Some Problems On Banach Lattice Algebras

Vector lattices and ordered structures (Madrid 13-17 May)

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Remark: $M > 1$. When $M = 1$, $(e_\lambda)_{\lambda\in\Lambda}$ is referred to a contractive

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Let A be a Banach lattice algebra and M a positive real number.

1. *There exists a positive left M-approximate identity if and only if for every $\varepsilon > 0$ and $a \in A$ we can find a positive element $e \in A$ satisfying $\|e\| \leq M$ and $\|ea - a\| < \varepsilon$.*

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3. *There exists a positive M-approximate identity if, and only if for every $\varepsilon > 0$ and $a, b \in A$, we can find a positive element $e \in A$ satisfying $\|e\| \leq M$, $\|ea - a\| < \varepsilon$ and $\|be - b\| < \varepsilon$.*

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Proof.

[[Idea of the proof]]
1.:Suppose that for all $\varepsilon > 0$ and $a \in A$ we can find a **positive** element $e \in A$ satisfying $\|e\| \leq M$ and $\|ea - a\| < \varepsilon$.

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2. Use (i) in the Banach lattice algebra $(A, *)$ where $a * b = b.a$. □

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Theorem (Bernau, Huijsmans 1990)

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If $(e_\lambda)_\lambda$ be a B-A-I then $(e_\lambda^+)_\lambda$ is a positive B-A-I. Let $0 \leq x \in A$,

$$\left\|e_\lambda^+ x - x\right\| = \left\|(e_\lambda x)^+ - x^+\right\| \leq \|e_\lambda x - x\| \longrightarrow 0$$

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- $\|\tau_x\| \leq \|x\|$ and $\|\tau_F\| \leq \|F\|$.

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We get $\tau_{e^-}(f) = 0$ for all $f \in A'$.So $f(e^-) = \tau_{e^-}(f)(e) = 0 \Longrightarrow e^+ = e \geq 0$. □

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Indeed, Let $(e_\alpha)_{\alpha\in\Delta}$ be a left B-A-I and $(f_\beta)_{\beta\in\Gamma}$ be a right B-A-I. Then $(u_{(\alpha,\beta)})_{(\alpha,\beta)}$

$$u_{(\alpha,\beta)} = e_\alpha + f_\beta - f_\beta e_\alpha.$$

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Problem

If a Banach lattice algebra A possesses both left and right positive approximate identities, does A necessarily admit a positive approximate identity?

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Fact (Cohen's factorization property)

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$$\|u\|_{|\pi|} = \sup\{\|\widehat{\varphi}(u)\| : \varphi \in \mathcal{L}\} \text{ for all } u \in E \otimes F,$$

where $\mathcal{L}$ is the set of all positive bilinear maps from $E \times F$ to all Banach lattices G with norm ≤ 1 and $\widehat{\varphi} : E \otimes F \longrightarrow G$ is the linear map corresponding to φ.

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Positive projective tensor product

Theorem (Fremlin 1974)

Let $E \otimes_{|\pi|} F$ *denote the completion of* $E \otimes F$ *with respect to the positive projective norm* $\|u\|_{|\pi|}$.

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1. $E \otimes_{|\pi|} F$ *is a Banach lattice.*
2. *(Universal property): For any Banach lattice* G *and any positive bilinear map* $\varphi : E \times F \longrightarrow G$ *there exist a unique positive linear map* $T : E \otimes_{|\pi|} F \longrightarrow G$ *such that* $\varphi(x, y) = T(x \otimes y)$ *for all* $(x, y) \in E \times F$.

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- $C_0(X) \otimes_{|\pi|} C_0(Y) = C_0(X \times Y)$.

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Moreover, $A \otimes_{|\pi|} B$ satisfy the universal property:
For any Banach lattice algebra G and any bilinear and multiplicative $\varphi : E \times F \longrightarrow G$ there exist a unique positive multiplicative linear map $T : E \otimes_{|\pi|} F \longrightarrow G$ such that $\varphi(x, y) = T(x \otimes y)$ for all $(x, y) \in E \times F$.

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- $(e_\lambda \otimes f_\beta)u \longrightarrow u$ for all $u \in A \otimes_{|\pi|} B$

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$(T_{f.x}(u_\lambda))_{\lambda \in \Delta}$ is a left B-A-I on B.

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$$T_{f.x}(a \otimes b)y = T_f((a \otimes b)(x \otimes y)) \text{ for all } (a, b) \in A \times B. \quad (1)$$

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$(T_{f.x}(u_\lambda))_{\lambda \in \Delta}$ is a left B-A-I on B. Since $u_\lambda \geq 0$ and $T_{f.x} \geq 0$, then B has a positive B-A-I. □

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- A is isometrically isomorphic to a closed subalgebra of $\mathcal{L}^r(A')$.

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Let A be a approximately unital Banach f-algebra, then A is isomorphic to a closed subalgebra and sublattice of $\mathcal{L}^r(A')$.

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[Idea] If A is an f-algebra then for all $0 \leq f \in A'$, $(f.x) = f.x^+$.

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Thank you very much
Muchas gracias