Inclusions of variable Lebesgue spaces: weak compactness and disjoint strict singularity

Francisco L. Hernandez

Madrid Complutense University

Workshop Banach spaces and Banach lattices (ICMAT 2024)

Joint work with Cesar Ruiz and Mauro Sanchiz

May 22, 2024

1.- **SINGULARITY OF SYMMETRIC BANACH LATTICE INCLUSIONS**

2.- **VARIABLE LEBESGUE SPACE INCLUSIONS**

1.- Singularity of symmetric Banach lattice inclusions

Def: Let E and F Banach spaces , $T : E \to F$ **strictly singular** if T is not invertible on any ∞-dim. (closed) subspace E_1 of E

if $c > 0, ,\quad c\|x\| \le \|Tx\|, \quad x \in E_1 \subset E \implies \dim E_1 < \infty$

$\mathcal{K}$. compact op. $\mathcal{K} \subset \mathcal{S}$ closed operator ideal

1.- Singularity of symmetric Banach lattice inclusions

Def: Let E and F Banach spaces , $T : E \to F$ **strictly singular** if T is not invertible on any ∞-dim. (closed) subspace E_1 of E

if $c > 0,$, $\quad c\,\|x\| \leq \|Tx\|, \quad x \in E_1 \subset E \implies \dim E_1 < \infty$

$\mathcal{K}$. compact op. $\quad \mathcal{K} \subset \mathcal{S}$ closed operator ideal

Ex: every $T : \ell^p \to \ell^q$, $p \neq q$ strictly singular

$T : E \to F$ is strictly singular $\Leftrightarrow$ for every $E_1 \subset E$ there exists $E_2 \subset E_1$ s.t. $T_{|E_2}$ compact

- spectral properties of strictly singular op. like compact op.

- Fredholm perturbation: R Fredholm, T strictly singular $\Rightarrow R + T$ Fredholm

Bad behavior: no stable by duality, no interpolation in general (yes for Lp-spaces)

1.- Singularity of symmetric Banach lattice inclusions

Def: Let E and F Banach spaces , $T : E \to F$ **strictly singular** if T is not invertible on any ∞-dim. (closed) subspace E_1 of E

if $c > 0,$, $c\|x\| \leq \|Tx\|$, $x \in E_1 \subset E \implies \dim E_1 < \infty$

$\mathcal{K}$. compact op. $\mathcal{K} \subset \mathcal{S}$ closed operator ideal

Ex: every $T : \ell^p \to \ell^q$, $p \neq q$ strictly singular

$T : E \to F$ is strictly singular $\Leftrightarrow$ for every $E_1 \subset E$ there exists $E_2 \subset E_1$ s.t. $T_{|E_2}$ compact

- spectral properties of strictly singular op. like compact op.

- Fredholm perturbation: R Fredholm, T strictly singular $\Rightarrow R + T$ Fredholm

Bad behavior: no stable by duality, no interpolation in general (yes for Lp-spaces)

$\mathcal{K}(L^p) = S(L^p) \iff p = 2$

1.- Singularity of symmetric Banach lattice inclusions

Def: Let E and F Banach spaces , $T : E \to F$ **strictly singular** if T is not invertible on any ∞-dim. (closed) subspace E_1 of E

if $c > 0,,\quad c\,\|x\| \leq \|Tx\|, \quad x \in E_1 \subset E \implies \dim E_1 < \infty$

$\mathcal{K}$. compact op. $\mathcal{K} \subset \mathcal{S}$ closed operator ideal

Ex: every $T : \ell^p \to \ell^q$, $p \neq q$ strictly singular

$T : E \to F$ is strictly singular $\Leftrightarrow$ for every $E_1 \subset E$ there exists $E_2 \subset E_1$ s.t. $T_{|E_2}$ compact

- spectral properties of strictly singular op. like compact op.

- Fredholm perturbation: R Fredholm, T strictly singular $\Rightarrow R + T$ Fredholm

Bad behavior: no stable by duality, no interpolation in general (yes for Lp-spaces)

$\mathcal{K}(L^p) = S(L^p) \iff p = 2$

$\star$ if μ-finite, $L^\infty(\mu) \hookrightarrow L^p(\mu)$ is strictly singular $1 \leq p < \infty$ (Grothendieck 1962)

$$c\,\|x\|_{L^\infty} \leq \|x\|_{L^p} \quad x \in E_1 \implies \dim E_1 < M(c)$$

Banach lattices of measurable functions

$(E(\mu), \| \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \quad \Rightarrow \quad \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\} \qquad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\quad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

Banach lattices of measurable functions

$(E(\mu), \| \, \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \;\Rightarrow\; \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\} \qquad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\quad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

$E(\mu)$ symmetric , finite μ , $\quad L^\infty(\mu) \hookrightarrow E(\mu) \hookrightarrow L^1(\mu)$

$(E(\mu), \| \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \;\Rightarrow\; \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\} \qquad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\quad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

$E(\mu)$ symmetric , finite μ , $\quad L^\infty(\mu) \hookrightarrow E(\mu) \hookrightarrow L^1(\mu)$

$(E(\mu), \| \, \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \;\Rightarrow\; \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\}$ $\qquad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\qquad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

$E(\mu)$ symmetric , finite μ , $\qquad L^\infty(\mu) \hookrightarrow E(\mu) \hookrightarrow L^1(\mu)$

★ $L^\infty(\mu) \hookrightarrow E(\mu)$ strictly singular for symmetric $E(\mu)(\neq L^\infty)$ (Novikov 1992)

Banach lattices of measurable functions

$(E(\mu), \| \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \Rightarrow \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\}$ $\quad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\quad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

$E(\mu)$ symmetric , finite μ , $\quad L^\infty(\mu) \hookrightarrow E(\mu) \hookrightarrow L^1(\mu)$

★ $L^\infty(\mu) \hookrightarrow E(\mu)$ strictly singular for symmetric $E(\mu)(\neq L^\infty)$ (Novikov 1992)

$L^p(\mu) \hookrightarrow L^1(\mu)$ no strictly singular $1 \leq p < \infty$

Khintchine Inequalty:

$$\Big(\int_0^1 |\sum_{n=1}^\infty a_n r_n|^p \, d\mu\Big)^{1/p} \sim \Big(\sum_{n=1}^\infty |a_n|^2\Big)^{1/2}$$

Rademacher $r_n(t) = sign(sin(2^n \pi t))$ $1 \leq p < \infty$

$$Rad(L^p) := [(r_n)]_{L^p} \cong \ell^2 \cong Rad(L^q) \quad , 1 \leq q, p < \infty$$

Banach lattices of measurable functions

$(E(\mu), \| \, \|)$ **symmetric** (or r.i.) if $\mu_f(\cdot) = \mu_g(\cdot) \;\Rightarrow\; \|f\| = \|g\|$

$\mu_f(s) := \mu\{|f| > s\} \qquad f^*(t) := \inf\{s \geq 0 : \mu_f(s) \leq t\}$

ϕ_E fundamental function $\quad \phi_E(t) = \|\chi_A\|$, $\mu(A) = t$

Examp.: L^p-spaces, Orlicz , Lorentz, Marcinkiewicz, mixed-norm, interpolation s., ...

$E(\mu)$ symmetric , finite μ , $\quad L^\infty(\mu) \hookrightarrow E(\mu) \hookrightarrow L^1(\mu)$

★ $L^\infty(\mu) \hookrightarrow E(\mu)$ strictly singular for symmetric $E(\mu)(\neq L^\infty)$ (Novikov 1992)

$L^p(\mu) \hookrightarrow L^1(\mu)$ no strictly singular $1 \leq p < \infty$

Khintchine Inequalty:

$$\Big(\int_0^1 |\sum_{n=1}^\infty a_n r_n|^p \, d\mu \Big)^{1/p} \;\sim\; (\sum_{n=1}^\infty |a_n|^2)^{1/2}$$

Rademacher $\quad r_n(t) = sign(sin(2^n \pi t)) \;\; 1 \leq p < \infty$

$$Rad(L^p) := [(r_n)]_{L^p} \cong \ell^2 \cong Rad(L^q) \quad , 1 \leq q, p < \infty$$

★ $E(\mu) \hookrightarrow L^1(\mu)$ strictly singular $\iff$ $L_0^{exp\, x^2}(\mu) \not\subseteq E(\mu)$

(H., Novikov and Semenov 2003)

Def.: Let E a Banach lattice , $T : E \to F$ **disjointly strictly singular (DSS)** if there is no disjoint sequence (f_n) s. t. $T_{|[(f_n)]}$ is an isomorphism

strict singularity $\Rightarrow$ *DSS*

($\not\Leftarrow$) $L^p(\mu) \hookrightarrow L^q(\mu),$, $1 \leq q < p < \infty$ DSS

Def.: Let E a Banach lattice , $T : E \to F$ **disjointly strictly singular (DSS)** if there is no disjoint sequence (f_n) s. t. $T_{|[(f_n)]}$ is an isomorphism

strict singularity $\Rightarrow$ *DSS*

($\not\Leftarrow$) $L^p(\mu) \hookrightarrow L^q(\mu)$, , $1 \leq q < p < \infty$ DSS

stable closed class by addition and right composition, (no by left composition)

Def.: Let E a Banach lattice , $T : E \to F$ **disjointly strictly singular (DSS)** if there is no disjoint sequence (f_n) s. t. $T_{|[(f_n)]}$ is an isomorphism

strict singularity $\Rightarrow$ *DSS*

($\not\Leftarrow$) $L^p(\mu) \hookrightarrow L^q(\mu), , 1 \leq q < p < \infty$ DSS

stable closed class by addition and right composition, (no by left composition)

★ for every symmetric $E(\mu)\,(\neq L1)$, μ finite , $E(\mu) \hookrightarrow L^1(\mu)$ *DSS*

Kadec-Pelcynski sets $A_p(\varepsilon)$

Def.: Let E a Banach lattice , $T : E \to F$ **disjointly strictly singular (DSS)** if there is no disjoint sequence (f_n) s. t. $T_{|[(f_n)]}$ is an isomorphism

strict singularity $\Rightarrow$ *DSS*

($\not\Leftarrow$) $L^p(\mu) \hookrightarrow L^q(\mu)$, , $1 \leq q < p < \infty$ DSS

stable closed class by addition and right composition, (no by left composition)

★ for every symmetric $E(\mu)\,(\neq L1)$, μ finite , $E(\mu) \hookrightarrow L^1(\mu)$ *DSS*

Kadec-Pelcynski sets $A_p(\varepsilon)$

♠ $E(\mu) \hookrightarrow F(\mu)$ strictly singular $\iff$ $DSS +$ $Rad(E) \neq Rad(F)$
(Astashkin, H., Semenov 2008)

Def.: Let E a Banach lattice , $T : E \to F$ **disjointly strictly singular (DSS)** if there is no disjoint sequence (f_n) s. t. $T_{|[(f_n)]}$ is an isomorphism

strict singularity $\Rightarrow$ *DSS*

($\not\Leftarrow$) $L^p(\mu) \hookrightarrow L^q(\mu)$, , $1 \le q < p < \infty$ DSS

stable closed class by addition and right composition, (no by left composition)

★ for every symmetric $E(\mu)\,(\neq L1)$, μ finite , $E(\mu) \hookrightarrow L^1(\mu)$ *DSS*

Kadec-Pelcynski sets $A_p(\varepsilon)$

♠ $E(\mu) \hookrightarrow F(\mu)$ strictly singular $\iff$ $DSS +$ $Rad(E) \neq Rad(F)$
(Astashkin, H., Semenov 2008)

Ex: if $0 < q < p < \infty$ and $p > 2$, $(L^{exp\, x^p}) \hookrightarrow (L^{exp\, x^q})$ strictly singular

$T : L^p \to L^p$

strictly singular $\Leftrightarrow$ ℓ_p-singular + ℓ_2-singular ($\Leftrightarrow$ DSS + ℓ_2-singular)
(L. Weis 1978)

$T : L^p \to L^p$

strictly singular $\Leftrightarrow$ ℓ_p-singular + ℓ_2-singular ($\Leftrightarrow$ DSS + ℓ_2-singular)
(L. Weis 1978)

extensions to Banach lattices:

- regular $T : E \to F$, E q-concave , F order continuous

strictly singular $\iff$ DSS $+$ ℓ_2-singular
(Flores, H., Kalton, Tradacete 2.009)

$T : L^p \to L^p$

strictly singular $\Leftrightarrow$ ℓ_p-singular + ℓ_2-singular ($\Leftrightarrow$ DSS + ℓ_2-singular)
(L. Weis 1978)

extensions to Banach lattices:

- regular $T : E \to F$, E q-concave , F order continuous

strictly singular $\iff$ DSS $+$ ℓ_2-singular
(Flores, H., Kalton, Tradacete 2.009)

No true for arbitrary operators in general.
(Yes, if F has 2-lower estimation)

$T : L^p \to L^p$

strictly singular $\Leftrightarrow$ ℓ_p-singular + ℓ_2-singular ($\Leftrightarrow$ DSS + ℓ_2-singular)
(L. Weis 1978)

extensions to Banach lattices:

- regular $T : E \to F$, E q-concave , F order continuous

strictly singular $\iff$ DSS $+$ ℓ_2-singular
(Flores, H., Kalton, Tradacete 2.009)

No true for arbitrary operators in general.
(Yes, if F has 2-lower estimation)

E q-concave , F order continuous ,
$0 \leq T : E \to F$ ℓ_2-singular $\Rightarrow$ AM-compact

($T : E \to F$ *AM*-**compact** if image of order intervals are relatively compact)

E q-concave $0 \leq T : E \to F$ DSS $+$ AM-compact $\Rightarrow$ strictly singular

2.- Variable Lebesgue space inclusions and DSS

exponent $p(\cdot)$: measurable f. $(\Omega, \mu) \mapsto [1, \infty)$
modular

$$m_{p(\cdot)}(f) := \int_{\Omega} |f(t)|^{p(t)} d\mu(t)$$

$L^{p(\cdot)}(\Omega)$ Variable Lebesgue (or Nakano) space : measurable f. s.t. $m_{p(\cdot)}(f/r) < \infty$ for some $r > 0$

Luxemburg norm $\|f\|_{p(\cdot)} = \inf.\{r > 0 : m_{p(\cdot)}(f/r) \leq 1\}$

2.- Variable Lebesgue space inclusions and DSS

exponent $p(\cdot)$: measurable f. $(\Omega, \mu) \mapsto [1, \infty)$
modular

$$m_{p(\cdot)}(f) := \int_\Omega |f(t)|^{p(t)} d\mu(t)$$

$L^{p(\cdot)}(\Omega)$ Variable Lebesgue (or Nakano) space : measurable f. s.t. $m_{p(\cdot)}(f/r) < \infty$ for some $r > 0$

Luxemburg norm $\|f\|_{p(\cdot)} = \inf .\{r > 0 : m_{p(\cdot)}(f/r) \leq 1\}$

$1 \leq p^- = \operatorname{ess\,inf}\{p(t) : t \in \Omega\} \leq p^+ = \operatorname{ess\,sup}\{p(t) : t \in \Omega\} \leq \infty$

$1 \leq p|_A^- \leq p|_A^+ \leq \infty$

exponent $p(\cdot)$: measurable f. $(\Omega, \mu) \mapsto [1, \infty)$
modular

$$m_{p(\cdot)}(f) := \int_{\Omega} |f(t)|^{p(t)} d\mu(t)$$

$L^{p(\cdot)}(\Omega)$ Variable Lebesgue (or Nakano) space : measurable f. s.t. $m_{p(\cdot)}(f/r) < \infty$ for some $r > 0$

Luxemburg norm $\|f\|_{p(\cdot)} = \inf.\{r > 0 : m_{p(\cdot)}(f/r) \leq 1\}$

$1 \leq p^{-} = \operatorname{ess\,inf}\{p(t) : t \in \Omega\} \leq p^{+} = \operatorname{ess\,sup}\{p(t) : t \in \Omega\} \leq \infty$
$1 \leq p|_{A}^{-} \leq p|_{A}^{+} \leq \infty$

$L^{p(\cdot)}(\Omega)$ non-symmetric, no translation invariant spaces

dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\mu)$ when $p^{+} < \infty$, $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ $a.e.$
$L^{p(\cdot)}(\Omega)$ reflexive $\iff 1 < p^{-} \leq p^{+} < \infty$
Nakano sequence spaces ℓ_{p_n}

exponent $p(\cdot)$: measurable f. $(\Omega,\mu)\mapsto[1,\infty)$
modular

$$m_{p(\cdot)}(f) := \int_\Omega |f(t)|^{p(t)}d\mu(t)$$

$L^{p(\cdot)}(\Omega)$ Variable Lebesgue (or Nakano) space : measurable f. s.t. $m_{p(\cdot)}(f/r)<\infty$ for some $r>0$

Luxemburg norm $\qquad \|f\|_{p(\cdot)} = \inf.\{r>0 : m_{p(\cdot)}(f/r)\leq 1\}$

$1\leq p^- = \operatorname{ess\,inf}\{p(t) : t\in\Omega\}\leq p^+ = \operatorname{ess\,sup}\{p(t) : t\in\Omega\}\leq\infty$
$1\leq p|_A^-\leq p|_A^+\leq\infty$

$L^{p(\cdot)}(\Omega)$ non-symmetric, no translation invariant spaces

dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\mu)$ when $p^+<\infty$, $\frac{1}{p(t)}+\frac{1}{p'(t)}=1\ a.e.$
$L^{p(\cdot)}(\Omega)$ reflexive $\Longleftrightarrow 1<p^-\leq p^+<\infty$
Nakano sequence spaces ℓ_{p_n}

- Essential range set

$$R_{p(\cdot)} := \{q\in[1,\infty) : \qquad \mu(p^{-1}(q-\epsilon,q+\epsilon))>0\,,\ \ for\ all\ \epsilon>0\}$$

$R_{p(\cdot)}$ is closed (compact if $p^+<\infty$)
$R_{p(\cdot)}$ is a lattice-isomorphic invariant

$$(g_k) := (\frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}})$$

$q \in R_{p(\cdot)} \iff \ell_q$ is lattice-embedding into $L^{p(\cdot)}(\mu)$

- if $p+ = \infty \quad L^{p(\cdot)}(\mu) \supsetapprox \ell_\infty$

$$(g_k) := \left(\frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}}\right)$$

$q \in R_{p(\cdot)} \iff \ell_q$ is lattice-embedding into $L^{p(\cdot)}(\mu)$

- if $p+ = \infty \quad L^{p(\cdot)}(\mu) \supsetsim \ell_\infty$

, if $p^+ < \infty$.

- $L^{p(\cdot)}(\mu) \supsetsim \ell_q \iff q \in R_{p(\cdot)} \cup \{2\}$, (if $p^- > 2$)
- $L^{p(\cdot)}(\mu) \supsetsim \ell_q \iff q \in R_{p(\cdot)} \cup [p^-, 2]$. , (if $1 \leq p^- \leq 2$)

$$(g_k) := (\frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}})$$

$q \in R_{p(\cdot)} \iff \ell_q$ is lattice-embedding into $L^{p(\cdot)}(\mu)$

- if $p+ = \infty \quad L^{p(\cdot)}(\mu) \supsetapprox \ell_\infty$

, if $p^+ < \infty$.

- $L^{p(\cdot)}(\mu) \supsetapprox \ell_q \iff q \in R_{p(\cdot)} \cup \{2\}$, (if $p^- > 2$)
- $L^{p(\cdot)}(\mu) \supsetapprox \ell_q \iff q \in R_{p(\cdot)} \cup [p^-, 2]$. , (if $1 \leq p^- \leq 2$)

$L^{p(\cdot)}(\mu) \supsetapprox \ell_q$-*complemented* sublattice for every $q \in R_{p(\cdot)}$
there exists orthogonal projection T_A bounded for "suitable" (A_k)

$$T_A(f)(t) = \sum_{k=1}^{\infty} \left(\int_{A_k} \frac{f(s)}{\mu(A_k)^{\frac{1}{p'(s)}}} d\mu(s) \right) \frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}},$$

$$(g_k) := (\frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}})$$

$q \in R_{p(\cdot)} \iff \ell_q$ is lattice-embedding into $L^{p(\cdot)}(\mu)$

- if $p+ = \infty \quad L^{p(\cdot)}(\mu) \gtrsim \ell_\infty$

, if $p^+ < \infty$.

- $L^{p(\cdot)}(\mu) \gtrsim \ell_q \iff q \in R_{p(\cdot)} \cup \{2\}$, (if $p^- > 2$)
- $L^{p(\cdot)}(\mu) \gtrsim \ell_q \iff q \in R_{p(\cdot)} \cup [p^-, 2]$. , (if $1 \leq p^- \leq 2$)

$L^{p(\cdot)}(\mu) \gtrsim \ell_q$-*complemented* sublattice for every $q \in R_{p(\cdot)}$
there exists orthogonal projection T_A bounded for "suitable" (A_k)

$$T_A(f)(t) = \sum_{k=1}^{\infty} \left(\int_{A_k} \frac{f(s)}{\mu(A_k)^{\frac{1}{p'(s)}}} d\mu(s) \right) \frac{\chi_{A_k}(t)}{\mu(A_k)^{\frac{1}{p(t)}}},$$

If , $T : L^{p(\cdot)}(\Omega) \mapsto L^{p(\cdot)}(\Omega)$, with $p^+ < \infty$
strictly singular $\iff$ ℓ_q-singular for $q \in R_{p(\cdot)} \cup \{2\}$
(extension of Weis's result)

variable Lebesgue space inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff$ (*) $q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$(**) \qquad \int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

variable Lebesgue space inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff$ (*) $q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$(**) \qquad \int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.
$r(t) := \frac{p(t)\,q(t)}{p(t)-q(t)}, \quad \frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}$ so By Holder norm inequality.

$$\|f\,\chi_\Omega\|_{q(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\,\|\chi_\Omega\|_{\frac{p\,q}{p-q}(\cdot)}$$

variable Lebesgue space inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff$ (*) $q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$(**) \qquad \int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.
$r(t) := \frac{p(t)\,q(t)}{p(t)-q(t)}$, $\quad \frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}$ so By Holder norm inequality.

$$\|f\,\chi_\Omega\|_{q(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\,\|\chi_\Omega\|_{\frac{p\,q}{p-q}(\cdot)}$$

if μ finite $L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu) \iff$ condition (*)

$$L^\infty(\mu) \hookrightarrow L^{p(\cdot)}(\mu) \hookrightarrow L^1(\mu)$$

variable Lebesgue space inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff$ (*) $q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$(**) \qquad \int_{\Omega_d} \lambda^{-\frac{pq}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.
$r(t) := \frac{p(t)\,q(t)}{p(t)-q(t)}, \quad \frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}$ so By Holder norm inequality.

$$\|f\,\chi_\Omega\|_{q(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\,\|\chi_\Omega\|_{\frac{pq}{p-q}(\cdot)}$$

if μ finite $L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu) \iff$ condition (*)

$$L^{\infty}(\mu) \hookrightarrow L^{p(\cdot)}(\mu) \hookrightarrow L^{1}(\mu)$$

Questions: Find suitable criteria on the exponents for these inclusions be *DSS* or be strictly singular ???

Theorem (H, Ruiz, Sanchiz 2.021)

Let $q(\cdot) \le p(\cdot) \le p^+ < \infty$ *.* μ *finite,* $L^{p(\cdot)}(\mu) \subset L^{q(\cdot)}(\mu)$ *weakly compact* $\Longleftrightarrow$

$$\lim_{\lambda \to 0} \sup_{\|f\|_{p(\cdot)} \le 1} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{q(t)} d\mu = 0$$

$$\lim_{\mu(A) \to 0} \sup_{\|f\|_{p(\cdot)} \le 1} \int_{A \cap \Omega_1} |f(t)| d\mu = 0,$$

for $\Omega_1 = q^{-1}(\{1\})$*.*

Ando type criterion. (Dunford-Pettis Thm for L^1)

Theorem (H, Ruiz, Sanchiz 2.021)

Let $q(\cdot) \leq p(\cdot) \leq p^+ < \infty$. μ finite, $L^{p(\cdot)}(\mu) \subset L^{q(\cdot)}(\mu)$ weakly compact $\Longleftrightarrow$

$$\lim_{\lambda \to 0} \sup_{\|f\|_{p(\cdot)} \leq 1} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{q(t)} d\mu = 0$$

$$\lim_{\mu(A) \to 0} \sup_{\|f\|_{p(\cdot)} \leq 1} \int_{A \cap \Omega_1} |f(t)| d\mu = 0,$$

for $\Omega_1 = q^{-1}(\{1\})$.

Ando type criterion. (Dunford-Pettis Thm for L^1)

- Def: $T : E \to F$, is **L-weakly compact** if $T(B_E)$ is equi-integrable in F for B_E the unit ball of E i.e.for every measurable (A_n) s.t. $\chi_{A_n} \to 0$ μ-a.e.

$$\lim_{n \to \infty} \sup_{f \in B_E} \{||T(f)\chi_{A_n}||_F\} = 0$$

Theorem (H, Ruiz, Sanchiz 2.021)

Let $q(\cdot) \leq p(\cdot) \leq p^+ < \infty$. μ finite, $L^{p(\cdot)}(\mu) \subset L^{q(\cdot)}(\mu)$ weakly compact $\Longleftrightarrow$

$$\lim_{\lambda \to 0} \sup_{\|f\|_{p(\cdot)} \leq 1} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{q(t)} d\mu = 0$$

$$\lim_{\mu(A) \to 0} \sup_{\|f\|_{p(\cdot)} \leq 1} \int_{A \cap \Omega_1} |f(t)| d\mu = 0,$$

for $\Omega_1 = q^{-1}(\{1\})$.

Ando type criterion. (Dunford-Pettis Thm for L^1)

- Def: $T : E \to F$, is **L-weakly compact** if $T(B_E)$ is equi-integrable in F for B_E the unit ball of E i.e.for every measurable (A_n) s.t. $\chi_{A_n} \to 0$ μ-a.e.

$$\lim_{n \to \infty} \sup_{f \in B_E} \{||T(f)\chi_{A_n}||_F\} = 0$$

(De la Vallée-Poussin type)

finite μ , $q(\cdot) \leq p(\cdot) < p^+ < \infty$.

$L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$ L-weakly compact $\Longleftrightarrow$ there exists Orlicz f. φ with $\lim_{x \mapsto \infty} \frac{\varphi(x)}{x} = \infty$ s. t. for $\Phi(t, x) = (\varphi(x))^{q(t)}$

$$L^{p(\cdot)}(\mu) \hookrightarrow L^{\Phi}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$$

Theorem (Edmunds,Gogathisvili, Nekvinda 2023)

bounded open $\Omega \subset \mathbf{R^n}$ $q(\cdot) \leq p(\cdot) \leq p^+ < \infty$. $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *L-weakly compact* $\Longleftrightarrow$ *for every* $a > 1$ $\int_0^{|\Omega|} a^{\left(\frac{1}{p-q}\right)^*(x)} dx < \infty$.

Theorem (Edmunds,Gogathisvili, Nekvinda 2023)

bounded open $\Omega \subset \mathbf{R^n}$ $q(\cdot) \leq p(\cdot) \leq p^+ < \infty$. $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *L-weakly compact* $\iff$ *for every* $a > 1$ $\int_0^{|\Omega|} a^{\left(\frac{1}{p-q}\right)^*(x)} dx < \infty$.

compactness of Sobolev embeddings $W^{1,p(.)} \hookrightarrow L^{q(.)}$

$$\begin{array}{ccccc} W^{1,p(.)} & & \hookrightarrow & & L^{q(.)} \\ & \searrow & & \nearrow & \\ & & L^{p(.)} & & \end{array}$$

Theorem (Edmunds,Gogathisvili, Nekvinda 2023)

bounded open $\Omega \subset \mathbf{R^n}$ $q(\cdot) \le p(\cdot) \le p^+ < \infty$. $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *L-weakly compact* $\iff$ *for every* $a > 1$ $\int_0^{|\Omega|} a^{\left(\frac{1}{p-q}\right)^*(x)} dx < \infty$.

compactness of Sobolev embeddings $W^{1,p(.)} \hookrightarrow L^{q(.)}$

$$
\begin{array}{ccccc}
W^{1,p(.)} & & \hookrightarrow & & L^{q(.)} \\
& \searrow & & \nearrow & \\
& & L^{p(.)} & &
\end{array}
$$

Def: $T : E \to F$ is **M-weakly compact** whenever $\lim_{n\to\infty} ||T(f_n)||_F = 0$ for bounded disjoint sequence (f_n) of E.

M-weakly compact $\Rightarrow$ DSS

$(\not\Leftarrow)$

Inclusion $L^{\infty}(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$ **, finite** μ .
when is strictly singular ? (obvious if $p^{+} < \infty$)
how quickly $p(.)$ can tend to ∞ preserving strict singularity ?

Inclusion $L^{\infty}(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$ **, finite** μ .
when is strictly singular ? (obvious if $p^+ < \infty$)
how quickly $p(.)$ can tend to ∞ preserving strict singularity ?

Theorem

finite μ *, and* $i : L^{\infty}(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$ *ṪFAE :*

1. *i is L-weakly compact.*
2. *i is M-weakly compact.*
3. *i is strictly singular*
4. *i is DSS.*
5. $\int_0^{\mu(\Omega)} a^{p^*(x)} dx < \infty$ *for every* $a > 1$.

$$(5) \iff p(\cdot) \in L_0^{\exp x}(\mu)$$

Inclusion $L^\infty(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$ **, finite** μ .
when is strictly singular ? (obvious if $p^+ < \infty$)
how quickly $p(.)$ can tend to ∞ preserving strict singularity ?

Theorem

finite μ , and $i : L^\infty(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$ ṪFAE :

1. *i is L-weakly compact.*
2. *i is M-weakly compact.*
3. *i is strictly singular*
4. *i is DSS.*
5. $\int_0^{\mu(\Omega)} a^{p^*(x)} dx < \infty$ *for every $a > 1$.*

$$(5) \iff p(\cdot) \in L_0^{\exp x}(\mu)$$

Ex:
- $p_\alpha(t) := \frac{1}{t^\alpha}$, $\alpha > 0$ $(0,1)$ $L^\infty \hookrightarrow L^{p_\alpha(\cdot)}$ no strictly singular

- $p_\alpha(t) := \ln^\alpha(\frac{1}{t})$, $(0, \frac{1}{e})$ $L^\infty \hookrightarrow L^{p_\alpha(\cdot)}$ strictly singular $\iff$ $0 < \alpha < 1$

(1) $\Rightarrow$ (2)

disjoint normalized (f_n) in L^∞ , so $\mu(supp(f_n)) \to 0$, hence

$$\lim_{n\to\infty} \|f_n\|_{p(\cdot)} \leq \lim_{n\mapsto\infty} \sup_k \|f_k\, \chi_{supp(f_n)}\|_{p(\cdot)} \leq \lim_{\mu(A)\to 0} \sup_k \|f_k \chi_A\|_{p(\cdot)} = 0.$$

it is M-weakly compact so weakly compact.

(2) $\Rightarrow$ (3) Dunford-Pettis property of L^∞.

(3)$\Rightarrow$ (4) it is obvious

$(4) \Rightarrow (5)$

Assume there exists ,$a > 1$ s.t. $\int_0^{\mu(\Omega)} a^{p^*(x)} dx = \infty$

$(4) \Rightarrow (5)$

Assume there exists ,$a > 1$ s.t. $\int_0^{\mu(\Omega)} a^{p^*(x)} dx = \infty$

- There exists $\beta > 0$ and disjoint $(E_n)_{n=1}^{\infty}$ s.t. $||\chi_{E_n}||_{p(\cdot)} \geq \beta > 0$

take $0 < \beta < 1$ s.t. $1/\beta > a$. Let $t_n \searrow 0$ s.t. $\int_{t_{n+1}}^{t_n} a^{p^*(x)}\, dx > 1$.
Take $F_n \subset \Omega$ with $\mu(F_n) = t_n$ s. t.

$$\int_{F_n} a^{p(t)}\, d\mu(t) = \int_0^{t_n} (a^{r(\cdot)})^*(x)\, dx = \int_0^{t_n} a^{p^*(x)}\, dx.$$

The sets (F_n) s. t. $F_{n+1} \subset F_n$ since $t_{n+1} < t_n$ Define disjoint $E_n := F_n \backslash F_{n+1}$.

$$\int_\Omega (\frac{\chi_{E_n}}{\beta})^{p(t)}\, d\mu(t) \geq \int_{E_n} (a\chi_{E_n})^{p(t)} d\mu(t) = \int_{F_n} (a\chi_{F_n})^{p(t)} d\mu(t) - \int_{F_{n+1}} (a\chi_{F_{n+1}})^{r(t)} d\mu(t)$$

$$= \int_0^{t_n} a^{p^*(x)}\, dx - \int_0^{t_{n+1}} a^{p^*(x)} dx = \int_{t_{n+1}}^{t_n} a^{p^*(x)}\, dx > 1$$

- $(\chi_{E_n})_n$ is equivalent in L^∞ and in $L^{p(\cdot)}$ to the basis of c_0.

$$||\sum_{n=1}^{k} a_n \chi_{E_n}||_\infty = \max_{1 \leq n \leq k} |a_n| \leq \frac{1}{\beta} ||\sum_{n=1}^{k} a_n \chi_{E_n}||_{p(\cdot)} \leq \frac{C}{\beta} ||\sum_{n=1}^{k} a_n \chi_{E_n}||_\infty$$

(5) $\Rightarrow$ (1)

• Every $\chi_{E_n} \to 0$ μ-a.e. verifies $||\chi_{E_n}||_{p(\cdot)} \to 0$
Assume no, there exists (E_n) with $\chi_{E_n} \to 0$ μ-a.e. (thus $\mu(E_n) \to 0$) and $0 < \delta < 1$ s.t. $||\chi_{E_n}||_{p(\cdot)} \geq \delta$, For $0 < \beta < \delta$,

$$1 < \int_\Omega (\frac{\chi_{E_n}}{\beta})^{p(t)} d\mu$$

$$\int_\Omega (\frac{\chi_\Omega}{\beta})^{p(t)} d\mu = \int_0^{\mu(\Omega)} (\frac{1}{\beta})^{p^*(x)} dx < \infty.$$

so ,by Lebesgue dom.th., $\int_\Omega (\frac{\chi_{E_n}}{\beta})^{p(t)} d\mu \to 0$

Now ,

$$\lim_{\mu(E_n)\to 0} \sup_{f \in B_{L^\infty}} \|f\chi_{E_n}\|_{p(\cdot)} \leq \lim_{\mu(E_n)\to 0} \|\chi_{E_n}\|_{p(\cdot)} = 0$$

Inclusions $i : L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$, finite μ

Inclusions $i : L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$ **, finite** μ

never i is strictly singular ($1 \leq q^{-} \leq p^{-} < \infty$)

Inclusions $i : L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$ **, finite** μ

never i is strictly singular ($1 \leq q^- \leq p^- < \infty$)

Theorem

TFAE:

1. *i is L-weakly compact.*
2. *i is DSS .*
3. *The restriction of i to $[\frac{\chi_{B_n}}{\mu(B_n)^{\frac{1}{p(t)}}}]$ (disjoint B_n) is not isomorphism.*
4. $\lim_{x \to \mu(\Omega)^-} (\mu(\Omega) - x)^{(\frac{p-q}{p\,q})^*(x)} = 0.$
5. $\int_0^{\mu(\Omega)} a^{(\frac{p\,q}{p-q})^*(x)} dx < \infty$ *for every $a > 1$.*

Inclusions $i : L^{p(\cdot)}(\mu) \hookrightarrow L^{q(\cdot)}(\mu)$ **, finite** μ

never i is strictly singular ($1 \leq q^- \leq p^- < \infty$)

Theorem

TFAE:

1. *i is L-weakly compact.*
2. *i is DSS .*
3. *The restriction of i to $[\frac{\chi_{B_n}}{\mu(B_n)^{\frac{1}{p(t)}}}]$ (disjoint B_n) is not isomorphism.*
4. $\lim_{x \to \mu(\Omega)^-} (\mu(\Omega) - x)^{(\frac{p-q}{p\,q})^*(x)} = 0.$
5. $\int_0^{\mu(\Omega)} a^{(\frac{p\,q}{p-q})^*(x)} dx < \infty$ *for every $a > 1$.*

-If $ess \inf(p - q)(\cdot) > 0$ and $p^+ < \infty \implies L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ DSS

($\not\Longleftarrow$)

(1)$\Longleftrightarrow$(2) fails for Orlicz space inclusions

-extends L-weak compactness criteria of [E-G-N]

(5) $\Rightarrow$ (1)

$s(t) := \frac{p(t)\,q(t)}{p(t)-q(t)}$, so

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{s(t)}$$

By Holder norm inequality.

$$\|f\,\chi_A\|_{q(\cdot)} \;\leq\; 2\,\|f\|_{p(\cdot)}\,\|\chi_A\|_{\frac{p\,q}{p-q}(\cdot)}$$

Hence

$$\lim_{\mu(A_n)\to 0}\;\sup_{f\in B_{L^{p(\cdot)}(\Omega)}}\|f\chi_{A_n}\|_{q(\cdot)} \;\leq\; 2\lim_{\mu(A_n)\to 0}\|\chi_{A_n}\|_{\frac{p\,q}{p-q}(\cdot)} \;=\; 0$$

$(5) \Rightarrow (1)$

$s(t) := \frac{p(t)\,q(t)}{p(t)-q(t)}$, so

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{s(t)}$$

By Holder norm inequality.

$$\|f\,\chi_A\|_{q(\cdot)} \;\leq\; 2\,\|f\|_{p(\cdot)}\,\|\chi_A\|_{\frac{p\,q}{p-q}(\cdot)}$$

Hence

$$\lim_{\mu(A_n)\to 0}\;\sup_{f\in B_{L^{p(\cdot)}(\Omega)}}\|f\chi_{A_n}\|_{q(\cdot)} \;\leq\; 2\lim_{\mu(A_n)\to 0}\|\chi_{A_n}\|_{\frac{p\,q}{p-q}(\cdot)} \;=\; 0$$

$(1) \Rightarrow (2)$ (similar before showing that *i* is M-weakly compact

(3) $\Rightarrow$ (4) (sketch) Assume there exists $(x_n) \nearrow \mu(\Omega)$ s.t. $(\frac{p-q}{p\,q})^*(x_n) \mapsto 0$ with

$$(\mu(\Omega) - x_n)^{(\frac{p-q}{p\,q})^*(x_n)} \geq r > 0$$

wlog $\frac{x_n+\mu(\Omega)}{2} < x_{n+1}$, $(\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}) > (\frac{p-q}{p\,q})^*(x_{n+1})$

(3) $\Rightarrow$ (4) (sketch) Assume there exists $(x_n) \nearrow \mu(\Omega)$ s.t. $(\frac{p-q}{p\,q})^*(x_n) \mapsto 0$ with

$$(\mu(\Omega) - x_n)^{(\frac{p-q}{p\,q})^*(x_n)} \geq r > 0$$

wlog $\frac{x_n+\mu(\Omega)}{2} < x_{n+1}$, $(\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}) > (\frac{p-q}{p\,q})^*(x_{n+1})$

$$A_n := \left(\left(\frac{p-q}{p\,q}\right)^*\right)^{-1}\left([(\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}), (\frac{p-q}{pq})^*(x_n)]\right)$$

$$B_n := \{t \in \Omega \;:\; (\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}) \leq \frac{p-q}{p\,q}(t) \leq (\frac{p-q}{p\,q})^*(x_n)\ \}.$$

$\mu(B_n) = |A_n|$.

$$s_n(t) := \frac{\chi_{B_n}}{\mu(B_n)^{\frac{1}{p(t)}}}$$

(3) $\Rightarrow$ (4) (sketch) Assume there exists $(x_n) \nearrow \mu(\Omega)$ s.t. $(\frac{p-q}{p\,q})^*(x_n) \mapsto 0$ with

$$(\mu(\Omega) - x_n)^{(\frac{p-q}{p\,q})^*(x_n)} \geq r > 0$$

wlog $\frac{x_n+\mu(\Omega)}{2} < x_{n+1}$, $(\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}) > (\frac{p-q}{p\,q})^*(x_{n+1})$

$$A_n := (\,(\frac{p-q}{p\,q})^*)^{-1}\left([(\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}),(\frac{p-q}{pq})^*(x_n)]\right)$$

$$B_n := \{t \in \Omega \;:\; (\frac{p-q}{p\,q})^*(\frac{x_n+\mu(\Omega)}{2}) \leq \frac{p-q}{p\,q}(t) \leq (\frac{p-q}{p\,q})^*(x_n)\;\}.$$

$\mu(B_n) = |A_n|$.

$$s_n(t) := \frac{\chi_{B_n}}{\mu(B_n)^{\frac{1}{p(t)}}}$$

(s_n) and (is_n) are equivalent basic sequences.

$$\rho_{p(\cdot)}(\sum \lambda y_n s_n) = \sum \int_{B_n} |\lambda y_n|^{p(t)} \frac{\chi_{B_n}}{\mu(B_n)} d\mu = \sum \int_{B_n} |\lambda y_n|^{q(t)} |\lambda y_n|^{p-q(t)} \frac{\chi_{B_n}}{|A_n|^{\frac{q(t)}{p(t)}} |A_n|^{1-\frac{q(t)}{p(t)}}} d\mu$$

$$\leq \sum \int_{B_n} |\lambda y_n|^{q(t)} \frac{\chi_{B_n}}{|A_n|^{\frac{q(t)}{p(t)}}} \left[\frac{1}{\lambda_0(|A_n|)^{\frac{p(t)-q(t)}{p(t)q(t)}}}\right]^{q(t)} d\mu$$

$$\leq \rho_{q(\cdot)}(\sum \lambda y_n s_n)$$

inclusions $i: L^{p(\cdot)}(\mu) \hookrightarrow L^1(\mu)$ **,** μ**-finite**

inclusions $i: L^{p(\cdot)}(\mu) \hookrightarrow L^1(\mu)$ **,** μ**-finite**

Corollary

finite μ *, exponent* $p(\cdot)$ *TFAE;*

1. *i is weakly compact.*
2. *i is DSS.*
3. $\lim_{x \to \mu(\Omega)^-} (\mu(\Omega) - x)^{\left(\frac{p-1}{p}\right)^*(x)} = 0$
4. $\int_0^{\mu(\Omega)} a^{\left(\frac{p}{p-1}\right)^*(x)} dx < \infty$ *for every* $a > 1$

inclusions $i: L^{p(\cdot)}(\mu) \hookrightarrow L^1(\mu)$ **,** μ**-finite**

Corollary

finite μ *, exponent* $p(\cdot)$ *TFAE;*

1. *i is weakly compact.*
2. *i is DSS.*
3. $\lim_{x\to\mu(\Omega)^-} (\mu(\Omega)-x)^{(\frac{p-1}{p})^*(x)} = 0$
4. $\int_0^{\mu(\Omega)} a^{(\frac{p}{p-1})^*(x)}\,dx < \infty$ *for every* $a > 1$

$$L^\infty(\mu) \hookrightarrow L^{p(\cdot)}(\mu) \;\; SS \iff \lim_{x\to\mu(\Omega)^-} (\mu(\Omega)-x)^{(\frac{1}{p})^*(x)} = 0$$

(Ω, μ) infinite m.,

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$\int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

.

Inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for infinite measures

(Ω, μ) infinite m.,

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff q(\cdot) \le p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$\int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

Theorem

Any inclusion $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for μ infinite, is not DSS.

.

Inclusions $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for infinite measures

(Ω, μ) infinite m.,

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$\int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

Theorem

Any inclusion $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *for* μ *infinite, is not DSS.*

(No true for Orlicz spaces)

.

(Ω, μ) infinite m.,

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$\int_{\Omega_d} \lambda^{-\frac{pq}{p-q}(t)} d\mu < \infty, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

Theorem

Any inclusion $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for μ infinite, is not DSS.

(No true for Orlicz spaces)

1.the case of bounded exponents $q^+ \leq p^+ < \infty$.

(*) for every $\varepsilon > 0$, $D_\varepsilon := \{t \in \Omega_d : p(t) < q(t) + \varepsilon\}$. has infinite measure

(Ω, μ) infinite m.,

$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \iff q(\cdot) \leq p(\cdot)$ μ-a.e. and for some $\lambda > 1$

$$\int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu < \infty\,, \qquad \Omega_d := \{t \in \Omega : q(t) < p(t)\}$$

.

Theorem

Any inclusion $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ for μ infinite, is not DSS.

(No true for Orlicz spaces)

1.the case of bounded exponents $q^+ \leq p^+ < \infty$.

(*) for every $\varepsilon > 0$, $D_\varepsilon := \{t \in \Omega_d : p(t) < q(t) + \varepsilon\}$. has infinite measure

if no , for every $\lambda > 1$,

$$\int_{\Omega_d} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu \geq \int_{\Omega_d \setminus D_\varepsilon} \lambda^{-\frac{p\,q}{p-q}(t)} d\mu \geq \lambda^{-\frac{p^+q^+}{\varepsilon}} \mu(\Omega_d \setminus D_\varepsilon) = \infty$$

• there exists disjoint (A_n) with $\mu(A_n) = 1$ s.t. $p^+_{|_{A_n}} - q^-_{|_{A_n}} < \frac{1}{n}$.

Now

$$\ell_{p^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{p(\cdot)} \hookrightarrow \ell_{p^+_{|_{A_n}}} \qquad \ell_{q^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{q(\cdot)} \hookrightarrow \ell_{q^+_{|_{A_n}}}.$$

(by Nakano sequence criteria) $\ell_{p^-_{|_{A_n}}} \cong \ell_{p^+_{|_{A_n}}} \cong \ell_{q^-_{|_{A_n}}} \cong \ell_{q^+_{|_{A_n}}}$ so

$[\chi_{A_n}]_{p(\cdot)} \cong [\chi_{A_n}]_{q(\cdot)}$

• there exists disjoint (A_n) with $\mu(A_n) = 1$ s.t. $p^+_{|_{A_n}} - q^-_{|_{A_n}} < \frac{1}{n}$.
Now

$$\ell_{p^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{p(\cdot)} \hookrightarrow \ell_{p^+_{|_{A_n}}} \qquad \ell_{q^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{q(\cdot)} \hookrightarrow \ell_{q^+_{|_{A_n}}} .$$

(by Nakano sequence criteria) $\ell_{p^-_{|_{A_n}}} \cong \ell_{p^+_{|_{A_n}}} \cong \ell_{q^-_{|_{A_n}}} \cong \ell_{q^+_{|_{A_n}}}$ so
$[\chi_{A_n}]_{p(\cdot)} \cong [\chi_{A_n}]_{q(\cdot)}$

2,. unbounded exponents ($q^+ \leq p^+ = \infty$)

Reduce to $\mu_{q(\cdot)}(n) = \infty$ for every n .

there exist $(n_k)_k$ and disjoint $(A_k)_k$ with $\mu(A_k) = 1$ s.t.

$$n_k \leq q^-_{|A_k} \leq p^-_{|A_k} \qquad \text{and} \qquad q^+_{|A_k} \leq p^+_{|A_k} \leq n_{k+1}.$$

$[\chi_{A_k}]_{p(.)}, \quad \ell_{(n_k)} \cong \ell_\infty \cong \ell_{(n_{k+1})}, \ \Rightarrow \ [\chi_{A_k}]_{p(\cdot)} \cong c_0.$
Similarly $[\chi_{A_k}]_{q(\cdot)} \cong c_0$

• there exists disjoint (A_n) with $\mu(A_n) = 1$ s.t. $p^+_{|_{A_n}} - q^-_{|_{A_n}} < \frac{1}{n}$.
Now

$$\ell_{p^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{p(\cdot)} \hookrightarrow \ell_{p^+_{|_{A_n}}} \qquad \ell_{q^-_{|_{A_n}}} \hookrightarrow [\chi_{A_n}]_{q(\cdot)} \hookrightarrow \ell_{q^+_{|_{A_n}}}.$$

(by Nakano sequence criteria) $\ell_{p^-_{|_{A_n}}} \cong \ell_{p^+_{|_{A_n}}} \cong \ell_{q^-_{|_{A_n}}} \cong \ell_{q^+_{|_{A_n}}}$ so
$[\chi_{A_n}]_{p(\cdot)} \cong [\chi_{A_n}]_{q(\cdot)}$

2,. unbounded exponents ($q^+ \leq p^+ = \infty$)

Reduce to $\mu_{q(\cdot)}(n) = \infty$ for every n.

there exist $(n_k)_k$ and disjoint $(A_k)_k$ with $\mu(A_k) = 1$ s.t.

$$n_k \leq q^-_{|A_k} \leq p^-_{|A_k} \qquad \text{and} \qquad q^+_{|A_k} \leq p^+_{|A_k} \leq n_{k+1}.$$

$[\chi_{A_k}]_{p(.)}, \quad \ell_{(n_k)} \cong \ell_\infty \cong \ell_{(n_{k+1})}, \Rightarrow [\chi_{A_k}]_{p(\cdot)} \cong c_0.$
Similarly $[\chi_{A_k}]_{q(\cdot)} \cong c_0$

$\star$ Any inclusion $L^\infty(\mu) \hookrightarrow L^{p(\cdot)}(\mu)$, μ-infinite, is no strictly singular.

THANK YOU VERY MUCH ¡¡¡