Multiple polylogarithms and periods of massless Feynman integrals

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Feynman integrals in Schwinger parameters

$$I(G) = \sum_k \epsilon^k I_k = \int_0^\infty \psi^{\omega - D/2} \cdot \varphi^{-\omega} \cdot \delta(1 - \alpha_N)$$

Graph polynomials:

$$\psi = \mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \mathcal{F} = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e$$

Hyperlogarithms

Definition (Poincaré, Lappo-Danilevsky)

To words $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n}$ with $\sigma_i \in \mathbb{C}$ associate *hyperlogarithms*

$$L_{\omega_0^n}(z) := \frac{\log^n z}{n!} \quad \text{and} \quad L_{\omega_\sigma w}(z) := \int_0^z \frac{\mathrm{d}z'}{z'-\sigma} L_w(z').$$

Goncharov polylogarithms, generalized harmonic polylogarithms

$$L_{\omega_{\sigma_1} \ldots \omega_{\sigma_n}}(z) = G(\sigma_1, \ldots, \sigma_n; z).$$

Polylogarithms $L_{\omega_0^{n-1}\omega_\sigma}(z) = -\operatorname{Li}_n\left(\frac{z}{\sigma}\right)$ and multiple polylogarithms:

$$L_{\omega_0^{n_r-1}\omega_{\sigma_r} \cdots \omega_0^{n_2-1}\omega_{\sigma_2}\omega_0^{n_1-1}\omega_{\sigma_1}}(z) = (-1)^r \operatorname{Li}_{n_1,\ldots,n_r}\left(\frac{\sigma_2}{\sigma_1}, \ldots, \frac{\sigma_r}{\sigma_{r-1}}, \frac{z}{\sigma_r}\right).$$

Primitive sixth roots of unity

David Broadhurst [2] massive Feynman integrals, $\xi_6 := e^{i\pi/3}$

Theorem (Deligne [6])

$$\mathcal{Z}_D^{(6)} := \mathbb{Q}\left[L_w(1)\colon w \in \{0,1,\xi_6\}^{\times} \cup \{0,1,\xi_6^*\}^{\times}\right] \tag{0.1}$$

Each element of $\mathcal{Z}_D^{(6)}$ is a rational linear combination of products of $i\pi$ and $\mathrm{Li}_{n_1,\ldots,n_r}(1,\ldots,1,\xi_6)$ for Lyndon words (with $n_1,\ldots,n_r > 1$), with at most the same total weight and depth:

$$\mathcal{Z}_D^{(6)} = \mathbb{Q}\left[(i\pi), \mathrm{Li}_n(1,\ldots,1,\xi_6)\colon n = (n_1,\ldots,n_r) \in \mathrm{Lyn}(\mathbb{N} \setminus \{1\})\right].$$

Primitive sixth roots of unity

$$\mathrm{Li}_n\left(e^{2\pi i x}\right) + (-1)^n \mathrm{Li}_n\left(e^{-2\pi i x}\right) = -\frac{(2\pi i)^n}{n!} B_n(x)$$

$$\mathrm{Re}\left(\mathrm{Li}_{2n+1}(\xi_6)\right) = \frac{1}{2}\left(1-2^{-2n}\right)\left(1-3^{-2n}\right)\zeta_{2n+1}.$$

Primitive sixth roots of unity

Proposition

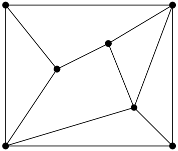
Abbreviate $\mathrm{Li}_{\vec{n}}(\xi_6) := \mathrm{Li}_{n_1,\ldots,n_r}(1,\ldots,1,\xi_6)$ *and write* $|\vec{n}| := n_1 + \cdots + n_r$ *for its weight. Then Deligne's subalgebra coincides with*

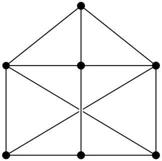
$$\mathcal{Z}_D^{(6)} = \mathbb{Q}\left[(i\pi), i^{r+|\vec{n}|}\,\mathrm{Re}\left(i^{r+|\vec{n}|}\,\mathrm{Li}_{\vec{n}}(\xi_6)\right) \colon \vec{n} = (n_1,\ldots,n_r) \in \mathrm{Lyn}\left(\mathbb{N}\setminus\{1\}\right)\right]$$

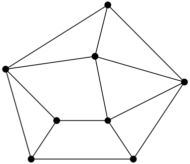
and every $\mathrm{Li}_{\vec{n}}(\xi_6)$ *has a representation as a polynomial in these generators with less or equal weight and depth.*

These generators $\mathrm{Re}\left(\mathrm{Li}_{\vec{n}}(\xi_6)\right)$ $(r + |\vec{n}|$ even) and $i\,\mathrm{Im}\left(\mathrm{Li}_{\vec{n}}(\xi_6)\right)$ $(r + |\vec{n}|$ odd) have the benefit that their products split into generators (conjecturally bases) of the subspaces $\mathcal{Z}_D^{(6)} = \mathrm{Re}\,\mathcal{Z}_D^{(6)} \oplus i\,\mathrm{Im}\,\mathcal{Z}_D^{(6)}$.

Broadhurst, Kreimer, Schnetz [4, 10]:

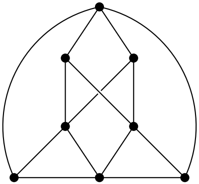
$P_{6,2} =$  $8\zeta_3^3 + \frac{1063}{9}\zeta_9$

$P_{6,3} =$  $252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25\,056}{875}\zeta_2^4$

$P_{7,2} =$  $\frac{62\,957}{192}\zeta_{11} - 9\left(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}\right) + 35\zeta_3^2\zeta_5$

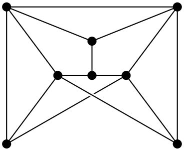
$P_{7,8}$ and $P_{7,9}$ have exceptional prime 2 [7, 11]. David Broadhurst [3]:

$P_{7,8} =$



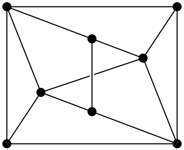
$$\frac{22\,383}{20}\zeta_{11} + \frac{4572}{5}\left(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}\right) - 700\zeta_3^2\zeta_5 + 1792\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

$P_{7,9} =$



$$\frac{92\,943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 + 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

$P_{7,11} =$



$$\approx 200{,}357\,566\,429$$

Primitive sixth roots of unity: $P_{7,11}$

exceptional prime 3 [11], irreducible quadratic denominator of $I_{-1}^{(10)}$:

$$d_{10} = \alpha_3^2(\alpha_1+\alpha_2+\alpha_4)^2 + \alpha_2\alpha_3(\alpha_1+\alpha_2+\alpha_4)(2\alpha_1-\alpha_4) \\ + \alpha_1\alpha_2(\alpha_1\alpha_2 - \alpha_2\alpha_4 - \alpha_4^2).$$

Changing variables $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2+\alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$d_{10}' = (\alpha_2+\alpha_3')(\alpha_2+\alpha_2\alpha_4'-\alpha_1')(\alpha_1'\alpha_4'+\alpha_2+\alpha_2\alpha_4'+\alpha_3'\alpha_4').$$

Last integrand then becomes

$$I_{-1}^{(12)} = \frac{L_{12}'}{1-\alpha_1'+{\alpha_1'}^2} = \frac{L_{12}'}{i\sqrt{3}}\left(\frac{1}{\alpha_1'-\xi_6} - \frac{1}{\alpha_1'-\xi_6^*}\right) = \frac{2L_{12}'}{\sqrt{3}}\operatorname{Im}\frac{1}{\alpha_1'-\xi_6},$$

for a harmonic polylogarithm (HPL) $L_{12}'(\alpha_1)$.

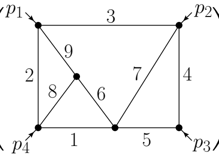
$\sqrt{3}\,\mathcal{P}(P_{7,11})$

$$
\begin{aligned}
&= \operatorname{Im}\left(\tfrac{19\,285}{6}\zeta_9 \operatorname{Li}_2 - \tfrac{1029}{2}\zeta_7 \operatorname{Li}_4 + 240\zeta_3^2(9\operatorname{Li}_{2,3} - 7\zeta_3 \operatorname{Li}_2)\right) - \tfrac{93\,824}{9675}\pi^3\zeta_{3,5} \\
&+ \tfrac{2592}{215}\operatorname{Im}\Big(36\operatorname{Li}_{2,2,2,5} + 27\operatorname{Li}_{2,2,3,4} + 9\operatorname{Li}_{2,2,4,3} + 9\operatorname{Li}_{2,3,2,4} + 3\operatorname{Li}_{2,3,3,3} \\
&\qquad - 43\zeta_3(\operatorname{Li}_{2,3,3} + 3\operatorname{Li}_{2,2,4})\Big) - \tfrac{96\,393\,596\,519\,864\,341\,538\,701\,979}{790\,371\,465\,315\,684\,594\,157\,620\,000}\pi^{11} \\
&+ \tfrac{216}{14\,755\,731\,798\,995}\operatorname{Im}\Big(2\,539\,186\,130\,125\,890\operatorname{Li}_8\zeta_3 - 1\,269\,593\,065\,062\,945\operatorname{Li}_{2,9} \\
&\qquad - 413\,965\,317\,054\,502\operatorname{Li}_6\zeta_5 - 996\,412\,983\,391\,539\operatorname{Li}_{3,8} \\
&\qquad - 546\,306\,741\,059\,841\operatorname{Li}_{4,7} - 156\,228\,639\,992\,955\operatorname{Li}_{5,6}\Big) \\
&+ \tfrac{2592}{10\,945\,435}\pi^2\operatorname{Im}\left(287\,205\operatorname{Li}_{2,7} - 574\,410\operatorname{Li}_6\zeta_3 + 55\,687\operatorname{Li}_{4,5} + 168\,941\operatorname{Li}_{3,6}\right) \\
&+ \pi\left(\tfrac{11\,613\,751}{9030}\zeta_5^2 + \tfrac{267\,067}{602}\zeta_{3,7} - \tfrac{31\,104}{215}\operatorname{Re}(3\operatorname{Li}_{4,6} + 10\operatorname{Li}_{3,7})\right)
\end{aligned}
$$

Massless on-shell four-point graphs

Linearly reducible four-loop example

$$\Phi\left(\right.$$



$$\left. \right) = \frac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\frac{s}{u}\right) \cdot \varepsilon^n$$

is linearly reducible along the sequence $1, 2, 8, 6, 9, 7, 5, 4$ of edges ($\alpha_3 = 1$). All f_n are harmonic polylogarithms of $\frac{s}{u}$:

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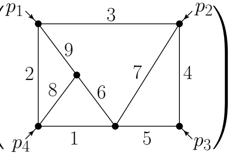
$$\Phi\left(\right.$$ 

$$\left. \right) = \frac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\frac{s}{u}\right) \cdot \varepsilon^n$$

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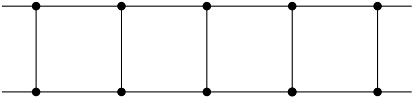
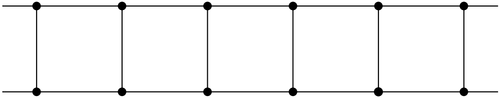
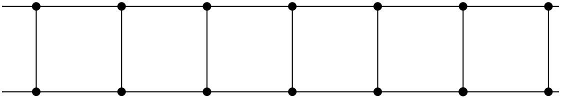
$$\begin{aligned} f_{-1} = &-\frac{79}{70}\zeta_2^3 H_{-1} - \zeta_3\big(15\zeta_2 H_{-1,-1} - 9\zeta_2 H_{-1,0} - H_{-1,-2,-1} + H_{-1,-1,-2} + 6H_{-1,-1,0,0}\big) \\ &-6\zeta_3^2 H_{-1} - \frac{3}{2}\zeta_5(11H_{-1,-1} - 5H_{-1,0}) - \frac{3}{10}\zeta_2^2(H_{-1,-2} - 17H_{-1,-1,0} - 10H_{-1,-1,-1}) \\ &-\zeta_2\Big(H_{-1,-2,0,0} - 2H_{-1,-1,-2,0} + 3H_{-1,-1,-2,-1} - H_{-1,-1,-1,0,0} + 6H_{-1,-1,-3} \\ &\qquad -3H_{-1,-2,-1,-1} - 2H_{-1,-1,0,0,0}\Big) + H_{-1,-2,-1,0,0,0} - H_{-1,-1,-2,-1,0,0} \\ &+H_{-1,-1,-2,0,0,0} - 2H_{-1,-1,-3,0,0} + H_{-1,-2,-1,-1,0,0} \end{aligned}$$

Massless ladder boxes with two off-shell legs

All ladder boxes are linearly reducible. For on-shell kinematics these are HPL of $x = \frac{u}{s}$ which we computed in $D = 6$ for $n \leq 6$.

$B_4 =$ 

$B_5 =$ 

$B_6 =$ 

Massless ladder boxes with two off-shell legs

All ladder boxes are linearly reducible. For on-shell kinematics these are HPL of $x = \frac{u}{s}$ which we computed in $D = 6$ for $n \leq 6$.
The $u \to 0$ limit $B_n = c_n/s + \mathcal{O}(x)$ is

$$c_4 = -56\zeta_7 - 32\zeta_2\zeta_5 + 32\zeta_3^2 + \tfrac{8}{5}\zeta_3\left(4\zeta_2^2 - 15\right) + \tfrac{992}{35}\zeta_2^3 - 8\zeta_2^2 - 18\zeta_2$$

$$\begin{aligned} c_5 = {} & 56\zeta_7(-5+\zeta_3) + 26\zeta_5^2 + 4\zeta_5(-40\zeta_2 - 49 + 8\zeta_2\zeta_3 + 35\zeta_3) \\ & - \tfrac{4}{5}\zeta_3^2\left(-140 + 25\zeta_2 + 4\zeta_2^2\right) + 8\zeta_3\left(7\zeta_2 + 4\zeta_2^2 - 14\right) \\ & - \tfrac{1168}{385}\zeta_2^5 - \tfrac{24}{7}\zeta_2^4 + \tfrac{496}{5}\zeta_2^3 + 4\zeta_2\left(-21 + 2\zeta_{3,5}\right) + 20\zeta_{3,5} + 4\zeta_{3,7} \end{aligned}$$

$$\begin{aligned} c_6 = {} & -1320\zeta_{11} - 12\zeta_9(20\zeta_2 + 161) + \tfrac{8}{5}\zeta_7\left(104\zeta_2^2 + 35\zeta_2 + 840\zeta_3 - 1120\right) \\ & + 624\zeta_5^2 + \tfrac{16}{35}\zeta_5\left(1680\zeta_2\zeta_3 - 3675 - 12\zeta_2^3 - 2240\zeta_2 + 490\zeta_2^2 + 5145\zeta_3\right) \\ & - \tfrac{48}{5}\zeta_3^2\left(35\zeta_2 + 8\zeta_2^2 - 60\right) - \tfrac{32}{5}\zeta_3\left(105 - 32\zeta_2^2 + 3\zeta_2^3 - 75\zeta_2\right) + 96\zeta_2^2 \\ & + 24\zeta_2\left(-21 + 8\zeta_{3,5}\right) - \tfrac{28032}{385}\zeta_2^5 - \tfrac{288}{5}\zeta_2^4 + \tfrac{18864}{35}\zeta_2^3 + 336\zeta_{3,5} + 96\zeta_{3,7} \end{aligned}$$

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