Recent advances in free Banach lattices with upper p-estimates

Enrique García-Sánchez

Instituto de Ciencias Matemáticas, Madrid

Workshop Banach spaces and Banach lattices, May 2024

Joint work with Denny H. Leung, Mitchell A. Taylor and Pedro Tradacete.

Grant CEX2019-000904-S-21-3 funded by

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

Cofinanciado por la Unión Europea

AGENCIA ESTATAL DE INVESTIGACIÓN

Overview

1. Preliminaries

2. Free Banach lattices

3. Embeddings into ∞-sums

4. Free Banach lattices with upper p-estimates

Overview

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X,\|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);
- if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);
- if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

Definition

An operator $T : X \to Y$ between two Banach lattices is called a **lattice homomorphism** if it is linear and preserves the lattice operations.

Banach lattices

Definition

A **Banach lattice** is a Banach space $(X, \|\cdot\|)$ equipped with a partial order $\leq$ satisfying:

- if $x \leq y$, then $x + z \leq y + z$ for every $z \in X$;
- if $x \leq y$, then $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$;
- for every $x, y \in X$ there exist the least upper bound $x \vee y$ (supremum) and the greatest lower bound $x \wedge y$ (infimum);
- if $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = x \vee (-x)$.

Definition

An operator $T : X \to Y$ between two Banach lattices is called a **lattice homomorphism** if it is linear and preserves the lattice operations.

Examples: ℓ_p, $L_{p,q}(\Omega)$, $C(K)$...

p-convexity

Definition

A Banach lattice X is said p-**convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$,

p-convexity

Definition

A Banach lattice X is said p-**convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$, where

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^{n} a_i x_i : \left(\sum_{i=1}^{n} |a_i|^{p^*} \right)^{\frac{1}{p^*}} \leq 1 \right\} \text{ and } \frac{1}{p} + \frac{1}{p^*} = 1.$$

p-convexity

Definition

A Banach lattice X is said **p-convex** for $1 \leq p \leq \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$, where

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^{n} a_i x_i : \left(\sum_{i=1}^{n} |a_i|^{p^*} \right)^{\frac{1}{p^*}} \leq 1 \right\} \text{ and } \frac{1}{p} + \frac{1}{p^*} = 1.$$

The lowest constant M satisfying the above inequality is called the **p-convexity constant of** X, and is denoted by $M^{(p)}(X)$.

Upper p-estimates

Definition

A Banach lattice X satisfies an **upper p-estimate** for $1 < p < \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$ pairwise disjoint,

Upper p-estimates

Definition

A Banach lattice X satisfies an **upper p-estimate** for $1 < p < \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$ pairwise disjoint, or equivalently, if

$$\left\| \bigvee_{i=1}^{n} |x_i| \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$.

Upper p-estimates

Definition

A Banach lattice X satisfies an **upper p-estimate** for $1 < p < \infty$ if there exists a constant $M \geq 1$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$ pairwise disjoint, or equivalently, if

$$\left\| \bigvee_{i=1}^{n} |x_i| \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

holds for any $x_1, \ldots, x_n \in X$. The lowest constant M satisfying any of the above inequalities is called the **upper p-estimates constant of X**, and is denoted by $M^{(p,\infty)}(X)$.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.
- X p-convex $\Rightarrow$ X with upper p-estimate.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.
- X p-convex $\Rightarrow$ X with upper p-estimate.
- X with upper p-estimate $\Rightarrow$ X q-convex for every $q < p$.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.
- X p-convex $\Rightarrow$ X with upper p-estimate.
- X with upper p-estimate $\Rightarrow$ X q-convex for every $q < p$.
- c_0, ℓ_∞, $L_\infty(\Omega)$ and $C(K)$ are ∞-convex with constant 1.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.
- X p-convex $\Rightarrow$ X with upper p-estimate.
- X with upper p-estimate $\Rightarrow$ X q-convex for every $q < p$.
- c_0, ℓ_∞, $L_\infty(\Omega)$ and $C(K)$ are ∞-convex with constant 1.
- ℓ_p and $L_p(\Omega)$ are p-convex with constant 1.

p-convexity and upper p-estimates

- Every Banach lattice is 1-convex with constant 1.
- X p-convex $\Rightarrow$ X with upper p-estimate.
- X with upper p-estimate $\Rightarrow$ X q-convex for every $q < p$.
- c_0, ℓ_∞, $L_\infty(\Omega)$ and $C(K)$ are ∞-convex with constant 1.
- ℓ_p and $L_p(\Omega)$ are p-convex with constant 1.
- $\ell_{p,\infty}$ and $L_{p,\infty}(\Omega)$ satisfy an upper p-estimate with constant 1.

Weak-L_p

Let $1 < p < \infty$ and (Ω, Σ, μ) be a measure space.

$$L_{p,\infty}(\mu) = \left\{ f : \Omega \to \mathbb{R} \, : \, f \text{ measurable, } \|f\|^*_{L_{p,\infty}} < \infty \right\},$$

where

$$\|f\|^*_{L_{p,\infty}} = \sup_{t>0} t\mu(\{\,|f| > t\})^{\frac{1}{p}}.$$

Weak-L_p

Let $1 < p < \infty$ and (Ω, Σ, μ) be a measure space.

$$L_{p,\infty}(\mu) = \left\{ f : \Omega \to \mathbb{R} \,:\, f \text{ measurable, } \|f\|^*_{L_{p,\infty}} < \infty \right\},$$

where

$$\|f\|^*_{L_{p,\infty}} = \sup_{t>0} t\mu(\{\, |f| > t\})^{\frac{1}{p}}.$$

In general, $\|\cdot\|^*_{L_{p,\infty}}$ is a quasinorm but not a norm. However, for $1 < p < \infty$, $L_{p,\infty}(\mu)$ can be renormed with the norm

$$\|f\|_{L^r_{p,\infty}} = \sup_{0<\mu(A)<\infty} \mu(A)^{\frac{1}{p}-\frac{1}{r}} \left(\int_A |f|^r \, d\mu \right)^{\frac{1}{r}},$$

for each $1 \leq r < p$. Since all these norms are equivalent, we will chose $r = 1$ and denote the norm by $\| \cdot \|_{L_{p,\infty}}$.

Overview

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space.

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding,

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \to X$

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \to X$ there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$

Free Banach lattices

The main motivation of this work was to understand the properties of free Banach lattices associated to special classes of Banach lattices (p-convex, upper p-estimates...).

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice over a Banach space** E is a pair $(\mathrm{FBL}[E], \phi_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and $\phi_E : E \to \mathrm{FBL}[E]$ is a linear isometrical embedding, satisfying that for every Banach lattice X and every bounded linear operator $T : E \to X$ there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} \mathrm{FBL}[E] \arrow[r, "\hat{T}"] & X \\ E \arrow[u, "\phi_E"] \arrow[ur, "T"'] & \end{tikzcd}$$

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$.

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E**

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free** p**-convex Banach lattice over** E is a p**-convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$,

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a p-**convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$,

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a **p-convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a **p-convex** Banach lattice with constant 1,

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over E** is a **p-convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a **p-convex** Banach lattice with constant 1, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \leq p \leq \infty$. The **free p-convex Banach lattice over** E is a p-**convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p-**convex** Banach lattice with constant 1, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[E] \arrow[r, "\hat{T}"] & X \\ E \arrow[u, "\phi_E"] \arrow[ur, "T"'] & \end{tikzcd}$$

Free p-convex Banach lattices

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 \le p \le \infty$. The **free p-convex Banach lattice over** E is a p**-convex** Banach lattice with constant 1, denoted by $\mathrm{FBL}^{(p)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a p**-convex** Banach lattice with constant 1, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd}
\mathrm{FBL}^{(p)}[E] \arrow[r, "\hat{T}"] & X \\
E \arrow[u, "\phi_E"] \arrow[ur, "T"'] &
\end{tikzcd}$$

Note that every Banach lattice is 1-convex with constant 1 (triangular inequality), so for $p = 1$ we recover $\mathrm{FBL}[E]$.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup \left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup\left\{\left(\sum_{k=1}^{n} |f(x_k^*)|^p\right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E}\left(\sum_{k=1}^{n} |x_k^*(x)|^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

The set $H_p[E] := \{f \in H[E] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup \left\{ \left(\sum_{k=1}^{n} |f(x_k^*)|^p \right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^{n} |x_k^*(x)|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

The set $H_p[E] := \{f \in H[E] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

We define $\phi_E : E \to H_p[E]$ as $x \mapsto \delta_x(x^*) = x^*(x)$, $x^* \in E^*$.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Let $H[E] := \{f : E^* \to \mathbb{R} : f(\lambda x^*) = \lambda f(x^*)\ \forall x^* \in E^*, \lambda \geq 0\}$ be the set of positively homogeneous functions over E^*. For every $f \in H[E]$ we define

$$\|f\|_p = \sup\left\{ \left(\sum_{k=1}^n |f(x_k^*)|^p\right)^{\frac{1}{p}} : (x_k^*) \subset E^*, \sup_{x \in B_E} \left(\sum_{k=1}^n |x_k^*(x)|^p\right)^{\frac{1}{p}} \leq 1 \right\}.$$

The set $H_p[E] := \{f \in H[E] : \|f\|_p < \infty\}$ is a Banach lattice when endowed with the pointwise ordering and lattice operations.

We define $\phi_E : E \to H_p[E]$ as $x \mapsto \delta_x(x^*) = x^*(x)$, $x^* \in E^*$.

The space $\mathrm{FBL}^{(p)}[E] := \overline{\mathrm{FVL}[E]} \subset H_p[E]$ is a representation of the free p-convex Banach lattice over E, where $\mathrm{FVL}[E] = \mathrm{lat}(\phi_E(E))$ denotes the free vector lattice generated by E.

Free Banach lattices with convexity conditions

The explicit construction of $\mathrm{FBL}^{(p)}[E]$ allows to show that $\mathrm{FBL}^{(p)}[E] \subseteq C(B_{E^*}, w^*)$, so in particular lattice homomorphism (evaluations on $x^* \in B_{E^*}$) separate the points of $\mathrm{FBL}^{(p)}[E]$.

Free Banach lattices with convexity conditions

The explicit construction of $\mathrm{FBL}^{(p)}[E]$ allows to show that $\mathrm{FBL}^{(p)}[E] \subseteq C(B_{E^*}, w^*)$, so in particular lattice homomorphism (evaluations on $x^* \in B_{E^*}$) separate the points of $\mathrm{FBL}^{(p)}[E]$.

In the same paper [JLTTT], the authors were able to extend the notion of free Banach lattice to more restricted classes of Banach lattices with "convexity conditions" by means of an abstract completion argument. These convexity conditions include as a particular case p-convexity, for which they were able to provide an explicit representation, or upper p-estimates, for which they stated the following problem:

Free Banach lattices with convexity conditions

The explicit construction of $\mathrm{FBL}^{(p)}[E]$ allows to show that $\mathrm{FBL}^{(p)}[E] \subseteq C(B_{E^*}, w^*)$, so in particular lattice homomorphism (evaluations on $x^* \in B_{E^*}$) separate the points of $\mathrm{FBL}^{(p)}[E]$.

In the same paper [JLTTT], the authors were able to extend the notion of free Banach lattice to more restricted classes of Banach lattices with "convexity conditions" by means of an abstract completion argument. These convexity conditions include as a particular case p-convexity, for which they were able to provide an explicit representation, or upper p-estimates, for which they stated the following problem:

Problem

Find an explicit formula for the norm of the free Banach lattice with upper p-estimates.

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$.

Free Banach lattices with upper p-estimates

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over E**

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over E** is a Banach lattice satisfying an **upper p-estimate** with constant 1, denoted by $\mathrm{FBL}^{(p,\infty)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p,\infty)}[E]$,

Free Banach lattices with upper p-estimates

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over E** is a Banach lattice satisfying an **upper p-estimate** with constant 1, denoted by $\mathrm{FBL}^{(p,\infty)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p,\infty)}[E]$, satisfying that for every bounded linear operator $T : E \to X$,

Free Banach lattices with upper p-estimates

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over E** is a Banach lattice satisfying an **upper p-estimate** with constant 1, denoted by $\mathrm{FBL}^{(p,\infty)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p,\infty)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a Banach lattice satisfying an **upper p-estimate** with constant 1,

Free Banach lattices with upper p-estimates

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over** E is a Banach lattice satisfying an **upper p-estimate** with constant 1, denoted by $\mathrm{FBL}^{(p,\infty)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p,\infty)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a Banach lattice satisfying an **upper p-estimate** with constant 1, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p,\infty)}[E] \to X$

Free Banach lattices with upper p-estimates

Definition (Jardón-Sánchez, Laustsen, Taylor, Tradacete, Troitsky, 2022)

Let E be a Banach space and $1 < p < \infty$. The **free Banach lattice with upper p-estimates over E** is a Banach lattice satisfying an **upper p-estimate** with constant 1, denoted by $\mathrm{FBL}^{(p,\infty)}[E]$, along with a linear isometrical embedding $\phi_E : E \to \mathrm{FBL}^{(p,\infty)}[E]$, satisfying that for every bounded linear operator $T : E \to X$, where X is a Banach lattice satisfying an **upper p-estimate** with constant 1, there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}^{(p,\infty)}[E] \to X$ such that $\hat{T} \circ \phi_E = T$, and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} \mathrm{FBL}^{(p,\infty)}[E] \arrow[r, "\hat{T}"] & X \\ E \arrow[u, "\phi_E"] \arrow[ur, "T"'] & \end{tikzcd}$$

Overview

Factorization Theorems

Theorem (Maurey's Factorization Theorem, 1974)

Let $1 < p < \infty$, $A \subseteq L_1(\mu)$ and $0 < M < \infty$. The following are equivalent.

1. For all finitely supported sequences $(\alpha_i)_{i\in I}$ of real numbers and $(f_i)_{i\in I} \subseteq A$,
$$\left\| \left(\sum |\alpha_i f_i|^p \right)^{\frac{1}{p}} \right\|_{L_1(\mu)} \leq M \left(\sum |\alpha_i|^p \right)^{\frac{1}{p}}.$$
2. There exists $g \in L_1(\mu)_+$, $\|g\|_{L^1(\mu)} = 1$, such that for any $f \in A$,
$$\left\| \frac{f}{g} \right\|_{L_p(g\cdot\mu)} \leq M.$$

Factorization Theorems

Theorem (Pisier's Factorization Theorem, 1986)

Let $1 < p < \infty$, $A \subseteq L_1(\mu)$, $0 < M < \infty$ and $\gamma_p = (1 - \frac{1}{p})^{\frac{1}{p}-1}$. Consider the following conditions.

1. For all finitely supported sequences $(\alpha_i)_{i \in I}$ of real numbers and $(f_i)_{i \in I} \subseteq A$,
$$\left\| \bigvee |\alpha_i f_i| \right\|_{L_1(\mu)} \leq M \left(\sum |\alpha_i|^p \right)^{\frac{1}{p}}.$$
2. There exists $g \in L^1(\mu)_+$, $\|g\|_{L^1(\mu)} \leq 1$, such that for any $f \in A$ and any μ-measurable set U,
$$\|f \chi_U\|_{L_1(\mu)} \leq \gamma_p M \left(\int_U g \, d\mu \right)^{1-\frac{1}{p}}.$$

Then (1) $\implies$ (2).

Factorization Theorems

In particular, if X is a p-**convex** Banach lattice (respectively, a Banach lattice with **upper** p-**estimates**) and $T : X \to L_1(\mu)$ is a **lattice homomorphism**, considering $A = T(B_X)$ we obtain that T **factors as** $T = M_g R$ **through** $L_p(g\mu)$ (respectively, $L_{p,\infty}(g\mu)$) for some $0 \leq g \in B_{L_1(\mu)}$, where $Rx = \frac{Tx}{g}$ and M_g is the multiplication by g operator.

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & L_1(\mu) \\ & L_p(g\mu) \arrow[ur, "M_g"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "R"'] & & L_1(\mu) \\ & L_{p,\infty}(g\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

Embeddings into ∞-sums

Proposition

The following statements hold:

1. Let X be a p-convex Banach lattice with constant M. There is a family Γ of probability measures and a lattice isomorphism
$$J : X \to \left(\oplus_{\mu\in\Gamma} L_p(\mu)\right)_\infty$$
such that $\|x\| \leq \|Jx\| \leq M\|x\|$ for all $x \in X$.
2. Let X be a Banach lattice with upper p-estimates with constant M. There is a family Γ of probability measures and a lattice isomorphism
$$J : X \to \left(\oplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$$
such that $\|x\| \leq \|Jx\| \leq \gamma_p M\|x\|$ for all $x \in X$.

Embeddings into ∞-sums

Sketch of the proof.

For every positive $x \in S_X$, find a normalized functional $x^* \in X_+^*$ such that $x^*(x) = 1$ and define the lattice seminorm $\rho_x(z) = x^*(|z|)$.

Embeddings into ∞-sums

Sketch of the proof.

For every positive $x \in S_X$, find a normalized functional $x^* \in X^*_+$ such that $x^*(x) = 1$ and define the lattice seminorm $\rho_x(z) = x^*(|z|)$. The completion of the quotient $X/\ker \rho_x$ under the norm ρ_x is an AL-space (i.e., a Banach lattice with an additive norm on positive elements), so it is lattice isometric to $L_1(\mu_x)$ for some probability measure μ_x (Kakutani's Representation Theorem). Denote by $T_x = i_x q_x$, where q_x denotes the quotient map and i_x the inclusion of the quotient space into $L_1(\mu)$.

Embeddings into ∞-sums

Sketch of the proof.

For every positive $x \in S_X$, find a normalized functional $x^* \in X_+^*$ such that $x^*(x) = 1$ and define the lattice seminorm $\rho_x(z) = x^*(|z|)$. The completion of the quotient $X/\ker \rho_x$ under the norm ρ_x is an AL-space (i.e., a Banach lattice with an additive norm on positive elements), so it is lattice isometric to $L_1(\mu_x)$ for some probability measure μ_x (Kakutani's Representation Theorem). Denote by $T_x = i_x q_x$, where q_x denotes the quotient map and i_x the inclusion of the quotient space into $L_1(\mu)$. Apply the corresponding Factorization Theorem to obtain $T_x = M_{g_x} R_x$ and a probability measure $\nu_x = g_x \mu_x$.

Embeddings into ∞-sums

Sketch of the proof.

For every positive $x \in S_X$, find a normalized functional $x^* \in X_+^*$ such that $x^*(x) = 1$ and define the lattice seminorm $\rho_x(z) = x^*(|z|)$. The completion of the quotient $X/\ker \rho_x$ under the norm ρ_x is an AL-space (i.e., a Banach lattice with an additive norm on positive elements), so it is lattice isometric to $L_1(\mu_x)$ for some probability measure μ_x (Kakutani's Representation Theorem). Denote by $T_x = i_x q_x$, where q_x denotes the quotient map and i_x the inclusion of the quotient space into $L_1(\mu)$. Apply the corresponding Factorization Theorem to obtain $T_x = M_{g_x} R_x$ and a probability measure $\nu_x = g_x \mu_x$. The operator

$$Jz = (R_x z)_{x \in (S_X)_+}$$

satisfies the properties of the statement. □

Embeddings into ∞-sums

The constant $\gamma_p > 1$ cannot be made 1 in the upper p-estimates case.

Embeddings into ∞-sums

The constant $\gamma_p > 1$ cannot be made 1 in the upper p-estimates case.

Lemma

There exists a Banach lattice X of dimension 3 such that X satisfies an upper p-estimate with constant 1 but X does not embed lattice isometrically into any space of the form $\left(\bigoplus_{\mu\in\Gamma} L_{p,\infty}(\mu)\right)_\infty$.

Overview

Free Banach lattices associated to a class of Banach lattices

Definition

Let $\mathcal{C}$ be a class of Banach lattices and E a Banach space. The **free Banach lattice associated to the class** $\mathcal{C}$ is the completion of $\mathrm{FVL}[E]$ under the following norm:

$$\rho_{\mathcal{C}}(f) = \sup\left\{ \|\widehat{T}f\|_X : X \in \mathcal{C},\ T : E \to X \text{ is a linear contraction} \right\},$$

where $f \in \mathrm{FVL}[E]$ and $\widehat{T} : \mathrm{FVL}[E] \to X$ denotes the unique extension of T to $\mathrm{FVL}[E]$ as a lattice homomorphism.

Free Banach lattices associated to a class of Banach lattices

Definition

Let $\mathcal{C}$ be a class of Banach lattices and E a Banach space. The **free Banach lattice associated to the class** $\mathcal{C}$ is the completion of $\mathrm{FVL}[E]$ under the following norm:

$$\rho_{\mathcal{C}}(f) = \sup \left\{ \| \widehat{T} f \|_X : X \in \mathcal{C},\ T : E \to X \text{ is a linear contraction} \right\},$$

where $f \in \mathrm{FVL}[E]$ and $\widehat{T} : \mathrm{FVL}[E] \to X$ denotes the unique extension of T to $\mathrm{FVL}[E]$ as a lattice homomorphism.

Given a class of Banach lattices $\mathcal{C}$, we denote by $\overline{\mathcal{C}}$ the enlarged class of all Banach lattices that embed lattice isometrically into a Banach lattice of the form $\left(\oplus_{\gamma\in\Gamma} X_\gamma\right)_\infty$, where the Banach lattices X_γ all belong to the class $\mathcal{C}$ (repetitions allowed).

By means of this new approach to free Banach lattices and the embedding result, we were able to provide an equivalent explicit norm for $\mathrm{FBL}^{(p,\infty)}[E]$, solving the question raised in [JLTTT] and later in [OTTT].

Free Banach lattices with upper p-estimates

By means of this new approach to free Banach lattices and the embedding result, we were able to provide an equivalent explicit norm for $\mathrm{FBL}^{(p,\infty)}[E]$, solving the question raised in [JLTTT] and later in [OTTT].

Theorem

Let E be a Banach space, $1 < p < \infty$ and denote by $\mathcal{C}_{p,\infty}$ the class of Banach lattices satisfying an upper p-estimate with constant 1, so that $\mathrm{FBL}^{\mathcal{C}_{p,\infty}}[E] = \mathrm{FBL}^{(p,\infty)}[E]$. Define

$$\|f\|_{p,\infty} = \sup\left\{ \|\,(f(x_k^*))_{k=1}^n\,\|_{\ell_{p,\infty}^n} : (x_k^*) \subset E^*, \sup_{x\in B_E} \|\,(x_k^*(x))_{k=1}^n\,\|_{\ell_{p,\infty}^n} \leq 1 \right\}.$$

Then

$$\|\cdot\|_{p,\infty} \leq \rho_{\mathcal{C}_{p,\infty}}(\cdot) \leq \gamma_p \|\cdot\|_{p,\infty},$$

where $\gamma_p = (1-\frac{1}{p})^{\frac{1}{p}-1}$.

Sketch of the proof.

The embedding result implies that $\rho_{\mathcal{C}_{p,\infty}}(\cdot)$ is equivalent to $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot)$ with constant γ_p, where $\mathcal{X}_{p,\infty}$ denotes the class of $L_{p,\infty}(\mu)$ spaces with μ a probability measure.

Free Banach lattices with upper p-estimates

Sketch of the proof.

The embedding result implies that $\rho_{\mathcal{C}_{p,\infty}}(\cdot)$ is equivalent to $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot)$ with constant γ_p, where $\mathcal{X}_{p,\infty}$ denotes the class of $L_{p,\infty}(\mu)$ spaces with μ a probability measure.

In general, $\rho_{\overline{\mathcal{C}}}(\cdot) = \rho_{\mathcal{C}}(\cdot)$ for every class $\mathcal{C}$, so $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot) = \rho_{\mathcal{X}_{p,\infty}}(\cdot)$.

Free Banach lattices with upper p-estimates

Sketch of the proof.

The embedding result implies that $\rho_{\mathcal{C}_{p,\infty}}(\cdot)$ is equivalent to $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot)$ with constant γ_p, where $\mathcal{X}_{p,\infty}$ denotes the class of $L_{p,\infty}(\mu)$ spaces with μ a probability measure.

In general, $\rho_{\overline{\mathcal{C}}}(\cdot) = \rho_{\mathcal{C}}(\cdot)$ for every class $\mathcal{C}$, so $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot) = \rho_{\mathcal{X}_{p,\infty}}(\cdot)$.

The norm $\rho_{\mathcal{X}_{p,\infty}}(\cdot)$ coincides with the supremum of the norms associated to the singleton classes $\left\{L_{p,\infty}(\mu)\right\}$ with μ non-atomic probability measure.

Sketch of the proof.

The embedding result implies that $\rho_{\mathcal{C}_{p,\infty}}(\cdot)$ is equivalent to $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot)$ with constant γ_p, where $\mathcal{X}_{p,\infty}$ denotes the class of $L_{p,\infty}(\mu)$ spaces with μ a probability measure.

In general, $\rho_{\overline{\mathcal{C}}}(\cdot) = \rho_{\mathcal{C}}(\cdot)$ for every class $\mathcal{C}$, so $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot) = \rho_{\mathcal{X}_{p,\infty}}(\cdot)$.

The norm $\rho_{\mathcal{X}_{p,\infty}}(\cdot)$ coincides with the supremum of the norms associated to the singleton classes $\{L_{p,\infty}(\mu)\}$ with μ non-atomic probability measure.

Each of these singleton norms satisfy $\rho_{\{L_{p,\infty}(\mu)\}}(\cdot) = \rho_{\mathcal{F}_\mu}(\cdot)$, where

$$\mathcal{F}_\mu = \left\{\operatorname{span}\{\chi_{V_i}\}_{i=1}^m : \{V_i\}_{i=1}^m \text{ pairwise disjoint, } \mu(V_i) = \frac{1}{r}, \frac{m}{r} \leq 1\right\}$$

Sketch of the proof.

The embedding result implies that $\rho_{\mathcal{C}_{p,\infty}}(\cdot)$ is equivalent to $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot)$ with constant γ_p, where $\mathcal{X}_{p,\infty}$ denotes the class of $L_{p,\infty}(\mu)$ spaces with μ a probability measure.

In general, $\rho_{\overline{\mathcal{C}}}(\cdot) = \rho_{\mathcal{C}}(\cdot)$ for every class $\mathcal{C}$, so $\rho_{\overline{\mathcal{X}}_{p,\infty}}(\cdot) = \rho_{\mathcal{X}_{p,\infty}}(\cdot)$.

The norm $\rho_{\mathcal{X}_{p,\infty}}(\cdot)$ coincides with the supremum of the norms associated to the singleton classes $\{L_{p,\infty}(\mu)\}$ with μ non-atomic probability measure.

Each of these singleton norms satisfy $\rho_{\{L_{p,\infty}(\mu)\}}(\cdot) = \rho_{\mathcal{F}_\mu}(\cdot)$, where

$$\mathcal{F}_\mu = \left\{ \mathrm{span}\, \{\chi_{V_i}\}_{i=1}^m : \{V_i\}_{i=1}^m \text{ pairwise disjoint, } \mu(V_i) = \frac{1}{r}, \frac{m}{r} \leq 1 \right\}$$

Finally, note that $\rho_{\mathcal{F}_\mu}(\cdot)$ coincides with $\|\cdot\|_{p,\infty}$ for every μ. □

Push outs

Theorem

Let $\mathcal{C}$ be the class of p-convex Banach lattices or Banach lattices with upper p-estimates (with constant 1). Given Banach lattices X_0, X_1, X_2 in $\mathcal{C}$ and lattice homomorphisms $T_i : X_0 \to X_i$ for $i = 1, 2$, there is a Banach lattice $PO^{\mathcal{C}}$ in $\mathcal{C}$ and lattice homomorphisms S_1, S_2 so that the following is an isometric push-out diagram in $\mathcal{C}$:

$$\begin{tikzcd} X_1 \arrow[r, "S_1"] & PO^{\mathcal{C}} \\ X_0 \arrow[u, "T_1"] \arrow[r, "T_2"'] & X_2 \arrow[u, "S_2"'] \end{tikzcd}$$

Push outs

Theorem

Let $\mathcal{C}$ be the class of p-convex Banach lattices or Banach lattices with upper p-estimates (with constant 1). Given Banach lattices X_0, X_1, X_2 in $\mathcal{C}$ and lattice homomorphisms $T_i : X_0 \to X_i$ for $i = 1, 2$, there is a Banach lattice $PO^{\mathcal{C}}$ in $\mathcal{C}$ and lattice homomorphisms S_1, S_2 so that the following is an isometric push-out diagram in $\mathcal{C}$:

$$\begin{tikzcd} X_1 \arrow[r, "S_1"] & PO^{\mathcal{C}} \\ X_0 \arrow[u, "T_1"] \arrow[r, "T_2"'] & X_2 \arrow[u, "S_2"'] \end{tikzcd}$$

Moreover, if $\|T_1\| \leq 1$ and T_2 is an isometric embedding, then S_1 is a $K_{\mathcal{C}}$-isomorphic embedding, with $K_{\mathcal{C}}$ depending only on p.

Subspace problem

Given an operator $T : F \to E$, denote by $\overline{T} = \widehat{\phi_E T}$:

$$\begin{CD} \mathrm{FBL}^{\mathcal{C}}[F] @>{\overline{T}}>> \mathrm{FBL}^{\mathcal{C}}[E] \\ @A{\phi_F}AA @AA{\phi_E}A \\ F @>>{T}> E \end{CD}$$

Subspace problem

Given an operator $T : F \to E$, denote by $\overline{T} = \widehat{\phi_E T}$:

$$\begin{tikzcd}
\mathrm{FBL}^{\mathcal{C}}[F] \arrow[r, "\overline{T}"] & \mathrm{FBL}^{\mathcal{C}}[E] \\
F \arrow[u, "\phi_F"] \arrow[r, "T"'] & E \arrow[u, "\phi_E"']
\end{tikzcd}$$

It is well known that

- $\overline{T}$ is injective if and only if T is injective,
- $\overline{T}$ has dense range if and only if T has dense range,
- $\overline{T}$ is onto if and only if T is onto.

Subspace problem

Problem (Subspace problem)

If T is an embedding, characterize when $\overline{T}$ is an embedding.

Subspace problem

Problem (Subspace problem)

If T is an embedding, characterize when $\overline{T}$ is an embedding.

Theorem (Oikhberg, Taylor, Tradacete, Troitsky)

Let $\iota : F \to E$ be an isometric embedding and $C > 0$. The following are equivalent.

1. $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a C-isomorphic lattice embedding.
2. Every $T : F \to L_p(\mu)$ (μ σ-finite) extends to $\widetilde{T} : E \to L_p(\mu)$ with $\|\widetilde{T}\| \leq C\|T\|$.
3. For every $n \in \mathbb{N}$ and $\epsilon > 0$, any $T : F \to \ell_p^n$ extends to $\widetilde{T} : E \to \ell_p^n$ with $\|\widetilde{T}\| \leq C(1+\epsilon)\|T\|$.

Subspace problem

Theorem

Suppose that $\iota : F \to E$ is an isometric embedding and let $\mathcal{C}$ be the class of p-convex Banach lattices or Banach lattices with upper p-estimates (with constant 1). The following are equivalent.

Subspace problem

Theorem

Suppose that $\iota : F \to E$ is an isometric embedding and let $\mathcal{C}$ be the class of p-convex Banach lattices or Banach lattices with upper p-estimates (with constant 1). The following are equivalent.

1. $\bar{\iota} : \mathrm{FBL}^{\mathcal{C}}[F] \to \mathrm{FBL}^{\mathcal{C}}[E]$ is a c_1-lattice embedding.

Subspace problem

Theorem

Suppose that $\iota : F \to E$ is an isometric embedding and let $\mathcal{C}$ be the class of p-convex Banach lattices or Banach lattices with upper p-estimates (with constant 1). The following are equivalent.

1. $\bar{\iota} : \mathrm{FBL}^{\mathcal{C}}[F] \to \mathrm{FBL}^{\mathcal{C}}[E]$ is a c_1-lattice embedding.
2. For every operator $T : F \to X$ with X in $\mathcal{C}$, there is Y in $\mathcal{C}$, a $K_{\mathcal{C}}$-lattice embedding $j : X \to Y$ and $S : E \to Y$ so that $jT = S\iota$ and $\|S\| \leq c_2\|T\|$.

$$\begin{tikzcd} X \arrow[r, "j"] & Y \\ F \arrow[u, "T"] \arrow[r, "\iota"'] & E \arrow[u, "S"'] \end{tikzcd}$$

Here, $c_2 \leq c_1 \leq K_{\mathcal{C}} c_2$.

Subspace problem

The fact that ℓ_p is injective in the class of p-convex Banach lattices

Subspace problem

The fact that ℓ_p is injective in the class of p-convex Banach lattices (that is, every sublattice of a p-convex Banach lattice that is lattice isomorphic to ℓ_p is complemented with a positive projection) allows to recover the statement from [OTTT] (up to constants).

Subspace problem

The fact that ℓ_p is injective in the class of p-convex Banach lattices (that is, every sublattice of a p-convex Banach lattice that is lattice isomorphic to ℓ_p is complemented with a positive projection) allows to recover the statement from [OTTT] (up to constants).

However, there seems to be a fundamental obstruction that prevents to obtain a similar statement for the upper p-estimates case:

Subspace problem

The fact that ℓ_p is injective in the class of p-convex Banach lattices (that is, every sublattice of a p-convex Banach lattice that is lattice isomorphic to ℓ_p is complemented with a positive projection) allows to recover the statement from [OTTT] (up to constants).

However, there seems to be a fundamental obstruction that prevents to obtain a similar statement for the upper p-estimates case:

Theorem

For any $1 < p < \infty$ there is a Banach lattice X with an upper p-estimate (with constant 1) and an uncomplemented closed sublattice Y of X that is lattice isomorphic to $\ell^{p,\infty}$.

Main references

- A. Avilés, J. Rodríguez, P. Tradacete, *The free Banach lattice generated by a Banach space*. J. Funct. Anal. **274** (10) (2018), 2955–2977.
- A. Avilés, P. Tradacete, *Amalgamation and injectivity in Banach lattices*. Int. Math. Res. Not. IMRN (2) (2023), 956–997.
- E. García-Sánchez, D. H. Leung, M. A. Taylor, P. Tradacete, *Banach lattices with upper p-estimates: free and injective objects*. Preprint (2024).
- H. Jardón-Sánchez, N. J. Laustsen, M. A. Taylor, P. Tradacete, V. G. Troitsky, *Free Banach lattices under convexity conditions*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (15) (2022).

Main references

B. Maurey, *Théorèmes de factorisation pour les opérateurs à valeurs dans les espaces L_p*. Astérisque 11 (1974), 1–150.

T. Oikhberg, M. A. Taylor, P. Tradacete, V. G. Troitsky, *Free Banach lattices*. Preprint (2022).

G. Pisier, *Factorization of operators through $L_{p\infty}$ or L_{p1} and non-commutative generalizations*. Math. Ann. 276 (1986), 105–136.

Thank you for your attention!