Higher Level Generalizations of Multiple Zeta Values

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Outline

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2. Some known identities of MZVs
3. Double shuffle relations
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5. Standard and non-standard relations
6. Higher levels, II: MZVs at level N
7. Multiple Eisenstein series
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Multiple zeta values

Definition.

The multiple zeta values (MZVs) are defined by

$$\zeta(n_1, \ldots, n_d) = \sum_{k_1 > \cdots > k_d > 0} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}}$$

when $n_1, \ldots, n_d$ are positive integers and $n_1 \geq 2$.

- d: depth
- $|(n_1, \ldots, n_d)| = n_1 + \cdots + n_d$: weight

A double shuffle relation

Another form of shuffle.

For positive integers $r, s \geq 2$ we have

$$\begin{aligned}\zeta(r)\zeta(s) &= \zeta(r,s) + \zeta(s,r) + \zeta(s+r) \\ &= \sum_{i=0}^{s-1} \binom{r+i-1}{r-1} \zeta(r+i, s-i) \\ &+ \sum_{j=0}^{r-1} \binom{s+j-1}{s-1} \zeta(s+j, r-j).\end{aligned}$$

Proof.

Use either iterated integrals or partial fractions.

A double shuffle relation

Another form of double shuffle. (Gangl, Kaneko & Zagier, 2005)

Define generating function

$$\mathcal{Z}(X,Y) = \sum_{r\geq 2, s\geq 1} \zeta(r,s) X^{r-1} Y^{s-1}.$$

Then

$$\mathcal{Z}(X,Y)+\mathcal{Z}(Y,X)+\frac{\mathcal{Z}(X)-\mathcal{Z}(Y)}{X-Y} = \mathcal{Z}(X+Y,Y)+\mathcal{Z}(X+Y,X).$$

MZV: more identities

Restricted sum formula. (Cai & Shen, 2011, Hoffman, 2012)

Let n and d be two positive integers with $n > 1$. Then

$$\sum_{\substack{j_1+\cdots+j_d=n\\ j_1,\dots,j_d\geq 1}} \zeta(2j_1,\dots,2j_d)$$

$$= \binom{2d-1}{d}\frac{\zeta(2n)}{2^{2(d-1)}} - \sum_{j=1}^{\lfloor\frac{d-1}{2}\rfloor}\binom{2d-2j-1}{d}\frac{\zeta(2j)\zeta(2n-2j)}{2^{2d-3}(2j+1)B_{2j}}.$$

MZV: more identities

Restricted sum formula. (Yuan and Z., 2013).

For any positive integers $n \geq d \geq 3$,

$$\sum_{\substack{j_1+\cdots+j_d=n \\ j_1,\ldots,j_d>0}} \zeta(4j_1,\ldots,4j_d)$$

$$= \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} \frac{2^{k+2}(-1)^{\lfloor \frac{k}{2} \rfloor + j + d}}{(2k+1)!} \binom{2k+1}{j} \binom{\frac{j-2}{4}}{d} \zeta(4n-2k)\pi^{2k}$$

$$+ \sum_{k=0}^{\lfloor \frac{d-2}{4} \rfloor} \sum_{j=0}^{4k+2} \frac{2^{2k+5}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \binom{\frac{j-2}{4}}{d}$$

$$\cdot \left(Q(4n-4k,2) - \frac{7}{8}\zeta(4n-4k) \right) \pi^{4k} \in \pi^{4n}\mathbb{Q}.$$

Weighted restricted sum formula. (Guo, Lei and Z., 2014)

Let n be a positive integer. If $n \geq 2$ then

$$\sum_{i+j=n} i^2 j^2 \zeta(2i,2j) = \frac{3}{32}(3n-2)(n-1)\zeta(2n) - \frac{3}{4}\zeta(4)\zeta(2n-4).$$

If $n \geq 3$ then

$$\begin{aligned}&\sum_{i+j+k=n} \left(i^2 j^2 k + j^2 k^2 i + k^2 i^2 j\right) \zeta(2i,2j,2k) \\ &= \frac{n}{256}\zeta(2n) + \frac{2n^2-6n+3}{64}\zeta(2)\zeta(2n-2) \\ &+ \frac{(5n-9)(2n-5)}{16}\zeta(4)\zeta(2n-4) - \frac{3(2n-7)}{8}\zeta(6)\zeta(2n-6).\end{aligned}$$

Remark. We have a conjecture for the general case.

MZV: more identities

2-1 Formula Conjecture. (Ohno & Zudilin, 2008)

For $k \in \mathbb{N}_0$ and $\mathbf{s} = (s_1, \ldots, s_\ell) \in \mathbb{N}^\ell$, $s_1 > 1$ we have

$$\zeta^\star(\{2\}^{a_1}, 1, \ldots, \{2\}^{a_\ell}, 1) = \sum_{\mathbf{p}} 2^{\mathrm{depth}(\mathbf{p})} \zeta(\mathbf{p}),$$

where $\mathbf{p}$ runs through the form $(2a_1 + 1) \circ \cdots \circ (2a_\ell + 1)$ with "$\circ$"="," or "$\circ$"="$+$".

Theorem. ($\{$Hessami-Pilehrood$\}^2$ & Tauraso, 2012)

For $a \geq 1$ we have

$$\zeta^\star(\{2\}^a, 1) = 2\zeta(2a + 1).$$

MZV: more identities

Theorem. 2-1 Formula. (Z., 2013)

For $a_1, \dots, a_\ell \in \mathbb{N}_0$ with $a_1 \geq 1$ we have

$$\zeta^\star(\{2\}^{a_1}, 1, \dots, \{2\}^{a_\ell}, 1) = \sum_{\mathbf{p}} 2^{\mathrm{depth}(\mathbf{p})} \zeta(\mathbf{p}),$$

where $\mathbf{p}$ runs through the form $(2a_1 + 1) \circ \cdots \circ (2a_\ell + 1)$ with "$\circ$"="," or "$\circ$"="$+$".

Multiple harmonic sums

Key idea to prove 2-1 formula.

Study multiple harmonic sums

$$H_n(s_1,\dots,s_d)=\sum_{n\ge k_1>\cdots>k_d\ge 1}\frac{1}{k_1^{s_1}\cdots k_d^{s_d}},$$

$$H_n^*(s_1,\dots,s_d)=\sum_{n\ge k_1\ge\cdots\ge k_d\ge 1}\frac{1}{k_1^{s_1}\cdots k_d^{s_d}}.$$

Multiple harmonic sums

Theorem. ($\{$Hessami-Pilehrood$\}^2$ & Tauraso, 2012)

Let $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$. Then for any $n \in \mathbb{N}$

$$H_n^\star(\{2\}^a, 1) = 2 \sum_{k=1}^{n} \frac{\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}},$$

$$H_n^\star(\{2\}^a, 1, \{2\}^b) = -2 \sum_{k=1}^{n} \frac{(-1)^k \binom{n}{k}}{k^{2a+1+2b}\binom{n+k}{k}} - 4 \sum_{k=1}^{n} \frac{H_{k-1}(\overline{2b})\binom{n}{k}}{k^{2a+1}\binom{n+k}{k}}.$$

Multiple harmonic sums: congruences

Theorem. ($\{$Hessami-Pilehrood$\}^2$ & Tauraso, 2012)

For all prime $p \geq 13$ we have

$$H_{p-1}(3,1,3,1) \equiv -\frac{31}{72} p B_{p-9} \pmod{p^2},$$

$$H_{p-1}(1,3,1,3) \equiv -\frac{1}{72} p B_{p-9} \pmod{p^2},$$

$$H^\star_{p-1}(1,1,1,6) \equiv \frac{1889}{648} B_{p-9} + \frac{1}{54} B^3_{p-3} \pmod{p}.$$

Going up now!

Multiple polylogarithms (multi-polyog)

Definition.

For any positive integers $n_1, \dots, n_d$, we define the multi-polyog function as follows:

$$\mathrm{Li}_{n_1,\dots,n_d}(x_1,\dots,x_d) = \sum_{k_1>k_2>\cdots>k_d>0} \frac{x_1^{k_1} x_2^{k_2} \dots x_d^{k_d}}{k_1^{n_1} k_2^{n_2} \dots k_d^{n_d}},$$

where $|x_1 \cdots x_j| < 1$ for all j.

Definition.

- N: positive integer, called the level.
- $\boldsymbol{\mu}_N = \{\exp(2k\pi i/N) : k = 0, 1, \ldots, N-1\}$.

For $\mu_1, \ldots, \mu_d \in \boldsymbol{\mu}_N$, the special value of multi-polylog

$$\mathrm{Li}_{n_1,\ldots,n_d}(\mu_1, \ldots, \mu_d) = \sum_{k_1 > k_2 > \cdots > k_d > 0} \frac{\mu_1^{k_1} \cdots \mu_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}},$$

is called a multi-polylog value (MPV). Note $(n_1, \mu_1) \neq (1, 1)$.

- d: depth
- $|(n_1, \ldots, n_d)| = n_1 + \cdots + n_d$: weight

First example of MPV: classical cyclotomy

Example 1. Weight one case: cyclotomic numbers.

The cyclotomic units can be constructed using $1-\mu$ for $1 \neq \mu \in \boldsymbol{\mu}_N$. We have

$$-\log(1-\mu) = \sum_{k>0} \frac{\mu^k}{k} := \mathrm{Li}_1(\mu).$$

Branch cut of log: $(-\infty, 0]$.

I: Symmetry relations.

Let $\mu = \exp(2\pi i/N)$. For all $1 < j < N/2$

$$\mathrm{Li}_1(\mu^j) - \mathrm{Li}_1(\mu^{-j}) = \frac{N-2j}{N-2}\big(\mathrm{Li}_1(\mu) - \mathrm{Li}_1(\mu^{-1})\big).$$

First example of MPV: classical cyclotomy

II: Distribution relations. (Bass, 1966)

Let $\mu = \exp(2\pi i/N)$. For any divisor b of N and $1 \le a < b$ we have

$$\sum_{0 \le j < N/b} \mathrm{Li}_1(\mu^{a+bj}) = \mathrm{Li}_1(\mu^{aN/b})$$

since

$$\sum_{j=0}^{N-1} \mu^{Mj} = \begin{cases} N, & \text{if } N|M; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem. (Bass, 1966)

All $\mathbb{Q}$-linear relations among $\{\mathrm{Li}_1(\mu) : 1 \ne \mu \in \boldsymbol{\mu}_N\}$ are generated by relations I and II.

MPV: more examples, levels 1 and 2

Example 2. MPV at level 1: MZVs

When level $N = 1$ we get the multiple zeta values (MZV)

$$\zeta(n_1, \dots, n_d) = \sum_{k_1 > \cdots > k_d > 0} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}}.$$

Example 3. MPV at level 2: alternating Euler sums.

At level $N = 2$ we get

$$\mathrm{Li}_{2,1}(-1, 1) = \sum_{m > n > 0} \frac{(-1)^m}{m^2 n} = \zeta(\bar{2}, 1).$$

MPV: more examples, level 2

Theorem. (Borwein, Broadhurst, & Bradley, 1996)

When level $N = 2$ we have

$$\zeta(3) = 8\zeta(\bar{2}, 1).$$

Conjecture. (Borwein, Broadhurst, & Bradley, 1996)

For all $n \geq 1$

$$\zeta(\{3\}^n) = 8^n \zeta(\{\bar{2}, 1\}^n).$$

MPV: more examples, level 2

Theorem. (Borwein, Broadhurst, & Bradley, 1996)

When level $N = 2$ we have

$$\zeta(3) = 8\zeta(\bar{2}, 1).$$

Theorem. (Z., 2007)

For all $n \geq 1$

$$\zeta(\{3\}^n) = 8^n \zeta(\{\bar{2}, 1\}^n).$$

MPV: main problems

Problems.

- Study the $\mathbb{Q}$-linear dependence among MPVs.
- Let $\mathcal{MPV}_{\mathbb{Q}}(w, N)$ be the $\mathbb{Q}$-vector space generated by MPVs of weight w and level N. Determine its dimension
$$d(w, N) := \dim_{\mathbb{Q}} \mathcal{MPV}_{\mathbb{Q}}(w, N).$$
- Study the mixed Tate motives over $\mathbb{Z}[\mu_N]\left[\frac{1}{N}\right]$.

Conjecture.

MPVs of different weights are $\mathbb{Q}$-linearly independent.

MPV: a key result

Dimension bounds on MPV. (Deligne & Goncharov, 2005)

For all w and N, $d(w,N) \le DG(w,N)$ where $1+\sum_{w=1}^{\infty} DG(w,N)t^w$ is the formal power series

$$\begin{cases} \dfrac{1}{1-t^2-t^3}, & \text{if } N=1; \\ \dfrac{1}{1-t-t^2}, & \text{if } N=2; \\ \dfrac{1}{1-a(N)t+b(N)t^2}, & \text{if } N\ge 3, \end{cases}$$

where $a(N)=\frac{\varphi(N)}{2}+\nu(N)$, $b(N)=\nu(N)-1$, φ is the Euler's totient function and $\nu(N)$ is the number of distinct prime divisors.

Example.

Let $p\ge 3$ be a prime. Then $d(2,p) \le \left(\frac{p+1}{2}\right)^2$.

MPV: standard $\mathbb{Q}$-linear relations (Racinet, 2002)

1. Weight One Relations

Type I and II relations in Bass Theorem.

2. Regularized distribution relations

For all positive integer d and $(n_1, x_1) \neq (1, 1)$

$$\mathrm{Li}_{n_1,\dots,n_d}(x_1, \dots, x_d) = a^{n_1+\cdots+n_d-d} \sum_{y_j^a = x_j, 1 \le j \le d} \mathrm{Li}_{n_1,\dots,n_d}(y_1, \dots, y_d),$$

3. Regularized double shuffle relations

series representation = integral representation

4. Lifted relations

Let $1 \le k < w$. (Weight $w - k$ MPV) $\times$ (weight k relation).

Upper bound of $d(w, N)$.

$w \backslash N$	1	2	3
1		1	2
2	1	2	4
3	1	3	8
4	1	5	16
5	2	8	32
6	2	13	

Table: Upper bounds of $d(w, N)$ by the standard relations and DG.

Note: DG=Deligne and Goncharov.

MPV: $\mathbb{Q}$-linear relations

Upper bound of $d(w, N)$.

N	4	5	6	7	8	9	10	11	12	13
SR_1	2	3	3	4	3	4	4	6	4	7
DG_1	2	3	3	4	3	4	4	6	4	7
SR_2	4	8	8	14	10	16	16	31	18	42
DG_2	4	9	8	16	9	16	15	36	15	49
SR_3	9	22	23	50	38	67	70	157	94	246
DG_3	8	27	21	64	27	64	56	216	56	343
SR_4	21	61	69							
DG_4	16	81	55							

Table: Upper bounds of $d(w, N)$ by the standard relations and DG.

MPV: $\mathbb{Q}$-linear relations

Upper bound of $d(w, N)$.

N	4	5	6	7	8	9	10	11	12	13
SR_1	2	3	3	4	3	4	4	6	4	7
DG_1	2	3	3	4	3	4	4	6	4	7
SR_2	4	**8**	8	**14**	10	16	16	**31**	18	**42**
DG_2	4	9	8	16	**9**	16	**15**	36	**15**	49
SR_3	9	**22**	23	**50**	38	67	70	**157**	94	**246**
DG_3	**8**	27	**21**	64	**27**	**64**	**56**	216	**56**	343
SR_4	21	**61**	69							
DG_4	**16**	81	**55**							

Table: Some bounds by DG are better.
Some bounds by the standard relations are better.

MPV: nonstandard relations at level 4

Theorem. (Z., 2008)

When $N = 4$, there are non-standard relations resulting from the octahedral symmetry of $0, \pm 1, \pm i$ and ∞.

An example of a nonstandard relation at level 4.

We have

$$\begin{aligned} 5\,\mathrm{Li}_{1,2}(-1,-i) =& 46\,\mathrm{Li}_{1,1,1}(i,1,1) - 7\,\mathrm{Li}_{1,1,1}(-1,-1,i) \\ & -13\,\mathrm{Li}_{1,1,1}(i,i,i) + 13\,\mathrm{Li}_{1,2}(-i,i) \\ & - \mathrm{Li}_{1,1,1}(-i,-1,1) + 25\,\mathrm{Li}_{1,1,1}(-i,1,1) \\ & -8\,\mathrm{Li}_{1,1,1}(i,i,-1) + 18\,\mathrm{Li}_{2,1}(-i,1). \end{aligned}$$

MPV: a result at prime levels

Theorem. (Z., 2008)

Let $p \geq 5$ be a prime. Then

$$d(2,p) \leq \frac{(5p+7)(p+1)}{24} = \left(\frac{p+1}{2}\right)^2 - \frac{p^2-1}{24}.$$

If Grothendieck's period conjecture is true then the equality holds and all the $\mathbb{Q}$-linear relations in $\mathcal{MPV}_{\mathbb{Q}}(2,p)$ follow from the standard relations.

MPV: conjectures at prime-power levels

Conjecture 2.

Let $p \geq 5$ be a prime. Then

$$d(2,p^2) = \frac{5p^4 - 6p^3 + 19p^2 - 18p + 24}{24} = \left(\frac{p^2 - p + 2}{2}\right)^2 - \frac{p(p-1)(p-2)(p-3)}{24}$$

Conjecture 3.

Let $p \geq 5$ be a prime. Then

$$d(3,p) = \frac{p^3 + 4p^2 + 5p + 14}{12} = \left(\frac{p+1}{2}\right)^3 - \frac{(p-1)(p+1)^2 - 24}{24}$$

Higher level generalization II: MZVs at level N

Definition.

Suppose $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$. The MZVs at level N is defined by

$$\zeta_N^{\alpha}(\mathbf{s}) = \sum_{\substack{k_1 > \cdots > k_d > 0 \\ k_j \equiv \alpha_j \pmod N \ \forall 1 \le j \le d}} \frac{1}{k_1^{s_1} \cdots k_d^{s_d}}.$$

Remark.

We have

$$\mathcal{MZV}_{\mathbb{Q}}(w, N) \otimes \mathbb{Q}(\boldsymbol{\mu}_N) = \mathcal{MPV}_{\mathbb{Q}}(w, N) \otimes \mathbb{Q}(\boldsymbol{\mu}_N).$$

Theorem. (Yuan & Z., 2014)

The double zeta values at level N satisfy the double shuffle relations:

$$\begin{aligned}\zeta_N^a(r)\zeta_N^b(s) &= \zeta_N^{a,b}(r,s) + \zeta_N^{b,a}(s,r) + \delta_{a,b}\zeta_N^a(s+r)\\ &= \sum_{i=0}^{s-1}\binom{r+i-1}{r-1}\zeta_N^{a+b,b}(r+i,s-i)\\ &\quad + \sum_{j=0}^{r-1}\binom{s+j-1}{s-1}\zeta_N^{a+b,a}(s+j,r-j)\end{aligned}$$

Motivation: DZV and modular forms at level 1

Theorem. (Gangl, Kaneko & Zagier, 2005)

The values $\zeta(\mathrm{odd}, \mathrm{odd})$ of weight k satisfy at least $\dim S_k$ linearly independent relations, where S_k is denotes the space of cusp forms of weight k on $\mathrm{PSL}_2(\mathbb{Z})$.

Example.

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3).$$

Review: Eisenstein series at level 1

Definition.

For integer $k \geq 2$, for $\tau \in \mathbb{H}$ we define the classical Eisenstein series

$$E_k(\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k}.$$

Theorem.

For every **even** integer $k \geq 4$ we have

$$E_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where $q = e^{2\pi i\tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Definition.

For any $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ we define the multiple Eisenstein series at level N by

$$E_{\mathbf{s}}^{\mathbf{a}}(\tau) = E_{\mathbf{s}}^{\mathbf{a};N}(\tau) := \sum_{\substack{m_1 N\tau + c_1 \succ \cdots \succ m_d N\tau + c_d \succ 0 \\ m_j, c_j \in \mathbb{Z},\ c_j \equiv a_j \pmod{N}\ \forall j}} \frac{1}{(m_1 N\tau + c_1)^{s_1} \cdots (m_d N\tau + c_d)^{s_d}}$$

where $m\tau + e \succ n\tau + f$ if $m > n$, or, $m = n$ and $e > f$.

Eisenstein series at level N

Chronology.

Gangl, Kaneko and Zagier: $N = 1$ and $d = 2$ (2005),
Kaneko and Tasaka: $N = d = 2$ (2011),
Bachmann: $N = 1$ (2013),
Yuan and Z.: general case (2014).

Theorem. Gangl, Kaneko and Zagier: $N = 1$ (2005),
Kaneko and Tasaka: $N = 2$ (2011)

The double Eisenstein series at levels 1 and 2 satisfy the double shuffle relations.

Eisenstein series at level N

Theorem. (Yuan & Z., 2014)

For every fixed level N the double Eisenstein series satisfy the double shuffle relations:

$$\mathcal{P}^{a,b}(X,Y) = \mathcal{E}^{a,b}(X,Y) + \mathcal{E}^{b,a}(Y,X) + \delta_{a,b}\frac{\mathcal{E}^{a}(X) - \mathcal{E}^{a}(Y)}{X-Y}, \\ = \mathcal{E}^{a+b,b}(X+Y,Y) + \mathcal{E}^{a+b,a}(X+Y,X),$$

where $\mathcal{E}^{a,b}(X,Y)$ is the generating series for $E^{a,b}_{r,s}(\tau)$

Consider Finite Sums

Definition.

Let $\mathcal{P}$ be the set of prime numbers and let ℓ be a positive integer. Define

$$\mathcal{A}_\ell = \Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p^\ell\mathbb{Z}\Big)\Big/\Big(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\ell\mathbb{Z}\Big)$$

with componentwise addition and multiplication. For convenience we put $\mathcal{A} = \mathcal{A}_1$.

Finite alternating Euler sums

Definition.

For nonzero $s_1, \ldots, s_d \in \mathbb{Z}$, we define

$$\zeta_{\mathcal{A}_\ell}(s_1, \ldots, s_d) = \sum_{p > k_1 > \cdots > k_d \geq 1} \frac{\operatorname{sgn}(s_1)^{k_1} \cdots \operatorname{sgn}(s_d)^{k_d}}{k_1^{|s_1|} \ldots k_d^{|s_d|}} \in \mathcal{A}_\ell.$$

These are called *finite alternating Euler sums* of modulus level ℓ.

Definition.

When all $s_j \in \mathbb{N}$ we get *finite MZVs*.

Finite MZVs

Theorem. (Z., 2003, Hoffman, 2004)

For all positive integers s, t

$$\zeta_{\mathcal{A}}(s,t) = \frac{(-1)^s}{s+t}\binom{s+t}{s}\beta_{s+t}.$$

where $\beta_k = (B_{p-k})_{p\in\mathcal{P}}$.

Theorem. (Z., 2003, Hoffman, 2004)

Let $(r,s,t)\in\mathbb{N}^3$. If the weight $w=r+s+t$ is an *odd* number then

$$\zeta_{\mathcal{A}}(r,s,t) = \left[(-1)^t\binom{w}{t} - (-1)^s\binom{w}{s}\right]\frac{\beta_w}{2w}.$$

Finite alternating Euler sums

Theorem. (Tauraso & Z., 2010)

Let $p > 5$ be a prime and $q_p = (2^{p-1} - 1)/p$ be the Fermat quotient. Then

$$\begin{aligned}
\zeta_{\mathcal{A}}(\bar{1}) &= -2q_p, \\
\zeta_{\mathcal{A}}(\{\bar{1}\}^2) &= 2q_p^2, \\
\zeta_{\mathcal{A}}(\{\bar{1}\}^3) &= -\frac{4}{3}q_p^3 - \frac{1}{6}\beta_3, \\
\zeta_{\mathcal{A}}(\{\bar{1}\}^4) &= \frac{2}{3}q_p^4 + \frac{1}{3}q_p\beta_3, \\
\zeta_{\mathcal{A}}(\{\bar{1}\}^5) &= -\frac{4}{15}q_p^5 - \frac{1}{3}q_p^2\beta_3, \\
\zeta_{\mathcal{A}}(\{\bar{1}\}^6) &= \frac{4}{45}q_p^6 + \frac{2}{9}q_p^3\beta_3 + \frac{1}{72}\beta_3^2.
\end{aligned}$$

Finite alternating Euler sums

Theorem. (Tauraso & Z., 2010)

In weight 4 we have

$$\mathcal{FAES}_{4,1} = \Big\langle \frac{1}{2} q_p \beta_3,\ \zeta_{\mathcal{A}}(\bar{3},1),\ \zeta_{\mathcal{A}}(\bar{2},\bar{1},1) \Big\rangle_{\mathbb{Q}}.$$

In particular,

$$\begin{aligned}
\zeta_{\mathcal{A}}(1,1,\bar{2}) &= -\frac{1}{2}\zeta_{\mathcal{A}}(\bar{3},1), \\
\zeta_{\mathcal{A}}(\bar{2},2) &= -2\zeta_{\mathcal{A}}(\bar{3},1), \\
\zeta_{\mathcal{A}}(2,\bar{1},1) &= -\frac{3}{2}\zeta_{\mathcal{A}}(\bar{3},1), \\
\zeta_{\mathcal{A}}(\bar{3},\bar{1}) = \zeta_{\mathcal{A}}(\bar{1},3) &= \frac{1}{2} q_p \beta_3, \\
\zeta_{\mathcal{A}}(\bar{1},\bar{1},\bar{2}) &= -\zeta_{\mathcal{A}}(\bar{3},1) - q_p \beta_3.
\end{aligned}$$

Finite MZVs

Definition.

We denote by $\mathcal{FMZV}_{w,\ell}$ be the $\mathbb{Q}$-vector space of $\mathcal{A}_\ell$ generated by all FMZVs of weight w and modulus level ℓ. Set

$$d_f(w,\ell) = \dim_{\mathbb{Q}} \mathcal{FMZV}_{w,\ell}.$$

Table of dimensions.

w	0	1	2	3	4	5	6	7	8	9	10	11	12
$d(w)$	1	0	1	1	1	2	2	3	4	5	7	9	12
$d_f(w,1)$	1	0	0	1	0	1	1	1	2	2	3	4	5
$d_f(w,2)$	1	0	1	1	1	2	2	3	4	5	7	9	12
$d_f(w,3)$	1	1	1	2	2	3	4	5	7	9	12	16	21
$d_f(w,4)$	2	1	**1**	3	3	5	6	8	11	14	19	25	
$d_f(w,5)$	2	**1**	**2**	**3**	5	7	9	12	16	21			

Finite MZVs

Definition.

We denote by $\mathcal{FMZV}_{w,\ell}$ be the $\mathbb{Q}$-vector space of $\mathcal{A}_\ell$ generated by all FMZVs of weight w and modulus level ℓ. Set $d(w) = \dim_{\mathbb{Q}} \mathcal{MZV}_w$ and $d_f(w,\ell) = \dim_{\mathbb{Q}} \mathcal{FMZV}_{w,\ell}$.

Conjecture.

For every modulus level ℓ

$$d_f(w,\ell) = d_f(w-2,\ell) + d_f(w-3,\ell)$$

for all sufficiently large w (depending on ℓ).

Finite MZVs

Conjecture. (Kaneko & Zagier, 2014?)

There is an isomorphism

$$f_{KZ} : \mathcal{FMZV}_{w,1} \xrightarrow{\sim} \frac{\mathcal{MZV}_w}{\zeta(2)\mathcal{MZV}_{w-2}}$$

$$\zeta_{\mathcal{A}_1}(\mathbf{s}) \longmapsto \zeta_f^{Ш}(\mathbf{s})$$

where

$$\zeta_f^{Ш}(\mathbf{s}) = \sum_{i=0}^{d} (-1)^{s_1+\cdots+s_i} \zeta^{Ш}(s_i, \ldots, s_1; T)\zeta^{Ш}(s_{i+1}, \ldots, s_d; T).$$

Theorem. (Yasuda, 2014)

The map f_{KZ} is surjective.

Finite alternating Euler sums

Conjecture. (Z., 2013)

There is an isomorphism

$$\mathcal{FAES}_{w,1} \xrightarrow{\sim} \frac{\mathcal{AES}_w}{\zeta(2)\mathcal{AES}_{w-2}}$$
$$\zeta_{\mathcal{A}_1}(\mathbf{s}) \longmapsto \zeta_f^{\amalg}(\mathbf{s})$$

where

$$\zeta_f^{\amalg}(s_1,\dots,s_d) = \sum_{i=0}^{d}\left(\prod_{j=1}^{i}(-1)^{s_j}\operatorname{sgn}(s_j)\right)\zeta^{\amalg}(s_i,\dots,s_1)\zeta^{\amalg}(s_{i+1},\dots,s_d).$$

Finite alternating Euler sums

Table of dimensions.

w	0	1	2	3	4	5	6	7
$\dim_{\mathbb{Q}} \mathcal{AES}_w$	1	1	2	3	5	8	13	21
$\dim_{\mathbb{Q}} \mathcal{FAES}_{w,1}$	1	1	1	2	3	5	8	13
$\dim_{\mathbb{Q}} \mathcal{FAES}_{w,2}$	1	1	2	4	7	12	20	33

Figure: Numerically verified conjectural dimensions of $\mathcal{AES}_w$ and $\mathcal{FAES}_{w,\ell}$ for $\ell \leq 2$.

Conjecture.

For all $w \geq 3$ we have

$$\dim_{\mathbb{Q}} \mathcal{FAES}_{w,2} = \dim_{\mathbb{Q}} \mathcal{FAES}_{w-1,2} + \dim_{\mathbb{Q}} \mathcal{FAES}_{w-2,2} + 1.$$