Approximate Morse-Sard type results for non-separable Banach spaces. Smooth functions with no critical points

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Bump functions and Rolle's theorem

- For a Banach space X a bump $f : X \to \mathbb{R}$ is a non-zero continuous function with bounded support.

Theorem (Asplund, Ekeland, Kurzweil, Lebourg, Leach and Whitfield, Namioka, Phelps, Preiss, Stegall, etc...)

In a separable Banach space X, T.F.A.E.:

1. *X has a C^1 smooth norm,*
2. *X has a C^1 smooth bump,*
3. *X is an Asplund space (every continuous convex function on X is Fréchet differentiable on a dense G_δ subset of X),*
4. *X^* has the Radon-Nikodym property (every bounded subset B of X^* is dentable: for all $\varepsilon > 0$, there are $F \in X^{**}$ and $\delta \in \mathbb{R}$ such that the set $\{g \in B : F(g) > \delta\}$ is non-empty with diameter smaller than ε).*
5. *X^* is separable.*

- In a general Banach space X, (1)⇒(2)⇒(3)⇔(4)⇔(5*): Every separable subspace of X has separable dual.

Bump functions and Rolle's theorem

Theorem (Fabian, Whitfield, Zizler, 1983)

- *A Banach space X is superreflexive $\iff$ X has a bump with uniformly continuous derivative.*
- *A Banach space X is superreflexive $\iff$ X has a bump with locally uniformly continuous derivative and X does not contain c_0.*

Question

Does Rolle's theorem hold in infinite dimension? If X is a infinite dimensional Banach space $f : X \to \mathbb{R}$ is a C^1 smooth bump, with $\operatorname{supp}_0 f = \{x \in X : f(x) \neq 0\} = U$, does there exist a point $x_0 \in U$ such that $f'(x_0) = 0$ (that is, a critical point)?

- A body (closed, bounded with non empty interior) subset $A \subset X$ is a starlike body provided there exists a point $x_0 \in \mathring{A}$ (we may assume $x_0 = 0$ by translations) such that each ray emanating from 0 meets ∂A once.

- A starlike body A is C^p smooth whenever its Minkowski functional μ_A is C^p smooth on $X \setminus \{0\}$.

Theorem (Shkarin 1992; Azagra, Dobrowolski 1997; Azagra, J.S., 2001)

Let X be a Banach space, $\dim X = \infty$. T.F.A.E:

1. *X has a C^p smooth bump ($p \in \mathbb{N} \cup \{\infty\}$).*
2. *There is no a "Rolle's theorem" in X: There is a C^p smooth bump $f : X \to \mathbb{R}$ such that $\operatorname{supp}_0 f = \{x \in X : f(x) \neq 0\} := U$ is contractible and yet $f'(x) \neq 0$ for all $x \in U$.*
3. *For every C^p smooth bounded starlike body A, there exists a C^p smooth bump f on X with $\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}} = A$ and yet $f'(x) \neq 0$ for all $x \in \mathring{A}$.*
4. *There exists a non-empty (contractible) closed subset D of the unit ball B_X of X and a C^p diffeomorphism $H : X \to X \setminus D$ so that $H|_{X \setminus B_X} = Id_X|_{X \setminus B_X}$. This kind of diffeomorphisms are called "deleting" or "extracting".*

- If, in addition, the bump in (1) is Lipschitz, then the bump in (2) can be constructed to be Lipschitz as well.
- Moreover, the bump in (2) can be constructed to satisfy $f'(U) \cap W = \emptyset$ for any pre-fixed finite-dimensional vector subspace $W \subset X^*$.

Let us recall the following equivalences:

Theorem (Bonic and Frampton, Kurzweil, Torunzcyk, Godefroy, Troyanski, Withfield, Zizler, etc...)

Let X be an infinite dimensional Banach space WCG (Weakly Compactly Generated; for example, separable or reflexive spaces) and $k \in \mathbb{N} \cup \{\infty\}$. T.F.A.E.:

1. *X has a C^k smooth bump.*
2. *X has uniform approximations by C^k smooth functions: For any pair of continuous functions $f : X \to Y$ (Y any Banach space) and $\varepsilon : X \to (0, \infty)$ there is a C^k smooth function $g : X \to Y$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in X$.*
3. *There is a homeomorphic embedding $H : X \to c_0(\Gamma)$ for some set of indexes Γ such that each "coordinate function" $H_\gamma : X \to \mathbb{R}$ is C^k smooth, where $\gamma \in \Gamma$.*
4. *X has C^k-partitions of unity.*

- In a general Banach space (1)$\Leftarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4).
- Not every Banach space with condition (2) is WCG.

Differentiable functions with no critical points

Question

- Let us consider $p \in \mathbb{N} \cup \{\infty\}$ and Banach spaces X, Y with the property that every continuous function $f : X \to \mathbb{R}^n$ can be uniformly approximated by C^p smooth functions, that is, for every $\varepsilon : X \to (0, \infty)$ there is $g : X \to \mathbb{R}^n$ such that

$$\|f(x) - g(x)\| < \varepsilon(x), \text{ for all } x \in X.$$

Can we uniformly approximate every continuous function $f : X \to \mathbb{R}^n$ by C^p smooth functions with no critical points?

- We say that $x \in X$ is a critical point of a differentiable function $g : X \to Y$, where X, Y are Banach spaces, if $g'(x) : X \to Y$ is a non surjective (bounded) operator.

A positive answer to this problem provides "approximate versions of the Morse-Sard theorem" for certain infinite dimensional Banach spaces.

Theorem (Morse-Sard Theorem, 1942)

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^r smooth function, with $r > \max\{n-m, 0\}$, and $\mathcal{C}_f$ is the set of critical points of f, then the set of critical values $f(\mathcal{C}_f)$ has Lebesgue measure zero in $\mathbb{R}^m$.

- An infinite-dimensional version of the Morse-Sard's theorem:

Theorem (Smale's Theorem, 1965)

If X, Y are Banach spaces and $f : X \to Y$ a C^r Fredholm mapping (that is, $\dim(\ker f'(x)) < \infty$, $f'(x)(X)$ is closed and $\operatorname{codim}(f'(x)(X)) < \infty$ for every $x \in X$).

Then, $f(\mathcal{C}_f)$ is of first Baire category and, in particular $f(\mathcal{C}_f)$ has no interior points, provided that $r > \max\{\operatorname{index}(f'(x)), 0\}$ for all $x \in X$, where $\operatorname{index}(f'(x)) := \dim(\ker f'(x)) - \operatorname{codim}(f'(x)(X))$.

- If $\dim X = \infty$ the above assumptions imply that $\dim Y = \infty$. Thus, Smale's theorem cannot be applied to functions $f : X \to \mathbb{R}$.

Theorem (Kupka's Theorem, 1965)

(A counterexample to the Morse-Sard theorem on infinite dimensional spaces). There are C^∞ smooth functions $f : \ell_2 \to \mathbb{R}$ such that their set of critical values $f(\mathcal{C}_f)$ contain intervals (and in particular, they have positive measure).

Example (Bates, Moreira 2001)

There are polynomials (of degree three) $p : \ell_2 \to \mathbb{R}$ such that their set of critical values $p(\mathcal{C}_p) = [0, 1]$.

Theorem (Eells, McAlpin, 1968)

For every pair of continuous functions $f : \ell_2 \to \mathbb{R}$ and $\varepsilon : \ell_2 \to (0, \infty)$, there is a C^1 smooth function $g : \ell_2 \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon(x), \text{ for all } x \in \ell_2,$$

and the set of critical values $g(\mathcal{C}_g)$ is of (Lebesgue) measure zero.

Approximate results in separable Banach spaces

Theorem (Azagra, Cepedello, 2004)

For every pair of continuous functions $f : \ell_2 \to \mathbb{R}^n (n \in \mathbb{N})$ and $\varepsilon : \ell_2 \to (0, \infty)$ there exists a C^∞ smooth function $g : X \to \mathbb{R}^n$ such that

$$|f(x) - g(x)| < \varepsilon(x) \text{ for all } x \text{ and } g \text{ has no critical points.}$$

Theorem (Hajek, 1998)

Any (Fréchet) differentiable function $f : c_0 \to \mathbb{R}$ with locally uniformly continuous derivative has locally compact derivative (thus $f'(X)$ is contained in a K_σ subset).

Theorem (Hajek, Johanis, 2003)

Let X be a separable Banach space with a C^p smooth bump ($p \in \mathbb{N} \cup \{\infty\}$) and containing c_0.

Then, for every pair of continuous functions $f : X \to \mathbb{R}$, $\varepsilon : X \to (0, \infty)$ and any pre-fixed countable set $N \subset X^*$, there exists a C^p smooth function $g : X \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon(x) \text{ for all } x \in X,$$

$$g'(X) \text{ is of first Baire category and } g'(X) \cap N = \emptyset.$$

Definition

A norm $\|\cdot\|$ in a Banach space X is **LUR** (locally uniformly rotund) if $\|x_n - x\| \xrightarrow{n} 0$ whenever $\|\frac{x_n+x}{2}\| \xrightarrow{n} 1$ and $\|x_n\| = \|x\| = 1$ for all $n \in \mathbb{N}$.

- Recall that LUR $\Rightarrow$ **Rotund (or strictly convex)** (the unit sphere of the norm does not have segments).

Theorem (2007, Azagra, J.S.)

Let X be a separable Banach space, $\dim X = \infty$, with a LUR and C^p smooth norm $||\cdot||$ ($p \in \mathbb{N} \cup \{\infty\}$).

For every pair of continuous functions $f : X \to \mathbb{R}$ and $\varepsilon : X \to (0, \infty)$, there exists $g : X \to \mathbb{R}$ C^p smooth such that

$$|f(x) - g(x)| < \varepsilon(x) \text{ and } g \text{ has no critical points.}$$

- If $p > 1$ the above conditions imply superreflexivity of X.

Examples

1. $X = \ell_r, L_r[0,1]$ $(1 < r < \infty)$, where $p = \infty$ if r even, $p = r - 1$ if r is odd and $p = [r]$ is not an integer.
2. X Banach space with separable dual (i.e. X separable Asplund space), $\dim X = \infty$ and $p = 1$.
3. (Azagra, J.S. 2007; Moulis 1971) X separable Banach space, $\dim X = \infty$, X with a unconditional basis and a C^p smooth and Lipschitz bump with $p > 1$.

Approximate results in separable spaces: vector-valued case

Theorem (Azagra, Dobrowolski, García-Bravo, 2019)

Let $X = c_0, \ell_r, L_r[0,1]$ $(1 < r < \infty)$, $p = \infty$ if $X = c_0$ or r is even, $p = r - 1$ if r is odd and $p = [r]$ if r is not an integer and F any (non-zero) quotient of X.

Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0, \infty)$ there is a C^p smooth function $g : X \to F$ such that

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

Theorem (Azagra, Dobrowolski, García-Bravo, 2019)

Let X be a separable reflexive such that $X \simeq X \oplus X$ and F is any (non-zero) quotient of X. Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0, \infty)$ there is a C^1 smooth function $g : X \to F$ such that

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

- The condition $X \simeq X \oplus X$ is only required when $\dim F = \infty$.

Theorem (Azagra, Dobrowolski, García-Bravo, 2019.)

Let X be a separable space with

1. *a C^1 smooth and LUR norm $\|\cdot\|$,*
2. *a 1-unconditional Schauder basis $\{e_j\}_{j\in\mathbb{N}}$ ($\|\sum_{j\in H}\lambda_j e_j\| \leq \|\sum_{j\in H'}\lambda_j e_j\|$ whenever $H \subset H' \subset \mathbb{N}$ are finite subsets and any set $\{\lambda_j\}_{j\in H'} \subset \mathbb{R}$),*
3. *F is any (non-zero) quotient of X.*
4. *If F is infinite-dimensional, there exists a subset $\mathbb{E}$ of $\mathbb{N}$ with $|\mathbb{E}| = |\mathbb{N}\setminus\mathbb{E}|$ and, for every infinite subset $J \subset \mathbb{E}$, F is a quotient of $\overline{\operatorname{span}}\{e_j : j \in J\}$.*

Then, for every pair of continuous functions $f : X \to F$ and $\varepsilon : X \to (0,\infty)$ there is a C^1 smooth function $g : X \to F$ such that

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in X \text{ and } g \text{ has no critical points.}$$

Examples

More technical results are obtained by Azagra, Dobrowolski and García-Bravo (2019) in separable Banach spaces. As a consequence, uniform approximation by C^p smooth functions with no critical points is obtained for

1. X separable Banach space containing c_0 with a shrinking Schauder basis, $p = 1$ and F any quotient of X.
2. $X = J, J^*$, where J is the James space, $p = 1$, and F any quotient of X.
3. X is a finite direct sum of the classical Banach spaces c_0, ℓ_r or $L_r[0,1]$, $(1 < r < \infty)$, p depending on the minimum r in the decomposition, F is a quotient of X.
4. $X = C(K)$ separable with K countable compact, $p = 1$ and F any quotient of X.

Approximate/extension results in separable spaces

Definition

- An infinite-dimensional Banach space is said to be polyhedral if the unit ball of any of its finite-dimensional subspaces is a polyhedron.

- **(Fabian, Fonf, Whitfield, Zizler)** Polyhedral spaces are know to be c_0-saturated and Asplund spaces.
- **(Hajek)** Every separable polyhedral space X has an equivalent C^∞ smooth and LFC-norm $\|\cdot\|$, that is, the norm locally depends on a finite number of functionals (except at 0), i.e. the norm is locally of the form

$$\|y\| = \varphi(f_1(y), \dots, f_n(y))$$

for certain C^∞ smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and functionals $\{f_1, \dots, f_n\} \subset X^*$. More precisely, for every $x \in X \setminus \{0\}$ there is a neighborhood V_x of x such that $\|y\| = \varphi(f_1(y), \dots, f_n(y))$ for all $y \in V_x$ for suitable function φ and functionals $\{f_1, \dots, f_n\}$ (determined locally).

Approximate/extension results in separable spaces

Theorem (García-Bravo, 2020)

Let X be a separable isomorphically polyhedral Banach space, $\dim X = \infty$, with unconditional basis.

Then, for every C^1 smooth function $f : X \to \mathbb{R}^n$ and every continuous function $\varepsilon : X \to (0, \infty)$, every open set $U \subset X$ such that $\mathcal{C}_f \subset U$, where $\mathcal{C}_f$ is the set of critical points of f, there exists a C^1 smooth function $g : X \to \mathbb{R}^n$ such that

- $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for all* $x \in X$;
- $f(x) = g(x)$ *for all* $x \in X \setminus U$,
- $\|f'(x) - g'(x)\| \leq \varepsilon(x)$ *for all* $x \in X$, *and*
- g *has no critical points.*

Theorem (García-Bravo, 2020)

Let X be a separable Banach space with a C^1 smooth and strictly convex equivalent norm $\|\cdot\|$ and 1-unconditional basis $\{e_j\}_{j\in\mathbb{N}}$ ($\|\sum_{j\in H}\lambda_j e_j\| \leq \|\sum_{j\in H'}\lambda_j e_j\|$ whenever $H \subset H' \subset \mathbb{N}$ are finite subsets and any set $\{\lambda_j\}_{j\in H'} \subset \mathbb{R}$).

Then, for every C^1 smooth function $f : X \to \mathbb{R}^n$ ($n \in \mathbb{N}$), every continuous function $\varepsilon : X \to (0, \infty)$, every open set $U \subset X$ such that $\mathcal{C}_f \subset U$, where $\mathcal{C}_f$ is the set of critical points of f, there exists a C^1 smooth function $g : X \to \mathbb{R}^n$ such that

- $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for all* $x \in X$;
- $f(x) = g(x)$ *for all* $x \in X \setminus U$, *and*
- g *has no critical points.*

Examples

- c_0, $d_*(\omega, 1)$ (preduals of Lorentz spaces).
- ℓ_p $(1 < p < \infty)$.

Approximate results for non-separable Banach (general target space)

Theorem (Azagra, García-Bravo, J.S.)

Consider $E = c_0(\Gamma), \ell_p(\Gamma)$ $(1 < p < \infty)$, where Γ is an infinite set and F a quotient of E.

Then, for every pair of continuous functions $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ there is a C^k smooth $g : E \to F$ such that

$$\|f(x) - g(x)\| \leq \varepsilon(x) \text{ and } g \text{ has no critical points}$$

where $k = \infty$ if $E = c_0(\Gamma)$ or p is even, $k = p - 1$ if p is odd and $k = [p]$ if p is not integer.

Definition

A Banach space Y has a decomposition of the form $Y = \bigoplus_{n\in\mathbb{N}} Y_n$ if

1. Y_n is a closed subspace of Y, for every n,
2. $Y_n \cap Y_m = \{0\}$ for $n \neq m$,
3. every $y \in Y$ can be written in a unique way $y = \sum_{n=1}^{\infty} y_n$ with $y_n \in Y_n$ for all n,
4. the canonical projections $P_n : Y \to Y_n$, $P_n(y) = y_n$ are continuous for all n. Thus, Y_n is complemented in Y for all n.

Definition

A Banach space X has C^k-partitions of unity ($k \in \mathbb{N} \cup \{\infty\}$) if for every open cover $\{U_\alpha\}_{\beta \in \Omega}$ of X, there is a family of C^k smooth functions $\{\psi_i\}_{i \in \Delta}$, $\psi_i : X \to [0, \infty)$ for all i, such that

- $\{\psi_i\}_{i \in \Delta}$ is locally finite: for every $x \in X$ there is a neighbourhood V_x of x such that V_x intersects $\operatorname{supp} \psi_i$ only for finitely many $i \in \Delta$,
- $\{\psi_i\}_{i \in \Delta}$ is subordinated to $\{U_\alpha\}_{\alpha \in \Omega}$: for every i there is α such that $\operatorname{supp} \psi_i \subset U_\alpha$,
- $\sum_{i \in \Delta} \psi_i(x) = 1$ for all $x \in X$.

- A Banach space X has C^k-partitions of unity in a Banach space $X \Leftrightarrow$ for every pair of continuous function $f : X \to Z$ (Z any Banach space) and $\varepsilon : X \to (0, \infty)$ there is a C^k smooth function $g : X \to Z$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in X$.

Theorem (Azagra, García-Bravo, J.S.)

Let X, Y, F be Banach spaces, $E = X \oplus Y$, and $k \in \mathbb{N} \cup \{\infty\}$ such that

1. *X has C^k-partitions of unity,*
2. $\dim Y = \infty$ *and has a C^k smooth and LUR norm.*
3. *Y is reflexive, $Y = \oplus_{n \in \mathbb{N}} Y_n$,*
4. *F is a (non-zero) quotient of Y_n for every n.*
5. *The canonical projection $Q : Y \to \oplus_{i \ odd} Y_i$, given by $Q(y) = \sum_{i \ odd} y_i$ for every $y = \sum_{i \in \mathbb{N}} y_i \in Y$ (with $y_i \in Y_i$ for all i) is well defined and continuous.*

Then, for every pair of continuous functions $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ there is $g : E \to F$ C^k smooth such that

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

- For $k > 1$, condition (2) $\Rightarrow$ Y is superreflexive.

Definition

Let Y be a Banach space. A family $\{e_i, e_i^*\}_{i\in\Gamma} \subset Y \times Y^*$ is a M-basis (Markushevich basis) if $e_i^*(e_j) = \delta_{ij}$ for all $i, j \in \Gamma$ and $\overline{\text{span}}\{e_i : i \in \Gamma\} = Y$. It is called shrinking provided $Y^* = \overline{\text{span}}\{e_i^* : \in \Gamma\}$.

Theorem (Azagra, García-Bravo, J.S.)

Let X, Y, F be Banach spaces, $E = X \oplus Y$, such that

1. *X has C^1-partitions of unity,*
2. $\dim Y = \infty$ *and Y has a C^1 smooth and LUR norm.*
3. *$Y = \oplus_{n\in\mathbb{N}} Y_n$, each Y_n has a shrinking M-basis and the union of all these M-bases is a shrinking M-basis in Y*
4. *F is a (non-zero) quotient of Y_n for every n.*
5. *The canonical projections $Q_m : Y \to (\oplus_{i=1}^m Y_i) \oplus (\oplus_{\substack{i \text{ odd} \\ i>m}} Y_i)$, given by $Q(y) = \sum_{i=1}^m y_i + \sum_{\substack{i \text{ odd} \\ i>m}} y_i$ for every $y = \sum_i y_i \in Y$ ($y_i \in Y_i$ for all i) are well defined and have norm one for all m.*

Then, for every pair of continuous functions $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ there is $g : E \to F$ C^1 smooth such that

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

Example

$E = X \oplus Y$ satisfies the approximate property by C^k ($k \in \mathbb{N} \cup \{\infty\}$) smooth functions with no critical points for

1. $E = X \oplus c_0(\Gamma)$ (Γ infinite) and F any quotient of $c_0(\Gamma)$, where X has C^k smooth partitions of unity.
2. $E = X \oplus \ell_p(\Gamma)$ (Γ infinite) and F any quotient of $\ell_p(\Gamma)$, where X has C^k smooth partitions of unity.
3. $E = X \oplus (\oplus_p Y)$ and F any quotient of Y, where Y is a **reflexive** space with $\boldsymbol{C^k}$ **smooth LUR** norm, where X has C^k smooth partitions of unity.
4. $E = X \oplus (\oplus_p Y)$ and F any quotient of Y, where Y is a Banach space with a C^1 smooth LUR norm and a shrinking M-basis (for example **Asplund WCG** Banach spaces), where X has C^1 smooth partitions of unity.

In (2) and (3) $k \leq p - 1$ if p is odd and $k \leq [p]$ if p is not an integer.

Approximate results for non-separable Banach (finite dimensional target space)

Theorem (Azagra, García-Bravo, J.S.)

Let X, Y be Banach spaces, $E = X \oplus Y$ and $k \in \mathbb{N} \cup \{\infty\}$, such that

1. *X has C^k smooth partitions of unity,*
2. $\dim Y = \infty$ *and has a C^k smooth and LUR norm,*
3. *Additionally, for $k = 1$, Y has an infinite-dimensional complemented and separable subspace.*

Then, for every pair of continuous functions $f : E \to \mathbb{R}^n$ and $\varepsilon : E \to (0, \infty)$ there is C^k smooth $g : E \to \mathbb{R}^n$ such that

$$\|f(z) - g(z)\| < \varepsilon(z) \text{ for all } z \in E \text{ and } g \text{ has no critical points.}$$

Example

- For $k = 1$, $E = X \oplus Y$, X with C^1 smooth partitions of unity and Y **a Banach space with separable dual**, $\dim Y = \infty$.
- More generally, for $k = 1$, $E = X \oplus Y$, X with C^1 smooth partitions of unity and Y any **Asplund and WCG (Weakly Compactly Generated)** Banach space, $\dim Y = \infty$.
- So, in particular, for $k = 1$, $E = X \oplus Y$, X with C^1 smooth partitions of unity and Y any **reflexive** space, $\dim Y = \infty$.
- For $k > 1$, Condition (2) $\Rightarrow$ Y is superreflexive.

Some consequences

For E Banach space satisfying the approximate property by C^p smooth functions with no critical points,

1. A "non-linear C^p smooth Hahn-Banach separation result": For any pair of disjoint closed subset C_1, $C_2 \subset E$ there is a C^p smooth function $g : E \to \mathbb{R}$ with no critical points "separating C_1 and C_2": that is, $g^{-1}(0)$ is a 1-codimensional C^p smooth submanifold in E separating C_1 from C_2 (that is $C_1 \subset g^{-1}((0,\infty))$ and $C_2 \subset g^{-1}((-\infty,0))$.

2. C^p smooth approximation of closed sets: For any closed subset $C \subset E$ and any open subset $W \subset E$ with $C \subset W$, there is an open subset U such that $C \subset U \subset W$ and U is C^p smooth (that is, ∂U is a 1-codimensional C^p smooth submanifold of E).

Some consequences

3. The support of bumps not satisfying Rolle's theorem: If X is **separable**, for every bounded and open subset $U \subset E$, there is a Fréchet differentiable bump $f : E \to \mathbb{R}$ with $\operatorname{supp} f = \overline{U}$, f is C^1 **smooth on** U and yet f has no critical points in U.
4. The support of bumps not satisfying Rolle's theorem: If X is **separable**, has an **inconditional Schauder basis** and a $\boldsymbol{C^p}$ **smooth Lipschitz bump** ($p > 1$ or $p = \infty$) for every bounded and open set $U \subset E$, there is a Fréchet differentiable bump $f : E \to \mathbb{R}$ with $\operatorname{supp} f = \overline{U}$, f is $\boldsymbol{C^p}$ **smooth on** U and yet f has no critical points in U.

- (West, 1969) For E separable Banach space with a C^1 smooth bump function, and for every bounded and open subset $U \subset E$ there is a C^1 smooth bump on E such that $\operatorname{supp} f = \overline{U}$.

Tools for the case of $c_0(\Gamma)$

The proof for functions $f : E := c_0(\Gamma) \to F$, being F a quotient of $c_0(\Gamma)$ relays on: **(Toruńczyk)** the existence of **LFC-$\{e_i^*\}_{i\in\Gamma}$ and C^∞- partitions of unity** $\{\psi_i\}_{i\in\Delta}$ in $c_0(\Gamma)$, where $\{e_i^*\}_{i\in\Gamma}$ are the functional associated with the canonical basis of $c_0(\Gamma)$, that is, each mapping ψ_i is **locally** of the form

$$\psi_i(y) = \varphi(e_{i_1}^*(y), \ldots, e_{i_n}^*(y))$$

$(y \in E)$ for a suitable C^∞ smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and a finite number of functionals $\{e_{i_1}^*, \ldots, e_{i_n}^*\}$ (locally determined). The approximating functions with no critical points are of the form

$$g(x) = \sum_{i\in\Delta} (f(x_i) + T(x - x_i))\psi_i(x), \text{ where}$$

- $x_i \in \operatorname{supp} \psi_i$, $T = \sum_{n\in\mathbb{N}} T_n \circ P_n$ bounded operator,
- $T_n : c_0(\Gamma_n) \to F$ surjective bounded operator,
- $P_n : c_0(\Gamma) \to c_0(\Gamma_n)$ are the canonical projections,
- $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$ (disjoint union) with $|\Gamma| = |\Gamma_n|$ for all $n \in \mathbb{N}$.

Tools for the case of $\ell_p(\Gamma)$

The proof for functions $f : E = l_p(\Gamma) \to F$, being F a quotient of E $(1 < p < \infty)$ relays on: **(Toruńczyk)** the existence of **C^k-partitions of unity $\{\psi_i\}_{i\in\Delta}$** in $\ell_p(\Gamma)$, where each function ψ_i is **locally** of the form

$$\boldsymbol{\psi_i(y) = \varphi(\|y\|^p, e^*_{i_1}(y), \dots, e^*_{i_n}(y))}$$

$(y \in E)$ for a suitable C^k smooth function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ and a finite number of functionals $\{e^*_{i_1}, \dots, e^*_{i_n}\}$ (locally determined), being $\{e^*_i\}_{i\in\Gamma}$ the functionals associated with the canonical basis of $\ell_p(\Gamma)$. The approximating functions with no critical points are of the form

$$\boldsymbol{g = h \circ d}, \text{ where}$$

- $h : E \to F$, $h(x) = \sum_{i\in\Delta}(f(x_i) + T(x - x_i))\psi_i(x)$,
- $x_i \in \operatorname{supp}\psi_i$, $T = \sum_{m \text{ odd}} T_m \circ P_m$ is a bounded operator,
- $T_m : \ell_p(\Gamma_m) \to F$ surjective bounded operator,
- $P_m : \ell_p(\Gamma) \to \ell_p(\Gamma_m)$ are the canonical projections,
- $\Gamma = \cup_{n\in\mathbb{N}}\Gamma_n$ (disjoint union) with $|\Gamma| = |\Gamma_n|$,

Tools for the case of $\ell_p(\Gamma)$

- **(Azagra, Dobrowolski, García-Bravo)** $d : E \to E \setminus \mathcal{C}_h$ is a $\boldsymbol{C^k}$ **"deleting (or extracting) diffeomorphism"** (close to the identity), being $\mathcal{C}_h$ the closed subset of critical points of h.
- So, by the chain rule $g'(x) = h'(d(x)) \circ d'(x)$ is a surjective operator for all $x \in E$.
- It is crucial that $\mathcal{C}_h$ is locally contained in $\ell_p((\cup_{i=1}^m \Gamma_i) \cup (\cup_{i \text{ odd}} \Gamma_i))$ for a suitable m (locally determined), that is, $\mathcal{C}_h$ is **locally contained in a complemented closed subspace of infinite codimension**.
- Recall that $k = \infty$ if p is even, $k = p - 1$ if p is odd and $k = [p]$ if p is not an integer.

Tools for the general cases

For continuous functions $f : E = X \oplus Y \to F$, with X, Y, F under any of the assumptions given above, we need:

- **(Toruńczyk) C^k-partitions of unity $\{\psi_i\}_{i\in\Delta}$** (k depending on X and Y), where each ψ_i is locally of the form,

$$\psi_i(x,y) = \varphi(x, \theta_{k_1}(\|y\|), \dots, \theta_{k_m}(\|y\|), e^*_{i_1}(y), \dots, e^*_{i_n}(y))$$

(here $(x,y) \in X \oplus Y$) for a suitable C^k smooth function $\varphi : X \oplus \mathbb{R}^{n+m} \to \mathbb{R}$, a finite number of functionals $\{e^*_{i_1}, \cdots, e^*_{i_n}\}$ and functions $\{\theta_{k_1}, \cdots, \theta_{k_m}\}$, being

★ $\{e^*_i\}_{i\in\Gamma} \subset Y^*$ funcionals associated with a shrinking M-basis in Y,

★ $\{\theta_j\}_{j\in\mathbb{N}}$ C^∞ smooth non-decreasing functions $\theta_j : \mathbb{R} \to [0,\infty)$ with $\theta_j(t) = t$ if $t \geq \frac{1}{j}$, $\theta(t) = 0$ if $t \leq \frac{1}{2j}$ and $|\theta'(t)| \leq 3$ for all $t \in \mathbb{R}$ and all $j \in \mathbb{N}$.

Tools for the general cases

The approximating functions with no critical points are of the form

$$\boldsymbol{g} = \boldsymbol{h} \circ \boldsymbol{d}, \text{ where}$$

- $h : E = X \oplus Y \to F$,

$$h(x, y) = \sum_{i \in \Delta} (f(x_i, y_i) + T(y - y_i))\psi_i(x, y),$$

- $(x_i, y_i) \in \operatorname{supp} \psi_i$,
- $T : Y \to F$ is a suitable surjective bounded operator (several technical conditions are required, depending on F, finite or infinite dimensional and the decomposition of $Y = \sum_{n \in \mathbb{N}} Y_n$),
- **(Azagra, Dobrowolski, García-Bravo) C^k "Deleting diffeomorphism"** $d : E \to E \setminus \mathcal{C}_h$, where $\mathcal{C}_h$ is the closed subset of critical points of h. Here, it is crucial that $\mathcal{C}_h$ **is locally contained in the graph of a continuous function** $c : M \to N$, being M, N closed subspaces and $E = M \oplus N$, M with C^k-partitions of unity, N with a C^k smooth norm and $\dim N = \infty$ (locally determined).

Tools for the general cases (finite dimensional target space)

- For $f : E = X \oplus Y \to \mathbb{R}^n$, Y non-reflexive we need a renorming result:

Proposition (Azagra, García-Bravo, J.S.)

Let $(Y, \|\cdot\|)$ be a Banach space and let W be a K_σ subset in the unit sphere of Y^*. Then, the set of (equivalent) norms $|||\cdot|||$ on Y such that their dual norms $|||\cdot|||^*$ are Fréchet differentiable at the points of W is residual in $(\mathcal{N}_Y, \rho)$, the metric space of all equivalent norms on Y with the usual metric

$$\rho(p, q) = \sup\{|p(x) - q(x)| : \|x\| = 1\}, \; p, q \in \mathcal{N}_Y.$$

In particular, for any of these norms $|||\cdot|||$, every functional $f \in W$ attains its $|||\cdot|||^*$-norm.

Corollary (Fabian, Zajicek, Zizler 1982; Azagra, García-Bravo, J.S.)

Let Y be a Banach space with a LUR norm $\|\cdot\|$ whose dual norm $\|\cdot\|^*$ is LUR and let W be a K_σ subset in the unit sphere of Y^*.

Then, the set of (equivalent) norms $|||\cdot|||$ on Y such that $|||\cdot|||$ and $|||\cdot|||^*$ are LUR and $|||\cdot|||^*$ is Fréchet differentiable at the points of W is residual in $(\mathcal{N}_Y, \rho)$. In particular, for any of these norms $|||\cdot|||$, every functional $f \in W$ attains its $|||\cdot|||^*$-norm.

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Thank you