Presenting p-multinormed spaces on Banach lattices

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Category of p-multinormed spaces

Definition

p-multinormed space ($1 \leqslant p \leqslant \infty$) is a Banach space X equipped with a sequence of norms $\|\cdot\|_n$ on $\ell_n^p \otimes X \sim X^n$, satisfying:

1. $\forall\ a \in \ell_n^p$ and $x \in X$, $\|a \otimes x\|_n = \|a\|\|x\|$ (cross-norm).
2. $\forall\ u : \ell_n^p \to \ell_m^p$, $\|u \otimes I_X : \ell_n^p \otimes X \to \ell_m^p \otimes X\| \leqslant \|u\|$ (left tensoriality).

Equivalently: a left tensorial cross-norm on $\ell^p \otimes X$ ($c_0 \otimes X$ for $p = \infty$).

Convention. Field of scalars $= \mathbb{R}$.

Why study p-multinormed spaces?

- “Abstract” characterization of subspaces of Banach lattices.

[Casazza, Nielsen 2001]: GL-type properties of Banach spaces via embeddings into Banach lattices.

- [Dales, Daws, Pham, Ramsden 2012]: connection between injectivity of Banach modules and amenability of locally compact groups.

Examples of p-multinorms

- Minimal p-multinorm on a Banach space X, $\mathrm{MIN}_p(X)$: arising from injective product $\ell^p \check{\otimes} X$.
- Maximal p-multinorm on a Banach space X, $\mathrm{MAX}_p(X)$: arising from projective product $\ell^p \hat{\otimes} X$.
- p-power multinorm: $\ell^p(X)$ (when X is a subquotient of an L^p space [Kwapien]).
- X is a Banach lattice: identify $\ell^p_n(X)$ with $X(\ell^p_n)$; $\| \sum_j \delta_j \otimes x_j \| = \| (\sum_j |x_j|^p)^{1/p} \|$.

Operators between p-multinormed spaces

Definition (p-multiboudned operators)

Suppose X and Y are p-multinormed spaces. $T \in B(X, Y)$ is p-multibounded if $\|T\|_p = \|I_{\ell^p} \otimes T\| = \sup_n \|I_{\ell_n^p} \otimes T\| < \infty$.

$\|T\|_p \geqslant \|T\|$, and the inequality may be strict.

Example. $p = 2$, $X = \mathrm{MIN}_2(\ell_n^2)$, $Y = \mathrm{MAX}_2(\ell_n^2)$.

$\ell_m^2 \otimes X = \ell_m^2 \check{\otimes} \ell_n^2 \sim B(\ell_m^2, \ell_n^2)$ (injective product).

$\ell_m^2 \otimes Y = \ell_m^2 \hat{\otimes} \ell_n^2 \sim N(\ell_m^2, \ell_n^2)$ (projective product).

For $m \geqslant n$, $T : \ell_m^2 \to \ell_n^2$ with singular values $\lambda_1, \ldots, \lambda_n$, $\|T\|_{\ell_m^2 \otimes X} = \max_i \lambda_i$, $\|T\|_{\ell_m^2 \otimes Y} = \sum_i \lambda_i$. Thus, $\|id : X \to Y\|_2 = n$.

Definition (p-multiisometry, p-multiisomorphism)

$T : X \to Y$ is a p-multiisometry if $\|T\|_p = 1 = \|T^{-1}\|_p$, p-multiisomorphism if T, T^{-1} are p-multibounded.

Ordered spaces

$\mathcal{C} \subset X$ is a cone if $\mathcal{C} = [0, \infty) \cdot \mathcal{C} = \mathcal{C} + \mathcal{C}$. We assume all cones are closed.
A cone $\mathcal{C} \subset X$ determines order: $x \leqslant y$ if $y - x \in \mathcal{C}$.

Definition (Regular spaces (normal + generating))

An ordered Banach space $(X, \mathcal{C}, \|\cdot\|)$ is called:
- Normal if $\|x\| \leqslant \|y\|$ whenever $-y \leqslant x \leqslant y$.
- Generating if $\forall x \in X$ with $\|x\| < 1$ $\exists y$ with $\|y\| < 1$, $-y \leqslant x \leqslant y$.
- Regular if it is normal and generating.

Note: normality implies that $\mathcal{C}$ is pointed: $-\mathcal{C} \cap \mathcal{C} = \{0\}$.

Examples of regular spaces.

- Normed lattices.
- Space of affine functions on convex compact set, with $\|\cdot\|_\infty$.
- $[S_p]_h$ – the space of Hermitian matrices with p-Schatten norm.

Multinorms on ordered spaces

Throughout, (δ_j) is the canonical basis of ℓ^p (or c_0 if $p=\infty$). $\frac{1}{p}+\frac{1}{p'}=1$.
X is a regular Banach space.

Definition (Canonical p-multinorm on a regular space X)

For $\overline{x}=\sum_{j=1}^n \delta_j\otimes x_j$, define $\|\overline{x}\|_{p,X}$ as $\inf\|u\|$, taken over all $u\in X$ s.t. $u\geqslant \sum_j \alpha_j x_j$ whenever $\sum_j |\alpha_j|^{p'}=1$.

Remarks. **(1)** For any $\overline{x}$, u as above exists.

(2) $\big\|\sum_j \delta_j\otimes a_j x\big\|_{p,X}=\big(\sum_j |a_j|^p\big)^{1/p}\|x\|$. So, $\|\cdot\|_{p,X}$ is a cross-norm.

(3) $\|\cdot\|_{p,X}$ is left tensorial (see below for more).

(4) If X is a Banach lattice, then
$\|\overline{x}\|_{p,X}=\big\|\vee\big\{\sum_j \alpha_j x_j : \sum_j |\alpha_j|^{p'}=1\big\}\big\|=\big\|\big(\sum_j |x_j|^p\big)^{1/p}\big\|$.

Which p-multinorms come from ordered spaces?

Definition

We say that a p-multinormed space E is presented on a regular ordered space X if there exists a p-multiisometric embedding $J : E \to X$.

Question

Which p-multinormed spaces can be presented (on Banach lattices)?

Example: 2-multinormed spaces with no Banach lattice presentation.

Take $E = \mathrm{MAX}_2(\ell_n^2)$. Show: if X a Banach lattice, $T : E \to X$ is a contraction, then $\|T^{-1}\|_2 \succ \sqrt{n}$.

$\|\sum_{i=1}^n \delta_i \otimes \delta_i\| = n$. Let $x_i = T\delta_i$.
Want: $\|\sum_{i=1}^n \delta_i \otimes x_i\| = \|(\sum_i |x_i|^2)^{1/2}\| \prec \sqrt{n}$.

Khintchine: $(\sum_i |x_i|^2)^{1/2} \leqslant \sqrt{2} \cdot 2^{-n} \sum_{\varepsilon_i = \pm 1} \big| \sum_i \varepsilon_i x_i \big|$.

$\|(\sum_i |x_i|^2)^{1/2}\| \leqslant \sqrt{2} \cdot 2^{-n} \sum_{\varepsilon_i = \pm 1} \big\| \sum_i \varepsilon_i x_i \big\| \leqslant \sqrt{2n}$, since
$\big\| \sum_i \varepsilon_i x_i \big\| = \big\| T(\sum_i \varepsilon_i \delta_i) \big\| \leqslant \big\| \sum_i \varepsilon_i \delta_i \big\| = \sqrt{n}$.

Strong and super-strong multinorms

Definition (Strong multinorms)

A p-multinorm on a Banach space X is strong if for any $E \subset \ell_n^p$, and any $u : E \to \ell_m^p$, we have $\|u \otimes I_X : E \otimes X \to \ell_m^p \otimes X\| = \|u\|$ (the norm on $E \otimes X$ is inherited from $\ell_n^p \otimes X$). Stronger than left tensoriality.

Observation. For $p \in \{2, \infty\}$, any p-multinorm is strong.

Definition (Super-strong multinorms)

A p-multinormed space X is super-strong if the following holds. Suppose $J : \ell^p \to \ell^\infty$ is an isometric embedding. Suppose $\overline{x}, \overline{x_1}, \ldots, \overline{x_n} \in \ell^p \otimes X$ are finitely supported, and $S_1, \ldots, S_n : \ell^p \to \ell^\infty$ satisfy $J \cdot \overline{x} = \sum_i S_i \cdot \overline{x_i}$. Then $\|\overline{x}\| \leqslant \sum_i \|S_i\| \|\overline{x_i}\|$.

Proposition

Any super-strong p-multinorm is strong. The converse is false.

Why super-strength?

Strong p-multinorm on X: $\forall E \subset \ell^p_n$, $u : E \to \ell^p_m$, $\overline{y} \in E \otimes X$, we have $\|u \cdot \overline{y}\| \leqslant \|u\|\|\overline{y}\|$.

$J : \ell^p_m \to \ell^\infty$ isometric embedding. Find extension $S : \ell^p_n \to \ell^\infty$ s.t. $\|S\| = \|u\|$, $S|_E = Ju$. Let $\overline{x} = u \cdot \overline{y}$, then $\boxed{J \cdot \overline{x} = S \cdot \overline{y},\ \|\overline{x}\| \leqslant \|S\|\|\overline{y}\|}$

Super-strong p-multinorm on X: $J : \ell^p \to \ell^\infty$ isometric embedding. Suppose $\overline{x}, \overline{x_1}, \dots, \overline{x_n} \in \ell^p \otimes X$ are finitely supported, and $S_1, \dots, S_n : \ell^p \to \ell^\infty$ satisfy $J \cdot \overline{x} = \sum_i S_i \cdot \overline{x_i}$. Then $\|\overline{x}\| \leqslant \sum_i \|S_i\|\|\overline{x_i}\|$.

Proposition

Any super-strong p-multinorm is strong. The converse is false.

Presentation and super-strength

Theorem

For a p-multinormed space X, TFAE:

1. *X can be presented on a regular ordered space.*
2. *X can be presented on a Banach lattice.*
3. *X is super-strong.*

Proposition

Every ∞-multinorm is super-strong, hence it can be presented on a Banach lattice.

Case of $p = \infty$: [L. McClaran 1994], [J. Marcolino-Nhany 2001], [Casazza, Nielsen 2001].

Some other cases: [Dales, Laustsen, O., T., 2017].

p-convexity

Definition (p-convexity)

A p-multinorm on X is called p-convex if, for any $k < n$, and any $x_1, \ldots, x_n \in X$, we have
$\|\sum_{i=1}^{n} \delta_i \otimes x_i\|^p \leqslant \|\sum_{i=1}^{k} \delta_i \otimes x_i\|^p + \|\sum_{i=k+1}^{n} \delta_i \otimes x_i\|^p$.

- Any 1-multinorm is 1-convex (triangle ineq.).
  For $p > 1$, $\exists$ p-multinorms which are not p-convex.
- The canonical p-multinorm on a Banach lattice X is p-convex iff X is p-convex as a Banach lattice, with constant 1.

Proposition

Every p-convex strong p-multinorm is super-strong.

Theorem (Dales, Laustsen, O., T., 2017)

Every p-convex strong p-multinormed space can be presented on a p-convex Banach lattice.

Non-uniqueness of presentation

Suppose $(E, \|\cdot\|)$ is a super-strong p-multinorm, X is a Banach lattice, and $J : E \to X$ is a presentation of E (a p-multiisometry) – that is, for every n, $I_{\ell_n^p} \otimes J : \ell_n^p \otimes E \to X(\ell_n^p)$ is an isometry. What can we say about $\mathrm{Lat}(J(E))$ – the Banach lattice generated in X by $J(E)$?

We say that the presentations J_1, J_2 are equivalent if $\mathrm{Lat}(J_1(E))$, $\mathrm{Lat}(J_2(E))$ are lattice isomorphic (lattice isomorphism between them need not take $J_1(E)$ to $J_2(E)$).

Theorem

If E is a super-strong p-multinormed space with $\dim E \geqslant 2$, then it has infinitely many non-equivalent presentations.

The proof produces presentations s.t. $C(\mathbb{T})$ factors through $\mathrm{Lat}(J(E))$ via lattice homomorphisms.

Question

For what E can $\mathrm{Lat}(J(E))$ be "small" (say a KB-space)?

Open question: super-strength and π_1-majorization

An operator $T : E \to F$ is 1-summing if $\exists C > 0$ s.t. $\forall x_1, \ldots, x_n \in E$, $\sum_i \|Tx_i\| \leqslant C \max_{\varepsilon_i = \pm 1} \|\sum_i \varepsilon_i x_i\|$. $\pi_1(T) := \inf C$.

Suppose X is a p-multinormed space. For $\overline{x} = \sum_j \delta_j \otimes x_j \in \ell_n^p \otimes X$, define $O_{\overline{x}}^p : X^* \to \ell_n^p : x^* \mapsto \sum_j \langle x^*, x_j \rangle \delta_j$.

Proposition (Majorization by π_1)

If X is super-strong, then, $\forall \overline{x} \in \ell_n^p \otimes X$, $\|\overline{x}\| \leqslant \pi_1(O_{\overline{x}}^p)$.

Proof. Suppose Z is a BL, $J : X \to Z$ presentation, $\phi : X \to \mathrm{FBL}[X]$, lattice homomorphism $\widehat{J} : \mathrm{FBL}[X] \to Z$ extends J. $\forall x_1, \ldots, x_n$,
$\|\sum_j \delta_j \otimes x_j\| = \|(\sum_j |Jx_j|^p)^{1/p}\|_Z = \|(\sum_j |\widehat{J}\phi x_j|^p)^{1/p}\|_Z$
$\leqslant \|(\sum_j |\phi x_j|^p)^{1/p}\|_{\mathrm{FBL}} = \pi_1(O_{\overline{x}}^p)$. ∎

Question (π_1-dominated + strong $\Rightarrow$ super-strong?)

Suppose X is a strong p-multinormed space, s.t. $\forall \overline{x} \in \ell_n^p \otimes X$, $\|\overline{x}\| \leqslant \pi_1(O_{\overline{x}}^p)$. Is X super-strong?

Operator "idealist" looks at p-multinorms

Suppose, for simplicity, $\dim E < \infty$. $\ell^p \otimes E \leftrightsquigarrow B(E^*, \ell^p)$:

For $\overline{x} = \sum_{j=1}^n \delta_j \otimes x_j$, $O^p_{\overline{x}} : E^* \to \ell^p : x^* \mapsto \sum_j \langle x^*, x_j \rangle \delta_j$.

p-multinorm $\| \cdot \|$ on $E \leftrightsquigarrow$ norm $\beta(\cdot)$ on $B(E^*, \ell^p)$: $\beta(O^p_{\overline{x}}) := \|\overline{x}\|$.

- $\| \cdot \|$ is a p-multinorm $\Leftrightarrow \beta$ is a left ideal norm: for $E^* \xrightarrow{T} \ell^p \xrightarrow{u} \ell^p$, $\beta(uT) \leqslant \|u\| \beta(T)$.

- $\| \cdot \|$ is a strong p-multinorm $\Leftrightarrow \forall Z \overset{i}{\hookrightarrow} \ell^p$, $E^* \xrightarrow{T} Z \xrightarrow{u} \ell^p$, $\beta(uT) \leqslant \|u\| \beta(iT)$ (we say β is strong).

$J : \ell^p \hookrightarrow \ell^\infty$ isometric embedding; extend Ju to $v : \ell^p \to \ell^\infty$, $\|v\| = \|u\|$.

$$\begin{tikzcd} E^* \arrow[r, "T"] \arrow[dr, "S"] & Z \arrow[r, "u"] \arrow[d, hook, "i"] & \ell^p \arrow[d, hook, "J"] \\ & \ell^p \arrow[r, dashed, "v"] & \ell^\infty \end{tikzcd}$$

Let $T' = uT : E^* \to \ell^p$,
$S = iT : E^* \to \ell^p$;
$JT' = vS$, $\beta(T') \leqslant \|v\| \beta(S)$.

Super-strength of p-multinorms from operator standpoint

Recall: $\|\cdot\|$ is a p-multinorm on E, β is the corresponding norm on $B(E^*, \ell^p)$. $J : \ell^p \hookrightarrow \ell^\infty$ is an isometric embedding.

$$\begin{tikzcd} E^* \arrow[r, "T"] \arrow[d, "S"] & \ell^p \arrow[d, hook, "J"] \\ \ell^p \arrow[r, dashed, "v"] & \ell^\infty \end{tikzcd}$$

- $\|\cdot\|$ (or β) is strong if $\beta(T) \leqslant \|v\|\beta(S)$ whenever $JT = vS$.

- $\|\cdot\|$ is super-strong $\Leftrightarrow \forall T : E^* \to \ell^p$, $S_i : E^* \to \ell^p, v_i : \ell^p \to \ell^\infty$, $\beta(T) \leqslant \sum_i \|v_i\|\beta(S_i)$ if $JT = \sum_i v_i S_i$.

Question

If β is strong, and $\beta(\cdot) \leqslant \pi_1(\cdot)$, is β super-strong?

Possible approach. For $U : E^* \to \ell^\infty$, let $\beta'(U) = \inf\{\sum_i \|v_i\|\beta(S_i) : U = \sum_i v_i S_i\}$. Prove that $\beta'(JT) = \beta(T)$.

Note: then $\beta(T) = \beta'(JT) \leqslant \nu_1(JT) = \pi_1(JT) = \pi_1(T)$.

Open question: non-linear maps

Definition (Pleasant maps)

Suppose X, Y are Banach spaces, $A \subset X^*, B \subset Y^*$ satisfy $\mathbb{R}_+ A = A$, $\mathbb{R}_+ B = B$. A map $\Phi : A \to B$ is called pleasant if:

- $\Phi(ta) = t\Phi(a)$ for $a \in A$, $t \geqslant 0$ (positive homogeneity).
- Φ is weak* to weak* continuous on bounded sets (hence bounded).

A, B are pleasantly homeomorphic if $\exists \Phi : A \to B$ s.t. Φ, Φ^{-1} are pleasant.

Theorem (O., T., Taylor, Tradacete)

If the Banach spaces X, Y have FDD, then X^*, Y^* are pleasantly homeomorphic.

Questions about pleasant maps

Suppose X, Y are Banach spaces, $A \subset X^*$, $B \subset Y^*$ satisfy $\mathbb{R}_+ A = A$, $\mathbb{R}_+ B = B$. A map $\Phi : A \to B$ is called pleasant if:

- $\Phi(ta) = t\Phi(a)$ for $a \in A$, $t \geqslant 0$ (positive homogeneity).
- Φ is weak* to weak* continuous on bounded sets (hence bounded).

A, B are pleasantly homeomorphic if $\exists \Phi : A \to B$ s.t. Φ, Φ^{-1} are pleasant.

Question

If X, Y are separable, are X^*, Y^* pleasantly homeomorphic?

In the non-separable case, the answer can be negative.

Question

Suppose X is a Banach space, $\dim X = \infty$, $x \in X$, $A = \{x^* \in X^* : \langle x^*, x \rangle \geqslant 0\}$. Are X^*, A pleasantly homeomorphic?

Banach-Mazur distances between fin. dim. spaces

Recall: if $\dim E = n = \dim F$, define the Banach-Mazur distance: $d(E, F) = \inf\{\|T\|\|T^{-1}\| : T \in B(E, F)\}$ (multiplicative distance: $d(E, G) \leqslant d(E, F)d(F, G)$).

Set of all n-dim spaces (or: centrally symmetric convex bodies) = Minkowski compactum (which is indeed compact).

Geometry of Minkowski compactum.

• [F. John 1948]: $\forall$ n-dim E, $d(E, \ell_n^2) \leqslant \sqrt{n}$. $\Rightarrow d(E, F) \leqslant n$.

Sharpness: $d(\ell_n^2, \ell_n^1) = d(\ell_n^2, \ell_n^\infty) = \sqrt{n}$. $d(\ell_n^1, \ell_n^\infty) \sim \sqrt{n}$.

• [E. Gluskin 1980]: $\sup\left\{d(E, F) : \dim E = n = \dim F\right\} \sim n$ (random construction).

Open problems: p-multinormed Minkowski compactum

Definition (p-Banach-Mazur distance)

For p-multinormed n-dim spaces E and F, define
$d_p(E, F) = \inf\{\|T\|_p \|T^{-1}\|_p : T \in B(E, F)\}$.

$\mathcal{M}(p, n)$ = set of n-dim p-multinormed spaces, with this metric.

Question

Geometry of $\mathcal{M}(p, n)$?

Theorem (O. '18)

For $n \in \mathbb{N}, \varepsilon > 0$ let $N = \lceil 8n^3/\varepsilon \rceil^n$. If $T : E \to F$ has rank n, then
$\|T\|_p \leqslant (1 + \varepsilon)\|I_{\ell_n^p} \otimes T : \ell_N^p \otimes E \to \ell_N^p \otimes F\|$.

Tentative corollary. $\mathcal{M}(p, n)$ is compact.

Beyond this, see **[Marcolino-Nhani 2001]**.

Duality of p-multinormed spaces

The dual of a p-multinormed space is p'-multinormed $(1/p + 1/p' = 1)$.

Let (δ_i) and (δ_i^*) be the canonical bases in $\ell^p, \ell^{p'}$.

Definition (Duality bracket)

$\langle \sum_i \delta_i \otimes x_i, \sum_i \delta_i^* \otimes x_i^* \rangle = \sum_i \langle x_i, x_i^* \rangle$.

Proposition (Duality of MIN and MAX spaces)

$\mathrm{MIN}_p(E)^* = \mathrm{MAX}_{p'}(E^*)$, $\mathrm{MAX}_p(E)^* = \mathrm{MIN}_{p'}(E^*)$.

Proposition

For $T : X \to Y$, $\|T\|_p = \|T^*\|_{p'}$.

Weakness: duality changes category

The dual of a p-multinormed space is p'-multinormed, $1/p + 1/p' = 1$.

Theorem (L. McClaran '94: need to change category)

There is no way to assign, to each fin. dim. ∞-multinormed space X, an ∞-multinormed space X^* in such a way that:

1. $X = X^{**}$ *for any* X;
2. *For any* $T : X \to Y$, $\|T\|_\infty = \|T^*\|_\infty$.
3. $\forall n$, $\mathrm{MAX}_\infty(\ell_n^1)^* = \mathrm{MIN}_\infty(\ell_n^\infty)$.

Question

Does a similar result hold for other $p \neq 2$*?*

Question

Develop duality theory for 2-multinormed spaces.

Thank you for your attention! Questions welcome!