Descriptive set theoretic applications in Banach lattices: a survey of results and open questions

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Descriptive complexity of Banach lattice classes

- Preliminaries on Polish and standard Borel spaces
- Borel classes and relations of Banach lattices
- Non Borel classes and relations of Banach lattices

Polish and standard Borel spaces

- A **Polish space** X is a complete, separable metrizable space.
- Sometimes, we can "forget" the Polish topology on X and focus on the Borel σ-algebra $\beta(X)$. A measure space (X, σ) is a **standard Borel space** if there exists a Polish Y such that (X, σ) is Borel isomorphic to $(Y, \beta(Y))$
  - Ex: For X Polish, Let $F(X)$ be the Effros Borel space comprised of closed subsets of X, equipped with the σ-algebra generated by sets of the form
$$\{F \in F(X) : \ F \cap U \neq \emptyset\}$$
with $U \subseteq X$ open.
  - Ex: For X standard Borel, any Borel subset of X is also standard Borel.

Theorem 1 (Kuratowski-Ryll-Nardsewski)

Let X be a Polish space. Then there exists a sequence $\psi_n : F(X) \to X$ of Borel functions such that for all $F \in F(X)$, $\overline{\{\psi_n(F) : n \in \mathbb{N}\}} = F$.

Standard Borel space of Banach lattices

We can construct a standard Borel space of separable Banach lattices:

- Let E be a separable Banach lattice and let $BL(E) \subseteq F(E)$ be the (infinite dimensional) Banach sublattices of E.

$$F \in BL(E) \iff \forall m, n \in \mathbb{N}, p, q \in \mathbb{Q}, \ (q\psi_m(F) + p\psi_n(F) \in F) \bigwedge$$
$$(\psi_n(F) \vee \psi_m(F) \in F) \bigwedge \Bigg(\forall k \exists n_1, ..., n_k \in \mathbb{N}^k$$
$$\neg(\forall M \in \mathbb{N} \ \exists q_1, ...q_k(\vee q_i = 1 \wedge \| \sum q_i \psi_{n_i}(F)\| < \frac{1}{M} \ )\Bigg)$$

- Let $\mathcal{U} = C(\Delta, L_1(0,1))$. $\mathcal{U}$ is isometrically universal for separable lattices, so $BL(\mathcal{U}) := BL$ is (at least up to isometry) a standard Borel space of all separable Banach lattices.

Borel maps between sets and lattices

Use KRN Theorem to define maps from Banach lattices to commonly used closed subsets of lattices, which can be used to characterize Banach lattice properties:

Theorem 2

Let X be a separable Banach lattice. Then the following maps are Borel:

- $_+ : SF(X) \to F(X)$, *mapping solid closed sets to the subset of positive elements.*
- $\mathbf{B} : BL(X) \to F(X)$, *mapping X to its unit ball*
- $\mathbf{S} : BL(X) \to F(X)$, *mapping X to its unit sphere.*
- $\overline{S}. : F(X) \times BL(X) \rightharpoonup F(X)$ *mapping $F \subseteq E \in BL(X)$ to its closed solid hull $\overline{S}_E(F)$ in E.*

Borel classes of Banach lattices

- Order continuous lattices
- Classes of Atomic lattices
- classes with weak Fatou Property conditions
- Rearrangement invariant lattices and their sums

Order continuous lattices

Theorem 3

The class of order continuous lattices is Borel.

sketch of proof:

- Let $\Lambda \subseteq BL \times \mathbf{B}(\mathcal{U})_+ \times \mathbb{R}_+ \times \mathbb{N}$ be defined by $\Lambda(E, x, \lambda, n) \iff$ there exist mutually disjoint $0 \leq y_1, ..., y_n \leq x$ in E such that $\|y_i\| > \lambda$.
- Lemma: E is not order continuous iff there exists an $x \in \mathbf{B}(E)_+$ and $\lambda > 0$ such that for all $n \in \mathbb{N}$, $\Lambda(E, x, \lambda, n)$ holds.
- Can show Λ is Borel by quantifying over countable dense subsets $\psi_{k_1}(\overline{S}_E(\{x\})_+), ..., \psi_{k_n}(\overline{S}_E(\{x\})_+)$ instead of over all $y_1, ..., y_n \leq x$.
- Quantify over $\psi_k(\mathbf{B}(E)_+)$ and $(1/n)_n$ instead of all $x \in \mathbf{B}(E)_+$ and $\mathbb{R}_+$.

Abstract L_p lattices

Theorem 4

The classes both of isomorphically and isometrically abstract $\mathcal{L}_p$ lattices is Borel for $1 \leq p < \infty$. similarly, the classes of isometric and isomorphic AM spaces is Borel.

- Can be shown using Borel statements characterizing p-convexity and p-concavity, but quantifying over countable dense set of $\{\psi_n(X)\}$ for $X \in BL$.

Atomic lattices

Theorem 5

- *For any $n = 0, 1, ..., \infty$, the class A_n of lattices with exactly n atoms is Borel.*
- *The class $\mathcal{A}t\mathcal{L}(X)$ of atomic sublattices of a lattice X is co-analytic.*

- Resorting to band of atoms generating all of X make complexity upper bound too high (order convergence is itself generally co-analytic).
- Instead, show co-analycity by arguing that X is atomic iff every closed ideal in X contains at least one atom.

Atomic lattices and Fatou norms

- A (separable) lattice E has a **weak Fatou norm** if there exists $M \geq 1$ such that for all increasing sequences $0 \leq x_n \uparrow x$, we have $\|x\| \leq M \sup_n \|x_n\|$.
- Ex: AM space setting: $X \subseteq \mathbf{c}$, such that for all $x \in X$, $x_1 = 2 \cdot \lim_n x_n$.

Theorem 6

Let $\mathcal{B}$ be a Borel class of lattices with weak Fatou norms. Then $\mathcal{B} \cap \mathcal{A}t\mathcal{L}(\mathcal{U})$ is also Borel.

Idea of proof: show that $X \in \mathcal{B} \cup \mathcal{A}t\mathcal{L}(\mathcal{U})$ iff there exists a sequence of atoms $(e_n)_n$ in X such that for all $\psi_n(X_+)$,

$$M \cdot \lim_k \|(k \sum_m 2^{-m} e_m) \wedge \psi_n(X_+)\| \geq M\|\psi_n(X_+)\|.$$

Implications:

- For $1 \leq p < \infty$ the isomorphism and isometry classes of ℓ_p are Borel.
- The isomorphism and isometry equivalence classes of c_0 are Borel.
- The class of order continuous atomic lattices A_{OC} is Borel

Approximating Atomicity

- Various properties of lattices are defined in terms of atoms... KRN functions ψ_n can only approximate atoms in the norm, but may lack the needed properties of atoms.
- Instead, define a property that approximates the property of atomicity: we say that $y \in S(X)_+$ is *ε-atomic* if for all $x \in X_+$, there exists $r \in [0,1]$ such that $\|x \wedge y - ry\| < \varepsilon$.
  - NB: if $\|x - e\| < \varepsilon$, then x is 2ε-atomic.

Theorem 7

Suppose X has an weak Fatou norm, and let M be the Fatou constant. Then if $y \in X$ is ε-atomic in X, and $\varepsilon \leq \frac{1}{13M}$, then there exists an atom $e_i \in X$ such that $\|y - e_i\| < 4\varepsilon$.

Atomic lattices and Fatou norms

Theorem 8

Let X be a Banach Lattice and $\mathcal{B}$ be a Borel class of weak Fatou sublattices of X. Then the partial map $A_{-} : \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \to F(X)$ with $E \mapsto A_E$, where A_E is the set of atoms in E, is Borel.

Idea of proof: given some closed $F \subseteq E$, $F = A_E$ iff every $\psi_n(F)$ is an atom in E, and any ε-atomic $\psi_n(E)$, for small enough ε (dependent on some weak Fatou constant), can be sufficiently approximated by some $\psi_m(F)$.

Theorem 9

Suppose $\mathcal{B}$ is as above. Then there exists a Borel map $\mathbf{e} : \mathbb{N} \times \mathcal{B} \cap \mathcal{A}t\mathcal{L}(\mathcal{U}) \to \mathcal{U}$ such that for each $E \in \mathcal{B} \cap \mathcal{A}t\mathcal{L}(\mathcal{U})$ and $n \in \mathbb{N}$, $\mathbf{e}(n, E) := \mathbf{e}_{n,E}$ is an atom in E, and furthermore, the map $\mathbf{e}(\cdot, E)$ is a bijection between $\mathbb{N}$ and the atoms of E.

open questions

Descriptive complexity and types of convergence:

- given a lattice X what is the complexity of various kinds of convergences in X?
  - Ex: for $(x_n)_n \in X^{\mathbb{N}}$, what is the complexity of $\mathcal{C}_0 \subseteq X^{\mathbb{N}}$, with

    $$(x_n)_n \in \mathcal{C}_0 \text{ iff } x_n \downarrow 0?$$

  - observe that when X is order continuous or X is some $C(K)$ space with K compact metric, $\mathcal{C}_0$ is Borel, but otherwise the relation is at most co-analytic.
  - Ex: for $(x_n)_n \in X^{\mathbb{N}}$, $x \in X$, what is the complexity of $x_n \xrightarrow{uo} x$?
- What are the complexities of properties related to types of convergences of X? For example, the complexity of Fatou/ Weak Fatou properties
- Is $\mathcal{A}t\mathcal{L}$ actually a Borel space, or is it complete co-analytic?

Isomorphically rearrangement invariant lattices

An atomic order continuous lattice E with generating atoms $(e_n)_n$ is **isomorphically rearrangement invariant** (r.i.) if for all permutations $\sigma \in S_\infty$, the map induced by $e_n \mapsto e_{\sigma(n)}$ generates a lattice isomorphism over E.

- Fact: $\sup_{\sigma \in S_\infty} \|e_n \mapsto e_{\sigma(n)}\| < \infty$.
- The class $\mathcal{R}$ of r.i. lattices is Borel: Define the relation $Q_n \subseteq BL \times \mathbb{N}$ by

$$Q_n(E, M) \iff E \in \mathcal{A}t\mathcal{L}(\mathcal{U}) \cap OC \bigwedge_{\substack{\sigma \in S_n \\ q_1, \dots, q_n \in \mathbb{Q}}} \left(\| \sum_m q_m \mathbf{e}_{m,E} \| \leq M \| \sum_m q_m \mathbf{e}_{\sigma(m),E} \| \right).$$

- Then $E \in \mathcal{R} \iff \bigvee_M \bigwedge_n (Q_n(E, M))$.

The lattice isomorphism relation restricted to r.i. lattices is Borel:

$$X \sim Y \iff X, Y \in \mathcal{R} \bigwedge \exists M \ \forall n \in \mathbb{N}, q_1, ..., q_n \in \mathbb{Q}$$
$$\left(\frac{1}{M} \| \sum_m q_m \mathbf{e}_{m,X} \| \leq \| \sum_m q_m \mathbf{e}_{m,Y} \| \leq M \| \sum_m q_m \mathbf{e}_{m,X} \| \right).$$

sums of r.i. lattices

Theorem 10

1. *For all n, the class $\mathcal{R}_n$ of Banach lattices of the form $E = E_1 \oplus ... \oplus E_n$ for "minimal" n with each E_i r.i. is Borel.*
2. *the lattice isomorphism equivalence relation restricted to $\cup_n \mathcal{R}_n$ is Borel.*

Sketch of proof:

- By strong induction: suppose for $m < n$, $\mathcal{R}_m$ is Borel.
- for $k \in \mathbb{N}$, Let $C_1, ..., C_n$ be a "suitable n-partition" of k:
    - for each $i < n$, $\min C_i < \min C_{i+1}$, and empty sets are indexed last.
    - There is an M such that for all $\tau = \sigma_1 ... \sigma_n \in S_{C_1} ... S_{C_n} \subseteq S_\infty$, the map $e_i \mapsto e_{\tau(i)}$ induces a lattice isomorphism over E with distortion at most M.
- Call the above relation $P^M_{n,k}(E)$. This relation can be defined in a Borel way.
- $E \in \mathcal{R}_n$ iff $E \notin \mathcal{R}_m$ for $m < n$ and there is $M \in \mathbb{N}$ such that for all k, $P^M_{n,k}(E)$ holds.

sums of r.i. lattices

Proof continued:

- If $E \in \mathcal{R}_n$, let $D_i = \{i \in \mathbb{N} : e_i \in E_i\}$. Can assume that for all $i < n$, $\min D_i < \min D_{i+1}$. Let M be appropriate maximum distortion constant for all lattice isomorphisms $\sigma_1 ... \sigma_n \in S_{D_1} ... S_{D_n}$.
- E then satisfies $P^M_{n,k}(E)$ for all k, with suitable partition $(D_i \cap \{1, ..., k\})$.
- Suppose $E \notin \mathcal{R}_m$ for $m < n$ and there is $M \in \mathbb{N}$ such that for all k, $P^M_{n,k}(E)$ holds.
- Can create a tree T on $\{1, ..., n\}$ whose branches are of the form $(n_1, ..., n_k)$ with $C_i = \{j : n_j = i\}$, and $C_1, ..., C_n$ is a suitable n-partition over k.
- T has infinitely many branches, so by König's Lemma, it has a branch $(n_i)_i$ of infinite length. Let E_i be the lattice generated by $\{e_j : n_j = i\}$. Each E_i is r.i., and $E = E_1 \oplus ... \oplus E_n$.

Open questions

Isomorphism and Isometry classes of Banach Lattices

- various kinds of isomorphism and isometry classes were shown to Borel. under improved coding techniques, How precisely in the Borel hierarchy can these classes of lattices be determined?
- Examples from Banach spaces:
  - (Cuth, Dolezal, Doucha, Kurka, '22) Isomorphism classes of Banach spaces: Hilbert spaces is Borel (in fact F_σ), l_p is Borel for $p > 1$ (in fact $G_{\delta\sigma}$)
  - Isometry classes of Banach spaces: Hilbert spaces have a closed isometry class, and ℓ_p for $p \geq 1$ and c_0 is $F\sigma\delta$.
  - (Kurka, '19) isomorphism class of c_0 is analytic non-Borel, complexity of ℓ_1 is unknown
  - (Bossard, '02) Isomorphism class L_p is non-Borel for any $p \neq 2$
- Among those that are determinable,how much can we improve upon the corresponding Banach space complexities?

Complete analytic sets

- A subset $A \subseteq X$ with X a standard Borel space is **analytic** if there is standard Borel space Y and a Borel function $f : Y \to X$ such that $f(Y) = A$. In addition, we say that $X \backslash A$ is **co-analytic**.
- Not all analytic sets are Borel!
  - Ex: Let Tr be the set of trees on $\mathbb{N}$. Tr is a Polish space, and the set $IF \subseteq Tr$ of ill-founded trees is analytic and not Borel (similarly, the set WF of well-founded trees is co-analytic and not Borel).
- One way to show some analytic set A is not Borel: find a Borel map $f : Tr \to X$ such that $f^{-1}(A) = IF$.

The Pelczynski lattice $\mathcal{V}$

Theorem 11

There exists an atomic order continuous lattice $\mathcal{V}$ equipped with atoms $(e_n)_n$ such that for any atomic order continuous lattice E with generating atoms $(u_k)_k$ and $M > 1$, there exists a subsequence $(e_{n_k})_k$ such that the map generated by $u_k \mapsto e_{n_k}$ is a lattice isomorphic embedding with distortion at most M. In addition, $\mathcal{V}$ is unique up to lattice isomorphism.

- Can be constructed using approach similar to that of Pelczynski's universal conditional basis
- alternate construction exists which enables $\mathcal{V}$ to have an approximate homogeneity property. If this construction is used, then $\mathcal{V}$ is "almost" isometrically unique: any two Pelczynski lattices with this approximate homogeneity are C-isometric for all $C > 1$.

The Pelczynski lattice $\mathcal{V}$

Theorem 12

For $i \in \{1, 2\}$, there exists a Banach lattice X_i and Borel maps $f_i : Tr \to BL(X_i)$ such that:

1. *If $T \in IF$, then $f_i(T) \sim \mathcal{V}$*
2. *If $T \in WF$, then $f_1(T)$ has the Schur property and $f_2(T)$ is reflexive*

In particular, the lattice isomorphism class $< \mathcal{V} >$ of $\mathcal{V}$ is analytic non-Borel.

- Let $I = \{i_1, ..., i_k\}$ be a finite set of intervals in $\mathcal{S} := \mathbb{N}^{<\mathbb{N}}$. We then say that I is an **admissible set of intervals** if every branch (finite or infinite) in $\mathcal{S}$ intersects at most one $i_j \in I$.
- Given $x = \sum_s x(s)u_s \in c_{00}(\mathcal{S})$, let

$$|||x|||_i = \sup \left\{ \left(\sum_{j=1}^{k} \left\| \sum_{s \in I_j} x(s) e_{|s|} \right\|^i \right)^{1/i} \ : \ \{I_1, ..., I_k\} \text{ admissable set of intervals} \right\}$$

- For a tree $T \subseteq \mathcal{S}$, Define $f_i(T)$ as the $||| \cdot |||_i$-closure of the lattice generated by the unit elements $\{u_s : s \in T\}$

Corollary 13

The following classes of Banach lattices are complete co-analytic:

1. *Reflexive lattices.*
2. *Dual lattices.*
3. *Lattices not containing an arbitrary infinite dimensional order continuous atomic lattice Z.*
4. *KB spaces.*

Applications:

proofs:

- Reflexivity: X reflexive iff it does not contain c_0 or ℓ_1, so reflexivity is co-analytic.
- Duality: (Talagrand) X is a separable dual Banach lattice iff it is *order dentable*: for every closed convex subset C of X_+, if $C \subseteq I_X(e)$ for some $e \in X_+$, then

$$C \neq \bigcap_n \overline{CH}(\{y \in C : \|y \wedge e\| \leq \frac{1}{n}\}),$$

- non-embeddability of infinite dimensional Z is clearly co-analytic: non-Borelness is determined by two cases depending on whether or not Z is reflexive

Corollary 14

If C is a non-Borel co-analytic class of lattices, and C is also isomorphically closed under sublattices, then there is no X in C that is isomorphically universal for all lattices in C. In particular, there is no separable KB lattice that is isomorphically universal for separable KB lattices.

Proof:

- If X were isomorphically universal in $\mathcal{C}$, C is also analytic, since an alternate characterization of C would be isomorphic embeddability into X.
- Being KB is co-analtyic, since it is equivalent to there being no isomorphic copy of c_0 in X.

Complexity of Equivalence relations

- Reducibility of equivalence relations
- The lattice isometry equivalence relation
- The lattice isomorphism equivalence relation
- Open questions

Equivalence relations

Definitions

- Suppose E is an equivalence relation on a standard Borel space X.we say E is **Borel** if E is Borel as a subset of $X \times X$.
- Let (E, X), (F, Y) be equivalence relations on standard Borel X and Y. We say that E is *Borel reducible* to F, and denote it by $(X, E) \leq_B (Y, F)$ (or $E \leq_B, F$), if there exists a Borel map $\phi : X \to Y$ such that for all $x, x' \in X$,

$$x \; E \; x' \iff \phi(x) \; F \; \phi(x')$$

- E is *Borel Bi-reducible* to F if $E \leq_B F$ and $F \leq_B E$.

Examples

- Equality: $xE_{=}x' \iff x = x'$. If $(X, E) \leq (\Delta, E_{=})$, we say that E is *smooth.*
- E_0: $X = \mathbb{R}$, with $x \; E_0 \; x' \iff x - x' \in \mathbb{Q}$.

Equivalence Relations

Examples

- $E_{\mathcal{L}}$: $Y = \prod_k 2^{\mathbb{N}^{n_k}}$ is a space encoding all possible $\mathcal{L}$-structures M with countable relational language $= \mathcal{L}((R_k)_k)$ with R_k a n_k-ary relation. Then $M\ E_{\mathcal{L}} M'$ iff there exists an isomorphism of structures $\phi : M \to M'$ such that for all $(x_1, ..., x_{n_k}), k \in \mathbb{N}$,

$$R_k^M(x_1, ..., x_k) \iff R^{M'}(\phi(x_1), ..., \phi(x_k))$$

  If $(X, E) \leq (Y, E_{\mathcal{L}})$, then we say that E is **classifiable by countable structures**
- relations induced by Polish group actions: E_G: let G be a Polish group acting on X a standard Borel space, with $x\ E_G\ x' \iff$ there is a $g \in G$ such that $g(x) = x'$.
- E_{max}: Analytic equivalence relation that is universal for all analytic equivalence relations on a standard Borel spaces.

Descriptive complexity of relations

Theorem 15

The following relations are analytic:

1. $Emb_{\sim}(X, Y)$: *X lattice isomorphically embeds into Y*
2. $Emb_{\cong}(X, Y)$: *X lattice isometrically embeds into Y*
3. $I_{\sim}(X, Y)$: *X is lattice isomorphic to Y*
4. $I_{\cong}(X, Y)$: *X is lattice isometric to Y*

Idea of proof:

- X isomorphically embeds into Y if there exist $(x_i)_i \subseteq X$ with $(x_i)_i$ dense in X, $(y_i)_i \subseteq Y$, $M \geq 1$, such that for any n and n-ary operation τ generated by linear and and lattice operations, $\frac{1}{M}\|\tau(x_1, ..., x_n)\| \leq \|\tau(y_1, ..., y_n)\| \leq M\|\tau(x_1, ..., x_n)\|$.
- For isometries, fix $M = 1$, and for surjectivity, also require y_i's to be dense in Y.

Theorem 16

The lattice isometry equivalence relation is Borel bi-reducible to E_{U_G}, the universal relation over equivalence relations induced by Polish groups on standard Borel spaces.

sketch of proof:

- First part: Show that $(BL, \cong) \leq_B (\mathbb{U}, \equiv)$, the orbit equivalence relation induced on $\mathbb{U}$ by its group of isometries, so the isometries on BL can be reduced to an equivalence relation induced by a Polish group over a standard Borel space
- Second part: Show that $(BL, \cong) \geq_B (\mathcal{K}(\mathbb{H}), \sim)$, which is itself bi-reducible with E_{U_G}.
    - Use fact that $K \sim L$ iff $C(K) \cong C(L)$, just need a suitable, at least Borel map $K \mapsto C(K)$ within an ambient lattice like $C(\Delta)$.

Part 1: $(BL, \cong) \leq_B (\mathbb{U}, \equiv)$

Theorem 17 (Elliot, et.al. ('13))

If T is a theory of the logic of metric structures in a separable language, then the isometry relation of models of T is Borel-reducible to an orbit equivalence relation of Iso$(\mathbb{U})$.

- the theory of Banach lattices is axiomatizable in the manner above using the countable language $\mathcal{L} := (+, 0, -, \wedge, \vee, \|\cdot\|, \mathbb{Q})$.
- embedding of $\mathcal{U}$ into $\mathbb{U}$ induces Borel map of BL into the space of Polish $\mathcal{L}$-structures $\mathcal{M}(\mathcal{L})$, where

$$\mathcal{M}(\mathcal{L}) := \{(X, (F_n)_n : X \in \mathcal{F}(\mathbb{U}) \text{ and } F_n \in X^{l_n}\}$$

- The above induce an equivalence relation $\approx^{\mathcal{M}(\mathcal{L})}$ on $\mathcal{M}(\mathcal{L})$, with $\mathcal{X} \approx^{\mathcal{M}(\mathcal{L})} \mathcal{Y}$ iff they are isomorphic as Polish $\mathcal{L}$-structures.
- Clearly $\cong \leq_B \approx^{\mathcal{M}(\mathcal{L})}$
- Any $\sigma \in ISO(\mathbb{U})$ can be extended to $\mathcal{M}(\mathcal{L})$.
- Can be shown that $\approx^{\mathcal{M}(\mathcal{L})} \leq_B \equiv$

other reducibilities

- Let (R_1, R_2) and (S_1, S_2) be two pairs of binary relations on standard Borel spaces X and Y respectively. A Borel map $f : X \to Y$ is a **Borel homomorphism from** (R_1, R_2) **to** (S_1, S_2) if for all $x, y \in X$, $xR_1y \to f(x)S_1f(y)$, and $xR_2y \to f(x)S_2f(y)$.
- We say that (R_1, R_2) is **Borel hom-reducible** to (S_1, S_2), and write $(R_1, R_2) \preccurlyeq_B (S_1, S_2)$.
- Note that given equivalence relations R and S,

$$R \leq_B S \iff (R, \neg R) \preccurlyeq_B (S, \neg S)$$

- A pair (R, S) is **analytically complete** if R and S are analytic relations which hom-reduce any other pair of analytic relations

Analytically complete relations

Example:

- Let $\mathcal{T}$ be the set of pruned, *normal* trees on $2 \times \mathbb{N}$.
  - $T \in \mathcal{T}$ is normal iff $(u, s) \in T$ and $t \geq s$ implies (u, t) in T as well.
- For $S, T \in \mathcal{T}$, we let:

$$S \leq_{\Sigma^1_1} T \iff \exists \alpha \in \mathcal{N}\ \forall (u, s)((u, s) \in S \to (u, s + \alpha_{|s|}) \in T)$$

$$S \equiv_{\Sigma^1_1} T \iff S \leq_{\Sigma^1_1} T \bigwedge T \leq_{\Sigma^1_1} S$$

$$S \neq_{\Sigma^1_1} T \iff A(S) \neq A(T),$$

  where

$$A(T) = \{\alpha \in \Delta : \exists \beta \in \mathcal{N}\ \big((\alpha, \beta) \in [T]\big)\ \},$$

- The pair $(\equiv_{\Sigma^1_1}, \neq_{\Sigma^1_1})$ is analytically complete.

Analytically complete relations

Theorem 18

The lattice isomorphism equivalence relation is universal for analytic equivalence relations on Standard Borel Spaces

Sketch of Proof

- Start with a **Cantor Scheme** $(I_u)_{u \in 2^{<\omega}}$ of closed, mutually disjoint sub-intervals of $(1,2)$ such that the following conditions hold:
  1. $I_{u\frown 0} \cup I_{u\frown 1} \subseteq I_u$
  2. $\max I_{u\frown 0} < \min I_{u\frown 1}$
  3. $\min I_u = \min I_{u\frown 0}$ and $\max I_u = \min I_{u\frown 1}$,
  4. For all $u \neq \emptyset$, the standard unit bases for $\ell^{|u|}_{\min I_u}$ and $\ell^{|u|}_{\max I_u}$ are 2-equivalent.
- Let $\mathbf{s} \subseteq \mathbf{T} := (2 \times \mathbb{N})^{<\mathbb{N}}$ denote finite segments of $\mathbf{T}$, that is sets of the form $\mathbf{s} = \{t \in \mathbf{T} : t_0 \subseteq t \subseteq t_1\}$, and for a finitely supported vector $x = \sum_{t \in \mathbf{T}} \lambda_t e_t$ and $\mathbf{s} = ((u_0, s_0), ..., (u_n, s_m))$, let

$$\| \sum_{t \in \mathbf{T}} \lambda_t e_t \|_{\mathbf{s}} = \sup_{m \leq n} \| \sum_{i=0}^{m} \lambda_{(u_i, s_i)} v \delta_i \|_{\min I_{u_m}},$$

- let

$$\Big\|\sum_{t\in\mathbf{T}}\lambda_t e_t\Big\| = \sup\left\{\left(\sum_{i=1}^{n}\Big\|\sum_{t\in\mathbf{s}_i}\lambda_t e_t\Big\|_{\mathbf{s}_i}^2\right)^2 : (\mathbf{s}_i)_{i=1}^n \text{ is an admissable set of segments}\right\},$$

- let $\mathbf{T}_2$ be the closure of $c_{00}(\mathbf{T})$ under $\|\cdot\|$.

Theorem 19

Let S, T be subtrees of $\mathbf{T}$ and let $\phi : S \to T$ be an isomorphism of trees such that for all $(u,s)\in\mathbf{T}$, $\phi(u,s) = (u,s')$ with $(u,s')\in T$. Let also Z_T and Z_S be the sublattices of $\mathbf{T}_2$ generated by the atoms e_t with $t\in T$ and S respectively. Then the map

$$M_\phi : e_{(u,s)} \mapsto e_{\phi(u,s)}$$

induces a lattice isometry between Z_T and Z_S.

- Finally, isomorphically embed $\mathbf{T}_2$ as an ideal in $\mathcal{V}$. The proof is completed by showing that $\sim_{\mathcal{V}}$ hom-reduces $(\equiv_{\Sigma_1^1}, \neq_{\Sigma_1^1})$

Open questions

- Examine the complexity of the isomorphism and isometry relations between sublattices of a given lattice:
- so far examined cases:
  - If $X = \mathcal{V}$, then $\sim_{\mathcal{V}} \simeq_B E_{max}$
  - If $X = L_p$ for $p \geq 1$, then $\sim_{L_p} \simeq_B \mathbb{N}$.
  - If $X = C(\Delta)$, then $\cong_{L_p} \simeq_B E_{\mathcal{U}_G}$
- Some places to look: isomorphism and isometry equivalence relations in $C(K)$ spaces in light of the underlying homeomorphism equivalence relations in K

Lattice positions

Set up in Banach spaces

- Suppose X, Y are Banach spaces, and let $Emb(X, Y) \subseteq B(X, Y)$ be the collection of infinite co-dimensional linear isomorphic (or isometric?) embeddings. It can be shown that this collection is a Borel subset of a standard Borel space.
- Can deduce equivalence relation for "positionings" of X in Y with $X \sim_Y X'$ iff $\exists \sigma \in ISO(X)$ such that $\sigma(X) = X'$.

Theorem 20 (Anisca, Ferenczi, Moreno, '17)

- *Suppose X is a Banach space that is not uniformly finitely extensible (UFO). Then there exists Y such that $E_0 \leq_B \sim_X$.*
- *The equivalence relation of positions of ℓ_p in ℓ_p for $p \neq 2$ reduces E_1. Hence it is not reducible to $E_{G_{\mathcal{U}}}$.*

Lattice positions

positionings for complimented subspaces

- can examine specifically space $Emb_c(X, Y)$ of complimented positionings of X inside Y: this is also a standard Borel space if it is interpreted as a subspace of $Emb(X, Y) \times P(Y)$, with

$$(T, P) \in Emb_c(X, Y) \iff TX = PY$$

- Provides a lower bound on complexity over $Emb(X, Y)$.

Theorem 21 (Anisca, Ferenczi, Moreno, '17)

The complixity of the relation of complimented positionings for the Pelczynski space $\mathcal{U}$ within itself is E_{max}

questions

- Is there any Banach space X for which for any Y, $\sim_X$ is smooth?
- Investigate the complexity of isometric positionings of X in Y for various cases.

Lattice positions (cntd)

Corresponding setup in Banach lattices

- X, Y Banach lattices,the space $Emb_L(X, Y)$ of Banach lattice embeddings is also standard Borel inducing corresponding "lattice positioning".
- On one hand, possible positionings are more restrictive, and on the other hand, possible automorphisms are also more restricted.

Additional Questions:

- Determine lower and upper bounds for non-injective Banach lattices vs injective lattices.
- Is there any relationship between the relation of linear positionings and lattice positionings?
- Investigate the lattice isometric version of positionings.

Lattice positions (cntd)

Ideal or Band positionings

- Corresponding to the question of complimented positionings in Banach spaces, we can also investigate the complexity of positionings $Emb_L^I(X, Y)$ of a lattice X in Y when X is a lattice ideal in Y:
- Ideal positionings also forms a standard Borel space:

$$T \in Emb_L^I(X, Y) \iff T \in Emb_L(X, Y) \wedge TX \in Id(Y)$$

- Initial observation: if $Y = L_p$, then there are only two positionings for any ideal in L_p
- Ideal positionings of X in Y when Y is a $C(K)$ space: X is ideal in $C(K)$ iff for some closed $F \subseteq K$ $X = \{f \in C(K) : f(F) = 0\}$.

Lattice isometry groups

Positioning for one dimension

- Let X be a separable Banach Lattice,and consider the group $ISO_L(X)$ of lattice isometries over X, equipped with the SOT. $ISO(X)$ is a Polish group.
- Can look specifically at the orbit equivalence relations on $S(X)_+$ induced by $ISO_L(X)$
- Observe for L_p lattices, there are exactly two orbits, one for elements with full support (which is co-meager), and one for elements without full support (which is meager).

questions

- Given a lattice X, what is the complexity of the orbit equivalence relation induced by $ISO_L(X)$ on $S(X)_+$?
- Under what conditions are there generic orbits induced by $ISO_L(X)$?
- Motivation for the above: $ISO(\mathfrak{G})$ induces a generic orbit over smooth points in the unit ball.

Lattice isometry groups

Universal Polish groups:

- (Uspenskij, '86) The hoemeomorphism group on the Hilbert cube is a universal Polish group , meaning that every Polish subgroup is homeomorphic to a closed subgroup of $\mathbb{U}$.
- (Uspenskij, '90) The isometry group of the Urysohn space $\mathbb{U}$ is a universal Polish group.
- (Ben Yaacov, '14) The linear isometry group of the Gurarij space $\mathfrak{G}$ is also a universal Polish group.

question:

- Is the lattice isometry group of the Gurarij lattice a universal Polish group?

Lattice isometry groups

Conjugacy equivalence relations

- Let G be a Polish group. G induces an equivalence relation by conjugacy, where $g \sim g'$ iff there is an $h \in G$ such that $g = hg'h^{-1}$.
- The conjugacy equivalence relation also admits certain complexities. Relevant example on groups of homeomorphisms $\mathcal{H}(K)$ over compact metric K:
  - $(\mathcal{H}(K), \sim)$ is classifiable by countable structures when K is 0-dimensional, or $K = [0, 1]$
  - $(\mathcal{H}(K), \sim)$ is not classifiable by countable structures when $K = [0, 1]^2$ (Hjorth '00) or when $K=$ the Sierpinski carpet ((Kulshreshtha, Panagiatopoulos, '24).
- The group of invertible measure preserving transformations is not classifiable by countable structures (Hjorth, '01)

Question

- Given a lattice X, what is the complexity of the conjugacy class of its lattice isometry group?

THANK YOU FOR WATCHING!