} + \tilde{\varrho}$ will be considered.

Remark: The strategy used can be applied to a more general class of monotone shear flows. The proof works for small perturbations in some sense of our steady state with $\Theta'(y) < 0$. However, a highly non-trivial problem is to extend this to the case of possibly degenerate shear flows where $\Theta'(y) = 0$ at some value.

A more precise statement of our result is presented as Theorem 2.5.1, where we also illustrate its proof through a bootstrap argument. Despite the apparent simplicity, understanding the stability of this flow is far from being trivial.

2.1.1 Motivation

The study of partial differential equations arising in fluid mechanics has been an active field in the past century, but many important and physically relevant questions remain wide open from the point of view of mathematical analysis. Among the problems that attracted recently renewed interest, *active scalar* equations that arise in fluid dynamics present a challenging set of problems in PDE. Maybe the best example is the Surface Quasi-Geostrophic equation (SQG), introduced in the mathematical literature in [13]. The inviscid SQG equation in $\mathbb{R}^2$ takes the form

$$\begin{cases} \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \mathbf{u} = \mathbf{R}^{\perp} \theta, \end{cases}$$

where $\mathbf{R} = (R_1, R_2)$ denote the 2D Riesz transforms. This problem has been widely investigated due to its mathematical analogies with the 3D Euler equation, but little is known. Local well-posedness and regularity criteria in various functional settings have been established, see [8] as a survey. The global regularity problem for the Cauchy problem with a general smooth initial data remains open. Besides radially symmetric solutions, which are all stationary, the first examples of non-trivial global smooth solutions we are aware of were recently provided in [8]. An alternative construction of smooth families of global special solutions can be found in [33], where the authors focus on travelling-wave solutions to the inviscid SQG. On the other hand, whether finite time blow up can happen for smooth initial data remains completely open.

It is important to note that, for both IPM and SQG, the operator relating the velocity and the active scalar is a *singular integral operator* of zero order. Even more, in the whole space, the velocity (2.1) can be rewritten in a more convenient way as $\mathbf{u} = \mathbf{R}^{\perp} R_1 \; \varrho$. Despite the fact that there are great similarities between the inviscid versions of SQG and IPM equations, there are also important differences. This work appears to be the first to find a scenario to prove the global existence of smooth solutions with bounded density and finite energy for the inviscid IPM equation.

2.1.1.1 The question of long-time behavior

A fundamental challenge in mathematical physics is to understand the dynamics of physical systems as they evolve over long times. This is particularly true when it comes to the study of the long-time behavior of such systems without dissipation. Depending upon the specific physical situation that a given fluid equation models, we find vastly different mathematical objects arising. In recent years, researchers have discovered numerous interesting phenomena such as the existence of solutions whose long-time behavior is determined entirely by

- some linear or dispersive effect, for example in water waves [31], [37] and [62];
- some linear mixing effect, for the Couette flow in Navier–Stokes and Euler equations [2], [3];
- some hypocoercive dissipative mechanisms, for kinetic theory [21] and [22].

The idea of taking a non-linear equation where global well-posedness is unknown and to prove it for a perturbation "close" to a stationary solution of the equation is natural. For small enough initial data, one might conjecture that solutions to the nonlinear problem behave asymptotically like solutions of the corresponding linear problem.

As in [27], where the author gives in $\mathbb{R}^2$ the first construction of a non-trivial global smooth solution for the inviscid IPM equation, the main idea is that stratification can be a stabilizing force. One can imagine that a fluid with density that is proportional to depth is in some sense "stable". The mechanism behind the stability is that the linearized IPM equation around the stratified state exhibits certain damping properties. This convergence back to equilibrium, despite the lack of dissipative mechanisms, is known as *inviscid* *damping* and is a close relative of Landau damping in plasma physics. It was proved that Landau damping provides a similar stability for Vlasov–Poisson in Mouhot and Villani's breakthrough work [47].

2.1.1.2 Previous results for IPM with smooth initial data

In [18], the local existence and uniqueness in Hölder space C^δ with $\delta \in (0,1)$ was shown by the particle-trajectory method for the whole space case. By a similar approach, the local well-posedness in Besov and Triebel-Lizorkin spaces was proved in [64] and [65].

For the Lagrangian formulation, in [16], the authors show that as long as the solution of this equation is in a class of regularity that assures Hölder continuous gradients of the velocity, the corresponding Lagrangian paths are real analytic functions of time.

In the class of weaker solutions, the results of [17] and [58] establish the non-uniqueness of $L^\infty_{t,x}$ weak solutions to the inviscid IPM equation starting from the zero solution. Recently, in [38] the authors were able to construct global weak solutions to the inviscid IPM equation which are of class $C^\delta_{t,x}$ with $\delta < 1/9$ starting from a smooth initial data. All these works are based on a variant of the method of convex integration.

In the direction of classical solutions, the only result known, due to Elgindi [27], shows that solutions which are "close" to certain stable stratified solutions exist globally in time, but since he works in the whole space, such solutions have unbounded density. He considers perturbations in two settings which are fundamentally different:

- On the whole space $\mathbb{R}^2$: In this case the stationary solution does not belong to $L^2(\mathbb{R}^2)$. However, the author can perturb the stationary solution by a sufficiently small H^s function, and to prove that the perturbation decays to equilibrium as $t \to +\infty$.

- On the two dimensional torus $\mathbb{T}^2$: Similarly, the stationary solution is not periodic but the author may perturb it by a periodic function and once more the perturbation will remain periodic. The result here is quite different for the main reason that ϱ itself does not decay. Even so, smooth perturbations of the stationary solution are stable for all time in Sobolev spaces.

We now motivate our attack setting. We start with the observation that the gravity term in Darcy's law (2.1) converts IPM in an anisotropic problem, which implies different properties in different directions. In our case, the vertical direction pointing in the direction of gravity will play a key role. By this anisotropic property, it seems natural that $\mathbb{T} \times [-1,1]$ might be an adequate scenario to set our equations.

In order to solve our problem in the bounded domain Ω, in certain Sobolev spaces, we have to overcome the following new difficulties:

i) To be able to handle the boundary terms that appear in the computations;

ii) The lack of higher order boundary conditions at the boundaries, due to the fact that we work in Sobolev spaces.

Indeed, both difficulties i) and ii) can be bypassed if our initial perturbation has a special structure. We introduce the following spaces to characterize our initial data:

$$X^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 0,2,4,\dots,k^\star\}, \tag{2.5}$$

$$Y^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 1,3,5,\dots,k_\star\}, \tag{2.6}$$

where we defined the auxiliary values of $k^\star$ and $k_\star$ as follows:

$$k^\star := \begin{cases} k-2 & k \quad \text{even}, \\ k-1 & k \quad \text{odd}, \end{cases} \qquad \text{and} \qquad k_\star := \begin{cases} k-1 & k \quad \text{even}, \\ k-2 & k \quad \text{odd}. \end{cases}$$

Lastly, we remember that the Trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ defined by $T[f] := f|_{\partial\Omega}$ is bounded for all $f \in H^1(\Omega)$. Consequently, both spaces are well defined.

2.1.2 The equations

In this section, we describe the equation that a perturbation (2.4) of the stratified solution must satisfy. In order to prove our goal, we plug into the system (2.3) the following ansatz:

$$\begin{aligned} \varrho\,(x,y,t) &= -y + \rho(x,y,t), \\ p(x,y,t) &= \Pi(x,y,t) - \tfrac{1}{2}y^2 + \int_0^y \tilde{\rho}(y',t)dy', \end{aligned}$$

where for a general function $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, we define

$$\tilde{f}(y,t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x',y,t)dx' \quad \text{and} \quad \bar{f}(x,y,t) := f(x,y,t) - \tilde{f}(y,t).$$

Then, for the perturbation ρ, we obtain the system

$$\begin{cases} \partial_t\rho + \mathbf{u}\cdot\nabla\rho &= u_2, \\ \mathbf{u} &= -\nabla\Pi - (0,\rho), \\ \nabla\cdot\mathbf{u} &= 0, \end{cases} \tag{2.7}$$

besides the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Note that in Ω, our perturbation ρ does not have to decay in time. Indeed, if we perturb the stationary solution by a function of y only there is no decay. More specifically, $\rho \equiv \rho(y)$ and $u \equiv 0$ is a stationary solution of (2.7). To overcome this difficulty, the orthogonal decomposition $\rho = \bar{\rho} + \tilde{\rho}$ will be considered.

The system (2.7) can be rewritten in terms of $\bar{\rho}$ and $\tilde{\rho}$ as follows:

$$\begin{cases} \partial_t\bar{\rho} + \overline{\mathbf{u}\cdot\nabla\bar{\rho}} + \partial_y\tilde{\rho}\,u_2 &= u_2, \\ \partial_t\tilde{\rho} + \widetilde{\mathbf{u}\cdot\nabla\bar{\rho}} &= 0, \\ \mathbf{u} &= -\nabla\Pi - (0,\rho), \\ \nabla\cdot\mathbf{u} &= 0. \end{cases} \tag{2.8}$$

Notice that $\tilde{\rho}$ is always a function of y only and $\bar{\rho}$ has zero average in the horizontal variable. It is expected that $\bar{\rho}$ will decay on time and $\tilde{\rho}$ will just remain bounded. The systems (2.7) and (2.8) are the same, but depending on what we need, we will work with one or the other.

2.1.3 Notation & Organization

We shall denote by (f,g) the $L^2(\Omega)$ inner product of f and g. As usual, we use bold for vector valued functions. Let $\mathbf{u} = (u_1,u_2)$ and $\mathbf{v} = (v_1,v_2)$, we define $\langle\mathbf{u},\mathbf{v}\rangle = (u_1,v_1) + (u_2,v_2)$. Also, we remember that the natural norm in Sobolev spaces is defined by

$$||f||^2_{H^k(\Omega)} := ||f||^2_{L^2(\Omega)} + ||f||^2_{\dot{H}^k(\Omega)}, \qquad ||f||^2_{\dot{H}^k(\Omega)} := ||\partial^k f||^2_{L^2(\Omega)}.$$

For convenience, in some place in this chapter, we may use L^2, $\dot{H}^k$ and H^k to stand for $L^2(\Omega)$, $\dot{H}^k(\Omega)$ and $H^k(\Omega)$, respectively. Moreover, to avoid clutter in computations, function arguments (time and space) will be omitted whenever they are obvious from context. Finally, we use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \leqslant Cg$.

Organization of the chapter: In Section 2.2, we introduce the functional spaces $X^k(\Omega)$ and $Y^k(\Omega)$ where we will work. The key point of working with initial perturbations with the structure given by these spaces is showed in Section 2.3. Section 2.4 contains the proof of the local existence in time for initial data in $X^k(\Omega)$ for the *confined* problem, together with a blow-up criterion. The core of the article is the proof of the main theorem in Section 2.5. We commence by the *a priori* energy estimates given in Section 2.5.1. This is followed by an explanation of the decay given by the linear semigroup of our system in Section 2.5.2. Finally, in Section 2.5.3 we exploit a bootstrapping argument to prove our theorem.

2.2 Mathematical setting and preliminares

In this section, we will see the importance of working with initial perturbations belonging to $X^k(\Omega)$. We also consider an adapted orthonormal basis for working with these perturbations, together with their eigenfunction expansion.

2.2.1 Motivation of the spaces $X^k(\Omega)$ and $Y^k(\Omega)$.

By the no-slip condition $u_2(t)|_{\partial\Omega} = 0$, the solution $\rho(t)$ of (2.7) satisfies the following transport equation on the boundary:

$$\partial_t \rho(t)|_{\partial\Omega} + u_1(t)\partial_x \rho(t)|_{\partial\Omega} = 0. \tag{2.9}$$

As our objective is to obtain global stability and decay to equilibrium of sufficiently small perturbations, it seems natural to consider $\rho(0)|_{\partial\Omega} = 0$. Then, by the transport character of (2.9) the initial condition is preserved in time $\rho(t)|_{\partial\Omega} = 0$ as long as the solution exists. In addition, taking derivatives in Darcy's law, using the incompressibility condition, and restricting to the boundary we have

$$\partial_y u_1(t)|_{\partial\Omega} = 0 \quad \text{and} \quad \partial_y^2 u_2(t)|_{\partial\Omega} = 0, \tag{2.10}$$

given that $\rho(t)|_{\partial\Omega} = u_2(t)|_{\partial\Omega} = 0$. Relations (2.10) give rise to the following equation for the derivative in time of $\partial_y^2\rho(t)$ at the boundary:

$$\partial_t \partial_y^2 \rho(t)|_{\partial\Omega} = -u_1(t)\partial_x(\partial_y^2\rho)(t)|_{\partial\Omega} - \partial_y u_2(t)\partial_y^2\rho(t)|_{\partial\Omega}.$$

Thus, we find that $\partial_y^2\rho(0)|_{\partial\Omega} = 0$ implies that $\partial_t\partial_y^2\rho(t)|_{\partial\Omega} = 0$, and consequently the condition on the boundary is preserved in time.

Iterating this procedure we can check that the conditions $\partial_y^n\rho(0)|_{\partial\Omega} = 0$, for $n = 2, 4, \ldots$ are preserved in time. This is the reason why we can look for solutions $\rho(t)$ in the space $X^k(\Omega)$, if the initial data belongs to it. Moreover $u_1(t)$ will belong to $Y^k(\Omega)$ and $u_2(t)$ will belong to $X^k(\Omega)$.

2.2.2 Biot-Savart law and stream formulation

In the whole space $\mathbb{R}^2$ we have a simple expression for $\nabla\Pi$ in terms of $\bar{\rho}$:

$$\nabla\Pi = \nabla(-\Delta)^{-1}\partial_y\bar{\rho},$$

so we can write the velocity in terms of $\bar{\rho}$ as

$$\mathbf{u} = -\nabla\Pi - (0,\bar{\rho}) = \mathbf{R}^{\perp}R_1\bar{\rho}$$

where $\mathbf{R}^{\perp} = (-R_2, R_1)$, being R_i the Riesz's transform.

In our setting $\Omega = \mathbb{T} \times [-1,1]$, to obtain an analogous expression we proceed as follow: due to the incompressibility of the flow, by taking the divergence of Darcy's law we find that

$$\Delta \Pi = -\partial_y \bar{\rho}. \tag{2.11}$$

Moreover, the no-slip conditon (2.2) give us the boundary condition

$$\partial_y \Pi|_{\partial\Omega} = -\bar{\rho}|_{\partial\Omega} = 0, \tag{2.12}$$

which vanishes as $\rho \in X^k(\Omega)$. Then, putting together (2.11) and (2.12) (notice that we look for a periodic in the x-variable Π), we recover the velocity field, in terms of $\bar{\rho}$, by the expression $\mathbf{u} = -\nabla\Pi - (0, \bar{\rho})$.

Another way to reach this expression it by following these steps: as $\nabla \cdot \mathbf{u} = 0$, we can write the velocity as the gradient perpendicular of a *stream function* ψ, that is

$$\mathbf{u} = \nabla^{\perp}\psi, \tag{2.13}$$

with $\nabla^{\perp} \equiv (-\partial_y, \partial_x)$. Then, applying the *curl* operator on (2.1), we get the Poisson equation for ψ:

$$\Delta\psi = -\partial_x \bar{\rho}.$$

Taking into account (2.13) and the no-slip condition (2.2) we obtain the boundary condition

$$\partial_x \psi|_{\partial\Omega} = 0.$$

Thus, we need to impose $\psi|_{\{y=\pm 1\}} = c_{\pm}$ where c_+ could be, in principle, different from c_-. However, the periodicity in the x-variable of Π forces to take $c_+ = c_-$, and since we are only interested in the derivatives of ψ we will take $c_{\pm} = 0$.

To sum up, in order to close the system of equations, we first solve either

$$\begin{cases} \Delta\Pi & = -\partial_y \bar{\rho} \qquad \text{in} \qquad \Omega, \\ \partial_y \Pi & = 0 \qquad\quad\;\; \text{on} \qquad \partial\Omega, \end{cases}$$

or

$$\begin{cases} \Delta\psi & = -\partial_x \bar{\rho} \qquad \text{in} \qquad \Omega, \\ \psi & = 0 \qquad\quad\;\; \text{on} \qquad \partial\Omega, \end{cases} \tag{2.14}$$

and after that write

$$\mathbf{u} = -\nabla\Pi - (0, \bar{\rho}) \qquad \text{or} \qquad \mathbf{u} = \nabla^{\perp}\psi.$$

In the rest of the chapter we will use the *stream formulation* to recover the velocity field. In the next section, we present an orthonormal basis of $X^k(\Omega)$ in order to solve (2.14), which allows to write the velocity in terms of the "Fourier coefficients" of $\bar{\rho}$.

2.2.3 An orthonormal basis for $X^k(\Omega)$

Our goal is to solve (2.14). In order to do this, we define

$$a_p(x) := \frac{1}{\sqrt{2\pi}} \exp(ipx) \quad \text{with } x \in \mathbb{T} \quad \text{for } p \in \mathbb{Z}$$

and

$$b_q(y) := \begin{cases} \cos\left(qy\frac{\pi}{2}\right) & q \quad \text{odd} \\ \sin\left(qy\frac{\pi}{2}\right) & q \quad \text{even} \end{cases} \quad \text{with } y \in [-1,1] \quad \text{for } q \in \mathbb{N},$$

where $\{a_p\}_{p\in\mathbb{Z}}$ and $\{b_q\}_{q\in\mathbb{N}}$ are orthonormal basis for $L^2(\mathbb{T})$ and $L^2([-1,1])$ respectively. Indeed, $\{b_q\}_{q\in\mathbb{N}}$ consists of eigenfunctions of the operator $S=(1-\partial_y^2)$ with domain $\mathscr{D}(S)=\{f\in H^2[-1,1] \,:\, f(\pm 1)=0\}$. Consequently, the product of them $\omega_{p,q}(x,y):=a_p(x)\,b_q(y)$ with $(p,q)\in\mathbb{Z}\times\mathbb{N}$ is an orthonormal basis for the product space $L^2(\mathbb{T}\times[-1,1])\equiv L^2(\Omega)$.

Now, we define an auxiliary orthonormal basis for $L^2([-1,1])$ given by

$$c_q(y):=\begin{cases}\sin\left(qy\frac{\pi}{2}\right) & q \quad \text{odd}\\ \cos\left(qy\frac{\pi}{2}\right) & q \quad \text{even}\end{cases} \qquad \text{with } y\in[-1,1] \quad \text{for } q\in\mathbb{N}\cup\{0\},$$

consisting of eigenfunctions of the operator S with domain $\mathscr{D}(S)=\{f\in H^2[-1,1] \,:\, (\partial_y f)(\pm 1)=0\}$. In the same way as before, the product $\varpi_{p,q}(x,y):=a_p(x)\,c_q(y)$ with $(p,q)\in\mathbb{Z}\times(\mathbb{N}\cup\{0\})$ is again an orthonormal basis for $L^2(\Omega)$.

Remark: Let us describe the analogue of the Fourier expansion in terms of our eigenfunctions expansion. This is, for $f\in L^2(\Omega)$, we have the $L^2(\Omega)$-convergence given by

$$f(x,y)=\sum_{\substack{p\in\mathbb{Z}\\ q\in\mathbb{N}}}\mathcal{F}_{\omega}[f](p,q)\,\omega_{p,q}(x,y) \quad \text{where} \quad \mathcal{F}_{\omega}[f](p,q):=\int_{\Omega} f(x',y')\,\overline{\omega_{p,q}(x',y')}\,dx'dy' \tag{2.15}$$

or

$$f(x,y)=\sum_{\substack{p\in\mathbb{Z}\\ q\in\mathbb{N}\cup\{0\}}}\mathcal{F}_{\varpi}[f](p,q)\,\varpi_{p,q}(x,y) \quad \text{where} \quad \mathcal{F}_{\varpi}[f](p,q):=\int_{\Omega} f(x',y')\,\overline{\varpi_{p,q}(x',y')}\,dx'dy'. \tag{2.16}$$

The main result of this part is to see that $\{\omega_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}}$ is an orthonormal basis not only for $L^2(\Omega)$ but for $X^k(\Omega)$, and that $\{\varpi_{p,q}\}_{(p,q)\in\mathbb{Z}\times(\mathbb{N}\cup\{0\})}$ is basis of $Y^k(\Omega)$. The sequence $\{a_p\}_{p\in\mathbb{Z}}$ is the standard Fourier basis in $H^k(\mathbb{T})$. Then, we will focus only on the convergence properties of *span*$\{b_1,b_2,b_3,\ldots\}$ and *span*$\{c_0,c_1,c_2,\ldots\}$.

As we will see below, the relation between derivatives of $\{b_q\}_{q\in\mathbb{N}}$ and $\{c_q\}_{q\in\mathbb{N}\cup\{0\}}$ plays a key role in the convergence properties. An easy computation gives us

$$(\partial_y b_q)(y)=(-1)^q q\tfrac{\pi}{2}c_q(y) \qquad \text{for} \quad q\in\mathbb{N} \tag{2.17}$$

and

$$(\partial_y c_q)(y)=\begin{cases}-(-1)^q q\frac{\pi}{2}b_q(y) & q\in\mathbb{N},\\ 0 & q=0.\end{cases} \tag{2.18}$$

Then, as a consequence of (2.17) and (2.18), for $q\in\mathbb{N}$ we have

$$(\partial_y^2 b_q)(y)=-\left(q\tfrac{\pi}{2}\right)^2 b_q(y) \qquad \text{and} \qquad (\partial_y^2 c_q)(y)=-\left(q\tfrac{\pi}{2}\right)^2 c_q(y). \tag{2.19}$$

Hence, for each $f\in L^2([-1,1])$, as $\{b_q\}_{q\in\mathbb{N}}$ and $\{c_q\}_{q\in\mathbb{N}\cup\{0\}}$ are orthonormal bases for $L^2([-1,1])$ we have

$$P_M f\xrightarrow{M\to\infty} f \qquad \text{and} \qquad Q_M f\xrightarrow{M\to\infty} f \qquad \text{in} \quad L^2([-1,1]) \tag{2.20}$$

where the partial sums are given by

$$P_M f(y)=\sum_{m=1}^{M}\langle f,b_m\rangle\, b_m(y) \qquad \text{and} \qquad Q_M f(y)=\sum_{m=0}^{M}\langle f,c_m\rangle\, c_m(y). \tag{2.21}$$

Remark: Here, the notation $\langle \cdot, \cdot \rangle$ refers to the inner product in $L^2([-1,1])$.

We are now ready to present the main lemmas of this section. Let us recall first definitions (2.5) and (2.6), which give us

$$X^k([-1,1]) = \{f \in H^k([-1,1]) : (\partial_y^n f)(\pm 1) = 0 \quad \text{for } n = 0, 2, 4, \ldots, k^\star\}$$

and

$$Y^k([-1,1]) = \{f \in H^k([-1,1]) : (\partial_y^n f)(\pm 1) = 0 \quad \text{for } n = 1, 3, 4, \ldots, k_\star\}.$$

Lemma 2.2.1. *$\{b_q\}_{q\in\mathbb{N}}$ is an orthonormal basis of $X^k([-1,1])$.*

Proof. Since the orthogonality is trivial, we will give the details of the completeness of the basis. For a function $f \in X^k([-1,1])$ we know that $f \in H^k([-1,1])$. Then, by (2.20) we have that

$$P_m \partial_y^n f \xrightarrow{M\to\infty} \partial_y^n f \qquad \text{in} \quad L^2([-1,1]) \qquad \text{for} \quad n = 0, 2, 4, \ldots, \text{either } k \text{ or } k-1.$$

By (2.21) we get:

$$P_M \partial_y^n f = \sum_{m=1}^{M} \left\langle \partial_y^n f, b_m \right\rangle b_m(y) \tag{2.22}$$

where, by integration by parts and (2.19), we have

$$\begin{aligned}
\left\langle \partial_y^n f, b_m \right\rangle = \int_{-1}^{+1} \partial_y^n f(y') b_m(y')\, dy' &= \int_{-1}^{+1} f(y') \partial_y^n b_q(y')\, dy' \\
&= (-1)^n \left(q\tfrac{\pi}{2}\right)^n \int_{-1}^{+1} f(y') b_q(y')\, dy' \\
&= (-1)^n \left(q\tfrac{\pi}{2}\right)^n \langle f, b_m \rangle\,.
\end{aligned} \tag{2.23}$$

We must note that, thanks to $b_q(\pm 1) = 0$ and the boundary conditions, the boundary terms in the integration by parts vanish. Therefore, putting (2.23) in (2.22) and applying (2.19) we arrive at $P_M \partial_y^n f \equiv \partial_y^n P_M f$ and we obtain

$$\partial_y^n P_M f \xrightarrow{M\to\infty} \partial_y^n f \qquad \text{in} \quad L^2([-1,1]) \qquad \text{for} \quad n = 0, 2, 4, \ldots, \text{either } k \text{ or } k-1.$$

Moreover, by (2.20) we have:

$$Q_M \partial_y^{n+1} f \xrightarrow{M\to\infty} \partial_y^{n+1} f \qquad \text{in} \quad L^2([-1,1]) \qquad \text{for} \quad n = 0, 2, 4, \ldots, k^*,$$

where by (2.21), we get

$$Q_M \partial_y^{n+1} f = \sum_{m=0}^{M} \left\langle \partial_y^{n+1} f, c_m \right\rangle c_m(y). \tag{2.24}$$

We notice that $\left\langle \partial_y^{n+1} f, c_0 \right\rangle = 0$ due to the fact that $(\partial_y^n f)(\pm 1) = 0$ by hypothesis. In addition, by integration by parts and (2.18) for $m \geqslant 1$ we obtain

$$\begin{aligned}
\left\langle \partial_y^{n+1} f, c_m \right\rangle = \int_{-1}^{+1} \partial_y^{n+1} f(y') c_q(y')\, dy' &= -\int_{-1}^{+1} \partial_y^n f(y') (\partial_y c_q)(y')\, dy' \\
&= (-1)^q \left(q\tfrac{\pi}{2}\right) \int_{-1}^{+1} \partial_y^n f(y') b_q(y')\, dy' \\
&= (-1)^q \left(q\tfrac{\pi}{2}\right) \left\langle \partial_y^n f, b_m \right\rangle.
\end{aligned} \tag{2.25}$$

Here, the boundary term vanishes because by hypothesis we have that $(\partial_y^n f)(\pm 1) = 0$. Hence, putting (2.25) in (2.24) and applaying again (2.17) we arrive to $Q_M \partial_y^{n+1} f \equiv \partial_y P_M \partial_y^n f \equiv \partial_y^{n+1} P_M f$. Therefore,

$$\partial_y^{n+1} P_M f \xrightarrow{M\to\infty} \partial_y^{n+1} f \qquad \text{in} \quad L^2([-1,1]) \qquad \text{for} \quad n = 0, 2, 4, \dots, k^\star.$$

□

Lemma 2.2.2. *$\{c_q\}_{q\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $Y^k([-1,1])$.*

Proof. This results follows from the same ideas than the proof of the above Lemma 2.2.1. □

Because of Lemmas 2.2.1 and 2.2.2 one has the following expressions for both the $X^k(\Omega)$ and $Y^k(\Omega)$ norms.

Corollary 2.2.3. *Let $f \in X^k(\Omega)$ and $g \in Y^k(\Omega)$. For $s_1, s_2 \in \mathbb{N}\cup\{0\}$ such that $s_1 + s_2 \leqslant k$, we have:*

$$||\partial_x^{s_1}\partial_y^{s_2} f||^2_{L^2(\Omega)} = \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}} |p|^{2s_1}|q\tfrac{\pi}{2}|^{2s_2}\, |\mathcal{F}_\omega[f](p,q)|^2\,,$$
$$||\partial_x^{s_1}\partial_y^{s_2} g||^2_{L^2(\Omega)} = \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}\cup\{0\}} |p|^{2s_1}|q\tfrac{\pi}{2}|^{2s_2}\, |\mathcal{F}_\varpi[f](p,q)|^2\,,$$

where $\mathcal{F}_\omega[f](p,q)$ and $\mathcal{F}_\varpi[f](p,q)$ are given by (2.15) and (2.16) respectively.

Introducing a threshold number $m \in \mathbb{N}$, we define the projections $\mathbb{P}_m$ and $\mathbb{Q}_m$ of $L^2(\Omega)$ onto the linear span of eigenfunctions generated by $\{\omega_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}}$ and $\{\varpi_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}\cup\{0\}}$ respectively, such that $\{|p|,\ q\} \leqslant m$. That is, we have that

$$\mathbb{P}_m[f](x,y) := \sum_{\substack{|p|\leqslant m\\ p\in\mathbb{Z}}}\sum_{\substack{q\leqslant m\\ q\in\mathbb{N}}} \mathcal{F}_\omega[f](p,q)\, w_{p,q}(x,y),$$
$$\mathbb{Q}_m[f](x,y) := \sum_{\substack{|p|\leqslant m\\ p\in\mathbb{Z}}}\sum_{\substack{q\leqslant m\\ q\in\mathbb{N}\cup\{0\}}} \mathcal{F}_\varpi[f](p,q)\, \varpi_{p,q}(x,y). \tag{2.26}$$

These projections have the following properties:

Lemma 2.2.4. *For $f \in L^2(\Omega)$, we have that $\mathbb{P}_m[f]$ and $\mathbb{Q}_m[f]$ are $C^\infty(\Omega)$ functions such that:*

- *For $f \in H^1(\Omega)$ we have that:*

$$\partial_x\mathbb{P}_m[f] = \mathbb{P}_m[\partial_x f], \qquad \partial_x\mathbb{Q}_m[f] = \mathbb{Q}_m[\partial_x f],$$
$$\partial_y\mathbb{P}_m[f] = \mathbb{Q}_m[\partial_y f], \qquad \partial_y\mathbb{Q}_m[f] = \mathbb{P}_m[\partial_y f].$$

  As a consequence, for $f \in H^2(\Omega)$, we have:

$$\partial_y^2\mathbb{P}_m[f] = \mathbb{P}_m[\partial_y^2 f] \qquad \textit{and} \qquad \partial_y^2\mathbb{Q}_m[f] = \mathbb{Q}_m[\partial_y^2 f].$$

- *The projectors are self-adjoint in $L^2(\Omega)$:*

$$(\mathbb{P}_m[f], g) = (f, \mathbb{P}_m[g]) \quad \textit{and} \quad (\mathbb{Q}_m[f], g) = (f, \mathbb{Q}_m[g]) \qquad \forall f, g \in L^2(\Omega).$$

- *For $f \in X^k(\Omega)$ and $g \in Y^k(\Omega)$:*

$$||\mathbb{P}_m[f]||_{H^k(\Omega)} \leqslant ||f||_{H^k(\Omega)}, \quad \mathbb{P}_m[f] \to f \quad \textit{in } X^k(\Omega)$$
$$||\mathbb{Q}_m[g]||_{H^k(\Omega)} \leqslant ||g||_{H^k(\Omega)}, \quad \mathbb{Q}_m[f] \to f \quad \textit{in } Y^k(\Omega).$$

Proof. The proof is based in the arguments of the proof of Lemma 2.2.1. □

2.3 Poisson's problem in a bounded strip

With all this in mind, it is time to solve Poisson's system with homogeneous Dirichlet condition (2.14).

Lemma 2.3.1. *Let $\rho \in X^k(\Omega)$. The solution of Poisson's problem*

$$\begin{cases} \Delta\psi &= -\partial_x \bar{\rho} \qquad \textit{in} \qquad \Omega, \\ \quad \psi &= 0 \qquad\qquad \textit{on} \qquad \partial\Omega, \end{cases}$$

satisfies that $\psi \in X^{k+1}(\Omega)$ with $||\psi||_{H^{k+1}(\Omega)} \lesssim ||\bar{\rho}||_{H^k(\Omega)}$ and its Fourier expansion is given by

$$\psi(x,y) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \left(\frac{ip}{p^2 + \left(q\frac{\pi}{2}\right)^2} \right) \mathcal{F}_\omega[\bar{\rho}](p,q)\, \omega_{p,q}(x,y). \tag{2.27}$$

Proof. We consider the sequence of problems

$$\begin{cases} \Delta\psi^{[m]} &= -\mathbb{P}_m[\partial_x \bar{\rho}] \qquad \text{in} \qquad \Omega, \\ \quad \psi^{[m]} &= 0 \qquad\qquad\qquad \text{on} \qquad \partial\Omega. \end{cases}$$

Taking n-derivatives with $n = 0, \dots, k$, testing against $\partial^n \psi^{[m]}$, integrating by parts and applying Young's inequality yields $||\psi^{[m]}||_{H^{k+1}(\Omega)} \leqslant C||\mathbb{P}_m[\bar{\rho}]||_{H^k(\Omega)} \leqslant ||\bar{\rho}||_{H^k(\Omega)}$, since $\rho \in X^k(\Omega)$ (the constant C does not depend on m). In addition, it is easy to check that $\partial_y^n \psi^{[m]}|_{\partial\Omega} = 0$, for any even number n (this is because of the definition of $\mathbb{P}_m$ and the boundary condition $\psi^{[m]}|_{\partial\Omega} = 0$). These two facts allow us to pass to the limit in m to find $\psi \in X^{k+1}(\Omega)$ solving (2.14).

As $\bar{\rho} \in X^k(\Omega)$ and $\psi \in X^{k+1}(\Omega)$ we can expand

$$\bar{\rho}(x,y) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \mathcal{F}_\omega[\bar{\rho}](p,q)\, \omega_{p,q}(x,y) \qquad \text{and} \qquad \psi(x,y) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \mathcal{F}_\omega[\psi](p,q)\, \omega_{p,q}(x,y),$$

then

$$\begin{aligned} -\partial_x \bar{\rho}(x,y) &= -\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} (ip)\, \mathcal{F}_\omega[\bar{\rho}](p,q)\, \omega_{p,q}(x,y), \\ \Delta\psi(x,y) &= -\sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \left(p^2 + \left(q\tfrac{\pi}{2}\right)^2\right) \mathcal{F}_\omega[\psi](p,q)\, \omega_{p,q}(x,y). \end{aligned}$$

Consequently, the following relation between the coefficients must be verified:

$$\mathcal{F}[\psi](p,q) = \frac{ip}{p^2 + \left(q\frac{\pi}{2}\right)^2} \mathcal{F}_\omega[\bar{\rho}](p,q). \tag{2.28}$$

$\square$

Corollary 2.3.2. *The velocity $\mathbf{u} = (u_1, u_2) = \nabla^\perp \psi$ from (2.14) satisfies:*

$$u_1 \in Y^k(\Omega), u_2 \in X^k(\Omega) \quad \textit{and} \quad ||\mathbf{u}||_{H^k(\Omega)} \lesssim ||\bar{\rho}||_{H^k(\Omega)}.$$

2.4 Local solvability of solutions in $X^k(\Omega)$

To obtain a local existence result for a general smooth initial data in a general bounded domain for an *active scalar* is far from being trivial. The presence of boundaries makes the well-posedness issues become more delicate (see for example [51] and [14], in the case of SQG).

Here, we only focus on our setting Ω. Apart from working with the spaces $X^k(\Omega)$ and as a consequence, being careful with the special boundary conditions they impose, the proof in this section is a standard application of Galerkin approximations. For the sake of completeness we write the details below.

We return to the equations for the perturbation of the *confined* IPM in Ω:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= u_2, \\ \mathbf{u} &= \nabla^{\perp} \psi, \\ \nabla \cdot \mathbf{u} &= 0, \end{cases} \tag{2.29}$$

where ψ solves (2.14) together with the no-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and initial data $\rho(0) \in X^k(\Omega)$. Hence, we will prove the following result:

Theorem 2.4.1. *Let $k \in \mathbb{N}$ with $k \geqslant 3$ and an initial data $\rho(0) \in X^k(\Omega)$. Then, there exists a time $T > 0$ and a constant C, both depending only on $||\rho||_{H^3(\Omega)}(0)$ and a unique solution $\rho \in C\left(0, T; X^k(\Omega)\right)$ of the equations (2.29) such that*

$$\sup_{0 \leqslant t < T} ||\rho||_{H^k(\Omega)}(t) \leqslant C \, ||\rho||_{H^k(\Omega)}(0).$$

Moreover, for all $t \in [0, T)$ the following estimate holds:

$$||\rho||_{H^k(\Omega)}(t) \leqslant ||\rho||_{H^k(\Omega)}(0) \, \exp\left[\widetilde{C} \int_0^t \left(||\nabla\rho||_{L^\infty(\Omega)}(s) + ||\nabla \mathbf{u}||_{L^\infty(\Omega)}(s)\right) \, ds\right]. \tag{2.30}$$

The general method of the proof is similar to that for proving the existence of solutions to the Navier–Stokes and Euler equations which can be found in [46].

The strategy of this section has two parts. First we find an approximate equation and approximate solutions that have two properties: (1) the approximate solutions exist for all time, (2) the solutions satisfy an analogous energy estimate. The second part is the passage to a limit in the approximation scheme to obtain a solution to the original equations.

Before embarking on the proof, we will need some basic properties of the Sobolev spaces in bounded domains. In the next lemma, $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary ∂D.

Lemma 2.4.2. *For $s \in \mathbb{N}$, the following estimates hold:*

- *If $f, g \in H^s(D) \cap \mathcal{C}(D)$, then*

$$||f\, g||_{H^s(D)} \lesssim \left(||f||_{H^s(D)} \, ||g||_{L^\infty(D)} + ||f||_{L^\infty(D)} \, ||g||_{H^s(D)}\right); \tag{2.31}$$

- *If $f \in H^s(D) \cap \mathcal{C}^1(D)$ and $g \in H^{s-1}(D) \cap \mathcal{C}(D)$, then for $|\alpha| \leqslant s$ we have that*

$$||\partial^\alpha (fg) - f \partial^\alpha g||_{L^2(D)} \lesssim ||f||_{W^{1,\infty}(D)} \, ||g||_{H^{s-1}(D)} + ||f||_{H^s(D)} \, ||g||_{L^\infty(D)}. \tag{2.32}$$

Moreover, the following Sobolev embeddings hold:

- $W^{s,p}(D) \subseteq L^q(D)$ *continuously if* $s < n/p$ *and* $p \leqslant q \leqslant np/(n-sp)$*;*
- $W^{s,p}(D) \subseteq \mathcal{C}^k(\overline{D})$ *continuously is* $s > k + n/p$.

Proof. See [29, p. 280] and references therein. □

Proof of Theorem 2.4.1. We firstly construct approximate equations by using a smoothing procedure called Galerkin method. The m^{th}-Galerkin approximation of (2.29) is the following system:

$$\begin{cases} \partial_t \rho^{[m]} + \mathbb{P}_m \left[\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]} \right] &= u_2^{[m]} \\ \mathbf{u}^{[m]} &= \nabla^{\perp} \psi^{[m]} \\ \rho^{[m]}|_{t=0} &= \mathbb{P}_m[\rho](0), \end{cases} \tag{2.33}$$

where

$$\begin{cases} \Delta \psi^{[m]} = -\partial_x \rho^{[m]} & \text{in} \quad \Omega \\ \psi^{[m]} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.34}$$

and with $\rho(0) \in X^k(\Omega)$. Since the initial data in (2.33) belongs to $\mathbb{P}_m L^2(\Omega)$ and because of the structure of the equations, we look for solutions of the form

$$\rho^{[m]}(t) = \sum_{\substack{|p| \leqslant m \\ p \in \mathbb{Z}}} \sum_{\substack{q \leqslant m \\ q \in \mathbb{N}}} \mathfrak{c}_{p,q}^{[m]}(t) \omega_{p,q}(x,y).$$

Then, by Lemma 2.3.1 we get:

$$\psi^{[m]}(t) = \sum_{\substack{|p| \leqslant m \\ p \in \mathbb{Z}}} \sum_{\substack{q \leqslant m \\ q \in \mathbb{N}}} \left(\frac{ip}{p^2 + (q\frac{\pi}{2})^2} \right) \mathfrak{c}_{p,q}^{[m]}(t) \omega_{p,q}(x,y).$$

Thereby, (2.33) is reduced to a finite dimensional ODE system for the coefficients $\mathfrak{c}_{p,q}^{[m]}(t)$ for $\{|p|, q\} \leqslant m$, and we can apply Picard's theorem to find a solution on a time of existence depending on m. Next, we will use energy estimates to prove that there is a time of existence T, uniform in m, for every solution $\rho^{[m]}(t)$ of (2.33) and a limit $\rho(t)$ which will solve (2.29). To do this, we recall that

$$\rho^{[m]} = \mathbb{P}_m \left[\rho^{[m]} \right] \qquad \text{and} \qquad \mathbf{u}^{[m]} = \left(u_1^{[m]}, u_2^{[m]} \right) = \left(\mathbb{Q}_m \left[u_1^{[m]} \right], \mathbb{P}_m \left[u_2^{[m]} \right] \right).$$

Taking derivatives ∂^s, with $|s| \leqslant k$ on the first equation of (2.33) and then taking the $L^2(\Omega)$ inner product with $\partial^s \rho^{[m]}$, we obtain

$$\left(\partial_t \partial^s \rho^{[m]}, \partial^s \rho^{[m]} \right) = \left(\partial^s u_2^{[m]}, \partial^s \rho^{[m]} \right) - \left(\partial^s \mathbb{P}_m \left[\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]} \right], \partial^s \rho^{[m]} \right) = \mathrm{I} - \mathrm{II}.$$

For the first term, since $\psi^{[m]}$ solves Poisson's problem (2.34), integrations by parts give us

$$\mathrm{I} = \left(\partial^s \partial_x \psi^{[m]}, \partial^s \rho^{[m]} \right) = \left(\partial^s \psi^{[m]}, \partial^s \Delta \psi^{[m]} \right) = -\|\partial^s \mathbf{u}^{[m]}\|_{L^2(\Omega)}^2 \tag{2.35}$$

thanks to the fact that $\partial_y^n \psi^{[m]}|_{\partial\Omega} = 0$, for any even number n. For the second one, we need to distinguish between an even or odd number of y-derivatives. In any case, the properties of $\mathbb{P}_m, \mathbb{Q}_m$ given by Lemma 2.2.4 and the commutator estimate (2.32) with $f = \mathbf{u}^{[m]}$ and $g = \nabla \rho^{[m]}$ give us the inequality

$$\mathrm{II} \lesssim \|\partial^s \rho^{[m]}\|_{L^2(\Omega)} \left(\|\nabla \mathbf{u}^{[m]}\|_{L^\infty(\Omega)} \|\rho^{[m]}\|_{H^k(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)} \|\nabla \rho^{[m]}\|_{L^\infty(\Omega)} \right). \tag{2.36}$$

Summing over $|s| \leqslant k$ and putting together (2.35) and (2.36) we obtain

$$\tfrac{1}{2}\partial_t \|\rho^{[m]}\|^2_{H^k(\Omega)} \lesssim \|\rho^{[m]}\|_{H^k(\Omega)} \left(\|\nabla \mathbf{u}^{[m]}\|_{L^\infty(\Omega)} \|\rho^{[m]}\|_{H^k(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)} \|\nabla \rho^{[m]}\|_{L^\infty(\Omega)} \right)$$

and as $\mathbf{u}^{[m]} = \nabla^\perp \psi^{[m]}$ where $\psi^{[m]}$ solves (2.34) by Lemma 2.3.1 we get $\|\mathbf{u}^{[m]}\|_{H^k(\Omega)} \lesssim \|\rho^{[m]}\|_{H^k(\Omega)}$. Therefore, we finally obtain that

$$\tfrac{1}{2}\partial_t \|\rho^{[m]}\|^2_{H^k(\Omega)} \lesssim \|\rho^{[m]}\|^2_{H^k(\Omega)} \left(\|\nabla \mathbf{u}^{[m]}\|_{L^\infty(\Omega)} + \|\nabla \rho^{[m]}\|_{L^\infty(\Omega)} \right) \lesssim \|\rho^{[m]}\|^2_{H^k(\Omega)} \|\rho^{[m]}\|_{H^3(\Omega)} \tag{2.37}$$

where the last inequality is true provided that $k \geqslant 3$ due to the Sobolev embedding $L^\infty(\Omega) \hookrightarrow H^2(\Omega)$. Hence, for all m and $0 \leqslant t < T \leqslant \left(c \, \|\rho\|_{H^3(\Omega)}(0)\right)^{-1}$ we have that

$$\|\rho^{[m]}\|_{H^3(\Omega)}(t) \leqslant \frac{\|\mathbb{P}_m[\rho]\|_{H^3(\Omega)}(0)}{1 - c\,t\,\|\mathbb{P}_m[\rho]\|_{H^3(\Omega)}(0)} \leqslant \frac{\|\rho\|_{H^3(\Omega)}(0)}{1 - c\,t\,\|\rho\|_{H^3(\Omega)}(0)} \tag{2.38}$$

and, in particular, that

$$\sup_{0 \leqslant t < T} \|\rho^{[m]}\|_{H^3(\Omega)}(t) \leqslant \frac{\|\rho\|_{H^3(\Omega)}(0)}{1 - c\,T\,\|\rho\|_{H^3(\Omega)}(0)}.$$

Applying (2.38) in the last term of (2.37), we obtain for all m and $0 \leqslant t < T$ by Gronwall's lemma that

$$\begin{aligned} \|\rho^{[m]}\|_{H^k(\Omega)}(t) &\leqslant \|\mathbb{P}_m[\rho^{[m]}]\|_{H^k(\Omega)}(0) \, \exp\left[c \int_0^t \frac{\|\rho\|_{H^3(\Omega)}(0)}{1 - c\,s\,\|\rho\|_{H^3(\Omega)}(0)} \, ds \right] \\ &\leqslant \|\rho\|_{H^k(\Omega)}(0) \, \exp\left[c \int_0^t \frac{\|\rho\|_{H^3(\Omega)}(0)}{1 - c\,s\,\|\rho\|_{H^3(\Omega)}(0)} \, ds \right] \end{aligned}$$

and, in particular, that

$$\sup_{0 \leqslant t < T} \|\rho^{[m]}\|_{H^k(\Omega)}(t) \leqslant C\, \|\rho\|_{H^k(\Omega)}(0) \tag{2.39}$$

where C is a constant depending only on $\|\rho\|_{H^3(\Omega)}(0)$.

Therefore, the family $\rho^{[m]}$ is uniformly bounded, with respect to m, in $L^\infty\left(0, T; H^k(\Omega)\right)$. One consequence of the Banach–Alaoglu theorem (see [55]) is that a bounded sequence $\|\rho^{[m]}\|_{H^k(\Omega)} \leqslant K$ has a subsequence that converges weakly to some limit in $H^k(\Omega)$, which is the dual of a separable Banach space. This is $\rho^{[m]}(t) \rightharpoonup \rho(t)$ in $H^k(\Omega)$ for $0 \leqslant t < T$.

Moreover, the family $\partial_t \rho^{[m]}$ is uniformly bounded in $L^\infty\left(0, T; H^{k-2}(\Omega)\right)$. By (2.33) we have that

$$\begin{aligned} \sup_{0 \leqslant t < T} \|\partial_t \rho^{[m]}\|_{H^{k-2}(\Omega)}(t) &= \sup_{0 \leqslant t < T} \|u_2^{[m]} - \mathbb{P}_m\left[\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]}\right]\|_{H^{k-2}(\Omega)}(t) \\ &\leqslant \sup_{0 \leqslant t < T} \|u_2^{[m]}\|_{H^{k-2}(\Omega)}(t) + \sup_{0 \leqslant t < T} \|\mathbb{P}_m\left[\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]}\right]\|_{H^{k-2}(\Omega)}(t). \end{aligned}$$

We need to show that $\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]} \in X^{k-1}(\Omega)$ in order to apply Lemma 2.2.4, for $k \geqslant 3$, and to get

$$\begin{aligned} \|\mathbb{P}_m\left[\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]}\right]\|_{H^{k-2}(\Omega)}(t) &\leqslant \|\mathbf{u}^{[m]} \cdot \nabla \rho^{[m]}\|_{H^{k-2}(\Omega)} \\ &\lesssim |\mathbf{u}^{[m]}\|_{H^{k-2}(\Omega)} \, \|\nabla \rho^{[m]}\|_{L^\infty(\Omega)} + \|\mathbf{u}^{[m]}\|_{L^\infty(\Omega)} \, \|\nabla \rho^{[m]}\|_{H^{k-2}(\Omega)} \\ &\lesssim \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}(t) \, \|\rho^{[m]}\|_{H^k(\Omega)}(t) \end{aligned}$$

where in the last inequalities we used (2.31) and the Sobolev embedding $L^\infty(\Omega) \hookrightarrow H^2(\Omega)$.

Checking that $\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]} \in X^{k-1}(\Omega)$ reduces to see that $\partial_y^n \left(\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}\right)|_{\partial\Omega} = 0$ for any even natural number n. We start with the observation

$$\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]} = \mathbb{Q}_m\left[u_1^{[m]}\right]\mathbb{P}_m\left[\partial_x\rho^{[m]}\right] + \mathbb{P}_m\left[u_2^{[m]}\right]\mathbb{Q}_m\left[\partial_y\rho^{[m]}\right]$$

and the fact that, due to (2.17) and (2.18),

$$\begin{aligned}\partial_y^2(b_q\, c_q)(y) &= (\partial_y^2 b_q)(y)\, c_q(y) + 2(\partial_y b_q)(y)\,(\partial_y c_q)(y) + b_q(y)\,(\partial_y^2 c_q)(y) \\ &= (-1)(q\,\pi)^2 b_q(y)\, c_q(y).\end{aligned}$$

Iterating this procedure and using that $b_q(\pm 1) = 0$ we prove the boundary conditions for the derivatives of even order of the non-linear term.

As before, by Lemma 2.3.1 we obtain the bound $\|\mathbf{u}^{[m]}\|_{H^k(\Omega)}(t) \lesssim \|\rho^{[m]}\|_{H^k(\Omega)}(t)$ and putting all together we obtain

$$\begin{aligned}\sup_{0\leqslant t<T}\|\partial_t\rho^{[m]}\|_{H^{k-2}(\Omega)}(t) &\lesssim \sup_{0\leqslant t<T}\|\rho^{[m]}\|_{H^k(\Omega)}(t)\left[1 + \|\rho^{[m]}\|_{H^k(\Omega)}(t)\right] \\ &\leqslant C\,\|\rho\|_{H^k(\Omega)}(0)\left[1 + C\,\|\rho\|_{H^k(\Omega)}(0)\right]\end{aligned}$$

thanks to (2.39). So, the family of time derivatives $\partial_t\rho^{[m]}(t)$ is uniformly bounded in $L^\infty\left(0, T; H^{k-2}(\Omega)\right)$.

Therefore, as we have seen above, the family of time derivatives $\partial_t\rho^{[m]}(t)$ is uniformly bounded in $L^\infty\left(0, T; H^{k-2}(\Omega)\right)$. Then, by Banach–Alaoglu theorem, $\partial_t\rho^{[m]}(t)$ has a subsequence that converges weakly to some limit in $H^{k-2}(\Omega)$ for $0 \leqslant t < T$.

Moreover, by virtue of Aubin–Lions's compactness lemma (see for instance [43]) applied with the triple $H^k(\Omega) \subset\subset H^{k-1}(\Omega) \subset H^{k-2}(\Omega)$ we obtain that the convergence $\rho^{[m]} \to \rho$ is strong in $C(0, T; H^{k-1}(\Omega))$. As $\mathbf{u}^{[m]} = \nabla^\perp\psi^{[m]}$ where $\psi^{[m]}$ solves (2.34) and the convergence $\rho^{[m]} \to \rho$ is strong in $C(0, T; H^{k-1}(\Omega))$, we obtain the strong convergence $\mathbf{u}^{[m]} \to \mathbf{u}$ in $C(0, T; Y^{k-1}(\Omega) \times X^{k-1}(\Omega))$. Using these facts, we may pass to the limit in the non-linear part of (2.33) to see that $\mathbb{P}_m[\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}] \to \mathbf{u} \cdot \nabla\rho$ in $C(0, T; H^{k-2}(\Omega))$ as follows:

$$\begin{aligned}&\|\mathbb{P}_m[\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}] - \mathbf{u} \cdot \nabla\rho\|_{H^{k-2}(\Omega)} \\ &\quad = \|\mathbb{P}_m[\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}] \pm \mathbf{u}^{[m]} \cdot \nabla\rho^{[m]} \pm \mathbf{u}^{[m]} \cdot \nabla\rho - \mathbf{u} \cdot \nabla\rho\|_{H^{k-2}(\Omega)} \\ &\quad \leqslant \left|\left|(\mathbb{P}_m - \mathbb{I})[\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}]\right|\right|_{H^{k-2}(\Omega)} + \left|\left|\mathbf{u}^{[m]} \cdot \nabla(\rho^{[m]} - \rho)\right|\right|_{H^{k-2}(\Omega)} \\ &\qquad + \left|\left|(\mathbf{u}^{[m]} - \mathbf{u}) \cdot \nabla\rho\right|\right|_{H^{k-2}(\Omega)} \to 0 \qquad \text{as} \quad m \to \infty.\end{aligned}$$

In the limit, we use the fact that $\lim_{m\to\infty}\|\mathbb{P}_m[f] - f\|_{H^s(\Omega)} = 0$ for $f \in X^s(\Omega)$, together with the convergences of $\mathbf{u}^{[m]} \to \mathbf{u}$ and $\rho^{[m]} \to \rho$ and (2.31), for $k \geqslant 3$.

Now, from (2.33), we have that $\partial_t\rho^{[m]} = u_2^{[m]} - \mathbb{P}_m\left[\mathbf{u}^{[m]} \cdot \nabla\rho^{[m]}\right] \to u_2 - \mathbf{u}\cdot\nabla\rho$ in $C(0, T; H^{k-2}(\Omega))$. Since $\rho^{[m]} \to \rho$ in $C(0, T; H^{k-1}(\Omega))$, the distribution limit of $\partial_t\rho^{[m]}$ must be $\partial_t\rho$ for the Closed Graph theorem [5]. Thus it follows that $\rho(t)$ is the unique classical solution of (2.29) which lies in $C(0, T; H^{k-1}(\Omega))$. Then, to show that $\rho \in C(0, T; H^k(\Omega))$ we follow [46, p. 110].

Firstly, we recall that $\rho \in L^\infty(0, T; H^k(\Omega)) \subset L^2(0, T; H^k(\Omega))$ and we start proving that $\rho(t)$ is continuous on $[0, T)$ in the weak topology of $H^k(\Omega)$. To prove that $\rho \in C_W(0, T; H^k(\Omega))$, we define the dual pairing of $(H^s)^\star(\Omega)$ and $H^s(\Omega)$ as $[\cdot,\cdot] : (H^s(\Omega))^\star \times H^s(\Omega) \to \mathbb{R}$ given by $[\varphi, f] := \varphi[f]$. Hence, because $\rho^{[m]} \to \rho$ in $C(0,T;H^{k-1}(\Omega))$, it follows that $[\varphi, \rho^{[m]}(t)] \to [\varphi, \rho(t)]$ uniformly on $[0,T)$ for any $\varphi \in (H^{k-1}(\Omega))^\star$.

Using that $(H^{k-1}(\Omega))^\star$ is dense in $(H^k(\Omega))^\star$ by means of an ϵ-argument together with (2.39), we have $[\varphi, \rho^{[m]}] \to [\varphi, \rho]$ uniformly on $[0,T)$ for any $\varphi \in (H^k(\Omega))^\star$. This fact implies that $\rho \in C_W(0,T;H^k(\Omega))$.

By virtue of the fact that $\rho \in C_W(0,T;H^k(\Omega))$, it suffices to show that the norm $\|\rho\|_{H^k(\Omega)}(t)$ is a continuous function of time to get that $\rho \in C(0,T;H^k(\Omega))$.

Recall the relation for the uniform $H^k(\Omega)$ norm for the approximations

$$\|\rho^{[m]}\|_{H^k(\Omega)}(t) \leqslant \frac{\|\rho\|_{H^k(\Omega)}(0)}{1 - C\,t\,\|\rho\|_{H^k(\Omega)}(0)} = \|\rho\|_{H^k(\Omega)}(0) + \frac{C\,t\,\|\rho\|^2_{H^k(\Omega)}(0)}{1 - C\,t\,\|\rho\|_{H^k(\Omega)}(0)} \qquad \text{for all} \quad 0 \leqslant t < T.$$

For fixed time $t \in [0,T)$ we have $\|\rho\|_{H^k(\Omega)}(t) \leqslant \liminf_{m\to\infty} \|\rho^{[m]}\|_{H^k(\Omega)}(t)$. Using this in the above expression, we obtain

$$\|\rho\|_{H^k(\Omega)}(t) \leqslant \|\rho\|_{H^k(\Omega)}(0) + \frac{C\,t\,\|\rho\|^2_{H^k(\Omega)}(0)}{1 - C\,t\,\|\rho\|_{H^k(\Omega)}(0)}.$$

On one hand, by the fact that $\rho \in C_W(0,T;H^k(\Omega))$, we get that $\|\rho\|_{H^k(\Omega)}(0) \leqslant \liminf_{t\to 0^+} \|\rho\|_{H^k(\Omega)}(t)$. On the other hand, the above expression gives us that $\limsup_{t\to 0^+} \|\rho\|_{H^k(\Omega)}(t) \leqslant \|\rho\|_{H^k(\Omega)}(0)$. Then, in particular, $\lim_{t\to 0^+} \|\rho\|_{H^k(\Omega)}(t) = \|\rho\|_{H^k(\Omega)}(0)$. This gives us strong right continuity at $t = 0$.

It remains to prove continuity of the $\|\cdot\|_{H^k(\Omega)}(t)$ norm of the solution at times other than the initial time. Consider a time $t^\star \in (0,T)$ and the solution $\rho(t^\star) \in H^k(\Omega)$. At this fixed time, we define $\rho^\star(0) := \rho(t^\star)$, so we can take $\rho^\star(0)$ as initial data and construct a solution as above by solving the regularized equation (2.33). Following the argument we used above to show that $\|\rho\|_{H^k(\Omega)}(t)$ is continuous at $t = 0$, we also conclude that it is continuous as $t = t^\star$. Because $t^\star \in (0,T)$ is arbitrary, we have just showed that $\|\rho\|_{H^k(\Omega)}(t)$ is a continuous function on $[0,T)$. As a consequence, we have proved that $\rho \in C(0,T;H^k(\Omega))$.

Since for every $m \in \mathbb{N}$ we have $\rho^{[m]} = \mathbb{P}_m[\rho^{[m]}] \in X^k(\Omega)$, that is $\partial_y^n \rho^{[m]}|_{\partial\Omega} = 0$ for any even number n and this property is closed, we obtain that the limiting function also has the desired property, which concludes that the solution ρ lies in $C\left(0,T;X^k(\Omega)\right)$.

Finally, applying Gronwall's lemma on the above estimate (2.37) and the previous convergence results, for all $t \in [0,T)$ we deduce that

$$\begin{aligned}\|\rho^{[m]}\|_{H^k(\Omega)}(t) &\leqslant \|\rho^{[m]}\|_{H^k(\Omega)}(0)\ \exp\left[\widetilde{C}\int_0^t \left(\|\nabla\rho^{[m]}\|_{L^\infty(\Omega)}(s) + \|\nabla\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}(s)\right)\,ds\right]\\ &\leqslant \|\rho\|_{H^k(\Omega)}(0)\ \exp\left[\widetilde{C}\int_0^t \left(\|\nabla\rho\|_{L^\infty(\Omega)}(s) + \|\nabla\mathbf{u}\|_{L^\infty(\Omega)}(s)\right)\,ds\right]\end{aligned}$$

and by lower semicontinuity we obtain (2.30).

□

Theorem 2.4.3. *If $\rho(t)$ is a solution of (2.29) in the class $C\left(0,T,X^k(\Omega)\right)$ with $\rho(0) \in X^k(\Omega)$, and if $T = T^\star$ is the first time such that $\rho(t)$ is not contained in this class, then*

$$\int_0^{T^\star} \left(\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}(s) + \|\nabla\rho\|_{L^\infty(\Omega)}(s)\right)\,ds = \infty.$$

Proof. This result follows from estimate (2.30). □

2.5 Global regularity for small initial data

This section is devoted to prove the main result of this chapter:

Theorem 2.5.1. *Let $\Theta(y) := -y$. There exists $\varepsilon_0 > 0$ and a parameter $\gamma \in \mathbb{N}$ with $\gamma > 4$ such that if we solve (2.3) with initial data $\varrho\,(0) = \Theta + \rho(0)$ and $\rho(0) \in X^{\kappa}(\Omega)$ with $\|\rho\|_{H^{\kappa}(\Omega)}(0) < \varepsilon \leqslant \varepsilon_0$ where $\kappa \geqslant 5 + \gamma$ then, the solution exists globally in time and satisfies the following:*

i) $\|\bar{\varrho}\|_{H^3(\Omega)}(t) \equiv \|\bar{\rho}\|_{H^3(\Omega)}(t) \lesssim \varepsilon(1+t)^{-\frac{\gamma}{4}}$

ii) $\|\tilde{\varrho} - \Theta\|_{H^{\kappa}(\Omega)}(t) \equiv \|\tilde{\rho}\|_{H^{\kappa}(\Omega)}(t) \leqslant 2\,\varepsilon$

where $\varrho := \bar{\varrho} + \tilde{\varrho}$ such that $\bar{\varrho} \perp \tilde{\varrho}$ and $\bar{\varrho}$ is given by the projection operator onto the subspace of functions with zero average in the horizontal variable.

In the next three sections we give the proof of this result.

2.5.1 Energy methods for the confined IPM equation

From what we have seen, we know that for $\rho(0) \in X^k(\Omega)$ there exists $T > 0$ such that $\rho(t) \in X^k(\Omega)$ is a solution of (2.7) for all $t \in [0, T)$. Moreover, if $T^\star$ is the first time such that $\rho(t)$ is not contained in this class, then

$$\int_0^{T^\star} \left(\|\nabla \mathbf{u}\|_{L^\infty(\Omega)}(s) + \|\nabla\rho\|_{L^\infty(\Omega)}(s)\right)\, ds = \infty.$$

2.5.1.1 A priori energy estimate

In what follows, we assume that $\rho(t) \in X^k(\Omega)$ is a solution of (2.7) for any $t \geqslant 0$. Then, the following estimate holds for $k \geqslant 6$:

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{H^k(\Omega)}(t) \lesssim \|\partial u_2\|_{L^\infty(\Omega)}(t)\, \|\rho\|^2_{H^k(\Omega)}(t) - \left(1 - \|\rho\|_{H^k(\Omega)}(t)\right)\, \|\mathbf{u}\|^2_{H^k(\Omega)}(t).$$

In this section we will perform the basic energy estimate for

$$\partial_t\rho + \mathbf{u}\cdot\nabla\rho = u_2. \tag{2.40}$$

$L^2(\Omega)$**-estimate:** We begin with the $L^2(\Omega)$ bound. We multiply (2.40) by ρ and integrate over Ω. Then,

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{L^2(\Omega)} = \int_\Omega \rho\,\partial_t\rho\, dxdy = \int_\Omega \rho\, u_2\, dxdy - \int_\Omega \rho\, (\mathbf{u}\cdot\nabla)\,\rho\, dxdy.$$

By the incompressibility of the velocity and the boundary conditions, we have that the second term vanishes, so by (2.13) we get:

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{L^2(\Omega)} = \int_\Omega \rho\, u_2\, dxdy = \int_\Omega \rho\,\partial_x\psi\, dxdy.$$

Finally, applying integration by parts and using that ψ solves (2.14) we achieve:

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{L^2(\Omega)} = \int_\Omega \Delta\psi\,\psi\, dxdy = -\int_\Omega (\nabla\psi)^2\, dxdy + \int_\Omega \partial_y[\partial_y\psi\,\psi]\, dxdy.$$

As $\psi|_{\partial\Omega} = 0$, it is clear that the boundary term vanishes, and consequently we have that

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{L^2(\Omega)} = -\|\nabla\psi\|^2_{L^2(\Omega)}. \tag{2.41}$$

$\dot{H}^k(\Omega)$**-estimate:** We next take ∂^k to (2.40), we multiply it by $\partial^k\rho$ and integrate over Ω. Then,

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\rho\|^2_{\dot{H}^k(\Omega)} = \int_\Omega \partial^k\rho\,\partial_t\partial^k\rho\,dxdy &= \int_\Omega \partial^k\rho\,\partial^k u_2 dxdy - \int_\Omega \partial^k\rho\,\partial^k(\mathbf{u}\cdot\nabla)\rho\,dxdy\\ &= I_1 + I_2.\end{aligned}$$

First of all, we study I_1. By (2.13), (2.14) and integration by parts, we get:

$$\begin{aligned}I_1 &= \int_\Omega \partial^k\rho\,\partial^k\partial_x\psi dxdy = -\int_\Omega \partial^k\partial_x\rho\,\partial^k\psi dxdy = \int_\Omega \Delta\partial^k\psi\,\partial^k\psi dxdy\\ &= -\int_\Omega \left(\nabla\partial^k\psi\right)^2 dxdy + \int_\Omega \partial_y\left[\partial_y\partial^k\psi\,\partial^k\psi\right] dxdy.\end{aligned}$$

As $\psi\in X^{k+1}(\Omega)$ due to Lemma 2.3.1, the boundary term vanishes and we have proved that

$$I_1 = -\|\nabla\psi\|^2_{\dot{H}^k(\Omega)}. \tag{2.42}$$

Secondly, we study I_2. The singular term vanishes by the incompressibility and the boundary conditions,

$$\begin{aligned}I_2 &= -\int_\Omega \partial^k\rho\,\partial^k(\mathbf{u}\cdot\nabla)\rho\,dxdy\\ &= -\int_\Omega \partial^k\rho\,\left(\partial\mathbf{u}\cdot\nabla\partial^{k-1}\rho\right) dxdy - \sum_{i=1}^{k-1}\binom{k}{i}\int_\Omega \partial^k\rho\,\left(\partial^{i+1}\mathbf{u}\cdot\nabla\partial^{k-i-1}\rho\right) dxdy.\end{aligned}$$

Now, we want to distinguish between two kinds of terms, first for the case where $i=0$ and then the case where $1\leqslant i\leqslant k-1$. The term for $i=0$ is bounded directly as

$$-\int_\Omega \partial^k\rho\,\left(\partial\mathbf{u}\cdot\nabla\partial^{k-1}\rho\right) dxdy \leqslant \|\partial\mathbf{u}\|_{L^\infty(\Omega)}\|\rho\|^2_{H^k(\Omega)}$$

but working a little bit harder, we achieve

$$\begin{aligned}-\int_\Omega \partial^k\rho\,\left(\partial\mathbf{u}\cdot\nabla\partial^{k-1}\rho\right) dxdy &= -\int_\Omega \partial^k\rho\,\left(\partial u_1\,\partial_x\partial^{k-1}\rho + \partial u_2\,\partial_y\partial^{k-1}\rho\right) dxdy\\ &\leqslant \int_\Omega \partial^k\rho\,\left(\partial u_1\,\partial_x\partial^{k-1}\rho\right) dxdy + \|\partial u_2\|_{L^\infty(\Omega)}\|\rho\|^2_{H^k(\Omega)}\end{aligned}$$

where, for the first integral, we consider two cases:

- $\boxed{\partial u_1 \equiv \partial_x u_1}$ By the incompressibility of the flow it is clear that

$$\int_\Omega \partial^k\rho\,\left(\partial_x u_1\,\partial_x\partial^{k-1}\rho\right) dxdy = -\int_\Omega \partial^k\rho\,\left(\partial_y u_2\,\partial_x\partial^{k-1}\rho\right) dxdy \leqslant \|\partial u_2\|_{L^\infty(\Omega)}\|\rho\|^2_{H^k(\Omega)}.$$

- $\boxed{\partial u_1 \equiv \partial_y u_1}$ In this case, by (2.13) we have that

$$\begin{aligned}\int_\Omega \partial^k\rho\,\left(\partial_y u_1\,\partial_x\partial^{k-1}\rho\right) dxdy &= \int_\Omega \partial^k\rho\,\left(\partial_x u_2\,\partial_x\partial^{k-1}\rho\right) dxdy + \int_\Omega \partial^k\rho\,\left(\partial_x\rho\,\partial_x\partial^{k-1}\rho\right) dxdy\\ &\leqslant \|\partial u_2\|_{L^\infty(\Omega)}\|\rho\|^2_{H^k(\Omega)} + \|\rho\|_{H^k(\Omega)}\|\nabla\psi\|^2_{H^k}.\end{aligned}$$

To sum up, we have proved that

$$-\int_\Omega \partial^k\rho\,\left(\partial\mathbf{u}\cdot\nabla\partial^{k-1}\rho\right)\,dxdy \leqslant ||\partial u_2||_{L^\infty(\Omega)}\,||\rho||^2_{H^k(\Omega)} + ||\rho||_{H^k(\Omega)}\,||\nabla\psi||^2_{H^k}. \tag{2.43}$$

Indeed, this is the only term that cannot be absorbed by the linear part. This term is the reason why we need to have a integrable time decay of $||\partial u_2||_{L^\infty(\Omega)}$. Precisely, the main goal of the next Section 2.5.2 is to obtain a time decay rate for it.

On the other hand, for $i = 1, \dots, k-1$ we separate the other term as follows:

$$\begin{aligned}\int_\Omega \partial^k\rho\,(\partial^{i+1}\mathbf{u}\cdot\nabla\partial^{k-i-1}\rho)\,dxdy &= \int_\Omega \partial^k\rho\partial^{i+1}u_1\partial_x\partial^{k-i-1}\rho dxdy + \int_\Omega \partial^k\rho\partial^{i+1}u_2\partial_y\partial^{k-i-1}\rho dxdy \\ &= J_1(i) + J_2(i) \qquad i = 1,\dots,k-1.\end{aligned}$$

In view of (2.13) and (2.14), we have that $J_1(i)$ can be rewritten as

$$J_1(i) = \int_\Omega \partial^k\rho\,\partial^{i+1}\partial_y\psi\,\partial^{k-i-1}\Delta\psi\,dxdy$$

and we clearly have

$$\begin{aligned}\sum_{i=1}^{k-1} J_1(i) &\leqslant ||\partial^k\rho||_{L^2}\left[\sum_{i=1}^{k-3}||\partial^{i+1}\partial_y\psi||_{L^\infty}||\partial^{k-i-1}\Delta\psi||_{L^2} + \sum_{i=k-2}^{k-1}||\partial^{i+1}\partial_y\psi||_{L^2}||\partial^{k-i-1}\Delta\psi||_{L^\infty}\right] \\ &\leqslant ||\rho||_{H^k(\Omega)}\,||\nabla\psi||^2_{H^k(\Omega)} \qquad \text{for} \qquad k\geqslant 4.\end{aligned} \tag{2.44}$$

For $J_2(i)$, by (2.13) we obtain that

$$J_2(i) = \int_\Omega \partial^k\rho\,\partial^{i+1}\partial_x\psi\,\partial_y\partial^{k-i-1}\rho\,dxdy$$

and for $i = 1,\dots,k-1$ we need to distinguish two situations:

- We have at least one derivative in x. This is $\partial^k \equiv \partial^{k-1}\partial_x$. Then, by (2.14) we can write $J_2(i)$ as follows:

$$J_2(i) = -\int_\Omega \partial^{k-1}\Delta\psi\,\partial^{i+1}\partial_x\psi\,\partial_y\partial^{k-i-1}\rho\,dxdy$$

and as before, we clearly have

$$\begin{aligned}\sum_{i=1}^{k-1} J_2(i) &\leqslant ||\partial^{k-1}\Delta\psi||_{L^2}\left[\sum_{i=1}^{k-3}||\partial^{i+1}\partial_x\psi||_{L^\infty}||\partial_y\partial^{k-i-1}\rho||_{L^2} + \sum_{i=k-2}^{k-1}||\partial^{i+1}\partial_x\psi||_{L^2}||\partial_y\partial^{k-i-1}\rho||_{L^\infty}\right] \\ &\leqslant ||\rho||_{H^k(\Omega)}\,||\nabla\psi||^2_{H^k(\Omega)} \qquad \text{for} \qquad k\geqslant 4;\end{aligned} \tag{2.45}$$

- All derivatives are in y. This is $\partial^k \equiv \partial_y^k$. In this case, we have that

$$J_2(i) = \int_\Omega \partial_y^k\rho\,\partial_y^{i+1}\partial_x\psi\,\partial_y^{k-i}\rho\,dxdy$$

and by integration by parts we achieve

$$\begin{aligned}J_2(i) = &\int_\Omega \partial_y^{k-1}\partial_x\rho\,\partial_y^{k-i+1}\rho\,\partial_y^{i+1}\psi\,dxdy + \int_\Omega \partial_y^{k-1}\partial_x\rho\,\partial_y^{k-i}\rho\,\partial_y^{i+2}\psi\,dxdy - \int_\Omega \partial_y^k\rho\,\partial_y^{k-i}\partial_x\rho\,\partial_y^{i+1}\psi\,dxdy \\ &- \int_\Omega \partial_y\left[\partial_y^{k-1}\partial_x\rho\,\partial_y^{k-i}\rho\,\partial_y^{i+1}\psi\right]\,dxdy + \int_\Omega \partial_x\left[\partial_y^k\rho\,\partial_y^{i+1}\psi\,\partial_y^{k-i}\rho\right]\,dxdy.\end{aligned}$$

By the periodicity in the horizontal variable, it is clear that the only boundary term that needs to be study carefully is the first one, which vanishes because $\rho \in X^k(\Omega)$ and $\psi \in X^{k+1}(\Omega)$. Therefore, we get

$$\begin{aligned} J_2(i) &= \int_\Omega \partial_y^{k-1}\partial_x\rho\, \partial_y^{k-i+1}\rho\, \partial_y^{i+1}\psi\, dxdy + \int_\Omega \partial_y^{k-1}\partial_x\rho\, \partial_y^{k-i}\rho\, \partial_y^{i+2}\psi\, dxdy - \int_\Omega \partial_y^{k}\rho\, \partial_y^{k-i}\partial_x\rho\, \partial_y^{i+1}\psi\, dxdy \\ &= -\int_\Omega \partial_y^{k-1}\Delta\psi\, \partial_y^{k-i+1}\rho\, \partial_y^{i+1}\psi\, dxdy - \int_\Omega \partial_y^{k-1}\Delta\psi\, \partial_y^{k-i}\rho\, \partial_y^{i+2}\psi\, dxdy + \int_\Omega \partial_y^{k}\rho\, \partial_y^{k-i}\Delta\psi\, \partial_y^{i+1}\psi\, dxdy \end{aligned}$$

where in the last equality we have used (2.14). Repeatedly applying Hölder's inequality we obtain that

$$\begin{aligned} \sum_{i=1}^{k-1} J_2(i) \leqslant{}& \|\partial_y^{k-1}\Delta\psi\|_{L^2} \left[\sum_{i=1}^{k-2} \|\partial_y^{k-i+1}\rho\|_{L^2}\, \|\partial_y^{i+1}\psi\|_{L^\infty} + \|\partial_y^{2}\rho\|_{L^\infty}\, \|\partial_y^{k}\psi\|_{L^2}\right] \\ &+ \|\partial_y^{k-1}\Delta\psi\|_{L^2} \left[\sum_{i=1}^{k-3} \|\partial_y^{k-i}\rho\|_{L^2}\, \|\partial_y^{i+2}\psi\|_{L^\infty} + \sum_{i=k-2}^{k-1} \|\partial_y^{k-i}\rho\|_{L^\infty}\, \|\partial_y^{i+2}\psi\|_{L^2}\right] \\ &+ \|\partial_y^{k}\rho\|_{L^2} \left[\sum_{i=1}^{k-2} \|\partial_y^{k-i}\Delta\psi\|_{L^2}\, \|\partial_y^{i+1}\psi\|_{L^\infty} + \|\partial_y\Delta\psi\|_{L^\infty}\, \|\partial_y^{k}\psi\|_{L^2}\right]. \end{aligned}$$

Then, by the Sobolev embedding, we clearly have

$$\sum_{i=1}^{k-1} J_2(i) \leqslant \|\rho\|_{H^k(\Omega)}\, \|\nabla\psi\|^2_{H^k(\Omega)} \qquad \text{for} \qquad k \geqslant 6. \tag{2.46}$$

Putting together (2.43), (2.44), (2.45) and (2.46) we have proved that

$$I_2 \lesssim \|\partial u_2\|_{L^\infty(\Omega)}\, \|\rho\|^2_{H^k(\Omega)} + \|\rho\|_{H^k(\Omega)}\, \|\nabla\psi\|^2_{H^k(\Omega)} \qquad \text{for} \qquad k \geqslant 6. \tag{2.47}$$

To sum up, we have obtained the next energy estimate.

Theorem 2.5.2. *Let $\rho(t) \in X^k(\Omega)$ be a solution of (2.7) for any $t \geqslant 0$. Then, the following estimate holds for $k \geqslant 6$:*

$$\tfrac{1}{2}\partial_t\|\rho\|^2_{H^k(\Omega)}(t) \leqslant C\, \|\partial u_2\|_{L^\infty(\Omega)}(t)\, \|\rho\|^2_{H^k(\Omega)}(t) - \Big(1 - C\, \|\rho\|_{H^k(\Omega)}(t)\Big)\, \|\mathbf{u}\|^2_{H^k(\Omega)}(t). \tag{2.48}$$

Proof. First of all, we remember that $\mathbf{u} = \nabla^\perp\psi$ and

$$\begin{aligned} \tfrac{1}{2}\partial_t\|\rho\|^2_{L^2(\Omega)} &= -\|\nabla\psi\|^2_{L^2(\Omega)}, \\ \tfrac{1}{2}\partial_t\|\rho\|^2_{\dot{H}^k(\Omega)} &= -\|\nabla\psi\|^2_{\dot{H}^k(\Omega)} + I_2, \end{aligned}$$

so summing and applying (2.47), we have achieved our target. □

As we want to prove a global existence in time result for small data, this is $\|\rho\|_{H^k(\Omega)}(t) \ll 1$. Then, the second term in the energy estimate (2.48) is a "good" one, because it has the right sign. In consequence, we fix our attention in the first term. If we have a "good" time decay of the $L^\infty(\Omega)$-norm of ∂u_2, then we will be able to prove that $\|\rho\|_{H^k(\Omega)}(t)$ remains small for all time by a boostraping argument.

2.5.2 Linear & Non-Linear estimates

Our goal for the rest of the chapter is to obtain time decay estimates for $\|\partial u_2\|_{L^\infty(\Omega)}(t)$. As we will see in the next Section 2.5.3, to close the energy estimate and finish the proof is enough to get an integrable rate.

We approach the question of global well-posedness for small initial data from a perturbative point of view, that is, we see (2.8) as a non-linear perturbation of the linear problem. The linearized system of (2.8) around the trivial solution $(\rho, \mathbf{u}) = (0, 0)$ reads

$$\begin{cases} \partial_t \bar{\rho} &= u_2, \\ \partial_t \tilde{\rho} &= 0, \\ \mathbf{u} &= -\nabla \Pi - (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \end{cases}$$

together with the no-slip condition on $\partial\Omega$ and initial data $\rho(0) \in X^k(\Omega)$ such that $\rho(0) = \bar{\rho}(0) + \tilde{\rho}(0)$.

It is not difficult to prove that $\bar{\rho}$ will decay in time and $\tilde{\rho}$ will just remain bounded at linear order. Consequently, the linearized problem has a very large set of stationary (undamped) modes.

Now, we return to our non-linear problem:

$$\begin{cases} \partial_t \bar{\rho} + \overline{\mathbf{u} \cdot \nabla \bar{\rho}} + \partial_y \tilde{\rho}\, u_2 &= u_2, \\ \partial_t \tilde{\rho} + \widetilde{\mathbf{u} \cdot \nabla \bar{\rho}} &= 0, \\ \mathbf{u} &= -\nabla \Pi - (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \end{cases}$$

together with the no-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Since $\bar{\rho}$ is decaying, the term $\overline{\mathbf{u} \cdot \nabla \bar{\rho}}$ should be small and controllable. The term $\partial_y \tilde{\rho}\, u_2$, however, acts like a second linear operator since $\tilde{\rho}$ is not decaying. It is conceivable that this extra quasi-linear operator could compete with the damping coming from the linear term. This makes the problem of long-time behavior more difficult.

As in [27] we solve this by, more or less, doing a second linearization around the undamped modes and showing that the stationary modes can be controlled. Then we wish to prove decay estimates for $\bar{\rho}$ in the following system:

$$\begin{cases} \partial_t \bar{\rho} &= (1 - \partial_y \tilde{\rho})\, u_2, \\ \partial_t \tilde{\rho} &= 0, \\ \mathbf{u} &= -\nabla \Pi - (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \end{cases}$$

assuming that the initial data $\rho(0) = \bar{\rho}(0) + \tilde{\rho}(0)$ is sufficiently small. By showing this, we find the decay mechanism is "stable" with respect to the sort of perturbations which this second linear operator introduces, and we are able to keep the decay mechanism and close a decay estimate for $\bar{\rho}$ and show that $\tilde{\rho}$, while not decaying, converges as $t \to \infty$.

Note that the second equation $\partial_t \tilde{\rho}(t) = 0$ reduces to a condition at time $t = 0$, that is $\tilde{\rho}(y, t) = \tilde{\rho}(y, 0)$. As consequence $\tilde{\rho}$ will just remain bounded and our goal is to solve the following system in Ω:

$$\begin{cases} \partial_t \bar{\rho} &= (1 - \partial_y \tilde{\rho})\, u_2, \\ \mathbf{u} &= -\nabla \Pi - (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \end{cases} \tag{2.49}$$

besides the no-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Using the *stream formulation* (2.13), we can rewrite (2.49) in a more adequate way as

$$\begin{cases} \partial_t \bar{\rho} &= (1 - \partial_y \tilde{\rho})\, \partial_x \psi, \\ \bar{\rho}|_{t=0} &= \bar{\rho}(0), \end{cases} \tag{2.50}$$

where ψ is the solution of Poisson's problem (2.14) and $\rho(0) \in X^k(\Omega)$.

2.5.2.1 Quasi-Linear Decay

In this subsection we prove $L^2(\Omega)$ decay estimates for the quasi-linear equation

$$\partial_t \bar{\rho} = (1 - G(y,t))\, \partial_x \psi, \tag{2.51}$$

where ψ is the solution of (2.14) given by (2.27) and $G(y,t)$ plays the role of $\partial_y \tilde{\rho}(y,t)$, is sufficiently small.
Remark: We cannot extract a formula for the solution by taking the analog of the Fourier transform given by the eigenfunction expansion, because the $G(y,t)$ term mixes the effects of all the Fourier coefficients.

Lemma 2.5.3. *There exists $\varepsilon > 0$ small enough such that if $\|G\|_{H^2([-1,1])}(t) \leqslant \varepsilon$ for all time, then the solution of equation (2.51) satisfies that*

$$\partial_t \|\bar{\rho}\|^2_{L^2(\Omega)}(t) \lesssim -\|\nabla\psi\|^2_{L^2(\Omega)}(t),$$

where ψ is the solution of (2.14).

Proof. Upon multiplying (2.51) by $\bar{\rho}$ and integrating we see that

$$\tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{L^2(\Omega)} = \int_\Omega (1 - G(y))\, \partial_x \psi \bar{\rho}\, dxdy.$$

After integrating by parts and using the *stream function* ψ, we arrive at

$$\begin{aligned} \tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{L^2(\Omega)} &= \int_\Omega (1 - G(y))\ \psi\Delta\psi\ dxdy \\ &= -\int_\Omega (1 - G(y))\ |\nabla\psi|^2\, dxdy + \int_\Omega G'(y)\, \psi\, \partial_y \psi\, dxdy. \end{aligned}$$

Now, applying the Sobolev embedding $L^\infty([-1,1]) \hookrightarrow H^1([-1,1])$ and the Poincaré inequality, we get

$$\tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{L^2(\Omega)}(t) \leqslant -\left[1 - C\, \|G\|_{H^2([-1,1])}(t)\right]\ \|\nabla\psi\|^2_{L^2(\Omega)}(t).$$

As $\|G\|_{H^2([-1,1])}(t)$ is small enough for all time, we get that $\|\bar{\rho}\|_{L^2(\Omega)}(t)$ is bounded by its initial data. □

As in [27], due to the fact that the Laplacian has discrete spectrum on Ω we can actually deduce that $\bar{\rho}$ decays in $L^2(\Omega)$ so long as its higher derivatives are controlled.

Lemma 2.5.4. *Let $\alpha \in \mathbb{N}$ and $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. The solution of (2.14) satisfies the following lower bound:*

$$\|\nabla\psi\|^2_{L^2(\Omega)}(t) \geqslant \frac{1}{N(t)} \|\bar{\rho}\|^2_{L^2(\Omega)}(t) - \frac{1}{N(t)^{1+\alpha}} \|\bar{\rho}\|^2_{H^\alpha(\Omega)}(t) \tag{2.52}$$

Proof. The solution of (2.14) is given by

$$\psi(x,y) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \left(\frac{ip}{p^2 + \left(q\frac{\pi}{2}\right)^2} \right) \mathcal{F}_\omega[\bar{\rho}](p,q)\, \omega_{p,q}(x,y).$$

Moreover, as $\|\nabla\psi\|^2_{L^2(\Omega)} = -(\psi, \Delta\psi) = (\psi, \partial_x \bar{\rho})$, it is clear that

$$\|\nabla\psi\|^2_{L^2(\Omega)}(t) = \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{N}} \frac{p^2}{p^2 + \left(q\frac{\pi}{2}\right)^2} \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2.$$

Now, on one hand, we introduce the auxiliary function $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ to obtain that

$$\begin{aligned}\|\nabla\psi\|_{L^2(\Omega)}^2(t) &\geqslant \frac{1}{N(t)}\|\bar{\rho}\|_{L^2(\Omega)}^2(t) + \sum_{(p,q)\in\mathbb{Z}_{\neq 0}\times\mathbb{N}} \left(\frac{1}{p^2+\left(q\frac{\pi}{2}\right)^2} - \frac{1}{N(t)}\right) \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 \\ &\geqslant \frac{1}{N(t)}\left(\|\bar{\rho}\|_{L^2(\Omega)}^2(t) - \sum_{p^2+q^2(\pi/2)^2\geqslant N(t)} \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2\right). \end{aligned} \tag{2.53}$$

On the other hand, by Corollary 2.2.3 we have that

$$\begin{aligned}\sum_{p^2+q^2(\pi/2)^2\geqslant N(t)} \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 &\leqslant \frac{1}{N(t)^\alpha} \sum_{p^2+q^2(\pi/2)^2\geqslant N(t)} \left(p^2+\left(q\frac{\pi}{2}\right)^2\right)^\alpha \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 \\ &\leqslant \frac{1}{N(t)^\alpha}\|\bar{\rho}\|_{H^\alpha(\Omega)}^2(t). \end{aligned} \tag{2.54}$$

Combining the estimates (2.53) and (2.54) we arrive at (2.52). □

This gives that

$$\partial_t\|\bar{\rho}\|_{L^2(\Omega)}^2(t) \lesssim -\frac{1}{N(t)}\|\bar{\rho}\|_{L^2(\Omega)}^2(t) + \frac{1}{N(t)^{1+\alpha}}\|\bar{\rho}\|_{H^\alpha(\Omega)}^2(t)$$

and assuming that $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies that $N'(t)N(t) \geqslant 1$ we obtain

$$\|\bar{\rho}\|_{L^2(\Omega)}^2(t) \lesssim e^{-(N(t)-N(0))}\|\bar{\rho}\|_{L^2(\Omega)}^2(0) + \int_0^t \frac{e^{-(N(t)-N(s))}}{N(s)^{1+\alpha}}\|\bar{\rho}\|_{H^\alpha(\Omega)}^2(s)\,ds. \tag{2.55}$$

For simplicity, we take $N(t) := 2\sqrt{1+t}$ in (2.55), which give us

$$\|\bar{\rho}\|_{L^2(\Omega)}^2(t) \lesssim e^{-2\sqrt{1+t}}\|\bar{\rho}\|_{L^2(\Omega)}^2(0) + \left(\int_0^t \frac{e^{-2(\sqrt{1+t}-\sqrt{1+s})}}{(1+s)^{\frac{1+\alpha}{2}}}\,ds\right)\|\bar{\rho}\|_{L^\infty([0,t],H^\alpha(\Omega))}^2.$$

Now, we use the following calculus lemma:

Lemma 2.5.5. *Let $\alpha \in \mathbb{N}$, we have that*

$$\int_0^t \frac{e^{-2(\sqrt{1+t}-\sqrt{1+s})}}{(1+s)^{\frac{1+\alpha}{2}}}\,ds \lesssim \frac{1}{(1+t)^{\frac{\alpha}{2}}}.$$

Proof. The proof of this lemma is simple and basically follows after we split the integral into two pieces: one from 0 to $t/2$ and the other from $t/2$ to t. The integral from 0 to $t/2$ decays exponentially. The second integral decays like $(1+t)^{-\frac{1+\alpha}{2}}$ multiplied by the following factor:

$$\int_{t/2}^t e^{-2(\sqrt{1+t}-\sqrt{1+s})}\,ds = \int_0^{2\left(\sqrt{1+t}-\sqrt{1+t/2}\right)} e^{-\tau}\left(\sqrt{1+t}-\frac{\tau}{2}\right)d\tau \lesssim \sqrt{1+t}.$$

This completes the proof. □

Then, if $\|G\|_{H^2([-1,1])}(t)$ is small enough for all time, applying the previous lemma we see that

$$\|\bar{\rho}\|^2_{L^2(\Omega)}(t) \lesssim \frac{\|\bar{\rho}\|^2_{L^\infty([0,t],H^\alpha(\Omega))}}{(1+t)^{\frac{\alpha}{2}}}. \tag{2.56}$$

Now, we prove a similar decay for higher derivatives. The idea is then to show that $\|\bar{\rho}\|^2_{H^\alpha(\Omega)}(t)$ is bounded by its initial data; this would then give (2.56) with $L^2(\Omega)$ replaced by $H^k(\Omega)$ and $H^\alpha(\Omega)$ replaced by $H^{k+\alpha}(\Omega)$.

Lemma 2.5.6. *Let $k \in \mathbb{N} \cup \{0\}$ and fix an auxiliary parameter $\alpha \in \mathbb{N}$. There exists $\varepsilon > 0$ small enough such that if $\|G\|_{H^{k+\alpha+2}([-1,1])}(t) \leqslant \varepsilon$ for all time and $\bar{\rho}(0) \in H^{k+\alpha}(\Omega)$, then the solution of equation (2.51) satisfies:*

$$\|\bar{\rho}\|^2_{H^k(\Omega)}(t) \lesssim \frac{\|\bar{\rho}\|^2_{H^{k+\alpha}(\Omega)}(0)}{(1+t)^{\frac{\alpha}{2}}}.$$

Proof. Fix $n \in \mathbb{N} \cup \{0\}$ such that $n \leqslant k+\alpha$. First, we will prove that $\|\bar{\rho}\|^2_{H^n(\Omega)}(t) \leqslant \|\bar{\rho}\|^2_{H^n(\Omega)}(0)$. Proceeding as before, after integrating by parts and using the *stream function* ψ, we arrive at

$$\tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{H^n(\Omega)} = \int_\Omega \partial^n \left[(1-G(y))\psi\right] \partial^n \Delta\psi \, dxdy.$$

By Leibniz's rule we have that

$$\tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{H^n(\Omega)} = \int_\Omega (1-G(y))\partial^n\psi \partial^n \Delta\psi \, dxdy + \sum_{i=1}^{n} \binom{n}{i} \int_\Omega \partial^i(1-G(y))\partial^{n-i}\psi \, \partial^n \Delta\psi \, dxdy.$$

As before, applying the Sobolev embedding $L^\infty([-1,1]) \hookrightarrow H^1([-1,1])$ and Poincaré's inequality, we get

$$\tfrac{1}{2}\partial_t \|\bar{\rho}\|^2_{H^n(\Omega)}(t) \leqslant -\left[1 - C\,\|G\|_{H^{n+2}([-1,1])}(t)\right] \|\nabla\psi\|^2_{H^n(\Omega)}(t).$$

Then, as $\|G\|_{H^{n+2}([-1,1])}(t)$ is small enough for all time, we get that $\|\bar{\rho}\|_{H^n(\Omega)}(t)$ is bounded by its initial data. Applying this in (2.56), we have proved our goal for the case $k=0$. Arguing as we did above when we proved the $L^2(\Omega) \equiv H^0(\Omega)$ decay, we can extend the result for general $k \in \mathbb{N}$. □

2.5.2.2 Non-Linear Decay

Next, we will show how this decay of the quasi-linear solutions can be used to establish the stability of the stationary solution $(\rho, \mathbf{u}) = (0,0)$ for the general problem (2.8). When perturbing around the stationary solution, we get the following system:

$$\begin{cases} \partial_t \bar{\rho} - (1-\partial_y \tilde{\rho})u_2 &= -\overline{\mathbf{u}\cdot\nabla\bar{\rho}}, \\ \partial_t \tilde{\rho} &= -\widetilde{\mathbf{u}\cdot\nabla\bar{\rho}}, \end{cases} \tag{2.57}$$

where $\mathbf{u} = \nabla^\perp\psi$ and ψ is the solution of (2.14).

Using Duhamel's formula, with $G(y,t) \equiv \partial_y \tilde{\rho}(y,t)$ small enough in the adequate norm, we write the solution of (2.57) as

$$\bar{\rho}(t) = e^{\mathscr{L}(t,0)}\bar{\rho}(0) - \int_0^t e^{\mathscr{L}(t,s)} \left[\overline{\mathbf{u}\cdot\nabla\bar{\rho}}\right](s)\, ds \qquad \text{and} \qquad \tilde{\rho}(t) = \tilde{\rho}(0) - \int_0^t \widetilde{\mathbf{u}\cdot\nabla\bar{\rho}}(s)\, ds$$

where $e^{\mathscr{L}(t,s)}$ denotes the solution operator of the associated quasi-linear problem (2.50) from s to t. Therefore, we have

$$\|\bar{\rho}\|_{H^n(\Omega)}(t) \lesssim \frac{\|\bar{\rho}\|_{H^{n+\alpha}(\Omega)}(0)}{(1+t)^{\frac{\alpha}{4}}} + \int_0^t \frac{1}{(1+(t-s))^{\frac{\alpha}{4}}} \|\overline{\mathbf{u}\cdot\nabla\bar{\rho}}\|_{H^{n+\alpha}(\Omega)}(s)\, ds.$$

2.5.3 The Bootstraping

We now demonstrate the bootstrap argument used to prove our goal. The general approach here is a typical continuity argument that has been used successfully in a plethora of other cases. Theorem 2.5.2 tells us that the following estimate holds for $k \geqslant 6$:

$$\frac{1}{2}\partial_t \|\rho\|^2_{H^k(\Omega)}(t) \leqslant C\,\|\partial u_2\|_{L^\infty(\Omega)}(t)\,\|\rho\|^2_{H^k(\Omega)}(t) - \Big(1 - C\,\|\rho\|_{H^k(\Omega)}(t)\Big)\,\|\mathbf{u}\|^2_{H^k(\Omega)}(t). \tag{2.58}$$

We need to prove:

Lemma 2.5.7. *If $\|\rho\|_{H^\kappa(\Omega)}(0) < \varepsilon$ and $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 4\,\varepsilon$ on the interval $[0,T]$ with $0 < \varepsilon \leqslant \varepsilon_0$ small enough. Then $\|\rho\|_{H^\kappa(\Omega)}(t)$ remains uniformly bounded by $2\,\varepsilon$ on the same interval $[0,T]$.*

We will prove Lemma 2.5.7 through a bootstrap argument, where the main ingredient is the estimate (2.58). We will work with a bootstrap hypothesis to assume that $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 4\varepsilon$ on the interval $[0,T]$, where κ is big enough and $0 < \varepsilon << 1$ such that

$$\big(1 - C\,\|\rho\|_{H^\kappa(\Omega)}(t)\big) \geqslant 0 \quad \text{on} \quad [0,T].$$

Then, by Grönwall's inequality we have

$$\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant \|\rho\|_{H^\kappa(\Omega)}(0)\,\exp\left(C\int_0^t \|\partial u_2\|_{L^\infty(\Omega)}(s)\,ds\right) \qquad t \in [0,T].$$

Our goal is to prove that $\|\partial u_2\|_{L^\infty(\Omega)}(t)$ decays on time at an integrable rate. As $L^\infty(\Omega) \hookrightarrow H^2(\Omega)$ by the Sobolev embedding, it is enough to prove this for $\|u_2\|_{H^3(\Omega)}(t)$. This will allow us to close the energy estimate and finish the proof.

2.5.3.1 Integral decay of $\|u_2\|_{H^3(\Omega)}$

In order to control $\|u_2\|_{H^3(\Omega)}$ in time it is enough to control $\|\bar{\rho}\|_{H^3(\Omega)}$. We have the following result:

Lemma 2.5.8. *Assume that $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 4\,\varepsilon$ for all $t \in [0,T]$ where $\kappa \geqslant 5 + 2\gamma$ with $\gamma > 4$. Then*

$$\|\bar{\rho}\|_{H^3(\Omega)}(t) \lesssim \frac{\varepsilon}{(1+t)^{\frac{\gamma}{4}}} \qquad \textit{for all} \quad t \in [0,T].$$

Proof. By assumption, $\partial_y \tilde{\rho}(t)$ is small in $H^{\kappa-1}(\Omega)$ for all $t \in [0,T]$. This implies that $e^{\mathscr{L}(t,s)}$ has nice decay properties for $s \leqslant t$ and $t \in [0,T]$ in $H^3(\Omega)$ if $\kappa \geqslant 6+\gamma$. Hence, Duhamel's formula gives us

$$\|\bar{\rho}\|_{H^3(\Omega)}(t) \lesssim \frac{\|\bar{\rho}\|_{H^{3+\gamma}(\Omega)}(0)}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{1}{(1+(t-s))^{\frac{\gamma}{4}}}\,\|\overline{\mathbf{u}\cdot\nabla\bar{\rho}}\|_{H^{3+\gamma}(\Omega)}(s)\,ds$$

and we have that

$$\|\overline{\mathbf{u}\cdot\nabla\bar{\rho}}\|_{H^{3+\gamma}} \leqslant \|\mathbf{u}\cdot\nabla\bar{\rho}\|_{H^{3+\gamma}} \lesssim \|\mathbf{u}\|_{H^{3+\gamma}(\Omega)}\,\|\bar{\rho}\|_{H^{4+\gamma}(\Omega)} \lesssim \|\bar{\rho}\|^2_{H^{4+\gamma}(\Omega)}.$$

Hence,

$$\|\bar{\rho}\|_{H^3(\Omega)}(t) \lesssim \frac{\|\bar{\rho}\|_{H^{3+\gamma}(\Omega)}(0)}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{1}{(1+(t-s))^{\frac{\gamma}{4}}}\,\|\bar{\rho}\|^2_{H^{4+\gamma}(\Omega)}(s)\,ds$$

and, in conclusion, we need a control in time of $\|\bar{\rho}\|_{H^{4+\gamma}(\Omega)}$.

However, due to the well-known Gagliardo–Nirenberg interpolation inequalities

$$\|D^j f\|_{L^2(\Omega)} \leqslant C\,\|D^{2j} f\|^{1/2}_{L^2(\Omega)}\,\|f\|^{1/2}_{L^2(\Omega)} + \tilde{C}\,\|f\|_{L^2(\Omega)}$$

we get

$$\|\bar\rho\|_{H^{4+\gamma}(\Omega)} \lesssim \|\bar\rho\|^{1/2}_{H^{3+2(1+\gamma)}(\Omega)}\,\|\bar\rho\|^{1/2}_{H^3(\Omega)}. \tag{2.59}$$

Therefore, if we apply (2.59) in the previous inequalities, we get

$$\|\bar\rho\|_{H^3(\Omega)}(t) \lesssim \frac{\|\bar\rho\|_{H^{3+\gamma}(\Omega)}(0)}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{\|\bar\rho\|_{H^\kappa(\Omega)}(s)}{(1+(t-s))^{\frac{\gamma}{4}}}\,\|\bar\rho\|_{H^3(\Omega)}(s)\,ds$$

where we have defined $\kappa \in \mathbb{N}$ so that $\kappa \geqslant \max\{5+2\gamma, 6+\gamma\}$.

By hypothesis, we have that $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 4\varepsilon$ on the interval $[0,T]$. Then, we obtain that

$$\|\bar\rho\|_{H^3(\Omega)}(t) \leqslant \frac{C\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{C\varepsilon}{(1+(t-s))^{\frac{\gamma}{4}}}\,\|\bar\rho\|_{H^3(\Omega)}(s)\,ds.$$

In particular, there exists $0 < T^\star(C) \leqslant T$ such that for $t \in [0, T^\star(C)]$ we have

$$\|\bar\rho\|_{H^3}(t) \leqslant 4\,\frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}}.$$

The following basic lemma is stated without proof (for a proof see [27, p. 584]):

Lemma 2.5.9. *Let $\delta, \tau > 0$, then*

$$\int_0^t \frac{ds}{(1+(t-s))^{\delta}\,(1+s)^{1+\tau}} \leqslant \frac{\mathcal{C}_{\delta,\tau}}{(1+t)^{\min\{\delta,1+\tau\}}}.$$

If we restrict to $0 \leqslant t \leqslant T^\star(C)$ and we apply the previous Lemma 2.5.9, we have

$$\begin{aligned}\|\bar\rho\|_{H^3(\Omega)}(t) &\leqslant \frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{C\,\varepsilon}{(1+(t-s))^{\frac{\gamma}{4}}}\,\frac{4\,C\,\varepsilon}{(1+s)^{\frac{\gamma}{4}}}\,ds \\ &\leqslant \frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \frac{4\,\tilde{C}\,\varepsilon^2}{(1+t)^{\frac{\gamma}{4}}}.\end{aligned}$$

The last term in the expression above is quadratic in ε, it is enough to find $0 < \epsilon \ll 1$ small enough so that

$$\|\bar\rho\|_{H^3(\Omega)}(t) \leqslant 2\,\frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}}$$

for all $t \in [0, T^\star(C)]$ and, by continuity, for all $t \in [0,T]$. □

Thus, with $\gamma > 4$ we have proved the integrable decay of u_2, and we are able to close our energy estimate. We are now in the position to show how the bootstrap can be closed. This is merely a matter of collecting the conditions established above and showing that they can indeed be satisfied.

In conclusion, if $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 4\,\varepsilon$ for all $t \in [0,T]$ we have that

$$\begin{aligned}\|\rho\|_{H^\kappa(\Omega)}(t) &\leqslant \|\rho\|_{H^\kappa(\Omega)}(0)\,\exp\left(C\int_0^t \|\partial u_2\|_{L^\infty(\Omega)}(s)\,ds\right) \\ &\leqslant \varepsilon\,\exp\left(C\int_0^t \frac{\tilde{C}\varepsilon}{(1+s)^{\frac{\gamma}{4}}}\,ds\right) \leqslant \varepsilon\exp\left(C^\star\varepsilon\right)\end{aligned}$$

and $\|\rho\|_{H^\kappa(\Omega)}(t) \leqslant 2\,\varepsilon$ for all $t \in [0,T]$ if we consider ε small enough, which allows us to prolong the solution and then repeat the argument for all time.

CHAPTER 3

THE DAMPING BOUSSINESQ SYSTEM

ABSTRACT: In this chapter, we consider the 2D inviscid Boussinesq equations with a velocity damping term in a strip $\mathbb{T} \times [-1, 1]$, with impermeable walls. In this physical scenario, where the *Boussinesq approximation* is accurate when density/temperature variations are small, our main result is the asymptotic stability for a specific type of perturbations of a stratified solution. To prove this result, we use a suitably weighted energy space combined with linear decay, Duhamel's formula and "bootstrap" arguments.

3.1 Introduction

The fundamental issue of regularity vs finite time blow up question for the 3D Euler equation remains outstandingly open and the study of the 2D Boussinesq equations may shed light on this extremely challenging problem. As pointed out in [46], the 2D Boussinesq equations are identical to the 3D Euler equations under the hypothesis of axial symmetry with swirl. The behavior of solutions to the 2D Boussinesq system and the axi-symmetric 3D Euler equations away from the symmetry axis $r = 0$ should be "identical".

The Boussinesq equations for inviscid, incompressible 2D fluid flow are given by

$$[\text{2D Boussinesq}] \qquad \begin{cases} \partial_t \varrho + \mathbf{u} \cdot \nabla \varrho &= 0, \qquad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+ \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla p &= g(0, \varrho), \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \\ \varrho\, |_{t=0} &= \varrho\, (0), \end{cases} \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2)$ is the incompressible velocity field, p is the pressure, g is the acceleration due to gravity and ϱ corresponds to the temperature transported without diffusion by the fluid.

3.1.1 Motivation and state-of-the-art

Boussinesq systems are widely used to model the dynamics of the ocean or the atmosphere, see e.g. [45] or [50]. They arise from the density dependent fluid equations by using the so-called *Boussinesq approximation* which consists in neglecting the density dependence in all the terms but the one involving the gravity. We refer to [56] for a rigorous justification.

Global regularity of solutions is known when classical dissipation is present in at least one of the equations (see [11], [36]), or under a variety of more general conditions on dissipation (see e.g. [7] for more information).

In contrast, the global regularity problem on the inviscid 2D Boussinesq equations appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria. Several analytic and numerical results on the inviscid 2D Boussinesq equations are available in [12], [26], [61], [28] [34].

In the class of temperature-patch type solutions with no diffusion and viscosity in the whole space, there is a vast literature, see for example [19], [30] and references therein.

This work is partially aimed to understand the global existence problem by examining how damping affects the regularity of the solutions to the inviscid 2D Boussinesq equations. In the present chapter we investigate the following system:

$$
\text{[2D damping Boussinesq]} \qquad \begin{cases} \partial_t \varrho + \mathbf{u}\cdot\nabla \varrho &= 0, \qquad (\mathbf{x},t)\in\Omega\times\mathbb{R}^+ \\ \partial_t\mathbf{u}+\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\nabla p &= g(0,\varrho), \\ \nabla\cdot\mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \\ \varrho\,|_{t=0} &= \varrho\,(0). \end{cases} \tag{3.2}
$$

Since these equations are studied on a bounded domain, we take $\mathbf{u}$ to satisfy the no-penetration condition $\mathbf{u}\cdot\mathbf{n}=0$ on the boundary of the domain $\partial\Omega$ where $\mathbf{n}$ denotes the normal exterior vector.

From the mathematical point of view, the interest to study the 2D Boussinesq system with a velocity damping term follows from the fact that (3.2) can be seen as the limiting case of fractional dissipation on the velocity equation without buoyancy diffusion.

From a physical point of view, the previous system appears in the field of electrowetting (EW), which is the modification of the wetting properties of a surface (which is typically hydrophobic) with an applied electric field. It was developed from electrocapillarity by Lippmann in 1875 [44] in his PhD thesis, but did not attract much attention until the 1990's, when the applications increased.

Through rigorous theory and experiments, Lippmann proves a relationship between electrical and surface tension phenomena. This relationship allows for controlling the shape and motion of a liquid meniscus through the use of an applied voltage. The liquid surface changes shape when a voltage is applied in order to minimize the total energy of the system.

More specifically, the system (3.2) (without nonlinear term) models droplet motion in a device driven by Electrowetting-On-Dielectric (EWOD), which consists of two closely spaced parallel plates with a droplet bridging the plates and a grid of square electrodes embedded in the bottom plate [60]. Applying voltages to the grid allows the droplet to move, split, and rejoin within the narrow space of the plates. They model the fluid dynamics by using Hele-Shaw type equations, but an extra term beyond the usual Hele-Shaw flow appears: a time derivative term is included because it may have a large magnitude due to rapidly varying pressure boundary conditions if high-frequency voltage is used to modulate the droplet's contact angles.

3.1.2 Hydrodynamic stability

In our physical system, where there is gravity and stratification ($\mathbf{u}=0$ and $\varrho\equiv\varrho\,(y)$ is an stationary solution), vertical movement may be penalized while horizontal movement is not. This opens up the perspective of treating the corresponding initial value problems from a perturbative point of view. The basic problem is to consider $\Theta(y)$ a given equilibrium for (3.2), and to study the dynamics of solutions which are close to it in a suitable sense. Hence, if we write the solution as

$$
\varrho\,(x,y,t)=\Theta(y)+\rho(x,y,t) \tag{3.3}
$$

and the pressure term is written as

$$
p(x,y,t)=P(x,y,t)+g\int_0^y \Theta(s)\,ds.
$$

Then, the exact evolution equations for the perturbation become

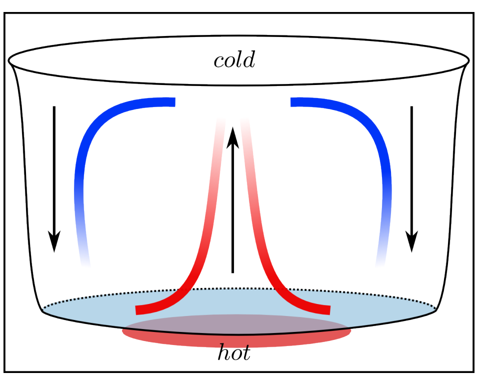
$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= -\partial_y \Theta \, u_2, \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \\ \partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u} \cdot \nabla) \, \mathbf{u} + \nabla P &= g(0, \rho), \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \\ \rho|_{t=0} &= \rho(0), \end{cases} \tag{3.4}$$

besides the no-slip condition on $\partial\Omega$. For hydrodynamic stability questions, $\rho(0)$ is assumed initially small in certain norm. This work is focused on *laminar equilibria*, simple equilibria in which the fluid is moving in well-ordered layers. However, even for these simple configurations, surprisingly little is understood about the near-equilibrium dynamics.

The field of hydrodynamic stability has a long history starting in the 19th century. One of the oldest problems considered is the stability and instability of shear flows, dating back to, for example, Rayleigh [53] and Kelvin [39], as well as by many modern authors with new perspectives (see [25] and references therein). In recent years, this type of problems has attracted renewed interest. For example, the stability of the planar Couette flow in 2D Euler [3] or in the 2D and 3D Navier-Stokes equations at high Reynolds number [2]. Very recently, this was done for the ideal MHD system (where there is viscosity in the momentum equation but there is no resistivity in the magnetic equation), for the two-dimensional case in [41] (see also some further results in [6], [54]). The three-dimensional case was then solved in [63], see also [42]. In the context of the 2D Boussinesq system when dissipation is present in at least one of the equations see [24] and [40], where the authors study the global well-posedness and stability/instability of perturbations near a special type of hydrostatic equilibrium. Finally, for other type of problems as the β-plane equation or the IPM equation see [28], [52] and [27], [10] respectively.

3.1.3 The Rayleigh-Bénard Stability

The phenomenon known as Rayleigh–Bénard convection has been studied by a number of authors for many years. The idea is simple: take a container filled with water which is at rest.



Now heat the bottom of the container and cool the top of the container. It has been observed experimentally in [32] and mathematically in [26] that if the temperature difference between the top and the bottom goes beyond a certain critical value, the water will begin to move and convective rolls will begin to form. This effect is called *Rayleigh–Bénard instability*.

In one sentence, we are going to study the opposite of the Rayleigh–Bénard instability. Now, in the inverse case, when one cools the bottom and heats the top, it is expected that the system remains stable. Here the temperature and density are assumed to be proportionally related, so that the cooler fluid is more dense. The gravitational force is thus expected to stabilize such a density (or temperature) distribution. In the presence of viscosity it is not difficult to prove this fact, see [23]. However, without the effects of viscosity (or temperature dissipation), it is conceivable for such a configuration to be unstable.

Under the *Boussinesq approximation*, a physical relevant scenario to study (3.2) is where the fluid is confined between infinite planar walls and density/temperature variations are smalls. For this reason, in the present article, we focus on the stability in Sobolev spaces of the steady state $\Theta(y) := y$ for the 2D damping

Boussinesq system setting on the two-dimensional flat strip $\Omega = \mathbb{T} \times [-1, 1]$ with no-slip condition on $\partial\Omega$. In our scenario, this only means that $u_2|_{\partial\Omega} = 0$. It is equivalent to assume impermeable boundary conditions for the velocity in top and bottom, together with periodicity conditions in left and right sides.

The main result of the chapter is the asymptotic stability of this particular stratified state $\Theta(y)$ for a specific type of perturbations. A more precise statement of our result is presented as Theorem 3.7.1, where we also illustrate its proof through a bootstrap argument. Despite the apparent simplicity, understanding the stability of this flow is far from being trivial. As in [27] and in [61], in this chapter a key idea is that stratification can be a stabilizing force. It is clear that a fluid with temperature that is inversely proportional to depth is, in some sense, stable. In fact, we will be able to prove that smooth perturbations of stratified stable solutions are stable for all time in Sobolev spaces.

In order to solve our problem in the bounded domain Ω, in certain Sobolev space, we have to overcome the following new difficulties:

i) To be able to handle the boundary terms that appear in the computations.

ii) The lack of higher order boundary conditions at the boundaries, due to the fact that we work in Sobolev spaces.

Indeed, both difficulties i) and ii) can be bypassed if our initial perturbation and velocity have a special structure. We introduce the following spaces, which we used in chapter 2 to characterize our initial data:

$$X^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 0, 2, 4, \dots, k^\star\}, \tag{3.5}$$

$$Y^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 1, 3, 5, \dots, k_\star\} \tag{3.6}$$

where we define the auxiliary values of $k^\star$ and $k_\star$ as follows:

$$k^\star := \begin{cases} k-2 & k \quad \text{even,} \\ k-1 & k \quad \text{odd,} \end{cases} \qquad \text{and} \qquad k_\star := \begin{cases} k-1 & k \quad \text{even,} \\ k-2 & k \quad \text{odd.} \end{cases}$$

Moreover, for our initial velocity field, we will use the notation $H^k(\Omega)$ for $H^k(\Omega; \Omega)$ and we also define the following functional space:

$$\mathbb{X}^k(\Omega) := \left\{ \mathbf{v} \in H^k(\Omega) : \mathbf{v} = (v_1, v_2) \in Y^k(\Omega) \times X^k(\Omega) \right\} \tag{3.7}$$

Lastly, we remember that the Trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ defined by $T[f] := f|_{\partial\Omega}$ is bounded for all $f \in H^1(\Omega)$. Consequently, all these spaces are well defined.

3.1.4 Notation & Organization

We shall denote by (f, g) the L^2 inner product of f and g. As usual, we use bold for vectors valued functions. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, we define $\langle \mathbf{u}, \mathbf{v} \rangle = (u_1, v_1) + (u_2, v_2)$.
Also, we remember that the natural norm in Sobolev spaces is defined by:

$$||f||_{H^k(\Omega)} := ||f||^2_{L^2(\Omega)} + ||\partial^k f||^2_{L^2(\Omega)}, \qquad ||f||_{\dot{H}^k(\Omega)} := ||\partial^k f||^2_{L^2(\Omega)}.$$

For convenience, in some places of this chapter, we may use L^2, $\dot{H}^k$ and H^k to stand for $L^2(\Omega)$, $\dot{H}^k(\Omega)$ and $H^k(\Omega)$, respectively. Moreover, to avoid clutter in computations, function arguments (time and space) will be omitted whenever they are obvious from context. Whenever a parameter is carried through inequalities explicitly, we assume that constants in the corresponding $\lesssim$ are independent of it. Finally, for a general function $f : \Omega \to \mathbb{R}$, we define:

$$\tilde{f}(y) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x', y)\, dx' \qquad \text{and} \qquad \bar{f}(x, y) := f(x, y) - \tilde{f}(y). \tag{3.8}$$

Organization of the chapter: In Section 3.2 we begin by setting up the perturbated problem. We go on to motivate the functional spaces $X^k(\Omega)$ and $Y^k(\Omega)$ where we will work. The key point of working with initial perturbations with the structure given by these spaces is showed in Section 3.3. Section 3.4 contains the proof of the local existence in time for initial data in these spaces, together with a blow-up criterion. The core of the chapter is the proof of the energy estimates in Section 3.5. In Section 3.6 we embark on the proof of a Duhamel's type formula for our system together with the study of the decay given by the linearized problem. Finally, in Section 3.7 we exploit a bootstrapping argument to prove our theorem.

3.2 The Equations

For our particular choice of $\Theta(y) = y$ and $g = 1$, the system (3.4) reduces to

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= -u_2, \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \\ \partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla P &= (0, \rho), \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \\ \rho|_{t=0} &= \rho(0), \end{cases}$$

besides the no-slip condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Note that our perturbation ρ does not have to decay in time. Indeed, if we perturb the stationary solution by a function of y only there is no decay. More specifically, $\rho \equiv \rho(y)$ and $\mathbf{u} = 0$ are stationary solutions of this system.

As our goal is the asymptotic stability and decay to equilibrium of sufficiently small perturbations, this could be a problem. To overcome this difficulty, the orthogonal decomposition of $\rho = \bar{\rho} + \tilde{\rho}$ given by (3.8) will be considered.

In order to prove our goal, we plug into the system (3.2) the following ansatz:

$$\begin{aligned} \varrho\,(x, y, t) &= y + \rho(x, y, t), \\ p(x, y, t) &= \Pi(x, y, t) + \frac{1}{2} y^2 + \int_0^y \tilde{\rho}(y', t) dy'. \end{aligned}$$

Then, for the perturbation ρ, we obtain the system

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= -u_2, \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \\ \partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + \nabla \Pi &= (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \\ \rho|_{t=0} &= \rho(0), \end{cases} \tag{3.9}$$

besides the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. The evolution equation for the perturbation ρ of the previous system (3.9) can be rewritten in terms of $\bar{\rho}$ and $\tilde{\rho}$ as follows:

$$\begin{cases} \partial_t \bar{\rho} + \overline{\mathbf{u} \cdot \nabla \bar{\rho}} = -(1 + \partial_y \tilde{\rho})\, u_2, \\ \partial_t \tilde{\rho} + \widetilde{\mathbf{u} \cdot \nabla \bar{\rho}} = 0. \end{cases} \tag{3.10}$$

Notice that $\tilde{\rho}$ is always a function of y only and $\bar{\rho}$ has zero average in the horizontal variable. It is expected that $\bar{\rho}$ will decay in time and $\tilde{\rho}$ will just remain bounded. The systems (3.9) and (3.10) are the same, but depending on what we need, we will work with one or the other.

3.3 Mathematical setting and preliminares

In this section, we will see the importance of selecting carefully our initial perturbation $\rho(0) \in X^k(\Omega)$ and our initial velocity $\mathbf{u}(0) \in \mathbb{X}^k(\Omega)$. Moreover, two adapted orthonormal basis for them are considered, together with their eigenfunction expansion.

3.3.1 Motivation of the spaces $X^k(\Omega)$, $Y^k(\Omega)$ and $\mathbb{X}^k(\Omega)$

By the no-slip condition $u_2(t)|_{\partial\Omega} = 0$, the solution $\rho(t)$ of (3.9) satisfies on the boundary of our domain the following transport equation

$$\partial_t \rho(t)|_{\partial\Omega} + u_1(t)\partial_x \rho(t)|_{\partial\Omega} = 0 \tag{3.11}$$

As our objective is the global stability and decay to equilibrium of sufficiently small perturbations, it seems natural to consider $\rho(0)|_{\partial\Omega} = 0$. Then, by the transport character of (3.11) the initial condition is preserved in time $\rho(t)|_{\partial\Omega} = 0$ as long as the solution exists. In addition, applying the *curl* on the evolution equation of the velocity field, using the incompressibility condition, and restricting to the boundary, we have that

$$\partial_t(\partial_y u_1)(t)|_{\partial\Omega} = -(\partial_y u_1)(t)|_{\partial\Omega} - u_1(t)\partial_x(\partial_y u_1)(t)|_{\partial\Omega}$$

because $\rho(t)|_{\partial\Omega} = u_2(t)|_{\partial\Omega} = 0$. Hence, we find that $\partial_y u_1(0)|_{\partial\Omega} = 0$ implies that $\partial_t(\partial_y u_1)(t)|_{\partial\Omega} = 0$, and consequently the condition on the boundary is preserved in time. Hence, by the incompressibility of the velocity, we get

$$\partial_y u_1(t)|_{\partial\Omega} = 0 \quad \text{and} \quad \partial_y^2 u_2(t)|_{\partial\Omega} = 0. \tag{3.12}$$

Previous relations (3.12) give the following equation for the restriction to the boundary of the derivative in time of $\partial_y^2\rho(t)$:

$$\partial_t\partial_y^2\rho(t)|_{\partial\Omega} = -u_1(t)\partial_x(\partial_y^2\rho)(t)|_{\partial\Omega} - \partial_y u_2(t)\partial_y^2\rho(t)|_{\partial\Omega}.$$

Therefore we find that $\partial_y^2\rho(0)|_{\partial\Omega} = 0$ implies that $\partial_t\partial_y^2\rho(t)|_{\partial\Omega} = 0$, and consequently the condition on the boundary is preserved in time.

Iterating this procedure we check that the conditions $\partial_y^n\rho(0)|_{\partial\Omega} = \partial_y^n u_2(0)|_{\partial\Omega} = 0$ for $n = 2, 4, \ldots$ and $\partial_y^n u_1(0)|_{\partial\Omega}$ for $n = 1, 3, \ldots$ are preserved in time. This is the reason why we can look for perturbations $\rho(t)$ in the space $X^k(\Omega)$ and velocity fields $\mathbf{u}(t)$ in $\mathbb{X}^k(\Omega)$, if the initial data belongs to them.

3.3.2 An orthonormal basis for $X^k(\Omega)$ and $Y^k(\Omega)$

Let us start by defining the following:

$$a_p(x) := \frac{1}{\sqrt{2\pi}} \exp(ipx) \quad \text{with } x \in \mathbb{T} \quad \text{for } p \in \mathbb{Z}$$

and

$$b_q(y) := \begin{cases} \cos\left(qy\frac{\pi}{2}\right) & q \quad \text{odd} \\ \sin\left(qy\frac{\pi}{2}\right) & q \quad \text{even} \end{cases} \qquad \text{with } y \in [-1, 1] \quad \text{for } q \in \mathbb{N},$$

where $\{a_p\}_{p\in\mathbb{Z}}$ and $\{b_q\}_{q\in\mathbb{N}}$ are orthonormal basis for $L^2(\mathbb{T})$ and $L^2([-1,1])$ respectively. Indeed, $\{b_q\}_{q\in\mathbb{N}}$ consists of eigenfunctions of the operator $S=(1-\partial_y^2)$ with domain $\mathscr{D}(S)=\{f\in H^2[-1,1]\,:\,f(\pm l)=0\}$. Consequently, the product of them $\omega_{p,q}(x,y):=a_p(x)\,b_q(y)$ is an orthonormal basis for $L^2(\Omega)$.

Moreover, we define an auxiliary orthonormal basis for $L^2([-1,1])$ given by

$$c_q(y):=\begin{cases}\sin\left(qy\frac{\pi}{2}\right) & q \quad \text{odd}\\ \cos\left(qy\frac{\pi}{2}\right) & q \quad \text{even}\end{cases} \qquad \text{with } y\in[-1,1] \quad \text{for } q\in\mathbb{N}\cup\{0\},$$

cosisting of eigenfunctions of the operator S with domain $\mathscr{D}(S)=\{f\in H^2[-1,1]\,:\,(\partial_y f)(\pm l)=0\}$. In the same way as before, the product $\varpi_{p,q}(x,y):=a_p(x)\,c_q(y)$ is again an orthonormal basis for $L^2(\Omega)$.

Remark: Let us describe the analogue of Fourier expansion with our eigenfunctions expansion. This is, for $f\in L^2(\Omega)$, we have the $L^2(\Omega)$-conergence given by:

$$f(x,y)=\sum_{(p,q)\in\mathbb{Z}\times\mathbb{N}}\mathcal{F}_\omega[f](p,q)\,\omega_{p,q}(x,y) \qquad \text{whit} \quad \mathcal{F}_\omega[f](p,q):=\int_\Omega f(x',y')\,\overline{\omega_{p,q}(x',y')}\,dx'dy' \tag{3.13}$$

or

$$f(x,y)=\sum_{(p,q)\in\mathbb{Z}\times\mathbb{N}\cup\{0\}}\mathcal{F}_\varpi[f](p,q)\,\varpi_{p,q}(x,y) \qquad \text{whit} \quad \mathcal{F}_\varpi[f](p,q):=\int_\Omega f(x',y')\,\overline{\varpi_{p,q}(x',y')}\,dx'dy'. \tag{3.14}$$

In the next lemma, we collect the main properties of our basis.

Lemma 3.3.1. *The following holds:*

- $\{\omega_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}}$ *is an orthonormal basis of* $X^k(\Omega)$.
- $\{\varpi_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}\cup\{0\}}$ *is an orthonormal basis of* $Y^k(\Omega)$.

Moreover, let $f\in X^k(\Omega)$ *and* $g\in Y^k(\Omega)$. *For* $s_1,s_2\in\mathbb{N}\cup\{0\}$ *such that* $s_1+s_2\leqslant k$, *we have that:*

$$\begin{aligned}||\partial_x^{s_1}\partial_y^{s_2}f||^2_{L^2(\Omega)}&=\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}}|p|^{2s_1}|q\tfrac{\pi}{2}|^{2s_2}\,|\mathcal{F}_\omega[f](p,q)|^2\,,\\ ||\partial_x^{s_1}\partial_y^{s_2}g||^2_{L^2(\Omega)}&=\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}\cup\{0\}}|p|^{2s_1}|q\tfrac{\pi}{2}|^{2s_2}\,|\mathcal{F}_\varpi[f](p,q)|^2\,,\end{aligned}$$

where $\mathcal{F}_\omega[f](p,q)$ *and* $\mathcal{F}_\varpi[f](p,q)$ *are given by (3.13) and (3.14) respectively.*

Introducing a threshold number $m\in\mathbb{N}$, we define the projections $\mathbb{P}_m$ and $\mathbb{Q}_m$ of $L^2(\Omega)$ onto the linear span of eigenfunctions generated by $\{\omega_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}}$ and $\{\varpi_{p,q}\}_{(p,q)\in\mathbb{Z}\times\mathbb{N}\cup\{0\}}$ respectively, such that $\{|p|,\ q\}\leqslant m$. This is, we have that:

$$\mathbb{P}_m[f](x,y):=\sum_{\substack{|p|\leqslant m\\ p\in\mathbb{Z}}}\sum_{\substack{q\leqslant m\\ q\in\mathbb{N}}}\mathcal{F}_\omega[f](p,q)\,w_{p,q}(x,y), \tag{3.15}$$

$$\mathbb{Q}_m[f](x,y):=\sum_{\substack{|p|\leqslant m\\ p\in\mathbb{Z}}}\sum_{\substack{q\leqslant m\\ q\in\mathbb{N}\cup\{0\}}}\mathcal{F}_\varpi[f](p,q)\,\varpi_{p,q}(x,y).$$

These projectors have the following properties:

Lemma 3.3.2. *Let $\mathbb{P}_m, \mathbb{Q}_m$ be the projectors defined in (3.15). For $f \in L^2(\Omega)$, we have that $\mathbb{P}_m[f]$ and $\mathbb{Q}_m[f]$ are $C^\infty(\Omega)$ functions such that:*

- *For $f \in H^1(\Omega)$ we have that:*

  $$\partial_x \mathbb{P}_m[f] = \mathbb{P}_m[\partial_x f], \quad \partial_x \mathbb{Q}_m[f] = \mathbb{Q}_m[\partial_x f], \quad \partial_y \mathbb{P}_m[f] = \mathbb{Q}_m[\partial_y f] \quad \textit{and} \quad \partial_y \mathbb{Q}_m[f] = \mathbb{P}_m[\partial_y f].$$

  In consequence, for $f \in H^2(\Omega)$ we have that:

  $$\partial_y^2 \mathbb{P}_m[f] = \mathbb{P}_m[\partial_y^2 f] \qquad \textit{and} \qquad \partial_y^2 \mathbb{Q}_m[f] = \mathbb{Q}_m[\partial_y^2 f].$$

- *The projectors are self-adjoint in $L^2(\Omega)$:*

  $$(\mathbb{P}_m[f], g) = (f, \mathbb{P}_m[g]) \quad \textit{and} \quad (\mathbb{Q}_m[f], g) = (f, \mathbb{Q}_m[g]) \qquad \forall f, g \in L^2(\Omega).$$

- *For $f \in X^k(\Omega)$ and $g \in Y^k(\Omega)$:*

  $$\begin{aligned} ||\mathbb{P}_m[f]||_{H^k(\Omega)} &\leqslant ||f||_{H^k(\Omega)}, \quad \mathbb{P}_m[f] \to f \quad \textit{in } X^k(\Omega), \\ ||\mathbb{Q}_m[g]||_{H^k(\Omega)} &\leqslant ||g||_{H^k(\Omega)}, \quad \mathbb{Q}_m[f] \to f \quad \textit{in } Y^k(\Omega). \end{aligned}$$

- *Leray projector $\mathbb{L} := \mathbb{I} + \nabla(-\Delta)^{-1}\mathrm{div}$ commutes with the pair $(\mathbb{Q}_m, \mathbb{P}_m)$ and with derivatives.*

Proof of Lemmas 3.3.1 and 3.3.2. See Section 2 of previous chapter. □

3.4 Local solvability of solutions

To obtain a local existence result for a general smooth initial data in a general bounded domain for an *active scalar* is far from being trivial. The presence of boundaries makes the well-posedness issues become more delicate. (See for example [15] and [51], in the case of SQG). As in the previous chapter, we focus only on *our setting* and in *our specific class* of initial data.

Then, we prove local existence and uniqueness of solutions using the Galerkin approximations. We return to the equations for the perturbation of the damping Boussinesq in Ω:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho &= -u_2, \\ \partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} &= -\nabla P - (0, \rho), \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0) \in \mathbb{X}^k(\Omega), \\ \rho|_{t=0} &= \rho(0) \in X^k(\Omega), \end{cases} \tag{3.16}$$

besides the no-slip conditions $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Hence, we will prove the following result:

Theorem 3.4.1. *Let $k \in \mathbb{N}$ and an initial data $(\rho(0), \mathbf{u}(0)) \in X^k \times \mathbb{X}^k$. Then, there exists a time $T > 0$ and a constant C, both depending only on $e_3(0)$ and a unique solution $(\rho, \mathbf{u}) \in C\left(0, T; X^k(\Omega) \times \mathbb{X}^k(\Omega)\right)$ of the system (3.16) such that:*

$$\sup_{0 \leqslant t \leqslant T} e_k(t) \leqslant C\, e_k(0)$$

where

$$e_k(t) := ||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\rho||^2_{H^k(\Omega)}(t).$$

Moreover, for all $t \in [0, T)$ the following estimate holds:

$$e_k(t) \leqslant e_k(0)\, \exp\left[\widetilde{C} \int_0^t \left(||\nabla \rho||_{L^\infty(\Omega)}(s) + ||\nabla \mathbf{u}||_{L^\infty(\Omega)}(s)\right)\, ds\right]. \tag{3.17}$$

The general method of the proof is similar to that for proving existence of solutions to the Navier–Stokes and Euler equations which can be found in [46]. The strategy of this section has two parts. First we find an approximate equation and approximate solutions that have two properties: (1) the existence theory for all time for the approximating solutions is easy, (2) the solutions satisfy an analogous energy estimate. The second part is the passage to a limit in the approximation scheme to obtain a solution to the original equations.

We begin with some basic properties of the Sobolev spaces in bounded domains. In the rest, $D \subset \mathbb{R}^d$ is a bounded domain with smooth boundary ∂D.

Lemma 3.4.2. *For $s \in \mathbb{N}$, the following estimates holds:*

- *If $f, g \in H^s(D) \cap \mathcal{C}(D)$, then*

$$\|f\, g\|_{H^s(D)} \lesssim \left(\|f\|_{H^s(D)} \|g\|_{L^\infty(D)} + \|f\|_{L^\infty(D)} \|g\|_{H^s(D)}\right). \tag{3.18}$$

- *If $f \in H^s(D) \cap \mathcal{C}^1(D)$ and $g \in H^{s-1}(D) \cap \mathcal{C}(D)$, then for $|\alpha| \leqslant s$ we have that:*

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2(D)} \lesssim \|f\|_{W^{1,\infty}(D)} \|g\|_{H^{s-1}(D)} + \|f\|_{H^s(D)} \|g\|_{L^\infty(D)}. \tag{3.19}$$

Moreover, the following Sobolev embedding holds:

- $W^{s,p}(D) \subseteq L^q(D)$ *continuously if* $s < n/p$ *and* $p \leqslant q \leqslant np/(n-sp)$.
- $W^{s,p}(D) \subseteq \mathcal{C}^k(\overline{D})$ *consinuously is* $s > k + n/p$.

Proof. See [29, p. 280] and references therein. □

Proof of Theorem 3.4.1. We firstly construct approximate equations by using a smoothing procedure called Galerkin method. The m^{th}-Galerkin approximation of (3.16) is the following system:

$$\left\{\begin{aligned} \partial_t \rho^{[m]} + \mathbb{P}_m \left[\left(\mathbf{u}^{[m]} \cdot \nabla\right) \rho^{[m]}\right] &= -u_2^{[m]}, \\ \partial_t \mathbf{u}^{[m]} + \mathbf{u}^{[m]} + (\mathbb{Q}_m, \mathbb{P}_m) \left[\left(\mathbf{u}^{[m]} \cdot \nabla\right) \mathbf{u}^{[m]}\right] &= -\nabla P^{[m]} + (0, \rho^{[m]}), \\ \nabla \cdot \mathbf{u}^{[m]} &= 0, \\ \mathbf{u}^{[m]}|_{t=0} &= \left(\mathbb{Q}_m[u_1], \mathbb{P}_m[u_2]\right)(0), \\ \rho^{[m]}|_{t=0} &= \mathbb{P}_m[\rho](0), \end{aligned}\right. \tag{3.20}$$

with $\rho(0) \in X^k$ and $\mathbf{u}(0) \in \mathbb{X}^k$.

Equations (3.20) explicitly contain the pressure term $P^{[m]}$. We eliminate $P^{[m]}$ and the incompressibility condition $\nabla \cdot \mathbf{u}^{[m]} = 0$ by projecting these equations onto the space of divergence-free functions:

$$\mathbb{V}^k(\Omega) := \left\{\mathbf{v} \in \mathbb{X}^k(\Omega) : \nabla \cdot \mathbf{v} = 0\right\}.$$

Because the Leray operator $\mathbb{L}$ commutes with the pair $(\mathbb{Q}_m, \mathbb{P}_m)$ and $\mathbb{L}\left[\mathbf{u}^{[m]}\right] = \mathbf{u}^{[m]}$, we have

$$\partial_t \mathbf{u}^{[m]} + \mathbf{u}^{[m]} + \mathbb{L}\left(\mathbb{Q}_m, \mathbb{P}_m\right)\left[\left(\mathbf{u}^{[m]} \cdot \nabla\right) \mathbf{u}^{[m]}\right] = \mathbb{L}\left[(0, \rho^{[m]})\right] \tag{3.21}$$

or equivalently

$$\left\{\begin{aligned} \partial_t u_1^{[m]} + u_1^{[m]} + \mathbb{Q}_m \mathbb{L}_1 \left[\left(\mathbf{u}^{[m]} \cdot \nabla\right) u_1^{[m]}\right] &= \mathbb{Q}_m \left[(-\Delta)^{-1} \partial_x \partial_y \rho^{[m]}\right], \\ \partial_t u_2^{[m]} + u_2^{[m]} + \mathbb{P}_m \mathbb{L}_2 \left[\left(\mathbf{u}^{[m]} \cdot \nabla\right) u_2^{[m]}\right] &= \mathbb{P}_m \left[(-\Delta)^{-1} \partial_y^2 \rho^{[m]} + \rho^{[m]}\right]. \end{aligned}\right.$$

Since $\rho^{[m]}|_{t=0} = \mathbb{P}_m[\rho](0)$ belongs to $\mathbb{P}_m L^2(\Omega)$, the initial velocity $\mathbf{u}^{[m]}|_{t=0} = (\mathbb{Q}_m[u_1], \mathbb{P}_m[u_2])(0)$ belongs to $\mathbb{Q}_m L^2(\Omega) \times \mathbb{P}_m L^2(\Omega)$, and because of the structure of the equations, we look for solutions of the form

$$\rho^{[m]}(t) = \sum_{\substack{|p|\leqslant m \\ p\in\mathbb{Z}}} \sum_{\substack{q\leqslant m \\ q\in\mathbb{N}}} \mathfrak{a}_{p,q}^{[m]}(t)\omega_{p,q}(x,y)$$

and

$$\mathbf{u}^{[m]}(t) = \left(\sum_{\substack{|p|\leqslant m \\ p\in\mathbb{Z}}} \sum_{\substack{q\leqslant m \\ q\in\mathbb{N}\cup\{0\}}} \mathfrak{b}_{p,q}^{[m]}(t)\varpi_{p,q}(x,y), \sum_{\substack{|p|\leqslant m \\ p\in\mathbb{Z}}} \sum_{\substack{q\leqslant m \\ q\in\mathbb{N}}} \mathfrak{c}_{p,q}^{[m]}(t)\omega_{p,q}(x,y) \right).$$

In this way, (3.20) is reduced to a finite dimensional ODE system for the coefficients $\mathfrak{a}_{p,q}^{[m]}(t)$, $\mathfrak{b}_{p,q}^{[m]}(t)$ and $\mathfrak{c}_{p,q}^{[m]}(t)$ for $\{|p|, q\} \leqslant m$, and we can apply Picard's theorem to find a solution with a time of existence depending on m. Next, we will use energy estimates to show a time of existence T, uniform in m, for every solution $\left(\rho^{[m]}(t), \mathbf{u}^{[m]}(t)\right)$ of (3.20) and a limit $(\rho(t), \mathbf{u}(t))$ which will solve (3.16).

Taking derivatives ∂^s, with $|s| \leqslant k$ on (3.21) and then taking the $L^2(\Omega)$ inner product with $\partial^s \mathbf{u}^{[m]}$, we obtain using the properties of the Leray projector that:

$$\tfrac{1}{2}\partial_t \|\partial^s \mathbf{u}^{[m]}\|^2_{L^2(\Omega)} = \left(\partial^s\rho^{[m]}, \partial^s u_2^{[m]}\right) - \|\partial^s \mathbf{u}^{[m]}\|^2_{L^2(\Omega)} - \left\langle \partial^s(\mathbb{Q}_m, \mathbb{P}_m)\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\mathbf{u}^{[m]}\right], \partial^s\mathbf{u}^{[m]}\right\rangle. \tag{3.22}$$

Moreover, as $\partial_t\rho^{[m]} + \mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right] = -u_2^{[m]}$, we obtain that:

$$\left(\partial^s\rho^{[m]}, \partial^s u_2^{[m]}\right) = -\tfrac{1}{2}\partial_t\|\partial^s\rho^{[m]}\|^2_{L^2(\Omega)} - \left(\partial^s\rho^{[m]}, \partial^s\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]\right). \tag{3.23}$$

By putting together (3.22) and (3.23), we achieve that:

$$\begin{aligned}
\tfrac{1}{2}\partial_t\left(\|\partial^s\mathbf{u}^{[m]}\|^2_{L^2(\Omega)} + \|\partial^s\rho^{[m]}\|^2_{L^2(\Omega)}\right) = &- \|\partial^s\mathbf{u}^{[m]}\|^2_{L^2(\Omega)} \\
&- \left(\partial^s\rho^{[m]}, \partial^s\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]\right) \\
&- \left(\partial^s u_1^{[m]}, \partial^s\mathbb{Q}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]}\right]\right) \\
&- \left(\partial^s u_2^{[m]}, \partial^s\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]}\right]\right) \\
= &-\|\partial^s\mathbf{u}^{[m]}\|^2_{L^2(\Omega)} + \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Now, we need to distinguish between an even or odd number of y-derivatives. In any case, the properties of $\mathbb{P}_m, \mathbb{Q}_m$ given by Lemma 3.3.2 and the commutator estimate (3.19) with $f = \mathbf{u}^{[m]}$ and $g = \nabla\rho^{[m]}$ give us the first inequality:

$$\mathrm{I} \lesssim \|\partial^s\rho^{[m]}\|_{L^2(\Omega)}\left(\|\nabla\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}\|\rho^{[m]}\|_{H^k(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}\|\nabla\rho^{[m]}\|_{L^\infty(\Omega)}\right). \tag{3.24}$$

For the rest, we proceed as before with $f = \mathbf{u}^{[m]}$ and $g = \nabla u_1^{[m]}$ or $g = \nabla u_2^{[m]}$ respectively to obtain the inequalities:

$$\mathrm{II} \lesssim \|\partial^s u_1^{[m]}\|_{L^2(\Omega)}\left(\|\nabla\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}\|u_1^{[m]}\|_{H^k(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}\|\nabla u_1^{[m]}\|_{L^\infty(\Omega)}\right),$$
$$\mathrm{III} \lesssim \|\partial^s u_2^{[m]}\|_{L^2(\Omega)}\left(\|\nabla\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}\|u_2^{[m]}\|_{H^k(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}\|\nabla u_2^{[m]}\|_{L^\infty(\Omega)}\right),$$

and in consequence:

$$\mathrm{II}+\mathrm{III} \lesssim \|\partial^s \mathbf{u}^{[m]}\|_{L^2(\Omega)}^2 \|\nabla \mathbf{u}^{[m]}\|_{L^\infty(\Omega)}. \tag{3.25}$$

Remark: In the previous computations, we have used that $\mathbf{u}^{[m]}$ is divergence-free and vanishes at the boundary $\partial\Omega$. Then, integration by parts gives that the singular terms disappear.

Summing over $|s| \leqslant k$ and putting together (3.24) and (3.25) we obtain:

$$\begin{aligned} \dot{e}_k^{[m]}(t) &\lesssim e_k^{[m]}(t)\, \left(\|\nabla \rho^{[m]}\|_{L^\infty(\Omega)}(t) + \|\nabla \mathbf{u}^{[m]}\|_{L^\infty(\Omega)}(t)\right) \\ &\lesssim e_k^{[m]}(t)\, \left(\|\rho^{[m]}\|_{H^3(\Omega)}(t) + \|\mathbf{u}^{[m]}\|_{H^3(\Omega)}(t)\right) \end{aligned} \tag{3.26}$$

thanks to the Sobolev embedding, where

$$e_k^{[m]}(t) := \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}^2(t) + \|\rho^{[m]}\|_{H^k(\Omega)}^2(t).$$

Hence, assuming that $k \geqslant 3$ in (3.26), for all m and $0 \leqslant t < T \leqslant \left(\frac{1}{2}\,[e_3^{[m]}(0)]^{1/2}\right)^{-1}$ we have that:

$$e_3^{[m]}(t) \leqslant \frac{[e_3^{[m]}(0)]^{1/2}}{1-\frac{t}{2}\,[e_3^{[m]}(0)]^{1/2}} \leqslant \frac{[e_3(0)]^{1/2}}{1-\frac{t}{2}\,[e_3(0)]^{1/2}} \tag{3.27}$$

and, in particular

$$\sup_{0\leqslant t<T} e_3^{[m]}(t) \leqslant \frac{[e_3(0)]^{1/2}}{1-\frac{T}{2}\,[e_3(0)]^{1/2}}.$$

Applying (3.27) in the last term of (3.26), we obtain for all m and $0 \leqslant t < T$ by Gronwall's lemma that:

$$\begin{aligned} e_k^{[m]}(t) &\leqslant e_k^{[m]}(0)\exp\left[\int_0^t \frac{[e_3(0)]^{1/2}}{1-\frac{s}{2}\,[e_3(0)]^{1/2}}\,ds\right] \\ &\leqslant e_k(0)\exp\left[\int_0^t \frac{[e_3(0)]^{1/2}}{1-\frac{s}{2}\,[e_3(0)]^{1/2}}\,ds\right], \end{aligned} \tag{3.28}$$

and, in particular

$$\sup_{0\leqslant t<T} e_k^{[m]}(t) \leqslant C\, e_k(0), \tag{3.29}$$

where C is a constant depending only on $e_3(0)$.

Remark: In the last inequality of (3.27) and (3.28), we have used in a crucial way the bound $e_k^{[m]}(0) \leqslant e_k(0)$ which, is a consequence of the fact that $\rho(0) \in X^k(\Omega)$ and $\mathbf{u}(0) \in \mathbb{X}^k(\Omega)$ together with the Lemma 3.3.2.

In view of (3.29), we have that the sequences $\rho^{[m]}$ and $\mathbf{u}^{[m]}$ are uniformly bounded in $L^\infty\left(0,T;H^k(\Omega)\right)$ and $L^\infty\left(0,T;H^k(\Omega)\times H^k(\Omega)\right)$ respectively. As a consequence of the Banach-Alaoglu theorem (see [55]), each of these sequences has a subsequence that converges weakly to some limit. This is $\rho^{[m]}(t) \rightharpoonup \rho(t)$ in $H^k(\Omega)$ and $\mathbf{u}^{[m]}(t) \rightharpoonup \mathbf{u}(t)$ in $H^k(\Omega)\times H^k(\Omega)$ for $0 \leqslant t < T$.

Furthermore, something similar can be obtained for the sequences of time derivatives. On one hand, the family $\partial_t \rho^{[m]}$ is uniformly bounded in $L^\infty\left(0,T;H^{k-2}(\Omega)\right)$. On the other hand, the family $\partial_t \mathbf{u}^{[m]}$ is uniformly bounded in $L^\infty\left(0,T;H^{k-2}(\Omega)\times H^{k-2}(\Omega)\right)$.

By (3.20) and the properties of Leray projector, we have that:

- $$\begin{aligned}\|\partial_t \rho^{[m]}\|_{L^\infty(0,T;H^{k-2}(\Omega))} &= \|u_2^{[m]} + \mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]\|_{L^\infty(0,T;H^{k-2}(\Omega))} \\ &\leqslant \sup_{0\leqslant t<T}\left\{\|u_2^{[m]}\|_{H^{k-2}(\Omega)} + \|\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]\|_{H^{k-2}(\Omega)}\right\}(t);\end{aligned}$$
- $$\begin{aligned}\|\partial_t \mathbf{u}^{[m]}\|_{L^\infty(0,T;H^{k-2}(\Omega))} &= \|\mathbb{L}[(0,\rho^{[m]})] - \mathbb{L}(\mathbb{Q}_m,\mathbb{P}_m)\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\mathbf{u}^{[m]}\right] - \mathbf{u}^{[m]}\|_{L^\infty(0,T;H^{k-2}(\Omega))} \\ &\leqslant \sup_{0\leqslant t<T}\left\{\|\rho^{[m]}\|_{H^{k-2}(\Omega)} + \|\mathbf{u}^{[m]}\|_{H^{k-2}(\Omega)}\right\}(t) \\ &+ \sup_{0\leqslant t<T}\left\{\|\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]}\right]\|_{H^{k-2}(\Omega)} + \|\mathbb{Q}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]}\right]\|_{H^{k-2}(\Omega)}\right\}(t).\end{aligned}$$

Now, we need to show that $\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\in X^{k-1}(\Omega)$ and $\left(\mathbf{u}^{[m]}\cdot\nabla\right)\mathbf{u}^{[m]}\in Y^{k-1}(\Omega)\times X^{k-1}(\Omega)$ to apply Lemma 3.3.2 for $k\geqslant 3$, and to get:

- $$\begin{aligned}\|\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]\|_{H^{k-2}(\Omega)} &\leqslant \|\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\|_{H^{k-2}(\Omega)} \\ &\lesssim \|\mathbf{u}^{[m]}\|_{H^{k-2}(\Omega)}\|\nabla\rho^{[m]}\|_{L^\infty(\Omega)} + \|\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}\|\nabla\rho^{[m]}\|_{H^{k-2}(\Omega)} \\ &\lesssim \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}\|\rho^{[m]}\|_{H^k(\Omega)};\end{aligned}$$
- $\|\mathbb{Q}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]}\right]\|_{H^{k-2}(\Omega)} \leqslant \|\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]}\|_{H^{k-2}(\Omega)} \lesssim \|\mathbf{u}^{[m]}\|^2_{H^k(\Omega)};$
- $\|\mathbb{P}_m\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]}\right]\|_{H^{k-2}(\Omega)} \leqslant \|\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]}\|_{H^{k-2}(\Omega)} \lesssim \|\mathbf{u}^{[m]}\|^2_{H^k(\Omega)},$

where we have used (3.18) and the Sobolev embedding $L^\infty(\Omega)\hookrightarrow H^2(\Omega)$.

Check that $\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\in X^{k-1}(\Omega)$ and $\left(\mathbf{u}^{[m]}\cdot\nabla\right)\mathbf{u}^{[m]}\in Y^{k-1}(\Omega)\times X^{k-1}(\Omega)$ is to see that:

$$\partial_y^n\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]}\right]|_{\partial\Omega} = \partial_y^n\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]}\right]|_{\partial\Omega} = \partial_y^{n+1}\left[\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]}\right]|_{\partial\Omega} = 0,$$

for any even natural number n. We start, with the following observations:

- ◇ $\left(\mathbf{u}^{[m]}\cdot\nabla\right)\rho^{[m]} = \mathbb{Q}_m\left[u_1^{[m]}\right]\mathbb{P}_m\left[\partial_x\rho^{[m]}\right] + \mathbb{P}_m\left[u_2^{[m]}\right]\mathbb{Q}_m\left[\partial_y\rho^{[m]}\right];$
- ◇ $\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_1^{[m]} = \mathbb{Q}_m\left[u_1^{[m]}\right]\mathbb{Q}_m\left[\partial_x u_1^{[m]}\right] + \mathbb{P}_m\left[u_2^{[m]}\right]\mathbb{P}_m\left[\partial_y u_1^{[m]}\right];$
- ◇ $\left(\mathbf{u}^{[m]}\cdot\nabla\right)u_2^{[m]} = \mathbb{Q}_m\left[u_1^{[m]}\right]\mathbb{P}_m\left[\partial_x u_2^{[m]}\right] + \mathbb{P}_m\left[u_2^{[m]}\right]\mathbb{Q}_m\left[\partial_y u_2^{[m]}\right],$

and the facts that

$$\begin{aligned}\partial_y(b_{p_1}b_{p_2})(y) &= (-1)^{p_1}p_1\tfrac{\pi}{2}c_{p_1}(y)\,b_{p_2}(y) + (-1)^{p_2}p_2\tfrac{\pi}{2}b_{p_1}(y)\,c_{p_2}(y);\\ \partial_y(c_{q_1}c_{q_2})(y) &= (-1)^{q_1+1}q_1\tfrac{\pi}{2}b_{q_1}(y)\,c_{q_2}(y) + (-1)^{q_2+1}q_2\tfrac{\pi}{2}c_{q_1}(y)\,b_{q_2}(y),\end{aligned}$$

and

$$\begin{aligned}\partial_y^2(b_{p_1}c_{q_2})(y) &= (\partial_y^2 b_{p_1})(y)\,c_{q_2}(y) + 2(\partial_y b_{p_1})(y)\,(\partial_y c_{q_2})(y) + b_{p_1}(y)\,(\partial_y^2 c_{q_2})(y)\\ &= (-1)\left[\left(p_1\tfrac{\pi}{2}\right)^2 + \left(q_2\tfrac{\pi}{2}\right)^2\right]b_{p_1}(y)\,c_{q_2}(y) + (-1)(-1)^{p_1+q_2}2p_1q_2\left(\tfrac{\pi}{2}\right)^2c_{p_1}(y)\,b_{q_2}(y).\end{aligned}$$

Iterating this procedure and using that $b_p(\pm 1) = 0$ we prove the boundary conditions for the derivatives of even and odd order of the non-linear terms.

Therefore, putting all together and using (3.29) we obtain:

- $\|\partial_t \rho^{[m]}\|_{L^\infty(0,T;H^{k-2}(\Omega))} \lesssim \sup\limits_{0\leqslant t<T} \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}(t) \left(1 + \|\rho^{[m]}\|_{H^k(\Omega)}(t)\right) \lesssim 1 + C\, e_k(0);$
- $\|\partial_t \mathbf{u}^{[m]}\|_{L^\infty(0,T;H^{k-2}(\Omega))} \lesssim \sup\limits_{0\leqslant t<T} \|\rho^{[m]}\|_{H^k(\Omega)}(t) + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}(t) \left(1 + \|\mathbf{u}^{[m]}\|_{H^k(\Omega)}(t)\right)$
$\lesssim 1 + C\, e_k(0).$

Hence, the family of time derivatives $\partial_t \rho^{[m]}(t)$ is uniformly bounded in $L^\infty\left(0,T;H^{k-2}(\Omega)\right)$ and the same for the familiy $\partial_t \mathbf{u}^{[m]}$ in $L^\infty\left(0,T;H^{k-2}(\Omega)\times H^{k-2}(\Omega)\right)$. Then, by Banach-Alaoglu theorem, $\partial_t \rho^{[m]}(t)$ has a subsequence that converges weakly to some limit in $H^{k-2}(\Omega)$ for $0 \leqslant t < T$ and analogously $\partial_t \mathbf{u}^{[m]}(t)$ has a subsequence that converges weakly to some limit in $H^{k-2}(\Omega)\times H^{k-2}(\Omega)$ for $0 \leqslant t < T$.

Moreover, by virtue of Aubin-Lions's compactness lemma (see for instance [43]) applied with the triples $H^k(\Omega) \Subset H^{k-1}(\Omega) \subset H^{k-2}(\Omega)$ and $H^k(\Omega)\times H^k(\Omega) \Subset H^{k-1}(\Omega)\times H^{k-1}(\Omega) \subset H^{k-2}(\Omega)\times H^{k-2}(\Omega)$ we obtain that the convergences of $\rho^{[m]} \to \rho$ and $\mathbf{u}^{[m]} \to \mathbf{u}$ are in fact strong in $C(0,T;H^{k-1}(\Omega))$ and in $C(0,T;H^{k-1}(\Omega)\times H^{k-1}(\Omega))$ respectively.

Using these facts, we may pass to the limit in the non-linear part of (3.20) to see the convergences of $\mathbb{P}_m[(\mathbf{u}^{[m]}\cdot\nabla)\rho^{[m]}] \to (\mathbf{u}\cdot\nabla)\,\rho$ and $(\mathbb{Q}_m,\mathbb{P}_m)[(\mathbf{u}^{[m]}\cdot\nabla)\mathbf{u}^{[m]}] \to (\mathbf{u}\cdot\nabla)\,\mathbf{u}$ in $C(0,T;H^{k-2}(\Omega))$ and in $C(0,T;H^{k-2}(\Omega)\times H^{k-2}(\Omega))$ respectively, as follows:

$$
\begin{aligned}
&\|\mathbb{P}_m[(\mathbf{u}^{[m]}\cdot\nabla)\rho^{[m]}] - (\mathbf{u}\cdot\nabla)\,\rho\|_{H^{k-2}(\Omega)} \\
&\qquad = \|\mathbb{P}_m[(\mathbf{u}^{[m]}\cdot\nabla)\rho^{[m]}] \pm (\mathbf{u}^{[m]}\cdot\nabla)\rho^{[m]} \pm (\mathbf{u}^{[m]}\cdot\nabla)\rho - (\mathbf{u}\cdot\nabla)\rho\|_{H^{k-2}(\Omega)} \\
&\qquad \leqslant \left\|(\mathbb{P}_m - \mathbb{I})[(\mathbf{u}^{[m]}\cdot\nabla)\rho^{[m]}]\right\|_{H^{k-2}(\Omega)} + \left\|(\mathbf{u}^{[m]}\cdot\nabla)(\rho^{[m]}-\rho)\right\|_{H^{k-2}(\Omega)} \\
&\qquad\quad + \left\|([\mathbf{u}^{[m]}-\mathbf{u}]\cdot\nabla)\rho\right\|_{H^{k-2}(\Omega)} \to 0 \qquad \text{as} \quad m\to\infty.
\end{aligned}
$$

In the limit, we use the fact that $\lim\limits_{m\to\infty}\|\mathbb{P}_m[f]-f\|_{H^s(\Omega)} = 0$ for $f \in X^s(\Omega)$, together with the convergences of $\mathbf{u}^{[m]} \to \mathbf{u}$ and $\rho^{[m]} \to \rho$ and (3.18), for $k \geqslant 3$. For the other, we repeat the same procedure using that $\lim\limits_{m\to\infty}\|\mathbb{Q}_m[g]-g\|_{H^s(\Omega)} = 0$ for $g \in Y^s(\Omega)$ and the fact that:

$$
\begin{aligned}
\|(\mathbb{Q}_m,\mathbb{P}_m)[(\mathbf{u}^{[m]}\cdot\nabla)\mathbf{u}^{[m]}] - (\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^{k-2}(\Omega)\times H^{k-2}(\Omega)} &= \|\mathbb{Q}_m[(\mathbf{u}^{[m]}\cdot\nabla)u_1^{[m]}] - (\mathbf{u}\cdot\nabla)\,u_1\|_{H^{k-2}(\Omega)} \\
&\quad + \|\mathbb{P}_m[(\mathbf{u}^{[m]}\cdot\nabla)u_2^{[m]}] - (\mathbf{u}\cdot\nabla)\,u_2\|_{H^{k-2}(\Omega)}.
\end{aligned}
$$

We have that $\partial_t\rho^{[m]} \to -c\,u_2 - \mathbf{u}\cdot\nabla\rho$ and $\partial_t\mathbf{u}^{[m]} \to \mathbb{L}\left[(0,\rho)\right] - \mathbf{u} - \mathbb{L}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]$ in $C(0,T;H^{k-2}(\Omega))$ and in $\partial_t\mathbf{u}^{[m]} \to \mathbb{L}\left[(0,\rho)\right] - \mathbf{u} - \mathbb{L}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]$ respectively. Since $\rho^{[m]} \to \rho$ and $\mathbf{u}^{[m]} \to \mathbf{u}$ in $C(0,T;H^{k-1}(\Omega))$ and in $C(0,T;H^{k-2}(\Omega)\times H^{k-2}(\Omega))$ respectively, the limit distributions of $\partial_t\rho^{[m]}$ and $\partial_t\mathbf{u}^{[m]}$ must be $\partial_t\rho$ and $\partial_t\mathbf{u}$ by the Closed Graph theorem [5].

So, in particular, it follows that the pair $(\rho(t),\mathbf{u}(t))$ is the unique classical solution of (3.16) which lies in $C(0,T;H^{k-1}(\Omega)) \times C(0,T;H^{k-1}(\Omega)\times H^{k-1}(\Omega))$. Moreover, we can follow the same ideas of [46, p. 110] to prove, as we did in [10], that $(\rho(t),\mathbf{u}(t)) \in C(0,T;H^k(\Omega)) \times C(0,T;H^k(\Omega)\times H^k(\Omega))$. Note that $\mathbb{L}\left[\partial_t\mathbf{u} + \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (0,\rho)\right] = 0$ implies

$$
\partial_t\mathbf{u} + \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P + (0,\rho)
$$

for some scalar function $P(\mathbf{x},t)$.

Since for all $m \in \mathbb{N}$ we have that $\rho^{[m]} = \mathbb{P}_m[\rho^{[m]}] \in X^k(\Omega)$ and $\mathbf{u}^{[m]} = (\mathbb{Q}_m[u_1^{[m]}], \mathbb{P}_m[u_2^{[m]}]) \in \mathbb{X}^k(\Omega)$ and this property is closed, we obtain that the limiting function also has the desired property. In consequence, the solution $(\rho, \mathbf{u})$ lies in $C\left(0, T; X^k(\Omega)\right) \times C\left(0, T; \mathbb{X}^k(\Omega)\right)$.

Finally, applying the Gronwall's lemma on the above estimate (3.26) and the previous convergence results, for all $t \in [0, T)$ we deduce:

$$\begin{aligned} e_k^{[m]}(t) &\leqslant e_k^{[m]}(0)\, \exp\left[\widetilde{C}\int_0^t \left(\|\nabla\rho^{[m]}\|_{L^\infty(\Omega)}(s) + \|\nabla\mathbf{u}^{[m]}\|_{L^\infty(\Omega)}(s)\right)\, ds\right] \\ &\leqslant e_k(0)\, \exp\left[\widetilde{C}\int_0^t \left(\|\nabla\rho\|_{L^\infty(\Omega)}(s) + \|\nabla\mathbf{u}\|_{L^\infty(\Omega)}(s)\right)\, ds\right] \end{aligned}$$

and by lower semicontinuity we obtain (3.17). □

Theorem 3.4.3. *Let $(\rho(t), \mathbf{u}(t))$ be a solution of (3.16) in the class $C\left(0, T, X^k(\Omega)\right) \times C\left(0, T, \mathbb{X}^k(\Omega)\right)$ with initial data $\rho(0) \in X^k$ and $\mathbf{u}(0) \in \mathbb{X}^k$. If $T = T^\star$ is the first time such that $(\rho(t), \mathbf{u}(t))$ is not contained in this class, then*

$$\int_0^{T^\star} \left(\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}(s) + \|\nabla\rho\|_{L^\infty(\Omega)}(s)\right)\, ds = \infty.$$

Proof. This result follows from estimate (3.17). □

3.5 Energy methods for the damping Boussineq equations

From what we have seen, we know that for $(\rho(0), \mathbf{u}(0)) \in X^k \times \mathbb{X}^k$ there exists $T > 0$ such that $(\rho(t), \mathbf{u}(t))$ is a solution of (3.16) for all $t \in [0, T)$. Moreover, if $T^\star$ is the first time such that $(\rho(t), \mathbf{u}(t))$ is not contained in this class $X^k \times \mathbb{X}^k$, then

$$\int_0^{T^\star} \left(\|\nabla\mathbf{u}\|_{L^\infty(\Omega)}(s) + \|\nabla\rho\|_{L^\infty(\Omega)}(s)\right)\, ds = \infty.$$

Therefore, to control $e_3(T)$ allows us to extend the solution smoothly past time T, where we remember that

$$e_k(t) := \|\mathbf{u}\|^2_{H^k(\Omega)}(t) + \|\rho\|^2_{H^k(\Omega)}(t).$$

Finally, note that $\rho(t) \in X^k$ implies that $\rho(t) \in H^k(\Omega)$, so the term "$\partial^{k-1}\rho$ restricted to $\partial\Omega$" has perfect sense, as long as the solution exists. Analogously, as $\mathbf{u}(t) \in \mathbb{X}^k$, can we talk about "$\partial^{k-1}\mathbf{u}$ restricted to $\partial\Omega$".

3.5.1 Energy Space

To motivate the energy space in which we will work, we present the linearized problem of (3.9). This is:

$$\begin{cases} \partial_t\rho = -u_2 \\ \partial_t\mathbf{u} + \mathbf{u} = -\nabla\Pi^L + (0, \bar{\rho}) \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

besides the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$. It is easy to check that:

$$\tfrac{1}{2}\partial_t\left\{\|\mathbf{u}\|^2_{L^2}(t) + \|\rho\|^2_{L^2}(t)\right\} = -\|\mathbf{u}\|^2_{L^2}(t) \quad \text{and} \quad \tfrac{1}{2}\partial_t\left\{\|\partial_t\mathbf{u}\|^2_{L^2}(t) + \|u_2\|^2_{L^2}(t)\right\} = -\|\partial_t\mathbf{u}\|^2_{L^2}(t).$$

By attending to this, for $k \in \mathbb{N}$ we define the energy

$$E_k(t) := \frac{1}{2}\left\{||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\rho||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k(\Omega)}(t)\right\}$$

and the auxiliar weighted energy

$$\dot{\mathcal{E}}_k(t) := \frac{1}{2}\left\{||\rho||^2_{\dot{H}^k(\Omega)}(t) + \int_\Omega |\partial^k \mathbf{u}(x,y,t)|^2 \,\left(1 + \partial_y \tilde{\rho}(y,t)\right)\, dxdy\right\}.$$

The introduction of the weight $1 + \partial_y \tilde{\rho}(y,t)$ in the last term of $\dot{\mathcal{E}}_k(t)$ is not obvious and plays a crucial role. We are forced to do it in order to control all the terms. Finally, our energy space will be

$$\mathfrak{E}_{k+1}(t) := E_k(t) + \dot{\mathcal{E}}_{k+1}(t). \tag{3.30}$$

Note that if our weight $1 + \partial_y \tilde{\rho}(y,t)$ is non-negative then our energy is positive definite. So, our energy space is perfectly well defined if $\tilde{\rho}$ is small enough. Moreover, it is clear that $e_k(t) \leqslant \mathfrak{E}_{k+1}(t)$.

3.5.2 A Priori Energy Estimates

In what follows, we assume that $(\rho(t), \mathbf{u}(t)) \in X^{k+1}(\Omega) \times \mathbb{X}^{k+1}(\Omega)$ is a solution of (3.9) for any $t \geqslant 0$. Then, this section is devoted to prove the following result.

Theorem 3.5.1. *There exist $0 < C < 1$ and $\tilde{C} > 0$ large enough such that for $k \geqslant 6$ the following estimate holds:*

$$\begin{aligned}\partial_t \mathfrak{E}_{k+1}(t) \leqslant& -(C - \tilde{C}\,\Psi_1(t))\left[||\nabla\Pi - (0,\bar{\rho})||^2_{H^k}(t) + ||\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}||^2_{H^k}(t) + ||\mathbf{u}||^2_{H^k}(t) + ||\partial_t \mathbf{u}||^2_{H^k}(t)\right]\\ &- \left(1 - \tilde{C}\,\Psi_2(t)\right)\left(\int_\Omega |\partial^{k+1}\mathbf{u}(x,y,t)|^2\,(1 + \partial_y \tilde{\rho}(y,t))\,dxdy\right)\\ &+ ||\mathbf{u}||_{H^4}\,\mathfrak{E}_{k+1}(t)\end{aligned} \tag{3.31}$$

with

$$\begin{aligned}\Psi_1(t) &:= ||\rho||_{H^{k+1}} + ||\mathbf{u}||_{H^k} + \left(\frac{1 + ||\partial_y \tilde{\rho}||_{L^\infty}}{1 - ||\partial_y \tilde{\rho}||_{L^\infty}}\right)^{1/2}\left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1 + \partial_y \tilde{\rho})\,dxdy\right)^{1/2},\\ \Psi_2(t) &:= \frac{||\rho||_{H^{k+1}} + ||\mathbf{u}||_{H^k} + ||\rho||_{H^{k+1}}||\mathbf{u}||_{H^k}}{1 - ||\partial_y \tilde{\rho}||_{L^\infty}}.\end{aligned}$$

As we want to prove a global existence result for small data, this is $\mathfrak{E}_{k+1}(t) \ll 1$, the first two terms in the energy estimate (3.31) are "good" ones, because they have the right sign. In consequence, we fix our attention in the last term. If we have a "good" time decay of $||\mathbf{u}||_{H^4}(t)$, we will be able to prove that $\mathfrak{E}_{k+1}(t)$ remains small for all time by a boostraping argument.

Then, we are now in a position to obtain the previous energy estimate. To do this, we study the time evolution of $E_k(t)$ and $\dot{\mathcal{E}}_{k+1}(t)$ independently.

3.5.2.1 $E_k(t)$ Energy Estimate

To do this we use the system (3.9). We start proving the following statement.

Lemma 3.5.2. *The next equality holds:*

$$\begin{aligned}\partial_t E_k(t) = &- \|\mathbf{u}\|^2_{H^k} - \|\partial_t \mathbf{u}\|^2_{H^k} \\ &- (\mathbf{u}\cdot\nabla\rho, \rho) - (\partial^k(\mathbf{u}\cdot\nabla\rho), \partial^k\rho) \\ &- \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - \langle \partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle \\ &- (\partial_t u_2, \mathbf{u}\cdot\nabla\rho) - (\partial^k\partial_t u_2, \partial^k(\mathbf{u}\cdot\nabla\rho)) \\ &- \langle \partial_t\mathbf{u}, \partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle - \langle \partial^k\partial_t\mathbf{u}, \partial^k\partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned} \tag{3.32}$$

Proof. First of all, we remember the definition of $E_k(t)$. Then, we split the proof in two parts. On one hand, we are able to prove that

$$\begin{aligned}\tfrac{1}{2}\partial_t \left\{\|\mathbf{u}\|^2_{H^k} + \|\rho\|^2_{H^k}\right\} = &- \|\mathbf{u}\|^2_{H^k} \\ &- (\mathbf{u}\cdot\nabla\rho, \rho) - (\partial^k(\mathbf{u}\cdot\nabla\rho), \partial^k\rho) \\ &- \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - \langle \partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned} \tag{3.33}$$

On the other hand, we will prove that

$$\begin{aligned}\tfrac{1}{2}\partial_t\{\|\partial_t\mathbf{u}\|^2_{H^k} + \|u_2\|^2_{H^k}\} = &- \|\partial_t\mathbf{u}\|^2_{H^k} \\ &- (\partial_t u_2, \mathbf{u}\cdot\nabla\rho) - (\partial^k\partial_t u_2, \partial^k(\mathbf{u}\cdot\nabla\rho)) \\ &- \langle \partial_t\mathbf{u}, \partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle - \langle \partial^k\partial_t\mathbf{u}, \partial^k\partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned} \tag{3.34}$$

By putting together (3.33) and (3.34), we achieve our goal. To prove (3.33), we start with the L^2 norm. One can check that

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\mathbf{u}\|^2_{L^2} &= \langle \mathbf{u}, \partial_t\mathbf{u}\rangle = \langle \mathbf{u}, -\nabla P + (0,\rho) - \mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle \\ &= \langle \mathbf{u}, -\nabla P + (0,\rho)\rangle - \|\mathbf{u}\|^2_{L^2} - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle.\end{aligned}$$

Then, by the incompressibility we get

$$\tfrac{1}{2}\partial_t\|\mathbf{u}\|^2_{L^2} = (u_2, \rho) - \|\mathbf{u}\|^2_{L^2} - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle.$$

As $\partial_t\rho + \mathbf{u}\cdot\nabla\rho = -u_2$, we obtain that

$$\tfrac{1}{2}\partial_t\|\mathbf{u}\|^2_{L^2} = -\tfrac{1}{2}\partial_t\|\rho\|^2_{L^2} - (\mathbf{u}\cdot\nabla\rho, \rho) - \|\mathbf{u}\|^2_{L^2} - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle$$

and consequently, we have proved that

$$\tfrac{1}{2}\partial_t\left\{\|\mathbf{u}\|^2_{L^2} + \|\rho\|^2_{L^2}\right\} = -\|\mathbf{u}\|^2_{L^2} - (\mathbf{u}\cdot\nabla\rho, \rho) - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle. \tag{3.35}$$

Doing the same computation in $\dot{H}^k$ we get

$$\begin{aligned}\tfrac{1}{2}\partial_t\left\{\|\mathbf{u}\|^2_{\dot{H}^k} + \|\rho\|^2_{\dot{H}^k}\right\} = &- \|\mathbf{u}\|^2_{\dot{H}^k} \\ &- (\partial^k(\mathbf{u}\cdot\nabla\rho), \partial^k\rho) - \langle \partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned} \tag{3.36}$$

By putting together (3.35) and (3.36), we obtain (3.33). To prove (3.34), we start again with the L^2 norm. One can check that

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\partial_t\mathbf{u}\|^2_{L^2} &= \langle \partial_t\mathbf{u}, \partial_t^2\mathbf{u}\rangle = \langle \partial_t\mathbf{u}, \partial_t[-\nabla P + (0,\rho)] - \partial_t\mathbf{u} - \partial_t\,(\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle \\ &= \langle \partial_t\mathbf{u}, \partial_t[-\nabla P + (0,\rho)]\rangle - \|\partial_t\mathbf{u}\|^2_{L^2} - \langle \partial_t\mathbf{u}, \partial_t\,(\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle.\end{aligned}$$

As before, by the incompressibility we get

$$\tfrac{1}{2}\partial_t \|\partial_t \mathbf{u}\|_{L^2}^2 = (\partial_t u_2, \partial_t \rho) - \|\partial_t \mathbf{u}\|_{L^2}^2 - \langle \partial_t \mathbf{u}, \partial_t [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.$$

As $\partial_t \rho = -u_2 - \mathbf{u}\cdot\nabla\rho$, we obtain that

$$\tfrac{1}{2}\partial_t \left\{\|\partial_t \mathbf{u}\|_{L^2}^2 + \|u_2\|_{L^2}^2\right\} = -\|\partial_t \mathbf{u}\|_{L^2}^2 - (\partial_t u_2, \mathbf{u}\cdot\nabla\rho) - \langle \partial_t \mathbf{u}, \partial_t [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle. \tag{3.37}$$

We can proceed similarly in $\dot{H}^k$ and get

$$\begin{aligned}\tfrac{1}{2}\partial_t \left\{\|\partial_t \mathbf{u}\|_{\dot{H}^k}^2 + \|u_2\|_{\dot{H}^k}^2\right\} =& -\|\partial_t \mathbf{u}\|_{\dot{H}^k}^2 \\ & - (\partial^k\partial_t u_2, \partial^k(\mathbf{u}\cdot\nabla\rho)) - \langle \partial^k\partial_t \mathbf{u}, \partial^k\partial_t [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned} \tag{3.38}$$

By putting together (3.38) and (3.37), we obtain (3.34). In consequence, we have proved our estimation. □

Next, we manipulate the quadratic terms of (3.32) to be able to control the cubic ones. Our goal here is to use our velocity evolution equation to control the signed term $-\left[\|\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right]$ by the following one, $-C\left[\|\mathbf{u}\|_{H^k}^2 + \|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2 + \|(\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right]$ with $0 < C < 1$. To do this, we have to pay with a remainder, which we will be able to control for small data.

More specifically, we can prove the following lemma, which is a key step in our proof.

Lemma 3.5.3. *There exists* $0 < C < 1$ *such that:*

$$\begin{aligned}-\left[\|\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right] \leqslant& -C\left[\|\mathbf{u}\|_{H^k}^2 + \|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2 + \|(\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right] \\ &+ \langle [(\mathbf{u}\cdot\nabla)\,\mathbf{u}], -\nabla\Pi + (0,\bar{\rho})\rangle - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle \\ &+ \langle \partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}], \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle - \langle \partial^k\mathbf{u}, \partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle.\end{aligned}$$

Proof. First of all, we use $\partial_t \mathbf{u} = -\nabla\Pi + (0,\bar{\rho}) - \mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{u}$, so we can rewrite:

$$\begin{aligned}-\|\partial_t \mathbf{u}\|_{H^k}^2 =& -\|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2 - \|\mathbf{u}\|_{H^k}^2 - \|(\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^k}^2 \\ &+ 2\langle \mathbf{u}, -\nabla\Pi + (0,\bar{\rho})\rangle + 2\langle (\mathbf{u}\cdot\nabla)\,\mathbf{u}, -\nabla\Pi + (0,\bar{\rho})\rangle - 2\langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle \\ &+ 2\langle \partial^k\mathbf{u}, \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle + 2\langle \partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}], \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle - 2\langle \partial^k\mathbf{u}, \partial^k(\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle.\end{aligned} \tag{3.39}$$

Then, we split the linear part as follows

$$-\left[\|\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right] = -\left[\|\mathbf{u}\|_{H^k}^2 + \tfrac{1}{2}\|\partial_t \mathbf{u}\|_{H^k}^2 + \tfrac{1}{2}\|\partial_t \mathbf{u}\|_{H^k}^2\right], \tag{3.40}$$

and combining equation (3.39) with (3.40) in an adequate way, we get:

$$\begin{aligned}-\left[\|\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right] =& -\tfrac{3}{2}\|\mathbf{u}\|_{H^k}^2 - \tfrac{1}{2}\|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2 - \tfrac{1}{2}\|(\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^k}^2 - \tfrac{1}{2}\|\partial_t \mathbf{u}\|_{H^k}^2 \\ &+ \langle \mathbf{u}, -\nabla\Pi + (0,\bar{\rho})\rangle + \langle (\mathbf{u}\cdot\nabla)\,\mathbf{u}, -\nabla\Pi + (0,\bar{\rho})\rangle - \langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle \\ &+ \langle \partial^k\mathbf{u}, \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle + \langle \partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}], \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle \\ &- \langle \partial^k\mathbf{u}, \partial^k(\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle.\end{aligned} \tag{3.41}$$

By Young's inequality it is clear that there exists $0 < \epsilon < 1$ such that:

$$\langle \mathbf{u}, -\nabla\Pi + (0,\bar{\rho})\rangle + \langle \partial^k\mathbf{u}, \partial^k[-\nabla\Pi + (0,\bar{\rho})]\rangle \leqslant \frac{1}{2}\left(\frac{\|\mathbf{u}\|_{H^k}^2}{\epsilon} + \epsilon\|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2\right).$$

Combining this with the above estimate when $1/3 < \epsilon < 1$ yields our lemma. □

Now, we combine (3.32) and Lemma 3.5.3 to get:

$$\begin{aligned}\partial_t E_k(t) \leqslant & -C\left[\|\mathbf{u}\|_{H^k}^2 + \|-\nabla\Pi + (0,\bar{\rho})\|_{H^k}^2 + \|(\mathbf{u}\cdot\nabla)\,\mathbf{u}\|_{H^k}^2 + \|\partial_t \mathbf{u}\|_{H^k}^2\right] \\ & + I^1 + I^2 + I^3 + I^4 + I^5\end{aligned} \tag{3.42}$$

with

$$\begin{aligned}
I^1 &:= -(\partial_t u_2, \mathbf{u}\cdot\nabla\rho) - (\partial^k \partial_t u_2, \partial^k(\mathbf{u}\cdot\nabla\rho)),\\
I^2 &:= -\langle \partial_t \mathbf{u}, \partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle - \langle \partial^k\partial_t\mathbf{u}, \partial^k\partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle,\\
I^3 &:= -2\,\langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - 2\,\langle \partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle,\\
I^4 &:= -\langle(\mathbf{u}\cdot\nabla)\,\mathbf{u}, \nabla\Pi - (0,\bar{\rho})\rangle - \ \langle \partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}], \partial^k[\nabla\Pi - (0,\bar{\rho})]\rangle,\\
I^5 &:= -(\mathbf{u}\cdot\nabla\rho, \rho) - (\partial^k(\mathbf{u}\cdot\nabla\rho), \partial^k\rho).
\end{aligned}$$

3.5.2.2 $\dot{\mathcal{E}}_{k+1}(t)$ Energy Estimate

To do this we use the system (3.10). We start proving the following statement.

Lemma 3.5.4. *The next equality holds:*

$$\begin{aligned}
\partial_t \dot{\mathcal{E}}_{k+1}(t) = & -\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\\
& -\int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]\ (1+\partial_y\tilde{\rho})\,dxdy\\
& +\int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\Pi\,\partial_y^2\tilde{\rho}\,dxdy\\
& -\int_\Omega \partial^{k+1}\left(\mathbf{u}\cdot\nabla\bar{\rho}\right)\ \partial^{k+1}\bar{\rho}\,dxdy\\
& +\int_\Omega \big((1+\partial_y\tilde{\rho})\,\partial^{k+1}u_2 - \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\big)\ \partial^{k+1}\bar{\rho}\,dxdy\\
& +\tfrac{1}{2}\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,\partial_t\partial_y\tilde{\rho}\,dxdy\\
& +\int_\Omega \partial^{k+2}\tilde{\rho}\,\partial^{k+1}\left(u_2\,\bar{\rho}\right)\ dxdy.
\end{aligned}$$

Proof. First of all, we start with the weighted term of $\dot{\mathcal{E}}_{k+1}(t)$. The estimation of such a term requires a long splitting into several controlled terms.

$$\tfrac{1}{2}\,\partial_t\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy = \int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\partial_t\mathbf{u}\,(1+\partial_y\tilde{\rho})\,dxdy + \tfrac{1}{2}\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,\partial_t\partial_y\tilde{\rho}\,dxdy.$$

As $\partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla\Pi + (0,\bar{\rho})$ we obtain that:

$$
\begin{aligned}
\tfrac{1}{2}\,\partial_t \int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy = &-\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\\
&-\int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]\,(1+\partial_y\tilde{\rho})\,dxdy\\
&-\int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\nabla\Pi\,(1+\partial_y\tilde{\rho})\,dxdy\\
&+\int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\bar{\rho}\,(1+\partial_y\tilde{\rho})\,dxdy\\
&+\tfrac{1}{2}\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,\partial_t\partial_y\tilde{\rho}\,dxdy.
\end{aligned}
$$

Since $\nabla\cdot\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, using integration by parts in the third term gives:

$$
-\int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\nabla\Pi\,(1+\partial_y\tilde{\rho})\,dxdy = \int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\Pi\,\partial_y^2\tilde{\rho}\,dxdy. \tag{3.43}
$$

By the periodicity in the $x-$variable, it is clear that the only boundary term that needs to be studied carefully is the one associated with the $y-$variable, which vanishes because $u_2 \in X^{k+1}(\Omega)$ and $\Pi \in Y^{k+1}(\Omega)$. Now, we focus in the fourth term, which can be written as:

$$
\begin{aligned}
\int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\bar{\rho}\,(1+\partial_y\tilde{\rho})\,dxdy = &\int_\Omega \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\,\partial^{k+1}\bar{\rho}\,dxdy\\
&+\int_\Omega \left((1+\partial_y\tilde{\rho})\,\partial^{k+1}u_2 - \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\right)\,\partial^{k+1}\bar{\rho}\,dxdy
\end{aligned}
$$

and, as $\partial_t\bar{\rho} + \overline{\mathbf{u}\cdot\nabla\bar{\rho}} = -(1+\partial_y\tilde{\rho})\,u_2$ we get:

$$
\begin{aligned}
\int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\bar{\rho}\,(1+\partial_y\tilde{\rho})\,dxdy = &-\tfrac{1}{2}\partial_t\int_\Omega |\partial^{k+1}\bar{\rho}|^2\,dxdy - \int_\Omega \partial^{k+1}\left(\mathbf{u}\cdot\nabla\bar{\rho}\right)\,\partial^{k+1}\bar{\rho}\,dxdy\\
&+\int_\Omega \left((1+\partial_y\tilde{\rho})\,\partial^{k+1}u_2 - \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\right)\,\partial^{k+1}\bar{\rho}\,dxdy
\end{aligned} \tag{3.44}
$$

where in the second integral, we have used $\widetilde{\mathbf{u}\cdot\nabla\bar{\rho}} \perp \bar{\rho}$. Therefore, putting (3.43) and (3.44) together, we obtain:

$$
\begin{aligned}
\tfrac{1}{2}\partial_t\left\{\|\bar{\rho}\|^2_{H^{k+1}} + \int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\right\} = &-\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\\
&-\int_\Omega \partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]\,(1+\partial_y\tilde{\rho})\,dxdy\\
&+\int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\Pi\,\partial_y^2\tilde{\rho}\,dxdy\\
&-\int_\Omega \partial^{k+1}\left(\mathbf{u}\cdot\nabla\bar{\rho}\right)\,\partial^{k+1}\bar{\rho}\,dxdy\\
&+\int_\Omega \left((1+\partial_y\tilde{\rho})\,\partial^{k+1}u_2 - \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\right)\,\partial^{k+1}\bar{\rho}\,dxdy\\
&+\tfrac{1}{2}\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,\partial_t\partial_y\tilde{\rho}\,dxdy.
\end{aligned} \tag{3.45}
$$

To prove the desired inequality we need to study the evolution in time of $\tilde{\rho}$. Note that $\tilde{\rho}(y,t)$ doesn't depend on the horizontal variable. And by the orthogonality, as $\bar{\rho} \perp \tilde{\rho}$ it is clear that:

$$\|\rho\|^2_{\dot{H}^{k+1}(\Omega)} = \|\bar{\rho}\|^2_{\dot{H}^{k+1}(\Omega)} + 2\pi \|\tilde{\rho}\|^2_{\dot{H}^{k+1}([-1,1])}.$$

As $\nabla \cdot \mathbf{u} = 0$, it is simple to see that $\widetilde{\mathbf{u} \cdot \nabla\bar{\rho}} = \partial_y \widetilde{(u_2\,\bar{\rho})}$, and by integration by parts we get:

$$\begin{aligned}
\tfrac{1}{2}\partial_t \int_{-1}^{1} |\partial_y^{k+1}\tilde{\rho}|^2\, dy &= \int_{-1}^{1} \partial_y^{k+1}\tilde{\rho}\, \partial_y^{k+1}\partial_t\tilde{\rho}\, dy = -\int_{-1}^{1} \partial_y^{k+1}\tilde{\rho}\, \partial_y^{k+1}(\widetilde{\mathbf{u}\cdot\nabla\bar{\rho}})\, dy \\
&= \int_{-1}^{1} \partial_y^{k+2}\tilde{\rho}\, \partial_y^{k+1}\widetilde{(u_2\,\bar{\rho})}\, dy - \partial_y^{k+1}\tilde{\rho}\, \partial_y^{k+1}\widetilde{(u_2\,\bar{\rho})}\Big|_{y=-1}^{y=1} \\
&= \int_{-1}^{1} \partial_y^{k+2}\tilde{\rho}\, \partial_y^{k+1}\widetilde{(u_2\,\bar{\rho})}\, dy
\end{aligned}$$

where, in the last step, we have used that $\tilde{\rho} \in X^k([-1,1])$ and $\widetilde{u_2\,\bar{\rho}} \in Y^k([-1,1])$. As $\overline{(u_2\,\bar{\rho})} \perp \tilde{\rho}$ we have proved that:

$$2\pi\,\partial_t\|\tilde{\rho}\|^2_{\dot{H}^{k+1}([-1,1])} = \tfrac{1}{2}\partial_t \int_{\mathbb{T}}\int_{-1}^{1} |\partial_y^{k+1}\tilde{\rho}|^2\, dxdy = \int_{\Omega} \partial_y^{k+2}\tilde{\rho}\, \partial_y^{k+1}\left(u_2\,\bar{\rho}\right)\, dxdy. \tag{3.46}$$

If we put (3.46) in (3.45) we obtain the claimed equality. □

Combining the estimates for $E_k(t)$ and $\dot{\mathcal{E}}_{k+1}(t)$ given by (3.42) and Lemma 3.5.4, we have proved that there exists $0 < C < 1$ such that:

$$\begin{aligned}
\partial_t\mathfrak{E}_{k+1}(t) \leqslant &-C\left[\|\mathbf{u}\|^2_{H^k} + \|\nabla\Pi - (0,\bar{\rho})\|^2_{H^k} + \|\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\|^2_{H^k} + \|\partial_t\mathbf{u}\|^2_{H^k}\right] - \int_{\Omega} |\partial^{k+1}\mathbf{u}|^2\left(1+\partial_y\tilde{\rho}\right) dxdy \\
&+ I^1 + I^2 + I^3 + I^4 + I^5 + I^6 + I^7 + I^8 + I^9
\end{aligned}$$

with

$$\begin{aligned}
I^1 &:= -(\partial_t u_2, \mathbf{u}\cdot\nabla\rho) - (\partial^k\partial_t u_2, \partial^k(\mathbf{u}\cdot\nabla\rho)), \\
I^2 &:= -\langle\partial_t\mathbf{u}, \partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle - \langle\partial^k\partial_t\mathbf{u}, \partial^k\partial_t[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle, \\
I^3 &:= -2\,\langle\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - 2\,\langle\partial^k\mathbf{u}, \partial^k\,[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle, \\
I^4 &:= -\langle(\mathbf{u}\cdot\nabla)\,\mathbf{u}, \nabla\Pi - (0,\bar{\rho})\rangle - \langle\partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}], \partial^k[\nabla\Pi - (0,\bar{\rho})]\rangle, \\
I^5 &:= -(\mathbf{u}\cdot\nabla\rho, \rho) - (\partial^k(\mathbf{u}\cdot\nabla\rho), \partial^k\rho), \\
I^6 &:= \int_{\Omega}\left((1+\partial_y\tilde{\rho})\,\partial^{k+1}u_2 - \partial^{k+1}\left((1+\partial_y\tilde{\rho})\,u_2\right)\right)\,\partial^{k+1}\bar{\rho}\, dxdy + \int_{\Omega}\partial^{k+2}\tilde{\rho}\,\partial^{k+1}\left(u_2\,\bar{\rho}\right)\, dxdy, \\
I^7 &:= \int_{\Omega}\partial^{k+1}u_2\,\partial^{k+1}\Pi\,\partial_y^2\tilde{\rho}\, dxdy + \tfrac{1}{2}\int_{\Omega}|\partial^{k+1}\mathbf{u}|^2\,\partial_t\partial_y\tilde{\rho}\, dxdy, \\
I^8 &:= -\int_{\Omega}\partial^{k+1}\mathbf{u}\cdot\partial^{k+1}\left[(\mathbf{u}\cdot\nabla)\,\mathbf{u}\right]\,\left(1+\partial_y\tilde{\rho}\right) dxdy, \\
I^9 &:= -\int_{\Omega}\partial^{k+1}\left(\mathbf{u}\cdot\nabla\bar{\rho}\right)\,\partial^{k+1}\bar{\rho}\, dxdy.
\end{aligned}$$

Before moving on to study each term $\{I^m\}_{m=1}^9$ separately, we make the following simple observations:

1. Let $f \in L^2(\mathbb{T})$ with zero average. Then, we have that:

$$\|f\|_{L^2(\mathbb{T})} \leqslant \|\partial_x f\|_{L^2(\mathbb{T})}. \tag{3.47}$$

Proof. The proof is an immediate consequence of Plancherel's theorem. As f has zero average, in the *Fourier side*, this means that $\hat{f}(0) = 0$. Then

$$||f||^2_{L^2(\mathbb{T})} = \sum_{k \in \mathbb{Z} \neq 0} |\hat{f}(k)|^2 \leqslant \sum_{k \in \mathbb{Z} \neq 0} |(ik)\hat{f}(k)|^2 = ||\partial_x f||^2_{L^2(\mathbb{T})}.$$

$\square$

2. As $\bar{\rho} := \rho - \tilde{\rho}$ has zero average in the horizontal variable, for $n \in \mathbb{N} \cup \{0\}$ we get:

$$||\bar{\rho}||_{H^n(\Omega)} \leqslant ||\nabla\Pi - (0, \bar{\rho})||_{H^{n+1}(\Omega)}. \tag{3.48}$$

Proof. For simplicity, we do the computation in $L^2(\Omega) \equiv H^0(\Omega)$, but the same argument can be repeated in $H^n(\Omega)$ with $n \in \mathbb{N}$. By (3.47) we obtain that $||\bar{\rho}||_{L^2(\Omega)} \leqslant ||\partial_x \bar{\rho}||_{L^2(\Omega)}$ and consequently we get:

$$\begin{aligned} ||\bar{\rho}||_{L^2(\Omega)} \leqslant ||\partial_x \bar{\rho}||_{L^2(\Omega)} &= ||\partial_x \left(\bar{\rho} \pm \partial_y \Pi\right)||_{L^2(\Omega)} \leqslant ||\partial_x (\bar{\rho} - \partial_y \Pi)||_{L^2(\Omega)} + ||\partial_y \partial_x \Pi||_{L^2(\Omega)} \\ &\leqslant ||\bar{\rho} - \partial_y \Pi||_{H^1(\Omega)} + ||\partial_x \Pi||_{H^1(\Omega)} = ||\nabla\Pi - (0, \bar{\rho})||_{H^1(\Omega)}. \end{aligned}$$

$\square$

3. The second component of the velocity $u_2(t)$ has zero average in the horizontal variable. This is:

$$u_2(t) = \bar{u}_2(t) \quad \text{or} \quad \tilde{u}_2(t) = 0. \tag{3.49}$$

Proof. By the periodicity in the horizontal variable and the incompressibility of the velocity, we get:

$$0 = \int_{\mathbb{T}} (\nabla \cdot \mathbf{u})\,(x', y, t)\, dx' = \partial_y \tilde{u}_2(y, t) \quad \Longrightarrow \quad \tilde{u}_2(y, t) = \beta(t).$$

Moreover, by the no-slip condition, we have $\tilde{u}_2(t)|_{\partial\Omega} = 0$ and in consequence $\beta(t) = 0$. $\square$

With all these tools in mind, it is time to prove:

Lemma 3.5.5. *The following estimates hold for $k \geqslant 5$:*

1. $I^1 \lesssim \left(||\partial_t \mathbf{u}||^2_{H^k} + ||\mathbf{u}||^2_{H^k}\right) ||\rho||_{H^{k+1}}$
2. $I^2 \lesssim ||\partial_t \mathbf{u}||^2_{H^k} ||\mathbf{u}||_{H^{k+1}}$
3. $I^3 \lesssim ||\mathbf{u}||^3_{H^k}$
4. $I^4 \lesssim \left(||\nabla\Pi - (0, \bar{\rho})||^2_{H^k} + ||\mathbf{u}||^2_{H^k}\right) ||\mathbf{u}||_{H^{k+1}}$
5. $I^5 \lesssim ||\rho||_{H^{k+1}} \left(||\mathbf{u}||^2_{H^k} + ||\nabla\Pi - (0, \bar{\rho})||^2_{H^k}\right) + ||\nabla\Pi - (0, \bar{\rho})||_{H^k} ||\mathbf{u}||_{H^{k+1}} ||\rho||_{H^k}$

Proof.

(1) If we add and subtract $\mathbf{u} \cdot \nabla \partial^k \rho$ in the second term, we obtain that:

$$I^1 = -(\partial_t u_2, \mathbf{u} \cdot \nabla\rho) - (\partial^k \partial_t u_2, \partial^k(\mathbf{u} \cdot \nabla\rho) - \mathbf{u} \cdot \nabla \partial^k \rho) - (\partial^k \partial_t u_2, \mathbf{u} \cdot \nabla \partial^k \rho).$$

Using (3.19) with $f = \mathbf{u},\ g = \nabla\rho$ and the Sobolev embedding $L^\infty(\Omega) \hookrightarrow H^2(\Omega)$ it is easy to see for $k \geqslant 3$ that:

$$\begin{aligned}
I^1 &\leqslant \|\partial_t u_2\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^2} + \|\partial^k\partial_t u_2\|_{L^2}\|\partial^k(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot\nabla\partial^k\rho\|_{L^2} + \|\partial^k\partial_t u_2\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla\partial^k\rho\|_{L^2}\\
&\lesssim \|\partial_t\mathbf{u}\|_{H^k}\|\mathbf{u}\|_{H^k}\|\rho\|_{H^{k+1}} \leqslant \left(\|\partial_t\mathbf{u}\|^2_{H^k} + \|\mathbf{u}\|^2_{H^k}\right)\|\rho\|_{H^{k+1}}.
\end{aligned}$$

(2) It is clear that we can rewrite I^2 as follows:

$$\begin{aligned}
I^2 = &-\langle\partial_t\mathbf{u}, (\partial_t\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - \langle\partial_t\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\partial_t\mathbf{u}\rangle\\
&- \langle\partial^k\partial_t\mathbf{u}, \partial^k[(\partial_t\mathbf{u}\cdot\nabla)\,\mathbf{u}] - (\partial_t\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\rangle - \langle\partial^k\partial_t\mathbf{u}, (\partial_t\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\rangle\\
&- (\partial^k\partial_t\mathbf{u}, \partial^k[(\mathbf{u}\cdot\nabla)\,\partial_t\mathbf{u}] - (\mathbf{u}\cdot\nabla)\,\partial^k\partial_t\mathbf{u}) - (\partial^k\partial_t\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\partial^k\partial_t\mathbf{u})
\end{aligned}$$

and since $\nabla\cdot\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, the last term vanishes. Then, we have:

$$\begin{aligned}
I_2 \lesssim{}& \|\partial_t\mathbf{u}\|^2_{L^2}\|\nabla\mathbf{u}\|_{L^\infty} + \|\partial_t\mathbf{u}\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty} + \|\partial^k\partial_t\mathbf{u}\|_{L^2}\|\partial_t\mathbf{u}\|_{L^\infty}\|\nabla\partial^k\mathbf{u}\|_{L^2}\\
&+ \|\partial^k\partial_t\mathbf{u}\|_{L^2}\left(\|\partial^k[(\partial_t\mathbf{u}\cdot\nabla)\,\mathbf{u}] - (\partial_t\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\|_{L^2} + \|\partial^k[(\mathbf{u}\cdot\nabla)\,\partial_t\mathbf{u}] - (\mathbf{u}\cdot\nabla)\,\partial^k\partial_t\mathbf{u}\|_{L^2}\right).
\end{aligned}$$

As before, by (3.19) with $f = \partial_t\mathbf{u},\ g = \nabla\mathbf{u}$ or $f = \mathbf{u},\ g = \nabla\partial_t\mathbf{u}$ and the Sobolev embedding for $k \geqslant 3$ we get:

$$I^2 \lesssim \|\partial_t\mathbf{u}\|^2_{H^k}\|\mathbf{u}\|_{H^{k+1}}.$$

(3) By definition, we have that $I^3 = -2\,\langle\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - 2\,\langle\partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\rangle$. If we add and subtract $\mathbf{u}\cdot\nabla\partial^k\mathbf{u}$ in the second term we obtain that:

$$I^3 = -2\,\langle\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\mathbf{u}\rangle - 2\,\langle\partial^k\mathbf{u}, (\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\rangle - 2\,\langle\partial^k\mathbf{u}, \partial^k\ [(\mathbf{u}\cdot\nabla)\,\mathbf{u}] - (\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\rangle$$

and since $\nabla\cdot\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, the first two terms vanish. Again by (3.19) with $f = \mathbf{u},\ g = \nabla\mathbf{u}$ and the Sobolev embedding we get for $k \geqslant 3$ that:

$$I^3 \lesssim \|\mathbf{u}\|^3_{H^k}.$$

(4) We rewrite I^4 in a more adequate way:

$$\begin{aligned}
I^4 = &-\langle(\mathbf{u}\cdot\nabla)\,\mathbf{u}, \nabla\Pi - (0,\bar\rho)\rangle - \langle(\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}, \partial^k[\nabla\Pi - (0,\bar\rho)]\rangle\\
&- \langle\partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}] - (\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}, \partial^k[\nabla\Pi - (0,\bar\rho)]\rangle.
\end{aligned}$$

Then, we have:

$$\begin{aligned}
I^4 \leqslant{}& \|\mathbf{u}\|_{L^\infty}\left(\|\nabla\mathbf{u}\|_{L^2}\|\nabla\Pi - (0,\bar\rho)\|_{L^2} + \|\nabla\partial^k\mathbf{u}\|_{L^2}\|\partial^k\,[\nabla\Pi - (0,\bar\rho)]\,\|_{L^2}\right)\\
&+ \|\partial^k[(\mathbf{u}\cdot\nabla)\,\mathbf{u}] - (\mathbf{u}\cdot\nabla)\,\partial^k\mathbf{u}\|_{L^2}\|\partial^k[\nabla\Pi - (0,\bar\rho)]\|_{L^2}.
\end{aligned}$$

As before, by (3.19) with $f = \mathbf{u},\ g = \nabla\mathbf{u}$ and the Sobolev embedding we get for $k \geqslant 3$:

$$I^4 \lesssim \|\nabla\Pi - (0,\bar\rho)\|_{H^k}\|\mathbf{u}\|_{H^k}\|\mathbf{u}\|_{H^{k+1}} \leqslant \left(\|\nabla\Pi - (0,\bar\rho)\|^2_{H^k} + \|\mathbf{u}\|^2_{H^k}\right)\|\mathbf{u}\|_{H^{k+1}}.$$

(5) Again, since $\nabla\cdot\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ we obtain that

$$I^5 = -(\partial^k(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot\nabla\partial^k\rho, \partial^k\rho)$$

and by (3.19) with $f = \mathbf{u},\ g = \nabla\rho$ and the Sobolev embedding we get for $k \geqslant 3$:

$$I^5 \lesssim \|\mathbf{u}\|_{H^k}\|\rho\|^2_{H^k}.$$

The above estimate is too crude, we will need to carry out the energy estimates carefully to ensure that we get the desired estimate. We shall see below that the property (3.48) is the key to close the right estimates.

The usual method of using the Leibniz's rule gives us:

$$\begin{aligned} I^5 =& -\sum_{j=0}^{k-1}\binom{k}{j}(\partial^{j+1}u_1\,\partial^{k-1-j}\partial_x\rho,\partial^k\rho)-\sum_{j=0}^{k-1}\binom{k}{j}(\partial^{j+1}u_2\,\partial^{k-1-j}\partial_y\rho,\partial^k\rho)\\ =&A_1+A_2. \end{aligned}$$

For the first one, as $\partial_x\rho=\partial_x\bar{\rho}$, Hölder's inequality for $k\geqslant 4$ gives us:

$$\begin{aligned} A_1 &\lesssim \|\rho\|_{H^k}\sum_{j=0}^{k-1}\|\partial^{j+1}u_1\,\partial^{k-1-j}\partial_x\bar{\rho}\|_{L^2}\\ &=\|\rho\|_{H^k}\left[\sum_{j=0}^{1}\|\partial^{j+1}u_1\|_{L^\infty}\|\partial^{k-1-j}\partial_x\bar{\rho}\|_{L^2}+\sum_{j=2}^{k-1}\|\partial^{j+1}u_1\|_{L^2}\|\partial^{k-1-j}\partial_x\bar{\rho}\|_{L^\infty}\right]\\ &\leqslant \|\rho\|_{H^k}\|u_1\|_{H^k}\|\partial_x\bar{\rho}\|_{H^{k-1}}\lesssim\|\rho\|_{H^k}\left(\|\mathbf{u}\|_{H^k}^2+\|\nabla\Pi-(0,\bar{\rho})\|_{H^k}^2\right). \end{aligned} \tag{3.50}$$

For the other one, as $u_2\equiv\bar{u}_2$ by (3.49), we have that $u_2\perp\tilde{\rho}$ and consequently:

$$\begin{aligned} A_2 &= -\sum_{j=0}^{k-1}\binom{k}{j}\left\{(\partial^{j+1}u_2\,\partial^{k-1-j}\partial_y\bar{\rho},\partial^k\rho)+(\partial^{j+1}u_2\,\partial_y^{k-j}\tilde{\rho},\partial_y^{k-1-j}\partial^{j+1}\bar{\rho})\right\}\\ &=A_2^1+A_2^2 \end{aligned}$$

where

$$\begin{aligned} A_2^1 &:= -\sum_{j=1}^{k-1}\binom{k}{j}(\partial^{j+1}u_2\,\partial^{k-1-j}\partial_y\bar{\rho},\partial^k\rho),\\ A_2^2 &:= -(\partial u_2\,\partial^{k-1}\partial_y\bar{\rho},\partial^k\rho)-\sum_{j=0}^{k-1}\binom{k}{j}(\partial^{j+1}u_2\,\partial_y^{k-j}\tilde{\rho},\partial_y^{k-1-j}\partial^{j+1}\bar{\rho}). \end{aligned}$$

Now, for A_2^1 repeatedly applying Hölder's inequality, we get:

$$A_2^1\lesssim\|\rho\|_{H^k}\left[\sum_{j=1}^{2}\|\partial^{j+1}u_2\|_{L^\infty}\|\partial^{k-1-j}\partial_y\bar{\rho}\|_{L^2}+\sum_{j=3}^{k-1}\|\partial^{j+1}u_2\|_{L^2}\|\partial^{k-1-j}\partial_y\bar{\rho}\|_{L^\infty}\right].$$

So, for $k\geqslant 5$ by the Sobolev embedding it is clear that:

$$A_2^1\leqslant\|\rho\|_{H^k}\|u_2\|_{H^k}\|\bar{\rho}\|_{H^{k-1}}\lesssim\|\rho\|_{H^k}\left(\|\mathbf{u}\|_{H^k}^2+\|\nabla\Pi-(0,\bar{\rho})\|_{H^k}^2\right), \tag{3.51}$$

where we have used (3.48) in a crucial way. For A_2^2, after integration by parts, we get:

$$\begin{aligned} A_2^2 =& \tfrac{1}{c}(\partial_y \left[\partial u_2\, \partial^k \rho\right], \partial^{k-1}\bar{\rho}) - \int_\Omega \partial_y \left[\partial u_2\, \partial^{k-1}\bar{\rho}\, \partial^k \rho\right]\, dxdy \\ &+ \sum_{j=0}^{k-1} \binom{k}{j} (\partial \left[\partial^{j+1} u_2\, \partial_y^{k-j} \tilde{\rho}\right], \partial_y^{k-1-j}\partial^j \bar{\rho}) - \sum_{j=0}^{k-1} \binom{k}{j} \int_\Omega \partial \left[\partial^{j+1} u_2\, \partial_y^{k-j} \tilde{\rho}\, \partial_y^{k-1-j}\partial^j \bar{\rho}\right]\, dxdy \\ =& \tfrac{1}{c}(\partial_y \left[\partial u_2\, \partial^k \rho\right], \partial^{k-1}\bar{\rho}) + \sum_{j=0}^{k-1} \binom{k}{j} (\partial \left[\partial^{j+1} u_2\, \partial_y^{k-j} \tilde{\rho}\right], \partial_y^{k-1-j}\partial^j \bar{\rho}), \end{aligned}$$

because the boundary terms are equal to zero since at least one of the terms that they contains vanishes thanks to the fact $\rho \in X^k(\Omega)$ and in consequence $\bar{\rho} \in X^k(\Omega)$ and $\tilde{\rho} \in X^k([-1,1])$.

After this, for A_2^2 repeatedly applying Hölder's inequality, we get:

$$\begin{aligned} A_2^2 \lesssim& ||\bar{\rho}||_{H^{k-1}} \left(||\partial_y \partial u_2||_{L^\infty} ||\partial^k \rho||_{L^2} + ||\partial u_2||_{L^\infty} ||\partial_y \partial^k \rho||_{L^2}\right) \\ &+ ||\bar{\rho}||_{H^{k-1}} \left(\sum_{j=0}^{k-3} ||\partial^{j+2} u_2||_{L^\infty} ||\partial_y^{k-j} \tilde{\rho}||_{L^2} + \sum_{j=k-2}^{k-1} ||\partial^{j+2} u_2||_{L^2} ||\partial_y^{k-j} \tilde{\rho}||_{L^\infty} \right) \\ &+ ||\bar{\rho}||_{H^{k-1}} \left(\sum_{j=0}^{k-3} ||\partial^{j+1} u_2||_{L^\infty} ||\partial_y^{k-j+1} \tilde{\rho}||_{L^2} + \sum_{j=k-2}^{k-1} ||\partial^{j+1} u_2||_{L^2} ||\partial_y^{k-j+1} \tilde{\rho}||_{L^\infty} \right) \end{aligned}$$

and by (3.48) and the Sobolev embedding $L^\infty(\Omega) \hookrightarrow H^2(\Omega)$ for $k \geqslant 4$ we achieve:

$$\begin{aligned} A_2^2 &\lesssim ||\bar{\rho}||_{H^{k-1}} \left(||\mathbf{u}||_{H^{k+1}} ||\rho||_{H^k} + ||\mathbf{u}||_{H^k} ||\rho||_{H^{k+1}}\right) \\ &\lesssim ||\rho||_{H^{k+1}} \left(||\mathbf{u}||_{H^k}^2 + ||\nabla \Pi - (0,\bar{\rho})||_{H^k}^2\right) + ||\nabla \Pi - (0,\bar{\rho})||_{H^k} ||\mathbf{u}||_{H^{k+1}} ||\rho||_{H^k}. \end{aligned} \tag{3.52}$$

Collecting everything, this is (3.50), (3.51) and (3.52) we have obtained that:

$$I^5 \lesssim ||\rho||_{H^{k+1}} \left(||\mathbf{u}||_{H^k}^2 + ||\nabla \Pi - (0,\bar{\rho})||_{H^k}^2\right) + ||\nabla \Pi - (0,\bar{\rho})||_{H^k} ||\mathbf{u}||_{H^{k+1}} ||\rho||_{H^k}.$$

□

Up to here, we have not used our weighted energy at all. Note that $E_k(t)$ give us control of $||\rho||_{H^k}(t)$ and $||\mathbf{u}||_{H^k}$, so it is natural to define

$$\dot{\mathcal{E}}_{k+1}^{\#}(t) := \tfrac{1}{2} \left\{ ||\rho||_{\dot{H}^{k+1}}^2(t) + \int_\Omega |\partial^{k+1} \mathbf{u}(x,y,t)|^2\, dxdy \right\},$$

but it is not difficult to see that it is impossible to close the energy estimates with it. For this reason, to work with the weighted energy space $\dot{\mathcal{E}}_{k+1}(t)$ is decisive to close the energy estimates. Before that, let's see what we have up to now. As $||\mathbf{u}||_{H^{k+1}} = ||\mathbf{u}||_{H^k} + ||\partial^{k+1}\mathbf{u}||_{L^2}$ we get:

$$||\mathbf{u}||_{H^{k+1}} \leqslant ||\mathbf{u}||_{H^k} + \frac{1}{(1 - ||\partial_y \tilde{\rho}||_{L^\infty})^{1/2}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right)^{1/2} \tag{3.53}$$

and we have that:

$$I^1 + \ldots + I^5 \lesssim \left(||\nabla \Pi - (0,\bar{\rho})||_{H^k}^2 + ||\mathbf{u}||_{H^k}^2 + ||\partial_t \mathbf{u}||_{H^k}^2\right) \tilde{\Theta}_1(t) + \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right) \tilde{\Theta}_2(t)$$

where

$$\tilde{\Theta}_1(t) := \|\rho\|_{H^{k+1}} + \|\mathbf{u}\|_{H^k} + \frac{1}{(1-\|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2}} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \, (1+\partial_y \tilde{\rho}) \, dxdy \right)^{1/2},$$
$$\tilde{\Theta}_2(t) := \frac{\|\rho\|_{H^k}}{1-\|\partial_y \tilde{\rho}\|_{L^\infty}}.$$

In consequence, we have proved that for $k \geqslant 5$ there exists $0 < C < 1$ and $\tilde{C} > 0$ large enough such that:

$$\begin{aligned}
\partial_t \mathfrak{E}_{k+1}(t) \leqslant &- \left(C - \tilde{C}\, \tilde{\Theta}_1(t) \right) \left[\|\mathbf{u}\|^2_{H^k} + \|\nabla \Pi - (0,\bar{\rho})\|^2_{H^k} + \|\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\|^2_{H^k} + \|\partial_t \mathbf{u}\|^2_{H^k} \right] \\
&- \left(1 - \tilde{C}\, \tilde{\Theta}_2(t) \right) \int_\Omega |\partial^{k+1}\mathbf{u}|^2 \, (1+\partial_y \tilde{\rho}) \, dxdy \\
&+ I^6 + I^7 + I^8 + I^9.
\end{aligned} \tag{3.54}$$

The aim of the next part is to make appear nice controlled terms via the use of a useful decomposition of each term $\{I^m\}_{m=6}^9$. Thanks to the weight $1+\partial_y \tilde{\rho}(y,t)$ in the definition of $\dot{\mathcal{E}}_{k+1}(t)$ we are able to control each term in our estimations. This is the goal of the next lemma, which is crucial to prove the main theorem of this section.

Lemma 3.5.6. *The following estimates hold for* $k \geqslant 6$*:*

$$\begin{aligned}
(1) \quad I^6 \lesssim\ & \|\mathbf{u}\|_{H^4} \|\bar{\rho}\|^2_{H^{k+1}} + \|\bar{\rho}\|_{H^{k+1}} \left(\|\mathbf{u}\|^2_{H^k} + \|\nabla\Pi - (0,\bar{\rho})\|^2_{H^k} \right) \\
&+ \frac{\|\bar{\rho}\|_{H^{k+1}}}{1-\|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \, (1+\partial_y \tilde{\rho}) \, dxdy \right); \\
(2) \quad I^7 \lesssim\ & \|\bar{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|^2_{H^k} + \|\mathbf{u}\|^2_{H^k} \right) \\
&+ \frac{(1+\|\mathbf{u}\|_{H^k})\|\rho\|_{H^{k+1}}}{1-\|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \, (1+\partial_y \tilde{\rho}) \, dxdy \right); \\
(3) \quad I^8 \lesssim\ & (1+\|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 (1+\partial_y \tilde{\rho}) \, dxdy \right)^{1/2} \|\mathbf{u}\|^2_{H^k} \\
&+ \frac{\|\mathbf{u}\|_{H^k}\,(1+\|\rho\|_{H^{k+1}})}{1-\|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 (1+\partial_y \tilde{\rho}) \, dxdy \right); \\
(4) \quad I^9 \lesssim\ & \|\mathbf{u}\|_{H^4} \, \|\bar{\rho}\|^2_{H^{k+1}} + \|\bar{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|^2_{H^k} + \|\mathbf{u}\|^2_{H^k} \right) \\
&+ \frac{\|\bar{\rho}\|_{H^{k+1}}}{1-\|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \, (1+\partial_y \tilde{\rho}) \, dxdy \right).
\end{aligned}$$

Proof.

(1) Obviously, the more singular term is I^6. The estimation of such a term requires a long splitting into several controlled terms. By definition we have that:

$$\begin{aligned}
I^6 &= \int_\Omega \left((1+\partial_y \tilde{\rho})\, \partial^{k+1} u_2 - \partial^{k+1}\left((1+\partial_y \tilde{\rho})\, u_2 \right) \right) \partial^{k+1}\bar{\rho} \, dxdy + \int_\Omega \partial^{k+2}\tilde{\rho}\, \partial^{k+1}\left(u_2\, \bar{\rho} \right) \, dxdy \\
&= I^6_1 + I^6_2.
\end{aligned}$$

Applying the chain rule, the terms becomes:

$$I^6_1 = -\int_\Omega \partial_y^{k+2}\tilde{\rho}\ u_2\, \partial_y^{k+1}\bar{\rho} \, dxdy - \sum_{j=1}^{k} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial^{k+1-j} u_2\, \partial^{k+1-j}\partial_y^j \bar{\rho} \, dxdy$$

and

$$I_2^6 = \int_\Omega \partial_y^{k+2}\tilde{\rho}\, u_2\, \partial_y^{k+1}\bar{\rho}\, dxdy + \sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y^{k+2}\tilde{\rho}\, \partial_y^j u_2\, \partial_y^{k+1-j}\bar{\rho}\, dxdy,$$

where we show that the first terms cancel each other out. This cancellation is the key step, that is the crucial point for which we need to work with the weighted energy space $\dot{\mathcal{E}}_{k+1}(t)$. Now, if we work a little more carefully with I_2^6, after integration by parts in the summation we get:

$$\begin{aligned} I_2^6 &= \int_\Omega \partial_y^{k+2}\tilde{\rho}\, u_2\, \partial_y^{k+1}\bar{\rho}\, dxdy - \sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y^{k+1}\tilde{\rho}\, \partial_y \left[\partial_y^j u_2\, \partial_y^{k+1-j}\bar{\rho}\right] dxdy \\ &+ \sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y \left[\partial_y^{k+1}\tilde{\rho}\, \partial_y^j u_2\, \partial_y^{k+1-j}\bar{\rho}\right] dxdy. \end{aligned}$$

Again, the boundary terms vanish because $\rho \in X^k(\Omega)$ and in consequence $\bar{\rho} \in X^k(\Omega)$ and $\tilde{\rho} \in X^k([-1,1])$. From this, I^6 is simply:

$$\begin{aligned} I^6 &= -\sum_{j=1}^{k} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial^{k+1-j} u_2\, \partial^{k+1-j}\partial_y^j \bar{\rho}\, dxdy \\ &- \sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y^{k+1}\tilde{\rho}\, \partial_y^{j+1} u_2\, \partial_y^{k+1-j}\bar{\rho}\, dxdy - \sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y^{k+1}\tilde{\rho}\, \partial_y^{j} u_2\, \partial_y^{k+2-j}\bar{\rho}\, dxdy \\ &= B_1 + B_2 + B_3. \end{aligned}$$

We analyze separately the terms in the previous expression. First of all, we split the term B_1 as follows:

$$\begin{aligned} B_1 &= -\sum_{j=1}^{k-2} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial^{k+1-j} u_2\, \partial^{k+1-j}\partial_y^j \bar{\rho}\, dxdy \\ &- \sum_{j=k-1}^{k} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial^{k+1-j} u_2\, \partial^{k+1-j}\partial_y^j \bar{\rho}\, dxdy \\ &= B_1^1 + B_1^2. \end{aligned}$$

Indeed, B_1^2 are the only terms in B_1 that cannot be absorbed by the linear part. These type of terms are the reason why we need to have an integrable time decay of the velocity. Precisely, the main goal of the next section 3.6 is to obtain a time decay rate for it. Then, for B_1^2 we have that:

$$B_1^2 \lesssim \left(||\partial^k\tilde{\rho}||_{L^\infty}\, ||\partial^2 u_2||_{L^2} + ||\partial^{k+1}\tilde{\rho}||_{L^2}\, ||\partial u_2||_{L^\infty}\right) ||\bar{\rho}||_{H^{k+1}} \leqslant ||u_2||_{H^3}\, ||\rho||_{H^{k+1}}^2 \tag{3.55}$$

where we have used the Sobolev embedding $L^\infty([-1,1]) \hookrightarrow H^1([-1,1])$. In particular, as $\tilde{\rho}$ only depend on the vertical variable, we have the bound $||\partial^k\tilde{\rho}||_{L^\infty([-1,1])} \leqslant ||\tilde{\rho}||_{H^{k+1}([-1,1])} \leqslant ||\rho||_{H^{k+1}(\Omega)}$.

To study the term B_1^1, we distinguish two cases:

i) All derivatives are in y, i.e. $\partial^{k+1-j} \equiv \partial_y^{k+1-j}$ and in consequence we get:

$$B_1^1 = -\sum_{j=1}^{k-2} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial_y^{k+1-j} u_2\, \partial_y^{k+1}\bar{\rho}\, dxdy.$$

By integration by parts and the fact that $\partial_y u_2 = -\partial_x u_1$ we get:

$$
\begin{aligned}
B_1^1 =& \sum_{j=1}^{k-2} \int_\Omega \partial_y^2 \left(\partial_y^{j+1}\tilde{\rho}\, \partial_y^{k-j} u_1\right) \partial_y^{k-1}\partial_x \bar{\rho}\, dxdy + \int_\Omega \partial_y \left[\partial_y^{j+1}\tilde{\rho}\, \partial_y^{k+1-j} u_2\, \partial_y^k \bar{\rho}\right] dxdy \\
& - \int_\Omega \partial_y \left[\partial_y \left[\partial_y^{j+1}\tilde{\rho}\, \partial_y^{k+1-j} u_2\right] \partial_y^{k-1}\bar{\rho}\right] dxdy - \int_\Omega \partial_x \left[\partial_y^2 \left(\partial_y^{j+1}\tilde{\rho}\, \partial_y^{k-j} u_1\right) \partial_y^{k-1}\bar{\rho}\right] dxdy \\
=& \sum_{j=1}^{k-2} \int_\Omega \left[\partial_y^{j+3}\tilde{\rho}\, \partial_y^{k-j} u_1 + \partial_y^{j+1}\tilde{\rho}\, \partial_y^{k+2-j} u_1 + 2\, \partial_y^{j+2}\tilde{\rho}\, \partial_y^{k+1-j} u_1\right] \partial_y^{k-1}\partial_x \bar{\rho}\, dxdy \\
=& B_1^{1,1} + B_1^{1,2} + B_1^{1,3}.
\end{aligned}
$$

Once again, the boundary terms vanish because the structure of our initial data is preserved in time. For the rest, repeatedly applying Hölder's inequality we arrive to our goal.

For the first one, with $k \geqslant 4$ and the Sobolev embedding $L^\infty([-1,1]) \hookrightarrow H^1([-1,1])$, we have the bound:

$$
\begin{aligned}
B_1^{1,1} &\leqslant \|\partial_x \bar{\rho}\|_{H^{k-1}} \left[\sum_{j=1}^{k-3} \|\partial_y^{j+3}\tilde{\rho}\|_{L^\infty} \|\partial_y^{k-j} u_1\|_{L^2} + \|\partial_y^{k+1}\tilde{\rho}\|_{L^2} \|\partial_y^2 u_1\|_{L^\infty}\right] \\
&\lesssim \|\nabla\Pi - (0,\bar{\rho})\|_{H^k} \|\tilde{\rho}\|_{H^{k+1}} \|u_1\|_{H^k} \leqslant \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right).
\end{aligned}
$$

For the second one with $k \geqslant 4$, we have:

$$
\begin{aligned}
B_1^{1,2} &\leqslant \|\partial_x \bar{\rho}\|_{H^{k-1}} \left[\|\partial_y^2\tilde{\rho}\, \partial_y^{k+1} u_1\|_{L^2} + \sum_{j=2}^{k-2} \|\partial_y^{j+1}\tilde{\rho}\|_{L^\infty} \|\partial_y^{k+2-j} u_1\|_{L^2}\right] \\
&\lesssim \|\nabla\Pi - (0,\bar{\rho})\|_{H^k} \left[\|\partial_y^2\tilde{\rho}\, \partial_y^{k+1} u_1\|_{L^2} + \|\tilde{\rho}\|_{H^k} \|u_1\|_{H^k}\right]
\end{aligned}
$$

where

$$
\|\partial_y^2\tilde{\rho}\, \partial_y^{k+1} u_1\|_{L^2} \leqslant \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y\tilde{\rho}\|_{L^\infty})^{1/2}} \left(\int_\Omega |\partial_y^{k+1} u_1|^2 (1 + \partial_y\tilde{\rho})\, dxdy\right)^{1/2}.
$$

Therefore, for $k \geqslant 4$ we have that:

$$
B_1^{1,2} \lesssim \|\tilde{\rho}\|_{H^k} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right) + \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y\tilde{\rho}\|_{L^\infty})} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 (1 + \partial_y\tilde{\rho})\, dxdy\right).
$$

The last one is the simplest, if $k \geqslant 4$ we have:

$$
B_1^{1,3} \lesssim \|\partial_x \bar{\rho}\|_{H^{k-1}} \sum_{j=1}^{k-2} \|\partial_y^{j+2}\tilde{\rho}\|_{L^\infty} \|\partial_y^{k+1-j} u_1\|_{L^2} \lesssim \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right).
$$

And finally, for $k \geqslant 4$ we have proved that:

$$
\sum_{n=1}^{3} B_1^{1,n} \lesssim \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right) + \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y\tilde{\rho}\|_{L^\infty})} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 (1 + \partial_y\tilde{\rho})\, dxdy\right).
$$

ii) We have at least one derivative in x, i.e. $\partial^{k+1-j} \equiv \partial^{k-j}\partial_x$ and in consequence we get:

$$
B_1^1 = -\sum_{j=1}^{k-2} \binom{k+1}{j} \int_\Omega \partial_y^{j+1}\tilde{\rho}\, \partial^{k-j}\partial_x u_2\, \partial^k \partial_x \bar{\rho}\, dxdy.
$$

By integration by parts and the fact that $\partial_y u_2 = -\partial_x u_1$ we get:

$$
\begin{aligned}
B_1^1 &= \sum_{j=1}^{k-2} \int_\Omega \partial \left(\partial_y^{j+1}\tilde{\rho}\, \partial^{k-j}\partial_x u_2\right)\, \partial^{k-j-1}\partial_y^j \partial_x \bar{\rho}\, dxdy + \int_\Omega \partial \left[\partial_y^{j+1}\tilde{\rho}\, \partial^{k-j}\partial_x u_2\, \partial^{k-j-1}\partial_y^j \partial_x \bar{\rho}\right]\, dxdy \\
&= \sum_{j=1}^{k-2} \int_\Omega \left[\partial_y^{j+2}\tilde{\rho}\, \partial^{k-j}\partial_x u_2 + \partial_y^{j+1}\tilde{\rho}\, \partial^{k+1-j}\partial_x u_2\right]\, \partial^{k-j-1}\partial_y^j \partial_x \bar{\rho}\, dxdy \\
&= B_1^{1,1} + B_1^{1,2}.
\end{aligned}
$$

In this case, the first one is the simplest. For $k \geqslant 4$ we have:

$$
B_1^{1,1} \lesssim \|\partial_x \bar{\rho}\|_{H^{k-1}} \sum_{j=1}^{k-2} \|\partial_y^{j+2}\tilde{\rho}\|_{L^\infty} \|\partial^{k-j}\partial_x u_2\|_{L^2} \lesssim \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right)
$$

by the Sobolev embedding $L^\infty([-1,1]) \hookrightarrow H^1([-1,1])$ in dimension one. For the next, also if $k \geqslant 4$ we have:

$$
\begin{aligned}
B_1^{1,2} &\lesssim \|\partial^{k-j-1}\partial_y^j \partial_x \bar{\rho}\|_{L^2} \left[\|\partial_y^2 \tilde{\rho}\, \partial^k \partial_x u_2\|_{L^2} + \sum_{j=2}^{k-2} \|\partial_y^{j+1}\tilde{\rho}\|_{L^\infty} \|\partial^{k+1-j}\partial_x u_2\|_{L^2}\right] \\
&\lesssim \|\nabla\Pi - (0,\bar{\rho})\|_{H^k} \left[\|\partial_y^2 \tilde{\rho}\, \partial^k \partial_x u_2\|_{L^2} + \|\tilde{\rho}\|_{H^k} \|u_2\|_{H^k}\right]
\end{aligned}
$$

where

$$
\|\partial_y^2 \tilde{\rho}\, \partial^k \partial_x u_2\|_{L^2} \leqslant \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2}} \left(\int_\Omega |\partial^k \partial_x u_2|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy\right)^{1/2}.
$$

Therefore, for $k \geqslant 4$, we have that:

$$
B_1^{1,2} \lesssim \|\tilde{\rho}\|_{H^k} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right) + \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y \tilde{\rho}\|_{L^\infty})} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy\right).
$$

And finally, for $k \geqslant 4$ we have proved that:

$$
\sum_{n=1}^{2} B_1^{1,n} \lesssim \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right) + \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y \tilde{\rho}\|_{L^\infty})} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy\right).
$$

In either case, for both *i)* and *ii)* together with (3.55), if $k \geqslant 4$ we have proved

$$
\begin{aligned}
B_1 \lesssim\ & \|u_2\|_{H^3} \|\rho\|_{H^{k+1}}^2 + \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla\Pi - (0,\bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2\right) \\
&+ \frac{\|\tilde{\rho}\|_{H^k}}{(1 - \|\partial_y \tilde{\rho}\|_{L^\infty})} \left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy\right).
\end{aligned} \tag{3.56}
$$

On the other hand, by the incompressibility of the velocity and the periodicity, for $B_2 + B_3$ we get that:

$$
\begin{aligned}
B_2 + B_3 &= -\sum_{j=1}^{k+1} \binom{k+1}{j} \left[\int_\Omega \partial_y^{k+1}\tilde{\rho}\, \partial_y^j (\partial_y u_2)\, \partial_y^{k+1-j}\bar{\rho}\, dxdy + \int_\Omega \partial_y^{k+1}\tilde{\rho}\, \partial_y^j u_2\, \partial_y^{k+1-j}(\partial_y \bar{\rho})\, dxdy\right] \\
&= -\sum_{j=1}^{k+1} \binom{k+1}{j} \int_\Omega \partial_y^{k+1}\tilde{\rho}\, \left[\partial_y^j \mathbf{u} \cdot \nabla \partial_y^{k+1-j}\bar{\rho}\right]\, dxdy
\end{aligned}
$$

and Hölder's inequality gives us

$$
\begin{aligned}
B_2 + B_3 &\lesssim \|\partial_y^{k+1}\tilde{\rho}\|_{L^2} \sum_{j=1}^{k+1} \|\partial_y^j \mathbf{u}\cdot\nabla\partial_y^{k+1-j}\bar{\rho}\|_{L^2} \\
&\leqslant \|\tilde{\rho}\|_{H^{k+1}}\left[\sum_{j=1}^{4}\|\partial_y^j\mathbf{u}\|_{L^\infty}\|\nabla\partial_y^{k+1-j}\bar{\rho}\|_{L^2} + \sum_{j=5}^{k}\|\partial_y^j\mathbf{u}\|_{L^2}\,\|\nabla\partial_y^{k+1-j}\bar{\rho}\|_{L^\infty} + \|\partial_y^{k+1}\mathbf{u}\cdot\nabla\bar{\rho}\|_{L^2}\right]
\end{aligned}
\tag{3.57}
$$

where

$$
\|\partial_y^{k+1}\mathbf{u}\cdot\nabla\bar{\rho}\|_{L^2} \leqslant \frac{\|\nabla\bar{\rho}\|_{L^\infty}}{(1-\|\partial_y\tilde{\rho}\|_{L^\infty})^{1/2}}\left(\int_\Omega |\partial_y^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\right)^{1/2}.
$$

Moreover, for $k \geqslant 6$ we have that:

$$
\sum_{j=1}^{4}\|\partial_y^j\mathbf{u}\|_{L^\infty}\|\nabla\partial_y^{k+1-j}\bar{\rho}\|_{L^2} \lesssim \|\mathbf{u}\|_{H^4}\|\bar{\rho}\|_{H^{k+1}} + \left(\|\mathbf{u}\|_{H^k}^2 + \|\nabla\Pi-(0,\bar{\rho})\|_{H^k}^2\right)
$$

and

$$
\sum_{j=5}^{k}\|\partial_y^j\mathbf{u}\|_{L^2}\,\|\nabla\partial_y^{k+1-j}\bar{\rho}\|_{L^\infty} \lesssim \|\mathbf{u}\|_{H^k}^2 + \|\nabla\Pi-(0,\bar{\rho})\|_{H^k}^2.
$$

In conclusion, putting all this in (3.57), for $k \geqslant 6$ we have proved that:

$$
\begin{aligned}
B_2 + B_3 \lesssim \|\mathbf{u}\|_{H^4}\|\bar{\rho}\|_{H^{k+1}}^2 + \|\tilde{\rho}\|_{H^{k+1}}\left(\|\mathbf{u}\|_{H^k}^2 + \|\nabla\Pi-(0,\bar{\rho})\|_{H^k}^2\right) \\
+ \frac{\|\tilde{\rho}\|_{H^{k+1}}}{1-\|\partial_y\tilde{\rho}\|_{L^\infty}}\left(\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\right).
\end{aligned}
\tag{3.58}
$$

We finally arrive at the claimed bound putting together (3.56) and (3.58).

(2) To work with I^7, first of all we must remember that $\partial_t\tilde{\rho} = -\partial_y\left(\widetilde{u_2\,\bar{\rho}}\right)$, then:

$$
\begin{aligned}
I^7 &= \int_\Omega \partial^{k+1}u_2\,\partial^{k+1}\Pi\,\partial_y^2\tilde{\rho}\,dxdy - \tfrac{1}{2}\int_\Omega |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,\frac{\partial_y^2\left(\widetilde{u_2\,\bar{\rho}}\right)}{1+\partial_y\tilde{\rho}}\,dxdy \\
&= I_1^7 + I_2^7
\end{aligned}
$$

so

$$
I_2^7 \lesssim \frac{\|\partial_y^2\left(\widetilde{u_2\,\bar{\rho}}\right)\|_{L^\infty}}{1-\|\partial_y\tilde{\rho}\|_{L^\infty}}\left(\int_{\mathbb{T}\times\mathbb{R}} |\partial^{k+1}\mathbf{u}|^2\,(1+\partial_y\tilde{\rho})\,dxdy\right)
$$

and in particular $\|\partial_y^2\left(\widetilde{u_2\,\bar{\rho}}\right)\|_{L^\infty} \lesssim \|u_2\|_{H^4}\,\|\bar{\rho}\|_{H^4}$.

As before, for the I_1^7-term, we also distinguish between two cases:

i) We have at least one derivative in x, i.e. $\partial^{k+1} \equiv \partial^k\partial_x$ and in consequence, for $k \geqslant 2$ we get:

$$
\begin{aligned}
I_1^7 = \int_\Omega \partial^k\partial_x u_2\,\partial^k\partial_x\Pi\,\partial_y^2\tilde{\rho}\,dxdy &\leqslant \|\partial_y^2\tilde{\rho}\|_{L^\infty}\,\|\partial^k\partial_x\Pi\|_{L^2}\,\|\partial^k\partial_x u_2\|_{L^2} \\
&\leqslant \|\tilde{\rho}\|_{H^{k+1}}\|\nabla\Pi-(0,\bar{\rho})\|_{H^k}\|\partial^{k+1}\mathbf{u}\|_{L^2}.
\end{aligned}
$$

Together with (3.53), we finally get:

$$I_1^7 \leqslant \|\tilde{\rho}\|_{H^{k+1}} \|\nabla \Pi - (0, \bar{\rho})\|_{H^k}^2 + \frac{\|\tilde{\rho}\|_{H^{k+1}}}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right).$$

ii) All derivatives are in y, i.e. $\partial^{k+1} \equiv \partial_y^{k+1}$. By integration by parts and the fact that $\partial_y u_2 = -\partial_x u_1$ we get:

$$\begin{aligned} I_1^7 = \int_\Omega \partial_y^{k+1} u_2\, \partial_y^{k+1} \Pi\, \partial_y^2 \tilde{\rho}\, dxdy &= -\int_\Omega \partial_y \left[\partial_y^k u_1 \partial_y^2 \tilde{\rho}\right] \partial_y^k \partial_x \Pi\, dxdy \\ &\quad + \int_\Omega \partial_y \left[\partial_y^k u_1\, \partial_y^k \partial_x \Pi\, \partial_y^2 \tilde{\rho}\right] dxdy - \int_\Omega \partial_x \left[\partial_y^k u_1\, \partial_y^{k+1} \Pi\, \partial_y^2 \tilde{\rho}\right] dxdy \\ &= -\int_\Omega \partial_y^{k+1} u_1\, \partial_y^k \partial_x \Pi\, \partial_y^2 \tilde{\rho}\, dxdy - \int_\Omega \partial_y^k u_1\, \partial_y^k \partial_x \Pi\, \partial_y^3 \tilde{\rho}\, dxdy \end{aligned}$$

where the boundary terms vanish by the periodicity in the horizontal variable and the fact that $\rho \in X^k(\Omega)$. Then, for $k \geqslant 3$ we have:

$$\begin{aligned} I_1^7 &\leqslant \|\partial_y^k \partial_x \Pi\|_{L^2} \left(\|\partial_y^{k+1} u_1\|_{L^2} \|\partial_y^2 \tilde{\rho}\|_{L^\infty} + \|\partial_y^k u_1\|_{L^2} \|\partial_y^3 \tilde{\rho}\|_{L^\infty} \right) \\ &\leqslant \|\tilde{\rho}\|_{H^{k+1}} \|\nabla \Pi - (0, \bar{\rho})\|_{H^k} \left(\|\partial^{k+1} \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{H^k} \right), \end{aligned}$$

and as before, by (3.53) we get:

$$I_1^7 \lesssim \frac{\|\tilde{\rho}\|_{H^{k+1}}}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right) + \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla \Pi - (0, \bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2 \right).$$

In any case, for $k \geqslant 4$ we have that:

$$I^7 \lesssim \frac{(1 + \|\mathbf{u}\|_{H^k}) \|\rho\|_{H^{k+1}}}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right) + \|\tilde{\rho}\|_{H^{k+1}} \left(\|\nabla \Pi - (0, \bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2 \right).$$

(3) Applying the chain rule, I^8 becomes:

$$\begin{aligned} I^8 &= -\int_\Omega \partial^{k+1} \mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\, \partial^{k+1} \mathbf{u}\, (1 + \partial_y \tilde{\rho})\, dxdy \\ &\quad - \sum_{j=2}^{k} \binom{k+1}{j} \int_\Omega \partial^{k+1} \mathbf{u} \cdot \left(\partial^j \mathbf{u} \cdot \nabla\right) \partial^{k+1-j} \mathbf{u}\, (1 + \partial_y \tilde{\rho})\, dxdy \\ &\quad - \binom{k+1}{1} \int_\Omega \partial^{k+1} \mathbf{u} \cdot (\partial \mathbf{u} \cdot \nabla)\, \partial^k \mathbf{u}\, (1 + \partial_y \tilde{\rho})\, dxdy - \int_\Omega \partial^{k+1} \mathbf{u} \cdot \left(\partial^{k+1} \mathbf{u} \cdot \nabla\right) \mathbf{u}\, (1 + \partial_y \tilde{\rho})\, dxdy \\ &= F_1 + F_2 + F_3. \end{aligned}$$

In the first term, since $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, we get:

$$F_1 = \frac{1}{2} \int_\Omega |\partial^{k+1} \mathbf{u}|^2\, u_2\, \partial_y^2 \tilde{\rho}\, dxdy \lesssim \frac{\|u_2\|_{L^\infty} \|\partial_y^2 \tilde{\rho}\|_{L^\infty}}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right).$$

For the second one, we need to work a bit harder:

$$F_2 \lesssim (1 + \|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2} \|\partial^{k+1} \mathbf{u}\, (1 + \partial_y \tilde{\rho})^{1/2}\|_{L^2} \sum_{j=2}^{k} \| \left(\partial^j \mathbf{u} \cdot \nabla\right) \partial^{k+1-j} \mathbf{u}\|_{L^2}$$

where, for $k \geqslant 5$ we have that:

$$\sum_{j=2}^{k} \| \left(\partial^j \mathbf{u} \cdot \nabla\right) \partial^{k+1-j} \mathbf{u} \|_{L^2} \leqslant \sum_{j=2}^{3} \|\partial^j \mathbf{u}\|_{L^\infty} \|\nabla \partial^{k+1-j} \mathbf{u}\|_{L^2} + \sum_{j=4}^{k} \|\partial^j \mathbf{u}\|_{L^2} \|\nabla \partial^{k+1-j} \mathbf{u}\|_{L^\infty} \lesssim \|\mathbf{u}\|_{H^k}^2.$$

Therefore, for $k \geqslant 5$ we have proved that:

$$F_2 \lesssim (1 + \|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2} \|\mathbf{u}\|_{H^k}^2 \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right)^{1/2}.$$

In the last one, we have that:

$$F_3 \lesssim \|\nabla \mathbf{u}\|_{L^\infty} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy \right).$$

Therefore, putting together the estimates, for $k \geqslant 5$ we have proved that:

$$\begin{aligned} I^8 \lesssim\ & (1 + \|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right)^{1/2} \|\mathbf{u}\|_{H^k}^2 \\ & + \frac{\|\mathbf{u}\|_{H^k} (1 + \|\rho\|_{H^{k+1}})}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 (1 + \partial_y \tilde{\rho})\, dxdy \right). \end{aligned}$$

(4) We note that $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, so that in estimating $(\partial^{k+1}(\mathbf{u} \cdot \nabla \bar{\rho}), \partial^{k+1} \bar{\rho})$ we only have to bound terms of the form $\|\partial^{j+1} \mathbf{u} \cdot \nabla \partial^{k-j} \bar{\rho}\|_{L^2}$, where $j = 0, 1, \ldots, k$. We use Hölder inequality to conclude then, for $k \geqslant 5$ that:

$$\begin{aligned} \sum_{j=0}^{k} \|\partial^{j+1} \mathbf{u} \cdot \nabla \partial^{k-j} \bar{\rho}\|_{L^2} &\leqslant \sum_{j=0}^{k-3} \|\partial^{j+1} \mathbf{u}\|_{L^\infty} \|\nabla \partial^{k-j} \bar{\rho}\|_{L^2} + \sum_{j=k-2}^{k-1} \|\partial^{j+1} \mathbf{u}\|_{L^2} \|\nabla \partial^{k-j} \bar{\rho}\|_{L^2} + \|\partial^{k+1} \mathbf{u} \cdot \nabla \bar{\rho}\|_{L^2} \\ &\lesssim \|\mathbf{u}\|_{H^4} \|\bar{\rho}\|_{H^{k+1}} + \|\mathbf{u}\|_{H^k} \|\nabla \Pi - (0, \bar{\rho})\|_{H^k} + \|\partial^{k+1} \mathbf{u} \cdot \nabla \bar{\rho}\|_{L^2}. \end{aligned}$$

Here, in the last term, to close the estimate we need to proceed as follows:

$$\begin{aligned} \|\partial^{k+1} \mathbf{u} \cdot \nabla \bar{\rho}\|_{L^2} &\leqslant \|\partial^{k+1} \mathbf{u}\|_{L^2} \|\nabla \bar{\rho}\|_{L^\infty} \\ &\leqslant \frac{\|\bar{\rho}\|_{H^{k-1}}}{(1 - \|\partial_y \tilde{\rho}\|_{L^\infty})^{1/2}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy \right)^{1/2} \end{aligned}$$

and finally, we obtain that:

$$\|\partial^{k+1} \mathbf{u} \cdot \nabla \bar{\rho}\|_{L^2} \leqslant \|\nabla \Pi - (0, \bar{\rho})\|_{H^k}^2 + \frac{1}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy \right).$$

Therefore, for $k \geqslant 5$ we have proved that:

$$\begin{aligned} I^9 \lesssim\ & \|\mathbf{u}\|_{H^4} \|\bar{\rho}\|_{H^{k+1}}^2 + \|\bar{\rho}\|_{H^{k+1}} \left(\|\nabla \Pi - (0, \bar{\rho})\|_{H^k}^2 + \|\mathbf{u}\|_{H^k}^2 \right) \\ & + \frac{\|\bar{\rho}\|_{H^{k+1}}}{1 - \|\partial_y \tilde{\rho}\|_{L^\infty}} \left(\int_\Omega |\partial^{k+1} \mathbf{u}|^2 \,(1 + \partial_y \tilde{\rho})\, dxdy \right). \end{aligned}$$

□

Putting it all together, by Lemma 3.5.6 and (3.54), we have proved Theorem 3.5.1.

3.6 Linear & non-linear estimates

Our goal for this and the following section is to obtain a time decay estimate for $\|\mathbf{u}\|_{H^4(\Omega)}(t)$. As we will see in Section 3.7, to close the energy estimate and finish the proof it is enough to get an integrable rate.

We approach the question of global well-posedness for a small initial data from a perturbative point of view, i.e., we see (3.9) as a non-linear perturbation of the linear problem. Therefore, a finer understanding of the linearized system allows us to improve their time span.

3.6.1 The Quasi-Linearized Problem

In view of a descomposition of this system into linear and nonlinear part, we split the pressure as

$$\Pi = \Pi^L + \Pi^{NL}$$

where

$$\begin{aligned} \Pi^L &:= -(-\Delta)^{-1}\partial_y \bar{\rho}, \\ \Pi^{NL} &:= (-\Delta)^{-1}\mathrm{div}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]. \end{aligned} \tag{3.59}$$

The linearized equation of (3.9) around the trivial solution $(\rho, \mathbf{u}) = (0,0)$ reads

$$\begin{cases} \partial_t \bar{\rho} &= -u_2, \\ \partial_t \tilde{\rho} &= 0, \\ \partial_t \mathbf{u} + \mathbf{u} &= -\nabla \Pi^L + (0, \bar{\rho}), \\ \nabla\cdot\mathbf{u} &= 0, \end{cases} \tag{3.60}$$

together with the no-slip condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and initial data $(\rho(0), \mathbf{u}(0)) \in X^k(\Omega)\times\mathbb{X}^k(\Omega)$ such that $\rho(0) = \bar{\rho}(0) + \tilde{\rho}(0)$. It is not difficult to prove that $\bar{\rho}(t)$ will decay in the time and $\tilde{\rho}(t)$ will just remain bounded at linear order. In consequence, the linearized problem has a very large set of stationary (undamped) modes. Now, we return to our non-linear problem:

$$\begin{cases} \partial_t \bar{\rho} + \overline{\mathbf{u}\cdot\nabla\bar{\rho}} + \partial_y \tilde{\rho}\, u_2 &= -u_2, \\ \partial_t \tilde{\rho} + \widetilde{\mathbf{u}\cdot\nabla\bar{\rho}} &= 0, \\ \partial_t \mathbf{u} + \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\Pi^{NL} &= -\nabla\Pi^L + (0,\bar{\rho}), \\ \nabla\cdot\mathbf{u} &= 0, \end{cases}$$

together with the no-slip condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Since $\bar{\rho}$ is decaying, the term $\overline{\mathbf{u}\cdot\nabla\bar{\rho}}$ should be very small and should be controllable. The term $\partial_y \tilde{\rho}\, u_2$, however, acts like a second linear operator since $\tilde{\rho}$ is not decaying. It is conceivable that this extra linear operator could compete with the damping coming from the linear term. This makes the problem of long-time behavior more difficult.

We solve this by, more or less, doing a second linearization around the undamped modes and showing that the stationary modes can be controlled. Then, we wish to prove decay estimates for $\bar{\rho}$ in the following system:

$$\begin{cases} \partial_t \bar{\rho} &= -(1+\partial_y\tilde{\rho})\, u_2, \\ \partial_t \tilde{\rho} &= 0, \\ \partial_t \mathbf{u} + \mathbf{u} &= -\nabla \Pi^L + (0, \bar{\rho}), \\ \nabla\cdot\mathbf{u} &= 0, \end{cases} \tag{3.61}$$

assuming that the initial data is sufficiently small. By showing that, the decay mechanism is "stable" with respect to the sort of perturbations which this second linear operator introduces, we are able to keep the decay mechanism and close a decay estimate for $\bar{\rho}$ and show that $\tilde{\rho}$, while not decaying, converges as $t\to\infty$.

3.6.2 The Quasi-Linear Decay

We prove $L^2(\Omega)$ decay estimates for the quasi-lineared system (3.61). To do it, let $w : [-1,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. We consider the w-weighted $L^2(\Omega)$ space defined as

$$\|f\|_{L^2_w(\Omega)}(t) := \left(\int_\Omega |f(x,y)|^2 w(y,t)\, dx dy\right)^{\frac{1}{2}}$$

and their analogous Sobolev space

$$\|f\|_{H^k_w(\Omega)}(t) := \|f\|_{L^2_w(\Omega)}(t) + \|\partial^k f\|_{L^2_w(\Omega)}(t).$$

After recalling this definition, we notice that the second equation $\partial_t \tilde{\rho}(t) = 0$ of (3.61) reduces to a condition at time $t = 0$, i.e. $\tilde{\rho}(y,t) = \tilde{\rho}(y,0)$. However, for the non-linear problem it is expected that $\tilde{\rho}$ will just remain bounded and in consequence, our goal is to solve the following system in Ω:

$$\begin{cases} \partial_t \bar{\rho} &= -(1 + G(y,t))\, u_2, \\ \partial_t \mathbf{u} + \mathbf{u} &= -\nabla \Pi^L + (0, \bar{\rho}), \\ \nabla \cdot \mathbf{u} &= 0, \\ \bar{\rho}|_{t=0} &= \bar{\rho}(0), \\ \mathbf{u}|_{t=0} &= \mathbf{u}(0), \end{cases} \tag{3.62}$$

where $\bar{\rho}(0) \in X^k(\Omega)$ and $\mathbf{u}(0) \in \mathbb{X}^k(\Omega)$. Note that the auxiliary function $G(y,t)$, which plays the role of $\partial_y \tilde{\rho}(y,t)$, will be sufficiently small in the appropriate space.

Remark: By taking the analog of Fourier transform given by the eigenfunction expansion, we cannot obtain an exact formula for the solution because the $G(y,t)$ term mixes the effect of all the Fourier coefficients.

In the following, we fix our attention in the quasi-linear problem (3.62). By the previous comment we can not extract an exact formula for the solution. For this reason we need to work a little harder to obtain the decay for the quasi-linear problem.

Lemma 3.6.1. *Let $k \geqslant 2$ and $(\rho(0), \mathbf{u}(0)) \in X^k(\Omega) \times \mathbb{X}^k(\Omega)$. Then, for $w := 1 + G(y,t)$ and $w^\star := w - \frac{1}{2}\partial_t w$ the solution of equation (3.62) satisfies that:*

$$\begin{aligned} \tfrac{1}{2}\partial_t \left\{ \|\mathbf{u}\|^2_{H^k_w(\Omega)}(t) + \|\bar{\rho}\|^2_{H^k(\Omega)}(t) \right\} \leqslant & -\|\mathbf{u}\|^2_{H^k_{w^\star}(\Omega)}(t) \\ & + C\, \|G'\|_{H^k([-1,1])}(t)\, \|\bar{\rho}\|_{H^{k-1}(\Omega)}(t)\, \|u_2\|_{H^k(\Omega)}(t) \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} \tfrac{1}{2}\partial_t \left\{ \|\partial_t \mathbf{u}\|^2_{H^k(\Omega)}(t) + \|u_2\|^2_{H^k_w(\Omega)}(t) \right\} \leqslant & -\|\partial_t \mathbf{u}\|^2_{H^k(\Omega)}(t) + \|u_2\|^2_{H^k_{w-w^\star}(\Omega)}(t) \\ & + C\, \|G'\|_{H^{k-1}([-1,1])}(t)\, \|u_2\|_{H^k(\Omega)}(t)\, \|\partial_t u_2\|_{H^k(\Omega)}(t) \end{aligned} \tag{3.64}$$

for some positive constant C.

Proof. We start with (3.63). Using the incompressibility and the boundary conditions it is clear that:

$$\tfrac{1}{2}\partial_t \|\mathbf{u}\|^2_{H^k_w(\Omega)} = -\|\mathbf{u}\|^2_{H^k_{w^\star}(\Omega)} + \int_\Omega \bar{\rho}\, u_2\, w\, dx dy + \int_\Omega \partial^k \bar{\rho}\, \partial^k u_2\, w\, dx dy.$$

Now, as $\partial_t \bar{\rho} = -(1 + G(y,t))\, u_2$, we have that:

$$\tfrac{1}{2}\partial_t \|\mathbf{u}\|^2_{H^k_w(\Omega)} = -\|\mathbf{u}\|^2_{H^k_{w^\star}(\Omega)} - \int_\Omega \bar{\rho}\, \partial_t\bar{\rho}\, dxdy - \int_\Omega \partial^k\bar{\rho}\, \partial^k(\partial_t\bar{\rho}\, w^{-1})\, w\, dxdy \pm \int_\Omega \partial^k\bar{\rho}\, \partial^k\partial_t\bar{\rho}\, dxdy$$

and we arrive to:

$$\tfrac{1}{2}\partial_t \left\{ \|\mathbf{u}\|^2_{H^k_w(\Omega)} + \|\bar{\rho}\|^2_{H^k(\Omega)} \right\} = -\|\mathbf{u}\|^2_{H^k_{w^\star}(\Omega)} - \int_\Omega \partial^k\bar{\rho}\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy.$$

Applying integration by parts in the last term, we obtain that:

$$-\int_\Omega \partial^k\bar{\rho}\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy = \int_\Omega \partial^{k-1}\bar{\rho}\, [\partial^{k+1}, w]u_2\, dxdy - \int_\Omega \partial^{k-1}\bar{\rho}\, \partial w\, \partial^k u_2\, dxdy$$

and using the commutator estimate (3.19) we have the bound:

$$\begin{aligned} -\int_\Omega \partial^k\bar{\rho}\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy &\leqslant \|\partial^{k-1}\bar{\rho}\|_{L^2} \left(\|[\partial^{k+1}, w]u_2\|_{L^2} + \|\partial w\|_{L^\infty}\|\partial^k u_2\|_{L^2}\right) \\ &\lesssim \|\partial^{k-1}\bar{\rho}\|_{L^2} \left(\|\partial w\|_{L^\infty}\|\partial^k u_2\|_{L^2} + \|\partial^{k+1} w\|_{L^2}\|u_2\|_{L^\infty}\right). \end{aligned}$$

Applying the Sobolev embedding in the previous inequality, we have for $k \geqslant 2$ that:

$$-\int_\Omega \partial^k\bar{\rho}\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy \lesssim \|\partial w\|_{H^k([-1,1])}\|u_2\|_{H^k(\Omega)}\|\bar{\rho}\|_{H^{k-1}(\Omega)}$$

and, in consequence, we have proved the first inequality.

To prove (3.64) we proceed as before, using the incompressibility and the boundary conditions to get:

$$\tfrac{1}{2}\partial_t \|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} = -\|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} + \int_\Omega \partial_t\bar{\rho}\, \partial_t u_2\, dxdy + \int_\Omega \partial^k\partial_t\bar{\rho}\, \partial^k\partial_t u_2\, dxdy.$$

Again, as $\partial_t \bar{\rho} = -(1 + G(y,t))\, u_2$, we have that:

$$\tfrac{1}{2}\partial_t \|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} = -\|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} - \int_\Omega w\, u_2\, \partial_t u_2\, dxdy - \int_\Omega \partial^k(w\, u_2)\, \partial^k\partial_t u_2\, dxdy$$

and finally we arrive to:

$$\begin{aligned} \tfrac{1}{2}\partial_t \left\{ \|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} + \|u_2\|^2_{H^k_w(\Omega)} \right\} = &- \|\partial_t\mathbf{u}\|^2_{H^k(\Omega)} + \|u_2\|^2_{H^k_{w-w^\star}(\Omega)} \\ &- \int_\Omega \partial_t\partial^k u_2\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy. \end{aligned}$$

Using the commutator estimate (3.19) and the Sobolev embedding in the last term, for $k \geqslant 2$ we have that:

$$\begin{aligned} -\int_\Omega \partial_t\partial^k u_2\, \left[\partial^k(u_2\, w) - w\, \partial^k u_2\right]\, dxdy &\lesssim \|\partial_t\partial^k u_2\|_{L^2} \left(\|\partial w\|_{L^\infty}\|\partial^{k-1} u_2\|_{L^2} + \|\partial^k w\|_{L^2}\|u_2\|_{L^\infty}\right) \\ &\lesssim \|\partial w\|_{H^{k-1}([-1,1])}\|u_2\|_{H^k(\Omega)}\|\partial_t u_2\|_{H^k(\Omega)} \end{aligned}$$

and, in consequence, we have proved the second inequality. $\square$

Plugging together (3.63) with (3.64) and using $||\bar{\rho}||_{H^{k-1}(\Omega)}(t) \leqslant ||\nabla\Pi^L - (0,\bar{\rho})||_{H^k(\Omega)}(t)$ we get:

$$
\begin{aligned}
&\tfrac{1}{2}\partial_t \left\{ ||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t) \right\} \\
&\qquad \leqslant -\left(||\mathbf{u}||^2_{H^k_{w^\star}(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right) + ||u_2||^2_{H^k_{w-w^\star}(\Omega)}(t) \\
&\qquad\quad + \frac{C}{2}\, ||\partial w||_{H^k([-1,1])}(t) \left(||\nabla\Pi^L - (0,\bar{\rho})||^2_{H^k(\Omega)}(t) + 2\, ||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right).
\end{aligned}
$$

Therefore, we are in position to prove the main result of this section. To do it, we consider some smallness assumptions over the auxiliary function G.

Lemma 3.6.2. *Let* $k \geqslant 2$ *and* $(\rho(0), \mathbf{u}(0)) \in X^k(\Omega) \times \mathbb{X}^k(\Omega)$. *Assume that* $G : [-1,1] \times \mathbb{R}^+ \to \mathbb{R}$ *satisfies that* $G \in L^\infty(0,\infty; H^{k+1}([-1,1]))$ *and* $\partial_t G \in L^\infty(0,\infty; L^\infty([-1,1]))$ *with:*

$$
\max\{||G||_{H^{k+1}([-1,1])}(t), ||\partial_t G||_{L^\infty([-1,1])}(t)\} \leqslant \epsilon \qquad \textit{for all} \quad t \geqslant 0.
$$

Then, for $w := 1 + G(y,t)$ *and* $w^\star := w - \frac{1}{2}\partial_t w$ *the solution of equation (3.62) satisfies that:*

$$
\tfrac{1}{2}\partial_t \left\{ ||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t) \right\} \lesssim -||\partial_t \mathbf{u} + \mathbf{u}||^2_{H^k(\Omega)}(t). \quad (3.65)
$$

Proof. First of all, due to the smallness conditions over G, for all $(y,t) \in [-1,1] \times \mathbb{R}^+$ we have that:

$$
1 - \tfrac{3}{2}\epsilon \leqslant |w^\star(y,t)| \leqslant 1 + \tfrac{3}{2}\epsilon \qquad \text{and} \qquad |w(y,t) - w^\star(y,t)| \leqslant \tfrac{1}{2}\epsilon.
$$

In consequence, we get:

$$
-\left(||\mathbf{u}||^2_{H^k_{w^\star}(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right) + ||u_2||^2_{H^k_{w-w^\star}(\Omega)}(t) \lesssim -\left(||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right).
$$

Now, considering the linear version of the Lemma 3.5.3 we have that there exists $0 < \tilde{C} < 1$ such that:

$$
||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \geqslant \tilde{C} \left(||\mathbf{u}||^2_{H^k(\Omega)}(t) + || - \nabla\Pi^L + (0,\bar{\rho})||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right).
$$

Hence, thanks to the fact that $||G||_{H^{k+1}([-1,1])}(t)$ is small enough for all time, we arrive to:

$$
\begin{aligned}
&\tfrac{1}{2}\partial_t \left\{ ||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t) \right\} \\
&\qquad \leqslant -C^\star \left(||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right)
\end{aligned}
$$

for some $0 < C^\star < \tilde{C} < 1$. Hence, by Young's inequality it is clear that there exists $0 < \gamma < 1$ such that:

$$
\begin{aligned}
&\tfrac{1}{2}\partial_t \left\{ ||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t) \right\} \\
&\leqslant -C^\star \left(||\mathbf{u}||^2_{H^k(\Omega)}(t) + ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right) \\
&\leqslant -C^\star \left(||\mathbf{u} + \partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) + 2\, ||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t) \right) + 2\, C^\star \left(\frac{\gamma\, ||\mathbf{u} + \partial_t \mathbf{u}||^2_{H^k(\Omega)}(t)}{2} + \frac{||\partial_t \mathbf{u}||^2_{H^k(\Omega)}(t)}{2\gamma} \right).
\end{aligned}
$$

Considering for simplicity $\gamma = 1/2$, we have proved our goal. □

3.6.2.1 The Stream Formulation

Because of the incompressibility of the flow $\nabla \cdot \mathbf{u} = 0$, we write the velocity as the gradient perpendicular of a *stream function* ψ^L, i.e.,

$$\mathbf{u} = \nabla^{\perp}\psi^L \tag{3.66}$$

with $\nabla^{\perp} \equiv (-\partial_y, \partial_x)$. Then, computing the *curl* of the evolution equation of the velocity, we get the following Poisson equation:

$$\Delta\left(\partial_t\psi^L + \psi^L\right) = \partial_x\bar{\rho}. \tag{3.67}$$

Taking in account (3.66) and the no-slip condition we obtain the boundary condition:

$$\partial_x\psi^L|_{\partial\Omega} = 0.$$

Thus, we need to impose $\psi^L|_{\{y=\pm 1\}} = b_{\pm}$ where b_+ could be, in principle, different from b_-. However the periodicity in the x-variable of Π force to take $b_+ = b_-$, and since we are only interested in the derivatives of ψ^L we will take $b_{\pm} = 0$.

To sum up, in order to close the system of equations, we first solve

$$\begin{cases} \Delta\left(\partial_t\psi^L + \psi^L\right) & = \partial_x\bar{\rho} \qquad \text{in} \qquad \Omega, \\ \partial_t\psi^L + \psi^L & = 0 \qquad \text{on} \qquad \partial\Omega. \end{cases} \tag{3.68}$$

and then, we will use the *stream formulation* to recover the velocity field $\mathbf{u} = \nabla^{\perp}\psi^L$. To solve (3.68) with $\rho \in X^k(\Omega)$ and $\mathbf{u} \in \mathbb{X}^k(\Omega)$ we use the orthonormal basis introduced in section 3.3.2, which allows us to write the velocity in terms of the "Fourier coefficients" of $\bar{\rho}$.

Lemma 3.6.3. *Let $\rho(t) \in X^k(\Omega)$. The solution of Poisson's problem*

$$\begin{cases} \Delta(\partial_t\psi^L + \psi^L) & = \partial_x\bar{\rho} \qquad \textit{in} \qquad \Omega, \\ \partial_t\psi^L + \psi^L & = 0 \qquad \textit{on} \qquad \partial\Omega, \end{cases}$$

satisfies that $\left(\partial_t\psi^L + \psi^L\right)(t) \in X^{k+1}(\Omega)$ with $\|\partial_t\psi^L + \psi^L\|_{H^{k+1}(\Omega)}(t) \lesssim \|\bar{\rho}\|_{H^k(\Omega)}(t)$ and its Fourier expansion is given by

$$\left(\partial_t\psi^L + \psi^L\right)(x,y,t) = \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}}\left(\frac{(-1)\,ip}{p^2 + \left(q\frac{\pi}{2}\right)^2}\right)\mathcal{F}_{\omega}[\bar{\rho}(t)](p,q)\,\omega_{p,q}(x,y). \tag{3.69}$$

Proof. See section 2.3 of previous chapter. □

In particular, using the *stream formulation* we can rewrite $\partial_t\mathbf{u} + \mathbf{u} = \nabla^{\perp}(\partial_t\psi^L + \psi^L)$ where $\partial_t\psi^L + \psi^L$ is the solution of (3.68) given by (3.69). Then, we have that:

$$\begin{aligned} \|\partial_t\mathbf{u} + \mathbf{u}\|^2_{L^2(\Omega)} &= (\Delta(\partial_t\psi^L + \psi^L), \partial_t\psi^L + \psi^L) = (\partial_x\bar{\rho}, \partial_t\psi^L + \psi^L) \\ &= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{N}}\left(\frac{p^2}{p^2 + \left(q\frac{\pi}{2}\right)^2}\right)\left|\mathcal{F}_{\omega}[\bar{\rho}(t)](p,q)\right|^2. \end{aligned} \tag{3.70}$$

Lemma 3.6.4. *Let $\alpha \in \mathbb{N}$ and $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Then, the following lower bound holds:*

$$\|\partial_t\mathbf{u} + \mathbf{u}\|^2_{L^2(\Omega)}(t) \geqslant \frac{1}{N(t)}\|\bar{\rho}\|^2_{L^2(\Omega)}(t) - \frac{1}{N(t)^{1+\alpha}}\|\bar{\rho}\|^2_{H^{\alpha}(\Omega)}(t). \tag{3.71}$$

Proof. First of all, we introduce the auxiliary function $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ into (3.70) to obtain that:

$$\begin{aligned}||\partial_t \mathbf{u} + \mathbf{u}||^2_{L^2(\Omega)}(t) &\geqslant \frac{1}{N(t)}||\bar{\rho}||^2_{L^2(\Omega)}(t) + \sum_{(p,q)\in\mathbb{Z}_{\neq 0}\times\mathbb{N}} \left(\frac{1}{p^2+\left(q\frac{\pi}{2}\right)^2} - \frac{1}{N(t)}\right)\left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 \\ &\geqslant \frac{1}{N(t)}\left(||\bar{\rho}||^2_{L^2(\Omega)}(t) - \sum_{p^2+q^2(\pi/2)^2\geqslant N(t)} \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2\right). \end{aligned} \tag{3.72}$$

On the other hand, by Lemma (3.3.1) we have that:

$$\begin{aligned}\sum_{p^2+q^2(\pi/2)^2\geqslant N(t)} \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 &\leqslant \frac{1}{N(t)^\alpha}\sum_{p^2+\left(q\frac{\pi}{2}\right)^2\geqslant N(t)} \left(p^2+q^2(\pi/2)^2\right)^\alpha \left|\mathcal{F}_\omega[\bar{\rho}](p,q)\right|^2 \\ &\leqslant \frac{1}{N(t)^\alpha}||\bar{\rho}||^2_{H^\alpha(\Omega)}(t). \end{aligned} \tag{3.73}$$

Combining the preceding estimates (3.72) and (3.73) we arrive to (3.71). □

This gives for some $0 < C < 1$ that:

$$\begin{aligned}\partial_t \left\{||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t\mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t)\right\} &\leqslant -C\,||\partial_t\mathbf{u}+\mathbf{u}||^2_{H^k(\Omega)}(t) \\ &\leqslant -\frac{C}{N(t)}||\bar{\rho}||^2_{H^k(\Omega)}(t) + \frac{C}{N(t)^{1+\alpha}}||\bar{\rho}||^2_{H^{k+\alpha}(\Omega)}(t).\end{aligned}$$

It is enough to assume that $N : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies that $N'(t)\,N(t) \geqslant 1$ to obtain:

$$E_k(t) \lesssim e^{-(N(t)-N(0))}E_k(0) + \int_0^t \frac{e^{-(N(t)-N(s))}}{N(s)^{1+\alpha}}||\bar{\rho}||^2_{H^{k+\alpha}(\Omega)}(s)\,ds \tag{3.74}$$

where

$$E_k(t) := ||\mathbf{u}||^2_{H^k_w(\Omega)}(t) + ||\bar{\rho}||^2_{H^k(\Omega)}(t) + ||\partial_t\mathbf{u}||^2_{H^k(\Omega)}(t) + ||u_2||^2_{H^k_w(\Omega)}(t).$$

For simplicity, we take $N(t) := 2\sqrt{1+t}$ in (3.74), which gives us:

$$E_k(t) \lesssim e^{-2\sqrt{1+t}}\,E_k(0) + \left(\int_0^t \frac{e^{-2(\sqrt{1+t}-\sqrt{1+s})}}{(1+s)^{\frac{1+\alpha}{2}}}\,ds\right)||\bar{\rho}||^2_{L^\infty([0,t],H^{k+\alpha}(\Omega))}.$$

Now, we use the following calculus lemma, whose proof can be found in Lemma 2.5.5.

Lemma 3.6.5. *Let $\alpha \in \mathbb{N}$, we have that:*

$$\int_0^t \frac{e^{-2(\sqrt{1+t}-\sqrt{1+s})}}{(1+s)^{\frac{1+\alpha}{2}}}\,ds \lesssim \frac{1}{(1+t)^{\frac{\alpha}{2}}}.$$

Then, applying the previous inequality we see that:

$$E_k(t) \lesssim e^{-2\sqrt{1+t}}\,E_k(0) + \frac{||\bar{\rho}||^2_{L^\infty([0,t],H^{k+\alpha}(\Omega))}}{(1+t)^{\frac{\alpha}{2}}}.$$

Using that $\|\mathbf{u}\|_{H^n_w(\Omega)}(t) \approx \|\mathbf{u}\|_{H^n(\Omega)}(t)$ are equivalent norms together with the fact that $E_n(t)$ decays in time by (3.65), we have proved that:

$$E_k(t) \lesssim \frac{E_{k+\alpha}(0)}{(1+t)^{\frac{\alpha}{2}}}.$$

In particular, we have that:

$$\|\mathbf{u}\|^2_{H^k(\Omega)}(t) + \|\bar{\rho}\|^2_{H^k(\Omega)}(t) \lesssim \frac{\|\mathbf{u}\|^2_{H^{k+\alpha}(\Omega)}(0) + \|\bar{\rho}\|^2_{H^{k+\alpha}(\Omega)}(0)}{(1+t)^{\frac{\alpha}{2}}}.$$

3.6.3 Non-Linear Decay

Next, we will show how this decay of the quasi-linear solutions can be used to establish the stability of the stationary solution $(\rho, \mathbf{u}) = (0,0)$ for the general problem (3.9). When perturbing around it, as we have seen in Section 3.6.1, we get the following system:

$$\begin{cases} \partial_t \bar{\rho} + (1 + \partial_y \tilde{\rho})\, u_2 & = -\overline{\mathbf{u} \cdot \nabla \bar{\rho}} \\ \partial_t \tilde{\rho} & = -\widetilde{\mathbf{u} \cdot \nabla \bar{\rho}} \\ \partial_t \mathbf{u} + \mathbf{u} - \left(-\nabla \Pi^L + (0, \bar{\rho})\right) & = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \Pi^{NL} \\ \nabla \cdot \mathbf{u} & = 0 \end{cases} \tag{3.75}$$

with $(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \Pi^{NL} \equiv \mathbb{L}[(\mathbf{u} \cdot \nabla)\mathbf{u}]$, where $\mathbb{L}$ is the Leray's proyector.

Using Duhamel's formula, with $G(y,t) \equiv \partial_y \tilde{\rho}(y,t)$ small enough in the adequate space, we can write the solution of (3.75) as:

$$\begin{pmatrix} \bar{\rho}(t) \\ \mathbf{u}(t) \end{pmatrix} = e^{\mathscr{L}(t,0)} \begin{pmatrix} \bar{\rho}(0) \\ \mathbf{u}(0) \end{pmatrix} - \int_0^t e^{\mathscr{L}(t,s)} \begin{pmatrix} \overline{\mathbf{u} \cdot \nabla \bar{\rho}}(s) \\ \mathbb{L}[(\mathbf{u} \cdot \nabla)\mathbf{u}](s) \end{pmatrix} ds \qquad \text{and} \qquad \tilde{\rho}(t) = \tilde{\rho}(0) - \int_0^t \widetilde{\mathbf{u} \cdot \nabla \bar{\rho}}(s)\, ds$$

where $e^{\mathscr{L}(t,s)}$ denotes the solution operator of the associated quasi-linear problem (3.62) from s to t. Hence, we have:

$$\begin{aligned} \|\bar{\rho}\|_{H^n(\Omega)}(t) + \|\mathbf{u}\|_{H^n(\Omega)}(t) \lesssim & \frac{\|\bar{\rho}\|_{H^{n+\alpha}(\Omega)}(0) + \|\mathbf{u}\|_{H^{n+\alpha}(\Omega)}(0)}{(1+t)^{\frac{\alpha}{4}}} \\ & + \int_0^t \frac{\|\overline{\mathbf{u} \cdot \nabla \bar{\rho}}\|_{H^{n+\alpha}(\Omega)}(s) + \|\mathbb{L}[(\mathbf{u} \cdot \nabla)\mathbf{u}]\|_{H^{n+\alpha}(\Omega)}(s)}{(1+(t-s))^{\frac{\alpha}{4}}}\, ds \end{aligned} \tag{3.76}$$

and

$$\|\tilde{\rho}\|_{H^n(\Omega)}(t) \leqslant \|\tilde{\rho}\|_{H^n(\Omega)}(0) + \int_0^t \|\widetilde{\mathbf{u} \cdot \nabla \bar{\rho}}\|_{H^n(\Omega)}(s)\, ds.$$

3.7 The boostraping

We now demonstrate the bootstrap argument used to prove our goal. Theorem 3.5.1 tell us that the following estimate holds for $k \geqslant 6$:

$$\begin{aligned} \partial_t \mathfrak{E}_{k+1}(t) \leqslant & -(C - \tilde{C}\,\Psi_1(t)) \left[\|\nabla \Pi - (0,\bar{\rho})\|^2_{H^k}(t) + \|(\mathbf{u} \cdot \nabla)\,\mathbf{u}\|^2_{H^k}(t) + \|\mathbf{u}\|^2_{H^k}(t) + \|\partial_t \mathbf{u}\|^2_{H^k}(t)\right] \\ & - \left(1 - \tilde{C}\,\Psi_2(t)\right) \left(\int_\Omega |\partial^{k+1}\mathbf{u}(x,y,t)|^2\, (1 + \partial_y \tilde{\rho}(y,t))\, dx dy\right) \\ & + \|\mathbf{u}\|_{H^4}\, \mathfrak{E}_{k+1}(t). \end{aligned}$$

The last section is devoted to prove the main result of this chapter:

Theorem 3.7.1. *There exists $\epsilon_0 > 0$ and parameters $\gamma, \kappa \in \mathbb{N}$ with $\gamma > 4$ and $\kappa \geqslant 6 + 2\gamma$ so that if we solve (3.2) with initial data $\varrho\ (0) = \Theta + \rho(0)$ and velocity $\mathbf{u}(0)$ such that $(\rho(0), \mathbf{u}(0)) \in X^{\kappa+1}(\Omega) \times \mathbb{X}^{\kappa+1}(\Omega)$ and $\mathfrak{E}_{\kappa+1}(0) < \epsilon^2 \leqslant \epsilon_0^2$. Then, the solution exists globally in time and satisfies the following:*

i) $||\bar{\varrho}||_{H^4}(t) \equiv ||\bar{\rho}||_{H^4}(t) \lesssim \dfrac{\varepsilon}{(1+t)^{\gamma/4}}$;

ii) $||\mathbf{u}||_{H^4}(t) \lesssim \dfrac{\varepsilon}{(1+t)^{\gamma/4}}$;

iii) $||\tilde{\varrho} - \Theta||_{H^{\kappa+1}}(t) \equiv ||\tilde{\rho}||_{H^{\kappa+1}}(t) \leqslant 6\epsilon^2$.

We need to prove:

Lemma 3.7.2. *If $\mathfrak{E}_{\kappa+1}(0) \leqslant \varepsilon^2$ and $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ on the interval $[0, T]$ with $0 < \varepsilon \leqslant \varepsilon_0$ small enough. Then $\mathfrak{E}_{\kappa+1}(t)$ remains uniformly bounded by $3\varepsilon^2$ on the interval $[0, T]$.*

We will prove Lemma 3.7.2 through a bootstrap argument, where the main ingredient is the estimate (3.31). We will work with the following bootstrap hypothesis, to assume that $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ on the interval $[0, T]$ where κ is big enough and $0 < \varepsilon \ll 1$ such that:

$$\left(C - \tilde{C}\,\Psi_1(t)\right) > 0 \quad \text{and} \quad \left(1 - \tilde{C}\,\Psi_2(t)\right) > 0 \qquad \text{on} \quad [0, T].$$

Then, by Grönwall's inequality we have:

$$\mathfrak{E}_{\kappa+1}(t) \leqslant \mathfrak{E}_{\kappa+1}(0) \exp\left(\int_0^t ||\mathbf{u}||_{H^4}(s)\, ds\right) \qquad t \in [0, T].$$

Our aim here is to show that the interval on which the a priori estimates hold can be extended to infinity. Using a continuity argument it will suffice to prove that $||\mathbf{u}||_{H^4}(t)$ decays at an integrable rate. An immediate consequence of this and the previous inequality is that there exists $T^\star > T$ such that $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ on it. Therefore, we can repeat iteratively this process, in order to extend our result for all time.

3.7.1 Integral Decay of $||\mathbf{u}||_{H^4(\Omega)}$

In order to control $||\mathbf{u}||_{H^4(\Omega)}$ in time we have the following result.

Lemma 3.7.3. *Assume that $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ for all $t \in [0, T]$ where $\kappa \geqslant 5 + 2\gamma$ with $\gamma \in \mathbb{N}$. Then, the solution satisfies that:*

$$(||\bar{\rho}||_{H^4} + ||\mathbf{u}||_{H^4})\,(t) \lesssim \frac{(||\bar{\rho}||_{H^{4+\gamma}} + ||\mathbf{u}||_{H^{4+\gamma}})\,(0)}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{(||\bar{\rho}||_{H^4} + ||\mathbf{u}||_{H^4})\,(s)}{(1+(t-s))^{\frac{\gamma}{4}}}\,(||\bar{\rho}||_{H^{\kappa+1}} + ||\mathbf{u}||_{H^{\kappa+1}})\,(s)\, ds.$$

Proof. By assumption $\partial_y \tilde{\rho}(t)$ is small in $H^\kappa(\Omega)$ and $\partial_t \partial_y \tilde{\rho}(t)$ is small in $H^{\kappa-1}(\Omega)$ for all $t \in [0, T]$. This implies that $\mathscr{L}(t, s)$ has nice decay properties from s to t with $t \in [0, T]$ in $H^4(\Omega)$ if $\kappa \geqslant 5 + \gamma$. Hence, Duhamel's formula (3.76) gives us:

$$\begin{aligned}||\bar{\rho}||_{H^4}(t) + ||\mathbf{u}||_{H^4}(t) \lesssim\ & \frac{||\bar{\rho}||_{H^{4+\gamma}}(0) + ||\mathbf{u}||_{H^{4+\gamma}}(0)}{(1+t)^{\frac{\gamma}{4}}} \\ &+ \int_0^t \frac{1}{(1+(t-s))^{\frac{\gamma}{4}}}\,\left\{||\overline{\mathbf{u}\cdot\nabla\bar{\rho}}||_{H^{4+\gamma}}(s) + ||\mathbb{L}[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]||_{H^{4+\gamma}}(s)\right\}\, ds\end{aligned}$$

and we have that:

$$\|\overline{\mathbf{u}\cdot\nabla\bar{\rho}}\|_{H^{4+\gamma}} + \|\mathbb{L}[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]\|_{H^{4+\gamma}} \leqslant \|\mathbf{u}\|_{H^{4+\gamma}}\left(\|\bar{\rho}\|_{H^{5+\gamma}} + \|\mathbf{u}\|_{H^{5+\gamma}}\right).$$

To sum up, we obtain that:

$$\left(\|\bar{\rho}\|_{H^4} + \|\mathbf{u}\|_{H^4(\Omega)}\right)(t) \lesssim \frac{\left(\|\bar{\rho}\|_{H^{4+\gamma}} + \|\mathbf{u}\|_{H^{4+\gamma}}\right)(0)}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{\|\mathbf{u}\|_{H^{4+\gamma}}(s)}{(1+(t-s))^{\frac{\gamma}{4}}}\left(\|\bar{\rho}\|_{H^{5+\gamma}} + \|\mathbf{u}\|_{H^{5+\gamma}}\right)(s)\,ds.$$

However, due to the well-known Gagliardo-Nirenberg interpolation inequalities:

$$\|D^j f\|_{L^2(\Omega)} \leqslant C\,\|D^{2\,j} f\|_{L^2(\Omega)}^{1/2}\,\|f\|_{L^2(\Omega)}^{1/2} + \tilde{C}\,\|f\|_{L^2(\Omega)}$$

we obtain

$$\|\bar{\rho}\|_{H^{5+\gamma}} \lesssim \|\bar{\rho}\|_{H^{6+2\gamma}}^{1/2}\,\|\bar{\rho}\|_{H^4}^{1/2} \qquad \text{and} \qquad \|\mathbf{u}\|_{H^{5+\gamma}} \lesssim \|\mathbf{u}\|_{H^{6+2\gamma}}^{1/2}\,\|\mathbf{u}\|_{H^4}^{1/2}. \tag{3.77}$$

Therefore, if we apply (3.77) in the previous inequality, we get:

$$\begin{aligned}\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4}(t) \lesssim{}& \frac{\|\bar{\rho}\|_{H^{4+\gamma}}(0) + \|\mathbf{u}\|_{H^{4+\gamma}}(0)}{(1+t)^{\frac{\gamma}{4}}} \\ &+ \int_0^t \frac{\|\bar{\rho}\|_{H^4}(s) + \|\mathbf{u}\|_{H^4}(s)}{(1+(t-s))^{\frac{\gamma}{4}}}\,\|\mathbf{u}\|_{H^{4+2\gamma}}^{1/2}(s)\left(\|\bar{\rho}\|_{H^{6+2\gamma}}^{1/2}(s) + \|\mathbf{u}\|_{H^{6+2\gamma}}^{1/2}(s)\right)ds.\end{aligned}$$

In particular, for $\kappa \in \mathbb{N}$ such that $\kappa \geqslant 5 + 2\gamma$ we have proved our goal. □

The following basic lemma is stated without proof (for a proof see [27], Lemma 2.4).

Lemma 3.7.4. *Let* $\delta, \tau > 0$, *then:*

$$\int_0^t \frac{ds}{(1+(t-s))^{\delta}\,(1+s)^{1+\tau}} \leqslant \frac{\mathcal{C}_{\delta,\tau}}{(1+t)^{\min\{\delta,1+\tau\}}}.$$

Lemma 3.7.5. *Assume that* $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ *for all* $t \in [0,T]$ *where* $\kappa \geqslant 5 + 2\gamma$ *with* $\gamma \in \mathbb{N}$. *Then, we have:*

$$\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4(\Omega)}(t) \lesssim \frac{\varepsilon}{(1+t)^{\frac{\gamma}{4}}} \qquad \textit{for all} \quad t \in [0,T].$$

Proof. By hypothesis, $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\varepsilon^2$ on the interval $[0,T]$. Then, we obtain that:

$$\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4}(t) \leqslant \frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{C\,\varepsilon}{(1+(t-s))^{\frac{\gamma}{4}}}\left(\|\bar{\rho}\|_{H^4}(s) + \|\mathbf{u}\|_{H^4}(s)\right)\,ds$$

and in particular, there exist $0 < T^{\star}(C) \leqslant T$ such that for $t \in [0, T^{\star}(C)]$ we have:

$$\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4}(t) \leqslant 6\,\frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}}.$$

If we restrict to $0 \leqslant t \leqslant T^{\star}(C)$ and we apply the previous Lemma 3.7.4, we have:

$$\begin{aligned}\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4}(t) &\leqslant \frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \int_0^t \frac{C\,\varepsilon}{(1+(t-s))^{\frac{\gamma}{4}}}\,\frac{6\,C\,\varepsilon}{(1+s)^{\frac{\gamma}{4}}}\,ds \\ &\leqslant \frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}} + \frac{\tilde{C}\,\varepsilon^2}{(1+t)^{\frac{\gamma}{4}}}.\end{aligned}$$

The last term in the expression above is quadratic in ε, it is enough to find $0 < \epsilon \ll 1$ small enough so that

$$\|\bar{\rho}\|_{H^4}(t) + \|\mathbf{u}\|_{H^4}(t) \leqslant 3\,\frac{C\,\varepsilon}{(1+t)^{\frac{\gamma}{4}}}$$

for all $t \in [0, T^\star(C)]$ and, by continuity, for all $t \in [0, T]$.

$\square$

So, with $\gamma > 4$ we have proved the integrable decay of $\|\mathbf{u}\|_{H^4(\Omega)}(t)$. Then we are able to close our energy estimate. We are now in the position to show how the bootstrap can be closed. This is merely a matter of collecting the conditions established above and showing that they can indeed be satisfied.

In conclusion, if $\mathfrak{E}_{\kappa+1}(t) \leqslant 6\,\varepsilon^2$ for all $t \in [0, T]$ we have that

$$\begin{aligned}\mathfrak{E}_{\kappa+1}(t) &\leqslant \mathfrak{E}_{\kappa+1}(0)\exp\left(\int_0^t \|\mathbf{u}\|_{H^4}(s)\,ds\right)\\ &\leqslant \varepsilon^2\,\exp\left(\int_0^t \frac{C\varepsilon}{(1+s)^{\gamma+(1/4)^-}}\,ds\right) \leqslant \varepsilon^2\exp(\tilde{C}\varepsilon)\end{aligned}$$

and $\mathfrak{E}_{\kappa+1}(t) \leqslant 3\,\varepsilon^2$ for all $t \in [0, T]$ if we consider ε small enough, which allows us to prolong the solution and then repeat the argument for all time.

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Para Carlota: nos queda todo.