Saturation Recovery and Phase Retrieval

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Acknowledgments

Projects:

- *Declipping and the recovery of vectors from saturated measurements*
  with Wedad Alhardi, Daniel Freeman, Brody Johnson and Lova Randrianarivony
- *Stable phase retrieval and perturbations of frames,*
  with Wedad Alhardi, Daniel Freeman, Clair Lois, and Shanea Sebastian
- *Discretizing the L_p norms and frame theory*
  with Daniel Freeman

Discrete Frames

Frame
A family of vectors $(x_j)_{j\in J}$ in a Hilbert space $\mathbb{H}$ is a *frame* if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}$

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_j\rangle|^2 \leq B\|x\|^2$$

where A and B are the *lower and upper frame bounds*.

Analysis operator
For a frame $(x_j)_{j\in J}$, *analysis operator* is defined as an operator $\Theta : \mathbb{H} \to \ell_2(J)$ to be

$$\Theta x = (\langle x, x_j\rangle)_{j\in J} \qquad \text{for all } x \in \mathbb{H}.$$

Continuous setting

The notion of a frame can be generalized to a **continuous frame** by changing the summation to integration over a measure space.

Continuous Frame
A family of vectors $(x_j)_{j\in\Omega}$ is a **continuous frame** of $\mathbb{H}$ over a measure space (Ω, μ) if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}$,

$$A\|x\|^2 \leq \int_{j\in\Omega} |\langle x, x_j\rangle|^2 d\mu(j) \leq B\|x\|^2$$

The **analysis operator** of a continuous frame $(x_j)_{j\in\Omega}$ of $\mathbb{H}$ is the map $\Theta : \mathbb{H} \to L_2(\Omega)$ given by

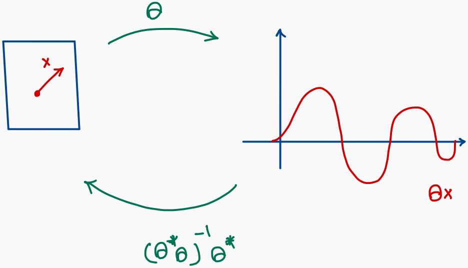
$$\Theta x = (\langle x, x_j\rangle)_{j\in\Omega}$$

$(x_j)_{j\in\Omega}$ is a continuous frame of $\mathbb{H}$ is equivalent to say that Θ is an embedding of $\mathbb{H}$ into $L_2(\Omega)$.

Recovery of vectors from linear measurements

Let $(x_j)_{j\in\Omega}$ be a continuous frame in $\mathbb{H}$ and assume we are given the image of a signal x in $\mathbb{H}$ under the analysis operator. We can reconstruct x from these linear measurements by applying the linear operator $(\Theta^*\Theta)^{-\mathbf{1}}\Theta^*$ such that:

$$(\Theta^*\Theta)^{-\mathbf{1}}\Theta^*(\langle x, x_j\rangle)_{j\in\Omega} = (\Theta^*\Theta)^{-\mathbf{1}}\Theta^*\Theta x = x$$



Phase retrieval and λ-saturation recovery

Phase retrieval is the problem of recovering a vector $x \in \mathbb{H}$ from the magnitude of the frame coefficients,

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λ-saturation recovery is the problem of recovering a vector $x \in \mathbb{H}$ from saturated frame coefficients,

$$\Phi_\lambda \Theta x = (\phi_\lambda(\langle x, x_j \rangle))_{j \in \Omega}$$

when for $\lambda > 0$, the function $\phi_\lambda : \mathbb{R} \to [-\lambda, \lambda]$ given by:

$$\phi_\lambda(t) = \begin{cases} \lambda & \text{if } t > \lambda \\ t & \text{if } -\lambda \leq t \leq \lambda \\ -\lambda & \text{if } t < -\lambda \end{cases}$$

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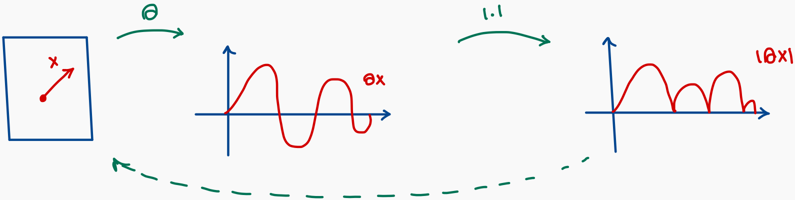

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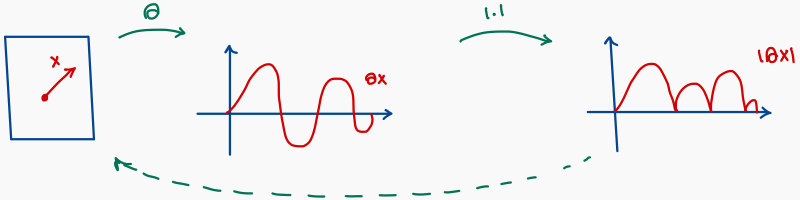
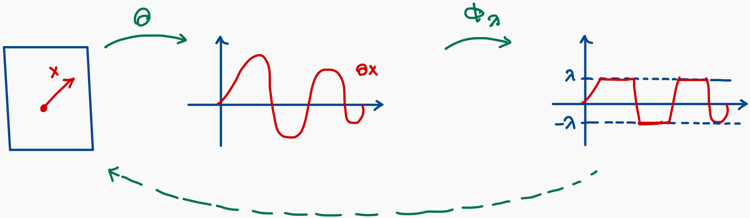
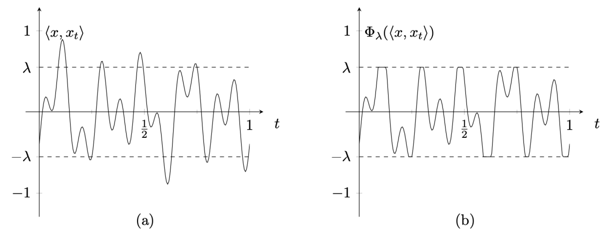


λ-saturation recovery is the problem of recovering a vector $x \in \mathbb{H}$ from saturated frame coefficients, $\Phi_\lambda \Theta(x) = (\phi_\lambda(\langle x, x_j \rangle))_{j \in J}$

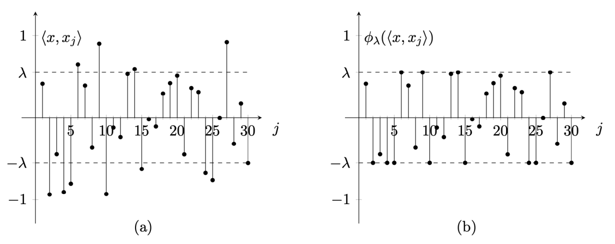


λ-saturation

Saturation of continuous frame coefficients:



Saturation of discrete frame coefficients:



(a) Unsaturated frame coefficients (b) Saturated frame coefficients.

Phase retieval and λ-saturation recovery

Phase retrieval: A frame $(x_j)_{j\in J}$ yields *phase retrieval* when for all $x, y \in \mathbb{H}$ we have

$$|\Theta x| = |\Theta y|$$

if and only if $x = \lambda y$ for some $|\lambda| = 1$.

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λ-saturation recovery: A frame $(x_j)_{j \in J}$ yields λ-saturation recovery on the unit ball $B_{\mathbb{H}}$ when for all $x, y \in \mathbb{H}$ we have

$$\phi_\lambda(\langle x, x_j \rangle)_{j \in J} = \phi_\lambda(\langle y, x_j \rangle))_{j \in J}$$

if and only if $x = y$.

When can we do phase retrieval and λ-saturation recovery?

In phase retrieval

A frame $(x_j)_{j\in J}$ in Hilbert space $\mathbb{H}$ satisfies the **complement property** if for all subsets $J_0 \subset J$, either $\text{span}(x_j)_{j\in J_0} = \mathbb{H}$ or $\text{span}(x_j)_{j\in J_0^c} = \mathbb{H}$.

Theorem (Balan, Casazza, Edidin '06)

If a frame does phase retrieval then it satisfies the complement property. In the $\mathbb{R}^n$, if a frame satisfies the complement property then it does phase retrieval.

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In λ-saturation recovery

For $x \in \mathbb{H}$, we denote the coordinates corresponding to the unsaturated frame coefficients of x by $J_\lambda(x) = \{j \in J : |\langle x, x_j\rangle| \leq \lambda\}$.

Theorem (Alharbi, Freeman, G., Johnson, Randrianarivony '23)

Let $(x_j)_{j\in J}$ be a frame for a finite dimensional Hilbert space $\mathbb{H}$ and let $\lambda, \alpha > 0$ then $(x_j)_{j\in J}$ does λ-saturation recovery on $\alpha B_{\mathbb{H}}$ if and only if for all $x \in \mathbb{H}$ with $\|x\| \leq \alpha$, $(x_j)_{j\in J_\lambda(x)}$ is a frame of $\mathbb{H}$.

Minimal structure for the recovery

In phase retrieval

In $\mathbb{R}^n$, generic frames do phase retrieval if they have at least $2n-1$ vectors.
In $\mathbb{C}^n$, generic frames do phase retrieval if they have at least $4n-4$ vectors. (not sharp)

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The number of unit vectors required to do λ-saturation recovery on the ball $B_{\mathbb{R}^n}$ depends on both the value $\lambda > 0$ and the dimension $n \in \mathbb{N}$. We propose the following problems:

- **Problem**: Let $n \in \mathbb{N}$ and $\lambda > 0$. What is the smallest $m \geq n$ so that there exists a frame of m unit vectors which does λ-saturation recovery on $B_{\mathbb{R}^n}$?

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- **Problem:** Let $m \geq n$. What is the smallest $\lambda > 0$ so that there exists a frame of m unit vectors $\underbrace{\text{which does } \lambda - \text{saturation recovery on } B_{\mathbb{R}^n}}_{\text{such that } (x_j)_{j \in J_\lambda(x)} \text{ is a frame for all } x \in B_{\mathbb{R}^n}}$?

Minimal structure for λ-saturation recovery

Problem: Let $m \geq n$. What is the smallest $\lambda > 0$ so that there exists a frame of m unit vectors such that $(x_j)_{j \in J_\lambda(x)}$ is a frame for all $x \in B_{\mathbb{R}^n}$. We considered two cases:

- If $m = n$, in this case we have 1-saturation recovery on $B_{\mathbb{R}^n}$ for every basis of unit vectors of $\mathbb{R}^n$. If $1 > \lambda > 0$ then no basis of unit vectors for $\mathbb{R}^n$ does λ-saturation recovery on $B_{\mathbb{R}^n}$.
- If $m = n + 1$, then we have $2^{-1/2}(1 + 1/n)^{1/2}$-saturation recovery on $B_{\mathbb{R}^n}$ for an equiangular frame of $(x_j)_{j=1}^{n+1}$ in $\mathbb{R}^n$.
  If $2^{-1/2}(1 + 1/n)^{1/2} > \lambda > 0$, then no frame of $n + 1$ unit vectors for $\mathbb{R}^n$ does λ-saturation recovery on $B_{\mathbb{R}^n}$.

Stability of the recovery

- We say that a frame does *C-stable phase retrieval* if the recovery of x from $|\Theta x| \in \ell_2(J)$ is C-Lipschitz.

C-stable phase retrieval: A frame $(x_j)_{j\in J}$ of $\mathbb{H}$ which satisfies phase retrieval for $\mathbb{H}$ with analysis operator $\Theta : \mathbb{H} \to \ell_2(J)$ yields *C-stable phase retrieval* if for all $x, y \in \mathbb{H}$,

$$\min_{|\lambda|=1} \|x - \lambda y\|_{\mathbb{H}} \leq C\||\Theta x| - |\Theta y|\|_{\ell_2(J)} = C\Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^2\Big)^{1/2}$$

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• We say that a frame does *C-stable λ-saturation recovery* if the recovery of x from $\Phi_\lambda \Theta x \in \ell_2(J)$ is C-Lipschitz.

C-stable λ-saturation recovery: A frame $(x_j)_{j\in J}$ of $\mathbb{H}$ which does λ-saturation recovery for $B_{\mathbb{H}}$, with analysis operator $\Theta : H \to \ell_2(J)$ yields *C-stable λ-saturation recovery on $B_{\mathbb{H}}$* if for all $x, y \in B_{\mathbb{H}}$ we have that

$$\|x - y\|_{\mathbb{H}} \leq C\big\|\Phi_\lambda \Theta x - \Phi_\lambda \Theta y\big\|_{\ell_2(J)} = C\Big(\sum_{j\in J} |\phi_\lambda(\langle x, x_j\rangle) - \phi_\lambda(\langle y, x_j\rangle)|^2\Big)^{1/2}$$

Stability of the recovery

Theorem (Bandeira, Cahill, Mixon, and Nelson '14)(Balan and Wang '15)

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Critical level of saturation: consider λ_c to be the critical level of saturation,

$$\lambda_c = inf\{\lambda : (x_j)_{j=1}^m \text{ does } \lambda\text{-saturation recovery on } B_{\mathbb{R}^n}\}$$

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Theorem (Alharbi, Freeman, G., Johnson, Randrianarivony '24)

Let $(x_j)_{j\in J}$ be a frame for $\mathbb{R}^n$, and let λ_c be the critical value for $(x_j)_{i\in J}$ to do λ_c-saturation recovery on $B_{\mathbb{R}^n}$. Then for all $\lambda > \lambda_c$ there exists $C_\lambda > 0$ so that $(x_j)_{j\in J}$ does C_λ-stable λ-saturation recovery.

Problem: Does $(x_j)_{j\in J}$ do stable λ_c-saturation recovery on $B_{\mathbb{R}^n}$?

Does the stability preserve under perturbation?

A frame $(y_j)_{j\in J}$ is called an ε-perturbation of a frame $(x_j)_{j\in J}$ if

$$\sum_{j\in J} \|y_j - x_j\|^2 < \varepsilon$$

- If a frame does stable phase retrieval then any sufficiently small perturbation of the frame vectors will do stable phase retrieval but the stability bound gets worse.
- How is the stability constant for phase retrieval affected by a small perturbation of the frame vectors?

Perturbing frames and stability

Theorem (Christensen '95)

Let $(x_j)_{j\in J}$ be a frame of a Hilbert space $\mathbb{H}$ with frame bounds $0 < A \leq B$. Let $0 < \varepsilon < A$ and $(y_j)_{j\in J} \subseteq \mathbb{H}$ is an ε-perturbation of $(x_j)_{j\in J}$. Then, $(y_j)_{j\in J}$ is a frame of $\mathbb{H}$ with upper frame bound $B\left(1+\sqrt{\frac{\varepsilon}{B}}\right)^{\mathbf{2}}$ and lower frame bound $A\left(1-\sqrt{\frac{\varepsilon}{A}}\right)^{\mathbf{2}}$.

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Corollary (Balan '17)

Let $(x_j)_{j=\mathbf{1}}^m$ be a frame for $\mathbb{H}^n$ with upper and lower frame bound $0 < A \leq B$ which does C-stable phase retrieval. Let $A, B, C > 0$ and $m \in \mathbb{N}$ then there exists $\varepsilon > 0$ so that $(y_j)_{j\in J} \subseteq \mathbb{H}$ is an ε-perturbation of $(x_j)_{j\in J}$. Then $(y_j)_{j=\mathbf{1}}^m$ is a frame of $\mathbb{H}^n$ which does 2C-stable phase retrieval.

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Let $(x_j)_{j=1}^{m}$ be a frame for $\mathbb{H}^n$ with upper and lower frame bound $0 < A \leq B$ which does C-stable phase retrieval. Let $A, B, C > 0$ and $m \in \mathbb{N}$ then there exists $\varepsilon > 0$ so that $(y_j)_{j\in J} \subseteq \mathbb{H}$ is an ε-perturbation of $(x_j)_{j\in J}$. Then $(y_j)_{j=1}^{m}$ is a frame of $\mathbb{H}^n$ which does 2C-stable phase retrieval.

Theorem (Alharbi, Freeman, G., Lois, Sebastian '23)

Let $(x_j)_{j\in J}$ be a frame of a finite-dimensional Hilbert space $\mathbb{H}$ with frame bounds $0 < A \leq B$ which does C-stable phase retrieval. Let $\varepsilon > 0$ satisfy $\varepsilon < 2^{-4}C^{-4}B^{-1}$ and let $(y_j)_{j\in J} \subseteq \mathbb{H}$ be an ε-perturbation of $(x_j)_{j\in J}$. Then, $(y_j)_{j\in J}$ is a frame of $\mathbb{H}$ with upper frame bound $B\left(1+\sqrt{\frac{\varepsilon}{B}}\right)^{2}$ and lower frame bound $A\left(1-\sqrt{\frac{\varepsilon}{B}}\right)^{2}$ which does $C(1-4\,C^{2}\sqrt{\varepsilon B})^{-1/2}$-stable phase retrieval for $\mathbb{H}$.

Perturbing frames and stability of λ-saturation recovery

Can we say that if $(x_j)_{j\in J}$ does λ-saturation recovery, then every small perturbation of the frame would yield λ-saturation recovery?

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- Suppose λ_c is the critical level of saturation for the frame $(x_j)_{j\in J}$ then if $\lambda > \lambda_c$, then there exists and $\varepsilon > 0$ such that every ε-perturbation of $(x_j)_{j\in J}$ does λ-saturation recovery.

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- However, for all $\varepsilon > 0$, $\left((1 + \frac{\sqrt{\varepsilon}}{|J|\|x_j\|})x_j\right)_{j\in J}$ is ε-perturbations of $(x_j)_{j\in J}$ that does not do λ_c-saturation recovery.

How to construct frames which do C-stable phase retrieval

- Recall: almost every family of m vectors in $\mathbb{R}^n$ satisfy *phase retrieval* as long as $m \geq 2n - 1$.
- But it is very difficult to create frames which satisfy *C-stable phase retrieval* for higher dimensions.

How to construct frames which do C-stable phase retrieval

- Recall: almost every family of m vectors in $\mathbb{R}^n$ satisfy *phase retrieval* as long as $m \geq 2n-1$.
- But it is very difficult to create frames which satisfy *C-stable phase retrieval* for higher dimensions.

How do we create frames that satisfy C-stable phase retrieval?

We are looking at the subspaces of $L_2(\Omega)$:

Suppose X^n is an n-dimensional subspace in $L_2(\Omega)$. For almost every $t \in \Omega$, point evaluation at t is a bounded linear functional on X^n. That is, there exists $(x_t) \in X^n$ such that

$$\langle f, x_t \rangle = f(t) \qquad \text{for all } f \in X^n$$

Then $(x_t)_{t\in\Omega}$ is a **continuous Parseval frame** of X^n.

How to construct frames which do C-stable phase retrieval

- In order to create a frame for $\mathbb{H}^n$ which does C-stable phase retrieval and is close to being a Parseval frame, we need to have a continuous Parseval frame $(x_t)_{t\in\Omega}$ which does C-stable phase retrieval.

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- In order to create a frame for $\mathbb{H}^n$ which does C-stable phase retrieval and is close to being a Parseval frame, we need to have a continuous Parseval frame $(x_t)_{t\in\Omega}$ which does C-stable phase retrieval.

- Then, we would like to sample m points $(t_j)_{j=1}^m \subseteq \Omega$ so that $(\frac{1}{\sqrt{m}}x_{t_j})_{j=1}^m$ does C'-stable phase retrieval and has frame bounds close to 1.

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- Notice that discrete frames are much better suited for computations than continuous frames. Therefore, creating a discrete frame by sampling the continuous frame and then use the discrete frame for computations instead of the entire continuous frame is widely used by researchers.

How to construct frames which do C-stable phase retrieval

Theorem (E. J. Candès, T. Strohmer, and V. Voroninski '13)

There exists $C, C' > 0$ so that for all $n \in \mathbb{N}$, if $(x_t)_{t\in\Omega}$ is uniformly distributed in $\sqrt{n}S_{\mathbb{R}^n}$ then $(x_t)_{t\in\Omega}$ is a continuous Parseval frame which does C-stable phase retrieval.

If m is on the order of $n\log(n)$ and $(t_j)_{j=1}^m$ is randomly sampled in Ω then with high probability $(\frac{1}{\sqrt{m}}x_{t_j})_{j=1}^m$ does C'-stable phase retrieval.

Theorem (E. J. Candès and X. Li '14, Y. C. Eldar and S. Mendelson '13)

There exists $C, C' > 0$ so that for all $n \in \mathbb{N}$, if $(x_t)_{t\in\Omega}$ is uniformly distributed in $\sqrt{n}S_{\mathbb{R}^n}$ or has Gaussian distribution then $(x_t)_{t\in\Omega}$ does C-stable phase retrieval.

If m is on the order of n and $(t_j)_{j=1}^m$ is randomly sampled in Ω then with high probability $(\frac{1}{\sqrt{m}}x_{t_j})_{j=1}^m$ does C'-stable phase retrieval.

[F. Krahmer and Y. Liu '18, F. Krahmer and D. Stöger '20] Greatly extend these results to very general classes of sub-Gaussian distributions.

When does this work for phase retrieval?

Suppose that $(x_t)_{t\in\Omega}$ is a continuous Parseval frame of $\mathbb{H}^n$ which does C-stable phase retrieval.

When can we choose m on the order of n sampling points $(t_j)_{j=1}^m$ so that $(\frac{1}{\sqrt{m}}x_{t_j})_{j=1}^m$ does C'-stable phase retrieval?

Every known result for obtaining m on the order of n is for sub-Gaussian distributions.

Discretization examples

Problem (S. Ali, J. Antoine, J. Gazeau '00)

When can a continuous frame be sampled to obtain a discrete frame?

Theorem (D. Freeman, D. Speegle '18)

Every bounded continuous frame can be discretized.

Theorem (I. Limonova, V. Temlyakov '22)

Every bounded continous frame for an n-dimentional Hilbert space can be discretized using m on the order of n sampling points.

When can we have the discretization?

We want to know when can a continuous frame which does C-stable phase retrieval for ℓ_2^n be discretized using m on the order of n sampling points to obtain a frame which does C'-stable phase retrieval?

Let $(x_t)_{t\in\Omega}$ be a continuous Parseval frame for $\mathbb{H}^n$ over a probability space Ω.
Let $\Theta : \mathbb{H}^n \to L_2(\Omega)$ be the analysis operator.

Discretizing $(x_t)_{t\in\Omega}$ to do phase retrieval requires both:

1. Discretizing the L_2 norm on $\Theta(\mathbb{H}^n) \subseteq L_2(\Omega)$.
2. Discretizing the L_1 norm on $\Theta(\mathbb{H}^n) \subseteq L_1(\Omega)$.

C-stable phase retrieval and sampling

Theorem (D. Freeman, G. '23)

Let $(x_t)_{t\in\Omega}$ be a continuous Parseval frame for ℓ_2^n over a probability space Ω which does κ-stable phase retrieval and $\|x_t\| \leq \beta\sqrt{n}$ for all $t \in \Omega$. Let $\Theta : \ell_2^n \to L_2(\Omega)$ be the analysis operator. Suppose that $(\frac{1}{\sqrt{m}}x_{t_j})_{j=1}^m$ is a frame of ℓ_2^n with upper frame bound B and lower frame bound A which does C-stable phase retrieval. Then both the L_2 norm and the L_1 norm on the range of the analysis operator are discretized in the following way for all $x \in \ell_2^n$,

1. $$A\|\Theta x\|_{L_2(\Omega)}^2 \leq \frac{1}{m}\sum_{j=1}^m |\langle x, x_{t_j}\rangle|^2 \leq B\|\Theta x\|_{L_2(\Omega)}^2$$

2. $$\frac{A^{1/2}}{B^{3/2}C^3(1+A^{-1}\beta^2)^{3/2}}\|\Theta x\|_{L_1(\Omega)} \leq \frac{1}{m}\sum_{j=1}^m |\langle x, x_{t_j}\rangle| \leq B^{1/2}\kappa^3(1+\beta^2)^{3/2}\|\Theta x\|_{L_1(\Omega)}$$

- In order to sample a continuous Parseval frame to obtain a frame which does stable phase retrieval, it is necessary to simultaneously discretize both the L_1-norm and the L_2-norm on the range of the analysis operator.

Construction frames satistfying C-stable λ-saturation recovery

Theorem (J. Laska, P. Boufounos, M. Davenport, R. Baraniuk '13)(S. Foucart, T. Needham '17)

There exists $1 > \alpha > 0$ and a constant $C_\alpha > 0$ such that for all $n \in \mathbb{N}$ if $\lambda \geq \alpha/\sqrt{n}$ and $(x_j)_{j=1}^m$ are chosen randomly and independently with uniform distribution in $S_{\mathbb{R}^n}$ and m is chosen on the order of n then with high probability, for all $x, y \in S_{\mathbb{R}^n}$,

$$\|x - y\|^2 \leq C_\alpha^2 \frac{n}{m} \sum_{j=1}^{m} |\phi_\lambda(\langle x, x_j \rangle) - \phi_\lambda(\langle y, x_j \rangle)|^2$$

Theorem (W. Alharbi, D. Freeman, G., B. Johnson, N. Randrianarivony '24)

For all $\alpha > 0$ there exists $C_\alpha > 0$ such that for all $n \in \mathbb{N}$ if $\lambda \geq \alpha/\sqrt{n}$ and $(x_j)_{j=1}^m$ are chosen randomly and independently with uniform distribution in $S_{\mathbb{R}^n}$ and m is chosen on the order of $n \log(n)$ then with high probability , for all $x, y \in S_{\mathbb{R}^n}$

$$\|x - y\|^2 \leq C_\alpha^2 \frac{n}{m} \sum_{j=1}^{m} |\phi_\lambda(\langle x, x_j \rangle) - \phi_\lambda(\langle y, x_j \rangle)|^2$$

Implementation of λ-saturation recovery

- For a frame $(x_j)_{j\in J}$, λ-saturation recovery is the problem of reconstructing a vector $x \in \mathbb{H}$ from the measurements $\phi_\lambda \Theta x = (\phi_\lambda(\langle x, x_j\rangle)_{j\in J})$.

We are constructing $y \in \mathbb{H}$ such that:

$$
\begin{aligned}
\langle y, x_j\rangle &\geq \lambda && \text{if} \quad \phi_\lambda(\langle x, x_j\rangle) = \lambda \\
\langle y, x_j\rangle &= \phi_\lambda(\langle x, x_j\rangle) && \text{if} \quad -\lambda \leq \phi_\lambda(\langle x, x_j\rangle) \leq \lambda \\
\langle y, x_j\rangle &\leq -\lambda && \text{if} \quad \phi_\lambda(\langle x, x_j\rangle) = -\lambda
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In (Foucart, Needham '17)(Foucart, Li '18) the conditions above are expressed as a linear programming problem.

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Frame Algorithm (no saturation)
Let $(x_j)_{j\in J}$ be a frame of a Hilbert space $\mathbb{H}$ with frame bounds A, B and analysis operator $\Theta : \mathbb{H} \to \ell_2(J)$. Let $0 < \alpha < B/2$. Given an element in the range of the analysis operator $\Theta x \in \ell_2(J)$, define a sequence $(y_k)_{k=0}^\infty$ in $\mathbb{H}$ by $y_0 = 0$ and

$$y_{k+1} := y_k + \alpha\, \Theta^*(\Theta x - \Theta y_k) = y_k + \alpha \sum_{j\in J} \langle x - y_k, x_j\rangle x_j \qquad \text{for all } k \geq 0.$$

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Then $\|x - y_{k+1}\| \leq C_\alpha \|x - y_k\|$ for all $k \geq 0$, where $C_\alpha := \max\{|1-\alpha A|, |1-\alpha B|\}$.

Thus, $(y_k)_{k=0}^\infty$ converges to x and satisfies

$$\|x - y_k\| \leq C_\alpha^k \|x\| \qquad \text{for all } k \geq 0.$$

Note that $\frac{B-A}{B+A}$ is the optimal value for C_α and it occurs when $\alpha = \frac{2}{A+B}$

λ-saturated frame algorithm

• Our goal is to adapt the frame algorithm to the non-linear problem of recovering a vector from saturated measurements.

Let $(x_j)_{j\in J}$ be a frame of a Hilbert space $\mathbb{H}$ and let $\lambda > 0$. For $x \in \mathbb{H}$ we denote the following sets,

$$
\begin{aligned}
&\text{Unsaturated coordinates:} && J_\lambda(x) = \{j \in J : |\langle x, x_j\rangle| \le \lambda\},\\
&\text{Positively saturated coordinates:} && J_\lambda^+(x) = \{j \in J : \langle x, x_j\rangle > \lambda\},\\
&\text{Negatively saturated coordinates:} && J_\lambda^-(x) = \{j \in J : \langle x, x_j\rangle < -\lambda\}.
\end{aligned}
$$

Define a subsets of $J_\lambda^+(x)$ and $J_\lambda^-(x)$, respectively, relative to a fixed element $y \in \mathbb{H}$,

$$
\begin{aligned}
J_\lambda^+(x,y) &= \{j \in J_\lambda^+(x) : \langle y, x_j\rangle < \lambda\},\\
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Recursively define the λ-*saturated frame algorithm*. We set $y_{\mathbf{0}} = 0$ and for $k \in \mathbb{N} \cup \{0\}$ and $y_k \in H$ we choose $\alpha_k, \beta_k \ge 0$ and let

$$\begin{aligned}
y_{k+\mathbf{1}} = y_k + \alpha_k \sum_{j\in J_\lambda(x)} \Big(\langle x, x_j\rangle - \langle y_k, x_j\rangle\Big)x_j + \beta_k \sum_{j\in J_\lambda^+(x,y_k)} \Big(\lambda - \langle y_k, x_j\rangle\Big)x_j\\
+ \beta_k \sum_{j\in J_\lambda^-(x,y_k)} \Big(-\lambda - \langle y_k, x_j\rangle\Big)x_j.
\end{aligned}$$

λ-saturated frame algorithm

- Vector $y_k \in \mathbb{H}$ is known and we can use the actual frame coefficients $(\langle y_k, x_j \rangle)_{j\in J}$ rather than the saturated frame coefficients $(\phi_\lambda(\langle y_k, x_j \rangle))_{j\in J}$.
- In the frame algorithm, the optimal choice for the scalar α is $\alpha = 2/(A+B)$ where A and B are the frame bounds.
- In λ-saturated frame algorithm, the optimal choice for α_k and β_k can change at each step.

Theorem (Alharbi, Freeman, G., Johnson, Randrianarivony '23)

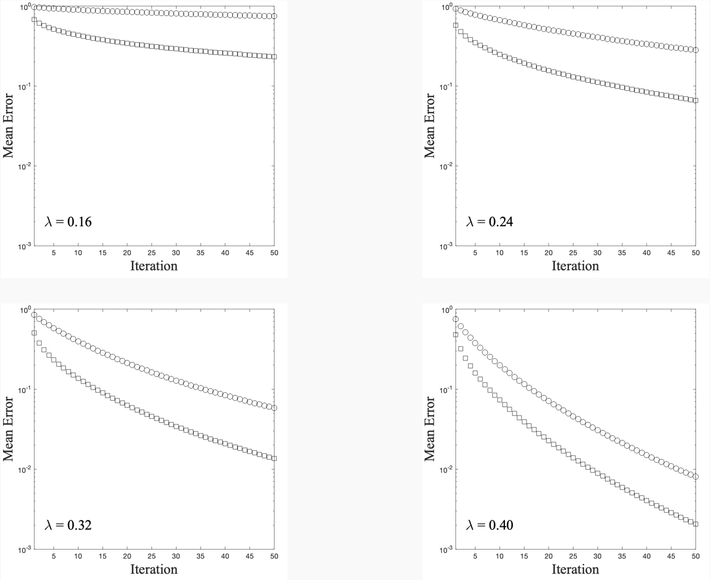
Let $(x_j)_{j\in J}$ be a frame for a Hilbert space $\mathbb{H}$ with frame bounds $A \le B$. Let $x \in \mathbb{H}$ and suppose that $(y_k)_{k=\mathbf{0}}^{\infty} \subseteq \mathbb{H}$ is constructed by the λ-saturated frame algorithm with $\alpha_k = \beta_k = 2/(A+B)$ for all $k \in \mathbb{N} \cup \{0\}$. For each $k \in \mathbb{N} \cup \{0\}$, if the optimal lower frame bound for $(x_j)_{j\in J_\lambda(x)}$ is strictly less than the optimal lower frame bound for $(x_j)_{j\in J_\lambda(x)\cup J_\lambda^+(x,y_k)\cup J_\lambda^-(x,y_k)}$ then there exists $\varepsilon_{x,y_k} > 0$ so that

$$\|x - y_{k+\mathbf{1}}\| \le (1 - \varepsilon_{x,y_k}) C_{\mathbf{2}/(A+B)} \|x - y_k\|,$$

where $C_{\mathbf{2}/(A+B)}$ is the constant for applying the frame algorithm to $(x_j)_{j\in J_\lambda(x)}$ with coefficient $\alpha = 2/(A+B)$.

Numerical implementation: different saturation levels

Using random frames of 30 vectors for $\mathbb{R}^{\mathbf{10}}$. Mean error $\|y_k - x\|$ for frame algorithm ($\circ$) and λ-saturation frame algorithm ($\square$)



High and low levels of saturation

- In all cases, the λ-saturated frame algorithm outperforms the frame algorithm based only on unsaturated coefficients.
- Considering the saturated frame coefficients improves the convergence in the first few iterations, (steeper slope between consecutive iterations).
- **Question:** When saturation is high, the frame algorithm fails, and using the saturation recovery algorithm does a better job at the recovery, how does the percentage of saturation change the recovery?

Thank you!