1 Classical and Exceptional Orthogonal Polynomials

1.1 Orthogonal Polynomials

An orthogonal polynomial system is an infinite sequence of univariate polynomials that are orthogonal with respect to a positive measure on the real line. Let $W(z)$ be a positive weight function on an open interval $I = (a, b)$ and assume that the moments

$$m_n = \int_a^b z^n W(z) \, dz, \quad n = 0, 1, 2, \ldots$$

are finite.

For $n \geq 0$, the Hankel matrix is the symmetric matrix of the moments:

$$A_n = \begin{bmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{bmatrix}$$

This matrix is positive definite, meaning that for any non-zero vector $a = [a_0, a_1, \ldots, a_{n+1}]^t \in \mathbb{R}^{n+1}$ we have

$$a^t A_n a = \int_I (a_0 z^0 + a_1 z + \cdots + a_n z^n)^2 W(z) \, dz > 0.$$ 

Consequently, the corresponding Hankel determinants,

$$\Delta_n = \det A_n > 0, \quad n \geq 0$$

are all strictly positive.

Consider the inner product on the weighted Hilbert space $L^2(W, I)$

$$\langle f, g \rangle_W = \int_I f(z) g(z) W(z) \, dz.$$  (1)
Let $P_n$ be the $n$-th degree polynomial defined by the following $(n+1) \times (n+1)$ determinant
\[
P_n = \begin{vmatrix}
m_0 & m_1 & \ldots & 1 \\
m_1 & m_2 & \ldots & z \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n+1} & \ldots & z^n
\end{vmatrix}
\] (2)

Proposition 1. The polynomials $P_n$, $n = 0, 1, 2, \ldots$ are orthogonal relative to the inner product (1).

Proof. Fix an $n \geq 1$. Since the determinant is linear with respect to the last column, we have that
\[
\langle P_n, z^i \rangle_W = \begin{vmatrix}
m_0 & m_1 & \ldots & m_i \\
m_1 & m_2 & \ldots & m_{i+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n+1} & \ldots & m_{i+n}
\end{vmatrix}, \quad i = 0, 1, 2, \ldots
\]

Since a determinant with repeated columns is zero, we deduce that
\[
\langle P_n, z^i \rangle_W = 0, \quad i = 0, 1, 2, \ldots, n-1.
\]

Again, by multilinearity of the determinant, every $P_i(z)$ is a linear combination of $1, z, \ldots, z^i$. It follows that
\[
\langle P_n, P_i \rangle_W = 0, \quad i = 0, 1, 2, \ldots, n-1.
\]

Exercise 1. Prove that the sequence of polynomials $M_n^{-1}P_n$, $n = 0, 1, 2, \ldots$ are precisely the sequence obtained by applying the Gram-Schmidt algorithm to $1, z, z^2, \ldots$.

Since $zP_n(z)$ is a polynomial of degree $n+1$, it is a linear combination of $P_{n+1}, P_n, \ldots, P_0$. For $j = 0, 1, 2, \ldots$, we have
\[
\langle zP_n, P_j \rangle_W = \langle P_n, zP_j \rangle_W = \int zP_n(z)P_j(z)dz.
\]

It follows that $zP_n$ is orthogonal to $P_j$ for $j = 0, 1, 2, \ldots, n-2$. Therefore, there exists real $A_n, B_n, C_n$ such that
\[
zP_0(z) = A_1P_1(z) + B_1P_0(z),
zP_n(z) = A_nP_{n+1}(z) + B_nP_n + C_nP_{n-1}, \quad n \geq 1.
\] (3)

Relation (3) is called a 3-term recurrence relation, and it fully determines the sequence $P_n$, $n = 0, 1, 2, \ldots$ once $P_0(z)$ is chosen.

If $K_n \neq 0, n = 0, 1, 2, \ldots$ is a sequence of non-zero real numbers, then the polynomials $\hat{P}_n(z) = K_nP_n(z)$ are also orthogonal relative to the $W$-weighted inner product. Thus, a particular weight does not fully determine the orthogonal polynomials and the recurrence relation; one also requires a choice of normalization.

The 3-term recurrence relation is just as fundamental to the theory of orthogonal polynomials as the inner product because of the following result.
**Theorem 1.** Let $P_n(z)$, $n = 0, 1, 2, \ldots$ be a sequence of polynomials with $\deg P_n = n$ and positive leading coefficients that satisfies a 3-term recurrence relation (3). Let $\lambda_n$, $n = 0, 1, 2, \ldots$ be a sequence of positive real numbers. Then, there exists a measure $\mu$ on $\mathbb{R}$ such that

$$
\int_{\mathbb{R}} P_i(z)P_j(z)d\mu(z) = \begin{cases} 
0 & \text{if } i \neq j \\
\lambda_i & \text{if } i = j.
\end{cases}
$$

The above is known as Favard’s Theorem — the proof is available in many places. Note that we are not asserting uniqueness of the measure. There are some sufficient conditions that guarantee uniqueness, and some examples where uniqueness fails. Also note that the measure in question may not be absolutely continuous. Indeed, some of the most interesting orthogonal polynomials are defined relative to a discrete measure.

### 1.2 Sturm-Liouville problems

A Sturm-Liouville problem is a second-order boundary value problem of the form

$$
-(P(z)y')' + R(z)y = \lambda W(z)y, \quad y = y(z),
$$

$$
\alpha_0 y(a) + \alpha_1 y'(a) = 0,
$$

$$
\beta_0 y(b) + \beta_1 y'(b) = 0
$$

where $I = (a, b)$ is an interval, where $\lambda$ is a spectral parameter, where $P(z), W(z), R(z)$ are suitably smooth real-valued functions with $P(z), W(z) > 0$ for $z \in I$.

Dividing (4) by $W(z)$ re-expresses the underlying differential equation in an operator form:

$$
-T[y] = \lambda y,
$$

where

$$
T[y] = p(z)y'' + q(z)y' + r(z)y,
$$

and where

$$
p(z) = \frac{P(z)}{W(z)} \quad P(z) = \exp \int \frac{q(z)}{p(z)}dz
$$

$$
q(z) = \frac{P'(z)}{W(z)} \quad W(z) = \frac{P(z)}{p(z)},
$$

$$
r(z) = -\frac{R(z)}{W(z)} \quad R(z) = -r(z)W(z)
$$

---

1In the case of an unbounded interval with $a = -\infty$ and/or $b = +\infty$, or if solutions $y(z)$ of (4) have no defined value at the endpoints, one has to consider the asymptotics of the corresponding solutions and impose boundary conditions of a more general form:

$$
\alpha_0 y(z) + \alpha_1 y'(z) \to 0 \quad \text{as } z \to a^-
$$

$$
\beta_0 y(z) + \beta_1 y'(z) \to 0 \quad \text{as } z \to b^+
$$

where $\alpha_0(z), \alpha_1(z), \beta_0(z), \beta_1(z)$ are continuous functions defined on $I$. 

3
If \( y_1(z), y_2(z) \) are two sufficiently smooth real-valued functions, then integration by parts gives Lagrange's identity:

\[
\int (T[y_1]y_2 - T[y_2]y_1)(z) W(z) dz = P(z)(y_1'(z)y_2(z) - y_2'(z)y_1(z)).
\]

(9)

Suppose that the boundary conditions entail (i) the square integrability of eigenfunctions with respect to \( W(z) dz \) over the interval \( I \); and (ii) the vanishing of the right side of (9) at the endpoints of the interval. With some suitable regularity assumptions on \( P(z), W(z), R(z) \) one can then show that the eigenvalues of \(-T\) can be ordered so that

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty.
\]

If \( y_i, y_j, i \neq j \) are two eigenfunctions corresponding to eigenvalues \( \lambda_i, \lambda_j \), respectively, then (9) reduces to

\[
(\lambda_i - \lambda_j) \int_I y_i(z)y_j(z)W(z) dz = P(z)(y_i'(z)y_j(z) - y_j'(z)y_i(z)) \bigg|_{a}^{b} = 0.
\]

(10)

Therefore, the eigenfunctions are orthogonal with respect to the inner product

\[
\langle f, g \rangle_W = \int_I f(z)g(z)W(z) dz.
\]

**Example 1.** Let’s work out the weight and boundary conditions for the Hermite differential equation

\[
y'' - 2zy + \lambda y, \quad y = y(z).
\]

(11)

We apply (8) and rewrite the above in Sturm-Liouville form

\[
-(W(z)y') = \lambda W(z)y, \quad y \in L^2(\mathbb{R}, W dz)
\]

(12)

where the weight has the form

\[
W(z) = \exp \left( \int^z (-2z) \right) = e^{-z^2}
\]

In this case, the boundary conditions are that \( e^{-z^2}y(z)^2 \) be integrable near \( \pm \infty \).

A basis of solutions to (11) are

\[
\phi_0(z; \lambda) = \Phi \left( -\frac{\lambda}{4}, \frac{1}{2}, z^2 \right)
\]

(13)

\[
\phi_1(z; \lambda) = z \Phi \left( \frac{1}{2} - \frac{\lambda}{4}, \frac{3}{2}, z^2 \right)
\]

(14)

where

\[
\Phi(a, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n,
\]
is the confluent hypergeometric function. This function has the asymptotic behaviour
\[ \Phi(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)} e^{x a - c} (1 + O(|x|^{-1})), \quad x \to +\infty, \]

This implies that
\[ e^{-z^2} \phi_0(z; \lambda)^2 = \frac{\pi e^{z^2} z^{-2-\lambda}}{\Gamma(-\lambda/4)^2} (1 + O(z^{-2})), \quad z \to \pm\infty, \]
\[ e^{-z^2} \phi_1(z; \lambda)^2 = \frac{\pi e^{z^2} z^{-2-\lambda}}{4\Gamma(1/2 - \lambda/4)^2} (1 + O(z^{-2})), \quad z \to \pm\infty. \]

are not integrable for generic values of \( \lambda \) near \( z = \pm\infty \). We now introduce two other solutions of (11),
\[ \psi_R(z; \lambda) = \Psi \left( -\frac{\lambda}{4}, \frac{1}{2}, z^2 \right), \quad z > 0 \quad (15) \]
\[ \psi_L(z; \lambda) = \Psi \left( -\frac{\lambda}{4}, \frac{1}{2}, z^2 \right), \quad z < 0 \quad (16) \]

where
\[ \Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} \Phi(a-c+1, 2-c; x), \quad x > 0. \quad (17) \]

Note that \( \psi_R(z) \) and \( \psi_L(z) \) are different functions, because \( \Psi \) is a branch of a multi-valued function defined by taking a branch cut over the negative real axis. However, \( \psi_L, \psi_R \) may be continued to solutions of (11) over all of \( \mathbb{R} \) by means of connection formulae (18), below.

We have the asymptotics
\[ x^a \Psi(a, c; x) = 1 + O(x^{-1}), \quad x \to +\infty \]
\[ e^{-z^2} \psi_R(z; \lambda)^2 = e^{-z^2} z^\lambda (1 + O(z^{-2})), \quad z \to +\infty \]
\[ e^{-z^2} \psi_L(z; \lambda)^2 = e^{-z^2} z^\lambda (1 + O(z^{-2})), \quad z \to -\infty \]

Hence, \( \psi_R, \psi_L \) each satisfy a one-sided boundary conditions at \( \pm\infty \).

From (17) we get the connection formulae
\[ \psi_R(z; \lambda) = \frac{\sqrt{\pi}}{\Gamma(1/2 - \lambda/4)} \phi_0(z; \lambda) - \frac{2\sqrt{\pi}}{\Gamma(-\lambda/4)} \phi_1(z; \lambda), \quad (18) \]
\[ \psi_L(z; \lambda) = \frac{\sqrt{\pi}}{\Gamma(1/2 - \lambda/4)} \phi_0(z; \lambda) + \frac{2\sqrt{\pi}}{\Gamma(-\lambda/4)} \phi_1(z; \lambda). \]

Therefore, our boundary conditions amount to imposing the condition that \( \psi_L \) be proportional to \( \psi_R \). By inspection of (18), this can happen in exactly two ways: \( \psi_L = \psi_R \) and \( \psi_L = -\psi_R \). The first case occurs when \( \Gamma(-\lambda/4) \to \infty \) that is when \( \lambda/2 = 2n, \quad n = \)
The second possibility occurs when $\Gamma(1/2 - \lambda/4) \to \infty$ which occurs when $
/2 = 2n + 1$, $n = 1, 2, \ldots$. In the first case, we recover the even Hermite polynomials; in the second the odd Hermite polynomials. This last observation can be restated as the following identity

$$2^{-n}H_n(z) = \sqrt{\pi} \left( \frac{\phi_0(z; 2n)}{\Gamma(1/2 - n/2)} - \frac{2\phi_1(z; 2n)}{\Gamma(-n/2)} \right), \quad n = 0, 1, 2, \ldots.$$

Therefore the Hermite polynomials are precisely the solutions of (11) that satisfy the boundary conditions of (12), namely they are the only solutions of (11) that are square-integrable with respect to $e^{-z^2}$ over all of $\mathbb{R}$.

### 1.3 Classical Orthogonal Polynomials

The notion of a Sturm-Liouville system with polynomial eigenfunctions is the cornerstone idea in the theory of classical orthogonal polynomials. The reason is simple; if the eigenfunctions of a Sturm-Liouville problem (4) are polynomials, then they will be orthogonal with respect to the corresponding weight $W(z)$.

The following three types of polynomials — bearing the names of Hermite, Laguerre, and Jacobi — are known as the classical orthogonal polynomials.

The Hermite polynomials obey the following 3-term recurrence relation:

$$2zH_n = H_{n+1} + 2nH_{n-1}, \quad H_{-1} = 0, \quad H_0 = 1. \quad (19)$$

They are orthogonal with respect to

$$W_H(z) = e^{-z^2}, \quad z \in (-\infty, \infty),$$

and satisfy the following differential equation

$$y'' - 2zy' + 2ny = 0, \quad y = H_n(z), \quad n \in \mathbb{N}, \quad (20)$$

Laguerre polynomials $L_n = L_n^{(\alpha)}(z)$ have one parameter $\alpha$, and satisfy the following 3-term recurrence relation:

$$2zL_n = (n + 1)L_{n+1} - (2n + \alpha + 1)L_n + (n + \alpha)L_{n-1}, \quad L_{-1} = 0, L_0 = 1. \quad (21)$$

For $\alpha > -1$, Laguerre polynomials are orthogonal with respect to

$$W_L = e^{-z}z^\alpha, \quad z \in (0, \infty).$$

They satisfy the following differential equation

$$zy'' + (\alpha + 1 - z)y' + ny = 0, \quad y = L_n^{(\alpha)}(z), \quad n \in \mathbb{N}, \quad (22)$$
Jacobi polynomials $P_n = P_n^{(\alpha, \beta)}(z)$ have two parameters, $\alpha, \beta$ and are defined by:

$$zP_n = \frac{2(n + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}P_{n+1} + \frac{(\beta^2 - \alpha^2)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}P_n + \frac{2(n + \alpha)(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}P_{n-1},$$

$$P_{-1} = 0, \quad P_0 = 1 \quad (23)$$

These polynomials obey the differential equation

$$(1 - z^2)P_n'' + (\beta - \alpha - z(\alpha + \beta + 2))P_n' + n(\alpha + \beta + n + 1)P_n = 0 \quad (24)$$

For $\alpha, \beta > -1$ they are orthogonal with respect to

$$W_H = (1 - z)^\alpha (1 + z)^\beta, \quad z \in (-1, 1).$$

**Exercise 2.** Rewrite the above differential equations in Sturm-Liouville form. In each case, work out the boundary conditions that pick out the polynomial solutions.

The class of Sturm-Liouville problems with polynomial eigenfunctions is was studied and classified by Solomon Bochner in the following fundamental result. Bochner’s Theorem was subsequently refined by Lesky to show that the three classical families of Hermite, Laguerre, and Jacobi give a full classification of such Sturm-Liouville problem.

**Theorem 2** (Bochner). Suppose that an operator

$$T[y] = p(z)y'' + q(z)y' + r(z)y \quad (25)$$

admits eigenpolynomials of every degree; that is, there exist polynomials $y_k(z)$ with $\deg y_k = k$ and constants $\lambda_k$ such that

$$-T[y_k] = \lambda_k y_k, \quad k = 0, 1, 2, \ldots. \quad (26)$$

Then, necessarily $p, q, r$ are polynomials with

$$\deg p \leq 2, \quad \deg q \leq 1, \quad \deg r = 0.$$

Moreover, if these polynomials are the orthogonal eigenfunctions of a Sturm-Liouville system, then up to an affine transformation of the independent variable $z$, they are the classical polynomials of Hermite, Laguerre, and Jacobi.

**Proof.** Applying (26) to $k = 0, 1, 2$, we obtain

$$-\lambda_0 y_0 = r$$
$$-\lambda_1 y_1 = qy_1' + ry_1$$
$$-\lambda_2 y_2 = py_2'' + qy_2' + ry_2.$$
Über Sturm-Liouville'sche Polynomsysteme.

Von

S. Bochner in München.

Wir betrachten irgendeine Differentialgleichung der Form

\[ p_0(x) y'' + p_1(x) y' + p_2(x) y + \lambda y = 0. \]

Die Koeffizienten \( p_0(x), p_1(x), p_2(x) \) sind irgendwelche reell- oder komplexwertige Funktionen der Variablen \( x \); von denen wir in erster Linie nur anzunehmen brauchen, daß sie in einem gemeinsamen Intervall \( J \) der \( x \)-Achse definiert sind; und \( \lambda \) bedeutet einen Parameter, der aller komplexen Werte fähig ist.

Wir setzen nunmehr voraus, daß es eine Folge von Parameterwerten

\[ \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \]

und eine Folge von Polynomen

\[ P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots \]

gibt, von denen das \( n \)-te, \( P_n(x) \), von genau \( n \)-tem Grade ist, und die so beschaffen sind, daß das Polynom \( P_n(x) \) im Intervall \( J \) der Gleichung (1) für den Wert \( \lambda = \lambda_n \) genügt:

\[ \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \]

Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouville'sche Differentialgleichungen

PETER LESKY

Vorgelegt von J. SERRIN

Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouville'sche Differentialgleichungen gelingt besonders einfach, wenn die Verschiedenheit der zu diesen Polynomlösungen gehörigen Parameterwerte vorausgesetzt wird (vgl. E. KAMKE [7], S. 273). Im folgenden zeigen wir, daß diese Voraussetzung insofern überflüssig ist, als die erwähnten Parameterwerte gerade dann voneinander verschieden sind, wenn die Orthogonalität der entsprechenden Polynomlösungen verlangt wird.

Wir folgen zunächst der Arbeit von S. BOCHNER [2], indem wir die notwendigen Bedingungen dafür angeben, daß eine Sturm-Liouville'sche Differentialgleichung Polynomlösungen aller Grade \( n \) \((n = 0, 1, 2, \ldots)\) besitzt. Darauf aufbauend, leiten wir eine notwendige Bedingung dafür ab, daß diese Polynomlösungen einem
By inspection, $r$ is a constant, while $q, p$ are polynomials with $\deg q \leq 1$ and $\deg p \leq 2$.

Up to an affine transformation $z \mapsto sz + t$, the leading coefficient $p(z)$ can assume one of the following normal forms:

$$1, z, z^2, 1 - z^2, 1 + z^2.$$ 

Write $q(z) = az + b$. Applying (8), the corresponding weights have the form

(i) $W(z) = e^{\frac{b^2}{2(z+b/a)^2}}$

(ii) $W(z) = e^{az^b-1}$

(iii) $W(z) = e^{-\frac{b}{2}z^{a-2}}$

(iv) $W(z) = (1 - z)^{-(a+b)/2-1}(1 + z)^{(b-a)/2-1}$

(v) $W(z) = e^{b \arctan(z)}(1 + z^2)^{a/2-1}$.

For normal form (i), the case $a = 0$ is excluded. If not, the resulting operator would be strictly degree lowering, which would preclude the existence eigenpolynomials of degrees $\geq 2$. Since $p(z) = 1$ is invariant with respect to scaling and translation, no generality is lost by setting $b = 0, a = \pm 2$. The case of $a = -2$ corresponds to the classical Hermite polynomials. The case $a = 2$ can be excluded because there is no choice of boundary conditions that result in the vanishing of the right side of (10).

The form $p(z) = z$ is preserved by scaling. Hence, without loss of generality $a = -1$. This case corresponds to the classical Laguerre polynomials.

Normal form (iii) is a bit tricky. The case $b = 0$ can be ruled out because of the absence of suitable boundary conditions. The analysis of $b < 0$ and $b > 0$ is the same, so suppose that $b > 0$. Here the only possible boundary conditions are at the endpoints of the interval $(0, \infty)$. If $a < 0$ then a finite number of polynomials can be made to be square integrable with respect to the weight in question. These constitute the so-called Bessel orthogonal polynomials, which however fall outside the range of our definition — we require that all $y_k$ be square-integrable with respect to $W(z)$.

Normal form (iv) corresponds to the Jacobi orthogonal polynomials.

Normal form (v) corresponds to the so-called twisted Jacobi polynomials. If $a < 0$ then a finite number of initial degrees are square-integrable with respect to the indicated weight over the interval $(-\infty, \infty)$. As above, this violates our requirement that all the $y_k$ be square-integrable with respect to $W(z)dz$.

\[\square\]

**1.4 Exceptional Polynomials and Operators**

We now modify the assumption of Bochner’s Theorem 2 to arrive at the following.

**Definition 1.** We say that $T[y] = p(z)y'' + q(z)y' + r(z)y$ is an exceptional operator if it admits polynomial eigenfunctions for a cofinite number of degrees; that is, there exist polynomials $y_k(z), k \notin \mathbb{N} \setminus \{d_1, \ldots, d_m\}$ with $\deg y_k = k$ and with $d_1, \ldots, d_m \in \mathbb{N}$ a finite number of exceptional degrees, and constants $\lambda_k$ such that

$$-T[y_k] = \lambda_k y_k, \quad k \in \mathbb{N} \setminus \{d_1, \ldots, d_m\}.$$
Moreover, if it is possible to impose boundary conditions so that the polynomials \( y_k \) become eigenfunctions of the corresponding Sturm-Liouville problem, then we call the \( \{ y_k \}_{k \not\in \{ d_1, ..., d_m \}} \) exceptional orthogonal polynomials.

The relaxed assumption that permits for a finite number of missing degrees allows to escape the constraints of Bochner’s theorem and characterizes a large and interesting new class of operators and polynomials.

**Example: codimension 2 exceptional Hermite polynomials**  
Recall the classical Hermite polynomials defined by (19). Introduce a family of exceptional Hermite polynomials defined by

\[
\hat{H}_n = \frac{\text{Wr}[H_1, H_2, H_n]}{8(n-1)(n-2)} = H_n + 4nH_{n-2} + 4(n-3)H_{n-4}, \quad n \neq 1, 2
\]  
(27)

where the \( H_i(z) \) are classical Hermite polynomials and where \( \text{Wr} \) denotes the usual Wronskian determinant:

\[
\text{Wr}[H_1, H_2, H_n] = \begin{vmatrix}
H_1 & H'_1 & H''_1 \\
H_2 & H'_2 & H''_2 \\
H_n & H'_n & H''_n
\end{vmatrix}
\]

**Exercise 3.** Using the following identity for the classical Hermite polynomials:

\[
H'_n = 2nH_{n-1}
\]

and the 3-term recurrence relation (19) reduce the Wronskian expression

\[
\hat{H}_n = \frac{\text{Wr}[H_1, H_2, H_n]}{8(n-1)(n-2)}, \quad n \neq 1, 2
\]

to the right-hand side expression shown in (27).

Observe that \( \deg \hat{H}_n = n \). We call the resulting sequence of polynomials exceptional because the degree sequence \( \deg \hat{H}_n \) is missing two the degrees – the exceptional degrees \( n = 1 \) and \( n = 2 \). We call the \( \hat{H}_n(z) \) exceptional Hermite polynomials because they furnish polynomial solutions of the following modified version of the Hermite differential equation:

\[
y'' - \left( 2z + \frac{8z}{1+2z^2} \right) y' + 2ny = 0, \quad y = \hat{H}_n(z), \quad n \neq 1, 2.
\]

(28)

**Exercise 4.** Prove the above relation.

At first glance, the exceptional modification of Hermite’s differential equation (28) has a rather peculiar form; indeed it is slightly paradoxical that a differential equation with rational coefficients admits polynomial solutions. However, some of the underlying
structure of the equation comes to light once we “clear denominators” and re-express (28) using the following, bilinear formulation:

\[(\eta y'' - 2\eta' y' + \eta'' y) - 2z (\eta y' - \eta' y) + 2(n - 2) \eta y = 0 \]  
(29)

where 
\[\eta = \text{Wr}[H_1, H_2] = 4 + 8z^2\]

Now the equation is bilinear in \(\eta\), which is fixed and \(y = y(z)\) the dependent variable, and nearly symmetric with respect to the two variables. We will have more to say about the bilinear formulation of exceptional differential equations in the sequel.

We can also express (28) using Sturm-Liouville form, as

\[\left(\hat{W}y'\right)' = \lambda \hat{W} y,\]  
(30)

where 
\[\hat{W}(z) = \frac{e^{-z^2}}{\eta(z)^2}, \quad \lambda = -2n.\]

As before, the Sturm-Liouville form implies the orthogonality of the eigenpolynomials:

\[\int_{-\infty}^{\infty} \hat{H}_m(z)\hat{H}_n(z)\hat{W}(z)dz = 0, \quad m \neq n, \quad m, n \neq 1, 2\]

It is also possible to show that the exceptional polynomials satisfy recurrence relations. However, now there are multiple relations of higher order:

\[4z(3 + 2z^2)\hat{H}_n = \hat{H}_{n+3} + 6n \hat{H}_{n+1} + 12n(n - 3) \hat{H}_{n-1} + 8n(n - 4)(n - 5) \hat{H}_{n-3},\]  
(31)

\[16z^2(1 + z^2)\hat{H}_n = \hat{H}_{n+4} + 8n\hat{H}_{n+2} + 4(6n^2 - 14n + 1)\hat{H}_n + 32n(n - 3)(n - 4)\hat{H}_{n-2} + 16n(n - 3)(n - 5)(n - 6)\hat{H}_{n-4}\]  
(32)

Table 1 lists the degrees of the exceptional polynomials involved in the above recurrence at the values of \(n = 0, 3, 4, \ldots\). By inspection, \(\hat{H}_0\) determines \(\hat{H}_3, \hat{H}_4, \hat{H}_6, \ldots, \hat{H}_{2k}, k \geq\)
Relation (31) with \( n = 5 \) then determines \( \hat{H}_5 \). After that the \( \hat{H}_{2k+1}, \ k \geq 3 \) are established. Observe that \( \hat{H}_n(z), \ n \geq 7 \) are determined by both relations (31) and (32). Remarkably, the relations are coherent, in the sense that both relations give the same value of \( \hat{H}_n(z), \ n \geq 7 \). This may be explained by the fact that the finite-order difference operators that describe the RHS of (31) and (32) commute with one another.

**Exercise 5.** Verify the above recurrence relations using a computer algebra system.

**Exercise 6.** Prove that the finite-difference operators that define the right-hand sides of (31) and (32) commute.

Finally, many of the properties of exceptional polynomials are explained by the fact that there is a hidden relation between them and their classical counterparts. Let us define second order operators

\[
T[y] = y'' - 2zy',
\]

\[
\hat{T}[y] = y'' - \left(2z + \frac{8z}{1 + 2z^2}\right)y'
\]

and re-express the classical and exceptional Hermite differential equations in operator form, respectively, as

\[-T[H_n] = 2nH_n, \ n \in \mathbb{N} \quad -\hat{T}[\hat{H}_n] = 2n\hat{H}_n, \ n \neq 1, 2,\]

Let us also introduce the second order operator

\[
A[y] = \text{Wr}[H_1, H_2, y] = 4(1 + 2z^2)y'' - 16zy' + 16y.
\]

A direct calculation shows that the three operators are joined by an intertwining relation:

\[
\hat{T}A = AT.
\]  

**Exercise 7.** Verify the above relation.

Note that up to a normalization constant, the exceptional polynomials are given by applying the intertwiner \( A \) to the classical polynomials:

\[\hat{H}_n \propto A[H_n].\]

If we take the intertwining relation as proven, we obtain that

\[\hat{T}[A[H_n]] = (\hat{T}A)[H_n] = (AT)[H_n] = -2nA[H_n].\]

Thus, the intertwining relation "explains" why the \( \hat{H}_n \) are eigenpolynomials of the exceptional operator \( \hat{T} \).

**2 The structure theorem**

In this section we formulate and prove a far-ranging generalization of Bochner’s Theorem that gives a normal form for all exceptional operators and weights.
2.1 Preliminaries: rings of functions and operators

Let $\mathcal{P} = \mathbb{C}[z]$ and $\mathcal{Q} = \mathbb{C}(z)$ denote rings of univariate polynomial and rational functions, respectively. In both cases, a $\mathcal{P}^\times, \mathcal{Q}^\times$ superscript indicates the corresponding subset of non-zero elements. We define the degree of a rational function to be

$$\deg p/q = \deg p - \deg q, \quad p, q \in \mathcal{P}. $$

Equivalently, if $p/q = r + p_0/q$ where $\deg p_0 < \deg q$, then $\deg p/q = \deg r$; i.e. the degree of a rational function is the degree of its rational part. Set

$$\mathcal{P}_n = \{ y \in \mathcal{P} : \deg y \leq n \}, \quad \mathcal{Q}_n = \{ y \in \mathcal{Q} : \deg y \leq n \}$$

We call a differential expression

$$L = \sum_{i=0}^{r} a_i(z)D^i, \quad a_r \neq 0 \quad (34)$$

an $r$-th order differential operator because it defines the linear transformation

$$L[y](z) = \sum_{i=0}^{r} a_i(z)y^{(i)}(z), \quad y^{(i)}(z) = (D^i y)(z). \quad (35)$$

Let

$$\text{Diff} \mathcal{P} = \mathbb{C}[z, D], \quad \text{Diff} \mathcal{Q} = \mathbb{C}(z)[D]$$

denote rings of differential operators with the indicated coefficients. We use Diff, $\mathcal{P}$, Diff, $\mathcal{Q}$ to denote the subsets of $r$-th order operators.

**Proposition 2.** Let $L \in \text{Diff} \mathcal{Q}$. Then, $L[\mathcal{P}] \subset \mathcal{P}$ if and only if $L \in \text{Diff} \mathcal{P}$.

In other words, Diff $\mathcal{P}$ may be characterized as the subring of Diff $\mathcal{Q}$ that preserves $\mathcal{P} \subset \mathcal{Q}$.

**Proof.** The proof in one direction is obvious; we focus on the converse. Let $L$ be as in (34) with $a_i(z) \in \mathcal{Q}$ and suppose that $L[\mathcal{P}] \subset \mathcal{P}$. For $0 \leq k \leq r$, we have

$$L[z^k] = k!a_k(z) + \sum_{i=0}^{k-1} k^i z^{k-i} a_i(z), \quad \text{where } k^i = k(k-1) \cdots (k-1+i).$$

In particular, $a_0(z) = L[1]$ is a polynomial. By induction on $k$ it follows that $a_1(z), \ldots, a_r(z)$ are polynomials also. \hfill $\square$

**Definition 2.** For an operator $L \in \text{Diff} \mathcal{Q}$, we define the degree

$$\deg L = \max_i \deg(a_i) - i,$$

where $a_0, \ldots, a_r \in \mathcal{Q}$ are the coefficients:

$$L = \sum_{i=0}^{r} a_i(z)D^i.$$
Throughout, we adopt the convention that the degree of the zero polynomial and the zero operator is $-\infty$.

**Proposition 3.** For $L \in \text{Diff } \mathcal{P}$ and $y \in \mathcal{P}^\times$ we have
\[
\deg L[y] \leq \deg L + \deg y
\]
with equality for all but finitely many $n = \deg y$.

**Proof.** Write $L[y](z) = \tilde{L}[y](z)/\xi(z)$ where $\tilde{L} \in \text{Diff } \mathcal{P}$ and $\xi \in \mathcal{P}$. Thus,
\[
\deg L = \deg \tilde{L} - \deg \xi
\]
and $a_i(z) = \tilde{a}_i(z)/\xi(z)$, where $\tilde{a}_i \in \mathcal{P}, \ i = 0, \ldots, \rho$ are the coefficients of $\tilde{L}$. Set $k = \deg \tilde{L}$ and write
\[
\tilde{a}_i(z) = \sum_{j=0}^{i+k} a_{ij} z^j, \quad a_{ij} \in \mathbb{C}.
\]
Hence, $\tilde{L} = L_0 + L_1$, where
\[
L_0 = \sum_{i=0}^{\rho} a_{i,i+k} z^{i+k} D_i
\]
and where $\deg L_1 < k$. Observe that
\[
L_0[z^n] = \sigma(n) z^{n+j}.
\]
where
\[
\sigma(n) = \sum_{i=0}^{\rho} a_{i,i+k} n^i
\]
is polynomial in $n$. Hence, if $\sigma(n) \neq 0, \ n \in \mathbb{N}$ then
\[
\deg \tilde{L}[z^n] = n + k.
\]
Hence, if $\deg y = n$ and $\sigma(n) \neq 0$, then
\[
\deg L[y] = \deg \tilde{L}[y] - \deg \xi = k + n - \deg \xi = \deg L + n.
\]

**Proposition 4.** Let $A, B \in \text{Diff } \mathcal{P}$ and let $[A, B] = AB - BA \in \text{Diff } \mathcal{P}$ be the corresponding commutator. If $\deg A = \deg B = 0$, then $\deg [A, B] < 0$.

**Proof.** By assumption, $A = A_0 + A_1$ where $A_0 \in \text{Diff } \mathcal{P}$ is a homogeneous degree 0 operator, that is a linear combination of $z^i D_i, \ i = 0, 1, 2 \ldots$, and where $A_1 \in \text{Diff } \mathcal{P}$ with $\deg A_1 < 0$. Let $B = B_0 + B_1, \ B_0, B_1 \in \text{Diff } \mathcal{P}$ be the analogous decomposition of $B$. Observe that
\[
[z^i D_i, z^j D^j] = 0.
\]
It follows that
\[
[A, B] = [A_0, B_1] + [A_1, B_0] + [A_1, B_1].
\]
By inspection, each of the above 3 terms has negative degree. $\square$
2.2 The formulation

**Definition 3.** We will say that a \( T \in \text{Diff}_2 \mathcal{Q} \) has a natural formulation if there exist polynomials \( \eta(z) \in \mathcal{P}^\times, \ p(z) \in \mathcal{P}_2^\times \) and constants \( a, b, c \in \mathbb{C} \) such that \( T \) admits the following form, bilinear in \( y \) and \( \eta \):

\[
\eta T[y] = p(\eta y'' - 2\eta'y' + \eta''y) + (az + b + p')(\eta y' - \eta'y) + (p'\eta' + c\eta)y
\]  

(36)

Dividing through by \( \eta \), and letting \( p, q, r \) denote the coefficients of \( T \) as per (25), we obtain the following formulation, which is equivalent to (36):

\[
q = p' + az + b - 2p \frac{\eta'}{\eta}
\]  

(37a)

\[
r = p \frac{\eta''}{\eta} - (az + b) \frac{\eta'}{\eta} + c.
\]  

(37b)

**Definition 4.** A gauge transformation is the conjugation of a differential operator by a multiplication operator. In the present context, we will say that two operators \( T, \hat{T} \in \text{Diff}_2 \mathcal{Q} \) are gauge-equivalent, if there exists a rational function \( \sigma \in \mathbb{Q} \) such that

\[
\hat{T} = \sigma T \sigma^{-1}.
\]  

(38)

**Theorem 3.** Every exceptional operator is gauge equivalent to a exceptional operator \( T \in \text{Diff}_2 \mathcal{Q} \) with natural form (36). Moreover,

\[
\eta(z) = \prod_{i=1}^{N} (z - \zeta_i)^{\nu_i}
\]  

(39)

where \( \zeta_i, i = 1 \ldots, N \) are the poles of \( \hat{T} \), and \( \nu_i = \nu_{\zeta_i} \) are the corresponding gap cardinalities.

In light of the above theorem, when an exceptional operator has form (36), we will refer to it as being in the natural gauge.

Note that if \( \eta(z) = 1 \) is trivial, then (36) reduces to the form of a Bochner operator. Theorem 3 should therefore be regarded as a generalization of Bochner’s normal form theorem to a much larger and richer class of operators and polynomials.

Applying (8), we immediately obtain a normal form for exceptional weights:

\[
W(z) = \frac{W_0(z)}{\eta(z)^2}, \quad \text{where} \quad W_0(z) = \int \frac{az + b}{p(z)} dz
\]  

(40)

is a classical weight.

**Corollary 1.** Up to an affine transformation of the \( z \) variable, there are three types of exceptional polynomials:

(i) Exceptional Hermite: \( y_n = \widehat{H}_n, \ p(z) = 1; \)
(ii) Exceptional Laguerre: \( y_n = \hat{L}_n, \ p(z) = z \);

(iii) Exceptional Jacobi: \( y_n = \hat{P}_n, \ p(z) = 1 - z^2 \).

Let \( \nu \geq 0 \) be the codimension; that is the number of missing degrees. In the natural gauge, the corresponding eigenvalue relations have the form:

\[
\begin{align*}
(\eta \hat{H}'_n - 2(\eta' \hat{H}_n + \eta'' \hat{H}_n) - 2(\eta \hat{H}'_n - \eta' \hat{H}_n) + 2(n - \nu) \eta \hat{H}_n = 0 \quad (41) \\
z(\eta \hat{L}'_n - 2(\eta' \hat{L}_n + \eta'' \hat{L}_n) + (1 + \alpha - z)(\eta \hat{L}'_n - \eta' \hat{L}_n) + (\eta' + (n - \nu) \eta) \hat{L}_n = 0, \quad (42) \\
(1 - z^2)(\eta \hat{P}'_n - 2(\eta' \hat{P}_n + \eta'' \hat{P}_n) + ((-2 + \alpha + \beta)z + \beta - \alpha)(\eta \hat{P}'_n - \eta' \hat{P}_n) + \\
(-2z\eta' + (n - \nu)(\alpha + \beta + 1 + n - \nu) \eta) \hat{P}_n = 0. \quad (43)
\end{align*}
\]

The corresponding weights have the form:

\[
\begin{align*}
\hat{W}_H &= \frac{e^{-z^2}}{\eta^2}, \quad z \in (-\infty, \infty) \\
\hat{W}_L &= \frac{x^\alpha e^{-z}}{\eta^2}, \quad \alpha > -1, \quad z \in (0, \infty) \\
\hat{W}_J &= \frac{(1 - z)^\alpha(1 + z)^\beta}{\eta^2}, \quad \alpha, \beta > -1, \quad z \in (-1, 1).
\end{align*}
\]

2.3 Polynomial subspaces

A one-point differential functional of order \( r \) to is a linear mapping \( \phi: \mathcal{P} \rightarrow \mathbb{C} \) defined by

\[
\phi: y \mapsto \sum_{i=0}^{r} a_i(\zeta)y^{(i)}(\zeta), \quad a_i \in \mathcal{P}, \ a_r \neq 0, \ \zeta \in \mathbb{C}.
\]

In other words,

\[
\phi = \text{ev}\zeta \circ L,
\]

where \( L \in \text{Diff} \mathcal{P} \) and \( \text{ev}\zeta: \mathcal{P} \rightarrow \mathbb{C} \) is evaluation at \( z = \zeta \). Let \( \mathcal{P}^* \) be the dual vector space of linear mappings \( \mathcal{P} \rightarrow \mathbb{C} \), and let \( \mathcal{P}^*_\zeta \subset \mathcal{P}^* \) denote the subspace one-point differential functionals with support at \( z = \zeta \).

For \( \zeta \in \mathbb{C} \) and \( y(z) \) meromorphic, the expression

\[
y(z) = O((z - \zeta)^k), \quad z \rightarrow \zeta
\]

will be taken to mean that \( y(z)(z - \zeta)^{-k} \) is either analytic, or has a removable singularity at \( z = \zeta \). We define \( k = \text{ord}_\zeta y \) to be the leading order of the Laurent expansion of \( y(z) \) at \( z = \zeta \); that is the smallest \( k \) such that (44) holds.

Thus, the notion of order applies to operators, functionals, and to functions, albeit with different meanings. The two notions are somewhat related via the following.

**Proposition 5.** Let \( y \in \mathcal{Q} \) and \( \phi \in \mathcal{P}^*_\zeta, \ \zeta \in \mathbb{C} \). If \( \text{ord}_\zeta y > \text{ord} \phi \), then \( \phi[y] = 0 \).
Let $U \subset \mathcal{P}$ be a polynomial subspace of finite codimension. This means that
\[
\text{codim} U = \dim \mathcal{P}/U < \infty.
\]

**Proposition 6.** Let $U \subset \mathcal{P}$ be a polynomial subspace and let $I = \{\deg y : y \in U^k\}$ be the corresponding degree sequence. Then $\text{codim} U < \infty$ if and only if $\#(\mathbb{N} \setminus I)$ is finite; i.e. if and only if only a finite number of degrees are missing from the degree sequence.

One way to generate polynomial subspaces of finite codimension is to consider the joint kernel of a finite number of one-point differential functionals. For $U \subset \mathcal{P}$, let
\[
\text{Ann}_\zeta U = \{\phi \in \mathcal{P}^* : U \subset \ker \phi\}, \quad \zeta \in \mathbb{C}.
\]
We will say that $U$ is of **generated by 1-point conditions** if
\[
U = \bigcap_{\zeta \in \mathbb{C}} \ker \text{Ann}_\zeta U.
\]
Of course $U \subset \cap_{\zeta} \ker \text{Ann}_\zeta U$, by definition. However, $U$ is generated by 1-point conditions precisely when the inclusion becomes an equality.

**Exercise 8.** Not all polynomial subspace can be generated by 1-point conditions. Consider
\[
U = \{p \in \mathcal{P} : p(0) = p(1)\},
\]
which is the kernel of a 2-point differential condition. Show that $U$ is not in the kernel of any non-trivial 1-point condition.

For a fixed $U \subset \mathcal{P}$ and $\zeta \in \mathbb{C}$, call
\[
I_\zeta = \{\text{ord}_\zeta y : U\}
\]
the **pivot orders** and
\[
\nu_\zeta = \#(\mathbb{N} \setminus I_\zeta)
\]
the **gap cardinalities** of the subspace at $z = \zeta$. The reason for this terminology has to with a choice of basis of $U$ adapted to $\text{ord}_\zeta$. Imagine an arbitrary basis of $U$ converted into an infinite matrix by looking at the coefficients of the basis expanded in powers $(z - \zeta)^k$, $k \in \mathbb{N}$. We wish to construct a basis by row-reducing this matrix to echelon form. The elements of $I_\zeta$ enumerate the pivot columns of the reduced matrix, while $\nu_\zeta$ measures the co-rank, that is to say the number of columns without a pivot. There is a complication. Since there is an infinite number of columns, we have to be careful, and to suitably modify the usual row-reduction procedure.

**Proposition 7.** For $U \subset \mathcal{P}$ be a polynomial subspace and $\zeta \in \mathbb{C}$, there exists a basis $\{y_k\}_{k \in I_\zeta}$ of $U$ such that $\text{ord}_\zeta y_k = k$, $k \in I_\zeta$. 

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Proof. For every $k \in I_\zeta$ choose a polynomial $y_k \in \mathcal{U}$ such that $\text{ord}_\zeta y_k = k$ and such that $\deg y_k$ is as small as possible. We claim that $\{y_k\}_{k \in I_\zeta}$ spans $\mathcal{U}$. Let a non-zero $g_1 \in \mathcal{U}$ be given. Set

$$k_1 := \text{ord}_\zeta g_1, \quad g_2 = g_1 - c_1 y_{k_1},$$

where $c_1 \in \mathbb{C}$ is chosen so that $\text{ord}_\zeta g_2 > \text{ord}_\zeta g_1$.

Observe that since $\deg y_{k_1} \leq \deg g_1$ we have $\deg g_2 \leq \deg g_1$. If $g_2 \neq 0$ then continue the above construction inductively:

$$k_i = \text{ord}_\zeta g_i, \quad g_{i+1} = g_i - c_i y_{k_i},$$

where $c_i \in \mathbb{C}$ is chosen so that $\text{ord}_\zeta g_{i+1} > \text{ord}_\zeta g_i$, $\deg g_{i+1} \leq \deg g_i$.

Since order cannot exceed degree, eventually we will arrive at $i$ such that $g_{i+1} = 0$, thereby proving our claim. \qed

**Proposition 8.** For every $\zeta \in \mathbb{C}$, we have $\nu_\zeta \leq \text{codim} \mathcal{U}$.

**Proof.** We claim that the $\nu_\zeta$ polynomials $\{(z - \zeta)^n\}_{n \notin I_\zeta}$ are linearly independent modulo $\mathcal{U}$. Indeed, the only way for

$$\sum_{n \notin I_\zeta} a_n (z - \zeta)^n \in \mathcal{U}, \quad a_n \in \mathbb{C}$$

is if $a_n = 0$ for every $n \notin I_\zeta$. Our conclusion follows once we observe that, by definition, it is not possible to choose more than $\text{codim} \mathcal{U}$ linearly independent polynomials in $\mathcal{P}/\mathcal{U}$.

Paradoxically, even if $\mathcal{U}$ has finite codimension, $\nu_\zeta$ in general is **strictly less than** the codimension; indeed $\nu_\zeta = 0$ for generic $\zeta$. \qed

**Exercise 9.** Consider the the codimension 1 subspace $\mathcal{U} = \{p \in \mathcal{P} : p(1) = 0\}$. One choice of basis is $(z - 1), (z - 1)^2, (z - 1)^3, \ldots$, so that $\nu_1 = 1$. For $\zeta \neq 1$ construct a basis adapted to $\text{ord}_\zeta$ and demonstrate that $\nu_\zeta = 0$.

The following proposition gives an important interpretation of the co-rank.

**Proposition 9.** If $\text{codim} \mathcal{U} < \infty$, then $\nu_\zeta = \dim \text{Ann}_\zeta \mathcal{U}$.

**Proof.** Let $n = \max(\mathbb{N} \setminus I_\zeta)$ be the largest gap value. By Proposition 7, we can choose a basis of $\mathcal{U}$ of the form

$$y_k(z) = (z - \zeta)^k + \sum_{j=k+1}^{n-1} a_{jk}(z - \zeta)^j + O((z - \zeta)^{n+1}), \quad k \in I_\zeta, k < n$$

$$y_k(z) = (z - \zeta)^k + O((z - \zeta)^{k+1}), \quad k > n$$
where both relations hold as $z \to \zeta$. We refine our basis by setting
\[
\tilde{y}_k = y_k - \sum_{k < j < n \atop j \in I_\zeta} a_{jk} y_j, \quad k \in I_\zeta, k < n
\]
\[
\tilde{y}_k = y_k, \quad k \in I_\zeta, k > n.
\]
This has the effect of “knocking out” the pivots $j \in I_\zeta, k < j < n$ from $\tilde{y}_k$ so that
\[
\tilde{y}_k(z) = (z - \zeta)^k + \sum_{j > k \atop j \in I_\zeta} a_{jk} (z - \zeta)^j + O((z - \zeta)^{n+1}), \quad k \in I_\zeta, k < n
\]
\[
\tilde{y}_k(z) = (z - \zeta)^k + O((z - \zeta)^{k+1}), \quad k \in I_\zeta, k > n.
\]
For $j \in N \setminus I_\zeta$, define the differential functionals
\[
\phi_j[y] = \frac{y^{(j)}(\zeta)}{j!} - \sum_{k < j \atop k \in I_\zeta} a_{jk} \frac{y^{(k)}(\zeta)}{k!}, \quad y \in P.
\]
Thus,
\[
\phi_j[(z - \zeta)^k] = \begin{cases} 0 & \text{if } k > j, \\ 1 & \text{if } k = j, \\ -a_{kj} & \text{if } k < j, k \in I_\zeta. \end{cases}
\]
Hence, $\phi_j[\tilde{y}_k] = 0$ if $k > j$. For $k \in I_\zeta$, $k < j$ we have
\[
\phi_j[\tilde{y}_k] = a_{jk} - a_{jk} = 0.
\]
Hence, by construction $\phi_j \in \text{Ann}_\zeta U$.

Our claim will be proven once we show that the $\phi_j$, $j \notin I_\zeta$ give a basis for $\text{Ann}_\zeta U$. Since $\text{ord} \phi_j = j$ are distinct, the $\phi_j$ are are linearly independent. Let $\alpha \in \text{Ann}_\zeta U$ be given. Write
\[
\alpha[y] = \sum_{j=0}^N \alpha_j y^{(j)}(\zeta).
\]
Then, $\alpha = \alpha - \sum_{j \notin I_\zeta} \alpha_j j! \phi_j$ is also in $\text{Ann}_\zeta U$ but has the form
\[
\hat{\alpha}[y] = \sum_{k \in I_\zeta} \hat{\alpha}_k y^{(k)}(\zeta)
\]
We claim that $\hat{\alpha} = 0$. Suppose not. Observe that $k = \text{ord} \hat{\alpha}$ is the largest $k$ such that $\hat{\alpha}_k \neq 0$. Hence,
\[
\hat{\alpha}[(z - \zeta)^k] = \hat{\alpha}[(z - \zeta)^k] = \hat{\alpha}_k k! \neq 0,
\]
a contradiction. 

\begin{exercise}
Let $\eta \in P$ be a polynomial with $\deg \eta \geq 1$. Define $U_\eta = \{ y \in P : \eta \mid y \}$ be the subspace of all those polynomials that are divisible by $\eta$. Prove that $\text{codim} U_\eta = \deg \eta$.

Then, prove that $U_\eta$ is generated by $1$-point conditions. Hint: show that
\[
\text{Ann}_\zeta U_\eta = \text{span}\{ y \mapsto y^{(j)}(\zeta) : 0 \leq j < \text{ord}_\zeta \eta \}, \quad \zeta \in \mathbb{C}.
\]
\end{exercise}
2.4 The maximal invariant polynomial subspace.

One seemingly paradoxical property of exceptional operators is the fact that an operator with rational coefficients possesses invariant subspaces consisting of polynomials. To address this mystery systematically, for a fixed \( T \in \text{Diff}_2 \) define

\[
U^{(j)} = \{ y \in P : T^j[y] \in P \}, \quad j = 1, 2, \ldots,
\]

and set

\[
U = \bigcap_{j \geq 1} U^{(j)}.
\]

**Proposition 10.** The subspace \( U \subset P \) is invariant with respect to \( T \). Moreover, if \( V \subset P \) is invariant with respect to \( T \), then necessarily \( V \subset U \).

**Proof.** By definition, a \( y \in U \) has the property that \( T^j[y] \in P \) for all \( j = 1, 2, \ldots \). Hence, the same is true for \( T[y] \). Hence, \( T[y] \in U \), and therefore \( U \) is \( T \)-invariant. Next, if \( y \in V \), then \( T^j[y] \in V \subset P \) for all \( j \). Hence \( y \in U \), which proves that \( V \subset U \). \( \square \)

In light of the above Proposition, we call \( U \) the maximal invariant polynomial subspace of \( T \).

For generic \( T \in \text{Diff}_2 \) the maximal invariant subspace \( U \) is trivial. However if \( T \) is exceptional, then \( U \) contains all the eigenpolynomials of \( T \); and hence \( \text{codim} U < \infty \).

There is another important property of \( U \) when \( T \) is exceptional.

**Proposition 11.** If \( T \in \text{Diff}_2 \) is exceptional, then \( U \) is generated by 1-point conditions. Moreover,

\[
\text{codim} U = \sum_\zeta \nu_\zeta,
\]

where \( \nu_\zeta > 0 \) if and only if \( z = \zeta \) is a pole of the operator \( T \).

**Proof.** Let \( \zeta_1, \ldots, \zeta_N \in \mathbb{C} \) be the poles of the operator \( T \). Express \( T \) as

\[
T[y](z) = L[y](z) \prod_{i=1}^N (z - \zeta_i)^{-n_i},
\]

where \( L \in \text{Diff} P \) has polynomial coefficients and where the exponents \( n_i \in \mathbb{N} \) are as small as possible. Similarly, for \( k = 2, 3, \ldots \), express \( k \)-fold operator composition \( T^k \) as

\[
T^k[y](z) = L_k[y](z) \prod_{i=1}^N (z - \zeta_i)^{-n_{ki}}
\]

where \( L_k \in \text{Diff} P \), and again \( n_{ki} \in \mathbb{N} \) are as small as possible. The condition that \( T^k[y] \in P \) is then equivalent to the condition that \( \text{ord}_{\zeta_i} L_k[y] \geq n_{ki} \); i.e., that \( L_k[y](z) \) be divisible by \( \prod_{i=1}^N (z - \zeta_i)^{n_{ki}} \). Hence,

\[
U^{(k)} = \bigcap_{i=1}^N \ker \text{Ann}_{\zeta_i} U^{(k)},
\]

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where,

\[ \text{Ann}_\zeta U^{(k)} = \text{span}\{ y \mapsto L_k[y]^{(j)}(\zeta_i): 0 \leq j < n_{ki} \}. \]

Since \( \text{codim} U < \infty \), there exists an \( M \) such that

\[ U = \bigcap_{k \geq 1} U^{(k)} = \bigcap_{k=1}^{M} U^{(k)}. \]

Hence,

\[ U = \bigcap_{k=1}^{M} \bigcap_{i=1}^{N} \ker \text{Ann}_\zeta U^{(k)}, \]

is the joint kernel of a finite number of differential functionals with support at the poles of \( T \).

By Proposition 9, \( \nu_\zeta > 0 \) if and only if there are differential functionals with support at \( z = \zeta \) that annihilate \( U \). Therefore, \( \sum_\zeta \nu_\zeta \) is a finite sum whose value counts the number of independent differential conditions that serve to “carve out” \( U \).

\[ \square \]

2.5 More preliminaries.

2.5.1 Rational coefficients.

Our first observation is that an exceptional polynomial must have rational coefficients. Let \( T \) be an exceptional operator and \( p, q, r \) its coefficients as per (25). By assumption, there exist three linearly independent polynomials of \( y_1, y_2, y_3 \in \mathcal{P} \) such that \( u_i = T[y_i] \in \mathcal{P}, i = 1, 2, 3 \). Applying Cramer’s rule to the matrix equation

\[
\begin{bmatrix}
  y'_1 & y'_1 & y_1 \\
  y'_2 & y'_2 & y_2 \\
  y'_3 & y'_3 & y_3
\end{bmatrix}
\begin{bmatrix}
  p \\
  q \\
  r
\end{bmatrix}
= \begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
\]

we conclude that \( p, q, r \in \mathcal{Q} \).

2.5.2 More on gauge transformations.

Let’s begin by working out the transformation laws for a gauge transformation (38).

**Proposition 12.** Let \( T, \hat{T} \in \text{Diff}_2 \mathcal{Q} \) be gauge equivalent operators as in (38). Denoting the corresponding operator coefficients using \( p, q, r, \hat{p}, \hat{q}, \hat{r} \in \mathcal{Q} \), as per (25), we then have

\[
\begin{align*}
p &= \hat{p} \\
q &= \hat{q} + \frac{2\sigma'}{\sigma} \hat{p} \\
r &= \hat{r} + \frac{\sigma'}{\sigma} \hat{q} + \frac{\sigma''}{\sigma} \hat{p}\end{align*}
\]

(47a)

(47b)

(47c)
Proof. Exercise.

Observe that, a gauge transformation only effects the first and zeroth-order coefficients \(q, r\); the leading coefficient \(p\) is left invariant.

**Proposition 13.** Let \(T, \hat{T} \in \text{Diff}_2 Q\) be gauge-equivalent operators as per (38). If \(T\) is exceptional, and if \(\sigma \in \mathcal{P}^\times\) is a polynomial, then \(\hat{T}\) is also exceptional.

**Proof.** Let \(y_k \in \mathcal{P}_k, k \in I\) be the eigenpolynomials of \(T\) with \(I \subset \mathbb{N}\) a cofinite index set. Let \(\delta = \text{deg} \sigma\) and set \(\hat{y}_{\delta+k} = \sigma y_k\). By construction, the \(\hat{y}_{\delta+k}\) are eigenpolynomials of \(\hat{T}\). Since \(I + \delta\) is a cofinite subset of \(\mathbb{N}\), the operator \(\hat{T}\) is also exceptional.

Indeed, gauge transformations allow us to create “new” exceptional polynomials in a rather trivial fashion. For example, consider the classical Hermite differential equation

\[
y'' - 2zy' + 2ny = 0, \quad n = 0, 1, 2, \ldots
\]

whose polynomial solutions are the classical Hermite polynomials \(y = H_n(z)\). One could instead consider the polynomials \(\hat{H}_n(z) = (1 + z^2)H_{n-2}(z), n \geq 2\). By construction, \(y = \hat{H}_n\) is a solution of the differential equation

\[
y'' - 2 \left( z + \frac{2z}{1 + z^2} \right) y' + \left( 4 + 2n + \frac{2}{1 + z^2} - \frac{8}{(1 + z^2)^2} \right) y = 0,
\]

which is obtained by conjugating the classical Hermite operator by the multiplication operator \(1 + z^2\). The ordinary Hermite polynomials are orthogonal on \((-\infty, \infty)\) relative to the weight \(e^{-z^2}\), and hence by construction the modified polynomials \(\hat{H}_n(z)\) are orthogonal relative to the weight \(e^{-z^2}/(1 + z^2)^2\). Thus, \(\hat{H}_n, n \geq 2\) constitute a family of exceptional orthogonal polynomials with 2 missing degrees. This type of construction is quite general, but does not produce genuinely new orthogonal polynomials.

It therefore makes sense to consider exceptional operators only up to gauge-equivalence. However, it is possible to “fix the gauge” by asking that the polynomials in \(\mathcal{U}\) not share a common root.

**Definition 5.** Let \(T \in \text{Diff}_2 Q\) be an exceptional operator and \(\mathcal{U} \subset \mathcal{P}\) its maximal invariant polynomial subspace. We call \(T\) a reduced operator if the elements of \(\mathcal{U}\) do not share a common root.

**Proposition 14.** Every exceptional operator is gauge-equivalent to a reduced, exceptional operator.

**Proof.** Suppose that \(T\) is exceptional, but not reduced. Let \(\sigma \in \mathcal{P}\) be the greatest common divisor of \(\mathcal{U}\); i.e. every \(y \in \mathcal{U}\) is divisible by a \(\sigma\) with \(\text{deg} \sigma \geq 1\) as large as possible. Consider the operator

\[
\hat{T} = \sigma^{-1} T\sigma,
\]

which by construction leaves invariant the polynomial subspace

\[
\hat{\mathcal{U}} = \{ \sigma^{-1} y : y \in \mathcal{U} \}.
\]

By construction the GCD of \(\mathcal{U}\) is trivial, and therefore \(\hat{T}\) is reduced exceptional.
2.5.3 When natural isn’t reduced

The following example illustrates the difference between the natural and reduced gauge of an exceptional operator. The example is based on the following family of two-step exceptional Laguerre polynomials [1]. Let \( L_n^{(\alpha)}(z) \) denote the classical Laguerre polynomial of degree \( n \). For \( n \geq 2 \) set

\[
\widehat{L}_n^{(\alpha)}(z) := e^{-z} \text{Wr} \left[ L_n^{(\alpha)}(z), L_1^{(\alpha)}(z), e^z L_2^{(\alpha)}(-z) \right].
\]  

(48)

By construction, \( \widehat{L}_3^{(\alpha)}(z) = 0 \), and so we obtain a codimension-3 family of polynomials with degrees \( n = 2, 4, 5, 6, \ldots \). These polynomials can also be given using the following form introduced by Durán [2]

\[
\widehat{L}_n^{(\alpha)}(z) = \begin{vmatrix}
L_{n-2}^{(\alpha)}(z) & -L_{n-3}^{(\alpha+1)}(z) & L_{n-4}^{(\alpha+2)}(z) \\
L_1^{(\alpha)}(z) & -L_0^{(\alpha+1)}(z) & 0 \\
L_2^{(\alpha)}(-z) & L_2^{(\alpha+1)}(-z) & L_2^{(\alpha+2)}(-z)
\end{vmatrix}, \quad n = 2, 4, 5, \ldots \tag{49}
\]

where \( L_j^{(\alpha)}(z) \) is understood to be zero for \( j < 0 \).

Let

\[
\eta^{(\alpha)}(z) = e^{-z} \text{Wr} \left[ L_1^{(\alpha)}(z), e^z L_2^{(\alpha)}(-z) \right] = \begin{vmatrix}
L_1^{(\alpha)}(z) & -1 \\
L_2^{(\alpha)}(-z) & L_2^{(\alpha+1)}(-z)
\end{vmatrix} = -\frac{1}{2} (z^3 + (\alpha + 4)z^2 - (\alpha + 4)(\alpha + 1)z - (\alpha + 1)(\alpha + 2)(\alpha + 4)).
\]

The polynomial family \( \widehat{L}_n^{(\alpha)}(z), \; n = 2, 4, 5, \ldots \) is exceptional and in the natural gauge, because of the following bilinear relations:

\[
z \left( \eta^{(\alpha)} \widehat{L}_n^{(\alpha)} - 2 \eta^{(\alpha)'} \widehat{L}_n^{(\alpha)'} + \eta^{(\alpha)''} \widehat{L}_n^{(\alpha)} \right) + (-z + \alpha + 3) \left( \eta^{(\alpha)} \widehat{L}_n^{(\alpha)''} - \eta^{(\alpha)'} \widehat{L}_n^{(\alpha)'} \right) + (\eta^{(\alpha)''} + \eta^{(\alpha)}(n - 3)) \widehat{L}_n^{(\alpha)} = 0 \tag{50}
\]

It is easy to check that \( \eta^{(\alpha)}(z) \neq 0 \) for \( z \in [0, \infty) \) if and only if \( \alpha \in (-\infty, -4) \cup (-2, -1) \). Hence, for \( \alpha \in (-2, -1) \) the polynomials \( \widehat{L}_n^{(\alpha)}(z) \) are orthogonal with respect to the inner product

\[
\langle f, g \rangle = \int_0^\infty \frac{z^{\alpha+2} e^{-z}}{(\eta^{(\alpha)}(z))^2} f(z) g(z) dz.
\]

The discriminant of \( \eta^{(\alpha)}(z) \) is \( \frac{1}{8}(\alpha + 1)(\alpha + 4)^2(4\alpha + 7)^2 \). Hence, for \( \alpha = -\frac{7}{4} \) the denominator polynomial has a multiple root. Indeed,

\[
\eta^{(-\frac{7}{4})}(z) = -\frac{1}{2} \left( z + \frac{3}{4} \right)^3
\]

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there is a single root with a triple multiplicity. Moreover,

\[ L_2(-\frac{7}{4}) (z) = \frac{1}{2} (z + \frac{3}{4}) (z - \frac{1}{4}) \]

\[ L_1(-\frac{7}{4}) (z) = - (z + \frac{3}{4}). \]

Hence,

\[ \hat{L}_n(-\frac{7}{4}) = - e^{-z} \left( z + \frac{3}{4} \right)^{\frac{3}{2}} \text{Wr} \left[ \frac{L_{n-\frac{1}{2}}(z)}{z + \frac{3}{4}}, 1, \frac{1}{2} e^z (z - \frac{1}{4}) \right] \]

\[ = -\frac{1}{2} \left( z + \frac{3}{4} \right)^{\frac{3}{2}} L_n^{\frac{1}{4}}(z) - \frac{1}{2} \left( z + \frac{3}{4} \right)^2 \left( z + \frac{15}{4} \right) L_n^{-\frac{3}{4}}(z) - \frac{1}{2} \left( z + \frac{3}{4} \right) \left( z + \frac{15}{4} \right) L_n^{-\frac{3}{4}}(z) \]

has a root at \( z = -\frac{3}{4} \) for every \( n \). Thus, for \( \alpha = -\frac{7}{4} \) the natural gauge does not agree with reduced gauge.

Let us therefore introduce the reduced family of polynomials

\[ \tilde{L}_n(z) = \left( z + \frac{3}{4} \right)^{-1} \hat{L}_n(-\frac{7}{4}) (z), \quad n = 1, 3, 4, \ldots \]

This family of polynomials is exceptional and reduced. The reduced inner product is

\[ \langle f, g \rangle = \int_0^\infty \frac{z^\frac{1}{4} e^{-z}}{(z + \frac{3}{4})^2} f(z) g(z) dz. \]

To obtain the corresponding differential equation we conjugate (50) by \( z + \frac{3}{4} \). Applying the gauge-transformation law (47), we obtain the differential equation

\[ z y'' + \left( \frac{5}{4} - z \right) y' + (n - 1)y - \frac{4zy' + y}{z + \frac{3}{4}} = 0, \quad y = \tilde{L}_n(z), \]

which after clearing denominators becomes

\[ z(\tilde{\eta} y'' - 2\tilde{\eta}' y' + \tilde{\eta}'' y) + \left( -z + 1 + \frac{1}{4} \right) (\eta y' - \eta' y) + \]

\[ + (\eta' + (n - 3)\eta) y + \left( \frac{3/2}{(z + 3/4)^2} - \frac{1}{z + 3/4} \right) y\eta = 0, \quad \eta(z) = \left( z + \frac{3}{4} \right)^2, \]

which is no longer natural form.

This example is another illustration of the principle that codimension very much depends on the choice of gauge. The generic family described above has codimension 3. However, for one particular value of the parameter, the “true” codimension, that is the codimension of the corresponding reduced family, is actually 2.
2.5.4 Laurent and partial fraction expansions.

The proof of Theorem 3 requires the notions of a Laurent expansion and a partial fraction expansion of an operator $T \in \text{Diff} \mathcal{Q}$.

Recall that every rational function $\phi(z) \in \mathcal{Q}$ can be expressed as a series about $z = \zeta$. If $\text{ord}_z \phi \geq 0$ the expansion is a power series, but if the order is negative, the expansion is a Laurent series with the leading term a multiple of $(z - \zeta)^d$ where $d = \text{ord}_z \phi$. We now extend this notion to operators.

Let $T \in \text{Diff}_2 \mathcal{Q}$, and let $p, q, r \in \mathcal{Q}$ be the coefficients as per (25). Fix a $\zeta \in \mathbb{C}$ and set

$$d = \min\{\text{ord}_\zeta p - 2, \text{ord}_\zeta q - 1, \text{ord}_\zeta r\}.$$  

Expand $p, q, r$ as Laurent series about $z = \zeta$, but with the indices shifted as follows:

$$p(z) = \sum_{k \geq d+2} p_{k-2}(z - \zeta)^k, \quad q(z) = \sum_{k \geq d+1} q_{k-1}(z - \zeta)^k, \quad r(z) = \sum_{k \geq d} r_k(z - \zeta)^k.$$  

We are now able to write

$$T = \sum_{k=d}^\infty T_k, \quad d \in \mathbb{Z},$$  

where

$$T_k[y] = (z - \zeta)^k (p_k(z - \zeta)^2 y'' + q_k(z - \zeta) y' + r_k y), \quad p_k, q_k, r_k \in \mathbb{C}. $$

The summands $T_k$ are degree-homogeneous operators in the sense that

$$T_k[(z - \zeta)^j] = \sigma_k(j)(z - \zeta)^{j+k},$$  

where

$$\sigma_k(j) = p_k j^2 + q_k j + r_k$$

is a second-degree polynomial in the degree $j \in \mathbb{Z}$.

The other way to express a rational function $\phi(z) \in \mathcal{Q}$ is by means of partial fraction expansion

$$\phi(z) = \sum_{i=1}^N \sum_{j=d_i}^{-1} \phi_{ij} (z - \zeta_i)^j + \phi_\infty(z),$$

where $\zeta_1, \ldots, \zeta_N \in \mathbb{C}$ are the poles of $\phi(z)$, where $d_i = \text{ord}_\zeta \phi$, where $\phi_{ij} \in \mathbb{C}$ and where the remainder $\phi_\infty(z)$ is a polynomial. The Laurent polynomial $\sum_{j=d_i}^{-1} \phi_{ij}(z - \zeta_i)^j$ is just the singular part of the Laurent expansion of $\phi(z)$ about $z = \zeta_i$, $i = 1, \ldots, N$.

We now extend this notion to operators $T \in \text{Diff}_2 \mathcal{Q}$ as follows. Let $\zeta_1, \ldots, \zeta_N \in \mathbb{C}$ be the poles of $T$; i.e., the poles of the coefficients $p, q, r$. Set

$$d_i = \min\{\text{ord}_\zeta p - 2, \text{ord}_\zeta q - 1, \text{ord}_\zeta r\}.$$  

For each $i = 1, \ldots, N$ let

$$T_{ij}[y] = (z - \zeta_i)^{d_i} (p_{ij}(z - \zeta_i)^2 y'' + q_{ij}(z - \zeta_i) y' + r_{ij} y), \quad d_i \leq j \leq -1.$$  

$$y$$
be the initial summands of the Laurent expansion of $T$ about $z = \zeta_i$. Then,

$$T = T_s + T_\infty$$

(54)

where

$$T_s = \sum_{i=1}^{N} \sum_{j=d_i}^{-1} T_{ij}$$

is the singular part of the operator, and where $T_\infty \in \text{Diff}_2 \mathcal{P}$ has polynomial coefficients.

In essence, the proof of Theorem 3 rests on establishing certain necessary conditions on the orders $d_i$ and the form of the $p_{ij}, q_{ij}, r_{ij}$, and then showing that, up to a gauge transformations, these conditions correspond to the singularity structure inherent in the bilinear form (36).

### 3 Part I of the Proof

Here is our first effort in the direction of a normal form.

**Proposition 15.** Suppose that $T \in \text{Diff}_2 \mathcal{Q}$ is reduced and exceptional. Then,

1. $p$ is a polynomial,
2. $q$ has simple poles,
3. the poles of $q$ and the roots of $p$ are distinct,
4. $r$ also has simple poles and these coincide with the poles of $q$.
5. A point $z = \zeta$ is a pole of $q(z)$ if and only if $\nu_\zeta > 0$; c.f., (46).
6. If this is the case, then

$$T_{-2} = p(\zeta) \left( D_{zz} - 2\nu_\zeta (z - \zeta)^{-1}D_z \right)$$

(55)

The proof of the above assertions will be broken up as a sequence of Lemmas. Throughout, we consider the Laurent expansion (51) of an exceptional $T$ about some fixed $\zeta \in \mathbb{C}$. As usual, $\mathcal{U} \subset \mathcal{P}$ is the maximal invariant polynomials subspace, $I_\zeta$ is the order sequence, and $\nu_\zeta$ are the gap cardinalities as per (45) (46).

**Lemma 1.** If $k \in I_\zeta$ and $T_d[(z - \zeta)^k] \neq 0$, then $k + d \in I_\zeta$.

**Proof.** By Proposition 7, $\mathcal{U}$ has a basis of the form

$$y_k(z) = (z - \zeta)^k + O((z - \zeta)^{k+1}), \quad z \to \zeta, \quad k \in I_\zeta.$$

Hence, by (52),

$$T[y_k] = T_d[(z - \zeta)^k] + O((z - \zeta)^{k+d+1}), \quad z \to \zeta = \sigma_d(k)(z - \zeta)^{k+d} + O((z - \zeta)^{k+d+1})$$

Hence, if $\sigma_d(k) \neq 0$ then $k + d \in I_\zeta$ because $T$ preserves $\mathcal{U}$. \qed
Lemma 2. The leading order is bounded below by \( d \geq -2 \).

Proof. Suppose that \( d \leq -3 \). For \( i = 0, 1, 2 \), set

\[
 n_i = \min \{ n \in \mathbb{N} : i + dn \in I_\zeta \}.
\]

The definition is sound because \( \nu_\zeta \) is finite. By Lemma 1

\[
 T_d[(z - \zeta)^{i+dn_i}] = 0, \quad i = 0, 1, 2.
\]

Since \( d \leq -3 \), the integers \( n_1, n_2, n_3 \) are distinct. This is impossible, because \( \sigma_d(n) \) has at most two distinct roots.

The above Lemma establishes Claims 1 and 2. If \( z = \zeta \) were a pole of \( p(z) \) or a higher-order pole of \( q(z) \), then \( d \leq 3 \). Therefore, \( p(z) \) is polynomial and \( q(z) \) has simple poles only.

Lemma 3. We have \( 0 \in I_\zeta \).

Proof. Since \( T \) is reduced, there is a \( y_0 \in \mathcal{U} \) such that

\[
 y_0(z) = 1 + O(z - \zeta), \quad z \to \zeta.
\]

Lemma 4. If \( d < 0 \), then \( r_d = 0 \).

Proof. Since \( 0 \in I_\zeta \) and since \( 0 + d \notin I_\zeta \), Lemma 1 implies \( T_d[1] = 0 \). Hence, \( \sigma_d(0) = 0 \), which implies that \( r_d = 0 \).

Lemma 5. Suppose that \( \nu_\zeta > 0 \). Then, \( d = -2 \). Furthermore, \( T_{-2} \) annihilates 1 and \( (z - \zeta)^{2\nu_\zeta + 1} \), and \( I_\zeta = \mathbb{N} \setminus \{1, 3, \ldots, 2\nu_\zeta - 1\} \).

Proof. Since \( d \geq -2 \) it suffices to show that \( d = -1 \) is impossible. Suppose otherwise. Since \( q_{-2} = 0 \), we have that \( q(z) \) is analytic at \( z = \zeta \). By the preceding Lemma, \( r_{-1} = 0 \). Hence \( T \) has no pole at \( z = \zeta \), which contradicts Proposition 11.

Thus, \( d = -2 \). By assumption, \( \mathbb{N} \setminus I_\zeta \neq \emptyset \). Let \( k = \max(\mathbb{N} \setminus I_\zeta) \) be the largest gap. Since \( 0, k + 2 \in I_\zeta \), but \(-2, k \notin I_\zeta \), Lemma 1 implies that \( T_{-2} \) annihilates 1 and \( (z - \zeta)^{k+2} \). Since \( T_{-2} \) is second-order it does not annihilate any other monomials.

By Lemma 1, if \( 0 < j < k + 2 \) and \( j - 2 \notin I_\zeta \), then \( j \notin I_\zeta \). Observe that \(-1 \notin I_\zeta \). Hence, if \( j \) is odd and if \( 0 < j < k + 2 \), then \( j \notin I_\zeta \). It follows that \( k \) is odd. If \( k \) were even, then \( k + 1 \notin I_\zeta \), which contradicts the maximality of \( k \). Similarly, if \( 0 < j < k + 2 \) and if \( j \in I_\zeta \), then \( j - 2 \in I_\zeta \). Since \( k + 1 \) is even and \( k + 1 \in I_\zeta \), it follows that if \( j \) is even and \( 0 < j < k + 2 \), then \( j \in I_\zeta \). Hence, \( I_\zeta = \mathbb{N} \setminus \{1, 3, \ldots, k\} \). Since there are \( \nu_\zeta \) gaps, we must have \( k = 2\nu_\zeta - 1 \).

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Claims 3–6 follow by the preceding Lemma. Suppose that \( z = \zeta \) is a pole of \( q(z) \). By Proposition 11, \( \nu_\zeta > 0 \). By the preceding Lemma, \( \sigma_{-2}(j) \) has roots \( j = 0 \) and \( j = 2\nu_\zeta + 1 \). Hence,
\[
\sigma_{-2}(j) = p_{-2} j(j - 2\nu_\zeta - 1).
\]
Claim 3 follows because \( p(z) \) is polynomial and hence \( p_{-2} = p(\zeta) \). By (53)
\[
q_{-2} = -2p_{-2}\nu_\zeta,
\] which establishes Claim 6.

We now prove Claims 4 and 5. By Proposition 11, if \( z = \zeta \) is the pole of either \( q(z) \) or \( r(z) \), then \( \nu_\zeta > 0 \). Conversely, suppose that \( \nu_\zeta > 0 \). By (57), \( q_{-2} \neq 0 \), which implies that \( q(z) \) has a pole at \( z = \zeta \). Claim 5 is established. Claim 4 follows because \( r_{-2} = 0 \).

### 3.1 Trivial monodromy. Part II of the Proof

Proposition 15 shows that \( d = -2 \) and establishes the form of \( T_{-2} \). The partial fraction decomposition of \( T \) requires that we describe \( T_{-1} \). To do so requires that we introduce the notions of an apparent singularity and of trivial monodromy.

Express the differential equation \( T[y] = 0 \) in the equivalent form
\[
y'' + \frac{q(z)}{p(z)} y' + \frac{r(z)}{p(z)} y = 0
\]
and inquire about formal series solution in the sense of the method of Frobenius.

A regular singularity of (58) corresponds to points \( z = \zeta \) where \( q(z)/p(z) \) has a simple pole and \( r(z)/p(z) \) a pole of order \( \leq 2 \). At a regular singular point (58) admits two independent series-type solutions which usually feature power-type singularities. If the roots of the indicial equation differ by an integer, then generically the general solution also features a logarithmic singularity.

However, it may also happen that a singular point is only an apparent singularity; this means that there exist two linearly independent meromorphic solutions. Such a possibility comes about if the indicial equation has integer solutions but that somehow the second solution avoids a logarithmic singularity. If moreover, at a fixed \( z = \zeta \) the differential equation \( T[y] = \lambda y \) has an apparent singularity for all \( \lambda \in \mathbb{C} \) then we speak of trivial monodromy at the point in question.

**Proposition 16.** Suppose that \( T \) is reduced and exceptional. If \( z = \zeta \) is a pole of \( q(z) \), then \( T \) has trivial monodromy there.

**Proof.** First note if \( T \) is reduced, exceptional then so is \( T + \lambda \) for all \( \lambda \in \mathbb{C} \). Thus, it suffices to show that \( z = \zeta \) is an apparent singularity. Without loss of generality \( p(\zeta) = 1 \). We take is as proven that there exists a basis of \( y_i \in \mathcal{U}, i \in I_\zeta \) of the form
\[
y_i(z) = (z - \zeta)^i + O((z - \zeta)^{i+1}), \quad z \to \zeta.
\]
Observe that a formal series
\[ a(z) = \sum_{i \in I_\zeta} a_i y_i(z), \quad a_i \in \mathbb{C} \]
defines a power series in \( z - \zeta \), with the coefficient of \((z - \zeta)^n\), \( k \in \mathbb{N} \) being a finite linear combination of the \( a_i, \ i \in I_\zeta \) such that \( i \leq n \). Since \( \mathcal{U} \) is \( T \)-invariant and \( d_\zeta = -2 \), for a given \( i \in I_\zeta \) we have
\[ T[y_i] = \sum_{j \geq i-2} B_{ij} y_j, \quad B_{ij} \in \mathbb{C}, \]
with \( B_{ij} = 0 \) for \( j \) sufficiently large. Thus, \( T[a(z)] = 0 \) is equivalent to the recurrence relations
\[ \sum_{i \in I_\zeta} a_i B_{ij} = 0, \quad j \in I_\zeta. \]

By Proposition 15 (f),
\[ B_{i,i-2} = i(i - 1 - 2\nu_\zeta), \quad i \in I_\zeta. \]
Hence, the recurrence relations take the form
\[ (j + 2)(j + 1 - 2\nu_\zeta)a_{j+2} + \sum_{i \in I_\zeta} B_{ij} a_i = 0, \quad j \in I_\zeta. \tag{59} \]

By Proposition 15 (c),
\[ (j + 2)(j + 1 - 2\nu_\zeta) \neq 0, \quad j \in I_\zeta, \]
Therefore relations (59) recursively define \( a_j, \ j \in I_\zeta \) with \( a_0, a_{2\nu_\zeta+1} \) being free parameters. \( \square \)

**Proposition 17** (Duistermaat-Grünbaum). Let \( U(x) \) be meromorphic in a neighbourhood of \( x = 0 \) with Laurent expansion
\[ U(x) = \sum_{j \geq -2} c_j x^j, \quad c_{-2} \neq 0. \]
Then the Schrödinger operator \(-D_{xx} + U(x)\) has trivial monodromy at \( x = 0 \) if and only if there exists an integer \( \nu \geq 1 \) such that
\[ c_{-2} = \nu(\nu + 1), \quad c_{2j-1} = 0, \quad 0 \leq j \leq \nu. \tag{60} \]

**Proof.** Suppose that (60) hold. Set
\[ T[y] = -y'' + U(x)y, \]

\[ 29 \]
Differential Equations in the Spectral Parameter

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Abstract. We determine all the potentials $V(x)$ for the Schrödinger equation $(-\frac{d^2}{dx^2} + V(x))\phi = k^2\phi$ such that some family of eigenfunctions $\phi$ satisfies a differential equation in the spectral parameter $k$ of the form $B(k, \partial_k)\phi = \Theta(x)\phi$. For each such $V(x)$ we determine the algebra of all possible operators $B$ and the corresponding functions $\Theta(x)$.

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0. Introduction

In this paper we study the following question: For which linear ordinary differential operators $L = \sum_{j=0}^1 L_j(x) \cdot \left(\frac{d}{dx}\right)^j$ is there a non-zero family of eigenfunctions $\phi(x, \lambda)$,

\[(L\phi)(x, \lambda) = \lambda \cdot \phi(x, \lambda),\]

(0.1)
depending smoothly on the eigenfunction parameter $\lambda$, which is also an eigenfunction of a linear ordinary differential operator $A = \sum_{r=0}^m A_r(\lambda) \cdot \left(\frac{d}{d\lambda}\right)^r$

\[(A\phi)(x, \lambda) = \Theta(x) \cdot \phi(x, \lambda)\]

(0.2)

for an eigenvalue $\Theta$ which is a function of $x$?

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Remark. The last argument also shows that no non-zero multiple \( \phi \) of the recessive solution at \( p \) satisfies an equation of the form \((B - \Theta)\phi = 0\).

**Proposition 3.3.** Let \( V \) be an arbitrary meromorphic function in a neighborhood of \( p \in \mathbb{C} \), with Laurent expansion

\[
V(x) = \sum_{r \geq -2} c_r \cdot (x - p)^r.
\]

Then all eigenfunctions of \( L = -\partial_x^2 + V(x) \) are single-valued around \( p \) if and only if

\[
c_{-2} = v_p(v_p + 1) \quad \text{for some} \quad v_p \in \mathbb{Z}_{>0},
\]

and

\[
c_{2j - 1} = 0 \quad \text{for all integers} \quad j \quad \text{such that} \quad 0 \leq j \leq v_p.
\]

Finally, if this is the case, then every eigenfunction \( \phi \) has a Laurent expansion of the form

\[
\phi(x) = (x - p)^{-v_p} \cdot \sum_{r = 0}^{v_p} d_r \cdot (x - p)^r,
\]

with

\[
d_{2j - 1} = 0 \quad \text{for all integers} \quad j \quad \text{such that} \quad 1 \leq j \leq v_p.
\]

**Proof.** The proof of Lemma 3.2 has already shown us that \( l = \frac{1}{2} + v_p \) for some \( v_p \in \mathbb{Z}_{>0} \), so \( c_{-2} = l^2 - \frac{1}{4} = v_p(v_p + 1) \). Also that \( d_{2j} \) for \( 0 \leq j \leq v_p \) is a polynomial in \( \lambda \) of degree \( j \), with positive leading coefficient.

Let \( j_0 \) be the smallest integer \( j \) such that \( 0 \leq j \leq v_p \) and \( c_{2j - 1} \neq 0 \). On the right-hand side of (3.12) for \( r = 2j - 1 \) there appear only terms with \( s \) odd \( \geq 2j_0 - 1 \), so \( r = 2s - 2 \) even \( \leq 2 (2j_0_0 - 1) \), or with \( s \) even \( \geq 0 \), \( r - s - 2 \) odd \( \leq 2j - 3 \). By introduction over \( j \) it follows that \( d_{2j - 1} = 0 \) for \( 1 \leq j \leq j_0 \) and that \( d_{2j - 1} \) is a polynomial in \( \lambda \) of degree \( \leq j - j_0 - 1 \) for \( j_0 < j \leq v_p \). The only terms containing \( \lambda^{j - j_0 - 1} \) are \( c_{2j - 1}, d_{2j - 1}, j_0 - j_0 - 1, \) and \((c_0 - \lambda) \cdot d_{2j - 1}\). It follows, again by induction over \( j \), that the leading coefficient in \( d_{2j - 1} \) is equal to \( c_{2j - 1} \) times a strictly negative number, using that the coefficient in front of the left-hand side of (3.12) \( - \) is negative if \( r = 2j - 1, 1 \leq j \leq v_p \).

Now the right-hand side of (3.17) for \( l = \frac{1}{2} + v_p \) contains terms for \( s \) odd \( \geq 2j_0 - 1 \), the leading one being \( c_{2j_0 - 1} \cdot d_{2j_0 - 1} \); and for \( s \) even \( \geq 0 \), for which the leading one is \( (c_0 - \lambda) \cdot d_{2j_0 - 1} \). We see that \( c \) is a polynomial of degree \( v_p - j_0 \) in \( \lambda \), with a leading coefficient equal to a positive multiple of \( c_{2j_0 - 1} \). If \( j_0 = v_p \), \( d_{2v_p - 1} = 0 \), but still \( c \) is a non-zero constant. The conclusion is that \( c \equiv 0 \) implies (3.21), which in turn implies (3.23).

If finally (3.21), and thus (3.23), holds, then on the right-hand side of (3.17) for \( l = \frac{1}{2} + v_p \) either \( s \) is odd \( \leq 2v_p - 1 \), or \( 2l - 2 - s \) is odd \( \leq 2v_p - 1 \). In both cases \( c_{2j - 2, -1} \cdot d_{2j - 2, -1} = 0 \), so \( c \equiv 0 \).

**Theorem 3.4.** Let \( V \) be a rational function with \( V(\infty) = 0 \). Then the following properties are equivalent.

a) All eigenfunctions of \( -\partial_x^2 + V(x) \) are meromorphic in \( \mathbb{C} \).

b) All eigenfunctions of \( -\partial_x^2 + V(x) \) are of the form \( e^{ikx} \cdot a^+(x, k) + e^{-ikx} \cdot a^-(x, k) \) rational and bounded at infinity.

c) At each pole \( p \) of \( V \), (3.19)-(3.21) hold.

d) \( V \) is obtained from \( V = 0 \) by finitely many rational Darboux transformations.

e) The potentials in the KdV-flow starting at \( V \) remain rational.
and consider a series solution of that satisfies \((T - \lambda)[y] = 0\). The indicial equation is

\[-d(d - 1) + \nu(\nu + 1) = 0,
\]

and hence \(d = -\nu\) or \(d = \nu + 1\). We must show that there exist two independent series solution of the form

\[y = \sum_{j=0}^{\infty} a_j x^{j-\nu}\]  (61)

The series coefficients must satisfy

\[-(j - \nu)(j - \nu - 1)a_j + \sum_{k=0}^{j} a_k c_{j-2-k} - \lambda a_{j-2} = 0, \quad j \geq 1.\]

Expanding the sum gives the recurrences

\[j(j - 2\nu - 1)a_j = a_{j-2}(c_0 - \lambda) + a_{j-4}c_2 + \cdots + a_{j-2\nu-2}c_{2\nu} + a_{j-2\nu-3}c_{2\nu+1} + \cdots + a_0 c_{j-2}, \quad j \geq 1.\]  (62)

It follows that \(a_1 = a_3 = \ldots = a_{2\nu-1} = 0\). This implies that the recurrence for \(j = 2\nu + 1\) is satisfied automatically. Since the recurrences for \(j = 0\) and \(j = 2\nu + 1\) are satisfied identically, \(a_0\) and \(a_{2\nu+1}\) function as free variables. Therefore, there exist two independent series solutions.

Conversely, suppose that the trivial monodromy condition holds. Hence, the indicial equation

\[-d(d - 1) + c_{-2} = 0\]

has integer solutions. It follows that \(c_{-2}\) is a triangular number; say \(c_{-2} = \nu(n + 1)\) with \(\nu \geq 1\). Hence, for every \(\lambda\) there exists a series solution of \((T - \lambda)[y] = 0\) having the form (61) with \(a_0 \neq 0\). Without loss of generality, \(c_0 = 0\). Hence, the solution coefficient satisfies recurrence relations

\[j(j - 2\nu - 1)a_j = a_{j-1}c_{-1} - a_{j-2}\lambda + a_{j-3}c_1 + \cdots + a_0 c_{j-2}, \quad j \geq 1.\]  (63)

The first \(2\nu\) recurrences fix \(a_1, \ldots, a_{2\nu}\). The coefficients \(a_j\) are polynomial in \(\lambda\). By inspection,

\[a_{2j} = A_j \lambda^\nu a_0 + \text{lower degree terms}, \quad j = 0, 1, \ldots, \nu,\]

\[a_{2j+1} = B_j \lambda^{\nu-1} a_0 c_{-1} + \text{lower degree terms}, \quad j = 0, 1, \ldots, \nu - 1\]

where \(A_j > 0\) and \(B_j < 0\). The next recurrence, with \(j = 2\nu + 1\), has the form

\[a_{2\nu} c_{-1} - a_{2\nu-1}\lambda + a_{2\nu-2} c_1 + \cdots + a_0 c_{2\nu-1} = 0\]  (64)

The left hand side is a polynomial

\[(A_{2\nu} - B_{2\nu-1}) a_0 c_{-1} \lambda^\nu + \text{lower degree terms}\]

does not vanish. It follows that \(c_{-1} = 0\). Recursion relation \(j = 1\) then implies that \(a_1 = 0\). The left-side of (64) then reduces to a \(\lambda\)-polynomial of degree \(\nu - 1\) whose leading coefficient is a multiple of \(c_1\). This implies that \(c_1 = 0\). Continuing in like fashion we conclude that \(c_{-1} = c_1 = \ldots = c_{2\nu-1} = 0\) and that \(a_1 = \ldots = a_{2\nu-1} = 0\) also. \(\Box\)
We are now ready to prove the following.

**Proposition 18.** Suppose that $T$ is reduced, exceptional. If $z = \zeta$ is a pole of $q(z)$, then

$$ T_{-1} = p_{-1} \left( (z - \zeta) D_{zz} - \frac{1}{2} \nu \zeta (3 \nu \zeta - 1) \right) + q_{-1} \left( D_z - \frac{\nu \zeta}{z - \zeta} \right), $$

where $p_{-1} = p'(\zeta)$, $q_{-1} = q(\zeta)$.

**Proof.** Since $p(\zeta) \neq 0$ we can define an analytic change of variables $z = \phi(x)$ via

$$ \phi'(x)^2 = p(\phi(x)), \quad \phi(0) = \zeta. $$

Explicitly,

$$ x = \int_{\zeta}^{\phi(x)} \frac{dz}{\sqrt{p(z)}}. $$

In this way

$$ D_{xx} = p(z) D_{zz} + \frac{1}{2} p'(z) D_z. $$

Set

$$ \mu(z) = \exp \left( \frac{1}{2} \int \frac{q(z) - \frac{1}{2} p'(z)}{p(z)} dz \right). $$

Observe that $\mu(z)$ is analytic at $z = \zeta$. A direct calculation shows that

$$ \mu T \mu^{-1} = p(z) D_{zz} + \frac{1}{2} p'(z) D_z + V(z), $$

where

$$ V(z) = \frac{p''(z)}{4} - \frac{q'(z)}{2} - \frac{(q(z) - \frac{1}{2} p'(z))(q(z) - \frac{3}{2} p'(z))}{4p(z)} + r(z). $$

Set

$$ H = -D_{xx} - V(\phi(x)), $$

so that $T[y] = \lambda y$ if and only if $H[\psi] = -\lambda \psi$, where

$$ \psi(x) = \mu(\phi(x)) y(\phi(x)). $$

Hence, $T$ has trivial monodromy at $z = \zeta$ if and only if $H$ has trivial monodromy at $x = 0$. In terms of our notation for the Laurent expansion of $T$, we have

$$ p(z) = p_{-2} + p_{-1}(z - \zeta) + p_0(z - \zeta)^2 + O((z - \zeta)^3), \quad z \to \zeta $$

$$ q(z) = \frac{q_{-2}}{z - \zeta} + q_{-1} + O((z - \zeta)), \quad z \to \zeta, $$

$$ r(z) = \frac{r_{-1}}{z - \zeta} + O(1), \quad z \to \zeta. $$

We already established that

$$ q_{-2} = -2 p_{-2} \nu \zeta. $$
So what we are trying to do here is to give the form for $r_{-1}$.

A direct calculation gives

$$V(z) = -\frac{\nu_\zeta(\nu_\zeta + 1)p_{-2}}{(z - \zeta)^2} + \frac{\nu_\zeta q_{-1} + r_{-1} + p_{-1}\nu_\zeta(\nu_\zeta - 1)}{(z - \zeta)} + O(1), \quad z \to \zeta.$$ 

Relation (66) implies

$$\phi(x) - \zeta = \phi'(0)x + \phi''(0)x^2 + O(x^3), \quad x \to 0,$$

where

$$\phi'(0)^2 = p(\zeta) = p_{-2}$$

$$\phi''(x) = \frac{1}{2}p'(\phi(x))$$

$$\phi''(0) = \frac{1}{2}p'(\zeta) = \frac{1}{2}p_{-1}.$$

Hence,

$$(\phi(x) - \zeta)^{-1} = \frac{x^{-1}}{\phi'(0)} + O(1),$$

$$(\phi(x) - \zeta)^{-2} = \frac{x^{-2}}{\phi'(0)^2} - \frac{\phi''(0)x^{-1}}{\phi'(0)^3} + O(1),$$

$$= \frac{x^{-2}}{p_{-2}} - \frac{p_{-1}x^{-1}}{2p_{-2}\phi'(0)} + O(1),$$

with all relations holding as $x \to 0$. Hence,

$$U(x) \equiv \nu_\zeta(\nu_\zeta + 1)x^{-2} - \frac{1}{\phi'(0)}(\nu_\zeta q_{-1} + r_{-1} + \frac{1}{2}p_{-1}\nu_\zeta(3\nu_\zeta - 1))x^{-1} + O(1), \quad x \to 0.$$

By the Duistermaat-Grunbaum result the coefficient of $x^{-1}$ must vanish. From this (65) follows directly.

### 3.2 Conclusion of the Proof

To complete the proof we need to say something about the polynomial part $\hat{T}$ of an exceptional operator.

**Proposition 19.** If $T$ is exceptional, then $\deg p \leq 2$, $\deg q \leq 1$, $\deg r \leq 0$.

**Proof.** Consider the partial fraction decomposition of $T$ shown in (54). By construction,

$$\deg T_s[y] < \deg y, \quad y \in \mathcal{P}.$$ 

Hence, if $y_k \in \mathcal{P}$ is an eigenpolynomial of degree $k$ we must have

$$\deg T_\infty[y_k] \leq k.$$ 

The desired conclusion follows because this is true for infinitely many $k$. 

$34$
Proposition 20. Suppose that \( T \) is reduced and exceptional, and let

\[
\eta(z) = \prod_{i=1}^{N} (z - \zeta_i)^{\nu_i}, \quad (67)
\]

\[
\mu(z) = \prod_{i=1}^{N} (z - \zeta_i)^{\nu_i(\nu_i-1)/2}, \quad (68)
\]

where \( \zeta_i, i = 1, \ldots, N \) are the poles of the operator, and \( \nu_i = \nu_{\zeta_i} \) the gap cardinalities as defined in (46). Then, for some \( a, b, c \in \mathbb{C} \) we have

\[
q = p' + az + b - 2p \frac{\eta'}{\eta}, \quad (69a)
\]

\[
r = p \frac{\eta''}{\eta} - (az + b) \frac{\eta'}{\eta} + 2p \left( \frac{\mu''}{\mu} - \left( \frac{\mu'}{\mu} \right)^2 \right) + \frac{p'\mu'}{\mu} + c. \quad (69b)
\]

Note that if \( T \) if \( \mu \) were a constant, then (69) reduces to the natural form shown in (36). Thus, if \( \mu \) is not a constant, or equivalently if \( \nu_i > 1 \) for some \( i \), then a gauge transformation is needed to make the transformation to the natural form.

Proof. By Proposition 15 we have

\[
q(z) = -\frac{2p_0\nu_i}{z - \zeta_i} + O(1), \quad z \to \zeta_i \quad (70)
\]

where \( p_0 = p(\zeta_i) \). By (67),

\[
\frac{\eta'(z)}{\eta(z)} = \frac{\nu_i}{z - \zeta_i} + O(1), \quad z \to \zeta_i.
\]

Hence, \(-2p\eta'/\eta\) agrees with the singular part of \( q \). By Form (69a) follows because by Proposition 19 \( q \leq 1 \).

Let \( r(z) \) denote the right side of (69b). Since \( \deg p \leq 2 \), by inspection, \( \deg \hat{r} \leq 0 \). By Proposition 19, \( \deg r \leq 0 \). By Proposition 15, \( r(z) \) has simple poles at \( z = \zeta_i \). Hence, relation (69b) will follow once we show that \( r(z) \) and \( \hat{r}(z) \) have the same residues. Set

\[
\tau_i = \sum_{j \neq i} \frac{\nu_j}{\zeta_i - \zeta_j}, \quad i = 1, \ldots, N,
\]

so that

\[
\frac{\eta'(z)}{\eta(z)} = \frac{\nu_i}{z - \zeta_i} + \tau_i + O((z - \zeta_i)),
\]

\[
p(z) \frac{\eta'(z)}{\eta(z)} = (p_{i0} + p_{i1}(z - \zeta_i)) \left( \frac{\nu_i}{z - \zeta_i} + \tau_i \right) + O((z - \zeta_i)),
\]

\[
= p_{i0} \nu_i + p_{i0} \tau + p_{i1} \nu_i + O((z - \zeta_i)), \quad z \to \zeta_i.
\]
From (69a), which we have already established, it follows that
\[ q_{i0} = p_{1i} \left( \frac{1}{2} - 2\nu_i \right) + s_{i0} - 2p_{i0}\tau_i, \quad s_{i0} = s(z_i). \]
and by (65) of Proposition 18 we have
\[ r(z) = \frac{\frac{1}{2}p_{1i}\nu_i(1 - 3\nu_i) - (p_{1i} \left( \frac{1}{2} - 2\nu_i \right) + s_{i0} - 2p_{i0}\tau_i)\nu_i}{z - z_i} + O(1), \quad z \to z_i \quad (71) \]
Hence by (68) and a direct calculation we obtain
\[
\begin{align*}
\frac{\mu'(z)}{\mu(z)} &= \frac{1}{2}\nu_i(\nu_i - 1) + O(1) \\
\frac{\mu''(z)}{\mu(z)} - \left( \frac{\mu'(z)}{\mu(z)} \right)^2 &= -\frac{1}{2}\nu_i(\nu_i - 1) + O(1), \\
\frac{\eta'(z)}{\eta(z)} + 2\frac{\mu'(z)}{\mu(z)} &= \frac{\nu_i^2}{z - z_i} + O(1), \\
\frac{\eta''(z)}{\eta(z)} &= \frac{\nu_i(\nu_i - 1)}{(z - z_i)^2} + \frac{2\tau_i\nu_i}{z - z_i} + O(1), \\
\tilde{r}(z) &= p \left( \frac{\eta''}{\eta} + 2 \left( \frac{\mu''}{\mu} - \left( \frac{\mu'}{\mu} \right)^2 \right) \right) + p' \left( \frac{\eta'}{2\eta} + \frac{\mu'}{\mu} \right) - s\frac{\eta'}{\eta} \\
&= \frac{2p_{i0}\tau_i\nu_i + \frac{1}{2}p_{1i}\nu_i^2 - s_{i0}\nu_i}{z - z_i} + O(1), \quad z \to z_i,
\end{align*}
\]
which agrees with (71).

We now derive the effect of a gauge transformation on the form (69).

**Proposition 21.** Let \( T = p(z)D_{zz} + q(z)D_z + r(z) \) where \( p \in \mathcal{P}_2 \) and \( q, r \) have the form shown in (69). Let \( \sigma \in \mathcal{Q} \), and let \( \tilde{T} = \sigma T \sigma^{-1} \) be the indicated, gauge-equivalent operator. Then the coefficients \( \tilde{q}(z), \tilde{r}(z) \) of \( \tilde{T} \) have the form shown in (69), but with \( \tilde{\eta} = \sigma \eta, \quad \tilde{\mu} = \sigma^{-1}\mu \), \quad (72)

in place of \( \eta, \mu \).

**Proof.** Set
\[
\begin{align*}
H &= \frac{\eta'}{\eta}, \quad M = \frac{\mu'}{\mu}, \quad S = \frac{\sigma'}{\sigma}, \\
\tilde{H} &= \frac{\tilde{\eta}'}{\tilde{\eta}} = H + S, \quad \tilde{M} = \frac{\tilde{\mu}'}{\tilde{\mu}} = M - S.
\end{align*}
\]
Applying (47), we have
\[ \tilde{q} = q - 2pS = \frac{p'}{2} + s - 2pH - 2pS \]
\[ = \frac{p'}{2} + s - 2\tilde{H}, \]
\[ \tilde{r} = r - qS + p(-S' + S^2), \]
\[ = p(H' + H^2) + \left( \frac{p'}{2} - s \right)H + 2pM' + p'M - \left( \frac{p'}{2} + s - 2pH \right)S + p(-S' + S^2), \]
\[ = p(H' + S' + (H + S)^2 + 2M' - 2S') + p' \left( \frac{H'}{2} + \frac{S}{2} + M - S \right) - s(H + S), \]
\[ = p(\tilde{H}' + \tilde{H}^2) + \left( \frac{p'}{2} - s \right)\tilde{H} + 2p\tilde{M}' + p'\tilde{M}, \]
which is the form shown in (69) but with \( \eta, \mu \) replaced by \( \tilde{\eta}, \tilde{\mu} \).

We now show that the gap cardinalities may be recovered from the residues of the coefficients of an exceptional operator.

**Proposition 22.** Let \( \tilde{T} \in \text{Diff}_2 Q \) be an exceptional operator, and let \( p, \tilde{q}, \tilde{r} \in Q \) be the corresponding coefficients as per (25). Let \( \tilde{\eta}(z) = \prod_{i=1}^{N}(z - \zeta_i)^{\tilde{\nu}_i} \), where \( \zeta_i, i = 1, \ldots, \tilde{N} \) are the poles of \( \tilde{T} \) and \( \tilde{\nu}_i = \tilde{\nu}_{\zeta_i} \) the gap cardinalities. Then
\[ \tilde{q} = p' + az + b - 2p \frac{\eta'}{\eta}, \quad a, b \in \mathbb{C}. \]  

**Proof.** Let \( \mathcal{U} \) be the maximal invariant polynomial subspace of \( \tilde{T} \). Let \( \sigma \in \mathcal{P} \) be a GCD of all polynomials in \( \mathcal{U} \) so that \( T = \sigma^{-1}\tilde{T}\sigma \) is reduced. Proposition 20 gives the form of \( T \). By Proposition 21, \( \tilde{T} \) has the same form. Let \( \zeta_1, \ldots, \zeta_N \) be the poles of \( T \) and \( \nu_1, \ldots, \nu_N \) the gap cardinalities as per (46). Write
\[ \sigma(z) = \prod_{i=1}^{N}(z - \zeta_i)^{\alpha_i} \prod_{i=1}^{M}(z - \xi_i)^{\beta_i}, \]
where \( \zeta_1, \ldots, \zeta_M \) are the zeros of \( \sigma(z) \) distinct from the \( \zeta_i \), and \( \alpha_i \geq 0, \beta_i \geq 0 \) the corresponding multiplicities. Let \( \mathcal{U} \) be the maximal invariant polynomial subspace of \( T \).

We claim that \( \mathcal{U} = \sigma\mathcal{U} \). The inclusion \( \sigma\mathcal{U} \subset \mathcal{U} \) is obvious. We now prove that \( \mathcal{U} \subset \sigma\mathcal{U} \). By definition, every element of \( \mathcal{U} \) is divisible by \( \sigma \). Let \( \tilde{y} \in \mathcal{U} \) be given and set \( y = \sigma^{-1}\tilde{y} \). Observe that
\[ T^k[y] = (\sigma^{-1}\tilde{T}^k\sigma)[y] = \sigma^{-1}\tilde{T}^k[\tilde{y}], \quad k \in \mathbb{N}. \]
By definition, $\tilde{T}^{k}[\tilde{y}] \in \tilde{U}$ for all $k \in \mathbb{N}$. Hence, $T^{k}[y] \in P$ for all $k$. Therefore, $y \in U$.

Hence, the poles and the gap cardinalities of $T$ are

$$\tilde{\zeta}_i = \begin{cases} 
\zeta_i, & i = 1, \ldots, N \\
\xi_{i-N}, & i = N + 1, \ldots, N + M 
\end{cases}$$

$$\tilde{\nu}_i = \begin{cases} 
\nu_i + \alpha_i, & i = 1, \ldots, N \\
\beta_{i-N}, & i = N + 1, \ldots, N + M 
\end{cases}$$

By Propositions 20 and 21, $\deg \tilde{q} \leq 1$ with

$$\tilde{q}(z) \equiv -2 \sum_{i=1}^{N} \frac{p(\tilde{\zeta}_i)\nu_i}{z - \tilde{\zeta}_i} - 2 \sum_{i=1}^{N} \frac{p(\tilde{\zeta}_i)\alpha_i}{z - \tilde{\zeta}_i} - 2 \sum_{i=1}^{M} \frac{p(\xi_i)\beta_i}{z - \xi_i} \mod P_1$$

$$\equiv -2 \sum_{i=1}^{N} \frac{p(\tilde{\zeta}_i)(\nu_i + \alpha_i)}{z - \tilde{\zeta}_i} - 2 \sum_{i=1}^{M} \frac{p(\xi_i)\beta_i}{z - \xi_i} \mod P_1$$

$$\equiv -2 \sum_{i=1}^{N+M} \frac{p(\tilde{\zeta}_i)\tilde{\nu}_i}{z - \tilde{\zeta}_i} \mod P_1.$$ 

This establishes (73).

\[ \square \]

**Proof of Theorem 3.** Let $T \in \text{Diff}_2 Q$ be an exceptional operator. By Proposition 14 no generality is lost if we assume that $T$ is reduced. From there, Proposition 20 gives the form of $T$. Set $\tilde{T} = \mu T \mu^{-1}$, with $\mu \in P$ as per (68). Let $\tilde{q}, \tilde{r}$ be the corresponding first- and zero-order coefficients. By Proposition 21, $\tilde{q}, \tilde{r}$ have the form shown in (69), but with $\tilde{\eta} = \mu \eta$ and $\tilde{\mu} = 1$ in place of $\eta, \mu$. Hence,

$$\tilde{q} = p' + az + b - 2p \frac{\tilde{\eta}'}{\tilde{\eta}},$$

$$\tilde{r} = \frac{p\tilde{\eta}''}{\tilde{\eta}} - (az + b) \frac{\tilde{\eta}'}{\tilde{\eta}} + c,$$

which means that $\tilde{T}$ has a natural formulation. By Proposition 13, $\tilde{T}$ is exceptional; this proves the first assertion of the Theorem. The second assertion, shown in (39), follows from Proposition 22.

\[ \square \]

### 4 Rational factorizations

#### 4.1 Rational Intertwining relations

Gauge-equivalence (38) is the same thing as a 0th order intertwining relation

$$\sigma T = \hat{T} \sigma$$

Higher-order intertwining relations lead to an expanded notion of equivalence.

**Definition 6.** For $\hat{T}, T \in \text{Diff}_2(Q)$ say that $T$ is conjugate to $\hat{T}$ and write $T \sim \hat{T}$ if there exists an $L \in \text{Diff}(Q)^\times$ such that

$$LT = \hat{T}L.$$ (74)
Note: given an intertwining relation (74), we will refer to the order of \( L \) will be referred to as the order of the intertwining relation.

**Proposition 23.** *Conjugacy is an equivalence relation on \( \text{Diff}_2 \mathcal{Q} \).*

**Proof.** Reflexivity is obvious. Let’s prove transitivity. Suppose that
\[
L_0 T_0 = T_1 L_0, \quad L_1 T_1 = T_2 L_2.
\]
It follows that
\[
L_1 L_0 T_0 = L_1 T_1 L_0 = T_2 L_1 L_0,
\]
as was to be shown. The hardest part is reflexivity. We have to show that if (74) holds, then there also exists an \( M \in \text{Diff} \mathcal{Q} \) such that
\[
TM = MT_1.
\]
The remainder of this section is devoted to the proof of this fact. \( \square \)

**Definition 7.** A rational factorization is a relation of the form
\[
T = BA + \lambda, \tag{75}
\]
where \( T \in \text{Diff}_2(\mathcal{Q}) \), \( A, B \in \text{Diff}_1(\mathcal{Q}) \), and \( \lambda \in \mathbb{C} \).

**Definition 8.** A quasi-rational eigenfunction of \( T \) is a function \( \phi(z) \) such that
\[
T[\phi] = \lambda \phi \tag{76}
\]
for some \( \lambda \in \mathbb{C} \) and such that \( \phi'/\phi \in \mathcal{Q} \).

**Proposition 24.** Suppose that (75) holds. Then a non-zero \( \phi \in \ker A \) is a quasi-rational eigenfunction of \( T \) with eigenvalue \( \lambda \).

**Proof.** Applying both sides of (75) to \( \phi \) shows that (76) holds. Write
\[
A = b(D - w), \quad b, w \in \mathcal{Q}, \tag{77}
\]
and observe that \( \phi'/\phi = w \). Therefore, \( \phi \) is quasi-rational. \( \square \)

As we now show, a rational factorization of \( T \) is fully determined by a quasi-rational eigenfunction and the leading coefficient \( b \in \mathcal{Q} \) in (77). For reasons explained below, we call \( b \) the factorization gauge.

**Proposition 25.** Let \( \phi \) be a quasi-rational eigenfunction of \( T \) and \( w = \phi'/\phi \) its rational log-derivative. Let \( b \in \mathcal{Q}^\times \) be arbitrary and set
\[
A = b(D - w), \quad B = \frac{p}{b} \left( D + w + \frac{q}{p} - \frac{b'}{b} \right). \tag{78}
\]
Then, (75) holds.

Furthermore, suppose that \( b_1, b_2 \in \mathcal{Q}^\times \) are two choices of factorization gauge, and that \( A_1, B_1, A_2, B_2 \in \text{Diff}_1 \mathcal{Q} \) and \( T_1, T_2 \in \text{Diff}_2 \mathcal{Q} \) are the corresponding rational factorizations. Then, \( T_1 \) and \( T_2 \) are gauge-equivalent with \( T_2 = \sigma T_1 \sigma^{-1} \), where \( \sigma = b_1/b_2 \).
Proof. Exercise.

We begin by proving the symmetry property of 1st order intertwining relations.

**Proposition 26.** Let $T, \hat{T} \in \text{Diff}_2 \mathcal{Q}$ and suppose that there exists a non-zero $A \in \text{Diff}_1 \mathcal{Q}$ such that

\[ AT = \hat{T}A. \]  

(79)

Then, there also exists a non-zero $B \in \text{Diff}_1 \mathcal{Q}$ such that

\[ TB = B\hat{T}. \]

**Proof.** Write $A$ as in (77), and let $\phi \in \ker A$ be the quasi-rational function given by

\[ \phi(z) = \exp \int^z w. \]  

(80)

By (79), $A[T[\phi]] = 0$. Since $\ker A$ is 1-dimensional, it follows that $\phi$ is a quasi-rational eigenfunction of $T$. Let $\lambda \in \mathbb{C}$ be the corresponding eigenvalue. Then, $T = BA + \lambda$ with $B$ defined as in (78). It follows that

\[ A(T - \lambda) = ABA = (\hat{T} - \lambda)A. \]

Hence, $(AB + \lambda - \hat{T})A = 0$. The ring of differential operators has no zero divisors (exercise). It follows that $\hat{T} = AB + \lambda$. Therefore,

\[ TB = BAB + \lambda B = B\hat{T}. \]

Note: the construction of $\hat{T} = AB + \lambda$ from $T = BA + \lambda$ by the interchange of the first order factors $A, B$ is usually called a Darboux transformation. In the physics literature, the second-order $T, \hat{T}$ related by a Darboux transformation are typically referred to as supersymmetric partners.

**Definition 9.** A 1st-order conjugation chain is a finite sequence of operators $T_0, T_1, \ldots, T_n \in \text{Diff}_2 \mathcal{Q}$ and first-order intertwiners $A_1, \ldots, A_n \in (\text{Diff}_1 \mathcal{Q})^\times$ such that

\[ A_i T_{i-1} = T_i A_i, \quad i = 1, \ldots, n. \]

By Proposition 26, there also exist $B_1, \ldots, B_n \in \text{Diff}_1 \mathcal{Q}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

\[ T_{i-1} = B_i A_i + \lambda_i, \quad i = 1, \ldots, n \]

\[ T_i = A_i B_i + \lambda_i \]  

(81)

For this reason we will often use the term factorization chain in place of 1st-order conjugation chain. These relations maybe combined to conclude that $T_0 \sim T_n$ because

\[ T_n A_n \cdots A_0 = A_n \cdots A_0 T_0, \quad B_0 \cdots B_n T_n = T_0 B_0 \cdots B_n. \]

The symmetry of the conjugation relation $\sim$ is a consequence of the following.
Proposition 27. Suppose that the intertwining relation (74) holds. Then either $T, \hat{T}$ are gauge-equivalent, or there exists a rational factorization chain with $T_0 = T$ and $T_n = \hat{T}$.

Proof. If $\text{ord} L = 0$, then $T$ and $\hat{T}$ are gauge-equivalent. Thus, suppose that (74) holds and that $\text{ord} L \geq 1$.

1. No generality is lost if we assume that $L$ does not have a right factor of the form $T - \lambda$. Indeed, suppose that
   \[ L = \tilde{L}(T - \lambda), \quad \lambda \in \mathbb{C}. \]
   Since $T$ commutes with $T - \lambda$, it follows that
   \[ \hat{T}\tilde{L} = \tilde{L}T \]
   is a lower order intertwining relation between $\hat{T}$ and $T$. Repeating this argument a finite number of times yields an intertwiner $L$ with the desired property.

2. Claim: $T$ leaves $\ker L$ invariant. By relation (74), if $y \in \ker L$, then
   \[ L[T[y]] = \hat{T}[L[y]] = 0, \]
   so $T[y] \in \ker L$ also.

3. Claim: if $T[y] = \lambda y$, then
   \[ L[y] = F(z, \lambda)y + G(z, \lambda)y', \tag{82} \]
   where $F, G$ are polynomial in $\lambda$ and rational in $z$. By assumption,
   \[ y'' = \frac{q(z)}{p(z)}y' + \frac{\lambda - r(z)}{p(z)}y, \]
   Hence, $y^{(k)}$, $k \geq 2$ can always be written as a linear combination of $y$ and $y'$ with coefficients that are polynomial in $\lambda$ and rational in $z$.

4. Since $\ker L$ is finite-dimensional and invariant with respect to $T$, let us choose an eigenvector $\phi \in \ker L$ of $T$ with eigenvalue $\lambda_0$. It follows that
   \[ F(z, \lambda_0)\phi + G(z, \lambda_0)\phi' = 0, \]
   with $F, G$ defined above.

5. Claim: $G(z, \lambda_0)$ is not identically zero. If $\text{ord} L = 1$ the claim is trivial. For $\text{ord} L \geq 2$ we argue by contradiction and suppose that $G(z, \lambda_0) = 0$. Then, $F(z, \lambda_0) = 0$ also, which implies that
   \[ \ker(T - \lambda_0) \subset \ker L. \]
   It can then easily be shown [3, Section 5.4] that
   \[ L = \tilde{L}(T - \lambda_0), \]
   which violates the reducibility assumption established by Claim (a). QED
6. Thus, \( G(z, \lambda_0) \) is not identically zero, and therefore,

\[
\frac{\phi'(z)}{\phi(z)} = -\frac{F(z, \lambda_0)}{G(z, \lambda_0)}
\]

is a rational function. Set \( T_0 = T \) and \( L_0 = L \). Since we have a quasi-rational eigenfunction we can form the rational factorization

\[
T_0 = B_0 A_0 + \lambda_0
\]

with \( A_0[\phi] = 0 \). Since \( \phi \in \ker L \) we also have a rational factorization

\[
L = L_1 A_0, \quad L_1 \in \text{Diff}(\mathbb{Q}).
\]

Setting

\[
T_1 = A_0 B_0 + \lambda_0
\]

we have

\[
(\hat{T}L_1 - L_1 T_1)A_0 = 0
\]

which implies that

\[
\hat{T}L_1 = L_1 T_1.
\]

7. Claim: \( L_1 \) has no right factors of the form \( T_1 - \lambda \). Suppose otherwise, so that

\[
L_1 = \tilde{L}(T_1 - \lambda).
\]

Then, setting \( \tilde{\lambda} = \lambda - \lambda_0 \) we have

\[
L = L_1 A_0 = \tilde{L}(A_0 B_0 - \tilde{\lambda})A_0 = \tilde{L}A_0(B_0 A_0 - \tilde{\lambda}) = \tilde{L}A_0(T - \lambda),
\]

which again violates the irreducibility assumption of Claim (a).

8. Continuing by induction, we have

\[
\hat{T}L_i = L_i T_i, \quad i = 0, 1, \ldots
\]

with \( L_i \) reduced. Repeating the above argument, we construct rational factorizations

\[
T_i = B_i A_i + \lambda_i, \quad T_{i+1} = A_i B_i + \lambda_i,
\]

so that

\[
L = L_{i+1} A_i \cdots A_0
\]

and

\[
\hat{T}L_{i+1} = L_{i+1} T_{i+1},
\]

and so that \( L_{i+1} \) is reduced as per Claim (a). This process terminates when \( L_i \) is a first-order operator, because then we can take \( L_i = A_i \), which gives \( \hat{T} = T_{i+1} \), and completes the factorization chain that connects \( \hat{T} \) and \( T \).
4.2 The fundamental theorem

The structure theorem 3 only provides a normal form for exceptional operators; the polynomial $\eta$ and the constants $a, b$ cannot be freely chosen. On the other hand, quasi-rational eigenfunctions may be used to construct exceptional operators from their classical parts. Borrowing some terminology from the domain of integrable systems, we will say that exceptional operators can be constructed by “dressing” classical, Bochner operators. The fundamental theorem, which we prove in this section, asserts that every exceptional operators is conjugate to a classical operator. What we are asserting is that, in a sense, all exceptional operators and polynomials arise by suitably dressing the classical objects.

**Definition 10.** We define a Bochner operator to be a $T \in \text{Diff}_2 P$ such that $\deg T = 0$.

**Proposition 28.** Every Bochner operator is exceptional.

**Proof.** $T$ has triangular action on $P$ with at most finitely many equalities on the diagonal.

**Theorem 4.** If $T \in \text{Diff}_2 Q$ is exceptional, then there exists a Bochner operator $T_B \in \text{Diff}_2 P$ and a polynomial intertwiner $A \in \text{Diff}_P$ such that $TA = AT_B$.

The rest of this section is devoted to the proof. The converse, though, is much easier.

**Proposition 29.** Suppose that $TA = AT_B$ where $T \in \text{Diff}_Q, A \in (\text{Diff}_P)^\times$ and $T_B \in \text{Diff}_2 P$ is a Bochner operator. Then $T$ is an exceptional operator. Moreover, $A[P] \subset U$, where $U$ is the maximal invariant subspace of $T$.

**Proof.** Let $u_k, \deg u_k = k$ be the eigepolynomials of $T_B$. By Proposition 28, the eigenpolynomial degrees $k$ belong to a co-finite set. Set $\delta = \deg A$ and $y_{k+\delta} = A[u_k]$. The intertwining relation implies that $A[u_k]$ is an eigenpolynomial of $T$, and hence $A[P] \subset U$. Since $\ker A$ is finite dimensional, we conclude that $\text{codim } A[P] < \infty$.

**Definition 11.** Let $T \in \text{Diff}_Q$ and $U = \{ y \in P : T^k[y] \in P \text{ for all } k \}$ the maximal invariant polynomial subspace. We call

$$A = \{ A \in \text{Diff}_P : A[P] \subset U \}$$

the vector space of candidate intertwiners.

**Proposition 30.** An $A \in \text{Diff}_P$ is a candidate intertwiner if and only if $T^k A \in \text{Diff}_P$ for all $k = 1, 2, \ldots$.

**Proof.** Suppose that $A[P] \subset U$. Then, $(T^k A)[P] \subset P$ for all $k$. Hence, $T^k A \in \text{Diff}_P$ by Proposition 2. Conversely, suppose that $T^k A \in \text{Diff}_P$ for all $k$. For a given $f \in P$, write $y = A[f]$. Hence, $T^k[y] = (T^k A)[f] \in P$. Hence, $y \in U$, and therefore $A \in A$.

Next, define the vector space

$$A_0 = \{ A \in A : \deg A \leq 0 \}$$
Proposition 31. The vector space \( \mathcal{A}_0 \) is non-trivial.

Proof. Let \( \zeta_i, i = 1, \ldots N \) be the poles of \( T \) and let \( \nu_i = \nu_{\zeta_i} \) denote the corresponding gap cardinalities as defined in (46). Set

\[
f(z) = \prod_{i=1}^{N} (z - \zeta_i)^{2\nu_i}.
\]

By Lemma 5, the largest gap in \( I_{\zeta_i} \) occurs at \( 2\nu_i - 1 \). Hence, by the proof of Proposition 9, \( \text{Ann}_\mathcal{U} \mathcal{U} \) is spanned by differential functionals of order \( \leq 2\nu_i - 1 \). Hence, if \( y \in \mathcal{P} \), then

\[
D^k [f y](\zeta_i) = 0, \quad k = 1, \ldots, 2\nu_i - 1.
\]

Hence, multiplication by \( f \) is a linear transformation \( \mathcal{P} \to \mathcal{U} \), and hence \( f \in \mathcal{A} \). Therefore \( f D^n \in \mathcal{A}_0 \) where \( n = \deg f \).

The next proposition shows that every candidate intertwiner may be related to an operator in \( \mathcal{A}_0 \).

Proposition 32. Let \( A \in \mathcal{A}_\infty \) and set \( \deg A = k \). If \( k > 0 \), then \( A D^k \in \mathcal{A}_0 \). If \( k < 0 \), then there exists a \( \tilde{A} \in \mathcal{A}_0 \) such that \( A = \tilde{A} D^{-k} \).

Proof. Suppose that \( k > 0 \). Then, \( \deg A D^k = 0 \) by Proposition 3. Since \( D[\mathcal{P}] = \mathcal{P} \), it follows that \( (A D^k)[\mathcal{P}] \subset \mathcal{U} \), and hence \( A D^k \in \mathcal{A} \).

Suppose that \( k < 0 \). Then \( A \) is a linear combination of terms \( z^j D^i \) where \( j - i \leq k \), with at least one term where \( j - i = k \) exactly. Observe that

\[
z^j D^i = (z^j D^{i+k}) D^{-k}.
\]

where \( \deg z^j D^{i+k} \leq 0 \) with at least one term where the degree is 0, exactly. Thus, \( A = \tilde{A} D^k \) for some \( \tilde{A} \in \text{Diff}_2 \mathcal{P} \) with \( \deg \tilde{A} = 0 \). By assumption, \( A[\mathcal{P}] \subset \mathcal{U} \). Let \( f \in \mathcal{P} \).

Since, \( D[\mathcal{P}] = \mathcal{P} \), there exists a \( g \in \mathcal{P} \) such that \( D^k [g] = f \). Hence, \( \tilde{A} [f] = (\tilde{A} D^k)[g] \in \mathcal{U} \).

Hence, \( A[\mathcal{P}] \subset \mathcal{U} \). Therefore \( A \in \mathcal{A}_0 \).

Let \( \rho_{\text{min}} = \min \{ \text{ord} A : A \in \mathcal{A}_0 \} \) and let \( \mathcal{A}_{\text{min}} = \{ A \in \mathcal{A}_0 : \text{ord} A \leq \rho_{\text{min}} \} \).

Proposition 33. We have \( \dim \mathcal{A}_{\text{min}} \leq \rho_{\text{min}} + 1 \).

Proof. We can decompose every \( A \in \mathcal{A}_\infty \) as \( A = A_0 + A_1 \) where

\[
A_0 \in \text{span}\{ z^j D^i : j = 0, 1, \ldots, \rho_{\text{min}} \}
\]

is degree homogeneous, and where \( \deg A_1 < 0 \). If \( \dim \mathcal{A}_{\rho_{\text{min}}} > \rho_{\text{min}} + 1 \), then there exists an \( A_1 \in \mathcal{A}_{\rho_{\text{min}}} \) such that \( \deg A_1 < 0 \). Then, \( A_1 = \tilde{A}_1 D^k, k > 0 \) with \( \tilde{A}_1 \in \mathcal{A} \) by Proposition 32. However, \( \text{ord} \tilde{A}_1 = \rho_{\text{min}} - k \), contradicting the minimality of \( \rho_{\text{min}} \).

Proposition 34. Suppose that \( T \in \text{Diff}_2 \mathcal{Q} \) is an exceptional operator. Then, \( T = T_0 + T_s \) where \( T_0 \in \text{Diff}_2 \mathcal{P} \) is a Bochner operator and where \( T_s \in \text{Diff}_1 \mathcal{Q} \) is such that \( \deg T_s < 0 \).
Proof. If $T$ is in the natural gauge, then the claim follows by the Theorem 3 and (37).

Let $S, R : \mathcal{P} \to \mathcal{P}$ be the linear operators defined by

$$\frac{f}{\eta} = S[f] + \frac{R[f]}{\eta}, \quad f \in \mathcal{P}; \quad \deg R[f] < \deg \eta.$$ 

Observe that

$$\deg S[f] = \deg f - \deg \eta.$$ 

Then, just take

$$T_0[y] = p(z)y'' + (p'(z) + az + b - 2S[p\eta'])y' + (c + S[p\eta'' - az\eta'])y$$

$$T_s[y] = -2R[p\eta']y' + R[p\eta'' - (az + b)\eta']y$$

In general, $T$ is gauge-equivalent to an $T_0$ in natural form. Our result then follows by (47).

In general, the $T_0$ given by the decomposition is not the $T_B$ we are after. Indeed, the choice of $T_0$ is not unique. For any $\gamma \in \mathbb{C}$ we have

$$T = (T_0 + \gamma D) + (T_s - \gamma D),$$

where the first summand is still a Bochner operator and where the second summand if first-order and negative degree. The idea of the proof is that we make an initial choice of $T_0$ and then determine a value of $\gamma$ so that $T \sim T_0 + \gamma D$

Proposition 35. If $A \in \mathcal{A}_{\min}$, then there exists a $\tilde{A} \in \mathcal{A}_{\min}$ such that

$$TA - AT_0 = \tilde{A}D.$$ \hspace{1cm} (83)

Proof. By Proposition 4, $\deg(T_0A - AT_0) < 0$. Since $\deg A = 0$ and $\deg T_s < 0$, we have that $\deg(TA - AT_0) < 0$ also. By Proposition 32 there exists a $\tilde{A} \in \mathcal{A}_0$ such that (83) holds. The leading order terms of $T_0A$ and $AT_0$ are the same. Hence,

$$\ord(T_0A - AT_0) \leq \ord T_0 + \ord A - 1 = \ord A + 1$$

because $T_0$ is a 2nd order operator. Since $T_s$ is a 1st order operator, we have that

$$\ord(TA - AT_0) \leq \ord A + 1$$

also. Hence, $\ord \tilde{A} \leq \rho_{\min}$, and therefore $\tilde{A} \in \mathcal{A}_{\min}$. \hfill \Box

Proof of Theorem 4. By Proposition 35, there exists a linear transformation $\phi : \mathcal{A}_{\min} \to \mathcal{A}_{\min}$ such that $\phi(A) = \tilde{A}$ where $A, \tilde{A} \in \mathcal{A}_{\min}$ are related by (83). By Proposition 33, $\mathcal{A}_{\min}$ is finite-dimensional, and hence $\phi$ has at least one eigenvalue. Let $\gamma \in \mathbb{C}$ be such an eigenvalue. This means that

$$TA - AT_0 = \gamma AD.$$ 

Therefore, 

$$TA - A(T_0 - \gamma D) = 0.$$ 

Therefore, $T_B = T_0 - \gamma D$ is the desired Bochner operator. \hfill \Box

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4.3 Seed eigenfunctions

Proposition 36. Let $T \in \text{Diff}_2 \mathbb{Q}$ be an exceptional operator in the natural gauge (36). Let $\phi(z)$ be a quasi-rational eigenfunction of $T$ and let $w = \phi'/\phi$ be the corresponding rational log-derivative. Then, there exists a factorization gauge $b \in \mathbb{Q}$ and an exceptional $\hat{T} \in \text{Diff}_2 \mathbb{Q}$, also in the natural gauge, such that $AT = \hat{T}A$ where $A = b(D - w)$.

Proof. Let $y_k \in \mathcal{P}_k$, $k \in I$, where $I \subset \mathbb{N}$ is co-finite, be the eigenpolynomials of $T$. Choose a $b \in \mathbb{Q}$ such that $A[y_k] = b(y_k - wy_k) \in \mathcal{P}$ for all $k \in I$. Such a $b$ exists; one possibility would be to take the denominator of $w$. Set $\delta = \deg A$ and $\hat{y}_k = A[y_k]$. By construction, if $T[y_k] = \lambda_k y_k$, then

$$\hat{T}[\hat{y}_k] = \hat{T}[A[y_k]] = A[T[y_k]] = \lambda \hat{y}_k.$$  

Since $\ker A$ is 1-dimensional, it follows that $\{\deg \hat{y}_k : k \in I\}$ is cofinite in $\mathbb{N}$. Therefore $\hat{T}$ is exceptional. By Theorem 3 and by Proposition 25, the gauge factor $b$ may be chose so that $\hat{T}$ is in the natural gauge. \hfill \Box

We now generalize the above to the construction of a multi-step factorization chain. Recall that, by Proposition 27, every intertwining relation $LT = \hat{T}L$, $T, \hat{T} \in \text{Diff}_2 \mathbb{Q}$, $L \in \text{Diff}_2 \mathbb{Q}$ may be realized as a factorization chain (81) with

$$L = A_n \cdots A_1.$$  

Proposition 37. Let $T \in \text{Diff}_2 \mathbb{Q}$ be an exceptional operator in the natural gauge. Let $\phi_i(z)$, $i = 1, \ldots, n$ be a sequence of quasi-rational eigenfunctions. Suppose that the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are distinct. Then, there exists a factorization gauge $b \in \mathbb{Q}$ and an exceptional $\hat{T} \in \text{Diff}_2 \mathbb{Q}$, also in the natural gauge, such that

$$LT = \hat{T}L$$  

(84)

where

$$L[y] = \frac{b \text{Wr}[\phi_1, \ldots, \phi_n, y]}{\text{Wr}[\phi_1, \ldots, \phi_n]}$$  

(85)

and where $\text{Wr}$ denotes the usual Wronskian determinant.

Proof. Let $T_0 = T$ and define a factorization chain (81) recursively as follows. Set $w_1 = \phi'_1/\phi_1$ and choose $b_1$ as per Proposition 36 so that

$$A_1T_0 = T_1A_1$$  

(86)

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where $T_1$ is exceptional in the natural gauge. The key claim is that
\[ \phi_{1,i} := A_1[\phi_i] = b_1 \text{Wr}[\phi_1, \phi_i]/\phi_1, \quad i = 2, \ldots, n \]
are quasi-rational eigenfunctions of $T_1$. Observe that $\phi_1, \phi_i$ are linearly independent, since $\lambda_1 \neq \lambda_i$. Hence, $\phi_{1,i} \neq 0$. A direct calculation shows that
\[ \frac{\phi'_{1,i}}{\phi_{1,i}} = \frac{b'}{b} + w_i + \frac{w'_i - w'_i}{w_1 - w_i}, \quad i = 2, \ldots, n. \]
Hence, $\phi_{1,i}$ is quasi-rational. Finally, the intertwining relation (120) implies that
\[ T_1[\phi_{1,i}] = \lambda_i \phi_{1,i}. \]
Continue in the same fashion to construct a factorization chain where each $T_i, i = 2, \ldots, n$ is an exceptional operator in the natural gauge such that at each step
\[ A_i[\phi_{i-1,j}] = 0, \quad i = 1, \ldots, n \]
and such that
\[ \phi_{i,j} = A_i[\phi_{i-1,j}], \quad i = 1, \ldots, n; j = i + 1, \ldots, n \]
are quasi-rational eigenfunctions of $T_i$.
Setting $L = A_n \cdots A_1$ and $\hat{T} = T_n$ gives (84). By construction,
\[ (A_i \cdots A_1)[\phi_i] = A_i[\phi_{i-1,i}] = 0, \quad i = 1, \ldots, n. \]
It follows that $\phi_1, \ldots, \phi_n \in \ker L$. Since $L$ is an order $n$ operator, it follows that it is a multiple of $y \mapsto \text{Wr}[\phi_1, \ldots, \phi_n, y]$. Also observe that
\[ \frac{\text{Wr}[\phi_1, \ldots, \phi_n, y]}{\text{Wr}[\phi_1, \ldots, \phi_n]} = y^{(n)} + \text{terms with derivatives of lower order} \]
Since $L$ has rational coefficients, there exist a $b \in \mathbb{Q}$ such that (85) holds.

We now have a recipe for constructing exceptional operators. Begin with a classical Bochner operator $T_0$ and choose a finite number of quasi-rational eigenfunctions (often called seed eigenfunctions in the literature). Apply the algorithm detailed in the preceding Proposition to construct a factorization chain of exceptional operators. It is also worth observing that the order of the seed eigenfunctions is not really important, because the $L$ operator in (85) is invariant with respect to permutations of $\phi_1, \ldots, \phi_n$. In the section below, we see how this construction plays out for the Hermite case.

## 5 Exceptional Hermite Polynomials

In this section we focus on the class of exceptional Hermite polynomials in order to explore some key concepts and constructions of exceptional OP theory.
Figure 4: Three equivalent Maya diagrams together with their Frobenius representation. The third diagram is in standard form.

5.1 Maya diagrams

Definition 12. A labelled Maya diagram is a set of integers $M \subset \mathbb{Z}$ that $\{m < 0: M \in \mathbb{Z} \setminus M\}$ and $\{m \geq 0: m \in M\}$ are both finite sets. If $M$ is a Maya diagram, then for $k \in \mathbb{Z}$ so is $M + k = \{m + k: m \in M\}$. We define an unlabelled Maya diagram to be the equivalence class of Maya diagrams related by such shifts.

Definition 13. Let $M \subset \mathbb{Z}$ be a Maya diagram and let $s_1 > \cdots > s_p \geq 0$ be the descending enumeration of

$$M_- = \{-m - 1: m < 0, m \notin M\}$$

and $0 \leq t_1 < \cdots < t_q$ be the ascending enumeration of

$$M_+ = \{m \geq 0: m \in M\}.$$  

The tuple $(s_1, \ldots, s_p | t_1, \ldots, t_q)$ is called the Frobenius symbol of $M$.

It is helpful to visualize a Maya diagram as a horizontally extended sequence of filled and empty boxes, with an origin placed between the box in position $-1$ and box in position $0$, and with filled boxes indicating membership in $M$. In this formulation, the defining assumption of a Maya diagram is that all boxes sufficiently far to the left are filled, and that all boxes sufficiently far to the right are empty. The Frobenius symbol indicates the positions of the empty boxes to the left of the origin and the filled boxes to the right of the origin, and therefore fully and faithfully represents the corresponding Maya diagram.

Definition 14. We say that a Maya diagram $M \subset \mathbb{Z}$ is in standard form if $M_- = \emptyset$ and $0 \notin M$.

Visually, a standard diagram has a gap just to the right of the origin, and no gaps to the left of the origin.

Proposition 38. Let $M \subset \mathbb{Z}$ be a Maya diagram. Then there exists a unique $k \in \mathbb{Z}$ such that $M - k$ is in standard form.
Proof. The desired shift is given by \( k = \min Z \setminus M \).

**Definition 15.** A partition \( \lambda \) is a non-increasing sequence of integers \( \lambda_1 \geq \lambda_2 \geq \cdots \) such that \( \lambda_i = 0 \) for sufficiently large \( i \). Let \( \ell \) be the largest index such that \( \lambda_\ell > 0 \). We call \( \ell \) the length of the partition. We call \( |\lambda| = \lambda_1 + \cdots + \lambda_\ell \) the size of the partition.

Let \( M \subset \mathbb{Z} \) be a Maya diagram and let \( m_1 > m_2 > \cdots \) be its elements ordered in decreasing order. Consider the partition defined by

\[
\lambda_i = \# \{ m \in M : m < m_i \}, \quad i = 1, 2, \ldots
\]

(89)

In other words, \( \lambda \) is formed by counting the gaps to the left of each filled box.

**Proposition 39.** Let \( \lambda \) be a partition, and define

\[
M_\lambda = \{ m_i : i = 1, 2, \ldots \}, \quad \text{where} \quad m_i = \lambda_i + \ell - i.
\]

(90)

Then \( M_\lambda \) is a Maya diagram in standard form. Moreover, (89) holds.

**Proof.** Exercise.

**Corollary 2.** The correspondence \( M \mapsto \lambda \) given by (89) defines a bijection between the set of unlabelled Maya diagrams and the set of partitions.

There is a visually appealing description of the bijection between unlabelled Maya diagrams and partitions.

**Definition 16.** We define a bent Maya diagram to be a doubly infinite sequence \( B = \{(i_n, j_n) \in \mathbb{N} \times \mathbb{N} : n \in \mathbb{Z}\} \) such that

\[
(i_{n+1}, j_{n+1}) - (i_n, j_n) \in \{(1,0), (0,-1)\}, \quad n \in \mathbb{Z}
\]

and such that \( i_n j_n = 0 \) for all but finitely many \( n \).

Note that since \((i_n, j_n) \mapsto (i_{n+1}, j_{n+1})\) is a unit displacement, either down or to the right, the above definition implies that \( i_n = 0 \) for all \( n \) sufficiently small and that \( j_n = 0 \) for all \( n \) sufficiently large.

**Proposition 40.** For a Maya diagram \( M \subset \mathbb{Z} \) set

\[
i_n = \# \{ m \notin M : m < n \}, \quad j_n = \# \{ m \in M : m \geq n \}, \quad n \in \mathbb{Z}.
\]

(91)

Then, the doubly infinite sequence \( B = \{(i_n, j_n)\}_{n \in \mathbb{Z}} \) is a bent Maya diagram.

**Proof.** By assumption, there exists an \( N > 0 \) such that \( n \notin M \) for all \( n \geq N \) and such that \( n \in M \) for all \( n \leq -N \). Hence, \( j_n = 0 \) for all \( n \geq N \) and \( i_n = 0 \) for all \( n \leq -N \). If \( n \in M \), then \((i_{n+1}, j_{n+1}) = (i_n, j_n - 1)\). If \( n \notin M \), then \((i_{n+1}, j_{n+1}) = (i_n + 1, j_{n+1})\). Therefore, in both cases the defining condition of a bent diagram is satisfied.

\[\square\]
A Young diagram is a visual representation of a partition as a finite collection of squares arranged in left-justified rows of length $\lambda_i$, $i = 1, \ldots, \ell$, with the length of each row not exceeding the length of the row beneath it; that is, $\lambda_{i+1} \leq \lambda_i$.

Visually, a bent Maya diagram is a 2-dimensional representation of a Maya diagram, with a filled box at position $n$ corresponding to a unit downward displacement $(0, -1)$ and an empty box corresponding to a unit rightward displacement $(1, 0)$, as depicted in Figure 5. A translation $M' = M - k$ corresponds to an index shift in the bent diagram:

$$ (i'_n, j'_n) = (i_{n+k}, j_{n+k}). \quad (92) $$

**Proposition 41.** The broken line connecting the points of a bent Maya diagram is the union of an infinite horizontal half-line, an infinite vertical half-line, and the top–right border of the Young diagram associated with corresponding partition.

**Proof.** Exercise.

### 5.2 Pseudo-Wronskians

We base the explicit description of the exceptional Hermite operator and polynomials on the notion of a pseudo-Wronskian.

**Definition 17.** Let $M \subset \mathbb{Z}$ be a Maya diagram and $(s_1, \ldots, s_p \mid t_1, \ldots, t_q)$ the corresponding Frobenius symbol. The corresponding Hermite pseudo-Wronskian is defined to be the polynomial

$$ H_M = e^{-pz^2} \text{Wr}[e^{z^2} \tilde{H}_{s_1}, \ldots, e^{z^2} \tilde{H}_{s_p}, H_{t_1}, \ldots, H_{t_q}]. \quad (93) $$
The polynomial nature of $H_M$ becomes evident once we represent it using a slightly different determinant.

**Proposition 42.** A Hermite pseudo-Wronskian admits the following alternative determinantal representation

$$H_M = \begin{vmatrix} \tilde{H}_s & \tilde{H}_{s+1} & \ldots & \tilde{H}_{s+p+q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{H}_s & \tilde{H}_{s+1} & \ldots & \tilde{H}_{s+p+q-1} \\ H_t & D\tilde{H}_t & \ldots & D^{p+q-1}H_t \\ \vdots & \vdots & \ddots & \vdots \\ H_t & D\tilde{H}_t & \ldots & D^{p+q-1}H_t \end{vmatrix}$$

(94)

**Proof.** The desired conclusion follows by the fundamental identities

\begin{align*}
DH_n &= 2nH_{n-1}, \quad n \geq 0, \\
D\tilde{H}_n &= 2n\tilde{H}_{n-1}, \quad n \geq 0, \\
2zH_n &= H_{n+1} + 2nH_{n-1}, \\
2z\tilde{H}_n &= \tilde{H}_{n+1} - 2n\tilde{H}_{n-1}, \\
D(e^{z^2}\tilde{H}_n) &= e^{z^2}\tilde{H}_{n+1}(x), \\
D(e^{-z^2}h_n) &= -e^{-z^2}h_{n+1}.
\end{align*}

(95)

and the Wronskian identity

$$\text{Wr}[g_1, \ldots, g_s] = g^* \text{Wr}[f_1, \ldots, f_s].$$

(96)

We refer to $H_M$ as a pseudo-Wronskian because it is constructed by means of a modified Wronskian operator that replaces the derivative $D$ with an indicial shift for the rows with the conjugate Hermites. More specifically, if $j \in M_+$, then the $(j, k)$ entry is $D^{(k-1)}H_j$, as in the ordinary Wronskian. If $i \in M_-$, then the $(i, k)$ entry is the conjugate Hermite polynomial $\tilde{H}_{i+k-1}$.

**Example 2.** Consider the first Maya diagram in Figure 4. The Frobenius symbol is $(5, 2, 1 | 1, 2)$. The Hermite pseudo-Wronskian $H_M$ associated to $M$ is given by

$$H_M = e^{-3z^2} \text{Wr}[e^{z^2}\tilde{H}_5, e^{z^2}\tilde{H}_2, e^{z^2}\tilde{H}_1, H_1, H_2] = \begin{vmatrix} \tilde{H}_5 & \tilde{H}_6 & \tilde{H}_7 & \tilde{H}_8 & \tilde{H}_9 \\ \tilde{H}_2 & \tilde{H}_3 & \tilde{H}_4 & \tilde{H}_5 & \tilde{H}_6 \\ \tilde{H}_1 & \tilde{H}_2 & \tilde{H}_3 & \tilde{H}_4 & \tilde{H}_5 \\ H_1 & H_1' & H_1'' & H_1'^{(4)} & \tilde{H}_1^{(4)} \\ H_2 & H_2' & H_2'' & H_2'^{m} & H_2^{(4)} \end{vmatrix}$$
The main result of this section is the following class of fundamental determinantal identities enjoyed by these polynomials. Let $M \subset \mathbb{Z}$ be a Maya diagram and $(s_1, \ldots, s_p \mid t_1, \ldots, t_q)$ its Frobenius symbol. Define

$$
\sigma = \sum_{i=1}^{p} s_i - \frac{1}{2} p(p - 1),
$$

$$
\tau = \sum_{i=1}^{q} t_i - \frac{1}{2} q(q - 1),
$$

$$
K_M = (-1)^{\sigma + \tau} \prod_{i<j} (2s_i - 2s_j) \prod_{i<j} (2t_i - 2t_j).
$$

**Theorem 5.** Let $M \subset \mathbb{Z}$ be a Maya diagram, $k \in \mathbb{Z}$ and $M' = M - k$ a shift-equivalent Maya diagram. Then,

$$
\frac{H_{M'}}{K_{M'}} = \frac{H_M}{K_M}, \quad (97)
$$

**Proof.** Once the following two Lemmas are established, it suffices to verify that the factors shown in (98) and (100) below are equal to the above-defined $\gamma_i$ and $\epsilon_i$ symbols, respectively.

Throughout, $s_1 > \cdots > s_p$ and $t_1 < \cdots < t_q$ are, respectively, the enumerations of $M_-$ and $M_+$, respectively.

**Lemma 6.** Suppose that $M' = M - 1$ and that $0 \in M$. Then,

$$
H_M = (-1)^p 2^{q - 1} \left( \prod_{b=1}^{q-1} t_b \right) H_{M'}. \quad (98)
$$

**Proof.** By assumption, $t_1 = 0$ and

$$
M_- = \{s_1 + 1, \ldots, s_p + 1\}, \quad M_+ = \{t_1 - 1, \ldots, t_{q-1} - 1\}.
$$

The identity

$$
\text{Wr}[1, f_1, \ldots, f_s] = \text{Wr}[D f_1, \ldots, D f_s], \quad (99)
$$

together with (95) implies that

$$
H_M = e^{-px^2} \text{Wr}[e^{x^2} \hat{H}_{s_1}, \ldots, e^{x^2} \hat{H}_{s_p}, 1, H_{t_2}, \ldots, H_{t_q}],
$$

$$
= (-1)^p e^{-px^2} \text{Wr}[D(e^{x^2} \hat{H}_{s_1}), \ldots, D(e^{x^2} \hat{H}_{s_p}), DH_{t_2}, \ldots, DH_{t_q}]
$$

$$
= (-1)^p 2^{q - 1} \left( \prod_{b=2}^{q} t_b \right) e^{-px^2} \text{Wr}[e^{x^2} \hat{H}_{s_1+1}, \ldots, e^{x^2} \hat{H}_{s_p+1}, H_{t_2-1}, \ldots, H_{t_q-1}]
$$

$\square$

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Lemma 7. Suppose that $M' = M + 1$ and that $-1 \notin M$. Then,

$$H_M = (-1)^{p+q-1}2^{p-1} \left( \prod_{a=1}^{p-1} s_a \right) H_{M'}$$

(100)

Proof. By assumption, $s_p = 0$ and

$$M'_- = \{ s_1 - 1, \ldots, s_{p-1} - 1 \}, \quad M'_+ = \{ t_1 + 1, \ldots, t_q + 1 \}.$$

The identity (96) together with (99) implies

$$H_M = e^{-px^2} \text{Wr}[e^{x^2} \tilde{H}_{s_1}, \ldots, e^{x^2} \tilde{H}_{s_{p-1}}, e^{x^2} h_{t_1}, \ldots, H_{t_1}]$$

$$= e^{qx^2} \text{Wr}[\tilde{H}_{s_1}, \ldots, \tilde{H}_{s_{p-1}}, 1, e^{-x^2} h_{t_1}, \ldots, e^{-x^2} H_{t_1}]$$

$$= (-1)^{p-1} e^{qx^2} \text{Wr}[D\tilde{H}_{s_1}, \ldots, D\tilde{H}_{s_{p-1}}, De^{-x^2} h_{t_1}, \ldots, De^{-x^2} H_{t_1}]$$

$$= (-1)^{p+q-1}2^{p-1} \left( \prod_{a=1}^{p-1} s_a \right) e^{qx^2} \text{Wr}[\tilde{H}_{s_1-1}, \ldots, \tilde{H}_{s_{p-1}-1}, e^{-x^2} h_{t_1+1}, \ldots, e^{-x^2} H_{t_1+1}]$$

$$= (-1)^{p+q-1}2^{p-1} \left( \prod_{a=1}^{p-1} s_a \right) e^{-(p-1)x^2} \text{Wr}[e^{x^2} \tilde{H}_{s_1-1}, \ldots, e^{x^2} \tilde{H}_{s_{p-1}-1}, h_{t_1+1}, \ldots, H_{t_1+1}]$$

Example 3. Consider the three equivalent Maya diagrams in Figure 4. We have

$$M' = M + 3 : \quad M'_- = \{ 2 \}, \quad M'_+ = \{ 2, 4, 5 \}$$

$$M'' = M + 6 : \quad M''_+ = \emptyset, \quad M''_+ = \{ 1, 2, 3, 5, 7, 8 \}$$

Hence,

$$H_{M'} = \begin{bmatrix}
\tilde{H}_2 & \tilde{H}_3 & \tilde{H}_4 & \tilde{H}_5 \\
\tilde{H}_2' & \tilde{H}_3' & \tilde{H}_4' & \tilde{H}_5' \\
\tilde{H}_4' & \tilde{H}_5' & H_4' & H_5' \\
\tilde{H}_5' & H_5' & H_4' & H_3'
\end{bmatrix},$$

$$H_{M''} = \text{Wr}[H_1, H_2, H_5, H_7, H_8]$$

By (97),

$$-483840 H_{M''} = -1935360 H_{M'} = H_M.$$
Proposition 43. The quasi-rational solutions of (101) fall into two classes. One class of solutions is
\[ y = H_n(z), \quad \lambda = 2n, \quad n \in \mathbb{N} \]
where \( H_n(z) \) is the classical Hermite polynomial of degree \( n \). There is other class of quasi-rational solutions is
\[ y = e^{z^2} \tilde{H}_n(z), \quad \lambda = -2n - 2 \]
where
\[ \tilde{H}_n(z) = i^{-n} H_n(iz). \]

The polynomials \( \tilde{H}_n(z) \) are sometimes referred to as the twisted Hermite polynomials. They are the polynomial solutions of the twisted Hermite differential equation
\[ y'' + 2zy' + 2ny = 0, \quad y = \tilde{H}_n(z), \]
(102)
obtained from (20) by the change of variables \( z \mapsto iz \). The twisted Hermite polynomials resemble the ordinary Hermites, but have all positive coefficients. For example:
\[ H_2(z) = 4z^2 - 2 \quad \tilde{H}_2(z) = 4z^2 + 2 \]
\[ H_3(z) = 32z^5 - 160z^3 + 120z \quad \tilde{H}_3(z) = 32z^5 + 160z^3 + 120z \]

Exercise 11. Verify all this.

Proof. Let \( y = y(z) \) be a quasi-rational solution of (101). Then, \( w = y'/y \) is a rational solution of the Ricatti equation
\[ w' + w^2 - 2zw + \lambda = 0. \]

Write
\[ w(z) = az^k + O(z^{k-1}), \quad z \to \infty, \quad k \in \mathbb{N}, \quad a \in \mathbb{C}. \]
It follows that the highest degree term of \( w^2 \) must match the highest-degree term of \(-2z\). This implies that either \( k = 1, a = 2 \) or that \( w(z) \) has negative degree.

Let \( z = \zeta \) be a pole of \( w(z) \) and write
\[ w(z) = a(z - \zeta)^{-k} + O((z - \zeta)^{-k-1}), \quad z \to \zeta, \quad n \quad k \geq 1. \]
Hence,
\[ w'(z) = -ka(z - \zeta)^{-k-1} + O((z - \zeta)^{-k}), \quad z \to \zeta \]
\[ w(z)^2 = a^2(z - \zeta)^{-2k} + O((z - \zeta)^{-2k+1}) \]
It follows that \(-2k = -k - 1\) and that \( a = k \). It follows that all poles of \( w(z) \) are simple and have unit residues.

In the first case
\[ w(z) = 2z + \sum_{i=1}^{n} \frac{1}{z - \zeta_i}, \quad n \in \mathbb{N} \]
where \( \zeta_1, \ldots, \zeta_n \in \mathbb{C} \) are distinct. Hence,
\[
y = e^{z^2} p(z),
\]
where \( p(z) \) is a polynomial and \( \lambda = -2n - 2 \). Hence \( y = p(z) \) must satisfy (102), and hence \( p(z) \propto H_n(z) \).

In the second case, \( y(z) \) is a polynomial that satisfies (20), and therefore \( y(z) \propto H_n(z) \).

It is very convenient to represent a finite set of quasi-rational Hermite eigenfunctions by means of a Maya diagram. The holes to the left of the origin represent the eigenfunctions \( e^{z^2} \tilde{H}_{s_i}(z), i = 1, \ldots, p \), while the filled boxes to the right of the origin represent the eigenfunctions \( H_{t_j}(z), j = 1, \ldots, q \). It follows by Proposition 37 that every Maya diagram corresponds to an exceptional operator which is conjugate to the classical Hermite operator.

For every Maya diagram \( M \), define the operator
\[
T_M[y] = y'' - 2 \left( z + \frac{H'_M}{H_M} \right) y' + \left( \frac{H''_M}{H_M} + 2z \frac{H'_M}{H_M} \right) y. \tag{103}
\]
Note that \( T_M \) is an natural operator of Hermite type (41) with \( \eta = H_M \). By Theorem 5, the pseudo-Wronskians of two shift-equivalent Maya diagrams differ by a multiplicative factor. Therefore, no generality is lost if we consider only Maya diagrams in standard forms, and label our operators using partitions, rather than Maya diagrams.

Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a partition and \( M \subset \mathbb{Z} \) the corresponding standard Maya diagram with \( m_1 > m_2 > \cdots > m_\ell > 0 \), its positive elements, determined by (90). Set
\[
H_\lambda = \text{Wr}[H_{m_\ell}, \ldots, H_{m_1}], \tag{104}
\]
\[
H_n^{(\lambda)} = \text{Wr}[H_{m_\ell}, \ldots, H_{m_1}, H_{n-d_\lambda}], \tag{105}
\]
where
\[
d_\lambda = |\lambda| - \ell. \tag{106}
\]
and where \( n \geq 0 \) and \( n \notin M + d_\lambda \). Both of the above polynomials are “pure” pseudo-Wronskian polynomials, where by construction
\[
\deg H_\lambda = |\lambda| \tag{107}
\]
\[
\deg H_n^{(\lambda)} = \sum_{i=1}^\ell (m_i - i + 1) + n - d_\lambda - \ell = n \tag{108}
\]
The degree sequence for corresponding to partition \( \lambda \) is
\[
I_\lambda = \mathbb{Z} \setminus (M + d_\lambda).
\]
Thus, \( 0, 1, \ldots, d_\lambda - 1 \) and \( m_1 + d_\lambda, \ldots, m_\ell + d_\lambda \) are the exceptional, missing degrees. Hence, the degree sequence \( I_\lambda \) is missing a total of \( d_\lambda + \ell = |\lambda| \) degrees.
Proposition 44. Let \( \lambda \) be a partition of length \( \ell \). Then, the operator \( T_\lambda + \ell \) is conjugate to the classical Hermite operator
\[
T[y] = y'' - 2zy.
\]

Specifically,
\[
AT = (T_\lambda - \ell)A,
\]
where
\[
A[y] = \text{Wr}[H_{m_1}, \ldots, H_{m_1}, y].
\]
Moreover, we have the eigenvalue relations
\[
T_\lambda [H^{(\lambda n)}_n] = 2(|\lambda| - n)H^{(\lambda)}_n, \quad n \in I_\lambda.
\] (109)

We prove this Proposition by constructing a factorization chain that links \( T \) with \( T_\lambda - 2\ell \).

To give a nice form to the exceptional Hermite equation, introduce the following bilinear differential expression:
\[
\chi[f, y] = fy'' - 2fy' + f''y - 2z(fy' - f'y), \quad f = f(z), \ y = y(z),
\] (110)
and observe that (109) is equivalent to the relation
\[
\chi [H_\lambda, H^{(\lambda)}_n] = 2(|\lambda| - n)H_\lambda H^{(\lambda)}_n.
\] (111)

Lemma 8. Fix polynomials \( \xi, \eta \in \mathcal{P} \), and define the operators \( A, B \in \text{Diff}_1 \mathbb{Q} \) by
\[
A[y] = \eta^{-1} \text{Wr}[\xi, y] = \frac{\xi}{\eta} \left( y' - \frac{\xi'}{\xi} y \right)
\] (112)
\[
B[y] = e^{-z^2} \xi^{-1} \text{Wr}[e^{z^2} \eta, y] = \frac{\eta}{\xi} \left( y' - \left( 2z + \frac{\eta'}{\eta} \right) y \right).
\] (113)

Then
\[
(AB)[y] = \xi^{-1} \chi[\xi, y] - \eta^{-1} \xi^{-1} \chi[\eta, \xi] y - 2y,
\] (114a)
\[
(BA)[y] = \eta^{-1} \chi[\eta, y] - \eta^{-1} \xi^{-1} \chi[\eta, \xi] y
\] (114b)

Proof. We prove (114b); relation (114a) can be proved using similar methods. By inspection, both the left- and the right-side expression in (114b) are 2nd-order operators that annihilate \( \xi \). It suffices to show that the 2nd- and 1st-order coefficients of both sides agree. By inspection,
\[
B = e^{z^2} \frac{\eta^2}{\xi} \left( D + \frac{\xi'}{\xi} \right) e^{-z^2} \xi \eta
\]
\[
A = \frac{\xi}{\eta} \left( D - \frac{\xi'}{\xi} \right)
\]
Hence,

\[ BA = e^{z^2 \eta^2} \left( D + \frac{\xi'}{\xi} \right) e^{-\frac{z^2}{\eta^2}} \left( D - \frac{\xi'}{\xi} \right) = e^{z^2 \eta^2} D e^{-\frac{z^2}{\eta^2}} D + \text{ a zeroth order expression} \]

By inspection,

\[ e^{z^2 \eta^2} D e^{-\frac{z^2}{\eta^2}} D = D - \left( 2z + \frac{2\eta'}{\eta} \right) \]

Hence,

\[ e^{z^2 \eta^2} D e^{-\frac{z^2}{\eta^2}} D = D^2 - \left( 2z + \frac{2\eta'}{\eta} \right) D + \ldots, \]

which agrees with

\[ \eta^{-1} \chi[\eta, y] = y'' - \left( 2z + \frac{2\eta'}{\eta} \right) y' + \ldots \]

\[ \square \]

**Lemma 9.** Let \( \lambda \) be a partition of length \( \ell > 0 \) and let \( \lambda' \) denote the truncated partition of length \( \ell - 1 \) with elements \( \lambda_2 \geq \cdots \geq \lambda_\ell > 0 \); i.e., \( \lambda'_i = \lambda_{i+1} \). Define

\[ A[y] = \frac{H_\lambda}{H_{\lambda'}} \left( y' - \frac{H'_\lambda}{H_\lambda} y \right) \quad (115) \]

Then,

\[ A[\text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_m]] = \text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_m, H_m] \quad (116) \]

Moreover,

\[ A \left[ H_{n+1-\lambda}^{(\lambda')} \right] = H_n^{(\lambda)}, \quad n \in I_\lambda. \quad (117) \]

**Proof.** The proof is based on the “Wronskian of Wronskians” differential identity:

\[ \text{Wr}[\text{Wr}[f_1, \ldots, f_n], g], \text{Wr}[f_1, \ldots, f_n, h]] = \text{Wr}[f_1, \ldots, f_n] \text{Wr}[f_1, \ldots, f_n, g, h] \quad (118) \]

Observe that

\[ A[y] = \frac{1}{H_\lambda} \text{Wr}[H_\lambda, y]. \]

Since

\[ H_\lambda = \text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_m] \]

\[ H_\lambda = \text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_m] \]

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Assertion (116) follows immediately.

Next, define Let $n \in I_\lambda$, and set $m = n - d_\lambda$ so that

$$H^{(\lambda)}_n = \text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_{m_1}, H_m]$$

Observe that

$$n + 1 - \lambda_1 - d_{\lambda'} = n + 1 - \lambda_1 - |\lambda'| + \ell - 1 = n - d_\lambda = m.$$ 

Hence,

$$H^{(\lambda')}_{n+1-\lambda_1} = \text{Wr}[H_{m_\ell}, \ldots, H_{m_2}, H_m]. \quad (119)$$

Assertion (117) now follows from (116).

Proof of Proposition 44. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ be the partition in question, and $m_1 > m_2 > \ldots > m_\ell$ be the corresponding positive elements of $M_\lambda$ defined by (90). Let $\lambda^{(i)}$ denote the truncated partition $\lambda_j \geq \ldots \geq \lambda_\ell$ of length $\ell - j$ obtained by dropping the first $j$ elements of $\lambda$. Set

$$T_i = T_{\lambda^{(\ell-i)}}, \quad \eta_i = H_{\lambda^{(\ell-i)}}, \quad i = 1, \ldots, \ell.$$ 

With this definition, we have $T_\ell = T_\lambda$ and $T_0 = T$, the classical Hermite operator. By construction,

$$M_{\lambda^{(i)}} = \{m_k : k \geq j\}.$$ 

Hence,

$$\eta_i = \text{Wr}[H_{m_\ell}, \ldots, H_{m_{\ell-i+1}}], \quad i = 1, \ldots, \ell,$$

with $\eta_1 = H_{m_\ell}$ and $\eta_\ell = H_\lambda$. Define

$$A_1[y] = \eta_1 \left( y' - \frac{\eta_1'}{\eta_1} \right),$$

$$B_1[y] = \frac{1}{\eta_1} (y' - 2zy).$$

These operators have the form shown in (112) with $\xi = \eta_1 = H_{m_\ell}$ and $\eta = 1$. By (110) and (103)

$$\eta^{-1}\chi[\eta, y] = T[y],$$

$$\xi^{-1}\chi[\xi, y] = T_1[y].$$

Hence,

$$\chi[\eta, \xi] = T[\xi] = -2m_\ell \xi.$$ 

Hence, by (114)

$$B_1 A_1 = T_0 + 2m_\ell,$$

$$A_1 B_1 = T_1 + 2m_\ell - 2.$$
Hence, \( T_0 \sim T_1 - 2 \) because 
\[
  A_1 T_0 = (T_1 - 2) A_1. 
\] (120)

Furthermore, by Lemma 9, 
\[
  A_1[H_{m_{\ell-1}}] = \text{Wr}[H_{m_{\ell}}, H_{m_{\ell-1}}] = \eta_2. 
\]

Since 
\[
  T_0[H_{m_{\ell-1}}] = -2m_{\ell-1} H_{m_{\ell-1}}, 
\]
we have by (120)
\[
  \eta_1^{-1} \chi[\eta_1, \eta_2] = T_1[\eta_2] = T_1[A_1[H_{m_{\ell-1}}]] = (A_1(T_0 + 2))[H_{m_{\ell-1}}] = (2 - 2m_{\ell-1}) \eta_2. 
\]

We can therefore continue in the same fashion to conclude that 
\[
  A_i T_{i-1} = (T_i - 2) A_i, \quad i = 1, \ldots, \ell 
\]
The details are left as an exercise. The proof of (109) is also left as an exercise. \( \square \)

6 Review of relevant literature

Exceptional orthogonal polynomials are complete systems of orthogonal polynomials that satisfy a Sturm-Liouville problem. They differ from the classical families of Hermite, Laguerre and Jacobi in that there are a finite number of exceptional degrees for which no polynomial eigenfunction exists. The total number of gaps in the degree sequence is the codimension of the exceptional family. As opposed to their classical counterparts [4, 5], the differential equation contains rational instead of polynomial coefficients, yet the eigenvalue problem has an infinite number of polynomial eigenfunctions that form the basis of a weighted Hilbert space. Because of the missing degrees, exceptional polynomials circumvent the strong limitations of Bochner’s classification theorem, which characterizes classical Sturm-Liouville orthogonal polynomial systems [6, 7].

The recent development of exceptional polynomial systems has received contributions both from the mathematics community working on orthogonal polynomials and special functions, and from mathematical physicists. Among the physical applications, exceptional polynomial systems appear mostly as solutions to exactly solvable quantum mechanical problems, describing both bound states [8, 9, 10, 11, 12, 13, 14, 15, 16] and scattering amplitudes [17, 18, 19, 20]. But there are also connections with super-integrability [21, 22] and higher order symmetry algebras [23, 24, 25], diffusion equations and random processes [26, 27, 28], quantum information entropy [29], exact solutions to Dirac equation [30] and finite-gap potentials [31].

Some examples of exceptional polynomials were investigated back in the early 90s, [32] but their systematic study started a few years ago, where a full classification was given for codimension one, [33, 34]. Soon after that, Quesne recognized the role of Darboux transformations in the construction process and wrote the first codimension two examples, [35], and Odake & Sasaki showed families for arbitrary codimension, [13,
The role of Darboux transformations was further clarified in a number of works, [37, 38, 14], and the next conceptual step involved the generation of exceptional families by multiple-step or higher order Darboux transformations, leading to exceptional families labelled by multi-indices, [1, 39, 40]. Other equivalent approaches to build exceptional polynomial systems have been developed in the physics literature, using the prepotential approach [41] or the symmetry group preserving the form of the Rayleigh-Schrödinger equation [42], leading to rational extensions of the well known solvable potentials.

In the mathematical literature, two main questions have centred the research activity in relation to exceptional polynomial systems: describing their mathematical properties and achieving a complete classification. Among the mathematical properties, the study of their zeros deserve particular attention. Zeros of exceptional polynomials are classified into two classes: regular zeros which lie in the interval of orthogonality and exceptional zeros, which lie outside this interval. Their interlacing, asymptotic behaviour, monotonicity as a function of parameters and electrostatic interpretation have been investigated in a number of works, [43, 44, 45, 46, 47], but there are still open problems in this direction.

A fundamental object in the theory of orthogonal polynomials is the recurrence relation. Classical orthogonal polynomials have a three term recurrence relation, but exceptional polynomial systems have recurrence relations whose order is higher than three. There is a set of recurrence relations of order $2N + 3$ where $N$ is the number of Darboux steps [48, 8] with coefficients that are functions of $x$ and $n$, and another set of recurrence relations whose coefficients are just functions of $n$ (as in the classical case) and whose order is $2m + 3$ where $m$ is the codimension, [49, 50, 51]. While the former relations are generally of lower order and thus more convenient for an efficient computation, the latter are more amenable to a theoretical interpretation in terms of the usual theory of Jacobi matrices and bispectrality.

The spectral theoretic aspects of exceptional differential operators were first addressed in [52, 53] and developed more recently in a series of papers [54, 55, 56].

The quest for a complete classification of exceptional polynomials has been fundamental problem that is now close to being solved, and the results in the present paper are a key step towards this goal. The first attempts to classify exceptional polynomial systems proceeded by increasing codimension. Codimension one systems were classified in [33] and they included just one $X_1$-Laguerre and one $X_1$-Jacobi family. The classification for codimension two was performed in [57], based on an exhaustive case-by-case enumeration of invariant flags under a given symmetry group. Due to the combinatorial growth of complexity with increasing codimension, this original approach proved to be unfeasible for the purpose of achieving a complete classification. However, a fundamental idea towards the full classification was that every exceptional polynomial operator can be obtained “dressing” a classical operator, that is by applying a finite number of Darboux transformations. This idea was launched in as a conjecture [57], and proved in [58]. It appears in these notes as Theorem 4.

Now that the conjecture has been proved, the program to classify exceptional polynomial systems is almost constructive: start from the three classical systems of Hermite, Laguerre and Jacobi and apply all possible Darboux transformations to describe the
entire exceptional class. Only rational Darboux transformations need to be considered, i.e. those that map polynomial eigenfunctions into polynomial eigenfunctions, and this type of transformations are well understood and catalogued, and they are indexed by sequences of integers. This constructive approach has already been used to generate large classes of exceptional polynomial systems. The most general class obtained in this way can be labelled by two sets of indices or partitions (for the Laguerre and Jacobi classes) [2] or just one set (for the Hermite class) [8, 59] which can be conveniently represented in a Maya diagram [60], a representation that takes naturally into account a number of equivalent sets of indices that lead to the same exceptional system, [61, 62].

However, the classification program is not complete owing to an unresolved complication. We now know that there exist of irregular rational factorization chains, that is factorization chain were two of the factorization eigenvalues are equal [63, 64]. The papers just cited derive and exhibit a large class of such irregular factorization chains, but at this point there is no classification.

A corollary of the structure Theorem 3 is now the assertion that the weight for the exceptional system $W(z)$ is a rational modification of a classical weight $W_0(z)$ having the following form:

$$W(z) = \frac{W_0(z)}{\eta(z)^2},$$

(121)

where $\eta(z)$ is a polynomial in $z$ (a Wronskian-like determinant) whose degree coincides with the codimension of the system.

To obtain genuine Sturm-Liouville problems and genuine orthogonal polynomial, whoever, once cannot admit all possible exceptional weights because may of them are singular. This is known as the weight regularity problem. This means studying the sequence of Darboux transformations and the range of parameters for which:

i) the weight has the right asymptotic behaviour at the endpoints

ii) $\eta(z)$ has no zeros inside the interval of orthogonality.

The regularity problem has been solved for the exceptional Hermite class [8, 65] based on results by Krein [66], and Adler [67]. Another approach to regularity is developed in a series of papers by Duran, who studies exceptional difference operators and obtains the exceptional differential operators discussed here by means of a limit process [68, 69]. The method is a remarkable correspondence between exceptional polynomials and discrete Krall type polynomials [2] and naturally yields regularity criteria for the Hermite [65], Laguerre [2] and the Jacobi [70] classes.

A key technical tool utilized in the proof of Theorem 3 is the demonstration that an exceptional operator has trivial monodromy at its poles. This result was already known for the exceptional Hermite class [8, 71], and is extended in these notes to the class of general exceptional operators. The connection between trivial monodromy, bispectrality, Darboux transformations and the solvable character of Schrödinger operators has been discussed in a number of papers (see for instance [72, 71, 73, 74, 75, 76] and the references therein), and the results in [58] are one further piece of evidence of the close relationship among these concepts.
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