SPR subspaces and Kadec-Pełczysńki dichotomy

Jesús Illescas Fiorito (UCM-ICMAT)

jillesca@ucm.es

Research term Lattice Structures in Analysis and Applications

Banach lattices

$\mathcal{C}(K,\mathbb{R})$, $\mathrm{L}_p(\mu,\mathbb{R})$, c_0, ℓ_p, $\mathbb{R} \curvearrowright$ **partial order** defined pointwise

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A vector space equipped with a partial order $(E,\geq)$ is an **ordered vector space** if

- $\forall\, x,y,z \in X,\ x \leq y:\ x+z \leq y+z,$
- $\forall\, x,y \in X,\ x \leq y,\ \forall\, \lambda \in \mathbb{R}_+:\ \lambda x \leq \lambda y.$

In this case we say that $\geq$ is a **linear order**.

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Definition

A partial ordered set $(\mathcal{S},\geq)$ is called a **lattice** if for all $x,y \in \mathcal{S}$, both $x \wedge y := \inf\{x,y\}$ and $x \vee y := \sup\{x,y\}$ exist in $\mathcal{S}$.

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OVS + lattice ≡ vector lattice

We can define **lattice operations**

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x).$$

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We say that a vector lattice which is also a normed space $(X, \geq, \|\cdot\|)$ is a **normed lattice** if

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If $(X, \|\cdot\|)$ is also complete, we say it is **Banach lattice.**

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PR problems

- Wolfgang Pauli and Norbert Straumann. *Die allgemeinen Prinzipien der Wellenmechanik.* Springer Berlin Heidelberg, 1990.

Die mathematische Frage, ob bei gegebenen Funktionen $W(\vec{x})$ und $W(\vec{p})$ die Wellenfunktionen Φ stets eindeutig bestimmt ist, wenn es eine solche zugehörige Wellenfunktion überhaupt gibt [d. h. wenn $W(\vec{x})$ und $W(\vec{p})$ physikalisch vereinbar sind], ist noch nicht all-gemein untersucht worden.

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This raised

Question (Pauli problem)

Given both the amplitude of a complex valued square integrable function and the amplitude of its Fourier transform, can we recover the function?

From $|f|$ and $|\mathcal{F}f|$ is not possible to difference between f and $e^{i\theta}f$.

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Example (Corbett '77)

If $\varphi \in \mathrm{L}_2(\mathbb{R})$ verifies that $\varphi(-x) = \pm\varphi(x)$ for all $x \in \mathbb{R}$ then

$$\left|\overline{\mathcal{F}[\varphi]}\right| = \left|\mathcal{F}\left[\overline{\varphi}\right]\right|.$$

Take for example,

$$\varphi(x) = e^{-(1\pm i)\pi x^2}.$$

So we must restrict our attention to a **proper** subset/subspace of $\mathrm{L}_2(\mathbb{R})$.

If $E \subseteq \mathrm{L}_2(\mathbb{R})$ does this recovery, then on

$$\mathrm{Graph}(\mathcal{F}|_E) = (E, \mathcal{F}E) := \{(\varphi, \mathcal{F}[\varphi]),\ \varphi \in E\} \subseteq \mathrm{L}_2(\mathbb{R}) \times \mathrm{L}_2(\mathbb{R})$$

we can recover elements therein from $|\cdot|$.

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Question (Pauli problem for subspaces)

For which subspaces E of $\mathrm{L}_2(\mathbb{R})$ the map

$$\begin{array}{rccc} |\cdot|_{\mathsf{P}} : & (E, \mathcal{F}E)/\mathbb{T} & \longrightarrow & \mathrm{L}_2(\mathbb{R}) \times \mathrm{L}_2(\mathbb{R}) \\ & \varphi & \mapsto & (|\varphi|, |\mathcal{F}[\varphi]|) \end{array}$$

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Question (Fourier transform problem)

Given the amplitude of the Fourier transform of a real/complex valued square integrable function, can we recover the function?

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For which subspaces E of $\mathrm{L}_2(\mathbb{R})$ the map

$$\begin{array}{rccc} |\cdot|_{\mathsf{G}}: & E/\mathbb{T} & \longrightarrow & \mathrm{L}_2(\mathbb{R}) \\ & \varphi & \mapsto & \left|\mathcal{V}_g[\varphi]\right| \end{array}$$

is injective?

Definition

Let H be a Hilbert space. A collection $\Phi = \{\varphi_j\}_{j\in\mathcal{J}} \subseteq H$ is called a **frame** if there are uniform constants $B \geq A > 0$ called the **frame bounds** such that

$$A\,\|f\|_H^2 \leq \sum_{j\in\mathcal{J}} |\langle f, \varphi_j\rangle|^2 \leq B\,\|f\|_H^2 \quad \forall\, f \in H.$$

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We also define the **analysis operator of** Φ as

$$\begin{array}{rccc} \mathcal{T}_\Phi : & H & \longrightarrow & \ell^2(\mathcal{J}) \\ & f & \mapsto & (\ldots, \langle f, \varphi_n\rangle, \ldots) \end{array}$$

We can provide a linear, stable, and unconditional reconstruction formula:

$$f = \sum_{j\in\mathcal{J}} \langle f, \varphi_j\rangle \widetilde{\varphi_j}, \quad \forall\, f \in H.$$

Question (Frame problem for subspaces)

For which subspaces E of H the map

$$\begin{array}{rccc} |\cdot|_{\mathcal{T}_\Phi}: & E/\mathbb{T} & \longrightarrow & \ell_2 \\ & f & \mapsto & (\ldots, |\langle f, \varphi_n\rangle|, \ldots)_n \end{array}$$

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Definition

Let $E \subseteq X$ be a subspace of a given Banach lattice X. We say that E **does PR** *when the map $|\cdot| : E/\mathbb{T} \longrightarrow X$ is injective. Equivalently, if*

$$\forall\, f, g \in E,\ |f| = |g|\,,\ \exists\, \lambda \in \mathbb{K} : f = \lambda g.$$

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Definition

Let $E \subseteq X$ be a subspace of a given Banach lattice X. We say that E **does SPR with constant** *C, or just that E* **is a C-SPR subspace**, *if*

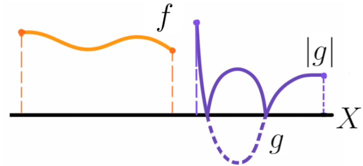
$$\min_{\lambda \in \mathbb{T}} \|f - \lambda g\| \leq C \cdot \Big\| |f| - |g| \Big\| \quad \forall\, f, g \in E,$$

that is, if the inverse of the map $|\cdot|$ is C-Lipschitz.

Observation

If we can find $f, g \in E \subseteq X$, so that $f \perp g$... then

$$|f + g| = |f - g| = |f| + |g|$$



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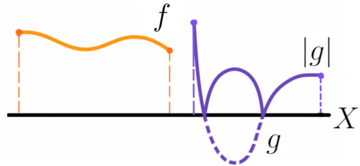

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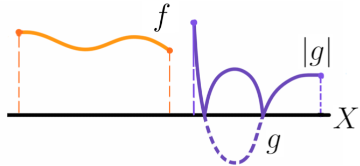

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$\exists$ **Pairs of disjoint vectors** $\Longrightarrow$ **no PR**

?? $\Longrightarrow$ **no SPR**

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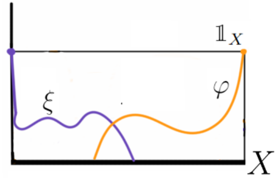
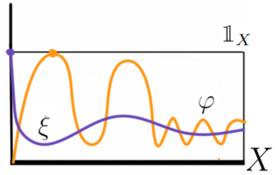
*We say that $f, g \in X : \|f\| = \|g\| = 1$ are an ε-**almost disjoint pair** if*

$$\Big\| \, |f| \wedge |g| \, \Big\| < \varepsilon,$$

which would be denoted by $f \perp_\varepsilon g$.





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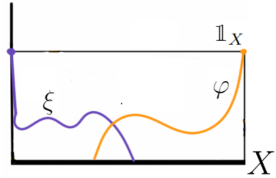

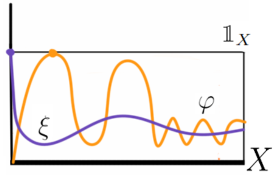

Observation

If for any $1 > \varepsilon > 0$, $f, g \in S_E \subseteq X : f \perp_\varepsilon g$, then E **is not** a $1/\varepsilon$-SPR subspace of X.

Observation (for $\mathbb{R}$-Banach lattices)

If for any $1 > \varepsilon > 0$, $f, g \in S_E \subseteq X : f \perp_\varepsilon g$, as

$$\Big| |f+g| - |f-g| \Big| = 2\Big(|f| \wedge |g| \Big)$$

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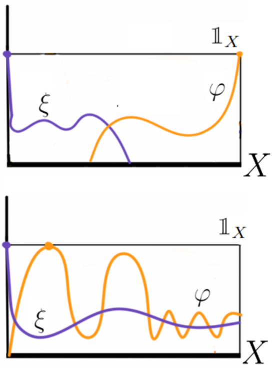

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we have

$$\min_{\lambda=\pm 1} \|f - \lambda g\| = 2 > \frac{1}{\varepsilon} \Big\| |f+g| - |f-g| \Big\|.$$

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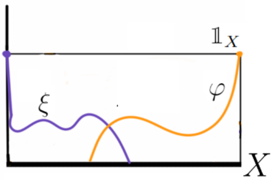

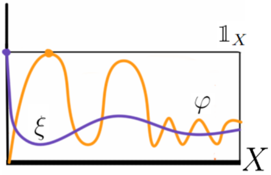

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$\exists$ some ε-almost disjoint pair of vectors $\implies$ no $1/\varepsilon$-SPR

Definition

E **contains almost disjoint pairs of vectors** *if $\forall\, \varepsilon > 0$ we can find f_ε, $g_\varepsilon \in S_E$ so that $f_\varepsilon \perp_\varepsilon g_\varepsilon$.*

$\exists$ **almost disjoint pairs of vectors** $\implies$ **no SPR**

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Theorem (FOPTB)

Let E be a subspace of a $\mathbb{R}$-Banach lattice X. Then the following conditions are equivalent.

- *E does C-SPR,*
- *E does not contain $1/C$-almost disjoint pairs.*

In particular,

$$E \text{ does SPR} \iff E \text{ does not contain almost disjoint pairs.}$$

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Definition

A sequence $\{x_n\}_n \subseteq X$ *in a Banach lattice* X *is called:*

- **normalized** *if*

$$\|x_n\| = 1, \quad \forall n \geq 1.$$

- **almost disjoint** *if*

$$\exists\ \{d_n\}_n \subseteq X,\ d_i \perp d_k \text{ if } i \neq k,\ \|x_n - d_n\| \xrightarrow{n\to\infty} 0.$$

Observation

If $\{x_n\}_{n=1}^{\infty} \subseteq E$ is a normalized almost disjoint sequence, with

$$\|x_n - d_n\|_X \xrightarrow{n\to\infty} 0, \quad \{d_n\}_{n=1}^{\infty} \subseteq X \textbf{ disjoint,}$$

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then

$$\begin{aligned}\|x_n \wedge x_m\| &= \|(x_n - d_n) \wedge x_m + d_n \wedge x_m\| \\ &\leq \|(x_n - d_n) \wedge x_m\| + \|d_n \wedge x_m\| \\ &\leq \|x_n - d_n\| + \|d_n \wedge x_m\| \\ &= \|x_n - d_n\| + \|d_n \wedge (x_m - d_m)\| \leq \|x_n - d_n\| + \|x_m - d_m\| .\end{aligned}$$

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E does SPR $\implies$ E lacks normalized almost disjoint sequences

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Definition

We say that a subspace E of a Banach lattice is **dispersed** *if it* **fails** *to contain normalized almost disjoint sequences.*

Observation

If $\{x_n\}_{n=1}^{\infty} \subseteq E$ is a normalized almost disjoint sequence, with

$$\|x_n - d_n\|_X \xrightarrow{n\to\infty} 0, \quad \{d_n\}_{n=1}^{\infty} \subseteq X \textbf{ disjoint,}$$

then

$$\|x_n \wedge x_m\| \leq \|x_n - d_n\| + \|x_m - d_m\| \xrightarrow{n,\, m\to\infty} 0.$$

E **does SPR** $\Longrightarrow$ E **lacks normalized almost disjoint sequences**

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- Kadec-Pełczyński: *Bases, lacunary sequences and complemented subspaces in the spaces L_p.* $\sim$ **isomorphic structure of subspaces of** $\mathrm{L}_p(\mu)$

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Definition

For any $p \geq 1$ and $\varepsilon > 0$ we define a **Kadec-Pełczysńki class** as

$$\mathrm{KP}^p_\varepsilon = \Big\{ f \in \mathrm{L}_p[0,1],\ \ \mathrm{m}\{\mathtt{t},\ |f(\mathtt{t})| \geq \varepsilon \, \|f\|_p\} \geq \varepsilon \Big\}.$$

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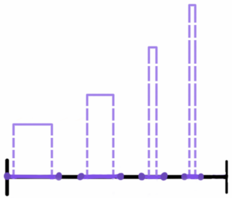
- If $\varphi \notin \mathrm{KP}^p_\varepsilon$, then $\exists\, A \subseteq [0,1]$ with $\mathrm{m}(A) < \varepsilon$ and

$$\int_A \left| \frac{f(\mathtt{t})}{\|f\|_p} \right|^p \mathrm{dm}(\mathtt{t}) > 1 - \varepsilon.$$

Theorem

$$\forall (x_n)_{n=1}^{\infty} \subseteq S_{\mathrm{L}_p[0,1]},\ \forall\, \varepsilon > 0,\ \exists x_{n_\varepsilon} \notin \mathrm{KP}_\varepsilon^p$$

we can find a subsequence $(x_{n_k})_{k=1}^{\infty}$ *equivalent to the canonical basis of* ℓ_p *and whose closed span is complemented on* $\mathrm{L}_p[0,1]$.



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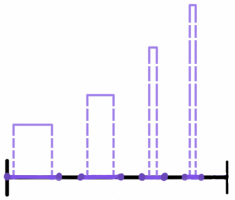

For $p > 2$ KP classes are *stronger*

Lemma

Let $p > 2$ *and* $\mathrm{KP}_\varepsilon^p$ *any class. Then*

$$\varepsilon^{3/2} \|f\|_{\mathrm{L}_p[0,1]} \leq \|f\|_{\mathrm{L}_2[0,1]} \leq \|f\|_{\mathrm{L}_p[0,1]}, \quad \forall f \in \mathrm{L}_p[0,1].$$

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Corollary

On any subspace $E \subseteq \mathrm{L}_p[0,1]$, $p > 2$, *contained on any* KP*class the norms* $\|\cdot\|_p$ *and* $\|\cdot\|_2$ *are equivalent.*

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Theorem

Let $p > 2$ and let E be an infinite dimensional subspace of $\mathrm{L}_p[0,1]$. Then the following conditions are equivalent:

1. *$E \subseteq \mathrm{KP}_\varepsilon^p$ for some $\varepsilon > 0$,*
2. *E is isomorphic to ℓ_2,*
3. *no subspace of E is isomorphic to ℓ_p,*
4. *the norms $\|\cdot\|_2$ and $\|\cdot\|_p$ are equivalent on E.*

Theorem (L_1-rep's.)

Let X be an order continuous Banach lattice with weak unit. Then we can view X as a norm and order dense ideal of some $\mathrm{L}_1(\mu)$, with μ a probability measure, so that both

$$\mathrm{L}_\infty(\mu) \xrightarrow[\text{Dense ideal}]{} X \xrightarrow[\text{Dense ideal}]{} \mathrm{L}_1(\mu)$$

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Theorem (Kadec-Pełczysńki)

Let X be an order continuous Banach lattice cont. embedded as an ideal of some $\mathrm{L}_1(\mu)$, for some probability m. Let $(x_n) \subseteq X$ be a bounded sequence in X. If $x_n \xrightarrow{\mu} 0$ on $\mathrm{L}_1(\mu)$ then (x_n) has a normalized almost disjoint subsequence on X.

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Corollary (Kadec-Pełczyński dichotomy for sequences)

Let X be an order continuous Banach lattice with a weak unit represented as an ideal of some $\mathrm{L}_1(\mu)$-space, with μ a probability measure, and let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in $X_+ \setminus \{0\}$. Then:

- *either (x_n) is semi-normalized when viewed in $\mathrm{L}_1(\mu)$,*
- *or (x_n) has an almost disjoint subsequence in X.*

$$(x_n)_n \subseteq E \; + \; x_n \xrightarrow{\mu} 0 \implies E \textbf{ not dispersed} \implies E \textbf{ not SPR}$$

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Let X be an order continuous Banach lattice with a weak unit represented as an ideal of some $\mathrm{L}_1(\mu)$-space, with μ a probability measure. Then, for any closed subspace $E \subseteq X$, either

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SPR subspaces $\overset{oc}{\sim}$ Dispersed subspaces

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- E is separable,
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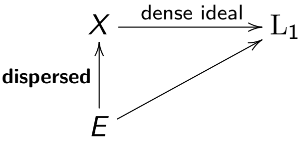

E must be dispersed when viewed on $\mathrm{L}_1(\mu)$.

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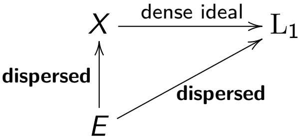

E must be dispersed on $L_1(\mu)$. If not, E contains an almost disjoint sequence $(f_n)_{n=1}^{\infty}$, so that

$$\|f_n - d_n\|_{L_1(\mu)}, \text{ for some disjoint sequence } (d_n)_{n=1}^{\infty} \subseteq L_1(\mu).$$

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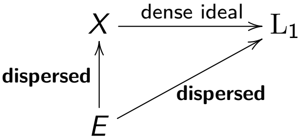

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Thus, $f_n \xrightarrow{\mu} 0$. But then $(f_n)_{n=1}^{\infty}$ has an almost disjoint sequence on X.

$$\boxed{\mathrm{L}_1(\mu)\textbf{-case}} \overset{\textbf{FOPT}}{\Longleftarrow} \text{Maurey-Krivine results + Clarkson SPR.} \qquad \blacksquare$$

Kadec-Pełczysńki dichotomy is stronger *here* thanks to KP classes.

Theorem (Kadec-Pełczysńki)

Let $1 \leq p < \infty$ and μ a probability measure. For a closed subspace $E \subseteq \mathrm{L}_p(\mu)$ the following are equivalent:

1. *E is dispersed,*
2. *there exists $0 < q < p$ such that $\|\cdot\|_{\mathrm{L}_p} \sim \|\cdot\|_{\mathrm{L}_q}$ on E,*
3. *for all $0 < q < p$, $\|\cdot\|_{\mathrm{L}_p} \sim \|\cdot\|_{\mathrm{L}_q}$,*
4. *E is strongly embedded on $\mathrm{L}_p(\mu)$.*

Proof. $\boxed{2 \iff 3 \iff 4}$ are well known.

$\boxed{3 \iff 1}$ Follows from KP dichotomy for subspaces.

Theorem (KP)

Let $1 \leq p < \infty$ and μ a probability measure. For a closed subspace $E \subseteq \mathrm{L}_p(\mu)$ the following are equivalent:

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3. *for all $0 < q < p$, $\|\cdot\|_{\mathrm{L}_p} \sim \|\cdot\|_{\mathrm{L}_q}$,*
4. *E is strongly embedded on $\mathrm{L}_p(\mu)$.*

Moreover,

- *if $p \neq 2$, a closed subspace of L_p is dispersed $\iff$ it does not contain ℓ_p as an isomorphic copy,*
- *for $p > 2$, a closed subspace of L_p is dispersed $\iff$ it is isomorphic to a Hilbert space.*

Observation

Being dispersed *passes up* between L_p-spaces.

$$\begin{array}{ccccc}
\mathrm{L}_p[0,1] & \longrightarrow & \mathrm{L}_q[0,1] & \longrightarrow & \mathrm{L}_1[0,1] \\
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SPR on L_p-spaces

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What happens with SPR?

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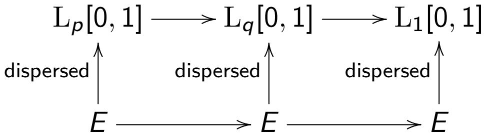

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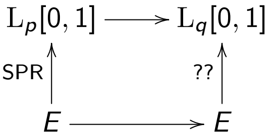

This is no longer true.

Theorem (FOPT '23)

For all $2 \leq p < +\infty$, there exists a closed subspace $E \subseteq \mathrm{L}_p[0,1]$ such that E is an SPR-subspace of $\mathrm{L}_p[0,1]$, but fails to be an SPR-subspace for each $1 \leq q < p < \infty$.

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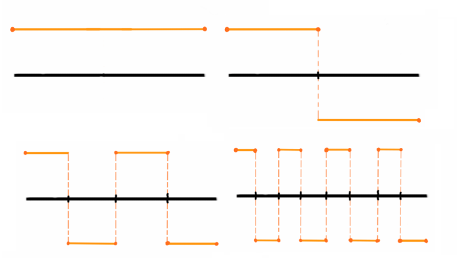
Observation

- *We know that E must be isomorphic to a Hilbert space. Moreover, thanks to KP we also know that E can not a dispersed subspace of any $\mathrm{L}_{p'}[0,1]$ with $p < p'$.*

Sketch of the proof. Recall that for the Rademacher system on $\mathrm{L}_p[0,1]$, Khintchine inequality says

$$A_p\left(\sum_{n\in\omega}|a_n|^2\right)^{1/2}\leq\left\|\sum_{n\in\omega}a_nr_n\right\|_{\mathrm{L}_p[0,1]}\leq B_p\left(\sum_{n\in\omega}|a_n|^2\right)^{1/2}$$



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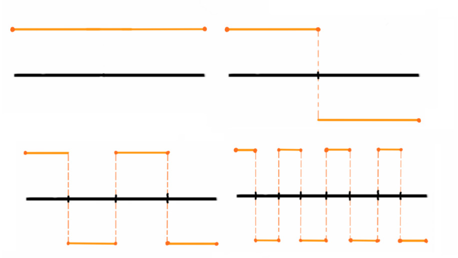
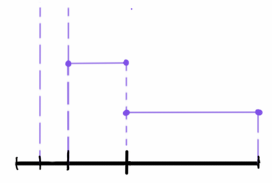


We have that $\overline{\mathrm{span}}^{\|\cdot\|_p}\{r_n\} \cong \ell_2$. Thus, their span is dispersed, but it can not make PR. Idea: perturb (on a smart way) this system.

Observation

The functions



$$t_n = 2^{n/p} \mathbb{1}_{\left[1+\frac{1}{2^n}, 1+\frac{1}{2^{n-1}}\right]}$$

verify that

$$\|t_n\|^r_{\mathrm{L}_r[1,2]} = \int_{[1,2]} 2^{r\frac{n}{p}} \mathbb{1}_{\left[1+\frac{1}{2^n}, 1+\frac{1}{2^{n-1}}\right]} = 2^{r\frac{n}{p}} \cdot 2^{-n} = 2^{n(\frac{r}{p}-1)}$$

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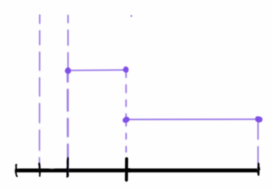

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Thus,

$$\begin{cases} \|t_n\|_{\mathrm{L}_r[1,2]} \to \infty, & \text{if } r > p, \\ \|t_n\|_{\mathrm{L}_r[1,2]} = 1 \quad \forall\, n, & \text{if } r = p, \\ \|t_n\|_{\mathrm{L}_r[1,2]} \to 0, & \text{if } r < p. \end{cases}$$

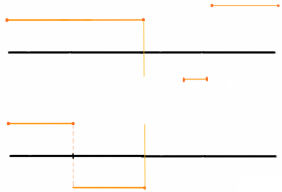
and then for $p > r$

$$\lim_{n\to\infty} \||t_n| - |t_{n+1}||^r_{\mathrm{L}_r[0,2]} = \lim_{n\to\infty} \|t_n\|^r_{\mathrm{L}_r[1,2]} + \|t_{n+1}\|^r_{\mathrm{L}_r[1,2]} = 0 + 0 = 0.$$

Set

$$g_n(\mathtt{t}) := r_n(\mathtt{t}) + t_n(\mathtt{t}),\ \mathtt{t} \in [0,2],$$

$$E := \overline{\text{span}}^{\|\cdot\|_p}\{g_n\}_{n=1}^{\infty}.$$



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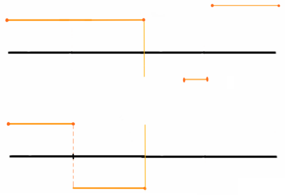

E fails SPR for $q < p$:

- $\lim_{n\to\infty} \| |g_n| - |g_{n+1}| \|^q_{\mathrm{L}_q[0,2]} = \lim_n 2^{n\left(\frac{q}{p}-1\right)} + 2^{(n+1)\left(\frac{q}{p}-1\right)} = 0,$
- If $m > n$, then

$$\|g_n \pm g_m\|^q_{\mathrm{L}_q[0,2]} \geq \|r_n \pm r_m\|^q_{\mathrm{L}_q[0,2]} = \int_{[0,1]} |r_n \pm r_m|^q$$
$$= \frac{2^m}{2} \cdot \frac{1}{2^m} \cdot 2^q = 2^{q-1} > 0.$$

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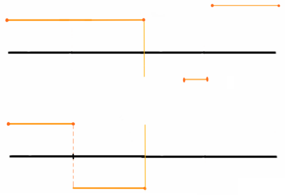

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- $\text{lím}_{n\to\infty} \||g_n| - |g_{n+1}|\|^q_{\mathrm{L}_q[0,2]} = \text{lím}_n 2^{n\left(\frac{q}{p}-1\right)} + 2^{(n+1)\left(\frac{q}{p}-1\right)} = 0,$
- If $m > n$, then

$$\|g_n \pm g_m\|^q_{\mathrm{L}_q[0,2]} \geq \|r_n \pm r_m\|^q_{\mathrm{L}_q[0,2]} = \int_{[0,1]} |r_n \pm r_m|^q$$
$$= \frac{2^m}{2} \cdot \frac{1}{2^m} \cdot 2^q = 2^{q-1} > 0.$$

It is less easy, but possible, to check that

$$\boxed{E \text{ does SPR on } \mathrm{L}_p[0,2] \Longleftarrow \text{Hölder SPR!}}$$

■

What happens with SPR?

We need **something more!** (namely, dispersion info)

$$\begin{array}{ccc} L_p[0,1] & \longrightarrow & L_q[0,1] \\ \text{SPR} \uparrow & & \uparrow ?? \\ E & \longrightarrow & E \end{array}$$

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\end{array}
$$

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$$
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Then

$$\begin{array}{ccccccccc} \mathrm{L}_s[0,1] & \longrightarrow & \mathrm{L}_r[0,1] & \longrightarrow & \mathrm{L}_p[0,1] & \longrightarrow & \mathrm{L}_r[0,1] & \longrightarrow & \mathrm{L}_1[0,1] \\ \text{Dispersed} \uparrow & & \text{SPR} \uparrow & & \text{SPR} \uparrow & & \text{SPR} \uparrow & & \text{SPR} \uparrow \\ E & \longrightarrow & E & \longrightarrow & E & \longrightarrow & E & \longrightarrow & E \end{array}$$

Proof.

$$\begin{array}{ccccc}
\mathrm{L}_s[0,1] & \longrightarrow & \mathrm{L}_r[0,1] & \longrightarrow & \mathrm{L}_p[0,1] \\
\text{Dispersed}\Big\uparrow & & \textbf{¿SPR?}\Big\uparrow & & \text{SPR}\Big\uparrow \\
E & \longrightarrow & E & \longrightarrow & E
\end{array}$$

Recall that

$$\|\cdot\|_s \sim \|\cdot\|_p \text{ on } E, \quad \|\cdot\|_p \leq \|\cdot\|_r \leq \|\cdot\|_s\,.$$

Thus, $\forall\, f,\, g \in E$

$$\begin{aligned}
\min_{|\lambda|=1} \|f - \lambda g\|_r &\leq \min_{|\lambda|=1} \|f - \lambda g\|_p \\
&\leq C^{(p)} \Big\| \, |f| - |g| \, \Big\|_p \leq C^{(p)} \Big\| \, |f| - |g| \, \Big\|_r\,.
\end{aligned}$$

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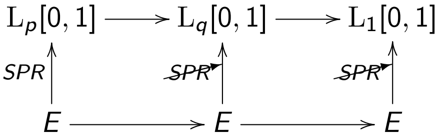
&\leq C^{(p)} \Big\| \, |f| - |g| \, \Big\|_s \not\leq C^{(p)} \Big\| \, |f| - |g| \, \Big\|_r \,.
\end{aligned}$$

For fixing this... Hölder SPR!

In contrast with

Theorem (FOPT '23)

For all $2 \leq p < +\infty$, there exists a closed subspace $E \subseteq \mathrm{L}_p[0,1]$ such that E is an SPR-subspace of $\mathrm{L}_p[0,1]$, but fails to be an SPR-subspace for each $1 \leq q < p < \infty$.



the range $1 \leq p < 2$ behaves in a different way, as

Theorem (FOPT '23)

If $1 \leq p < 2$, a closed subspace $E \subseteq \mathrm{L}_p[0,1]$ does SPR if, and only if, it does SPR on $\mathrm{L}_q[0,1]$ for $1 \leq q < p < 2$.