Elliptic Analogues of Multiple Zeta Values

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1. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and multiple zeta values
2. Elliptic parallel transport and elliptic multiple zeta values

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Chen's theorem for $\mathbb{P}^1 \setminus \{0,1,\infty\}$

- $M = \mathbb{P}^1 \setminus \{0,1,\infty\}$, $a \in M$
- $\mathbb{C}\pi_1(M;a)$ group algebra, $\varepsilon : \mathbb{C}\pi_1(M;a) \to \mathbb{C}$ augmentation map
- $\mathbb{C}\pi_1(M;a)\widehat{}$ completion w.r.t. to $\ker(\varepsilon)$ (augmentation ideal)
- $\omega_{KZ} = \frac{dz}{z}X_0 + \frac{dz}{z-1}X_1 \in \Omega^1(M) \otimes \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$

Parallel transport isomorphism (Chen)

$$\mathcal{T}_a : \mathbb{C}\pi_1(M;a)\widehat{} \xrightarrow{\cong} \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$$
$$\gamma \mapsto 1 + \sum_{k=1}^{\infty} \int_\gamma \omega_{KZ}^k$$

(well-defined because $d\omega_{KZ} + \omega_{KZ} \wedge \omega_{KZ} = 0$).

- works also for $\mathbb{C}\pi_1(M;b,a)\widehat{}$ instead of $\mathbb{C}\pi_1(M;a)\widehat{}$:

$$\rightsquigarrow \mathcal{T}_{b,a} : \mathbb{C}\pi_1(M;b,a)\widehat{} \xrightarrow{\cong} \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$$

Chen's theorem with tangential base points

- also works if $a, b \in T_z\mathbb{P}^1 \setminus \{0\}$, $z \in \{0, 1, \infty\}$.
- specifically, let

$$\overrightarrow{01} := \frac{\partial}{\partial \xi} \in T_0\mathbb{P}^1, \ \overrightarrow{10} := -\frac{\partial}{\partial \xi} \in T_1\mathbb{P}^1.$$

Chen's theorem with tangential base points

have an isomorphism

$$\mathcal{T}_{\overrightarrow{10},\overrightarrow{01}} : \mathbb{C}\pi_1(M; \overrightarrow{10}, \overrightarrow{01})\widehat{\ } \to \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$$

$$\gamma \mapsto \lim_{t \to 0} t^{-X_1} \mathcal{T}_{\gamma(t),\gamma(1-t)}(\gamma_t^{1-t}) t^{X_0}.$$

Multiple zeta values

- $[0,1] \subset \mathbb{R}$ canonical path from 0 to 1

Drinfel'd associator

$$dch := \mathcal{T}_{\vec{10},\vec{01}}([0,1]) = \sum_{w \in \langle X_0, X_1 \rangle} \zeta(w) w,$$

- We have

$$dch = \sum_{w \in \langle X_0, X_1 \rangle} \zeta^{sh}(w) w \in \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle,$$

where $\zeta^{sh} = (-1)^r \zeta(k_1, ..., k_r)$ for $w = X_0^{k_1-1} X_1 ... X_0^{k_r-1} X_1$, $k_1 \geq 2$.

- In general, $\zeta^{sh} \in \mathbb{Q}[MZV]$.

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Towards an elliptic analogue multiple zeta values

- Fix $\tau \in \mathbb{H} = \{\xi \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, $E_\tau^\times := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}$.
- Want elliptic transport function.
- need an *Eisenstein-Kronecker series* $F_\tau(\xi, \alpha) = \frac{\theta_\tau'(0)\theta_\tau(\xi+\alpha)}{\theta_\tau(\xi)\theta_\tau(\alpha)}$, where $\theta_\tau(\xi)$ standard odd elliptic theta function.
- Let $\xi = s + r\tau$ be the canonical coordinate on $E_\tau^\times$, and consider

$$\Omega_\tau(\xi, \alpha) := e^{2\pi i r\alpha} F_\tau(\xi, \alpha) = \sum_{k=0}^{\infty} \omega^{(k)} \alpha^{k-1}$$

- Let $\nu = 2\pi i dr$ and

$$J = \nu X_0 - \mathrm{ad}(X_0)\Omega(\xi, -ad(X_0))(X_1) \in \Omega^1(E_\tau^\times) \otimes_{\mathbb{Q}} \mathbb{Q}\langle\langle X_0, X_1 \rangle\rangle.$$

(satisfies $dJ + J \wedge J = 0$)

Towards an elliptic analogue multiple zeta values

Elliptic parallel transport (Brown& Levin; 1110.6917)

For $\rho, \xi \in E_\tau^\times$, we have an isomorphism

$$\mathcal{T}^{ell}_{\rho,\xi} : \mathbb{C}\pi_1(E_\tau^\times; \rho, \xi)^\wedge \to \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$$
$$\gamma \mapsto 1 + \sum_{k=1}^{\infty} \int_\gamma J^k$$

- Can also be defined for $\rho, \xi \in T_0E_\tau \setminus \{0\}$. Specifically, let

$$\vec{v} = (-2\pi i)^{-1} \frac{\partial}{\partial \xi} \in (T_0 E_\tau)^\times$$

$$\mathcal{T}^{ell}_{-\vec{v},\vec{v}} : \mathbb{C}\pi_1(E_\tau^\times; -\vec{v}, \vec{v})^\wedge \to \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$$
$$\gamma \mapsto \lim_{t\to 0} (-2\pi i t)^{[X_0,X_1]} \mathcal{T}^{ell}_{\gamma(1-t),\gamma(t)}(\gamma_t^{1-t})(-2\pi i t)^{-[X_0,X_1]}$$

Relation with elliptic associators

- On $E_\tau^\times$, have two canonical paths $[0,1],[0,\tau]$.

Proposition (N.M.)

Let $A(\tau), B(\tau) \in \mathbb{C}\langle\langle X_0, X_1\rangle\rangle$ denote the elliptic associators of Enriquez (Selecta 2014). We have

$$\mathcal{T}^{ell}([0,1]) = e^{\pi i[X_0,X_1]}A(\tau),\ \mathcal{T}^{ell}([0,\tau]) = e^{-\pi i[X_0,X_1]}B(\tau).$$

- In this talk, we will consider only the elliptic associator $A(\tau)$.

An elliptic analogue of multiple zeta values

Elliptic analogue of multiple zeta values

Let w be a word in the letters X_0, X_1. Define

$$I_w(\tau) = \mathcal{T}^{ell}([0,1])_w \in \mathbb{C}.$$

- The elliptic associator satisfies $A(\tau+1) = A(\tau)$ $\rightsquigarrow$ the $I_w(\tau)$ admit expansions

$$I_w(\tau) = \sum_{n \in \mathbb{N}} a_n q^n$$

  where $q = e^{2\pi i \tau}$.
- one can show: $a_n \in \mathbb{Q}[MZV, (2\pi i)^{-1}]$.

Examples of elliptic analogues of multiple zeta values

Length 1

$I_d(\tau) = \frac{(2\pi i)^d B_d}{d!}$

Length 2

Let $\alpha_d = 2\frac{(2\pi i)^d}{d!}, \quad \beta_d = \frac{(2\pi i)^d B_d}{d!}$

$$I_{d_1,d_2}(\tau) = \begin{cases} \dfrac{\beta_{d_1}\beta_{d_2}}{2} & \text{if } d_1 + d_2 \in 2\mathbb{Z} \\ \dfrac{\alpha_{d_2}\beta_{d_1}}{2\pi i} \displaystyle\sum_{k=1}^{\infty} \sum_{k=ab} \frac{b^{d_2}}{a} q^k & \text{if } d_1 \text{ odd and} \\ + \dfrac{\alpha_{d_1}\alpha_{d_2}}{2(2\pi i)} \displaystyle\sum_{k=1}^{\infty} \sum_{\substack{k=a(b_1+b_2) \\ b_2 \neq 0}} \frac{b_1^{d_1} b_2^{d_2}}{a} q^k & d_2 \neq 0 \text{ even} \end{cases}$$

An elliptic analogue of multiple zeta values

- $\mathbb{Q}[eMZV_\tau]$ the $\mathbb{Q}$-vector space spanned by the $I_w(\tau)$, for fixed τ (algebra with the shuffle product).

Proposition (N.M.)

We have a surjection of $\mathbb{Q}$-algebras

$$\mathbb{Q}[eMZV_\tau, (2\pi i)^{-1}] \to \mathbb{Q}[MZV, (2\pi i)^{-1}]$$
$$\sum_{n \in \mathbb{N}} a_n q^n \mapsto a_0.$$

- Problem: how to describe explicitly a section of this surjection?
- Related problem: find a good "'numerology"' for elliptic multiple zeta values.

Elliptic multiple zeta values

- Now consider $I_w(\tau)$ as a complex function on the upper-half plane $\mathbb{H}$. It is holomorphic for every w (Enriquez, Selecta 2014).
- Let $\mathbb{Q}[eMZV] \subset \mathcal{O}(\mathbb{H})$ be the $\mathbb{Q}$-algebra spanned by the I_w.
- For a word $w \in \langle X_0, X_1 \rangle$ define its *complexity* $c(w)$ as the number of X_1's appearing, and denote its *length* by $l(w)$.
- Let
$$\mathbb{Q}[eMZV]_{c,l} = \mathrm{Span}_{\mathbb{Q}}\{I_w \mid c(w) = c \ \ l(w) = l\}.$$
Note that $d_{c,l} := \dim_{\mathbb{Q}} \mathbb{Q}[eMZV]_{c,l} < \infty$.

Goal

understand $\mathbb{Q}[eMZV]_{c,l}$; in particular compute $d_{c,l}$.

First steps in computing $d_{c,l}$

- $c = 0$
  - Have
    $$I_{X_0^n} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$
    hence $d_{0,l} = \delta_{0,l}$ for all $l \in \mathbb{N}$.
  - $\rightsquigarrow \mathbb{Q}[eMZV]_0 = \mathbb{Q}$.
- $c = 1$
  - Have
    $$I_{X_0^n X_1} = \begin{cases} -2\zeta(n) & \text{if } n \text{ is even} \\ 0 & \text{else} \end{cases}$$
  - In general, $I_{X_0^m X_1 X_0^n} \in \mathbb{Q}\pi^{2(m+n)}$, i.e. in particular, they are all constant.
  - $\rightsquigarrow \mathbb{Q}[eMZV]_1 = \mathbb{Q}[\pi^2]$.

First steps in computing $d_{c,l}$

- $c = 2$
  - Have
$$\mathbb{Q}[eMZV]_2 = \underbrace{\mathbb{Q}[eMZV]_2^{even}}_{=\mathbb{Q}[\pi^2]} \bigoplus \mathbb{Q}[eMZV]_2^{odd}.$$
  - $\mathbb{Q}[eMZV]_2^{odd}$ contains non-constant I_w's.
  - I can prove
$$\left\lfloor \frac{l}{4} \right\rfloor \leq d_2^l - 1 \leq \left\lfloor \frac{l}{2} \right\rfloor$$
for l odd.

Conjecture

We have
$$d_2^l = \left\lfloor \frac{l}{3} \right\rfloor + 1$$
for all odd l

- verified with a computer up to length 200.