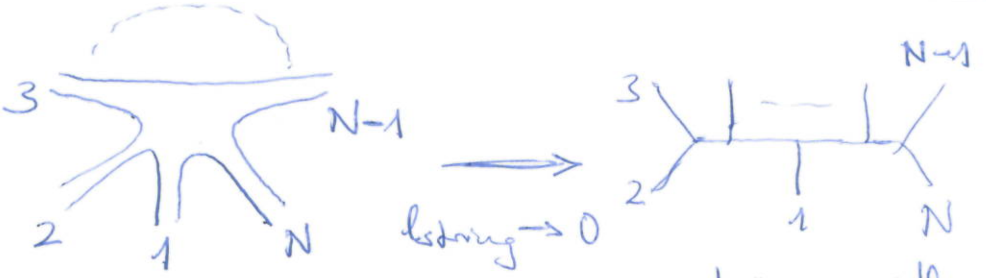
Superstring amplitudes as a laboratory for MZV's

based on: 1205.1516 OS, Stieberger, 1304.7304
Brödel, OS, Stieberger, Terasoma

String amplitude



Field theory

+ many others

tree level: can factor out "stringyness" [1106.2645]

$$A_{open}(1, \Sigma(2, \ldots N-2), N-1, N; l_{string}) = \sum_{\sigma \in S_{N-3}} F^{\sigma}_{\Sigma}(l_{string}) \underbrace{A_{SYM}(1, \sigma(2 \ldots N-2), N-1, N)}_{(N-3)!\ \text{basis}}$$

$$F^{\sigma}_{\Sigma}(s_{ij} = l_{string}^2 k_i \cdot k_j) = \int_{0 \leq z_{\Sigma(i)} < z_{\Sigma(i+1)} \leq 1} dz_2 \ldots dz_{N-2} \prod_{i<j}^{N-1} |z_i - z_j|^{s_{ij}} \prod_{k=2}^{N-2} \sum_{j=1}^{k-1} \frac{s_{\sigma(j),\sigma(k)}}{z_{\sigma(j)} - z_{\sigma(k)}} \Bigg|_{\substack{z_1 = 0 \\ z_{N-1} = 1 \\ z_N \to \infty}}$$

ext. momenta
$k_i^2 = 0$

same blackboard

Outline

I) l_{string}^2 expansion of F^{σ} via Drinfeld associator → MZV's

II) pattern in l_{string}^2 expansion via motivic MZV's

III) closed string & single-valued map

more material (e_0, e_1, P_w, M_w)

http://mzv.mpp.mpg.de

I) Drinfeld associator → MZV's [Le Murakami]

iterated integral $\int \frac{dz_i}{z_i - v_i}$, shuffle-reg.

$$\Phi(e_0, e_1) = \sum_{v \in \{0,1\}^\times} \zeta_{\{v_1 v_2 \dots v_j \dots\}} \, e_{\{\dots v_j \dots v_2 v_1\}} \qquad \to e_{\{v\}} = e_{v_1} e_{v_2} \dots$$

$$= 1 + \zeta_2 [e_0, e_1] + \zeta_3 [e_0 - e_1, [e_0, e_1]] + \dots$$

monodromy of KZ-equation

$$\frac{df}{dz} = \left(\frac{e_0}{z} + \frac{e_1}{1-z} \right) f, \quad C_0 = \lim_{z \to 0} z^{-e_0} f(z), \quad C_1 = \lim_{z \to 1} (1-z)^{e_1} f(z)$$

$$\boxed{C_1 = \Phi(e_0, e_1)\, C_0}$$

Recursion for F^σ: $\quad C_0 \equiv (N-1)\,\mathrm{pt}, \quad C_1 \equiv N\,\mathrm{pt}$

$$\begin{array}{l} (N-3)! \\ (N-2)! \end{array} \underbrace{\begin{pmatrix} F^\sigma \\ \hline \vdots \\ \\ \end{pmatrix}}_{C_1} = \left[\Phi(e_0, e_1) \right]_{(N-2)! \times (N-2)!} \underbrace{\begin{pmatrix} F^\sigma \big|_{k_{N-1} \to 0} \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{C_0} \begin{array}{l} (N-3)! \\ (N-2)! \end{array}$$

$(N-2)! \times (N-2)!$ matrix, linear in s_{ij} ⇒ uniform transcend.

Ex (i) $3 \to 4$ with $e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix}$

$$\begin{pmatrix} F^{(2)} \\ \hline * \end{pmatrix} = \Phi_{2\times 2}(e_0, e_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \text{reproduce Veneziano}$$

$$F^{(2)} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} \left[s_{12}^n + s_{23}^n - (s_{12} + s_{23})^n \right] \right)$$

no depth > 1, $\mathrm{ad}_{0,1}[e_0, e_1] \sim \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ nilpotent | table 2

Ex (ii) $4 \to 5$ with e_0, e_1 $\quad 6 \times 6$

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \hline \vdots \end{pmatrix} = \Phi_{6\times 6}(e_0, e_1) \underbrace{\begin{pmatrix} F^{(2)} \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{(N-2)!\ \text{cpt.}}$$

Proof: by deformation $F^\sigma \to \hat{F}^\sigma_{\nu}(z_0)$, $\nu = 1, 2 \ldots N-2$, subject to KZ in z_0

II) $F^\sigma \to F_{\Sigma}^{\sigma}$ square matrix; in MZV basis $\left\{ \begin{array}{c} P_w, M_w \\ \updownarrow \\ O(s_{ij}^{\,w}) \\ \text{rat coeff} \end{array} \right.$

$\mapsto$ 5pt $\qquad$ separate line $\leftarrow$

$$F = 1\!\!1 + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_2^2 P_4 + \zeta_5 M_5$$

$$+ \zeta_2 \zeta_3 P_2 M_3 + \zeta_2^3 P_6 + \frac{1}{2} \zeta_3^2 M_3^2 + \zeta_7 M_7$$

$$+ \zeta_2^2 \zeta_3 P_4 M_3 + \zeta_2 \zeta_5 P_2 M_5 \; \Big| + \zeta_2^4 P_8$$

Henrik conv $\quad \sum_{0<k<l} k^{-3} l^{-5}$

$w = 11$

$$+ \zeta_3 \zeta_5 M_5 M_3 + \frac{1}{2} \zeta_2 \zeta_3^2 P_2 M_3^2 - \frac{1}{5} \zeta_{3,5} [M_3, M_5] + \ldots$$

$\hookrightarrow \overset{!}{\geq} 2$

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motivic MZV's $\zeta^m_{n_1, n_2 \ldots n_r}$ $\to$ all known $\mathbb{Q}$ rel's of MZV's

- bypass unsettled $\mathbb{Q}$-bases questions
- automatically build in $\mathbb{Q}$ rel's

$$\{\zeta^m\} \xrightarrow{\text{isomorphism } \phi} \underbrace{\mathbb{Q}\langle f_3, f_5, \ldots f_{2k+1} \ldots \rangle}_{\text{non-commutative Hopf algebra: ш \& deconcat. } \Delta} \otimes_{\mathbb{Q}} \overbrace{\mathbb{Q}[f_2]}^{\text{commutative}}$$

ϕ preserves Hopf algebra structure

- normalization $\phi(\zeta^m_w) = f_w$
- product $\phi(\zeta^m_{\{n_i\}}\zeta^m_{\{p_i\}}) = \phi(\zeta^m_{\{n_i\}}) ⧢ \phi(\zeta^m_{\{p_i\}})$
- coproduct $\phi(\Delta_G \zeta^m_{\{n_i\}}) = \Delta_{dec.}\phi(\zeta^m_{\{n_i\}})$

e.g. $\phi(\zeta^m_3\zeta^m_5) = f_3 f_5 + f_5 f_3$ & $\phi(\zeta^m_{3,5}) = -5 f_5 f_3\ [+\alpha f_8]$

[clean up $w \leq 8$] [show $w = 11$]

$$\Rightarrow \boxed{\phi(F^m) = \left(\sum_{k=0}^{\infty} f_2^k P_{2k}\right) \underbrace{\sum_{p=0}^{\infty} \sum_{\substack{i_1 i_2 \ldots i_p \\ \in 2\mathbb{N}+1}} M_{i_1} M_{i_2} \ldots M_{i_p} f_{i_p} \ldots f_{i_2} f_{i_1}}_{\frac{1}{1-\sum_{\ell=1}^{\infty} f_\ell M_\ell}}}$$

III) Closed string tree $\equiv$ $(A_{open})^2$ "KLT-relations"

$$M^n_{closed} = \underset{(N-3)!\ cpt.}{A_{YM}}\ \underset{M^t_w S_0 = S_0 M_w}{S_0}\ G(s_{ij})\ \tilde{A}_{YM}$$

$G(s_{ij}) = \underset{FT}{1} + 2\zeta_3 M_3 + \ldots$ no ζ_2

$$\phi(G^m) = \sum_{p=0}^{\infty} \sum_{\substack{i_1 \ldots i_p \\ \in 2\mathbb{N}+1}} M_{i_1} M_{i_2} \ldots M_{i_p} \sum_{k=0}^{p} (f_{i_1} f_{i_2} \ldots f_{i_k}) ⧢ (f_{i_p} \ldots f_{i_{k+1}})$$

sv-projection @ $f_{i_p} \ldots f_{i_2} f_{i_1}$

$$= \mathrm{sv}\left[\phi(F^m)\right]$$

[Brown, Stieberger]

Preparatory material

blackboard 1 | $\mathfrak{s}$pt e_0, e_1 "6×6"

$$e_0 = \begin{pmatrix} s_{123} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{123} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{234} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{234} \end{pmatrix}$$

blackboard $\cancel{2}$ 3 | $w=11$

$$F\big|_{w=11} = \zeta_{11} M_{11} + \frac{1}{2}\zeta_3^2 \zeta_5 M_5 M_3^2 + \zeta_2 \zeta_9 P_2 M_9$$
$$+ \frac{1}{6}\zeta_2 \zeta_3^3 P_2 M_3^3 + \zeta_2^2 \zeta_7 P_4 M_7 + \zeta_2^3 \zeta_5 P_6 M_5 + \zeta_2^4 \zeta_3 P_8 M_3$$
$$+ \frac{1}{5}\zeta_{3,5}\zeta_3 [M_5, M_3] M_3 + \begin{pmatrix} \frac{1}{5}\zeta_{3,3,5} + 9\zeta_2\zeta_9 \\ + \frac{6}{25}\zeta_2^2\zeta_7 - \frac{4}{35}\zeta_2^3\zeta_5 \end{pmatrix} [M_3, [M_5, M_3]]$$

$$\phi(\zeta^{\mathfrak{m}}_{3,3,5}) = -5 f_5 f_3^2 + \frac{4}{7} f_5 f_2^3 - \frac{6}{5} f_7 f_2^2 - 45 f_9 f_2$$

$$\phi(F^{\mathfrak{m}}\big|_{w=11}) = f_{11} M_{11} + f_3 f_3 f_5 M_5 M_3 M_3 + f_3 f_5 f_3 M_3 M_5 M_3$$
$$+ f_5 f_3 f_3 M_3 M_3 M_5 + f_2 f_9 P_2 M_9 + f_2 f_3^3 P_2 M_3^2 +$$
$$+ f_2^2 f_7 P_4 M_7 + f_2^3 f_5 P_6 M_5 + f_2^4 f_3 P_8 M_3$$

blackboard ~~3~~ 2 | Example of a 5pt matrix

$$P_2 = \begin{pmatrix} s_{12}s_{34} - s_{34}s_{45} - s_{51}s_{12} & s_{13}s_{24} \\ s_{12}s_{34} & s_{13}s_{24} - s_{24}s_{45} - s_{51}s_{13} \end{pmatrix}$$