A counterexample to the complemented subspace problem in Banach lattices

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joint work with G. Martínez Cervantes, A. Salguero Alarcón, and P. Tradacete

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One of the most important problems in the theory of Banach lattices, which is still open, is whether any complemented subspace of a Banach lattice must be linearly isomorphic to a Banach lattice.

P. Casazza, N. Kalton and L. Tzafriri (1987)

The Complemented Subspace Problem

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- If P is a lattice homomorphism (that is, $P(x \vee y) = Px \vee Py$), then E is a sublattice of X.
- If P is positive (that is, $Px \geq 0$ whenever $x \geq 0$), then E with the order inherited, its lattice operations given by

$$x \vee_E y = P(x \vee y), \quad x \wedge_E y = P(x \wedge y) \quad \text{and} \quad |x|_E = P(|x|),$$

  and with the renorming $|||x||| = \|P|x|\|$ (for $x \in E$) is a Banach lattice.

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Free Banach lattices provide a (not very tractable) criterion to distinguish between the two:

- X is complemented in a Banach lattice $\iff$ X is complemented in FBL$[X]$.
- X is isomorphic to a Banach lattice $\iff$ there is an ideal $I \subset$ FBL$[X]$ such that FBL$[X] = I \oplus X$.

Some answers and some open questions

Positive answers

- Every 1-complemented subspace of an L_p-space $(1 \leq p < \infty)$ is an L_p-space (Bernau-Lacey 1974).

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Actually, PS_2 cannot be isomorphic to a Banach lattice.

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Definition. A Banach lattice X is said to be an AM-space if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for any $x, y \in X^+$. An AL-space is a Banach lattice such that $\|x + y\| = \|x\| + \|y\|$ for every $x, y \in X^+$.

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$|||\cdot|||$ is an AL-norm (compatible with the lattice order of X) and is related with the original norm by

$$\|x\| \leq |||x||| \leq (K_G \lambda)^2 \|x\|, \quad x \in X.$$

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Proof. X^* Banach lattice and $\mathcal{L}_1$-space, hence X^* is lattice isomorphic to an AL-space. Then, X^{**} is lattice isomorphic to certain $C(K)$-space, so X is lattice embeddable into that $C(K)$-space. □

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Corollary 2. If the CSP had a positive answer in the separable setting:

1. Every complemented subspace of $L_1[0,1]$ would be isomorphic to ℓ_1 or to $L_1[0,1]$.
2. Every complemented subpace of $\mathcal{C}[0,1]$ would be isomorphic to a $C(K)$-space.

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Let $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\} \subset \mathcal{P}(\mathbb{N})$ be an almost disjoint family, that is, $|A_\xi|$ is infinite for every ξ and $|A_\xi \cap A_{\xi'}|$ is finite whenever $\xi \neq \xi'$.

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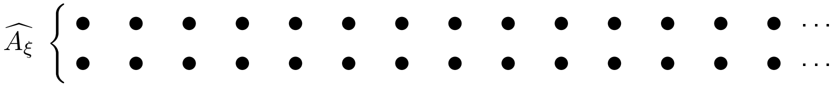

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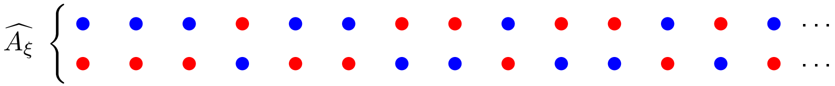

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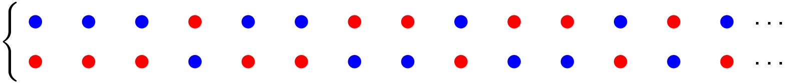

We define:

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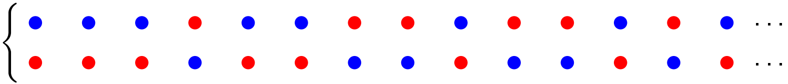

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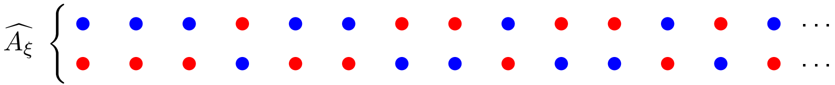

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Moreover, we can define a norm-one projection

$$P : \mathsf{JL}(\mathcal{B}) \longrightarrow \mathsf{JL}(\mathcal{A})$$

$$f \longmapsto Pf(n,0) = Pf(n,1) = \frac{f(n,0) + f(n,1)}{2}$$

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then $\|Q\| = 1$.

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$$Qf(n,0) = -Qf(n,1) = \frac{f(n,0) - f(n,1)}{2},$$

then $\|Q\| = 1$.

G. Plebanek and A. Salguero Alarcón show, through an inductive process of cardinality $\mathfrak{c}$, that there exist almost disjoint families $\mathcal{A}$, $\mathcal{B}$ such that X is not isomorphic to a $C(K)$-space. This X was christened PS_2.

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Consequently, the following statements are equivalent:

1. PS_2 is isomorphic to a Banach lattice.
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3. There exists a norming sequence $(x_n^*)_{n=0}^\infty$ in $B_{\mathsf{PS}_2^*}$ such that for every $f \in \mathsf{PS}_2$ there is an element $g \in \mathsf{PS}_2$ such that

$$x_n^*(g) = |x_n^*(f)|, \text{ for every } n \in \mathbb{N}.$$

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With slight modifications, it is possible to construct a $\widetilde{\mathsf{PS}_2}$ space such that $\widetilde{\mathsf{PS}_2} \oplus i\widetilde{\mathsf{PS}_2}$ is not isomorphic to a complex Banach lattice.

Ultrapowers

In 1983, S. Heinrich, C. Henson and L. Moore constructed a Banach space $X \subset \ell_\infty$ not isometric to a Banach lattice such that:

1. X is not isometric to a Banach lattice;
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It is an *immediate consequence* of a result of Heinrich, Henson and Moore (1986) that PS_2 satisfies the second property.

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1. This is trivial if $\mathrm{Hom}(X, \mathbb{R}) \neq \{0\}$.
2. Gowers' solution to *Banach's Hyperplane problem* (E non-isomorphic to $E \oplus \mathbb{R}$) *does not work for us.*

Thank you!