Banach lattices with disjointness preserving isometries: Linear versus vector lattice structures

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Basic definitions

Definition

Let $(E_i)_{i\in I}$ be a family of Banach spaces indexed by I and $\mathcal{U}$ be an ultrafilter over the index set I. The *ultraproduct* $\prod_{i,\mathcal{U}} E_i$ is defined as the quotient Banach space

$$\ell_\infty(E_i; i \in I)/c_{0,\mathcal{U}}(E_i; i \in I)$$

with $\ell_\infty(E_i; i \in I)=$ space of bounded families in $\prod_{i\in I} E_i$ with sup-norm and $c_{0,\mathcal{U}}(E_i; i \in I)=$ the subspace of families $\mathcal{U}$-converging to zero.
If $E_i = E$ for all $i \in I$ then we speak of an *ultrapower* of E, denoted by $E_{\mathcal{U}}$.

Notation

If $(x_i) \in \ell_\infty(E_i; i \in I)$ denote by $[x_i]_{\mathcal{U}}$ the element it defines in $\prod_{i,\mathcal{U}} E_i$.

Remark

$$\|[x_i]_{\mathcal{U}}\| = \lim_{i,\mathcal{U}} \|x_i\|$$

Lattice ultraproducts

These concepts extend easily to other categories of *normed structures*, i. e. Banach spaces with additional structures: Banach lattices, modulared Banach spaces, Banach algebras, operator spaces...

In this talk, beside the category of Banach spaces we shall consider only that of Banach lattices.

If $(E_i) =$ Banach lattices, then $\ell_\infty(E_i; i \in I)$ is also a Banach lattice and $c_{0,\mathcal{U}}(E_i; i \in I)$ is a closed order ideal in this Banach lattice.
Thus the quotient $\prod_{\mathcal{U}} E_i$ has a natural Banach lattice structure too.

The inf operation on the ultraproduct $\prod_{\mathcal{U}} E_i$ is given by

$$[x_i]_{\mathcal{U}} \wedge [y_i]_{\mathcal{U}} = [x_i \wedge y_i]_{\mathcal{U}}$$

Some classes which are closed under ultraproducts

- $C(K)$ spaces (as Banach lattices or as Banach algebras)
- L_p-spaces, $1 \leq p < \infty$ (Banach lattices) [Krivine,Henson-Moore]
- Nakano spaces $L_{p(\cdot)}$ (or Lebesgue spaces with variable exponents). The class $\mathcal{N}_K$ of Nakano spaces with exponent function taking values in a given compact set $K \subset [1, \infty)$ is closed by ultraproducts. (as Banach lattices)
- Preduals of von Neumann algebras. [U. Groh]
- General non-commutative L_p-spaces, $1 \leq p < \infty$ (Operator spaces) [YR].

Classes which are not closed under ultrapowers, but...

but a suitable enlargement is closed under ultraproducts:

- $L_p + L_q$-spaces, $1 \leq p \neq q < \infty$; but the class of "generalized sums" $L_p(\Omega_1) + L_q(\Omega_2)$ (Ω_1, Ω_2 subsets of the same measure space) is closed under ultraproducts. [YR]
- Orlicz spaces. But the class of Musielak-Orlicz spaces (generalized Orlicz spaces with variable Orlicz function) satisfying a prescribed uniform Δ_2-estimate is closed under ultraproducts. [Dacunha-Castelle]
- $L_p(L_q)$-spaces. But the class BL_pL_q of Banach lattices isomorphic to a band in some $L_p(L_q)$-space, $1 \leq p \neq q < \infty$ is closed under ultraproducts. [M. Levy, Y.R]

How similar are a normed structure and its ultrapowers?

- A normed structure and its ultrapowers have "approximatively" the same set of finite dimensional substructures.
- Indeed
  - the canonical embedding $D_E : E \to E_{\mathcal{U}}, x \mapsto [x]_{\mathcal{U}}$ preserves the norm and the operations.
  - $E_{\mathcal{U}}$ is finitely representable in E .
- In fact the similarity between E and $E_{\mathcal{U}}$ goes far beyond finite representability.
  - To clarify this question C. W. Henson introduced his "logic of positive bounded formulas and approximate satisfaction" , for B. space setting.
  - It was later adapted to normed space structures by Henson and Iovino.
  - Later on "Continuous Logic" was designed for model theoretical purposes (but is more adapted to bounded metric structures). [Ben Yaacov, Berenstein, Henson, Usviatsov]

Sentences in Henson's language

Such sentences have the form

$$Q^1_{r_1} x_1 Q^2_{r_2} x_2 \dots Q^n_{r_n} x_n \ \varphi(x_1, \dots x_n)$$

The variables $x_1, \dots x_n$ represent elements of the normed structure (but never heigher level objects like subsets, functions ...).

Each Q^i is a quantifier ($\forall$ or $\exists$). $Q^i_{r_i}$ means that the scope of the quantifier Q^i is limited to the ball of radius r_i.

φ is a logical formula which is constructed iteratively from basic formulas using the logical connectives $\wedge$ and $\vee$ (but never $\neg$).

Basic formulas

The basic formulas have the form

$$F(\|t_1(x_1,\dots,x_n)\|,\|t_2(x_1,\dots,x_n)\|,\dots,\|t_m(x_1,\dots,x_n)\|)\le r$$

where r is a real constant, $F:\mathbb{R}^m\to\mathbb{R}$ is a continuous function and the $t_j(x_1,\dots x_n)$ are "terms".

- In the language of Banach spaces such terms are simply linear combinations: $t(x_1,\dots,x_n)=\sum_{j=1}^n a_jx_j$ (the a_j's are real constants).
- In the language of Banach lattices the terms are more complicated, since their writing may involve the lattice operations $\wedge$ and $\vee$.

Approximate satisfaction

- Given a sentence A we define a set of weakenings of A by "relaxing all the conditions appearing in basic formulas and quantifiers".
- A is approximately satisfied in a normed structure E if all of its weakenings are satisfied in E.

Two classical theorems revisited

If X is a normed structure the *theory* of X is the set $Th(X)$ of all sentences that are approximately satisfied in X.

Theorem (Loś)

X and any of its ultrapowers have the same theory.

Theorem (Shelah; Henson; Henson-Iovino)

X and Y have the same theory iff they have two isomorphic ultrapowers (that is, $X_{\mathcal{U}} \simeq Y_{\mathcal{U}}$, for some ultrafilter $\mathcal{U}$).

Remark

An isomorphism preserves not only the operations of the category under consideration *but also the norm* so it must be isometric.

Axiomatizable classes

Let $\mathcal{C}$ be a class of normed structures.

- The theory of $\mathcal{C}$ is the set $Th(\mathcal{C})$ of all sentences approximately satisfied by all elements of $\mathcal{C}$.
- A normed structure which satisfies approximately all sentences in $Th(\mathcal{C})$ is called a *model* of $Th(\mathcal{C})$.
- $\mathcal{C}$ is called axiomatizable if it contains all the models of $Th(\mathcal{C})$.

Theorem

A normed structure X is a model of $Th(\mathcal{C})$ iff some ultrapower of X is isomorphic to some ultraproduct of members of $\mathcal{C}$.

Corollary

$\mathcal{C}$ is axiomatizable iff it is closed under isomorphisms, ultraproducts, and ultraroots.

(X is an ultraroot of Y if for some ultrafilter $X_{\mathcal{U}} \simeq Y$.)

Ultraroots: the Banach space case

Within the list of classes of Banach spaces which are closed under ultraproduct, very few are known to be closed under ultraroots (and thus axiomatizable):

- L_p-spaces, $1 \leq p < \infty$ (Henson)
- L_1-preduals, and various subclasses (in particular, $C(K)$ spaces) (Heinrich, 1981)
- p-direct sums of spaces $L_p(H_i)$, $1 < p < \infty$, H_i Hilbert (Y.R., 2004)

Ultraroots: the Banach lattice case

In the Banach lattice setting, the question is easier to settle. Classes known to be axiomatizable in the Banach lattice setting are

- L_p-spaces, $1 \leq p < \infty$ (easy), $C(K)$ spaces.
- The class $\mathcal{MO}_K$ of Musielak-Orlicz spaces, satisfying an uniform Δ_2-estimate with constant K (easy).
- The class $\mathcal{N}_K$ of Nakano spaces, $K \subset [1, \infty)$ (Poitevin, 2006)
- BL_pL_q-spaces, $1 \leq p, q < \infty$ (C.W. Henson, Y. R., 2007.)

Tools for showing closure under ultraroots

- the class is closed under substructures [$E \subset E_{\mathcal{U}}$ *as a substructure*]
- the class is closed under contractive projections (on a substructure) and consists of reflexive B. spaces or of B. lattices not containing c_0 [*A contractive projection* $E_{\mathcal{U}} \to E$ *exists in both cases*]
- the class equals its script-class (example: class of L_p spaces = class $\mathcal{SL}_{p,1}$ of isometric script L_p spaces)
- duality (for a class of superreflexive normed structures)
- convexification/concavification (Banach lattices setting) [*Both operations preserve ultraproducts*]

All these tools are internal to the category to which belongs the class (B. spaces, B. lattices...) In this talk we shall introduce a new tool which links distinct categories:

Property DPIU

We introduce now a property which provides a direct link between axiomatizability in Banach lattice sense and in Banach space sense.

Definition

We say that a Banach lattice L has property DPIU if every linear isometric embedding of L into any of its ultrapowers preserves disjointness.

Example

For $1 \leq p < \infty$, $p \neq 2$, L_p spaces have property DPIU.

Indeed for $p \in [1, 2) \cap (2, \infty)$, disjointness of two elements x, y in a given L_p-space is characterized by the equation:

$$\|x + y\|^p + \|x - y\|^p = 2(\|x\|^p + \|y\|^p)$$

which involves only Banach spaces operations and the norm.
Thus any isometry from a L_p-space to another one is disjointness preserving.

Other classes of Banach lattices with DPIU

Let us say that a Banach lattice X is exactly s-convex (resp. s-concave) iff it is r-convex (resp.concave) with constant one, i.e.

$$\forall x_1, \ldots, x_n \in X \quad \|(\sum_i |x_i|^s)^{1/s}\| \le (\sum_i \|x_i\|^s)^{1/s} \quad \text{(resp. } \ge)$$

Proposition

Let X, Y be exactly r-convex Banach lattices, $r > 2$, with stricly monotone norms. Every linear isometry from X to Y preserves disjointness.

Corollary

Every exactly r-convex, s-concave Banach lattice, with $2 < r \le s < \infty$ has property (DPIU).

Proposition

Let $1 \le p, q < \infty$ with $p, q \ne 2$. If $p > 2$ **or** *$q > 2$, the class BL_pL_q has property (DPIU).*

From Lattice to Banach axiomatizability

Theorem (Banach ultraroots of DPIU lattices)

Let L be an order continuous Banach lattice satisfying (DPIU). Assume that X is a Banach space which has an ultrapower $X_{\mathcal{U}}$ linearly isometric to L. Then X itself is linearly isometric to a closed sublattice Y of L. Furthermore the ultrapower $Y_{\mathcal{U}}$ is lattice isomorphic to L.

As an immediate corollary we obtain the main result of this section:

Theorem (Transfer of axiomatizability: Y.R.)

Let $\mathcal{C}$ be an axiomatizable class of Banach lattices consisting of order continuous Banach lattices with property (DPIU). Then the class $\mathcal{C}^B$ of Banach spaces linearly isometric to members of $\mathcal{C}$ is axiomatizable.

Vector sublattices "up to a sign change"

Let X be a Banach lattice.

- A sign change on X is a modulus preserving operator $U : X \to X$.
- Equivalently $U = P - Q$, where P, Q are complementary band projections. If X can be represented as a Köthe function space, U is a sign multiplication operator.
- A linear subspace E of X is called a "sublattice up to a sign change" if for some sign change U, UX is a vector sublattice of X.

An intrinsic characterization of vector sublattices up to a sign change

Notation (Lacey's b-function)

For $x, y \in X$ set $b(x, y) = |x| \wedge y_+ - |x| \wedge y_-$.

Lemma

Assume that X is an order continuous Banach lattice. Then for a closed linear subspace E of X the following assertions are equivalent:
i) E is a closed vector sublattice up to a sign change.
ii) The function b maps $E \times E$ into E.

Lemma

The function b is preserved under bounded linear disjointness preserving maps: i. e. $Tb(x, y) = b(Tx, Ty)$ for such a map $T : X \to Y$.

Banach ultraroots of Banach lattices: sketch of proof

Let L be a lattice with DPIU property, X a Banach space such that $X_{\mathcal{U}}$ is linearly isometric to L.
X is linear isometric to a subspace of $X_{\mathcal{U}}$, and thus to a subspace X_0 of L. We may assume w.l.o.g. $X = X_0$. We prove that X is a vector sublattice up to a sign change.

Then the isometry $J : L \to X_{\mathcal{U}}$ fixes points of X, this gives a commutative diagramme, that we insert in a second one. We complete by setting $S = i_{\mathcal{U}} J$.

$$
\begin{tikzcd}
 & X_{\mathcal{U}} \arrow[dr, "i_U"] & \\
X \arrow[ur, "D_X"] \arrow[r, "i"'] \arrow[dr, "i"'] & L \arrow[u, "J"'] \arrow[r, "S"'] & L_{\mathcal{U}} \\
 & L \arrow[ur, "D_L"'] &
\end{tikzcd}
$$

Proof cont'd

$$
\begin{tikzcd}
 & X_{\mathcal{U}} \arrow[dr, "i_U"] & \\
X \arrow[ur, "D_X"] \arrow[r, "i"'] \arrow[dr, "i"'] & L \arrow[u, "J"'] \arrow[r, "S"'] & L_{\mathcal{U}} \\
 & L \arrow[ur, "D_L"'] &
\end{tikzcd}
$$

Note that
- the isometry S is disjointness preserving (DPIU property for L).
- For the natural position of L in $L_{\mathcal{U}}$ we have $X = L \cap X_{\mathcal{U}}$ (*)
(i.e. $i_{\mathcal{U}} X_U \cap D_L L = i_{\mathcal{U}} D_X X$).

Since S is disjointness preserving and D_L is a vector lattice isomorphism, we have for $x, y \in X$

$$i_{\mathcal{U}} Jb(ix, iy) = Sb(ix, iy) = b(Six, Siy) = b(D_L ix, D_L iy) = D_L b(ix, iy)$$

which implies by (*) that $Jb(ix, iy) = D_X z = Jiz$, for some $z \in X$, hence $b(ix, iy) = iz$ and thus $iX = X$ is a (closed) vector sublattice up to a sign change in L.

Final step

We have $X \simeq E \subset L$, with E a Banach sublattice of L.
Thus passing to ultrapowers:

$$L \simeq X_{\mathcal{U}} \simeq E_{\mathcal{U}} \subset L_{\mathcal{U}}$$

Consider the maps

$$L \overset{V}{\twoheadrightarrow} E_{\mathcal{U}} \xrightarrow{j_{\mathcal{U}}} L_{\mathcal{U}}$$

V linear isometry (onto), $j_{\mathcal{U}}$ inclusion map.
DPIU for $L \implies j_{\mathcal{U}}V$ is disjointness preserving
$j_{\mathcal{U}}$ is a normed lattice embedding $\implies V$ also disjointness preserving.
By a general theorem it has a modulus $|V|$ which is an isometric lattice isomorphism.
Thus E is a Banach lattice ultraroot of L. $\square$

After reading our work, Henson found an improvement of our result.

Definition (C.W. Henson)

A Banach lattice X has property DPA [*Disjointness preserving automorphisms*] if every surjective linear isometry from X to itself preserves disjointness.

Theorem (C. W. Henson, unpublished manuscript)

If $\mathcal{C}$ is an axiomatizable class of order continuous Banach lattices and every member of $\mathcal{C}$ has DPA, then $\mathcal{C}^B$ is an axiomatizable class of Banach spaces.