Questions and results around James' Theorem

based on joint works with S. Dantas, J.D. Rodríguez Abellán, A. Rueda Zoca and M. Jung

Workshop Banach spaces and Banach lattices

Gonzalo Martínez Cervantes

University of Murcia, Spain

May 20th, 2024

This work was partially supported by grants 21955/PI/22, funded by Fundación Séneca - ACyT Región de Murcia; PID2021-122126NB-C32, funded by MCIN/AEI/10.13039/501100011033 and FEDER A way of making Europe, and CIGE/2022/9, funded by Generalitat Valenciana

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Abstract

The well-known **James' Theorem** states that a Banach space is reflexive if and only if every bounded linear functional on it attains its norm. In this talk we will investigate operator and lattice versions of this result.

PART I.

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Lattice versions of James' Theorem.

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Definition

$T : X \to Y$ is a lattice homomorphism if it preserves suprema and infima, i.e. $T(x \vee y) = T(x) \vee T(y)$ and $T(x \wedge y) = T(x) \wedge T(y)$ for every $x, y \in X$.

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Any Banach lattice X can be embedded into a Banach lattice Y with no nontrivial lattice homomorphisms. In particular, no nontrivial lattice homomorphism $x^* \in X^*$ can be extended to a lattice homomorphism on Y.

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$$\bigvee_{n\in\mathbb{N}} \delta_\infty(f_n) = 0 \neq 1 = \delta_\infty(1) = \delta_\infty\left(\bigvee_{n\in\mathbb{N}} f_n\right).$$

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Let E be a Banach space. The *free Banach lattice* generated by E is a Banach lattice, denoted by $FBL[E]$, together with a bounded operator $\phi\colon E \longrightarrow FBL[E]$

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It exists and is unique up to isometries. It can be constructed as a sublattice of

$$\{f\colon E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0 \text{ and } f|_{B_{E^*}} \text{ is } w^*\text{-continuous}\}$$

with a suitable norm.

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We have some partial answers.

Definition

Let E be a Banach space. A functional $x^* \in E^*$ not attaining the norm satisfies property (P) if the set

$$C := \{y^* \in E^* : |x^*(x)| + |y^*(x)| \leq \|x^*\| \textit{ for every } x \in B_E\}$$

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Theorem (S. Dantas, G.M.C., J.D. Rodríguez Abellán and A. Rueda Zoca, 2020)

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Since $C \neq \{0\}$ for a point $x^* \in S_{X^*}$ if and only if x^* is not an extreme point of the sphere, we conclude that E does not have property (P).

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Is the property "*every lattice homomorphism attains its norm*" invariant under lattice isomorphisms?

PART II.

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Operator versions of James' Theorem.

Theorem (James' Theorem)

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In particular, if X is reflexive and Y has the Schur property then every operator in $\mathcal{L}(X, Y)(= \mathcal{K}(X, Y))$ attains the norm.

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Theorem (Holub, 1973 and **Mujica, 2001**)

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Indeed, the implications $(i)\Rightarrow(ii)\Rightarrow(iii)\iff(iv)$ always hold (under the reflexivity of X and Y)

Problem 3

If every operator in $\mathcal{L}(X, Y)$ attains the norm, then $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$?

Theorem (Holub, 1973 and **Mujica, 2001**)

Let X and Y be both reflexive spaces.

(a) *If every operator in $\mathcal{L}(X, Y)$ attains the norm, then $\mathcal{L}(X, Y)$ is reflexive.*

(b) *If X or Y has the* **compact approximation property (CAP)**, *then the following statements are equivalent.*

   (i) $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$;

   (ii) *Every operator in $\mathcal{L}(X, Y)$ attains the norm;*

   (iii) $\mathcal{L}(X, Y)$ *is reflexive;*

   (iv) $(\mathcal{L}(X, Y), \|\cdot\|)^* = (\mathcal{L}(X, Y), \tau_c)^*$, *where τ_c denotes the topology of compact convergence.*

Indeed, the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \iff (iv)$ always hold (under the reflexivity of X and Y) and the CAP is only used to prove $(iv) \Rightarrow (i)$.

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$$\boxed{\mathcal{L}(X,Y)=\mathcal{K}(X,Y)} \Longrightarrow \boxed{\mathcal{L}(X,Y)=\mathrm{NA}(X,Y)}$$

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Theorem (S. Dantas, M. Jung, G.M.C., 2021)

If there exists a relatively WOT-compact set $K \subseteq \mathcal{L}(X, Y)$ *such that* $0 \in \overline{K}^{WOT}$ *but* $0 \notin \overline{\mathrm{co}}^{\|\cdot\|}(K)$,

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Recall that a net (T_α) in $\mathcal{L}(X, Y)$ converges in the **SOT** to T (resp. in the **WOT**) if and only if $(T_\alpha(x))$ converges in norm (resp. in the weak topology) to $T(x)$ for every $x \in X$.

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The proof of the previous theorem uses the fact that if every operator in $\mathcal{L}(X, Y)$ attains the norm then

$$B := \left\{ x \otimes y^* : x \in S_X, y^* \in S_{Y^*} \right\} \subseteq \mathcal{L}(X, Y)^*$$

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is a **James boundary** of $\mathcal{L}(X, Y)$. Then $\overline{K}^{WOT}$ is norm-bounded and $w(\mathcal{L}(X, Y), B)$-compact. By a **Theorem of Pfitzner**, $\overline{K}^{WOT} = \overline{K}^{w}$ is weakly compact . Since $0 \in \overline{K}^{WOT} = \overline{K}^{w}$, then $0 \in \overline{\mathrm{co}}^{\|\cdot\|}(K)$, which gives the desired contradiction.

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We say that (X, Y) has **James property** if it satisfies the hypothesis of the previous theorem, i.e. if there exists a relatively WOT-compact set $K \subseteq \mathcal{L}(X, Y)$ such that $0 \in \overline{K}^{WOT}$ but $0 \notin \overline{\text{co}}^{\|\cdot\|}(K)$.

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Theorem (S. Dantas, M. Jung, G.M.C., 2021)

If $B_{\mathcal{K}(X,Y)}$ is not WOT-closed in $\mathcal{L}(X, Y)$ then (X, Y) has **James property** *and therefore there exists a not norm-attaining operator in $\mathcal{L}(X, Y)$.*

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We say that a pair (X,Y) has the **pointwise-BCAP property** if $\mathcal{L}(X,Y) = \bigcup_{\lambda>0} \lambda \overline{B_{\mathcal{K}(X,Y)}}^{\tau_c}$. For such a pair we have that

$$\mathcal{L}(X,Y) = \mathcal{K}(X,Y) \iff (X,Y) \text{ fails James property.}$$

If X is reflexive:

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Notice that in this case we have $\mathcal{L}(X) \neq \mathcal{K}(X)$. If such X exists, then both X and $\mathcal{L}(X)$ must be reflexive. But if $\mathcal{L}(X)$ is reflexive then X is separable (Kalton, 1974).

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If there is an infinite-dimensional Banach space X such that every operator on $\mathcal{L}(X)$ attains its norm, then X does not have the bounded $\mathcal{A}$-approximation property for any nontrivial ideal $\mathcal{A}$ (i.e. for any ideal $\mathcal{A} \neq \mathcal{L}(X)$).

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Thank you for your attention.