Countable unions of operator ranges and spaceability

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Index

Index

Definition (Operator range)

A linear subspace $R \subset E$ is an operator range if there exist a Banach space F and an operator $T : F \to E$ such that

$$R = T(F).$$

Theorem (Rosenthal, 1969; Saxon and Wilansky, 1979)

Let E be a Banach space. Then TFAE:

1. There exists $X \subset E$ closed such that E/X is separable.
2. There exists a proper dense operator range in E.
3. There exists a strictly increasing chain of closed subspaces $\{X_m\}_m$ in E such that $\bigcup_m X_m$ is dense in E.
4. There exists a pair of proper **quasicomplements** $Y_1, Y_2 \subset E$; i. e.,

$$Y_1 \cap Y_2 = \{0\}, \quad \overline{Y_1 + Y_2} = E \quad \text{and} \quad Y_1 + Y_2 \neq E.$$

Theorem (Plichko, 1981; Drewnowski, 1984)

If E is a Banach space, then for every infinite-codimensional operator range R in E, the set $(E \setminus R) \cup \{0\}$ is **spaceable**, i. e., there exists $X \subset E$ closed with $\dim X = \infty$ s. t.

$$R \cap X = \{0\}.$$

Theorem (Kitson and Timoney, 2011)

Let E be a Banach space. Let R_m be operator ranges and set

$$R = \operatorname{span}\left(\bigcup_m R_m\right).$$

If R is not closed in E, then

$$(E \setminus R) \cup \{0\} \quad \text{is spaceable.}$$

Notation

$$\mathcal{R}(E) = \{R \subset E :\ R \text{ operator range with } \operatorname{codim}_E R = \infty\}$$

$$\mathcal{R}_d(E) = \{R \subset E :\ R \ \text{ proper dense operator range}\}$$

$$\mathcal{S}(E) = \left\{\bigcup_m R_m \subset E :\ \{R_m\}_m \subset \mathcal{R}(E)\right\}$$

Index

Theorem (Bennett and Kalton, 1973)

Let $\{X_m\}_m$ be a strictly increasing chain of closed subspaces in E such that

$$\overline{\bigcup_m X_m} = E.$$

Then there exists $R \in \mathcal{R}_d(E)$ such that

$$\bigcup_m X_m \subset R.$$

Theorem

Let $\{R_m\}_{m\geq 1} \subset \mathcal{R}(E)$ be a strictly increasing sequence such that

$$\overline{\bigcup_m R_m} = E.$$

Then there exists $R \in \mathcal{R}_d(E)$ such that

$$\bigcup_m R_m \subset R.$$

Index

Theorem

If $S \in \mathcal{S}(E)$, then there is $X \subset E$ closed such that

$$S \cap X = \{0\}.$$

Lemma (Jiménez Sevilla and Lajara, 2023)

Let $S \in \mathcal{S}(E)$ and $\{x_n\}_n \subset E$ minimal. Then, there exist an isomorphism $\varphi : E \to E$ and $\{u_n\}_n \subset B_E$ minimal such that

1. $\varphi(x_n)$ and u_n are collinear and thus
$$\varphi([\{x_n\}_n]) = [\{u_n\}_n].$$
2. $\{u_n\}_n$ satisfies **property** $(*)$ with respect to S: if $\{\alpha_n\}_n \in \ell_1$ satisfies that $\sum_n \alpha_n u_n \in S$, then
$$\alpha_n = 0 \text{ for all } n \geq 1.$$

Lemma (Generalization of a result by Drewnowski)

Let $\{w_j\}_j \subset E$ be a sequence of linearly independent elements and let $\{C_n\}_n$ be a sequence of bounded closed convex subsets such that

$$C_n \cap \operatorname{span}\{w_j\}_j = \emptyset \quad \text{for each} \quad n \geq 1.$$

Then there is a block sequence $\{z_j\}_j \subset E \setminus \{0\}$ of $\{w_j\}_j$ such that

$$C_n \cap [\{z_j\}_j] = \emptyset.$$

A dual counterpart

Theorem

Let E be a Banach space which has a quotient with separable dual, then for every $S \in \mathcal{S}(E^*)$ there exists $Z \subset E^*$ w^*-closed such that

$$S \cap Z = \{0\}.$$

Theorem (Johnson and Rosenthal, 1972

If E^* is separable, then there exists $\{f_n\}_n \subset E^*$ w^*-basic such that $[\{f_n\}_n]$ is w^*-closed.

Index

Theorem (Cross and Shevchik, 1998)

Let E be a separable Banach space and let $R \in \mathcal{R}(E)$. Then there exists a pair of proper quasicomplements $X, Y \subset E$ such that

$$R \cap (X + Y) = \{0\}.$$

Definition (Nuclear operator)

A bounded operator $T : E \to F$ is **nuclear** if there exist $\{x_n\}_{n\geq 1} \subset E$ and $\{f_n\}_{n\geq 1} \subset E^*$ such that $\sum_n \|f_n\|\|x_n\| < \infty$ and

$$T(u) = \sum_n f_n(u)x_n, \quad u \in E.$$

Definition (Nuclearly adjacent quasicomplements)

Let X, Y be two quasicomplements in E. Y is **nuclearly adjacent** to X if for the quotient map $Q_X : E \to E/X$, the restriction $Q_X|_Y$ is a nuclear map.

X and Y are **mutually** nuclearly adjacent if X is nuclearly adjacent to Y and vice versa.

Theorem

Let E be a separable Banach space and let $S \in \mathcal{S}(E)$. Then, for every $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ such that $\|\varphi - I_E\| < \varepsilon$, and a closed subspace $X \subset E$ such that

(1) $\varphi(X)$ and X are mutually nuclearly adjacent quasicomplements.

(2) $(\varphi(X) + X) \cap S = \{0\}$.

Definition (M-basis)

A biorthogonal system $\{x_n, f_n\}_n \subset E$ is a Markushevich basis (M-basis) if

$$[\{x_n\}_n] = E \quad \text{and} \quad \overline{[\{f_n\}]}^{w^*} = E^*.$$

Lemma (Jiménez Sevilla and Lajara, 2023)

Let $S \in \mathcal{S}(E)$ and $\{x_n\}_n \subset E$ minimal. Then, there exist an isomorphism $\varphi : E \to E$ and $\{u_n\}_n \subset B_E$ minimal such that

1. $\|\varphi - I_E\| < \varepsilon$.
2. $\varphi(x_n)$ and u_n are collinear and thus
$$\varphi([\{x_n\}_n]) = [\{u_n\}_n].$$
3. $\{u_n\}_n$ satisfies **property** $(*)$ with respect to S: if $\{\alpha_n\}_n \in \ell_1$ satisfies that $\sum_n \alpha_n u_n \in S$, then
$$\alpha_n = 0 \text{ for all } n \geq 1.$$

Theorem (Jiménez Sevilla and Lajara, 2023)

Let E be a separable Banach space, let $X \subset E$ and let $\{R_k\}_{k\geq 1} \subset \mathcal{R}(E)$ be such that

$$X \subset \bigcap_{k\geq 1} R_k.$$

Then, for every $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ with $\|\varphi - I_E\| < \varepsilon$ satisfying the following properties:

1. $\varphi(X)$ is a nuclearly adjacent quasicomplement of Y and $\varphi(X) \cap \left(\bigcup_{k\geq 1} R_k\right) = \{0\}$.
2. X is a nuclearly adjacent quasicomplement of $\varphi(Y)$.

Theorem

Suppose that E is separable and let $S \in \mathcal{S}(E)$. Then, for every $Y \subset E$ and every $\varepsilon > 0$ there exist two isomorphisms $\psi, \varphi : E \to E$ such that $\|\psi - I_E\| < \varepsilon$ and $\|\varphi - I_E\| < \varepsilon$, and a closed subspace $X \subset \psi(Y)$ such that

(1) $\varphi(X)$ and X are mutually nuclearly adjacent quasicomplements.

(2) $(\varphi(X) + X) \cap S = \{0\}$.

Theorem (dual version)

Suppose that E^* is separable and let $S \in \mathcal{S}(E^*)$. Then, for every $Y \subset E^*$ and every $\varepsilon > 0$ there exist two isomorphisms $\psi, \varphi : E \to E$ such that $\|\psi - I_E\| < \varepsilon$ and $\|\varphi - I_E\| < \varepsilon$, and $Z \subset \psi^*(Y)$ w^*-closed such that

(1) $\varphi^*(Z)$ and Z are mutually nuclearly adjacent quasicomplements.

(2) $(\varphi^*(Z) + Z) \cap S = \{0\}$.

Theorem (Johnson, 1973)

Let E be a Banach space s. t. E^* is separable and let $Y \subset E^*$. Then there exists a w^*-closed quasicomplement $Z \subset E$ of Y.

Theorem

Let E be a Banach space s. t. E^* is separable and let $\{R_m\}_{m\geq 1} \subset \mathcal{R}(E^*)$ such that

$$\bigcap_m R_m \quad \text{is spaceable.}$$

Then, for every $Z \subset \bigcap_m R_m$ closed there exists $Y \subset E^*$ w^*-closed which is a quasicomplement of Z and

$$\left(\bigcup_m R_m\right) \cap Y = \{0\}.$$

Index

Proposition (Jiménez Sevilla and Lajara, 2024)

Let E be a Banach space and $R \in \mathcal{R}_d(E)$. Then there exists $X \subset E$ closed such that E/X is separable and

$$\operatorname{codim}_E(R + X) = \infty.$$

Proposition

Let E be a Banach space with a separable quotient and let $\{R_m\}_{m\geq 1} \subset \mathcal{R}(E)$. Then TFAE:

(1) There exists $X \subset E$ closed such that E/X is separable and $\text{codim}_E(R_m + X) = \infty$ for all $m \geq 1$.

(2) There exists a pair of quasicomplements $X, Y \subset E$ such that $\text{codim}_E(R_m + X + Y) = \infty$ for all $m \geq 1$.

(3) There exists $R \in \mathcal{R}_d(E)$ such that $\text{codim}_E(R_m + R) = \infty$ for all $m \geq 1$.

Theorem

Let E be a Banach space with w^*-separable dual and let $\{R_m\}_{m\geq 1} \subset \mathcal{R}(E)$. Then TFAE:

1. There exists $X \subset E$ closed such that E/X is separable, $\text{codim}_E(R_m + X) = \infty$ and $R_m \cap X = \{0\}$ for all $m \geq 1$.
2. There exists $X \subset E$ closed such that E/X is separable, $\text{codim}_E(R_m + X) = \infty$ and $R_m \cap X = \{0\}$, and for every $\varepsilon > 0$ there exists an isomorphism $\varphi : E \to E$ with $\|\varphi - I_E\| < \varepsilon$ such that $\varphi(R_m + X) \cap (R_m + X) = \{0\}$ for all $m \geq 1$ and $\varphi(X)$ and X are mutually nuclearly adjacent quasicomplements.
3. There exist $X, Y \subset E$ mutually nuclearly adjacent quasicomplementary such that $R_m \cap (X + Y) = \{0\}$ for all $m \geq 1$.
4. There exists $R \in \mathcal{R}_d(E)$ such that $R_m \cap R = \{0\}$ for all $m \geq 1$.

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