Research Trimester on

Multiple Zeta Values, Multiple Polylogarithms and Quantum Field Theory[1]

ICMAT, September 15 - December 19, 2014

Trimester Seminar

ICMAT, September 22 – November 28, 2014

Speakers:

1. Robin de Jong (Leiden, The Netherlands)
2. Javier Fresán (ETH Zürich, Switzerland)
3. David Holmes (Leiden, The Netherlands)
4. Dominique Manchon (Clermont-Ferrand, France)
5. Erik Panzer (Paris, France)
6. Frédéric Patras (Nice, France)
7. Danylo Radchenko (MPIM Bonn, Germany)
8. Ismaël Soudères (Osnabrück, Germany)
9. Stephan Stierberger (MPI Munich, Germany)
10. Ivan Todorov (Sofia, Bulgaria)
11. Jianqiang Zhao (Bonn, Germany and St. Petersburg, USA)
12. Wadim Zudilin (Newcastle, Australia)

[1] www.icmat.es/RT/MZV2014

Titles + Abstracts:

Robin de Jong

Degenerations of jacobians of algebraic curves over high-dimensional bases, part 2

Let S be a smooth variety, and U an open dense subvariety of S. Let J/U be a family of jacobians over U. Let P in $J(U)$ be a section of J/U. Then to each test curve T into S we associate a numerical invariant on T called the height jump, which is an obstruction to extend P, or any multiple of P, into a semiabelian scheme over S. The height jump has been introduced in an analytic context by R. Hain. As an example we take U to be the universal elliptic curve over the modular curve $Y(N)$, and $(J/U, P)$ the tautological family over U, equipped with its tautological section. We compare the outcome with a recent analytic result in this context due to Burgos, Kramer and Kühn.

Javier Fresán

An introduction to mixed Tate motives

Mixed Tate motives constitute one of the few examples in which the motivic philosophy is more than a powerful and suggestive prediction tool. Thanks to Borel's computation of the rank of K-theory, we have at our disposal an unconditional abelian category of mixed Tate motives over a number field. In this series of lectures I will first recall the construction of this category. Then I will present some methods to exhibit interesting elements of it, building on hyperbolic geometry and modular curves.

David Holmes

Degenerations of jacobians of algebraic curves over high-dimensional bases, part 1

Every non-singular algebraic curve C has a jacobian J, which is an abelian variety. Choosing a point on the curve determines an *abel-jacobi* map from C to J. The same constructions can be made in families: given a family of non-singular curves (together with a section),

one obtains a family of abelian varieties, and an abel-jacobi map. We are interested in what happens when such a family of non-singular pointed curves degenerates to a singular pointed curve. In the case where the base-space of the family has dimension 1 (a '1-parameter family), this is completely understood due to work of André Néron in the 1960s. However, when the base space has higher dimension things become more difficult. We describe a seemingly-new combinatorial invariant which controls these degenerations. In particular, we can show both the non-existence of 'Néron models' even after alteration of the base, and also the existence of finite-type Néron models after a certain canonical 'infinite sequence of blowups' of the base space of the family.

Dominique Manchon

The double shuffle structure of the Ohno–Okuda–Zudilin q-multiple zeta values

Several q-analogues of multiple zeta values have been explored in the recent years. The model recently proposed by Y. Ohno, J.-I. Okuda and W. Zudilin shows particularly good algebraic properties, and readily extends to arguments running over all the integers, regardless to the sign. We exhibit the iterated Jackson integral representation of these qMZVs, and describe the q-double shuffle relations thus obtained. Joint work with Jaime Castillo-Medina and Kurusch Ebrahimi-Fard.

Erik Panzer

Open problems on relative periods of graph hypersurfaces

Feynman integrals of quantum field theories that contain non-scalar particles go beyond the well-studied leading period $\int \frac{\Omega}{\psi^2}$ associated to a Feynman graph via its graph polynomial ψ. Instead, all of its periods must be considered, corresponding to convergent integrals $\int \frac{P}{\psi^N}\Omega$ with arbitrary polynomials P.

Even for the simplest Feynman graphs, this total space of periods is not understood and carries non-trivial structures. We explain open

problems like gaps in even weights and absence of alternating sums in examples of linearly reducible graphs. For this class of graphs we suggest a method to compute the span of all periods explicitly in finite terms.

Frédéric Patras

Deformations of shuffles and quasi-shuffles

The talk will be concerned with deformations of the shuffle Hopf algebra structure which can be defined on the tensor algebra over a commutative algebra A. Such deformations, leading for example to the quasi-shuffle algebra $QSh(A)$, can be interpreted as natural transformations of the functor Sh, regarded as a functor from commutative nonunital algebras to coalgebras. The monoid of natural endomophisms of the functor Sh appears to be isomorphic to the monoid of formal power series in one variable without constant term under composition, so that in particular, its natural automorphisms are in bijection with formal diffeomorphisms of the line. These transformations can be interpreted as elements of the Hopf algebra of word quasi-symmetric functions $WQSym$, and in turn define deformations of its structure. This leads to a new embedding of free quasi-symmetric functions into $WQSym$, whose relevance is illustrated by a simple and transparent proof of Goldberg's formula for the coefficients of the Hausdorff series. Based on a joint work with L. Foissy and JY. Thibon.

Danylo Radchenko

Degenerations of jacobians of algebraic curves over high-dimensional bases, part 2

In a very rough form Zagier's conjecture states that for any integer $m > 1$ and any number field F, the special value $\zeta_F(m)$ of the Dedekind zeta function ζ_F can be expressed in terms of the m-th classical polylogarithm function evaluated at some elements of F. So far it has been proved only for $m = 2, 3$.

In this talk, I will give a formulation of this conjecture, survey some known results and report on some recent progress in the case $m = 4$.

Ismaël Soudères

Bloch's cycle complex over a base

In this talk we will discuss different approaches giving a multplicative structure to Bloch cycle complex over a base scheme X, which compute higher Chow groups and hence K-theory. After introducing Bloch–Kriz construction and presenting some more recent version of it, we will concentrate on the case of equidimensional cycles over the projective line minus three points and their relations to mixed Tate motives.

Stephan Stieberger

Motivic Multiple Zeta Values and Single-Valued Multiple Zeta Values in Superstring Theory

In physics amplitudes describing the interaction of physical states play an important role in determining physical observables. In string theory the physical states are given by vibrations of open and closed strings and their interactions are described (at tree-level) by iterated real integrals on $\mathbb{R}P^1\{0,1,\infty\}$ or complex integrals on $P^1\{0,1,\infty\}$, respectively.

The mathematical structure of these string amplitudes share many recent advances in arithmetic algebraic geometry and number theory like multiple zeta values, single-valued multiple zeta values, Drinfeld, Deligne associators and Lie algebra structures related to Grothendiecks Galois theory.

We review these results, with emphasis on a beautiful link between generalized hypergeometric functions describing the real integrals on $\mathbb{R}P^1\{0,1,\infty\}$ and the decomposition of motivic multiple zeta values. Furthermore, a relation expressing complex integrals on $P^1\{0,1,\infty\}$ as single-valued projection of iterated real integrals on $\mathbb{R}P^1\{0,1,\infty\}$. This work is based on the works: arXiv:1401.1218, arXiv:1310.3259, arXiv:1205.1516

Ivan Todorov

Renormalization theory and multiple zeta values

Far from being a liability of quantum field theory or a reason for divorce between mathematics and physics [D] (as it was viewed until not so long ago) renormalization theory has become a common playground for physicists and mathematicians. In the (Bogolubov–) Epstein–Glaser approach position space renormalization is reduced to the extension of distributions, originally defined for non-co-inciding arguments: one never has to speak of subtracting infinities. In a dilation invariant massless theory we are dealing with analytically regularized associate homogeneous distributions that are meromorphic functions of the regularization parameter. The residue of the pole of a primitively divergent amplitude is independent of the renormalization ambiguity and is a period in the sense of [KZ]. Most of the periods in low order Feynman amplitudes (in particular, all periods in the ϕ^4 theory up to including six loops [S]) are found to be linear combinations with integer coefficients of multiple zeta values. The periods of an infinite sequence of n-loop graphs (the "zig-zag graphs") were proven to be rational multiples of a single zeta value, $\zeta(2n-3)$, [BS]. We review and discuss these results and their relevance to the renormalization program for amplitudes that include integration over internal vertices.

References:

[BS] F. Brown, O. Schnetz, *Proof of the zig-zag conjecture*, arXiv:1208.1890v2.

[D] F.J. Dyson, *Missed opportunities*, Bull. Amer. Math. Soc. 78:5 (1972) 635-652.

[GGV] J.M. Gracia-Bondia, H. Gutiérrez-Garro, J.C. Várilly, *Improved Epstein-Glaser renormalization in x-space. III. Versus differential renormalization*, arXiv:1403.1785.

[KZ] M. Kontsevich, D. Zagier, *Periods*, in: Mathematics Unlimitted 2001 and beyond, B. Engquist, W. Schmid, eds., Springer, Berlin et al. 2001, pp. 771-808.

[NTS] N.M. Nikolov, R. Stora, I. Todorov, *Renormalization of massless Feynman amplitudes in configuration space*, Rev. Math. Phys. 26:4 (2014) 1430002 (65 pages); arXiv:1307.0854.

[S] O. Schnetz, *Quantum periods: A census of ϕ^4 transcendentals*, Commun. Number Theory Phys. 4:1 (2010) 1-46; arXiv:0801.2856v2.

[**T**] I. Todorov, *Polylogarithms and multizeta values in massless Feynman amplitudes*, Bures-sur-Yvette preprint, IHES/P/14/10.

Jianqiang Zhao

Finite multiple zeta values and finite alternating Euler sums

The multiple zeta values are defined as iterated sums generalizing the Riemann zeta values. There are many rational linear relations among these values and to determine all such relations is one of the key problems in this area. On the other hand we can study the van-Hamme type congruence related to the partial sums of the series defining the multiple zeta values. Putting into an adele-like setting these values are called the finite multiple zeta values. In the lectures I will summarize the known results concerning these values and describe a mysterious connection to the classical multiple zeta values. At the end I will consider the generalization of this problem to the alternating Euler sums and present some numerical results.

Wadim Zudilin

Radial asymptotics of (multiple) q-zeta values and independence questions

Multiple zeta values (MZVs) possess a rich algebraic structure of algebraic relations, which is conjecturally determined by two different (shuffle and harmonic) products of a certain algebra of noncommutative words. In a recent work, Bachmann constructed a q-analogue of the MZVs — the so-called bi-brackets — for which the two products are dual to each other, in a very natural way. In my talk I will discuss the asymptotics of Bachmann's bi-brackets at roots of unity, its links to MZVs, and related linear (in)dependence questions of the q-analogue.