Compact inclusions between variable Hölder spaces

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WORKSHOP BANACH SPACES AND BANACH LATTICES
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Introduction. Hölder spaces

Definition

Let $0 \leq \alpha \leq 1$. *The Hölder space* $C^\alpha(\Omega)$ *is the Banach space*

$$C^\alpha(\Omega) := \{f \in C(\Omega) : \rho_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty\},$$

endowed with the norm

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_\infty + \rho_\alpha(f).$$

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- If $\alpha = 1$, then $C^\alpha(\Omega) = Lip(\Omega)$ is the space of Lipschitz functions.
- If $\alpha = 0$, then $C^\alpha(\Omega) = C(\Omega)$ with an equivalent norm.

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Proposition

The inclusion $i : C^\alpha(\Omega) \hookrightarrow C^\beta(\Omega)$ *holds for* $\alpha \geq \beta$.

Inclusions between Hölder spaces

Theorem (Ascoli-Arzelà)

(f_n) *has an uniformly convergent subsequence* $\Leftrightarrow$ *both*

- (f_n) *is uniformly bounded, i.e.* $\|f_n\|_\infty \leq M$ *for all* n.
- (f_n) *is uniformly equicontinuous, i.e.* $|f_n(x) - f_n(y)| \leq \varepsilon$ *for* $d(x, y) \leq \delta$.

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The inclusion $i : C^\alpha(\Omega) \hookrightarrow C^\beta(\Omega)$ *is compact when* $\alpha > \beta$.

Inclusions between Hölder spaces

Theorem (Ascoli-Arzelà)

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- *(f_n) is uniformly bounded, i.e. $\|f_n\|_\infty \le M$ for all n.*
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Proposition

The inclusion $i : C^\alpha(\Omega) \hookrightarrow C^\beta(\Omega)$ is compact when $\alpha > \beta$.

Proof: Take $(u_n) \subset C^\alpha(\Omega)$ bounded.

$$\sup_{x,n} |u_n(x)| \le \sup_n \|u_n\|_\infty \le M < \infty.$$

Given $\varepsilon > 0$, for $\delta = \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}} > 0$, $n \in \mathbb{N}$ and $x, y \in \Omega$ with $0 < d(x,y) \le \delta$, we have

$$|u_n(x) - u_n(y)| \le M\, d(x,y)^\alpha \le \varepsilon.$$

Proof

Thus, by Ascoli-Arzelà, there exists a subsequence $u_{n_k} \to u$ uniformly, i.e. $\|u_{n_k} - u\|_\infty \to 0$. Also, for every $x \neq y$,

$$M \geq \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\alpha} \xrightarrow{n\to\infty} \frac{|u(x) - u(y)|}{d(x,y)^\alpha},$$

so $u \in C^\alpha(\Omega)$.

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so $u \in C^\alpha(\Omega)$. To prove that $\|u_{n_k} - u\|_\beta \to 0$, we just calculate (suppose $u \equiv 0$ or denote $u' = u_k - u$)

$$\begin{aligned}\rho_\beta(u_{n_k}) &= \sup_{x\neq y} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\beta} \\ &= \sup_{x\neq y} \left(\frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\alpha}\right)^{\frac{\beta}{\alpha}} |u_{n_k}(x) - u_{n_k}(y)|^{1-\frac{\beta}{\alpha}} \\ &\leq \rho_\alpha(u_{n_k})^{\frac{\beta}{\alpha}} (2\|u_{n_k}\|_\infty)^{1-\frac{\beta}{\alpha}} \to 0.\end{aligned}$$

Hence, the inclusion $C^\alpha(\Omega) \hookrightarrow C^\beta(\Omega)$ is compact. □

Variable Hölder spaces

We introduce now a generalization of Hölder spaces taking instead a function $\alpha : \Omega \to [0, 1]$.

Definition

Let $\alpha : \Omega \to [0, 1]$. *The variable order Hölder space is the space*

$$C^{\alpha(\cdot)}(\Omega) := \{f \in C(\Omega) : \rho_{\alpha(\cdot)}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^{\alpha(x)}} < \infty\},$$

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$$\|f\|_{C^{\alpha(\cdot)}(\Omega)} = \|f\|_\infty + \rho_{\alpha(\cdot)}(f).$$

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endowed with the norm

$$\|f\|_{C^{\alpha(\cdot)}(\Omega)} = \|f\|_\infty + \rho_{\alpha(\cdot)}(f).$$

- If $\alpha(\cdot) \equiv \alpha$, we have the classical Hölder spaces $C^\alpha(\Omega)$.
- If $\alpha(x) \equiv 0$, then $C^{\alpha(\cdot)}(\Omega) = C(\Omega)$ with an equivalent norm.
- If $\alpha(x) \equiv 1$, then $C^{\alpha(\cdot)}(\Omega) = Lip(\Omega)$.
- We can use $\alpha_{\max} = \max\{\alpha(x), \alpha(y)\}$ or $\alpha_+ = \frac{\alpha(x)+\alpha(y)}{2}$ instead of $\alpha(x)$ as the exponent of $d(x, y)$.

When is the inclusion compact?

Proposition

If $\alpha(\cdot) \geq \beta(\cdot)$*, then the inclusion* $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ *holds.*

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If $\alpha(\cdot) \geq \beta(\cdot)$, then the inclusion $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ holds.

Slightly different conditions for $\alpha(\cdot)$ and $\beta(\cdot)$ might be needed for the inclusion $C^{\alpha_{\max}(\cdot)}(\Omega) \hookrightarrow C^{\beta_{\max}(\cdot)}(\Omega)$ and $C^{\alpha_+(\cdot)}(\Omega) \hookrightarrow C^{\beta_+(\cdot)}(\Omega)$ be bounded.

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Proposition

If $\inf_{x,y\in\Omega}(\phi(x,y) - \psi(x,y)) \geq \varepsilon > 0$, then the inclusion $C^{\phi}(\Omega) \hookrightarrow C^{\psi}(\Omega)$ is compact.

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If $\inf_{x,y\in\Omega}(\phi(x,y) - \psi(x,y)) \geq \varepsilon > 0$*, then the inclusion* $C^{\phi}(\Omega) \hookrightarrow C^{\psi}(\Omega)$ *is compact.*

That conditions is, unfortunately, sufficient but not necessary.

The previous reasoning does not work

Thus, by Ascoli-Arzelà, there exists a subsequence $u_{n_k} \to u$ uniformly, i.e. $\|u_{n_k} - u\|_\infty \to 0$. Also, for every $x \neq y$,

$$M \geq \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\alpha} \xrightarrow{n\to\infty} \frac{|u(x) - u(y)|}{d(x,y)^\alpha},$$

so $u \in C^\alpha(\Omega)$. To prove that $\|u_{n_k} - u\|_\beta \to 0$, we just calculate (suppose $u \equiv 0$)

$$\begin{aligned}\rho_\beta(u_{n_k}) &= \sup_{x\neq y} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\beta} \\ &= \sup_{x\neq y} \left(\frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^\alpha} \right)^{\frac{\beta}{\alpha}} |u_{n_k}(x) - u_{n_k}(y)|^{1-\frac{\beta}{\alpha}} \\ &\leq \rho_\alpha(u_{n_k})^{\frac{\beta}{\alpha}} (2\|u_{n_k}\|_\infty)^{1-\frac{\beta}{\alpha}} \to 0.\end{aligned}$$

Hence, the inclusion $C^\alpha(\Omega) \hookrightarrow C^\beta(\Omega)$ is compact. □

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$$M \geq \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\alpha(x)}} \xrightarrow{n\to\infty} \frac{|u(x) - u(y)|}{d(x,y)^{\alpha(x)}},$$

so $u \in C^{\alpha(\cdot)}(\Omega)$. To prove that $\|u_{n_k} - u\|_{\beta(\cdot)} \to 0$, we just calculate (suppose $u \equiv 0$)

$$\begin{aligned}
\rho_{\beta(\cdot)}(u_{n_k}) &= \sup_{x\neq y} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\beta(x)}} \\
&= \sup_{x\neq y} \left(\frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\alpha(x)}} \right)^{\frac{\beta(x)}{\alpha(x)}} |u_{n_k}(x) - u_{n_k}(y)|^{1-\frac{\beta(x)}{\alpha(x)}} \\
&\leq \rho_\alpha(u_{n_k})^{c_{\alpha(\cdot),\beta(\cdot)}} (2\|u_{n_k}\|_\infty)^{1-\frac{\beta(x)}{\alpha(x)}} \overset{?}{\to} 0.
\end{aligned}$$

Hence, is the inclusion $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ compact?

A sufficient condition

Theorem

Let (Ω, d) be a metric space and $\alpha, \beta : \Omega \to [0,1]$ such that $\alpha(\cdot) \geq \beta(\cdot)$. If Ω is totally bounded and

$$\lim_{\delta \to 0} \sup_{0<d(x,y)\leq\delta} d(x,y)^{\alpha(x)-\beta(x)} = 0, \qquad (\diamond)$$

then the inclusion $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ is compact.

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Proposition

Let (Ω, d) be a metric space. If there exist order functions $\alpha, \beta : \Omega \to [0,1]$ such that the inclusion $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ is compact, then Ω is totally bounded.

The proof now

Given $\varepsilon > 0$, take $\delta > 0$ s.t. $\sup_{0<d(x,y)<\delta} d(x,y)^{\alpha(x)-\beta(x)} \leq \frac{\varepsilon}{M}$ (and suppose that $u_{n_k} \xrightarrow{\|\cdot\|_\infty} u \equiv 0$):

$$\rho_\psi(u_{n_k}) = \max\left\{ \sup_{0<d(x,y)<\delta} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\beta(x)}}, \sup_{d(x,y)>\delta} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\beta(x)}} \right\}$$

$$\text{(for large } k) = \sup_{0<d(x,y)<\delta} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\beta(x)}}$$

$$= \sup_{0<d(x,y)<\delta} \frac{|u_{n_k}(x) - u_{n_k}(y)|}{d(x,y)^{\beta(x)}} \cdot d(x,y)^{\alpha(x)-\beta(x)}$$

$$\leq \rho_\psi(u_{n_k}) \cdot \frac{\varepsilon}{C} \leq \varepsilon.$$

Hence, the inclusion $C^{\alpha(\cdot)}(\Omega) \hookrightarrow C^{\beta(\cdot)}(\Omega)$ is compact. □

Variable Hölder spaces (again)

We introduce now a different definition of variable Hölder spaces taking instead a two-variable function $\phi : \Omega \times \Omega \to [0,1]$.

Definition

Let $\phi : \Omega \times \Omega \to [0,1]$. The variable order Hölder space is the space

$$C^{\phi}(\Omega) := \{f \in C(\Omega) : \rho_{\phi}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^{\phi(x,y)}} < \infty\},$$

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The previous definitions are particular cases given $\alpha : \Omega \to [0,1]$:

- If $\phi(x,y) = \alpha(x)$, then $C^{\phi}(\Omega) = C^{\alpha(\cdot)}(\Omega)$.
- If $\phi(x,y) = \max\{\alpha(x), \alpha(y)\}$, then $C^{\phi}(\Omega) = C^{\alpha_{\max}(\cdot)}(\Omega)$.
- If $\phi(x,y) = \frac{\alpha(x)+\alpha(y)}{2}$, then $C^{\phi}(\Omega) = C^{\alpha_{+}(\cdot)}(\Omega)$.

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Also if $\phi(x, y) \geq \psi(x, y)$ for $d(x, y) \leq \delta$ (for some $\delta > 0$).

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Also if $\phi(x, y) \geq \psi(x, y)$ for $d(x, y) \leq \delta$ (for some $\delta > 0$).

Theorem

Let (Ω, d) be a metric space and $\phi, \psi : \Omega \times \Omega \to [0, 1]$ such that $\phi \geq \psi$. If Ω is totally bounded and

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) \leq \delta} d(x, y)^{\phi(x,y) - \psi(x,y)} = 0, \qquad (\diamond)$$

then the inclusion $C^{\phi}(\Omega) \hookrightarrow C^{\psi}(\Omega)$ is compact.

Is our condition necessary?

Definition

A function $\phi : \Omega \times \Omega \to \mathbb{R}$ *is log-Hölder continuous if there exists a constant* $C_{\log} > 0$ *such that, for all* $(x, y) \neq (x', y') \in \Omega \times \Omega$,

$$|\phi(x,y) - \phi(x',y')| \leq \frac{C_{\log}}{\log\left(e + \frac{1}{\max\{d(x,x'),d(y,y')\}}\right)}.$$

Theorem

Let (Ω, d) *be a metric space and* $\phi, \psi : \Omega \times \Omega \to [0,1]$ *such that* $\psi \leq \phi$. *If* ϕ *is log-Hölder continuous with* $\phi^- > 0$, *then the inclusion* $C^\phi(\Omega) \hookrightarrow C^\psi(\Omega)$ *is compact if and only if* Ω *is totally bounded and*

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) \leq \delta} d(x,y)^{\phi(x,y) - \psi(x,y)} = 0. \qquad (\diamond)$$

Sketch of the proof

(By contraposition) Take (x_n), (y_n) in Ω with $0 < d(x_n, y_n) \leq \delta_n$ and

$$d(x_n, y_n)^{\phi(x_n,y_n)-\psi(x_n,y_n)} \geq \varepsilon.$$

Define the sequence

$$u_n(t) = \begin{cases} d(t, y_n)^{\phi_n} = d(t, y_n)^{\phi(x_n,y_n)} & \text{if } t \in B_n, \\ d(x_n, y_n)^{\phi_n} = d(x_n, y_n)^{\phi(x_n,y_n)} & \text{if } t \in \Omega \setminus B_n. \end{cases}$$

(u_n) is bounded sequence in $C^{\phi}(\Omega)$ and $(u_n) \to 0$ in $C^{\psi}(\Omega)$.

- $$\frac{|u_n(x) - u_n(y)|}{d(x,y)^{\phi(x,y)}} = \frac{|d(x,y_n)^{\phi_n} - d(y,y_n)^{\phi_n}|}{d(x,y)^{\phi(x,y)}} \leq \frac{|d(x,y_n) - d(y,y_n)|^{\phi_n}}{d(x,y)^{\phi(x,y)}}$$
  $$\leq \frac{d(x,y)^{\phi_n}}{d(x,y)^{\phi(x,y)}} \leq \text{(by log-Hölder continuity) } C_{\log}.$$
- $$\rho_{\psi}(u_n) = \sup_{x \neq y} \frac{|u_n(x) - u_n(y)|}{d(x,y)^{\psi(x,y)}} \geq \frac{|d(x_n,y_n)^{\phi_n} - d(y_n,y_n)^{\phi_n}|}{d(x_n,y_n)^{\psi(x_n,y_n)}}$$
  $$= d(x_n,y_n)^{\phi(x_n,y_n)-\psi(x_n,y_n)} \geq \varepsilon.$$

How does it matter to be log-Hölder continuous?

Proposition

Let (Ω, d) *be a metric space,* $\phi : \Omega \times \Omega \to [0, 1]$ *and* $\alpha : \Omega \to [0, 1]$ *defined by* $\alpha(x) = \phi(x, x)$*. If* ϕ *is log-Hölder continuous, then*

$$C^{\phi}(\Omega) = C^{\alpha(\cdot)}(\Omega) = C^{\alpha_{\max}(\cdot)}(\Omega) = C^{\alpha_{+}(\cdot)}(\Omega).$$

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This means that, for log-Hölder continuous order functions, one definition for variable Hölder spaces was enough.

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Let (Ω, d) be a metric space, $\phi : \Omega \times \Omega \to [0,1]$ and $\alpha : \Omega \to [0,1]$ defined by $\alpha(x) = \phi(x,x)$. If ϕ is log-Hölder continuous, then

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This means that, for log-Hölder continuous order functions, one definition for variable Hölder spaces was enough. So, has this talk been continuously redundant? No, let us look at the theorem again:

Theorem

Let (Ω, d) be a metric space and $\phi, \psi : \Omega \times \Omega \to [0,1]$ such that $\psi \le \phi$. If ϕ is log-Hölder continuous with $\phi^- > 0$, then the inclusion $C^{\phi}(\Omega) \hookrightarrow C^{\psi}(\Omega)$ is compact if and only if Ω is totally bounded and

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) \le \delta} d(x,y)^{\phi(x,y) - \psi(x,y)} = 0. \qquad (\diamond)$$

Other results

(X, d) is *uniformly perfect* if there exists a constant $\lambda \in (0, 1)$ s.t., for each $x \in X$ and each $r > 0$, one has $B_x(r) \setminus B_x(\lambda r) \neq \emptyset$ whenever $X \setminus B_x(r) \neq \emptyset$.

Proposition

Let (X, d) be an uniformly perfect metric space and $\alpha, \beta : X \to [0, 1]$ such that $\beta \le \alpha$, α is log-Hölder continuous and $\alpha^- > 0$. Then, $C^{\alpha(\cdot)}(X) \hookrightarrow C^{\beta(\cdot)}(X)$ is compact $\Leftrightarrow$ X is totally bounded and $\inf(\alpha - \beta)(\cdot) > 0$.

Theorem

Let (X, d) be a compact metric space without isolated points and ϕ such that the function $(x, y) \mapsto d(x, y)^{\phi(x,y)}$ is continuous in $X \times X$. Then, $C^{\phi}(X) \hookrightarrow C(X)$ is compact if and only if

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) \le \delta} d(x, y)^{\phi(x,y)} = 0. \qquad (\diamond)$$

Conjecture

We could get rid of conditions ϕ be log-Hölder and ϕ^- if we just prove theoretically that there exists some sequence of functions (v_n) satisfying three simple properties:

(1) $v_n(x_n) = d(x_n, y_n)^{\phi_{\alpha(\cdot)}(x_n,y_n)}$ and $v_n(y_n) = 0$,

(2) (v_n) is bounded in $C^{\phi_{\alpha(\cdot)}}(\Omega)$, and

(3) $\|v_n\|_\infty \to 0$.

To back up this conjecture, we give an example where the order functions are not continuous, the condition $(\diamond)$ is not satisfied and the inclusion is not compact.

Exemple

Let (X, d) be a totally bounded metric space and $0 < \beta < \alpha < \gamma \leq 1$. Let $x_0 \in X$ be a non-isolated point and $\alpha, \beta : [0, 1] \to [0, 1]$ such that $\alpha(x_0) = \beta(x_0) := \gamma$ and $\alpha(x) := \alpha$, $\beta(x) := \beta$ for $x \neq x_0$. Then, the inclusion $C^{\alpha(\cdot)}(X) \hookrightarrow C^{\beta(\cdot)}(X)$ is not compact.

Proof.

Define the sequence (u_n)

$$u_n(x) = \begin{cases} d(x, x_0)^\gamma & \text{if } x \in B_{x_0}(1/n) \\ 1/n^\gamma & \text{if } x \in X \setminus B_{x_0}(1/n). \end{cases}$$

(u_n) is bounded in $C^{\alpha(\cdot)}(X)$ and $\|u_n\|_{C(X)} \xrightarrow{n\to\infty} 0$:

$$\rho_{\alpha(\cdot)}(u_n) = \sup_{x\neq y} \frac{|u_n(x) - u_n(y)|}{d(x,y)^{\alpha(x)}} \leq \sup_{x\neq y} \frac{d(x,y)^\gamma}{d(x,y)^{\alpha(x)}} \leq \max(1, \operatorname{diam}(X))^{\gamma-\alpha},$$

and

$$\|u_n\|_{C(X)} \leq 1/n^\gamma,$$

Also, no subsequence of (u_n) converges to 0 in $C^{\beta(\cdot)}(X)$. Take $y \in B_{x_0}(1/n) \setminus \{x_0\}$, we have

$$\rho_{\beta(\cdot)}(u_n) = \sup_{x\neq y} \frac{|u_n(x) - u_n(y)|}{d(x,y)^{\beta(x)}} \geq \frac{d(x_0,y)^\gamma}{d(x_0,y)^\gamma} = 1.$$

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THANK YOU VERY MUCH