Polish spaces of Banach lattices

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State of art in Banach spaces

Theorem 1

Every real separable Banach E is isometrically isomorphic to a closed subspace of $C(\Delta)$, where $\Delta = 2^{\omega}$.

Theorem 2

For every separable Banach space E, there exist a surjective contractive bounded linear $U\colon \ell_1(\omega) \to E$.

State of art in Banach lattices I

Theorem 3 (Leung, Li, Oikhberg, Tursi, 2018)

Every separable Banach lattice X embeds lattice isometrically into $C(\Delta; L_1(\Delta))$.

State of art in Banach lattices II

Free Banach lattice generated by a Banach space (Avilés, Rodríguez, Tradacete, 2017)

Let E be a real Banach space. We define a free Banach lattice generated by E, denoted by $\mathrm{FBL}[E]$ by following universal property. Let X be any Banach lattice and $\phi_E \colon E \to \mathrm{FBL}[E]$ be an isometric embedding. Then for any contractive linear operator $T \colon E \to X$, there exist a unique lattice homomorphism $\hat{T} \colon \mathrm{FBL}[E] \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{FBL}[E] & & \\ {\scriptstyle \phi_E}\uparrow & \overset{\hat{T}}{\searrow} & \\ E & \xrightarrow{\quad T \quad} & X \end{array}$$

and $\|T\|_{\mathcal{B}(E,X)} = \|\hat{T}\|_{\mathcal{B}(\mathrm{FBL}[E],X)}$.

State of art in Banach lattices III

The free Banach lattice FBL$[E]$ admits an explicit construction. By "The free Banach lattice generated by a Banach space" (Avilés, Rodríguez, Tradacete, 2017):

1. Let $H[E] = \{f : E' \to \mathbb{R} : f - \text{positively homogeneous}\}$
2. Consider a norm
$$\|f\|_{\mathrm{FBL}[E]} = \sup\left\{\sum_{i=1}^{m} |f(x_i^*)| : \{x_i^*\}_{i=1}^{m} \subset X', \sup_{x\in \overline{B}_E} \sum_{i=1}^{m} |x_i^*(x)| \leqslant 1\right\}.$$
3. Let $H_1[E] = \{f \in H[E] : \|f\|_{\mathrm{FBL}[E]} < \infty\}$.
4. Define $\delta_E : E \ni x \mapsto \delta_x \in H_1[E]$
5. A closure of a free vector lattice generated by $\delta_E(E)$, with respect to $\|\cdot\|_{\mathrm{FBL}[E]}$, satisfies the universal property of FBL$[E]$.

State of art in Banach lattices IV

Theorem 4 (Avilés, Rodríguez, Tradacete, 2017)

$\mathrm{FBL}[\ell_1(\omega)]$ is isometrically lattice isomophic to the $\mathrm{FBL}(\omega)$.

Theorem 5 (de Pagter, Wickstead, 2012)

Let E be a Banach lattice and assume that unit ball of E has a dense subset of cardinality $\mathfrak{a}$. Then there exist a closed ideal J of $\mathrm{FBL}(\mathfrak{a})$ such that E is isometrically lattice isomorphic to $\mathrm{FBL}(\mathfrak{a})/J$.

Polish space of separable Banach spaces I

Let X be a Polish space and let $\mathcal{F}(X)$ be the collection of all nonempty closed sets. Let τ be the topology on $\mathcal{F}(X)$

Admissible topology (Godefroy, Saint-Raymond, 2018)

1. A set $E^+(U) = \{E \in \mathcal{F}(X) : U \cap E \neq \emptyset\}$ is $\tau-$open for every open set U in X
2. There exist a subbase of τ, such that every element of that subbase can be represented as a countable union of sets of the form $E^+(U) \setminus E^+(V)$.
3. The set $\{(x, F) \in X \times \mathcal{F}(X) : x \in F\}$ is closed in product topology on $X \times \mathcal{F}(X)$.

Polish space of separable Banach spaces II

Theorem 6 (Godefroy, Saint-Raymond, 2018)

Let τ, τ' be the admissible topologies on $\mathcal{F}(X)$. Then every $\tau-$open set is $\mathbf{\Sigma}_2^0$ in τ'.

Examples of admissible topologies (Godefroy, Saint-Raymond, 2018)

1. Topology induced by a totally bounded distacnce on X
2. Wijsman topology

Theorem 7 (Godefroy, Saint-Raymond, 2018)

Consider a space $\mathcal{F}(C(\Delta))$. Let $SB(C(\Delta)) \subset \mathcal{F}(C(\Delta))$ will be the the set of all closed vector subspaces of $C(\Delta)$. Consider a dense sequence $\{\phi_j\}_{j\in\omega} \subset C(\Delta)$ with $\phi_0 = 0$. Then $F \in SB(C(\Delta))$ iff,

1. $\phi_0 \in F$
2. $\forall p, q, n \in \omega : F \cap B(\phi_p, 2^{-n}) = \emptyset$ $\vee F \cap B(\phi_q, 2^{1-n}) = \emptyset \vee F \cap B\left(\phi_p - \frac{\phi_q}{2}, 2^{1-n}\right) \neq \emptyset$

Polish space of separable Banach spaces IV

Theorem 8 (Godefroy, Saint-Raymond, 2018)

Consider a separable Banach space X. Then there exist a continuous functions $f_j \colon SB(X) \to X$ such that for all $F \in SB(X)$ we have $F = \overline{\{f_j(F) : j \in \omega\}}$.

Another approach to the Polish space of separable Banach spaces I

In article "Polish spaces of Banach spaces" (Cúth M, Doležal M, Doucha M, Kurka O) we are introduced to another approach to encode the separable Banach spaces:

1. $\mathcal{P}$
2. $\mathcal{P}_\infty$
3. $\mathcal{B}$

Another approach to the Polish space of separable Banach spaces II

Let V be the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers.
Consider a $\mathcal{P}$ a space of all seminorms on V. $\mathcal{P}$ is closed subset of $\mathbb{R}^V$, hence a Polish space. It can be shown that the subbasis of $\mathcal{P}$ consists of sets of form

$$U_{v,I} = \{\mu \in \mathcal{P} : \mu(v) \in I\},$$

where $v \in V$ and I is some open interval.

Another approach to the Polish space of separable Banach spaces III

We can identify $\mu \in \mathcal{P}$ with its extension onto c_{00}.
For every $\mu \in \mathcal{P}$ we define

$$X_\mu = X/N,$$

where $X = (c_{00}, \mu)$ and $N = \{x \in c_{00} : \mu(x) = 0\}$.
$\mathcal{P}_\infty$ denotes space of those $\mu \in \mathcal{P}$, for which X_μ is infinite-dimensional.
$\mathcal{B}$ denotes space of those $\mu \in \mathcal{P}$, for which extension of μ to c_{00} is a norm.

Another approach to the Polish space of separable Banach spaces IV

Theorem 9 (Cúth, Doležal, Doucha, Kurka, 2019)

Consider a infinite-dimensional separable Banach space X. Then

$$\{\nu \in \mathcal{B} : X_\nu \text{ is finitely representable in } X\} = \overline{\langle X \rangle} \cap \mathcal{B},$$

where $\langle X \rangle = \{\mu \in \mathcal{B} : X_\mu \equiv X\}$.

Theorem is true if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ and $\mathcal{P}$.

Consider a separable Banach lattice E. We can consider a space of its closed linear subspaces $SB(E)$ and the sequence of continuous functions $\{f_j\}_{j\in\omega}$ that arise from Michael's selection theorem. Let Ideal(E) denote the set of a closed lattice ideals of E.
$F \in$ Ideal(E) iff,

1. $\forall n \in \omega : f_n(F) \wedge -f_n(F) \in F$
2. $\forall m, n \in \omega : \neg(f_m(E) \wedge 0 = 0)$
$\vee \neg(f_m(E) \wedge f_n(F) = f_m(E)) \vee f_m(E) \in F$

Projective approach - open questions

$$T_j\colon \ \mathrm{FBL}[\ell_1(\omega)] \ni x \mapsto x \vee \delta_{e_j} \in \mathrm{FBL}[\ell_1(\omega)]$$

$$T\colon \ \mathrm{FBL}[\ell_1(\omega)] \ni x \mapsto |x| \in \mathrm{FBL}[\ell_1(\omega)]$$

Sources I

- Avilés A, Rodríguez, J, Tradacete P, *The free Banach lattice generated by a Banach space*. Journal of Functional Analysis 2018;274(10):2955-2977. doi:10.1016/j.jfa.2018.03.001
- Cúth M, Doležal M, Doucha M, Kurka O. *Polish spaces of Banach spaces*. Forum of Mathematics, Sigma. 2022;10:e26. doi:10.1017/fms.2022.16
- García-Sánchez E, de Hevia D, Tradacete P., *Free objects in Banach space theory*, 2023; arXiv 2302.10807
- Godefroy G, Saint-Raymond J. "Descriptive complexity of some isomorphism classes of Banach spaces." *Journal of Functional Analysis* 2018; 275(4) 1008-1022. 10.1016/j.jfa.2018.01.018
- Leung D, Li L, Oikhberg T, Tursi MA, *Separable universal Banach lattices*. Israel Journal of Mathematics. 2019;230(1) 141–152. doi: 10.1007/s11856-018-1811-0

Sources II

de Pagter B, Wickstead AW, *Free and projective Banach lattices*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics. 2015;145(1):105-143. doi:10.1017/S0308210512001709

Thank you for attention.