Iterated Binomial Sums and their Associated Iterated Integrals

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SFB F050 Algorithmic and Enumerative Combinatorics
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- Iterated Binomial Sums
- Iterated Integrals over Square-Root Valued Alphabets
- Mellin Transformation of D-finite Functions
- Inverse Mellin Transfromation
- Asymptotic Expansions of Nested Sums
- Generating functions for Iterated Integrals

Nested Sums

$$\sum_{i_1=1}^{n} s_1(i_1) \sum_{i_2=1}^{i_1} s_2(i_2) \sum_{i_3=1}^{i_2} s_3(i_3) \cdots \sum_{i_k=1}^{i_{k-1}} s_k(i_k)$$

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- $s_i(j) = \frac{(\pm 1)^j}{j^{c_i}},\ c_i \in \mathbb{N}$ harmonic sums

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- $s_i(j) = \frac{(\pm 1)^j}{j^{c_i}},\ c_i \in \mathbb{N}$ harmonic sums
- $s_i(j) = \frac{{x_i}^j}{j^{c_i}},\ x_i \in \mathbb{R}^*;\ c_i \in \mathbb{N}$ S-sums

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- $s_i(j) = \frac{(\pm 1)^j}{(a_i j + b_i)^{c_i}},\ a_i, c_i \in \mathbb{N};\ b_i \in \mathbb{N}_0$ cyclotomic sums

$$\sum_{i_1=1}^{n} s_1(i_1) \sum_{i_2=1}^{i_1} s_2(i_2) \sum_{i_3=1}^{i_2} s_3(i_3) \cdots \sum_{i_k=1}^{i_{k-1}} s_k(i_k)$$

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- $s_i(j) = \frac{{x_i}^j}{j^{c_i}} \binom{2i}{i}^{d_i},\ x_i \in \mathbb{R}^*;\ c_i \in \mathbb{N};\ d_i \in \mathbb{Z}$ binomial sums

Definition (Iterated Binomial Sums (B-Sums))

For $a_i, c_i \in \mathbb{N}$, $b_i \in \mathbb{N}_0$, $x_i \in \mathbb{R}^*$, $d_i \in \mathbb{Z}$ and $n \in \mathbb{N}$ we define

$$\mathsf{S}_{(a_1,b_1,c_1,d_1,x_1),\dots,(a_k,b_k,c_k,d_k,x_k)}(n) = \sum_{n \geq i_1 \geq \dots \geq i_k \geq 1} \frac{{x_1}^{i_1}}{(a_1 i_1 + b_1)^{c_1}} \binom{2i_1}{i_1}^{d_1} \cdots \frac{{x_k}^{i_k}}{(a_k i_k + b_k)^{c_k}} \binom{2i_k}{i_k}^{d_k}$$

k is called the depth and $w = \sum_{i=1}^{k} c_i$ is called the weight of the iterated binomial sum $\mathsf{S}_{(a_1,b_1,c_1,d_1,x_1),\dots,(a_k,b_k,c_k,d_k,x_k)}(n)$.

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$$\mathsf{S}_{(2,1,3,-1,\frac{1}{4}),(1,0,3,1,1)}(n) \quad = \quad \sum_{i=1}^{n} \frac{(\frac{1}{4})^i \sum_{j=1}^{i} \frac{\binom{2j}{j}}{j^3}}{(2i+1)^3 \binom{2i}{i}}$$

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$$\mathsf{S}_{(1,0,2,0,1),(1,0,3,0,1),(1,0,1,0,-1)}(n) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j}\frac{(-1)^k}{k}}{j^3}}{i^2}$$

$$\mathsf{S}_{\mathbf{p}}(n)\,\mathsf{S}_{\mathbf{q}}(n) = \sum_{\mathbf{r}=\mathbf{p}\sqcup\!\sqcup\mathbf{q}} \mathsf{S}_{\mathbf{r}}(n) + \text{sums of lower depth}$$

here $\mathbf{p} \sqcup\!\sqcup \mathbf{q}$ represent all merges of $\mathbf{p}$ and $\mathbf{q}$ in which the relative orders of the elements of $\mathbf{p}$ and $\mathbf{q}$ are preserved.

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$$\begin{aligned} \mathsf{S}_{a_1,a_2}(n)\,\mathsf{S}_{b_1,b_2}(n) &= \mathsf{S}_{a_1,a_2,b_1,b_2}(n) + \mathsf{S}_{a_1,b_1,a_2,b_2}(n) \\ &\quad + \mathsf{S}_{a_1,b_1,b_2,a_2}(n) + \mathsf{S}_{b_1,b_2,a_1,a_2}(n) \\ &\quad + \mathsf{S}_{b_1,a_1,b_2,a_2}(n) + \mathsf{S}_{b_1,a_1,a_2,b_2}(n) \\ &\quad + \text{sums of lower depth} \end{aligned}$$

$$\left(\sum_{i=1}^{n} \frac{\binom{2i}{i}}{i}\right) \sum_{i=1}^{n} \frac{\binom{2i}{i} \sum_{j=1}^{i} \frac{(-1)^j}{\binom{2j}{j} j^2}}{(2i+1)} =$$

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$$+ \sum_{i=1}^{n} \frac{\binom{2i}{i} \sum_{j=1}^{i} \frac{\binom{2j}{j} \sum_{k=1}^{j} \frac{(-1)^k}{\binom{2k}{k} k^2}}{2j+1}}{i} - \sum_{i=1}^{n} \frac{\binom{2i}{i}^2 \sum_{j=1}^{i} \frac{(-1)^j}{\binom{2j}{j} j^2}}{i}$$

$$+2 \sum_{i=1}^{n} \frac{\binom{2i}{i}^2 \sum_{j=1}^{i} \frac{(-1)^j}{\binom{2j}{j} j^2}}{2i+1} - \sum_{i=1}^{n} \frac{\binom{2i}{i} \sum_{j=1}^{i} \frac{(-1)^j}{j^3}}{2i+1}$$

$$\mathsf{S}_{(1,0,1,1,1)}(n)\,\mathsf{S}_{(2,1,1,1,1),(1,0,2,-1,0)}(n) =$$

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$$\begin{aligned}
&\mathsf{S}_{(2,1,1,1,1),(1,0,2,-1,0),(1,0,1,1,1)}(n)\\
&+\mathsf{S}_{(2,1,1,1,1),(1,0,1,1,1),(1,0,2,-1,0)}(n)\\
&+\mathsf{S}_{(1,0,1,1,1),(2,1,1,1,1),(1,0,2,-1,0)}(n)\\
&-\mathsf{S}_{(1,0,1,1,2),(1,0,2,-1,-1)}(n)\\
&+2\,\mathsf{S}_{(2,1,1,1,2),(1,0,2,-1,-1)}(n)\\
&-\mathsf{S}_{(2,1,1,1,2),(1,0,3,-1,0)}(n)
\end{aligned}$$

Iterated Integrals

$$\int_0^x f_1(y_1) \int_0^{y_1} f_2(y_2) \int_0^{y_2} f_3(y_3) \cdots \int_0^{y_{k-1}} f_k(y_k) dy_k \cdots dy_3 dy_2 dy_1$$

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- $f_i(y) = \frac{1}{y-a_i},\ a_i \in \{-1, 0, 1\}$ harmonic polylogarithms

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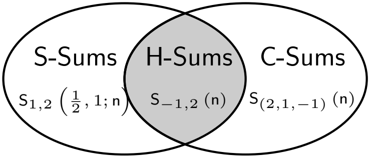
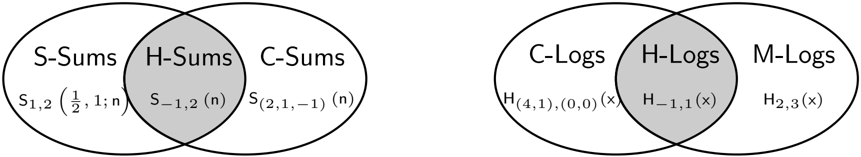
- $f_i(y) = \frac{y^{j_i}}{\Phi_{a_i}(y)},\ j_i \in \mathbb{N}$ cyclotomic polylogarithms


S-Sums
H-Sums
C-Sums
$S_{1,2}\left(\frac{1}{2},1;n\right)$
$S_{-1,2}(n)$
$S_{(2,1,-1)}(n)$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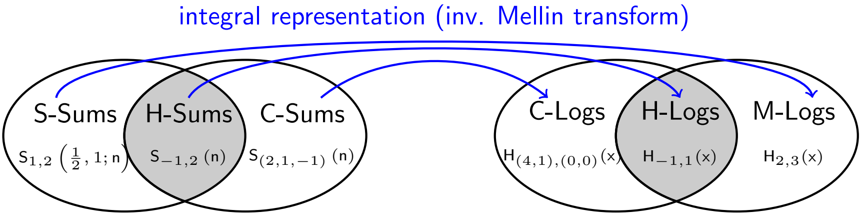
C-Logs
H-Logs
M-Logs
$H_{(4,1),(0,0)}(x)$
$H_{-1,1}(x)$
$H_{2,3}(x)$


integral representation (inv. Mellin transform)
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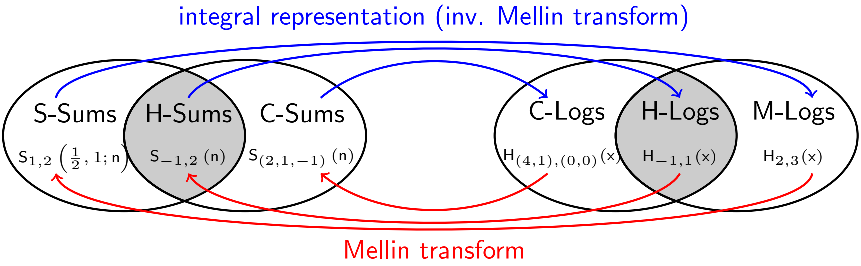

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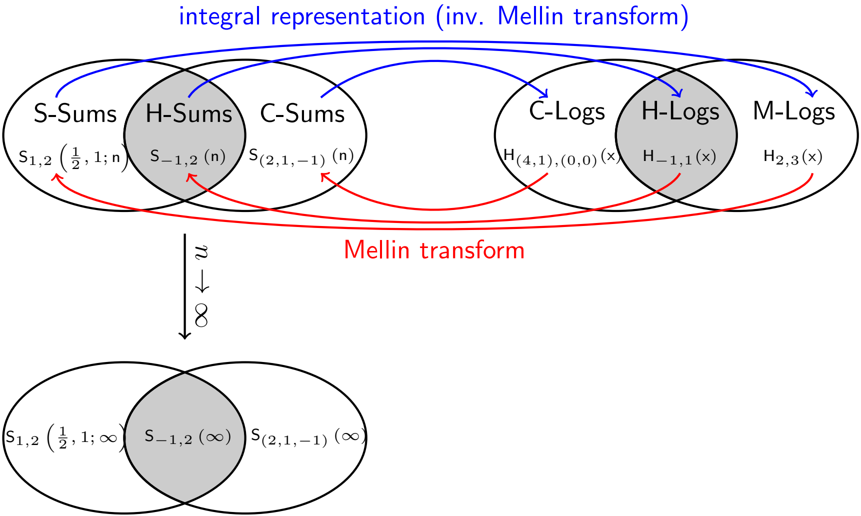
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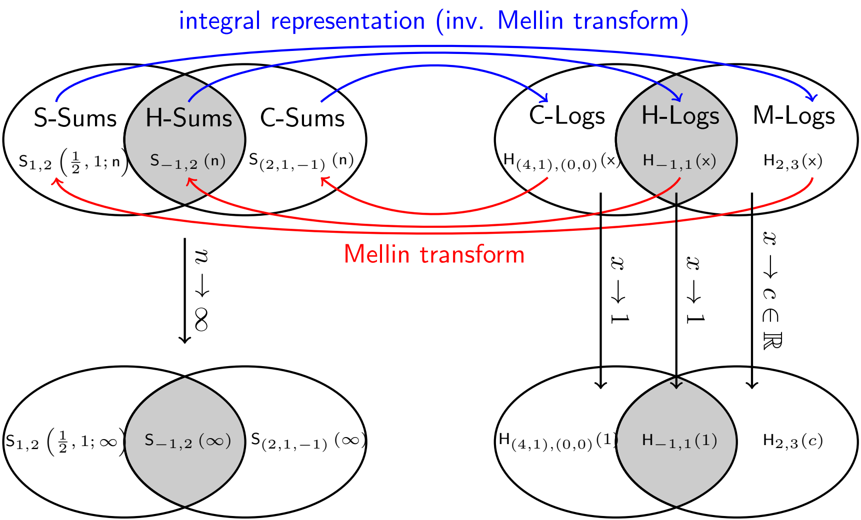
Mellin transform
$n \to \infty$
$S_{1,2}\left(\frac{1}{2},1;\infty\right)$
$S_{-1,2}(\infty)$
$S_{(2,1,-1)}(\infty)$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Mellin transform
$n\to\infty$
$x\to 1$
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$x\to c\in\mathbb{R}$
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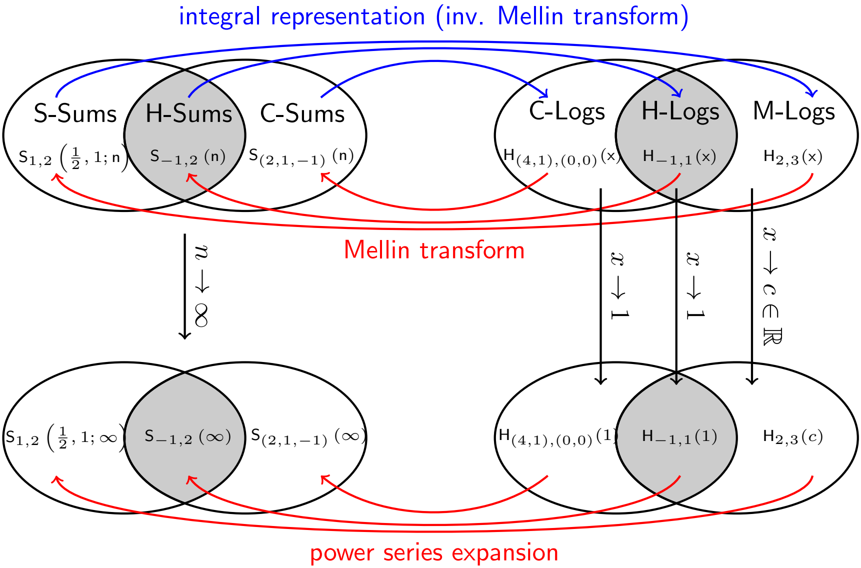
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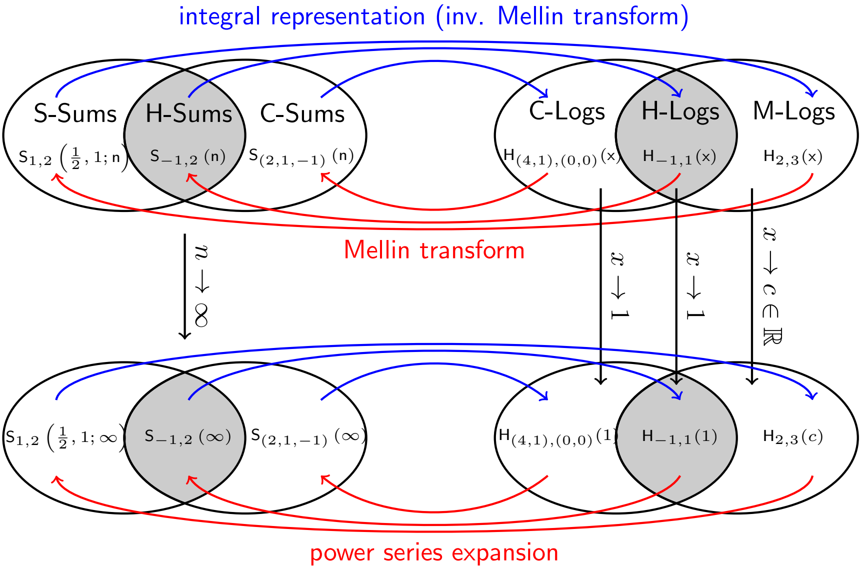
power series expansion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$$\int_0^x f_1(y_1)\int_0^{y_1} f_2(y_2)\int_0^{y_2} f_3(y_3)\cdots\int_0^{y_{k-1}} f_k(y_k)dy_k\cdots dy_3dy_2dy_1$$

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- $f_i(y) = \frac{y^{j_i}}{\Phi_{a_i}(y)},\ j_i \in \mathbb{N}$ cyclotomic polylogarithms

$$\begin{aligned} f_a(x) &:= \frac{\mathsf{sign}(1-a-0)}{x-a}, \\ f_{\{a_1,\ldots,a_k\}}(x) &:= f_{a_1}(x)^{1/2}\ldots f_{a_k}(x)^{1/2} \qquad k \geq 2, \\ f_{(a_0,\{a_1,\ldots,a_k\})}(x) &:= f_{a_0}(x) f_{a_1}(x)^{1/2}\ldots f_{a_k}(x)^{1/2} \qquad k \geq 1, \\ f_{(\{a_1,\ldots,a_k\},j)}(x) &:= x^j f_{(a_1,\ldots,a_k)}(x) \qquad j \in \{1,\ldots,k-2\}. \end{aligned}$$

Restricting to at most two root-singularities we are left with the following cases:

$$\begin{aligned} f_a(x) &:= \frac{\mathsf{sign}(1-a-0)}{x-a}, \\ f_{(a,\{b\})}(x) &:= f_a(x)\sqrt{f_b(x)}, \\ f_{\{a,b\}}(x) &:= \sqrt{f_a(x)}\sqrt{f_b(x)}, \\ f_{(a,\{b,c\})}(x) &:= f_a(x)\sqrt{f_b(x)}\sqrt{f_c(x)}. \end{aligned}$$

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$$\mathsf{H}_{(2,\{-4\}),(4,\{-1\})}(x) = \int_0^x \frac{1}{(2-x)\sqrt{x+4}} \int_0^y \frac{1}{(4-y)\sqrt{1+y}} dy dx$$

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$$\mathsf{H}_{(2),(\{-1\}),(\{-1\})}(x) = \int_0^x \frac{1}{(2-x)} \int_0^y \frac{1}{\sqrt{1+y}} \int_0^y \frac{1}{\sqrt{1+z}}dzdydx$$

$$\mathsf{H}_{\mathbf{p}}(x)\mathsf{H}_{\mathbf{q}}(x) = \sum_{\mathbf{r}=\mathbf{p}\sqcup\!\sqcup\mathbf{q}} \mathsf{H}_{\mathbf{r}}(x)$$

here $\mathbf{p} \sqcup\!\sqcup \mathbf{q}$ represent all merges of $\mathbf{p}$ and $\mathbf{q}$ in which the relative orders of the elements of $\mathbf{p}$ and $\mathbf{q}$ are preserved.

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here $\mathbf{p} \sqcup\!\sqcup \mathbf{q}$ represent all merges of **p** and **q** in which the relative orders of the elements of **p** and **q** are preserved.

$$\begin{aligned} \mathsf{H}_{a_1,a_2}(x)\mathsf{H}_{b_1,b_2}(x) \quad = \quad & \mathsf{H}_{a_1,a_2,b_1,b_2}(x) + \mathsf{H}_{a_1,b_1,a_2,b_2}(x) \\ & + \mathsf{H}_{a_1,b_1,b_2,a_2}(x) + \mathsf{H}_{b_1,b_2,a_1,a_2}(x) \\ & + \mathsf{H}_{b_1,a_1,b_2,a_2}(x) + \mathsf{H}_{b_1,a_1,a_2,b_2}(x) \end{aligned}$$

There are no additional algebraic relations among the iterated integrals if the alphabet is chosen carefully.

Mellin transform

- Let $\mathbb{K}$ be a field of characteristic 0.

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- A function $f = f(x)$ is called *D-finite* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

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  form a P-finite sequence.
- The generating function of a P-finite sequence $(f_n)_{n\geq 0}$ is D-finite.

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- We get a linear recurrence equation with polynomial coefficients, satisfied by $(f_n)_{n \geq 0}$.

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$$f(x) = \int_0^x \int_0^{\tau_1} \frac{1}{(1+\tau_1)(1-\tau_2)} d\tau_2 d\tau_1 = \sum_{n\geq 0} f_n x^n.$$

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$$-\sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)f_{n+3}x^n = 0$$

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Hence

$$(n+1)^3 f_{n+1} - (n+2)(n+1)f_{n+2} - (n+2)(n+3)(n+1)f_{n+3} = 0$$

holds for $(f_n)_{n\geq 0}$.

Let $f(x)$ be a D-finite function such that

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx$$

exists and let $p_i(x) \in \mathbb{K}[x]$ such that

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Since we have

$$\begin{aligned} \mathbf{M}[x^m f^{(p)}(x)](n) &= \frac{(-1)^p (n+m)!}{(n+m-p)!} \mathbf{M}[f(x)](n+m-p) \\ &\quad + \sum_{i=0}^{p-1} \frac{(-1)^i (n+m)!}{(n+m-i)!} f^{(p-1-i)}(1). \end{aligned}$$

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We can conclude:

Proposition

If the Mellin transform of a D-finite function is defined i.e., the integral $\int_0^1 x^n f(x)dx$ exist, then it is P-finite.

Given a D-finite function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

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4. Solve the recurrence to get a closed form representation for $\mathbf{M}[f(x)](n)$.

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Initial values can be computed easily and solving the recurrence leads to

$$\begin{aligned} \mathbf{M}[f(x)](n) \quad &= \quad (-1)^n \left(\frac{4^{n+1}}{(2n+1)(2n+3)\binom{2n}{n}} + \frac{\int_0^1 \frac{\sqrt{1-\tau}}{1+\tau}\, d\tau - 2}{n+1} \right) \\ &\quad - \frac{4(-1)^n \sum_{i=1}^n \frac{4^i}{(2i+1)\binom{2i}{i}}}{n+1} + \frac{\int_0^1 \frac{\sqrt{1-\tau}}{1+\tau}\, d\tau}{n+1}. \end{aligned}$$

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$$\int_0^{1-\varepsilon} x^N f(x)dx \quad = \quad \frac{1}{N+1}\left((1-\varepsilon)^{N+1} f(1-\varepsilon) - \int_0^{1-\varepsilon} dx x^{N+1} f'(x)\right)$$

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$$\begin{aligned}\int_0^{1-\varepsilon} \frac{x^N f(x)}{\sqrt{x-a}}dx = & \frac{(4a)^N}{(2N+1)\binom{2N}{N}}\left(\int_0^{1-\varepsilon} dx \frac{f(x)}{\sqrt{x-a}}\right.\\ & +2\sum_{i=1}^{N}\frac{\binom{2i}{i}}{(4a)^i}\left(\sqrt{1-a-\varepsilon}(1-\varepsilon)^i f(1-\varepsilon)\right.\\ & \left.\left.-\int_0^{1-\varepsilon} dx x^i \sqrt{x-a} f'(x)\right)\right).\end{aligned}$$

$$\mathbf{M}\left[\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)\right](N) = \frac{1}{N+1}\,\mathbf{M}\left[x h_1(x)\mathrm{H}^*_{\mathsf{h_2,\ldots,h_k}}(x)\right](N)$$

$$\mathbf{M}\left[\frac{\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)}{\sqrt{x}}\right](N) = \frac{1}{N+1/2}\,\mathbf{M}\left[\sqrt{x}h_1(x)\mathrm{H}^*_{\mathsf{h_2,\ldots,h_k}}(x)\right](N)$$

$$\mathbf{M}\left[\frac{\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)}{\sqrt{x-a}}\right](N) = \frac{(4a)^N}{(2N+1)\binom{2N}{N}}\left(\int_0^1 dx\frac{\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)}{\sqrt{x-a}} + \right.$$
$$\left. +2\sum_{i=1}^{N}\frac{\binom{2i}{i}}{(4a)^i}\,\mathbf{M}\left[\sqrt{x-a}h_1(x)\mathrm{H}^*_{\mathsf{h_2,\ldots,h_k}}(x)\right](i)\right)$$

$$\mathbf{M}\left[\frac{\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)}{\sqrt{x(x-a)}}\right](N) = \left(\frac{a}{4}\right)^N\binom{2N}{N}\left(\int_0^1 dx\frac{\mathrm{H}^*_{\mathsf{h_1,\ldots,h_k}}(x)}{\sqrt{x(x-a)}} + \right.$$
$$\left. +\sum_{i=1}^{N}\frac{(4/a)^i}{i\binom{2i}{i}}\,\mathbf{M}\left[\sqrt{\frac{x-a}{x}}h_1(x)\mathrm{H}^*_{\mathsf{h_2,\ldots,h_k}}(x)\right](i)\right)$$

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- As a consequence we have the following summation formula

$$\sum_{i=1}^{N} c^i \, \mathbf{M}[f(x)](i) = c^N \, \mathbf{M}\left[\frac{x}{x-\frac{1}{c}} f(x)\right](N) - \mathbf{M}\left[\frac{x}{x-\frac{1}{c}} f(x)\right](0) \, .$$

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- Furthermore, the following properties are immediate, where $a > 0$:

$$\begin{aligned} \mathbf{M}\left[\ln(x)^m f(x)\right](N) &= \frac{d^m}{dN^m} \mathbf{M}[f(x)](N), \\ \mathbf{M}[f(ax)](N) &= \frac{1}{a^{N+1}} \mathbf{M}[f(x)\theta(a-x)](N), \quad a \leq 1, \\ \mathbf{M}[f(x^a)](N) &= \frac{1}{a} \mathbf{M}[f(x)]\left(\frac{N+1-a}{a}\right). \end{aligned}$$

- The Mellin-convolution of two real functions is defined by

$$f(x) * g(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) f(x_1) g(x_2) \ .$$

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- The Mellin transformation for functions with $+$-prescription

$$\mathbf{M}[[f(x)]_+](N) = \int_0^1 dx (x^N - 1) f(x) \ .$$

has similar properties.

Inverse Mellin transform

Aim: represent our nested sums in terms of Mellin transforms in the form

$$c_0 + \sum_{j=1}^{k} c_j^N \, \mathbf{M}[f_j(x)](N), \qquad (3)$$

where the constants c_j and functions $f_j(x)$ do not depend on N.

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- achieved by virtue of the properties of the Mellin transform.
- as starting point we only need the following basic integral representations:

$$\begin{aligned}
\frac{1}{N} &= \mathbf{M}\left[\frac{1}{x}\right](N) \\
\binom{2N}{N} &= \frac{4^N}{\pi}\,\mathbf{M}\left[\frac{1}{\sqrt{x(1-x)}}\right](N) \\
\frac{1}{N\binom{2N}{N}} &= \frac{1}{4^N}\,\mathbf{M}\left[\frac{1}{x\sqrt{1-x}}\right](N).
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  we first set up an integral rep. for $a_j(N)$ of the form (3).

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  we first set up an integral rep. for $a_j(N)$ of the form (3).
- This may require the computation of Mellin convolutions.
- We obtain an integral representation of the same form of

$$a_j(N) \sum_{i_{j+1}=1}^{N} a_{j+1}(i_{j+1}) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k) \qquad (5)$$

  by Mellin convolution with the result for the inner sums.

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- For each intermediate sum

$$\sum_{i_j=1}^{N} a_j(i_j) \sum_{i_{j+1}=1}^{i_j} a_{j+1}(i_{j+1}) \cdots \sum_{i_k=1}^{i_{k-1}} a_k(i_k) \tag{4}$$

  we first set up an integral rep. for $a_j(N)$ of the form (3).
- This may require the computation of Mellin convolutions.
- We obtain an integral representation of the same form of

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  by Mellin convolution with the result for the inner sums.
- By the summation property we obtain an integral representation for (4).
- Repeat until the outermost sum has been processed.

$$\sum_{i=1}^{N} \frac{1}{i\binom{2i}{i}} = \int_0^1 dx \frac{(\frac{x}{4})^N - 1}{x-4} \frac{1}{\sqrt{1-x}}$$

$$\sum_{i=1}^{N} \frac{1}{i}\binom{2i}{i}(-1)^i = \frac{1}{\pi}\int_0^1 dx \frac{(-4x)^N - 1}{x+\frac{1}{4}} \mathrm{H}^*_{\mathsf{w}_1}(x)$$

$$\sum_{i=1}^{N} \frac{1}{i^2\binom{2i}{i}} \sum_{j=1}^{i} \binom{2j}{j}(-2)^j = -\int_0^1 dx \frac{(-2x)^N - 1}{x+\frac{1}{2}} \left(\ln(x) + \frac{\mathrm{H}^*_{\mathsf{w}_{28}}(x)}{6\sqrt{2}}\right)$$

$$-\frac{2}{3}\int_0^1 dx \frac{(\frac{x}{4})^N - 1}{x-4} \mathrm{H}^*_{\mathsf{w}_3}(x) \ .$$

Asymptotic Expansion of Nested Sums

We say that the function $f : \mathbb{R} \to \mathbb{R}$ is expanded in an asymptotic series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k},\ x \to \infty,$$

where a_k are constants, if for all $K \geq 0$

$$R_K(x) = f(x) - \sum_{k=0}^{K} \frac{a_k}{x^k} = o\left(\frac{1}{x^K}\right),\ x \to \infty.$$

Why do we need these expansions of nested sums?

E.g.,

- for limits of the form

$$\lim_{n\to\infty} n\left(S_2(n)-\zeta_2-S_{2,2}(n)+\frac{7\,\zeta_2^2}{10}\right)$$

- for the approximation of the values of analytic continued nested sums at the complex plane

$$S_{2,-3}(-2.5+2i)$$

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$$S_{-1,3}(n) \quad = \quad (-1)^n \int_0^1 x^n \frac{\mathsf{H}_{1,0,0}(x)}{1+x} dx + const$$

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$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

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$$S_{-1,3}(n) \sim (-1)^n \left(-\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6}\right) + \frac{3\ \log(2)\ \zeta_3}{4} + (-1)^n \left(\frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6}\right) \zeta_3 - \frac{19\ \zeta_2^2}{40}$$

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Generating Functions for Iterated Integrals

Given an iterated integral $f(x)$.
Find a generating series i.e., find $(f_n)_{n\geq 0}$ such that

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- Make use of the D-finitness of our integrals.
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- Compute a P-finite recurrence for f_n.
- Compute initial values for the recurrence.
- Solve the recurrence to get a closed form for f_n

Example

We want to compute the power series expansion of

$$f(x) := \int_0^x \frac{\sqrt{1-y}}{1+y}\, dy.$$

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Computing initial values and solving the recurrence leads to

$$f(x) = \sum_{i=1}^{\infty}(-x)^i\left(\frac{\sum_{j=1}^{i}\frac{\left(-\frac{1}{4}\right)^j\binom{2j}{j}}{2j-1}}{i} - \frac{\left(-\frac{1}{4}\right)^i\binom{2i}{i}}{i(2i-1)} - \frac{1}{i}\right).$$

Nested Sums at Infinity

Iterated Integrals at One

- Using the generating function we can convert the generalized harmonic polylogarithms at one to binomial sums at infinity.

$$\int_0^x \frac{\sqrt{1-y}}{1+y}\,dy \quad = \quad \sum_{i=1}^{\infty}(-x)^i\left(\frac{\sum_{j=1}^{i}\frac{\left(-\frac{1}{4}\right)^j\binom{2j}{j}}{2j-1}}{i} - \frac{\left(-\frac{1}{4}\right)^i\binom{2i}{i}}{i(2i-1)} - \frac{1}{i}\right)$$

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$$\int_0^1 \frac{\sqrt{1-y}}{1+y}\,dy = \sum_{i=1}^{\infty}(-1)^i\left(\frac{\sum_{j=1}^{i}\frac{\left(-\frac{1}{4}\right)^j\binom{2j}{j}}{2j-1}}{i}-\frac{\left(-\frac{1}{4}\right)^i\binom{2i}{i}}{i(2i-1)}-\frac{1}{i}\right)$$

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$$
\begin{aligned}
\int_0^x \frac{\sqrt{1-y}}{1+y}\,dy &= \sum_{i=1}^{\infty}(-x)^i\left(\frac{\sum_{j=1}^{i}\frac{\left(-\frac{1}{4}\right)^j\binom{2j}{j}}{2j-1}}{i} - \frac{\left(-\frac{1}{4}\right)^i\binom{2i}{i}}{i(2i-1)} - \frac{1}{i}\right)\\
\int_0^1 \frac{\sqrt{1-y}}{1+y}\,dy &= \sum_{i=1}^{\infty}(-1)^i\left(\frac{\sum_{j=1}^{i}\frac{\left(-\frac{1}{4}\right)^j\binom{2j}{j}}{2j-1}}{i} - \frac{\left(-\frac{1}{4}\right)^i\binom{2i}{i}}{i(2i-1)} - \frac{1}{i}\right)\\
&= \sum_{i=1}^{\infty}\frac{(-1)^i\sum_{j=1}^{i}\frac{(-4)^{-j}\binom{2j}{j}}{2j-1}}{i} + \sum_{i=1}^{\infty}\frac{4^{-i}\binom{2i}{i}}{i}\\
&\quad -2\sum_{i=1}^{\infty}\frac{4^{-i}\binom{2i}{i}}{2i-1} - \sum_{i=1}^{\infty}\frac{(-1)^i}{i}
\end{aligned}
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$$= \sum_{i=1}^{\infty}\frac{(-1)^i\sum_{j=1}^{i}\frac{(-4)^{-j}\binom{2j}{j}}{2j-1}}{i} + \sum_{i=1}^{\infty}\frac{4^{-i}\binom{2i}{i}}{i} - 2\sum_{i=1}^{\infty}\frac{4^{-i}\binom{2i}{i}}{2i-1} - \sum_{i=1}^{\infty}\frac{(-1)^i}{i}$$

- Relations between the constants due to the shuffle algebra of the iterated integrals.

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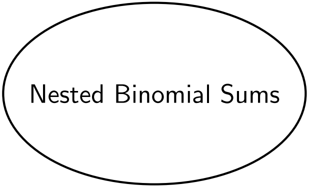
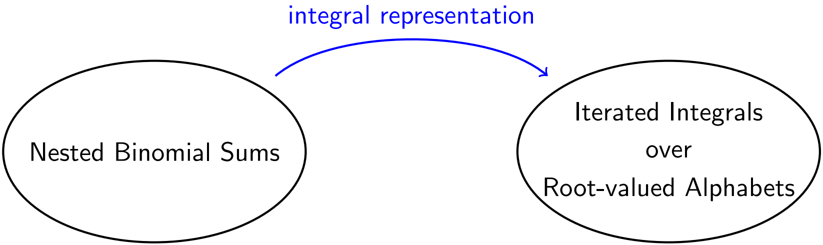
- Relations between the constants due to the shuffle algebra of the iterated integrals.
- Relations between the constants due to the quasi shuffle algebra of the nested sums.


Nested Binomial Sums


Nested Binomial Sums
Iterated Integrals
over
Root-valued Alphabets


integral representation
Nested Binomial Sums
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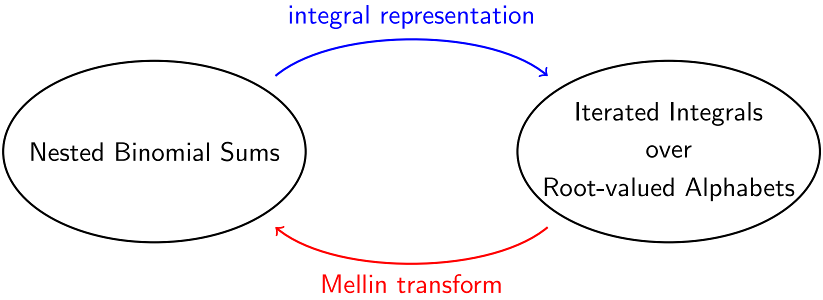
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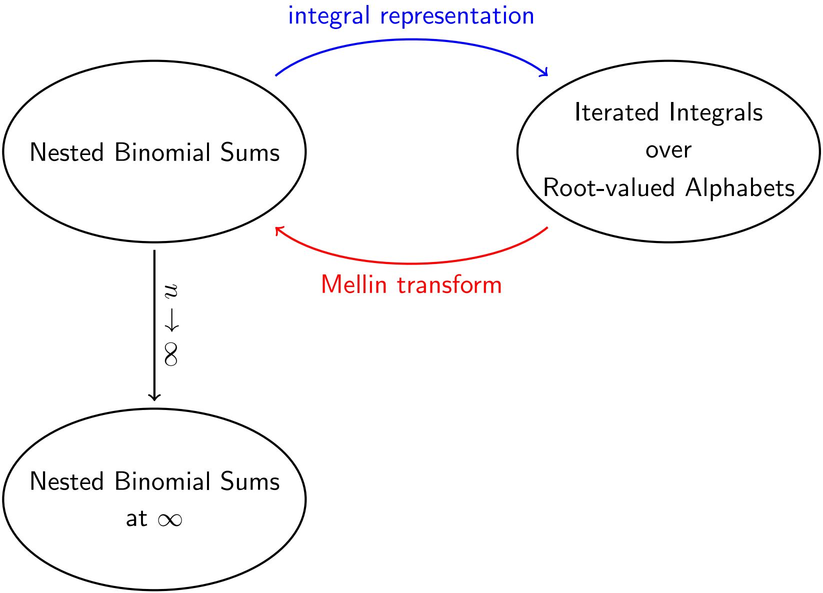
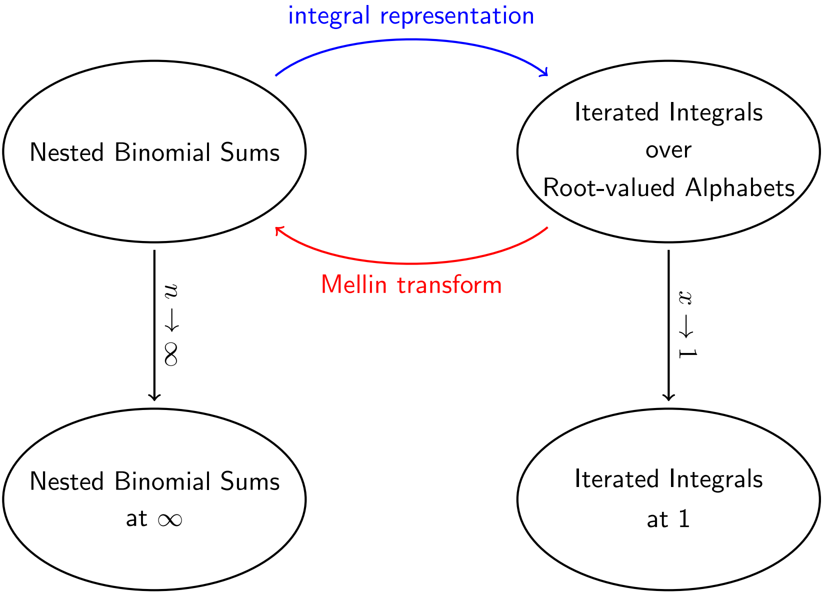

integral representation
Nested Binomial Sums
Iterated Integrals over Root-valued Alphabets
Mellin transform
$n \to \infty$
Nested Binomial Sums at ∞


integral representation
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over
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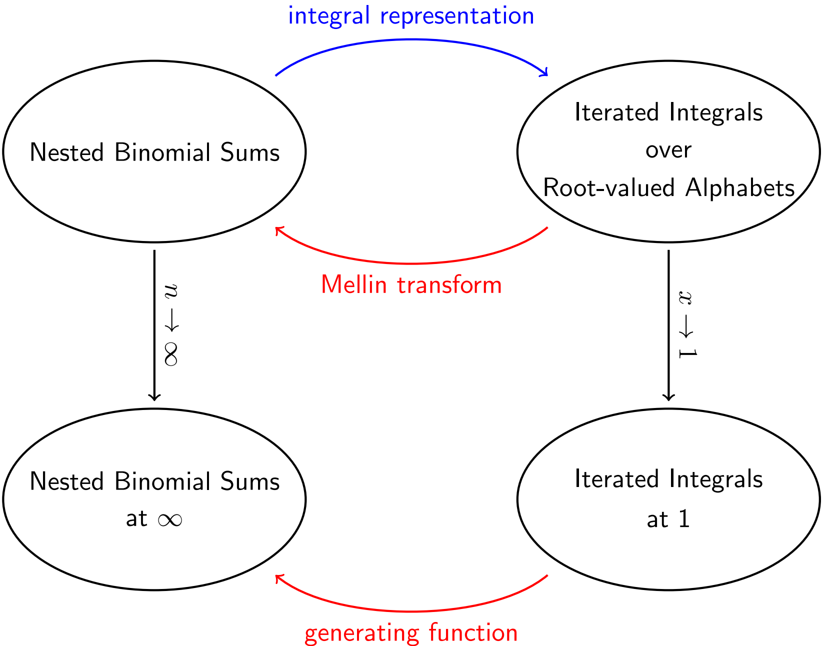
Mellin transform
$n \to \infty$
$x \to 1$
Nested Binomial Sums
at ∞
Iterated Integrals
at 1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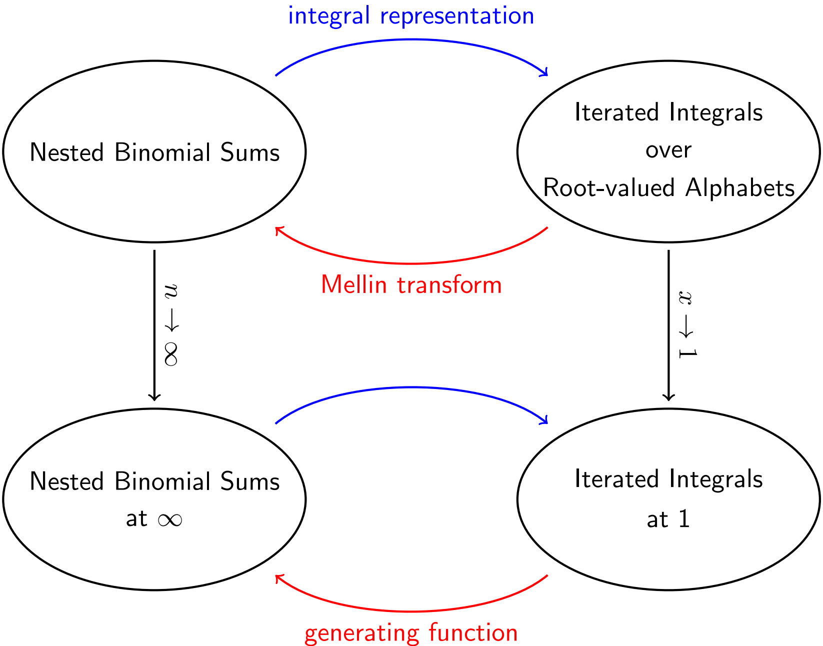
generating function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All the algorithms mentioned in this talk and many more are implemented in the package

`HarmonicSums`

which is available at

http://www.risc.jku.at/research/combinat/software/