Resonant tori of arbitrary codimension for quasi-periodically forced systems

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Consider a Hamiltonian system, described by the Hamiltonian function:

$$H(\varphi, J) = H_0(J) + \varepsilon f(\varphi, J), \qquad (\varphi, J) \in \mathbb{T}^n \times \mathbb{R}^n,$$

where (J, φ) are *action-angle* variables, both H_0 and f are analytic and ε is a small parameter (H_0 is the *unperturbed Hamiltonian* and f is the *perturbation*). Assume also H_0 to be convex.

The corresponding Hamilton equations are

$$\begin{cases} \dot{\varphi} = \partial_J H(\varphi, J), \\ \dot{J} = -\partial_\varphi H(\varphi, J). \end{cases}$$

For $\varepsilon = 0$ the system is *integrable*: all solutions have the form

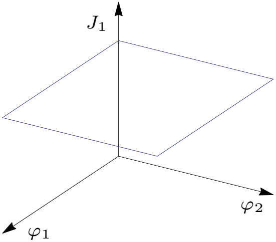
$$J(t) = J_0 = \text{const.}, \qquad \varphi(t) = \varphi_0 + \omega_0 t,$$

where $\omega_0 = \partial_J H_0(J_0)$ is the *frequency vector* $\Longrightarrow$ the full phase space foliated into n-dimensional *invariant tori* and on each torus the motion is a *quasi-periodic flow* $\varphi \to \varphi + \omega_0 t$.

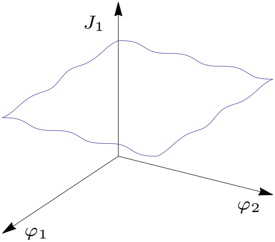
Let ω_0 satisfy some strong non-resonance condition, say a *Diophantine condition*

$$|\omega_0 \cdot \nu| > \frac{\gamma}{|\nu|^\tau} \qquad \forall \nu \in \mathbb{Z}^n \text{ such that } \nu \neq 0,$$

with $\gamma > 0$ and $\tau > n - 1$ (here $\cdot$ is the scalar product in $\mathbb{R}^n$ and $|\nu|$ is the Euclidean norm of ν). Then the corresponding unperturbed torus persists slightly deformed for $|\varepsilon| < \varepsilon_0 = O(\gamma^2)$.



$\varepsilon = 0$



$\varepsilon \neq 0$

For any open set $A \subset \mathbb{R}^n$ the set of $\omega_0 \in A$ satisfying the Diophantine condition for some $\gamma > 0$ has full measure in A.

For fixed (small) ε, any unperturbed torus with frequency vector satisfying the Diophantine condition with $\gamma = O(\sqrt{\varepsilon})$ persists $\Longrightarrow$ the relative measure of the tori which break down is $O(\sqrt{\varepsilon})$.

We say that the system is *quasi-integrable*: most of the tori persist slightly deformed ("most" means that the relative measure of the phase space filled by invariant tori goes to 1 as ε goes to 0).

In particular, if ω_0 is *resonant* (i.e. has rationally dependent components: $\omega_0 \cdot \nu = 0$ for some ν), the corresponding torus is destroyed.

There appear gaps between the persisting tori (infinitely many of them, but very thin: the smaller ε, the thinner the gaps), around the regions where the resonant tori break down.

A natural question is the following.

Problem

Even though the n-dimensional invariant torus disappears, maybe some submanifold still persists (it can be viewed as a "trace" or "ghost" of that torus).

We say that ω_0 is r-resonant (or resonant with multiplicity r) if there is a subgroup G of $\mathbb{Z}^n$ of rank r such that:

1. $\omega_0 \cdot \nu = 0$ for all $\nu \in G$,
2. $\omega_0 \cdot \nu \neq 0$ for all $\nu \in \mathbb{Z}^n \setminus G$.

If we look at a torus with r-resonant frequency vector ω_0, by a suitable change of coordinates, the Hamiltonian function can be written as

$$H(\alpha, \beta, A, B) = H_0(A, B) + \varepsilon f(\alpha, \beta, A, B),$$

where $(\alpha, A) \in \mathbb{T}^d \times \mathbb{R}^d$ and $(\beta, B) \in \mathbb{T}^r \times \mathbb{R}^r$, with $d + r = n$, and the frequency vector becomes $(\omega, 0)$, with $\omega \in \mathbb{R}^d$ non-resonant.

An unperturbed n-dimensional torus with r-resonant frequency vector ω_0 is foliated into a family of $(n-r)$-dimensional submanifolds on which the motion is quasi-periodic with frequency vector ω_0 (*lower-dimensional tori*) $\Longrightarrow$ in the new variables the motion is of the form

$$\alpha(t) = \alpha_0 + \omega t, \qquad \beta(t) = \beta_0,$$

that is d angles rotate, while the remaining r are just constant (the family is parametrised by $\beta_0 \in \mathbb{T}^r$).

Assume H_0 to be convex.

Conjecture

1. For any $r \leq n-1$ and for most families of r-resonant tori, if ε is small enough at least $r+1$ tori survive any perturbation f.

For $r = n-1$ (where the resonant tori are closed orbits) this has been proved [Bernstein & Katok 1987]. So we can confine to the case $1 \leq r < n-1$.

For $r = 1$ the result has been proved too [Cheng 1999].

The problem is open for $1 < r < n-1$: only partial results exist in such cases (that is *non-degeneracy* assumptions are made on the perturbation f).

Now fix the r-resonant frequency vector ω_0 and define ω as before.

Conjecture

2. For any $r \leq n-1$, if ω satisfies a Diophantine condition and if ε is small enough at least one torus with frequency vector ω_0 survives any perturbation f.

Again, this has been proved for $r = 1$ [Cheng 1996] and is still an open problem for $r > 1$ (without assuming any non-degeneracy condition).

Fix ω_0 to be r-resonant and pass to the variables (α, β, A, B) where ω_0 becomes $(\omega, 0)$, with $\omega \in \mathbb{R}^d$ and $0 \in \mathbb{R}^r$.

We shall consider the non-convex, partially isochronous Hamiltonian function

$$H(\alpha, \beta, A, B) = \omega \cdot A + \frac{1}{2} B^2 + \varepsilon f(\alpha, \beta).$$

The corresponding Hamilton equations for β are

$$\ddot{\beta} = -\varepsilon \partial_\beta f(\omega t, \beta),$$

and hence describe a forced system with quasi-periodic forcing.

For any $r \geq 1$ we have the following result [Corsi & G 2014].

Theorem

Assume ω to be Diophantine. Assume also f to satisfy the following parity condition: $f(-\alpha, \beta) = f(\alpha, \beta)$. Then if ε is small enough there exists at least one quasi-periodic solution $\beta(t)$ with frequency vector ω.

1. The parity condition on f is a *time-reversibility* condition (on a non-autonomous Hamiltonian).
2. Note that the parity condition is a symmetry property, not a non-degeneracy condition, and it aims to ensure a suitable cancellation that is needed in the proof. It is not clear whether such a cancellation hold in general.
3. For $r = 1$, the parity condition is not necessary and the result can also be obtained by adapting Cheng's proof to the case of non-convex unperturbed Hamiltonian.
4. For $r > 1$ the result is new: it can be considered as a first step in proving Conjecture 2 (existence of at least one lower-dimensional torus of arbitrary codimension r).
5. The Diophantine condition can be weakened into a weaker condition, the so-called *Bryuno condition*. If

$$\alpha_m(\omega) = \inf_{0<|\nu|\le 2^m} |\omega \cdot \nu|, \qquad \mathfrak{B}(\omega) = \sum_{m=0}^{\infty} \frac{1}{2^m} \log \frac{1}{\alpha_m(\omega)},$$

then the Bryuno condition reads $\mathfrak{B}(\omega) < \infty$. If ω is Diophantine, then it satisfies automatically the Bryuno condition.

Consider the equation $\ddot{\beta} = -\partial_\beta f(\omega t, \beta)$, that we rewrite as

$$\ddot{\beta} = -\varepsilon F(\omega t, \beta), \qquad F(\alpha, \beta) := \partial_\beta f(\alpha, \beta),$$

and look for a quasi-periodic solution $\beta(t) = \beta_0 + b(\omega t)$, where $\langle b(\cdot) \rangle = 0$, if $\langle \cdot \rangle$ denotes the average (on $\mathbb{T}^d$).

Then $\ddot{\beta} = (\omega \cdot \partial)^2 b$ and we can rewrite the equation as

$$\begin{cases} (\omega \cdot \partial)^2 b + \varepsilon \big(F(\alpha, \beta_0 + b) - \langle F(\cdot, \beta_0 + b(\cdot)) \rangle \big) = 0, & \text{(RE)} \\ \langle F(\cdot, \beta_0 + b(\cdot)) \rangle = 0, & \text{(BE)} \end{cases}$$

where RE = *range equation* and BE = *bifurcation equation*.

For $d = 1$ one solves first RE for any β_0 and then fixes β_0 by imposing that BE is satisfied too (Melnikov theory for subharmonic solutions [Corsi & G 2008]).

If $d > 1$, the inverse of the operator $(\omega \cdot \partial)^2$ is unbounded: in Fourier space it becomes $-(\omega \cdot \nu)^2$, which can be arbitrarily small (this is the so-called *small divisor problem*) $\Longrightarrow$ a fast iterative scheme is required, such as KAM, Nash-Moser or Renormalisation Group (RG). We follow a RG approach; see also [Bricmont, Gawędzki & Kupinanen 1999] and [Gallavotti & G 2005], inspired on previous works by Eliasson (1988), by Gallavotti (1994) and by G & Mastropietro (1996).

We work in Fourier space, by writing

$$b(\alpha) = \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \mathrm{e}^{\mathrm{i}\nu\cdot\alpha} b_\nu,$$

so that RE becomes

$$(\omega \cdot \nu)^2 b_\nu = \varepsilon \left[F(\cdot, \beta_0 + b(\cdot)) \right]_\nu, \qquad \nu \neq 0,$$

while BE reads

$$\left[F(\cdot, \beta_0 + b(\cdot)) \right]_0 = 0.$$

We say that $\nu \neq 0$ is on *scale* $n \geq 1$ if $2^{-n} \leq |\omega \cdot \nu| < 2^{-(n-1)}$ and on scale $n = 0$ if $|\omega \cdot \nu| \geq 1$ (actually a smooth partition is used). One writes

$$b(\alpha) = \sum_{n=0}^{\infty} b_n(\alpha), \qquad b_n(\alpha) = \sum_{\nu \text{ on scale } n} \mathrm{e}^{\mathrm{i}\nu\cdot\alpha} b_\nu,$$

in other words b_n has only the harmonics $b_{n,\nu} = b_\nu$ with ν on scale n.

To simplify the notation, we do not write explicitly the dependence of the function on $\alpha = \omega t$, so that the RE becomes $(\omega \cdot \nu)^2 b_\nu = \varepsilon [F(\beta_0 + b)]_\nu$.

Then one tries to solve RE "scale by scale", that is by fixing iteratively b_n in terms of the corrections $b_{\geq n+1} := b_{n+1} + b_{n+2} + \ldots$, so as to obtain a sequence of approximating solutions.

- One first writes $b = b_0 + b_{\geq 1}$ and, considering $b_{\geq 1}$ as a given function, one look for a solution b_0 depending on $b_{\geq 1}$, i.e. $b_0 = b_0(b_{\geq 1})$. For $b_{\geq 1} = 0$ one obtains the first approximation $b_{\leq 0} = b_0(0)$.
- Then one can write $b = b_0(b_{\geq 1}) + b_{\geq 1} = b_0(b_1 + b_{\geq 2}) + b_1 + b_{\geq 2}$ and look for a solution b_1 depending on $b_{\geq 2}$, i.e. $b_1 = b_1(b_{\geq 2})$. By setting $b_{\geq 2} = 0$ one has an approximate solution $b_{\leq 1} = b_0(b_1(0)) + b_1(0)$.
- And so on: at the n-th step one obtains an approximate solution $b_{\leq n}$. Of course, if the scheme is expected to work, one need the approximate solutions to converge (fast enough) to a limit function (and hence the corrections b_n to become smaller and smaller).

More concretely, by writing $b = b_0 + b_{\geq 1}$ one considers

$$(\omega \cdot \nu)^2 b_\nu = \varepsilon \left[F(\beta_0 + b)\right]_\nu , \qquad (*)$$

for all ν on scale 0 and look at it as an equation for b_0 to be solved in terms of the corrections $b_{\geq 1}$. By linearising at $b = 0$, this gives

$$(\omega \cdot \nu)^2 b_{0,\nu} = \varepsilon \left[F(\beta_0)\right]_\nu + \varepsilon[\partial F(\beta_0) b_0]_\nu + \varepsilon[\partial F(\beta_0) b_{\geq 1}]_\nu + O(b^2).$$

If we are able to solve that equation, we obtain $b_0 = b_0(b_{\geq 1})$ and then we can pass to the ν's on scale 1. By writing $b_{\geq 1} = b_1 + b_{\geq 2}$, we study $(*)$ as an equation for b_1 to be solved in terms of $b_{\geq 2}$. Again we linearise at $b_{\geq 1} = 0$ and look for a solution $b_1(b_{\geq 2})$.

By iterating, at each step n, linearisation of $(*)$ at $b_{\geq n} = 0$ gives

$$\begin{aligned}(\omega \cdot \nu)^2 b_{n,\nu} = \varepsilon[F(\beta_0 + b_{\leq n-1}(0))]_\nu + \varepsilon[\partial F(\beta_0 + b_{\leq n-1}(0)) \left(1 + \partial b_{\leq n-1}(0)\right) b_n]_\nu \\ + \varepsilon[\partial F(\beta_0 + b_{\leq n-1}(0)) \left(1 + \partial b_{\leq n-1}(0)\right) b_{\geq n+1}]_\nu + O(b_{\geq n}^2).\end{aligned}$$

However, we are not able to solve the equations in this form.

The problem is that the linear part $\varepsilon[\partial F(\beta_0 + b_{\le n-1}(0))\,\big(1 + \partial b_{\le n-1}(0)\big)\, b_n]_\nu$ contains terms which cause the *accumulation* of the small divisors (bad bounds preventing the convergence of the series). Such terms arise from

$$M_n(\nu) := \mathrm{diag}_\nu(\varepsilon \partial F(\beta_0 + b_{\le n-1}(0))\,\big(1 + \partial b_{\le n-1}(0)\big))$$

i.e. the diagonal part of $\partial F(\beta_0 + b_{\le n-1}(0))\,\big(1 + \partial b_{\le n-1}(0)\big)$.

Indeed, even taking only such terms, one obtains

$$(\omega \cdot \nu)^2 b_{n,\nu} = \varepsilon[F(\beta_0 + b_{\le n-1}(0))]_\nu + M_n(\nu)\, b_{n,\nu}$$

and one realises immediately that, if one tries to solve the equation above, for instance iteratively, one finds

$$\begin{aligned} b_{n,\nu} = {} & \frac{1}{(\omega\cdot\nu)^2}\varepsilon[F(\beta_0 + b_{\le n-1}(0))]_\nu + \frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \frac{1}{(\omega\cdot\nu)^2}\varepsilon[F(\beta_0 + b_{\le n-1}(0))]_\nu \\ & + \frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \left(\frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \frac{1}{(\omega\cdot\nu)^2}\varepsilon[F(\beta_0 + b_{\le n-1}(0))]_\nu \right) \\ & + \frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \left(\frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \left(\frac{1}{(\omega\cdot\nu)^2} M_n(\nu) \frac{1}{(\omega\cdot\nu)^2}\varepsilon[F(\beta_0 + b_{\le n-1}(0))]_\nu \right)\right) + \dots \end{aligned}$$

and this produces bounds growing like factorials.

Then one can try to add $M_n(\nu)$ to the differential operator $(\omega \cdot \nu)^2$.

In other words, by defining

$$\mathcal{D}_n(\nu) := (\omega \cdot \nu)^2 - M_n(\nu),$$
$$\mathcal{N}_n := \varepsilon \partial F(\beta_0 + b_{\leq n-1}(0)) \left(1 + \partial b_{\leq n-1}(0)\right) - M_n(\nu),$$

the n-th step equation becomes

$$\mathcal{D}_n(\nu)\, b_{n,\nu} = \varepsilon [F(\beta_0 + b_{\leq n-1}(0))]_\nu + [\mathcal{N}_n b_n]_\nu$$
$$+\varepsilon [\partial F(\beta_0 + b_{\leq n-1}(0)) \left(1 + \partial b_{\leq n-1}(0)\right) b_{\geq n+1}]_\nu + O(b_{\geq n}^2).$$

With respect to the previous equation we have:

1. Advantage: the source of accumulation of small divisors has been eliminated.
2. Disadvantage: we have modified the differential operator, so that we are not allowed anymore to bound $\mathcal{D}_n(\nu)$ through the Diophantine condition.

In other words, before that we could use the Diophantine condition to bound the small divisors as $(\omega \cdot \nu)^2 \geq \gamma |\nu|^{-\tau}$, but now in general we have no control on the corrected differential operator $\mathcal{D}_n(\nu)$.

If we assume that $\mathcal{D}_n(\nu)$ can still be bounded proportionally to the original differential operator, say

$$\|\mathcal{D}_n(\nu)\| = \left\|(\omega \cdot \nu)^2 - M_n(\nu)\right\| \geq \frac{(\omega \cdot \nu)^2}{2},$$

then the iterative scheme works.

One finds the solution in the form of graph expansion (*Feynman diagrams* with no loops): the propagators are the small divisors and admit bad dimensional bounds, but their product can be bounded by using the Diophantine condition.

The proof relies on *Siegel-Bryuno bounds* (essentially: the small divisors cannot accumulate too much if we eliminate the diagonal part of the linear term). This is a very important issue, from a technical point of view, but it is rather standard.

So, the problem is reduced to study if the bounds on the corrected differential operators hold, at least for some values of $\beta_0 \Longrightarrow$ the matrix $M_n(\nu)$ is of order ε, but, since $\omega \cdot \nu$ can be arbitrarily small (for ν arbitrarily large), it is not obvious that bounds like that might hold true for all n and all ν on scale n.

The bounds on the differential operators are easily found to be satisfied, for a careful choice of β_0, if one assumes a *non-degeneracy condition* on the perturbation f, more precisely if one assumes that $f_0(\beta) := \langle f(\cdot,\beta)\rangle$ has non-degenerate critical points.

Indeed, an explicit computation gives

$$M_n(\nu) = \varepsilon \partial^2_{\beta_0} f_0(\beta_0) + O(\varepsilon^2),$$

so that, if β_0 is a maximum point if $\varepsilon > 0$ and a minimum point if $\varepsilon < 0$, one has $M_n(\nu) = -c\,\varepsilon + O(\varepsilon^2)$, with $c\,\varepsilon > 0$ (positive definite), and hence

$$\mathcal{D}_n(\nu) = (\omega \cdot \nu)^2 + c\,\varepsilon + O(\varepsilon^2),$$

so that the bound $\|\mathcal{D}_n(\nu)\| \geq (\omega \cdot \nu)^2/2$ immediately follows.

Of course, we have to take into account also BE $[F(\beta_0 + b)]_0 = 0$. However

$$[F(\beta_0 + b)]_0 = \partial_{\beta_0} f_0(\beta_0) + O(\varepsilon),$$

so that, if β_0 is a non-degenerate critical point of $f_0(\beta)$ (so that $\partial_{\beta_0} f_0(\beta_0) = 0$), we can apply the implicit function theorem and find a value β_0 close enough to the critical point such that $[F(\beta_0 + b)]_0 = 0$ and still $M_n(\nu) = -c\,\varepsilon + O(\varepsilon^2)$.

Therefore the problem is solved when the non-degeneracy condition above is assumed. However, in the general case (no assumption on f), the argument above does not work.

For $r = 1$, by assuming some weaker non-degeneracy condition, something similar can still be obtained; see for instance [You 1998] and [Gallavotti, G & Giuliani 2006]. The general case for $r = 1$ has been solved by Cheng. However no result exists when $r > 1$. So, the problem is how to control $M_n(\nu)$ in the case of *general* (arbitrary) perturbations.

An important property is that the matrix $M_n(\nu)$, by construction, depends on ν only through the quantity $\omega \cdot \nu$, i.e. $M_n(\nu) = \mathcal{M}_n(\omega \cdot \nu)$.

Since $\omega \cdot \nu$ is small (where problems arise), we can expand

$$\mathcal{M}_n(\omega \cdot \nu) = \mathcal{M}_n(0) + \partial \mathcal{M}_n(0)\,(\omega \cdot \nu) + O(\varepsilon(\omega \cdot \nu)^2),$$

where we have used again that $\mathcal{M}_n(\omega \cdot \nu)$ is $O(\varepsilon)$.

The last term is negligible with respect to $(\omega \cdot \nu)^2$, so we have to control the first two contributions $\mathcal{M}_n(0)$ and $\partial \mathcal{M}_n(0)\,(\omega \cdot \nu)$

The parity assumption on the perturbation f ensures that $\mathcal{M}_n(\omega\cdot\nu)$ is even in $\omega\cdot\nu$, so that the linear term vanishes identically $\Longrightarrow$ one has to control the constant part $\mathcal{M}_n(0)$ only.

The following idendity holds for the matrix $\mathcal{M}_n(\omega\cdot\nu)$:

$$(\mathcal{M}_n(-\omega\cdot\nu))_{i,j} = (\mathcal{M}_n(\omega\cdot\nu))_{j,i}\,, \qquad i,j=1,\ldots,r,$$

For instance, up to second order (in the formal expansion), one has

$$(\mathcal{M}_n(\omega\cdot\nu))_{i,j} = \varepsilon\partial_{ij} f_0(\beta_0) + \sum_{\nu'\neq 0}\varepsilon^2\left(\frac{\partial_{ik} f_{-\nu'}(\beta)\,\partial_{kj} f_{\nu'}(\beta_0)}{(\omega\cdot\nu'+\omega\cdot\nu)^2} + \frac{\partial_{ijk} f_{-\nu'}(\beta)\,\partial_k f_{\nu'}(\beta_0)}{(\omega\cdot\nu')^2}\right),$$

where $\partial_i = \partial_{\beta_{0,i}}$.

Therefore, for $r=1$ the matrix $\mathcal{M}_n(\omega\cdot\nu)$ is even in its argument without any assumption on f. However, for $r>1$, we need $f_{-\nu}(\beta_0) = f_\nu(\beta_0)$ for $\mathcal{M}_n(\omega\cdot\nu)$ to be even in its argument $\Longrightarrow$ we require

$$f(-\alpha,\beta) = f(\alpha,\beta).$$

At a formal level, i.e. assuming that $\|\mathcal{D}_n(\nu)\|$ can be bounded proportionally to $(\omega \cdot \nu)^2$, one has that

1. $[F(\beta_0 + b)]_0 = -\partial_{\beta_0} L(\beta_0)$, where $L(\beta_0)$ is the average of the Lagrangian computed along the solution b to RE;
2. $\mathcal{M}_n(0) = -\partial^2_{\beta_0} L_n(\beta_0)$, where $L_n(\beta)$ is the average of the Lagrangian computed along the n-step approximate solution $b_{\leq n}$.

The second identity tells us that if, for every n, we could take β_0 as a critical point of $L_n(\beta_0)$, such that $\partial^2_{\beta_0} L_n(\beta_0) \geq 0$, then the bounds on $\mathcal{D}_n(\nu)$ would follow.

Of course, the identities above are only formal, because we are assuming that a solution exists (and is defined for all values of β_0).

However, there are problems:

1. In order to impose the bounds on $\mathcal{D}_n(\nu)$ (so as to define $L_n(\beta_0)$), one might be forced to restrict β_0 to some set $\mathcal{C} \subset \mathbb{T}^r$ and such a set could be empty.
2. Even if $\mathcal{C}$ were not empty, the critical points could be outside such a set.
3. Even if existing, the value of the stationary point should depend on n, so that in principle there could exist no value β_0 such that $\partial^2_{\beta_0} L_n(\beta_0) \geq 0$ for all n.

To overcome such a difficulty we define an auxiliary function $\overline{b}(\omega t)$, as the solution of the equation obtained by replacing iteratively $\mathcal{M}_n(0)$ with $-\partial^2_{\beta_0}\overline{L}_n(\beta_0)\,\xi_n(\beta_0)$, where $\overline{L}_n(\beta_0)$ is the average of the Lagrangian computed along the n-step approximation of the auxiliary function $\overline{b}$ and $\xi_n(\beta_0)$ is a suitable cut-off function.

The cut-off functions are such that $\xi_n(\beta_0)$ identically vanish when $\partial^2_{\beta_0}\overline{L}_n(\beta_0) \leq -(\omega\cdot\nu)^2/2 \implies$ so the bounds $\|\mathcal{D}_n(\nu)\| \geq (\omega\cdot\nu)^2/2$ hold trivially for the new function $\overline{b}(\omega t)$ and the latter function is defined for all $\beta_0 \in \mathbb{T}^r$.

Of course the drawback is that $\overline{b}$ is no longer a solution to RE (because it solves an equation that has been obtained by changing the range equation).

But now the iterative scheme works and the sequence $\overline{L}_n(\beta_0)$ converges to a limit function $\overline{L}_\infty(\beta_0)$. Then we can fix $\beta_0 = \overline{\beta}_0$ as a minimum point of $\overline{L}_\infty(\beta_0)$.

In particular one has

1. $\partial_{\beta_0}\overline{L}_\infty(\overline{\beta}_0) = 0,$
2. $\partial^2_{\beta_0}\overline{L}_\infty(\overline{\beta}_0) \geq 0.$
3. $\overline{L}_n(\beta_0) = \overline{L}_\infty(\beta_0) + \text{corrections} \implies \partial^2_{\beta_0}\overline{L}_n(\overline{\beta}_0) \geq -(\omega\cdot\nu)^2/2.$

Then one can show recursively that, for such a value $\overline{\beta}_0$, one has

$$\mathcal{M}_n(0) = -\partial^2_{\beta_0} \overline{L}_n(\overline{\beta}_0)\, \xi_n(\overline{\beta}_0),$$

so that the two functions b and $\overline{b}$ coincide for $\beta_0 = \overline{\beta}_0 \implies$ the auxiliary function $\overline{b}$ is defined for all β_0 and for $\beta_0 = \overline{\beta}_0$ solves RE.

Moreover, for $\beta_0 = \overline{\beta}_0$, one has

$$\overline{L}_\infty(\overline{\beta}_0) = L_\infty(\overline{\beta}_0) = \lim_{n\to\infty} L_n(\overline{\beta}_0) = L(\overline{\beta}_0),$$

where L and L_n are the averaged Lagragians computed along the formal solution and the formal approximate solution, respectively, and $L_\infty(\beta_0) = \lim_{n\to\infty} L_n(\beta_0)$.

As a consequence one has

$$[F(\overline{\beta}_0 + \overline{b})]_0 = \partial_{\beta_0} L(\overline{\beta}_0) = \partial_{\beta_0} \overline{L}_\infty(\overline{\beta}_0) = 0,$$

and hence BE is solved too (technical issue: $\partial_{\beta_0} L(\overline{\beta}_0)$ is defined as $\lim_{n\to\infty} \partial_{\beta_0} L_n(\overline{\beta}_0)$ and for any n the function $L_n(\overline{\beta}_0)$ is defined in a small neighbourhood of $\overline{\beta}_0$).

Summarising:

- One modifies RE by introducing suitable cut-off functions, so as to controll the small divisor problem.
- In this way, we find a solution (to the modified equation) $\overline{\beta}(t)$ which not only exists, but is defined for all $\beta_0 \in \mathbb{T}^r$.
- One fixes $\beta_0 = \overline{\beta}_0$ as the minimum point of a suitable well-defined function $\overline{L}(\beta_0)$, in such a way that that function $\overline{\beta}(t)$ reduces to the solution $\beta(t)$ to the original RE.
- One checks eventually that for such a value of β_0 also BE is satisfied, by showing that the BE equation expresses the condition that $\overline{\beta}_0$ is a critical point for the function $\overline{L}(\beta_0)$.
- The value $\overline{\beta}_0$ is fixed so as to correspond to the minimum of a smooth function defined on $\mathbb{T}^r$, so that there is always at least one such value (and there might be no more than one) $\Longrightarrow$ we conclude that there is at least one quasi-periodic solution with frequency vector ω (and hence at least one lower-dimensional torus with that frequency vector).

1. We have assumed f to satisfy a parity condition: it would be nice to remove such a condition (if possible).
2. Is there still a cancellation in general, or do we have to look for a completely different approach?
3. We have considered partially isochronous systems, physically describing a class of systems with quasi-periodic forcing. A natural extension would be: systems with forcing depending on both β and B (for $r = 1$ this has been done [Corsi & G, to appear]).
4. Of course, one would like to study systems with convex unperturbed hamiltonian (as considered in Conjecture 2).
5. The scheme described here works for non-convex Hamiltonians of the form [Plotnikov & Kuznetsov 2011]

$$H(\alpha, \beta, A, B) = -\frac{1}{2}A^2 + \frac{1}{2}B^2 + \varepsilon f(\alpha, \beta),$$

by relying on remarkable symmetries which do not hold if one has the sign + in front of A^2 (convex case).

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