Webs and polylogarithms

Mark Harley
The University of Edinburgh


Numbers and Physics, ICMAT, 18 September 2014
Outline

- Infrared Singularities
- Webs and subtracted webs
- Computing webs
- Interesting properties of subtracted webs
- Open questions
Singularities of Gauge Theories

Loop level scattering amplitudes suffer from divergences

\[ \int \frac{d^d k}{k^2 (k^2 + 2p_1 \cdot k) (k^2 - 2p_2 \cdot k)} \]

is log divergent as \( k^\mu \to 0 \)

Regularise with dimensional regularisation: \( d = 4 - 2\epsilon \)

\( \epsilon < 0 \) to regulate IR
Infrared Singularities — Cancellation

The virtual infrared singularities cancel with those from real emissions (unlike UV)

\[ \log \left( \frac{q^2}{\mu^2} \right) \quad \text{Large logarithm} \]

Phenomenology: necessary to compute for practical purposes

Theory: Interesting insights into perturbative series and computations
Eikonal Approximation

In order to study the infrared singularities of a scattering amplitude we approximate

\[ \Phi_\beta(0, \infty) = \mathcal{P} \exp \left( ig_s \int_0^\infty d\lambda \beta \cdot A(\lambda \beta^\mu) \right), \quad \beta^2 \neq 0 \]
Factorisation

This approximation gives the IR singularities of an amplitude through factorisation

\[ \mathcal{M} = S \otimes H \]

The **hard function**, \( H \), is a matching coefficient containing information from non-soft underlying amplitude

\[ S = \langle \Phi_{\beta_1} \Phi_{\beta_2} \ldots \Phi_{\beta_n} \rangle_0 \]

The universal **soft function** is a product of Wilson lines
Exponentiation — QED

The soft function exponentiates, drastically simplifying computation, e.g. QED form factor

\[ S_{\text{QED}} = \exp \left( \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 1}
\end{array}
\end{array} + \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 2}
\end{array}
\end{array} + \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 3}
\end{array}
\end{array} + \ldots \right) \right) \right) \]

\[ = 1 + \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 1}
\end{array}
\end{array} + \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 2}
\end{array}
\end{array} + \left( \begin{array}{c}
\begin{array}{c}
\text{diagram 3}
\end{array}
\end{array} + \ldots \right) \right) \right) \]

Exponent is formed from only “connected” diagrams
QCD is non-abelian (specifically SU(3) gauge theory) and therefore exponentiation is far less simple.

\[ S_{\text{QCD}} = \mathcal{P} \exp \left( \begin{array}{c} \left. \right\rangle + \left. \right\rangle + \left. \right\rangle + \cdots \end{array} \right) \]

Especially when considering more lines (matrix valued)
Webs

Webs are specific collections of diagrams which contribute to the exponent

E.g. 1-2-1:

\[ w^{(2,-1)}_{1-2-1} = \frac{1}{2} f^{abc} T^a_1 T^b_2 T^c_3 \left( \begin{array}{cc} \times & \times \end{array} \right) \]

1-1-1-3:

\[ w^{(3,-1)}_{1-1-1-3} = -\frac{1}{6} f^{ade} f^{bce} T^a_1 T^b_2 T^c_3 T^d_4 (2A - B - C + 2D - E - F) \]

\[ -\frac{1}{6} f^{abe} f^{cde} T^a_1 T^b_2 T^c_3 T^d_4 (A + B - 2C + D - 2E + F) \]

Webs appear in exponent with “connected” colour factors
Web combinatorics

The exponent takes the form of the diagrammatic colour and kinematic factors mixed by the “Mixing Matrix”

\[ W \equiv \sum_{D,D'} \mathcal{F}(D) \ R_{D,D'} \ C(D') \]

Recent studies have found interesting links between the combinatorics of these matrices and partially ordered sets (posets)

Dukes, Gardi, McAslan, Scott, White  [arXiv:1310.3127]

Dukes, Gardi, Steingrimsson, White  [arXiv:1301.6576]
Renormalising the soft function

The Eikonal approximation results in further UV divergences due to introduction of cusp

\[ \propto \int \frac{d^d k}{k^2 \beta_1 \cdot k \beta_2 \cdot k} \]

Need to renormalise the soft function

\[ S_{\text{ren.}}(\alpha_{ij}, \alpha_s(\mu^2), \epsilon_{\text{IR}}, \mu) = S_{\text{UV+IR}} Z(\alpha_{ij}, \alpha_s(\mu^2), \epsilon_{\text{UV}}, \mu) = Z(\alpha_{ij}, \alpha_s(\mu^2), \epsilon_{\text{UV}}, \mu) \]

Can determine the UV poles by introducing IR regulator

\[ S_{\text{ren.}}(\alpha_{ij}, \alpha_s(\mu^2), \mu, m) = S(\alpha_{ij}, \alpha_s(\mu^2), \epsilon, m) Z(\alpha_{ij}, \alpha_s(\mu^2), \epsilon, \mu) \]
Renormalisation and the exponent

In QCD the soft function and renormalisation factor are matrices

\[ S = \exp(w(\epsilon)), \quad Z = \exp(\zeta(\epsilon, \mu)) \]

\[ w = \sum_{n,k} w^{(n,k)} \alpha_s^n \epsilon^k \]

Applying BCH formula and some physical constraints,

\[ \Gamma^{(1)} = -2w^{(1,-1)} \]
\[ \Gamma^{(2)} = -4w^{(2,-1)} - 2 \left[ w^{(1,-1)}, w^{(1,0)} \right] \]
\[ \Gamma^{(3)} = \ldots \]

\[ \frac{dZ}{d \ln \mu} - Z\Gamma, \quad \Gamma = \sum_n \Gamma^{(n)} \alpha_s^n \]

Subtracted webs

A subtracted web is a web combined with a relevant set of commutators

\[ \tilde{w}_{1-2-1}^{(2)} = -4 \frac{1}{2} \left( \begin{array}{c} \text{a diagram} \\ \text{a diagram} \end{array} \right) + \begin{array}{c} \text{a diagram} \\ \text{a diagram} \end{array} \]

- Renormalisation factor, \( Z \), can not have dependance on IR regulator therefore neither do subtracted webs
- Owing to this physical symmetries hidden by regularisation are restored
- Free of subdivergences (only physically relevant single pole)
Multiple Gluon Exchange Webs (MGEWs)

We wish to specialise to a subclass of simple webs involving only multiple gluon exchanges.

Easily manifest themselves as iterated multiple-polylogarithmic integrals.

Lend themselves naturally to development of automated techniques.
Kinematics: Two line, colour-singlet case

IR regulated one loop

Exponential regulator

\[
\begin{align*}
\omega_{1-1}^{(1)} &= T_1^a T_2^a \alpha_s \mu^{2\varepsilon} \mathcal{N} \beta_1 \cdot \beta_2 \int_0^\infty ds \int_0^\infty dt \left( -(s \beta_1 - t \beta_2)^2 \right) \varepsilon^{-1} e^{-m \sqrt{\beta_1^2 s - m \sqrt{\beta_2^2 t}}} \\
&= T_1^a T_2^a \kappa \gamma_{12} \int_0^\infty d\tau \int_0^\infty d\sigma \left( \sigma^2 + \tau^2 - \gamma_{12} \tau \sigma \right) \varepsilon^{-1} e^{-\tau - \sigma} \\
&= T_1^a T_2^a \kappa \gamma_{12} \Gamma(2\varepsilon) \int_0^1 dx \ P(x, \gamma_{12}) \\
\lambda &= \tau + \sigma, \quad x = \frac{\sigma}{\sigma + \tau}
\end{align*}
\]

\[\gamma_{12} = \frac{2 \beta_1 \cdot \beta_2}{\sqrt{\beta_1^2 \beta_2^2}}\]

\[P(x, \gamma_{12}) = (x^2 + (1 - x)^2 - \gamma_{12} x (1 - x))^{\varepsilon^{-1}}\]

Korchemsky, Radyushkin (1987)

Recent formulation in terms of iterated integrals

Kidonakis (2009); Henn, Huber (2012)

Three loop results recently obtained

Grozin, Henn, Korchemsky, Marquard [arXiv:1409.0023]
Kinematics continued

Let’s choose a more convenient kinematic variable

\[ \gamma_{ij} = \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_1^2 \beta_2^2}} = -\alpha_{ij} - \frac{1}{\alpha_{ij}} \]

\[ w^{(1,-1)} = \frac{\gamma_{12}}{4\pi} \int_0^1 dx \ P_0(x, \gamma_{12}) \]

\[ = -\frac{1}{4\pi} \left( \alpha_{12} + \frac{1}{\alpha_{12}} \right) \int_0^1 dx \ \frac{x}{x^2 + (1-x)^2 + x(1-x)(\alpha_{12} + 1/\alpha_{12})} \]

\[ = \frac{1}{4\pi} \frac{1 + \alpha_{12}^2}{1 - \alpha_{12}^2} \int_0^1 dx \ \left( \frac{1}{x - \frac{1}{1-\alpha_{12}}} - \frac{1}{x + \frac{\alpha_{12}}{1-\alpha_{12}}} \right) \]

\[ = \frac{1}{4\pi} 2 \ r(\alpha_{12}) \ln(\alpha_{12}) \]

\[ \alpha \rightarrow 1/\alpha \text{ symmetry is realised through interplay between rational and logarithm} \]

This structure generalises to any MGEW: products of \( r(\alpha_{ij}) \) multiplying multiple-polylogs
General form of MGEW and methodology

Now for MGEW diagrams

\[ \mathcal{F}^{(n)} = \kappa^n \Gamma(2n\epsilon) \int_0^1 \left[ \prod_{k=1}^{n} dx_k \, \gamma_k \, P(x, \gamma_k) \right] \phi_D^{(n)}(\{x_i\}; \epsilon) \]

\[ \phi_D^{(n)}(\{x_i\}; \epsilon) = \int_0^1 \left[ \prod_{k=1}^{n-1} dy_k (1 - y_k)^{-1+2\epsilon} y_k^{-1+2k\epsilon} \right] \Theta_D[\{x_k, y_k\}] \]

Has a Laurent expansion in \( \epsilon \),

\[ \phi_D^{(n)}(x_i; \epsilon) = \sum_k \phi_D^{(n,k)}(\{x_i\}) \epsilon^k, \]

where \( \phi_D^{(n,k)}(\{x_i\}) \) is a purely transcendental function of weight \( n - 1 + k \) multiplying, in some cases, Heaviside functions of \( \{x_i\} \)

Gardi [arXiv:1310.5268]
Computing webs

We combine integrands to directly obtain subtracted web

\[
\tilde{w}^{(2)}_{1-2-1} = -4 \frac{1}{2} \left( \begin{array}{c}
\includegraphics{image1.png} \\
\includegraphics{image2.png}
\end{array} \right)^{(2,-1)} + \left[ \begin{array}{c}
\includegraphics{image3.png} \\
\includegraphics{image4.png}
\end{array} \right]^{(1,0)}, \quad \left[ \begin{array}{c}
\includegraphics{image5.png} \\
\includegraphics{image6.png}
\end{array} \right]^{(1,-1)}
\]

General subtracted MGEW:

\[
\tilde{w}^{(n)} = c_i^{(n)} \left( \prod_{k=1}^{n} r(\alpha_k) \right) \int_0^1 \left[ \prod_{k=1}^{n} dx_k \left( \frac{1}{x_k - \frac{1}{1-\alpha_k}} - \frac{1}{x_k + \frac{\alpha_k}{1-\alpha_k}} \right) \right] g(\{x_i\}) \Theta[\{x_i\}]
\]

Conjecture: Integrand factorises such that result can be written as sums of products of polylogarithms, each dependent upon a **single cusp angle**

\[
M_{k,l,m}(\alpha) = \frac{1}{r(\alpha)} \int_0^1 dx \gamma P_0(x, \gamma) \ln^k \left( \frac{q(x, \alpha)}{x^2} \right) \ln^l \left( \frac{x}{1-x} \right) \ln^m \tilde{q}(x, \alpha)
\]

\[
\ln q(x, \alpha) = \ln \left( x - \frac{1}{1-\alpha} \right) + \ln \left( x + \frac{\alpha}{1-\alpha} \right)
\]

\[
\ln \tilde{q}(x, \alpha) = \ln \left( x - \frac{1}{1-\alpha} \right) - \ln \left( x + \frac{\alpha}{1-\alpha} \right)
\]
MGEW Basis

\[ \tilde{w}^{(3)}_{(1,2,3)} = \ldots + c_4^{(3)} \frac{4}{3} r(\alpha_{13}) r^2(\alpha_{23}) \left[ M_{0,1,1}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) + \frac{1}{8} \left( M_{1,0,0}^2(\alpha_{23}) - M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{23}) \right. \right. \]
\[ \left. \left. - \frac{1}{12} M_{0,0,0}^4(\alpha_{23}) + 2M_{0,0,0}(\alpha_{23}) M_{0,2,0}(\alpha_{23}) \right) M_{0,0,0}(\alpha_{13}) \right] + \ldots \]

\[ M_{0,0,0}(\alpha) = 2 \ln(\alpha) \]

\[ M_{1,0,0}(\alpha) = 2 \text{Li}_2(\alpha^2) + 4 \log(\alpha) \log \left(1 - \alpha^2\right) - 2 \log^2(\alpha) - 2 \zeta(2) \]

\[ M_{0,1,1}(\alpha) = 2 \text{Li}_3(\alpha^2) - 2 \log(\alpha) \left[ \text{Li}_2(\alpha^2) + \frac{\log^2(\alpha)}{3} + \zeta(2) \right] - 2 \zeta(3) \]

\[ M_{0,2,0}(\alpha) = \frac{2}{3} \log^3(\alpha) + 4 \zeta(2) \log(\alpha) \]

\[ M_{2,0,0}(\alpha) = -4 \left[ \text{Li}_3(\alpha^2) + 2\text{Li}_3 \left(1 - \alpha^2\right) \right] - 8 \log \left(1 - \alpha^2\right) \log^2(\alpha) \]
\[ + \frac{8}{3} \log^3(\alpha) + 8 \zeta(2) \log(\alpha) + 4 \zeta(3) \]
Basis conjecture

Holds for every MGEW we have studied
Term from 1-2-3 (unsubtracted)

\[\int_0^1 dx_1 \int_0^1 dx_2 \gamma_1 P_0(x_1, \gamma_1) \gamma_2 P_0(x_2, \gamma_2) \text{Li}_2(-\frac{1-x_1}{x_2}) = \ldots + \]

\[G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1] + G[\frac{1}{1-a}, 1, 1, 1] + G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1] + G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1] - G[\frac{1}{1-a}, 1, 1, 1]
\]
Computing Webs — Method 2

Integrating webs (brute force):

• After shifting to $\alpha_{ij}$ variables, propagators factorise and integrals are in plain “dlog” form after expansion in $\epsilon$

$$F_D^{(n,-1)} = \mathcal{N}\left(\prod_{k=1}^{n} r(\alpha_k)\right) \int_0^1 \left[ \prod_{k=1}^{n} dx_k \left( \frac{1}{x_k - \frac{1}{1-\alpha_k}} - \frac{1}{x_k + \frac{\alpha_k}{1-\alpha_k}} \right) \right] T(\{x_i\}) \Theta_D[\{x_i\}]$$

$T(\{x_i\})$ MPLs from $\phi_D^{(n)}(\{x_i\}, \epsilon)$ and logs from expansion of $P(x_k, \gamma_k)$

• Integrate to get higher weight MPLs (depending on multiple angles)

• Combine with combinatoric factors and commutators to get sub. web

• Look for relations between MPLs which don’t factorise
\[
\int_0^1 dx_1 \int_0^1 dx_2 \, \gamma_1 P_0(x_1, \gamma_1) \, \gamma_2 P_0(x_2, \gamma_2) \, \text{Li}_2\left(-\frac{1-x_1}{x_2}\right) = \ldots +
\]

\[
G\left[\frac{1}{1-a}, 1, \frac{1}{1-b} - 1, 1\right] - G\left[\frac{1}{1-a}, 1, \frac{1}{1-b}, 0, 1\right] - G\left[\frac{1}{1-a}, 1, \frac{1}{1-b} - 1, 1\right] + G\left[\frac{1}{1-a}, 1, \frac{1}{1-b}, 0, 1\right] + G\left[\frac{1}{1-a}, 1, \frac{1}{1-b} - 1, 1\right] - G\left[\frac{1}{1-a}, 1, \frac{1}{1-b}, 0, 1\right] - G\left[\frac{1}{1-a}, 1, \frac{1}{1-b} - 1, 1\right] + \ldots
\]
### MGEW Basis symbols

<table>
<thead>
<tr>
<th>w</th>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( M_{0,0,0} )</td>
<td>( 2 (\otimes \alpha) )</td>
</tr>
<tr>
<td>2</td>
<td>( M_{1,0,0} )</td>
<td>( -4 \alpha \otimes \eta )</td>
</tr>
<tr>
<td>3</td>
<td>( M_{0,0,2} )</td>
<td>( 16 \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,1,1} )</td>
<td>( -4 \alpha \otimes \eta \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,2,0} )</td>
<td>( 4 \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,0,0} )</td>
<td>( 16 \alpha \otimes \eta \otimes \eta )</td>
</tr>
<tr>
<td>4</td>
<td>( M_{1,0,2} )</td>
<td>( -32 \alpha \otimes \alpha \otimes \alpha \otimes \eta )</td>
</tr>
<tr>
<td></td>
<td>( M_{1,1,1} )</td>
<td>( -16 \alpha \otimes \alpha \otimes \alpha \otimes \alpha + 8 \alpha \otimes \eta \otimes \alpha \otimes \eta + 8 \alpha \otimes \eta \otimes \eta \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{1,2,0} )</td>
<td>( -8 \alpha \otimes \alpha \otimes \alpha \otimes \eta - 8 \alpha \otimes \eta \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{3,0,0} )</td>
<td>( -96 \alpha \otimes \eta \otimes \eta \otimes \eta )</td>
</tr>
<tr>
<td>5</td>
<td>( M_{0,0,4} )</td>
<td>( 768 \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,1,3} )</td>
<td>( -96 \alpha \otimes \alpha \otimes \alpha \otimes \eta \otimes \alpha - 96 \alpha \otimes \alpha \otimes \eta \otimes \alpha \otimes \alpha - 96 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,2,2} )</td>
<td>( 96 \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha + 32 \alpha \otimes \eta \otimes \alpha \otimes \eta \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,3,1} )</td>
<td>( -24 \alpha \otimes \alpha \otimes \alpha \otimes \eta \otimes \alpha - 24 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{0,4,0} )</td>
<td>( 48 \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,0,2} )</td>
<td>( 128 \alpha \otimes \alpha \otimes \alpha \otimes \eta \otimes \eta )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,1,1} )</td>
<td>( 64 \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \eta + 32 \alpha \otimes \alpha \otimes \eta \otimes \alpha \otimes \alpha + 32 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,2,0} )</td>
<td>( -32 \alpha \otimes \eta \otimes \alpha \otimes \eta \otimes \eta - 32 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \eta - 32 \alpha \otimes \eta \otimes \eta \otimes \eta \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,3,0} )</td>
<td>( 32 \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \eta \otimes \eta + 32 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \eta + 32 \alpha \otimes \eta \otimes \alpha \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{2,4,0} )</td>
<td>( +32 \alpha \otimes \eta \otimes \eta \otimes \alpha \otimes \alpha )</td>
</tr>
<tr>
<td></td>
<td>( M_{4,0,0} )</td>
<td>( 768 \alpha \otimes \eta \otimes \eta \otimes \eta \otimes \eta )</td>
</tr>
</tbody>
</table>

\[
\eta = \frac{\alpha}{1 - \alpha^2}
\]

\[
M_{k,l,m}(\alpha) = \frac{1}{r(\alpha)} \int_0^1 dx \, \gamma \, P_0(x, \gamma) \, \ln^k \left( \frac{q(x, \alpha)}{x^2} \right) \ln^l \left( \frac{x}{1-x} \right) \ln^m \tilde{q}(x, \alpha)
\]
Physical constraints: The symbol

- Subtracted web: crossing symmetry $\alpha \to -\alpha$ is restored. Leaves result unchanged up to discontinuity.

- Expect a divergence at $\alpha \to 0, -1$ and a zero at $1$.

Coupled with knowledge of integrands constrains symbol entries.

Symbol conjecture: $\{\alpha^2, 1 - \alpha^2\}$
Outlook and open problems

• Still want to understand better the symbol alphabet constraints

• Why $\eta = \frac{\alpha}{1 - \alpha^2}$?

• Does the basis of functions describe all MGEWS?

• Can we prove factorisation to all orders?

• What can we say about non-MGEWs?