Gabor phase retrieval via semidefinite programming

Martin Rathmair

University of Vienna

07.05.2024

Outline

Part I:

- abstract phase retrieval – 2 viewpoints
- 1 concrete problem: Gabor phase retrieval
- 3 concepts reappearing in phase retrieval:

  lifting – holomorphicity – connectivity

Part II:

- quantitative & constructive arguments $\rightarrow$ reconstruction scheme

Jaming, P. and Rathmair, M. Gabor phase retrieval via semidefinite programming (2023) arXiv:2310.11214

Introduction

Abstract phase retrieval – 2 viewpoints

$$\boxed{g \sim f \quad \Leftrightarrow \quad \exists c \in \mathbb{C}, |c| = 1 : \; g = cf}$$

i) Hilbert space

- $\mathcal{X}$...Hilbert space over $\mathbb{C}$
- $(\phi_\omega)_{\omega\in\Omega} \subseteq \mathcal{X}$

$m : \mathcal{X} \ni f \mapsto (|\langle f, \phi_\omega\rangle|)_{\omega\in\Omega} \in \mathbb{R}_+^\Omega$

ii) Function space

- $\mathcal{X} \subseteq C(\mathbb{R}^d, \mathbb{C})$...linear space
- $\Omega \subseteq \mathbb{R}^d$

$m : \mathcal{X} \ni f \mapsto (|f(\omega)|)_{\omega\in\Omega} \in \mathbb{R}_+^\Omega$

Task:

Given $m(f)$, reconstruct f (up to $\sim$). Or equivalently, invert

$$m : (\mathcal{X}/\sim) \to \mathbb{R}_+^\Omega.$$

Three desirable properties

- **Uniqueness**
  Injectivity of $m : (\mathcal{X}/\sim) \to \mathbb{R}_+^{\Omega}$
- **Stability**
  (Lipschitz) continuity of m^{-1}
- **Reconstruction**
  algorithm
  accuracy guarantees
  robustness w.r.t. noise

Lifting

Consider: $\mathcal{X} = \mathbb{C}^d$ and $m(f) = (|\langle f, \varphi_\omega \rangle|)_{\omega \in \Omega}$

$\mathcal{X} \ni g \mapsto \|m(g) - m(f)\|_{\ell^p(\Omega)}$ is <u>not</u> convex!

$$f \in \mathbb{C}^d \quad \overset{\text{"lift"}}{\longrightarrow} \quad \mathcal{L}(f) := f \otimes \bar{f} = f \cdot f^H \in \mathbb{C}^{d \times d}$$

Idea: Rewrite

$$|\langle f, \varphi_\omega \rangle|^2 = (f^H \cdot \phi_\omega) \cdot (\phi_\omega^H \cdot f) = \ldots = \langle \mathcal{L}(f), \mathcal{L}(\phi_\omega) \rangle_F$$

$0 \leq G \in \mathbb{C}^{d \times d} \mapsto \left\| (\langle G, \mathcal{L}(\varphi_\omega) \rangle_F)_{\omega \in \Omega} - m(f)^2 \right\|_{\ell^p(\Omega)}$ is convex!

PhaseLift

Reformulation

find $G \geq 0$
s.t. $\langle G, \mathcal{L}(\varphi_\omega) \rangle_F = m(f)^2_\omega, \; \omega \in \Omega$
$\operatorname{rank}(G) \to \min$

convex Relaxation

find $G \geq 0$
s.t. $\langle G, \mathcal{L}(\varphi_\omega) \rangle_F = m(f)^2_\omega, \; \omega \in \Omega$
$\operatorname{tr}(G) \to \min$

not convex	vs.	convex (SDP even)
correspondence	vs.	?

PhaseLift: Exact and Stable Signal Recovery from Magnitude Measurements via Convex Programming. Candès, E., Strohmer, T. and Voroninski, V. Comm. Pure Appl. Math. (2013)

Holomorphic phase retrieval

Consider: $\mathcal{X} = \mathcal{O}(\mathbb{C})$ and $\Omega = \mathbb{C}$

$$f^*(z) := \overline{f(\bar{z})}, \qquad f \in \mathcal{O}(\mathbb{C})$$
$$\mathcal{L}(f) := f \otimes f^* \in \mathcal{O}(\mathbb{C}^2).$$

Lemma (Uniqueness)

Let $f \in \mathcal{O}(\mathbb{C})$. If $F \in \mathcal{O}(\mathbb{C}^2)$ is such that

$$\forall z \in \mathbb{C}: \; F(z, \bar{z}) = |f(z)|^2,$$

then $F = \mathcal{L}(f)$.

Specify the problem

Gabor phase retrieval

$$T_x f = f(\cdot - x), \quad M_\xi f = e^{2\pi i \xi \cdot} f(\cdot), \quad \pi\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right) = M_\xi \circ T_x$$

Definition (Short-time Fourier transform)

Let $\varphi = 2^{\frac{1}{4}} e^{-\pi \cdot^2}$. The STFT of $f \in L^2(\mathbb{R})$ is given by

$$\begin{aligned} \mathcal{V}f(x,\xi) &:= \int_{\mathbb{R}} f(t)\varphi(t-x)e^{-2\pi i \xi t}\, \mathrm{d}t, \qquad (x,\xi) \in \mathbb{R}^2 \\ &= \mathcal{F}[f(\cdot)\varphi(\cdot - x)](\xi) \\ &= \langle f, \pi\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)\varphi \rangle_{L^2(\mathbb{R})} \end{aligned}$$

$|\mathcal{V}f|^2$ (or $|\mathcal{V}f|$) is called the spectrogram of f.

From now on: $\mathcal{X} = \mathcal{V}(L^2(\mathbb{R}))$, $d = 2$ and $\Omega \subseteq \mathbb{R}^2$

Bargman relation

Definition (Fock space)

$$\mathfrak{F} = \left\{ F \in \mathcal{O}(\mathbb{C}) : \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} \, \mathrm{d}A(z) < \infty \right\}$$

Lemma

Let $\eta(x,y) = \exp(-\frac{\pi}{2} |\binom{x}{y}|^2 + \pi ixy)$. *T.f.s.a.e.:*

- $V \in \mathcal{X} = \mathcal{V}(L^2(\mathbb{R}))$
- $z = x + iy \mapsto \dfrac{V(x,-y)}{\eta(x,y)} \in \mathfrak{F}$

$$\boxed{\underline{\textit{in short:}} \qquad \mathcal{X} = \mathfrak{F} \cdot \eta}$$

USP of the gaussian window

Theorem (Ascensi, Bruna '08)

Let $g \in L^2(\mathbb{R})$ be a window function. Then the space

$$\left\{ F(z) = \int_{\mathbb{R}} f(t) e^{2\pi i t y} g(t-x)\, dt,\ f \in L^2(\mathbb{R}) \right\}$$

is a space of holomorphic functions modulo a multiplicative weight if and only if $g = \pi(\lambda)\varphi$ for some $\lambda \in \mathbb{R}^2$.

Reconstruction with phase: $\mathcal{V}f\big|_\Omega \rightsquigarrow f$

Gabor frame expansion (Seip, Wallstén, Lyubarskii, Janssen)

Let $\Lambda = a\mathbb{Z}^2$ with $a \in (0,1)$. There exists $\psi \in L^2(\mathbb{R})$ (dual window) s.t.

$$\begin{aligned} f &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\psi\rangle_{L^2} \cdot \pi(\lambda)\varphi \\ &= \sum_{\lambda\in\Lambda} \underbrace{\langle f, \pi(\lambda)\varphi\rangle_{L^2}}_{=\mathcal{V}f(\lambda)} \cdot \pi(\lambda)\psi \qquad f \in L^2(\mathbb{R}) \end{aligned}$$

For $a \in \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots\}$ $\longrightarrow$ explicit formula for ψ.

Reconstruction with phase: $\mathcal{V}f\big|_\Omega \rightsquigarrow f$

Gabor frame expansion (Seip, Wallstén, Lyubarskii, Janssen)

Let $\Lambda = a\mathbb{Z}^2$ with $a \in (0,1)$. There exists $\psi \in L^2(\mathbb{R})$ (dual window) s.t.

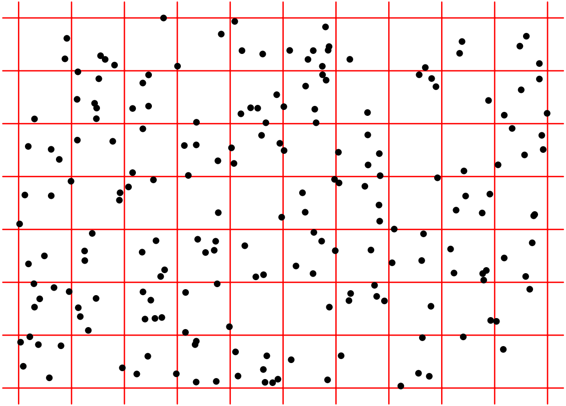
$$\begin{aligned} f &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\psi\rangle_{L^2} \cdot \boxed{\pi(\lambda)\varphi} \\ &= \sum_{\lambda\in\Lambda} \underbrace{\langle f, \pi(\lambda)\varphi\rangle_{L^2}}_{=\boxed{\mathcal{V}f(\lambda)}} \cdot \pi(\lambda)\psi \qquad\qquad f \in L^2(\mathbb{R}) \end{aligned}$$

For $a \in \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots\}$ $\quad\longrightarrow\quad$ explicit formula for ψ.

$\rightsquigarrow$ Break up the task:

$$\boxed{|\mathcal{V}f\big|_\Omega| \xrightarrow{?} \mathcal{V}f\big|_\Lambda} \xrightarrow{\checkmark} f$$

Reformulated task



$\Lambda = \frac{1}{\sqrt{2}}\mathbb{Z}^2$.. lattice
Ω.. sampling positions

$$(|\mathcal{V}f|)\big|_{\Omega} \quad \xrightarrow{?} \quad \mathcal{V}f\big|_{\Lambda} \text{ (up to a phase factor)}$$

What can we expect?

Notions of stability

- Given $f \in L^2(\mathbb{R})$ the Lipschitz stability constant $C_L(f)$ is the infimal $C > 0$ such that

$$\min_{|\tau|=1} \|g - cf\|_{L^2(\mathbb{R})} \leq C \||\mathcal{V}g| - |\mathcal{V}f|\|_{L^2(\mathbb{R}^2)}, \quad g \in L^2(\mathbb{R}).$$

- Given $f \in L^2(\mathbb{R})$ and $D \subseteq \mathbb{R}^2$ a domain the local Lipschitz stability constant $C_L(f, D)$ is the infimal $C > 0$ such that

$$\min_{|\tau|=1} \|\mathcal{V}g - \tau\mathcal{V}f\|_{L^2(D)} \leq C \||\mathcal{V}g| - |\mathcal{V}f|\|_{L^2(D)}, \quad g \in L^2(\mathbb{R}).$$

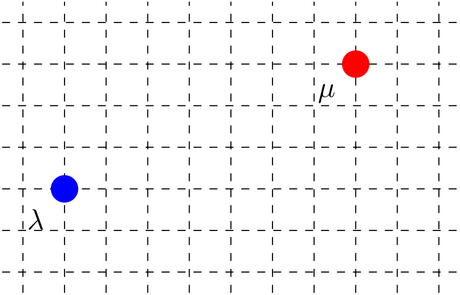
Remark: $C_L(f, \mathbb{R}^2) = C_L(f)$.

No uniform stability!

$$\sup_{f \in L^2(\mathbb{R})} C_L(f) = \infty$$

Cahill, J., Casazza, P., Daubechies, I. Phase retrieval in infinite-dimensional Hilbert spaces. Trans. Amer. Math. Soc. Ser. B3(2016)

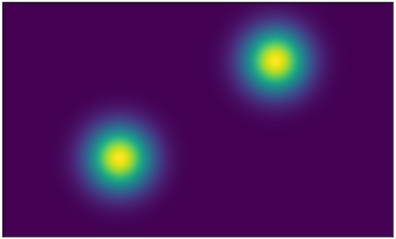
Alaifari, R., Grohs, P. Phase retrieval in the general setting of continuous frames for Banach spaces. SIAM J. Math. Anal.49(2017)

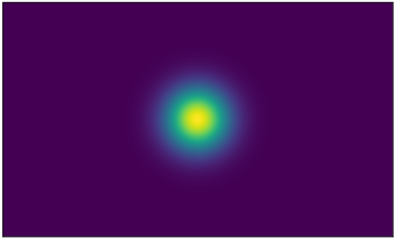


$$
\begin{aligned}
f &= \pi(\lambda)\varphi + \pi(\mu)\varphi \\
g &= \pi(\lambda)\varphi - \pi(\mu)\varphi
\end{aligned}
$$

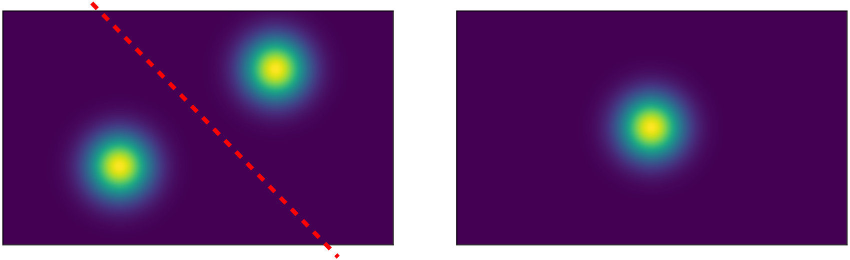
$$
\begin{aligned}
\Phi_\lambda(z) &:= \mathcal{V}[\pi(\lambda)\varphi](z) \\
&= \boxed{e^{-\frac{\pi}{2}|z-\lambda|^2}} e^{i\cdots}
\end{aligned}
$$

Quantifying connectivity of measurements





Quantifying connectivity of measurements



Definition (Cheeger constant)

Let $D \subseteq \mathbb{R}^2$ a domain and $w : D \to \mathbb{R}_+$ a continuous weight.

$$h(w, D) := \inf_{\substack{S \subseteq D \text{ open} \\ \partial S \cap D \text{ smooth}}} \frac{\mathrm{cut}(S)}{\min\{\mathrm{vol}(S), \mathrm{vol}(D \setminus S)\}} \quad \text{(Cheeger const.)}$$

where $\mathrm{vol}(S) = \int_S w(x) \mathrm{d}x$ and $\mathrm{cut}(S) = \int_{\partial S \cap D} w(y) \mathrm{d}\mathcal{H}^1(y)$.

Stability when connected

Theorem (Grohs, R.(informal) '19)

Let $D \subseteq \mathbb{R}^2$ be a domain. For all $f, g \in L^2(\mathbb{R})$ it holds that

$$\min_{|c|=1} \|\mathcal{V}g - c\mathcal{V}f\|_{L^1(D)} \leq (1 + h(|\mathcal{V}f|, D)^{-1}) \||\mathcal{V}g| - |\mathcal{V}f|\|_{W(D)}.$$

Stability when connected

Theorem (Grohs, R.(informal) '19)

Let $D \subseteq \mathbb{R}^2$ be a domain. For all $f, g \in L^2(\mathbb{R})$ it holds that

$$\min_{|c|=1} \|\mathcal{V}g - c\mathcal{V}f\|_{L^1(D)} \leq (1 + h(|\mathcal{V}f|, D)^{-1})\||\mathcal{V}g| - |\mathcal{V}f|\|_{W(D)}.$$

- ✓ Tells us when phase reconstruction problem is well-posed.
- ✗ Doesn't tell us how to reconstruct in this case.

Objective for part II

Formulate a reconstruction algorithm which

- comes with convergence gurantees
- which reflect what we know from stability analysis (i.e., accurate if connected)
- can deal with noisy measurements,
- is (reasonably) efficient.

Objective for part II

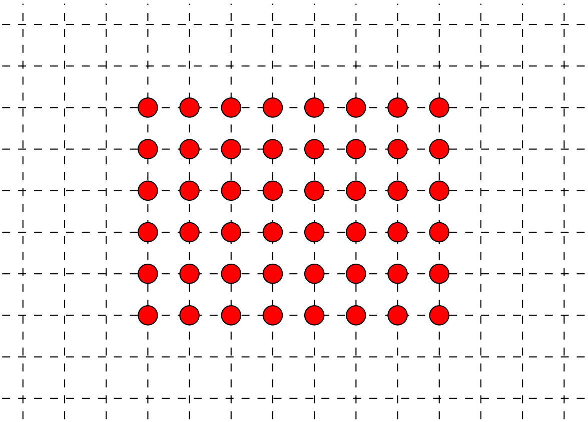
Formulate a reconstruction algorithm which

- comes with convergence gurantees
- which reflect what we know from stability analysis (i.e., accurate if connected)
- can deal with noisy measurements,
- is (reasonably) efficient.

For $\mathcal{X} = \mathcal{V}$(shift-inv. space):

Grohs, P., Liehr, L. Stable Gabor Phase Retrieval in Gaussian Shift-Invariant Spaces via Biorthogonality. Constr Approx 59, 61–111 (2024).

Reduction to finite dimension



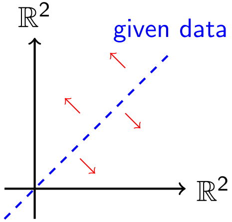
$\Lambda \subseteq \frac{1}{\sqrt{2}}\mathbb{Z}^2$ finite

$\boxed{x = (\mathcal{V}f(\lambda))_{\lambda\in\Lambda}} \in \mathbb{C}^{\Lambda}$

$xx^H = \mathcal{L}(\mathcal{V}f\big|_{\Lambda}) \in \mathbb{C}^{\Lambda\times\Lambda}$

Estimator for the tensor (step one)

The estimator $\mathcal{E}$: an extension problem



What? $\mathcal{E} : \mathbb{R}_+^\Omega \to C(\mathbb{R}^4, \mathbb{C})$ s.t.

$$\mathcal{E}(|\mathcal{V}f\big|_\Omega|) \approx \mathcal{L}(\mathcal{V}f) = \mathcal{V}f \otimes \overline{\mathcal{V}f}.$$

How? Recall: $$\mathcal{V}f \otimes \overline{\mathcal{V}f} = \sum_{\lambda,\lambda' \in \frac{1}{\sqrt{2}}\mathbb{Z}^2} c_\lambda \overline{c_{\lambda'}} \cdot \Phi_\lambda \otimes \overline{\Phi_{\lambda'}}.$$

$\rightsquigarrow$ Ansatz: $$\Psi_C = \sum_{\lambda,\lambda' \in \Gamma} C_{\lambda,\lambda'} \cdot \Phi_\lambda \otimes \overline{\Phi_{\lambda'}}, \quad (\Gamma \subseteq \tfrac{1}{\sqrt{2}}\mathbb{Z}^2,\ C \in \mathbb{C}^{\Gamma\times\Gamma}).$$

$$\text{find } C \geq 0 \ : \quad \forall \omega \in \Omega : \quad |\Psi_C(\omega,\omega) - |\mathcal{V}f(\omega)|^2| \leq \epsilon$$

convex domain

convex constraint

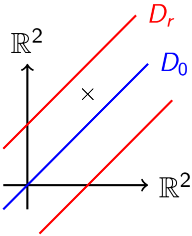
Why/to what extent is $\mathcal{E}$ accurate? [Asspt: $\Omega = \mathbb{R}^2$]

$$E := \Psi_C - \mathcal{V}f \otimes \overline{\mathcal{V}f}, \qquad D_r := \{(z,z') : |z - z'| = r\} \subseteq \mathbb{R}^4.$$

Lemma (Quant. uniqueness principle(QUP))

Let $(z,z') \in \mathbb{R}^4$ *and* $r = 2|z - z'|$. *Then,*

$$|E(z,z')| \leq \sqrt{\|E\|_{L^\infty(D_0)} \cdot \|E\|_{L^\infty(D_r)}} \cdot e^{\frac{\pi}{2}|z-z'|^2}.$$



Lemma (Controll error on D_r)

$$\|E\|_{L^\infty(D_r)} \leq \|\mathcal{V}f\|_{L^\infty}^2 + \|C\|_\infty.$$

$\|\operatorname{diag}(C)\|_\infty \to \min$

s.t. $\sup_{\omega \in \Omega} \left|\Psi_C(\omega,\omega) - |\mathcal{V}f(\omega)|^2\right| \leq \epsilon$

QUP – where does it come from?

$$M_f(y) := \sup_{x\in\mathbb{R}} |f(x+iy)|.$$

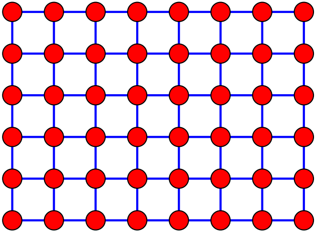
Lemma (Hadamard's three line theorem)

Let $\mathcal{S} = \{x+iy : x \in \mathbb{R}, y \in (0,1)\}$. *Suppose that* $f \in \mathcal{O}(\mathcal{S}) \cup C_b(\bar{\mathcal{S}})$. *Then it holds that*

$$M_f(t) \leq M_f(0)^{1-t} M_f(1)^t, \quad t \in (0,1).$$

Graph viewpoint

$\mathcal{E}$ allows us to estimate $\mathcal{V}f(z)\overline{\mathcal{V}f(z')}$ if $|z-z'| \leq r$ (say $r = \frac{1}{\sqrt{2}}$).



def $G = G^{f,\Lambda,r} = (V, E, \alpha)$ by

- $V = \Lambda$ (vertices)
- $e = (\lambda, \lambda') \in E \Leftrightarrow |\lambda - \lambda'| \leq r$ (edges)
- $\alpha(\lambda) = |\mathcal{V}f(\lambda)|^2$ (weights)

$$L(\lambda, \lambda') = \begin{cases} \sum_{\mu \sim \lambda} \alpha(\mu), & \lambda' = \lambda \\ -\sqrt{\alpha(\lambda)\alpha(\lambda')}, & \lambda \sim \lambda' \\ 0, & \text{otw.} \end{cases} \qquad \text{(Laplacian)}$$

Properties of the Laplacian

Lemma

- $L \geq 0$ *with eigenvalues* $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|\Lambda|}$
- $\lambda_2 > 0$ *if and only if* G *is connected (i.t.s. of vertex weighted graphs)*
- *let* $\tilde{L}$ *be def. by*

$$\tilde{L}(\lambda, \lambda') = \begin{cases} \sum_{\mu \sim \lambda} |\mathcal{V}f(\mu)|^2 & \lambda' = \lambda \\ -\mathcal{V}f(\lambda)\overline{\mathcal{V}f(\lambda')}, & |\lambda - \lambda'| \leq r \\ 0, & otw. \end{cases}$$

then, L *and* $\tilde{L}$ *are similar and* $x = \mathcal{V}f\big|_{\Lambda} \in \ker(\tilde{L})$

spectral gap measures "connectivity of $|\mathcal{V}f|$ on Λ"

$$\deg(u) := \sum_{v \sim u} \alpha_v, \quad (u \in V)$$

$$\mathrm{vol}(S) := \sum_{v \in S} \alpha_v, \qquad \mathrm{cut}(S) := \sum_{v \in S} \sum_{\substack{u \in V \setminus S \\ u \sim v}} \alpha_u \alpha_v, \quad (S \subseteq V)$$

$$h_G := \min_{S \subseteq V} \frac{\mathrm{cut}(S)}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}} \ldots \text{(Cheeger const.)}$$

Lemma (Cheeger-type result)

Let $G = (V, E, \alpha)$ be a vertex-weighted graph. It holds that

$$\frac{h_G^2}{2 \max_{v \in V} \deg(v)} \leq \lambda_2 \leq 2 h_G.$$

Matrix completion / null space extraction (step two)

Why do we benefit from a spectral gap?

step two: assemble $\tilde{L}$ and extract eigenvector v_1 corresponding to smallest eigenvalue

- in the disconnected case: $\geq$ 2-dim null space!
- if $\lambda_2 \approx 0$: small noise $(L + \nu)$ can lead to ambiguities
- if $\lambda_2 \gg \lambda_1 = 0$: $\tilde{L} \mapsto \operatorname{span} v_1$ is robust w.r.t. perturbations

Algorithm and results

Reconstruction algorithm

Parameters: $\varepsilon, \varepsilon', r > 0$ and $\Lambda, \Gamma \subseteq \frac{1}{\sqrt{2}}\mathbb{Z}^2$ finite

Input: $\sigma \in \mathbb{R}^\Omega$

1. Find C_* the solution of

$$\min_{0 \leq C \in \mathbb{C}^{\Gamma \times \Gamma}} \|\operatorname{diag}(C)\|_\infty \quad \text{s.t.} \quad \|\Psi_C - \sigma\|_{\ell^\infty(\Omega)} \leq \varepsilon.$$

2. Find $0 \leq Y \in \mathbb{C}^{\Lambda \times \Lambda}$ which satisfies the constraints

$$\forall (\lambda, \lambda') \in \Lambda^2, |\lambda - \lambda'| \leq r : \quad |Y_{\lambda,\lambda'} - \Psi_C(\lambda, \lambda')| \leq \varepsilon'.$$

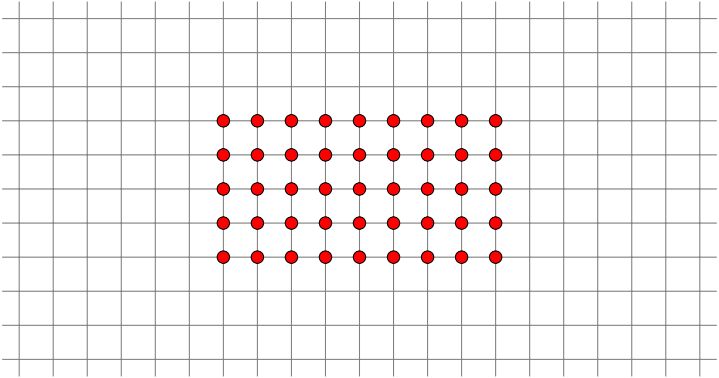
3. Extract $v \in \mathbb{C}^\Lambda$ a (normalized) leading eigenvector of Y and return

$$f_* := \sqrt{\operatorname{tr}(Y)} \cdot \sum_{\lambda \in \Lambda} v_\lambda \cdot \pi(\lambda)\psi.$$

Parameter configuration I

Given $T, S, R, s > 0$ we set

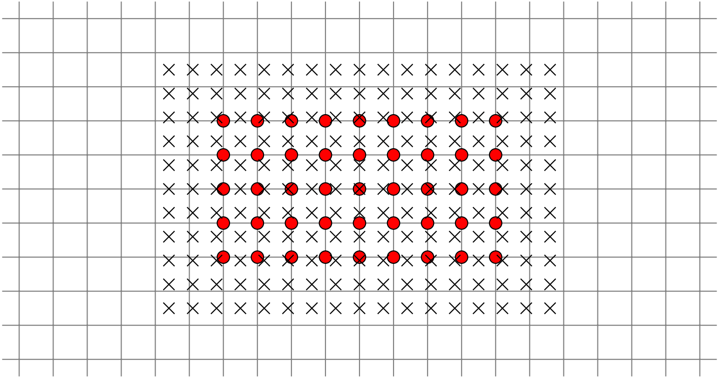
- $\Lambda = ([-T, T] \times [-S, S]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$
- $\Omega = ([-T-R, T+R] \times [-S-R, S+R]) \cap s\mathbb{Z}^2$
- $\Gamma = ([-T-2R, T+2R] \times [-S-2R, S+2R]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$



Parameter configuration I

Given $T, S, R, s > 0$ we set

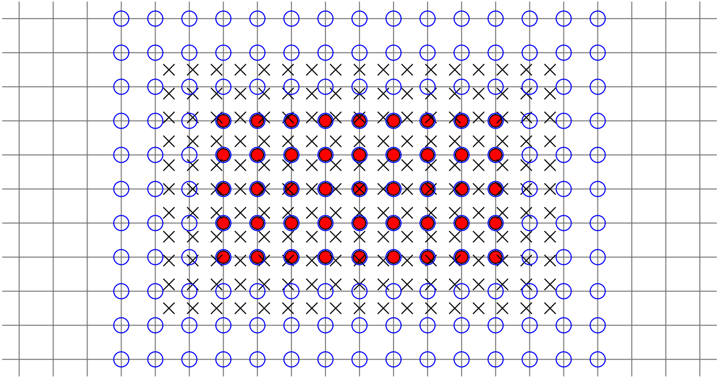
- $\Lambda = ([-T, T] \times [-S, S]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$
- $\Omega = ([-T-R, T+R] \times [-S-R, S+R]) \cap s\mathbb{Z}^2$
- $\Gamma = ([-T-2R, T+2R] \times [-S-2R, S+2R]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$



Parameter configuration I

Given $T, S, R, s > 0$ we set

- $\Lambda = ([-T, T] \times [-S, S]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$
- $\Omega = ([-T - R, T + R] \times [-S - R, S + R]) \cap s\mathbb{Z}^2$
- $\Gamma = ([-T - 2R, T + 2R] \times [-S - 2R, S + 2R]) \cap \frac{1}{\sqrt{2}}\mathbb{Z}^2$



Parameter configuration II

Definition

Let $f \in L^2(\mathbb{R})$. We say that $T, S, R, r, s, \varepsilon, \varepsilon' > 0$ are *well-calibrated* if

$$\boxed{\varepsilon} \leq \left[\frac{e^{-\frac{17\pi}{32} r^2}}{1.33 \times 10^5} \times \min \left\{ \frac{\|\mathcal{V} f\|^2_{\ell^2(\Lambda)}}{|\Lambda|^2}, \frac{\boxed{\lambda_2(f, \Lambda, r)}}{192 r^2} \right\} \right]^2,$$

$$\varepsilon' = (3.1 \times 10^4) \sqrt{\varepsilon} e^{\frac{17\pi}{32} r^2},$$

$$s \leq \frac{0.3}{\sqrt{\ln\left(\frac{2}{3\varepsilon}\right)}},$$

$$R \geq \max \left\{ 2.1 + 0.9 \sqrt{\ln\left(\frac{1}{\varepsilon}\right)}, \frac{r + s^{-1}}{2} \right\}$$

Main result (step one and two)

Theorem (Jaming, R. '23)

Let $f \in L^2(\mathbb{R})$ such that $\|\mathcal{V}f\|_{L^\infty(\mathbb{R}^2)} \leq 1$. Suppose that $T, S, R, r, s, \varepsilon, \varepsilon'$ are well calibrated. Let $\sigma \in \mathbb{R}^\Omega$ be such that

$$\forall \omega \in \Omega : \quad |\sigma_\omega - |\mathcal{V}f(\omega)|^2| \leq \frac{\varepsilon}{2}.$$

Then it holds that both CPs (step one and two) are feasible. Moreover, the top eigenvector $v \in \mathbb{C}^\Lambda$ of Y satisfies that

$$\min_{|c|=1} \left| \sqrt{\operatorname{tr}(Y)} v - c \cdot \mathcal{V}_f \big|_\Lambda \right| \leq C e^{0.84 r^2} \cdot \left(1 + r \sqrt{\frac{|\mathcal{V}f\big|_\Lambda|^2}{\lambda_2(f, \Lambda, r)}} \right) \cdot \sqrt[4]{\varepsilon}.$$

Main result (step one and two)

Theorem (Jaming, R. '23)

Let $f \in L^2(\mathbb{R})$ such that $\|\mathcal{V}f\|_{L^\infty(\mathbb{R}^2)} \leq 1$. Suppose that $T, S, R, r, s, \varepsilon, \varepsilon'$ are well calibrated. Let $\sigma \in \mathbb{R}^\Omega$ be such that

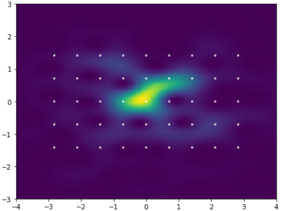
$$\forall \omega \in \Omega : \quad |\sigma_\omega - |\mathcal{V}f(\omega)|^2| \leq \frac{\varepsilon}{2}.$$

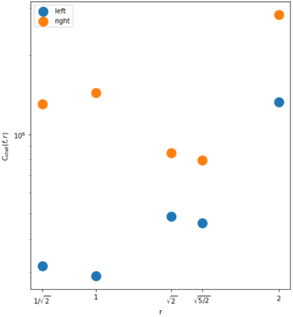
Then it holds that both CPs (step one and two) are feasible. Moreover, the top eigenvector $v \in \mathbb{C}^\Lambda$ of Y satisfies that

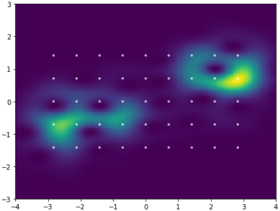
$$\min_{|c|=1} \left| \sqrt{\mathrm{tr}(Y)} v - c \cdot \mathcal{V}_f \big|_\Lambda \right| \leq C e^{0.84 r^2} \cdot \left(1 + r^2 \frac{|\mathcal{V}f|_\Lambda|}{h_G} \right) \cdot \sqrt[4]{\varepsilon}.$$

On the choice of r

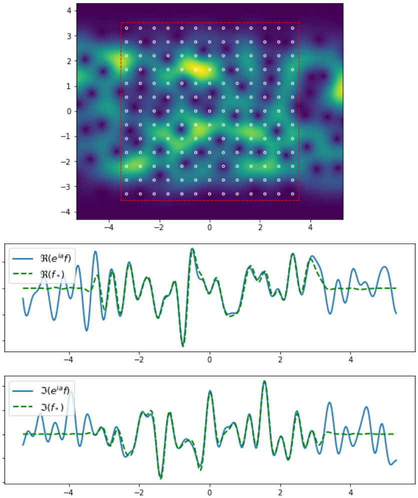
$$\text{minimize} \quad r \mapsto \frac{e^{0.84 r^2} r}{\sqrt{\lambda_2(f, \Lambda, r)}}$$



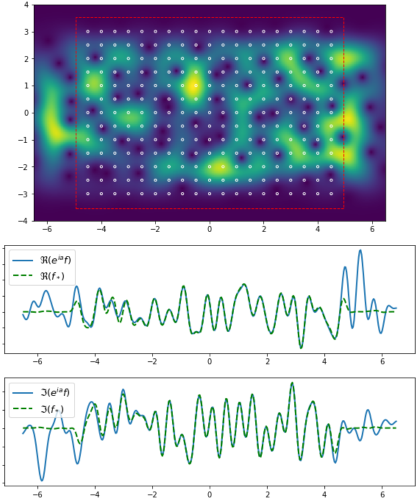




Numerical experiments



$\lvert\Omega\rvert$	ν	$\lvert\Lambda\rvert$	$\lvert\Gamma\rvert$	r	ε	ε'	∞-error	time(s)
169	0.03	81	121	1.414	0.05	0.08	0.513	233.0



$\lvert\Omega\rvert$	ν	$\lvert\Lambda\rvert$	$\lvert\Gamma\rvert$	r	ε	ε'	∞-error	time(s)
247	0.02	117	165	1	0.1	0.25	1.089	905.0

Questions and open problems

Is there a more appropriate notion of similarity?

Can we replace $\min_{|\tau|=1} \|g - \tau f\|_{L^2(\mathbb{R})}$ by a different metric $d(f,g)$ such that

$$d(f,g) \le C \| |\mathcal{V}f| - |\mathcal{V}g| \|_{L^2(\mathbb{R}^2)}, \quad f,g \in \mathcal{C} \subseteq L^2(\mathbb{R})$$

for a finite $C > 0$ and $\mathcal{C}$ a reasonably large subclass?

$\mathcal{C} = \{$set of probability distributions on $\mathbb{R}$ with gaussian decay$\}$,

$$f = c_1(\phi + \Re\{M_k\phi\}), \quad g = c_2(\phi - \Re\{M_k\phi\})$$

($c_1, c_2 > 0$ normalization constants, $k \gg 0$ large)
Then $\min_{|\tau|=1} \|g - \tau f\|_{L^2(\mathbb{R})} \asymp 1$ while

$$W_1(f,g) \to 0, \quad \text{as} \quad k \to \infty.$$

Quantitative uniqueness principle beyond holomorphic functions

Definition (Quantitative uniqueness principle)

Let $\mathcal{X} \subseteq L^2(\mathbb{R}^d)$ be a function space and $\Delta = \{(z,z), z \in \mathbb{R}^d\}$ the diagonal.

We say that $\mathcal{X}$ satisfies a QUP if there exist

- $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$,
- $\mu : \mathbb{R}^{2d} \to \mathbb{R}_+$,
- $\beta > 0$,

such that

$$\|\mathcal{L}f - \mathcal{L}g\|_{L^2(\mathbb{R}^{2d},\mu)} \leq K(f,g)\|\mathcal{L}f - \mathcal{L}g\|^{\beta}_{L^2(\Delta)}, \quad f,g \in \mathcal{X}.$$

$\longrightarrow$ Find function spaces $\mathcal{X}$ which satisfy a QUP.

Concentration implies stability?

- one can show: $\exists f \in L^2(\mathbb{R}) : C_L(f) = \infty$, e.g., the function

$$f = \sum_{k=1}^{\infty} k^{-2} \varphi(\cdot - 2^k).$$

- Poincaré inequalities as intermediate object

$$C_L(f) \leq C_P(|\mathcal{V}f|^2) \leq h(|\mathcal{V}f|^2)^{-1}$$

- exponential concentration of μ is necessary for $C_P(\mu) < \infty$ (Gromov, Milman '83)

Conjecture

If $\mathcal{V}f(z) \cdot e^{\varepsilon|z|}$ is bounded for some $\varepsilon > 0$, then $C_L(f) < \infty$.

Thank you!

Jaming, P., Rathmair, M. Gabor phase retrieval via semidefinite programming. preprint arXiv:2310.11214, 2023.