Guarded Fraïssé Banach spaces

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Motivation 1

Mazur rotation problem

Let X be a Banach space and let $\mathrm{Iso}(X)$ denote the group of all linear isometries of X. Observe that $\mathrm{Iso}(X)$ acts on $S_X := \{x \in X\colon \|x\| = 1\}$.

If X is a Hilbert space then the action $\mathrm{Iso}(X) \curvearrowright S_X$ is transitive, i.e. for any $x, y \in S_X$ there exists $\phi \in \mathrm{Iso}(X)$ such that $\phi(x) = y$.

Problem. If X is a Banach space such that the action $\mathrm{Iso}(X) \curvearrowright S_X$ is transitive, must X be (isometric to) a Hilbert space?

Remarks

If X is finite-dimensional, then the answer is yes. If X is allowed to be non-separable, then the answer is no.

It remains open in case X is separable and infinite-dimensional.

Homogeneity of Hilbert spaces

By the Hilbert space, denoted by ℓ_2, we shall always mean the separable infinite-dimensional one.

For Banach spaces E and X, denote by $\mathrm{Emb}(E, X)$ the space of all isometric embeddings of E into X. Moreover, for $\delta > 0$, denote by $\mathrm{Emb}_\delta(E, X)$ the space of all embeddings $\iota : E \to X$ satisfying

$$e^{-\delta}\|x\| \leq \|\iota(x)\| \leq e^{\delta}\|x\|, \quad x \in E.$$

Higher homogeneity

Let $E \subseteq \ell_2$ be finite-dimensional. Then the action $\mathrm{Iso}(\ell_2) \curvearrowright \mathrm{Emb}(E, \ell_2)$ is transitive.

The Hilbert space enjoys even a property where one both relaxes and strengthens the condition above: For every $k \in \mathbb{N}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that the for every k-dimensional $E \subseteq \ell_2$ the action $\mathrm{Iso}(\ell_2) \curvearrowright \mathrm{Emb}_\delta(E, \ell_2)$ is ε-transitive, i.e. for $\iota, \iota' \in \mathrm{Emb}_\delta(E, \ell_2)$ there is $\phi \in \mathrm{Iso}(\ell_2)$ such that $\|\phi \circ \iota - \iota'\| < \varepsilon$.

Fraïssé Banach spaces

Definition (Ferenczi, Lopez-Abad, Mbombo, Todorčević)

A Banach space X is *Fraïssé* if for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for every k-dimensional $E \subseteq X$ the action $\mathrm{Iso}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ is ε-transitive.

Examples

- The Hilbert space.
- $L_p[0,1]$ for $1 \leq p < \infty$ such that $p \neq 4, 6, 8, \ldots$.
- the Gurarii space.

Question (FLMT)

Are there any other separable infinite-dimensional Fraïssé Banach spaces?

Motivation 2 - descriptive set theory

A descriptive set theoretic study of the class of separable Banach spaces is usually performed via the standard Borel space of separable Banach spaces.

Recall that a standard Borel space is a set with a σ-algebra that is isomorphic to an algebra of Borel sets of $[0,1]$. Let X be some universal separable Banach space, e.g. $C(2^{\mathbb{N}})$. The set $F(X)$ of all closed subsets of X can be canonically equipped with a Borel structure making it a standard Borel space (however, no canonical Polish topology), and the subset $S(X) \subseteq F(X)$ consisting of linear subspaces is standard Borel as well and can serve as the standard Borel space of separable Banach spaces.

Polish spaces of separable Banach spaces

Recall that *Polish space* is a completely metrizable separable topological space (i.e. every separable Banach space is a Polish space). Every G_δ subset, in particular every closed subset, of a Polish space is Polish space itself. Also, a countable direct product of Polish spaces is a Polish space

Polish spaces of separable Banach spaces

Denote by V the unique countable infinite-dimensional vector space over $\mathbb{Q}$. By a pseudonorm (or seminorm) we mean a non-negative valued function on a vector space that satisfies all the axioms of a norm except that it may vanish on non-zero elements. Denote by $\mathcal{P}$ the set of all pseudonorms on V. We can identify $\mathcal{P}$ with a closed subset of $\mathbb{R}^V$, thus by the remark above, it is a Polish space.

Then we also define

- the space $\mathcal{P}_\infty \subseteq \mathcal{P}$ of pseudonorms λ such that the completion $\overline{(V, \lambda)}$ is infinite-dimensional;
- the space $\mathcal{B} \subseteq \mathcal{P}_\infty$ of norms $\lambda \in \mathcal{P}_\infty$ such that the extension of λ to the completion $\overline{(V, \lambda)}$ remains a norm on the completion.

Both $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ subspaces of $\mathcal{P}$, therefore they are Polish spaces.

Polish spaces of separable Banach spaces

Theorem (Cúth, Doležal, D., Kurka (2023))

- ℓ_2 is characterized as
  - the unique separable Banach space (up to isometry) whose isometry class is closed in $\mathcal{P}_\infty$ and $\mathcal{B}$;
  - the unique separable Banach space (up to isomorphism) whose isomorphism class is F_σ in $\mathcal{P}_\infty$ and $\mathcal{B}$.
- The isometry classes of the Gurarij space and of $L_p([0,1])$, for $p \in [1,\infty) \setminus \{2\}$, are G_δ in $\mathcal{P}_\infty$ and $\mathcal{B}$ (and not F_σ).
- The isometry classes of ℓ_p, for $p \in [1,\infty) \setminus \{2\}$, are $F_{\sigma\delta}$ in $\mathcal{P}_\infty$ and $\mathcal{B}$ (and not 'simpler').

Remark

We don't know any example of a Banach space whose isometry class is F_σ (except ℓ_2 whose class is closed).

Polish spaces of separable Banach spaces

Remark

Spaces that have G_δ isometry class in $\mathcal{B}$ also have it in $\mathcal{P}$. It turns out that every finite-dimensional Banach space has a G_δ isometry class in $\mathcal{P}$.

Question

Are there other separable Banach spaces with G_δ isometry classes?

Banach spaces with G_δ classes are generic

Proposition (CDDK)

Let X be a separable infinite-dimensional Banach space and let $[X]$ be its isometry class in $\mathcal{B}$ (i.e. the set of norms defining isometrically X). Then the closure $\overline{[X]} \subseteq \mathcal{B}$ is the set of all separable infinite-dimensional Banach spaces (or rather norms defining them) that are finitely-representable in X.

In particular, if X is a separable infinite-dimensional Banach space with a G_δ isometry class, then in $\overline{[X]}$, the space of Banach spaces finitely-representable in X, X is dense G_δ - that is, generic.

Definition

Let X be a separable Banach space. We call it *guarded Fraïssé* if

for all finite-dimensional $E \subseteq X$ and all $\varepsilon > 0$ there are

a finite-dimensional $E \subseteq F \subseteq X$ and $\delta > 0$ such that

$\mathrm{Iso}(X) \curvearrowright \{\iota \upharpoonright E \colon \iota \in \mathrm{Emb}_\delta(F, X)\}$ is ε-transitive.

Theorem

Let X be a (infinite-dimensional) separable Banach space. TFAE:

1. X is guarded Fraïssé.
2. The isometry class of X is G_δ.
3. The isometry class of X is comeager in the subspace $\overline{[X]}$ of $\mathcal{B}$ consisting of spaces finitely representable in X.
4. X admits game-theoretic characterization presented on the blackboard.

Corollary

If X and Y are separable guarded Fraïssé Banach spaces finitely representable in each other, then they are isometric.

Proof. We have $\overline{[X]} = \overline{[Y]}$, so $[X]$ and $[Y]$ are two dense G_δ subsets of $\overline{[X]}$, which must intersect by the Baire category theorem which implies the isometry. □

Connections with ω-categorical Banach spaces

Connections between guarded Fraïssé Banach spaces and ω-categorical Banach spaces, and how to construct the former from the latter, were presented on blackboard during the talk. The last next slide presents how to apply such ideas to spaces of the form $L_p(L_q)$ and how to prove the 'hard part' of the following theorem.

Theorem

Let $1 \leq p \neq q < \infty$. Then $L_p(L_q)$ is guarded Fraïssé if and only if $q \neq 2$ and $q \notin (p, 2)$.

Guarded Fraïssé $L_p(L_q)$

Theorem (CdRD)

Let X be an ω-categorical separable Banach space and let $\mathcal{F}$ be a family of finite-dimensional subspaces of X satisfying that

1. for any finite-dimensional $E \subset X$ and any $\varepsilon > 0$ there is $F \in \mathcal{F}$ and $\phi \in \mathrm{Emb}_\varepsilon(E, F)$ satisfying $\|\mathrm{Id}_E - \phi\| < \varepsilon$, and
2. $\forall F \in \mathcal{F}, \delta > 0$, $\mathrm{Iso}(X)$ acts δ-transitively on $\mathrm{Emb}(F, X)$.

Then X is guarded Fraïssé.

Theorem (Tursi, 2023)

Let $1 \leq p \neq q < \infty$. For every $E = \ell_p^n(\ell_q^m)$ and every $\varepsilon > 0$ the action $\mathrm{LIso}(L_p(L_q)) \curvearrowright \mathrm{LEmb}(E, L_p(L_q))$ is ε-transitive.

Theorem (Raynaud, 2018)

Suppose that $1 \leq p \neq q < \infty$, $q \neq 2$ with $q \notin [p, 2]$ are given. Then every linear isometry $U : \ell_p^n(\ell_q^m) \to L_p(L_q)$ with $n, m \in \mathbb{N}$, $m \geq 3$ preserves disjointness of supports.