ITERATED INTEGRALS, DEDEKIND SYMBOLS, 
AND ZETA POLYNOMIALS

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PLAN

• PART I: DEDEKIND SYMBOLS

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CLASSICAL, GENERALIZED, AND 
NON–COMMUTATIVE DEDEKIND SYMBOLS

- Classical Dedekind symbol := an “interesting part” of the difference $\log \eta(az) - \log \eta(z)$ where

$$\eta(z) := e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}), \quad a = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in PSL(2, \mathbb{Z})$$

After a normalisation, this interesting part $d(p, q)$ depends only on the first column $(p, q)$ of $a$, and satisfies functional equations

$$d(p, q) = d(p, q + p), \quad d(p, -q) = -d(p, q),$$

$$d(p, q) - d(q, -p) = \frac{p^2 + q^2 - 3pq + 1}{12pq}.$$
• Classical Dedekind symbols appeared in a variety of topological contexts:
  – signature defects of 3–dim lens spaces (F. Hirzebruch)
  – Casson invariants and quantum invariants of rational homology 3–spheres etc. (R. Kirby, P. Melvin et al.)

• T. Apostol, Sh. Fukuhara et al. defined and studied generalised Dedekind symbols with values in various commutative groups. The first part of this talk is dedicated to the generalisation of their constructions to non–commutative groups.
• A generalized Dedekind symbol with values in a commutative group: a map

$$D : W \to G, \quad W := \{(p, q) \in \mathbb{Z} \times \mathbb{Z} | \gcd(p, q) = 1\}$$

satisfying functional equations, a part of which is convenient to state in terms of reciprocity function for $D$:

$$f(p, q) := D(p, q) - D(q, -p).$$

• Functional equations:

$$D(p, q) = D(p, q + p), \quad D(p, -q) = D(-p, q),$$

$$f(p, -q) = f(-p, q), \quad f(p, q) + f(-q, p) = 0_G,$$

$$f(p, p + q) + f(p + q, q) = f(p, q).$$
• **Example**: The reciprocity function for the classical symbol:

\[ f(p,q) = \frac{p^2 + q^2 - 3pq + 1}{12pq}. \]

• **Example**: Generalized reciprocity functions from cusp forms.

Let \( \varphi(z) \) be a cusp form of even weight \( w+2 \) for the full modular group \( \text{PSL}(2, \mathbb{Z}) \). In particular, \( \varphi(z + 1) = \varphi_j(z) \) and \( \varphi(-z^{-1}) = \varphi_j(z)^w \) for all \( j \).

Then

\[ D(p,q) := \int_{p/q}^{i\infty} \varphi(z)(pz - q)^w dz, \quad f(p,q) := \int_0^{i\infty} \varphi(z)(pz - q)^w dz \]

is a \( \mathbb{C} \)-valued generalized Dedekind symbol, resp. its reciprocity function.
• **Generalized non commutative Dedekind symbols**: the group law of $G$ is written multiplicatively and is not assumed to be commutative. Basic relations for $D : W \to G$ are lifted to the non commutative context in the following way:

$$f(p, q) := D(p, q)D(q, -p)^{-1}.$$ 

• **Functional equations**:

\[
D(p, q) = D(p, q + p), \quad D(p, -q) = D(-p, q),
\]

\[
f(p, -q) = f(-p, q), \quad f(p, q)f(-q, p) = 1_G,
\]

\[
f(p, p + q)f(p + q, q) = f(p, q).
\]

• **First properties**:

\[
f(1, 1) = f(-1, 1) = 1_G.
\]

Moreover, $f(-p, -q) = f(p, q)$ so that $f(p, q)$ depends only on $q/p \in P^1(Q)$. 


RECONSTRUCTION OF A SYMBOL FROM ITS RECIPROCITY FUNCTION

- **Modified continued fractions.** Let $A_0, A_1, \ldots, A_n, \ldots$ be commuting independent variables. Define by induction polynomials $Q^{(n)}, P^{(n)} \in \mathbb{Z}[A_0, \ldots, A_n]:$

$$Q^{(0)}(A_0) = A_0, \ P^{(0)}(A_0) = 1,$$

$$Q^{(n+1)}(A_0, \ldots, A_{n+1}) = A_0 Q^{(n)}(A_1, \ldots, A_{n+1}) - P^{(n)}(A_1, \ldots, A_{n+1}),$$

$$P^{(n+1)}(A_0, \ldots, A_{n+1}) = Q^{(n)}(A_1, \ldots, A_{n+1})$$

One easily sees that

$$\frac{Q^{(n)}(A_0, \ldots, A_n)}{P^{(n)}(A_0, \ldots, A_n)} = \langle A_0, A_1, \ldots A_n \rangle := A_0 - \frac{1}{A_1 - \frac{1}{A_2 - \ldots}}$$

where the lowest layer in the continued fraction representation is $A_{n-1} - \frac{1}{A_n}$. For any integer values $a_i$ of $A_i$ and each $n$, values of $P^{(n)}, Q^{(n)}$ are coprime.
Let now \( f : W \to G \) be a map satisfying the functional equations for a reciprocity function.

- **Reconstruction of the symbol.** For a given \((p, q) \in W\), choose a presentation as above \( \frac{q}{p} = \langle a_0, \ldots, a_n \rangle \) that is,

\[
q = q_0 := Q^{(n)}(a_0, \ldots, a_n), \quad p = p_0 := P^{(n)}(a_0, \ldots, a_n)
\]

Furthermore, for \( i = 1, \ldots, n \), define \((q_i, p_i) \in \mathbb{Z}^2\) by

\[
q_i := Q^{(n-i)}(a_i, \ldots, a_n), \quad p_i := P^{(n-i)}(a_i, \ldots, a_n)
\]

so that \( \frac{q_i}{p_i} = \langle a_i, \ldots, a_n \rangle \).
Finally, put

$$D(p, q) := f(p_1, q_1)^{-1} f(p_2, q_2)^{-1} \cdots f(p_n, q_n)^{-1} \in G.$$ 

- **Theorem.** (i) $D(p, q)$ depends only on $(p, q)$ and not on the choice of the presentation $q = p^{a_0, \ldots, a_n}$.

(ii) The function $D$ thus defined is a generalized Dedekind symbol with reciprocity function $f$. 
• Generalized non commutative symbols from cusp forms.

Denote by \( \{ \varphi_j(z) \}, j = 1, \ldots, r \), a basis of the space of cusp forms of even weight \( w + 2 \) for the full modular group.

From now on, \((A_j), j = 1, \ldots, r\), denote independent associative but noncommuting formal variables. For \((p, q) \in W\), put

\[
\Omega(p, q) := \sum_{j=1}^{r} A_j \varphi_j(z)(pz - q)^w dz .
\]

Finally, consider iterated integrals along the geodesics from rational points to \( i\infty \) in the upper half plane:

\[
f(p, q) := J_{0}^{i\infty}(\Omega(p, q)), \quad D(p, q) := J_{p/q}^{i\infty}(\Omega(p, q))
\]
They take values in the multiplicative subgroup
\[ G := 1 + (A_1, \ldots, A_r) \]
of the ring of free associative formal series \( \mathbb{C}\langle\langle A_1, \ldots, A_r \rangle\rangle \).

- **Theorem.** The map \( D : W \rightarrow G \) is a generalised Dedekind symbol with reciprocity function \( f \).

- **Remark.** This construction can be generalised to Eisenstein series using regularised iterated integrals.
REMINDER ABOUT ITERATED INTEGRALS

Let \((\omega_1, \ldots, \omega_r)\) be a family of holomorphic 1–forms. Put

\[ \Omega := \sum_{v \in V} A_v \omega_v. \]

Then the most direct definition is:

\[ J^\omega_a(\Omega) := 1 + \sum_{n=1}^{\infty} \sum_{(v_1, \ldots, v_n) \in [1, \ldots, r]_n} A_{v_1} \cdots A_{v_n} I^\omega_a(\omega_{v_1}, \ldots, \omega_{v_n}), \]

where

\[ I^\omega_a(\omega_{v_1}, \ldots, \omega_{v_n}) = \int_a^{z_1} \omega_{v_1}(z_1) \int_a^{z_2} \omega_{v_2}(z_2) \cdots \int_a^{z_{n-1}} \omega_{v_n}(z_n) \]

are the usual iterated integrals.
DEDEKIND SYMBOLS
AND NONCOMMUTATIVE 1–COCYCLES

• 1–cocycles. Let $\Gamma$ be a group, and $G$ a group endowed with a left action of $\Gamma$ by automorphisms: $(\gamma, g) \mapsto \gamma g$.

The set of 1–cocycles is

$$Z^1(\Gamma, G) := \{ u : \Gamma \to G \mid u(\gamma_1 \gamma_2) = u(\gamma_1) \gamma_1 u(\gamma_2) \}.$$ 

Let $\Gamma = PSL(2, \mathbb{Z})$, and let $G$ be a possibly noncommutative $\Gamma$–module. It is known that $PSL(2, \mathbb{Z})$ is the free product of its two subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ generated respectively by

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$
• **Corollary.** Restriction to \((\sigma, \tau)\) of any cocycle in \(Z^1(\text{PSL}(2, \mathbb{Z}), G)\) belongs to the set

\[
\{(X, Y) \in G \times G \mid X \cdot \sigma X = 1, Y \cdot \tau Y \cdot \tau^2 Y = 1\}.
\]

(ii) Conversely, any element of this set comes from a unique 1–cocycle so that we can and will identify these two sets.

• **Definition.** A pair \((X, Y)\) as above is called (the representative of) a Dedekind cocycle, iff it satisfies the relation

\[Y = \tau X.\]

• We will now show that in certain situations there is a bijection between Dedekind cocycles and reciprocity functions of generalised Dedekind symbols.
• Reciprocity functions as cocycles. Let now $G_0$ be a group. Denote by $G$ the group of functions $f : \mathbb{P}^1(\mathbb{Q}) \to G_0$ with pointwise multiplication. Define the left action of $\Gamma$ upon $G$ by

$$(\gamma f)(x) = f(\gamma^{-1} x); \quad f \in G, \ x \in \mathbb{P}^1(\mathbb{Q}), \ \gamma \in \Gamma.$$

Let $f : W \to G_0$ be a $G_0$–valued reciprocity function.

Define elements $X_f, Y_f \in G$ as the following functions $\mathbb{P}^1(\mathbb{Q}) \to G_0$:

$$X_f(qp^{-1}) := f(p, q),$$

$$Y_f(qp^{-1}) := (\tau X_f)(qp^{-1}) = X_f(\tau^{-1}(qp^{-1})) = f(q, q - p).$$

• Theorem. The map $f \mapsto (X_f, Y_f)$ establishes a bijection between the set of $G_0$–valued reciprocity functions and the set of (representatives of) Dedekind cocycles from $Z^1(\Gamma, G)$.
**L–FUNCTIONS AND ZETA POLYNOMIALS**

- **Notations.**
  
  $U(z) \in \mathbb{R}[z]$ a polynomial of degree $e \geq 1$, $U(1) \neq 0$,
  
  $P(z) := \frac{U(z)}{(1-z)^d}, \; d > e$,
  
  $H(x) :=$ the polynomial of degree $d - 1$ such that if for $|z| < 1$ we have
  
  $$P(z) := \frac{U(z)}{(1-z)^d} = \sum_{n=0}^{\infty} h_n z^n,$$
  
  then

  $H(n) = h_n$ for all $n \geq \max \{0, e - d + 1\}$. 

• **Theorem.** (Popoviciu, Rodriguez–Villegas, et al.) Assume that all roots of $U$ lie on the unit circle. Then $H$ satisfies the zeta–type functional equation

$$H(x) = (-1)^{d-1} H(-d + e - x),$$

and vanishes at integer points $x = -1, \ldots, -d + e + 1$ inside its “critical strip”.

Moreover, all the remaining zeroes lie on the vertical line passing through the middle of the critical strip:

$$\text{Re}(x) = \frac{-d + e + 1}{2}.$$

• **Examples :** Hilbert polynomials $H$ of certain graded commutative rings, in particular anti-canonical rings of Fano varieties, and their versions for Calabi–Yau varieties.
• Heuristics: relationship between $H$ and $P$ as a “discrete Mellin transform”.

  – Classical Mellin transform: $f(z)$ a Fourier series in upper half-plane, e.g. a cusp form, $Z_f(s)$ the associated Dirichlet series:

    $$Z_f(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} f(z) \left(\frac{z}{i}\right)^{s-1} d\left(\frac{z}{i}\right).$$

  – Integral representation of the passage from $P(z)$ to $H(x)$:

    $$H_f(n) = \frac{1}{2\pi i} \int_\gamma P_f(z)z^{-(n+1)} dz,$$

    where $\gamma$ is a small contour around zero.

  – NB This explains also that $s$ corresponds to $-n$. 

ZETA POLYNOMIALS FROM MODULAR FORMS

• I will call polynomials $H$ as above zeta polynomials.

Below I will show how to obtain zeta polynomials from those $PSL(2,\mathbb{Z})$–cusp forms $f$ that are eigenforms for all Hecke operators.

**Heuristics:** These zeta polynomials correspond to Euler factors of the respective $L$–series “in characteristics one”.

• **Period polynomials.** Let $f$ be a cusp form of positive even weight $k$, $w := k + 2$. Put

$$r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau, \quad r^+_f(z) := \frac{r_f(z) \pm r_f(-z)}{2}.$$
• **Proposition.** (J. B. Conrey, D. W. Farmer, Ö. Imamoğlu).
Let \( f \) be a cusp Hecke eigenform of weight \( k \) with real Fourier coefficients. Then

\[
U_f(z) := \frac{r_f^-(z)}{z(z^2 - 4)(z^2 - 1/4)(z^2 - 1)^2} \in \mathbb{R}[z]
\]

is a polynomial of degree \( e := w - 10 \) without real zeros whose complex zeros all lie on the unit circle.
\textbf{Final result.} Fix an integer $d > e = w - 10$ and put

$$P_f(z) := \frac{U_f(z)}{(1 - z)^d}.$$ 

Let $H_f(x) \in \mathbb{R}[x]$ be the polynomial of degree $d - 1$ such that

$$P_f(z) = \sum_{n=0}^{\infty} H_f(n) z^n$$

for $|z| < 1$. This polynomial satisfies the functional equation

$$H_f(x) = (-1)^{d-1} H(-d + e - x)$$

and vanishes at $x = -1, \ldots, -d + e + 1$. All its remaining zeros lie on the vertical line $\text{Re} \, x = -(d - e - 1)/2$. 
References


THANK YOU FOR YOUR ATTENTION!