Embedded unbounded order convergent sequences in topologically convergent nets in vector lattices

Marcel de Jeu
Leiden University and University of Pretoria

Research term *Lattice Structures in Analysis and Applications*
ICMAT
16 May 2024

Joint work with Yang Deng

Example

Let (X, Σ, μ) be a σ-finite measure space.

- Known: If $(f_\alpha)_{\alpha \in A}$ is a net in $\mathrm{L}^0(X, \Sigma, \mu)$ that is convergent in measure to f on all measurable subsets of finite measure, then there exist indices $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots$ such that the sequence $(f_{\alpha_n})_{n \geq 1}$ converges almost everywhere to f.
- Rephrase: If $(f_\alpha)_{\alpha \in A}$ is a net in $\mathrm{L}^0(X, \Sigma, \mu)$ that is convergent to f in the uo-Lebesgue topology of $\mathrm{L}^0(X, \Sigma, \mu)$, then there exist indices $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots$ such that the sequence $(f_{\alpha_n})_{n \geq 1}$ is unbounded order convergent to f.

More general phenomenon

- Several similar results known: if a net converges in a locally solid topology, then there is a sequence ‘within the net’ that is unbounded order convergent to the same limit.
- Our paper in BJMA: general theorem that implies (improved versions of) these results.

Disclaimer and overview

Disclaimer

- In this talk: no attempts to give due credits.
- In the paper: detailed references and comments.

Overview

- Preliminaries
- Metrisability and submetrisability of locally solid topologies
- Main theorem on embedded uo-convergent sequences
- Combine the above: applications
- Summary and questions

Preliminaries

- Vector lattices are real and Archimedean
- Linear topologies are Hausdorff
- *uo-Lebesgue topology* on a vector lattice: a locally solid topology τ such that $x_\alpha \xrightarrow{\tau} x$ whenever $x_\alpha \xrightarrow{\mathrm{uo}} x$.
- *o-Lebesgue topology* on a vector lattice: a locally solid topology τ such that $x_\alpha \xrightarrow{\tau} x$ whenever $x_\alpha \xrightarrow{\mathrm{o}} x$
- *Fatou topology* on a vector lattice: a locally solid topology such that zero has a neighbourhood basis of solid order closed subsets
- uo-Lebesgue implies o-Lebesgue; o-Lebesgue implies Fatou
- There is at most one uo-Lebesgue topology

Embedded sequences in nets

Let S be a non-empty set, let A be a directed set, and let (x_α) be a net in S. Suppose that $\alpha_1, \alpha_2, \alpha_3, \dots$ in A are such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$, and such that $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ when A has no largest element. Then the sequence (x_{α_n}) is said to be *embedded in the net* (x_α).

Preliminaries

In the metrisable case, the main theorem about embedded sequences in nets is about subsequences:

Lemma

Let (E, τ) be a vector lattice with a metrisable linear topology τ. The following are equivalent:

1. *every τ-convergent net has an embedded sequence that is uo-convergent to the same limit;*
2. *every τ-convergent sequence has a subsequence that is uo-convergent to the same limit.*

Remark

Elementary proof. Have not tried to formulate this, but appropriate analogue should be true for arbitrary first countable topological space and (suitable?) convergence structure.

Metrisability and submetrisability of locally solid topologies

A topological vector space is metrisable if and only if it is first countable. For a locally solid metrisable topology, the metric can be chosen to be nice:

Theorem

Let (E, τ) be a locally solid vector lattice. If τ is metrisable, then there exists a metric $d\colon E \times E \to [0,1]$ on E with the following properties:

1. *d is compatible with τ;*
2. *d is translation invariant;*
3. *$d\,(0,x) \leq d\,(0,y)$ for $x, y \in E$ such that $|x| \leq |y|$;*
4. *the metric balls $\{x \in E : d(0,x) < r\}$ and $\{x \in E : d(0,x) \leq r\}$ are solid for all $r \geq 0$.*

If, in addition, τ is locally convex, then there exists a metric $d\colon E \times E \to [0,1]$ on E satisfying 1–4, as well as:

5. *the metric balls $\{x \in E : d(c,x) < r\}$ and $\{x \in E : d(c,x) \leq r\}$ are convex for all $c \in E$ and $r \geq 0$.*

Sketch of proof

1. Use the proof of metrisation theorem for first countable topological vector spaces in the book by Schaefer and Wolff.
2. Starts with a neighbourhood basis (V_n) of (not necessarily open) balanced neighbourhoods of zero such that

$$V_{n+1} + V_{n+1} \subseteq V_n \tag{1}$$

3. Metric is constructed from the V_n.
4. In our case: can take the V_n solid (or solid and convex).
5. Use that the sum of two solid (convex) subsets is again solid (convex).

Call such a metric a *Riesz metric*.

Metrisability and submetrisability of locally solid topologies

Definition

Let (E, τ) be a locally solid vector lattice. We say that τ is:

1. *solidly submetrisable* if it is finer than some metrisable locally solid topology on E;
2. *locally solidly submetrisable* if, for every $x \in E$, the restricted topology $\tau|_{B_x}$ is solidly submetrisable;
3. *locally interval complete solidly submetrisable* if, for every $x \in E^+$, there exists a metrisable locally solid topology $\widetilde{\tau}_x$ on B_x such that:
    1. the restricted topology $\tau|_{B_x}$ is finer than $\widetilde{\tau}_x$;
    2. every $\widetilde{\tau}_x$-Cauchy sequence in the order interval $[0, x]$ is $\widetilde{\tau}_x$-convergent to an element of $[0, x]$.

Crucial: metric can always be chosen to be a Riesz metric.
Crude version of main theorem: a convergent net in a locally interval complete solidly submetrisable topology has an embedded sequence that is uo-convergent to the same limit.

Three facts for perspective:

Proposition

Let (E, τ) be a locally solid vector lattice. Suppose that there exists a metrisable linear topology τ^* on E that is coarser than τ. Then there exists a metrisable locally solid topology $\widetilde{\tau}$ on E that is coarser than τ and finer than τ^*.
If τ is Fatou, then $\widetilde{\tau}$ can be chosen to be Fatou.

So: for locally solid topologies, being submetrisable (with the obvious definition) and being solidly submetrisable are equivalent.

Proposition

Let (E, τ) be a locally solid vector lattice. Take $x \in E$. The following are equivalent:

1. $\tau|_{B_x}$ *is solidly submetrisable;*
2. $\tau|_{I_x}$ *is solidly submetrisable.*

Sketch of proof

1. Clear that 1 implies 2.
2. For the converse: take a Riesz metric d_x on I_x such that its metric topology is a locally solid topology that is weaker than $\tau|_{I_x}$.
3. Define $\widetilde{d_x}\colon B_x \times B_x \to [0, \infty)$ by setting $\widetilde{d_x}(z_1, z_2) := d_x(0, |z_1 - z_2| \wedge |x|)$ for $z_1, z_2 \in B_x$.
4. This is a Riesz metric on B_x and its metric topology is a locally solid topology on B_x that is weaker than $\tau|_{B_x}$.

Let (E, τ) be a locally solid vector lattice. A sequence $(V_n)_{n\geq 1}$ of neighbourhoods of zero is *normal* if

$$V_{n+1} + V_{n+1} \subseteq V_n.$$

Let $\mathcal{N}$ denote the collection of all normal sequences of solid τ-neighbourhoods of zero. Define the *carrier* C_τ *of* τ by setting

$$C_\tau := \bigcup \{ N^{\mathrm{d}} : \text{there exists } (V_n)_{n\geq 1} \in \mathcal{N} \text{ such that } N = \bigcap_{n\geq 1} V_n \}.$$

C_τ is an ideal because the N^{d} form a directed family of ideals.

Clear: if τ is metrisable, then we can have $N = 0$, so that $C_\tau = E$.
Can be more precise:

Proposition

A locally solid topology τ is locally solidly submetrisable if and only if $C_\tau = E$.

Ingredients of proof

1. Known:

$$C_\tau = \bigcup_{\rho \in \mathcal{F}} (\ker \rho)^{\mathrm{d}}$$

   where $\mathcal{F}$ is the family of continuous Riesz pseudonorms on E.
2. If d_x is a Riesz metric on B_x, then $\rho_x(z) := d_x(0, |x| \wedge |z|)$ $(z \in E)$ is a Riesz pseudonorm on E with $\ker \rho_x = \{x\}^{\mathrm{d}}$.
3. If ρ is a Riesz pseudonorm on E, then $d(y_1, y_2) := \rho(|y_1 - y_2|)$ $(y_1, y_2 \in E)$ is a Riesz metric on E.

Some special topologies:

Proposition

Let E be a Banach lattice. Then the norm topology and the unbounded norm topology on E are locally interval complete solidly submetrisable.

Sketch of proof

Clear for norm topology τ. For $\mathrm{u}\tau$, take $x \in E^+$, and define $d_x \colon B_x \times B_x \to [0, \infty)$ by setting

$$d_x(x_1, x_2) \coloneqq \| |x_1 - x_2| \wedge (2x) \|$$

for $x_1, x_2 \in B_x$. Thus $\mathrm{u}\tau|_{B_x}$ is solidly submetrisable. Since $d_x(x_1, x_2) = \|x_1 - x_2\|$ for $x_1, x_2 \in [0, x]$, $([0, x], d_x)$ is a complete metric space.

Remark

Similar proof gives metric for which *one* given order interval in B_x with end points in I_x is complete. Does not work for all such order intervals at the same time, let alone all order intervals in B_x.

Metrisability and submetrisability of locally solid topologies

Known: If (E, τ) is a locally solid vector lattice with τ Fatou, there exists a unique Fatou topology τ^δ on E^δ extending τ. If τ is o-Lebesgue, so is τ^δ; if τ is metrisable, so is τ^δ; if $\tau_1 \subseteq \tau_2$, then $\tau_1^\delta \subseteq \tau_2^\delta$. These are ingredients for the following.

Theorem

Let (E, τ) be a locally solid vector lattice, where τ is a Fatou topology. Let τ^δ be the extension of τ to a Fatou topology on E^δ. The following are equivalent:

1. *τ is locally solidly submetrisable;*
2. *τ^δ is locally solidly submetrisable;*
3. *τ^δ is locally interval complete solidly submetrisable;*
4. *$C_\tau = E$;*
5. *$C_{\tau^\delta} = E^\delta$.*

When τ is an o-Lebesgue topology, then 1–5 are also equivalent to:

6. *E has the countable sup property.*

Remarks on the proof

1. Uses quite a bit of the results on (sub)metrisability.
2. Where does the completeness magically come from?
3. Suppose τ^{δ} is locally solidly submetrisable. Want to prove that it is locally solidly interval complete submetrisable.
4. Take $x \in E^{\delta}$. Then $\tau^{\delta}|_{B_x^{\delta}}$ is Fatou.
5. By assumption: there is a locally solid metrisable topology on B_x^{δ} that is weaker than $\tau^{\delta}|_{B_x^{\delta}}$.
6. Earlier results: this weaker locally solid metrisable topology can be chosen to be a Fatou topology, and the metric can be chosen to be translation invariant.
7. Now apply a result of Nakano's: in a Dedekind complete vector lattice with a Fatou topology, order intervals are complete.

Remark

For a Fatou topology, the countable sup property still implies that $C_\tau = E$. The reverse implication fails: $\ell^\infty(A)$ for an uncountable set A has a Fatou topology with maximal carrier (it is metrisable), but not the countable sup property.

So: concentrate on when $C_\tau = E$ for τ Fatou, which implies that τ^δ is locally interval complete solidly submetrisable (which, in turn, is what we want for the embedded sequences):

Theorem

Let (E, τ) be a locally solid vector lattice with τ Fatou. Suppose that $C_\tau = E$, which is certainly the case when E has the countable sup property, when C_τ has a countable order basis, and when τ is metrisable. Then τ^δ is locally interval complete solidly submetrisable.

Main theorem on embedded uo-convergent sequences

Theorem

Let (E, τ) be a locally solid vector lattice, where τ is locally interval complete solidly submetrisable, and let F be a regular vector sublattice of E. Let (x_α) be a net in F and let $x \in F$ be such that $x_\alpha \xrightarrow{\tau} x$ in E.
Suppose that A has a largest element $\alpha_{\mathrm{largest}}$. Then $x = x_{\alpha_{\mathrm{largest}}}$.
Suppose that A has no largest element. In this case, one can choose any $\widetilde{\alpha}_1 \in A$; then find an $\alpha_1 \in A$; then choose any $\widetilde{\alpha}_2 \in A$; then find an $\alpha_2 \in A$; etc., such that:

1. $\alpha_1 < \alpha_2 < \alpha_3 < \ldots;$
2. $\alpha_n > \widetilde{\alpha}_n$ *for* $n \geq 1$;
3. $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$ *as* $n \to \infty$ *in* F.

In particular, for arbitrary A, (x_α) has an embedded sequence (x_{α_n}) such that $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$ in F.

Contours of proof: where does uo-convergence come from?

1. Reduce to case where A has no largest element, $F = E$, $(x_\alpha) \subseteq E^+$ and where the limit is 0.
2. Choose any index $\widetilde{\alpha}_1 \in A$. Then we can find $\alpha_1 > \widetilde{\alpha}_1$. By assumption, we can find a Riesz metric d_1 on $B_{x_{\alpha_1}}$ such that:
    1. the metric topology on $B_{x_{\alpha_1}}$ induced by d_1 is coarser than $\tau|_{B_{x_{\alpha_1}}}$;
    2. $([0, x_{\alpha_1}], d_1)$ is a complete metric space.
3. For $n = 2, 3, \ldots$, we inductively choose any index $\widetilde{\alpha}_n$, and can then find an index α_n, together with a Riesz metric d_n on $B_{x_{\alpha_n}}$ such that:
    1. the metric topology on $B_{x_{\alpha_n}}$ induced by d_n is coarser than $\tau|_{B_{x_{\alpha_n}}}$;
    2. $([0, x_{\alpha_n}], d_n)$ is a complete metric space;
    3. $\alpha_n > \alpha_{n-1}$ and $\alpha_n > \widetilde{\alpha}_n$;
    4. $d_k\left(0, x_{\alpha_n} \wedge x_{\alpha_k}\right) \leq \dfrac{1}{2^n}$ for $k = 1, 2, \ldots, n-1$.
4. There is 'metric Cauchyness' for suitable sequences looming in the choice of the x_{α_n}.

5. Exploit this ‘metric Cauchyness’ , the completeness of the metrics and the fact that they are Riesz metrics to show that, for each fixed k, $x_{\alpha_n} \wedge x_{\alpha_k} \xrightarrow{\mathrm{o}} 0$ in E as $n \to \infty$.
6. Recall general result: if $S \subset E$ and (y_i) is a net in B_S such that $|y_i| \wedge |s| \xrightarrow{\mathrm{o}} 0$ in E for all $s \in S$, then $y_i \xrightarrow{\mathrm{uo}} 0$ in E.
7. Apply this with $S = \{x_{\alpha_n} : n = 1, 2, \ldots\}$.

Main theorem on embedded uo-convergent sequences

The proof of the following uses the ping-pong freedom in the main theorem:

Corollary

Let (E, τ) be a locally solid vector lattice, where τ is locally interval complete solidly submetrisable, and let F be a regular vector sublattice of E. Let (x_α) be a net in F and let $x \in F$ be such that $x_\alpha \xrightarrow{\tau} x$ in E. Suppose, furthermore, that $\tau_1, \ldots, \tau_k$ are metrisable linear topologies on E, and that $x_1, \ldots, x_k \in E$ are such that $x_\alpha \xrightarrow{\tau_i} x_i$ for $i = 1, \ldots, k$. Then the net (x_α) has an embedded sequence (x_{α_n}) such that:

1. $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$ *as* $n \to \infty$ *in* F.
2. $x_{\alpha_n} \xrightarrow{\tau_i} x_i$ *as* $n \to \infty$ *for* $i = 1, \ldots, k$.

For completeness: start by including a case where the main theorem does *not* yield an optimal statement.

Let (E, τ) be a locally solid vector lattice with τ metrisable and complete. Since order intervals are closed, τ is locally interval complete solidly submetrisable. Hence the main theorem yields embedded uo-convergent sequences in τ-convergent nets. But much more is true.

Proposition

Let (E, τ) be a locally solid vector lattice with τ metrisable and complete. Then every τ-convergent net in E has an embedded sequence that is relatively uniformly convergent to the same limit.

Proof.

It is known that every τ-convergent sequence has a subsequence that is relatively uniformly convergent to the same limit. Now apply the embedded result from the preliminaries (for relative uniform convergence) about subsequences and embedded sequences. □

Applications

The un-topology on a Banach lattice is locally interval complete solidly submetrisable, so the main theorem applies.
When the norm is order continuous, the un-topology is the uo-Lebesgue topology and the main theorem 'bites its own tail': the embedded uo-convergent sequence is un-convergent again.

Theorem

Let E be a Banach lattice. Then:

1. *every un-convergent net in E has an embedded sequence that is uo-convergent to the same limit.*

If E has an order continuous norm, then:

2. *every un-convergent net in E has an embedded sequence that is uo-convergent as well as un-convergent to the same limit;*
3. *a sequence in E is un-convergent to $x \in E$ if and only if every subsequence has a further subsequence that is uo-convergent to x.*

For Fatou topologies:

Theorem

Let (E, τ) be a locally solid vector lattice, where τ is Fatou. Suppose that $C_\tau = E$, which is certainly the case when E has the countable sup property, when C_τ has a countable order basis, and when τ is metrisable. Then every τ-convergent net in E has an embedded sequence that is uo-convergent to the same limit.

Proof.

Suppose that $x_\alpha \xrightarrow{\tau} x$ for some $x \in E$. If we let τ^δ denote the extension of τ to a Fatou topology on E^δ, then also $x_\alpha \xrightarrow{\tau^\delta} x$ in E^δ. Since τ^δ is locally interval complete solidly submetrisable, (x_α) has an embedded sequence that is uo-convergent to x in E^δ. As E is a regular vector sublattice of E^δ, this embedded sequence is also uo-convergent to x in E. □

For o-Lebesgue topologies, there is an equivalence involving the 'embedded uo-convergent sequence property':

Theorem

Let (E, τ) be a locally solid vector lattice with τ o-Lebesgue. The following are equivalent:

1. $C_\tau = E$;
2. *E has the countable sup property;*
3. *every τ-convergent net in E has an embedded sequence that is uo-convergent to the same limit;*
4. *every increasing τ-convergent net in E^+ has an embedded sequence that is uo-convergent to the same limit.*

How 4 implies 2: suppose $0 \leq x_\alpha \uparrow x$. Then $x_\alpha \xrightarrow{\tau} x$. By assumption, there exists an embedded sequence (x_{α_n}) such that $x_{\alpha_n} \xrightarrow{\mathrm{uo}} x$. Since the sequence is order bounded, we even have $x_{\alpha_n} \xrightarrow{\mathrm{o}} x$. As the α_n are increasing, so are the x_{α_n}. Hence $x_{\alpha_n} \uparrow x$.

Applications

What we could not solve for Fatou topologies:

1. For general Fatou topologies, the previous equivalence is not true.
2. In $\ell^\infty(A)$ with A uncountable, every norm convergent net has an embedded sequence that is uo-convergent (even: relatively uniformly convergent) to the same limit.
3. Yet ℓ^∞ does not have the countable sup property.
4. For a Fatou topology, it is still true that $C_\tau = E$ implies the existence of embedded uo-convergent sequences in τ-convergent nets.
5. *Conversely, does this existence imply that $C_\tau = E$?*
6. *This* is still the case for $\ell^\infty(A)$.
7. And that is as far as we got. . .

Just as for the un-topology on a Banach lattice with order continuous norm, the main theorem bites its tail for general uo-Lebesgue topologies.

Theorem

Let (E, τ) be a locally solid vector lattice with τ uo-Lebesgue. Suppose that $C_\tau = E$ or, equivalently, that E has the countable sup property; this is certainly the case when C_τ has a countable order basis, and when τ is metrisable. Then:

1. *every τ-convergent net in E has an embedded sequence that is uo-convergent as well as τ-convergent to the same limit;*
2. *a sequence in E is τ-convergent to $x \in E$ if and only if every subsequence has a further subsequence that is uo-convergent to x.*

Applications

If x is in the weak closure of a convex subset S of a normed space, then there is a sequence in S that is norm convergent to x. There is a result in this vein for a vector lattice E, based on the main theorem and Kaplan's theorem.

Preparation:
Suppose that I is an ideal in $E^{\sim}$ that separates the points of E. For $\varphi \in I$, define the Riesz seminorm p_{φ} by $p_{\varphi}(x) := |\varphi|(|x|)$. The p_{φ} define a locally convex-solid topology $|\sigma|(E, I)$ on E.
If $I \subseteq E_{\mathrm{oc}}^{\sim}$, then $|\sigma|(E, I)$ is o-Lebesgue, so that the unbounded topology $\mathrm{u}|\sigma|(E, I)$ is the uo-Lebesgue topology on E.

Theorem

Let E be a vector lattice, and suppose that $E_{\mathrm{oc}}^{\sim}$ has an ideal I that separates the points of E. Suppose that $C_{\mathrm{u}|\sigma|(E,I)} = E$ or, equivalently, that E has the countable sup property; this is certainly the case when $C_{\mathrm{u}|\sigma|(E,I)}$ has a countable order basis, and when $\mathrm{u}|\sigma|(E,I)$ is metrisable. Let S be a convex subset of E, and take $x \in \overline{S}^{\sigma(E,I)}$. Then there exists a sequence in S that is uo-convergent to x.

Proof.

The topological duals of the locally convex topological vector spaces $(E, |\sigma|(E,I))$ and $(E, \sigma(E,I))$ are both equal to I (Kaplan's theorem). Hence $\overline{S}^{\sigma(E,I)} = \overline{S}^{|\sigma|(E,I)}$, so that there exists a net (x_α) in S with $x_\alpha \xrightarrow{|\sigma|(E,I)} x$. Then certainly $x_\alpha \xrightarrow{\mathrm{u}|\sigma|(E,I)} x$. Now apply the existence result on embedded uo-convergent sequences for uo-Lebesgue topologies with maximal carrier. □

Applications

We conclude with an application to adherences and closures.

Let A be a subset of a vector lattice E. Define the *σ-uo-adherence* of A as

$$a_{\sigma\text{-uo}}(A) = \left\{ x \in E : \text{ there exists a sequence } (x_n) \text{ in } A \text{ with } x_n \xrightarrow{\text{uo}} x \text{ in } E \right\}$$

and the *uo-adherence* of A in E as

$$a_{\text{uo}}(A) = \left\{ x \in E : \text{ there exists a net } (x_\alpha) \text{ in } A \text{ with } x_\alpha \xrightarrow{\text{uo}} x \text{ in } E \right\}.$$

For a topology τ one similarly defines the σ-τ-adherence and the τ-adherence (the τ-closure) of A.

A is said to be *σ-uo-closed* when $a_{\sigma\text{-uo}}(A) = A$. The σ-uo-closed subsets are the closed subsets of a topology: the σ-uo-topology.

Write $\overline{A}^{\sigma\text{-uo}}$ for the closure of A in this topology. Then $A \subseteq a_{\sigma\text{-uo}}(A) \subseteq \overline{A}^{\sigma\text{-uo}}$ and $\overline{a_{\sigma\text{-uo}}(A)}^{\sigma\text{-uo}} = \overline{A}^{\sigma\text{-uo}}$.

Similarly for uo-adherences and σ-τ-adherences and the associated topologies.

Theorem

Let E be a vector lattice that admits a uo-Lebesgue topology τ, and let A be a subset of E. Suppose that $C_\tau = E$ or, equivalently, that E has the countable sup property; this is certainly the case when C_τ has a countable order basis, and when τ is metrisable. Then the following seven subsets of E are all equal:

1. *$a_{\sigma\text{-}\tau}(A)$ and $\overline{A}^{\sigma\text{-}\tau}$;*
2. *$a_{\sigma\text{-uo}}(A)$ and $\overline{A}^{\sigma\text{-uo}}$;*
3. *$a_{\mathrm{uo}}(A)$ and $\overline{A}^{\mathrm{uo}}$;*
4. *$\overline{A}^{\tau}$.*

In particular, the σ-τ-topology, the σ-uo-topology, and the uo-topology on E all coincide with τ.

Hence the somewhat elusive topologies for the uo-convergence structure and the σ-uo-convergence structures are quite concrete in this case.

Summary and questions

Summary

1. Results on various aspects of metrisability.
2. A good number of locally solid topologies are locally interval complete solidly submetrisable.
3. Main theorem: there is an embedded uo-convergent sequence in a net converging in a locally solid topology that is locally interval complete solidly submetrisable.
4. Combine to give (improved versions of) known results on embedded uo-convergent sequences (and new ones. . .)

Questions

1. A 'left over' within the current framework: if a Fatou topology has the property that every convergent net has an embedded uo-convergent sequence, does it then follow that it has maximal carrier?
2. A thematical one: is it possible to extend any of this to locally solid *convergence structures*?

Reference

Y. Deng and M. de Jeu, *Embedded unbounded order convergent sequences in topologically convergent nets in vector lattices*, Banach J. Math. Anal. **18** (2024), no. 2, Paper no. 22, 25pp.